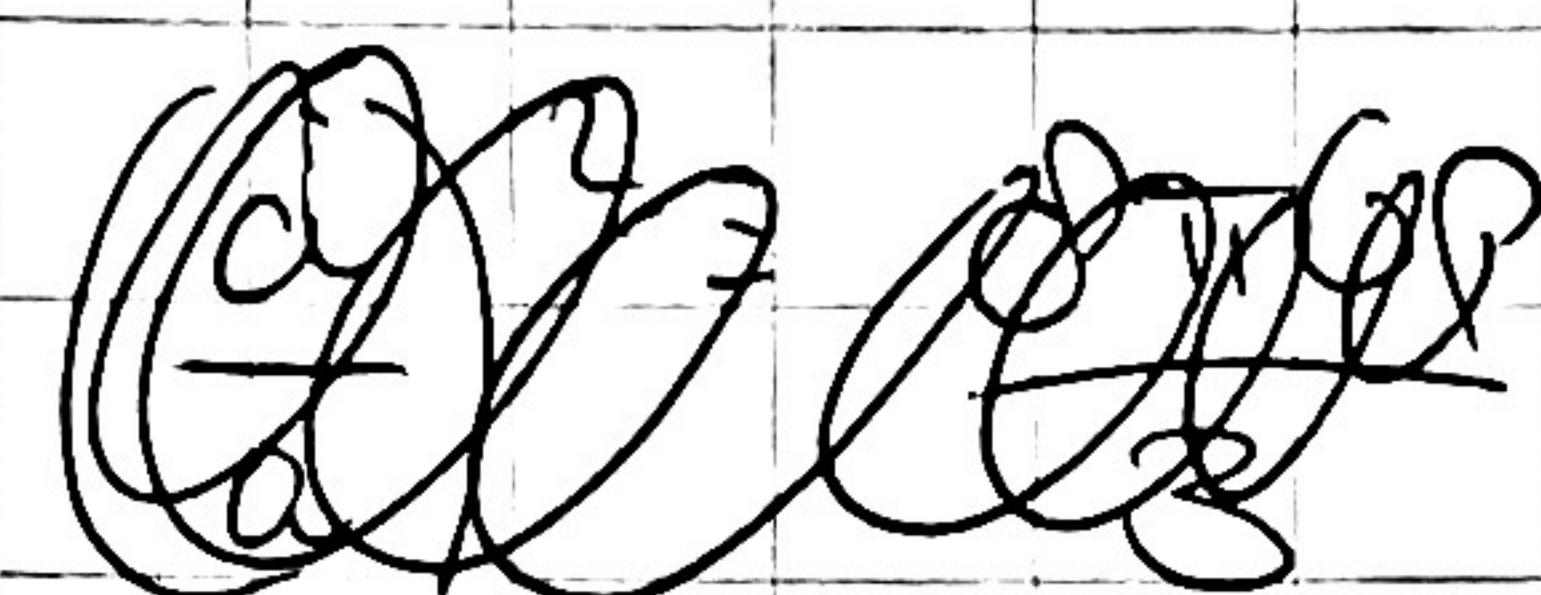


1) First Friedmann + continuity \rightarrow 2nd Friedmann

$$F1: \rho \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \quad k = \frac{K}{R_0^2}$$

$$\text{Cont: } \ddot{\rho} + 3H(\rho + p) = 0$$

~~Say terms~~



$$\dot{a}^2 = \frac{8\pi G \rho}{3} a^2 - k$$

derivative
time

$$\text{divide by } 2\dot{a}\ddot{a} \quad \ddot{a} = \frac{8\pi G}{3} (2\dot{a}\ddot{a} + \dot{a}^2 \ddot{\rho})$$

$$\ddot{a} = \frac{8\pi G}{3} \left(\dot{a} \ddot{\rho} + \frac{\dot{a}^2}{2\dot{a}} \ddot{\rho} \right)$$

$$\text{divide by } a \quad \frac{\ddot{a}}{a} = \frac{4\pi G}{3} \left(2\ddot{\rho} + \frac{\dot{a}}{a} \ddot{\rho} \right)$$

