

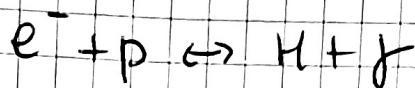
Phys 600 HW 4 Josephine Baggen

Recombination

1) Derive:

$$\frac{1-x_e}{x_e^2} = \frac{2\sqrt{3}}{\pi^2} n \left(\frac{2\pi T}{me} \right)^{3/2} e^{-EI/T}$$

recombination



In equilibrium the abundances for e, p, H are

** I use over lecture notes with constants from notes*

$$n_{eq} = g \left(\frac{m_H k_B T}{2\pi h^2} \right)^{3/2} \exp \left(-\frac{(mc^2 - EI)}{k_B T} \right)$$

Because $N_f = 0$ we have $N_e + N_p = N_H$

So this will fall out of the exponent when we take the ratios:

$$\left(\frac{n_H}{n_e n_p} \right)_{eq} = \left(\frac{g_H}{g_e g_p} \right) \left(\frac{m_H}{m_e m_p} \right)^{3/2} \left(\frac{k_B T}{2\pi h^2} \right)^{-3/2} \exp \left(-\frac{(m_H + m_p + m_e)c^2}{k_B T} \right)$$

Simplifying $g_H = 2 \times 2 = 4$

$$g_e = 2$$

$$g_p = 2$$

$$m_H \approx m_p$$

$$\text{call } EI = (m_p + m_e - m_H)c^2 = 13.6 \text{ eV}$$

$$\left(\frac{n_H}{n_e n_p} \right)_{eq} = \left(\frac{2\pi h^2}{k_B T me} \right)^{3/2} \exp \left(\frac{EI}{k_B T} \right)$$

$$\text{Call } X_e = \frac{n_e}{n_b}$$

fraction of free electrons

$$n_b = n_H + n_p \quad (\text{neglecting He})$$

$$n_e = n_p = n_b X_e$$

$$n_H = n_b - n_e = n_b(1 - X_e)$$

$$\text{so we have } n_H = n_b(1 - x_e)$$

$$n_e = n_p = n_b x_e$$

Plugging this in:

$$\left(\frac{n_H}{n_p n_e}\right)_{\text{eq}} = \left(\frac{2\pi\hbar^2}{k_B T m_e}\right)^{3/2} \exp\left(\frac{EI}{k_B T}\right)$$

$$\frac{1-x_e}{x_e^2} = n_b \left(\frac{2\pi\hbar^2}{k_B T m_e}\right)^{3/2} \exp\left(\frac{EI}{k_B T}\right)$$

~~Relating number density of baryons to photons.~~

$$n_b = \eta n_f = \eta \frac{2 J(3)}{\pi^2 (\hbar c)^3} (k_B T)^3$$

plugging this in gives

$$\frac{1-x_e}{x_e^2} = \eta \frac{2 J(3)}{\pi^2 (\hbar c)^3} (k_B T)^3 \left(\frac{2\pi\hbar^2}{m_e}\right)^{3/2} (k_B T)^{-3/2} \exp\left(\frac{EI}{k_B T}\right)$$

Setting constants to 1 ~~$\hbar=c=k_B=1$~~ to get the expression:

