MATC32 Notes

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September 2020

Abstract

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Graph Theory and Algorithms for its Applications

Contents

1 Introduction

In this course, we are analyzing graphs, not traditional graphs but tree and network structures.

1.1 Basic Terms

Vertex: aka. node, points, arcs are the points on a graph.

Edges: aka. bonds, lines are the connections that connect each vertex.

Degree: the number of connections per vertex.

Order: Number of vertices.

Size: Number of edges.

$\mathbf{K_{n,m}}$: is a graph with **order** $n + m$ and **size** $n * m$.

Neighbor: $u, v \in V$, u is a neighbor of v if $uv \in E$.

Complete graph: a graph in which all nodes have max degrees.

Adjacent edges: 2 edges are adjacent if they share a vertex.

Each graph has two sets:
V : the non-empty set that includes all the vertices.
E : the set that includes all the edges. Note: $ab = ba \in E$

Connected: A graph is connected if for every $u \neq v \in V$ there exists a $u - v$ walk. Connection implies:

$$u \leftrightarrow u \text{ reflexivity}$$
$$u \leftrightarrow v \Rightarrow v \leftrightarrow u \text{ symmetry}$$
$$u \leftrightarrow v, v \leftrightarrow w \Rightarrow u \leftrightarrow w$$

1.2 Subgraphs

Subgraph: H is a subgraph of G if $V_H \subset V_G$, $E_H \subset E_G$.

There are a few types of subgraphs: for $H \subset G$
Proper: If H is not equal to G.

$$H \neq G$$

Spanning: If all of G's vertices exist in H.

$$V_H = V_G$$

Induced: If an edge exists in G, it exists H provided vertices exists.

$$u, v \in V_H, uv \in E_G \Rightarrow uv \in E_H$$

1.3 Walks

A walk is a sequence of nodes that are connected.

Open/Closed: A walk is closed if the first node is also the last.

Trail: A walk with no repeated lines. Note: nodes can repeat.

Circuit: A closed trail.

Path: A walk with no repeated nodes.

Cycle: A closed path.

1.4 Operations on Graphs

Union:(Disjoint) Two graph are put next to each other; nodes are not connected.

Join: Nodes are connected.

$$V = V_{H1} \cup V_{H2}$$

$$E = E_{H1} \cup E_{H2} \cup \{uv : u \in V_{H1}, v \in V_{H2}\}$$

Cartesian Product: Use one graph as the other graph's nodes.

$$\text{for } G_1 \times G_2 = H$$

$$V_H = \{(v_1, u_1) : v_1 \in G_1, u_1 \in G_2\}$$

$$(v_1, u_1)(v_2, u_2) \in E_H \iff$$
$$1.\ v_1 = v_2, u_1u_2 \in E_2$$
$$2.\ u_1 = u_2, v_1v_2 \in E_1$$

1.5 Bipartite Graphs

Def: A union-ed graph where all connected nodes are from different sets.

G **is a bipartite graph if,**
for partitioned vertices $V = V_1 \cup V_2$

There is no edge from V_1 to V_1 or V_2 to V_2.

Thm : G is a **bipartite** graph iff it contains no odd cycle.

Proof:

Backwards
We know odd cycles are not bipartite thus if graph contains an odd cycle, it is not bipartite.

Forwards
If the graph is bipartite, prove it does not contain an odd cycle.

First assume that G contains an odd cycle. Proof component by component.

If G has a single component, fix vertex v.
Let V_1 : vertex of even distance from v and,
Let V_2 : vertex of odd distance from v.

This shows that $V(G)$ is a partition of V_1, V_2 that intersects v.

Now to show that there is no edge from V_1 to V_1 or V_2 to V_2.

Let $x, y \in V_1$, assume there is a edge xy.
$d(v, x) \leq d(v, y) + 1$ and $d(v, y) \leq d(v, x) + 1$. Since $x, y \in V_1$ should be even distance from v, thus $d(v, x) = d(v, y)$.

Now we've found a cycle of $v - x - y - v$, which it's distance is $d(v, x) + d(v, y) + d(xy) = 2n + 2n + 1 = 2(2n) + 1$ which is odd, thus a contradiction. Thus G contains no odd cycle.

QED.

2 Degree sequences, regular graphs

2.1 Properties of Degrees

Thm : First theorm of graph theory, for graph G

$$\sum_{v \in V} deg(v) = 2|E|$$

Every graph has even degree sum is equivalent to,
the number of odd degree vertices is even.

Proof: Count edges by their end points

$$\begin{aligned} \sum_{v \in V} deg(v) &= \sum_{v \in V} \sum_{e \in E}, \text{number of e for each v} \\ &= \sum_{e \in E} \sum_{v \in V}, \text{number of v for each e} \\ &= \sum_{e \in E} 2 = 2|E| \end{aligned}$$

QED

The minimal degree is leq than the average degree is leq than the maximum degree.

Degree Notation: $\delta(G) \leq \overline{d}(G) \leq \Delta(G)$

Crlly : If $\delta(G) \geq \frac{n-1}{2}$, then G is connected.

Proof: For $v, w \in V, deg(v) + deg(w) \geq \delta(G) + \delta(G) \geq n - 1$ QED.

Thm : If $deg(v) + deg(w) \geq n - 1$ for every pair of non-adjacent vertices v, w, then G is connected.

Proof: We need to show $v \neq w, \in V$, there exists v-w path.

Case 1: If v, w are adjacent, then vw is the path.

Case 2: If v, w are not adjacent, there are $n - 2$ other nodes who are distinct neighbours of v, w, so at most $deg(v) + deg(w) \leq n - 2$. Thus for the two to be connected there must be a common neighbour u of v, w with forms the vuw path. Thus $deg(v) + deg(w) \geq n - 1$ QED.

2.2 Regular Graphs

Def : If $\forall v \in V, deg(v) = r$, G is r-regular.

In regular graphs min degree = average degree = max degree = r.

Thm : A r-regular graph of order n can only exist if,

1. n or r is even.
2. $r < n$.

Proof: Use Harori graphs.
For r<n
Case 1: r is even, n is arbitrary
$V = \{1, \ldots, n\}$
$E = \{vw : w \in \{v - \frac{r}{2}, v - \frac{r}{2} + 1, \ldots v + \frac{r}{2} : \mod n\}\}$

There are r neighbours for each v.

