CONCEPT OF LINEAR ALGEBRA.

HOMEWORK 7.

(1) More practices with SMDS.) Consider the following ordered bases. $\varphi = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\beta = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$

Let P_β denote the transformation from β- coordinate into φ- coordinate.

a). Carefully draw and label the co-ordinate grids given by both ordered bases. (make ~~sure~~ ~~it~~ plenty big because you will be drawing lot of things on it.) You may do this by hand or use GeoGebra.

(b) Shade in S_β the β-unit square.

c). Let $A = P_\beta \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} P_\beta^{-1}$. Sketch on use GeoGebra to show how A transforms S_β

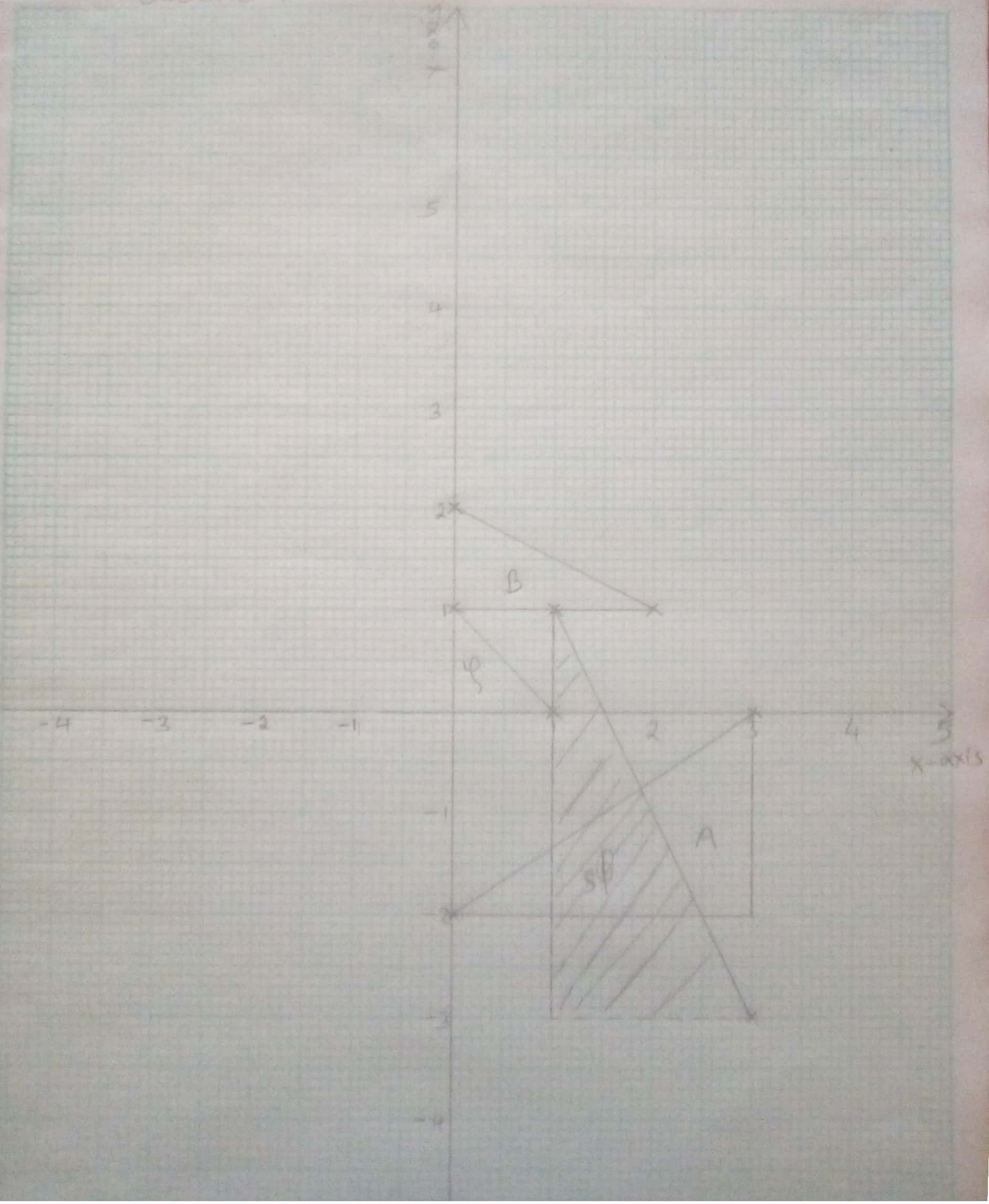
d). Using complete sentences and geometric words, explain how A transforms the plane.
A intersects the plane S_β

e). Let $B = P_\beta \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} P_\beta^{-1}$. Sketch or use GeoGebra to show how B transforms S_β.

f). Using complete sentences and geometric words, explain how B transforms the plane.

