Lie Group Analysis of a Nonlinear Coupled System of Korteweg-de Vries Equations

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14 May 2020

Submitted in partial fulfillment of a structured masters degree at AIMS South Africa

Abstract

Lie group analysis of nonlinear partial differential equations (NLPDEs) makes use of symmetries of the NLPDE to find solutions and conservation laws. The solutions describe the state of the system at all times and conservation laws give its invariant properties. NLPDEs have applications in many settings. However, finding exact solutions to such NLPDEs is a challenge because of the increased complexity since NLPDEs have many independent variables. In this dissertation, we propose to research on Lie group analysis of a nonlinear coupled system of Korteweg de-Vries (KdV) equations. To achieve this, we first introduce the mathematical concepts of Lie group analysis and conservation laws. As an illustrative example, we perform a Lie group analysis of the KdV equation. This means we find Lie point symmetries and compute invariant solutions. Conservation laws for mass, momentum, and energy are also constructed for the KdV equation by the method of multipliers. Next, we focus on Lie group analysis of a nonlinear system of coupled KdV equations. We find symmetries of the coupled system and use them for symmetry reductions. We also compute invariant solutions for the coupled KdV equations. Furthermore, conservation laws are constructed by using multipliers and a theorem proposed by Nail Ibragimov. The target application for this dissertation is in systems that can be described by a KdV model. Particularly systems behaving like solitons which are traveling waves that do not change shape after a pair-wise collision.

Keywords: coupled KdV equations; Lie group analysis; symmetry reductions; soliton; multipliers; conservation laws.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Joseph Owuor Owino, 14 May 2020

Contents

1. Introduction

Nonlinear partial differential equations (NLPDEs) are used in the formulation of many natural laws such as in fluid dynamics. NLPDEs have been very key to technological developments as we look for numerical approximations and simulations using computers. To be able to this, we must be able to obtain approximations to solutions or exact closed-form solutions. In order to understand NLPDEs and the systems they represent better, several methods have been proposed and are still being used to describe the state of NLPDE systems. Lie group analysis is one of those methods.

Lie group analysis was inspired was the work Sophus Lie (Lie, 1891) about applying Galois theory of algebraic equations to differential equations (DEs). He showed that Lie group-invariant DEs could be reduced to a simpler form that is easily solvable. Furthermore, group-invariant solutions of the equation could also be found. There are many books written on this subject such as (Ovsyannikov, 2013), (Olver, 1993), (Bluman and Kumei, 1989), (Ibragimov, 1999), (Ibragimov, 2009) and the method has been recently extensively used in solving NLPDEs.

Conservation laws, by which we mean a divergence expression vanishing on all the solutions of a system and describes invariant properties of the system, are very important in the understanding of real physical phenomena. They have applications in various mathematical sciences disciplines. Conservation laws have been used to explain the integrability of DEs and the effectiveness of numerical methods (LeVeque, 1992). Recently, conservation laws have been used to construct exact closed-form solutions of certain partial differential equations (PDEs). This needs a good understanding of conservation laws.

Emmy Noether (Noether, 1918) found a connection between conservation laws and symmetries for DEs that have a Lagrangian which is a function that describes the dynamics of the system. For those DEs that do not have a Lagrangian, methods which do not rely on the variational principle have been developed. Some of these methods include the multipliers approach and a conservation theorem proposed by Nail Ibragimov (Ibragimov, 2007).

To contribute to Lie group analysis of NLPDEs, we shall in this dissertation investigate an application to a nonlinear coupled system of Korteweg-de Vries (KdV) equations. The KdV equation describes the dynamics of solitons, and ion-acoustic waves in plamsas. We shall perform a Lie group analysis of the system, that is, finding Lie point symmetries as well as group-invariant solutions and conservation laws.

The application addressed in this dissertation is an understanding of a typical KdV equation which has uses in many areas of mathematical sciences. Some of the many applications include describing the dynamics of shallow-water waves, ion-acoustic waves in plasmas, and long internal waves in oceans.

To reach our goal of contributing to Lie group analysis of PDEs, we shall in Chapter 2 develop mathematical concepts of Lie group analysis and present some algorithms for computing conservation laws. In Chapter 3, we will present an illustrative example of Lie group analysis of the KdV equation. That means we find Lie point symmetries, group-invariant solutions including a one-soliton solution, and compute conservation laws using multipliers. Next, we will in Chapter 4 investigate a system of coupled KdV equations. We find Lie point symmetries of the coupled KdV system and use them for symmetry reductions. We also compute invariant solutions. Moreover, we find conservation laws for the nonlinear coupled system of KdV equations by using multipliers and a theorem proposed by Nail Ibragimov. We conclude the work in Chapter 5 and suggest further work.

2. Preliminaries

This chapter serves as a brief introduction of fundamental concepts on Lie group analysis (Ibragimov, 1999) and conservation laws Ibragimov (2007). The first section of the chapter reviews the notions of local Lie groups and infinitesimal transformations, prolongations of Lie groups, group invariants, and symmetry groups. In the second section, we provide an introduction to conservation laws as well as the method of multipliers and Ibragimov's theorem for computing conservation laws. The notations fixed here will be used in the rest of the dissertation.

2.1 Lie group analysis

This section introduces the basic concepts of the Lie group theory of PDEs and how one can obtain symmetries.

2.1.1 Local Lie groups.

The setting is in Euclidean spaces $\mathbb{R}^n$ of $x = x^i$ independent variables and $\mathbb{R}^m$ of $u = u^\alpha$ dependent variables. Consider the transformations

$$T_\epsilon : \quad \bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \tag{2.1.1}$$

involving the continuous parameter ϵ which ranges from a neighbourhood $\mathcal{N}' \subset \mathcal{N} \subset \mathbb{R}$ of $\epsilon = 0$ where the functions φ^i and ψ^α differentiable and analytic in the parameter ϵ.

Definition 2.1.2. The set $\mathcal{G}$ of transformations given by (2.1.1) is a local Lie group if it holds true that

(i) (Closure) Given $T_{\epsilon_1}, T_{\epsilon_2} \in \mathcal{G}$, for $\epsilon_1, \epsilon_2 \in \mathcal{N}' \subset \mathcal{N}$, then $T_{\epsilon_1}T_{\epsilon_2} = T_{\epsilon_3} \in \mathcal{G},\ \epsilon_3 = \phi(\epsilon_1, \epsilon_2) \in \mathcal{N}$.

(ii) (Identity) There exists a unique $T_0 \in \mathcal{G}$ if and only if $\epsilon = 0$ such that $T_\epsilon T_0 = T_0 T_\epsilon = T_\epsilon$.

(iii) (Inverse) There exists a unique $T_{\epsilon^{-1}} \in \mathcal{G}$ for every transformation $T_\epsilon \in \mathcal{G}$, where $\epsilon \in \mathcal{N}' \subset \mathcal{N}$ and $\epsilon^{-1} \in \mathcal{N}$ such that $T_\epsilon T_{\epsilon^{-1}} = T_{\epsilon^{-1}} T_\epsilon = T_0$.

Remark 2.1.3. Associativity property of the group $\mathcal{G}$ defined in (2.1.1) follows from (i).

2.1.4 Prolongations.

Since we shall be dealing with PDEs, we wish to describe how the dependant variables u^α transform. Consider the system PDEs

$$\Delta_\alpha \left(x^i, u^\alpha, u_{(1)}, \ldots, u_{(\pi)}\right) = \Delta_\alpha = 0. \tag{2.1.2}$$

The partial derivatives $u_{(1)} = \{u_i^\alpha\}, u_{(2)} = \{u_{ij}^\alpha\}, \ldots, u_{(\pi)} = \{u_{i_1 \ldots i_\pi}^\alpha\}$ are of the first, second, ..., up to the πth-orders. Denoting

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \tag{2.1.3}$$

the total differentiation operator with respect to the variables x^i and δ_i^j, the Kronecker delta, we have

$$D_i(x^j) = \delta_i^j, \quad u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j(D_i(u^\alpha)), \ldots, \tag{2.1.4}$$

where variables u_i^α defined in (2.1.4) are named as differential variables (Ibragimov, 1999). A discussion of prolonged groups and prolonged generator follows.

(a) **Prolonged groups**

Consider the local Lie group $\mathcal{G}$ given by the transformations

$$\bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \varphi^i\Big|_{\epsilon=0} = x^i, \quad \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \tag{2.1.5}$$

where the symbol $\Big|_{\epsilon=0}$ means evaluated on $\epsilon = 0$.

Definition 2.1.5. The construction of the group $\mathcal{G}$ given by (2.1.5) is an equivalence of the computation of infinitesimal transformations

$$\bar{x}^i \approx x^i + \xi^i(x^i, u^\alpha)\epsilon, \quad \varphi^i\Big|_{\epsilon=0} = x^i, \quad \bar{u}^\alpha \approx u^\alpha + \eta^\alpha(x^i, u^\alpha)\epsilon, \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \tag{2.1.6}$$

obtained from (2.1.1) by a Taylor series expansion of $\varphi^i(x^i, u^\alpha, \epsilon)$ and $\psi^i(x^i, u^\alpha, \epsilon)$ in ϵ about $\epsilon = 0$ and keeping only the terms linear in ϵ, where

$$\xi^i(x^i, u^\alpha) = \frac{\partial \varphi^i(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}, \quad \eta^\alpha(x^i, u^\alpha) = \frac{\partial \psi^\alpha(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}. \tag{2.1.7}$$

Remark 2.1.6. The symbol of infinitesimal transformations, X, is used to write (2.1.6) as

$$\bar{x}^i \approx (1+X)x^i, \quad \bar{u}^\alpha \approx (1+X)u^\alpha, \tag{2.1.8}$$

where

$$X = \xi^i(x^i, u^\alpha)\frac{\partial}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial}{\partial u^\alpha}, \tag{2.1.9}$$

is the generator of the group $\mathcal{G}$ given by (2.1.5).

Remark 2.1.7. To obtain transformed derivatives from (2.1.1), we use a change of variable formulae

$$D_i = D_i(\varphi^j)\bar{D}_j, \tag{2.1.10}$$

where $\bar{D}_j$ is the total differentiation in the variables $\bar{x}^i$. This means that

$$\bar{u}_i^\alpha = \bar{D}_i(\bar{u}^\alpha), \quad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha). \tag{2.1.11}$$

If we apply the change of variable formula given in (2.1.10) on $\mathcal{G}$ given by (2.1.5), we get

$$D_i(\psi^\alpha) = D_i(\varphi^j)\bar{D}_j(\bar{u}^\alpha) = \bar{u}_j^\alpha D_i(\varphi^j). \tag{2.1.12}$$

Expansion of (2.1.12) yields

$$\left(\frac{\partial \varphi^j}{\partial x^i} + u_i^\beta \frac{\partial \varphi^j}{\partial u^\beta}\right)\bar{u}_j^\beta = \frac{\partial \psi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \psi^\alpha}{\partial u^\beta}. \tag{2.1.13}$$

The variables $\bar{u}_i^\alpha$ can be written as functions of $x^i, u^\alpha, u_{(1)}$, that is

$$\bar{u}_i^\alpha = \Phi^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon), \quad \Phi^\alpha\Big|_{\epsilon=0} = u_i^\alpha. \tag{2.1.14}$$

Definition 2.1.8. The transformations in the space of the variables $x^i, u^\alpha, u_{(1)}$ given in (2.1.5) and (2.1.14) form the first prolongation group $\mathcal{G}^{[1]}$.

