Lebesgue Integration (from Wikipedia)

$A_f(t) = \{x \mid f(x) > t\}$

$\mu : A_f \to \mathbb{R}^+$, ~~$\Delta\mu = \mu(t+\Delta t) - \mu(t)$~~

size of strip: $\mu(A_f(t))\,dt$

$$\int_0^\infty \mu(A_f(t))\,dt$$

hyper planes have 0 measure in their ambient space

σ-finite : $\mu < \infty$ ie $\mu : \Sigma \to [0, \infty)$

measure able function: if pre-image of any measurable set is measurable
this way you can measure the domain size that results in a given output range. Eg, the number of events with a given probability

Lebesgue measure able function: pre-images of (t, ∞) $\forall t \in (0, \infty)$ are measurable
that is, $\{x \mid f(x) > t\} \in \Sigma$

1 Distribution function

1.1 Monotone functions

f is increasing : $(x_1 < x_2 \Rightarrow f(x_1) \le f(x_2))$

$t \uparrow x$: $t < x,\ t \to x$, $\quad t \downarrow x$: $t > x,\ t \to x$

For increasing $f : \mathbb{R} \to \mathbb{R}$ (monotone)

(i.) unilateral limits exist: $\lim_{t \uparrow x} f(t) = f(x-)$, $\lim_{t \downarrow x} f(t) = f(x+)$ and are finite $\forall x$

and (maybe not finite) limits at ∞ exist:

$\lim_{t \downarrow -\infty} f(t) = f(-\infty)$, $\lim_{t \uparrow \infty} f(t) = f(\infty)$

and $f(x-) = \sup_{-\infty < t < x} f(t)$, $f(x+) = \inf_{x < t < \infty} f(t)$ b/c of monotinicity

(ii.)

f is continuous at x iff $f(x-) = f(x) = f(x+)$

or equivalently, $\lim_{t \uparrow x} f(t) = f(x) = \lim_{t \downarrow x} f(t)$

~~and~~ $f(x-) \le f(x) \le f(x+)$ $\longleftarrow$ except still

"jump at x" if $f(x-) \ne f(x+)$ but both exist. (no condition on $f(x)$)

(iii.) only kind of discontinuity is a jump

"x is a jump point of f" and $f(x+) - f(x-)$ is "size of the jump"

Interlude: an open set means $\forall x, \exists \varepsilon$ s.t. $B_x(\varepsilon)$ is also in the set.
A closed set is one who's complement is open.
Note that $(a,b]$ (not closed or open) can be expressed as $(a,c) \cup [c,b]$
$\forall c \in (a,b]$

the set of jump points can be open

Example 1.

consider something like

accumulation, but not jump point
point

Example 2.

$\{a_n\} \subset$ domain , $\{b_n\}$ s.t. $\sum b_n < \infty$ and $b_n \ge 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \delta_{a_n}(x)$$

$$x_2 > x_1 \Rightarrow \delta_{a_n}(x_2) \ge \delta_{a_n}(x_1) \Rightarrow f(x_2) - f(x_1) = \sum_{n=1}^{\infty} b_n \left(\delta_{a_n}(x_2) - \delta_{a_n}(x_1)\right) \ge 0$$

absolute convergence b/c $|b_n \delta_{a_n}(x)| \le b_n$ and $\{b_n\}$ converges

continue $\longrightarrow$

/

Now for uniform convergence. Show $\forall \varepsilon > 0, \exists N > 0$ s.t. $|S_m(x) - S_\infty(x)| < \varepsilon$ for $m > N$

where $S_m = \sum_{n=1}^{m} b_n \delta_{a_n}(x)$, so $|S_\infty(x) - S_m(x)| = \sum_{n=m+1}^{\infty} b_n \delta_{a_n}(x)$

(but $|b_n \delta_{a_n}(x)| \leq b_n$ and $\{b_n\}$ converges, so by the Weierstrass test, the series converges uniformly.)

but continuing w/ other method, $\lim_{m\to\infty} \sup_x |S_\infty(x) - S_m(x)| = \lim_{m\to\infty} \sum_{n=m+1}^{\infty} b_n = 0$

