Lecture 17 Mar 15, 2021

Ex. LP

Max: $z = 4x_1 + 5x_2 + 2x_3$

S.t. $x_1 - x_2 - 4x_3 \geq -1$

$2x_1 + x_2 - 2x_3 \leq 4$ $\quad \vec{x} \geq 0$

-1	1	4	1	0	1
2	1	-2	0	1	4
4	5	2	0	0	0

-1	1	4	1	0	1
3	0	-6	-1	1	3
9	0	-18	-5	0	-5

0	1	2	$\frac{2}{3}$	$\frac{1}{3}$	2
1	0	-2	$-\frac{1}{3}$	$\frac{1}{3}$	1
0	0	0	-2	-3	-14

negative, end simplex.

Optimal = $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad z = 14$ //

x_3 "z-coef" is 0, what if we pivot x_3 ?

$$\begin{array}{ccccc|c} 0 & 1 & 2 & \frac{2}{3} & \frac{1}{3} & 2 \\ 1 & 0 & -2 & -\frac{1}{3} & \frac{1}{3} & 1 \\ \hline 0 & 0 & 0 & -2 & -3 & -14 \end{array}$$

$$\begin{array}{ccccc|c} 0 & \frac{1}{2} & 1 & \frac{1}{3} & \frac{1}{6} & 1 \\ 1 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & 3 \\ \hline 0 & 0 & 0 & -2 & -3 & 14 \end{array} \qquad \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad z = 14$$

We see that we get another optimal with $z = 14$.

Homework:

If we connect $(1\ 2\ 0)$ with $(3\ 0\ 1)$

to form line segment $\vec{x} = (1-\lambda)\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\lambda \in [0,1]$

We have $\vec{x} = \begin{pmatrix} 3(1-\lambda) + \lambda \\ 2\lambda \\ 1-\lambda \end{pmatrix}$.

$$z\begin{pmatrix} 3-2\lambda \\ 2\lambda \\ 1-\lambda \end{pmatrix} = 4(3-2\lambda) + 5(2\lambda) + 2(1-\lambda)$$

$$= 12 - 8\lambda + 10\lambda + 2 - 2\lambda$$

$$= 14$$

We see all points on $\vec{x}$ has $z = 14$.

Issues with Simplex.

1) Initial tableau does not exist. ($\vec{b} < 0$)

2) Cycling, tableau doesn't end

3) Picking basis variable in an unambiguous way

Cycling //

$$\begin{array}{cccccc|c} 2 & -4 & 6 & 1 & 0 & 0 & 3 \\ -1 & 3 & 4 & 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 1 & 1 \\ \hline 2 & -1 & 8 & 0 & 0 & 0 & 0 \end{array}$$

$x = \vec{0}$ $z = 0$ T_0

$$\begin{array}{cccccc|c} 2 & -4 & 0 & 1 & 0 & -3 & 0 \\ -1 & 3 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \hline 2 & -1 & 0 & 0 & 0 & -4 & -4 \end{array}$$

$\vec{x} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$ $z = 4$ T_1

$$
\begin{array}{cccccc|c}
1 & -2 & 0 & \frac{1}{2} & 0 & -\frac{3}{2} & 0 \\
0 & 1 & 0 & \frac{1}{2} & 1 & -\frac{7}{2} & 0 \\
0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
\hline
0 & 3 & 0 & -1 & 0 & -1 & -4
\end{array}
\qquad \vec{x} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad z = 4 \quad T_2
$$

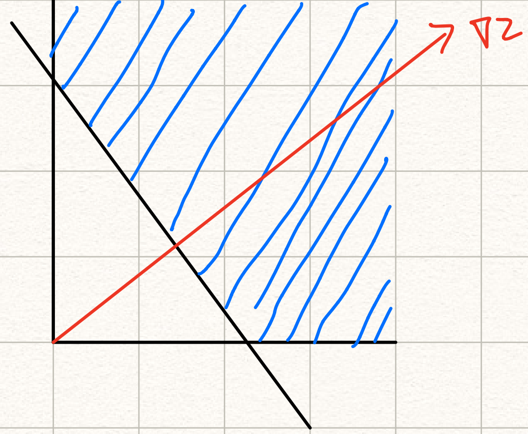
$$
\begin{array}{cccccc|c}
1 & 0 & 0 & \frac{3}{2} & 2 & -\frac{17}{2} & 0 \\
0 & 1 & 0 & \frac{1}{2} & 1 & -\frac{7}{2} & 0 \\
0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
\hline
0 & 0 & 0 & -\frac{5}{2} & -3 & \frac{19}{2} & -4
\end{array}
\qquad \vec{x} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad z = 4 \quad T_3
$$

$$
\begin{array}{cccccc|c}
1 & 0 & 17 & \frac{3}{2} & 2 & 0 & \frac{17}{2} \\
0 & 1 & 7 & \frac{1}{2} & 1 & 0 & \frac{7}{2} \\
0 & 0 & 2 & 0 & 0 & 1 & 1 \\
\hline
0 & 0 & -19 & -\frac{5}{3} & -3 & 0 & -\frac{27}{2}
\end{array}
\qquad \vec{x} = \begin{pmatrix} \frac{17}{2} \\ \frac{7}{2} \\ 0 \end{pmatrix} \quad z = \frac{27}{2} \quad T_4
$$

We see that there was possible <u>cycling</u> from $T_1 - T_3$; the value of z doesn't change.