B is an Inverse reflection of Plane y.

QUESTION 1



2. Suppose we know that $A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}^{-1}$

a). Recall $A^2 = A \cdot A$. In complete sentences and geometric world, verbally explain how A^2 transforms the plane. (Hint: Consider using SMD together with associative property of matrix multiplication.)

b) Generalize your reason to explain how A^n transforms the plane, A^n transforms the plane through an enlargement transformation.

a). using associative property of matrix multiplication.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \right).$$

$$A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}^{-1} \right)$$

$$\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}^{-1}$$

Let $\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} = A$, $A^{-1} = \frac{1}{\det(A)} \text{adj}$

$\det = (4 \times 4) - (-3 \times 3) = 25$

$$\frac{1}{25} \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{4}{25} & \frac{3}{25} \\ \frac{-3}{25} & \frac{4}{25} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{25} & \frac{3}{25} \\ \frac{-3}{25} & \frac{4}{25} \end{bmatrix} = \begin{bmatrix} \frac{8}{25} + \frac{6}{25} \\ \frac{-3}{25} + \frac{4}{25} \end{bmatrix} = \begin{bmatrix} \frac{14}{25} \\ \frac{1}{25} \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \frac{14}{25} \\ \frac{1}{25} \end{bmatrix} = \begin{bmatrix} \frac{56}{25} - \frac{3}{25} \\ \frac{42}{25} + \frac{4}{25} \end{bmatrix} = \begin{bmatrix} \frac{53}{25} \\ \frac{46}{25} \end{bmatrix}$$

b)

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3. Consider the matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$.

a). Find elementary matrix decomposition of A. Do this by hand and show how you get each elementary matrix.

b). Use your elementary matrix decomposition to describe in complete sentences, using geometric words, how A transforms the plane.

c). Using your decomposition to clearly sketch and label each step of how A transforms the plane. Do so by following the standard unit square. You may do this either carefully by hand or on GeoGebra.

d). Exploration. Is the elementary matrix decomposition you found unique? In other words, could you have found a different collection of elementary matrices whose product is A? Explain.

multiply A by identity matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

a). $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

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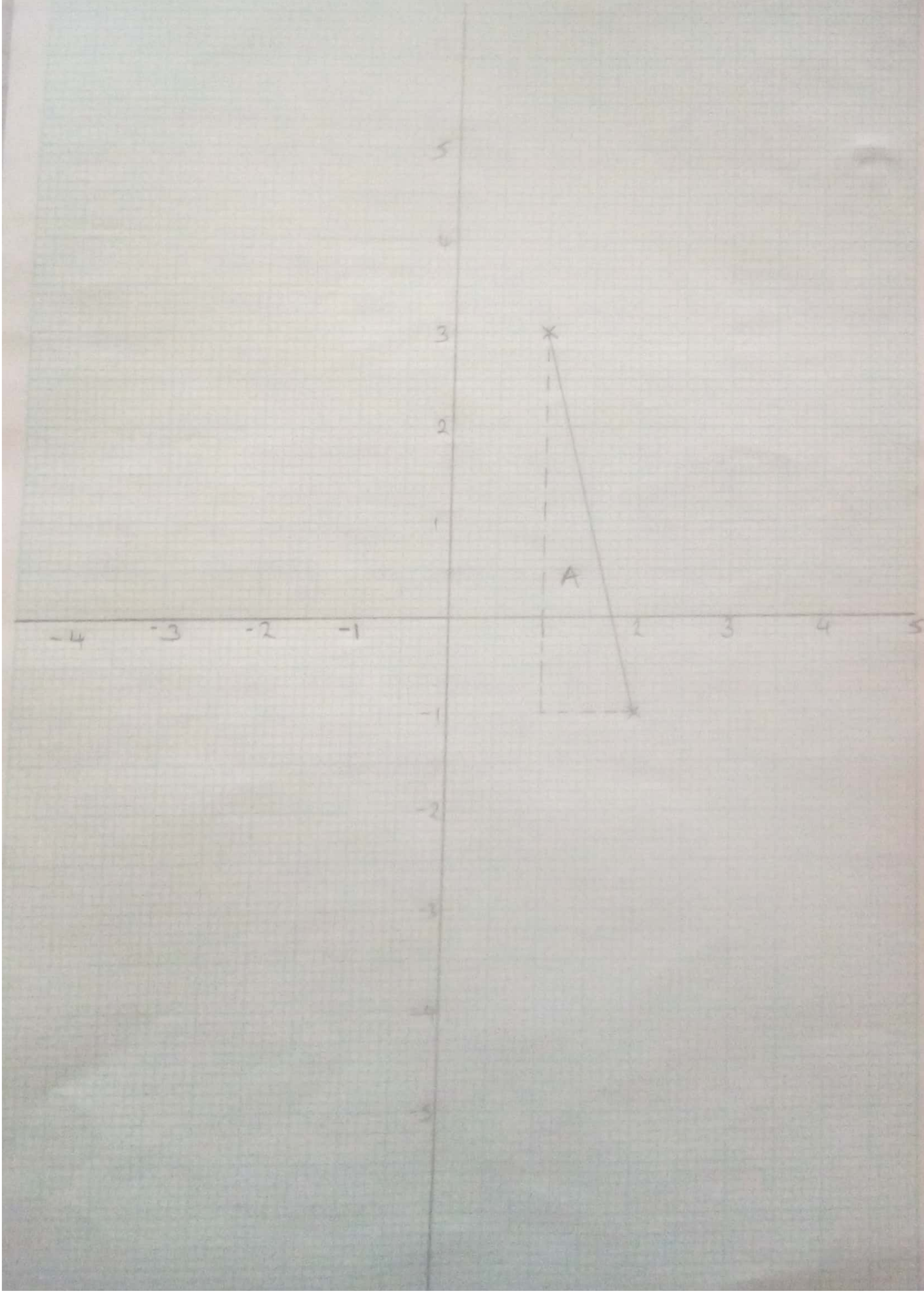
QUESTION 3



