Item 1

a. Given $\begin{bmatrix} 8 & 6 \\ -6 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ -34 \end{bmatrix}$, the solution is calculated as follows:

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 8 & 6 \\ -6 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ -34 \end{bmatrix} \\
&= \frac{1}{8(8)-(-6)(6)}\begin{bmatrix} 8 & -6 \\ 6 & 8 \end{bmatrix}\begin{bmatrix} 12 \\ -34 \end{bmatrix} \\
&= \frac{1}{100}\begin{bmatrix} 8(12)-6(-34) \\ 6(12)+8(-34) \end{bmatrix} \\
&= \frac{1}{100}\begin{bmatrix} 300 \\ -200 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -2 \end{bmatrix}
\end{aligned}$$

The solution set is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

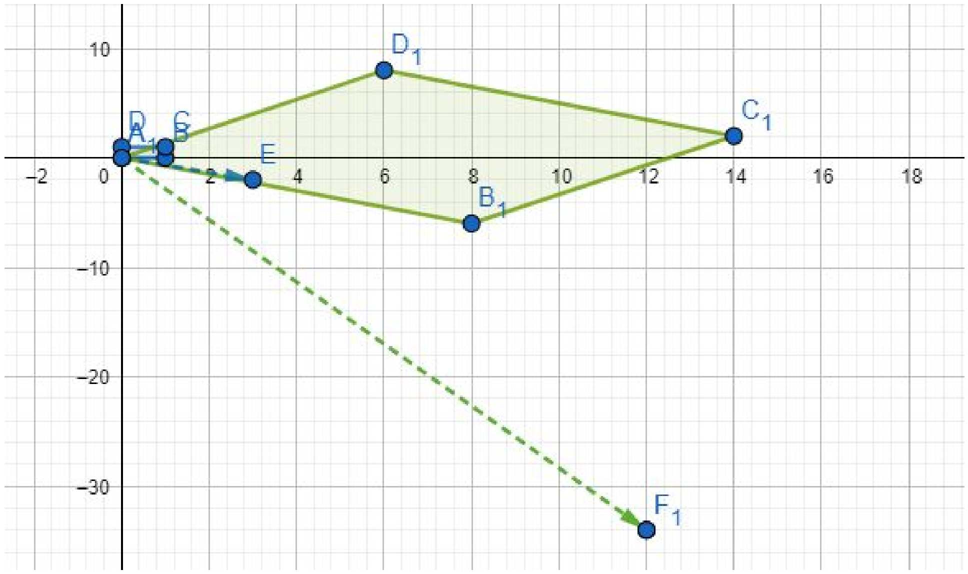
b.

i. The coefficients necessary to write $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ as a linear combination of vectors $\begin{bmatrix} 8 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$ are 3 and -2, respectively.

ii. The transformation of the plane $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 8 & 6 \\ -6 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ is a rotation of the plane by a clockwise angle of $\theta = \tan^{-1}\frac{6}{8} = 36.87°$ and a stretch factor of $\sqrt{6^2+8^2} = 10$ in all directions. The point that gets sent to (12, -34) under the transformation is the point (3, -2).

iii. The lines $8x+6y=12$ and $-6x+8y=-34$ intersect at the point (3, -2).

Visualization of geometric transformation via GeoGebra:



Item 2

a.

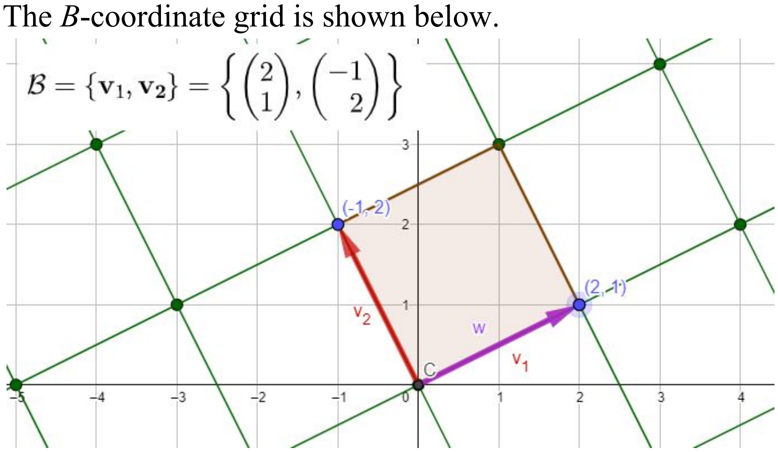
$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} R_1/2 \to R_1 = \begin{bmatrix} 1 & -0.5 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.5 \\ 1 & 2 \end{bmatrix} R_2 - R_1 \to R_2 = \begin{bmatrix} 1 & -0.5 \\ 0 & 2.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.5 \\ 0 & 2.5 \end{bmatrix} 2R_2/5 \to R_2 = \begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix} R_1 + R_2/2 \to R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