Plug in continuity $\ddot{\rho} = -3H(\rho + p)$ $H = \frac{\dot{a}}{a}$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3} \left(2\rho + -\frac{a}{\dot{a}} 3H(\rho + p) \right)$$

$$= \frac{4\pi G}{3} (2\rho - 3\rho - 3p)$$

$$= \frac{4\pi G}{3} (-\rho - 3p) = -\frac{4\pi G}{3} (\rho + 3p)$$

Friedmann II

$$2) F = \frac{L}{4\pi dL^2}$$

L in Watts, energy/s
flux energy passing through surface
 W/m^2

Intensity = Flux / solid angle $\text{W/m}^2/\text{sr}$

So Intensity and Flux evolve the same
over z (Steradian does not evolve)

$$I \propto \frac{L}{dL^2} \propto$$

From the lecture we saw that

L maps to $L \rightarrow L \cdot \frac{(1+z)^2}{(1+z)^2}$ because photon energy decreases with $(1+z)$ and the rate at which it arrives as $(1+z)$

$$dL = S u (1+z)$$

$$\text{so } dL^2 \propto (1+z)^2$$

From this we see

$$I \propto \frac{1}{(1+z)^4} \text{ or } I = I_0 (1+z)^{-4}$$

$$3) m = -2.5 \log_{10} \left(\frac{f}{f_0} \right)$$

M defined as m at 10 pc

$$f = \frac{L}{4\pi d^2}$$

$$m_1 - m_2 = -2.5 \log_{10} \left(\frac{f_1}{f_2} \right)$$

Say source 1 with apparent mag m_1
at distance d . $f_1 = \frac{L_1}{4\pi d^2}$

Source 2 with apparent mag M

at distance 10 pc $f_2 = \frac{L_2}{4\pi (10\text{pc})^2}$

$$m - M = -2.5 \log \left(\frac{\frac{L_1}{4\pi d^2}}{\frac{L_2}{4\pi (10\text{pc})^2}} \right)$$

L_1 and L_2 are the same, same source
but the different distances ~~with change~~
~~the apparent magnitudes make them look~~
brighter/dimmer.

$$\begin{aligned} m - M &= -2.5 \log \left(\left(\frac{10\text{pc}}{d} \right)^2 \right) \quad \text{taking out} \\ &= 5 \log \left(\left(\frac{d}{10\text{pc}} \right)^{-2} \right) \quad \text{the squared} \\ &\quad \text{and putting} \\ &\quad \text{in the} \\ &\quad \text{minus} \\ &\quad \text{sign} \end{aligned}$$

So indeed

$$m = M + 5 \log \left(\frac{d}{10\text{pc}} \right)$$

$d = D_L$, depends
on z so $D_L(z)$

call this
 $DM(z)$

- $L_v \leftarrow$ on frequency band v

$$f = \int f_v dv$$

$$f_v = \frac{df(v)}{dv} \quad L_v = \frac{dL(v)}{dv}$$

Photons emitted with frequency v_e in a band with bandwidth dv_e at redshift z will be observed with

$$v = \frac{v_e}{1+z} \quad v_e = v(1+z)$$

$$\text{and } dv = \frac{dv_e}{1+z}$$

$$\text{So } f_v = \frac{df}{dv} = \frac{df}{dv_e} \frac{dv_e}{dv} = (1+z) \frac{df}{dv_e}$$

$$f = \frac{L}{4\pi d_L^2}$$

$$\frac{df}{dv_e} = \frac{d}{dv_e} \left(\frac{L}{4\pi d_L^2} \right) = \frac{1}{4\pi d_L^2} \frac{dL}{dv_e}$$

$$= \frac{1}{4\pi d_L^2} L_{v_e}$$

this is how L_v is defined

$$\begin{aligned} \text{So } f_v &= (1+z) \frac{1}{4\pi d_L^2} L_{v_e} \quad v_e = v(1+z) \\ &= (1+z) \frac{1}{4\pi d_L^2} L_v (1+z)^c \end{aligned}$$

