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1 Introduction

In this project, we are modeling the temperature inside of a building, factoring in the outside temperature, heat produced by people machinery and lights inside of the building, as well as artificial heating and cooling by furnaces and air conditioners.

We are using differential equations and important related concepts in order to do this analysis and better understand the temperature of this building and what contributes to it. One of the important concepts we used was Newton's Law of Cooling on how the outside temperature affected the temperature inside of the building.

We are also using concepts such as the integrating factor method, Picard's theorem, finding equilibrium solutions, finding maxima and minima, and solving initial value problems to accurately analyze the effects of different parameters (machinery contributions, artificial air and conditioning contributions, etc.) on the inside temperature of the building.

2 Beginning Analysis

Our project started with a first order, linear, homogenous, variable coefficient differential equation given by equation 1:

$$\frac{dT}{dt} = \kappa[M(t) - T(t)] + H(t) + Q(t) \qquad (1)$$

In this equation, if Q(t) was changed to $Q(t) = tT$, the classification would change to non-homogeneous.

Using Picard's theorem we were able to guarantee a unique solution to the equation for any initial condition when M(t), Q(t), and H(t) are with no variable in the denominator with no asymptotes and no exponent variables. We found that if Q(t) = tT then the function would remain continuous and unique. To explain in the physical sense, the continuity property of the differential equation means the temperature is able to be increased or decreased continuously, at any point so that the temperature can exist at any value. The uniqueness reflects the rate of change of temperature only.

By using the integrating factor method, we were able to find the general solution to equation 3…

$$\frac{dT}{dt} + \kappa T(t) = H(t) + Q(t) - \kappa M(t)$$

$$\frac{dy}{dx} + P(x)y = Q(x) \quad ; \qquad I.F. = e^{\int P(x)dx}$$

$$P(x) = \kappa \qquad ; \qquad e^{\int k dt} = e^{kt}$$

$$T(t) = \frac{\int (H(t) + Q(t) - \kappa M(t))e^{kt} dt + c}{e^{kt}} \qquad (2)$$

We can call this general solution equation 2.

Now, assuming that there are no people in the building, the machinery and lights are off, there is no heating or cooling systems running, and the outside temperature is constant with M_0, we can arrive at a new differential equation…

$$\frac{dT}{dt} = \kappa[M_0 - T(t)] \quad (3)$$

$$where\ \frac{dT}{dt} = A(t)$$

This new differential equation (equation 3) had the equilibrium solution $M_0 = T(t)$. We know that if $A(t) > 0$, then $M_0 > T(t)$ and if $A(t) > 0$, then $M_0 < T(t)$ which means the function converges and is therefore stable. In physical terms, if the air surrounding the building is warmer than the temperature inside the building, then the temperature of the surroundings in which the object is located is greater than the temperature of the object. And vice versa.

Using equation 2 to solve the initial value problem consisting of equation 3 and the initial condition $T(t_0) = T_0$…

$$T(t) = \int 0 + 0 - \frac{\int \kappa M_0 e^{kt}}{e^{kt}} dt$$

$$T(t) = - \frac{\kappa M_0 (\frac{1}{\kappa}) e^{kt}}{e^{kt}}$$

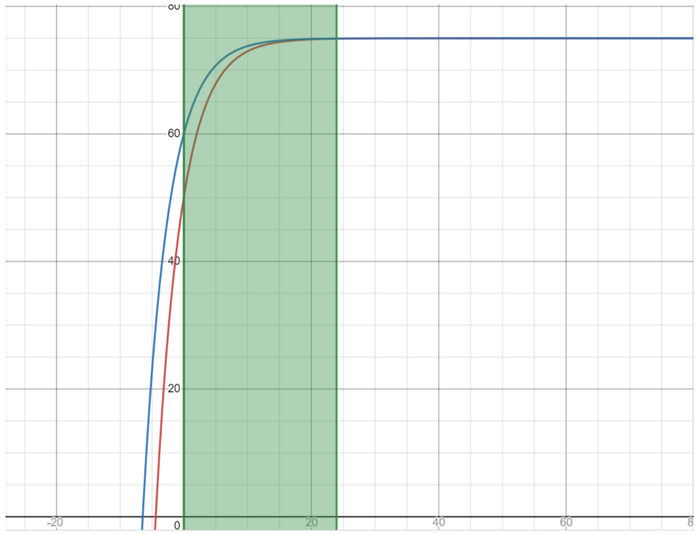
$$T(t) = \frac{M_0 e^{kt} + c}{e^{kt}}$$

