Cosmology HW11 Josephine Baggen

1) First friedmann + continuity → 2nd Friedmann

$$F1: \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2} \leftarrow \kappa = \frac{K}{R_0^2}$$

Cont: $\dot\rho + 3H(\rho + P) = 0$

$$\dot a^2 = \frac{8\pi G\rho a^2}{3} - \kappa$$

derivative time

$$2\dot a \ddot a = \frac{8\pi G}{3}\left(2a\dot a\rho + a^2\dot\rho\right)$$

divide $2\dot a$

$$\ddot a = \frac{8\pi G}{3}\left(a\rho + \frac{a^2}{2\dot a}\dot\rho\right)$$

divide by a

$$\frac{\ddot a}{a} = \frac{4\pi G}{3}\left(2\rho + \frac{a}{\dot a}\dot\rho\right)$$

Plug in continuity $\dot\rho = -3H(\rho+P)$ $H = \frac{\dot a}{a}$

$$\frac{\ddot a}{a} = \frac{4\pi G}{3}\left(2\rho + \frac{-a}{\dot a}3H(\rho+P)\right)$$

$$= \frac{4\pi G}{3}\left(2\rho - 3\rho - 3P\right)$$

$$= \frac{4\pi G}{3}\left(-\rho - 3P\right) = -\frac{4\pi G}{3}\left(\rho + 3P\right) = \text{Friedmann II}$$

2) $F = \dfrac{L}{4\pi d_L^2}$

L in Watts, J/s

flux energy passing through surface per sec
W/m^2

Intensity $= \dfrac{\text{Flux}}{\text{solid angle}}$ $\quad W/m^2/sr$

$$I = \frac{F}{\Omega}$$

$$I = \frac{L}{4\pi d_L^2} \frac{1}{\Omega}$$

~~$\Omega = 4\pi R_{surface}^2 / d$~~

~~$\Omega \ldots$~~

$$I = \frac{dE}{dA\, dt\, d\Omega}$$

$\Omega \sim \text{angular size}^2 \sim d_A^2$

L luminosity at emission, fixed value

$$I \propto \frac{L}{4\pi}\left(\frac{d_A}{d_L}\right)^2$$

observed intensity

we know that $\dfrac{d_A}{d_L} = \dfrac{1}{(1+z)^2}$

so $\quad I \propto \left(\dfrac{d_A}{d_L}\right)^2 = (1+z)^{-4}$

$$I = I_0 (1+z)^{-4}$$

3) $$m = -2.5 \log_{10}\left(\frac{f}{f_0}\right)$$

M defined as m at 10 pc

$$f = \frac{L}{4\pi d^2}$$

$$m_1 - m_2 = -2.5 \log_{10}\left(\frac{f_1}{f_2}\right)$$

Say source 1 with apparent mag m1 at distance d. $f_1 = \frac{L_1}{4\pi d^2}$

Source 2 with apperent mag ~~~~ M at distance 10 pc $f_2 = \frac{L_2}{4\pi (10pc)^2}$

$$m - M = -2.5 \log\left(\frac{L_1 \quad 4\pi \cdot (10pc)^2}{4\pi d^2 \quad L_2}\right)$$

L_1 and L_2 are the same, same source but the different distances ~~with change the apparent magnitudes~~ make them look brighter/dimmer.

$$m - M = -2.5 \log\left(\left(\frac{10pc}{d}\right)^2\right)$$

$$= 5 \log\left(\left(\frac{d}{10pc}\right)\right)$$

taking out the squared and putting in the minus sign

So indeed

$$m = M + \underbrace{5 \log\left(\frac{d}{10pc}\right)}_{\text{call this } DM(z)}$$

