Lecture 3 Jan 18th

Def // Slack Variable

A variable M to take up the "slack" in an inequality.

$$c_1x + c_2x \le b \iff c_1x + c_2x + M = b$$

We use slack variables to change inequalities to equalities.

Ex.

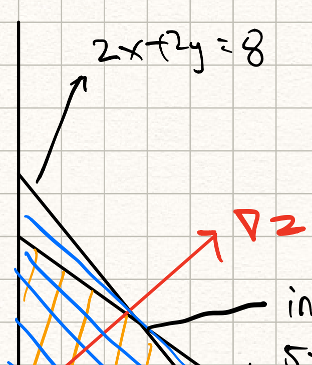
For constraints below, maximize $z = 120x + 100y$

$$\begin{aligned} 2x + 2y &\le 8 \\ 5x + 3y &\le 5 \\ x, y &\geq 0 \end{aligned} \Rightarrow \begin{aligned} 2x + 2y + M &= 8 \\ 5x + 3y + v &= 5 \end{aligned}$$

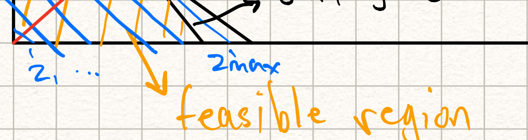
Thus

$$\begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} 8 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

$$\nabla z = \begin{bmatrix} \frac{dz}{dx} \\ \frac{dz}{dy} \end{bmatrix} = \begin{bmatrix} 120 \\ 100 \end{bmatrix}$$



intersection between constraints and z is optimal solution.



Main Topic

① Convex Set

② Feasible Region and Convex set

- using idea of convex set to find max/min objective func

Def // Convex Combination

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. The convex combination of $\vec{u}, \vec{v}$ is

$$\{ \vec{x} \mid \vec{x} = \alpha \vec{u} + (1-\alpha)\vec{v},\ 0 \le \alpha \le 1 \} \subseteq \mathbb{R}^n$$

↳ not linear combination

Def // Convex Set

A subset $C \subseteq \mathbb{R}^n$ is convex if for any $\vec{u}, \vec{v} \in C$,

$$\alpha \vec{u} + (1-\alpha)\vec{v} \in C, \quad 0 \le \alpha \le 1$$

Or the their convex combination $\in C$.

Ex. Is $\vec{c}^T \vec{x} \le \vec{b}$ is convex?

Let $\vec{x}_1, \vec{x}_2$ be solutions of $\vec{c}^T \vec{x} \le b$.

$\alpha \vec{c}^T \vec{x}_1 + (1-\alpha) \vec{c}^T \vec{x}_2 \le \alpha \vec{b} + (1-\alpha) \vec{b} = \vec{b}$

Yes, it is.

Homework // Is $x^2+y^2 \le 1$ convex?

$$S = \{ \vec{x} \mid \|x\| \le 1 \}$$

let $x_1, x_2 \in S$

$$\|\alpha x_1 + (1-\alpha)x_2\| \le \alpha\|x_1\| + (1-\alpha)\|x_2\|$$
$$= 1$$

Def // Hyperplane

A set of solutions to $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

or $\{ \vec{x} \mid \vec{a}^T\vec{x} = b \}$

A Hyperplane is convex. Let $\vec{x}_1, \vec{x}_2 \in H$.

$$\alpha \vec{a}^T\vec{x}_1 + (1-\alpha)\vec{a}^T\vec{x}_2 = \alpha b + (1-\alpha) b$$
$$= b \in H$$

Homework // Can n be changed?

Yes, as shown above.

Thm // Intersection of Convex Sets is Convex

If $S_i \subseteq \mathbb{R}^n$ is convex for $1 \le i \le k$, then

$\bigcap_{i=1}^{k} S_i$ is convex.

Proof:

If $S_i \subseteq \mathbb{R}^n$ is convex then $x_1, x_2 \in S_i$, $1 \le i \le k$ and

$$\alpha x_1 + (1-\alpha)x_2 \in S_i, \quad 1 \le i \le k.$$

Thus $\alpha x_1 + (1-\alpha)x_2 \in \bigcap_{i=1}^{k} S_i$ and

$\bigcap_{i=1} S_i$ is convex.

Ex. $\{A\vec{x}=\vec{b}\}$ is convex ?

$\{A\vec{x}=\vec{b}\}$ is all $\vec{x} : A\vec{x}=\vec{b}$.