Now

$$f(x+) - f(x-) = \sum_{n=1}^{\infty} b_n \left(\delta_{a_n}(x+) - \delta_{a_n}(x-)\right)$$

remember $\delta_{a_n}(x) = 0$ for $x < a_n$, and 1 for $x \geq a_n$

suppose $x = a_n$, then $\delta_{a_n}(x+) = 1$ and $\delta_{a_n}(x-) = 0$

suppose $x < a_n$, then $\delta_{a_n}(x+) = \delta_{a_n}(x-) = 0$

suppose $x > a_n$, then $\delta_{a_n}(x+) = \delta_{a_n}(x-) = 1$

thus $f(x+) - f(x-) = b_n$ if x is a_n, and 0 otherwise

so jumps only at every point a_n, a countable set of jump points

this shows the set of jump points might be dense in the domain (eg, a_n are rationals)

$$f(x+) - f(x-) = \lim_{t\downarrow x} f(t) - \lim_{t\uparrow x} f(t) = \lim_{t\downarrow x} \sum_{n=1}^{\infty} b_n \delta_{a_n}(t) - \lim_{t\uparrow x} \sum_{n=1}^{\infty} b_n \delta_{a_n}(t)$$

but since the series is uniformly convergent, we can transfer to the limit, which is $f(t)$, so we can operate on this limit (w/ our operator being another limit, $\lim_{t\Downarrow x}$) instead of operating on the series. so we get:

$$\sum_{n=1}^{\infty} b_n \left(\delta_{a_n}\left(\lim_{t\downarrow x} t\right) - \delta_{a_n}\left(\lim_{t\uparrow x} t\right)\right) = \sum_{n=1}^{\infty} b_n \left(\delta_{a_n}(x+) - \delta_{a_n}(x-)\right) \text{ as above}$$

theorem: The set of discontinuities of f is countable, monotone f

proof:

for jump point x, consider the bijection $g(x) = (f(x-), f(x+))$

we argue for $x_2 > x_1$, the intervals are disjoint, and since they are open they each contain a rational number, and thus they are countable so the jump points must be countable by bijection.

we have $x_1 < x_2 \Rightarrow f(x_1+) \leq f(x_2-)$

thus even if $f(x_1+) = f(x_2-)$, neither $g(x_1)$ nor $g(x_2)$ contain this point so they are disjoint

theorem: monotone f_1, f_2; dense D; $\forall x \in D\ f_1(x) = f_2(x) \Rightarrow$ {same jump points and same size and $f_1(x) = f_2(x)\ \forall$ points of continuity

proof:

$\forall x,\ f_1(x-) = \lim_{n\to\infty} f_1(t_n) = \lim_{n\to\infty} f_2(t_n) = f_2(x-)$ for $\{t_n\} \to x$

similar for $f_1(x+) = f_2(x+)$

then we get $f_1(x+) - f_1(x-) = f_2(x+) - f_2(x-)$

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Suppose $\tilde{f}(x) = f(x+)$

$\tilde{f}$ is right continuous because, show that $\lim_{t \downarrow x} \tilde{f}(t) = f(x+) \Rightarrow$

$\lim_{t \downarrow x} f(t+) = f(x+)$

For $\varepsilon > 0, \exists \delta > 0$ s.t. $t - x < \delta \Rightarrow f(t+) - f(x+) < \varepsilon \quad \cancel{\Longleftrightarrow}$

$\cancel{\forall s \in (x, x+\delta) \Rightarrow f(s) - f(x+) < \varepsilon \Rightarrow} f(s+)$ idk, too abstract

Def: f increasing on D, dense in domain

$\forall x: \tilde{f}(x) = \inf_{\substack{t \in D: \\ x < t}} f(t) \quad \Rightarrow \tilde{f}$ is increasing and right-continuous

proof:

increasing obvious. For right-continuity, show

$\lim_{s \downarrow x} \tilde{f}(s) = \lim_{s \downarrow x} \inf_{\substack{t \in D: \\ s < t}} f(t) = \inf_{\substack{t \in D: \\ x < t}} f(t)$

or show $\tilde{f}(x) - \tilde{f}(x_0) \leq \varepsilon$ for $x_0 < x$

Since D is dense, $\exists\, t_0 \in D$ arbitrarily close to x_0 such that

$\tilde{f}(t_0) - \tilde{f}(x_0) \leq \varepsilon$

then $\forall x$ s.t. $x_0 < x < t_0$, $\tilde{f}(x) - \tilde{f}(x_0) \leq \varepsilon$

