

# I Growth of matter perturbations

## Matter and radiation

$$\delta_m + 2H\delta_m = 4\pi G \bar{\rho}_m \delta_m$$

↓ show

$$\frac{d}{da} \left( a^3 H \frac{d\delta_m}{da} \right) = 4\pi G \bar{\rho}_{m,0} \frac{\delta_m}{Ha^2}$$

Friedmann:

$$H^2 = H_0^2 \left( \frac{\rho_{r,0}}{a^4} + \frac{\rho_{m,0}}{a^3} \right)$$

$$H^2 = \frac{8\pi G}{3} \rho \quad \rho = \rho_r + \rho_m$$

$$H = \dot{a}/a$$

$$\rho_m = \rho_{m,0} a^{-3} \quad (\text{matter density } \propto a^{-3})$$

$$\rho_r \propto a^{-4}$$

$$\frac{dx}{dt} = \frac{dx}{da} \frac{da}{dt} = \frac{dx}{da} \dot{a} = Ha \frac{dx}{da}$$

$$\delta_m = \frac{d^2 \delta_m}{dt^2} = \frac{d}{dt} \left( Ha \frac{d\delta_m}{da} \right)$$

$$= Ha \cancel{\frac{d}{da}} \left( Ha \frac{d\delta_m}{da} \right)$$

$$2H\delta_m = 2H \frac{d\delta_m}{dt} = 2H^2 a \frac{d\delta_m}{da}$$

$$\delta m + 2H\delta m = 4\pi G \rho_m \delta m$$

$$Ha \frac{d}{da} \left( Ha \frac{d\delta m}{da} \right) + 2H^2 a \frac{d\delta m}{da} = 4\pi G \rho_{m,0} \frac{\delta m}{a^3}$$

$$a^2 \frac{d}{da} \left( Ha \frac{d\delta m}{da} \right) + 2Ha^2 \frac{d\delta m}{da} = \frac{4\pi G \rho_{m,0}}{Ha^2} \delta m$$

now I'm going to show that this is

$$\frac{d}{da} \left( a^3 H \frac{d\delta m}{da} \right)$$

to prove the  
equation

product rule

$$\frac{d}{da} \left( a^3 H \frac{d\delta m}{da} \right) = \frac{d}{da} (a^3 H) \frac{d\delta m}{da} + a^2 \frac{d}{da} \left( Ha \frac{d\delta m}{da} \right)$$

↑ this is my first term above

$$\begin{aligned} \frac{d}{da} (a^3 H) &= 3a^2 H + a^3 \frac{dH}{da} \\ &= 3a^2 \cdot \frac{a}{a} - a^3 \cdot \frac{a \cdot \dot{a}}{a^2} = 3a\dot{a} - a\dot{a} = 2a\dot{a} \\ &= 2a^2 H \end{aligned}$$

So indeed I found  $\frac{d}{da} (a^3 H \frac{d\delta m}{da})$

$$= a^2 \frac{d}{da} \left( Ha \frac{d\delta m}{da} \right) + 2Ha^2 \frac{d\delta m}{da} = \frac{4\pi G \rho_{m,0} \delta m}{Ha^2}$$

$$\frac{d}{da} \left( a^3 H + \frac{dS_m}{da} \right) = \frac{d}{da} (a^3 H) \frac{dS_m}{da}$$

~~$a^2$~~   $+ a^2 \frac{d}{da} \left( H a \frac{dS_m}{da} \right)$

$$\frac{d}{da} (a^3 H) = 3a^2 \cancel{\frac{dH}{da}} + a^3 \frac{dH}{da}$$

$$= 3a^2 \frac{\dot{a}}{a} + a^3 \frac{\dot{a}}{a^2} = 2a\dot{a}$$

$$= 2 \cancel{R^2 a^2 H}$$

$$\frac{dH}{da} = -\frac{\dot{a}}{a^2}$$

•  $y = a/a_{eq}$        $a = a_{eq}y$

~~$a_{eq}$~~

$$H(a) = H_0 \sqrt{\left( \frac{S_{r,0}}{a^4} + \frac{S_{m,0}}{a^3} \right)}$$

$$a_{eq} = \frac{S_{r,0}}{S_{m,0}}$$

$$H(y) = H_0 \sqrt{\frac{S_{r,0}}{a_{eq}^4 y^4} + \frac{S_{m,0}}{a_{eq}^3 y^3}}$$