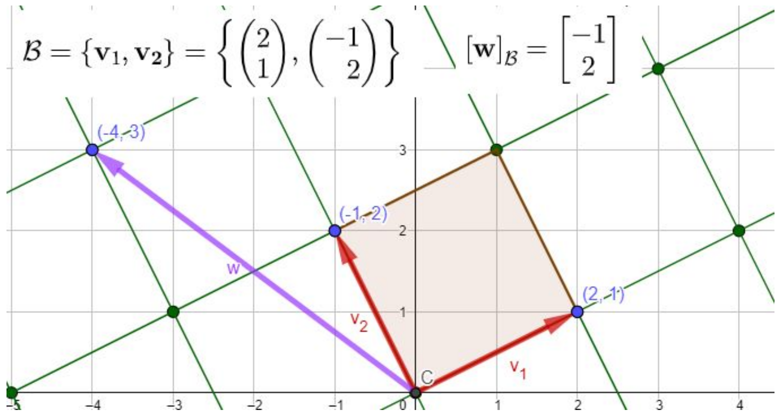
b. The *B*-coordinate grid is shown below.



Since each column contains a leading entry and $\text{rref}(A) = I$, then the vector set forms a basis.

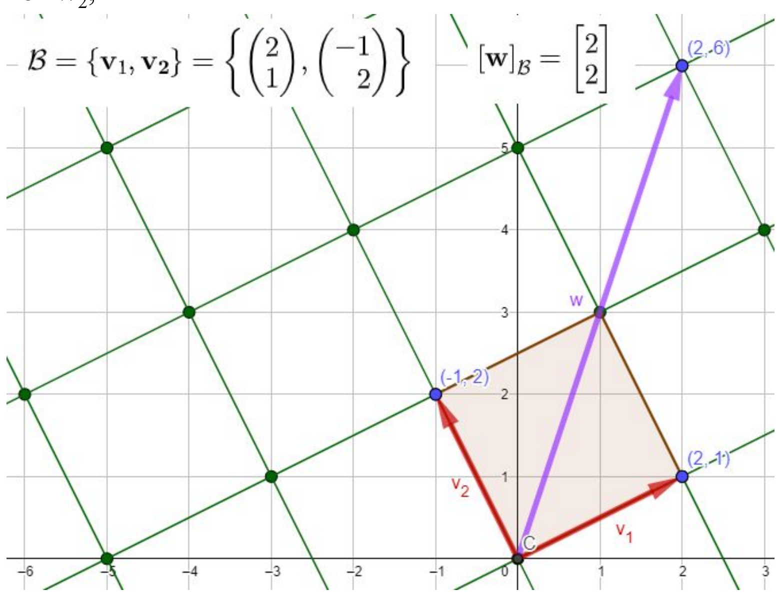
c.

i. For $\mathbf{w}_1$,



(See purple vector)

ii. For $\mathbf{w}_2$,



d.

i. The standard coordinates of $\mathbf{w}_1$ are (-4,3).

ii. The B-coordinates of $\mathbf{w}_2$ is (2, 2).

