**Concepts of Linear Algebra-Homework 6**

Question 1

a) Given that matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

The matrix is in the notation $A = UDV^T$

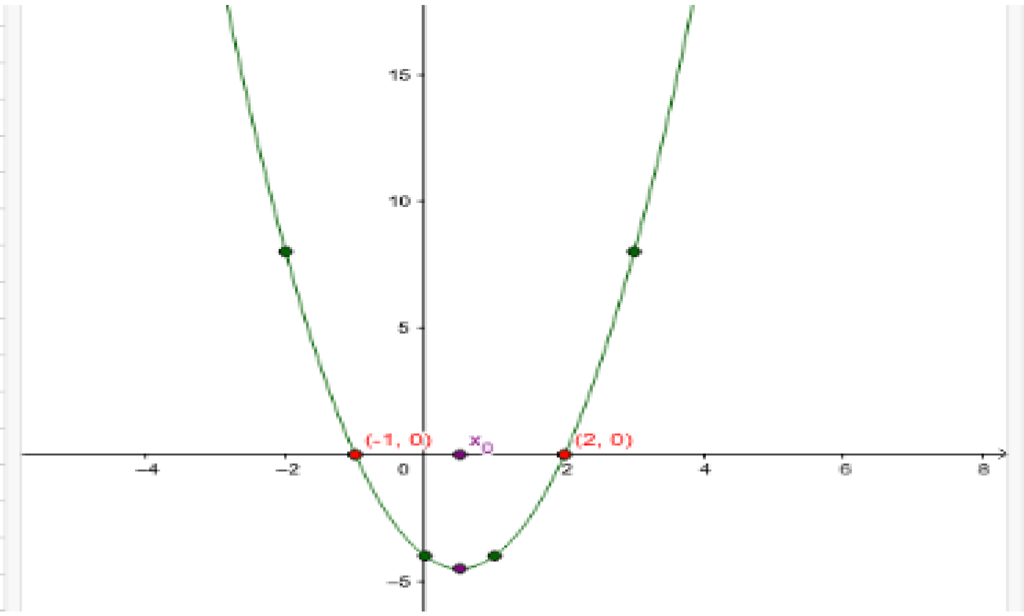
Where U is the left singular vector, D is the singular value and $V^T$ is the right singular vector.

U id defined as by the eigenvectors of $AA^T$, V corresponds to the eigenvector of $A^T A$, and D corresponds to eigenvectors of $A^T A$ and $AA^T$ that are the same.

To get the original matrix A, we first multiply the first two matrices or the last two since multiplication is associative.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 \\ -4 & 2 \end{bmatrix}$$




Question 2

a) Given that $\gamma = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$and $\beta = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$ the matrix $P_{\gamma \to \beta}$ transforms from basis $\gamma$ to basis $\beta$.

Then $P_{\gamma \to \beta} = [[y_1]_b [y_2]_b]$

$$[y_1]_\gamma : \begin{bmatrix} 4 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Forming an augmented matrix to solve the values of $C_1$ and $C_2$.

$$\left\{ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 4 \\ 1 \end{matrix} \right\}$$

This gives the solution as $[y_1]_\gamma = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $[y_2]_b = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Therefore, $P_{\gamma \to \beta} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$

b) The $\beta$ coordinate of the $W_1 = \begin{bmatrix} 10 \\ 9 \end{bmatrix}$

$$W_{1\beta} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 9 \end{bmatrix} = \begin{bmatrix} 31 \\ 37 \end{bmatrix}$$