Case 2: r is odd, n is even
$V = \{1, \ldots, n\}$

$E = \{vw : w \in \{v - \frac{r-1}{2}, v - \frac{r-1}{2} + 1, \ldots v + \frac{r-1}{2} : \mod n\} \text{ or } w = v + \frac{n}{2}\}$

There are r-1 neighbours for each v and v connected to the node opposite to it.

> **Thm :** The Bollobas model for generating n-order r-regular graphs is drawing n nodes with r edges and connecting those edges randomly. This algorithm generates an uniform distribution of graphs.

The problems with this graph is that there could be node that form its own component and there could be self loop where node connects with itself.
Usually in implementation, if that happens, just reset and start again.

> **Thm :** For any graph G and $r \geq \Delta(G)$, there exists a r-regular graph H such that G is a induced subgraph of H.

Proof:
Case 1: $\delta(G) = r$. Then we are done because $\delta(G) = r = \delta(H)$ and H is r-regular.

Case 2: $\delta(G) < r$. Take two copies of G denoted G_1, G_2.

For G_1, G_2, let v_1, v_2 denote the two copies of $v \in G$.
If $deg(v) < r$ connect v_1, v_2 and do this for any $v \in V$.

Now for the new graph $G^{'}$, $\delta(G^{'}) = \delta(G)+1$. If $G^{'}$ is regular is we are done. If not repeat this process with $G^{'}$.

Note: Every rep increases $\delta(G)$ by one so it will stop in $r - \delta(G)$ repetitions. QED.

2.3 Graphical Sequences

Def : Degree Sequence
A sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ is a graphical sequence if there is a graph with these degrees.

Not all sequences are degree sequences, ex:

1. sum of sequence is odd
2. odd number of odd degrees
3. biggest number = order of graph

Look for contradictions in the biggest and smallest numbers.

For n nodes, there are $2^{\binom{n}{2}}$ possible graphs.

Use the following algorithm to know a graphical sequence is a graph,

Thm : A sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ is a graphical iff $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots d_n$ is also graphical.

Proof:
Forwards
For seq1→seq2, assuming seq2 exists. We can add a node v_1 to seq2 such that $deg(v_1) > \Delta(\text{seq2})$. By connecting v_1 to the largest nodes in seq2, we can create seq1.

Backwards
We want to prove that the neighbors of v_1 has largest sum of degrees. (Eg. $d_2, \ldots, d_{d_1+1}, \ldots, d_n$ is in decreasing order).
Assume the contrary, there exists a neighbor v_s of v_1 which has lower degree than non-neighbor v_t.
Then there exists v_r such that $v_r v_t \in E$ but $v_r v_s \notin E$.

By switching the connections between the edges above creating $G^{'}$, the degree sequence doesn't change but there is a contradiction.

Since v_1 lost v_s but gained v_t and $deg(v_t) > deg(v_s)$, the total degree of neighbors of v_1 in G is less than $G^{'}$ but we assumed that G is the largest thus neighbors of v_1 must have the largest sum of degrees.

QED.

Thm : Any graph must have 2 vertices with the same degree.

Proof:

In any graph the possible degrees are: $0, 1 \ldots, n-1$.
The only way for this to happen is if each vertex has degree:$n-1, n-2, \ldots, 0$. This is impossible since a graph can't have $n-1$ connections and 0 at the same time.
Also the degree sequence $n-1, n-2, \ldots, 0 \rightarrow n-2-1, n-3-1, \ldots, -1$ is impossible. QED.

2.4 Adjacency Matrices

Def: For $V = \{v_1, v_2, \ldots, v_n\}$, the adjacency matrix of graph G is the $n \times n$ matrix $A : a_{ij} = \begin{cases} 1 & v_i v_j \in E \\ 0 & v_1 v_j \notin E \end{cases}$
Note: $a_{ii} = 0$

Rows and columns represent vertices. This matrix is always symmetric.
Row sum is degree of vertex.

Thm : Let A be adjacency matrix for graph G. $A \times A \times \cdots \times A = A^k$. A^k_{ij} is the number of v_i-v_j walks of length k.

Proof: For graph G of size n, $(A^{(1)} \times A^{(2)})_{ij} = \sum_{k=1}^{n} A_{ik}^{(1)} A_{kj}^{(2)}$

$((A^{(1)} \times A^{(2)}) \times A^{(3)})_{il} = \sum_{k=1}^{n} \sum_{j=1}^{n} A_{ik}^{(1)} A_{kj}^{(2)} A_{jl}^{(3)}$

Thus in general
$(A^{(1)} \times A^{(2)} \times \cdots \times A^{(k)})_{il} = \sum_{j_1, j_2, \ldots, j_{k-2}=1}^{n} A_{ij_1}^{(1)} A_{j_1 j_2}^{(2)} \ldots A_{j_{k-1} l}^{(k)}$

So $v_i v_{j_1} \ldots v_{j_l}$ is a walk. QED.

2.5 Incidence Matrix

2.6 Incidence Matrices

Def: The incidence matrix of graph G is the $n(order) \times m(size)$ matrix $B : b_{ij} = \begin{cases} 1 & v_i \in e_j \\ 0 & o/w \end{cases}$

Rows represent vertices and columns represent edges of the graph. Row sum is still degree for vertex. Column sum is always 2.

Prop: $BB^t = D + A$
Where D is a diagonal matrix with degrees, and A is the adjacency matrix.

Thm : G is a connected graph, the number of walks from $v - v$ of length $2k$ is,

$$\lambda_1^{2k(1+O(1))}$$

as $k \to \infty, O(1)$ quantity $\to 0$
λ_1 is the top eigenvalue of A.

Def : Signed incidence matrix J
$W_{2k} = \lambda_1^{2k(1+O(1))} \Leftrightarrow \sqrt[2k]{W_{2k}} \to \lambda_1$ as $k \to \infty$
For each edge $e_1, \ldots, e_m$, pick a direction for each $e_i = v_j v_k$

$$J : j_{ij} = \begin{cases} 1 & v_i \in e_j second \\ -1 & v_i \in e_j first \\ 0 & o/w \end{cases}$$

Rows represent vertices and columns represents incoming(1) and outgoing(-1) egdes.

Prop: $JJ^t = D - A$
Where D is a diagonal matrix with degrees, and A is the adjacency matrix.

Lemma: Determinate on Signed Incidence Matrix
$JJ^t[1, \ldots, 1] = 0 \Leftrightarrow JJ^t$ row sum $= 0 \to det(JJ^t) = 0$.

