Lecture 7 Feb 1

Midterm Structure (Up to Feb 3):

① Based on $\underbrace{\text{Lecture Notes, Lecture Homework}}_{40\%}$, $\underset{50\%}{\text{Assignments}}$

and $\underset{10\%}{\text{Quiz}}$.

Thm//

Let $A \in \mathbb{R}^{m \times n}$, $\vec{b} \in \mathbb{R}^n$, then exactly one of the following sets must be empty

1) $\{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{b},\ \vec{x} \geq 0 \}$

2) $\{ \vec{y} \in \mathbb{R}^m \mid A^T \cdot \vec{y} \leq 0,\ \vec{b}^T y > 0 \}$

2i) $A^T \cdot \vec{y} \geq 0,\ \vec{b}^T y < 0$ still true //

Variant of Above Theorem

① Let $A \in \mathbb{R}^{m \times n}$ matrix. Only one of the following set must be empty

1) $\{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} < 0 \}$

2 $\{ \vec{y} \in \mathbb{R}^m \mid \vec{y} \neq 0,\ A^T y = 0,\ y > 0 \}$

②

1) $\{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{b} \}$

2) $\{ \vec{y} \in \mathbb{R}^m \mid A^T \vec{y} = 0,\ \vec{b}^T \vec{y} \neq 0 \}$

Applications of Above Theorem //

Ex. Does this LP problem have a solution?

obj func : z

subject to: $3x_1 + 4x_2 + 5x_3 \geq 0$

$4x_1 \geq 0$

$2x_2 + 5x_3 \geq 0$

$x_1 + x_2 + 2x_3 < 0$

Too many contraints, $FR \neq \emptyset$.

First convert to matrix form:

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 0 & 0 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$ Where $A\vec{x} \geq 0$ and $\vec{b}^T\vec{x} < 0$ which is 2)

Then use theorem show 1)

$$A^T\vec{x} = b \Rightarrow \begin{bmatrix} 3 & 4 & 0 \\ 4 & 0 & 2 \\ 5 & 0 & 5 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Since $\{A^T\vec{x} = b\} \neq \emptyset$, $\{A\vec{x} \geq 0\} = \emptyset$. In other words, since 1) is non-empty, 2) must be empty.

Thus this LP has no solution //

Ex. Does this LP problem have a solution?

Subject to: $2x_1 + x_2 + x_3 + 4x_4 \geq 0$

$x_1 + x_3 \geq 0$

$x_1 + x_2 + x_3 + x_4 < 0$

Define $\begin{bmatrix} 2 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vec{x} \geq \vec{0}$

$[1\ 1\ 1\ 1]\vec{x} < 0$

is in form 2i)

Check if $\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & 1 \\ 4 & 0 \end{bmatrix} [y_1 \quad y_2] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{y} \geq \vec{0}$, has solution

is in form 1)

$2y_1 + y_2 = 1$ and $y_1 + y_2 = 1$ $y_1 = 0$ $y_2 = 1$

Then $4 \cdot 0 + 0 \cdot 1 \neq 1$, which is a contridiction.

Thus 1) has no solution, thus 2i) has solution.

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Proof:

Lem //

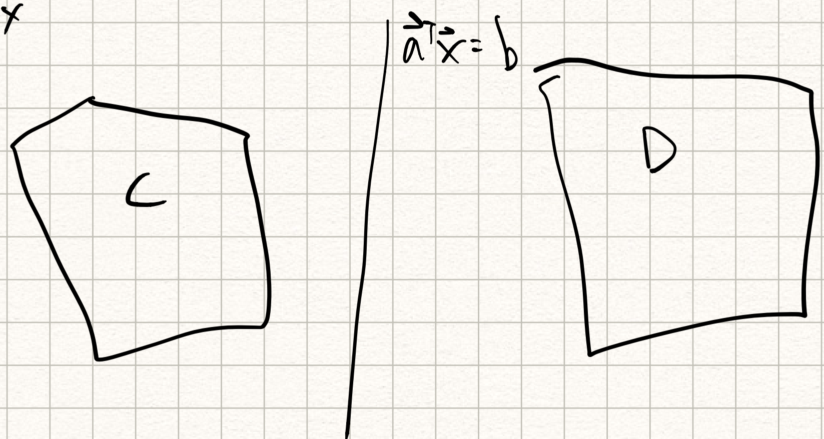
Let $C, D \in \mathbb{R}^n$ be two convex sets s.t. $C \cap D = \phi$. (disjoint)

$\exists\, \vec{a} \in \mathbb{R}^n, \vec{a} \neq \vec{0}, b \in \mathbb{R}$ s.t.

$$\vec{a}^T \vec{x}_1 \leq b \quad \forall \vec{x}_1 \in C$$

$$\vec{a}^T \vec{x}_2 \geq b \quad \forall \vec{x}_2 \in D$$

Ex



Eg. There always exists a hyperplane $\vec{a}^T\vec{x} = b$ that seperates C, D.

