Some for the Price of One: Vote Buying on a Network

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Abstract

Social structure matters for vote buying, yet most analyses do not account for interdependence between actors. We provide a formal model of elections in which candidates may offer private transfers to policy-motivated voters connected on a social network at the expense of a public good. Investigation of deep parameters governing social structure allows for comparison across societies. Contrary to much existing theory, equilibrium transfers are not determined by network density, but primarily by group fractionalization and homophily, while inequalities are driven by a disproportionate targeting of minorities. Additionally, we consider heterogeneous information between candidates, clarifying the role of density and demonstrating that homophily can endogenously generate in-group favoritism.

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1 Introduction

Why are varied levels of vote buying observed across similar institutional and cultural environments? The practice of vote buying has detrimental consequences for society, slowing development and undermining democracy (Schaffer 2007; Vicente and Wantchekon 2009). Vote buying allocates public resources inefficiently: as an illicit activity perpetrated by candidates to improve their electoral performance, redistribution via vote buying is unlikely to correspond to an optimal redistribution mechanism. It also violates the logic of democratic processes, creating a preference aggregation problem where voters no longer choose the candidates they believe are most fit for office, but those who provide them with the most private favors (Groseclose and Snyder 1996; Banks 2000; Lizzeri and Persico 2001; Dekel, Jackson, and Wolinsky 2008).

While empirically widespread, difficulties in implementation mean that the extent of peer monitoring and voters' social orientations determine the viability of vote buying as an electoral strategy (Rueda 2015, 2017; Cruz 2019). Furthermore, recent work in political science has emphasized the defining role of social network structure in both the form and efficacy of clientelist electoral linkages, especially vote buying (Holland and Palmer-Rubin 2015; Cruz, Labonne, and Querubin 2017). Even in the absence of direct inducements, social pressure can be a crucial determinant of vote choice (Fafchamps, Vaz, and Vicente 2020). The usefulness of localized targeting is therefore determined by the underlying social structure (Finan and Schechter 2012; Larson and Lewis 2017), but the exact mechanisms through which interdependence between actors influence the extent and distribution of private political gains have not been systematically theorized.

In this article, we study a networked model of a large election in which candidates[1] compete to influence policy-motivated voters with private transfers (i.e., bribes). Each

[1] While we use the term "candidate" to refer to the agents in our model, it may be more natural in many settings to think of them as local brokers acting on behalf of a candidate.

candidate is associated with a policy and can extend private transfers to voters at the expense of a public good, which both candidates and voters care about. Voters belong to a group with distinct preferences and care about the welfare of their neighbors on the network—voting decisions are informed not only by the received bribes and partisan policy preferences, but also by the electoral preferences of their social connections. The model admits a measure of centrality as the unique equilibrium, capturing the core tradeoff between targeting voters who influence others and those who are easily influenced.

The sensitivity of any agent's centrality to small changes on the network, however, complicates direct analysis of the role played by a society's underlying structural features. To overcome this problem, we eschew consideration of exact graphs in favor of a random generative model.[2] In particular, we apply techniques from random graph analysis (Chung and Radcliffe 2011) to a class of games on networks whose equilibria correspond to the vector of Katz-Bonacich centralities (Katz 1953; Bonacich 1987; Ballester, Calvó-Armengol, and Zenou 2006).[3] In doing so, we derive closed-form expressions for the dependence of equilibrium strategies on deep features of society that hold almost surely in large societies. Specifically, we focus on three network-level attributes that research on social networks has consistently highlighted:

1. density, or the ratio of realized to potential ties;
2. fractionalization, or the relative size of salient social groups; and
3. homophily, or the relative propensity of agents to form ties with members of their own group.

[2] In this regard, recent work by Dasaratha (2020) and Mostagir and Siderius (2021) provides a framework for studying the dependence of centrality measures on social structure.

[3] This approach has provided a good empirical fit when compared to alternative measures of centrality (Calvó-Armengol, Patacchini, and Zenou 2009; Battaglini and Patacchini 2018; Battaglini, Sciabolazza, and Patacchini 2020).

Our approach allows us to focus on the underlying generative process that gives rise to observed social structure while retaining much of the rich complexity of the setting.

We find that the extent of vote buying is greatest when groups differ in size and when candidates are well-informed, with equilibrium transfers growing as a society becomes homogeneous and more predictable. Social segregation between groups, however, reduces the extent of vote buying—minorities are disproportionately targeted and the incentive to bribe is moderated as minorities become less connected to others on the network. When the quality of information is allowed to vary across candidates and voters, we find that the extent of vote buying is increasing in fractionalization when the candidate has low-precision information on minorities and is typically decreasing in homophily. Inequality, on the other hand, is typically decreasing in fractionalization and has a highly nonlinear relationship to homophily.

These results suggest new directions for empirical studies of vote buying and clientelism. While simple network properties such as density may matter through their association with information structure, our findings underscore the importance of attending to less obvious network features that may have counterintuitive effects (e.g., homophily). In addition, our results speak directly to the literature on ethnic diversity and public goods provision. In line with recent work challenging the conventional wisdom that ethnic diversity has harmful effects on the provision of public goods, our model suggests that, all else being equal, the most public goods will be provided in maximally fractionalized societies (Alesina, Baqir, and Easterly 1999; Miguel and Gugerty 2005; Habyarimana, Humphreys, Posner, and Weinstein 2007; Singh and Vom Hau 2016). Additionally, similarly to Baldwin and Huber (2010), we emphasize the importance of uncoupling fractionalization from diversity more generally, as group share and social segregation typically have countervailing effects.

1.1 Vote Buying and Social Influence

Vote buying is a prevalent feature of unconsolidated democracies and is associated with negative economic, social, and institutional outcomes (Keefer 2007; Hicken 2011). While the use of private inducements to influence voting may be effective (Rueda 2015), prospective vote buyers face several key challenges in implementing these strategies in large elections (Dekel, Jackson, and Wolinsky 2008).