$d = D_L$, depends on z so $D_L(z)$

- $L_\nu \leftarrow$ on frequency band ν

$$f = \int f_\nu \, d\nu$$

$$f_\nu = \frac{df(\nu)}{d\nu} \qquad L_\nu = \frac{dL(\nu)}{d\nu}$$

photons emitted with frequency ν_e in a band with bandwidth $d\nu_e$ at redshift z will be observed with

$$\nu = \frac{\nu_e}{1+z} \qquad \nu_e = \nu(1+z)$$

and $$d\nu = \frac{d\nu_e}{1+z}$$

So $$f_\nu = \frac{df}{d\nu} = \frac{df}{d\nu_e}\frac{d\nu_e}{d\nu} = (1+z)\frac{df}{d\nu_e}$$

$$f = \frac{L}{4\pi d_L^2}$$

$$\frac{df}{d\nu_e} = \frac{d}{d\nu_e}\left(\frac{L}{4\pi d_L^2}\right) = \frac{1}{4\pi d_L^2}\underbrace{\frac{dL}{d\nu_e}}_{\text{this is how } L_\nu \text{ is defined}}$$

$$= \frac{1}{4\pi d_L^2} L_{\nu_e}$$

So $$f_\nu = (1+z)\frac{1}{4\pi d_L^2} L_{\nu_e}$$ $\nu_e = \nu(1+z)$

$$= (1+z)\frac{1}{4\pi d_L^2} L_{\nu(1+z)}$$

Specific flux, sometimes written as S_ν

we get exactly the expression as on the problem set with this new equation for flux f_ν

we still write

$$m = -2.5\log\left(\frac{f_\nu}{f_0}\right) = -2.5\log(f_\nu) + \text{const}$$

$\uparrow$ in 1 band

$$f_\nu = (1+z)\frac{1}{4\pi d_L^2} L_{\nu(1+z)}$$

$$m = -2.5\log\left(\frac{(1+z)}{4\pi d_L^2} L_{\nu(1+z)}\right) + \text{const}$$

4π into const and multiply with $\frac{L_\nu}{L_\nu}$

$$= -2.5\log\left((1+z)\frac{L_{\nu(1+z)}}{L_\nu}\frac{L_\nu}{d_L^2}\right) + \text{const}$$

$$= -2.5\log\left((1+z)\frac{L_{\nu(1+z)}}{L_\nu}\right) \cancel{-2.5\log\left(\frac{L_\nu}{d_L^2}\right)}$$

$$-2.5\log(L_\nu) - 2.5\log(d_L^{-2}) + \text{const}$$

$$= -2.5\log\left((1+z)\frac{L_{\nu(1+z)}}{L_\nu}\right) - 2.5\log(L_\nu) + 5\log(d_L) + \text{const}$$

$$= -2.5\log\left((1+z)\frac{L_{\nu(1+z)}}{L_\nu}\right) - 2.5\log(L_\nu) + 5\log\left(\frac{d}{10\text{pc}}\right) + \text{const}$$

$\uparrow$ changed const to plug in 10pc

For $d = 10\text{pc}$ $(z \simeq 0)$ m should be M

$$M = 0 - 2.5\log(L_\nu) + 0 + \text{const}$$
$$= -2.5\log(L_\nu) + \text{const}$$

So $m = \underbrace{-2.5\log\left((1+z)\frac{L_{\nu(1+z)}}{L_\nu}\right)}_{\text{this is indeed K}} + M + 5\log\left(\frac{d}{10\text{pc}}\right)$

$$= K + M + DM$$

4) $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho_m}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}$$

$$\rho_m = \rho_{m,0}\, a^{-3}$$

derive wrt t

$$\dot{a}^2 = \frac{8\pi G\rho_m a^2}{3} + \frac{\Lambda a^2}{3} - k$$

$$2\dot{a}\ddot{a} = \frac{d}{dt}\left(\frac{8\pi G}{3}\frac{\rho_{m,0}}{a}\right) + \frac{2\Lambda}{3}a\dot{a}$$

$$2\dot{a}\ddot{a} = -\frac{8\pi G\rho_{m,0}}{3}a^{-2}\dot{a} + \frac{2\Lambda}{3}a\dot{a}$$

divide by $2\dot{a}$

$$\ddot{a} = -\frac{4\pi G\rho_{m,0}}{3a^2} + \frac{\Lambda}{3}a$$

divide by a

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho_{m,0}}{3a^3} + \frac{\Lambda}{3}$$

new approach

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

now $\rho = \rho_m + \rho_\Lambda$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda) - \frac{k}{a^2}$$

