Lecture1 Jan 11 th

How to succeed

1. Midterm / Final material come from:
   - i. bi-weekly quizes - graded
   - ii. assignments from textbooks - not graded
   - iii. homework problems from lecture - not graded
2. Go to review seminar(s)
3. All proof questions come from lectures
   - or identical to lecture homework.

What IS Linear Programming and Optimization?

Def1.1 Linear Function

A function $l : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t.

$$l(x + \alpha y) = l(x) + \alpha l(y) \quad , \ \alpha \in \mathbb{R}$$

The process of minimizing/maximizing of l is called optimization.

l is also called an objective function.

The constrainst of l will define a feasible region that is a subset of $\mathbb{R}^n$.

objzctive

Ex. Maximize Profit

Bag manufactures produces two bags,

B1: profit is 25k per bag

B2: profit is 20k per bag

There are two constraints:

1. Material

   i: 20 material per B1

   ii: 12 material per B2

   Supply 1800 units of mat per year

2. Labour

   Labourers can only make 15 bags a month

   and they can only work 8 months a year

Step 1: Identify the decision variables

Step 2: Determine the objective and the objective function

Step 3: Determine the explicit constraints

Step 4: Determine the implicit constraints

Step 1 B1 B2 are the decision variables

Step 2 $\ell = 25B1 + 20B2$ is the function to maximize

Step 3 $B1 + B2 \leq 8 \cdot 15$

$20B1 + 12B2 \leq 1800$

Step 4 $B1, B2 \geq 0$, $B_1, B_2 \in \mathbb{Z}$

Homework: is ℓ linear?

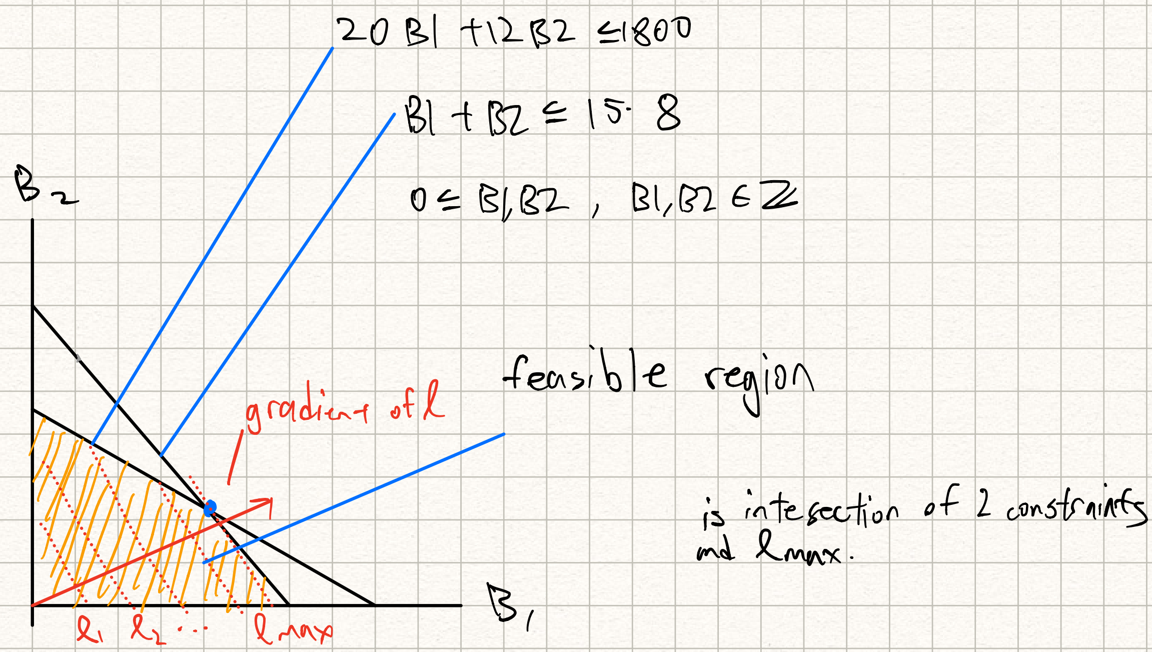
$\ell(a,c) + \ell(b,c) = 25a + 20c + 25b + 20c$

$= 25(a+b) + 20(2c) \neq \ell(a+b, c)$

$\ell(ra, b) = 25(ra) + 20$

$= r\,25(a) + 20b$

$\mp r\, l(a,b)$



First, we take the gradient of l since the gradient always points in the direction of increasing function values.

