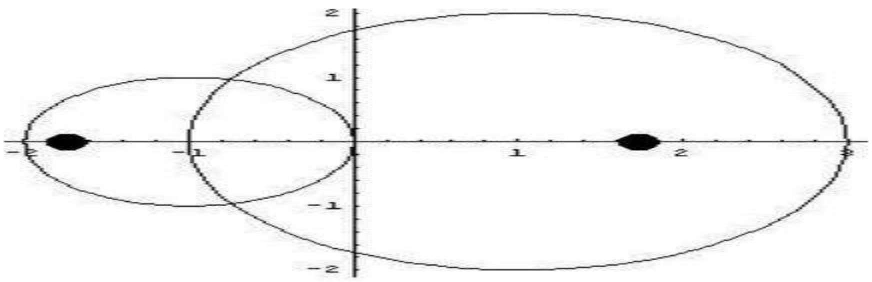
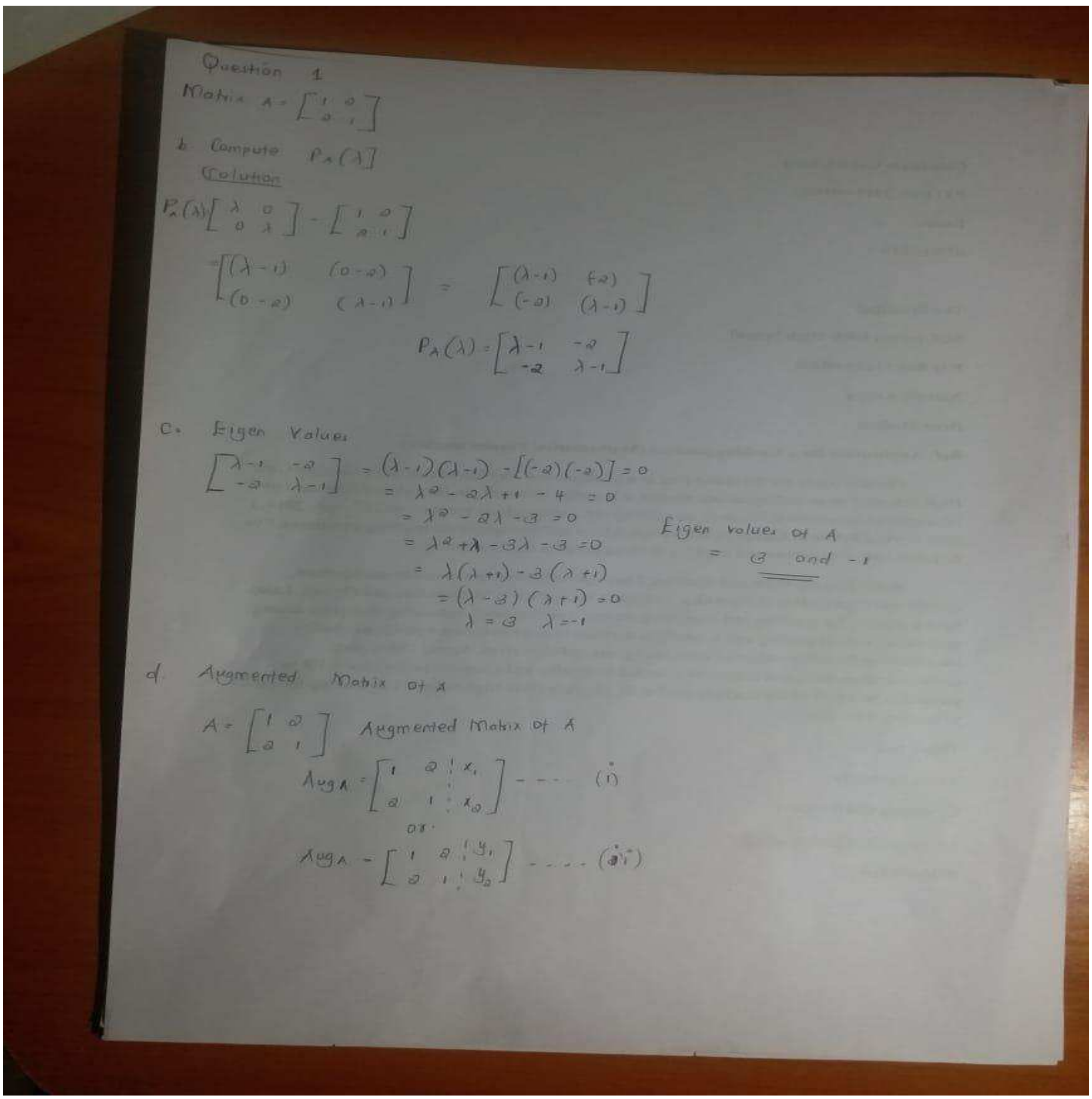
Question 1.

