Lecture 5 Jan 25 th.

Suppose $C \subseteq \mathbb{R}^n$ has the following properties.

① C is convex

② C has a non-empty interior

Lem 1

If $\vec{x}^0 \in int(C)$ and $\vec{x} \in C$, then

$$(1-\lambda)\vec{x}^0 + \lambda \vec{x} \in int(C) \quad , \quad \lambda \in [0,1)$$

Proof:

We need these four props proven earlier:

For $\alpha, \varepsilon \in \mathbb{R}$, $\vec{a}^0 \in \mathbb{R}$

① $\alpha B_\varepsilon(\vec{0}) = B_{\alpha \cdot \varepsilon}(\vec{0})$

② $B_\varepsilon(\vec{a}^0) = \vec{a}^0 + B_\varepsilon(\vec{0})$

③ $\alpha B_\varepsilon(\vec{a}^0) = B_{\alpha \cdot \varepsilon}(\vec{a}^0)$

④ If C is convex, then the closure of C

$$\overline{C} = Cl(C) = C \cup \{ \partial C \}$$

is convex

Thus to prove lemma above, we need to show

$$S = \lambda \overline{C} + (1-\lambda) int(C) \subset int(C)$$

Since $int(C)$ is defined as the maximal open subset of C, it is sufficient to show S is open and is a subset of C.

Since $int(C)$ is open, $\forall \vec{b} \in int(C)$, $\exists B_r(\vec{b}) \subset int(C)$.

Thus take $\vec{a} \in \overline{C}$, $\vec{b} \in int(C)$.

$\lambda \vec{a} + (1-\lambda) B_r(\vec{b})$

$= B_{(1-\lambda)r}(\lambda \vec{a} + \vec{b})$

$\in \lambda \bar{C} + (1-\lambda)\operatorname{int}(C)$ since $1-\lambda < 1$ and $\lambda \vec{a} + \vec{b} \in S$

Thus S is open.

Now we show S is a subset of C.

If $\lambda \bar{C} \subset C$ since $\lambda < 1$

$\lambda \bar{C} \subset C - (1-\lambda)\vec{c}$ for $\vec{c} \in \operatorname{int}(C)$

Thus $\lambda \bar{C} + (1-\lambda)\operatorname{int}(C) \subset C$.

Finally since S is open subset of C. $S \subset \operatorname{int}(C)$. ▨

★ Study above and below lemma for midterm

Lemma Interior also Convex

If $C \subseteq \mathbb{R}^n$ is convex, $\operatorname{int}(C)$ is also convex

Proof:

For $\vec{a}, \vec{b} \in \operatorname{int}(C)$, show $\alpha \vec{a} + (1-\alpha)\vec{b} \in \operatorname{int}(C)$.

Case 1: $\alpha < 1$

By lemma above $\alpha \vec{a} + (1-\alpha)\vec{b} \in \operatorname{int}(C)$.

Case 2: $\alpha = 1$

Then $\alpha \vec{a} = \vec{a} \in \operatorname{int}(C)$

Thus $\operatorname{int}(C)$ is convex. ▨

Lemma Extreme Point, No Line

Let $C \subseteq \mathbb{R}^n$ be a closed non-empty convex set.

C has an extreme point

$\iff$

C does not contain a line

Def || Extreme Point

The point $\bar{x}$ in convex set $C \subseteq \mathbb{R}^n$ s.t.

$$\nexists \vec{a}, \vec{b} \in C, \quad \bar{x} = \alpha \vec{a} + (1-\alpha)\vec{b}, \quad \alpha \in (0,1)$$

Proof:

Forwards

Suppose C is convex and has E.P. $\bar{y}$. Show C does not contain a line. Suppose it does.