Since l is linear func., the gradient is always constant.

$$\nabla(l) = \begin{pmatrix} \frac{\partial l}{\partial B1} \\ \frac{\partial l}{\partial B2} \end{pmatrix} = \begin{pmatrix} 25 \\ 20 \end{pmatrix}$$

Thus to maximize take the intesection between the constraints and the level set of l along its gradient, $\nabla l: B_2 = \frac{25}{20} B1$

Summary:

For $\mathbb{R}^2$ problems,

Step1: Graph the linear constraints / feasible region

Step2: Shade the feasible region

Step3: Draw the gradient vector of ℓ

Step4. Push ℓ along its gradient, $\nabla \ell$, until intersection with constraints.

the level set

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Def // General Linear Programming Problem

Find the values of $x_1, \ldots, x_n \in \mathbb{R}$ that will max/min the objective function:

$$Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = \vec{c}^{\,T} \vec{x}$$

s.t. $Z : \mathbb{R}^n \longrightarrow \mathbb{R}^1$.

They are also subject to constraints, $a_{ij} \in \mathbb{R}$ s.t.

$$\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &\ (\leq, \geq, =) \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &\ (\leq, \geq, =) \\
\vdots \qquad\qquad & \qquad \vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &\ (\leq, \geq, =)
\end{aligned}
\quad
\boxed{\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{matrix}} = \vec{b}$$

Note: The value of m depends on question.

Also notice the objective function and constrains are all linear.

Lecture 2 Jan 13th

Def // Standard Form of a Linear Programming Problem

Find the values of $x_1, \ldots, x_n \in \mathbb{R}$ that will maximize the objective function:

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

s.t. $Z : \mathbb{R}^n \longrightarrow \mathbb{R}^1$.

They are also subject to constraints, $a_{ij} \in \mathbb{R}$ s.t.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$

Finally $x_1, x_2, \cdots x_n \geq 0$.

Def // Canonical Form of a Linear Programming Problem

The same form as above but all the constraints are strictly $=$.

Prop // Changing Between Forms

1. Changing minimizing problem to maximizing

$$\min(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = \max(-a_1x_1 - a_2x_2 - \cdots - a_nx_n)$$

2. Changing $\leq$ to $\geq$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$$
$$-a_1x_1 - a_2x_2 - \cdots - a_nx_n \geq -b$$

3. Changing $=$ to $\geq / \leq$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$
$$\Longleftrightarrow$$
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$$

$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b$

4. Changing negative constrains to positive

if $x_i < 0$

$\Leftrightarrow$

$x_i = x_i^+ - x_i^-$, $x_i^+, x_i^- \in \mathbb{R}$ and $x_i^+, x_i^- > 0$

A negative number can always be written as the difference of two positive numbers.

We can represent the problems as a system of equations solvable by matrices.

Ex. Converting above example to matrix representing

$\ell(B1, B2) = 25B + 20C$

$20B1 + 12B2 \leq 1800$

$B1 + B2 \leq 8.15$

$$\ell: \begin{bmatrix} 25 & 20 \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix}$$

$$\begin{bmatrix} 20 & 12 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix} \leq \begin{bmatrix} 1800 \\ 120 \end{bmatrix}$$

$$[B \ C] \in \{\mathbb{Z}_{\geq 0}\}^2$$

Def// Feasible Solution

A vector $\vec{x} \in \mathbb{R}^n$ that satisfies a linear problem.

Def// Optimal Solution

A particular feasible solution that max / min the objective function.

Def'l Closed Set

A set that contains all its limit points

Def'l Open Set

A set where its complement is closed