e. The matrix expression that represents $[\mathrm{w}_1]_B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is

$$\left[\begin{array}{cc|c} 2 & -1 & x \\ 1 & 2 & y \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

f. Solving for the standard coordinates,

$$\left[\begin{array}{cc|c} 2 & -1 & x \\ 1 & 2 & y \end{array}\right] \begin{array}{l} 0.5r_1 \to r_1 \\ \end{array} \sim \left[\begin{array}{cc|c} 1 & -0.5 & 0.5x \\ 1 & 2 & y \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -0.5 & 0.5x \\ 1 & 2 & y \end{array}\right] \begin{array}{l} \\ r_2 - r_1 \to r_2 \end{array} \sim \left[\begin{array}{cc|c} 1 & -0.5 & 0.5x \\ 0 & 2.5 & y - 0.5x \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -0.5 & 0.5x \\ 0 & 2.5 & y - 0.5x \end{array}\right] \begin{array}{l} \\ 0.4r_2 \to r_2 \end{array} \sim \left[\begin{array}{cc|c} 1 & -0.5 & 0.5x \\ 0 & 1 & 0.4y - 0.2x \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -0.5 & 0.5x \\ 0 & 1 & 0.4y - 0.2x \end{array}\right] \begin{array}{l} r_1 + 0.5r_2 \to r_1 \\ \end{array} \sim \left[\begin{array}{cc|c} 1 & 0 & 0.2y + 0.4x \\ 0 & 1 & 0.4y - 0.2x \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0.2y + 0.4x \\ 0 & 1 & 0.4y - 0.2x \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

$$\left.\begin{array}{r} 0.4x + 0.2y = -1 \\ -0.2x + 0.4y = 2 \end{array}\right\} \to \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

g. The augmented matrix whose solution represents the B-coordinates of $\mathbf{w}_2$ is:

$$\left[\begin{array}{cc|c} 2 & -1 & 2 \\ 1 & 2 & 6 \end{array}\right]$$

h. Using rref,

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 6 \end{bmatrix} \begin{array}{l} 0.5r_1 \to r_1 \\ \end{array} \sim \begin{bmatrix} 1 & -0.5 & 1 \\ 1 & 2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.5 & 1 \\ 1 & 2 & 6 \end{bmatrix} \begin{array}{l} \\ r_2 - r_1 \to r_2 \end{array} \sim \begin{bmatrix} 1 & -0.5 & 1 \\ 0 & 2.5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.5 & 1 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{array}{l} \\ 0.4r_2 \to r_2 \end{array} \sim \begin{bmatrix} 1 & -0.5 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

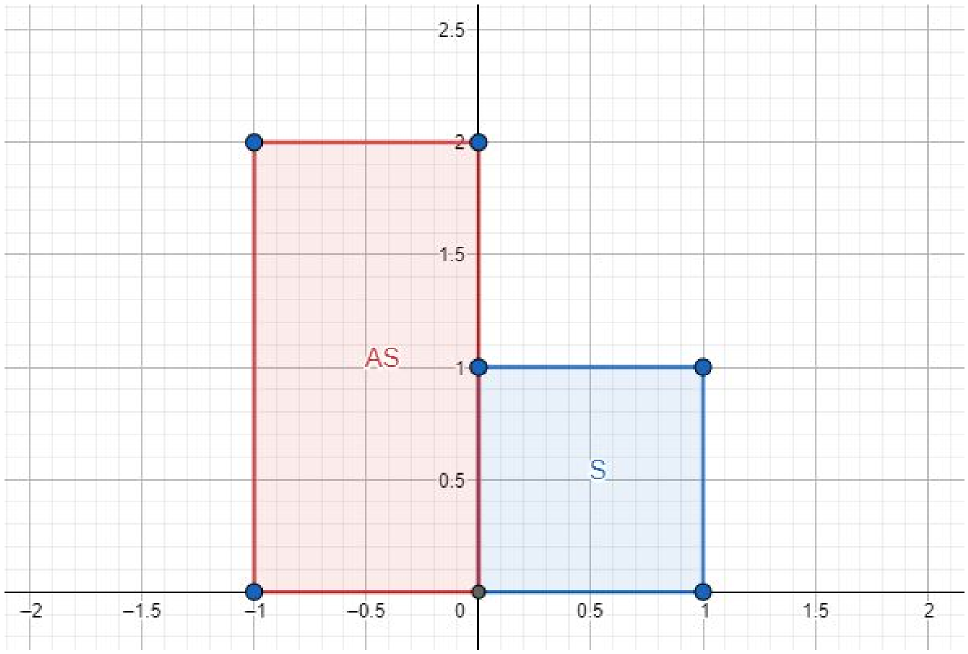
$$\begin{bmatrix} 1 & -0.5 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{array}{l} r_1 + 0.5r_2 \to r_1 \\ \end{array} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

As we can see $w_2 \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ is equivalent to $[w_2]_B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. At the standard Cartesian plane, the vector $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ is equivalent to the vector $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ in the *B*-coordinate system that are based on vectors $\{v_1, v_2\}$.