Candidate uncertainty over voter preferences can make it difficult to employ vote buying strategies effectively. In large elections, no candidate is likely to have enough information, resources, or bureaucratic capacity to reliably identify, approach, and provide private goods to the optimal voters. Further, since the exchange is illegal and ballots are typically secret, politicians cannot be certain that voters will follow through on their promise, nor can voters be sure that politicians will deliver on their promises (Brusco, Nazareno, and Stokes 2004; Nichter 2008; Finan and Schechter 2012; Keefer and Vlaicu 2017). Social network connections to voters, however, can facilitate information gathering and two-way monitoring (Calvo and Murillo 2013; Frye, Reuter, and Szakonyi 2019). The cultivation of political networks may therefore play a crucial role, as they relay information to politicians and pressure highly-connected voters to follow through on agreements (Cruz 2019; Fafchamps, Vaz, and Vicente 2020). They may also create an advantage for those voters with preexisting ties to candidates, as they are more likely to be targeted for high-value goods (Fafchamps and Labonne 2017; Cruz, Labonne, and Querubin 2017).

A consistent theme in studies of vote buying (and clientelism more generally) is thus that network linkages that connect elites to voters and voters to one another are crucial determinants of the efficacy of these electoral strategies. However, the specific features of social networks that are most conducive to vote buying remain largely unspecified. For instance, while a key claim of Stokes (2005, p. 318) is that clientelist parties are "effective to the extent that they insert themselves into the social networks of constituents," the mechanics of these networks are not modeled.

Another example of this difficulty can be found in the literature on the political economy of ethnicity, where the network density of ethnic groups is taken to be a defining feature that mediates between diversity and outcomes such as conflict, public goods provision, and the development of clientelism (Putnam 2000; Miguel and Gugerty 2005; Chandra 2007; Habyarimana, Humphreys, Posner, and Weinstein 2009; Gubler and Selway 2012). For instance, Fearon and Laitin (1996) explain the prevalence of cooperative equilibria in diverse societies by the relatively high probability of interaction among members of the same group, while Larson and Lewis (2017) find dense ties are actually associated with greater diversity. To generate reliable predictions, we need to engage explicitly with network structure.

Given the inherent complexity of social networks, this issue is well-suited to a formal modeling approach. Few models of vote buying, however, have considered the role of connections between players—most models of vote buying assume continuous distributions of voters (Groseclose and Snyder 1996; Lizzeri and Persico 2001), for which results do not necessarily generalize to finite populations (Banks 2000; Dekel, Jackson, and Wolinsky 2008). This literature has generated important insights into the possibility of vote buying to induce inefficient supermajority coalitions (Groseclose and Snyder 1996; Banks 2000), the difficulty of overcoming private incentives by providing public goods (Lizzeri and Persico 2001), the role of varying commitment structures and institutions in mitigating the inefficiencies introduced by vote buying (Dal Bó 2007; Dekel, Jackson, and Wolinsky 2008), and the institutional factors driving the mix of strategies chosen by clientelist machines (Gans-Morse, Mazzuca, and Nichter 2014). However, variation is driven either by individual factors or by institutional environments. Despite the prominent role afforded to ties between actors in empirical accounts, this aspect of the strategic environment has gone largely unexplored.

An important exception comes from Battaglini and Patacchini (2018), who study the problem of influencing members of a legislature through campaign contributions using a networked model.[4] This work finds, following Ballester, Calvó-Armengol, and Zenou (2006), that the equilibrium transfers to voters (legislators) are proportional to their Katz-Bonacich centrality (Bonacich 1987), weighted by the equilibrium probability of pivotality. While the basic structure of their model is similar to ours, this paper differs in that we draw on recent advances in the analysis of random graphs to avoid the combinatoric complexity of fixed networks. This makes it possible to draw sharp conclusions about the effects of social structure by shifting attention from realized networks to an underlying generative model.

2 Model

We begin by assuming that voters care about policy in a unidimensional space and the provision of a public good, but they can also be bribed with private transfers that influence their likelihood of voting for one candidate over another. Since we are primarily interested in elections where n is sufficiently large that the probability of pivotality is approximately zero, we assume expressive voting based on net preference after transfers. In order to retain our focus on the network-specific elements of the model, we assume that both candidates are endowed with full commitment power, so that transfers can be treated as either up-front or as campaign promises without consequence (Dekel, Jackson, and Wolinsky 2008). These bribes, however, are not a binding contract—voters may receive transfers from both candidates and will ultimately vote in accordance with their own preferences. As such, neither candidate can ever be certain that a vote has been bought, reflecting the commitment problem

[4] See also Battaglini, Sciabolazza, and Patacchini (2020).

highlighted in much of the literature.

The key feature of this model is network dependence. In addition to being influenced by direct transfers and campaign promises, voters place some weight on their neighbors' expected votes. While our primary interest is in the role played by aggregate network properties, in this section we take the network as a fixed realization of an arbitrary generative process in order to characterize equilibrium strategies conditional on the network. In the following section, we shift our focus to social structure, assuming that the network is generated according to a stochastic block model and studying the effect of changes in its parameters.

Substantively, the network spillover mechanism has two main interpretations. First, voters can be thought of as communicating with their acquaintances about their intent to vote, which provides information about the candidate's desirability. A bribe paid to any voter on the network will positively affect *all* voters' likelihood of voting for a candidate on a connected graph, albeit with diminishing returns in social distance. Second, network spillovers may be a consequence of social pressure. Even if voters do not gain any payoff relevant information from their neighbors, they may still be intrinsically motivated to take the same action as a majority of them.

An important observation regarding network dependence is that we normalize the total social influence of any voter's connections to 1, so that all voters place equal weight on their neighbors' vote probabilities. Intuitively, this implies that the influence exerted on voter i by their network neighbor j is greater if j is i's only neighbor than it would have been were i connected to a hundred other voters. Equivalently, voters can be thought of as making their decisions based on a weighted average of their neighbors' actions, and not a sum. This aspect of the model diverges somewhat from others in the literature who assume that the most connected voters are also the most easily influenced. We feel that our approach more accurately captures the relevant strategic environment, as there is no clear reason to suppose that more peripheral (in the sense of having lower in-degrees) voters are systematically less influenced by others.

