Lecture 19 Mar 22 2021

From last class we learned to prevent cycling, we alway pick the smallest ~~index~~ index in tie situations.

Artificial Variables

Ex: LP

$$\text{Max: } z = x_1 - x_2 + x_3$$

$$\text{s.t.} \quad 2x_1 - x_2 + 2x_3 \le 4$$

$$2x_1 - 3x_2 + x_3 \le -5$$

$$-x_1 + x_2 - 2x_3 \le -1 \qquad \vec{x} \ge 0$$

Since initial tableau is not feasible transform LP to LP_2

LP_2

$$\min \ x_0$$

$$-x_0 + 2x_1 - x_2 + 2x_3 \le 4$$

$$-x_0 + 2x_1 - 3x_2 + x_3 \le -5$$

$$-x_0 - x_1 + x_2 - 2x_3 \le -1 \qquad \vec{x} \ge 0$$

Then

$$\max \ -x_0$$

$$-x_0 + 2x_1 - x_2 + 2x_3 \le 4$$

$$-x_0 + 2x_1 - 3x_2 + x_3 \le -5$$

$$-x_0 - x_1 + x_2 - 2x_3 \le -1 \qquad \vec{x} \ge 0$$

Into tableau

$$\begin{array}{ccccccc|c}
 & & & & \multicolumn{3}{c|}{\overbrace{\qquad\qquad\qquad}^{\text{slack var for LP}}} & \\
-1 & 2 & -1 & 2 & 1 & 0 & 0 & 4 \\
-1 & 2 & -3 & 1 & 0 & 1 & 0 & -5 \\
-1 & -1 & 1 & -2 & 0 & 0 & 1 & -1 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

"b"s that are negative

← 2 in LP

← 2 in LP_2

To begin 2-phase algorithm, to get rid of negative b_i's, pick pivot row of most negative b_i. Pick pivot column containing -1 in -2 in LP_2

Initialization

$$\begin{array}{ccccccc|c}
-1 & 2 & -1 & 2 & 1 & 0 & 0 & 4 \\
-1 & 2 & -3 & 1 & 0 & 1 & 0 & -5 \\
-1 & -1 & 1 & -2 & 0 & 0 & 1 & -1 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

← most negative

↑

-1 in LP_2

Now compute simplex as usual.

$$
\begin{array}{c|ccccccc|c}
 & 0 & 0 & 2 & 1 & 1 & -1 & 0 & 9 \\
 & 1 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
\rightarrow & 0 & -3 & 4 & -3 & 0 & -1 & 1 & 4 \\
 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
\hline
 & 0 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
 & & & \uparrow & & & & &
\end{array}
$$

$\vdots$

We end phase1 since objective row ≤ 0

$$
\begin{array}{ccccccc|c}
-2 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & \frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{8}{5} \\
\frac{3}{5} & -\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{11}{5} \\
-\frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \frac{3}{5} \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Phase 2 just takes out th row and colum in from phase 1.

$$
\begin{array}{r|rrrrrr|r}
\rightarrow & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\
 & \frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{8}{5} \\
 & -\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{11}{5} \\
\hline
 & \frac{1}{5} & 0 & 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \frac{3}{5} \\
 & & & & & & \uparrow &
\end{array}
$$

We end Phase 2 since objective row negative.

$$
\begin{array}{rrrrrr|r}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
\hline
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{array}
\qquad \text{optimal} \begin{bmatrix} 0 \\ \frac{14}{5} \\ \frac{17}{5} \\ 0 \\ 0 \\ 3 \end{bmatrix} \qquad z = \frac{3}{5}
$$

Summary (2-Phase Simplex)

Phase 1: Formulat and solve LP_2

2 possible outcomes at the end of phase 1:

1. optimal value of LP_2 is positive
   $\Rightarrow$ LP is infeasible

2. optimal value of LP_2 is zero
   $\Rightarrow$ LP is feasible

Phase 2: If LP is feasible, apply simplex to phase I tableau with x_0 row/columns removed.

2 possible outcomes at the end of phase 2:

1. LP is unbounded
   $\Rightarrow$ column coefficient is negative

2. LP has optimal.

Fundamental Theorem of Linear Programming //

Every LP has 3 properties:

1) If it has no optimal solution, then it is either infeasible or unbounded

2) If it has an feasible solution, then it has a basic feasible solution

3) If it has a feasible solution and is bounded, then it has an optimal basic feasible solution.