Besides simple showing this algebraicly,

Since if $\vec{x}$ is a solution, then $\vec{b} = \begin{bmatrix} A_1\vec{x} \\ A_2\vec{x} \\ \vdots \\ A_n\vec{x} \end{bmatrix}$

which is an intersection of hyperplanes $A_1\vec{x}=b_1, \cdots, A_n\vec{x}=b_n$

and thus is convex.

Homework// Show $\{A \in M_{n\times n}(\mathbb{R}) \mid A^T=A,\ \vec{x}^TA\vec{x}\geq 0, \vec{x}\in\mathbb{R}\}$

is convex.

Let $A_1, A_2 \in S$, are positive s.d and symmetric.

$$\vec{x}^T(\alpha A_1 + (1-\alpha)A_2)\vec{x} = (\alpha\vec{x}^TA_1 + (1-\alpha)\vec{x}^TA_2)\vec{x}$$

$$= \alpha\vec{x}^TA_1\vec{x} + (1-\alpha)\vec{x}^TA_2\vec{x}$$

$$\geq 0 \text{ since } \alpha, (1-\alpha) \geq 0 \qquad \in S$$

Thus S is convex.

Def// Midpoint Convexity

A set $\Omega \subset \mathbb{R}^n$ is midpoint convex if for any $\vec{x}, \vec{y} \in \Omega$,

$\frac{1}{2}\vec{x} + \frac{1}{2}\vec{y} \in \Omega$ ⇐ subspace with just this is not convex.

Thm// Ω is Convex

A closed, midpoint convex set Ω is convex.

Proof:

Let $\vec{x}, \vec{y} \in \Omega$.

Define λ_k as the λ multiplier after the kth iteration.

$$\lambda_k = c_1 \tfrac{1}{2}^1 + c_2 \tfrac{1}{2}^2 + \dots + c_k \tfrac{1}{2}^k, \quad c \in \{0, 1\}$$

$\lambda = 0.12$ $\quad 1 - \lambda = .$

For non-boundry points:

$\lambda_k \vec{x} + (1-\lambda_k)\vec{y} \in \Omega$ as we can get λ_k by iterating midpoint convexity k times.

For boundary points:

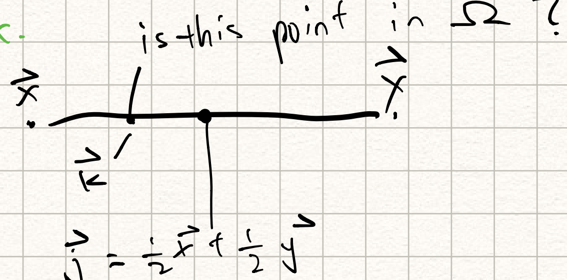
Since Ω is closed, $\lim_{k\to\infty} \lambda_k = \lambda$.

Thus $\lim_{k\to\infty} (\lambda_k \vec{x} + (1-\lambda_k)\vec{y})$

$= \lambda\vec{x} + (1-\lambda)\vec{y} \in \Omega$

Thus Ω is convex.

Ex.



$k = \frac{1}{2}\vec{x} + \frac{1}{2}\vec{j} = \frac{1}{2}\vec{x} + \frac{1}{4}\vec{x} + \frac{1}{4}\vec{y}$

$$= \frac{3}{4}x + \frac{1}{4}y \in \Omega$$

Thus Ω needs to be closed so we can get x_k for an appropiatly large k.

Ex Using middle point convexity to prove convexity.

Let $L = \{ \vec{x} \mid \sqrt{(\vec{x}-\vec{c})^T P(\vec{x}-\vec{c})} \leq r \}$, where $P \in \mathbb{R}^{n \times n}$, $\vec{x}^T P \vec{x} \geq 0 \ \forall x \in \mathbb{R}^n$, $\vec{x}, \vec{c} \in \mathbb{R}^n$, $r \in \mathbb{R}$

Let p-norm be $\|\vec{x}\|_p = \sqrt{\vec{x}^T P \vec{x}}$.

Show $(L \cdot \|\ \|_p)$ is convex. Additionally L is closed since $\leq r$.

Let $\vec{x}_1, \vec{x}_2 \in L$. Then show $\| \frac{\vec{x}_1 + \vec{x}_2}{2} + \vec{c} \|_p \in L$

$$\left\| \frac{\vec{x}_1}{2} + \frac{\vec{x}_2}{2} + c \right\|_p = \left\| \frac{\vec{x}_1}{2} + \frac{\vec{c}}{2} + \frac{\vec{x}_2}{2} + \frac{\vec{c}}{2} \right\|_p$$
$$\leq \left\| \frac{\vec{x}_1}{2} + \frac{\vec{c}}{2} \right\|_p + \left\| \frac{\vec{x}_2}{2} + \frac{\vec{c}}{2} \right\|_p$$
$$\leq \frac{1}{2} r + \frac{1}{2} r = r$$

Thus L is convex by Thm aforementioned.

