Linear Algebra- Homework 8

Question 1

1. Let A =$\begin{bmatrix} 7 & 5 \\ 2 & 4 \end{bmatrix}$

   a). Elementary matrix decomposition of 2 by 2 matrix A

   A = $\begin{bmatrix} 7 & 5 \\ 2 & 4 \end{bmatrix}$

   det(A)= 7det ($[4]$)- 5det ($[2]$)
   = (7$\times$ 4)- (5$\times$ 2)
   =28-10
   =18

   b). Area scaling factor of A
   det(A) = Area scaling factor
   18= Area scaling factor of A
   A.S.F= 18

   c).The sign of A
   det(A) =Sign A $\times$ A.S.F(A)
   18 = Sign A $\times$ 18
   Sign A =18/18 = 1

Question 2

2. Matrix A=$\begin{bmatrix} 2 & 1 & 4 \\ -2 & 0 & -1 \\ -4 & -5 & -12 \end{bmatrix}$

   a). Lu decomposition of A

   $\begin{bmatrix} 2 & 1 & 4 \\ -2 & 0 & -1 \\ -4 & -5 & -12 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 3 \\ 0 & -3 & -4 \end{bmatrix} \longrightarrow$ Add 3(row 2) and row 3 $\begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

   Add row 1
   Add row 2
   Add row 2(row 1) and row 3

   U = $\begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$, L = $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix}$

   A=LU

Therefore, $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ -2 & 0 & -1 \\ -4 & -5 & -12 \end{bmatrix}$

b). The volume scaling factor of A

Volume scaling factor = Determinant of a matrix

det A $\begin{bmatrix} 2 & 1 & 4 \\ -2 & 0 & -1 \\ -4 & -5 & -12 \end{bmatrix}$

det A= $\begin{bmatrix} 2 & 1 & 4 \\ -2 & 0 & -1 \\ -4 & -5 & -12 \end{bmatrix}\begin{matrix} 2 & 1 \\ -2 & 0 \\ -4 & -5 \end{matrix}$

$-$ 0+4+40= 44

0+10+24= 34

10

det A= 10

Volume scaling factor = 10

c). Sign of A

det(A) = Sign (A). v.s.f(A)

10 =Sign (A). 10

Sign A= 10/10=1

Question 3

Det(A) = Sign (A) v.s.f (A)

If A = P B P^{-1}

Then det(A) = det(P)det(B)det(P^{-1})

det(A)= Sign(A) v.s.f(A)

Sign (A) v.s.f (A)= $\cancel{\det}$(P)det(B)$\cancel{\det}$(P^{-1})

Sign (A) v.s.f (A)= det(B)

Hence det(A)=det(B)

a). Algebraic property of sign

Sign (A) is the square root of an identity matrix, its not equal to 1 or -1 unless the spectrum of A lies entirely in the open righthalf plane or open left half plane.

b). Geometric meaning of det(A)=det(B) in the context of 2 cordinates.

If $A = P B P^{-1}$ where p is an invertible matrix with columns.

$V_1, V_2, \ldots, Vn$ Let $B = \{ V_1, V_2, \ldots, Vn\}$

a basis for R^n . Let x be a vector in R^n

1. Multiply x by P^{-1} which changes to the B- cordinates

$$[x]_B = P^{-1} x$$

2. Multiply by B

$$B[x]_B = BP^{-1}x$$

3. Interpreting this vector as a B- coordinate vector , we multiply it by P to change back to the usual cordinate.

   $A_X = PBC^{-1} x = PB\,[x]_B$

Question 4

a) Determinant of the given matrix

$$A = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -7 & 1 \end{bmatrix}\begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (1-0)\,(5-0)(0-1)(-2)(1)(1)(1)(3)$$

$$= 1 \times 5 \times 0$$

$$= 0$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 1 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= (1 \times 1 \times 1) - (5 \times 1 \times 3)$$

$$= 1 - 15 = -14$$

$\det B$ =Product of leading diagonal $(1 \times 1 \times 1)$-product of leading diagonal $(5 \times 1 \times 3)$=-14

$$C = \begin{bmatrix} 3 & 7 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 7 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 7 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 7 \end{bmatrix}^{-1}$$

$$\det \begin{bmatrix} 3 & 7 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 7 \end{bmatrix} \begin{matrix} 3 & 7 \\ 4 & 5 \\ 1 & 2 \end{matrix}$$

$$= (5 + 12 + 196) - (105 + 14 + 8) = 86$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{matrix}$$

$$= 0 + 6 = 6$$

$$Det = (86)^{-1} = \frac{1}{86}$$

Matrix C

Det (C)=$86 \times \frac{1}{86} \times 6 = 6$

Question 5

5. A=$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

a) Lu- Decomposition

If =Lu then det (A) = det (Lu)