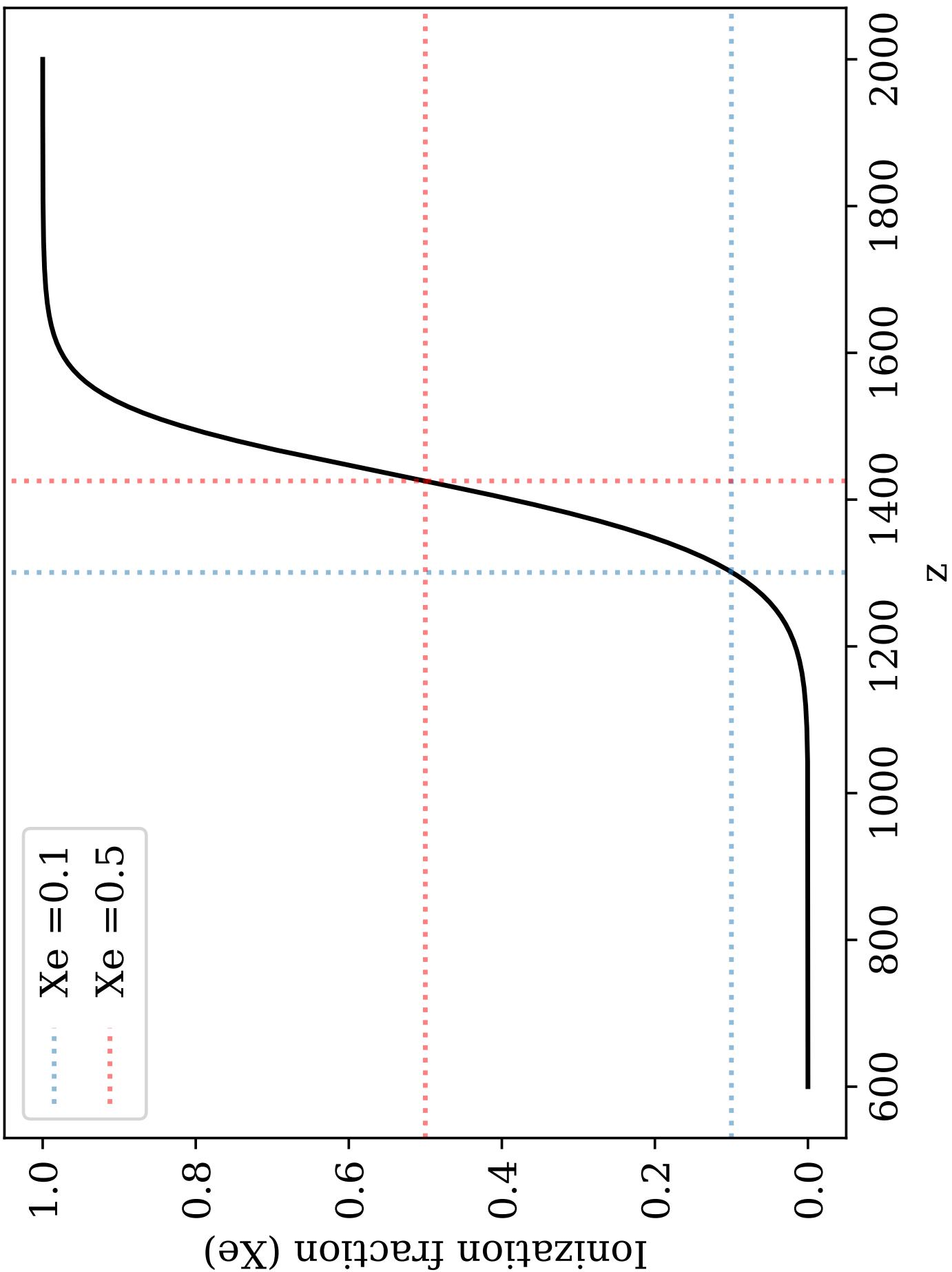
$$\frac{1-x_e}{x_e^2} = \eta \frac{2 J(3)}{\pi^2} \left(\frac{2\pi T}{m_e}\right)^{3/2} \exp\left(\frac{EI}{T}\right)$$

2) Solve for x_e as a function of z

$$a \propto \frac{1}{T} \quad 1+z \propto T$$

$$\frac{1+z}{1} = \frac{T}{T_0} \quad T_0 = 2.73 \text{ K} \\ = 0.23 \text{ meV}$$

