Tutorial 1 Jan 15, 2021

Linear Algebra Review

Importance of Linear Algebra

- Vectorizing code
- Images are in matrices
- Dimension reduction

Basic Properties

1. $(AB)^T = B^T A^T$
2. $(AB)^{-1} = B^{-1} A^{-1}$
3. $(A^{-1})^T = (A^T)^{-1}$
4. $|AB| = |A||B|$ - Determinant can tell if matrix is non-singular $\neq 0$ or singular $= 0$. ie. how many solutions.
5. $|A| = \frac{1}{|A^{-1}|}$

Determinant of 2D Matrix

$$\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = ad - bc$$

Orthogonal Vectors

2 vectors are orthogonal, $\vec{x}, \vec{y} \in \mathbb{R}^n$ if

$$\vec{x} \cdot \vec{y} = 0$$

Orthogonal Matrices

A square matrix U is orthogonal if

$$U^T U = I = U U^T \iff U^T = U^{-1}$$

and

$$U = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$$

where $x_i x_j = 0$ if $i \neq j$

$x_i x_j = 1$ if $i = j$

$\uparrow$

This is called orthonormal

More Properties

1. $\vec{x}^T x = \|\vec{x}\|_2^2$
2. $AB \neq BA$
3. $\vec{x}^T \vec{y} = \vec{y}^T \vec{x}$

Multivariate Calc Review

Chain Rule

$$\frac{d}{dx}\left[f(g(x)) \right] = f'(g(x))\, g'(x)$$

Ex.

$z = (x^2 + 2)^2$

$z = h^2$

$h = x^2 + 2$

$$\boxed{\frac{dz}{dx}} = \frac{dz}{dh} \cdot \frac{dh}{dx}$$

$$= 2h \cdot 2x$$

Partial Derivatives

Ex.

Let $f(x,y) = 2x^2 + y^2 + 4xy$

$$\nabla f = \left[\frac{df}{dx}, \frac{df}{dy} \right] = \left[4x + 4y, 2y + 4x \right]$$

Jacobian / Hessian Matrix

Jacobian = first order derivatives

Hessian = second order derivatives

$$H(f(x)) = J(\nabla f(x))^T$$

So if $\nabla f(x) = \left[\frac{df}{dx}, \frac{df}{dy} \right]$

$$H(f(x)) = \begin{bmatrix} \frac{d^2 f}{dx dx} & \frac{d^2 f}{dx dy} \\ \frac{d^2 f}{dy dx} & \frac{d^2 f}{dy dy} \end{bmatrix}$$

Some Identities.

1. $\frac{d \vec{a}^T \vec{x}}{d \vec{x}} = a \iff \nabla_{\vec{x}} (\vec{a}^T \vec{x}) = a$

where A is symmetric

2. $\frac{d \vec{x}^T A x}{d \vec{x}} = 2A\vec{x} \iff \nabla_{\vec{x}} (\vec{x}^T (A\vec{x}))$

where A is symmetric

3. $\frac{d \vec{x}^T A x}{dx} = (A + A^T)\vec{x}$