Lecture 21 Mar 29 2021

From last week: 1) max: $c^T\vec{x}$

s.t.: $A\vec{x} \leq b$, $0 \leq \vec{x}$

2) min $b^T\vec{y}$

s.t.: $A^Ty \geq c$, $0 \leq \vec{y}$

Weak Duality Thm//

$$c^Tx \leq y^TAx \leq b^Ty$$

"max" $\leq$ "min"

Strong Duality Thm//

If either 1) or 2) has finite optimal value, the so does the other.

Their values coincide (equal).

Homework ☆

Lemma //

Let $A \in \mathbb{R}^{m \times n}$. Only one has solution.

1) $\{ \vec{x} \mid A\vec{x} \leq \vec{b} \}$

2) $\{ \vec{\lambda} \mid A^T\lambda = 0 \quad \lambda^T b < 0 \}$

Proof //

Suppose 1) has optimal value $|p^*| < \infty$, we would be done if we can show

$$3) \longrightarrow \begin{bmatrix} y^T b \geq p^* \\ A^T y \leq c \end{bmatrix} \quad \text{is feasible.}$$

$\exists y$ s.t. $y^T b \leq p^*$, $A^T y \leq c$ by weak duality thm.

by proving 3), we can have $y^T b = p^*$ ////

Rewrite $3) = \begin{bmatrix} A^T \\ -b^T \end{bmatrix} y^T \leq \begin{bmatrix} c \\ -p^* \end{bmatrix}$

Prove by contridiction, assume $3) = \emptyset$.

By lemma $\exists \lambda = \begin{bmatrix} \tilde{\lambda} \\ \end{bmatrix} \geq 0$

$$\begin{bmatrix} \vdots \\ \lambda_0 \end{bmatrix}$$

Then $\tilde{\lambda}^T A^T - \lambda_0 b^T = 0$
$\tilde{\lambda}^T c - \lambda_0 p^* < 0$

4)

show 4) DNE

$\Rightarrow \tilde{\lambda}^T A^T = \lambda_0 b^T$
$\tilde{\lambda}^T c < \lambda_0 p^*$

Case 1: $\lambda_0 = 0$

$\tilde{\lambda}^T A^T = 0 \quad \Rightarrow \quad A\tilde{\lambda} = 0$

$\tilde{\lambda}^T c < 0 \quad \Rightarrow \quad c^T \tilde{\lambda} < 0$

Since p^* is a finite optimal, $\exists\, x^*$ that is a optimal value of 1).

Define $x = x^* + \tilde{\lambda}$, $x \geq 0$.

$$Ax = Ax^* + A\tilde{\lambda}$$
$$= Ax^* = b$$

$$c^T x = c^T(x^* + \tilde{\lambda})$$
$$= c^T x^* + c^T \tilde{\lambda}$$
$$= p^* + \underbrace{c^T \tilde{\lambda}}_{< 0}$$

$< p^*$ which is a contridiction since

we assumed p^* is minimum optimal value.

case 2 $\lambda_0 > 0$

let $x = \frac{\tilde{\lambda}}{\lambda_0}$

Then $Ax = A\left(\frac{\tilde{\lambda}}{\lambda_0}\right) = \frac{1}{\lambda_0} A\tilde{\lambda}$

$= \frac{1}{\lambda_0}\left(\lambda_0 \cdot b^T\right)$

$= b^T$

$c^T x = c^T\left(\frac{\tilde{\lambda}}{\lambda_0}\right)$

$< \frac{1}{\lambda_0} \lambda_0 p^*$

$= p^*$ which is another contridiction

Thus 4) is empty and 3) must be feasible.

$\blacksquare$