a)





Question 1

Matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

b. Compute $P_A(\lambda)$

Solution

$$P_A(\lambda) \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda - 1) & (0 - 2) \\ (0 - 2) & (\lambda - 1) \end{bmatrix} = \begin{bmatrix} (\lambda - 1) & (-2) \\ (-2) & (\lambda - 1) \end{bmatrix}$$

$$P_A(\lambda) = \begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{bmatrix}$$

c. Eigen Values

$$\begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{bmatrix} = (\lambda - 1)(\lambda - 1) - [(-2)(-2)] = 0$$

$$= \lambda^2 - 2\lambda + 1 - 4 = 0$$

$$= \lambda^2 - 2\lambda - 3 = 0$$

$$= \lambda^2 + \lambda - 3\lambda - 3 = 0$$

$$= \lambda(\lambda + 1) - 3(\lambda + 1)$$

$$= (\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = 3 \quad \lambda = -1$$

Eigen values of A = 3 and -1

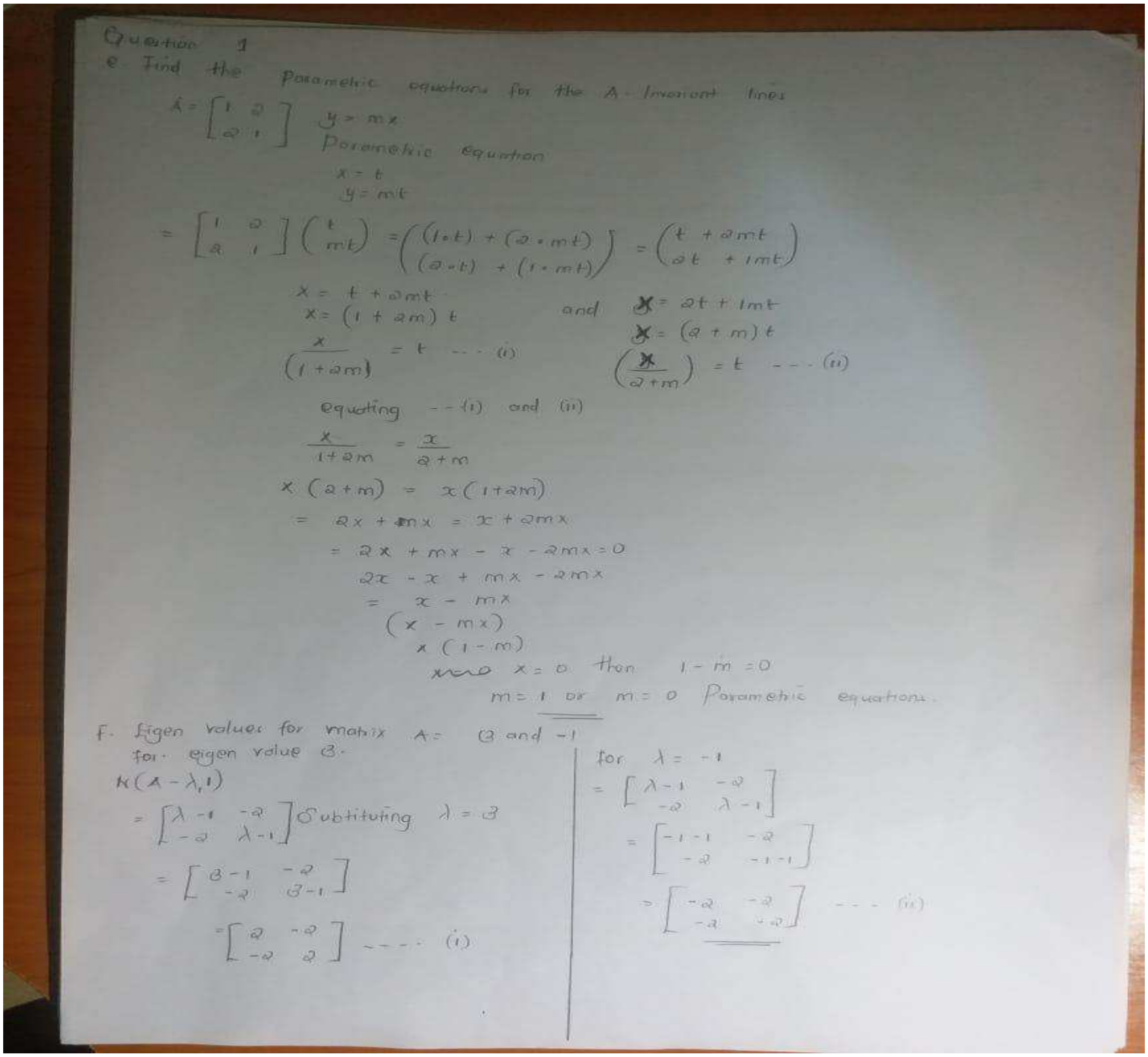
d. Augmented Matrix of A

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ Augmented Matrix of A

$$Aug\,A = \left[\begin{array}{cc|c} 1 & 2 & x_1 \\ 2 & 1 & x_2 \end{array}\right] \text{- - - - (i)}$$

or

$$Aug\,A = \left[\begin{array}{cc|c} 1 & 2 & y_1 \\ 2 & 1 & y_2 \end{array}\right] \text{- - - - - (ii)}$$



Question 1

e Find the Parametric equations for the A - Invariant lines

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $y = mx$

Parametric equation

$x = t$