$$= \frac{8\pi G}{3}\rho_m + \underbrace{\frac{8\pi G}{3}\rho_\Lambda} - \frac{k}{a^2}$$

call this $\frac{\Lambda}{3}$ to get the expression

so $\Lambda = 8\pi G\rho_\Lambda$

- Find k, Λ such that $\dot{a} = 0$, $\ddot{a} = 0$

$\dot{a} = 0$ gives

$$0 = \frac{8\pi G\rho_m}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}$$

$$\rho_m = \rho_{m,0}\, a^{-3}$$

~~Derive wrt t~~

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3} - \frac{k}{a^2}$$ derive wrt t

$$\dot{a}^2 = \frac{8\pi G}{3}\rho_m a^2 + \frac{\Lambda}{3}a^2 - k$$

$$2\dot{a}\ddot{a} = -\frac{8\pi G}{3}\rho_{m,0}\, a^{-2}\dot{a} + \frac{2\Lambda}{3} a\dot{a}$$

$$\ddot{a} = -\frac{4\pi G}{3}\rho_{m,0}\, a^{-2} + \frac{\Lambda}{3} a$$

$\ddot{a} = 0$ gives

$$0 = -\frac{4\pi G}{3}\rho_{m,0}\, a^{-2} + \frac{\Lambda}{3} a$$

$$\frac{4\pi G\rho_{m,0}}{3} a^{-2} = \frac{\Lambda}{3} a$$

$$\Lambda = 4\pi G\rho_{m,0}\, a^{-3}$$

$$\frac{\Lambda}{3} = \frac{-8\pi G\rho_{m,0}}{3a^3} + \frac{3k}{a^2}$$

$$12\pi G\rho_{m,0}\, a^{-3} = \frac{3k}{a^2} \rightarrow \frac{k}{a^2} = 4\pi G\rho_m$$

~~$k = \frac{4\pi G\rho_{m,0}}{a} > 0$ so it's open~~

comparing this with previous page I find

$\Lambda = 4\pi G\rho_{m,0}\, a^{-3}$
$= 4\pi G\rho_m$

$\Lambda = 8\pi G\rho_\Lambda$

so $\rho_\Lambda = \frac{1}{2}\rho_m$

also

$\Lambda = 4\pi G\rho_m = 4\pi G\Omega_{m,0}\, \rho_{crit,0}$

$= 4\pi G\Omega_{m,0}\frac{3H_0^2}{8\pi G} = \frac{3}{2}\Omega_{m,0} H_0^2$

Sorry for the messy writing, I found

$\Lambda = 4\pi G \rho_m$ so $\rho_\Lambda = \frac{1}{2}\rho_m$

$= 8\pi G \rho_\Lambda$

and

$\Lambda = \frac{3}{2}\Omega_{m,0} H_0^2$

Then I found $\frac{k}{a^2} = 4\pi G \rho_m$, since $\rho_m > 0$,

k must be +1, so a Closed universe

- Perturbation $a(t) = 1 + \delta a(t)$ $\delta_a(t)$

$\rho_m(t) = \rho_m(1 - 3\delta a(t))$

Second Friedmann eq

$$\frac{\ddot{a}}{a} = -\frac{4\pi G \rho_m}{3} + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}\rho_m(1 - 3\delta a(t)) + \frac{\Lambda}{3}$$

call $\delta_a(t) = \delta(t)$ (for my simplicity)

$a = 1 + \delta(t)$ $\qquad \ddot{a} = \ddot{\delta}$

$\rho_m = \rho_m(1 - 3\delta(t))$

$$\frac{\ddot{a}}{a} = \frac{-4\pi G\rho_m}{3} + \frac{\Lambda}{3} = \cancel{\ }$$

$$\frac{\ddot{\delta}}{1+\delta} = \frac{-4\pi G\rho_m}{3}(1-3\delta) + \frac{\Lambda}{3}$$