Definition 2.1.9. Infinitesimal transformation of the first derivatives is

$$\bar{u}_i^\alpha \approx u_i^\alpha + \zeta_i^\alpha \epsilon, \tag{2.1.15}$$

where $\zeta_i^\alpha = \zeta_i^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon)$.

Remark 2.1.10. In terms of infinitesimal transformations, the first prolongation group $\mathcal{G}^{[1]}$ is given by (2.1.6) and (2.1.15).

(b) **Prolonged generators**
A description of how to extend a group generator is given.

Definition 2.1.11. By using the relation given in (2.1.12) on the first prolongation group $\mathcal{G}^{[1]}$ given by Definition 2.1.8, we obtain (Ibragimov, 2009)

$$D_i(x^j + \xi^j\epsilon)(u_j^\alpha + \zeta_j^\alpha\epsilon) = D_i(u^\alpha + \eta^\alpha\epsilon), \tag{2.1.16}$$

which gives

$$u_i^\alpha + \zeta_j^\alpha\epsilon + u_j^\alpha\epsilon D_i\xi^j = u_i^\alpha + D_i\eta^\alpha\epsilon, \tag{2.1.17}$$

and thus

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \tag{2.1.18}$$

is the first prolongation formula.

Remark 2.1.12. Similarly, we get higher order prolongations (Ibragimov, 1999),

$$\begin{aligned} \zeta_{ij}^\alpha &= D_j(\zeta_i^\alpha) - u_{i\kappa}^\alpha D_j(\xi^\kappa), \\ &\vdots \\ \zeta_{i_1,\dots,i_\kappa}^\alpha &= D_{i_\kappa}(\zeta_{i_1,\dots,i_{\kappa-1}}^\alpha) - u_{i_1,i_2,\dots,i_{\kappa-1}j}^\alpha D_{i_\kappa}(\xi^j). \end{aligned} \tag{2.1.19}$$

Remark 2.1.13. The prolonged generators of the prolongations $\mathcal{G}^{[1]}, \dots, \mathcal{G}^{[\kappa]}$ of the group $\mathcal{G}$ are

$$\begin{aligned} X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \\ &\vdots \\ X^{[\kappa]} &= X^{[\kappa-1]} + \zeta_{i_1,\dots,i_\kappa}^\alpha \frac{\partial}{\partial \zeta_{i_1,\dots,i_\kappa}^\alpha}, \ \kappa \geq 1, \end{aligned} \tag{2.1.20}$$

where X is the group generator given by (2.1.9).

2.1.14 Group invariants.

This subsection presents some basic definitions and a theorem that are very important in Lie group analysis.

Definition 2.1.15. A function $\Gamma(x^i, u^\alpha)$ is called an invariant of the group $\mathcal{G}$ of transformations given by (2.1.1) if

$$\Gamma(\bar{x}^i, \bar{u}^\alpha) = \Gamma(x^i, u^\alpha). \tag{2.1.21}$$

Theorem 2.1.16. *A function $\Gamma(x^i, u^\alpha)$ is an invariant of the group $\mathcal{G}$ given by (2.1.1) if and only if it solves the following first-order linear PDE: (Ibragimov, 1999)*

$$X\Gamma = \xi^i(x^i, u^\alpha)\frac{\partial \Gamma}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial \Gamma}{\partial u^\alpha} = 0. \tag{2.1.22}$$

From Theorem (2.1.16), we have the following result.

Theorem 2.1.17. *The local Lie group $\mathcal{G}$ of transformations in $\mathbb{R}^n$ given by (2.1.1) (Ibragimov, 2009) has precisely $n-1$ functionally independent invariants. One can take, as the basic invariants, the left-hand sides of the first integrals*

$$\psi_1(x^i, u^\alpha) = c_1, \ldots, \psi_{n-1}(x^i, u^\alpha) = c_{n-1}, \tag{2.1.23}$$

of the characteristic equations for (2.1.22):

$$\frac{\mathrm{d}x^i}{\xi^i(x^i, u^\alpha)} = \frac{\mathrm{d}u^\alpha}{\eta^\alpha(x^i, u^\alpha)}. \tag{2.1.24}$$

2.1.18 Symmetry groups .

This subsection presents some vital definitions on Lie groups of PDEs.

Definition 2.1.19. The vector field X (2.1.9) is a Lie point symmetry of the PDE system given in (2.1.2) if the determining equations

$$X^{[\pi]}\Delta_\alpha\Big|_{\Delta_\alpha=0} = 0, \quad \alpha = 1, \ldots, m, \quad \pi \geq 1, \tag{2.1.25}$$

are satisfied, where $\Big|_{\Delta_\alpha=0}$ means evaluated on $\Delta_\alpha = 0$ and $X^{[\pi]}$ is the π-th prolongation of X.

Definition 2.1.20. The Lie group $\mathcal{G}$ is a symmetry group of the PDE system given in (2.1.2) if the PDE system (2.1.2) is form-invariant, that is

$$\Delta_\alpha\left(\bar{x}^i, \bar{u}^\alpha, \bar{u}_{(1)}, \ldots, \bar{u}_{(\pi)}\right) = 0. \tag{2.1.26}$$

Theorem 2.1.21. *Given the infinitesimal transformations in (2.1.5), the Lie group $\mathcal{G}$ in (2.1.1) is found by integrating the Lie equations*

$$\frac{\mathrm{d}\bar{x}^i}{\mathrm{d}\epsilon} = \xi^i(\bar{x}^i, \bar{u}^\alpha), \quad \bar{x}^i\Big|_{\epsilon=0} = x^i, \quad \frac{\mathrm{d}\bar{u}^\alpha}{\mathrm{d}\epsilon} = \eta^\alpha(\bar{x}^i, \bar{u}^\alpha), \quad \bar{u}^\alpha\Big|_{\epsilon=0} = u^\alpha. \tag{2.1.27}$$

2.1.22 Lie algebras.
We use this subsection to define a Lie algebra of operators.

Definition 2.1.23. A vector space $\mathcal{V}_r$ of operators (Ibragimov, 1999) X (2.1.9) is a Lie algebra if for any two operators, $X_i, X_j \in \mathcal{V}_r$, their commutator

$$[X_i, X_j] = X_iX_j - X_jX_i, \tag{2.1.28}$$

is in $\mathcal{V}_r$ for all $i, j = 1, \ldots, r$.

Remark 2.1.24. The commutator satisfies the properties of bilinearity, skew symmetry and the Jacobi identity (Ibragimov, 1999).

Theorem 2.1.25. *The set of solutions of the determining equation given by (2.1.25) forms a Lie algebra(Ibragimov, 1999).*

2.2 Conservation laws

This section serves as an introduction to conservation laws and the algorithms used to find them, that is, the method of multipliers and Ibragimov's conservation theorem. We will use these methods in the next part of the dissertation.

2.2.1 Fundamental operators.
Let a system of πth-order PDEs be given by (2.1.2).

Definition 2.2.2. The Euler-Lagrange operator $\delta/\delta u^\alpha$ is

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{\kappa\geq 1}(-1)^\kappa D_{i_1}, \ldots, D_{i_\kappa}\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{2.2.1}$$

and the Lie- Bäcklund operator in abbreviated form (Ibragimov, 1999) is

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \ldots. \tag{2.2.2}$$

Remark 2.2.3. The Lie- Bäcklund operator (2.2.2) in its prolonged form is

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \sum_{\kappa\geq 1}\zeta_{i_1\ldots i_\kappa}\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{2.2.3}$$

where

$$\zeta^\alpha_i = D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \quad j = 1, \ldots, n, \tag{2.2.4}$$

$$\vdots \tag{2.2.5}$$

$$\zeta^\alpha_{i_1\ldots i_\kappa} = D_{i_1\ldots i_\kappa}(W^\alpha) + \xi^j u^\alpha_{ji_1\ldots i_\kappa}, \tag{2.2.6}$$

and the Lie characteristic function is

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{2.2.7}$$

Remark 2.2.4. The characteristic form of Lie- Bäcklund operator (2.2.3) is

$$X = \xi^i D_i + W^\alpha\frac{\partial}{\partial u^\alpha} + D_{i_1\ldots i_\kappa}(W^\alpha)\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}. \tag{2.2.8}$$

2.2.5 The method of multipliers.

Definition 2.2.6. A function $\Lambda^\alpha\left(x^i, u^\alpha, u_{(1)}, \dots\right) = \Lambda^\alpha$, is a multiplier of the PDE system given by (2.1.2) if it satisfies the condition that (Olver, 1993)

$$\Lambda^\alpha \Delta_\alpha = D_i T^i, \tag{2.2.9}$$

where $D_i T^i$ is a divergence expression.

Definition 2.2.7. To find the multipliers Λ^α, one solves the determining equations (2.2.10) (Bluman and Anco, 2008),

$$\frac{\delta}{\delta u^\alpha}(\Lambda^\alpha \Delta_\alpha) = 0. \tag{2.2.10}$$

2.2.8 Ibragimov's conservation theorem .

The technique (Ibragimov, 2007) enables one to construct conserved vectors associated with each Lie point symmetry of the PDE system given by (2.1.2).

Definition 2.2.9. The adjoint equations of the system given by (2.1.2) are

$$\Delta^*_\alpha\left(x^i, u^\alpha, v^\alpha, \dots, u_{(\pi)}, v_{(\pi)}\right) \equiv \frac{\delta}{\delta u^\alpha}(v^\beta \Delta_\beta) = 0, \tag{2.2.11}$$

where v^α is the new dependent variable.

Definition 2.2.10. Formal Lagrangian $\mathcal{L}$ of the system (2.1.2) and its adjoint equations (2.2.11) is (Ibragimov, 2007)

$$\mathcal{L} = v^\alpha \Delta_\alpha(x^i, u^\alpha, u_{(1)}, \dots, u_{(\pi)}). \tag{2.2.12}$$

Theorem 2.2.11. *Every infinitesimal symmetry X of the system given by (2.1.2) leads to conservation laws (Ibragimov, 2007)*

$$D_i T^i\Big|_{\Delta_\alpha = 0} = 0, \tag{2.2.13}$$

where the conserved vector

$$\begin{aligned} T^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u^\alpha_i} - D_j\left(\frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}}\right) + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}}\right) - \dots\right] + \\ D_j(W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} - D_k\left(\frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}}\right) + \dots\right] + D_j D_k(W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} - \dots\right]. \end{aligned} \tag{2.2.14}$$

2.3 Concluding remarks

This chapter has introduced fundamental concepts of Lie group analysis of PDEs and conservation laws. We have defined local Lie groups, their generators, and infinitesimal transformations. We have also shown how one can obtain prolongations to a generator and prolonged groups. This is very key when solving determining equations for symmetries of a PDE. The ideas of group-invariants and characteristic equations will help to find group-invariant solutions. Also important to mention is the methods of obtaining conservation laws of PDEs. For this dissertation, we have considered only the method of multipliers and Ibragimov's conservation theorem.

3. Korteweg-de Vries (KdV) Equation

We introduced the concepts of Lie group analysis and conservation laws in Chapter 2, which we will rely on in this chapter. Our aim is to provide an illustrative example of Lie group analysis of a nonlinear partial differential equation (NLPDE). We find Lie point symmetries of the KdV equation and use them for symmetry reductions and construction of invariant solutions. In addition, we use the multiplier method to get its conservation laws.