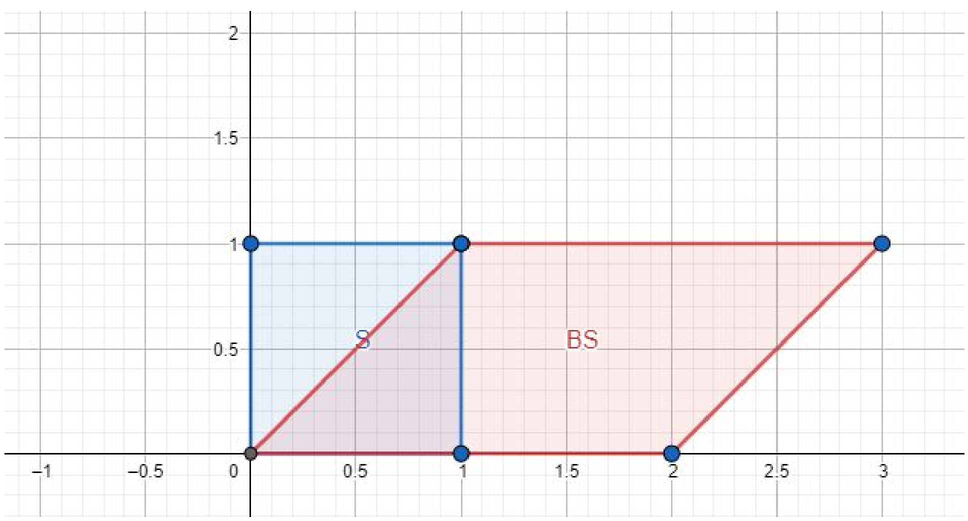
Item 3

a. The visual sketches S under different transformation matrices A to D are shown below.

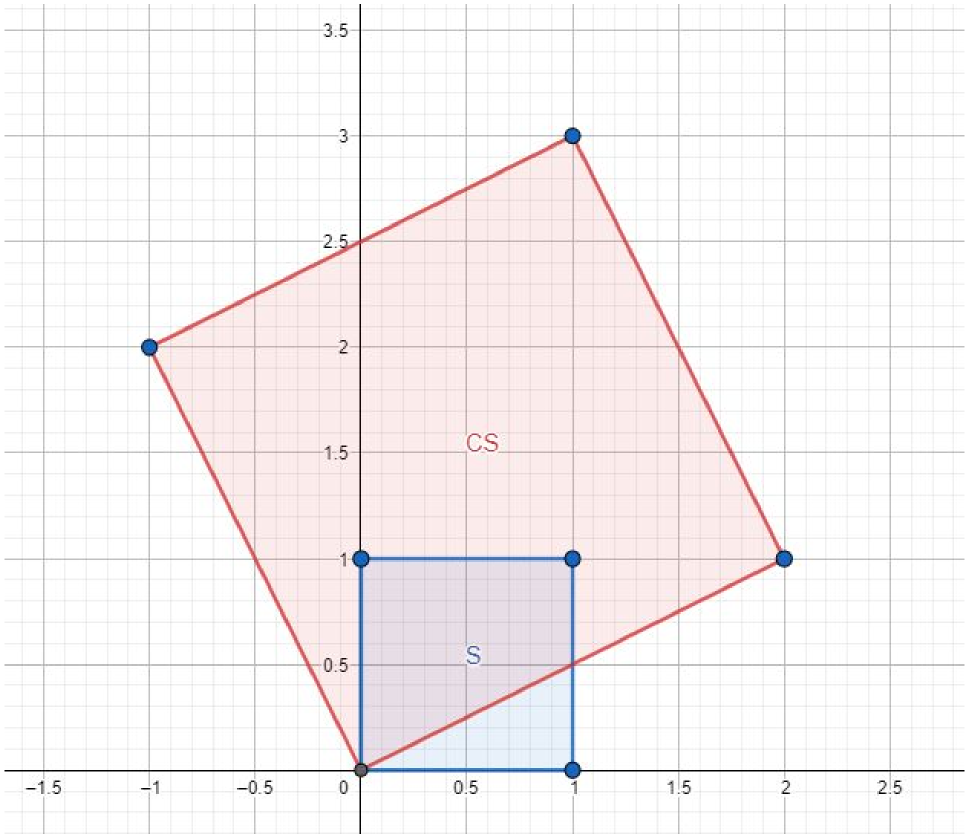
Matrix A:



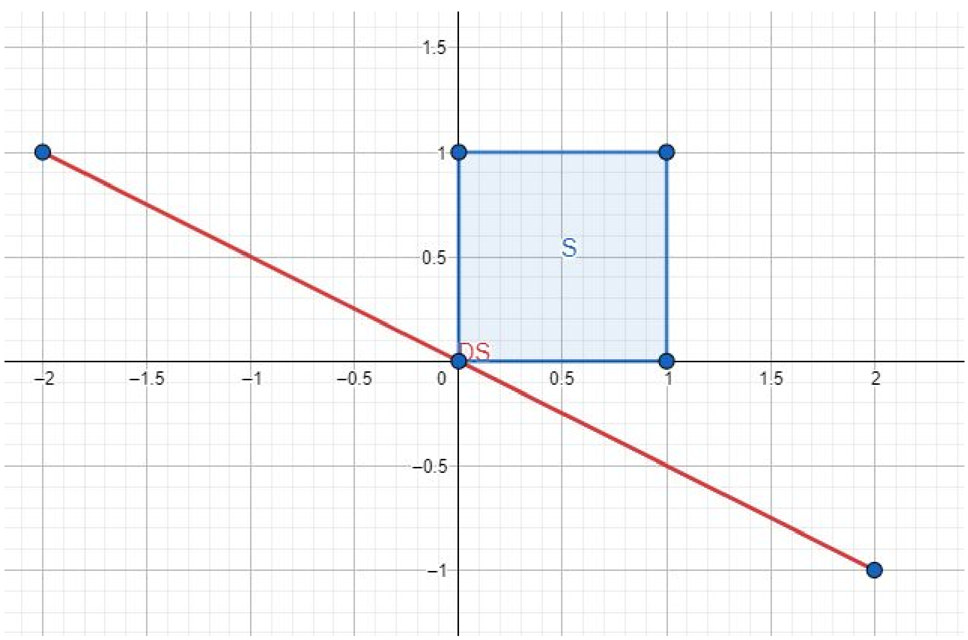
Matrix B:



Matrix *C*:



Matrix *D*:



b. Matrix *A* transforms *S* through (1) reflection at the *y*-axis and a (2) vertical stretch at a factor of 2.

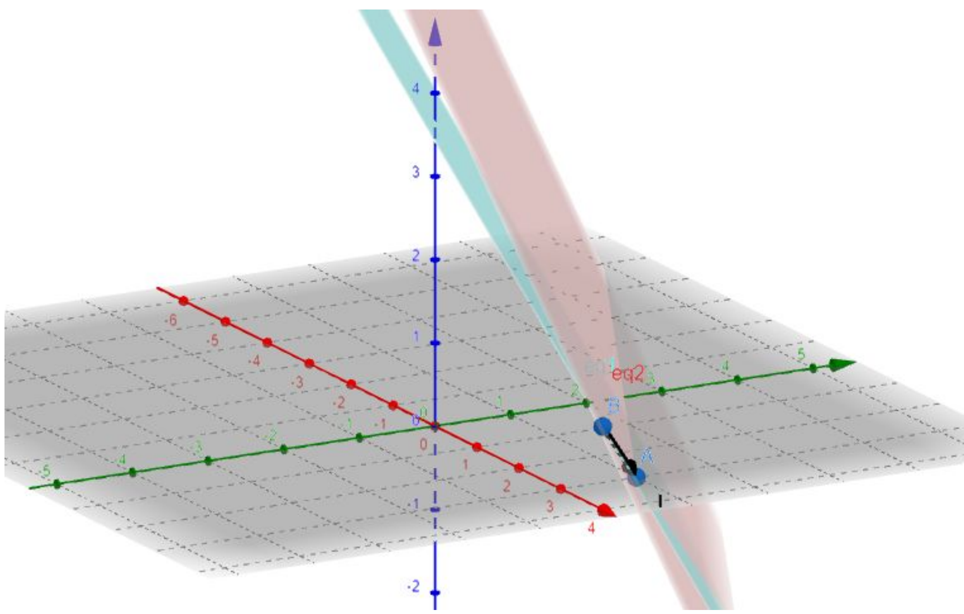
Matrix *B* transforms *S* through (1) horizontal stretch at a factor of 2 and (2) shearing of the top edge by 1 unit to the right.

