Item 1

a. Matrix A transformed the unit square plane by stretching the size of the edges by a factor of $\sqrt{10}$, then rotating the plane with A as pivot at a counterclockwise angle of 18.43°.

Matrix B transformed the unit square plane by stretching the size of the edges by a factor of 3, then shearing the top edge 1 unit towards the right.

b. For Matrix A, the symbolic matrix is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Based on the plot, $a = 3$ and $b = 1$, making each edge equal to $\sqrt{a^2 + b^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$. The angle of rotation (counterclockwise) is defined by $\tan^{-1}\frac{b}{a} = \tan^{-1}\frac{1}{3} = 18.43°$. Hence, $A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$.

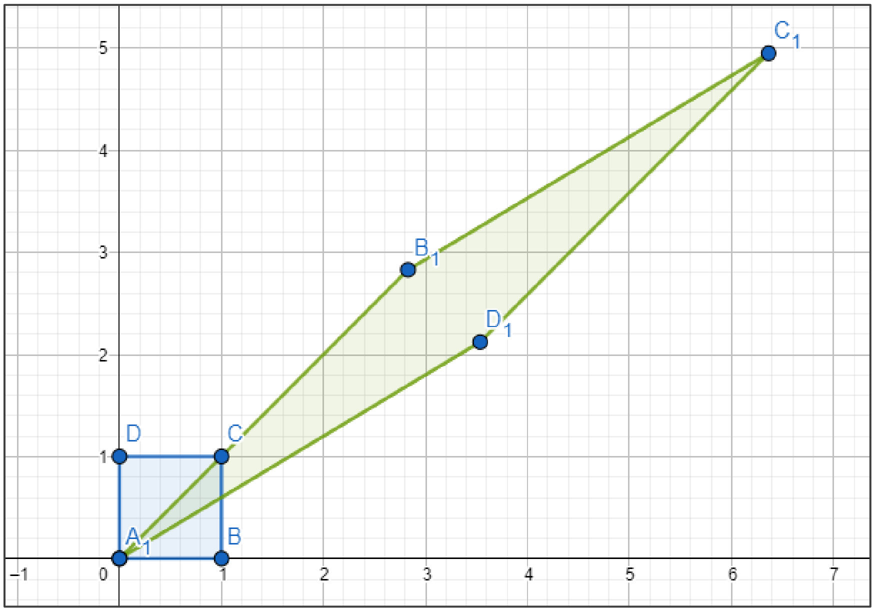
For Matrix B, the symbolic matrix is $\begin{bmatrix} c & d \\ 0 & c \end{bmatrix}$. Based on the plot, $c = 3$ and $d = 1$, making each edge equal to $\sqrt{c^2 + d^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$. The shearing effect is 1 unit in length, as defined by *d*. Hence, $B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.

Item 2

- The first transformation matrix $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is similar to the rotating matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a = b = \frac{1}{\sqrt{2}}$. This makes the default square plane stretch its edges by a factor of $\sqrt{a^2 + b^2} = 1$ and rotate at a counterclockwise angle of $\tan^{-1}\frac{b}{a} = \tan^{-1} 1 = 45°$ at the pivot point A.
- The second transformation matrix $\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ is similar to the stretching/reflecting matrix $\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$ where *c* stretches the plane horizontally by a factor of 4 and *d* reflects (negative value) the matrix with respect to the horizontal axis by a factor of 1.

- Finally, the third transformation matrix $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is similar to the shearing matrix $\begin{bmatrix} e & f \\ 0 & e \end{bmatrix}$ where e stretches the edges of the plane by a factor of 2 and f shears the edge CD by 1 unit.

When all of the effects of the three transformation matrices are combined, the results are as follows:



Item 3

a. The product AB is a 2×2 matrix where each cell (per A's columns) is a linear combination of column groups a_1 and a_2 with weight groups (per B's rows) b_1 and b_2.