Lem - Rephrased //

① Let C, D be a closed convex set $\in \mathbb{R}^n$

② At least one of them is bounded

③ C, D is disjoint. $C \cap D = \emptyset$

Then,

$\exists \vec{a} \in \mathbb{R}^n, \vec{a} \neq \vec{0}, b \in \mathbb{R}$ s.t.

$$\vec{a}^T \vec{x}_1 > b \quad \forall \vec{x}_1 \in C$$

$$\vec{a}^T \vec{x}_2 < b \quad \forall \vec{x}_2 \in D$$

Proof of Lem - Rephrased //

Def // Distance function

Let $Dist(C, D) = \inf \|\vec{u} - \vec{v}\|$ s.t. $u \in C, v \in D$

Let $\vec{c} \in C$, $\vec{d} \in D$ be the points of $Dist(C, D)$.

i.e. their distantce is minimum of any points between C and D.

Let $\vec{a} = \vec{d} - \vec{c}$ and $b = \dfrac{\|d\|^2 - \|c\|^2}{2}$

Define hyperplane $f(\vec{x}) = a^T \vec{x} - b$

Show that this f satisfies conditions in lemma

namely

$$f(\vec{x}_1) > 0 \quad \forall \vec{x}_1 \in D$$

$$f(\vec{x}_2) < 0 \quad \forall \vec{x}_2 \in C$$

Since we chose $\vec{a}, \vec{b}$ as above, we have property:

$$f\left(\frac{\vec{c} + \vec{d}}{2}\right) = (\vec{d} - \vec{c})^T \left(\frac{\vec{c} + \vec{d}}{2}\right) - \frac{\|d\|^2 - \|c\|^2}{2}$$

$$= \vec{b} - \vec{b} = \vec{0}$$

Assume the contrary, $\exists \bar{d} \in D$ s.t. $f(\bar{d}) \le 0$.

So $(\vec{d}-\vec{c})^T \bar{d} - \dfrac{\|\vec{d}\|^2 - \|\vec{c}\|^2}{2} \le 0$ ☆

Define func: $g(\vec{x}) = \|\vec{x} - \vec{c}\|^2$

Let $\vec{v} = \bar{d} - d$

$$\nabla g^T(x) \vec{v} \Big|_{x=\vec{d}} = \nabla_{\vec{v}}\, g^T(x) \Big|_{x=\vec{d}} \quad \text{by B41}$$

So $g(x) = [\vec{x} - \vec{c}]^T [\vec{x} - \vec{c}]$

$$= \vec{x}^T \vec{x} - 2\vec{x}^T \vec{c} + \vec{c}^T \vec{c}$$

And $\nabla g = 2\vec{x} - 2\vec{c}$

$$\nabla_{\vec{v}}\, g^T(x)\Big|_{x=\vec{d}} = [2\vec{d} - 2\vec{c}] \cdot [\bar{d} - \vec{d}]$$

$$= 2(\vec{d}^T \bar{d} - \|d\|^2 - \vec{c}^T \bar{d} + \vec{c}^T \vec{d})$$

$$= 2(-\|\vec{d}\|^2 + (\vec{d} - \vec{c})^T \bar{d} + \vec{c}^T \vec{d})$$

$$\le 2\left(-\|\vec{d}\|^2 + \frac{\|\vec{d}\|^2 - \|\vec{c}\|^2}{2} + \vec{c}^T \vec{d}\right) \quad ☆$$

$$= -\|\vec{d}\|^2 - \|\vec{c}\|^2 + 2\vec{c}^T \vec{d}$$

$$= -\|\vec{d} - \vec{c}\|^2 < 0$$

Since $\nabla_{\vec{v}}\, g^T(\vec{x})\big|_{x=\vec{d}} < 0$, this tells us

$g(\vec{d} + \alpha(\bar{d} - \vec{d}))$ decreases as $0 < \alpha \to \infty$.