Matrix *C* transforms *S* through (1) a stretch at all edges by a factor of $\sqrt{1^2+2^2}=\sqrt{5}$ and a counterclockwise rotation with the origin corner as the pivot at an angle of $\theta=\tan^{-1}\frac{1}{2}$ or approximately 27 degrees.

Finally, Matrix *D* transforms *S* through a horizontal stretch at a factor of (2), shearing off the rightmost edge 1 unit downward, and another (3) shearing of 2 units to the left.

Item 4

a. The system of equation pertains to the solution set described by line of intersection between the planes $2x+5y+3z=11$ and $x+2y+z=5$ at $\mathbb{R}^3$. The visualization made using GeoGebra is shown below.



Based on the graph above, I expect an infinite number of solutions (line) for the system.

b. The augmented matrix form of the system $\begin{bmatrix} 2 & 5 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$ is:

$$\begin{bmatrix} 2 & 5 & 3 & 11 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

c. Finding the rref for the system.

$$\begin{bmatrix} 2 & 5 & 3 & 11 \\ 1 & 2 & 1 & 5 \end{bmatrix} R_1/2 \to R_1 = \begin{bmatrix} 1 & 2.5 & 1.5 & 5.5 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2.5 & 1.5 & 5.5 \\ 1 & 2 & 1 & 5 \end{bmatrix} R_2 - R_1 \to R_2 = \begin{bmatrix} 1 & 2.5 & 1.5 & 5.5 \\ 0 & 0.5 & 0.5 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2.5 & 1.5 & 5.5 \\ 0 & 0.5 & 0.5 & 0.5 \end{bmatrix} 2R_2 \to R_2 = \begin{bmatrix} 1 & 2.5 & 1.5 & 5.5 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2.5 & 1.5 & 5.5 \\ 0 & 1 & 1 & 1 \end{bmatrix} R_1 - 2.5R_2 \to R_1 = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

d. For system A, the rref $\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ is read as: $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

The solution set is the linear combination of the vectors $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, with 1 and z as factors, respectively.

e. Yes, my expectations matched the outcome. The system has pivoted and 1 degree of freedom, indicating that one dimension of the solution vector is a free variable. Hence, the solution set is graphically a line $\vec{u}$ (see parametric form above).

Item 5

a. The system of equation can be written as follows:

$$\begin{aligned} 3x_1 + 3x_2 + 5x_3 &= c_1 \\ 7x_1 + 4x_2 + 3x_3 &= c_2 \\ 8x_1 + 8x_2 + 9x_3 &= c_3 \end{aligned}$$

where c_1 is the total number of three-bedroom units, c_2 is the total number of two-bedroom units, and c_1 is the total number of one-bedroom units.

As an augmented matrix, the expression at the left side is equivalent to

$$\begin{bmatrix} 3 & 3 & 5 \\ 7 & 4 & 3 \\ 8 & 8 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 3 & 5 \\ 7 & 4 & 3 \\ 8 & 8 & 9 \end{bmatrix} x$$

b. With $c = \begin{bmatrix} 61 \\ 105 \\ 144 \end{bmatrix}$, $\begin{bmatrix} 3 & 3 & 5 \\ 7 & 4 & 3 \\ 8 & 8 & 9 \end{bmatrix} x = \begin{bmatrix} 61 \\ 105 \\ 144 \end{bmatrix}$.

