Item 1

a. The system of hyperplanes described by matrix $\left[A \mid \mathrm{b}\right]$ is a collection of $\mathbb{R}^5$ hyperplanes contained in $\mathbb{R}^6$. The solution set is described by $\vec{x}$ which is the intersection of the said $\mathbb{R}^5$ hyperplanes, with 2 degrees of freedom.
b. Based on the augmented matrix expressed in reduced row echelon form, the determined (basic) variables are x_1, x_3, and x_5 while the free variables are x_2 and x_4.
c. Expressing the solution set in parametric vector form,

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to x_1 + 2x_2 + 4x_4 = 1 \\ \to x_3 + 2x_4 = -1 \\ \to x_5 = 1 \\ \end{array}$$

Hence,

$$\begin{aligned} x_1 &= 1 - 2x_2 - 4x_4 \\ x_2 &= x_2 \\ x_3 &= -1 - 2x_4 \\ x_4 &= x_4 \\ x_5 &= 1 \end{aligned}$$

Separating the constants and the coefficients,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Item 2

a. The interchange matrix E_1 should be a 2×2 matrix.

$$E_1 \begin{bmatrix} 2 & 2 & 4 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2a+b & 2a+2b & 4a+3b \\ 2c+d & 2c+2d & 4c+3d \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{bmatrix}$$

Comparing a_{11} to a_{12} and a_{21} to a_{22},

$$\left.\begin{aligned} 2a+2b&=2 \\ 2a+b&=1 \end{aligned}\right\} \rightarrow a=0 \text{ and } b=1$$

$$\left.\begin{aligned} 2c+2d&=2 \\ 2c+d&=2 \end{aligned}\right\} \rightarrow c=1 \text{ and } d=0$$

Hence,

$$E_1=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

b. The replacement matrix E_2 should be a 2×2 matrix. Performing the desired row operation first,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{bmatrix} -2r_1+r_2 \rightarrow r_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2-2(1) & 2-2(2) & 4-2(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \end{bmatrix}$$

Setting the equation,

$$E_2 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} a+2b & 2a+2b & 3a+4b \\ c+2d & 2c+2d & 3c+4d \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \end{bmatrix}$$

Comparing a_{11} to a_{12} and a_{21} to a_{22},

$$\left.\begin{aligned} 2a+2b&=2 \\ a+2b&=1 \end{aligned}\right\} \rightarrow a=1 \text{ and } b=0$$

$$\left.\begin{aligned} 2c+2d&=-2 \\ c+2d&=0 \end{aligned}\right\} \rightarrow c=-2 \text{ and } d=1$$

Hence,

$$E_2=\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

c. The scaling matrix E_3 should be a 2×2 matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \end{bmatrix} -\frac{1}{2}r_2 \to r_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Setting the equation,

$$E_3 \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & 2a-2b & 3a-2b \\ c & 2c-2d & 3c-2d \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Comparing a_{11} to a_{12} and a_{21} to a_{22},

$$\left.\begin{aligned} a &= 1 \\ 2a-2b &= 2 \end{aligned}\right\} \to a = 1 \text{ and } b = 0$$

$$\left.\begin{aligned} c &= 0 \\ 2c-2d &= 1 \end{aligned}\right\} \to c = 0 \text{ and } d = -\frac{1}{2}$$

Hence,

$$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

<u>Item 3</u>

a. Given that $n = 1000$,

$$\begin{aligned} \text{number of floating point operations} &= \frac{2}{3}n^3 \\ &= \frac{2}{3}(1000)^3 \\ &= 667\times 10^6 \text{ counts} \end{aligned}$$

Since a standard computer performs at roughly 10^{10} FLOPS,

$$\begin{aligned} t &= \frac{\text{number of floating point operations}}{\text{FLOPS}} \\ &= \frac{667\times 10^6}{10\times 10^9}\,\text{s} \\ &= 66.7\,\text{ms} \end{aligned}$$

I already take roughly 10-15 minutes just to solve a simple 3×3 matrix using Gaussian elimination. For a ridiculously more complex 1000×1000 matrix, it might take me months or even years, and that is assuming I do not make arithmetic mistakes along the way.

