I Growth of matter perturbations
Matter and radiation

$$\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G\bar{\rho}_m \delta_m$$

$\downarrow$ show

$$\frac{d}{da}\left(a^3 H \frac{d\delta_m}{da}\right) = 4\pi G\bar{\rho}_{m,0}\frac{\delta_m}{Ha^2}$$

Friedmann:

$$H^2 = H_0^2\left(\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}\right)$$

$$H^2 = \frac{8\pi G}{3}\rho \qquad \rho = \rho_r + \rho_m$$

$H = \dot{a}/a$

$\bar{\rho}_m = \bar{\rho}_{m,0}\, a^{-3}$ (matter density $\propto a^{-3}$)

$\rho_r \propto a^{-4}$

$$\frac{dx}{dt} = \frac{dx}{da}\frac{da}{dt} = \frac{dx}{da}\dot{a} = Ha\frac{dx}{da}$$

$$\ddot{\delta}_m = \frac{d^2}{dt^2}\delta_m = \frac{d}{dt}\left(Ha\frac{d}{da}\delta_m\right)$$

$$= Ha\frac{d}{da}\left(Ha\frac{d\delta_m}{da}\right)$$

$$2H\dot{\delta}_m = 2H\frac{d}{dt}\delta_m = 2H^2 a\frac{d\delta_m}{da}$$

$$\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G \bar{\rho}_m \delta_m$$

$$Ha\frac{d}{da}\left(Ha\frac{d\delta_m}{da}\right) + 2H^2 a\frac{d\delta_m}{da} = 4\pi G\frac{\bar{\rho}_{m,0}}{a^3}\delta_m$$

$$\underbrace{a^2\frac{d}{da}\left(Ha\frac{d\delta_m}{da}\right) + 2Ha^2\frac{d\delta_m}{da}} = \frac{4\pi G\bar{\rho}_{m,0}\delta_m}{Ha^2}$$

now I'm going to show that this is $\frac{d}{da}\left(a^3 H\frac{d\delta_m}{da}\right)$ to prove the equation

product rule

$$\frac{d}{da}\left(a^3 H\frac{d\delta_m}{da}\right) = \frac{d}{da}\left(a^3 H\right)\frac{d\delta_m}{da} + a^2\frac{d}{da}\left(Ha\frac{d\delta_m}{da}\right)$$

↑ this is my first term above

$$\frac{d}{da}\left(a^3 H\right) = 3a^2 H + a^3\frac{dH}{da}$$

$$= 3a^2\frac{\dot{a}}{a} \cancel{+} - a^3\cdot\frac{\dot{a}}{a^2} = 3a\dot{a} - a\dot{a} = 2a\dot{a}$$

$$= 2a^2 H$$

So indeed I found $\frac{d}{da}\left(a^3 H\frac{d\delta_m}{da}\right)$

$$= a^2\frac{d}{da}\left(Ha\frac{d\delta_m}{da}\right) + 2Ha^2\frac{d}{da}\delta_m = \frac{4\pi G\bar{\rho}_{m,0}\delta_m}{Ha^2}$$

$$\frac{d}{da}\left(a^3 H \frac{d\delta_m}{da}\right) = \overbrace{\frac{d}{da}\left(a^3 H\right)}^{\text{not used}} \frac{d\delta_m}{da} + a^2 \frac{d}{da}\left(Ha \frac{d\delta_m}{da}\right)$$

$$\frac{d}{da}\left(a^3 H\right) = 3a^2 \cancel{\frac{dH}{da}} + a^3 \frac{dH}{da}$$

$$= 3a^2 \frac{\dot a}{a} - a^3 \frac{\dot a}{a^2} = 2a\dot a$$

$$= 2a^2 H$$

$$\frac{dH}{da} = -\frac{\dot a}{a^2}$$

- $y = a/a_{eq} \qquad a = a_{eq}\, y$

$$H(a) = H_0 \sqrt{\left(\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}\right)}$$

$$a_{eq} = \frac{\Omega_{r,0}}{\Omega_{m,0}}$$

$$H(y) = H_0 \sqrt{\frac{\Omega_{r,0}}{a_{eq}^4 y^4} + \frac{\Omega_{m,0}}{a_{eq}^3 y^3}}$$