$$T = T_0 (1+z)$$



```

import numpy as np
import matplotlib.pyplot as plt
#from ccdproc import wcs_project
import matplotlib
matplotlib.use('TKAgg')
import sys
import os
import io
import math
import scipy
import astropy.io.fits as fits
import astropy.units as u
import astropy.coordinates as coord
import pandas as pd
import matplotlib.lines as mlines
from astropy.cosmology import Planck13, z_at_value
from astropy.coordinates import SkyCoord
from astropy.coordinates import SkyCoord
from astropy.cosmology import FlatLambdaCDM
from astropy import constants as const
from scipy.integrate import quad, dblquad
from scipy.special import zeta

plt.rcParams["font.family"] = "serif"
plt.rcParams["mathtext.fontset"] = "dejavuserif"
path ='/Users/josephine/Desktop/Yale_2/COSM/'

#(1-Xe)/Xe^2 = 2 zeta(3)/pi^2 eta (2piT/me)^3/2 e^E_I/T

def calculate_left_side(z):
    return 1e-9 * 2 *zeta(3)/np.pi**2 * (2*np.pi *
0.23e-3*(1+z)/511e3)**(3/2) * np.exp(13.6/(0.23e-3*(1+z)))

def calc_Xe(C):
    return (-1 + np.sqrt(1+4*C))/(2*C)

def find_nearest(array, value):
    array = np.asarray(array)
    idx = (np.abs(array - value)).argmin()
    return idx

z_range = np.linspace(600,2000,1000)
Xe = np.zeros(len(z_range))

for i in range(len(z_range)):
    outcome = calculate_left_side(z_range[i])
    Xe[i] = calc_Xe(outcome)

index = find_nearest(Xe, 0.1)
z_01 = z_range[index]
index = find_nearest(Xe, 0.5)
z_05 = z_range[index]

```

```

print('z=', z_01)
print('z=', z_05)
plt.plot(z_range, Xe, color='black')
plt.axhline(0.1, ls='dotted', alpha=0.5)
plt.axvline(z_01, ls='dotted', alpha=0.5, label='Xe =0.1')
plt.axhline(0.5, ls='dotted', alpha=0.5, color='red')
plt.axvline(z_05, ls='dotted', alpha=0.5, label='Xe =0.5',
color='red')
plt.xlabel('z', fontsize=12)
plt.ylabel('Ionization fraction (Xe)', fontsize=12)
plt.legend(fontsize=12)
plt.xticks(fontsize=12)
plt.yticks(fontsize=12)
plt.savefig(path+'Ionization_fraction_redshift.pdf')
plt.show()

def calculate_left_side2(T):
    return 1e-9 * 2 *zeta(3)/np.pi**2 * (2*np.pi * T/
511e3)**(3/2) * np.exp(13.6/(T))

def calc_Xe2(C):
    return (-1 + np.sqrt(1+4*C))/(2*C)

#For kB T =0.01 the exponent blows up, I need to start at 0.1 ev.
#Trying out different ranges, I see that the answer converges to
~0.3 eV so that is why I made my array specific for 0.1-0.5.
kBT_range = np.linspace(0.1,0.5,1000) #(eV)
left_side = np.zeros(len(kBT_range))
Xes = np.zeros(len(kBT_range))
for i in range(len(kBT_range)):
    outcome = calculate_left_side2(kBT_range[i])
    Xes[i] = calc_Xe2(outcome)
    left_side[i] = (calc_Xe2(outcome) * kBT_range[i]**(3) *
(kBT_range[i]/0.23e-3)**(-3/2))

index = find_nearest(left_side, (1.9e-9))
Tdec = (kBT_range[index])
Xe_at_Tdec = Xes[index]
print('Tdec (eV)', Tdec)
print('Xe(Tdec)', Xe_at_Tdec)

```

$$\frac{1-x_e}{x_e^2} = \eta \cdot 2 \cdot \frac{\Im(3)}{\pi^2} \left(\frac{2\pi T_0(1+z)}{m_e} \right)^{3/2} \exp\left(\frac{E_I}{T_0(1+z)}\right)$$

Plugging this in to python with

$$T_0 = 0.23 \text{ meV}$$

$$m_e = 510 \text{ keV}$$

$$E_I = 13.6 \text{ eV}$$

$$\eta = 10^{-9}$$

call it C

$$C x_e^2 = 1 - x_e$$

$$C x_e^2 + x_e - 1 = 0$$

ABC formula

$$a = C, b = 1, c = -1$$

$$x_e = \frac{-b \pm \sqrt{D}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1^2 - 4 \cdot C \cdot -1}}{2C}$$

$$= \frac{-1 \pm \sqrt{1 + 4C}}{2C}$$

only + sign here
means something
(positive x_e)

In python gives plot (next page)

- 3) For $x_e = 0.1$ I find $z \approx 1300$
for $x_e = 0.5$ I find $z \approx 1425$