Finally, candidates care about both their electoral performance and their "programmatic" commitment to provide a public good, but face a potential tradeoff between these two goals. The funds for private transfers to voters, which can provide an electoral advantage, must be diverted from the provision of the public good. In order to maximize the clarity of the results, we assume that candidates attempt to maximize vote share rather than to win a majority. This assumption is empirically appropriate in many cases, such as in authoritarian regimes, where incumbents frequently seek to achieve overwhelming vote shares (Reuter and Robertson 2012).

2.1 Setup

Consider a game with n voters that need to make a choice between two candidates. All voters are located on a network $\mathcal{G}$, which is assumed to be connected. We use the terms network and graph interchangeably throughout the paper to refer to an undirected graph, which is an ordered pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a set of n vertices and $\mathcal{E}$ is a set of m edges such that $\mathcal{E} \subseteq \{\{x, x'\} : x, x' \in \mathcal{V} \wedge x \neq x'\}$. Each candidate $k = 1, 2$ is associated with a policy $y_k = k$, where the policy space is a subset of $\mathbb{R}$, and each voter $i \in \mathcal{V}$ is endowed with a group membership $\ell_i = 1, 2$, which corresponds to an ideal policy $x_i = \ell_i$. Substantively, groups may be interpreted as corresponding to any grouping that is both socially and politically meaningful, such as political parties and ethnic or religious groups.

In attempt to gain vote share, candidate k can extend n private offers or bribes, $b_{ik} \geq 0$. These bribes, however, come at the expense of a public good, which the candidate also values. A candidate k's problem is to choose $\boldsymbol{b}_k$ that solves

$$\max_{\boldsymbol{b}_k} \ \alpha \sum_{i \in \mathcal{V}} \phi_{ik}(\boldsymbol{b}_k, \boldsymbol{b}_{-k}) - \boldsymbol{b}_k \cdot \mathbf{1}$$
$$\text{subject to} \ \ b_{ik} \geq 0 \text{ for all } i$$

where $\phi_{ik}(\cdot)$ is the probability voter i votes for candidate k and α represents the value placed on 1 unit of the public good by the candidate. We thus normalize the value placed on a unit of public good by the candidate to 1 so that α can be interpreted as the candidate's relative degree of office motivation. In particular, an α of 0 corresponds to a fully programmatic candidate, who trivially prefers to offer no bribes and promise the full amount of the public good.

Voters support the candidate that offers them a higher total payoff.[5] All voters care about policy according to a standard quadratic loss function and have $\gamma \geq 0$ value for a unit of public good, so that they incur a loss of γ for every unit of bribes offered by a candidate to any voter. Additionally, voters have private information unknown to the candidates and other voters in the form of a private valence shock for each candidate, $\varepsilon_{ik} \in \mathbb{R}$. Without loss of generality, we can normalize $\varepsilon_{i2} = 0$ and define $\varepsilon_i := \varepsilon_{i1}$, which we assume is an independent, uniformly distributed random variable with mean zero and density $\theta > 0$. We interpret θ as the candidates' information, with smaller θ indicating less-informed candidates. θ can also be taken as reflecting the intensity of the commitment problem facing candidates, as candidates with higher values can be more certain that a transfer to a voter will actually secure their vote.

Social structure also matters. In particular, voters prefer to vote for the same candidate as their neighbors as defined by the network. Denote by $\phi_{ik}(\cdot)$ the probability voter i votes for candidate k given all bribes, but before the realization of the valence shock ε_i. Then, each voter i places weight $w_{ij} \geq 0$ on voter j's probability of

[5] There is no obligation to vote for a candidate who offered them a bribe.

voting for candidate k if i and j are connected, and 0 otherwise. In the graph $\mathcal{G}$, the set of a voter i's social ties is denoted by $\mathcal{T}_i(\mathcal{G}) \subseteq \mathcal{V}$.[6] The total social influence on each voter is normalized to 1, so that the actual influence of each neighbor j on i's utility is equal to $\left(\sum_{h\in\mathcal{T}_i(\mathcal{G})} w_{ih}\right)^{-1} w_{ij}$, implying that more highly connected voters are less influenced by each individual neighbor.

The expected payoff voter i receives from candidate k can thus be expressed as

$$U_i(k) = -(x_i - y_k)^2 + u(b_{ik}) + \frac{\sum_{j\in\mathcal{T}_i(\mathcal{G})} w_{ij}\phi_{jk}(\boldsymbol{b})}{\sum_{h\in\mathcal{T}_i(\mathcal{G})} w_{ih}} - \gamma \sum_{m\in\mathcal{V}} b_{mk} + \varepsilon_{ik} \tag{1}$$

where $u(\cdot)$ is voter utility over bribes, which we assume is strictly increasing with diminishing marginal returns and that the rate of diminution is decreasing. That is, we require that the utility over bribes satisfy $u'(\cdot) > 0$, $u''(\cdot) < 0$, $u'''(\cdot) \geq 0$, $\lim_{b\to 0} u'(b) = \infty$, and $\lim_{b\to\infty} u'(b) = 0$. These assumptions include a wide range of plausible utilities, particularly logarithmic utility. Together, they ensure that all solution objects are well-defined and rule out the possibility of corner solutions.

Timing

The timing of the game is as follows.

1. Nature randomly chooses a private valence shock for each voter, $\varepsilon_i \sim \mathcal{U}\left[\frac{-1}{2\theta}, \frac{1}{2\theta}\right]$

2. For all voters $i \in \mathcal{V}$, each candidate $k = 0, 1$ offers a bribe $b_{ik} \geq 0$, which determines the residual public good offered

3. Each voter $i \in \mathcal{V}$ votes for candidate 1, $v_i = 1$, or candidate 2, $v_i = 2$

[6] Here, $\mathcal{G}$ is a fixed graph and may have any social structure provided that it is connected.