$$= \frac{H_0}{y^2} \sqrt{\frac{\Omega_{r,0}}{a_{eq}^4} + \frac{\Omega_{m,0}}{a_{eq}^3} y}$$

$$= \frac{H_0}{y^2} \sqrt{\frac{\Omega_{r,0}}{a_{eq}^4}} \sqrt{1 + \underbrace{\frac{\Omega_{m,0}}{a_{eq}^3} \frac{a_{eq}^4}{\Omega_{r,0}}}_{\text{this is 1}} y}$$

$$= \frac{H_0}{y^2} \frac{\sqrt{\Omega_{r,0}}}{a_{eq}^2} \sqrt{1+y}$$

$$\frac{\Omega_{m,0}}{\Omega_{r,0}} = a_{eq}$$

So $A = H_0 \sqrt{\Omega_{r,0}}\; a_{eq}^{-2}$

or $A = H_0 \frac{\sqrt{\Omega_{r,0}}}{\Omega_{r,0}^2} \Omega_{m,0}^2$ $\qquad A = \frac{\sqrt{\rho_{m,0} \frac{8\pi G}{3}}}{a_{eq}^{3/2}}$

$= H_0 \frac{\Omega_{m,0}^2}{\Omega_{r,0}^{3/2}}$

$= H_0 \frac{\Omega_{m,0}^2}{(a_{eq}\,\Omega_{m,0})^{3/2}} = H_0 \frac{\sqrt{\Omega_{m,0}}}{a_{eq}^{3/2}}$

- $\Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{crit,0}} = \frac{\rho_{m,0}\,8\pi G}{3H_0^2}$

$\ddot{\delta}_m$

$$\frac{d}{da}\left(a^3 H \frac{d\delta_m}{da}\right) = 4\pi G \bar{\rho}_{m,0} \frac{\delta_m}{Ha^2}$$

$y = a/a_{eq}$ $\qquad a = a_{eq} \cdot y$

$da = a_{eq}\,dy$

$$\frac{1}{a_{eq}^2}\frac{d}{dy}\left(a_{eq}^3 y^3 H \frac{d\delta_m}{dy}\right) = 4\pi G \bar{\rho}_{m,0} \frac{\delta_m}{H a_{eq}^2 y^2}$$

$$\frac{d}{dy}\left(y^3 H \frac{d\delta_m}{dy}\right) = \frac{d}{dy}\left(y^3 H\right)\frac{d\delta_m}{dy} + y^3 H \frac{d^2}{dy^2}\delta_m$$

$$\frac{d}{dy}\left(y^3 H\right) = 3y^2 H + y^3 \frac{dH}{dy}$$

$$\frac{dH}{dy} = -\frac{1}{2}\frac{A}{y^2}\frac{1}{2\sqrt{1+y}} - \frac{1}{3}\frac{A}{y^3}\sqrt{1+y}$$

$$= -\frac{A}{4y^2\sqrt{1+y}} - \frac{1}{3}\frac{A}{y^3}\sqrt{1+y}$$

$$\underbrace{\frac{d}{dy}\left(y^3 H \frac{d\delta m}{dy}\right)} = 4\pi G \rho_{\bar{m},0} \frac{\delta m}{H a_{eq}^3 y^2}$$

$$= \left(3y^2 H + y^3 \frac{dH}{dy}\right)\frac{d\delta m}{dy} + y^3 H \frac{d^2\delta m}{dy^2}$$

$$= \frac{d^2\delta m}{dy^2} + \left(\frac{3}{y} + \frac{1}{H}\frac{dH}{dy}\right)\frac{d\delta m}{dy}$$

$$= \frac{d^2\delta m}{dy^2} + \underbrace{\left(\frac{3}{y} - \frac{1}{4y^4(1+y)} - \frac{1}{3}\frac{1}{y^2}\right)}\frac{d\delta m}{dy}$$

$$\frac{3y}{y^2} - \frac{\frac{1}{3}}{y^2} - \frac{1}{4y^4(1+y)} = \frac{\left(3y - \frac{1}{3}\right)y^2(1+y)}{y^4(1+y)} - \frac{\frac{1}{4}}{y^4(1+y)}$$