Specific flux, sometimes written as S_v
we get exactly the expression as on the
problem set

with this new equation for flux

$$f_V$$

we still write

$$m = -2.5 \log\left(\frac{f_V}{f_0}\right) = -2.5 \log(f_V) + \text{const}$$

\uparrow
in I
band

$$f_V = (1+z) \frac{L}{4\pi d_L^2} (L_{V(1+z)})$$

$$m = -2.5 \log\left(\frac{(1+z)}{4\pi d_L^2} L_{V(1+z)}\right) + \text{const}$$

$$= -2.5 \log\left(\frac{(1+z)}{\cancel{4\pi}} \frac{L_{V(1+z)}}{L_V} \frac{L_V}{d_L^2}\right) + \text{const}$$

$$= -2.5 \log((1+z) \frac{L_{V(1+z)}}{L_V}) + 2.5 \log\left(\frac{L_V}{d_L^2}\right)$$

$$- 2.5 \log(L_V) - 2.5 \log(d_L^{-2}) + \text{const}$$

$$= -2.5 \log((1+z) \frac{L_{V(1+z)}}{L_V}) - 2.5 \log(L_V) + 5 \log(d_L^{-2}) + \text{const}$$

$$= -2.5 \log((1+z) \frac{L_{V(1+z)}}{L_V}) - 2.5 \log(L_V) + 5 \log\left(\frac{d}{10\text{pc}}\right) + \text{const}$$

For $d=10\text{pc}$ ($z=0$) m should be M

changed
const
to plug in
 10pc

$$\begin{aligned} M &= 0 - 2.5 \log(L_V) + 0 + \text{const} \\ &= -2.5 \log(L_V) + \text{const} \end{aligned}$$

$$\begin{aligned} \text{So } m &= -2.5 \log((1+z) \frac{L_{V(1+z)}}{L_V}) + M + 5 \log\left(\frac{d}{10\text{pc}}\right) \\ &\quad \text{this is indeed } k \\ &= k + M + DM \end{aligned}$$

$$4) \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho_m}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}$$

wrt t

$$\text{derivative} \quad \dot{a}^2 = \frac{8\pi G \rho_m a^2}{3} + \frac{\Lambda a^2}{3} - k$$

$$2\ddot{a}\dot{a} = \frac{d}{dt} \left(\frac{8\pi G \rho_{m,0}}{3} \frac{1}{a} \right) + 2\frac{\Lambda}{3} a \dot{a}$$

$$2\ddot{a}\dot{a} = -\frac{8\pi G \rho_{m,0}}{3} \dot{a}^2 + 2\frac{\Lambda}{3} a \dot{a}$$

divide by $2\dot{a}$

$$\ddot{a} = -\frac{4\pi G \rho_{m,0}}{3 a^2} + \frac{\Lambda a}{3}$$

divide by a

$$\frac{\ddot{a}}{a} = -\frac{4\pi G \rho_{m,0}}{3 a^3} + \frac{\Lambda}{3}$$

New approach

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2}$$

$$\text{now } \rho = \rho_m + \rho_n$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} (\rho_m + \rho_n) - \frac{k}{a^2}$$

$$= \frac{8\pi G \rho_m}{3} + \underbrace{\frac{8\pi G \rho_n}{3}}_{\sim} - \frac{k}{a^2}$$

call this $\frac{\Lambda}{3}$ to get the expression

$$\text{so } \Lambda = 8\pi G \rho_n$$

- Find μ, Λ such that $\ddot{a} = 0, \ddot{\dot{a}} = 0$

$\ddot{a} = 0$ gives

$$0 = \frac{8\pi G P_m}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}$$

$$P_m = P_{m,0} a^{-3}$$

a derive wst

$$\left(\frac{\ddot{a}}{a}\right)^2 = \frac{8\pi G P_m}{3} + \frac{\Lambda}{3} - \frac{k}{a^2} \quad \text{derive wst t}$$

$$\ddot{a}^2 = \frac{8\pi G P_m a^2}{3} + \frac{\Lambda a^2}{3} - k \quad \text{f}$$