Since D is the diagonal matrix of degrees and A's rows sum to the degrees, thus when multiplied by $[1, \ldots, 1]$, rows sum to 0.

3 Isomorphisms of graphs

A graph G_1 is isomorphic to G_2 if renaming the vertex set on G_1 gives G_2.

Def : Graph Isomorphism
G_1 is isomorphic to G_2 if for isomorphic function:

$$\alpha : V_1 \to V_2$$

For $u, v \in V_1$

1. α is a bijection(one to one, onto)
2. $\alpha(u)\alpha(v) \in E_2 \Leftrightarrow uv \in E_1$

Further if two graphs are isomorphic, we know that they have the same orders, sizes, degrees.

Def : Complementary Isomorphism
G_1 is isomorphic to G_2 if and only if $\bar{G_1}$ is isomorphic to $\bar{G_2}$, their complements.

Def : Graph Automorphism, $Aut(G)$
An automorphism of G is an isomorphism of G to itself.

If a graph is automorphic, it is symmetrical.

Observe that $Aut(G)$ is a group. Eg. $\alpha, \beta, \gamma \in Aut(G)$:

1. $\exists \epsilon \in G, \alpha \circ \epsilon = \epsilon \circ \alpha = \alpha$
2. $\alpha \circ \beta \in Aut(G)$
3. $\alpha^{-1} \in Aut(G)$
4. $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$

Thus this leads to,

Thm: Frucht Theorem
Every finite group Γ is the automorphism for some graph G.
(All group theory is graph theory.)

3.1 Graph Reconstruction

What is **Quasi-Polynomial** time?

Polynomial $e^{log(n)} = n, e^2 log(n) = n^2, e^3 log(n) = n^3 dots$

Quasi Polynomial $e^{log(n)^\alpha}, \alpha \in \mathbb{R}$

Exponential $e^n = e^n$

Graph Reconstruction Problem
With a graph G with at least 3 vertices, let there be $n = |V|$ subgraphs of G where each subgraph, G/e_i has one vertex deleted. Without knowing what G looks like, piece G/e_i together.

Def: Recognizable Parameter
A graph parameter is recognizable if it is determinable by the isomorphic classes of $G/v, v \in V$.

The **order** is recognizable in the Graph Reconstruction Problem.
Since the number of isomorphism classes n, of G/v is equal to $|V|$.

The **size** is recognizable.
For each isomorphic class the number of edges is $|E_i| = |E| - d_i$, where d_i is the degree of the ith vertex. Then $|E_1| + |E_2| + \cdots + |E_n| = |E| - d_1 + \cdots + |E| - d_i = n(|E|) - (d_1 + \cdots + d_n) = n|E| - 2|E| = (n-2)|E| \rightarrow |E| = \frac{|E_1| + |E_2| + \cdots + |E_n|}{n-2}$.

Note: $n > 2$.

The **degree sequence** is recognizable since $|E_i| = |E| - d_i \rightarrow d_i = |E| -$

$|E_i|$.

The **connectivity** is recognizable.

> **Thm** Recognizable Connectivity
> For a graph $G, n = |V| \geq 3$, is connected if and only if $\exists v, w$ such that $G/v, G/w$ are both connected.

Eg. If two subgraphs are connected, then the graph is connected. Otherwise not.

Proof:
$\Leftarrow$

Let there exist two vertices $v, w : G/v, G/w$ are connected. We must find a u_1-u_2 walk for any vertex u_1, u_2 in G. There must exists $u^{'} \in G/v, G/w$ such that $u^{'}$ is connected to all vertices not v, w as $G/v, G/w$ are connected. Lastly for v, w, we can form a vuw walk thus G is connected.

$\Rightarrow$
Let graph G be connected. Pick two vertices $v, w : dist(v, w) = diam(G)$.

We only need to prove one way by symmetry of argument. Show G/w is connected. Suppose not.
So that means there exists $u_1, u_2 \in G/w$ such that u_1, u_2 are not connected. Then there's no v, u_1 or v, u_2 path.

Then u_1, u_2 must be connected in G with a path u_1vwu_2 which is a path containing w that is longer than $dist(v, w)$. This is a contradiction since we assumed that $dist(v, w) = diam(G)$.

QED.

> **Thm:**
> If G is a connected graph and $e \in E$ is on a cycle of $G \Leftrightarrow e$ is not a bridge.

Proof:
Forwards
If G is connected and e is on a cycle.
Let that cycle be $c_1c_2\dots c_ic_{i+1}\dots c_l = c_1$ where $e = c_ic_{i+1}$. Then for any $u-v$ walk that contained e in G, $u-v$ is still connected by $c_{i+1}\dots c_lc_2\dots c_i$ or reverse in G/e. Thus G/e is still connected.

Backwards
If e is not a bridge. Then G/e is connected. Find arbitrary path $u_1\dots u_l$ on G/e. Let $e = u_1u_l$ and thus e is in a cycle in G and is also connected.
QED.

4 Trees

4.1 Definitions

Forest: An acyclic graph.

Tree: A connected forest.

Leaf: A vertex with degree 1.

Bridge: An edge $e \in E$ such that G/e is disconnected.

Thm: Recognizable Trees
Trees are recognizable.

4.2 Theorems On Trees

Thm: Trees are unique paths
If G is a tree if and only if there is a unique path between between any two vertices u, v.

Proof:
Forwards
If G is a tree, it is connected and has no cycles. Show unique path between between any two vertices u, v.

Assume the contrary, there are multiple paths from u, v, $Q, Q^{'}$. Then cycle $uQvQ^{'}u$ exists thus by contradiction there is only one path.

Backwards
If for any u, v, there exists a unique path that connects them thus by def G is a tree. QED.

> **Thm:** Edge and Order of Trees
> For trees $m = n - 1$

Proof:
We want to prove every tree with $n > 1$ has a leaf. Thus we can build a tree by adding leafs which proves the theorem.

So let $w_0 \dots w_l$ be the longest path on tree G. Claim w_0, w_l are leafs.
Let's assume the contrary and let $deg(w_l) > 1$, so it has neighbours w_{l-1}, v. v can not be on the path of $w_0 \dots w_l$ or it will be a cycle but if it's not, we found a longer path $w_0 \dots w_l v$ thus a contradiction. By symmetry, w_0, w_l are leaves.

> **Thm:** Leaves on Trees
> There are at least 2 leaves on every tree.

