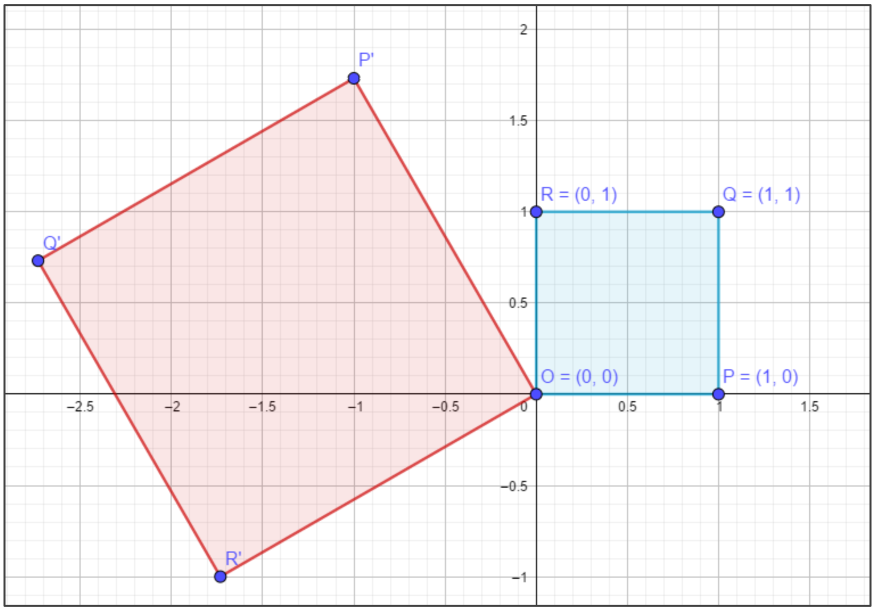
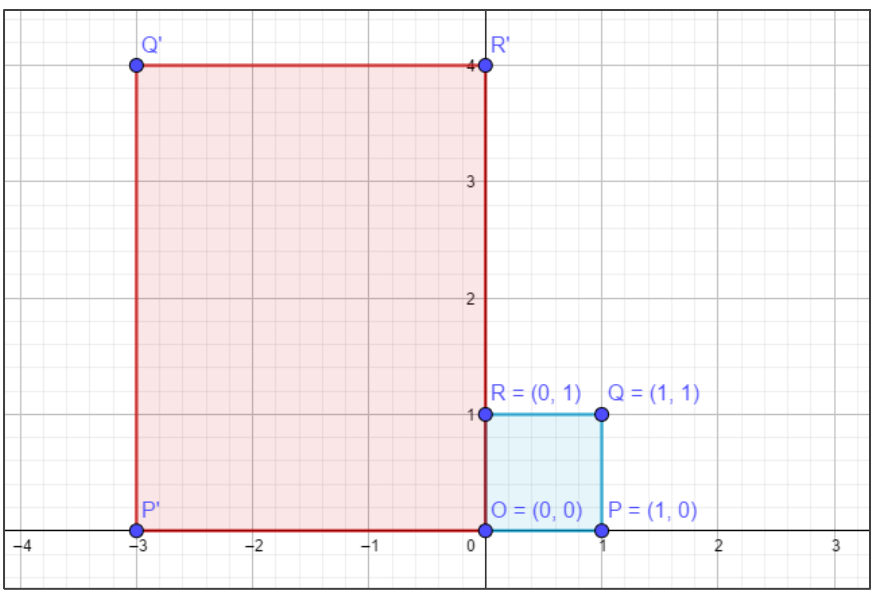
<u>Item 1</u>

Matrix *A* transforming *S*:



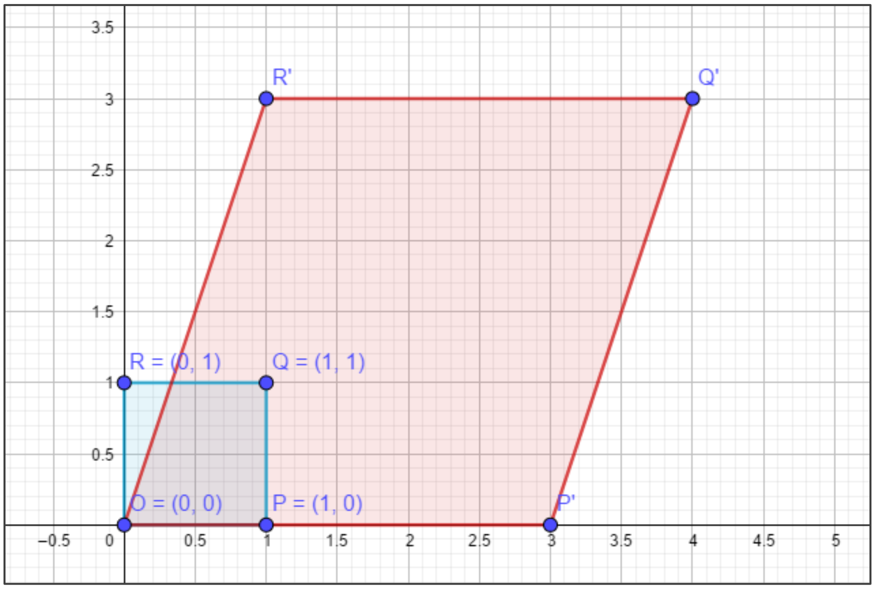
Transformation: The plane *S* rotated by about 120 degrees counterclockwise with point O as pivot. In addition, the radius increased by about twofold.

Matrix *B* transforming *S*:



Transformation: The plane *S* reflected horizontally with edge OR as the mirror. In addition, the horizontal edges stretched threefold while the vertical edges stretched fourfold.

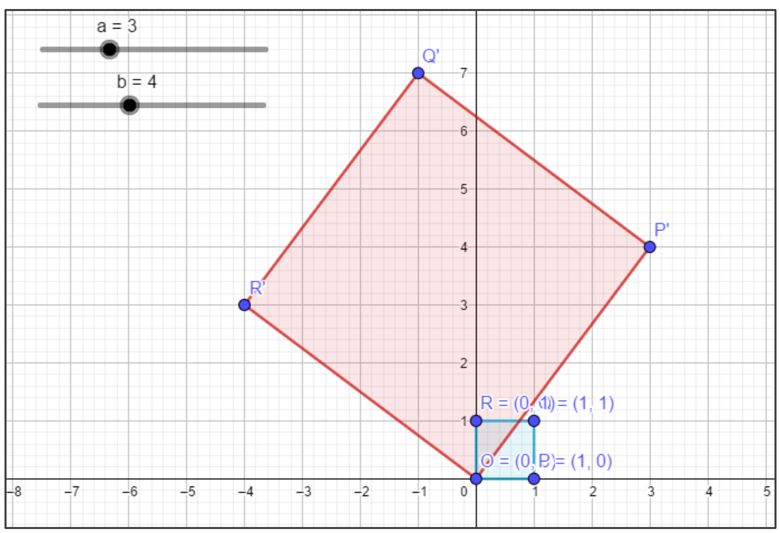
Matrix C transforming S:



Transformation: First, all edges stretched threefold. Then, shearing occurred to the right wherein the base edge OP remained in its place but the top edge QR displaced by about one unit to the right.

<u>Item 2</u>

a. The effect of matrices A and B is shown below.



b. Rotation and stretching geometric transformations occur to S due to Matrix A.

c. As I vary sliders a and b, two transformations via Matrix A happen. The first transformation stretches the edge of the square plane S by a factor of $\sqrt{a^2 + b^2}$. The second transformation rotates S at an angle of $\tan^{-1}\dfrac{b}{a}$ counterclockwise with point O as the pivot.

Derivation of factors:

The transformed plane S' in matrix form has the following new coordinates:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax-by \\ bx+ay \end{bmatrix}$$

For all four corners, the new coordinates are:

$$O=(0,0) \qquad P'=(a,b)$$
$$Q'=(a-b,a+b) \quad R'=(-b,a)$$

These are the same corners of the larger square shown in the figure above once we substitute a = 3 and b = 4.