3.1 Introduction

A Scottish civil engineer, John Scott Russell is credited with first work on solitons (Wazwaz, 2010). Russell observed that water waves maintained their shape and structure as they traveled in a canal. He went ahead to conduct experiments, which culminated in the discovery of solitons. In the year 1895, Diederik Korteweg and Gustav De Vries analytically derived the KdV equation. However, Joseph Boussinesq had earlier (1877) introduced the equation in his work on water waves. The KdV equation has applications in aerodynamics, fluid dynamics, and continuum mechanics among other disciplines of mathematical sciences. The equation has been used to describe the dynamics of solitons, ion-acoustic waves in plasmas, surface waves in shallow water, and long internal waves in oceans. The KdV equation also models shock wave formation, turbulence, boundary layer behavior, and mass transport. The simplest form of the KdV equation is given by

$$u_t + auu_x + u_{xxx} = 0. \tag{3.1.1}$$

The KdV equation combines a quadratic nonlinear dissipative term uu_x which localizes the wave and a linear dispersive term u_{xxx} which spreads out the wave. The physical meaning of $u(t, x)$ is the local elevation of the wave surface at time t and position x. With the help of scaling, we associate any equation of the form

$$\alpha u_t + \beta uu_x + \gamma u_{xxx} = 0, \tag{3.1.2}$$

to be of "KdV type". In this chapter, we use the form

$$\Delta \equiv u_t + 6uu_x + u_{xxx} = 0, \tag{3.1.3}$$

where the 6 factor is just conventional and of no great significance. Infact, most commonly used factors are $\pm 1, \pm 6$.

3.2 Solutions of the KdV Equation (3.1.3)

3.2.1 Lie point symmetries of (3.1.3) .

We use Lie's method to determine Lie point symmetries of the KdV equation. The infinitesimal transformations of a local Lie group with a parameter ϵ are

$$\bar{t} = t + \tau(t, x, u)\epsilon,\ \ \bar{x} = x + \xi(t, x, u)\epsilon,\ \ \bar{u} = u + \eta(t, x, u)\epsilon. \tag{3.2.1}$$

The vector field

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}, \tag{3.2.2}$$

is a Lie point symmetry of (3.1.3) provided that

$$\mathsf{X}^{[3]}\Delta\Big|_{\Delta=0} = 0, \tag{3.2.3}$$

where

$$\mathsf{X}^{[3]} = X + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{222}\frac{\partial}{\partial u_{xxx}}, \tag{3.2.4}$$

is the third prolongation of the Lie point symmetry X as defined in (2.1.20) of Chapter 2 and

$$\begin{aligned}
\zeta_1 &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), & (3.2.5)\\
\zeta_2 &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), & (3.2.6)\\
\zeta_{22} &= D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi), & (3.2.7)\\
\zeta_{222} &= D_x(\zeta_{22}) - u_{txx} D_x(\tau) - u_{xxx} D_x(\xi), & (3.2.8)
\end{aligned}$$

as defined in (2.1.19), and

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots, & (3.2.9)\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \ldots. & (3.2.10)
\end{aligned}$$

Applying the definitions of D_t and D_x given in (3.2.9) and (3.2.10), we obtain the expanded form of the ζ_s as

$$\begin{aligned}
\zeta_1 &= \eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x\xi_u,\\
\zeta_2 &= \eta_x + u_x\eta_u - u_t\tau_x - u_t u_x\tau_u - u_x\xi_x - u_x^2\xi_u,\\
\zeta_{22} &= \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu} - 2u_{xx}\xi_x - u_x\xi_{xx} - 2u_x^2\xi_{xu} - 3u_x u_{xx}\xi_u\\
&\quad - u_x^3\xi_{uu} - 2u_{tx}\tau_x - u_t\tau_{xx} - 2u_t u_x\tau_{xu} - (u_t u_{xx} + 2u_x u_{tx})\tau_u - u_t u_x^2\tau_{uu},\\
\zeta_{222} &= \eta_{xxx} - 3u_t u_x u_t u_{xx}\tau_{uu} + u_x^3\eta_{uuu} + 3u_x^2\eta_{uux} - u_t\tau_{xxx} - 3u_x^2\xi_{uux} - 3u_x^3\xi_{uux} \qquad (3.2.11)\\
&\quad - u_x^4\xi_{uuu} - 3u_{xx}^2\xi_u + 3u_{xx}\eta_{ux} + u_{xxx}\eta_u - 3u_{txx}\tau_x - 3u_{tx}\tau_{xx} - 3u_{xxx}\xi_x\\
&\quad - 3u_{xx}\xi_{xx} - 3u_t u_x\tau_{uxx} - 3u_t u_{xx}\tau_{ux} - 6u_{tx}u_x\tau_{ux} + 3u_{xx}u_x\eta_{uu} - 3u_{tx}u_x^2\tau_{uu}\\
&\quad + 3u_x\eta_{uxx} - u_x\xi_{xxx} - 3u_t u_x^2\tau_{uux} - u_t u_{xxx}\tau_u - 3u_{txx}u_x\tau_u - u_t u_x^3\tau_{uuu}\\
&\quad - 4u_{xxx}u_x\xi_u - 9u_x u_{xx}\xi_{ux} - 3u_{tx}u_{xx}\tau_u - 6u_x^2 u_{xx}\xi_{uu}.
\end{aligned}$$

From (3.2.3), we get

$$\Big[\tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{222}\frac{\partial}{\partial u_{xxx}}\Big]\Big(u_t + 6uu_x + u_{xxx}\Big)\Big|_{()} = 0. \tag{3.2.12}$$

After expanding (3.2.12), we obtain

$$(6u_x\eta + \zeta_1 + 6u\zeta_2 + \zeta_{222})\Big|_{u_{xxx}=-u_t-6uu_x} = 0. \tag{3.2.13}$$

Substitutions of values of ζ_1, ζ_2 and ζ_{222} in (3.2.13) yield

$$\begin{aligned}
&6u_x\eta + [\eta_t + u_t\eta_u - u_t\tau_u - u_t^2\tau_u - u_x\xi_t - u_tu_x\xi_u] + 6u[\eta_x + u_x\eta_u - u_t\tau_x - u_tu_x\tau_u \\
&- u_x\xi_x - u_x^2\xi_u] + [\eta_{xxx} - 3u_tu_xu_tu_{xx}\tau_{uu} + u_x^3\eta_{uuu} + 3u_x^2\eta_{uux} - u_t\tau_{xxx} - 3u_x^2\xi_{uux} \\
&- 3u_x^3\xi_{uux} - u_x^4\xi_{uuu} - 3u_{xx}^2\xi_u + 3u_{xx}\eta_{ux} + u_{xxx}\eta_u - 3u_{txx}\tau_x - 3u_{tx}\tau_{xx} - 3u_{xxx}\xi_x \\
&- 3u_{xx}\xi_{xx} - 3u_tu_x\tau_{uxx} - 3u_tu_{xx}\tau_{ux} - 6u_{tx}u_x\tau_{ux} + 3u_{xx}u_x\eta_{uu} - 3u_{tx}u_x^2\tau_{uu} \\
&- u_x\xi_{xxx} - 3u_tu_x^2\tau_{uux} - u_tu_{xxx}\tau_u - 3u_{txx}u_x\tau_u - u_tu_x^3\tau_{uuu} - 4u_{xxx}u_x\xi_u - 9u_xu_{xx}\xi_{ux} \\
&- 3u_{tx}u_{xx}\tau_u - 6u_x^2u_{xx}\xi_{uu}]\Big|_{u_{xxx}=-u_t-6uu_x} = 0.
\end{aligned} \tag{3.2.14}$$

Replacing u_{xxx} by $-u_t - 6uu_x$ in (3.2.14), we obtain

$$\begin{aligned}
&6u_x\eta + [\eta_t + u_t\eta_u - u_t\tau_u - u_t^2\tau_u - u_x\xi_t - u_tu_x\xi_u] + 6u[\eta_x + u_x\eta_u - u_t\tau_x - u_tu_x\tau_u \\
&- u_x\xi_x - u_x^2\xi_u] + [\eta_{xxx} - 3u_tu_xu_tu_{xx}\tau_{uu} + u_x^3\eta_{uuu} + 3u_x^2\eta_{uux} - u_t\tau_{xxx} - 3u_x^2\xi_{uux} \\
&- 3u_x^3\xi_{uux} - u_x^4\xi_{uuu} - 3u_{xx}^2\xi_u + 3u_{xx}\eta_{ux} + (-u_t - 6uu_x)\eta_u - 3u_{txx}\tau_x - 3u_{tx}\tau_{xx} \\
&- 3(-u_t - 6uu_x)\xi_x - 3u_{xx}\xi_{xx} - 3u_tu_x\tau_{uxx} - 3u_tu_{xx}\tau_{ux} - 6u_{tx}u_x\tau_{ux} + 3u_{xx}u_x\eta_{uu} \\
&- 3u_{tx}u_x^2\tau_{uu} + 3u_x\eta_{uxx} - u_x\xi_{xxx} - 3u_tu_x^2\tau_{uux} - u_t(-u_t - 6uu_x)\tau_u - 3u_{txx}u_x\tau_u \\
&- u_tu_x^3\tau_{uuu} - 4(-u_t - 6uu_x)u_x\xi_u - 9u_xu_{xx}\xi_{ux} - 3u_{tx}u_{xx}\tau_u - 6u_x^2u_{xx}\xi_{uu}] = 0.
\end{aligned} \tag{3.2.15}$$

Splitting (3.2.15) on derivatives of u gives an overdetermined system of eight PDEs, namely

$$u_{txx} \;:\; \tau_x = 0, \tag{3.2.16}$$
$$u_{tx}u_{xx} \;:\; \tau_u = 0, \tag{3.2.17}$$
$$u_{xx}^2 \;:\; \xi_u = 0, \tag{3.2.18}$$
$$u_xu_{xx} \;:\; \eta_{uu} = 0, \tag{3.2.19}$$
$$u_{xx} \;:\; \eta_{ux} - \xi_{xx} = 0, \tag{3.2.20}$$
$$u_t \;:\; 3\xi_x - \tau_t = 0, \tag{3.2.21}$$
$$u_x \;:\; 6\eta + 12u\xi_x - \xi_t + 3\eta_{uxx} - \xi_{xxx} = 0, \tag{3.2.22}$$
$$\text{Rest} \;:\; \eta_{xxx} + 6u\eta_x + \eta_t = 0. \tag{3.2.23}$$

From (3.2.16) and (3.2.17), we have

$$\tau = \tau(t), \tag{3.2.24}$$

where arbitrary function τ depends on t. Substituting the expression of τ given by (3.2.24) into (3.2.21), and integrating thereafter with respect to x while taking note of (3.2.18), we have

$$\xi = \frac{\tau_t(t)}{3}x + a(t), \tag{3.2.25}$$

for some arbitrary function a of t. Using the expression for ξ given by (3.2.25) in (3.2.22), we have

$$\eta = \frac{-2\tau_t(t)u}{3} + \frac{\tau_{tt}(t)}{18}x + \frac{a_t(t)}{6}. \tag{3.2.26}$$

The expressions for ξ and η as given by (3.2.25) and (3.2.26) respectively, admit (3.2.20). Substituting the expression of η given by (3.2.26) in (3.2.23) yields

$$-\frac{\tau_{tt}(t)}{3}u + \frac{\tau_{ttt}(t)}{18}x + \frac{a_{tt}(t)}{6} = 0. \tag{3.2.27}$$

If we split (3.2.27) on powers of u and x, we obtain

$$u \ : \ \tau_{tt}(t) = 0, \tag{3.2.28}$$
$$x \ : \ \tau_{ttt}(t) = 0, \tag{3.2.29}$$
$$\text{rest} \ : \ a_{tt}(t) = 0. \tag{3.2.30}$$

We observe that (3.2.28) satisfies (3.2.29). Integration of (3.2.28) and (3.2.30) twice with respect to t gives

$$\tau = 3C_1 t + C_2, \tag{3.2.31}$$
$$a(t) = 6C_3 t + C_4. \tag{3.2.32}$$

Finally, appropriate substitutions give

$$\tau = 3C_1 t + C_2, \tag{3.2.33}$$
$$\xi = C_1 x + 6C_3 t + C_4, \tag{3.2.34}$$
$$\eta = -2C_1 u + C_3. \tag{3.2.35}$$

The calculations prove that the KdV Equation (3.1.3) admits a four-dimensional Lie algebra generated by

$$X_1 = \frac{\partial}{\partial x}, \tag{3.2.36}$$
$$X_2 = \frac{\partial}{\partial t}, \tag{3.2.37}$$
$$X_3 = 6t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \tag{3.2.38}$$
$$X_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}. \tag{3.2.39}$$

Remark 3.2.2. The first two symmetries represent space and time translations respectively while the third represents Galilean boost and the fourth represents scaling symmetry.

