Lecture 11 Feb 22

What we've covered so far:

① Thm 11

Only on is non-empty:

1 $\{ x \mid Ax = b \quad \vec{x} \geq 0 \}$

2 $\{ y \mid A^T y \geq 0 \quad b^T y < 0 \}$

② Arbitrage

The difference in price of an item in two markets.

We can use arbitrage to make money

Equalibrium price = no arbitrage

③ De Finetti Arbitrage Theorem

Suppose that there's an experiment having m possible outcomes for which there are n possible wagers.

If you bet y on wager i, you win amany $y \cdot r_i(j)$; r is an indicator var.

If the outcome of this experiment is j

A betting strategy on "n" possible wagers

$$\vec{y} = (y_1, \dots, y_n) = (y_1 \dots y_i \dots y_n)$$

You may represent your gain/loss from strategy $\vec{y}$ is

$$\sum^{n} y_i \, r_i(j)$$

Thm//

Exactly one is true:

1) $\exists$ a probability vector $p = (p_1 \cdots p_m)$ with

$$\sum_{j=1}^{m} r_i(j) p_j = 0 \quad \text{for } i = 1, \dots, n$$

or

2) $\exists$ a stratigy $\vec{y}$

$$\sum_{i=1}^{n} y_i r_i(j) > 0$$

$$\text{for } j = 1, \dots, m$$

$$A = \begin{bmatrix} r_1(1) & \cdots & r_1(m) \\ \vdots & & \\ r_n(1) & \cdots & r_n(m) \\ -1 & \cdots & -1 \end{bmatrix} \in \mathbb{R}^{n+1 \times m} \qquad \vec{b} = (0, \dots, 0, -1) \in \mathbb{R}^{n+1 \times 1}$$

$$A\vec{p} = \vec{b}, \quad \vec{p} = (p_1, \dots, p_m) \in \mathbb{R}^m, \quad p_i \geq 0$$

$$\sum_{j=1}^{m} r_i(j) p_j = 0, \quad i = 1 \cdots n \qquad = \quad a_i^T \cdot p$$

where a_i^T is the ith row of A.

And $\sum_j^m -1\, p_j = -1 \implies \sum_j^m p = 1$

So by above theorem

$$\left.\begin{array}{l} ① \ \{ A\vec{p} = b \mid \vec{p} \geq 0 \} \\ \\ ② \ \{ \vec{y}^T A \geq 0, \ y^T b < 0 \} \end{array}\right] \text{only one is empty}$$

Then $y^T A = \sum_{i=1}^{n} y_i r_i(j) - y_{n+1} \geq 0$

for $j = 1 \ldots m$

$y^T \cdot b < 0 \quad \Rightarrow \quad -y_{n+1} < 0 \quad \Rightarrow \quad y_{n+1} > 0$

$\sum_{i=1}^{n} y_i \, r_i(j) \geq y_{n+1} > 0 \quad$ for $j = 1 \ldots m$

Introduction of terms //

Call option //

A right to buy a certain amount of shares.

Ex Apple stock right now 120 / share

Option = exp date : end of the week

strike price \$150 / premium

option price \$10 / premium

3 outcomes

① $p > 150$ (200) gain \$(200 − 150) · 100

② $p = 150$ lose option price, −\$10

③ $p < 150$ lose option price, −\$10

De Finetti's Theorem //

Unless $\alpha(1-p') + \beta p' = 0$

there will be some combination of wagers

that has a guaranteed profits.

Ex 11

Suppose there is a certain stock at time $t=0$ has the price $S(0)$.

Suppose at $t=1$ there are two possible outcomes

1) The value of stock ↑ by factor of u at $t=1$.

$$S(1) = u \cdot S(0)$$

2) The value of stock ↓ by factor of d at $t=1$

$$S(1) = d \cdot S(0)$$

Let r be the risk free rate.

If the current value of such stock is M, then $\frac{M}{r+1}$ is the real fair value of such stock.

The outcome $r(2)$:

The profit per dollor if the stocks goes up:

$$r(2) = \frac{u}{r+1} - 1 \qquad \text{set } (S(0)=1)$$

The outcome $r(1)$

The profit per dollor if the stocks goes down:

$$r(1) = \frac{d}{r+1} - 1 \qquad \text{set } (S(0)=1)$$

Lecture 12, Feb 24, 2021

Review

Call Option 11

Contract between buyer and seller granting the buyer the right to buy the underlying security from the seller at a certain price at a specified time in the future.

