Linear Algebra- Homework 9

1. Let $\beta = \{v_1,...,v_n\}$ be a basis for R^n, $P = [v_1|\cdots|v_n]$, D be a diagonal matrix with diagonal entries $\lambda_1,...,\lambda_n$, and let $A = PDP^{-1}$.

For each v_i, the line $Li = \{tv_i \mid t \in R\}$ is an A-invariant line with scaling factor λ_i.

Solution

The matrix A of the operator L relative to the basis β is a block matrix of the form

$A= \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$

Where 0 is the (n-m) X m zero matrix and B is the matrix of restriction L/w relative to the basis β

Also using the notation

$A = PDP^{-1}$.

$\det A = \det (P) \det (D) \det (P^{-1})$

more over $\det A = \det D$

$\det (A- \lambda I_n) = \det (A. \lambda I_n)$ for any scalar since λ n-dimensional vector space v and w is an m-dimensional subspace of v that is invariant under L

Then for each vector space v_i the line $Li = \{tv_i \mid t \in R=0\}$

Hence the scaling factor is λ_i

2. Suppose that $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}^{-1}$

(a) Using the correct notation for the parametric equations for lines, find all A-invariant lines.

Solution

If $A = BPB^{-1}$ then $A=P$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Using

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ we have}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

3x=x   x=0   y=0   z=0

x + y + z = 0

there is an invariant point with equation y=-x-z

(b) For each A-invariant line, determine the scaling factor.

Solution

For the invariant line y = -x -z if z = 0

Then y = -x

Therefore, the scaling factor is 4 i.e. 3 + 2 + (-1) = 4

3. Suppose that A is a 2×2 matrix which transforms the plane according to the following picture.

(a) Using the correct notation for the parametric equations for lines, find all A-invariant lines.

In this case, the lines passing through the origin are invariant lines

Solution

i) (0, 0) (3, 1)

$$\frac{\Delta y}{\Delta x} = \frac{1-0}{3-0} = \frac{1}{3}$$

$$y = \frac{1}{3}x + c \quad (3, 1)$$

$$y = 1 + c$$

C= 0 y=3x is the invariant line

ii) (0, 0) (-3, 3)

$$\frac{\Delta y}{\Delta x} = \frac{3-0}{-3-0} = \frac{3}{-3} = -1$$

$$y = -x + c \quad (-3, 3)$$

$3 = 3 + c$ c=0

y=-x is the invariant line

(b) For each A-invariant line, determine the scaling factor.

i) invariant line, y=3x has a scaling factor 3

ii) invariant line, y=-x has a scaling factor 4

4. Suppose that A is a 2×2 matrix with invariant lines$L_1 = \left\{\begin{bmatrix}2\\1\end{bmatrix} t | \text{t} \in \text{R}\right\}$ and $L_2 = \left\{\begin{bmatrix}-3\\1\end{bmatrix} t | \text{t} \in \text{R}\right\}$ and associated scaling factors λ_1=5 and λ_2=2. Determine the invariant lines and associated scaling factors for the matrix A^3. Be sure to justify why the lines you identify are invariant.

Solution

Invariant line L_1 (2, 1) must pass through the origin (0, 0)

The equation of a straight line y = mx + c

m is given by $\frac{\Delta y}{\Delta x} = \frac{1-0}{2-0} = \frac{1}{2}$

$$y = \frac{1}{2}x + c\ (2, 1)$$

$$1 = \frac{1}{2}(2) + c$$

Hence c = 0

$y = \frac{1}{2}x$ invariant line, scaling factor 2

Invariant line passing through (0, 0) (-3, 1)

$$\frac{\Delta y}{\Delta x} = \frac{1-0}{-3-0} = \frac{1}{-3}$$

$y = \frac{1}{-3}x$ scaling factor 2

The lines are invariant because the scaling factor are same and pass through the origin

5. Consider the matrix A= $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

(a) Use your Invariant Line GeoGebra Tool to explore the find reasonable guesses for the invariant lines L_1 and L_2 and their associated scaling factors. Include GeoGebra images for both lines.

Solution

Let A= $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $P = \begin{bmatrix} u \\ v \end{bmatrix}$

If P is the invariant point with respect to A, then AP = P

Its equivalent to:

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

3u + v = u

u + 3v = v   and

3u – u + v = 0

u + 3v – v = 0

2u + v = 0

(u + 2v = 0) x 2

Solving for u and v

$$\begin{array}{c} 2u + v = 0 \\ \underline{2u + 4v = 0} \\ -3v = 0 \end{array}$$

v = 0

2u + 0 = 0

u = 0

Therefore, u – v = 0, hence the invariant point of the matrix is at the origin

(0, 0) and the matrix has an invariant line u = v.

(b) Use this to write down a similar matrix decomposition for A.

Solution

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Det (A) = 3 X 3 – 1

det = 8

(c) Multiply the SMD out to verify that you do in fact recover A and the lines and scaling factors you found were correct.

Solution

BP = P

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \frac{1}{8} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

6. Show that if A is a 2×2 matrix with eigenvalues λ_1 and λ_2, then $\lambda_1 + \lambda_2 = tr(A)$ and $\lambda_1\lambda_2 = det(A)$.

