# 1 Useful Formulas

## 1.1 General

- Arithmetic Series: $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$
- Geometric Series: $\begin{cases} \sum_{i=0}^{n} a^i = \dfrac{1-a^{n+1}}{1-a}, & \forall a \in \mathbb{R} \\ \sum_{i=0}^{\infty} a^i = \dfrac{1}{1-a}, & \forall |a| < 1 \end{cases}$
- Binomial Thm: $\begin{cases} (x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \\ \forall\, x, y \in \mathbb{R},\ n \in \mathbb{N} \end{cases}$
- Exponential Fn: $\sum_{i=0}^{\infty} \dfrac{a^i}{i!} = e^a, \ \ \forall a \in \mathbb{R}$
- Gamma Fn: $\begin{cases} \Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx,\ a > 0 \\ \Gamma(n) = (n-1)!,\ n \in \mathbb{N} \end{cases}$

## 1.2 Probability

- Law of Total Probability: $\forall$ partition $\{A_i\}_{i=1}^{n}$

  $$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$
- Bayes Rule: $\forall$ partition $\{A_i\}_{i=1}^{n}$

  $$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$
- Multiplication Rule: $P\left(\bigcap_{i=1}^{n} A_i\right) =$

  $$P(A_1)P(A_2|A_1) \times \cdots \times P(A_n|A_1, \ldots, A_{n-1})$$
- Inclusion-Exclusion Principle:

  $$\begin{aligned} P\left(\bigcup_{i=1}^{n} A_i\right) &= \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j) + \\ &+ \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_j) \\ &\ \ \vdots \\ &+ (-1)^{n-1} P\left(A_1 \cap \cdots \cap A_n\right) \end{aligned}$$

## 1.3 Calculus

- Change of Variables:

  $$\int_a^b f(x)dx \overset{(x=h(u))}{=} \int_{h^{-1}(a)}^{h^{-1}(b)} f(h(u))h'(u)du$$
- Fundamental Theorem of Calculus:

  $$F'(x) = f(x) \Rightarrow \int_a^b f(x)dx = F(b) - F(a)$$

  $$F(x) = \int_a^x f(u)du \Rightarrow F'(x) = f(x)$$
- Integration by Parts:

  $$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$$

## 1.4 Multivariate Distributions

- Marginal PDF

  $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$
- Conditional PDF

  $$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$
- 2D PDF method, for $(Z,W) = h(X,Y) \Leftrightarrow$

  $\Leftrightarrow (X,Y) = h^{-1}(Z,W)$

  $$f_{Z,W}(z,w) = \frac{f_{X,Y}(h^{-1}(z,w))}{|J(h^{-1}(z,w))|},$$

  where $|J| = \begin{vmatrix} dz/dx & dz/dy \\ dw/dx & dw/dy \end{vmatrix}$
- Convolution method, for $Z = X + Y$

  $$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)dx$$
- Order Statistics, for $X_i \sim^{iid} F(x)$

  $$P(X_{(i)} \le x) = \sum_{k=i}^{n} \binom{n}{k} F(x)^k (1 - F(x))^{n-k}$$

## 1.5 Expectations

- Mean of Cont./Discr. RV $X$

  $$E(X) = \mu = \int_{-\infty}^{\infty} x f_X(x)dx \Big/ = \sum_x x p_X(x)$$
- Mean of function $h$ of RV $X$

  $$E(h(X)) = \int_{-\infty}^{\infty} h(x) f_X(x)dx \Big/ = \sum_x h(x) p_X(x)$$
- Variance of RV

  $$V(X) = \sigma^2 = E[(X-\mu)^2] = E(X^2) - [E(X)]^2$$

- Moment Generating Function of RV $X$

  $m_X(t) = E(e^{tX})$

  $E(X^k) = m_X^{(k)}(0)$
- Covariance of two RVs

  $Cov(X,Y) = E(XY) - E(X)E(Y)$
- Expectation of Linear Function of RVs

  $$E\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i E(X_i)$$
- Variance of Linear Function of RVs

  $$V\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i^2 V(X_i) + $$
  $$+ 2 \sum_{1\le i<j\le n} a_i a_j Cov(X_i, X_j)$$
- Law of Total Expectation/Variance