$y = mt$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} (1 \cdot t) + (2 \cdot mt) \\ (2 \cdot t) + (1 \cdot mt) \end{pmatrix} = \begin{pmatrix} t + 2mt \\ 2t + 1mt \end{pmatrix}$$

$x = t + 2mt$

$x = (1 + 2m)\, t$

$\dfrac{x}{(1 + 2m)} = t \quad \text{---} (i)$

and $\cancel{x} = 2t + 1mt$

$\cancel{x} = (2 + m)\, t$

$\left(\dfrac{\cancel{x}}{2 + m}\right) = t \quad \text{---} (ii)$

equating -- (i) and (ii)

$\dfrac{x}{1 + 2m} = \dfrac{x}{2 + m}$

$x\,(2 + m) = x\,(1 + 2m)$

$= 2x + mx = x + 2mx$

$= 2x + mx - x - 2mx = 0$

$2x - x + mx - 2mx$

$= x - mx$

$(x - mx)$

$x\,(1 - m)$

$x = 0$ then $1 - m = 0$

$m = 1$ or $m = 0$ Parametric equations.

F. Eigen values for matrix $A = 3$ and -1

for eigen value 3.

$K(A - \lambda_1 I)$

$= \begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{bmatrix}$ Substituting $\lambda = 3$

$= \begin{bmatrix} 3 - 1 & -2 \\ -2 & 3 - 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad \text{----} (i)$

for $\lambda = -1$

$= \begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{bmatrix}$

$= \begin{bmatrix} -1 - 1 & -2 \\ -2 & -1 - 1 \end{bmatrix}$

$= \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \quad \text{---} (ii)$

G: Using complete sentences and geometric words, verbally describe how A transforms the plane.

Answer:

The matrix A transforms the plane diagonally.