Now by induction, tree with $n = 1$ holds,
Assuming tree with order $n - 1$ holds, prove tree with order n,

We added a leaf v in this tree such that G/v has order $n - 1$ thus for G, $m = n - 2 + 1 = n - 1$. QED.

5 Connectivity and Menger's theorem

Thm: Connected Graph Contains Spanning Tree
Every connected graph G, contains a spanning tree T.

Proof:
By induction, n=1 is true, assume n=k is true,
Prove n=k+1 is true,

Case 1: all edges bridges, then by def G is already a tree.

Case 2: some edge, v, is not a bridge, then G/v is still connected and T is also a spanning tree.

Thm: Size $\geq$ Order -1
Every connected graph satisfies $m \geq n - 1$.

Proof:
Since every connected graph G contains a spanning tree T. $E(G) \geq E(T) = n - 1$ by Thm Edge and Order of Trees.

Thm: Size of Forest
For every forest, $m = n - k$ where k is the number of components

Proof:
Let n_i, m_i be the order and size of each component. Then,

$m = \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} n_i - 1 = \sum_{i=1}^{k} n_i - k = n - k$.

Thm: Three Conditions Theorem
G is a tree if at least 2 of the 3 condition hold.

1. G is connected
2. G is acyclic
3. m=n-1

Proof:
1,2 proven by definition
2,3 2 says G is a forest, 3 implies k $= 1$ and by above theorem, there is only one component thus G is connected
1,3 if G is connected. Assume G has a cycle with e which is not a bridge, then G/e is still connected and $E(G/e) = n - 2 < n - 1$ thus there are no cycles.
QED.

5.1 Matrices in Graph Theory(Not in textbook)

Let $det^*(M)$ be the determinant of M with the last row and column removed.

Recall J is the signed incidence matrix $n \times m$ indexed by $V \times E$.

5.2 Matrix Tree Theorem

Thm: Kirkhoff's Matrix Tree Theorem
$T(G) = det^*(D - A) = det^*(JJ^t)$
The number of spanning trees in G is the determinant* of JJ^T, the combinatorial Laplacian, where J is the signed incidence matrix.

Proof: Simple Cases

Case 1: G is disconnected. Then $T(G) = 0$ since there are no spanning *trees.*

Show $det^*(D - A) = 0$
If G is disconnected, there exists v_1, v_2 where there isn't an edge between them. Then A becomes a block diagonal matrix with components A_1, A_2 partitioned by v_1, v_2,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$
$$D - A = \begin{bmatrix} D_1 - A_1 & 0 \\ 0 & D_2 - A_2 \end{bmatrix}$$

Where A_1, A_2 is the adjacency matrix of components G_1, G_2 induced by v_1, v_2. Thus $det^*(D-A) = det(D_1-A_1) \times det^*(D_2-A_2) = 0 \times det^*(D_2-A_2) = 0$ by Lemma $det(D - A) = 0$.

Case 2: G is a tree, it is the only tree.
Show $det^*(D - A) = 1$

Thm: Cauchy-Binet Theorem
For $n \times m$ matrix A and $m \times n$ matrix B,

$$det(AB) = \sum_{I \in \{1,\ldots,m\}, |I|=n} det A_I det B^I$$

$det(AB)$ equals the sum of determinant of $n \times n$ sub-matricies A_I, B^I where A^I is the matrix given by columns of I and B^I is the matrix given by the rows of I.

Proof: Matrix Tree Theorem
If $m = n - 1$ then G is disconnected or a tree. Then $det(D - A) = 0, 1$.

Using Cauchy-Binet Theorem

$$\begin{aligned} det^*(D-A) &= det^*(JJ^t) \\ &= det(\hat{J}\hat{J}^t) \end{aligned}$$

Where $\hat{J}$ is the $n \times n$ matrix J with the last row removed. The two are equivalent since you have to remove the last row and column in det^*.

$$\begin{aligned} &= \sum_{I \in E, |I|=n-1} det\hat{J}_I det\hat{J}^{It} \\ &= \sum_{I \in E, |I|=n-1} det\hat{J}_I \hat{J}^{It} \\ &= \sum_{I \in E, |I|=n-1} det^*(D_I - A_I) \end{aligned}$$

So $(D_I - A_I)$ is the matrices for the subgraph with vertex set V and edge set I sized $n-1$. If the subgraph indeed is a tree, $det^*(D-A) = 1$, and if it's disconnected, then $det^*(D-A) = 0$ proven above. Thus the sum of all Is is the number of spanning trees. QED.

5.3 Cayley's Theorem

Example Carley's Theorem
Find $T(K_n)$, the number of spanning trees of an nth order complete graph.

First way, using the Matrix Tree Theorem
Then $T(K_n) = det^*(D-A)$ where D is the diagonal matrix with $n-1$ across the diagonal since every vertex is connected with every other vertex and A is a matrix of 1s except for the diagonal which is filled with 0s.

Then $\hat{D} - \hat{A}$ which are $(n-1) \times (n-1)$ matrices with the last row and columns removed equals $(n-1)I - (M-I) = nI - M$ where M is the matrix of all 1s. Thus $T(K_n) = det(nI - M)$ and to determine this value, we use the fact that the determinant equals the product of the eigenvalues.

Obeservation, adding the identity to a matrix adds 1 to all the eigenvalues. So we need to find the eigenvalues of $-M$, and since $-M$ has rank 1(all the columns are the same), it's eigenvalues is some $\lambda \neq 0$ and the rest are 0 with multiplicity $n-2$. Thus $-M$ has eigenvector $v = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and $-Mv = \begin{pmatrix} n-1 \\ \vdots \\ n-1 \end{pmatrix} = -(n-1)v$ and eigenvalue $0, \dots, 0, -n+1$. Finally the eigenvalues of $nI - M$ is $n, \dots, n, 1$ so $T(K_n) = det(nI - M) = n^{n-2}$. QED.

Thus,

Def: Cayley's Theorem
For complete graph of order n, K_n,

$$T(K_n) = n^{n-2}$$

Define a new graph operator,

Def: Graph Mod e
For graph G, for edge $e \in E$, define G/e a multi-graph of where the vertices u, v that form e are combined into w and the edges that went into uv, now go into w.

Def: Another Way to Find Spanning Trees
For any graph G and edge e,

$$T(G) = T(G-e) + T(G/e)$$

Proof:
To proof this we need a bijection between the two sets.