$$= \frac{H_0}{y^2} \sqrt{\frac{S_{r,0}}{a_{eq}^4} + \frac{S_{m,0}}{a_{eq}^3} y}$$

$$= \frac{H_0}{y^2} \sqrt{\frac{S_{r,0}}{a_{eq}^4}} \sqrt{1 + \frac{S_{m,0}}{a_{eq}^3} \frac{a_{eq}^4}{S_{r,0}} y}$$

$$= \frac{H_0}{y^2} \sqrt{\frac{S_{r,0}}{a_{eq}^2}} \sqrt{1+y}$$

this is 1  
 $\frac{S_{m,0}}{S_{r,0}} = q_{eq}$

So  $A = H_0 \sqrt{S_{r,0}} a_{eq}^{-2}$

$$\text{or } A = H_0 \frac{\sqrt{\Omega_{r,0}^2 + \Omega_{m,0}^2}}{\Omega_{r,0}^2} = H_0 \frac{\Omega_{m,0}^2}{\Omega_{r,0}^{3/2}} = H_0 \frac{\Omega_{m,0}^2}{(\Omega_{eq} \Omega_{m,0})^{3/2}} = H_0 \frac{\Omega_{m,0}^2}{\Omega_{eq}^{3/2}}$$

$$\Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{crit,0}} = \frac{\rho_{m,0}}{3H_0^2} 8\pi G$$

$$\frac{d}{da} \left( a^3 H \frac{d\delta_m}{da} \right) = 4\pi G \rho_{m,0} \frac{\delta_m}{H a^2}$$

$$y = a/a_{eq} \quad a = a_{eq} \cdot y \\ da = a_{eq} dy$$

$$\frac{1}{a_{eq}^2} \frac{d}{dy} \left( a_{eq}^3 y^3 H \frac{d\delta_m}{dy} \right) = 4\pi G \rho_{m,0} \frac{\delta_m}{H a_{eq}^2 y^2}$$

$$\frac{d}{dy} \left( y^3 H \frac{d\delta_m}{dy} \right) = \frac{d}{dy} \left( y^3 H \right) \frac{d\delta_m}{dy} + y^3 H \frac{d^2}{dy^2} \delta_m$$

$$\frac{d}{dy} \left( y^3 H \right) = 3y^2 H + y^3 \frac{dH}{dy}$$

$$\frac{dH}{dy} = -\frac{1}{2y^2} \frac{1}{2\sqrt{1+y}} + \frac{1}{3} \frac{A}{y^3} \sqrt{1+y}$$

$$= -\frac{A}{4y^2\sqrt{1+y}} - \frac{1}{3} \frac{A\sqrt{1+y}}{y^3}$$

$$\frac{d}{dy} \left( y^3 H \frac{d\delta m}{dy} \right) = 4\pi G \rho_{m,0} \frac{\delta m}{H a_{eq}^3 y^2}$$

~~$$= \frac{3y^2 H + A y^3 \frac{dH}{dy}}{4\sqrt{1+y}} \frac{d\delta m}{dy}$$~~

$$= \left( 3y^2 H + y^3 \frac{dH}{dy} \right) \frac{d\delta m}{dy} + y^3 H \frac{d^2 \delta m}{dy^2}$$

$$= \frac{d^2 \delta m}{dy^2} + \left( \frac{3}{y} + \frac{1}{H} \frac{dH}{dy} \right) \frac{d\delta m}{dy}$$

$$= \frac{d^2 \delta m}{dy^2} + \left( \frac{3}{y} + \frac{1}{A} \frac{1}{4y^4(1+y)} - \frac{1}{3} \frac{1}{y^2} \right) \frac{d\delta m}{dy}$$

$$\frac{3y}{y^2} - \frac{3}{y^2} - \frac{1}{4y^4(1+y)} = \frac{(3y - \frac{1}{3})y^2(1+y)}{y^4(1+y)} - \frac{1}{4y^2(4y)}$$

$$= \frac{(3y - \frac{1}{3})y^2(1+y)}{y^4(1+y)} - \frac{1}{4} =$$

$$\frac{d}{da} \left( a^3 H \frac{dm}{da} \right) = 4\pi G \rho_{m,0} \frac{dm}{Ha^2}$$

$$H(y) = \frac{A}{y^2} \sqrt{1+y} \quad y = a/a_{eq}$$

$$da = a_{eq} dy$$

$$\frac{d}{da} \left( a^3 H \frac{dm}{da} \right) = \frac{d}{da} \left( a^3 H \right) \frac{dm}{da} + a^3 H \frac{d^2 m}{da^2}$$