$$= \frac{\left(3y - \frac{1}{3}\right)y^2(1+y) - \frac{1}{4}}{y^4(1+y)} =$$

$$\frac{d}{da}\left(a^3 H \frac{d\delta_m}{da}\right) = 4\pi G \rho_{m,0} \frac{\delta_m}{Ha^2}$$

$$H(y) = \frac{A}{y^2}\sqrt{1+y} \qquad y = a/a_{eq}$$

$$da = a_{eq}\, dy$$

$$\frac{d}{da}\left(a^3 H \frac{d\delta_m}{da}\right) = \frac{d}{da}\left(a^3 H\right)\frac{d\delta_m}{da} + a^3 H \frac{d^2\delta_m}{da}$$

$$= \left(3a^2 H + a^3 \frac{dH}{da}\right)\frac{d\delta_m}{da} + a^3 H \frac{d^2\delta_m}{da}$$

$$= \left(3 a_{eq}^2 y^2 H + a_{eq}^3 y^3 \frac{dH}{dy}\frac{dy}{da}\right)\frac{d\delta_m}{da} + a^3 H \frac{d^2\delta_m}{da}$$

$$\frac{dH}{dy} = \frac{\cancel{a} A y^2 \cdot \left(\cancel{*}\tfrac{1}{2}(1+y)^{-1/2}\right) - 2yA\sqrt{1+y}}{y^4}$$

$$= \frac{\cancel{*}\tfrac{1}{2}A}{y^2\sqrt{1+y}} - \frac{2A\sqrt{1+y}}{y^3}$$

$$= \frac{\cancel{*}\tfrac{1}{2}A\,y\sqrt{1+y} - 2A(1+y)^{3/2}}{y^3\,\cancel{\sqrt{}}\,(1+y)}$$

$$\left[3a_{eq}^2 \cancel{y^2} \frac{A}{\cancel{y^2}}\sqrt{1+y} + a_{eq}^2 \cancel{y^3}\left(\frac{\cancel{*}\tfrac{1}{2}A\,y\sqrt{1+y} - 2A(1+y)^{3/2}}{1+y}\right)\right]$$

$$\cdot\frac{d\delta_m}{da} + a^3 H \frac{d^2\delta_m}{da^2} = 4\pi G \rho_{m,0} \frac{\delta_m}{Ha^2}$$

$$= \frac{a^3 H}{a_{eq}^2}\frac{d^2\delta_m}{dy^2} + \left[3 a_{eq}^2 A\sqrt{1+y} + a_{eq}^2\left(\frac{\frac{1}{2} A y\sqrt{1+y} - 2A(1+y)^{3/2}}{1+y}\right)\right]\frac{d\delta_m}{da} = 4\pi G\rho_{m,0}\frac{\delta_m}{Ha^2}$$

$$\frac{d^2\delta_m}{dy^2} + \left[\frac{3 a_{eq}^3 A\sqrt{1+y}}{a^3 H} + \frac{a_{eq}^3}{a^3 H}\left(\frac{\frac{1}{2} A y\sqrt{1+y} - 2A(1+y)^{3/2}}{1+y}\right)\right]\frac{d\delta_m}{dy}$$

$$= 4\pi G\rho_{m,0}\frac{\delta_m}{Ha^2}\frac{a_{eq}^2}{a^3 H}$$

$$\frac{d^2\delta_m}{dy^2} + \left[\frac{3}{y} + \frac{1}{y}\left(\frac{\frac{1}{2}y - 2(1+y)}{1+y}\right)\right]\frac{d\delta_m}{dy}$$

$$= 4\pi G\rho_{m,0}\frac{\delta_m a_{eq}^2}{H^2 a^5}$$

$$= \frac{d^2\delta_m}{dy^2} + \left[\frac{3(1+y) + (\frac{1}{2}y - 2(1+y))}{y(1+y)}\right]\frac{d\delta_m}{dy}$$

$$= \frac{d^2\delta_m}{dy^2} + \left[\frac{3+3y+\frac{1}{2}y - 2 - 2y}{y(1+y)}\right]\frac{d\delta_m}{dy}$$

$$= \frac{d^2\delta_m}{dy^2} + \left[\frac{1+\frac{3}{2}y}{y(1+y)}\right]\frac{d\delta_m}{dy}$$

$$= \frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_m}{dy}$$ left side !