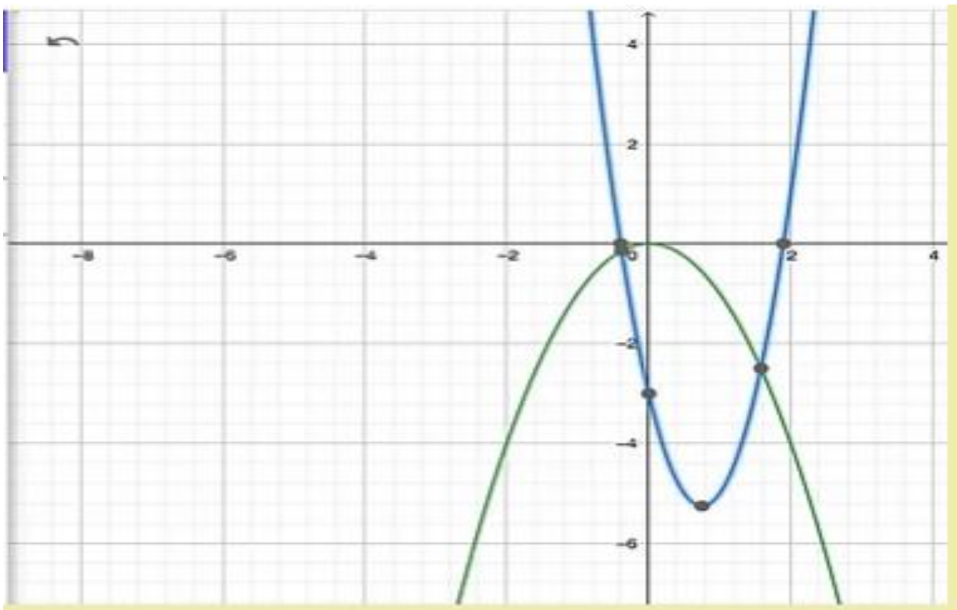
$$W_2 = \begin{bmatrix} -10 \\ 4 \end{bmatrix}$$

$$W_{2\beta} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -10 \\ 4 \end{bmatrix} = \begin{bmatrix} -44 \\ 2 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

$$W_{3\beta} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 11 \end{bmatrix} = \begin{bmatrix} 21 \\ 77 \end{bmatrix}$$

c)



Question 3

Given two basis $\beta_1 = \left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$ and $\beta_2 = \left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}\right\}$

The matrix $P_{1\to 2}$ Sends once from coordinate system one to two. Then $P_{1\to 2} = \{[b_1]_2[b_2]_2\}$

$$[b_1]_2: \begin{bmatrix}2\\1\end{bmatrix} = C_1\begin{bmatrix}3\\1\end{bmatrix} + C_2\begin{bmatrix}1\\2\end{bmatrix}$$

We then form and augmented matrix so as to evaluate the values of $C_1$ and $C_2$.

$$\left\{\begin{matrix}2 & 3\\1 & 1\end{matrix}\middle|\begin{matrix}2\\1\end{matrix}\right\}$$

To make the first entry of the second row to be a zero, the perform following row operation.

$$R_1 - 2R_2 = R_2$$

$$\left\{\begin{matrix}2 & 3\\0 & 1\end{matrix}\middle|\begin{matrix}2\\2\end{matrix}\right\}$$

Using the second row, $C_1 = 2$

Substituting the value of $C_1$ into the first row, we find that

$$2C_1 + 3C_2 = 2$$

$$3C_2 = 2 - 4 = 0$$

$$C_2 = -\frac{2}{3}$$

Therefore, $[b_1]_2 = \begin{bmatrix}2\\-\frac{2}{3}\end{bmatrix}$

$$[b_2]_2: \begin{bmatrix}-2\\1\end{bmatrix} = C_1\begin{bmatrix}3\\1\end{bmatrix} + C_2\begin{bmatrix}1\\2\end{bmatrix}$$

And the augmented matrix is;

$$\left\{\begin{matrix}2 & 3\\1 & 1\end{matrix}\middle|\begin{matrix}-2\\1\end{matrix}\right\}$$

Then $R_1 - 2R_2 = R_2$ gives

$$\left\{\begin{matrix}2 & 3\\0 & 1\end{matrix}\middle|\begin{matrix}-2\\2\end{matrix}\right\}$$