We found before that $\Lambda = 4\pi G\rho_m$, so

$$\frac{\ddot{\delta}}{1+\delta} = 4\pi G\rho_m \delta = \Lambda\delta$$

$$\ddot{\delta} = \Lambda(1+\delta)\delta$$ neglect higher order terms $\propto \delta^2$ because $|\delta| << 1$

$$\ddot{\delta} = \Lambda\delta$$

Initial conditions $\delta(0) = \delta_{a0}$ $\quad \frac{d(\delta)}{dt} = 0$

$\delta'' - \Lambda\delta = 0$ $\quad$ Guess $\delta = e^{rt}$ $\quad \ddot{\delta} = r^2 e^{rt}$

$r^2 e^{rt} - \Lambda e^{rt} = 0$

$(r^2 - \Lambda) = 0$ $\qquad r = \pm\sqrt{\Lambda}$ $\quad$ 2 solutions

So $\delta(t) = A e^{\sqrt{\Lambda}t} + B e^{-\sqrt{\Lambda}t}$

$\delta'(t) = A\sqrt{\Lambda} e^{\sqrt{\Lambda}t} - B\sqrt{\Lambda} e^{-\sqrt{\Lambda}t}$

$\delta'(0) = 0$ gives $A e^{\sqrt{\Lambda}t} = B e^{-\sqrt{\Lambda}t}$

So δ becomes easier

$\delta(t) = 2A e^{\sqrt{\Lambda}t} \rightarrow \delta(0) = \delta_{a0}$ so $2A = \delta_{a0}$

$$\delta(t) = \delta_{a0} e^{\sqrt{\Lambda}t}$$

This grows with time, so a small perturbation will grow and grow, making the static solution not static/unstable.

5 Redshift Drift

call $a(t_0)=a_0$
$a(t_1)=a_1$

$$1+z=\frac{a(t_0)}{a(t_1)} \qquad z=\frac{a_0}{a_1}-1 \qquad \frac{dz}{da_0}=\frac{1}{a_1}$$

$$\frac{dz}{da_1}=-\frac{a_0}{a_1^2}$$

$$\frac{\lambda_0}{\lambda_e}=1+z$$

t_1 emitted
t_0 observed

$$H=\frac{\dot{a}}{a}$$

$$\frac{dz}{dt_0}=\frac{dz}{da_0}\frac{da_0}{dt_0}+\frac{dz}{da_1}\frac{da_1}{dt_1}$$

$$=\frac{dz}{da_0}a_0\cdot\frac{da_0}{dt_0}\frac{1}{a_0}+\frac{dz}{da_1}a_1\frac{da_1}{dt_1}\frac{1}{a_1}$$

$$=\frac{dz}{da_0}a_0 H(t_0)+\frac{dz}{da_1}a_1 H(t_1)$$

$$=\frac{a_0}{a_1}H(t_0)-\frac{a_0}{a_1^2}a_1 H(t_1)$$

$$=(1+z)H(t_0)-(1+z)H(t_1)$$

$$\frac{1}{a}\frac{da}{dt}=H$$

$$\frac{da}{dt}=Ha$$

$$\frac{dz}{dt_0}=\frac{dz}{da_0}\frac{da_0}{dt_0}+\frac{dz}{da_1}\frac{da_1}{dt_0}$$