3.2.3 Commutator table.

The set of all solutions to the determining Equation (3.2.3) forms a Lie algebra. In this subsection , we evaluate the commutation relations for the symmetry generators. We use the definition of Lie bracket in (2.1.28). For example, we have that the commutator

$$[X_1, X_2] = \left(\frac{\partial}{\partial x}\frac{\partial}{\partial t}\right) - \left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}\right) = 0. \tag{3.2.40}$$

Remark 3.2.4. The remaining commutation relations are obtained similarly as displayed in Table 3.1.

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	X_1
X_2	0	0	6 X_1	$3X_2$
X_3	0	-6X_1	0	-2 X_3
X_4	-X_1	-3X_2	$2X_3$	0

Table 3.1: A commutator table for the Lie algebra generated by the symmetries of the KdV Equation (3.1.3).

3.2.5 Local Lie groups of transformations.

The corresponding one-parameter groups of transformations are determined by solving the Lie equations as given in Theorem (2.1.21). Let T_{ϵ_i} be the group of transformations for each $X_i, i = 1, 2, 3, 4$. To obtain T_{ϵ_1} from the infinitesimal generator X_1 (3.2.36), one integrates the Lie equations

$$\frac{\mathrm{d}\bar{t}}{\mathrm{d}\epsilon_1} = 0, \quad \bar{t}\Big|_{\epsilon_1=0} = t, \ \frac{\mathrm{d}\bar{x}}{\mathrm{d}\epsilon_1} = 1, \ \bar{x}\Big|_{\epsilon_1=0} = x, \ \frac{\mathrm{d}\bar{u}}{\mathrm{d}\epsilon_1} = 0, \ \bar{u}\Big|_{\epsilon_1=0} = u. \tag{3.2.41}$$

Solving the system (3.2.41), one obtains

$$T_{\epsilon_1} : \ \bar{t} = t, \ \bar{x} = x + \epsilon_1, \ \bar{u} = u. \tag{3.2.42}$$

The other three local Lie groups are obtained similarly and are given by

$$\begin{aligned} &T_{\epsilon_2} : \ \bar{t} = t + \epsilon_2, \ \bar{x} = x, \ \bar{u} = u, \\ &T_{\epsilon_3} : \ \bar{t} = t, \ \bar{x} = x + 6\epsilon_3 t, \ \bar{u} = u + \epsilon_3, \\ &T_{\epsilon_4} : \ \bar{t} = te^{3\epsilon_4}, \ \bar{x} = xe^{\epsilon_4}, \ \bar{u} = ue^{-2\epsilon_4}. \end{aligned} \tag{3.2.43}$$

Remark 3.2.6. In Lie group analysis, one can obtain solutions of PDEs either by transforming known solutions by the one-parameter groups or constructing group-invariant invariant solutions.

3.2.7 Transformation of known solutions.

We show how to transform known solutions. By the criterion of invariance, if $\bar{u} = \Gamma(\bar{t}, \bar{x})$ admits (3.1.3), then so does

$$\phi(t, x, u, \epsilon) = \Gamma(\varphi_1(t, x, u, \epsilon)\varphi_2(t, x, u, \epsilon)). \tag{3.2.44}$$

Using the Lie groups obtained in (3.2.42) and (3.2.43), one gets the transformed solutions

$$\begin{aligned} T_{\epsilon_1} :& u^{(1)} = \Gamma(t, x + \epsilon_1), \\ T_{\epsilon_2} :& u^{(2)} = \Gamma(t + \epsilon_2, x), \\ T_{\epsilon_3} :& u^{(3)} = \Gamma(t, x + 6t\epsilon_3) - \epsilon_3, \\ T_{\epsilon_4} :& u^{(4)} = e^{2\epsilon_4}\Gamma(te^{3\epsilon_4}, xe^{\epsilon_4}). \end{aligned} \tag{3.2.45}$$

3.2.8 Group-invariant solutions of (3.1.3).

We compute the group-invariant solutions of the KdV Equation (3.1.3) under all its Lie point symmetries.

(i) **Space Translation-Invariant Solutions.**
Consider the space translation operator $X_1 = \partial/\partial x$. The characteristic equations defined in (2.1.24) of Chapter 2 for the operator (3.2.36) are

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{0}, \tag{3.2.46}$$

which gives two invariants $J_1 = t$ and $J_2 = u$. Therefore, $J_2 = \psi(J_1)$ is the group-invariant solution for some arbitrary function ψ. Substitution of $u = \psi(t)$ into (3.1.3) yields $\psi'(t) = 0$, whose solution is $\psi(t) = C_1$, for some arbitrary constant C_1. Hence the space translation-invariant solution of (3.1.3) is

$$u(t, x) = C_1. \tag{3.2.47}$$

(ii) **Time translation-invariant (Stationary) solutions.**
Consider the time translation operator $X_2 = \partial/\partial t$. The Lagrangian system associated with the operator (3.2.37) yields the invariants $J_1 = x$ and $J_2 = u$. Thus, $u = \psi(x)$ is the group-invariant solution. Substituting of $u = \psi(x)$ into (3.1.3) yields

$$6\psi'(x)\psi(x) + \psi'''(x) = 0. \tag{3.2.48}$$

All the stationary solutions have the form $u = \psi(x)$ for some arbitrary function ψ satisfying (3.2.48) or

$$(\psi')^2 = -2\psi^3 + 2k\psi + l, \tag{3.2.49}$$

obtained from (3.2.48) by two integrations, for which k and l are constants of integration.
The general invariant solution takes the form (Ibragimov, 1994)

$$u = -2\Phi(x), \tag{3.2.50}$$

where $\Phi(x)$ is the Weierstrass elliptic function satisfying

$$\Phi(x)'^2 = 4\Phi(x)^3 - g_2\Phi(x) - g_3. \tag{3.2.51}$$

For some real roots r_1, r_2, r_3, of the cubic polynomial on the right-hand side of (3.2.49), the solutions (3.2.50) could be rewritten in the following forms: (Ibragimov, 1994)

(a) If $r_1 < r_2 < r_3$, then $u = u(x)$, is a limited function and

$$u = \frac{2a}{s^2}\mathrm{dn}^2\left(\sqrt{\frac{a}{s^2}}x, s\right) + \gamma, \tag{3.2.52}$$

is a cnoidal wave where dn^2(x,s) is the Jacobian elliptic function with modulus $s = \sqrt{\frac{r_3-r_2}{r_3-r_1}}$, where $a = \frac{r_3-r_2}{2}$ is the amplitude of a wave, and $\gamma = r_1$.

(b) If $r_1 = r_2 < r_3$, then $u \to r_1, u', u'' \to 0$ when $|x| \to \infty$

$$u = (r_3 - r_1)\,\mathrm{sech}^2\left(\sqrt{\frac{r_3-r_1}{2}}\right) + r_1, \tag{3.2.53}$$

is a solitary wave.

(c) If $r_1 = r_2 = r_3$, then

$$u = -\frac{2}{(x-c)^2}, \quad x \neq c. \tag{3.2.54}$$

(iii) **Galilean-invariant solutions.**
Consider the Galilean boost operator $X_3 = 6t\partial/\partial x + \partial/\partial u$. The characteristic equations associated to the operator (3.2.38) yield two invariants $J_1 = t$ and $J_2 = -u + x/(6t)$. As a result, the group-invariant solution of (3.1.3) for this case is $J_2 = \phi(J_1)$, for ϕ an arbitrary function. That is,

$$u(t,x) = -\phi(t) + \frac{x}{6t}, \quad t \neq 0. \tag{3.2.55}$$

Substitution of the expression for u from (3.2.55) into (3.1.3) yields a first order ordinary DE $\phi'(t) + \phi(t)/t = 0$, whose general solution is $\phi(t) = \delta/t$ for an arbitrary constant δ. Hence, we have a group-invariant solution for X_3 as

$$u(t,x) = \frac{x + \mathcal{C}}{6t}, \tag{3.2.56}$$

where $\mathcal{C} = -6\delta$ and $t \neq 0$.

(iv) **Scale-invariant solutions**.
Last but not least, we consider the scaling operator $X_4 = x\partial/\partial x + 3t\partial/\partial t - 2u\partial/\partial u$, whose associated Lagrangian equations yield two invariants, $J_1 = x^3/t$ and $J_2 = ux^2$. Thus,

$$u = x^{-2}\varphi(\lambda), \quad \lambda = \frac{x^3}{t}, \quad t \neq 0, \tag{3.2.57}$$

is the group-invariant solution under X_4 where φ satisfies

$$27\lambda^3\varphi''' + 18\lambda\varphi\varphi' + (24-\lambda)\lambda\varphi' - 12(2+\varphi)\varphi = 0. \tag{3.2.58}$$

3.2.9 Travelling wave solution.
To obtain travelling wave solution, in particular a soliton solution of the KdV equation, we consider a linear combination of the space and time translation symmetries, namely, $X = c\partial/\partial x + \partial/\partial t$ for some arbitrary constant c considered as the velocity of the wave. The characteristic equations give two invariants, $J_1 = u$ and $J_2 = x - ct$. So $u(t,x) = \varphi(x-ct)$, for some arbitrary function φ, is the invariant solution. Substitution of u into (3.1.3) yields a third-order ordinary DE

$$-c\varphi'(\xi) + 6\varphi(\xi)\varphi'(\xi) + \varphi'''(\xi) = 0, \;\; \xi = x - ct. \tag{3.2.59}$$

Integrating (3.2.59) with respect to ξ yields

$$-c\varphi(\xi) + 3\varphi^2(\xi) + \varphi''(\xi) = 0, \tag{3.2.60}$$

where we have chosen 0 as the constant of integration. The second integration is done after multiplying (3.2.60) by $2\varphi'$ and we get

$$(\varphi')^2 = c\varphi^2 - 2\varphi^3 \quad \text{or} \quad \frac{\mathrm{d}\varphi}{\sqrt{c\varphi^2 - 2\varphi^3}} = \mathrm{d}\xi. \tag{3.2.61}$$

In the second integration, again we took the constant of integration to be 0. By the change of variable $\varphi = (c/2)\,\mathrm{sech}^2(\eta)$ and integration of (3.2.61), we get the one-soliton solution

$$u(x,t) = \frac{c}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right). \tag{3.2.62}$$