Thus, $C_1 = 2,$ and $2C_1 + 3C_2 = -2$

$$C_2 = \frac{-2-4}{3} = -\frac{6}{3} = 2$$

$$[b_2]_2 = \frac{2}{2}$$

Hence, the matrix that can transform from basis one to two is:

$$P_{1\to 2} = \{[b_1]_2[b_2]_2\} = \begin{bmatrix} 2 & 2 \\ -\frac{2}{3} & 2 \end{bmatrix}$$

Question 4

$A = \begin{bmatrix} -3 & 1 & -2 \\ -12 & 10 & -6 \\ 2 & 4 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} -9 \\ -22 \\ 15 \end{bmatrix}$

a) The augmented matrix is

$$\begin{bmatrix} -3 & 1 & -2 & -9 \\ -12 & 10 & -6 \,| & -22 \\ 2 & 4 & 1 & 15 \end{bmatrix}$$

b) With the augmented matrix

$$\begin{bmatrix} -3 & 1 & -2 & -9 \\ -12 & 10 & -6 \,| & -22 \\ 2 & 4 & 1 & 15 \end{bmatrix}$$

Using the row operation $8R_1 - 2R_2 = R_2$, then we have

$$\begin{bmatrix} -3 & 1 & -2 & -9 \\ 0 & -12 & -4 \,| & -28 \\ 2 & 4 & 1 & 15 \end{bmatrix}$$

Then using $-2R_1 - 3R_3 = R_3$

$$\begin{bmatrix} -3 & 1 & -2 & -9 \\ 0 & -12 & -4 \,| & -28 \\ 0 & -14 & 1 & -27 \end{bmatrix}$$

To make the second entry of the third row zero, we use the following notation;$7R_2 - 6R_3 = R_3$ therefore, the matrix becomes;

$$\begin{bmatrix} -3 & 1 & -2 & -9 \\ 0 & -12 & -4 \,| & -28 \\ 0 & 0 & -34 & -34 \end{bmatrix}$$

Using the third row, the value of z can be determined to be $-34z = -34$

$$z = 1$$

Substituting the value of z in the second row, the value of y can be evaluated as; $-12y - 4(1) = -28$

$$-12y = -28 + 4$$

$$y = -\frac{24}{-12} = 2$$

And substituting the values of y and z in the first row, x can be evaluated as follows $-3x + 1(2) - 2(1) = -9$

$$-3x = -9 - 2 + 2$$

$$x = \frac{9}{3} = 3$$

The solution is (3,2,1)

c) Using the elimination method to find the inverse of the matrix A, we augment with a $3 \times 3$ identity matrix as shown below;

$$\begin{bmatrix} -3 & 1 & -2 & \vdots & 1 & 0 & 0 \\ -12 & 10 & -6 & \vdots & 0 & 1 & 0 \\ 2 & 4 & 1 & & 0 & 0 & 1 \end{bmatrix}$$

Dividing row 1 with -3 gives:

$$\begin{bmatrix} 1 & -0.333 & 0.667 & & -0.333 & 0 & 0 \\ -12 & 10 & -6 & \vdots & 0 & 1 & 0 \\ 2 & 4 & 1 & & 0 & 0 & 1 \end{bmatrix}$$

To have a zero as the first entry in the second roe, the following row operation is performed: $R_2 + 12R_1 = R_2$

$$\begin{bmatrix} 1 & -0.333 & 0.667 & & -0.333 & 0 & 0 \\ 0 & 6 & 2 & \vdots & -4 & 1 & 0 \\ 2 & 4 & 1 & & 0 & 0 & 1 \end{bmatrix}$$

To have a zero as the first entry of the third row, the following row operation is performed; $R_3 - 2R_1 = R_3$

$$\begin{bmatrix} 1 & -0.333 & 0.667 & \vdots & -0.333 & 0 & 0 \\ 0 & 6 & 2 & \vdots & -4 & 1 & 0 \\ 0 & 4.667 & -0.333 & & 0.666 & 0 & 1 \end{bmatrix}$$