Solution

By definition,

The characteristic polynomial of a 2X2 matrix is given by $p(t) = (A - tI) = (-1)^2 (t^2 – (tr A) t^{2-1}) + (-1)^2 \det A$

$= t^2 – (tr A) t + \det A$

On the other hand

$p(t) = (-1)^2\,(t - \lambda_1)\,(t - \lambda_2)$ where λ_1 and λ_2 are the eigen values of A

Therefore

$$P(t) = (t - \lambda_1)\,(t - \lambda_2)$$

So, $= t^2 - t\,(\lambda_1 + \lambda_2) + \lambda_1\lambda_2$

Comparing the coefficients, we have;

$$t^2 - (\mathrm{tr}\,A)\,t + \det A = t^2 - t\,(\lambda_1 + \lambda_2) + \lambda_1\lambda_2$$

$$(\mathrm{tr}\,A)\,t = t\,(\lambda_1 + \lambda_2)$$

$$\mathrm{tr}\,A = \lambda_1 + \lambda_2$$

$$\det A = \lambda_1\lambda_2$$

7. Using the previous problem, find a 2×2 matrix A with nonzero integer entries, with the specified eigen-values.

(a) $\lambda_1 = 2$ and $\lambda_2 = -2$

Solution

Then

$$(\lambda - 2)\,(\lambda + 2) = 0$$

$$\lambda^2 + 2\lambda - 2\lambda - 4 = 0$$

from problem 6

$$\lambda^2 + 2\lambda - 2\lambda - 4 = t^2 - t\,(\lambda_1 + \lambda_2) + \lambda_1\lambda_2$$

$\mathrm{tr}\,A = \lambda_1 + \lambda_2 = 0 \quad a + d = 0$

det A = $\lambda_1\lambda_2$ = -4   a x d = -4 where a = 2 and b = -2

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & b \\ c & -2 \end{bmatrix}$

ad – dc = -4

2d + 2c = -4

2c = -4 -2d

$$c = \frac{-2(2-d)}{-2} = 2 - d$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2a – 2b = 0

2c – 2d = 0

4 -2b = 0

4 = 2b

b = 2

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$

(b) λ_1 = 2 and $\lambda 2$ = 2

Solution

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow$ $(\lambda -2)(\lambda -2) = 0$

$\lambda^2 - 2\lambda - 2\lambda + 4$

$$\lambda_1 + \lambda_2 = -4$$

$$\lambda_1\lambda_2 = 4$$

$$4 - cd = 4$$

$$\begin{bmatrix} 2 & b \\ c & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

(c) $\lambda_1 = 3$ and $\lambda_2 = 4$

Solution

Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \quad (\lambda_1 - 3)(\lambda_2 - 4) = 0$$

$$\lambda^2 - 3\lambda - 4\lambda + 12 = 0$$

$$\lambda^2 - 3\lambda - 4\lambda + 12 = 0 = t^2 - t(\lambda_1 + \lambda_2) + \lambda_1\lambda_2$$

$\lambda = t$ and $\lambda_1 = a$

$\lambda_1 = 3$ $\lambda_2 = d$

$\lambda_2 = 4$

$$\lambda_1\lambda_2 = ad - cb$$

Therefore, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$

(d) $\lambda_1 = 3+4i$ and $\lambda_2 = 3-4i$

Solution

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \quad (\lambda_1 +3+4i)\ (\lambda_2 -3+4i) = 0$$

$$\lambda^2 - 3\lambda + 4i\lambda + - 3\lambda + 9\text{-}12i + 4i\lambda – 12i + 16$$

$$= \lambda^2 – 6\lambda + 8\ \lambda\ i) + 25$$

$$0 = \lambda^2 – \lambda\ (6 + 8i) + 25$$

$$a + d = 6 + 8i$$

$$\lambda_1\lambda_2 = ad – bc = \sqrt{25} = 5$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3+4i & 2+5i \\ 2-5i & 3-4i \end{bmatrix}$$

8. If A is an n×n matrix, a generalization of an A-invariant line in an A-invariant plane, i.e. a 2-dimensional plane W such that A(W) ⊂ W. (When this matrix is clear from context, we will just say an invariant plane). An A-invariant plane which is not built out of two A-invariant lines is irreducible.

Suppose $A = \begin{bmatrix} 5 & 3 & 0 \\ 0 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & 3 & 0 \\ 0 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}^{-1}$

(a) Using the correct notation for the parametric equations for planes, find all A-invariant planes.

Solution

$A = BPB^{-1}$

By multiplication property

$BB^{-1}=1$

Then, A=P

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ solving factors are 3, 2 and -1

Eigen vectors are $\lambda_1 = 3$

$\lambda_2 = 2$

$\lambda_3 = -1$

Eigen values associated with 2 turns out to be the line 3y = 2x in the xy plane i.e. z = 0 $y = \frac{2}{3}x$

(b) Which of these planes are irreducible? Justify.

Solution

$y = \frac{2}{3}x$ is not irreducible since for some eigen value, the algebraic multiplicity is not greater than the geometric multiplicity that is one for an invariant line. 2 for an invariant line is reducible since the sum of the dimension of its eigen spaces equals 3.