$$T(t) = M_0 + \frac{T_0 - M_0}{e^{kt}} \qquad (4)$$

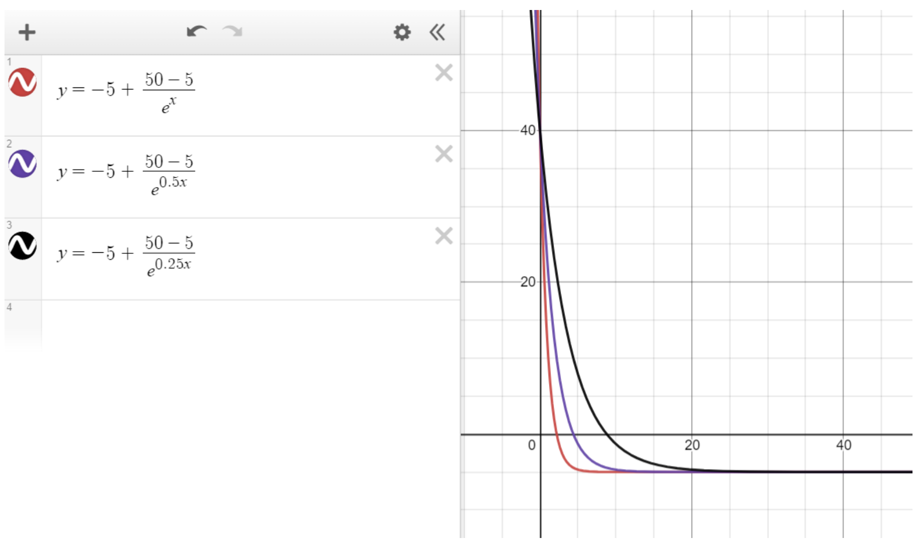
$$c = T_0 - M_0$$

$$Equilibrium \rightarrow T = M_0$$

If $M_0 = 75$, $t_0 = 0$, $T_0 = 50, 80$, and $\kappa = 0.25$ along the interval of $0 \leq t \leq 24$, we can graph their relationship using solution 4…



If $M_0 = 75$, $T_0 = 50$, κ = 1, ½, ¼, and $t_0 = 0$ along the interval of $0 \le t \le 24$, we can graph their relationship using solution 4. We can see from the graph below that the larger κ is, the larger the slope (rate of change) of temperature is. This means as κ increases, the temperature decreases at a faster rate. κ could change depending on air turbulence, humidity, and type of medium/gas.



When the difference between the buildings temperature and outside temperature is e^{-1} of the initial difference, we can use equation 4 to find the value of $\Delta t = t - t_0$. This is also known as the time constant, whose units would be hours per degree fahrenheit. When designing a building and considering temperature changes due only to the outside temperature, you would want the building to have a larger time constant. This would make it so the building responds slower.

$$T_0 - M_0 \rightarrow Initial\ Difference$$

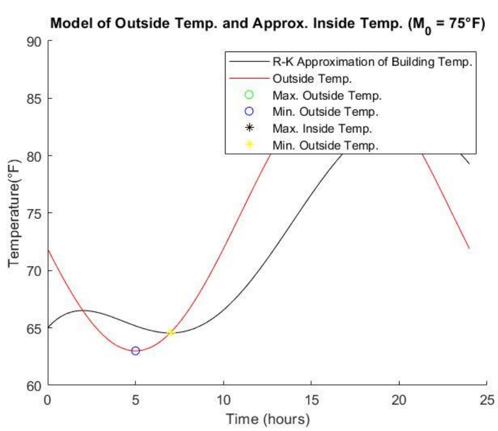
$$T_0 - M_0 = \frac{T_0 - M_0}{e^{\kappa \Delta t}}$$

