Lecture 13 Mar 1, 2021

Simplex Alogrithm II

Note: Initial $\vec{b}$ must be ≥ 0 for algorithm to work. Else, use artificial variables

Ex.

$$\max \; z = 5x_1 + 4x_2 + 3x_3$$

$$\text{s.t.} \quad \begin{aligned} 2x_1 + 3x_2 + x_3 &\leq 5 \\ 4x_1 + x_2 + 2x_3 &\leq 11 \\ 3x_1 + 4x_2 + 2x_3 &\leq 8 \\ \vec{x} &\geq 0 \end{aligned}$$

introduce slack variables (x_4, x_5, x_6)

$$\max \; z = 5x_1 + 4x_2 + 3x_3$$

$$\text{s.t.} \quad \begin{aligned} 2x_1 + 3x_2 + x_3 + x_4 &= 5 \\ 4x_1 + x_2 + 2x_3 + x_5 &= 11 \\ 3x_1 + 4x_2 + 2x_3 + x_6 &= 8 \\ \vec{x} &\geq 0 \end{aligned}$$

represent as matrix

$$\begin{bmatrix} 0 & A & I \end{bmatrix} \begin{bmatrix} z \end{bmatrix} = \begin{bmatrix} \vec{b} \end{bmatrix}$$ ← general form

$$\begin{bmatrix} -1 & c^T & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \quad I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix}$$

$$c = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 \\ -1 & 5 & 4 & 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 8 \\ 0 \end{bmatrix}$$

Initial dictionary //

$x_4 = 5 - 2x_1 - 3x_2 - x_3$

$x_5 = 11 - 4x_1 - x_2 - 2x_3$

$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$

$z = 5x_1 + 4x_2 + 3x_3$

① First solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 11 \\ 8 \end{bmatrix}$ → Basic Solution

For $Ax = b$, A is $\mathbb{R}^{m \times n}$ with $m \le n$, $r(A) = m$. A basic solution of A, $x \in \mathbb{R}^n$, is a vector that is a linear combination of m linearly independent columns of A and $n-m$ zeros.

A basic solution might not be feasible, a <u>feasible basic solution</u> is always feasible, they are extreme points.

② To iterate to the next dictionary, pick the non-basic <u>column</u> with the largest positive "z" coefficient. $\underline{\underline{x_1}}$ so column 1.

③ Pick the row in above column with the smallest θ-coefficient. $\theta = \frac{b}{c}$. s.t. b is the coefficient of b in given row, c is the column coefficient of given row.

$\min\left(\frac{5}{2}, \frac{11}{4}, \frac{8}{3}\right) = \underline{\underline{\frac{5}{2}}}$

so row 1.

So:

Initial dictionary //

$x_4 = 5 - 2x_1 - 3x_2 - x_3 \quad \rightarrow \quad \underline{x_1 = \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3}$

$x_5 = 11 - 4x_1 - x_2 - 2x_3$

$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$

$z = 5x_1 + 4x_2 + 3x_3$

We get new dictionary:

$x_1 = \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3$

$x_5 = 1 + 2x_4 + 5x_2$

$$x_6 = \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}x_3$$

$$z = \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3$$

By setting the non-basic vars to zero (x_2, x_3, x_4), we get basic solution $\begin{bmatrix} \frac{5}{2} \\ 0 \\ 0 \\ 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}$

Next iteration: pick x_3 column since it has largest "z coef".
pick x_6 row since $\frac{1}{2} / \frac{1}{2}$ is $\min(5, 1)$.

So
$$\left.\begin{aligned} x_3 &= 1 - 2x_6 + 3x_4 + x_2 \\ x_1 &= 2 + x_6 - 2x_4 - 2x_2 \\ x_5 &= 1 + 2x_4 + 5x_2 \\ z &= 13 - x_6 - x_4 - 3x_2 \end{aligned}\right]$$
new dictionary. with basic solution $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Since "z coefficients" are all negative, we cannot continue thus $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is the optimal solution with $z = 13$.

Using the table format:

$$
\begin{bmatrix}
0 & 2 & 3 & 1 & 1 & 0 & 0 \\
0 & 4 & 1 & 2 & 0 & 1 & 0 \\
0 & 3 & 4 & 2 & 0 & 0 & 1 \\
-1 & 5 & 4 & 3 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}
=
\begin{bmatrix} 5 \\ 11 \\ 8 \\ 0 \end{bmatrix}
$$

$\Downarrow$

1

z	x_1	x_2	x_3	x_4	x_5	x_6	
0	2	3	1	1	0	0	5
0	4	1	2	0	1	0	11
0	3	4	2	0	0	1	8
-1	5	4	3	0	0	0	0

min "θ-coefficient" row (pivot row)

max "z coefficient" column (pivot column)

2

z	x_1	x_2	x_3	x_4	x_5	x_6	
0	1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{5}{2}$
0	0	-5	0	-2	1	0	1
0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	0	1	$\frac{1}{2}$
-1	0	$-\frac{7}{2}$	$\frac{1}{2}$	$-\frac{5}{2}$	0	0	$-\frac{25}{2}$

From $1 \to 2$, we:

row2 − 2row1

row3 − $\frac{3}{2}$ row1

pivot row

row 4 − $\frac{5}{2}$ row1

pivot column

3

z	x_1	x_2	x_3	x_4	x_5	x_6	
0	1	2	0	2	0	-1	2
0	0	-5	0	-2	1	0	1

0	0	-1	1	-3	0	2	1
-1	0	-3	0	-1	0	-1	-13

since "z coef" are all negative, we have reached optimal solution.

$$\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{with } z = 13.$$

Lecture 14 Mar 3, 2021

Ex:

$$z = x_1 + 2x_2 - 3x_3$$

$$x_2 \leq 4$$

$$2x_1 + 3x_2 - 4x_3 \leq 24$$

$$\vec{x} \geq 0$$

0	0	1	0	1	0	4
0	2	3	-4	0	1	24
-1	1	2	-3	0	0	0

$$
\begin{array}{r}
\\
\rightarrow \\
\\
\\
\end{array}
\begin{array}{cccccc|c}
0 & 0 & 1 & 0 & 1 & 0 & 4 \\
0 & 2 & 0 & -4 & -3 & 1 & 12 \\
\hline
-1 & 1 & 0 & -3 & -2 & 0 & -8 \\
 & \uparrow & & & & &
\end{array}
\qquad
\begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 12 \end{bmatrix}
$$

$$
\begin{array}{cccccc|c}
0 & 0 & 1 & 0 & 1 & 0 & 4 \\
0 & 1 & 0 & -2 & -\frac{3}{2} & \frac{1}{2} & 6 \\
\hline
-1 & 0 & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & -14 \\
 & x_1 & x_2 & & & &
\end{array}
\qquad
\begin{bmatrix} 4 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$