Right side

Before:

$$A = \frac{\sqrt{\rho_{m,0}\frac{8\pi G}{3}}}{a_{eq}^{3/2}}$$

$$4\pi G \rho_{m,0} \frac{\delta_m}{H^2 a^{\cancel{5}}} a_{eq}^2$$

$$= \frac{4\pi G \rho_{m,0}}{A^2 (1+y)} y^4 \frac{a_{eq}^2}{a^5} \delta_m$$

$$= \frac{4\pi G \rho_{m,0}}{A^2 (1+y)} y^4 \frac{a_{eq}^2}{a_{eq}^5 y^5} \delta_m$$

$$= \frac{4\cancel{\pi}\cancel{G}\cancel{\rho_{m,0}}}{\frac{8\cancel{\pi}\cancel{G}}{3}\rho_{m,0}} \frac{\cancel{a_{eq}^3} a_{eq}^2}{\cancel{a_{eq}^5}} \frac{\delta_m}{y(1+y)}$$

$$= \frac{12}{8} \frac{\delta_m}{y(1+y)} = \frac{3}{2} \frac{\delta_m}{y(1+y)}$$

- Verify that solns are

(1) $\delta_m \propto 1 + \frac{3}{2} y$

and

(2) $\delta_m \propto \left(1+\frac{3}{2}y\right) \ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 3\sqrt{1+y}$

(1)

$$\frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)} \frac{d\delta_m}{dy} = \frac{3}{2} \frac{\delta_m}{y(1+y)}$$

$$\frac{d\delta_m}{dy} = \frac{3}{2} \qquad \frac{d^2\delta_m}{dy^2} = 0$$

$$0 + \frac{2+3y}{2y(1+y)} \cdot \frac{3}{2} = \frac{3}{2}\,\frac{\left(1+\frac{3}{2}y\right)}{y(1+y)}$$

$$\frac{3\left(1+\frac{3}{2}y\right)}{2y(1+y)} = \frac{3\left(1+\frac{3}{2}y\right)}{2y(1+y)}$$

indeed ① works

② $$\frac{d\delta m}{dy} = \frac{-3y\mp 1}{y\sqrt{1+y}} + \frac{3}{2}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right)$$

↑ Wolfram Alpha ↓

$$\frac{d^2\delta m}{dy^2} = \frac{1}{y^2(1+y)^{3/2}}$$

$$\frac{1}{y^2(1+y)^{3/2}} + \frac{2+3y}{2y(1+y)}\cdot\left(\frac{-3y\mp 1}{y\sqrt{(1+y)}} + \frac{3}{2}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right)\right)$$

$$= \frac{3}{2}\,\frac{1}{y(1+y)}\left(\left(1+\frac{3}{2}y\right)\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 3\sqrt{1+y}\right)$$

$$= \frac{3}{2}\,\frac{\left(1+\frac{3}{2}y\right)}{y(1+y}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - \frac{9\sqrt{1+y}}{2\,y(1+y)}$$

same

$$= \underbrace{\frac{1}{y^2(1+y)^{3/2}} + \frac{2+3y}{2y(1+y)}\,\frac{(-3y\mp 1)}{y\sqrt{(1+y)}}} + \frac{3\left(1+\frac{3}{2}y\right)}{2y(1+y)}\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right)$$

$$\cancel{= \frac{1}{y^2(1+y)^{3/2}} + \frac{(2+3y)(3y\mp 1)}{2y^2(1+y)^2}}$$

$$\cancel{= \frac{2\sqrt{1+y}}{2y^2(1+y)^{3/2}} + \frac{(2+3y)(-3y-1)}{2y^2(1+y)^2}}$$