Checking if $S = \left\{ \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 9 \end{bmatrix} \right\}$ forms the basis of $\mathbb{R}^3$,

$$\begin{bmatrix} 3 & 3 & 5 & 61 \\ 7 & 4 & 3 & 105 \\ 8 & 8 & 9 & 144 \end{bmatrix} \begin{matrix} R_1/3 \to R_1 \\ \\ \\ \end{matrix} = \begin{bmatrix} 1 & 1 & 5/3 & 61/3 \\ 7 & 4 & 3 & 105 \\ 8 & 8 & 9 & 144 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 5/3 & 61/3 \\ 7 & 4 & 3 & 105 \\ 8 & 8 & 9 & 144 \end{bmatrix} \begin{matrix} \\ R_2 - 7R_1 \to R_2 \\ R_3 - 8R_1 \to R_3 \end{matrix} = \begin{bmatrix} 1 & 1 & 5/3 & 61/3 \\ 0 & -3 & -26/3 & -112/3 \\ 0 & 0 & -13/3 & -56/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 5/3 & 61/3 \\ 0 & -3 & -26/3 & -112/3 \\ 0 & 0 & -13/3 & -56/3 \end{bmatrix} \begin{matrix} \\ -R_2/3 \to R_2 \\ \\ \end{matrix} = \begin{bmatrix} 1 & 1 & 5/3 & 61/3 \\ 0 & 1 & 26/9 & 112/9 \\ 0 & 0 & -13/3 & -56/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 5/3 & 61/3 \\ 0 & 1 & 26/9 & 112/9 \\ 0 & 0 & -13/3 & -56/3 \end{bmatrix} \begin{matrix} \\ \\ -3R_3/13 \to R_3 \end{matrix} = \begin{bmatrix} 1 & 1 & 5/3 & 61/3 \\ 0 & 1 & 26/9 & 112/9 \\ 0 & 0 & 1 & 56/13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 5/3 & 61/3 \\ 0 & 1 & 26/9 & 112/9 \\ 0 & 0 & 1 & 56/13 \end{bmatrix} \begin{matrix} \\ R_2 - 26R_3/9 \to R_2 \\ \\ \end{matrix} = \begin{bmatrix} 1 & 1 & 5/3 & 61/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 56/13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 5/3 & 61/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 56/13 \end{bmatrix} \begin{matrix} R_1 - R_2 \to R_1 \\ \\ \\ \end{matrix} = \begin{bmatrix} 1 & 0 & 5/3 & 61/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 56/13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5/3 & 61/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 56/13 \end{bmatrix} \begin{matrix} R_1 - 5R_3/3 \to R_1 \\ \\ \\ \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & 171/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 56/13 \end{bmatrix}$$

With each column containing a leading entry and $\text{rref}(A) = I$, then S is a basis. In addition, this automatically suggests that the basis is also linearly independent and spans $\mathbb{R}^3$.

The result suggests that it is theoretically possible to have any number of combinations of Plans A, B, and C to produce an exact number of three-bedroom, two-bedroom, and one-bedroom units. However, since we're talking about real-world scenarios, the solution set is limited only to counting numbers (no fractions, decimals, or negative values allowed).

Interpretation:

That being said, since the obtained solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 171/13 \\ 0 \\ 56/13 \end{bmatrix}$, then it is not possible to have exactly 61 three-bedroom, 105 two-bedroom, and 144 one-bedroom units. However, countless combinations may be possible as long as they are counting numbers.

Item 6

a. The most interesting thing that I have learned so far is the graphical visualization of matrix multiplication. I have encountered topics on matrices before, but seeing how each cell within the matrix can have a two- or three-dimensional equivalence such as stretching (expansion, compression), shearing, rotation, and reflection finally made me realize how industries like game development require an excellent background in linear algebra.

b. The most important thing that I have learned so far is the process of reducing a system of equations into an augmented matrix, and eventually into reduced row-echelon form. In algebra, methods like substitution and elimination are limited to two-variable and two-equation systems. For systems that include more than 2 variables and more than equations, the simplification process becomes more and more complicated and longer to perform. But with *rref*, I can learn more about the characteristics of a system of equation once its augmented form is properly reduced.

Item 7

I, Joseph Hyatt, certify that I understand the rules of this exam and have completed this exam without the use of any prohibited resources as outlined in the rules above.

Signature