1) Tree T that does not contain e:

T is a spanning tree of G that does not contain $e \leftrightarrow T$ is a spanning tree of $G - e$. This is obvious.

1) Tree T that contain e:
$T^{'}$ is a spanning tree of $G/e \Leftrightarrow T^{'} + e$ is a spanning tree of G containing e.
Where $T^{'}$ is a spanning tree of G/e.

Forwards
If $T^{'}$ spans $G/e \Leftrightarrow T^{'} + e$ spans G.
If $T^{'}$ has size $n-2$(because it's a spanning tree in G)$\rightarrow T^{'}+e$ has size $n-1$ and since $T^{'} + e$ is connected, it is also a tree.

Backwards
Is the same proof.QED.

5.4 More Spanning Tree Algorithms

5.4.1 Minimal Spanning Tree Problem

In a weighted graph(edges have cost), find the spanning tree with least cost.

> **Thm:** Kruskal's Algorithm
> For graph G to find the minimal spanning tree T,
> Keep adding lowest weight edges and skip edge if cycle is formed.
> This algorithm is not unique since multiple edges with the same weight have equal chance of being chosen.

Issue: Hard to decide is cycle is formed.

Proof:
Define $w(e)$ as the weight of an edge $e \in E$ and the weight of tree T, $w(T) = \sum_{e \in E} w(e)$ where $w(T) \leq w(S)$ for any other spanning tree S.

For graph G, create new minimal spanning tree T with Kruskal's Algorithm and let $e_1, \cdots, e_n$ be the sequences of edges added to it. Let S be a

minimal spanning tree of G so that it has the largest possible number of shared edges with T. Show $S = T$, assume not by contradiction.
Let index j be so that $e_1, \ldots e_{j-1}$ are edges used by S and e_j is the first edge used by T but not by S.

Now if we add e_j to S, we would create a cycle C and for every edge e in C, $w(e) \leq w(e_j)$ since S is minimal.
There must exist an edge $e^{'} \in C$ that's not in T but is in S. So $w(e^{'}) \leq w(e_j)$ as shown above.

We claim that graph $e_1, \ldots, e_{j-1}, e^{'}$ has no cycle. Thus given this claim, it's impossible $w(e^{'}) < w(e_j)$ since we chose edge e_j when $e^{'}$ was available since T is following Kruskal's Algorithm. Thus $w(e_j) = w(e^{'})$

Then $w(S + e_j - e^{'}) = w(S)$ and $S + e_j - e^{'}$ is also a minimal spanning tree but has one more edge in common with T than S thus a contradiction and $S = T$. QED.

Thm: Prim's Algorithm
For graph G to find the minimal spanning tree T,
Pick a starting vertex and add the smallest weighted edge to a neighbouring un-visited vertex. Then repeat with current set of nodes.

Proof:
For graph G, create new minimal spanning tree T and let $e_1, \cdots, e_{n-1}$ be the sequences of edges added to it. Let S be a minimal spanning tree so that it has the largest possible number of shared edges with T. Let index j be so that $e_1, \ldots e_{j-1}$ are edges used by S and e_j is the first edge used by T but not by S.

Now if we add e_j to S, we would create a cycle C and there must exist an edge $e^{'} \in C$ that's not in T but is in S. This cycle has some vertices of T_{j-1} and some not but contains all of S_{j-1}.

Now if we pick e_j as our next edge such that $T_j = e_1 \ldots e_j$ this implies $w(e_j) \leq w(e^{'})$. Then we have new graph $S^{'} = S - e^{'} + e_j$.

Finally $w(S^{'}) \leq w(S)$, so S' is minimal but it has one more edge in common with T that S thus a contradiction. QED.

Thm: Max Weighted Cycle Algorithm
For graph G where all weights are different, find the minimal spanning tree T,
For any cycle of G, erase the maximal weight in the cycle.

5.5 Cut Vertices

Def: Cut Vertex
A cut vertex $v \in V$ of a connected graph is a vertex v such that $G-v$ is disconnected.

If a vertex is on a bridge it's not always a cut vertex.

Thm: Cut Vertex Is Not Leaf
Given a bridge $e \in E$, a vertex $v \in V$ incident to e.
v is a cut vertex if and only if v is not a leaf.

Proof:
Let vertex v be incident to bridge $e = vw$.

Forwards
If v is a cut vertex, show v is not a leaf. Since v is a cut vertex, $G - v$ is disconnected.

If we assume v is a leaf, then $G - v$ is connected since $deg(v) > 1$ which is e. Thus v is not a leaf.

Backwards
If v is not leaf, show v is a cut vertex.

Show $G - v$ is disconnected. Since v is incident to a bridge and $deg(v) > 1$, there exists u connected to v not incident to $e = vw$. Thus w, u are not connected in $G - v$. So v is a cut vertex.

QED.

Thm: Cut Vertices in Connected Graph
Every connected graph with $|V| \geq 2$ has at least 2 vertices that are not cut vertices.

Proof:

Lem: Diameter of Cut Vertices
For $u, v \in V$, if $dist(u, v) = diam(u, v)$, then u, v are not cut vertices.

Thus by lemma, there are exist $u, v \in V$ such that $dist(u, v) = diam(G)$.

QED.

Col: Cut Vertices In Complete Graph
There are no cut vertices in a complete graph K_n since every pair of vertices has $diam(K_n) = 1$.

5.6 Separability

Def: Separable
A graph is separable if it contains a cut vertex.

Thm: Separable
A graph $n \geq 3$ is inseparable if and only if every 2 vertices lie on a common cycle.

Proof:

Backwards
If every vertex lie on a common cycle, show it contain no cut vertex. If $C = u_1 \dots u_{i-1}u_iu_{i+1} \dots u_n = u_1$, for arbitrary vertex $u_i \in V(G)$ show $G - u_i$ is connected.

Then on $G - u_i$, there exists a $u_{i+1} \dots u_n \dots u_{i-1}$ path for any v, w on C, thus is connected.

Forwards
If graph G with $n \geq 3$ is inseparable; $G - v$ is connected for any $v \in V$, show every 2 vertices lie on a cycle.

Prove by induction;

For $n = 3$, there is only one connected graph that is inseparable K_3, which is a cycle.

Assume $n = k$ on G_k holds, where every 2 vertices lie on a common cycle.

Prove if $n = k+1$ and is inseparable, then every 2 vertices lie on a common cycle.