We then divide row 2 throughout with 6 to have a one as the second entry; $\frac{R_2}{6}$ and the new matrix is;

$$\begin{bmatrix} 1 & -0.333 & 0.667 & & -0.333 & 0 & 0 \\ 0 & 1 & 0.333 & \vdots & -0.667 & 0.167 & 0 \\ 0 & 4.667 & -0.333 & & 0.666 & 0 & 1 \end{bmatrix}$$

To have a zero as the second entry in the first row, then $R_1 + 0.333R_2 = R_1$

$$\begin{bmatrix} 1 & 0 & 0.778 & & -0.555 & 0.055 & 0 \\ 0 & 1 & 0.333 & \vdots & -0.667 & 0.167 & 0 \\ 0 & 4.667 & -0.333 & & 0.666 & 0 & 1 \end{bmatrix}$$

To have a zero as the second entry of the third row, then $R_3 - 4.667R_2 = R_3$

$$\begin{bmatrix} 1 & 0 & 0.778 & & -0.555 & 0.055 & 0 \\ 0 & 1 & 0.333 & \vdots & -0.667 & 0.167 & 0 \\ 0 & 0 & -1.887 & & 3.779 & -0.779 & 1 \end{bmatrix}$$

The divide row three by -1.887 to get

$$\begin{bmatrix} 1 & 0 & 0.778 & & -0.555 & 0.055 & 0 \\ 0 & 1 & 0.333 & \vdots & -0.667 & 0.167 & 0 \\ 0 & 0 & 1 & & -2 & 0.413 & -0.53 \end{bmatrix}$$

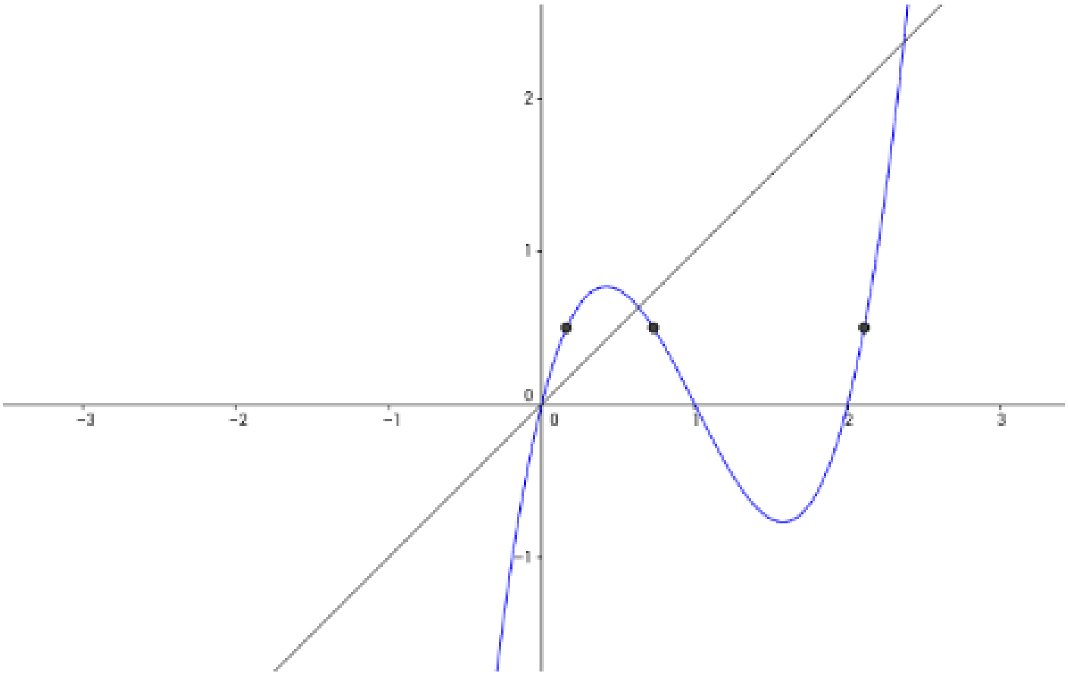
The new row one is obtained by $R_1 - 0.778R_3 = R_1$

$$\begin{bmatrix} 1 & 0 & 0 & & 1 & -0.264 & 0.413 \\ 0 & 1 & 0.333 & \vdots & -0.667 & 0.167 & 0 \\ 0 & 0 & 1 & & -2 & 0.413 & -0.53 \end{bmatrix}$$