3 Equilibrium

A voter will cast a ballot for candidate 1 if and only if $U_i(1) \geq U_i(2)$. Here, candidates will not be able to perfectly anticipate voting behavior due to their imperfect information over voter preferences. Using equation (1), we can rewrite this as a condition on the size of the valence shock,

$$\varepsilon_i \leq (-1)^{x_i - 1} + u(b_{i1}) - u(b_{i2}) + \frac{\sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij}(2\phi_j - 1)}{\sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij}} + \gamma \sum_{m \in \mathcal{V}} (b_{m2} - b_{m1}),$$

where we have denoted $\phi_i := \phi_{i1}(\boldsymbol{b}) = 1 - \phi_{i2}(\boldsymbol{b})$ the probability a voter i votes for candidate 1. Noting that $\varepsilon_i \sim \mathcal{U}\left[\frac{-1}{2\theta}, \frac{1}{2\theta}\right]$ implies $\Pr(\varepsilon_i \leq \varepsilon) = \frac{1}{2} + \theta\varepsilon$, we can correspondingly write each voter's probability for voting for candidate 1 as

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \theta\left((-1)^{x_1 - 1} + u(b_{11}) - u(b_{12}) + \frac{\sum_{j \in \mathcal{T}_1(\mathcal{G})} w_{1j}(2\phi_j - 1)}{\sum_{j \in \mathcal{T}_1(\mathcal{G})} w_{1j}} + \gamma \sum\limits_{m \in \mathcal{V}} (b_{m2} - b_{m1})\right) \\ \vdots \\ \frac{1}{2} + \theta\left((-1)^{x_n - 1} + u(b_{n1}) - u(b_{n2}) + \frac{\sum_{j \in \mathcal{T}_n(\mathcal{G})} w_{nj}(2\phi_j - 1)}{\sum_{j \in \mathcal{T}_n(\mathcal{G})} w_{nj}} + \gamma \sum\limits_{m \in \mathcal{V}} (b_{m2} - b_{m1})\right) \end{pmatrix}.$$

Here, $\boldsymbol{\phi}$ gives the unique vector of equilibrium vote probabilities. While each voter's utility is subject only to their neighbor's vote probabilities, this system of equations necessarily implies that a single voter's probability of supporting candidate 1 is a function of all other voter's probability of supporting 1. This occurs because, for example, a voter i's probability ϕ_i is affected by i's neighbor j's probability ϕ_j, which in turn is affected by j's neighbor m's probability ϕ_m. Since we rule out disconnected components, ϕ_i will both affect and be affected by all other voting probabilities throughout the entire network.

In equilibrium, each candidate chooses a vector of bribes that maximizes their utility, taking the other candidate's bribes as given. This gives rise to the n first-order conditions,

$$\sum_{j=1}^{n} \frac{\partial \phi_j}{\partial b_{ik}} = \frac{1 - \lambda_k}{\alpha}$$

where λ_k is the Lagrangian multiplier associated with k's nonnegativity constraint. Differentiating ϕ_i with respect to a bribe from candidate 1 to another voter h, we have

$$\frac{\partial \phi_i}{\partial b_{h1}} = \theta \left(u'(b_{h1}) \mathbb{1}(i = h) + \frac{2 \sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij} \phi'_j}{\sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij}} - \gamma \right).$$

Using this and the first-order conditions, we can rewrite the candidates' problem as

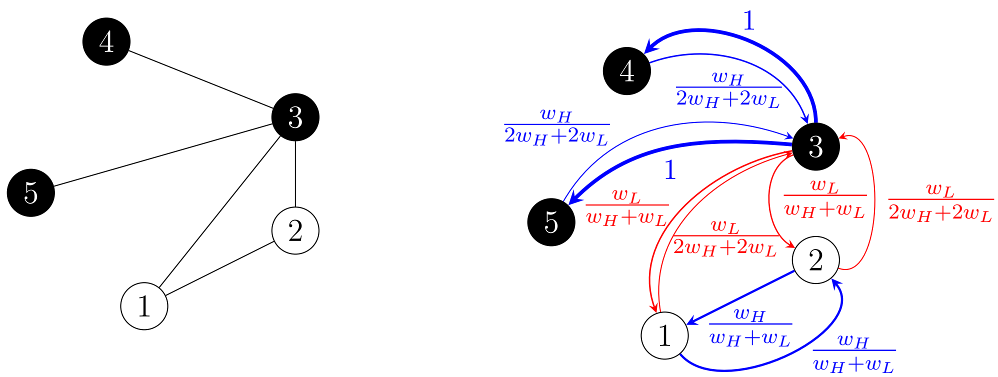
$$(\boldsymbol{J}[\boldsymbol{u}] - \boldsymbol{\Gamma})^{\top} \cdot \left(\boldsymbol{I} - 2\theta \tilde{\boldsymbol{A}} \right)^{-1} \cdot \mathbf{1} = \frac{(1 - \boldsymbol{\lambda})}{\alpha \theta} \tag{2}$$

where $\boldsymbol{J}[\cdot]$ is a diagonal matrix with $u'(b_i)$ as the nonzero entries, $\boldsymbol{\Gamma}$ is an $n \times n$ square matrix such that every element of $\boldsymbol{\Gamma}$ is γ, $\boldsymbol{I}$ denotes the identity matrix, $\mathbf{1}$ denotes an n-vector of 1s, $\boldsymbol{\lambda}$ is an n-vector of Lagrange multipliers, and $\tilde{\boldsymbol{A}}$ is a normalized weighted adjacency matrix.

Definition 1. *Consider a realized graph $\mathcal{G}$ and a corresponding adjacency matrix $\boldsymbol{A}$ such that for all $i, j \in \mathcal{V}$, $A_{ij} = 1$ if $j \in \mathcal{T}_i(\mathcal{G})$ and $A_{ij} = 0$ otherwise. Then, the* ***normalized weighted adjacency matrix $\tilde{\boldsymbol{A}}$*** *is given by, for all $i, j \in \mathcal{V}$,*

$$\tilde{A}_{ij} = \frac{w_{ij} A_{ij}}{\sum_{m \in \mathcal{V}} w_{im} A_{im}}.$$

By employing the normalized weighted adjacency matrix, we can account for several important features of social interaction. First, a voter i may be more influenced by one social tie than another. Second, it will be more difficult to influence a highly connected voter than a relatively disconnected one, e.g., an incremental change in the



(a) Example of a realized graph (b) Induced weighted directed graph

Figure 1: An example of a realized (undirected) network with $n = 5$ and group labels $\ell_1 = \ell_2 = 1$ and $\ell_3 = \ell_4 = \ell_5 = 2$ and the corresponding induced weighted directed network. In the weighted directed graph, thicker arrows indicate stronger influence, with $w_{ij} = w_H$ if $\ell_i = \ell_j$, $w_{ij} = w_L$ otherwise, and naturally $w_H > w_L$.

probability that i's neighbor j votes for candidate 1 will have less of an effect on i's vote probability if i has hundreds of neighbors than if j is i's only neighbor.