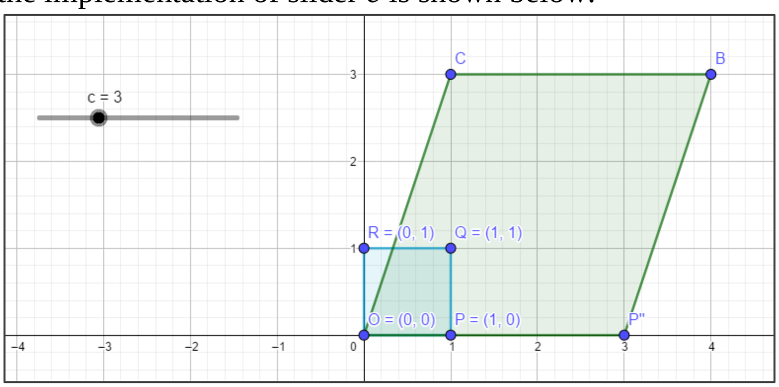
Solving for the edge,

$$s=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$s_{\overline{OP'}}=\sqrt{(a-0)^2+(b-0)^2}$$
$$=\sqrt{a^2+b^2} \rightarrow \text{Equation 1}$$

Solving for the angle of rotation,

$$\tan\theta=\frac{b}{a}$$
$$\theta=\tan^{-1}\frac{b}{a} \rightarrow \text{Equation 2}$$

d. For Matrix B, the implementation of slider c is shown below.



There are two transformations on S using Matrix B. The first transformation stretches the edges of S by a factor of c. The second transformation shears the upper base of S by one unit to the right. By playing with the slider, the matrix B has the following features:

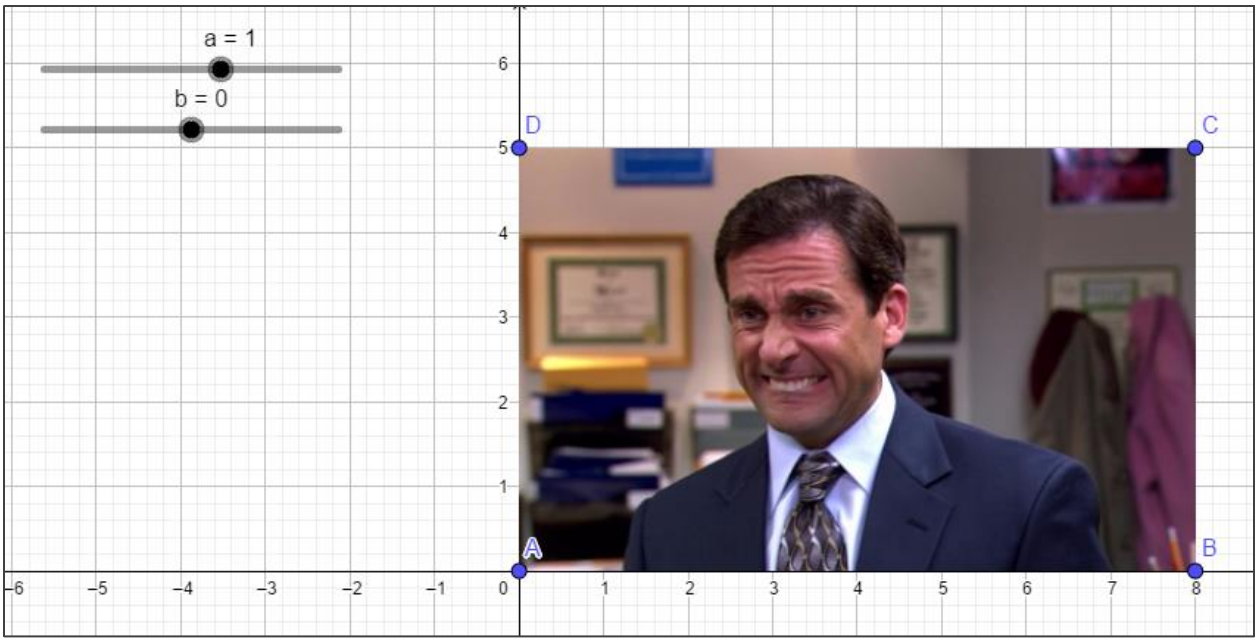
Shear factor

$$\begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx + y \\ cy \end{bmatrix}$$