And the new row tow is obtained by $R_2 - 0.333R_3 = R_2$

$$\begin{bmatrix} 1 & 0 & 0 & & 1 & -0.264 & 0.413 \\ 0 & 1 & 0 & \vdots & 0 & 0.03 & 0.176 \\ 0 & 0 & 1 & & -2 & 0.413 & -0.53 \end{bmatrix}$$

Therefore, the inverse of the matrix is $A^{-1} = \begin{bmatrix} 1 & -0.264 & 0.413 \\ 0 & 0.03 & 0.176 \\ -2 & 0.413 & -0.53 \end{bmatrix}$




d) Using the inverse, the linear equation can be solved $X = (A^{-1})b$

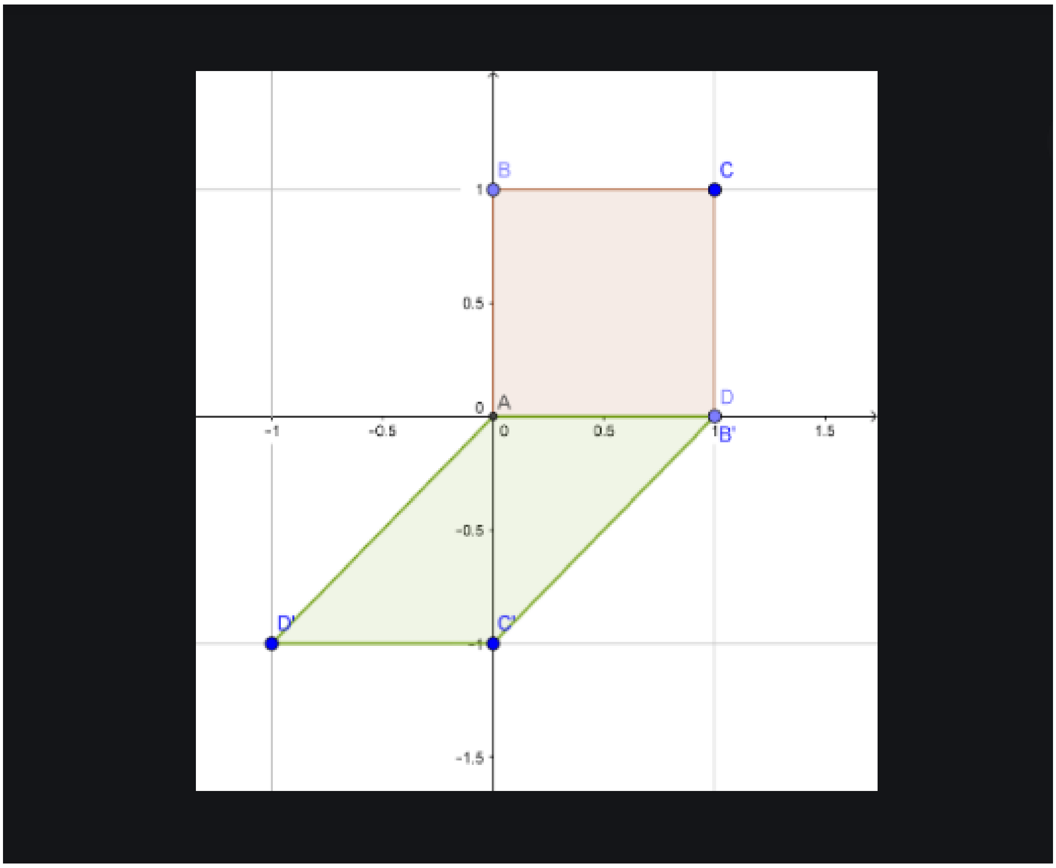
$$X = \begin{bmatrix} 1 & -0.264 & 0.413 \\ 0 & 0.03 & 0.176 \\ -2 & 0.413 & -0.53 \end{bmatrix} \begin{bmatrix} -9 \\ -22 \\ 15 \end{bmatrix}$$