Therefore: The Lu decomposition of A

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cong \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -3 & -6 \end{bmatrix} \cong \begin{bmatrix} 1 & 2 & 3 \\ 4 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

L = $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 7 & 1 & 1 \end{bmatrix}$

det A =det (L) det (U)

= (1×1×1) 1(-3×0) = 0

det A = 0

b) Rules of Sarrus

A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

1. Multiply the numbers in leading diagonal $1\times 5 \times 9$= 45
2. Multiply the numbers in the first triangle whose vertices don't lie on the diagonal $3\times 4 \times 8 = 96$.
3. Multiply the numbers in the second triangle whose vertices don't touch the diagonal. $7\times 2 \times 6 = 84$.

   Adding the three = Sum of diagonal + First triangle + 2nd triangle.

   = 45+96+84=225

   Product of numbers in the second diagonal $7\times 5 \times 3 = 105$.

4. Sum of the first triangle: $1\times 8 \times 6 = 48$.

5. Sum of the second triangle $4 \times 2 \times 9$ =72.

   Adding the three = Sum of diagonal of first triangle + Second triangle
   105+48+12=225
   det A = 225-225=0

C. Cofactor Expansion.

   It applies the formula:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \text{a.det} \begin{bmatrix} e & f \\ h & i \end{bmatrix} - \text{b.det} \begin{bmatrix} d & f \\ g & i \end{bmatrix} + \text{c.det} \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \text{1.det} \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} \text{- 2.det} \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + \text{3.det} \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

= 1.(45-48) – 2(36-42) + 3(32-35)

= -3+12+ -9
= -12+12=0

D. Geo Gebra

Given that A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Using GeoGebra,

$$\text{det (A)} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{matrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{matrix}$$

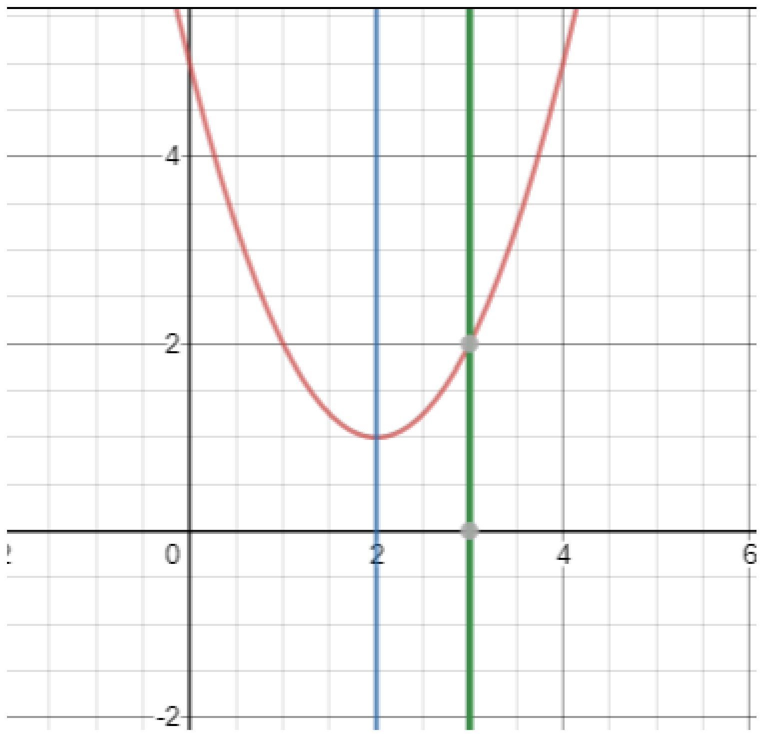
Multiply the three diagonals: $1 \times 5 \times 9 = 45$