From the candidates' problem in equation (2), we can recover the equilibrium bribe that voter i receives from candidate k,

$$b_{ik} = [u']^{-1}\left(\gamma n + \frac{1}{\alpha\theta c_i(\boldsymbol{w}, \theta; \mathcal{G})}\right), \tag{3}$$

where $c_i(\cdot)$, defined as the ith element of $\boldsymbol{c}$ and $\boldsymbol{c} = (\boldsymbol{I} - 2\theta\tilde{\boldsymbol{A}})^{-1} \cdot \mathbf{1}$ is our measure of centrality. This measure is analogous to Katz-Bonacich centrality on the weighted directed network corresponding to $\tilde{\boldsymbol{A}}$ with attenuation parameter 2θ. The nature of the strategic environment—specifically, the structure of social influence—can therefore be thought of as inducing a latent directed network with connections corresponding to the influence of i on j, which is decreasing in j's weighted degree and greatest when $\ell_i = \ell_j$. The value of a voter to a candidate is thus proportional to their centrality on this latent network, which captures the weighted sum of directed walks of any length that include that voter.

As the characterization makes clear, b_{ik} does not rely on policy considerations and hence, in equilibrium, both candidates bribe a given voter the same amount according to their centrality on the network.

Proposition 1 (Equality of bribes)**.** *In any equilibrium, $b_{i1} = b_{i2}$ for all $i \in \mathcal{V}$.*

To understand the intuition behind this result, recall that a voter's direct utility in bribes is increasing with diminishing marginal returns. Then, a candidate k will want to continue extending bribes to a voter i until the marginal gain in vote probability is equal to the marginal loss in public good. Since candidates want to maximize their total expected vote share, the point at which this occurs for voter i is the same for candidate 1 as it is for candidate 2. Voters will not receive more private inducements from candidates they are politically aligned with than from those they are not. Proposition 1 implies that the net direct gain from bribes is always zero, i.e., $u(b_{i1}) - u(b_{i2}) = 0$. Note also that $\gamma \sum_{i\in\mathcal{V}}(b_{i2} - b_{i1}) = 0$. Together, these facts lead to the next result.

Proposition 2 (Electoral outcomes)**.** *Vote buying does not affect electoral outcomes. In particular, the probability a voter i votes for candidate 1 is*

$$\phi_i = \frac{1}{2} + \theta\left((-1)^{x_i - 1} + \sum_{j\in\mathcal{T}_i(\mathcal{G})} 2w_{ij}(\phi_j - 1)\right)$$

for all $i \in \mathcal{V}$, which is independent of candidate strategies.

Since an individual voter receives the same bribe from each candidate, neither candidate is successful in effecting change in the voter's probability of supporting them in the election. As no voter is influenced by the bribes (or, more precisely, as all voters are influenced by the bribes equally in both directions), the equilibrium expected vote share is not altered by the candidates' bribes.

Finally, it is straightforward from the assumptions on the derivatives of $u(\cdot)$ stated in the previous section to derive the following comparative statics by taking derivatives of the equilibrium bribes defined by equation (3).

Proposition 3 (Comparative Statics)**.** *For any voter i, the equilibrium transfers offered by both candidates are*

1. *Weakly decreasing in γ*
2. *Weakly decreasing in n*
3. *Weakly increasing in α*
4. *Weakly increasing in θ*

These results are intuitively consistent with the basic strategic environment: the socially optimal bribes to voters would correspond to a transfer scheme such that the marginal value is equated with γ, but candidates provide additional transfers as a result of the network spillovers. The value of these spillovers is moderated by α—that is, more office-motivated candidates place higher value on the additional increase to expected vote share—and by θ, which determines the likelihood of their realization.

Thus far, we have taken the exact realized network as given. In order to study the dependence of equilibrium strategy on social structure, we must transition to considering the underlying generating model that gave rise to the observed network. We take up this issue in the next section.

4 Social Structure

This section studies the role of social structure. First, we present a set of novel results in the theory of random graphs that justify analysis of an average graph. This is needed for two reasons: since centrality is a highly complex function of the realized network, it is convenient to replace voters' expected centrality with their centrality *on the expected network*, which is far more tractable. In addition, even simple generative models may result in realized networks with arbitrarily high variance in emergent characteristics such as centrality. We therefore require bounds on the spectrum of the induced network in order to facilitate making claims that hold with high probability. Second, we employ these tools to study the implications of social structure on vote buying. In particular, we are able to derive closed-form expressions for the centrality of voters in each party, yielding sharp comparative statics in terms of the main features of social structure—namely, group fractionalization, density, and homophily.

These results are technical in nature and are therefore mainly reserved for the Appendix. However, it is worth noting that the main result—an analogue of the spectral theorems of Chung and Radcliffe (2011) applied to the case of weighted, directed networks—allows us to place tight bounds on the deviation of the realized normalized weighted adjacency matrix from its expected counterpart. A major limitation of this result is that the bound depends on n and can thus be arbitrarily loose in large societies. Under the assumption that the minimum degree grows in network size at a rate greater than $\ln(n)$, it is straightforward to show the following result. This lemma allows us to make asymptotic statements that will hold with high probability and therefore justifies the analysis of the expected, rather than the realized, network in the context of large elections.

Lemma 1. *Under the assumptions of Theorem 1, for any* $\epsilon > 0$, $\lim_{n\to\infty} \Pr(\|\boldsymbol{c}^{(n)}(\tilde{\boldsymbol{A}}) - \boldsymbol{c}^{(n)}(\bar{\tilde{\boldsymbol{A}}})\| > \epsilon) = 0$.

This result allows us to consider centrality on the average graph only, permitting analysis of comparative statics in terms of social structure rather than of a single realized graph. We now formalize the concept of an average network in the context of our model, which illustrated in Figure 2 and can be conveniently represented through its average adjacency matrix.