   The product BA is a 3×3 matrix where each cell (per B's columns) is a linear combination of column groups b_1, b_2, b_3 with weight groups (per A's rows) a_1, a_2, a_3.

b. Solving for AB,

$$AB = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 1+3\cdot 3+1\cdot 5 & 2\cdot 2+3\cdot 4+1\cdot 6 \\ 0\cdot 1+1\cdot 3+4\cdot 5 & 0\cdot 2+1\cdot 4+4\cdot 6 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 22 \\ 23 & 28 \end{bmatrix}$$

Solving for *BA*,

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 2 + 2\cdot 0 & 1\cdot 3 + 2\cdot 1 & 1\cdot 1 + 2\cdot 4 \\ 3\cdot 2 + 4\cdot 0 & 3\cdot 3 + 4\cdot 1 & 3\cdot 1 + 4\cdot 4 \\ 5\cdot 2 + 6\cdot 0 & 5\cdot 3 + 6\cdot 1 & 5\cdot 1 + 6\cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 9 \\ 6 & 13 & 19 \\ 10 & 21 & 29 \end{bmatrix}$$

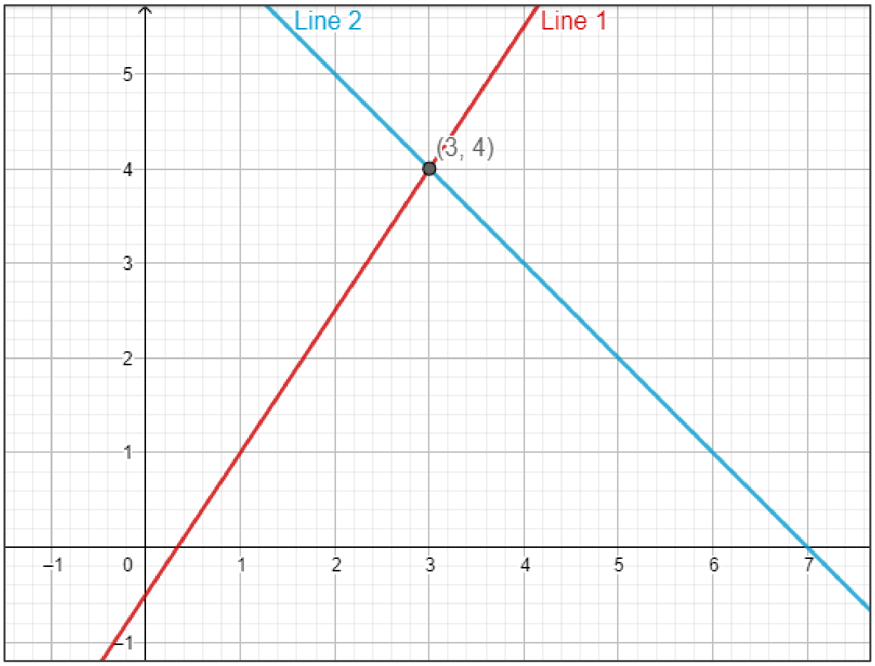
Clearly, $AB \neq BA$ mainly because the two product matrices have different dimensions.
Note: Matrix *AB* is a 2×2 matrix while Matrix *BA* is a 3×3 matrix.

c. The matrix *AB* represents the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^2$ of two 3D vectors $b_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ into corresponding 2D vectors $\begin{bmatrix} 16 \\ 23 \end{bmatrix}$ and $\begin{bmatrix} 22 \\ 28 \end{bmatrix}$.

The matrix *BA* represents the linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^3$ of three 2D vectors $a_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $a_3 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ into corresponding 3D vectors $\begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 13 \\ 21 \end{bmatrix}$ and $\begin{bmatrix} 9 \\ 19 \\ 29 \end{bmatrix}$.

Item 4

a. Graphing the two lines,



b. Based on the image above, the intersection of the two lines is the point (3, 4).

c. Solving for the intersection point using substitution method,

$$\ell_2 : x + y = 7 \rightarrow y = 7 - x$$
$$\ell_1 : 3x - 2y = 1 \rightarrow 3x - 2(7 - x) = 1$$

Hence,

$$3x - 14 + 2x = 1$$
$$5x = 15$$
$$x = 3$$

$$y = 7 - 3$$
$$y = 4$$

The intersection is indeed (3, 4).

d. Rewriting the system of equations as a matrix equation,

$$Ax = b$$
$$\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Matrix A is the coefficient matrix containing the coefficients of the two lines. Matrix $\mathbf{x}$ represents abscissa and ordinate or simply the point of intersection of the two lines.

Matrix $\mathbf{b}$ is the matrix representing the values on the right side of the equal sign of the standard form equations of the two lines.

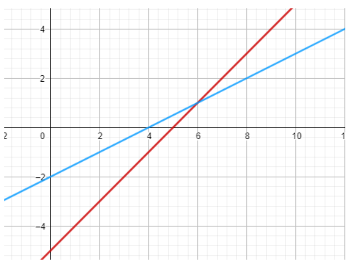
e. Combine x parts of $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and y parts of $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ to produce the mixture $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$.