1.2 Distribution Functions

$\forall x, \; -\infty \leq f(-\infty) \leq f(x) \leq f(\infty) \leq \infty$ then we can define "normalized" f as

$\tilde{f}(x) = \dfrac{f(x) - f(-\infty)}{f(\infty) - f(-\infty)}$, so $\tilde{f}(-\infty) = 0$, $\tilde{f}(\infty) = 1$

Def: Distribution Function

real valued, increasing, $F(-\infty) = 0$, $F(\infty) = 1$, right continuous, ~~abbreviated~~ $\Rightarrow$ d.f.

A point mass d.f. is "degenerate."

For jump points $\{a_j\}$ and sizes $\{b_j\}$ we have $F(a_j) - F(a_j -) = b_j$

Consider $F_d(x) = \sum_j b_j \delta_{a_j}(x)$, $\quad F_d(-\infty) = 0$, $F_d(\infty) = \sum_j b_j \leq 1$

F_d is not necessarily a d.f. only lacking $F_d(\infty) = 1$

but F_d can be the "discontinuous" jumping part of a d.f.

The other part can be "continuous", $F_c(x)$, s.t. $F_d(x) + F_c(x) = F(x)$

and we should have $F_c(x) \geq 0$, ie no jumping beyond F then retracting w/ F_c

Theorem 1.2.1

F_c is positive, increasing, and continuous

proof:

$x_1 < x_2$, $F_c(x_1) = F(x_1) - F_d(x_1) \leq F(x_2) - F_d(x_2) = F_c(x$

b/c $F(x_2) \geq F(x_1)$ and $F(x) \geq F_d(x)$ $\forall x$

thus $F(x_2) - F(x_1) \geq F_d(x_2) - F_d(x_1)$

For similar reason $F_c(x) \geq 0$

$$F_d(x) - F_d(x-) = \begin{cases} b_j, & x = a_j \\ 0, & \text{otherwise} \end{cases}$$

this should also hold for F, leaving $(F(x) - F(x-)) - (F_d(x) - F_d(x-)) = 0$

$\Rightarrow (F(x) - F_d(x)) - (F(x-) - F_d(x-)) = 0 \Rightarrow F_c(x) - F_c(x-) = 0$

so both left and right continuous.

Theorem 1.2.2

d.f. F, continuous G_c and $G_d = \sum_j b_j \delta_{a_j}(x)$

where $\sum_j |b_j| < \infty$ and $F = G_c + G_d = F_c + F_d$ $\Rightarrow G_c = F_c$, $G_d = F_d$

proof:

$G_d \neq F_d \Rightarrow \exists\, \alpha$ s.t. $[F_d(\alpha) - F_d(\alpha-)] \neq [G_d(\alpha) - G_d(\alpha-)]$

$\Rightarrow F_d(\alpha) - G_d(\alpha) \neq F_d(\alpha-) - G_d(\alpha-) \Rightarrow G_c(\alpha) - F_c(\alpha) \neq G_c(\alpha-) - F_c(\alpha-)$

$\Rightarrow$ for $D = G_c - F_c$, $D(\alpha) \neq D(\alpha-) \Rightarrow D$ is not continuous.

a contradiction, so $G_c = F_c$ and $G_d = F_c$.

Def: d.f. where $F = \sum_j b_j \delta_{a_j}$, for countable a_j and $\sum_j b_j = 1$ is a discrete d.f.

In other words, $F_c = 0 \Rightarrow$ discrete, $F_d = 0 \Rightarrow$ continuous

Take $\alpha = F_d(\infty)$; $F_d \neq 0 \Rightarrow \alpha > 0$; $F_c \neq 0 \Rightarrow$ I guess $\Rightarrow \alpha < 1$

$F = \alpha\left(\frac{1}{\alpha} F_d\right) + (1-\alpha)\left(\frac{1}{1-\alpha} F_c\right)$ so F is a convex combination of a discrete and continuous d.f.

