1. Given that $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

a) To set up an augumented matrix whose solution gives A^{-1}

The matrix is ivertible since its a 2x2 matrix therefore if X is the augumented matrix then

$$AX = XA = I$$

and $X = A^{-1}$

therefore

$$[A \mid I] = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

$$[I \mid A^{-1}] = \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

b) $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ its inverse.

$\det A = (2\times 2) - (3\times 1) = 1$

$= 1 \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

c) AA^{-1}

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} (2\times 2)+(3\times -1) & (2\times -3)+(3\times 2) \\ (2\times 1)+(2\times -1) & (1\times -3)+2\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$AA^{-1} = I$

2. Given $A = \begin{bmatrix} 9 & 20 \\ 4 & 9 \end{bmatrix}$

LU decomposition of A

$A \begin{bmatrix} 9 & 20 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = LU$

Multiplying LU

$L_{11} u_{11} = 9$

$L_{11} u_{12} = 20$

$L_{21} u_{11} = 4$

$L_{21} u_{12} + L_{22} u_{22} = 9$

Restricting the diagonals of L to be 1

then $\underline{L_{11} = 1}$

$\underline{L_{22} = 1}$

then $\underline{u_{11} = 9}$

$1 \times u_{12} = 20$

$u_{12} = \frac{20}{1} = 20$

$L_{21} u_{11} = 4$

$\frac{9}{9} L_{21} = \frac{4}{9}$

$L_{21} = \frac{4}{9}$

$\frac{4}{9}(u_{12}) + 1(u_{22}) = 9$

$\frac{4}{9}\left(\frac{20}{1}\right) + u_{22} = 9$

$u_{22} = \frac{9}{1} - \frac{80}{9} = \frac{1}{9}$

Therefore

$L = \begin{bmatrix} 1 & 0 \\ \frac{4}{9} & 1 \end{bmatrix}$

$U = \begin{bmatrix} 9 & 20 \\ 0 & \frac{1}{9} \end{bmatrix}$

$= LU = \begin{bmatrix} 1 & 0 \\ \frac{4}{9} & 1 \end{bmatrix} \begin{bmatrix} 9 & 20 \\ 0 & \frac{1}{9} \end{bmatrix}$

$= \begin{bmatrix} 9 & 20 \\ 4 & 9 \end{bmatrix}$

3. Given $$A = \begin{bmatrix} 4 & 4 & 5 \\ 16 & 17 & 23 \\ 20 & 26 & 45 \end{bmatrix}$$

LU Decomposition

$$A = \begin{bmatrix} 4 & 4 & 5 \\ 16 & 17 & 23 \\ 20 & 26 & 45 \end{bmatrix}$$ To obtain U, $-4R_1 + R_2$.

$$= \begin{bmatrix} 4 & 4 & 5 \\ 0 & 1 & 3 \\ 20 & 26 & 45 \end{bmatrix} \xrightarrow{-5R_1 + R_3} \begin{bmatrix} 4 & 4 & 5 \\ 0 & 1 & 3 \\ 0 & 6 & 20 \end{bmatrix} \xrightarrow{-6R_2 + R_3}$$

$$U = \begin{bmatrix} 4 & 4 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \qquad U = \begin{bmatrix} 4 & 4 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & 4 & 5 \\ 16 & 17 & 23 \\ 20 & 26 & 45 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

4. a) Determinant of matrices:

$$A = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The product of the determinants of a matrix factorization is equal to the determinant of the original matrix.

therefore $\det \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} = (1 \times 1 - 0) = 1$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}\begin{bmatrix} 6 & 1 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

$(1 \times 1 \times 1) + 6 \times -1 \times 5 = -30 + 1$
$= \underline{-29}$

$$C = \begin{bmatrix} 5 & 1 & 3 \\ 9 & -1 & 7 \\ 3 & -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 5 & 1 & 3 \\ 9 & -1 & 7 \\ 3 & -5 & 2 \end{bmatrix}$$

$CUC^{-1} = UI = U$

$C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & -3 \end{bmatrix}$ $\det = 2 \times 5 \times -3 = \underline{-30}$

4 b) To reduce computations, look at the LU-Decompositio
of a matrix where
the products of the diagonals = determinant
of the matrix

c) Matrix decomposition gives the elements
of the diagonals which are equivalent
to the product of each diagonal therefore
computing determinant.

d) The determinant of a matrix = to the
Area scale factor hence in 2D.

e) Each matrix, A - Reverses orientation.
B - Preserves orientation
C - Preserves orientation

5 Consider Matrices

$$A = \begin{bmatrix} 4 & -1 \\ 1 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ 6 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 10 & -3 \\ 15 & -2 \end{bmatrix}$$

a) disc (M) - Number of M Invariant lines In the Matrices M, i.e A, B and C are given by

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

The Matrices M does not have vertical invariant lines.

then for $A = \begin{bmatrix} 4 & -1 \\ 1 & 6 \end{bmatrix}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\begin{bmatrix} 4 & -1 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

$4x - y = X$

$x + 6y = Y$

we have

$y = mx + c$ and $Y = mX + c$.

Substituting for X and Y

$x + 6y = m(4x - y) + c$

Substituting for y,

$x + 6(mx + c) = m(4x - y) + c$.

rearranging,

$x + 6mx + 6c - [4mx - m(mx + c) + c$

$x + 6mx + 6c - 4mx + m^2x + mc + c$.