4) Age of the Universe at this epoch
(matter dominated)

$$z = 1300$$

$$z = 1425$$

$$\begin{aligned} a &\propto t^{2/3} \\ 1+z &\propto t^{-2/3} \\ (1+z)^{3/2} &\propto t \end{aligned}$$

$$t = t_0 (1+z)^{-3/2} \quad t_0 \approx \cancel{13.8} \text{ Gyr}$$

$$\text{At } z = 1300$$

$$t = 13.8 \text{ Gyr} (1+1300)^{-3/2} = 2.9 \cdot 10^{-4} \text{ Gyr}$$

$$= 2.9 \cdot 10^5 \text{ yr}$$

$$\text{At } z = 1425$$

$$t = 13.8 \text{ Gyr} (1+1425)^{-3/2} = 2.6 \cdot 10^5 \text{ yr}$$

The difference in time between

$x_e = 0.5$ and $x_e = 0.1$ is only ~ 30000 yr,
very short on astronomical timescales.

Recombination was a fast process.

5) Thompson scattering $e^- + f \leftrightarrow e^- + f$

$$\begin{aligned} F &= n_e v \\ &= n_e 0T C \end{aligned}$$

$$n_e = X_e n_b \quad n_b = \eta n_f = \eta \frac{2 J(3)}{\pi^2} T^3$$

or with constants:

$$n_b = \eta \frac{2 J(3)}{\pi^2 (hc)^3} (k_B T)^3$$

$$F_{\text{dec}} = X_e (T_{\text{dec}}) \eta \frac{2 J(3)}{\pi^2 (hc)^3} (k_B T_{\text{dec}})^3 c 0T$$

$$\frac{H^2}{H_0^2} = \frac{\rho_{f,0}}{a^4} + \frac{\rho_{m,0}}{a^3} + \frac{\rho_{h,0}}{a^2} + \rho_{n,0}$$

Assume only matter

$$\frac{H^2}{H_0^2} = \frac{\rho_{m,0}}{a^3} \quad H = H_0 \sqrt{\rho_{m,0}} a^{-3/2}$$

$$a \propto T^{-1}$$

$$H = H_0 \sqrt{\rho_{m,0}} \left(\frac{T}{T_0} \right)^{-3/2}$$

$$H(T_{dec}) = H_0 \sqrt{\rho_{m,0}} \left(\frac{T_{dec}}{T_0} \right)^{-3/2}$$

$T = H$ gives T_{dec}

$$X_e(T_{dec}) \eta \frac{2 J(3)}{\pi^2 (hc)^3} \left(\frac{k_B T_{dec}}{k_B T_0} \right)^{-3/2} c = H_0 \sqrt{\rho_{m,0}} \left(\frac{T_{dec}}{T_0} \right)^{-3/2}$$

$$X_e(T_{dec}) \left(\frac{k_B T_{dec}}{k_B T_0} \right)^3 \left(\frac{H_0 \sqrt{\rho_{m,0}} \pi^2 (hc)^3}{\eta 2 J(3) c} \right)^{-3/2}$$

= number

$$\text{number units: } \frac{s^{-1} \cdot (eV s \cdot m s^{-1})^3}{m s^{-1} m^2} = eV^3$$

that is exactly what I want, so we solve for
 $k_B T_{dec}$ in eV

$$\text{number} = \frac{2 \cdot 2 \cdot 10^{-10} \text{ s}^{-1} \sqrt{0.3} \pi^2 \left(\frac{4.136 \cdot 10^{15} \text{ eV s} \cdot 2.99 \cdot 10^8}{2\pi} \right)^3}{10^9 J(3) \cdot 2 \cdot 2.99 \cdot 10^8 \text{ m s}^{-1} \cdot 6.65 \cdot 10^{-29} \text{ m}^2}$$

$$= \frac{1.07 \cdot 10^{-7}}{1.09 \cdot 10^{-9}} (\text{eV})^3$$

So we have

$$X_e(T_{\text{dec}}) (h_B T_{\text{dec}})^3 \left(\frac{h_B T_{\text{dec}}}{0.23 \cdot 10^{-3}} \right)^{-3/2} = 1.07 \cdot 10^{-9} \text{ eV}^3$$