Thus $g(\vec{d} + \alpha(\bar{d} - \vec{d})) < g(\vec{d})$ for $\alpha \in (0, \infty)$.

or $\|\vec{d} + \alpha(\bar{d} - \vec{d})\|^2 < \|\vec{d} - \vec{c}\|^2$

Which is a contridiction since we defined c, d as $\mathrm{Dist}(C, D)$.

Therefore $f(\vec{d}) > 0 \quad \forall \vec{d} \in D$ as wanted.
$f(\vec{c}) < 0 \quad \forall \vec{c} \in C$ proven by symmetry.

Now use above lemma ☆ to prove above theorem. ○

Def // Cone

A set $K \subseteq \mathbb{R}^n$ is a cone if

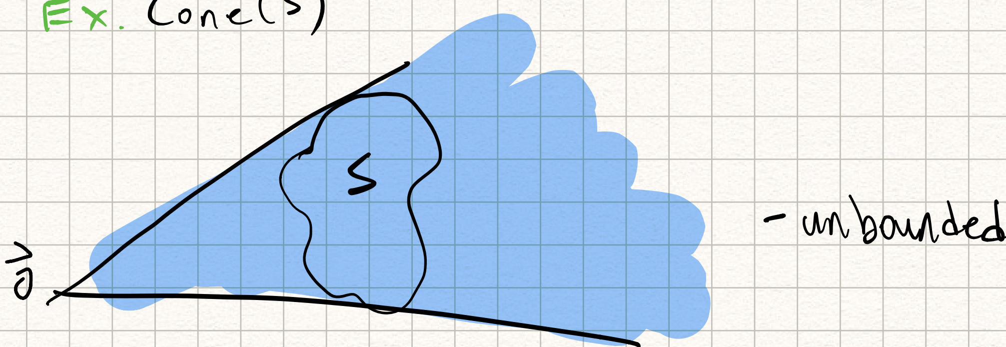
$$\vec{x} \in K \rightarrow \alpha \vec{x} \in K \quad \forall \alpha \in \mathbb{R}, \ \alpha \geq 0$$

Def // Cone hull

Given a set S, the cone hull of S, $\mathrm{cone}(S)$, is the set of all conic combinations of points in S.

$$\mathrm{cone}(S) = \{ \sum \alpha_i x_i \mid \alpha_i \geq 0, x_i \in S \}$$

Ex. Cone(S)



Continuing proof of theorem ○ 2) $\longrightarrow$ 1)

Prove:

if $\{ \vec{y} \in \mathbb{R}^m \mid A^T \vec{y} \leq 0 \quad b^T \vec{y} > 0 \}$ has solution 2) then $\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = b, \vec{x} \geq 0\}$ has no solution. i)

Suppose not, $\exists \vec{x} \geq 0 \in \mathbb{R}^n$ s.t $A\vec{x} = b$.

$$
\begin{aligned}
\text{Then } (A\vec{x})^T\vec{y} &= (\vec{x}^T A^T)\vec{y} \\
&= b^T\vec{y} \\
&\geq 0 \quad \text{by assumption}
\end{aligned}
$$

We have a contradiction. Since $\vec{x}^T A^T \vec{y} \geq 0$ and $\vec{x} \geq 0$, we must have $A^T\vec{y} \geq 0$ but we assumed $A^T\vec{y} \leq 0$.

Thus 1) has no solution

$\blacksquare$

Lecture 8 Feb 3

Continuing proof of 0. 1) $\longrightarrow$ 2).

$\Rightarrow$

Assume $\{\vec{x} \mid A\vec{x} = \vec{b}\} = \emptyset$. Show $\{\vec{y} \mid A^T\vec{y} \leq 0 \;\; \vec{b}^T\vec{y} > 0\} \neq \emptyset$

Let $\vec{a}_1, \vec{a}_2 \ldots \vec{a}_n$ be columns of A.

Let $C = \text{cone}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$. C is a closed, convex set.

Choose $\vec{b} \notin C$ and we know this exists since $\{\vec{x} \mid A\vec{x} = \vec{b}\} = \emptyset$

Since $\{\vec{b}\}$ is bounded, closed and convex.