Definition 2. *The* ***average normalized weighted adjacency matrix*** $\bar{\tilde{\boldsymbol{A}}}$ *is given*

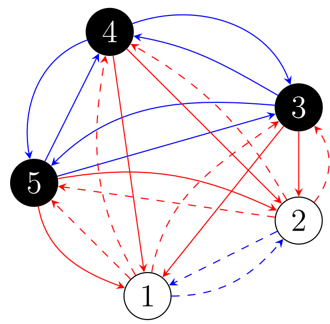


Figure 2: An example of an expected graph corresponding to the average normalized weighted adjacency matrix, from which the earlier example would have been generated. Blue lines indicate in-group influence, red lines indicate out-group influence, dashed lines indicate influence from a member of group 1, and solid lines indicate influence from a member of group 2.

by, for all $i, j \in \mathcal{V}$,

$$\bar{\tilde{A}}_{ij} = \frac{w_{ij} p_{ij}}{\sum_{m \in \mathcal{V}} w_{im} p_{im}}.$$

For the following results, we assume that the graph is drawn according to a two-group stochastic block model with share $s \geq \frac{1}{2}$ of group 1, a probability p_H of intra-group connection, and a probability $p_L \leq p_H$ of inter-group connection.[7] Further, assume for simplicity that $w_{ij} = w_H$ for in-group voters and $w_{ij} = w_L$ for out-group voters, with the natural assumption that $w_H \geq w_L$. Finally, we denote by $\delta \in (0, 1)$ the ratio $\frac{w_L p_L}{w_H p_H}$, which thus captures the degree of homophily on the network (lower δ corresponds to more homophily).

The main result, which draws on the asymptotic bounds on the average adjacency matrix derived in the previous section, allows us to obtain closed-form expressions for each voter's centrality that hold with high probability given large n.

Proposition 4 (Expected Centrality)**.** *Suppose that the assumptions of Theorem 1*

[7] Notably, the parameter s captures all of the information provided by the Herfindahl-Hirschman index, which is a widely used empirical measure of concentration.

hold. Then the centrality of a voter in party 1 *and* 2 *is asymptotically equivalent to*

$$c_1 = \frac{-\delta + (\delta - 1)s^2(\delta\theta + \delta + \theta - 1) - (\delta - 1)s(\delta\theta + \delta + \theta - 1)}{sn((\delta - 1)s - \delta)(-\theta + s(\delta + \theta - 1) + 1)}$$

and

$$c_2 = \frac{\delta - ((\delta - 1)s^2(\delta\theta + \delta + \theta - 1)) + (\delta - 1)s(\delta\theta + \delta + \theta - 1)}{n(s - 1)((\delta - 1)s + 1)(s(\delta + \theta - 1) - \delta)},$$

respectively, with probability approaching 1 *as* $n \to \infty$.

The proof of this result is presented in Appendix C. Unlike realized networks, the expected network is necessarily complete, since all voters have positive probability of being connected to all others. In particular, while the adjacency matrix can be arbitrarily large, it only contains four unique values that corresponding to directed connections within and between each group. Since this generates the structure of a block matrix, it is therefore possible to derive an explicit formula for its inverse, which in turn determines the value of each voter's centrality.

An immediate conclusion that follows from Proposition 4 is that, while centrality is a function of group sizes, information, and homophily, it is entirely *independent* of the actual density of the network.[8] A uniform increase in connection probabilities between all voters would not influence equilibrium bribes in any way. Despite the prominence of density in many informal accounts of network effects, our model suggests this need not be a major determinant of the efficacy of vote buying, further highlighting the value in formally considering network dependence.

Also of note is that $c_i \leq c_j$ when group i is larger than group j—i.e., each individual member of the minority group will always receive a higher equilibrium transfer than a member of the majority. Intuitively, this is driven by the fact that the influence of each member of the minority group increases as the group becomes smaller, making them more valuable to target.[9]

[8] While the centrality of each individual voter is decreasing in n, the total centrality is constant.

[9] Assuming any positive degree of homophily. In the case that $\delta = 1$, members of each group are,

These centralities are smoothly differentiable functions of all parameters almost everywhere. It is then straightforward to examine how the total spending of candidates, as well as the level of between-group inequality, depends on these parameters. Let B denote the sum of bribes across all voters and Q denote the level of inequality, specifically $Q := (1 - b_1/b_2)^2$. Since $u(\cdot)$ is assumed to be concave with convex first derivative, it is sufficient to consider the total centrality rather than examining bribes directly. Then we have the following result.

Proposition 5 (Total bribes)**.** *For all $\theta \leq 2 - \sqrt{2}$, $\frac{\partial B}{\partial \delta} > 0$, $\frac{\partial B}{\partial s} > 0$, and $\frac{\partial B}{\partial \theta} > 0$.*

Since θ needs to be small for the solution to be well-defined, the condition for these results will always hold for large n. The effect of θ here is intuitive: better information increases the expected return on bribes. The other two results are nonobvious, however. The effect of δ is counterintuitive: an increase in δ (i.e., a decrease in homophily) increases total spending. While the mechanism for this is straightforward—higher δ corresponds to stronger "weak ties," raising the value of transfers to all voters—it is again at odds with many informal descriptions of dense networks that are argued to be particularly amenable to vote buying due to their high degree of homophily.

Moreover, while it might be expected that more unequal group sizes lead to increased expenditure on vote buying, the mechanism is somewhat surprising. Increasing the size of the majority group leads to a relative increase in the individual-level transfers offered to members of the *minority* group (see below), which more than compensates for the reduction in its size.

It is similarly straightforward to study the effect of network parameters on inequality by taking derivatives of the negative of the ratio of the two centralities.

Proposition 6 (Group size and inequality)**.** *Let Q denote the total inequality, or $Q = \left(1 - \frac{b_1}{b_2}\right)^2$. Then $\frac{\partial Q}{\partial \theta} \geq 0$, with equality if $s = \frac{1}{2}$ or $s = 1$, and $\frac{\partial Q}{\partial s} > 0$ for all parameter values. Moreover, $\frac{\partial Q}{\partial \delta} < 0$ only if $\delta \geq \sqrt{1 - \theta}$, and is positive otherwise.*

on average, interchangeable and thus receive identical expected transfers.