$$X = \begin{bmatrix} 3.003 \\ 1.98 \\ 0.96 \end{bmatrix}$$

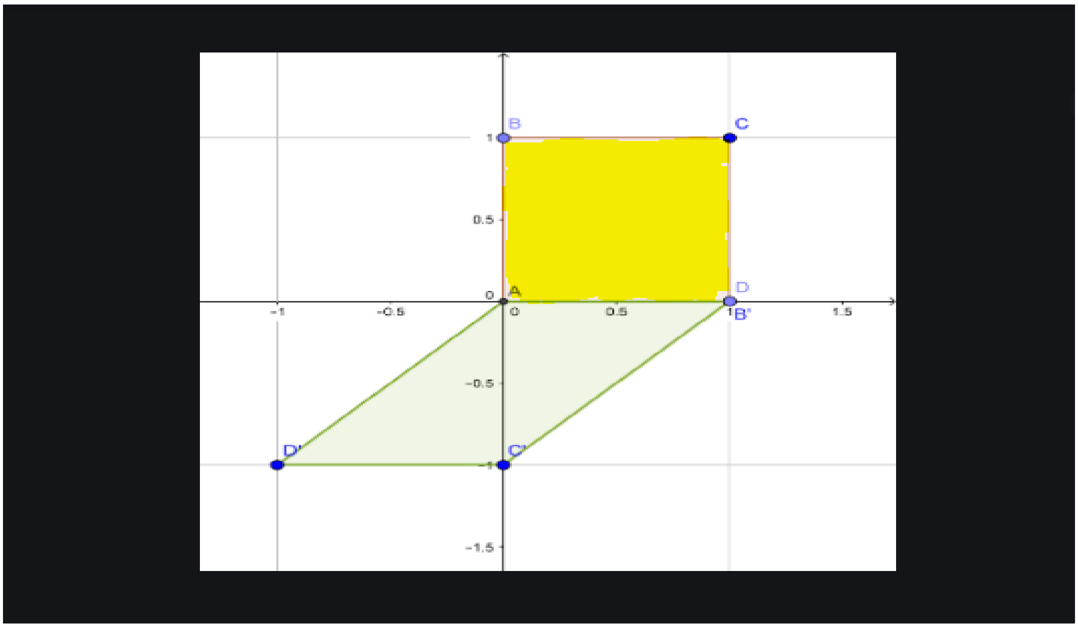
e) The answer found using the second method is not precise. The first method is quite straightforward and easy to calculate unlike the second which is tedious and a slight mistake, which is difficult to track, renders he whole solution wrong. I prefer the first method.