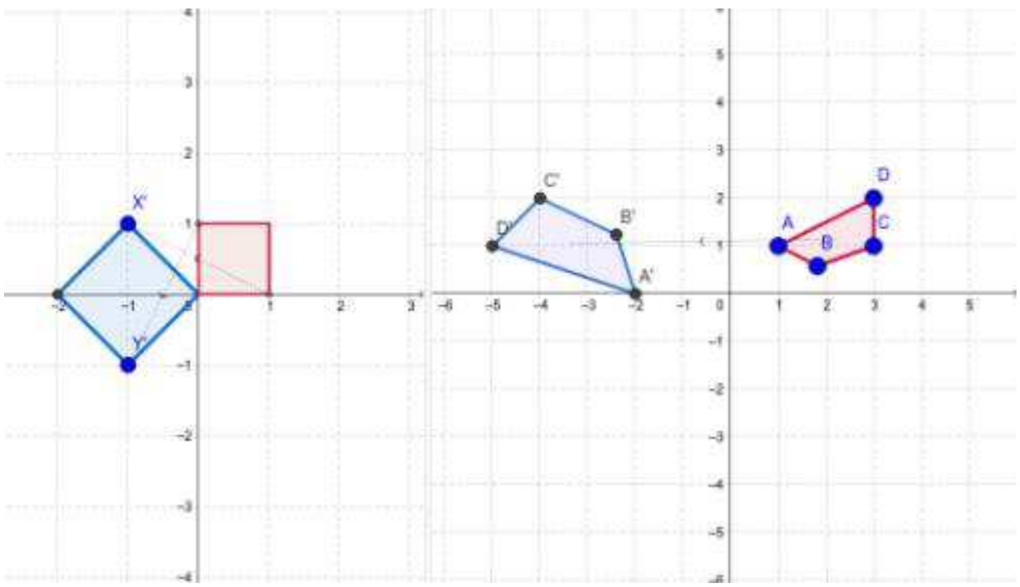
(h) Symbolically express your diagonalization of A.

1   2

1   2

(i) Using GeoGebra, visually represent the B-coordinate grid, the B-unit square, SB, and depict how A transforms S

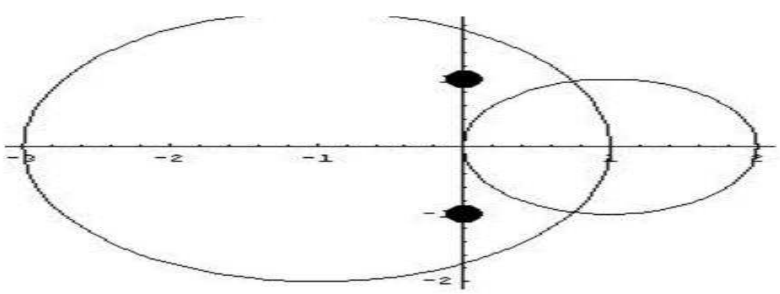
Answer:



Question 2

Consider the following matrix A = . = 1 4 −1 5 . (a) Compute disc(A) and then use this to determine the number of invariant lines A has in the plane.

Answer



Question 2

b. Compute $P_A(\lambda)$ $A = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix}$

Solution

$$P_A(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & 0 - 4 \\ 0 + 1 & \lambda - 5 \end{bmatrix}$$

$$P_A(\lambda) = \begin{bmatrix} \lambda - 1 & -4 \\ 1 & \lambda - 5 \end{bmatrix}$$

c. Eigen Values

$$\begin{bmatrix} \lambda - 1 & -4 \\ 1 & \lambda - 5 \end{bmatrix} = (\lambda - 1)(\lambda - 5) - ((-4) \cdot (1))$$
$$= \lambda^2 - 5\lambda - \lambda + 5 - -4$$
$$= \lambda^2 - 6\lambda + 9$$
$$= \lambda^2 - 3\lambda - 3\lambda + 9$$
$$\lambda(\lambda - 3) - 3(\lambda - 3)$$
$$(\lambda - 3)(\lambda - 3)$$
$$\lambda = 3 \quad \lambda = 3$$

d. $A = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix}$ Augmented Matrix of X

$$\text{Aug } A = \left[\begin{array}{cc|c} 1 & 4 & x_1 \\ -1 & 5 & x_2 \end{array}\right] \text{ --- (i)}$$

$$\text{Aug } A = \left[\begin{array}{cc|c} 1 & 4 & y_1 \\ -1 & 5 & y_2 \end{array}\right] \text{ --- (ii)}$$

e. Find the Parametric equations for A. Invariant lines

$A\begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix}$

$y = mx.$

$x = t$

$y = mt$

$$\begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} t \\ mt \end{bmatrix} = \begin{pmatrix} t + 4mt \\ -t + 5mt \end{pmatrix}$$