Let G_{k+1} be the graph where vertex w is attached to arbitrary vertex u on G_k. Let v be another arbitrary vertex on G_k. Then there are two unique paths P_1, P_2 that connect v, u on G_k as shown above. Since G_{k+1} inseparable, w can not be a leaf since if it was a leaf $G_{k+1} - u$ would be disconnected. Thus there must exist another path from v to w that does not contain u, call it P_3.

Thus finally for any two vertex on G_{k+1}, u, v, if one is on P_1, one is on P_2, the common cycle is P_1P_2'. If one is on P_1, one is on P_3, the common cycle is P_1uwP_3'. If one is on P_2, one is on P_3, the common cycle is P_2uwP_3'.

QED.

5.7 Blocks

Def: Block
A block is a maximal inseparable subgraph of G. **Note:** it is an equivalence class on edges.

This is maximal in the sense similarly to a component.

Def: Equivalence Relation of Cycles
If $v\tilde{}u$, means that u is on a cycle with v. Then,

1. $e\ \tilde{}e$
2. $v\ \tilde{}u \rightarrow u\ \tilde{}v$
3. if $a\ \tilde{}b$, $a\tilde{}c \rightarrow b\ \tilde{}c$

So along with the definition of inseparability, every two vertices on a block with $n \geq 3$ lie on a common cycle.

Thm: Sets on Blocks
If $B, B^{'}$ are two distinct blocks of G, then,

1. B and $B^{'}$ are edge disjoint
2. B and $B^{'}$ can have at most one vertex in common and is a cut vertex

Proof:

1. By def, blocks are an equivalence class on edges.

2. Assume the contrary that there are two vertex in common. Then we can form a bigger cycle for any two vertices in B and $B^{'}$, thus a contradiction since blocks are maximal.

QED.

Def: Vertex Cut
A set of vertices $U \subset V$ such that $G - U$ is disconnected.

Note: Complete graphs have no vertex cuts. $K_n - U = K_{n-|U|}$. Thus for any graph where there exists $u, v : uv \notin E$, there exists a vertex cut $V - u - v$.

Def: Minimum Vertex Cut
A vertex cut of smallest possible cardinality(size of set).

Example: How many hubs do we need to bring down in a network for disconnection.

Def: Vertex Connectivity
A vertex connectivity of G is the cardinality of a minimum vertex cut. (Kappa)

$$0 \leq \kappa(G) \leq n - 1$$

Define: Graph is k-connected if $\kappa(G) \geq k$

Note: Above is defined for non-complete graphs.

Def: Properties of Vertex Connectivity

1. $\kappa(G) = n - 1 \Leftrightarrow G$ is K_n
2. $\kappa(G) = 0 \Leftrightarrow G$ is disconnected
3. $\kappa(G) = 1 \Leftrightarrow G$ connected but separable, there is 1 cut vertex
4. $\kappa(G) \geq 2 \Leftrightarrow G$ is connected but non-separable

Def: Edge Cut
A set of edges $X \subset E$ such that $G - X$ is disconnected.

Def: Minimum Edge Cut
A edge cut of smallest possible cardinality(size of set).

Def: Edge Connectivity
A edge connectivity of G is the cardinality of a minimum edge cut.

$$0 \leq \lambda(G)$$

Define: Graph is k-edge connected if $\lambda(G) \geq k$

Note: If $\lambda(G) \leq \delta(G)$, X is edges incident to $v : deg(v) = \delta(G)$.

Def: Properties of Edge Connectivity

1. $\lambda(G) = 0 \Leftrightarrow G$ is disconnected
2. $\lambda(G) = 1 \Leftrightarrow G$ connected but has one bridge
3. $\lambda(G) \geq 2 \Leftrightarrow G$ is connected but with no bridges

Lem: Edge Connectivity of Complete Graph

$$\lambda(K_n) = n - 1$$

Proof:

Thm: Vertex vs Edge Connectivity

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

Since $\delta(K_n) = n-1$ and $\kappa(K_n) = n-1$, by Squeeze Theorem $\lambda(K_n) = n-1$.

Proof: In lecture 15

> **Thm:** Connectivity of Cubic Graphs
> In cubic graphs(3-regular), $\kappa = \lambda$

Proof:

We have $\kappa(G) \leq \lambda(G) \leq 3$

The two trivial cases are
if $\kappa = 0$, graph is disconnected then $\lambda = 0$
if $\kappa = 3$, $\lambda = 3$ by squeeze theorem

When $\kappa = 1, 2$, let $|U| = 1, 2 : U \subset V$ is the minimal vertex cut. Let G_1, G_2 be the two components of $G - U$. Either G_1 or G_2 contains exactly one neighbour w of u for each $u \in U$

Thus if $\kappa = 1$, then $\lambda = 1$ since u has one neighbour w.

If $\kappa = 2$ where vertex cut $= \{u, u^{'}\}$, then $\lambda = 2$ for neighbours $w, w^{'}$ where edge cut $= \{uw, u^{'}w^{'}\}$.

QED.

5.8 Menger's Theorem

Let $\kappa(G, u, v) = |S|$ where S is the minimal vertex set such that there is no $u - v$ path on $G - S$. $\kappa(G) \leq \kappa(G, u, v)$

Let $\Pi(G, u, v)$ be the maximal number of internally disjoint $u - v$ paths in G.

Thm: Menger's Theorem
For every G and u, v,

$$\kappa(G, u, v) = \Pi(G, u, v)$$

The minimal size of the u, v separating set = the maximal number of internally disjoint u, v paths.

So if $\Pi(G) = \min(\Pi(G, u, v))$ and $\kappa(G) = \min(\kappa(G, u, v))$, then by Menger's Theorem, $\Pi(G) = \kappa(G)$. So $\kappa(G) \geq k \Leftrightarrow \Pi(G) \geq k$. **Thus,**

G is k-connected $\Leftrightarrow$ Every u, v has k internally disjoint $u - v$ paths

Proof:

We want to show that $\kappa(G, u, v) = \Pi(G, u, v)$.

Note, $\Pi \leq \kappa$ since every u, v path has to use a vertex in a separating set.

We want to show $\Pi \leq \kappa$, so if $\kappa = k$, then there exists k internally disjoint u, v paths. Trivial $k = 1, 2$ always true. Assume $k \geq 2$,

Begin induction on size of G. Assume true for graph of size $< m$. Prove for graph size m.

Case 1.
There exists a u, v separating set with a vertex x adjacent to both u and v. So $G - x$ has a u, v separating set $U - x$ of cardinality $k - 1$. Thus by hypothesis, there $k - 1$ internally disjoint u, v paths.