b). The transformation undergone by matrix $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ to form matrix $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$. is a rotation about the centre, of 360°.

d). The elementary matrix found was very unique, I could not have found other collections of elementary matrix A that could map it to its original position

4. Let $A = \begin{bmatrix} 5 & 1 & 4 \\ 15 & 9 & 14 \\ 25 & 41 & 35 \end{bmatrix}$ and $b = \begin{bmatrix} 17 \\ 41 \\ 28 \end{bmatrix}$

a). Find an LU - decomposition of A.

b). Use Method #3 to solve the matrix equation
$Ax = b$.

a). Use gaussian elimination method to solve

$\cancel{A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}}$ $A = \begin{bmatrix} 5 & 1 & 4 \\ 15 & 9 & 14 \\ 25 & 41 & 35 \end{bmatrix} \Rightarrow A = LU., L = E_2^{-1} E_1^{-1}$

$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 4 \\ 15 & 9 & 14 \\ 25 & 41 & 35 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \\ 0 & 6 & 2 \\ 0 & 36 & 15 \end{bmatrix} \Rightarrow E_1 A = \begin{bmatrix} 5 & 1 & 4 \\ 0 & 6 & 2 \\ 0 & 36 & 15 \end{bmatrix}$

$E_2 (E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ -6 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 4 \\ 0 & 6 & 2 \\ 0 & 36 & 15 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \\ -30 & 0 & 22 \\ 0 & 36 & 15 \end{bmatrix} = U$

$U = E_2 E_1 A$

$LU = E_2^{-1} E_1^{-1} E_2 E_1 A = E_2^{-1} E_2 E_1^{-1} E_1 A$

$I I A = A$

$[E_1 | I] \sim [I | E_1^{-1}]$.

$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 4 \\ -30 & 0 & 22 \\ 0 & 36 & 15 \end{bmatrix}$.

$= \begin{bmatrix} 5 & 1 & 4 \\ -45 & -3 & 10 \\ -25 & 31 & -5 \end{bmatrix}$

b). Use method H3 to solve matrix equation $Ax=b$.

$$A = \begin{bmatrix} 5 & 1 & 4 \\ 15 & 9 & 14 \\ 25 & 41 & 35 \end{bmatrix} \quad b = \begin{bmatrix} 17 \\ 41 \\ 28 \end{bmatrix}.$$

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ 41 \\ 28 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 1 & 4 \\ 15 & 9 & 14 \\ 25 & 41 & 35 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ 41 \\ 28 \end{bmatrix}$$

$-3R_1+R_2$.

$$\begin{array}{rrrr} & -3(5 & 1 & 4) \\ + & 15 & 9 & 14 \\ \hline = & 0 & -6 & -2 \end{array} \Rightarrow \begin{pmatrix} 5 & 1 & 4 \\ 0 & -6 & -2 \\ 25 & 41 & 35 \end{pmatrix}$$

$-5R_1+R_3$.

$-5R_1 = -5(5\ 1\ 4) = (-25\ -5\ -20)$.

$$\begin{array}{rrrr} & -25 & -5 & -20 \\ + & 25 & 41 & 35 \\ \hline & 0 & 36 & +15 \end{array} \Rightarrow \begin{bmatrix} 5 & 1 & 4 \\ 0 & -6 & -2 \\ 0 & 36 & 15 \end{bmatrix}$$

Add $+6R_2$ to R_3.

$$\begin{bmatrix} 5 & 1 & 4 \\ 0 & 6 & -2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{matrix} 17 \\ 41 \\ 28. \end{matrix}$$

$3z = 28$.