$x = t + 4mt = t(1 + 4m)$

$\cancel{x} = -t + 5mt = t(-1 + 5m)$

$\frac{x}{1 + 4m} = t$ and $\frac{\cancel{x}}{-1 + 5m} = t$

$$\frac{x}{1 + 4m} = \frac{x}{-1 + 5m}$$

$$\frac{1 + 4m}{1} = \frac{1}{-1 + 5m}$$

$(1 + 4m)(-1 + 5m)$

$= -1 + 5m - 4m + 20m^2$

$20m^2 + m - 1$

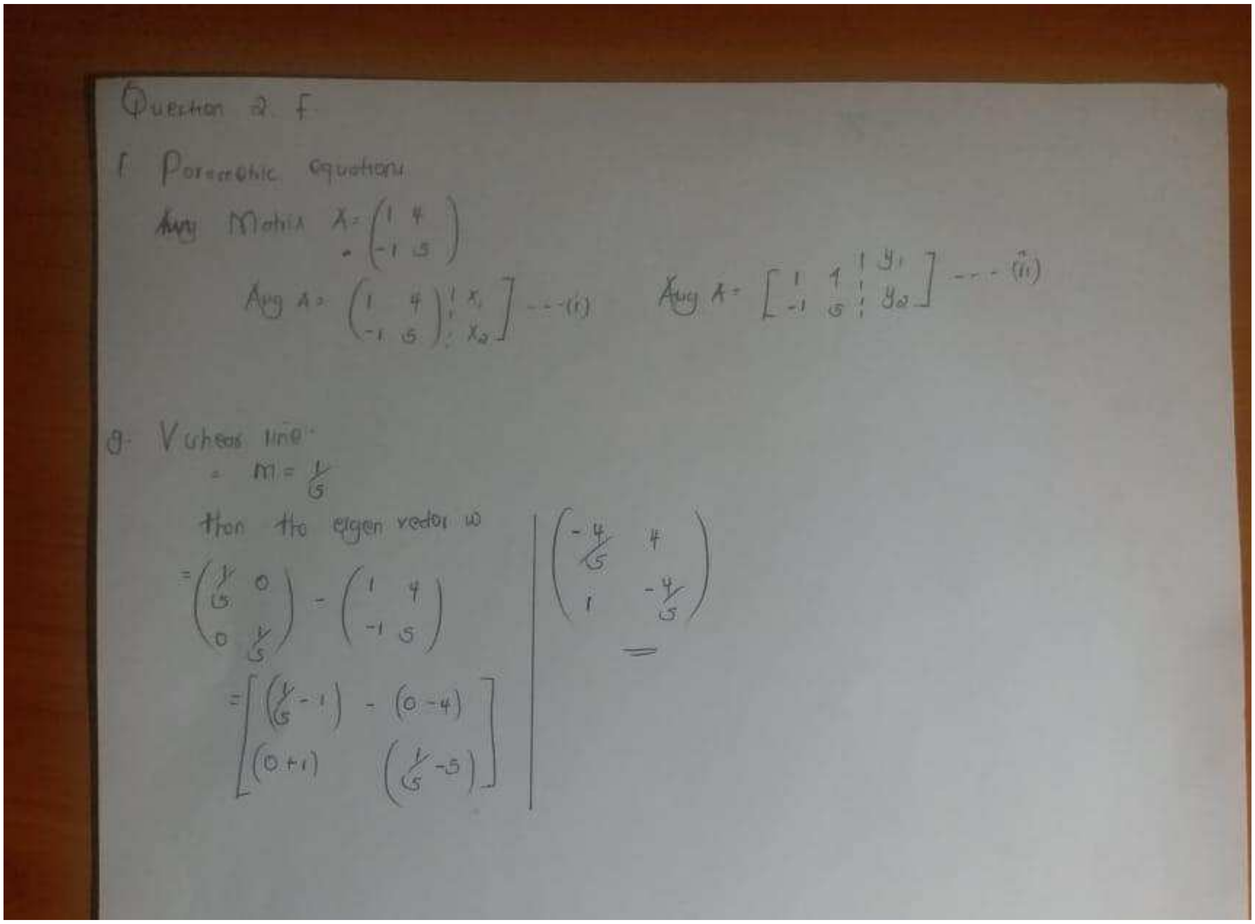
$20m^2 + 5m - 4m - 1$

$5m(4m + 1) - 1(4m + 1)$

$(5m - 1)(4m + 1)$

$m = \frac{1}{5} \quad m = \frac{-1}{4}$

Parametric Equations



Question 2 f

f. Paremetric equations

Aug Matrix $X = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix}$

$\text{Aug } A = \left[\begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}\right]$ ---(i) $\text{Aug } A = \left[\begin{matrix} 1 & 4 \\ -1 & 5 \end{matrix} \,\middle|\, \begin{matrix} y_1 \\ y_2 \end{matrix}\right]$ ---(ii)