3.3 Conservation laws of the KdV Equation (3.1.3)

Computation of conservation laws for KdV equation (3.1.3) is done by the using the method of multipliers (Olver, 1993).
3.3.1 Conservation Laws of the KdV Equation (3.1.3) using multipliers .
Using the Euler-Lagrange operator defined in (2.2.1), we search for a zeroth order multiplier $\Lambda(t,x,u) = \Lambda$. The determining equation for computing Λ is

$$\frac{\delta}{\delta u}\left[\Lambda\{u_t + 6uu_x + u_{xxx}\}\right] = 0. \tag{3.3.1}$$

Expansion of (3.3.1) yields

$$\Lambda_u(u_t + 6uu_x + u_{xxx}) + 6u_x\Lambda - D_t(\Lambda) - 6D_x(\Lambda u) - D_x^3(\Lambda) = 0. \tag{3.3.2}$$

Invoking the total derivatives defined in (3.2.9) and (3.2.10) on (3.3.2), we get

$$\Lambda_t + 6u\Lambda_x + \Lambda_{xxx} + 3\Lambda_{xxu}u_x + 3\Lambda_{xuu}u_x^2 + \Lambda_{uuu}u_x^3 + 3\Lambda_{xu}u_{xx} + 3\Lambda_{uu}u_x u_{xx} = 0. \tag{3.3.3}$$

Splitting (3.3.3) on derivatives of u produces a simplified system of 3 PDEs, namely,

$$u_{xx} : \Lambda_{xu} = 0, \tag{3.3.4}$$

$$u_x u_{xx} : \Lambda_{uu} = 0, \tag{3.3.5}$$

$$rest : \Lambda_t + 6u\Lambda_x + \Lambda_{xxx} = 0. \tag{3.3.6}$$

Integrating (3.3.5) with respect to u twice gives

$$\Lambda = \mathfrak{a}(t,x)u + \mathfrak{b}(t,x), \tag{3.3.7}$$

for some arbitrary functions $\mathfrak{a}$ and $\mathfrak{b}$ of t and x. Substitution of the value of Λ from (3.3.7) into (3.3.4) yields,

$$\mathfrak{a}_x(t,x) = 0, \tag{3.3.8}$$

from which we get that $\mathfrak{a} = \mathfrak{a}(t)$, and consequently

$$\Lambda = \mathfrak{a}(t)u + \mathfrak{b}(t,x). \tag{3.3.9}$$

If we substitute the expression for Λ given in (3.3.9) into (3.3.6), we obtain

$$\mathfrak{a}_t(t)u + \mathfrak{b}_t(t,x) + 6u\mathfrak{b}_x(t,x) + \mathfrak{b}_{xxx}(t,x) = 0. \tag{3.3.10}$$

Splitting (3.3.10) on powers of u, we find

$$u \quad : \quad \mathfrak{a}_t(t) + 6\mathfrak{b}_x(t,x) = 0, \tag{3.3.11}$$

$$u^0 \quad : \quad \mathfrak{b}_t(t,x) + \mathfrak{b}_{xxx}(t,x) = 0. \tag{3.3.12}$$

Equation (3.3.11) implies that

$$\mathfrak{b}_x(t,x) = -\frac{\mathfrak{a}_t(t)}{6}, \tag{3.3.13}$$

from which $\mathfrak{b}_{xxx}(t,x) = 0$. Thus, by (3.3.12), we deduce that $\mathfrak{b}_t(t,x) = 0$, meaning that $\mathfrak{b} = \mathfrak{b}(x)$. Upon integration of (3.3.13) with respect to x, we have that

$$\mathfrak{b}(x) = -\frac{\mathfrak{a}_t(t)}{6}x + C_1, \tag{3.3.14}$$

for some arbitrary constant C_1. Since $\mathfrak{a} = \mathfrak{a}(t)$ as shown by (3.3.8), we must have that $\mathfrak{a}_t(t) = C_2$ for some arbitrary constant C_2. This gives $\mathfrak{a}(t) = C_2 t + C_3$ and $\mathfrak{b}(x) = C_1 - C_2 x/6$ for an arbitrary constant C_3. Hence we have,

$$\Lambda = C_1 + C_2\left(tu - \frac{x}{6}\right) + C_3 u, \tag{3.3.15}$$

which means we have three non-trivial conservation law multipliers, namely,

$$\Lambda_1 = 1, \quad \Lambda_2 = tu - \frac{1}{6}x \quad \text{and} \quad \Lambda_3 = u. \tag{3.3.16}$$

Remark 3.3.2. Recall from (2.2.9) that a multiplier Λ for (3.1.3) has the property that for the density $T^t = T^t(t,x,u,u_x)$ and flux $T^x = T^x(t,x,u,u_x,u_{xx})$,

$$\Lambda\left(u_t + 6uu_x + u_{xxx}\right) = D_t T^t + D_x T^x. \tag{3.3.17}$$

We look for conserved vectors corresponding to each of the three multipliers obtained.

(a) **Conserved vectors for the multiplier** $\Lambda_1 = 1$. Expansion of (3.3.17) gives,

$$u_t + 6uu_x + u_{xxx} = T_t^t + u_t T_u^t + u_{tx} T_{u_x}^t + T_x^x + u_x T_u^x + u_{xx} T_{u_x}^x + u_{xxx} T_{u_{xx}}^x. \tag{3.3.18}$$

Splitting (3.3.18) on third derivatives of u yields;

$$\begin{aligned} u_{xxx} &: \quad T_{u_{xx}}^x = 1, && (3.3.19) \\ \text{Rest} &: \quad u_t + 6uu_x = T_t^t + u_t T_u^t + u_{tx} T_{u_x}^t + T_x^x + u_x T_u^x + u_{xx} T_{u_x}^x. && (3.3.20) \end{aligned}$$

By integrating (3.3.19) with respect to u_{xx}, we deduce that

$$T^x = u_{xx} + A(t, x, u, u_x), \tag{3.3.21}$$

for some arbitrary function A. Substituting the expression of T^x from (3.3.21) into (3.3.20), we have

$$u_t + 6uu_x = T_t^t + u_t T_u^t + u_{tx} T_{u_x}^t + A_x + u_x A_u + u_{xx} A_{u_x}. \tag{3.3.22}$$

Equation (3.3.22) splits on second derivatives of u to give;

$$\begin{aligned} u_{tx} &: \quad T_{u_x}^t = 0, && (3.3.23) \\ u_{xx} &: \quad A_{u_x} = 0, && (3.3.24) \\ \text{Rest} &: \quad u_t + 6uu_x = T_t^t + u_t T_u^t + A_x + u_x A_u. && (3.3.25) \end{aligned}$$

Integrating (3.3.23) and (3.3.24) with respect to u_x manifests that

$$\begin{aligned} T^t &= B(t, x, u), && (3.3.26) \\ A &= A(t, x, u), && (3.3.27) \end{aligned}$$

where B is an arbitrary function of its arguments. Substituting the expressions of T^t and A from (3.3.26) and (3.3.27) respectively, into (3.3.25) gives

$$u_t + 6uu_x = B_t + u_t B_u + A_x + u_x A_u, \tag{3.3.28}$$

which splits on the derivatives of u to yield

$$\begin{aligned} u_t &: \quad B_u = 1, && (3.3.29) \\ u_x &: \quad A_u = 6u, && (3.3.30) \\ \text{Rest} &: \quad A_x + B_t = 0. && (3.3.31) \end{aligned}$$

By integrating (3.3.29) and (3.3.30) with respect to u, we find that

$$\begin{aligned} B &= u + C(t, x), && (3.3.32) \\ A &= 3u^2 + D(t, x) && (3.3.33) \end{aligned}$$

where C and D are arbitrary functions of their arguments. Substitution of the expressions of A and B into (3.3.31) shows that $D_x + C_t = 0$. Since C and D contribute to the trivial part of the conservation law, we take $C = D = 0$. We get the conserved vector for the multiplier $\Lambda_1 = 1$ as

$$T_1^t = u, \quad T_1^x = 3u^2 + u_{xx}. \tag{3.3.34}$$

Remark 3.3.3. Similarly for the multipliers Λ_2 and Λ_3, we get the following conserved vectors.

(b) **Conserved vector for the multiplier** $\Lambda_2 = tu - x/6$ is given by

$$T_2^t = \frac{1}{2}tu^2 - \frac{1}{6}xu, \quad T_2^x = 2u^3t - \frac{1}{2}xu^2 + \frac{1}{6}u_x - \frac{1}{2}tu_x^2 + \left(tu - \frac{x}{6}\right)u_{xx}. \tag{3.3.35}$$

(c) **Conserved vector for the multiplier** $\Lambda_3 = u$ is given by

$$T_3^t = \frac{u^2}{2}, \quad T_3^x = 2u^3 + uu_{xx} - \frac{1}{2}u_x^2. \tag{3.3.36}$$

Remark 3.3.4. It can be verified that

$$D_tT_i^t + D_xT_i^x\Big|_{\Delta=0} = 0, \tag{3.3.37}$$

for $i = 1, 2, 3$.

Remark 3.3.5. The expressions in (3.3.37) are conservation laws for the mass, energy and momentum for the KdV Equation (3.1.3).

Remark 3.3.6. The presence of multiplier $\Lambda_1 = 1$ shows that the KdV Equation (3.1.3) is itself a conservation law.

3.4 Concluding remarks

In this chapter, we gave an illustrative example of Lie group analysis of a nonlinear PDE, namely the KdV Equation (3.1.3). We achieved that by first constructing time and space translations, Galilean boosts, and scaling symmetry. These symmetries have been used to reduce the KdV equation, a process which led to group-invariant solutions, including a soliton solution of (3.1.3). All the solutions found describe different states of the system. Finally, conservation laws for mass, momentum, and energy were obtained using the multiplier approach. That means those quantities are invariant in the evolution of the KdV system. The approach illustrated in this chapter will be very important for studying the nonlinear system of coupled KdV equations in Chapter 4.

4. Nonlinear Coupled Korteweg-de Vries (KdV) Equations

We have demonstrated in Chapter 3, how to find Lie point symmetries, symmetry reductions, and invariant solutions for a nonlinear PDE. As an illustrative example, we studied the nonlinear Korteweg-de Vries equation (3.1.3), for which we found invariant solutions which included a one-soliton solution. Conservation laws for mass, momentum, and energy were also constructed by using multipliers. This chapter will focus on Lie group analysis a nonlinear coupled system of Korteweg-de Vries equations. Our aim is to perform a Lie group analysis of this system. This we achieve by calculating Lie point symmetries, performing symmetry reductions, and constructing invariant solutions for the nonlinear coupled KdV system. Thereafter, we find conservation laws by use of both the method of multipliers and a theorem due to Ibragimov.

4.1 Introduction

Consider the Kortweg-de Vries equation

$$q_t + \alpha q q_x + \beta q_{xxx} = 0, \tag{4.1.1}$$

α and β are constants. Let

$$q(t,x) = u(t,x) + iv(t,x), \tag{4.1.2}$$

where $i^2 = -1$. Then substituting (4.1.2) into (4.1.1) and separating the real and imaginary parts, we obtain

$$\Delta_1 \equiv u_t + \alpha u u_x - \alpha v v_x + \beta u_{xxx} = 0, \tag{4.1.3}$$

$$\Delta_2 \equiv v_t + \alpha u v_x + \alpha v u_x + \beta v_{xxx} = 0, \tag{4.1.4}$$

which is a nonlinear system of coupled KdV equations. We perform Lie symmetry analysis on (4.1.3)-(4.1.4), that is , we obtain Lie point symmetries, invariant solutions and conservation laws of (4.1.3)-(4.1.4).