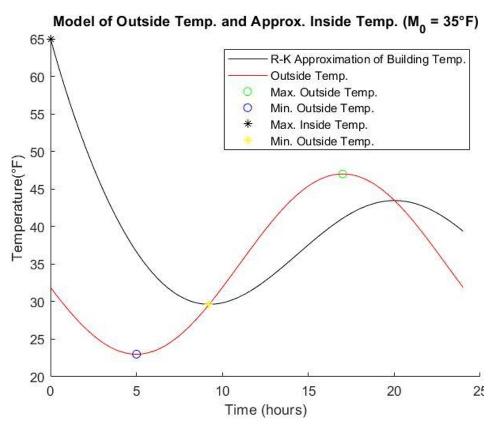
$$\frac{T_0 - M_0}{T(t) - M_0} = e^{\kappa \Delta t}$$

$$\frac{1}{e^{-1}} = e^{\kappa \Delta t}$$

$$\Delta t = \frac{1}{\kappa}$$

3 Refining Model

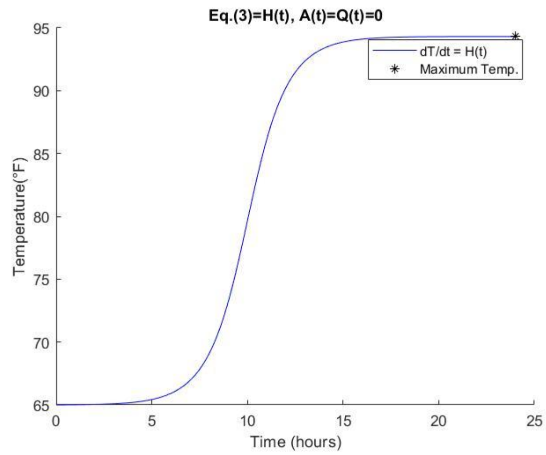




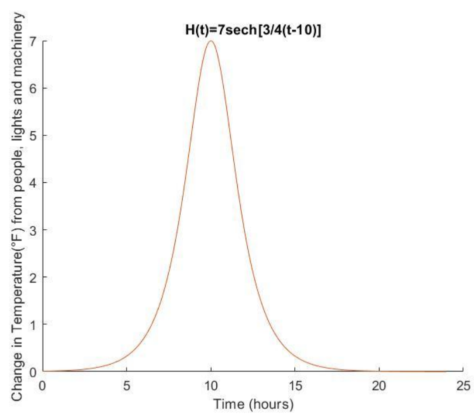
These two graphs represent a varying outside temperature with no other heating or cooling given by the equation:

$$M(t) = M_0 - 12cos[\frac{\pi(t-5)}{12}] \qquad (5)$$

When $M_o = 75$ (the graph on the left), the maximum outdoor temperature is 87°F and occurs at 17 hours and the minimum temperature is 63°F and occurs at 5 hours. The indoor temperature on the other hand hits its maximum temperature at 83.2°F at 20 hours and 6 minutes and its minimum temperature at 69.6°F and at 7 hours. When $M_0 = 35$ (the graph on the right) the maximum outdoor temperature is 47°F and it occurs at 17 hours. The minimum outdoor temperature is 29.6°F and it occurs at 9 hours and 12 minutes. The temperature of the building follows the trends of the outside temperature so as the outside temperature increases, so does the indoor temperature and vice versa.



This graph was created by using equation 1 and the initial condition $T(0) = 65$. This gives the maximum temperature of 94°F being achieved at about 24 hours.

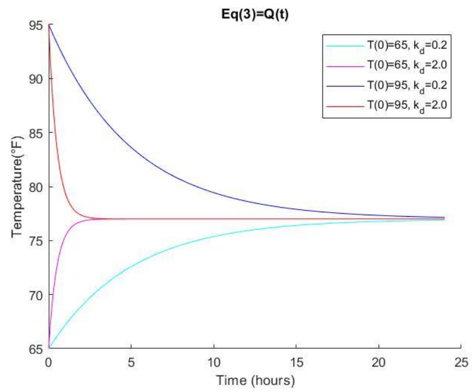


This graph is given by graphing the equation:

$$H(t) = 7sech[\frac{3}{4}(t - 10)] \qquad (6)$$

with $sech(t) = \frac{1}{cosh(t)} = \frac{2}{e^{t}+e^{-t}}$

A nonmathematical description for H(t) is the relationship where the more people that enter the building, the more the temperature will increase.
This does make sense for H(t) because the graph is increasing the quickest at t=10 which in the equation of H(t) would make it the maximum because the $(t - 10)$ term would go away.