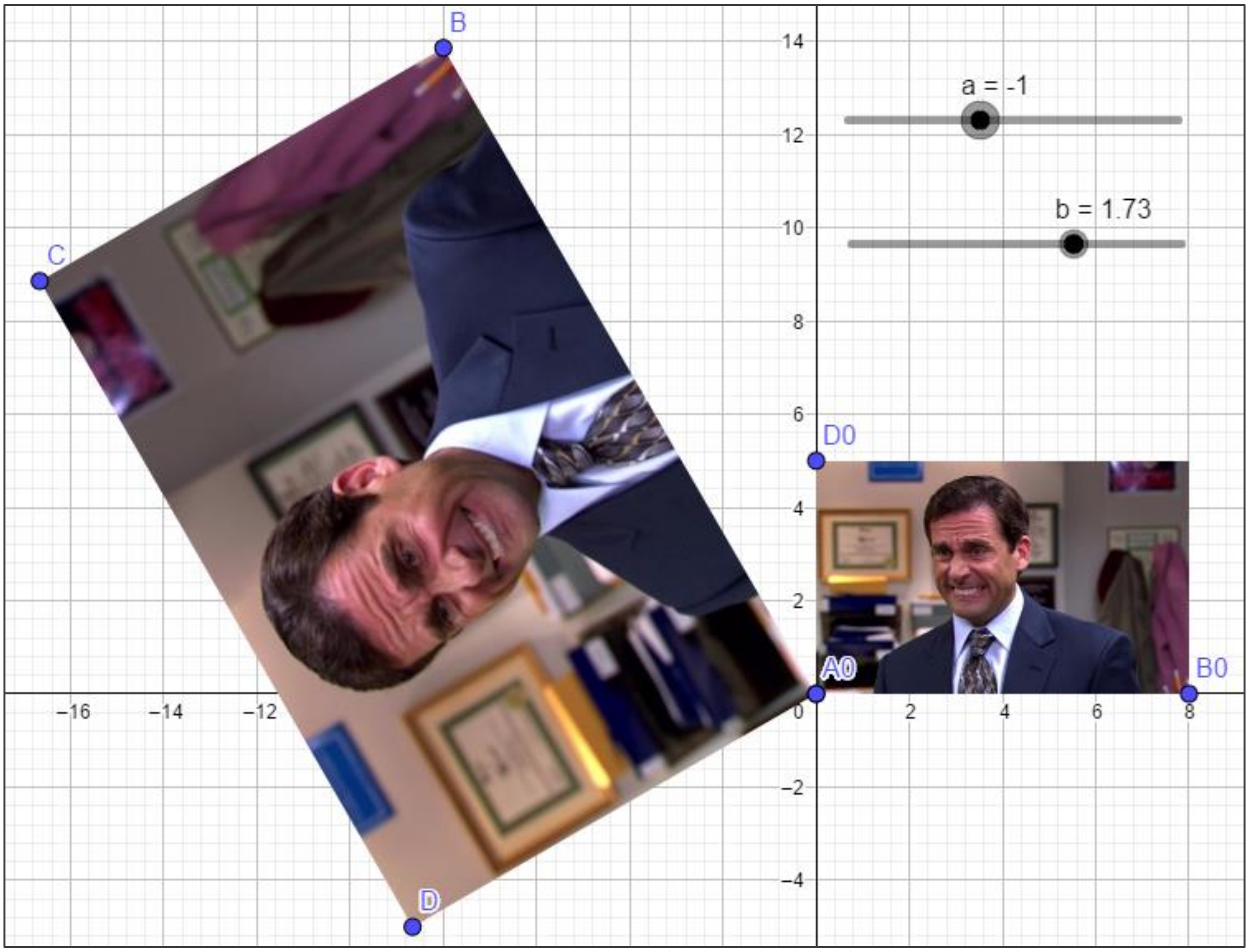
Stretch factor

<u>Item 3</u>

The image that I've chosen is:



The transformation matrix $M = \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ will take effect when the sliders are set at $a = -1$ and $b = \sqrt{3}$. Once these changes are made, the following transformation takes place:



Using the matrix $\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$, the original image "pic1" was underwent edge stretching at a stretching factor of 2 and counterclockwise rotation with point A as pivot at an angle of $120°$.