Question 5

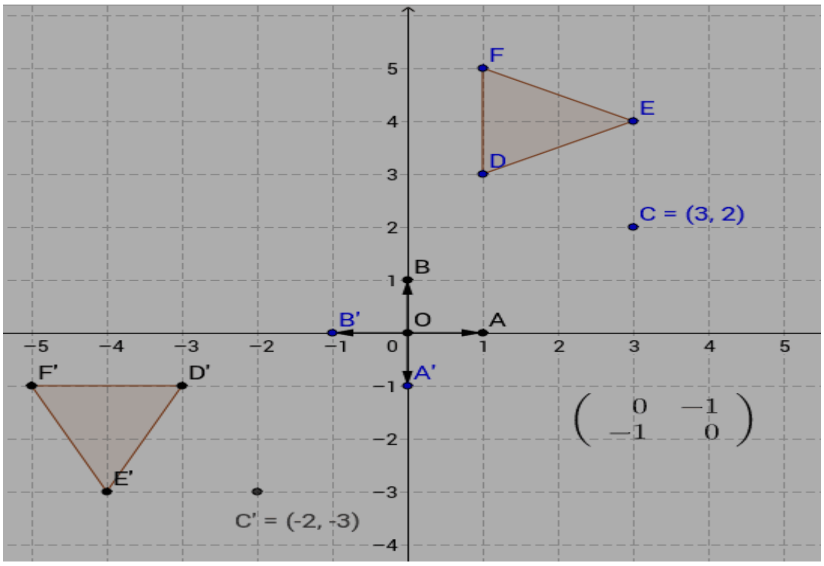
a)




b)



c)




d) If $A = P\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} P^{-1}$

Then A can be evaluated as $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}^{-1}$

But $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$

Therefore, $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$

$$A = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

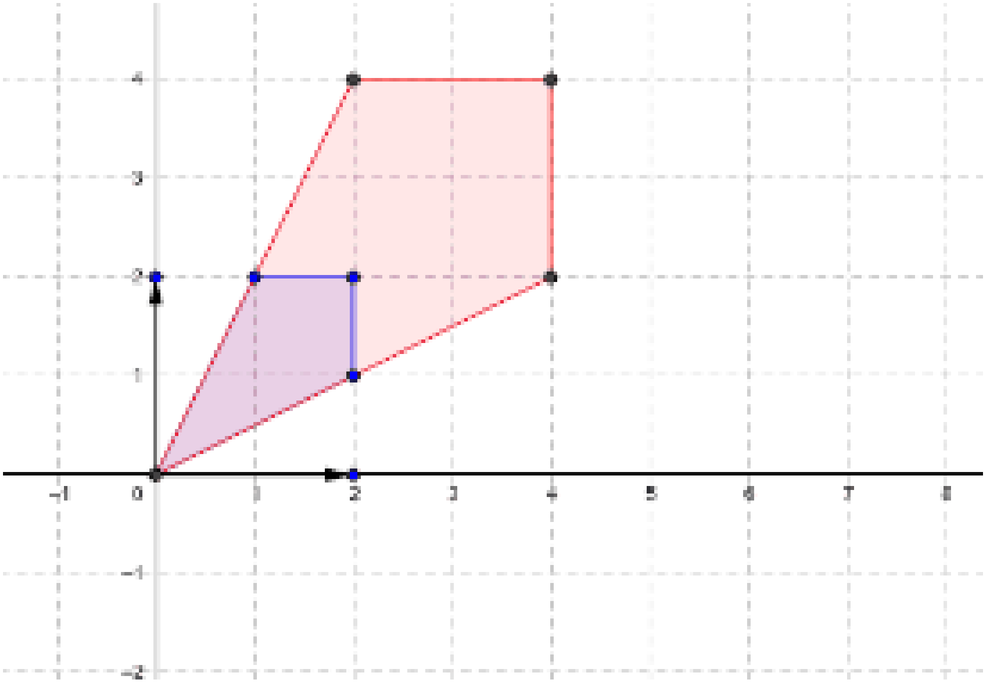
$$A = \begin{bmatrix} -\frac{4}{7} & -\frac{6}{7} \\ \frac{3}{7} & \frac{1}{7} \end{bmatrix}$$

e) $B = P\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} P^{-1}$

Hence $B = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$

$$B = \begin{bmatrix} -2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

$$B = \begin{bmatrix} -1.43 & 0.86 \\ -0.143 & -0.714 \end{bmatrix}$$



Question 6

a) A transforms $P_B$

b) $P_B = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

Question 7

a) **A** shears the direction of vector $\boldsymbol{V_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ one unit in the direction of $\boldsymbol{V_2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

b) $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$