b. Given n = 24 million,

$$\begin{aligned}\text{number of floating point operations} &= \frac{2}{3}n^3\\ &= \frac{2}{3}(24000000)^3\\ &= 9.216\times 10^{21}\ \text{counts}\end{aligned}$$

Since a standard computer performs at roughly 10^{10} FLOPS,

$$\begin{aligned}t &= \frac{\text{number of floating point operations}}{\text{FLOPS}}\\ &= \frac{9.216\times 10^{21}}{10\times 10^{9}}\,\text{s}\\ &= 921.6\ \text{Gs}\end{aligned}$$

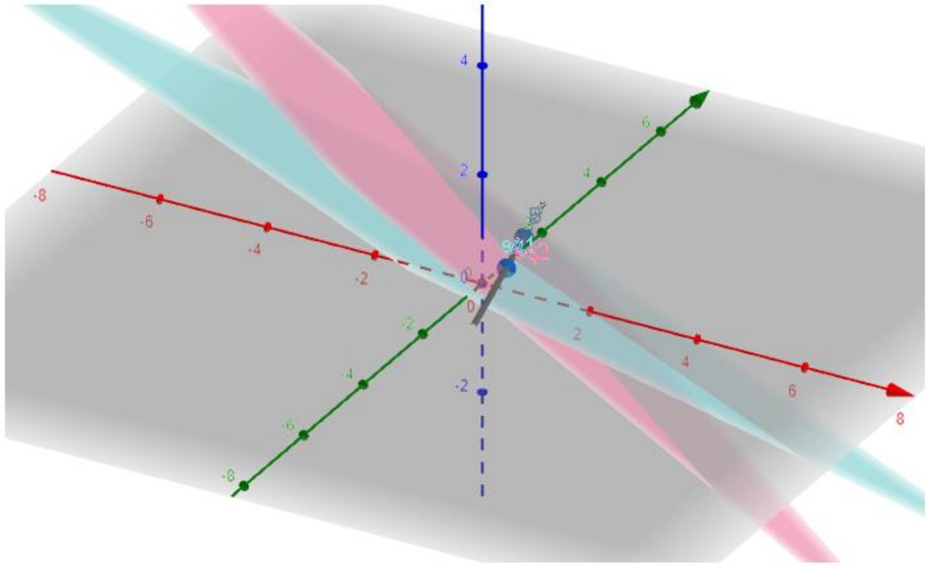
It would take roughly 1 trillion seconds or 29,000 years for a standard computer to perform Gaussian elimination and reduce the matrix into reduced row echelon form.

c. The unrealistic large amount of time calculated in b) suggests that we need to find a more efficient method of evaluating matrices in the shortest possible amount of time without sacrificing the accuracy.