$$= \left( 3a^2 H + a^3 \frac{dH}{da} \right) \frac{dm}{da} + a^3 H \frac{d^2 m}{da^2}$$

$$= \left( 3a_{eq}^2 y^2 H + a_{eq}^3 y^3 \frac{dH}{dy} \frac{dy}{da} \right) \frac{dm}{da} + a^3 H \frac{d^2 m}{da^2}$$

$$\frac{dH}{dy} = \frac{a^3 y^2 \cdot (\frac{1}{2}(1+y)^{-1/2}) - 2y A \sqrt{1+y}}{y^4}$$

$$= \frac{\frac{1}{2} A}{y^2 \sqrt{1+y}} - \frac{2 A \sqrt{1+y}}{y^3}$$

$$= \frac{\frac{1}{2} A y \sqrt{1+y} - 2 A (1+y)^{3/2}}{y^3 \cancel{(1+y)}}$$

$$\boxed{3a_{eq}^2 \frac{A \sqrt{1+y}}{y^2} + a_{eq}^3 \frac{\frac{1}{2} A y \sqrt{1+y} - 2 A (1+y)^{3/2}}{1+y}}$$

$$\frac{dm}{da} + a^3 H \frac{d^2 m}{da^2} = 4\pi G \rho_{m,0} \frac{dm}{Ha^2}$$

$$= \frac{d}{da} \left[ \frac{a^3 H d^2 \delta m}{a_{eq}^2 dy^2} + \left[ \frac{3 a_{eq}^2 A \sqrt{1+y}}{H} + \frac{a_{eq}^2}{H} \left( \frac{\frac{1}{2} A y \sqrt{1+y} - 2A(1+y)^{3/2}}{1+y} \right) \right] \frac{d\delta m}{da} \right] \frac{\delta m}{Ha^2}$$

$$\frac{d^2 \delta m}{dy^2} + \left[ \frac{3 a_{eq}^3}{a^3 H} A \sqrt{1+y} + \frac{a_{eq}^3}{a^3 H} \left( \frac{\frac{1}{2} A y \sqrt{1+y} - 2A(1+y)^{3/2}}{1+y} \right) \right] \frac{d\delta m}{dy} = 4\pi G P_{m,0} \frac{\delta m}{Ha^2} \frac{a_{eq}^2}{a^3 H}$$

$$\frac{d^2 \delta m}{dy^2} + \left[ \frac{3}{y} + \frac{1}{y} \left( \frac{\frac{1}{2} y - 2(1+y)}{1+y} \right) \right] \frac{d\delta m}{dy}$$

$$= 4\pi G P_{m,0} \frac{\delta m}{Ha^2} \frac{a_{eq}^2}{a^3 H}$$

$$= \frac{d^2 \delta m}{dy^2} + \left[ \frac{3(1+y)}{y(1+y)} + \left( \frac{\frac{1}{2} y - 2(1+y)}{1+y} \right) \right] \frac{d\delta m}{dy}$$

$$= \frac{d^2 \delta m}{dy^2} + \left[ \frac{3+3y + \frac{1}{2} y - 2 - 2y}{y(1+y)} \right] \frac{d\delta m}{dy}$$

$$= \frac{d^2 \delta m}{dy^2} + \left[ \frac{1 + \frac{3}{2} y}{y(1+y)} \right] \frac{d\delta m}{dy}$$

$$= \frac{d^2 \delta m}{dy^2} + \frac{2 + 3y}{2y(1+y)} \frac{d\delta m}{dy}$$

left side  
works!

Right side

Before:

$$A = \sqrt{\rho_{m,0} \sigma_{II} G}$$

$$4\pi G \rho_{m,0} \frac{dm}{R^2 a^8} a_{eq}^2$$

$$= \frac{4\pi G \rho_{m,0}}{A^2 (1+y)} y^4 \frac{a_{eq}^2 dm}{a^5}$$

$$= \frac{4\pi G \rho_{m,0}}{A^2 (1+y)} y^4 \frac{a_{eq}^2}{a_{eq}^5 y^5} dm$$

$$= \frac{4\pi G \rho_{m,0}}{\frac{2\pi G \rho_{m,0}}{3}} \frac{a_{eq}^3 c_{eq}^2}{a_{eq}^5} \frac{dm}{y(1+y)}$$

$$= \frac{12}{5} \frac{dm}{y(1+y)} = \frac{3}{2} \frac{dm}{y(1+y)}$$

• Verify that solns are

$$\textcircled{1} \quad dm \propto 1 + \frac{3}{2} y$$

and

$$\textcircled{2} \quad dm \propto \left(1 + \frac{3}{2} y\right) \ln \left( \frac{\sqrt{1+y} + 1}{\sqrt{1+y} - 1} \right) - 3\sqrt{1+y}$$