Theorem 1.2.3

Every d.f. can be a convex comb. of a discrete and continuous d.f. and this decomposition is unique.

(1.3) Absolutely continuous and singular distributions

notation: "m" is Lebesgue measure; S^c is S complement; L^1 means $L^1(-\infty,\infty)$

Def: F is absolutely continuous iff $\exists f \in L^1$ s.t. $x < x' \Rightarrow$

$$F(x') - F(x) = \int_x^{x'} f(t)\,dt$$

the derivative of F is equal to f a.e., and

$f \geq 0$ a.e. $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt = 1$

If an f satisfies this and is in L^1, then $F(x) = \displaystyle\int_{-\infty}^{x} f(t)\,dt$ is an absoulety continuous d.f.

Def: F is singular iff $F \neq 0$, F' (the derivative) exists and is zero a.e

maybe? WiKi says no → ~~(eg a discrete d.f. w/ jumps on rationals)~~ ↖ I think F' cant be 0 everywhere eg a constant is not singular

Theorem 1.3.1

F bounded, increasing; $F(-\infty) = 0$; F' derivative when it exists

(a) $S = \{x : F'(x) \text{ exists}\}$ (note $0 \leq F'(x) < \infty$), $m(S^c) = 0$

(b) $F' \in L^1$ and for $x < x'$

$$\int_{x}^{x'} F'(t)\,dt \leq F(x') - F(x)$$

(c) $F_{ac}(x) = \displaystyle\int_{-\infty}^{x} F'(t)\,dt$ is absolutely continuous and $F_{ac}' = F'$ a.e.

$F_s(x) = F(x) - F_{ac}(x)$ is singular (if not zero) and $F_s' = F' - F_{ac}'$ a.e.

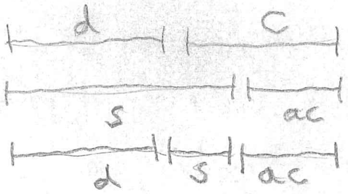
ie $F(x) = \displaystyle\int_{-\infty}^{x} F'(x)\,dt + F_s(x)$ can be decomposed into absolutely continuous and singular parts F_{ac} and F_s.

Def: an $f \geq 0$ where $f = F'$ a.e. is a density of F

F_{ac} is clearly increasing, and for $x < x'$, $F_s(x') - F_s(x) =$
$(F(x') - F(x)) - (F_{ac}(x') - F_{ac}(x)) = F(x') - F(x) - \displaystyle\int_{x}^{x'} F'(t)\,dt \geq 0$
by part (b). so F_s is also increasing

theorem 1.3.2

Every d.f. can be written as a convex comb. of discrete, singular continuous, and absolutely continuous d.f.s and this decomposition is unique.



? like this

singular d.f.s turn out to be left-continuous, so they are continuous

1

Make a singular (continuous?) distribution

Cantor set, after n steps have removed $1+2+\dots+2^{n-1} = 2^n - 1$ disjoint open intervals

Have 2^n closed disoint intervals left of length $\frac{1}{3^n}$. Removed $J_{n,k}$, $1 \le k \le 2^n - 1$

$U_n = \bigcup_k J_{n,k}$, $m(U_n) = \sum_{m=1}^{n} \frac{2^{m-1}}{3^m} = \frac{1}{3} + \frac{2}{3^2} + \dots + \frac{2^{n-1}}{3^n} = \frac{1}{3} \sum_{m=0}^{n-1} \left(\frac{2}{3}\right)^m$

$= \frac{1}{3}\left(\frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}}\right) = 1 - \left(\frac{2}{3}\right)^n$

$\lim_{n\to\infty} m(U_n) = 1$, so remainder is Cantor set C of measure 0.

Now have $c_{n,k} = \frac{k}{2^n}$, and define F on U (the removed set) as

$F(x) = c_{n,k}$ for $x \in J_{n,k}$

This is consistent iff for each x there is only one possible assignment

Suppose $x \in J_{n,k}$, $x \in J_{n',k'}$ show that $c_{n,k} = c_{n',k'}$

If $n \ne n'$, let $n = n'+1$ then $k = k'2$

More generally, $n = n'+d \Rightarrow k = k'2^d \Rightarrow c_{n,k} = \frac{k'2^d}{2^{n'+d}}$

$= \frac{k'}{2^{n'}} = c_{n',k'}$

If $n = n'$ then $k = k'$ because $J_{n,k}$ are disjoint in k.