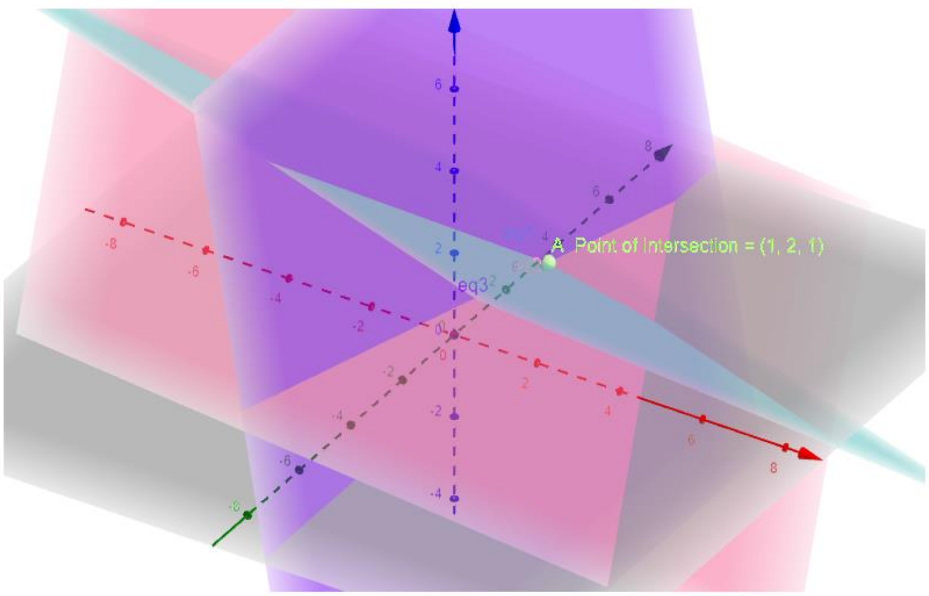
<u>Item 4</u>

a. The first system of equation pertain to the solution set described by line of intersection between the planes $x+2y+3z=2$ and $4x+5y+6z=5$ at $\mathbb{R}^3$. The second system of equations pertain to the solution set described by the point of intersection between the planes $x+3y+4z=11$, $2x+y+5z=9$, and $6x+z=7$ at $\mathbb{R}^3$. The visualizations made using GeoGebra are shown below.

System A:



System B:



b. Based on the graphs above, I expect an infinite number of solutions (line) for System A and a single solution (point) for System B.

c. See the images above.

d. The graphs matched my expectations. For System A, there are three variables and two equations, resulting in a minimum 1 free variable that hints an infinite solution set in the form of a line. For System B, there are three variables and three equations, resulting in a minimum 0 free variable that hints a single point of intersection.

e. The solution set for the first system of equation is the line described by the parametric vector equation $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. The solution for the second system of equation is the point (1, 2, 1).

f. Finding the rref for System A,

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 5 & 6 & 5 \end{bmatrix} R_2 - 4R_1 \to R_2 = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & -3 \end{bmatrix} -R_2/3 \to R_2 = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} R_1 - 2R_2 \to R_1 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

For System B,

$$\begin{bmatrix} 1 & 3 & 4 & 11 \\ 2 & 1 & 5 & 9 \\ 6 & 0 & 1 & 7 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \to R_2 \\ R_3 - 6R_1 \to R_3 \end{matrix} = \begin{bmatrix} 1 & 3 & 4 & 11 \\ 0 & -5 & -3 & -13 \\ 0 & -18 & -23 & -59 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 & 11 \\ 0 & -5 & -3 & -13 \\ 0 & -18 & -23 & -59 \end{bmatrix} -R_2/5 \to R_2 = \begin{bmatrix} 1 & 3 & 4 & 11 \\ 0 & 1 & 0.6 & 2.6 \\ 0 & -18 & -23 & -59 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 & 11 \\ 0 & 1 & 0.6 & 2.6 \\ 0 & -18 & -23 & -59 \end{bmatrix} \begin{matrix} R_1 - 3R_2 \to R_1 \\ \\ R_3 + 18R_2 \to R_3 \end{matrix} = \begin{bmatrix} 1 & 0 & 2.2 & 3.2 \\ 0 & 1 & 0.6 & 2.6 \\ 0 & 0 & -12.2 & -12.2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2.2 & 3.2 \\ 0 & 1 & 0.6 & 2.6 \\ 0 & 0 & -12.2 & -12.2 \end{bmatrix} \begin{matrix} \\ \\ -R_3/12.2 \to R_3 \end{matrix} = \begin{bmatrix} 1 & 0 & 2.2 & 3.2 \\ 0 & 1 & 0.6 & 2.6 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2.2 & 3.2 \\ 0 & 1 & 0.6 & 2.6 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 - 2.2R_3 \to R_1 \\ R_2 - 0.6R_3 \to R_2 \\ \\ \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

g. For system A, the rref $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ is read as: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

For system B, the rref $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is read as: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

h. The solutions found via intersections and via rref matrix agree for both systems.

