Final Review

Def // Hyperplane

A hyperplane in $\mathbb{R}^n$ is any set of the form

$$H_{\{a,b\}} = \{ \vec{x} \in \mathbb{R}^n \mid \vec{a}^T \vec{x} = \vec{b} \}$$

Facts about hyperplane:

- every H in $\mathbb{R}^n$ will generate 2 closed half spaces.

1. is $\{ \vec{x} \in \mathbb{R}^n \mid \vec{a}^T \vec{x} \geq \vec{b} \}$
2. is $\{ \vec{x} \in \mathbb{R}^n \mid \vec{a}^T \vec{x} \leq \vec{b} \}$

The constraints in any LP form a region of intersection of finitely many closed half spaces.

one constraint
one half space

$$H_i = \{ \vec{x} \mid \sum_{j=1}^{n} a_{ij} x_j \leq b_i \}$$

is a row of $A\vec{x} \leq \vec{b}$

$$\{ \vec{x} \mid A\vec{x} \leq \vec{b} \} \Leftrightarrow \bigcap^{n} H_i$$

$i=1$

Def// Convex Polyhedron (Feasible Region)

Any subset of $\mathbb{R}^n$ that can be represented as the intersection of finitely many closed half spaces is called convex polyhedron.

- recall intersection of convex is convex.

Thm// Representation Theorem for Convex Polyhedrons

Let $\vec{x}$ be in the Convex Polyhedron C.

$$C = \{ \vec{x} \in \mathbb{R}^s \mid T \in \mathbb{R}^{n \times s},\ T\vec{s} \leq \vec{g} \}$$

$\vec{X}$ is a vertex of C iff $\exists$ an index set $L \subseteq \{1, 2 \dots s\}$ with such that $\vec{x}$ is the unique solution to the system of equations

$$\sum_{j=1}^{n} t_{ij} x_j = g_i \qquad i \in L$$

moreover if x is a vertex then one can take $|L| = n$

cardinality of L.

Thm 11 A point $\vec{x}$ in the convex polyhedron defined by

$$A\vec{x} \leq b \quad \text{and} \quad \vec{x} \geq 0$$

where $A = [a_{ij}]_{m \times n}$.

$\vec{x}$ is a vertex of this polyhedron iff $\exists I \subseteq \{1, \dots, m\}$ and $\exists J \subseteq \{1, \dots, n\}$ with $|I + J| = n$ such that $\vec{x}$ is the unique solution to the system of equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad i \in I \qquad x_j = 0 \quad j \in J$$

Proof 11

$$T = \begin{bmatrix} A \\ -I \end{bmatrix} \qquad \vec{g} = \begin{bmatrix} \vec{b} \\ 0 \end{bmatrix}$$

Use representation thm of convex polyhedron.

$$T\vec{s} \leq \vec{g}$$

$\blacksquare$

Note: The FR of LP problem can be written as

$A\vec{x} \le \vec{b} \qquad \vec{x} \ge 0$

The optimal solution:

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \tag{1}$$

Let $\vec{x} = \{x_1, \dots, x_{m+n}\}$

and

let $\bar{x} = \{x_1, \dots, x_n\}$