The value is invariant under n, and strictly increasing with k.

Thus F is increasing, and $\lim_{x\downarrow 0} F(x) = 0$, $\lim_{x\uparrow 1} F(x) = \frac{2^n}{2^n} = 1$

$J_{n,k}$ is $1/3^n$ away from its neighbors. So for $0 \le x' - x \le 1/3^n \Rightarrow$

$0 \le F(x') - F(x) \le 1/2^n$ thus F is uniformly continuous

then $\tilde{F}(x) = \inf_{x<t\in U} F(t)$ is continuous but agrees w/ F, and $\tilde{F}' = 0$

where F is defined, U, which is a.e. Thus F can easily be extended to

(support of function is subset of domain where function is non-zero)

$(-\infty, \infty)$ to be a singular distribution.

actually definition is: x is in support iff $\forall \varepsilon > 0$

$F(x+\varepsilon) - F(x-\varepsilon) > 0$ for d.f. F

this looks partcular to a d.f. function, but it makes sense w/ right continuity. Eg a discrete d.f. is only defined at jump points

or: Since U is composed of open intervals and F' is 0 on each interval, U is not in the support of F, thus the support of F is in C so the support has measure 0. It follows that F is singular

(maybe measure 0 can be thought of as a set of isolated points)

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support of F measure 0 $\Rightarrow$ singular

singular $\Rightarrow F' = f = 0$ a.e. $\Rightarrow$ support of f is measure 0 $\not\Rightarrow$ support of F is measure 0

2. Measure Theory

2.1 Classes of Sets

Ω set of "point", $\omega \in \Omega$

Symmetric difference: $E \Delta F = (E \setminus F) \cup (F \setminus E)$

Singleton : $\{\omega\}$

~~$\forall n \in \mathbb{N}, j \in [n], E_j \in A \Rightarrow \bigcup_{j=1}^{n} E_j \in A$~~

For set of sub sets A there can be certain closure properties

eg $E \in A \Rightarrow E^c \in A$

Def: non-empty collection $\mathcal{F}$ of subsets of Ω is a $\boxed{\text{field}}$ iff

$E \in \mathcal{F} \Rightarrow E^c \in \mathcal{F}$ and $E_1, E_2 \in \mathcal{F} \Rightarrow E_1 \cap E_2 \in \mathcal{F}$

it is a $\boxed{\text{M.C.}}$ (mono-tone class) iff

$E_j \in \mathcal{F}, E_j \subset E_{j+1} \Rightarrow \bigcup_{j=1}^{\infty} E_j \in \mathcal{F}$

$E_j \in \mathcal{F}, E_j \supset E_{j+1} \Rightarrow \bigcup_{j=1}^{\infty} E_j \in \mathcal{F}$

it is a B.F. (Borel field) iff

$E \in \mathcal{F} \Rightarrow E^c \Rightarrow \mathcal{F}$

$E_j \in \mathcal{F} \Rightarrow \bigcup_{j=1}^{\infty} E_j \in \mathcal{F}$

Theorem 2.1.1

A field is a B.F. iff it is an M.C.

proof:

($\Rightarrow$) B.F. $\Rightarrow$ M.C. clearly

($\Leftarrow$) Suppose $E_j \in \mathcal{F} \Rightarrow \bigcup_j^n E_j \in \mathcal{F}$ and $E_j \in \mathcal{F}, E_j \subset E_{j+1} \Rightarrow \bigcup_j^{\infty} E_j \in \mathcal{F}$

Let $F_n = \bigcup_j^n E_j$, then $F_n \subset F_{n+1} \in \mathcal{F}$, and so $\bigcup_n^{\infty} F_n \in \mathcal{F}$

but $\bigcup_n^{\infty} F_n = \bigcup_j^{\infty} E_j \in \mathcal{F}$

From now on use notation $\mathcal{F}, \mathcal{A}, \mathcal{S}, \mathcal{C}, \mathcal{G}, \ldots$

probability triple $(\Omega, \mathcal{F}, P)$

Ω: sample space, $\mathcal{F}$: σ-algebra, P: probability measure

$P: \mathcal{F} \to [0,1]$, $P(\emptyset) = 0$, $P(\Omega) = 1$

$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$; $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \cap A_2, A_1 \cup A_2 \in \mathcal{F}$