g. Vuheas line

$= m = \frac{1}{5}$

then the eigen vector is

$= \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix}$

$= \begin{bmatrix} (\frac{1}{5} - 1) & -(0 - 4) \\ (0 + 1) & (\frac{1}{5} - 5) \end{bmatrix}$

$\begin{pmatrix} -\frac{4}{5} & 4 \\ 1 & -\frac{4}{5} \end{pmatrix}$

(h) Using complete sentences and geometric words, verbally explain how A transforms the plane.

Answer:

The matrix A transforms the plane by the use of its eigen values.

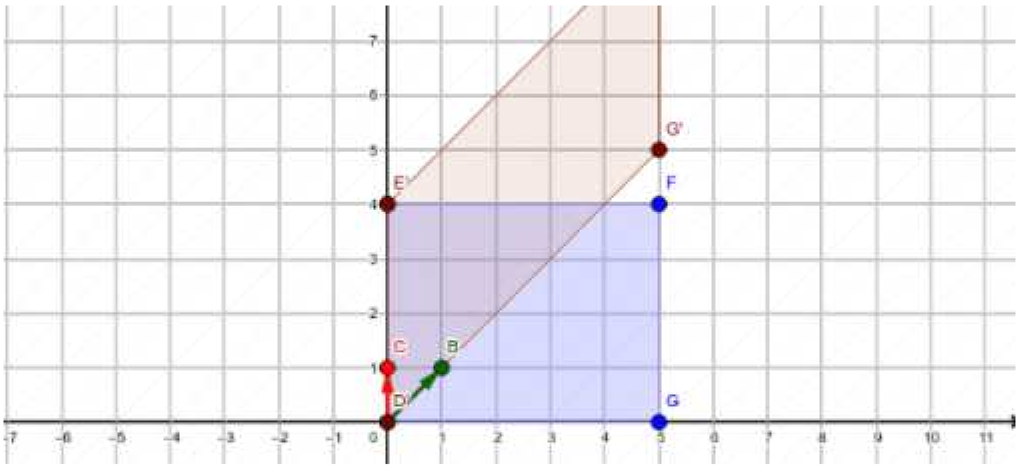
(i) Symbolically express your triangularization of A.

5 -1

4 1

(j) Using GeoGebra, visually represent the B-coordinate grid, the B-unit square, SB, and depict how A transforms SB

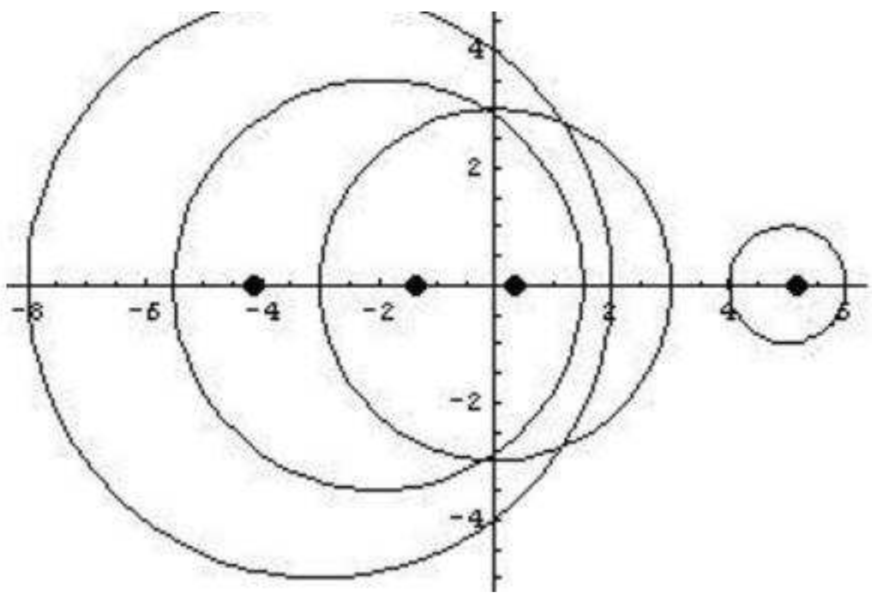
Answer:



Question 3

Consider the following matrix A = 2 −4 2 −2 .