Once again, the effects of information and group size are intuitive, suggesting that better informed candidates in more demographically uneven societies will concentrate their resources more intensely in the groups that provide the highest return. As with total spending, however, the effects of homophily are surprising. While higher homophily (lower δ) can increase inequality, this only holds for networks that exhibit extremely low degrees of homophily. In contrast, at all empirically plausible levels of homophily, increasing the relative influence of voters on members of their own group actually decreases the overall inequality of transfers from candidates.

This result is especially remarkable given that Dasaratha (2020) arrives at the opposite conclusion regarding Katz centrality on an undirected and unweighted network. In fact, the key to understanding this result is the tradeoff faced by candidates between targeting highly-connected voters, who influence many others, and voters whose neighbors are *not* highly connected, since they are more easily influenced. In the extreme, as δ approaches 0, the greater value of transfers to members of the minority is completely offset by their disconnectedness from the majority, such that the equilibrium bribes approach equality.

5 Heterogeneous Information

A key feature of the model studied thus far is that both candidates have identical and completely homogeneous information about the preferences of all voters, modeled as a single commonly known value of θ. Among the competing candidates, homogeneous information unsurprisingly results in homogeneous behavior. In practice, however, this is unlikely to hold true. In fact, empirical research has consistently emphasized the crucial intermediary role of brokers who possess superior knowledge about particular groups of voters, and in competitive settings informational asymmetries across candidates may account for divergent strategies (Stokes 2005; Stokes, Dunning, Nazareno, and Brusco 2013; Calvo and Murillo 2013; Fafchamps and Labonne 2017; Cruz, Labonne, and Querubin 2017).

In this section, we study the consequences of relaxing this assumption, allowing the precision of candidates' information to vary arbitrarily across candidates and voters.

5.1 Equilibrium

We begin from the setup of the baseline model, with the distinction that the information held by candidate k about voter i's preferences is allowed to vary. In particular, voter i's net preference for candidate 1, ε_i, is now drawn from one of two uniform distributions with density parameter $\theta_i \in \{\underline{\theta}, \overline{\theta}\}$ with $\overline{\theta} > \underline{\theta}$. We can think of θ_i as voter i's private type, which is unknown to candidates.

While the candidates do not know which distribution voter i's net preference was drawn from, they have common priors and receive signals about each voter's type $m_i \in \{\underline{\theta}, \overline{\theta}\}$ such that $m_i = \theta_i$ with a probability (assumed greater than half) that depends on the voter-candidate pair. In other words, candidates receive informative signals about the preferences of voters and those signals may be more precise for some voters than for others. After receiving signals $\boldsymbol{m} = (m_1, \ldots, m_n)$, candidates form posterior beliefs $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ where $\mu_i = \Pr(\theta_i = \underline{\theta} | m_i)$ and distribute bribes accordingly.

Incorporating uncertainty over voter types, we can now rewrite the candidates' problem as

$$\max_{\boldsymbol{b}_k} \ \alpha \sum_{i \in \mathcal{V}} \mathbb{E}_\mu[\phi_{ik}(\boldsymbol{b}_k, \boldsymbol{b}_{-k})] - \boldsymbol{b}_k \cdot \mathbf{1} \tag{4}$$

$$\text{subject to} \ \ b_{ik} \geq 0 \text{ for all } i. \tag{5}$$

Then, exactly as before, we can write each voter's probability for voting for candidate

1 as a function of all other voter's probability for voting for candidate 1,

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \theta_1 \left(V_1(\boldsymbol{b}) + \frac{\sum_{j \in \mathcal{T}_1(\mathcal{G})} w_{1j}(2\phi_j - 1)}{\sum_{j \in \mathcal{T}_1(\mathcal{G})} w_{1j}} \right) \\ \vdots \\ \frac{1}{2} + \theta_n \left(V_n(\boldsymbol{b}) + \frac{\sum_{j \in \mathcal{T}_n(\mathcal{G})} w_{nj}(2\phi_j - 1)}{\sum_{j \in \mathcal{T}_n(\mathcal{G})} w_{nj}} \right), \end{pmatrix}$$

where we denote i's net preference for candidate 1 short of network effects by $V_i(\boldsymbol{b})$ for notational convenience so that

$$V_i(\boldsymbol{b}) := (-1)^{x_i - 1} + u(b_{i1}) - u(b_{i2}) + \gamma \sum_{m \in \mathcal{V}} (b_{m2} - b_{m1}).$$

Unlike the baseline case, however, candidates maximize an expected utility that now relies on their posterior beliefs of voter types. In particular, we need to characterize the candidates' expected vote share conditional on their signals. Using the above equation, this can be expressed as

$$\mathbb{E}_\mu[\phi_i] = \frac{1}{2} + \mathbb{E}_\mu[\theta_i] V_i(\boldsymbol{b}) + \frac{\sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij}(2\mathbb{E}_\mu[\theta_i \phi_j] - \mathbb{E}_\mu[\theta_i])}{\sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij}}.$$

For the remainder of this section, we restrict all edge weights to $w_{ij} = 1$ for all $i, j \in \mathcal{V}$. This assumption is without loss of generality as Proposition 4 establishes that, on average for large n, it is only the ratio $\frac{w_L p_L}{w_H p_H}$ that determines outcomes. Hence, as long as tie formation probabilities can vary freely, edge weights have no independent effect in the expected network, which is our focus in this section.