\textcircled{1}

$$\frac{d^2 dm}{dy^2} + \frac{2+3y}{2y(1+y)} \frac{dm}{dy} = \frac{3}{2} \frac{dm}{y(1+y)}$$

$$\frac{dm}{dy} = \frac{3}{2} \quad \frac{d^2 dm}{dy^2} = 0$$

$$0 + \frac{2+3y}{2y(1+y)} \cdot \frac{3}{2} = \frac{3}{2} \frac{(1+\frac{3}{2}y)}{y(1+y)}$$

$$\frac{3(1+\frac{3}{2}y)}{2y(1+y)} = \frac{3(1+\frac{3}{2}y)}{2y(1+y)}$$

indeed ① works

$$② \frac{d\delta m}{dy} = \frac{-3y^{-1}}{y\sqrt{1+y}} + \frac{3}{2} \ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right)$$

wolfram Alpha

↓

$$\frac{d^2\delta m}{dy^2} = \frac{1}{y^2(1+y)^{3/2}}$$

$$\frac{1}{y^2(1+y)^{3/2}} + \frac{2+3y}{2y(1+y)} \cdot \left( \frac{-3y^{-1} + \frac{3}{2} \ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right)}{y\sqrt{1+y}} \right)$$

$$= \frac{3}{2} \frac{1}{y(1+y)} \left( \left(1+\frac{3}{2}y\right) \ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 3\sqrt{1+y} \right)$$

$$= \frac{3}{2} \frac{\left(1+\frac{3}{2}y\right)}{y(1+y)} \ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - \frac{3}{2} \frac{\sqrt{1+y}}{y(1+y)}$$

↙ Same ↘

$$= \frac{1}{y^2(1+y)^{3/2}} + \frac{2+3y}{2y(1+y)\sqrt{1+y}} (-3y^{-1}) + \frac{3\left(1+\frac{3}{2}y\right)}{2y(1+y)} \ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right)$$

~~$$= \frac{2+3y}{2y(1+y)^{3/2}} + \frac{(2+3y)(3y^{-1})}{2y^2(1+y)^2}$$~~

$$= \frac{2\sqrt{1+y}}{2y^2(1+y)^{3/2}} + \frac{(2+3y)(-3y-1)}{2y^2(1+y)^{3/2}}$$

$$= \frac{\cancel{2}\sqrt{1+y} \cdot -6y + gy^2 - 2\cancel{3}y}{2y^2(1+y)^{3/2}}$$

$$= \frac{2\sqrt{1+y} \cdot gy^2 - 3y + 2}{2y^2(1+y)^{3/2}}$$

$$\frac{1}{y^2(1+y)^{3/2}} + \frac{2+3y(-3y-1)}{2y(1+y)\cancel{y(1+y)}}$$

$$= \frac{2}{2y^2(1+y)^{3/2}} + \frac{(2+3y)(-3y-1)}{2y^2(1+y)^{3/2}}$$

$$= \frac{2 - 6y - gy^2 - 2 + 3y}{2y^2(1+y)^{3/2}} = \frac{-gy(y+1)}{2y^2(1+y)^{3/2}}$$

$$= -\frac{g\sqrt{1+y}}{2y(y+1)}$$

also the same term  
as before from the  
right hand side.