This graph was created by graphing the following equation with $T_d = 77$:

$$Q(t) = \kappa_d(T_d - T) \qquad (7)$$

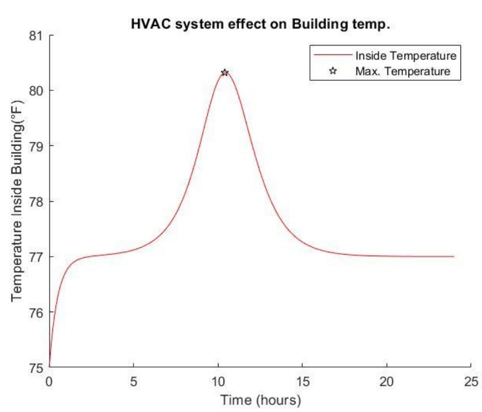
and when T(0) = 65, 95 and when $\kappa_d = 0.2, 2$. This resulted in converging lines showing κ_d is a constant that over time, causes the temperature to converge to the same value of around 77°F. Given Q(t) is the artificial heating and cooling due to furnaces and air conditioners, this 77°F could be the ideal temperature for the machines inside of the building, meaning the artificial heating and cooling will need to heat or cool the air in the building to this temperature and that is what κ_d would represent.

4 Putting it Together

Now we are looking at what would happen if the inside temperature of the building exceeded 81°F (the limit that equipment would be damaged) given the equation:

$$\frac{dT}{dt} = 7sech[\frac{3}{4}(t - 10)] + 2(77 - T)\,;\ T(0) = 75 \qquad (8)$$

This can be modeled with the following graph.

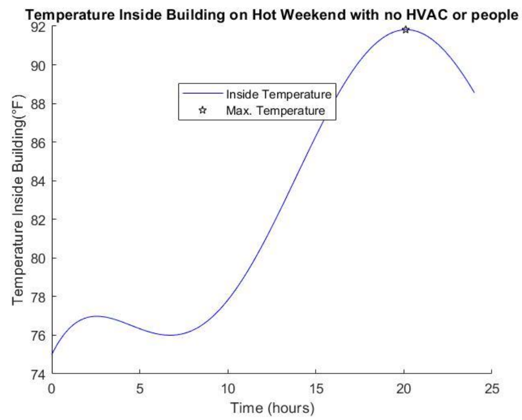


This graph shows the maximum temperature of the building is 80.3°F at 10 hours and 24 minutes. This does not exceed the limit of 81°F so the air conditioning system does prevent damage to the equipment. This graph also agrees with the previous finding that the air conditioning system will maintain a constant temperature of 77°F.

If there are no people in the building, the lights and machines are off, and there are no furnaces or air conditioners, we have the equation:

$$\frac{dT}{dt} = 0.25\{85 - 10cos[\frac{\pi(t-5)}{12}] - T\}\ ;\ T(0) = 75 \qquad (9)$$

In this case, a temperature of 81°F will still damage the equipment in the building. This equation can be modeled by the following graph.

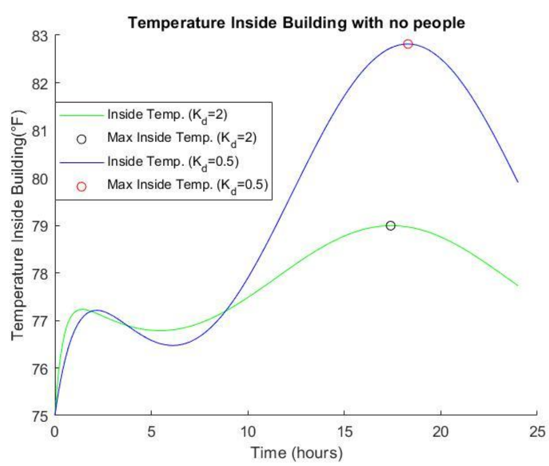


This graph shows that once the time has reached 12 hours and 12 minutes, the temperature will exceed 81°F and the equipment will be damaged. The maximum temperature that will be reached is 91.8°F.