Def // Convex Function

A convex function is a function,

$$f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$$

where S is convex and if

$$f(\alpha \vec{x} + (1-\alpha)\vec{y}) \leq \alpha f(\vec{x}) + (1-\alpha) f(\vec{y})$$

for $\vec{x}, \vec{y} \in S$, $0 \leq \alpha \leq 1$.

Note: Since the objective function, z, is linear,

$$z(\alpha\vec{x}_1 + (1-\alpha)\vec{x}_2) = \alpha z(\vec{x}_1) + (1-\alpha) z(\vec{x}_2)$$

All z is B61 is convex.

Thm 11 Local Extrema is Global Extrema

For L.P. question

$$z = f(x)$$

subject to $\vec{x} \in \Omega$

If f is a convex function and Ω is a convex set,

the local extrema is the global extrema.

Proof

Let $\vec{x}_0$ be a local min s.t. $\exists r \in R$, $f(\vec{x}_0) \leq f(\vec{x})$ $\forall \vec{x} \in B_r(x_0)$.

Suppose by contridiction $\exists \vec{z} \in \Omega$ s.t. $f(\vec{z}) < f(\vec{x}_0)$

Then $\alpha \vec{x}_0 + (1-\alpha)\vec{z} \in \Omega$ for $0 \leq \alpha \leq 1$

By convexity of f, $f(\alpha\vec{x}_0 + (1-\alpha)\vec{z}) \leq \alpha f(\vec{x}_0) + (1-\alpha) f(\vec{z})$

$$\leq \alpha f(\vec{x}_0) + (1-\alpha) f(\vec{x}_0)$$

$$= f(\vec{x}_0) \quad \text{for } 0 \leq \alpha \leq 1$$

Then as $\alpha \to 1$ $\alpha\vec{x}_0 + (1-\alpha)\vec{z} \to \vec{x}_0$

which is a point in $B_r(\vec{x}_0)$. This is a contridiction

since we assume $\vec{x}_0$ is a local min but

$f(\alpha\vec{x}_0 + (1-\alpha)\vec{z}) \leq f(\vec{x}_0)$ as $\alpha \to 1$.

Thus $\vec{x}_0$ must be the global minimum.

Lecture 4 Jan 19th

Every L.P. problem can be written as matrices:

ℓ: $\vec{c}^{T}\vec{x}$

subject to: $A\vec{x} = \vec{b}$ or $A\vec{x} \leq \vec{b}$

and $\vec{x} \geq \vec{0}$.

The feasible region is defined by $A\vec{x} = \vec{b}$, $\vec{x} \geq \vec{0}$.
and $A\vec{x} = (\leq)\vec{b}$ is convex.

1 From last lecture if Ω is closed, we can use middle point convexity to prove general convexity.

2 Finally using Local Extrema is Global Extrema (Thm) we can optimize more efficiently.

Note: If we have system $A\vec{x} < \vec{b}$, the feasible region isn't closed so we can't use midpoint convexity.

Thm II Closure is Convex

If S is convex, then the closure of S is convex.

$$Cl(S) = S \cup \{\partial S\}$$

Proof:

Suppose $\vec{x}, \vec{y} \in Cl(S)$

$\exists \{\vec{x}_k\}_{k=1}^{\infty}$, $\{\vec{y}_k\}_{k=1}^{\infty} \in S$ s.t. $\lim\limits_{k\to\infty} \vec{x}_k, \vec{y}_k = \vec{x}, \vec{y} \in Cl(S)$

$\lim\limits_{k\to\infty} (\alpha\vec{x}_k + (1-\alpha)\vec{x}_k) = \alpha \lim \vec{x}_k + (1-\alpha)\lim \vec{y}_k$

$= \alpha \vec{x} + (1-\alpha)\vec{y} \in Cl(S)$

Thus the closure of S is convex.

Lem// $B_\varepsilon(0) = \{\vec{x} \mid \sqrt{x^T x} \le \varepsilon\}$

a) For all $\alpha \ge 0$ $\quad \alpha \cdot B_\varepsilon(0) = B_{\varepsilon \cdot \alpha}(0)$

b) $B_\varepsilon(\vec{x}^\circ) = \vec{x}^\circ + B_\varepsilon(0)$

c) For all $\alpha \ge 0$, $\alpha B_\varepsilon(\vec{x}^\circ) = B_{\varepsilon \cdot \alpha}(\vec{x}^\circ)$

Homework: Proof above.