Finally add path uxv to get k internally disjoint paths in G.

Case 2.
There exists a u, v separating set W with no vertex adjacent to both u and v. Let $W = \{w_1, \ldots, w_k\}$

Let G_u be the sub-graph consisting of all $u - w_i$ graph in G that contain

no other vertices of W. Now let $G_u^{'}$ be G_u but all w_i are connect to new arbitrary vertex $v^{'}$.
The same can be done with v creating G_v and $G_v^{'}, v^{'}$.

Claim $G_v^{'} <$ size of G. $G_v^{'}$ has k extra edges since there are k connections from w_i to $u^{'}$. We threw away more than k edges since every w_i is connected to at least one edge. And since there exists no vertex adjacent to both u, v, there exists some $w_i - v$ of at least 2. So we thew away at least $k+1$ edges.

We need to prove above claim to use the induction hypothesis.

Since W is a minimal u, v separating set in $G_v^{'}$, by induction there are k internally disjoint u, v paths in $G_v^{'}$. Similarly this applies for $G_u^{'}$. Thus when you remove $u^{'}, v^{'}$ and join $G_u^{'}, G_v^{'}$, it's prove that there are k disjoint paths in G.

Case 3.
For each minimum u, v separating set S in G, either every vertex in S is adjacent to u or every vertex is S is adjacent to V.

Pick a u, v geodesic, $P = (u, x, y, \ldots, v)$ where $x \in S$. Consider $G - xy$, if $S^{'}$ is a u, v separating set in $G - xy$, then $\kappa(G - xy, u, v) \geq k - 1, k = |S|$.

Claim, if $S^{'}$ is a u, v separating set in $G - xy$, then $S^{'} \cup x$ and $S^{'} \cup y$ are u, v separating sets in G. By contradiction, if they are not, then there would exist a u, v path in $G - S^{'}$ not using x. This also implies that there exists a u, v path in $G - xy - S^{'}$ not using x which is a contradiction since $S^{'}$ is a u, v separating set for $G - xy$.

Finally since $S^{'} \cup x/y$ are separating sets, $\kappa(G - xy, u, v) \geq k$ and thus $\Pi(G - xy, u, v) \geq k \rightarrow \Pi(G, u, v) \geq k$

QED.

Note: Proof on lecture 16,17.

Thm: Edge Menger's Theorem
For every G and u, v, if $\Pi^*(G, u, v)$ is the number of edge-disjoint u, v paths

$$\lambda(G, u, v) = \Pi^*(G, u, v)$$

The minimal size of the u, v separating edge set = the maximal number of edge-disjoint u, v paths.

So if $\Pi^*(G) = \min(\Pi(G, u, v))$ and $\lambda(G) = \min(\lambda(G, u, v))$, then by Edge Menger's Theorem, $\Pi^*(G) = \lambda(G)$. So $\lambda(G) \geq k \Leftrightarrow \Pi^*(G) \geq k$. **Thus,**

G is k edge-connected $\Leftrightarrow$ Every u, v has k edge-disjoint $u - v$ paths

Thm K Cycles
If G is k-connected where $k \geq 2$, then every k vertices lie on a common cycle.

Proof:

If G is k-connected, then for every u, v, there are k internally disjoint u, v paths. Show every k vertices lie on a common cycle.

6 Euler's theorem and Traversibility

6.1 Eulerian Graphs

Def: Eulerian Trail is a trail that uses each edge once.

Def: Eulerian Circuit is a circuit that uses each edge once.

Def: Degree of Eulerian Trail
G has a Eulerian Trail if and only if all but two have even degrees.
G has a Eulerian Circuit if and only if all vertices have even degrees.

To traverse this trail, we can't take arbitrary edges. Use following algorithm.

Def: Fleury's Algorithm

1. Keep adding edges to the trail while removing them from G.
2. At every step, check that the remaining graph has a connected component containing the given end point, v. And all other components have order 1(just vertices).
3. Stop when there are no more edges.

Note: Works with directed graphs.

The components with order 1 have been traversed. The component containing v has not.

Proof:

Since the graph starts with all even or all even but two. At every step of the algorithm, those conditions still holds, thus at every step, the remaining graph G is still Eulerian. To proven algorithm, we only need to show one step.

6.2 Hamiltonian Graphs

Def: Hamiltonian Path is a path that uses each vertex once.

Def: Hamiltonian Cycle is a cycle that uses each vertex once.

Thm: Components of Hamiltonian
For Hamiltonian graph G, for every non-empty set of vertices S,

$$k(G - S) \leq |S|$$

where $k(G)$ is the number of components of G

Proof:

If G is Hamiltonian with cycle C, erase all edges in G except for those in C. Then $C - S$ has $|S|$ components. Now adding back the edges not incident to vertices in S will not increase the number of components.

QED.

Corr: Hamiltonian $K_{n,n}$
$K_{n,n}$ is Hamiltonian if $deg(u) = n, deg(v) = n$

$$\begin{aligned} deg(u) + deg(v) &\geq 2n \\ n + n &\geq 2n \end{aligned}$$

Thm: Degrees of Hamiltonian
For a graph G with $n \geq 3$, if

$$deg(u) + deg(v) \geq n$$

for all distinct pairs of $u, v \in V$, then G is Hamiltonian.

Proof:

If $deg(u) + deg(v) \geq n$ for all distinct pairs of $u, v \in V$. Prove by contradiction, assume G is not Hamiltonian.

Add the maximum number of edges to $G \to G^{'} \neq K_n$ such that if one more edge was added $G^{'}$ would become Hamiltonian. Thus there exists distinct non-adjacent u, v in G' where edge uv would make G' Hamiltonian.

This implies there exists Hamiltonian path in G' $P = (u = x_1, \ldots x_{i-1}x_i \ldots x_n = v)$.

Claim: this x_1x_i can only exist if x_1x_{i-1} doesn't exist since if they both exist, a Hamiltonian cycle would form. Thus $deg(x_1) + deg(x_n) \leq n - 1$.

Which is a contradiction. Thus G is Hamiltonian.

QED.

We can then conclude,

Cor: Regular Hamiltonian Graphs
If G is regular with $n \geq 3$

$$\delta(G) \geq \frac{n}{2}$$

G is Hamiltonian.

Also

Cor:
If $deg(u) + deg(v) \geq n$, then G is Hamiltonian $\Leftrightarrow G + uv$ is Hamiltonian.