The tableau from $T_1 - T_3$ also has basic variable equal to zero. A tableau with one or more basic variable equal to zero is call <u>degenerate</u>.

Ex. Unbounded Solution



$$x_i = \hat{b}_i - \sum_{j \in N} \hat{a}_{ij} x_j \qquad i \in B$$

$$z = \hat{z} + \sum_{j \in N} \hat{c}_j x_i \qquad i \in B$$

$$\hat{b}_i \geq 0 \qquad i \in B$$

Note: $\hat{b}_i$ $i \in B$ positive is equivalent to feasibility of dictionary D_B.

Thus x_{i_0} is the leaving variable, then

$$\frac{\hat{b}_{i_0}}{\hat{a}_{i_0}} = \min\left(\frac{\hat{b}_i}{\hat{a}_{ij}} \; : \; i \in B, a_{ij} > 0 \right)$$

and there are 2 potential problems with above ratio.

1) $\nexists$ $i \in B$ s.t. $a_{ij} > 0$

2) $\exists$ more than one minimum (multiple minimums)

1) Means we can increase the volume of entering variable x_{j_0} as much as we want.

FR is unbounded when pivot column ≤ 0.

To formalize: If $\exists j_0 \in N$ in dictionary D_B for $\hat{c}_{j_0} > 0$ and $\hat{a}_{ij} \leq 0$ for all $i \in B$, then LP is unbounded.

Lecture 18 Mar 17 2021

Last class review:

① Unbounded case, you cannot have an optimal solution for z.

② Degenerate case, pivot / tableau

Def Degenerate Tableau / Dictionary //
If a dictionary gives a basic feasible solution (BFS) with one or more basic variable = 0.

Def Degenerate Pivot//

If a pivot in degenerate tableau does not change the BFS.

Eg. Iterating tableau but BFS stays the same.

Degenerate pivots may give you cycling.

Fact //

We have 1:1 relation between dictionary and it's associated basic variables.

There are $\binom{n+m}{n}$ basic variables.

Suppose for LP we have dictionaries:

$D_1, \ldots, D_k$, D_k is opimal

but with cycling:

$D_1, \ldots, D_k = D_1$

Prop//

Every basis uniquely determines its associated dictionary.

Simplex algorithm fails to terminate iff $\exists$ cycling

Note: cycling only exist between degenerate dictionaries.

How to Prevent Cycling //

When there is degenerate pivot, make choice as follows:

choice of entering : x_{j_0} for $j_0 \in N$ is the entering variable. if $\hat{c}_{j_0} > 0$ and $j_0 \le j$ whenever $\hat{c}_j > 0$.

choice of leaving : x_{i_0} for $i_0 \in B$ is the leaving variable if

$$\frac{\hat{b}_{i_0}}{\hat{a}_{i_0 j_0}} = \min \left\{ \frac{\hat{b}_i}{\hat{a}_{ij}} : i \in B, a_{ij} > 0 \right\}$$

and $i_0 \le i$ whenever $\frac{\hat{b}_{i_0}}{\hat{a}_{i_0 j_0}} = \frac{\hat{b}_i}{\hat{a}_{ij}}$

Summary ☆

We choose the smallest subscript whenever there is a tie in entering / leaving variable. //

Ex.

Initially we have

T_1

→	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{5}{2}$	9	1	0	0	0

$\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	1	0	1	0	0
1	0	0	0	0	0	1	1
10	-57	-9	-24	0	0	0	0

T_2

1	-11	-5	18	2	0	0	0
0	4	2	-8	-1	1	0	0
0	11	5	-18	-2	0	1	1
0	53	41	-204	-20	0	0	0

T_3

1	0	$\frac{1}{2}$	-4	$-\frac{3}{4}$	$\frac{11}{4}$	0	0
0	1	$\frac{1}{2}$	-2	$-\frac{1}{4}$	$\frac{1}{4}$	0	0
0	0	$-\frac{1}{2}$	4	$\frac{3}{4}$	$-\frac{11}{4}$	1	1
0	0	$\frac{29}{2}$	-98	$-\frac{27}{4}$	$-\frac{53}{4}$	0	0

T_4

2	0	0	-8	$-\frac{3}{2}$	$\frac{11}{2}$	0	0
-1	1	0	2	$\frac{1}{2}$	$-\frac{5}{2}$	0	0
1	0	1	0	0	0	1	1
-29	0	0	18	15	-93	0	0

T_5

$$
\begin{array}{c|ccccccc|c}
\rightarrow & -2 & 4 & 1 & 0 & \frac{1}{2} & -\frac{9}{2} & 0 & 0 \\
 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & -\frac{5}{4} & 0 & 0 \\
 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline
 & -20 & -9 & 0 & 0 & \frac{21}{2} & -\frac{141}{2} & 0 & 0 \\
 & & & & & \uparrow & & &
\end{array}
$$

$$\vdots$$

$T_7 = T_1$

$$
\begin{array}{ccccccc|c}
\frac{1}{2} & -\frac{11}{2} & -\frac{5}{2} & 9 & 1 & 0 & 0 & 0 \\
\frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline
10 & -57 & -9 & -24 & 0 & 0 & 0 & 0
\end{array}
$$

Note: Cycling only happens when there is degenerate tableau.

Degenerate tableau does not necessarily mean cycling.