Item 5

a. I do expect that at least one linear combination of **v** and **w** results in **b**. While my usual answer to this question is the opposite, seeing that the components in all three vectors share the same <*n*, *n* + 1, *n* + 2> progression makes it more predictable that a linear combination is indeed possible.

b. The augmented matrix is as follows:

$$S = \{\mathrm{v}, \mathrm{w}, \mathrm{b}\}$$

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$$

$$S = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

c. Obtaining the reduced row echelon form of *S*,

Step 1:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{matrix} = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

Step 2:

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \begin{matrix} \\ -R_2/3 \to R_2 \\ -R_3/6 \to R_3 \end{matrix} = \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Step 3:

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 - 4R_2 \to R_1 \\ \\ R_3 - R_2 \to R_3 \end{matrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

d. Interpretation: The vectors **v**, **w**, and **b** are linearly dependent. The vector **b** can be represented using a linear combination of x**v** and y**w**, wherein x = -1 and y = 2. As an expression,

$$-\mathrm{v} + 2\mathrm{w} = \mathrm{b}$$

$$-1\begin{bmatrix}1\\2\\3\end{bmatrix} + 2\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}7\\8\\9\end{bmatrix}$$

Item 6

a. The null space Null(A) contains all of the vectors with dimensions $n \times 1$ that becomes nullified or reduced to the origin point when the transformation matrix A with dimensions $m \times n$ is applied to those unique set of vectors.

b. The nullity or degrees of freedom of A with dimensions $m \times n$ should be at least n – m. For example, a 3×3 matrix typically ends up with 0 free variables, hence 3 – 3 = 0 degrees of freedom. In some cases, a 3×3 matrix may end up with 1 or 2 free variables depending on the dynamics between each equation involved in the system.

Item 7

a. I expect the nullity of A to either 1 or 2. When expressed as a system of equations, there are 1 more unknowns compared to the number of equations. Usually, the augmented matrix and echelon form derived from the said system contains 2 or 3 pivot columns. Thus, it follows that the expected nullity of A is either 1 or 2.

b. Setting up the augmented matrix,

$$Ax = 0$$

$$\begin{bmatrix}1 & 2 & 5 & 7\\3 & 1 & 6 & 8\\3 & -4 & -3 & -5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Hence,

$$\left[\begin{array}{cccc|c}1 & 2 & 5 & 7 & 0\\3 & 1 & 6 & 8 & 0\\3 & -4 & -3 & -5 & 0\end{array}\right]$$

c. Finding the echelon form,

Step 1:

$$\begin{bmatrix} 1 & 2 & 5 & 7 & 0 \\ 3 & 1 & 6 & 8 & 0 \\ 3 & -4 & -3 & -5 & 0 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{matrix} = \begin{bmatrix} 1 & 2 & 5 & 7 & 0 \\ 0 & -5 & -9 & -13 & 0 \\ 0 & -10 & -18 & -26 & 0 \end{bmatrix}$$

Step 2:

$$\begin{bmatrix} 1 & 2 & 5 & 7 & 0 \\ 0 & -5 & -9 & -13 & 0 \\ 0 & -10 & -18 & -26 & 0 \end{bmatrix} \begin{matrix} \\ -R_2/5 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{matrix} = \begin{bmatrix} 1 & 2 & 5 & 7 & 0 \\ 0 & 1 & 1.8 & 2.6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there two pivot columns, rank A or the dimension of column space is equal to 2. With n = 4 (four columns),

$$\text{rank } A + \dim \text{Nul } A = n$$

$$\dim \text{Nul } A = n - \text{rank } A$$

$$\dim \text{Nul } A = 4 - 2$$

$$\dim \text{Nul } A = 2 \to \text{same as nullity}$$

To find Null(A), evaluate the augmented matrix further to its rref.

Step 3:

$$\begin{bmatrix} 1 & 2 & 5 & 7 & 0 \\ 0 & 1 & 1.8 & 2.6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_1 - 2R_2 \to R_1 = \begin{bmatrix} 1 & 0 & 1.4 & 1.8 & 0 \\ 0 & 1 & 1.8 & 2.6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The equivalent system of equation is:

$$x_1 + 1.4x_3 + 1.8x_4 = 0$$

$$x_2 + 1.8x_3 + 2.6x_4 = 0$$

Upon further evaluation,

$$x_1 = -1.4x_3 - 1.8x_4$$

$$x_2 = -1.8x_3 - 2.6x_4$$

$$x_3 = x_3$$

$$x_4 = x_4$$

$$\text{Null}(A) = \left\{ \begin{bmatrix} -1.4 \\ -1.8 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.8 \\ -2.6 \\ 0 \\ 1 \end{bmatrix} \right\}$$