4.2 Solutions of the nonlinear coupled KdV Equations (4.1.3)-(4.1.4)

4.2.1 Lie point symmetries of (4.1.3)-(4.1.4) .

To find Lie point symmetries of the coupled KdV Equations, we use Lie's method. The infinitesimal transformations of the Lie group with parameter ϵ are

$$\bar{t} = t + \xi^t(t,x,u,v)\epsilon,\ \ \bar{x} = x + \xi^x(t,x,u,v)\epsilon,\ \ \bar{u} = u + \eta^u(t,x,u,v)\epsilon,\ \ \bar{v} = v + \eta^v(t,x,u,v)\epsilon. \tag{4.2.1}$$

The vector field

$$X = \xi^t(t,x,u,v)\frac{\partial}{\partial t} + \xi^x(t,x,u,v)\frac{\partial}{\partial x} + \eta^u(t,x,u,v)\frac{\partial}{\partial u} + \eta^v(t,x,u,v)\frac{\partial}{\partial v}, \tag{4.2.2}$$

is a Lie point symmetry of (4.1.3)-(4.1.4) if

$$\mathsf{X}^{[3]}\Delta_1\Big|_{\Delta_1=0,\ \Delta_2=0} = 0, \tag{4.2.3}$$

$$\mathsf{X}^{[3]}\Delta_2\Big|_{\Delta_1=0,\ \Delta_2=0} = 0. \tag{4.2.4}$$

Expanding (4.2.3) and (4.2.4) and splitting on derivatives of v and u, we have an overdetermined system of ten PDEs, namely,

$$\begin{aligned} &\xi^t_u = 0, \quad \xi^t_v = 0, \quad \xi^t_x = 0, \quad \xi^x_u = 0, \quad \xi^x_v = 0, \quad \xi^t_{tt} = 0, \quad \xi^x_{tt} = 0, \quad 3\xi^x_x - \xi^t_t = 0, \\ &3\eta^v + 2\xi^t_t v = 0, \quad 3\alpha\eta^u + 2\alpha\xi^t_t u - 3\xi^x_t = 0. \end{aligned} \tag{4.2.5}$$

Solving the system (4.2.5) in the same way we did for (3.2.16)-(3.2.23), we obtain

$$\xi^t = A_1 + 3A_2 t, \tag{4.2.6}$$
$$\xi^x = A_2 x + \alpha A_3 t + A_4, \tag{4.2.7}$$
$$\eta^u = -2A_2 u + A_3, \tag{4.2.8}$$
$$\eta^v = -2A_2 v, \tag{4.2.9}$$

for arbitrary constants A_1, A_2, A_3, A_4. Hence from (4.2.6)-(4.2.9), the infinitesimal symmetries of the coupled KdV Equations (4.1.3)-(4.1.4) is a Lie algebra generated by the vector fields

$$X_1 = \frac{\partial}{\partial t}, \tag{4.2.10}$$
$$X_2 = \frac{\partial}{\partial x}, \tag{4.2.11}$$
$$X_3 = \alpha t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \tag{4.2.12}$$
$$X_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}. \tag{4.2.13}$$

4.2.2 Commutator table.

The set of all infinitesimal symmetries of coupled KdV equations forms a Lie algebra. By the definition of a commutator in Equation (2.1.28), we get the commutation relations, which we present in Table 4.1.

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	αX_2	$3X_1$
X_2	0	0	0	X_2
X_3	$-\alpha X_2$	0	0	$-2X_3$
X_4	$-3X_1$	$-X_2$	$2X_3$	0

Table 4.1: A commutator table for the Lie algebra generated by the symmetries of coupled KdV equation.

4.2.3 Local Lie groups.

In this subsection, we present the corresponding group of transformations relating to each Lie point symmetry of coupled KdV system. We obtain the groups by invoking the Lie equations as defined in Theorem 2.1.21. For the system (4.1.3)-(4.1.4), we have the Lie equations, for $i = 1, 2, 3, 4$,

$$\begin{aligned} &\frac{d\bar{t}}{d\epsilon_i} = \xi^t(\bar{t}, \bar{x}, \bar{u}, \bar{v}), \quad \bar{t}\Big|_{\epsilon_i=0} = t, \quad \frac{d\bar{x}}{d\epsilon_i} = \xi^x(\bar{t}, \bar{x}, \bar{u}, \bar{v}), \quad \bar{x}\Big|_{\epsilon_i=0} = x, \\ &\frac{d\bar{u}}{d\epsilon_i} = \eta^u(\bar{t}, \bar{x}, \bar{u}, \bar{v}), \quad \bar{u}\Big|_{\epsilon_i=0} = u, \quad \frac{d\bar{v}}{d\epsilon_i} = \eta^v(\bar{t}, \bar{x}, \bar{u}, \bar{v}), \quad \bar{v}\Big|_{\epsilon_i=0} = v. \end{aligned} \tag{4.2.14}$$

Taking for each $X_i, i = 1,2,3,4,\ T_{\epsilon_i}$ to be the corresponding group of transformations, we obtain from the Lie equations (4.2.14) the groups;

$$T_{\epsilon_1} : \bar{t} = t + \epsilon_1,\ \bar{x} = x,\ \bar{u} = u,\ \bar{v} = v, \tag{4.2.15}$$
$$T_{\epsilon_2} : \bar{t} = t,\ \bar{x} = x + \epsilon_2,\ \bar{u} = u,\ \bar{v} = v, \tag{4.2.16}$$
$$T_{\epsilon_3} : \bar{t} = t,\ \bar{x} = x + \alpha\epsilon_3 t,\ \bar{u} = u + \epsilon_3,\ \bar{v} = v, \tag{4.2.17}$$
$$T_{\epsilon_4} : \bar{t} = te^{3\epsilon_4},\ \bar{x} = xe^{\epsilon_4},\ \bar{u} = ue^{-2\epsilon_4},\ \bar{v} = ve^{-2\epsilon_4}. \tag{4.2.18}$$

4.2.4 Symmetry reductions of the coupled KdV Equations (4.1.3)-(4.1.4).

We use the symmetries obtained in **Subsection 4.2.1 Lie point symmetries of (4.1.3)-(4.1.4)** to perform symmetry reductions for the Coupled KdV Equations (4.1.3)-(4.1.4).

(i) **The time translation symmetry $X_1 = \partial/\partial t$.**

Solving the characteristic equations associated to the operator X_1 gives the invariants $J_1 = x, \quad J_2 = u, \quad$ and $\quad J_3 = v$. Hence, we have

$$u = \varphi(x), \quad v = \psi(x), \tag{4.2.19}$$

for arbitrary functions φ and ψ. Substituting the expressions for u and v given by (4.2.19) into the system (4.1.3)-(4.1.4), we get a system of third order ordinary DEs namely,

$$\alpha\left[\varphi(x)\varphi'(x) - \psi(x)\psi'(x)\right] + \beta\varphi'''(x) = 0, \tag{4.2.20}$$
$$\alpha\left(\varphi(x)\psi(x)\right)' + \beta\psi'''(x) = 0. \tag{4.2.21}$$

Integration of the system (4.2.20)-(4.2.21) yields;

$$\frac{\alpha}{2}\left[\varphi(x)^2 - \psi(x)^2\right] + \beta\varphi''(x) = C_1, \tag{4.2.22}$$
$$\alpha\left[\varphi(x)\psi(x)\right] + \beta\psi''(x) = C_2, \tag{4.2.23}$$

for arbitrary constants C_1 and C_2. If we take $C_1 = C_2 = 0$, the system (4.2.22)-(4.2.23) becomes

$$\frac{\alpha}{2}\left[\varphi(x)^2 - \psi(x)^2\right] + \beta\varphi''(x) = 0, \tag{4.2.24}$$
$$\alpha\left[\varphi(x)\psi(x)\right] + \beta\psi''(x) = 0, \tag{4.2.25}$$

To find more solutions of the system (4.2.24)-(4.2.25), we determine its Lie point symmetries. Using the Lie's algorithm for computing point symmetries, we see that the Lie point symmetries of (4.2.24)-(4.2.25) are

$$X_1^* = \frac{\partial}{\partial x},\ X_2^* = x\frac{\partial}{\partial x} - 2\varphi\frac{\partial}{\partial\varphi} - 2\psi\frac{\partial}{\partial\psi}. \tag{4.2.26}$$

Proceeding as above, we see that the symmetry X_1^* yields the trivial solution

$$u = 0,\ v = 0. \tag{4.2.27}$$

The second symmetry X_2^* has the characteristic equations

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}\varphi}{-2\varphi} = \frac{\mathrm{d}\psi}{-2\psi}, \tag{4.2.28}$$

which provides the invariants $J_1 = x^2\varphi,\ J_2 = x^2\psi$. Letting $\varphi = \lambda/x^2,\ \psi = \mu/x^2$, substituting the values of φ and ψ into (4.2.24)-(4.2.25) and solving the resulting equations yield:

(a) **Case one.** Taking $\mu = 0$ gives $\lambda = 0$ or $\lambda = -12\beta/\alpha$.
The case $\lambda = 0,\ \mu = 0$ also gives the trivial solution (4.2.27). One can easily see that $\lambda = -12\beta/\alpha,\ \mu = 0$ gives $\varphi = -12\beta/(\alpha x^2),\ \psi = 0$ which is a solution of the system (4.2.24)-(4.2.25). Hence

$$u_1(t,x) = -\frac{12\beta}{\alpha x^2},\ v_1(t,x) = 0, \tag{4.2.29}$$

is a solution of the coupled KdV system (4.1.3)-(4.1.4).

(b) **Case two.** Taking $\lambda = -6\beta/\alpha,\ \mu = \pm 6\beta i/\alpha$ with $i^2 = -1$. Consequently,

$$u_2(t,x) = -\frac{6\beta}{\alpha x^2},\ v_2(t,x) = \frac{6i\beta}{\alpha x^2} \tag{4.2.30}$$

and

$$u_3(t,x) = -\frac{6\beta}{\alpha x^2},\ v_3(t,x) = -\frac{6i\beta}{\alpha x^2} \tag{4.2.31}$$

are solutions of the coupled KdV system (4.1.3)-(4.1.4). Hence Lie group analysis has given us three steady-state solutions for the coupled KdV system (4.1.3)-(4.1.4) under the time translation symmetry $X_1 = \partial/\partial t$.

(ii) **The space translation symmetry** $X_2 = \partial/\partial x$.

Solving the characteristic equations associated to X_2 gives the invariants $J_1 = t,\quad J_2 = u$ and $J_3 = v$. Therefore, the group-invariant solution is

$$u = \phi(t), \quad v = h(t), \tag{4.2.32}$$

for arbitrary functions h and ϕ. Substitution of the solutions from (4.2.32) into (4.1.3)-(4.1.4), we get a system of first order ordinary DEs, namely,

$$\phi'(t) = 0, \quad h'(t) = 0, \tag{4.2.33}$$

which is integrated once with respect to t to yield,

$$\phi(t) = C_1, \quad h(t) = C_2, \tag{4.2.34}$$

for arbitrary constants C_1 and C_2. Consequently, the space translation group-invariant solution of the system (4.1.3)-(4.1.4) is

$$u(t,x) = C_1, \quad v(t,x) = C_2. \tag{4.2.35}$$

(iii) **The Galilean boost symmetry** $X_3 = \alpha t\partial/\partial x + \partial/\partial u$.