$$\cancel{= \frac{2\sqrt{1+y} - 6y + 9y^2 - 2 - 3y}{2y^2(1+y)^2}}$$

$$\cancel{= \frac{2\sqrt{1+y} + 9y^2 - 9y - 2}{2y^2(1+y)^2}}$$

$$\frac{1}{y^2(1+y)^{3/2}} + \frac{2+3y}{2y(1+y)}\,\frac{(-3y-1)}{y\sqrt{(1+y)}}$$

$$= \frac{2}{2y^2(1+y)^{3/2}} + \frac{(2+3y)(-3y-1)}{2y^2(1+y)^{3/2}}$$

$$= \frac{2 - 6y - 9y^2 - 2 + 3y}{2y^2(1+y)^{3/2}} = \frac{-9y(y+1)}{2y^2(1+y)^{3/2}}$$

$$= -\frac{9\sqrt{1+y}}{2y(y+1)}$$

also the same term as before from the right hand side.

- Early Times

$a \ll a_{eq}$ $\quad y \to$ small

Radiation

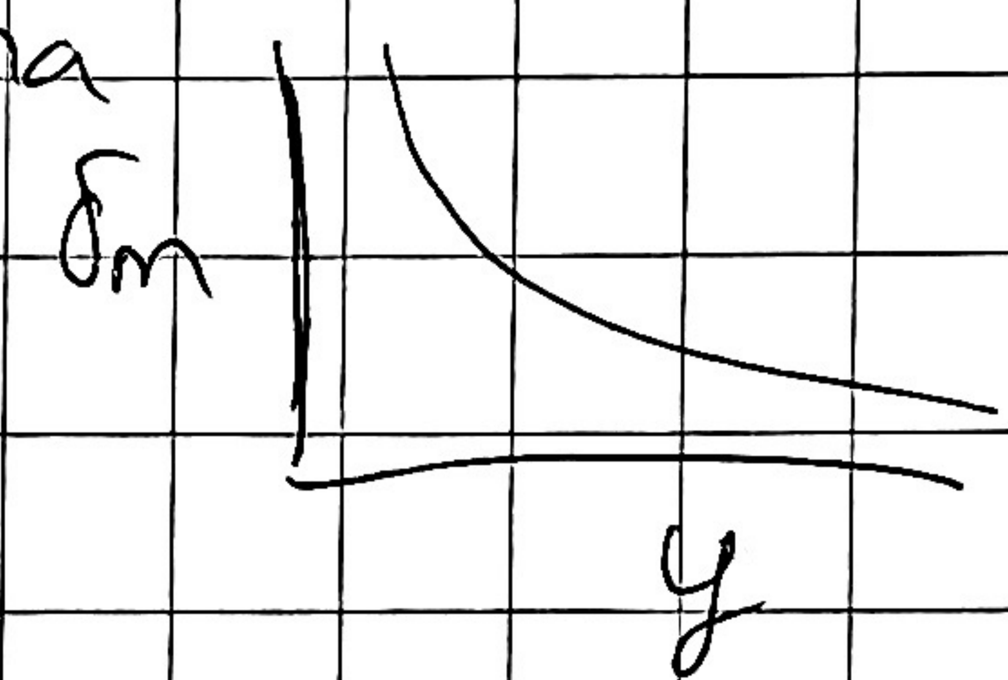
Late Times $a \gg a_{eq}$ $\quad y \to$ large

matter

The growing mode solution is

$$\delta_m \propto 1 + \frac{3}{2} y$$

Because the other one is the decaying mode (I looked up in Wolfram Alpha



Early times y small $\frac{3}{2} y \ll 1$

$$\delta_m \propto 1 \to \text{constant}$$

this is what we expect in the radiation dom epoch

Late times y large $\frac{3}{2} y \gg 1$

$$\delta_m \propto y \propto a$$

this is what we expect in matter dom epoch.

II Spherical Collapse

$$r(\theta) = A(1-\cos\theta) \qquad 0 \le \theta < 2\pi$$
$$t(\theta) = B(\theta - \sin\theta)$$
$$A = \frac{GM}{2|E|}$$
$$A^3 = GMB^2$$

From equation of motion

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} = E$$

$E<0 \rightarrow$ collapse. Solution in parametric form r, t as a function of θ.