$$2\ddot{a}\ddot{a} = -\frac{8\pi G P_{m,0} a^2 \ddot{a}}{3} + 2\frac{\Lambda a \ddot{a}}{3}$$

$$\ddot{a} = -\frac{4\pi G P_{m,0} a^{-2}}{3} + \frac{\Lambda a}{3}$$

$\ddot{a} = 0$ gives

$$0 = -\frac{4\pi G P_{m,0} a^{-2}}{3} + \frac{\Lambda a}{3}$$

comparing this
with previous
page I find

$$\Lambda = 4\pi G P_{m,0} a^{-3}$$
$$= 4\pi G P_m$$

$$\frac{4\pi G P_{m,0}}{3} a^{-2} = \frac{\Lambda a}{3}$$

$$\Lambda = 8\pi G P_m$$
$$\text{so } P_m = \frac{1}{2} P_{m,0}$$

$$\Lambda = 4\pi G P_{m,0} a^{-3}$$

$$\Lambda = -\frac{8\pi G P_{m,0}}{3a^3} + \frac{3k}{a^2}$$

also

$$\Lambda = 4\pi G P_m = 4\pi G S_{m,0}$$
$$= 4\pi G S_{m,0} \frac{3m_0^2}{8\pi G} \text{ per unit }$$

$$12\pi G P_{m,0} a^{-3} = \frac{3k}{a^2} \rightarrow \frac{k}{a^2} = 4\pi G P_m$$

~~cancel 2nd term~~

$$= \frac{3S_{m,0}}{2} \text{ m}$$

~~cancel 2nd term~~ so it's open

Sorry for the messy writing, I found

$$\Lambda = 4\pi G \rho_m \\ = 8\pi G \rho_n$$

$$\text{so } \rho_n = \frac{1}{2} \rho_m$$

and

$$\Lambda = \frac{3}{2} S m_0 H_0^2$$

Then I found $\frac{u}{a^2} = 4\pi G \rho_m$, since $\rho_m > 0$,

k must be +1, so an open universe

- Perturbations $a(t) = 1 + \delta a(t)$ $a(t)$
 $\rho_m(t) = \rho_m(1 - 3\delta a(t))$

Second Friedmann eq

$$\frac{\ddot{a}}{a} = -\frac{4\pi G \rho_m}{3} + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G \rho_m(1 - 3\delta a(t))}{3} + \frac{\Lambda}{3}$$

call $\delta_a(t) = \delta(t)$ (for my simplicity)

$$a = 1 + \delta(t)$$

$$\ddot{a} = \ddot{\delta}$$

$$\rho_m = \rho_m(1 - 3\delta(t))$$

$$\frac{\ddot{a}}{a} = \frac{-4\pi G\rho_m}{3} + \frac{1}{3} = \text{exp}$$

$$\frac{\ddot{\delta}}{1+\delta} = \frac{-4\pi G\rho_m(1-\delta)}{3} + \frac{1}{3}$$

We found before that $\Lambda = 4\pi G\rho_m$, so

$$\frac{\ddot{\delta}}{1+\delta} = 4\pi G\rho_m \approx \delta = \Lambda \delta$$

$$\ddot{\delta} = \Lambda(1+\delta)\delta$$

neglect higher order terms $\propto \delta^2$ because $|\delta| \ll 1$

$$\ddot{\delta} = \Lambda \delta$$

Initial conditions $\delta(0) = \delta_{a0}$ $\frac{d(\delta)}{dt} = 0$

$$\delta'' - \Lambda \delta = 0 \quad \text{Guess } \delta = e^{rt} \quad \ddot{\delta} = r^2 e^{rt}$$

$$r^2 e^{rt} - \Lambda e^{rt} = 0$$

$$(r^2 - \Lambda) = 0$$

$$r = \pm \sqrt{\Lambda}$$

2 solutions

$$\text{So } \delta(t) = A e^{\sqrt{\Lambda}t} + B e^{-\sqrt{\Lambda}t}$$

$$\delta'(t) = A \sqrt{\Lambda} e^{\sqrt{\Lambda}t} - B \sqrt{\Lambda} e^{-\sqrt{\Lambda}t}$$