Duality in LP

Given LP: $\max\ c^T x$

$\quad$ st $\ A\vec{x} \leq b \quad 0 \leq \vec{x}$

We have dual problem:

D: $\min\ b^T y$

$\quad$ st $\ A^T y \geq c, \quad 0 \leq \vec{y}$

LP: $\max\ c_1 x_1 + \dots + c_n x_n$

st $\quad a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \leq b_i, \quad i = 1, \dots, m$

So $A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m, \ c \in \mathbb{R}^n \qquad \vec{x} \geq 0$

D: $\min\ b_1 y_1 + \dots + b_m y_m$

st. $\quad a_{1j} y_1 + a_{2j} y_2 + \dots + a_{mj} y_m \leq c_j, \quad j = 1, \dots, n$

$\vec{y} \geq 0$

Rmk: LP is infeasible $\Rightarrow \{\vec{x} \mid A\vec{x} \le b, 0 \le \vec{x}\} = \emptyset$

LP is unbounded $\Rightarrow \exists \vec{x} \in \{\vec{x} \mid A\vec{x} \le b, 0 \le \vec{x}\}$

s.t. $c^T x \ge \alpha \quad \forall \alpha$

Homework:

If $LP^{\perp} = D$, is $(LP^{\perp})^{\perp} = LP$?

LP : $\max \; c^T x$

$Ax \le b \quad x \ge 0$

$LP^{\perp} = D$: $\min \; b^T y \quad = \max \; -b^T y$

$A^T y \ge c \rightarrow -A^T y \le -c \quad y \ge 0$

$(LP^{\perp})^{\perp} = \min \; -c^T x \quad = \max \; c^T x$

$-Ax \ge -b \longrightarrow Ax \le b \quad x \ge 0$

$= LP$

Weak Duality Theorem //

If $\vec{x} \in \mathbb{R}^m$ is feasible of LP and $\vec{y} \in \mathbb{R}^n$ is feasible for D, then

$$c^T \vec{x} \leq y^T A x \leq b^T \vec{y}$$

$\alpha = A^T$

$\begin{bmatrix} \max b^T y \\ \\ \\ \end{bmatrix}$

=>

1) If LP is unbounded, then D is necessarily infeasible

2) If D is unbounded, then LP is necessarily infeasible

3) If $c^T x = b^T y$ with x feasible in LP and y feasible in D, then $\vec{x}$ must solve LP and $\vec{y}$ must solve D

Matrix Form.

$$T_0 = \begin{bmatrix} 0 & A & I & b \\ -1 & c^T & 0 & 0 \end{bmatrix}$$

$$G \cdot T_0 = T_K = \begin{bmatrix} 0 & RA & R & Rb \\ -1 & \hat{c}^T - \hat{y}^T A & -\hat{y}^T & -\hat{y}^T b \end{bmatrix}$$

$$G = \begin{bmatrix} R & 0 \\ -\hat{y}^T & 1 \end{bmatrix}$$

LP: $\hat{c}^T - \hat{y}^T A \le 0$ $\quad A^T y \ge c \quad z = \hat{y}^T b$

$-\hat{y}^T \le 0 \quad \to \quad \hat{y} \ge 0$

BFS $\quad x \to \hat{x} = x$ / slack variables

D: $z = c^T \hat{x}$

$(A^T)^T \hat{x} \le b$

$\hat{x} \ge 0$

So $c^T \hat{x} = z = b^T \hat{y}$, so any optimal tableau gives optimal solution to its primal and its dual problem.

Ex.

Max

$z = x_1 - x_2 + x_3$

$2x_1 - x_2 + 2x_3 \le 4$

$2x_1 - 3x_2 + x_3 \le -5$

$-x_1 + x_2 - 2x_3 \le -1$

Phase 1

$$
\begin{array}{rrrrrrr|rl}
-1 & 2 & -1 & 2 & 1 & 0 & 0 & 4 & \\
-1 & 2 & -3 & 1 & 0 & 1 & 0 & -5 & \leftarrow \\
-1 & -1 & 1 & -2 & 0 & 0 & 1 & -1 & \\
0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\uparrow & & & & & & & &
\end{array}
$$

$$
\begin{array}{lrrrrrrr|r}
 & 0 & 0 & 2 & 1 & 1 & -1 & 0 & 9 \\
 & 1 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
\rightarrow & 0 & -3 & 4 & -3 & 0 & -1 & 1 & 4 \\
 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
\hline
 & 0 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
 & & & \uparrow & & & & &
\end{array}
$$

$$
\begin{array}{lrrrrrrr|r}
 & 0 & \frac{3}{2} & 0 & \frac{5}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 7 \\
\rightarrow & 1 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
 & 0 & -\frac{3}{4} & 1 & -\frac{3}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
\hline
 & 0 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
 & & & & \uparrow & & & &
\end{array}
$$

$$
\begin{array}{rrrrrrr|r}
-2 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & \frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{8}{5}
\end{array}
$$

$\frac{3}{5}$	$-\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{11}{5}$
$-\frac{1}{5}$	$\frac{1}{5}$	0	0	0	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$
-1	0	0	0	0	0	0	0