Solving the characteristic equations associated to Galilean boost gives the invariants

$$J_1 = t, \quad J_2 = v, \quad J_3 = -u + \frac{x}{\alpha t},\ t \neq 0. \tag{4.2.36}$$

Thus the invariant solution of (4.1.3)-(4.1.4) is

$$u = \frac{x}{\alpha t} - g(t),\ v = f(t),\ t \neq 0, \tag{4.2.37}$$

for arbitrary functions f and g. Substitution of the values of u and v from (4.2.37) into the System (4.1.3)-(4.1.4), we get a nonlinear system of coupled first order ordinary DEs, namely,

$$tg'(t) + g(t) = 0, \tag{4.2.38}$$
$$tf'(t) + f(t) = 0, \tag{4.2.39}$$

whose solutions are $g(t) = C_1/t$ and $f(t) = C_2/t$ for arbitrary constants C_1 and C_2. Hence the Galilean boost group-invariant solution of the system (4.1.3)-(4.1.4) is

$$u(t,x) = \frac{x+A}{\alpha t}, \quad v(t,x) = \frac{C_2}{t}, \tag{4.2.40}$$

where $A = -\alpha C_1$ and $t \neq 0$.

(iv) **The scaling** $X_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}$.
By solving of the characteristic equations associated to this symmetry, we obtain the invariants

$$J_1 = \frac{x^3}{t}, \quad J_2 = ux^2, \quad J_3 = vx^2. \tag{4.2.41}$$

Generally, the group-invariant solution pair is

$$u(t,x) = \frac{f(\lambda)}{x^2}, \quad v(t,x) = \frac{g(\lambda)}{x^2}, \quad \lambda = \frac{x^3}{t}, \tag{4.2.42}$$

where the functions f and g satisfy the system of third order nonlinear coupled ordinary DEs

$$2\alpha(g^2 - f^2) - \lambda^2 f' + 3\alpha\lambda(ff' - gg') + \beta(-24f + 24\lambda f' + 27\lambda^3 f''') = 0, \tag{4.2.43}$$
$$-4\alpha fg - \lambda^2 g' + 3\alpha\lambda(fg)' + \beta(-24g + 24\lambda g' + 27\lambda^3 g''') = 0. \tag{4.2.44}$$

(v) **Linear combination of time and space translations** $X_1 + cX_2$.
We consider a symmetry X, which is a linear combination of the time and space translations symmetries, that is, $X = \partial_t + c\partial_x$, for a constant c. The invariants associated to this symmetry X are $J_1 = x - ct,\ J_2 = u,\ J_3 = v$. Hence, the invariant solution for the symmetry X is

$$u = f(x-ct),\ v = g(x-ct), \tag{4.2.45}$$

for arbitrary functions f and g. Substitution of u and v from (4.2.45) into the system (4.1.3)-(4.1.4) yields a system of nonlinear third order ordinary DEs, namely

$$-cf'(\xi) + \alpha\{f(\xi)f'(\xi) - g(\xi)g'(\xi)\} + \beta f'''(\xi) = 0, \tag{4.2.46}$$
$$-cg'(\xi) + \alpha(f(\xi)g(\xi))' + \beta g'''(\xi) = 0, \tag{4.2.47}$$

which on integrating once with respect to ξ yields

$$-cf + \frac{1}{2}\alpha(f^2 - g^2) + \beta f'' + C_1 = 0, \tag{4.2.48}$$
$$-cg + \alpha fg + \beta g'' + C_2 = 0, \tag{4.2.49}$$

for arbitrary constants C_1 and C_2.

Remark 4.2.5. If we take the constants $C_1 = C_2 = 0$, then when the wave velocity $c = 0$, we can recover the stationary solutions given in (i).

Remark 4.2.6. Traveling wave solutions of the system (4.1.3)-(4.1.4) must satisfy the system (4.2.48)-(4.2.49).

4.3 Conservation laws of the coupled KdV Equations (4.1.3)-(4.1.4)

Computation of conservation laws for the coupled KdV Equations (4.1.3)-(4.1.4) is done using two methods; the method of multipliers and a theorem due to Ibragimov.

4.3.1 Conservation laws of (4.1.3)-(4.1.4) using the multipliers.

We seek local conservation law multipliers for the system (4.1.3)-(4.1.4), whose determining equations are

$$\frac{\delta}{\delta u}\left[\Lambda^1\Delta_1+\Lambda^2\Delta_2\right]=0, \tag{4.3.1}$$

$$\frac{\delta}{\delta v}\left[\Lambda^1\Delta_1+\Lambda^2\Delta_2\right]=0, \tag{4.3.2}$$

where

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}+D_x^2\frac{\partial}{\partial u_{xx}}-D_x^3\frac{\partial}{\partial u_{xxx}}+\ldots, \tag{4.3.3}$$

$$\frac{\delta}{\delta v}=\frac{\partial}{\partial v}-D_t\frac{\partial}{\partial v_t}-D_x\frac{\partial}{\partial v_x}+D_x^2\frac{\partial}{\partial v_{xx}}-D_x^3\frac{\partial}{\partial v_{xxx}}+\cdots, \tag{4.3.4}$$

are the Euler-Lagrange operators and

$$D_t=\frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+v_t\frac{\partial}{\partial v}+u_{tx}\frac{\partial}{\partial u_x}+v_{tx}\frac{\partial}{\partial v_x}+u_{tt}\frac{\partial}{\partial u_t}+v_{tt}\frac{\partial}{\partial v_t}+\cdots, \tag{4.3.5}$$

$$D_x=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+v_x\frac{\partial}{\partial v}+u_{xx}\frac{\partial}{\partial u_x}+v_{xx}\frac{\partial}{\partial v_x}+u_{tx}\frac{\partial}{\partial u_t}+v_{tx}\frac{\partial}{\partial v_t}+\cdots, \tag{4.3.6}$$

are total derivatives operators. We look for second order multipliers, that is,

$$\Lambda^n=\Lambda^n(t,x,u,u_x,u_{xx},v,v_x,v_{xx}),\qquad n=1,2. \tag{4.3.7}$$

The determining Equations (4.3.1)-(4.3.2) become

$$\frac{\delta}{\delta u}\left[\Lambda^1\{u_t+\alpha uu_x-\alpha vv_x+\beta u_{xxx}\}+\Lambda^2\{v_t+\alpha uv_x+\alpha vu_x+\beta v_{xxx}\}\right]=0, \tag{4.3.8}$$

$$\frac{\delta}{\delta v}\left[\Lambda^1\{u_t+\alpha uu_x-\alpha vv_x+\beta u_{xxx}\}+\Lambda^2\{v_t+\alpha uv_x+\alpha vu_x+\beta v_{xxx}\}\right]=0. \tag{4.3.9}$$

Expanding (4.3.8)-(4.3.9) and splitting on derivatives of u and v yields an overdetermined system of 22 PDEs, namely

$$\begin{gathered}
\Lambda^1_{xx}=0,\ \Lambda^2_{xx}=0,\ \Lambda^1_{vx}=0,\ \Lambda^2_{vx}=0,\ \Lambda^1_{xv_{xx}}=0,\ \Lambda^2_{xv_{xx}}=0,\ \beta\Lambda^1_{vv}-\alpha\Lambda^2_{v_{xx}}=0,\\
\beta\Lambda^2_{vv}+\alpha\Lambda^1_{vv_{xx}}=0,\ \Lambda^1_{vv_{xx}}=0,\quad \Lambda^2_{vv_{xx}}=0,\ \Lambda^1_{v_{xx}v_{xx}}=0,\ \Lambda^2_{v_{xx}v_{xx}}=0,\ \Lambda^1_u+\Lambda^2_v=0,\\
\Lambda^1_t+\alpha\left(\Lambda^2_xv+\Lambda^1_xu\right)=0,\ \Lambda^2_t+\alpha\left(\Lambda^2_xu-\Lambda^1_xv\right)=0,\ \Lambda^2_u-\Lambda^1_v=0,\ \Lambda^1_{u_x}=0,\\
\Lambda^2_{u_x}=0,\ \Lambda^1_{u_{xx}}+\Lambda^2_{v_{xx}}=0,\Lambda^2_{u_{xx}}-\Lambda^1_{v_{xx}}=0,\ \Lambda^2_{v_x}=0,\ \Lambda^1_{v_x}=0.
\end{gathered} \tag{4.3.10}$$

Calculations reveal the solution of the system (4.3.10) as

$$\begin{aligned}
\Lambda^1=&\frac{\alpha}{2\beta}\left(c_3\{u^2-v^2\}+2c_4uv\right)+(c_2t+c_5)u+(c_1t+c_6)v+c_3u_{xx}\\
&+c_4v_{xx}+c_7-\frac{1}{\alpha}c_2x,\\
\Lambda^2=&\frac{\alpha}{2\beta}\left(c_4\{u^2-v^2\}-2c_3uv+\right)+(c_1t+c_6)u-(c_2t+c_5)v+c_4u_{xx}\\
&-c_3v_{xx}+c_8-\frac{1}{\alpha}c_1x,
\end{aligned} \tag{4.3.11}$$

for arbitrary constants $c_1,\ldots,c_8$.

Remark 4.3.2. Essentially, the nonlinear coupled system of KdV Equations (4.1.3)-(4.1.4) have eight sets of local conservation law multipliers.

Remark 4.3.3. Recall that the multipliers must satisfy the property defined in (2.2.9). For the nonlinear coupled KdV system (4.1.3)-(4.1.4), we have

$$\Lambda^1\Delta_1 + \Lambda^2\Delta_2 = D_t T^t + D_x T^x, \tag{4.3.12}$$

where $T^t = T^t(t, x, u, v)$ and $T^x = T^x(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$.

Solving (4.3.12) in the same way we did in (3.3.1) for the KdV Equation (3.1.3), we obtain conserved vectors corresponding to each set of multipliers as shown below.

(i) The multiplier $(\Lambda_1^1, \Lambda_1^2) = (tv, tu - \frac{x}{\alpha})$ has the conserved vectors

$$T_1^t = tuv - \frac{xv}{\alpha}, \tag{4.3.13}$$

$$T_1^x = \beta\left[t\{vu_{xx} + uv_{xx} - v_x u_x\} + \frac{1}{\alpha}\{v_x - xv_{xx}\}\right] + \alpha\left[t\left(u^2 v - \frac{v^3}{3}\right)\right] - xuv. \tag{4.3.14}$$

(ii) The multiplier $(\Lambda_2^1, \Lambda_2^2) = (tu - \frac{x}{\alpha}, -tv)$ has the conserved vectors

$$\begin{aligned} T_2^t =& \frac{t}{2}\{u^2 - v^2\} - \frac{xu}{\alpha}, \\ T_2^x =& \beta\left[t\left(uu_{xx} - vv_{xx} + \frac{1}{2}\{v_x^2 - u_x^2\}\right) + \frac{1}{\alpha}\{u_x - xu_{xx}\}\right] + \alpha t\left[\frac{u^3}{3} - uv^2\right] \\ &+ \frac{x}{2}\{v^2 - u^2\}. \end{aligned} \tag{4.3.15}$$

(iii) The multiplier $(\Lambda_3^1, \Lambda_3^2) = \left(\frac{\alpha}{2\beta}\{u^2 - v^2\} + u_{xx}, -\{\frac{\alpha uv}{\beta} + v_{xx}\}\right)$ has the conserved vectors

$$T_3^t = \frac{\alpha}{2\beta}\left(\frac{u^3}{3} - uv^2\right), \tag{4.3.16}$$

$$T_3^x = \frac{\alpha}{2}\left[(u^2 - v^2)u_{xx} - v^2 v_{xx}\right] - \alpha uvv_{xx} + \frac{\beta}{2}\left[u_{xx}^2 - v_{xx}^2\right] + u_t u_x - v_t v_x \tag{4.3.17}$$

$$+ \frac{\alpha^2}{4\beta}\left[\frac{1}{2}\{u^4 + v^4\} - 3u^2v^2\right]. \tag{4.3.18}$$