- Early Times

$$a \ll a_{eq} \quad y \rightarrow \text{small}$$

Radiation

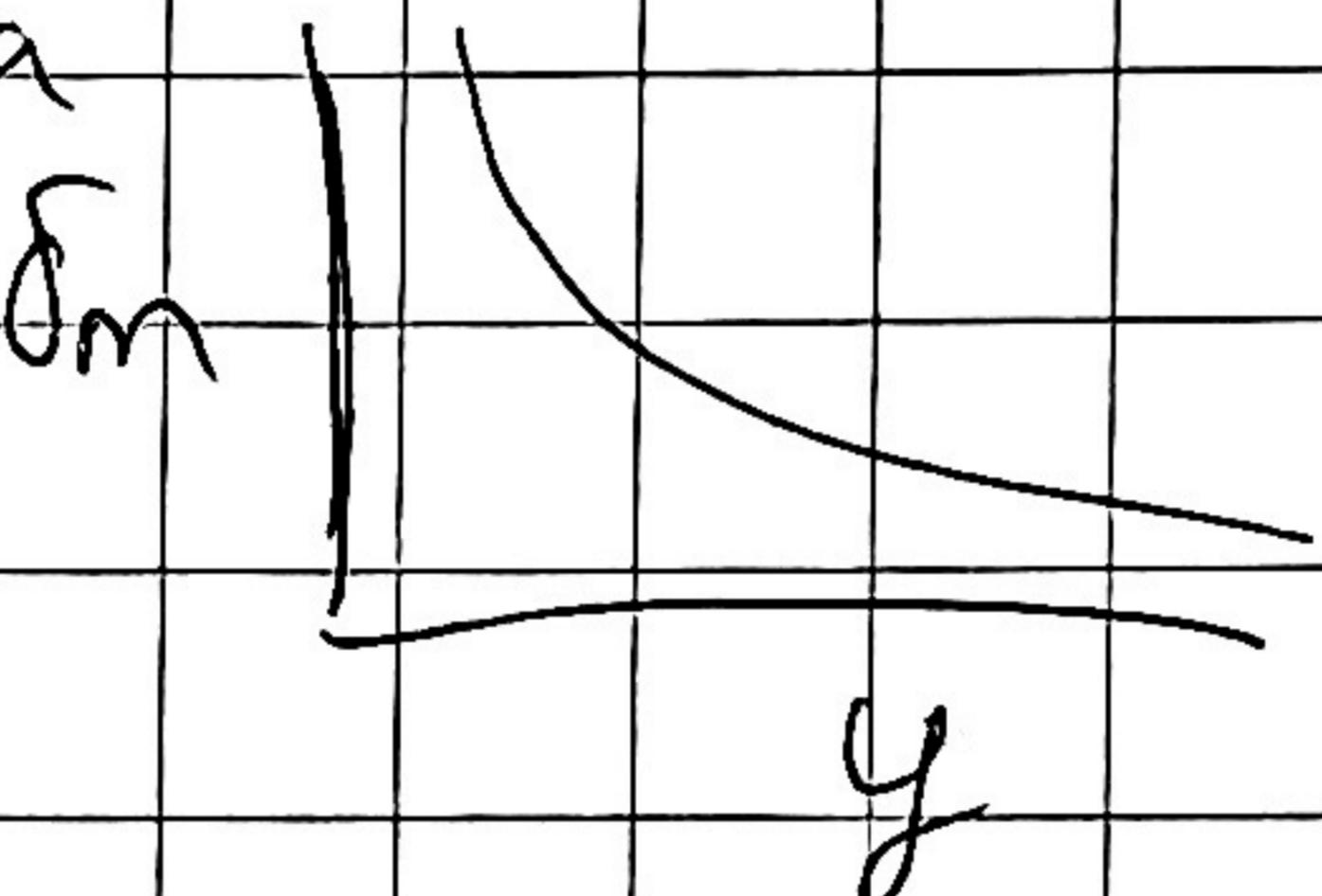
Late Times  $a \gg a_{eq}$   $y \rightarrow \text{large}$

Matter

The growing mode solution is

$$\delta_m \propto 1 + \frac{3}{2}y$$

Because the other one is the decaying mode (I looked up in Wolfram Alpha)



At Early times  $y$  small  $\frac{3}{2}y \ll 1$

$$\delta_m \propto 1 \rightarrow \text{constant}$$

this is what we expect in the radiation dom epoch

Late times  $y$  large  $\frac{3}{2}y \gg 1$

$$\delta_m \propto y + a$$

this is what we expect in matter dom epoch.

## II Spherical Collapse

$$r(\theta) = A(1 - \cos\theta) \quad 0 \leq \theta < 2\pi$$

$$t(\theta) = B(\theta - \sin\theta)$$

$$A = \frac{GM}{2IEI}$$

$$A^3 = GM B^2$$

From equation of motion

$$\frac{1}{2} \left( \frac{dr}{dt} \right)^2 - \frac{GM}{r} = E$$

$E < 0 \rightarrow$  collapse. Solution in parametric form  
 $r, t$  as a function of  $\theta$ .