This is an array in python for T_{dec} range

Find for which T_{dec} this equation holds

I do this with python and I find

$$h_B T_{\text{dec}} \approx 0.26 \text{ eV}$$

$$a \propto \frac{1}{T}$$

$$1+z \propto \frac{1}{T}$$

$$1+z \propto \frac{T_{\text{dec}}}{T_0}$$

$$1+z = \frac{0.26 \text{ eV}}{0.23 \cdot 10^{-3} \text{ eV}} = 1100 \quad (\text{as expected} \sim \text{CMB!})$$

Also this justified our choice for assuming matter dominated, it is $\ll z_m$ equaling



Het Baarnsch Lyceum

Naam: _____

Vak: _____

Datum: _____ Klas: _____



Age of Universe

$$t = t_0 (1+z)^{-3/2}$$

$$= 13.8 \text{ Gyr} (1+1100)^{-3/2}$$

$$= 320 \cdot 10^5 \text{ yr}$$

3.77

(or by hand)

Xe at this z, Tdec with python works too)

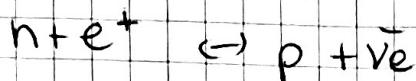
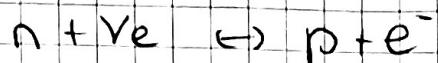
$$Xe(T_{dec}) = 0.004$$

(see code next page)

what-if BBN

$$\bullet \quad X_n = \frac{n_n}{n_n + n_p}$$

neutrons/protons coupled through



Again we have

$$n_{eq} = g \left(\frac{m_{UBT}}{2\pi\hbar^2} \right)^{3/2} \exp \left(- \frac{(mc^2 - \mu)}{k_B T} \right)$$

$$\frac{n_n}{n_p} = \frac{g_n}{g_p} \left(\frac{m_n}{m_p} \right)^{3/2} \exp \left(- \frac{(m_n + m_p) \mu}{k_B T} \right)$$

$$n_n \approx n_p$$

because

$\bar{v}e, ve$ very small

$$Q = (m_n - m_p)c^2 = 1.3 \text{ MeV}$$

$$g_n = g_p = 2 \quad m_n \approx m_p \quad \text{so}$$

$$\frac{n_n}{n_p} = \exp \left(- \frac{Q}{k_B T} \right) \quad (\text{neglect } k_B)$$

$$= \exp \left(- \frac{Q}{T} \right)$$

$$X_n = \frac{n_n}{n_n + n_p} = \frac{1}{1 + \frac{n_p}{n_n}} = \frac{1}{1 + \exp(Q/T)}$$

$$= \frac{\exp(-Q/T)}{\exp(-Q/T) + 1}$$

so we have our wanted expression



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- neutrinos decouple $T < H$

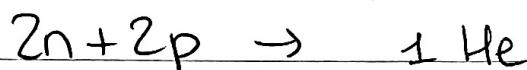
so the neutron-proton ratio freezes out

$X_n = \frac{n_n}{n_n + n_p}$ the fraction of neutrons of both that are

at $T = 0.8 \text{ MeV}$ $Q = 1.3 \text{ MeV}$

$$X_n = \frac{e^{-1.3/0.8}}{1 + e^{-1.3/0.8}}$$

$$= 0.165 \quad (16\% \text{ of nucleons are neutrons})$$



If we convert all neutrons to Helium nuclei, we have $n_{\text{He}} = \frac{1}{2} n_n$ because we need 2 neutrons for 1 He atom

$$Y_p = \frac{4 n_{\text{He}}}{n_H} = 2 \frac{n_n}{n_H} \quad \cancel{\text{not possible}} \quad \cancel{0.8 \text{ MeV}}$$

$$n_H = n_n + n_p \quad (\text{approx})$$

$$Y_p = \frac{2 n_n}{n_n + n_p} = \frac{2 X_n}{1 - X_n} = 0.33$$

(Actually some of the neutrons decayed before they could form He)

- Mass difference $Q = 2.6 \text{ MeV}$

$$X_n = \frac{e^{-2.6/0.8}}{1 + e^{-2.6/0.8}} = 0.04 \text{ much less neutrons!}$$

Then

$$Y = 2X_n = 0.08 \text{ (still not taking into account decay)}$$

So it is "lucky" that Q is smaller, otherwise there would not be that much helium or other higher elements.