Let line $L = \{ \vec{x} + a\vec{d} \mid \vec{x} \in C, \vec{d} \in \mathbb{R}^n, a \in \mathbb{R} \}$

Let $\vec{k}_n = (1-\frac{1}{n})\bar{y} + \frac{1}{n}(\vec{x} + n\vec{d}), \quad n \in \mathbb{Z}_{>0}$

Let $\vec{m}_n = (1-\frac{1}{n})\vec{y} + \frac{1}{n}(\vec{x} - n\vec{d}), \quad n \in \mathbb{Z}_{>0}$

Then $\vec{k}_n, \vec{m}_n \in C$ by convexity as $\bar{y}, \vec{x} \pm n\vec{d} \in C$.

Since $\vec{k}_n, \vec{m}_n \in C$.

$$\lim_{n\to\infty} \vec{k}_n = \bar{y} + \vec{d}, \quad \lim_{n\to} \vec{m}_n = \vec{y} - \vec{d} \quad \in C.$$

Thus $\bar{y}$ can be written as

$$\bar{y} = \frac{1}{2} \lim_{n\to\infty} (k_n + m_n)$$
$$= \frac{1}{2}(2\bar{y})$$
$$= \bar{y}$$

Which is a contridiction since $\bar{y}$ is an extreme point.

Backwards

Suppose there isn't a line in C. Show there exists a extreme point.

Prove by induction on dimensions of C.

Base case: $n=1$

$C=[x,y]$, C does not contain a line, x,y are the extreme points.

Induction hypo: Assume $n=k$ holds.

Induction step, prove $n=k+1$.

Let $\bar{y}$ be a boundary point of C.

Since C is convex, by the Supporting Hyperplane Theorem, $\exists \vec{c} \in \mathbb{R}^n$ and $\vec{x} \in C$, $\vec{x}$ is a boundary point s.t. $\vec{c}^T\vec{y} \le \vec{c}^T\vec{x}$ for all $\vec{y} \in C$.

Thus $C \cap \vec{c}^T\vec{x} \in \mathbb{R}^n$ contains an extreme point by IH. Let extreme point be called $\vec{e}$, now show $\vec{e}$ also in C.

Show $\nexists\, \vec{x}_1, \vec{x}_2 \in C$ s.t. $\vec{e} = \lambda\vec{x}_1 + (1-\lambda)\vec{x}_2 \quad \lambda \in (0,1)$

Let $\vec{x}_1, \vec{x}_2 \in C$. Then $\vec{c}^T\vec{x}_1 \le \vec{c}^T\vec{x}$, $\vec{c}^T\vec{x}_2 \le \vec{c}^T\vec{x}$ by SHPT. If $\vec{e} = \lambda\vec{x}_1 + (1-\lambda)\vec{x}_2$, then $\vec{c}^T\vec{e} = \lambda\vec{c}^T\vec{x}_1 + (1-\lambda)\vec{c}^T\vec{x}_2$

so $\vec{x}_1 = \vec{x}_2 = \vec{e}$

Which is a contridiction.

Thus $\vec{e}$ is also a extreme point in C. $\square$

Lecture 6 Jan 27

Thm 11 Extreme Point Theorem

Let $P = \{ \vec{x} : A\vec{x} \le \vec{b} \}$ be set of feasible solutions.

If P has an extreme point and LP has an optimal solution, then the optimal solution is an extreme point.

Proof:

Let $\alpha \in \mathbb{R}$ be the value of the optimal solution s.t. the set of all optimal solutions is

$$O = \{ \vec{x} \mid \vec{c}^T \vec{x} = \alpha \} \subset P.$$

If P doesn't contain a line, O doesn't either.

Thus O also has an extreme point, let it be $\vec{e} \in O$.

Show $\nexists\, \vec{x}_1, \vec{x}_2 \in P$ s.t. $\vec{e} = \lambda \vec{x}_1 + (1-\lambda)\vec{x}_2$, $\lambda \in (0,1)$

Then $\vec{c}^T \vec{e} = \lambda \vec{c}^T \vec{x}_1 + (1-\lambda)\vec{c}^T \vec{x}_2$

So since $\vec{c}^T \vec{x}_1, \vec{c}^T \vec{x}_2 \le \vec{c}^T \vec{e} = \alpha$

$\vec{e} = \vec{x}_1 = \vec{x}_2$

Therefore $\vec{e}$ is also an extreme point in P. ∎