At  $t=0$  shell expands from  $r=0$   
 $\theta=0$

$$r(\theta) = 0 \quad t(\theta) = 0$$

At  $t=\pi$   $r$  is maximum

$$r(\pi) = 2A$$

$$t(\pi) = \pi B$$

$\theta=2\pi$  shell collapses back  $r=0$ .

$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} = A \sin\theta \left( B(1 - \cos\theta) \right)^{-1}$$

$$= \frac{A \sin\theta}{B(1 - \cos\theta)}$$

$$\frac{1}{2} \frac{A^2}{B^2} \frac{\sin^2\theta}{(1 - \cos\theta)^2} = - \frac{GM}{A(1 - \cos\theta)} = E$$

Filling in  $A = \frac{GM}{2IEI}$   $A^3 = GM B^2$

$$\frac{1}{2} \frac{A^2}{B^2} \frac{\sin^2 \theta}{(1-\cos\theta)^2} - \frac{GM}{A(1-\cos\theta)} = E$$

$$\frac{1}{2} \frac{GM}{A} \frac{1-\cos^2 \theta}{(1-\cos\theta)^2} - \frac{GM}{A} \frac{1}{1-\cos\theta} = E \quad (E<0)$$

$$2 \frac{AE}{GM} = \frac{1}{2} \frac{1-\cos^2 \theta}{(1-\cos\theta)^2} - \frac{1}{1-\cos\theta}$$

$$1-\cos^2 \theta = -(\cos\theta-1)(\cos\theta+1)$$

$$= (1-\cos\theta)(1+\cos\theta)$$

$$\frac{AE}{GM} = \frac{1}{2} \frac{1+\cos\theta}{1-\cos\theta} - \frac{1}{1-\cos\theta}$$

$$= \frac{\frac{1}{2} + \frac{1}{2}\cos\theta}{1-\cos\theta} = \frac{-1}{2} \left( \frac{1-\cos\theta}{1-\cos\theta} \right)$$

$$= -\frac{1}{2}$$

$$A = -\frac{1}{2} \frac{GM}{E} \quad E < 0 \text{ so if we take } |E|, \text{ the minus sign disappears}$$

$$A = \frac{GM}{2|E|}$$

So indeed the solution given is correct.

### III Equating Scale

Show

$$h_{eq} = a_{eq} H_{eq} = H_0 \sqrt{\frac{2 \Omega_m}{a_{eq}}}^7$$

Friedmann eq

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}$$

At equality  $\frac{\Omega_{r,0}}{a^4} = \frac{\Omega_{m,0}}{a^3}$

$$a_{eq} = \frac{\Omega_{r,0}}{\Omega_{m,0}}$$

$$H = H_0 \sqrt{\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}} = H(a)$$

$$\Rightarrow H(a_{eq}) = H_0 \sqrt{\frac{2 \Omega_{m,0}}{a_{eq}^3}}$$

$$h_{eq} = a_{eq} H_{eq} = \sqrt{a_{eq}^2} H_{eq} = H_0 \sqrt{\frac{2 \Omega_{m,0}}{a_{eq}}}^7$$

Cosmology today (e.g. Planck, WMAP)

$$\Omega_{m,0} \approx 0.3$$

$$H_0 \approx 70 \text{ km/s/Mpc}$$

$$\Omega_{r,0} = \Omega_{f,0} + \Omega_{v,0}$$

$$= 0.5 \cdot 10^{-5}$$

$$a_{eq} = \frac{0.5 \cdot 10^{-5}}{0.3} = 2.0 \cdot 10^{-4}$$

(corresponding to  $z_{eq} \approx 3500$ )  $h_{eq} \approx 0.01 \text{ mpc}$

$$h_{eq} = \frac{70 \text{ km}}{5 \text{ Mpc}} \sqrt{\frac{2 \cdot 0.3}{2.0 \cdot 10^{-4}}} \approx 10^{-16} \text{ s}^{-1}$$

$$l_{eq} = \frac{2\pi}{c} = 6 \cdot 10^{16} \text{ s} \xrightarrow{\text{putting in } c} l_{eq} = 1.8 \cdot 10^{16} \text{ m}$$

$$\approx 600 \text{ Mpc}$$

**For the final exercises I uploaded an ipython notebook to github  
I unfortunately could not convert it into a PDF.  
But let me know if something does not work!**