$A \subset B \Rightarrow P(A) \le P(B)$ monotinicity

P is countably additive (w.r.t disjoint sets)

Theorem 2.2.1

countable Ω, $p: \Omega \to [0,1]$ w/ $\sum_{\omega \in \Omega} p(\omega) = 1$, $\mathcal{F} = \mathcal{P}(\Omega) \Rightarrow$
$(\Omega, \mathcal{F}, P)$ is a triple.

Example 2.2.2

finite Ω, $P(A) = |A|/|\Omega|$, $\mathcal{F} = \mathcal{P}(\Omega)$ is the discrete uniform dist.

Now for Uniform $[0,1]$

Ω should be $[0,1]$

$J = \{$all intervals in $[0,1]\} \subset \mathcal{F}$

J is a semialgebra

$\emptyset \in J$, $\Omega = [0,1] \in J$

$I_1, I_2 \in J \Rightarrow I_1 \cap I_2 \in J$

$I \in J \Rightarrow I^c = I_1 \cup I_2$ w/ $I_1, I_2 \in J$ and $I_1 \cap I_2 = \emptyset$

Maybe have $\mathcal{F}$ set of all unions of countable intervals

First try B_0, all finite unions of intervals

$\emptyset, \Omega \in B_0$

$J_0, J_1 \in B_0 \Rightarrow J_0 \cap J_1 \in B_0$ and $J_0 \cup J_1 \in B_0$

$J_0 \in B_0 \Rightarrow J_0^c \in B_0$

so its an algebra, but σ-algebra requires closure under countable,
not just finite, unions and intersections.

Now again consider B_1, all countable unions of intervals
not closed under complement, eg Cantor set is uncountable, but
complement is countable so in B_1.

Theorem 2.3.1: The Extension Theorem

J semialgebra of Ω, $P: J \to [0,1]$, $P(\emptyset) = 0$, $P(\Omega) = 1$

$P(\bigcup_i^K A_i) \geq \sum_i^K P(A_i)$ for disjoint A_i's $\in J$ (finite)

$P(\bigcup_n A_n) \le \sum_n P(A_n)$ for A_n's $\in J$ (countable)

then $\exists$ σ-algebra $M \supset J$ and probability measure P^* on M w/ $P^*(A) = P(A)$
$\forall A \in J$, s.t. (Ω, M, P^*) is a triple.

Look at proof:

$$P^*(A) = \inf_{\substack{\{A_i\}: \\ \cup A_i \subset A \\ A_i \in J}} \sum_i P(A_i) \qquad \text{for } A \subset \Omega$$

note P^* above may be improper probability measure

$P^*(\emptyset) = 0$,

$A \subset B \Rightarrow P^*(A) \leq P^*(B)$

$P^*(A) = P(A)\ \forall A \in J$ so P^* is an extension of P

Lemma 2.3.6: P^* is countably subadditive

$$P^*\left(\bigcup A_n\right) \leq \sum A_n, \quad \{A_n\} \subset \Omega$$

← Let $\mathcal{M} = \{A \subset \Omega : P^*(A \cap E) + P^*(A^c \cap E) = P^*(E)\ \forall E \subset \Omega\}$

Lemma 2.3.9

disjoint $\{A_i\} \Rightarrow P^*(\cup A_i) = \sum P^*(A_i)$

proof:

$$P^*(A_1 \cup A_2) = P^*(A_1 \cap (A_1 \cup A_2)) + P^*(A_1^c \cap (A_1 \cup A_2))$$
$$= P^*(A_1) + P^*(A_2)$$

then by induction

$$P^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P^*(A_i)$$

but $P^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geq P^*\left(\bigcup_{i=1}^{n} A_i\right)$, and $\sum_{i=1}^{\infty} P^*(A_i) \geq \sum_{i=1}^{n} P^*(A_i)$

so these imply

$$P^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P^*(A_i)$$

Show $\mathcal{M}$ is σ-algebra containing J