(iv) The multiplier $(\Lambda_4^1, \Lambda_4^2) = \left(\{\frac{\alpha uv}{\beta} + v_{xx}\}, \frac{\alpha[u^2 - v^2]}{2\beta} + u_{xx}\right)$ has the conserved vectors

$$T_4^t = \frac{\alpha}{2\beta}\left(u^2 v - \frac{v^3}{3}\right), \tag{4.3.19}$$

$$T_4^x = \frac{\alpha^2}{2\beta}\left[(u^3 v - uv^3)\right] + v_t u_x + u_t v_x + \frac{\alpha}{2}(u^2 - v^2)v_{xx} + \{\alpha uv + \beta v_{xx}\}u_{xx}. \tag{4.3.20}$$

(v) The multiplier $(\Lambda_5^1, \Lambda_5^2) = (u, -v)$ has the conserved vectors

$$T_5^t = \frac{1}{2}\{u^2 - v^2\}, \quad T_5^x = \beta\left(uu_{xx} - vv_{xx} + \frac{v_x^2 - u_x^2}{2}\right) + \alpha\left(\frac{u^3}{3} - uv^2\right). \tag{4.3.21}$$

(vi) The multiplier $(\Lambda_6^1, \Lambda_6^2) = (v, u)$ has the conserved vectors

$$T_6^t = uv, \quad T_6^x = \beta\,(vu_{xx} + uv_{xx} - u_x v_x) + \alpha\left(u^2 v - \frac{v^3}{3}\right). \tag{4.3.22}$$

(vii) The multiplier $(\Lambda_7^1, \Lambda_7^2) = (1, 0)$ has the conserved vectors

$$T_7^t = u, \quad T_7^x = \frac{\alpha}{2}\{u^2 - v^2\} + \beta u_{xx}. \tag{4.3.23}$$

(viii) The multiplier has $(\Lambda_8^1, \Lambda_8^2) = (0, 1)$ the conserved vectors

$$T_8^t = v, \quad T_8^x = \alpha uv + \beta v_{xx}. \tag{4.3.24}$$

Remark 4.3.4. It can be verified that

$$D_t T_i^t + D_x T_i^x \Big|_{\Delta_1=0,\ \Delta_2=0} = 0, \tag{4.3.25}$$

for $i = 1, \ldots, 8$.

Remark 4.3.5. The expressions in (4.3.25) are eight conservation laws for the coupled KdV system (4.1.3)-(4.1.4).

Remark 4.3.6. The presence of multipliers $(\Lambda_7^1, \Lambda_7^2) = (1, 0)$ and $(\Lambda_8^1, \Lambda_8^2) = (0, 1)$ manifest that the coupled KdV equations are themselves conservation laws.

4.3.7 Conservation laws of (4.1.3)-(4.1.4) using Ibragimov's theorem.

In this section, we derive conserved vectors for coupled KdV equations (4.1.3)-(4.1.4) by a new theorem due to Ibragimov. The concepts used here are from **Section 2.2.8 Ibragimov's conservation theorem** in Chapter 2. The adjoint equations for the nonlinear system coupled KdV Equations (4.1.3)-(4.1.4) are

$$\Delta_1^* \equiv f_t + \alpha\ u f_x + \alpha v g_x + \beta f_{xxx} = 0, \tag{4.3.26}$$
$$\Delta_2^* \equiv g_t - \alpha v f_x + \alpha u g_x + \beta g_{xxx} = 0. \tag{4.3.27}$$

The formal Lagrangian $\mathcal{L}$ for the nonlinear coupled system of the KdV Equations (4.1.3)-(4.1.4) and its adjoint Equations (4.3.26)-(4.3.27) is given by

$$\mathcal{L} = f\{u_t + \alpha u u_x - \alpha v v_x + \beta u_{xxx}\} + g\{v_t + \alpha u v_x + \alpha v u_x + \beta v_{xxx}\}, \tag{4.3.28}$$

where f and g are new variables. We shall use the Lie point symmetries of the system (4.1.3)-(4.1.4) ,namely

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \alpha t\partial_x + \partial_u, \quad X_4 = 3t\partial_t + x\partial_x - 2u\partial_u - 2v\partial_v, \tag{4.3.29}$$

to derive conserved vectors corresponding to each symmetry below.

Case (i) The symmetry $X_1 = \partial_t$ yields Lie characteristic functions given by $W_1^1 = -u_t$ and $W_1^2 = -v_t$. Hence the associated conserved vector is given by

$$\begin{aligned} T_1^t =&\alpha\,[f\{uu_x - vv_x\} + g\{vu_x + uv_x\}] + \beta\{fu_{xxx} + gv_{xxx}\},\\ T_1^x =&\alpha\,[f\{-uu_t + vv_t\} - g\{vu_t + uv_t\}]\\ &+ \beta\{f_x u_{tx} + g_x v_{tx} - u_t f_{xx} - v_t g_{xx} - f u_{txx} - g v_{txx}\}. \end{aligned} \tag{4.3.30}$$

Case (ii) The symmetry $X_2 = \partial_x$ yields Lie characteristic functions $W_2^1 = -u_x$, and $W_2^2 = -v_x$. Therefore the associated conserved vector is

$$T_2^t = -u_x f - v_x g, \tag{4.3.31}$$
$$T_2^x = fu_t + gv_t + \beta\{-u_x f_{xx} - v_x g_{xx} + f_x u_{xx} + g_x v_{xx}\}. \tag{4.3.32}$$

Case (iii) The symmetry $X_3 = \alpha t\partial_x + \partial_u$ yields Lie characteristic functions given by $W_3^1 = 1 - \alpha t u_x$, and $W_3^2 = -\alpha t v_x$. Hence the associated conserved vector is given by

$$\begin{aligned} T_3^t =& f - \alpha t\{u_x f + v_x g\}, \\ T_3^x =& \alpha\left[fu + gv + t\{u_t f + v_t g\} + \beta t\{\frac{f_{xx}}{\alpha t} - u_x f_{xx} - v_x g_{xx} + f_x u_{xx} + g_x v_{xx}\}\right]. \end{aligned} \tag{4.3.33}$$

Case (iv) The symmetry $X_4 = 3t\partial_t + x\partial_x - 2u\partial_u - 2v\partial_v$ yields the Lie characteristic functions: $W_4^1 = -2u - 3tu_t - xu_x$, $W_4^2 = -2v - 3tv_t - xv_x$. Consequently, the corresponding conserved vector is given by

$$\begin{aligned} T_4^t =& \alpha\left[3t\{fuu_x - fvv_x + guv_x + gvu_x\}\right] + \beta\left[3t\{fu_{xxx} + gv_{xxx}\}\right] \\ &- 2\{fu + gv\} - x\{fu_x + gv_x\}, \\ T_4^x =& x\{fu_t + gv_t\} + \beta\left[3\left(f_x u_x + g_x v_x + t\{f_x u_{tx} + g_x v_{tx}\}\right)\right] \\ &- \alpha\left[2\left(f\{u^2 - v^2\} + 2guv\right) + 3t\left(f\{uu_t - vv_t\} + g\{vu_t + uv_t\}\right)\right] \\ &- \beta\left[x\{u_x f_{xx} + v_x g_{xx} - f_x u_{xx} - g_x v_{xx}\} + 2\{uf_{xx} + vg_{xx}\}\right] \\ &- \beta\left[3t\{f_{xx}u_t + g_{xx}v_t + fu_{txx} + gv_{txx}\} + 4\{fu_{xx} + gv_{xx}\}\right]. \end{aligned} \tag{4.3.34}$$

Remark 4.3.8. The appearance of arbitrary functions $f(t,x)$ and $g(t,x)$ in the conserved vectors proves the existence of infinite conservation laws for coupled KdV system obtained by Ibagimov's method.

4.4 Concluding remarks

In this chapter, we have studied the nonlinear coupled system of KdV equations by use of Lie group analysis. Just like for the KdV Equation (3.1.3), we have symmetries that represent time and space translations, Galilean boost, and scaling for the nonlinear coupled KdV equation. For each symmetry, we performed a reduction of the nonlinear coupled system. All the group-invariant solutions describe the various states of the system. Last but not least, an infinite number of conservation laws were derived for the system by the multiplier approach and Ibragimov's conservation theorem.

Three of the conservation laws, just like those for the KdV equation in Chapter 3 manifest that mass, momentum, and energy are invariant quantities in the evolution of the coupled KdV system. In fact, only some of the first laws have a physical interpretation. Higher-order laws aid in understanding the qualitative properties of solutions. These conservation laws are very important in explaining the integrability of a system and the effectiveness of numerical methods used in approximating solutions. They can also be used to get exact solutions to the nonlinear coupled KdV system, which is a subject of future work.

5. Conclusion

This dissertation aimed to research on Lie group analysis of nonlinear partial differential equations (NLPDEs). It must be said that NLPDEs are very important in describing various phenomena and systems in real life. Our main concern in this study was the nonlinear coupled system of Korteweg-de Vries (KdV) equations that describe the dynamics of solitons. However, this powerful approach that uses invariant properties of an NLPDE to get exact solutions and conservation laws can be used for any NLPDE.

The first step was to develop mathematical concepts of Lie group analysis and conservation laws in Chapter 2. We developed pertinent notions of local Lie groups, prolongations, symmetry groups and invariants. Also introduced is the concept of conservation laws and two methods to compute them, that is, the method of multipliers and Ibragimov's conservation theorem.

Thereafter, we studied the KdV equation as an illustrative example in Chapter 3. This involved finding its space and time translations, Galilean boost, and scaling symmetries. We then performed symmetry reductions that resulted in a total of four group-invariant solutions. A linear combination of time and space translations also helped in finding the one-soliton solution for the KdV equation. Finally, we found conservation laws for mass, momentum, and energy by using the multipliers method.

Our third step was to investigate a system of coupled KdV equations in Chapter 4. Likewise to the KdV equation studied in Chapter 3, a four-dimensional Lie algebra of symmetries was found for the nonlinear coupled system KdV equations. This was also spanned by space and time translations, Galilean boost and scaling symmetries where the scaling symmetry acts on four variables. Associated to each symmetry, we obtained symmetry reductions that gave six nontrivial solutions for the coupled system. Lastly, we constructed infinite conservation laws of a nonlinear coupled KdV system by using multipliers and a theorem proposed by Nail Ibragimov. Three of these laws show that mass, momentum and energy are conserved quantities in the evolution of a nonlinear coupled KdV system.

The above results show a very interesting property of the KdV equation. Most important to note is that the infinite number of conservation laws for the coupled system show that the KdV equation is completely integrable, meaning that the behavior of the system can be determined by initial conditions and can be integrated from the prescribed initial conditions. Indeed, the KdV equation gives rise to multiple-soliton solutions thus emphasizing the importance of the KdV equation in the theory of integrable systems. The beautiful KdV equation is ubiquitous, having applications in various settings. The equation has been used to describe the dynamics of solitons, ion-acoustic waves in a plasma, shallow-water waves and nonlinear perturbations along internal surfaces between layers of different densities in stratified fluids, for example propagation of solitons of long internal waves in oceans. The KdV equation also models shock wave formation, turbulence, boundary layer behavior, and mass transport.

Further work

For future work, we are interested in applying the conservation laws to construct exact solutions to the nonlinear coupled system of Korteweg-de Vries equations.

Acknowledgements

I want to acknowledge AIMS and its funders for supporting this work, as well as my supervisor, Prof Chaudry Masood Khalique from North-West University.

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