$$\delta'(0) = 0 \text{ gives } A e^{\sqrt{\Lambda}t} = B e^{-\sqrt{\Lambda}t}$$

so δ becomes easier

$$\delta(t) = 2A e^{\sqrt{\Lambda}t} \rightarrow \delta(0) = \delta_{a0} \text{ so } 2A = \delta_{a0}$$

$$\delta(t) = \delta_{a0} e^{\sqrt{\Lambda}t}$$

This grows with time, so a small perturbation will grow and grow, making the static solution not static/unstable.

5 Redshift Drift

call $a(t_0) = a_0$
 $a(t_1) = a_1$

$$1+z = \frac{a(t_0)}{a(t_1)}$$

$$z = \frac{a_0 - 1}{a_1}$$

$$\frac{dz}{da_0} = \frac{1}{a_1}$$

$$\frac{\lambda_0}{\lambda_e} = 1+z$$

λ_e

$$H = \frac{\dot{a}}{a}$$

t_1 emitted

t_0 observed

$$\frac{dz}{da_1} = -\frac{a_0}{a_1^2}$$

$$\cancel{\frac{dz}{dt_0} = \frac{dz}{da_0} \frac{da_0}{dt_0} + \frac{dz}{da_1} \frac{da_1}{dt_1}}$$

$$\cancel{= \frac{dz}{da_0} a_0 \cdot \frac{da_0}{dt_0} + \frac{dz}{da_1} a_1 \frac{da_1}{dt_1}}$$

$$\cancel{= \frac{dz}{da_0} a_0 H(t_0) + \frac{dz}{da_1} a_1 H(t_1)}$$

$$\cancel{= \frac{a_0}{a_1} H(t_0) + a_0 a_1 H(t_1)}$$

$$\cancel{= (1+z) H_0^{(t_0)} - (1+z) H_1^{(t_1)}}$$

$$\frac{1}{a} \frac{da}{dt} = H$$

$$\frac{da}{dt} = H a$$

$$\cancel{\frac{dz}{dt_0} = \frac{dz}{da_0} \frac{da_0}{dt_0} + \frac{dz}{da_1} \frac{da_1}{dt_0}}$$

$$\cancel{= \frac{a_1}{a_0} H(t_0) - \frac{a_0}{a_1^2} \frac{da_1}{dt_1} \frac{dt_1}{dt_0}}$$

$$\cancel{= (1+z) H_0 - (1+z) H_1 \frac{dt_1}{dt_0}}$$

free time
observed

$$\frac{dt_1}{dt_0} = \frac{v_0}{v_1} = \frac{1}{1+z}$$

↑
emitted

$$H_1 = H(t_1)$$

$$\text{So } \frac{dz}{dt_0} = (1+z) H_0 - H_1$$

- Matter dom

$$H(z) = H_0(1+z)^{3/2}$$

So z_1

$$H(t) = H_0(2)^{3/2}$$

$$\frac{dz}{dt_0} = (1+z)H_0 - H(t_1)$$

or

$$\text{der } \frac{dz}{dt_0} = ((1+z)H_0 - H(t_1)) \cancel{\text{der}}$$

$$\frac{dz}{dt_0} = (6.2 \cancel{H_0}) 2H_0 - 2^{3/2} H_0$$

$$\approx -0.83 H_0$$

so for each dt_0 z changes by

$$-0.83 H_0$$

$$= -0.83 \cdot 100 \text{ km} = -2.7 \cdot 10^{-10} \text{ h}$$

$$8 \text{ Mpc} \quad \text{s}$$

So in 100 years we'll measure

(take e.g. $h=0.7$)

$$\Delta z = 100 \text{ yr.} \cdot -2.7 \cdot 10^{-10} \cdot 0.7 \perp$$

s

$$= -5.9 \cdot 10^{-9} \cancel{10^8}$$

This is exactly the same number as in the arXiv paper (Figure 6)