(a) Compute disc(A) and then use this to determine the number of invariant lines A has in the plane



Question 2

b. $A = \begin{pmatrix} 2 & -4 \\ 2 & -2 \end{pmatrix}$

$P_A(\lambda) =$

$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix} = \begin{pmatrix} \lambda - 2 & -4 \\ -2 & \lambda + 2 \end{pmatrix}$

$P_A(\lambda) = \begin{bmatrix} \lambda - 2 & -4 \\ -2 & \lambda + 2 \end{bmatrix}$

c. $\begin{bmatrix} \lambda - 2 & -4 \\ -2 & \lambda + 2 \end{bmatrix} = (\lambda - 2)(\lambda + 2) - (-4 \cdot -2)$

$= \lambda^2 + 2\lambda - 2\lambda - 4 - 8$

$\lambda^2 - 12$

$\frac{b^2 \pm 4ac}{b^2} = \frac{\sqrt{0^2 \pm 4 \times 1 \times 12}}{2} = \sqrt{48}$

$= \sqrt{3 \times 16}$

$\sqrt{3} \times \sqrt{16}$

$= \pm 4\sqrt{3}$

$\lambda_1 = -4\sqrt{3}$

$\lambda_2 = 4\sqrt{3}$

d. $A = \begin{pmatrix} 2 & -4 \\ 2 & -2 \end{pmatrix}$

Augmented Matrix

$\text{Aug } A = \left[\begin{array}{cc|c} 2 & -4 & x_1 \\ 2 & -2 & x_2 \end{array}\right]$

$\text{Aug } A = \left[\begin{array}{cc|c} 2 & -4 & y_1 \\ 2 & -2 & y_2 \end{array}\right]$

e. Find the Parametric equations for the A - Invariant

$\lambda \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix} \begin{pmatrix} t \\ mt \end{pmatrix}$

$y = mx$

$x = t$

$y = mt$

$x = 2t - 4mt$

$y = 2t - 2mt$

$x = t(2 - 4m)$

$y = t(2 - 2m)$

$\frac{x}{2 - 4m} = t$

$\frac{y}{2 - 2m} = t$

$\frac{x}{2 - 4m} = \frac{x}{2 - 2m}$

$\frac{2 - 4m}{x} = \frac{x}{2 - 2m}$

$(2 - 4m)(2 - 2m)$

$4 - 4m - 8m + 8m^2$

$8m^2 - 12m + 4$

$8m^2 - 8m - 4m + 4$

$8m(m - 1) - 4(m - 1)$

$(8m - 4)(m - 1)$

$8m = 4$

$m = \frac{4}{8} = \frac{1}{2}$

$m = \frac{1}{2}$

$m = 1$

(f) Use v to find an ordered rotation basis B.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

(g) Using complete sentences and geometric words, verbally explain how A transforms the plane.

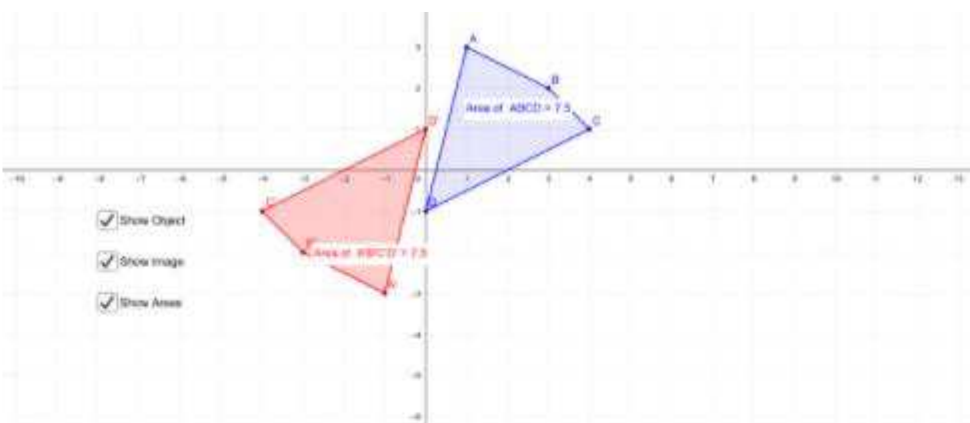
The matrix transforms the plane by rotation.

h) Symbolically express your rotationalization of A.