Def: Closure of Graph
The closure of G, $C(G)$ is the graph; for all distinct non-adjacent pairs of vertices $u, v \in V(G)$, if $deg(u) + deg(v) \geq n$ add edge uv.

Then by above Corollary, if $C(G)$ is Hamiltonian $\Leftrightarrow G$ is Hamiltonian.

Def: Degree Sequence of Hamiltonian Graph
For a graph with $n \geq 3$, if for any $j : 1 \leq j < \frac{n}{2}$, the number of vertices with degree j is strictly less than j, then G is Hamiltonian.

Thus no leaves, at most 1 degree 2, at most 2 degree 3...

Proof:

To show that G is Hamiltonian, show $C(G) = K_n$. Prove by contradiction so assume $C(G) \neq K_n$.

Pick non adjacent vertices u, w so that $deg(u)+deg(w)$ is maximal in $C(G)$. Since they are non-adjacent, $deg(u) + deg(w) \leq n - 1$.

Let $deg(u) \leq deg(w)$ where $k = deg(u) \leq \frac{n-1}{2}$.

Let W $= \{v : vw \notin E\}$, the set of vertices not adjacent to w. Claim: for all $v \in W, deg(v) \leq k$; since we assumed u, v are maximal. Thus by the theorem if $deg(v) \leq k$, then $|W| \leq k - 1$.

Then $deg(w) = n - 1 - |W| \geq n - k$. Finally this implies $deg(u) + deg(w) \geq k + n - k = n$ which is a contradiction from the earlier assumption of $deg(u) + deg(w) \leq n - 1$.

Therefore G is Hamiltonian. QED.

7 Planarity

Def: Planar Graphs
A graph is planar if it can be drawn in the plane such that edges do not cross.

Eg. Trees are planar.

Def: Plane Graph
A graph that is dawn in the plane.

Note: Isomorphic graphs are not necessarily isomorphic plane graphs.

Thm: Euler's Identity
Let $r =$ **number of regions** .

$$n + r = m + k + 1$$

where k is the number of components. Thus when G is **connected**.

$$n + r = m + 2$$

Proof:

Connected Case:

Proof by induction of number if edges.

Base case: m=n-1, true for trees.

Assume true for all graphs with n vertices and $< m$ edges. So $m > n - 1$.

Prove for m.

Consider a cycle in G, if we remove an edge e, the two regions neighbouring e become connected. Thus $n^{'} = n, m^{'} + 1 = m - 1, r^{'} = r - 1$. By hypothesis we have, $n^{'} + r^{'} = m^{'} + 2$, thus $n + r - 1 = m - 1 + 2 \rightarrow n + r = m + 2$.

QED.

Def: Dual of a Plane Graph
The dual graph G^* of plane graph G is a graph/multigraph drawn by connecting each planar region(as vertex in G^*) of G for each edge in G.

Edges of the dual are paired with edges of the original.

Prop: Spanning Tree of Dual
Let T be a spanning tree of G; the pairwise dual of the edges not used by T form a spanning tree of G^*, the dual of G.

Prop: Application of Euler's Identity
If G is planar with $n \geq 3$, then

$$m \leq 3n - 6$$

An immediate result is K_5 is not planar.

Proof:

Assume G is connected.

If $m < 3$, then $n = 3$. Thus $G = P_3$ which is planar.

If $m \geq 3$, the boundary of each region r_i has $m_i \geq 3$ edges.

Thus $M = \sum_{i=1}^{r} m_i \geq 3r$. Since some edges are counted twice while others counted once, $M \leq 2m$. Finally we have $3r \leq M \leq 2m \to 3r \leq 2m$.

Thus by Euler's Identity $n+r = m+2 \to 2 = n-m+r \to 6 = 3n-3m+3r \leq 3n - 3m + 2m = 3n - m \to m \leq 3n - 6$.

QED

Thm: Kuvatowski's Theorem
A graph is planar if and only if the graph does not contain a subgraph that is the subdivision of K_5 or $K_{3,3}$.

Or every non-planar graph contains a subdivision of K_5 or $K_{3,3}$.

Cor:
Every planar graph has a vertex of degree 5 or less.

$$\delta(G) \leq 5$$

Proof:

$$m \leq 3n - 6$$
$$\rightarrow 2m/nm \leq 6 - \frac{12}{n} < 6$$

If the average degree is less than 6, then the minimal degree is less than 6.

7.1 Embedded Graphs

Def: Shapes
A donut shape is called the torus (genus-1).
A figure 8 donus is called genus-2
A sphere is called genus-0

Def: Gamma Function
The smallest genus surface that G can be embedded in is

$$\gamma(G)$$

For example $\gamma(K_3), \gamma(K_4) = 0$ where $\gamma(K_5), \gamma(K_6) = 1$ since it's not planar.

Thus if G is a subgraph of H, then $\gamma(G) \leq \gamma(H)$.

Prop: Gamma of Complete Graph
For a complete graph K_n

$$\gamma(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$$

Thm: Euler's Identity for General Surfaces
If G is 2-cell embedded on a surface with genus-k, $k > 0$

$$n - m + r = 2 - 2k$$

for G with order n, size m and r regions.

Cor:
For any connected graph embedded in a surface of genus-$\gamma(G)$,

$$n - m + r = 2 - 2\gamma(G)$$

Since the minimal genus embedding is always 2-cell.

Cor:

$$\gamma(G) \geq \frac{m}{6} + \frac{n}{2} + 1$$

Proof:

From earlier proof of Application of Euler's Identity, we have $3r \leq 2m \to r \leq \frac{2}{3}m$

Thus by corollary above,

$$
\begin{aligned}
n - m + r &= 2 - 2\gamma(G) \\
\to 2\gamma(G) &= 2 - n + m - r \\
\to \gamma(G) &= 1 - n/2 + m/2 - r/2 \\
\to \gamma(G) &\geq 1 - n/2 + m/2 - 2/6m \\
\to \gamma(G) &\geq \frac{1}{6}m - \frac{1}{2}n + 1
\end{aligned}
$$

QED.

Cor:
For $n \geq 3$

$$
\begin{aligned}
\gamma(K_n) &\geq \frac{1}{6}\frac{n(n-1)}{n} + \frac{n}{2} + 1 \\
&= \frac{n^2 - 7n + 12}{12} = \frac{(n-3)(n-4)}{12}
\end{aligned}
$$