LP feasible

Cont. to Phase 2

1	0	0	1	0	1	3
$\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$
$-\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{11}{5}$
$-\frac{1}{5}$	0	0	0	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$

1	0	0	1	0	1	3
$\frac{4}{5}$	0	1	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{17}{5}$
$-\frac{2}{5}$	1	0	$\frac{1}{5}$	$-\frac{2}{5}$	0	$\frac{14}{5}$
$-\frac{1}{5}$	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	0	$-\frac{3}{5}$

done optimal: $\begin{bmatrix} 0 \\ \frac{14}{5} \\ \frac{17}{5} \\ 0 \\ 0 \\ 3 \end{bmatrix}$, $z = \frac{3}{5}$

Lecture 20 Mar 24 2021

Homework

Max: $z = -x_1 - x_2$

s.t. $-x_1 + x_2 \le -4$

$x_2 \ge 2$

Recall Pre-midterm //

Let $A \in \mathbb{R}^{m \times n}$, $\vec{b} \in \mathbb{R}^m$

Thm //

1) $\{ \vec{x} \mid A\vec{x} \le \vec{b},\ \vec{x} \ge 0 \}$

2) $\{ \vec{y} \mid A^T y \ge 0,\ b^T y > 0 \}$

Only one of these sets has a solution.

2) has variant $\{ \vec{y} \mid A^T y \ge 0,\ b^T y < 0 \}$

Thm has variant:

$\{ \vec{x} \mid A\vec{x} \le \vec{b} \}$ is empty

$\iff \exists \lambda \ge 0,\ \lambda^T A = 0$ and $\lambda^T b < 0$

is used to prove duality thm

Strong Duality Theorem //

Consider Primal-dual Pair :

$$1)\ \left[\begin{array}{c} \min c^T x \\ Ax = b \\ x \geq 0 \end{array}\right] \qquad 2)\ \left[\begin{array}{c} \max b^T y \\ A^T y \leq c \\ y \geq 0 \end{array}\right]$$

(2)

If 1) has a finite optimal value, then so does 2). The two values are the same

Rmk : difference between SDT and ☆

$$1)\ \left[\begin{array}{c} \min 0 \\ Ax = b \\ x \geq 0 \end{array}\right] \qquad 2)\ \left[\begin{array}{c} \max b^T y \\ A^T y \leq 0 \\ y \geq 0 \end{array}\right]$$

let $c=0$

If 1) is infeasible, then 2) is unbounded or else by SDT the two optimal values should match which is impossible.

Thus $\exists y$ s.t. $A^T y \leq 0$ $b^T y > 0$, which is $\star$.

Proof for SDT II

Assume $\star$

1) $\begin{bmatrix} \min c^T x \\ Ax = b \\ x \succeq 0 \end{bmatrix}$ 2) $\begin{bmatrix} \max b^T y \\ A^T y \preceq c \\ y \succeq 0 \end{bmatrix}$

If ① has optimal value $|p^*| < \infty$

if I can show $\begin{bmatrix} y^T b \geq p^* \\ A^T y \preceq c \end{bmatrix} \rightarrow$ 3)

if feasible, the proof is done. Why?

4) By weak duality theorem $A^T y \preceq c$, $y^T b \leq p^*$

So 4) and 3) $\Rightarrow$ $y^T b = p^*$

Rewrite 3) $\begin{pmatrix} -A^T \\ -b^T \end{pmatrix} y \preceq \begin{pmatrix} c \\ p^* \end{pmatrix}$

If 3) was infeasible, the varient of $\star$

implies $\exists \lambda = \begin{pmatrix} \tilde{x} \end{pmatrix} \geq 0$

$\left(\lambda_0 \right)$

s.t. $\tilde{\lambda}^T A^T - \lambda_0 b' = 0$ and $\tilde{\lambda}^T < -\lambda_0 p^* < 0$

$\Rightarrow A\tilde{\lambda} = \lambda_0 b$; is this possible?
$c^T \tilde{\lambda} < \lambda_0 p^*$

1) $\lambda_0 < 0$ by assumption
2) $\lambda_0 = 0$
3) $\lambda_0 > 0$

if 2), then $A\tilde{\lambda} = 0$ $c^T\tilde{\lambda} < 0$
let x^* be s.t. $c^T x^* = p^*$

let $x = x^* + \tilde{\lambda}$

$Ax = Ax^* + A\tilde{\lambda}$
$= b + 0$
$= b$ $\Rightarrow x$ in FR.

$c^T x = c^T x^* + c^T \tilde{\lambda}$
$p^* + < 0$

$\leq p^*$ which is a contridiction

since we assumed x^* is optimal. $\lambda_0 \neq 0$.