-2 2

-4 2

i) Using GeoGebra, visually represent the B-coordinate grid, the B-unit square, SB, and depict how A transforms S



(4) (Application: Mixing populations) In a certain region, about 20% of a city's population moves to the surrounding suburbs each year, and about 10% of the suburban population moves into the city. Suppose there are 300,000 people living in the region. Is

there an initial (city to suburb) ratio of the population such that there is no net change from year to year? Furthermore, is this a stable or unstable fixed value? Clearly explain your reasoning in complete sentences, showing computations where appropriate and attach clearly labeled GeoGebra images where appropriate.

Answer:

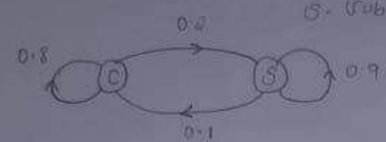
Question 4

City to Suburb migration = 20% → Percentage of City population remained in city = 80%

Suburbs to city migration = 10%

Percentage of Suburb population remained with suburbs is 90%

State of market chain are c - city
S - Suburbs



let $X_c(t)$ and $X_s(t)$ are population of city and suburbs in the year

In any year $X_c(t) + X_s(t) = 300{,}000$

$$x_s(t+1) = 0.9\, x_s(t) + 0.2\, x_c(t)$$

$$X_c(t+1) = 0.1\, x_s(t) + 0.8\, x_c(t)$$

$$[x_c(t+1) \;\; X_s(t+1)] = [x_c(t) \;\; x_s(t)] \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}$$

Suppose there exits an initial distribution $[X_c \;\; X_s]$ Such that there is no change from year to year.

$$[X_c \;\; X_s] = [X_c \;\; X_s] \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}$$

$$\Rightarrow X_c = 0.8 X_c + 0.1 X_s \rightarrow 2X_c = X_s$$

$$X_c + X_s = 300{,}000 \Rightarrow 3X_c = 300{,}000$$

$$X_c = 100{,}000 \quad \text{and} \quad X_s = 200{,}000$$

City population = 100,000

Suburbs population = 200,000

This is fixed stable point

(5) (Exploration: Coming full circle back to the Fibonacci sequence.) Recall that the matrix F = 0 1 1 1 is related to the Fibonacci sequence. (i.e. if v = 0 1 then the bottom terms of the vector F nv generate the sequence.) (a) Compute the eigenvalues of F. (b) One of these eigenvalues is very special and related to symmetry. Research and figure out which one of the eigenvalues is special, and what is special about it. (c) Write an expository paragraph discussing why you think this number should be related to the Fibonacci sequence

Answer:

Question 5.

Given that Matrix $F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is related to Fibonacci sequence.

$A\vec{v} = \lambda\vec{v}$ for non zero $\vec{v}$ if and only if:

$$\det(\lambda I_n - \lambda) = 0$$

Eigen values $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

To find

a. eigen Values of F $\det(\lambda I_n - F) = 0$

$$\det \left[\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right] = 0$$

det

$$\begin{bmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{bmatrix} = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = 1.61$$

wield Condition was not true.

Now taking that $A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$$

$$\lambda = 1$$

b. One of these eigen values is very special to related the symmetry so $\lambda = 1 \Rightarrow f$

If $\det(A) = 0$ Then A Condition is Invertible.

If $\det(A) \neq 0$ A Condition is Invertible.

Therefore f is invertible

C)
Write an expository paragraph discussing why you think this number should be related to the Fibonacci sequence.
Answer:
Recall that the matrix F= [0 1 1 1] is related to the Fibonacci sequence. (In particular, if v = [0 1] then the bottom terms of the vector Fnv generate the sequence.)