$z = \frac{28}{3}$.

$6y - 2z = 41$

$6y - 2\left(\frac{28}{3}\right) = 41$

$6y = 41 + \frac{56}{3}$

$y = \frac{179}{18}$.

$5x + \frac{179}{18} + 4\left(\frac{28}{3}\right) = 17$

$x = \underline{6}$

5). Consider matrix $G = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 6 & -8 & 3 \end{bmatrix}$

a). Why does G fail to have an LU- decomposition? Explain

In Complete sentences. It is impossible for the matrix to be written in the lower triangular and upper triangular form.

b). Write G as a product PLU where L is lower triangular, U is upper triangular and P is a permutation matrix (ie. a product of type 1 - elementary matrices).

c). In class we discussed a step-by-step produce for using LU- decomposition to solve systems. Create and clearly explain, a step-by-step procedure for using PLU-Decomposition to solve systems.

d). Use the procedure you created to solve matrix equation

$$GX = \begin{bmatrix} 5 \\ 18 \\ 17 \end{bmatrix}.$$

a). LU decomposition is not always possible. LU de-composition is only possible when the leading minors are are non-zero.

b). ~~PA~~ =

b). PA = LU.

$$G = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 6 & -8 & 3 \end{bmatrix}$$

for lower triangular matrix.
$a_{ij} = 0$ for $j > i$.

$\frac{PG}{\cancel{P}} = \frac{LU}{P}$.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 6 & -8 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 6 & -8 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 6 & -8 & 3 \end{bmatrix}$$

c). $\begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 6 & -8 & 3 \end{bmatrix}$ $-2R_2 + R_1 \Rightarrow$

$$\begin{matrix} 0 & 2 & 1 \\ -6 & -2 & -2 \\ \hline -6 & 0 & -1 \end{matrix} \Rightarrow \begin{bmatrix} 0 & 2 & 1 \\ -6 & 0 & -1 \\ 6 & -8 & 3 \end{bmatrix}$$

Add $R_2 + R_3$ $\begin{bmatrix} 0 & 2 & 1 \\ -6 & 0 & -1 \\ 0 & -8 & 2 \end{bmatrix}$ Add $6 + R_2$. $\begin{bmatrix} 0 & 2 & 1 \\ 0 & 6 & 5 \\ 0 & -8 & 2 \end{bmatrix}$ $4R_1 + R_3$.

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$L = \cancel{\begin{bmatrix} 0 & 2 & 1 \\ (2) & & \\ & & \end{bmatrix}}$ $L = \begin{bmatrix} 0 & 2 & 1 \\ -6 & (-2) & -1 \\ 6 & -8 & 3 \end{bmatrix}$ $U = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 & 1 \\ -6 & -2 & -1 \\ 6 & -8 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 6 & -8 & 3 \end{bmatrix}.$$

d). $Gx = \begin{bmatrix} 5 \\ 18 \\ 17 \end{bmatrix}$ $\begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 6 & -8 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 17 \end{bmatrix}$

$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 6 & 5 \\ 0 & -8 & 2 \end{bmatrix}$ $4R_1 + R_3$ $= \begin{bmatrix} 0 & 2 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{matrix} 5 \\ 18 \\ 17 \end{matrix}$.

$6z = 17$.
$z = \frac{17}{6}$
$6y + 5z = 18$.
$6y + 5\left(\frac{17}{6}\right) = 18$
$y = \frac{23}{36}$.
$x = 2\frac{13}{72}$

6). Consider the following ordered base $\varphi = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$. Recall that $P_{\mathcal{B}} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ is a

the transformation from $\mathcal{B}$-coordinates into φ-coordinates.

a). Carefully draw and label the coordinate grids given by both ordered bases. You may do this by hand or use GeoGebra.

b) Shade in $S_{\mathcal{B}}$, the $\mathcal{B}$-unit square and $S\varphi$, the standard unit square,

c). What is the ratio of their areas.

$\frac{\text{Area}(S\mathcal{B})}{\text{Area}(S\varphi)}$. (you may wish to use the Geogebra area tool, found under the angle. (ie. 8th) dropdown menu.)

Ratio of areas.

$\frac{\text{Area}(S\mathcal{B})}{\text{Area}(S\varphi)}$

Area of $S\mathcal{B} = \frac{1}{2}Bh$.

$\frac{1}{2} \times 4 \times 1 = 2$ ~~units~~ square units.

Area of ~~$S\varphi$~~ $(S\varphi)$

$\frac{1}{2} \times BH = \frac{1}{2} \times 1 \times 1 = 0.5$ sq. units

Ratios.

$\frac{2}{0.5}$

$= \underline{\underline{4}}$

QUESTION 6