At ~~[illegible]~~ $t=0$ shell expands from $r=0$
$\theta=0$

$r(0) = 0 \qquad t(0) = 0$

At $t=\pi$ r is maximum

$r(\pi) = 2A$

$t(\pi) = \pi B$

$\theta = 2\pi$ shell collapses back $r=0$.

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = A\sin\theta\left(B(1-\cos\theta)\right)^{-1}$$
$$= \frac{A\sin\theta}{B(1-\cos\theta)}$$

$$\frac{1}{2}\frac{A^2}{B^2}\frac{\sin^2\theta}{(1-\cos\theta)^2} - \frac{GM}{A(1-\cos\theta)} = E$$

Filling in $A = \frac{GM}{2|E|} \qquad A^3 = GMB^2$

$$\frac{1}{2}\frac{A^2}{B^2}\frac{\sin^2\theta}{(1-\cos\theta)^2} - \frac{GM}{A(1-\cos\theta)} = E$$

$$\frac{1}{2}\frac{GM}{A}\frac{1-\cos^2\theta}{(1-\cos\theta)^2} - \frac{GM}{A}\frac{1}{1-\cos\theta} = E \qquad (E<0)$$

$$\cancel{2}\,\frac{A\cdot E}{GM} = \frac{1}{2}\frac{1-\cos^2\theta}{(1-\cos\theta)^2} - \frac{1}{1-\cos\theta}$$

$$1-\cos^2\theta = -(\cos\theta-1)(\cos\theta+1) = (1-\cos\theta)(1+\cos\theta)$$

$$\frac{AE}{GM} = \frac{1}{2}\frac{1+\cos\theta}{1-\cos\theta} - \frac{1}{1-\cos\theta}$$

$$= \frac{-\frac{1}{2}+\frac{1}{2}\cos\theta}{1-\cos\theta} = -\frac{1}{2}\left(\frac{1-\cos\theta}{1-\cos\theta}\right)$$

$$= -\frac{1}{2}$$

$$A = -\frac{1}{2}\frac{GM}{E}$$

$E<0$ so if we take $|E|$, the minus sign disappears

$$A = \frac{GM}{2|E|}$$

So indeed the solution given is correct.

III Equality Scale

Show

$$k_{eq} = a_{eq} H_{eq} = H_0 \sqrt{\frac{2\Omega_m}{a_{eq}}}$$

Friedmann eq

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}$$

At equality $\frac{\Omega_{r,0}}{a^4} = \frac{\Omega_{m,0}}{a^3}$

$$a_{eq} = \frac{\Omega_{r,0}}{\Omega_{m,0}}$$

$$H = H_0 \sqrt{\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}} = H(a)$$

$$\Rightarrow H(a_{eq}) = H_0 \sqrt{\frac{2\Omega_{m,0}}{a_{eq}^3}}$$

$$k_{eq} = a_{eq} H_{eq} = \sqrt{a_{eq}^2}\, H_{eq} = H_0 \sqrt{\frac{2\Omega_{m,0}}{a_{eq}}}$$

Cosmology today (e.g. Planck, WMAP)

$\Omega_{m,0} \approx 0.3$ $\qquad H_0 \approx 70\ \mathrm{km/s/Mpc}$

$$\Omega_{r,0} = \Omega_{\gamma,0} + \Omega_{\nu,0} = 8.5 \cdot 10^{-5}$$

$$a_{eq} = \frac{8.5 \cdot 10^{-5}}{0.3} = 2.8 \cdot 10^{-4}$$

(corresponding to $z_{eq} \approx 3500$) $\quad k_{eq} \sim 0.01\ \mathrm{Mpc}^{-1}$

$$k_{eq} = 70 \frac{\mathrm{km}}{\mathrm{s\ Mpc}} \sqrt{\frac{2 \cdot 0.3}{2.8 \cdot 10^{-4}}} \approx 10^{-16}\ \mathrm{s}^{-1}$$

$$l_{eq} = \frac{2\pi}{k_{eq}} = 6 \cdot 10^{16}\ \mathrm{s} \xrightarrow{\text{putting in } c} l_{eq} = 1.8 \cdot 10^{25}\ \mathrm{m} \approx 600\ \mathrm{Mpc}$$