$$=\frac{1}{a_1}a_0 H(t_0)-\frac{a_0}{a_1^2}\frac{da_1}{dt_1}\frac{dt_1}{dt_0}$$

$$=(1+z)H_0-(1+z)H_1\frac{dt_1}{dt_0}$$

$H_1=H(t_1)$

freq ~ $\frac{1}{\text{time}}$ ↙observed

$$\frac{dt_1}{dt_0}=\frac{\nu_0}{\nu_1}=\frac{1}{1+z}$$

↑emitted

So
$$\frac{dz}{dt_0}=(1+z)H_0-H_1$$

- Matter dom

$$H(z) = H_0 (1+z)^{3/2}$$

So

$$H(\overset{z=1}{1}) = H_0 (2)^{3/2}$$

$$\frac{dz}{dt_0} = (1+z) H_0 - H(t_1)$$

$$\cancel{\text{det}}\ \frac{dz}{dt_0} = \left((1+z) H_0 - H(t_1)\right)$$

$$\frac{dz}{dt_0} = 2 H_0 - 2^{3/2} H_0$$

$$\approx -0.83\, H_0$$

So for each Δt_0 z changes by $-0.83 H_0$

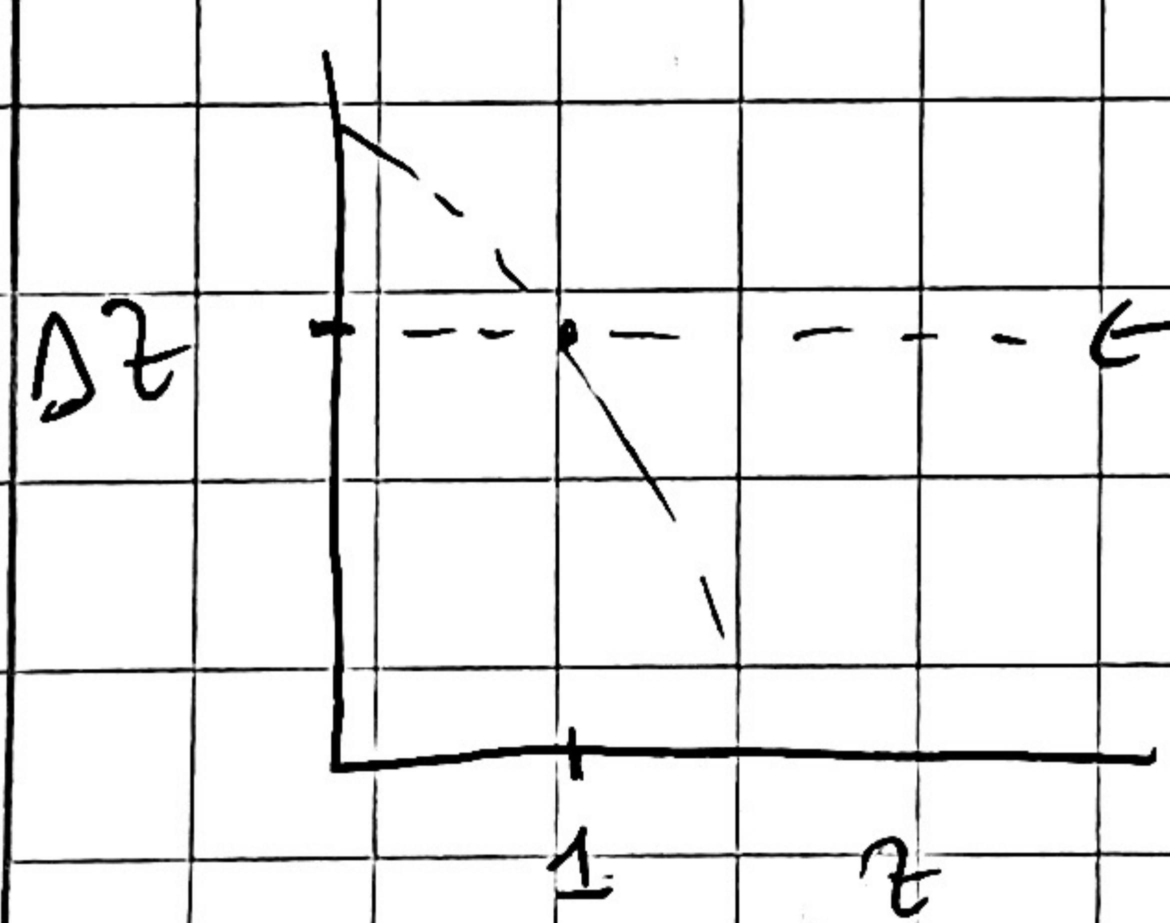
$$= -0.83 \cdot 100\, h \,\frac{km}{s\,Mpc} = -2.7 \cdot 10^{-18} \frac{h}{s}$$

So in 100 years we'll measure (take e.g. $h = 0.7$)

$$\Delta z = 100\,yr \cdot -2.7 \cdot 10^{-18} \cdot 0.7 \,\frac{1}{s}$$

$$= -5.9 \cdot 10^{-9}$$

This is exactly the same number as in the arxiv paper (Figure 6)



$\leftarrow -0.6 \cdot 10^{-8}$ for $\Omega_m = 1$