$2\times 6 \times 7 = 84$
$3\times 4 \times 8 = \underline{96}$
225
$7\times 5 \times 3 = 105$
$2\times 6 \times 1 = 48$
$9\times 4 \times 2 = \underline{72}$
225

det(A) = 225-225=0

Question 6

6. a) Plotting the parabola $P(x) = x^2 - tx + d$



b) 2 real roots of parabola, one real root and no real root. The roots of a parabola are the x-intercept. By definition, the y-co-ordinate of points lying on x-axis is zero.
Therefore

$$0 = x^2 - tx + d$$

Using the completing square method

$$x^2 - tx + d = 0$$
$$x^2 - tx = -d$$
$$x^2 - t + \left(\frac{-t}{2}\right)^2 tx = -d + \left(\frac{-t}{2}\right)^2$$
$$\sqrt{\left(x - \frac{t}{2}\right)^2} = \sqrt{-d - \frac{t^2}{4}}$$

c) $\left(x - \frac{t}{2}\right) = \pm\sqrt{-d - \frac{t^2}{4}}$

$$x = \frac{t}{2} \pm \sqrt{-d - \frac{t^2}{4}}$$

$$x = \frac{t}{2} \pm \sqrt{-d - \frac{t^2}{4}}$$

If $-d + t^2 > 0$ then the parabola has two real roots
If $-d + t^2 < 0$ then the parabola has no real roots
If $-d + t^2 = 0$ then the parabola has one real root

d) $P_1(x) = x^2 - 4x + 5$

$$b^2 - 4ac = (-4)^2 - 4(1)(5)$$

$$= +16 - 20 = -4$$
$$-4 < 0$$

Therefore, the parabola has no real roots

$$P_2(x) = x^2 - 4x + 4$$

$$b^2 - 4ac = (-4)^2 - 4(1)(4)$$

$$= +16 - 16 = 0$$
$$= 0$$

$b^2 - 4ac = 0$ therefore, the parabola has one real root.

$$P_3(x) = x^2 - 6x + 1$$

$$b^2 - 4ac = (-6)^2 - 4(1)(1)$$

$$= +36 - 4 = 32$$

Type equation here.

$b^2 - 4ac > 0$ therefore, the parabola has two real roots

QUESTION 7

Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

To verify $P_A(\lambda) = \det(A - \lambda I)$

Then since $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$P_A(\lambda) = \det(A - \lambda I)$$
$$= Det\left(\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\right)$$

$$= Det\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right]$$
$$= Det\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}$$

$$= (a - \lambda)(d - \lambda) - cd$$

$$= ad - a\lambda - d\lambda + \lambda^2 - cd$$
$$P_A(\lambda) = \lambda^2 - \lambda(a - \lambda) + ad - cd$$
$$a + d = tr(A)$$
$$ad - cd = \det A$$

Substitute

$$P_A(\lambda) = \lambda^2 - tr(A)\lambda + \det(A)$$

Question 8

Suppose A and B are similar 2 * 2 matrices i.e. $A = P\,B\,P^{-1}$

a) tr(A)=tr(B)

If the two matrices are similar, then they have the same

Let $A = P\,B\,P^{-1}$ then by an elementary property of the trace, we have

tr(A)=tr ($P\,B\,P^{-1}$)

=tr (($P\,B$) P^{-1})

=tr (B ($P\,P^{-1}$))

=tr (B I) then tr(A)=tr(B)

b) det(A)=det(B)

Suppose P is a non-singular matrix then

det (A) = det ($P\,B\,P^{-1}$)

= det (P) det (B) det (P^{-1})

By multiplicative properties of determinants then det (P^{-1})=$\frac{1}{\det P}$

=~~Det (P)~~ det (B) $\frac{1}{\cancel{\det P}}$

Det (A)=det (B)

c) $P_A(\lambda) = P_B(\lambda)$

Suppose A and B are similar, then A = P B P^{-1} where P is invertible

Let $P_A(\lambda)$ and $P_B(\lambda)$ denote the characteristic polynomials of A and B, then

$$P_A(\lambda) = \det(A - \lambda I) = \det(PBP^{-1} - \lambda I)$$

$$= \det(P(B - \lambda I)P^{-1})$$

$$= \det(P)\ det(B - \lambda I) \det\ P^{-1}$$

By multiplication properties of determinants then

$$P_A(\lambda) = \det(A - \lambda I) = \det(PBP^{-1} - \lambda I)$$

$$= P_B(\lambda)$$

Thus $P_A(\lambda) = P_B(\lambda)$