Define $B_\varepsilon(0) = \{\vec{x} \mid \sqrt{x^T x} \le \varepsilon\}$

a) $\alpha \cdot B_\varepsilon(0) = \{\alpha \vec{x} \mid \sqrt{x^T x} \le \varepsilon\}$

$= \{\vec{x} \mid \sqrt{(\frac{x}{\alpha})^T \frac{x}{\alpha}} \le \varepsilon\}$

$= \{\vec{x} \mid$

check this

Lem// Interior of S

Suppose $S \subseteq \mathbb{R}^n$ is convex and has non-empty interior.

If $\vec{x}^\circ \in int(S)$ (interior of S) and $\vec{x} \in S$

Then for any $(1-\alpha)\vec{x}^\circ + \alpha\vec{x} \in int(S)$, $\alpha \in [0,1)$

Prop // Interior is Convex

If $S \subseteq \mathbb{R}^n$ is convex, then $int(S)$ is also convex.

$A\vec{x} \leq b$ is convex $\Rightarrow$ $A\vec{x} < \vec{b}$ is convex

Homework: ① Let $c_1, c_2 \dots, c_n \in \mathbb{R}$ s.t. $\sum c_i = 1$, $c_i \geq 0$

Let $S \subseteq \mathbb{R}^n$ be a convex set.

If $x_1, x_2, \dots, x_n \in S$, show $\sum c_i x_i \in S$.

Proof: Since S is convex, we know

$X_i, X_j \in S$ $\alpha x_i + (1-\alpha)x_j \in S$

Prove using induction on n

$n=1$ $1 x_i \in S$

$n=2$ $\alpha x_i + (1-\alpha)x_j \in S$ $>$ by def

$n=3$ $\alpha_2(\alpha_1 x_i + (1-\alpha_1)x_j) + (1-\alpha_2)x_k \in S$ by convexity

if $x_k \in S$.

$\alpha_2\alpha_1 x_i + \alpha_2(1-\alpha_1)x_j + (1-\alpha_2)x_k$

$\cancel{\alpha_2\alpha_1} + \cancel{\alpha_2} - \cancel{\alpha_2\alpha_1} + 1 - \cancel{\alpha_2} = 1$

Thus $c_3 + c_2 + c_1 = 1$

Suppose true for $n = k$

$\sum_{i=1}^{k} c_i x_i \in S$ $\sum_{i=1}^{k} c_i = 1$

Prove $n = k+1$, $x_{k+1} \in S$

$c_{k+1}\sum_{i=1}^{k} c_i x_i + (1-c_{k+1})x_{k+1} \in S$ by convexity

where $c_{k+1} \in [0,1]$

Let $\sum^{k+1} d_i = c_{k+1}c_1 + c_{k+1}c_2 + \dots + c_{k+1}c_n + (1-c_{k+1})$

$$= c_{k+1} \sum_{i=1}^{k} c_i + (1 - c_{k+1})$$
$$= c_{k+1} + 1 - c_{k+1} = 1$$

Thus $\sum_{i=1}^{k+1} d_i x_i \in S$. QED

(2) Let $S_1, S_2, S_3 \subseteq \mathbb{R}^n$ be convex sets.

Let $c \in \mathbb{R}$

i) $cS_1 = \{\vec{z} \mid \vec{z} = c\vec{x}, \vec{x} \in S_1\}$ is convex

choose $z_1 = c\bar{x}_1$, $z_2 = c\bar{x}_2 \in Z$.

$$\alpha z_1 + (1-\alpha) z_2 = c(\alpha x_1 + (1-\alpha) x_2)$$
$\in Z$ since $\alpha x_1 + (1-\alpha) x_2 \in C$.

Thus cS_1 is convex.

ii) $S_2 + S_3 = \{\vec{z} \mid \vec{z} = \vec{x}_2 + \vec{x}_3, \vec{x}_2 \in S_2, \vec{x}_3 \in S_3\}$

Let $z_1 = x_2 + x_3$, $z_2 = x_4 + x_5 \in (S_2 + S_3)$

$$\alpha z_1 + (1-\alpha) z_2 = \alpha (x_2 + x_3) + (1-\alpha)(x_4 + x_5)$$
$$= \alpha x_2 + (1-\alpha) x_4 + \alpha x_3 + (1-\alpha) x_5$$
$$= x_5 + x_6, \quad x_5 = \alpha x_2 + (1-\alpha) x_4 \in S_2$$
$\in S_2 + S_3$, $\quad x_6 = \alpha x_3 + (1-\alpha) x_5 \in S_3$

Thus $S_2 + S_3$ is convex.

∎