If we want to analyze the previous scenario but with the change of parameters that the furnace and air conditioning systems are on, the equation becomes:

$$\frac{dT}{dt} = 0.25\{85 - 10cos[\frac{\pi(t-5)}{12}] - T\} + \kappa_d(77 - T)\ ;\ T(0) = 75 \qquad (10)$$

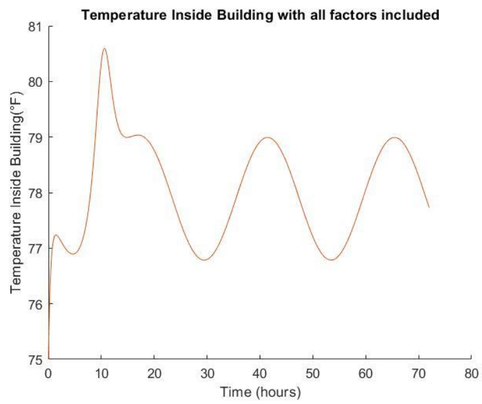
This can be modeled by the following graph.



The maximum temperature when $\kappa_d = 2$ will be 79°F and the maximum temperature when $\kappa_d = 0.5$ will be 82.8°F. After around 8 hours and 48 minutes, the temperature will exceed the limit of 81°F when $\kappa_d = 0.5$. When $\kappa_d = 2$ however, the temperature never exceeds 79°F so the equipment will not be damaged.

When all factors of temperature are included, we obtain the following graph which is given by the equation:

$$\frac{dT}{dt} = 0.25\{85 - 10cos[\frac{\pi(t-5)}{12}] - T\} + 7sech[\frac{3}{4}(t - 10)] + 2(77 - T)\,;\ T(0) = 75 \quad (11)$$

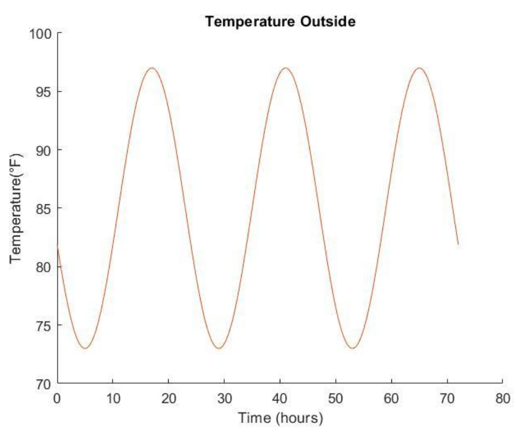


In this case, the equipment will not be damaged because the temperature will never exceed 81°F over the three day period. The graph approaches a steady state as it oscillates steadily between 77°F and 79°F.

Now we are plotting the equation:

$$M(t) = 85 - 10cos[\frac{\pi(t-5)}{12}]\,;\ 0 \le t \le 72 \quad (12)$$

Which gives the graph:



It is clear based on this graph that the temperature outside of the building greatly affects the temperature in the building as shown by the large oscillations. In non mathematical terms, as the outside temperature increases, so does the temperature inside the building which causes the air conditioning to turn on and temperature approaches that steady state when it oscillates between 77°F and 79°F.

5 Conclusion

In this project, we modeled the temperature inside of a building, factoring in the outside temperature, heat produced by people machinery and lights inside of the building, as well as artificial heating and cooling by furnaces and air conditioners by using differential equations and important related concepts in order to determine the temperature of this building and what contributes to it. By using concepts such as the integrating factor method, Picard's theorem, finding equilibrium solutions, finding maxima and minima, and solving initial value problems we were able to accurately analyze the effects of different parameters (machinery contributions, artificial air and conditioning contributions, etc.) on the inside temperature of the building. The models we created showed exactly what parameters (i.e. the machinery in the building, the lights, the people, etc.) would affect the temperature inside of the building based on the maximums found on the models of the differential equations. We were able to see that if the HVAC system was on, the equipment would not be damaged since the temperature would not exceed 81°F.

6 Appendix

Analysis Pages: Project01.pdf
MATLAB Code: Project1Code.pdf
https://drive.google.com/file/d/1QT1RLjRPHUANo1CSHkQd2KGh5Xink9tM/view
https://drive.google.com/file/d/1aCulC3Bb9l0x9z8IUsAIm7BzVaiZ3Hbc/view?usp=sharing