Lecture 15, Mar 8, 2021

Note: Simplex Q1 on Final

LPs: max $z = c^T x$ $\quad c \in \mathbb{R}^n,\ b \in \mathbb{R}^m\ A \in \mathbb{R}^{m\times n}$

s.t. $A\vec{x} \le b$

$\vec{x} \ge 0$

D_I: $x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j$ $\quad x_{n+i}$ is the ith slack variable $i \in \{1,\dots,m\}$

$z = \sum_{j=1}^{n} c_j x_j$ $\quad j \in N \quad N$ is collection of non basic variables

D_B: $x_i = \hat{b}_i - \sum \hat{a}_{ij} x_j$ (check 2.7)

$z = \hat{z} + \sum \hat{c}_j x_j$ $\quad i \in B \quad B$ is collection of basic variables

B, N are index set contained in the set of integers $\{1, \dots, n+m\}$

B contains m elements and $B \cap N = \emptyset$.

$B \cup N = \{1, 2, \dots, m+n\}$

D_I, D_B have identical solution sets. The set $\{x_j, j \in B\}$ is said to be basis associated with (D_B) and variables $\{x_j, j \in N\}$ are said to be non-basic variables associated with this dictionary

$$\star \begin{cases} x_j = 0 & j \in N \\ x_i = \hat{b}_i & i \in B \end{cases} = \text{Basic solution of dictionary}$$

The above dictionary is said to be feasible if $\hat{b}_i \geq 0$ for every $i \in B$. If D_B is feasible, then the basic feasible solution is $\star$.

The BFS is optimal if $\hat{c}_j \leq 0$ for every $j \in N$.

Represent simplex pivoting by matrix multiplication//

$\vec{v} \in \mathbb{R}^n$ $\quad \vec{v} = \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$ $\quad a \in \mathbb{R}^s,\ b \in \mathbb{R}^t,\ \alpha \in \mathbb{R},\ \alpha \neq 0$

$s + t + 1 = n$

You want to find G s.t. $G\vec{v} = e_{s+1}$ — elementary row $(s+1)$

$$G = \begin{bmatrix} I_{s\times s} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{t\times t} \end{bmatrix} \qquad G^{-1} = \begin{bmatrix} I & a & 0 \\ 0 & \alpha & 0 \\ a & b & I \end{bmatrix}$$

G^{-1} is invertible — Homework

$G\vec{w} = \vec{w}$ if $\vec{w} = \begin{pmatrix}\vec{x}\\0\\\vec{y}\end{pmatrix}$, $\vec{x} \in \mathbb{R}^s$, $\vec{y} \in \mathbb{R}^t$

G is called Gaussian matrix

Ex.

a, α

$$T_0 = \begin{array}{cccccc|c}
1 & 4 & 2 & 1 & 0 & 0 & 11 \\
3 & 2 & 1 & 0 & 1 & 0 & 5 \\
4 & 2 & 2 & 0 & 0 & 1 & 8 \\
\hline
4 & 5 & 3 & 0 & 0 & 0 & 0
\end{array}$$

b

column of coefficents for next interation of simplex

$$G_1 = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -\frac{5}{2} & 0 & 1 \end{bmatrix}$$

$s=1$ (since $a \in \mathbb{R}$) $t=2$ (since $\vec{b} \in \mathbb{R}^2$) $a=4$ $\alpha=2$ $b=\begin{bmatrix}2\\5\end{bmatrix}$

$$G_1T_0 = \begin{array}{cccccc|c}
-5 & 0 & 0 & 1 & -2 & 0 & 1 \\
\frac{3}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{5}{2} \\
1 & 0 & 1 & 0 & -1 & 1 & 3 \\
\hline
-\frac{7}{2} & 0 & \frac{1}{2} & 0 & -\frac{5}{2} & 0 & -\frac{25}{2}
\end{array} \quad T_1$$

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$s=2$ $t=1$ $a=\begin{bmatrix}0\\-\frac{1}{2}\end{bmatrix}$ $\alpha=1$ $b=\frac{1}{2}$

$G_2T_1 = G_2G_1T_0 = \begin{array}{cccccc|c} -5 & 0 & 0 & 1 & -2 & 0 & 1 \end{array}$

$$\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 1 & -\frac{1}{2} & 1 \\ 1 & 0 & 1 & 0 & -1 & 1 & 3 \\ \hline -4 & 0 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2} & -14 \end{array}$$

optimal
since z-coef
all nevative.

If $T_0 = \begin{bmatrix} 0 & A & I & b \\ -1 & c^T & 0 & 0 \end{bmatrix}$ and $G = G_2 G_1$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -\frac{5}{2} & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & -\frac{1}{2} & 1 \end{bmatrix}$$

And $T_k = \begin{bmatrix} 0 & A & Q & \hat{b} \\ -1 & \hat{c}^T & -\hat{y}^T & \hat{z} \end{bmatrix} = G T_0$

$G = \begin{bmatrix} M & \vec{u} \\ \vec{v}^T & \beta \end{bmatrix}$, where $M \in \mathbb{R}^{m \times m}$, $\vec{u}, \vec{v} \in \mathbb{R}^m$, $\beta \in \mathbb{R}$

$T_k = G \cdot T_0$

$$= \begin{bmatrix} M & \vec{u} \\ \vec{v}^T & \beta \end{bmatrix} \begin{bmatrix} 0 & A & I & b \\ -1 & c^T & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\vec{u} & MA + uc^T & M & Mb \\ -\beta & v^T A + \beta c^T & v^T & v^T b \end{bmatrix}$$

$$= \begin{bmatrix} 0 & A & R & b \\ -1 & \hat{c}^T & -\hat{y}^T & \hat{z} \end{bmatrix}$$