$x(m^2 + 2m + 1) + c(m + 7) = 0$

for all values of x

$m^2 + 2m + 1 = 0$ also $c(m + 7) = 0$

$(m+1)(m+1) = 0$

$m_1 = -1 \quad m_1 = -1 \qquad c = 0$

There are one invariant line $y = x$.

5) a) $B = \begin{bmatrix} 2 & 3 \\ 6 & -1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

$2x + 3y = X$

$6x - y = Y$

We have $y = mx + c$

$Y = mX + c$.

Substituting Y in X.

$6x - y = m(2x + 3y) + c$

$6x - (mx + c) = m(2x + 3(mx + c)) + c$

$6x - mx - c = 2mx + 3m^2x + 3mc + c$

$x(3m^2 + 3m - 6) + c(3m + 2) = 0$

$3m^2 + 3m - 6 = 0$ $\quad c = 0$

$(m - 1)(3m + 6) = 0$

$m_1 = 1 \quad m_2 = -3$

There are two invariant lines $\underline{y = x}$ and $\underline{y = -3x}$

$C = \begin{bmatrix} 10 & -3 \\ 15 & -2 \end{bmatrix}$

$$\begin{bmatrix} 10 & -3 \\ 15 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

$10x - 3y = X$

$15x - 2y = Y$

we have $y = mx + c$ and

$Y = mX + c$.

Substituting Y and X.

$15x - 2y = m(10x - 3y) + c$

Substituting y

$15x - 2(mx + c) = m(10x - 3(mx + c)) + c$

$= 15x - 2mx + 2c =$

$m(10x - 3mx - 3c) + c$

$= 15x - 2mx - 2c =$

$10mx + 3m^2x - 3mc$

$= x(3m^2 - 12m + 15) + c(3m - 3)$

$3m^2 - 12m + 15 = 0$

$(m - 1)(3m - 15) = 0$.

$M_1 = 1 \quad M_2 = 5 \quad c =$

It has two Invariant Lines

$\underline{y = -x}$ and $\underline{y = -5x}$

5 b) $p\,m(\lambda)$

$p\,m(\lambda) = \lambda^2 + a\lambda' + a_0$

$\det(\lambda I - m) = p\lambda$ therefore the λ represents the eigen values of the Matrix.

c) Eigen values of M

$A = \begin{bmatrix} 4 & -1 \\ 1 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 \\ 6 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 10 & -2 \\ 15 & -2 \end{bmatrix}$

$Av = \lambda v$

$A = \begin{pmatrix} 4 & -1 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$

$4x - y = \lambda x$

$x + 6y = \lambda y$

for non zero solution

$(4-\lambda)x - y = 0$

$x + (6-\lambda)y = 0$

$\mathrm{Det}(A) = 0$

$\det\begin{pmatrix} 4-\lambda & -1 \\ 1 & 6-\lambda \end{pmatrix} = 0$

$= \lambda^2 - 10\lambda + 25 = 0$

$(\lambda - 5)(\lambda - 5) = 0$

$\lambda_1 = 5 \quad \lambda_2 = 5$

The matrix has 1 repeated real eigen values

$B = \begin{pmatrix} 2 & 3 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$

$\det\begin{pmatrix} 2-\lambda & 3 \\ 6 & -1-\lambda \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix} =$

$\mathrm{Det}\,A = 0$

$\lambda^2 - 1 - 20 = 0$

$(\lambda + 4)(\lambda - 5) = 0$

$\lambda_1 = 4 \quad \lambda_2 = -5$

It has 2 distinct real roots

5 c)

$$C = \begin{bmatrix} 10 & -3 \\ 15 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -3 \\ 15 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det \begin{bmatrix} 10-\lambda & -3 \\ 15 & -2-\lambda \end{bmatrix} = 0$$

$\lambda^2 + 20 + 25 = 0$

It has two complex conjugate roots

5 d) M consists of

$A = \begin{bmatrix} 4 & -1 \\ 1 & 6 \end{bmatrix}$ $B \begin{bmatrix} 0 & 3 \\ 6 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 10 & -3 \\ 15 & -2 \end{bmatrix}$

$\det A = 24 - 1$ $\det B = -2 - 18$ $\det C = -20 + 45$

$= 25$ $= -20$ $= 25$

$\det A = C$

The SMD of M is real since it has non zero matrix

5 e) M transforms places, where for

A, has an invariant line $y = x$

B, has two invariant lines, $y = x$ and $y = -3x$

C has two Invariant lines $y = -x$ and $y = -5x$

for A the plane stretches on One line side

for B and C, the place streches diagonally

5 f) The B-coordinate grid, B unit square SP

M transforms the unit square in a stretch diagonall along the line $y = x$, $y = -x$, $y = -5x$ and $y = -3x$

5 g) The SMD for M

$$M = \begin{bmatrix} 4 & -1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 10 & -3 \\ 15 & -2 \end{bmatrix}$$

The SMD $= 25 + (-20) + 25$
$= 30$

a) The most Interesting thing Learned is finding the LU decomposition of 2×2 matrices and 3×3 matrices.
This is because its interesting, and on how to work out, jags ones mind.

b) The most important thing Learned :-
Is that the Linear algebra and mathematics as a whole is a continous learning process because every other unit in Linear algebra is a continuation of previous learned concepts.

I Joseph Hyatt certify that I understand the rules of this exam and have completed this exam without the use of any prohibited resources as outlined in the rules above.