By Lemma $\exists \vec{y} \in \mathbb{R}^m, y \neq 0$, $r \in \mathbb{R}$ s.t.

$$\vec{y}^T\vec{z} < r \quad \forall \vec{z} \in C$$

$$\vec{y}^T \vec{b} > r$$

Since $\vec{0} \in C$ we must have $r \geq 0$ and thus

$\vec{y}^T \vec{z} \leq 0$, $\vec{y}^T b > 0$

Since $\vec{a}_1, \vec{a}_2 \dots \vec{a}_n \in C$, and generates C. $\vec{A}^T \vec{y} \leq 0$

$\boxtimes$

Applications//

Arbitrarge Theorem//

$\exists$ a system of absolute priors for commodies s.t. the exchange rates (1) and price ratios (1) iff the market are arbitrage free.

Suppose an experiment has m possible autcomes and n possible wagers.

That's if you bet y on wager i, you win the $y \cdot r_i(j)$ if the outcome of the experiment is j.

A betting strategy : $\vec{y} = (y_1\ y_2 \cdots y_n)$. If the outcome of the experiment is j, then the gain/loss is amount $\Sigma\ \vec{y} \cdot \vec{r} = \hat{\Sigma}\ y_i \cdot r_i(j)$

Homework:

1) Show $A = \{ \vec{x} \in \mathbb{R}^2 \mid y \geq 0$, positive x-axis removed $\}$

is convex.

Let $\vec{x}_1, \vec{x}_2 \in A$

$\alpha(x_1, y_1) + (1-\alpha)(x_2, y_2)$

$= (\alpha x_1 + (1-\alpha)x_2 \,,\, \alpha y_1 + (1-\alpha)y_2)$

Since $x_1, x_2 \notin$ positive x-axis, $\alpha x_1 + (1-\alpha)x_2 \notin$ positive x-axis

Since $y_1, y_2 \geq 0$, $\alpha y_1 + (1-\alpha)y_2 \geq 0$

Thus A is convex. $\square$

2) Show $C \subseteq \mathbb{R}^n$ a closed convex set and $\vec{x} \in \mathbb{R}^n$ be a point not in C.
Show there exists a hyperplane that separates the two.

Since $\{\vec{x}\}$ is bounded, closed and convex, C is closed and convex and $C \cup \{\vec{x}\} = \emptyset$.

By Lemma ☆, $\exists\, \vec{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ s.t.

$$\vec{a}^T \vec{x} > b$$

$$\vec{a}^T \vec{c} < b \qquad \forall\, c \in C.$$

$\square$ b

3) Show $\text{cone}(S) = \{ \sum \alpha_i x_i \mid \alpha_i \geq 0, x_i \in S \}$ is convex.

Let $a = \sum \alpha_i x_i$, $b = \sum \lambda_i x_i$

s.t. $x_i \in S$. $\alpha_i, \lambda_i \geq 0$, $a, b \in \text{cone}(S)$.

$$\theta a + (1-\theta) b = \theta \sum \alpha_i x_i + (1-\theta) \sum \lambda_i x_i$$

$= \sum(\theta \alpha_i + (1-\theta)\lambda_i)x_i$

$\in$ Cone(S) as $(\theta\alpha_i + (1-\theta)\lambda_i) \in \mathbb{R}$

4) Show either $\{\vec{x} \mid A\vec{x} = \vec{b}\}$ has a solution ①

or $\{\vec{y} \mid A^T\vec{y} = 0,\ \vec{y}^T\vec{b} \neq 0\}$ has a solution ②

$\Rightarrow$ Assume $A\vec{x} = \vec{b}$ has a solution.

Assume by contridiction $\exists \vec{y}$ s.t. $A^T\vec{y} = 0$ $\vec{y}^T\vec{b} \neq 0$.

$(A\vec{x})^T = x^T A^T = b^T$.

$\vec{b}^T\vec{y} = x^T A^T \vec{y} = x^T \vec{0} = 0$ but $\vec{y}^T\vec{b} \neq 0$.

Thus by contridiction ② has no solution.

$\Leftarrow$ Assume $A^T\vec{y} = 0$, $\vec{y}^T\vec{b} \neq 0$ has a solution ②

Prove $A\vec{x} = \vec{b}$ has no solution. ①

Assume the contrary.

Then $(A\vec{x})^T = x^T A^T = b^T$

$b^T\vec{y} = \vec{x}^T A^T \vec{y} = \vec{x}^T\vec{0} = \vec{0}$ but $\vec{y}^T\vec{b} \neq 0$.

Thus ① has no solution

$\blacksquare$.