<u>Item 5</u>

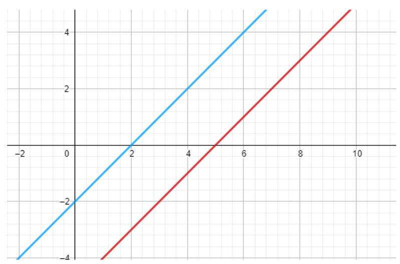
a. There are three types of intersection for two lines in a plane.

Type 1: Intersection at a single point; consistent-independent.
Type 2: No intersection at all; inconsistent.
Type 3: Intersection at all points; consistent-dependent.

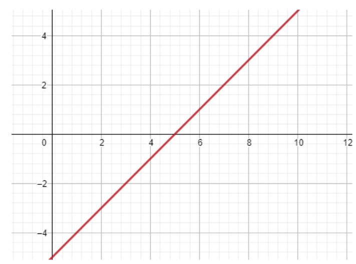
Graphical configuration:



Type 1



Type 2



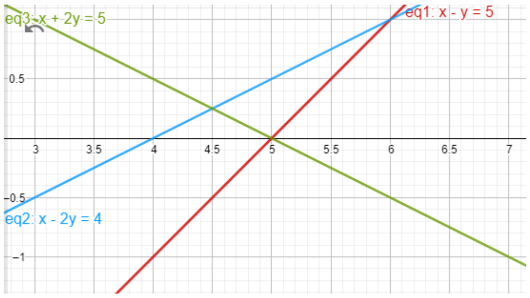
Type 3

b. The Type 1 (intersection at a single point) or consistent-independent system is most likely to occur for any 2 random lines.

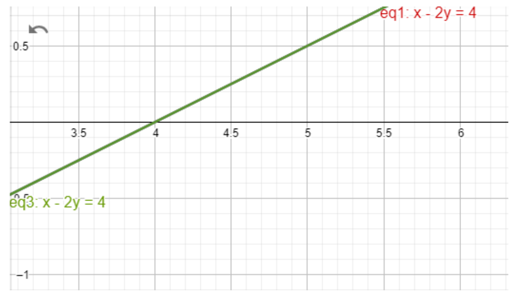
c. There are three major types of intersection for three lines in a plane.

Type 1: Intersection at a single point.
Type 2: No intersection at all (either all are parallel or at least one line is not parallel to the others).
Type 3: Intersection at all points.

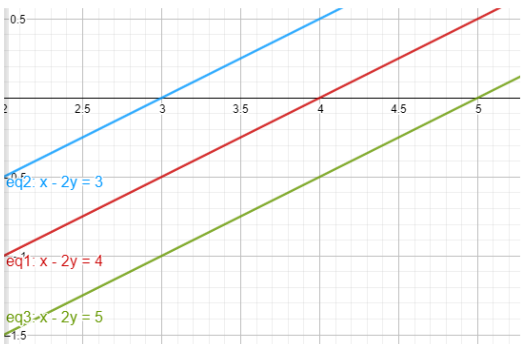
Graphical configuration:



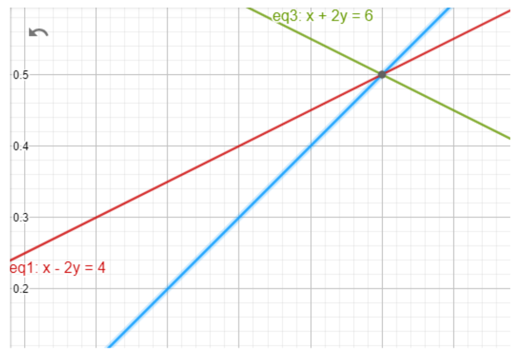
Type 2 A



Type 3



Type 2 B



Type 1

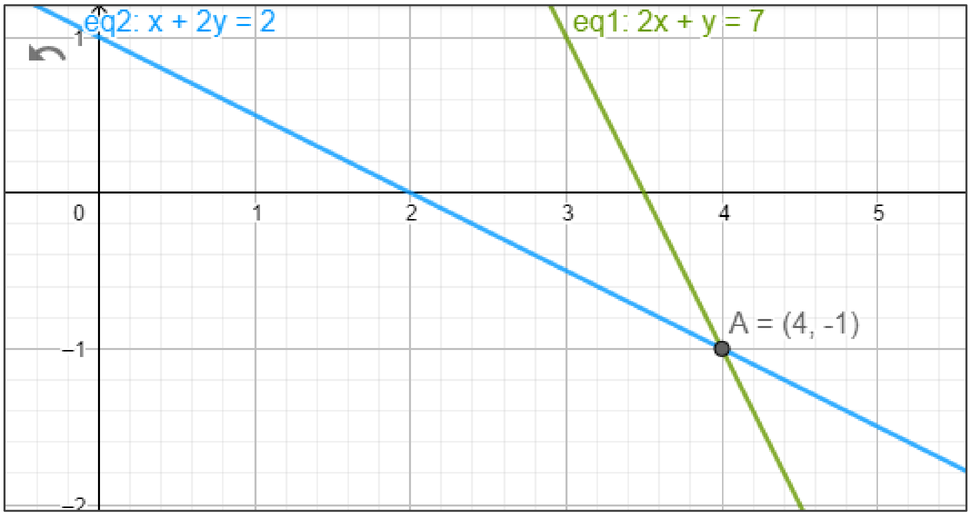
d. Type 2 intersection (where at least one line is not parallel to the others) is most likely to occur compared to the other types.

e. When there are *n* random lines, the intersection type where at least one line is not parallel to the others is still most likely to occur (same as d). This makes sense because it is highly unlikely to have *n* lines that have the same random slopes, have the exact equivalent standard forms, or all intersect at exactly one point.

<u>Item 6</u>

a. I expect a single point of intersection since it is the most likely scenario to occur (as discussed in item 5).

b. Graph:



Based on the graph above, the solution set is (4, -1).

c. There is only one point that was sent to **b** – located at (7, 2) – via the transformation matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and that point is (4, -1).

Item 7

a. Rewriting as a matrix equation,

$$Ax = b$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

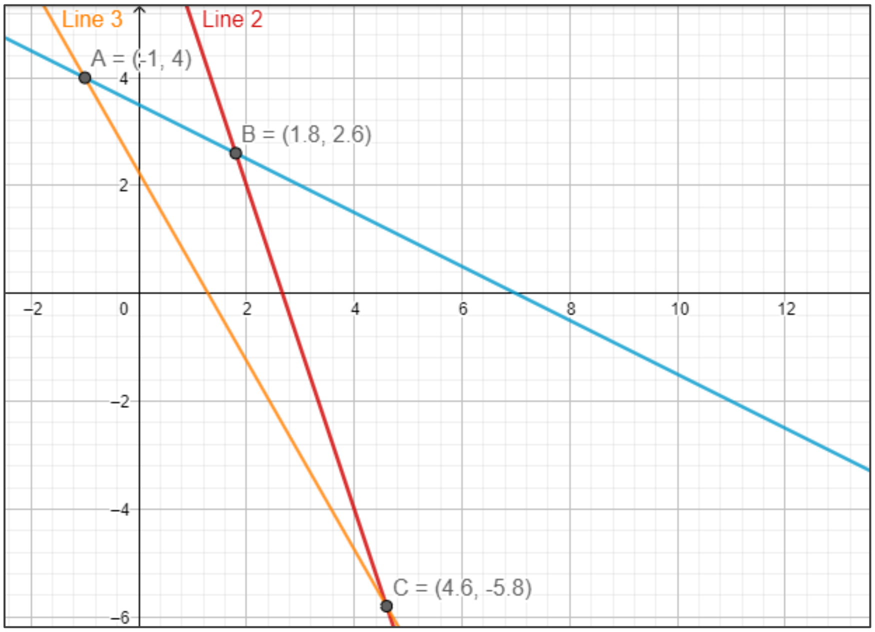
The matrix A is made of two vectors **v** and **w** at $\mathbb{R}^3$ written in each column. The cells in vector **x** represent the weights x and y necessary to linearly combine **v** and **w** to generate vector **b**.

b. Combine x parts **v** and y parts **w** to produce mixture **b**.

c. There are three distinct lines.

$$\ell_1 : x + 2y = 7$$
$$\ell_2 : 3x + y = 8$$
$$\ell_3 : 7x + 4y = 9$$

d. Graph:



e. No point exists wherein all three lines intersect. Lines 1 and 2 intersect at (1.8, 2.6) but not Line 3. Lines 2 and 3 intersect at (4.6, 5.8) but not Line 1. Finally, Lines 1 and 3 intersect at (-1, 4) but not Line 2.

f. Since there is no point of intersection that lies on all three lines, it is not possible to write matrix **b** as a linear combination of **v** and **w**.