Using Equations 1 and 2 to confirm these values,

$$\begin{aligned} s_{\overline{CD}} &= \sqrt{a^2 + b^2} \\ &= \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1} \frac{b}{a} \\ &= \tan^{-1} \frac{\sqrt{3}}{-1} \\ &= \frac{\pi}{3} \text{ or } 120° \end{aligned}$$

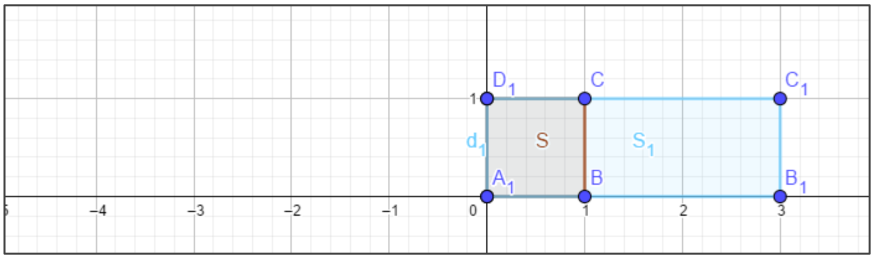
Item 4

I will try to use a unit square plane S just like in previous examples.

Transformation 1: Stretch in the x-direction by a factor of 3. This is equivalent to the matrix:

$$M_1 = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \text{ but } a = 3$$

$$M_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$



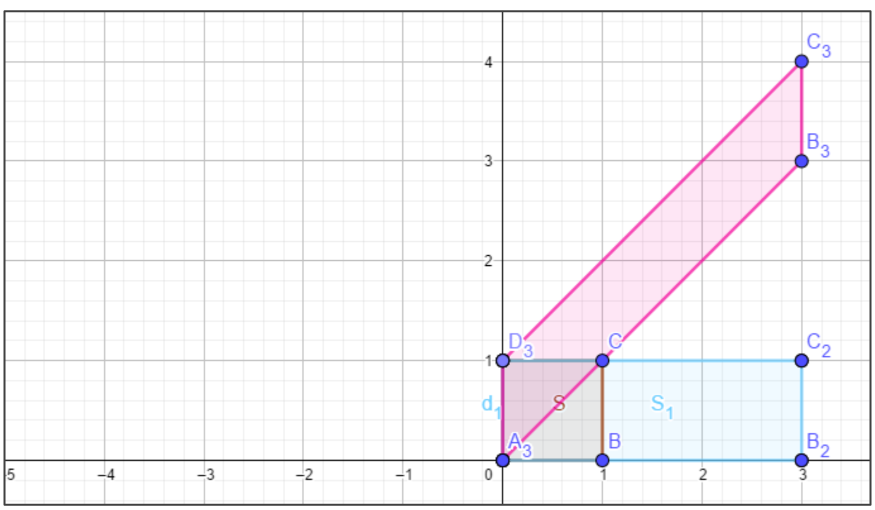
Transformation 2: Shear the x-axis in the y-direction to slope 1. This is equivalent to the matrix:

$$M_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

When combined with the effect of M_1,

$$M_2 M_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}$$



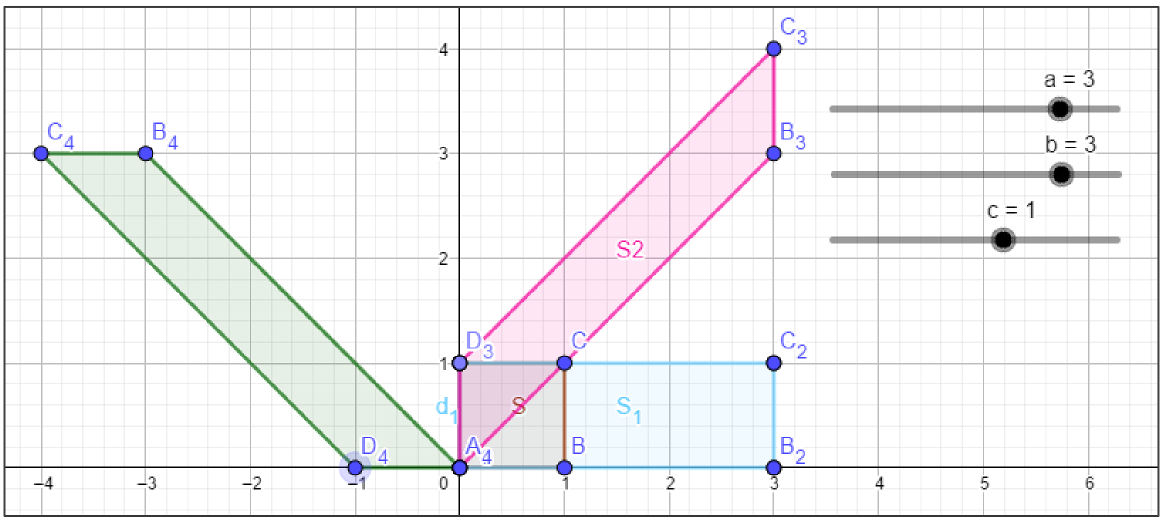
Transformation 3: Rotate the figure by 90 degrees counterclockwise. This is equivalent to the matrix:

$$M_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

When combined with the effect of $M_2 M_1$,

$$M_3(M_2 M_1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}$$

$$M_3 M_2 M_1 = \begin{bmatrix} -3 & -1 \\ 3 & 0 \end{bmatrix}$$



Hence, the transform matrix is $\begin{bmatrix} -3 & -1 \\ 3 & 0 \end{bmatrix}$.

<u>Item 5</u>

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{aligned} A^2 &= AA \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} \\ &= \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix} \end{aligned}$$

But $A^2 \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

Regardless,

$$c(a+d) = b(a+d) = 0 \rightarrow b = c = 0$$

Simplifying the expression for A^2,

$$\begin{aligned} A^2 &= \begin{bmatrix} a^2 + \cancel{bc} & \cancel{b}(a+d) \\ \cancel{c}(a+d) & \cancel{bc} + d^2 \end{bmatrix} \\ &= \begin{bmatrix} a^2 & 0 \\ 0 & d^2 \end{bmatrix} \end{aligned}$$

Solving for A^4,

$$\begin{aligned} A^4 &= A^2 A^2 \\ &= \begin{bmatrix} a^2 & 0 \\ 0 & d^2 \end{bmatrix} \begin{bmatrix} a^2 & 0 \\ 0 & d^2 \end{bmatrix} \\ &= \begin{bmatrix} a^4 & 0 \\ 0 & d^4 \end{bmatrix} \end{aligned}$$

Since $A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $a^4 = d^4 = 1$.

Listing down the possible combinations of *a* and *d* such that $A^2 \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

First set: $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ when $A = \begin{bmatrix} \pm i & 0 \\ 0 & \pm i \end{bmatrix}$

Second set: $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ when $A = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm i \end{bmatrix}$

Third set: $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ when $A = \begin{bmatrix} \pm i & 0 \\ 0 & \pm 1 \end{bmatrix}$

In short, there must be at least one element/cell in the matrix *A* that contains the imaginary number *i*.

Interpretation: The identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ transforms a plane into itself no matter how many times the identity matrix is used for transformation purposes. This is similar to the identity property of multiplication wherein any number multiplied by 1 is equal to the number itself.

However, the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (see A^2) are not identity matrices. Either using these matrices for transformation purposes will alternately flip/reflect the plane horizontally, vertically, or both. This is similar to getting the square root of -1, resulting in an imaginary number *i*. In short, these reflecting matrices do not fit with the identity property of multiplication. Only by transforming a plane using $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ four times each can the matrix be shifted/transformed to its original location.