  $E(X) = E(E(X|Y))$

  $V(X) = E(V(X|Y)) + V(E(X|Y))$

## 1.6 Inequalities

- Markov Inequality, for $X > 0$

  $P(X \geq a) \leq E(X)/a$
- Chebyshev Inequality

  $P(|X-\mu| \geq a) \leq V(X)/a^2$
- Cauchy-Schwarz Inequality

  $(E(XY))^2 \leq E(X^2)E(Y^2)$
- Jensen Inequality, for convex $g(\cdot)$

  $E(g(X)) \geq g(E(X))$

## 1.7 Limit Theorems

- Weak Law of Large Numbers for i.i.d. $X_i$

  $(X_1 + \cdots + X_n)/n = \bar{X}_n \to^P \mu$
- Central Limit Theorem for i.i.d. $X_i$

  $$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \to^D \mathrm{N}(0,1)$$

## 1.8 Important Distributions

| Name | PMF/PDF | CDF | Mean | Variance | MGF |
|---|---|---|---|---|---|
| Discrete Distributions | | | | | |
| Binomial$(n,p)$ | $\binom{n}{x}p^x(1-p)^{n-x}$<br>$x = 0,\ldots,n$ | | $np$ | $np(1-p)$ | $(1-p+pe^t)^n$ |
| Geometric$(p)$<br>(# trials) | $p(1-p)^{x-1}$<br>$x = 1,\ldots,\infty$ | $1-(1-p)^x$ | $1/p$ | $(1-p)/p^2$ | $\frac{pe^t}{1-(1-p)e^t}$<br>$t < -\ln(1-p)$ |
| Poisson$(\lambda)$ | $e^{-\lambda}\lambda^x/x!$<br>$x = 0,\ldots,\infty$ | | $\lambda$ | $\lambda$ | $\exp\{\lambda(e^t-1)\}$ |
| NegBinom$(r,p)$ | $\binom{x-1}{r-1}p^r(1-p)^{x-r}$<br>$x = r, r+1,\ldots,\infty$ | | $r/p$ | $r(1-p)/p^2$ | $\left(\frac{pe^t}{1-(1-p)e^t}\right)^r$<br>$t < -\ln(1-p)$ |
| HyperGeom<br>$(n,M,N,)$ | $\binom{M}{x}\binom{N-M}{n-x}/\binom{N}{n}$<br>$x = 0,\ldots,n,\ x \leq M,$<br>$x \geq n-N+M$ | | $nM/N$ | $n\frac{M}{N}\frac{(N-M)}{N}\frac{(N-n)}{(N-1)}$ | |
| Continuous Distributions | | | | | |
| Uniform$(\ell,u)$ | $1/(u-\ell),\ \ x \in [\ell,u]$ | $\frac{x-\ell}{u-\ell}$ | $\frac{u+\ell}{2}$ | $\frac{(u-\ell)^2}{12}$ | $\frac{e^{tu}-e^{t\ell}}{t(u-\ell)}$ |
| Exponential$(\lambda)$ | $\lambda e^{-\lambda x},\ \ x \geq 0$ | $1-e^{-\lambda x}$ | $1/\lambda$ | $1/\lambda^2$ | $\frac{\lambda}{\lambda-t},\ t<\lambda$ |
| Normal$(\mu,\sigma^2)$ | $\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$<br>$x \in \mathbb{R}$ | | $\mu$ | $\sigma^2$ | $\exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$ |
| Gamma$(a,\lambda)$ | $\frac{\lambda^a}{\Gamma(a)}x^{a-1}e^{-\lambda x},\ \ x \geq 0$ | | $a/\lambda$ | $a/\lambda^2$ | $\left(\frac{\lambda}{\lambda-t}\right)^a,\ t<\lambda$ |