1. Buyer has no obligation to buy
2. Seller must sell if buyer buys
3. The strike price is fixed
4. Price of option is called option price
5. Each option contract contains 100 options
   each option contains 100 shares
6. A option has an expiration date
   1) American : buyer can exercise call option any time before exp date
   2) European : buyer can exercise call option only on exp date

Define option: (P, S, E)
- Price
- Strike price
- Expiration date

Ex 1 Share of $(10, 220, \text{Feb } 28)$ in E

1) Share $= 250$ on Feb 28
   The profit $= (250 - 220)100 - 10$

2) Share $= 200$ on Feb 28
   The loss $= -10$

Applying Options and Arbitrage //

Exactly one is true:

1) $\exists$ a probability vector $p = (p_1 \dots p_m)$ with

$$\sum_{j=1}^{m} r_i(j) p_j = 0 \quad \text{for } i = 1, \dots, n$$

or

2) $\exists$ a strategy $\vec{y}$

$$\sum_{i=1}^{n} y_i r_i(j) > 0$$

for $j = 1, \dots, m$

In matrix form:

α 1) $\exists$ a probability vector $p \in \mathbb{R}^m$ s.t.

$$Ap = b \qquad p \geq 0$$

β 2) $\exists$ a strat $y \in \mathbb{R}^n$ s.t.

$$y^T A \geq 0 \qquad y^T b < 0$$

Ex//

$$A = \begin{bmatrix} r(1) & r(2) \\ -1 & -1 \end{bmatrix}$$

α) $Ap = b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

if A has inverse, then

$$p = A^{-1} b = \frac{1}{-r(1) + r(2)} \left[\begin{bmatrix} -1 & -r(2) \\ 1 & r(1) \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right]$$

$$= \frac{1}{r(2)-r(1)} \begin{bmatrix} r(2) \\ -r(1) \end{bmatrix}$$

two ways to find if p $\exists$

We also have

$$A p = \begin{bmatrix} r(1) & r(2) \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1-p \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$r(1)(1-p) + r(2)(p) = 0$

↳ if such p exists, then there does not exist a betting strat to make money.

else if p does not exist, then there is a way to make money

From last lec, we defined

$S(0)$ = stock price at $t=0$

If stock ↑ by factor u at $t=1$

$S(1) = u\,S(0)$

If stock ↓ by factor d at $t=1$

$S(1) = d(S0)$

Risk free rate, r.

↑ $r(2) = \frac{u}{r+1} - 1$: profit per \$ when stock goes up

↓ $r(1) = \frac{d}{r+1} - 1$: loss per \$ when stock goes down

$$p = \begin{pmatrix} 1-p \\ p \end{pmatrix} = \frac{1}{r(2)-r(1)} \begin{bmatrix} r(2) \\ -r(1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{r(2)}{r(2)-r(1)} \\ \frac{-r(1)}{r(2)-r(1)} \end{bmatrix}$$

Thus $(1-p) = \dfrac{\frac{u}{r+1} - 1}{\frac{u}{r+1} - \frac{d}{r+1}} = \dfrac{u-1-r}{u-d}$

$$p = \frac{1+r-d}{u-d}$$

What is the fair price for such call option?

Let K be the strike price

Let C be the current price of the call option.

Define param k, strike price vs. stock price

$K = k \cdot S(0)$

If stock $\uparrow$, you can buy the stock at $t=1$ for price

$K = k \cdot S(0)$

and sell immediatly at price $u \cdot S(0)$.

The profit gained per stock

is $\frac{(u-k)}{1+r} - C = r(2)$

If stock $\downarrow$, your loss per stock

is $C. \quad = r(1)$

By De-Finett's Thm //

$$(1-p)\, r(1) + p\, r(2) = 0$$

$$(1-p)(-C) + p\left(\frac{u-k}{1+r} - C\right) = 0$$

$$p \cdot \frac{u-k}{1+r} - C = 0$$

$$\frac{1-r-d}{u-d} \cdot \frac{u-k}{1+r} - C = 0$$

$$C = \frac{1+r-d}{u-d} \cdot \frac{u-k}{1+r}$$

C is the fair price of the option.

If C is different from the actual C,

$\exists$ arbitrage and $\exists$ betting strat to

make money.

If C is the same, there is no money

to be made.