$\beta = 1 \quad \vec{u} = \vec{0}$

Thus, $M = R \quad v^T = -\hat{y}^T$. and $G = \begin{bmatrix} R & \vec{0} \\ -\hat{y}^T & 1 \end{bmatrix}$

And $G^{-1} = \begin{bmatrix} R^{-1} & \vec{0} \\ \hat{y}^T R^{-1} & 1 \end{bmatrix}$

$$T_k = \begin{bmatrix} R & \vec{0} \\ -\vec{y}^T & 1 \end{bmatrix} \begin{bmatrix} 0 & A & I & b \\ -1 & c^T & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & RA & R & Rb \\ -1 & -y^T A + c^T & -\vec{y}^T & -\vec{y}^T b \end{bmatrix}$$

Optimal tableau of LP requires objective row to be negative

$\hat{c}^T - \hat{y}^T A \leq 0 \Rightarrow c^T \leq \hat{y}^T A \Rightarrow c \leq A^T \hat{y}$

$-\hat{y}^T \leq 0 \Rightarrow \hat{y}^T \geq 0$ with $z = b^T \hat{y}$

$A^T \hat{y} \geq c$ and $0 \leq \hat{y}$ with $z = b^T \hat{y}$

or

$Ax \leq b$ and $0 \leq \hat{x}$ with $z = c^T \hat{x}$

Thm: Let $\hat{x}$ be the basic feasible solution for the tableau (LP)

Lecture 16, Mar 10 2021

Example of D_I, D_B

Let LP be: max $5x_1+4x_2+3x_3=z$

s.t. $2x_1+3x_2+x_3 \le 5$

$4x_1+x_2+2x_3 \le 11$

$3x_1+4x_2+2x_3 \le 8$

$\vec{x} \ge 0$

To conntical: max $z=5x_1+4x_2+3x_3$

$2x_1+3x_2+x_3+x_4=5$

$4x_1+x_2+2x_3+x_5=11$

$3x_1+4x_2+2x_3+x_6=8$

$\vec{x}\ge 0$

So $A=\begin{bmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{bmatrix}$ $b=\begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix}$

Then D_I: $x_4 = 5-(2x_1+3x_2+x_3+x_4)$

$x_5 = 11-(4x_1+x_2+2x_3+x_5)$

$x_6 = 8-(3x_1+4x_2+2x_3+x_6)$

initial dic.

Since $n=3$

$$i=1 \quad x_4 = x_{3+1} = b_1 - \sum_{j=1}^{3} a_{ij} x_j$$

$$i=2 \quad x_5 = x_{3+2} = b_2 - \sum_{j=1}^{3} a_{ij} x_j$$

$$i=3 \quad x_6 = x_{3+3} = b_3 - \sum_{j=1}^{3} a_{ij} x_j$$

and $z = \sum_{j=1}^{3} c_j x_j = 5x_1 + 4x_2 + 3x_3$

Then D_B: $\quad x_i = \hat{b}_i - \sum_{j \in N} \hat{a}_{ij} x_j \quad i \in B$

iteration of D_I $\quad z = \hat{z} + \sum_{j \in N} \hat{c}_{ij} x_j$

Consider trivial case: do nothing

Then $N = \{1, 2, 3\}$, $B = \{4, 5, 6\}$

$$x_i = \hat{b}_i - \sum_{j \in N} \hat{a}_{ij} x_j \quad i \in B$$

$$\Rightarrow \hat{b}_i = b_i$$

Basic solution is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$

For non trivial case: $x_1 \to$ basis
$x_4 \to$ leaves basis

based on pivot column/row

So $N = \{2, 3, 4\}$, $B = \{1, 5, 6\}$

$$x_1 = \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \quad - a_{ij}$$
$$x_5 = 1 + 2x_4 - 5x_2$$
$$x_6 = \frac{1}{2} + \frac{3}{2}x_4 - \frac{1}{2}x_2 - \frac{1}{2}x_3$$

$$z = \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3$$

$$\left(-\frac{5}{2} \;\; -\frac{7}{2} \;\; \frac{1}{2}\right) = \hat{c}_{ij}$$

$$\left(\frac{5}{2} \;\; 1 \;\; \frac{1}{2}\right) = \hat{b}_i$$

$$\frac{25}{2} = \hat{z}$$

Thus we have basic solution

$$\begin{bmatrix} \frac{5}{2} \\ 0 \\ 0 \\ 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

Optimal Dictionary Tk.

$$T_k = \begin{bmatrix} 0 & RA & R & Rb \\ -1 & c^T - \hat{y}^T A & -\hat{y}^T & -\hat{y}^T b \end{bmatrix}$$

is optimal if:

1) T_k is feasible

2) $A^T \hat{y} \geq c$ and $\hat{y} \geq 0$ with $z = b^T \hat{y}$

The basic solution associated with this tableau $\vec{x}$, geilds a vector

$$\hat{x} = \hat{x}_j = x_j \quad j = 1, \dots n$$

$$\text{s.t. } A\hat{x} \leq b \text{ and } 0 \leq \hat{x}$$
$$\text{with } z = c^T \hat{x}$$

If you check ☆. $c^T \hat{x} = z = b^T \hat{y}$

Thm// Let x be the basic feasible solution for T_k. If T_k is optimal for given LP, then $\hat{y}$ is an optimal solution to the dual problem D and vector $\hat{x} \in \mathbb{R}^n$ given by $\hat{x}_j = x_j$, $j = 1, \dots n$ is an optimal solution to LP.

Consequence:

① $FR = \emptyset$

② FR unbounded

③ Simplex cannot find optimal solution with finite steps. (Looping)