First, note that $\mathbb{E}_\mu[\theta_i] = \overline{\theta} - \mu_i(\overline{\theta} - \underline{\theta})$. This can be thought of as the candidate's net "information" about voter i, taking into account both first-order uncertainty about i's vote choice and second-order uncertainty over her type. Second, by Lemmas 3 and 4 in the Appendix, we know that unilateral changes in a particular voter's type has a negligible influence on changes in other voter's vote probabilities as the network grows sufficiently large; i.e, $\phi_j(\cdot|\boldsymbol{\theta} : \theta_i = \underline{\theta}) \approx \phi_j(\cdot|\boldsymbol{\theta} : \theta_i = \overline{\theta})$ for $i \neq j$ as $n \to \infty$. Hence we can conclude that, asymptotically, $\mathbb{E}_\mu[\theta_i \phi_j] = \left(\overline{\theta} - \mu_i(\overline{\theta} - \underline{\theta})\right) \mathbb{E}_\mu[\phi_j]$. This allows us to recover n first-order conditions,

$$\frac{\partial \mathbb{E}_\mu[\phi_i]}{\partial b_{h1}} = \left(\overline{\theta} - \mu_i(\overline{\theta} - \underline{\theta})\right) \left(u'(b_{h1}) \mathbb{1}(i = h) + \frac{2 \sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij} [\mathbb{E}[\phi_j]]'}{\sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij}} - \gamma \right).$$

As before, the above equations allow us to rewrite the candidates' problem as

$$(\boldsymbol{J}[\boldsymbol{u}] - \boldsymbol{\Gamma})^\top \cdot \left(\boldsymbol{I} - 2\Theta \tilde{\boldsymbol{A}} \right)^{-1} \cdot \mathbf{1} = \Theta^{-1} \cdot \frac{(1 - \boldsymbol{\lambda})}{\alpha}, \tag{6}$$

where $\Theta := \overline{\theta} \boldsymbol{I} - (\overline{\theta} - \underline{\theta}) \boldsymbol{M}$ and $\boldsymbol{M}$ is an $n \times n$ diagonal matrix with posteriors μ_i as nonzero elements. Equilibrium bribes can therefore be explicitly expressed,

$$b_{ik} = [u']^{-1} \left(\gamma n + \frac{1}{\alpha \hat{\theta}_{ik} c_i(\boldsymbol{w}, \underline{\theta}, \overline{\theta}; \mathcal{G})} \right),$$

where the only difference from the previous section is that equilibrium bribes to a voter i from a candidate k rely not only on their centrality measure c_i, now the ith element of $\boldsymbol{c} = (\boldsymbol{I} - 2\Theta \tilde{\boldsymbol{A}})^{-1} \cdot \mathbf{1}$, but also on candidate k's belief about voter i's type, $\hat{\theta}_{ik} := \mathbb{E}_\mu[\theta_i]$. By the law of iterated expectations, since each candidate's posteriors are equal to their priors in expectation, they will act according to their (common) priors on average. Therefore, it is necessarily true that $\mathbb{E}[\hat{\theta}_{i1}] = \mathbb{E}[\hat{\theta}_{i2}] = \hat{\theta}_i$ for each voter i.

Now, because one candidate may have more informative signals than the other, equality of bribes as in Proposition 1 may no longer hold in this setting for a particular realization; however, equality of bribes will continue to hold in expectation. This remains true even when there is diverging quality of information among the candidates. For a given network, the candidate with more accurate signals will be more responsive to the realized voter types, which means they will have an advantage over their opponent in the sense that they can better anticipate whether they should spend more or less on specific voters. However, as long as each candidate has well-specified prior beliefs about voter types, then both candidates will spend the same amount on each voter on average.

Further, the baseline model's result on electoral outcomes (Proposition 2) continues to hold for any realization when candidates have the same quality of information. By introducing an informational advantage to one candidate, the better-informed candidate should be able to improve their electoral performance in expectation by more precisely allocating bribes to the voters with the greatest marginal return. Nonetheless, gains in electoral performance via vote buying remain orthogonal to group membership to the extent the informational structure is also orthogonal to group membership.

5.2 Comparative Statics

A natural question is how the information available to candidates relates to the structure of society. There are two main sources of variation in information: cross-group differences in the prior distribution of types and network-dependent variations in posterior information. In this section, we study the first type of variation analytically, while Appendix B examines the impact of network dependencies through simulations.

Variation in information may arise due to systematic differences between the two groups and thus affect both candidates symmetrically. For instance, if partisanship is stronger in one party than another, then candidates may view those associated with the "weaker" party as more likely to be swing voters. Intuitively, the first-order effect of this variation is to reduce the value of transfers to members of the less predictable group, as they are associated with a lower marginal value in expectation. Nevertheless, it is unclear *a priori* how this affects the comparative statics derived in the homogeneous case, as the reduced value of members of this groups also reduces the significance of all flow-on effects in the network.

We begin by assuming without loss of generality that $\hat{\theta}_1 < \hat{\theta}_2$. This may be interpreted as reflecting a difference in prior distributions across groups, affecting both candidates symmetrically. Applying the same approach as for the baseline model,[10] we can recover explicit expressions for centrality:

Proposition 7. *For n sufficiently large, the centrality of a voter in group 1 and 2 is asymptotically equivalent to*

$$c_1 = \frac{\delta\left(-\hat{\theta}_1 + \hat{\theta}_2 - 1\right) + (\delta-1)s^2\left(\delta\hat{\theta}_1 + \delta + \hat{\theta}_2 - 1\right) - s\left(\delta^2(\hat{\theta}_1+1) - 2\delta(\hat{\theta}_1 - \hat{\theta}_2 + 1) - \hat{\theta}_2 + 1\right)}{ns((\delta-1)s - \delta)(-\hat{\theta}_2 + s(\delta + \hat{\theta}_2 - 1) + 1)}$$

and

$$c_2 = \frac{\delta - \left((\delta-1)s^2(\delta\hat{\theta}_2 + \delta + \hat{\theta}_1 - 1)\right) + s\left(\delta^2(\hat{\theta}_2+1) - 2\delta - \hat{\theta}_1 + 1\right)}{n(s-1)((\delta-1)s+1)(s(\delta + \hat{\theta}_1 - 1) - \delta)},$$

respectively, with probability approaching 1.

Due to the greater complexity of these expressions, it is no longer feasible to provide explicit characterizations for the main comparative statics—in most cases, the sign of the relevant derivatives depends nonlinearly on the four parameters s, δ, $\hat{\theta}_1$, and $\hat{\theta}_2$. In this section, we therefore adopt the approach of evaluating each derivative at a fine grid of points in $(s, \delta, \hat{\theta}_1)$ space, holding $\hat{\theta}_2$ constant at a range of values, assuming without loss of generality that $\hat{\theta}_1 < \hat{\theta}_2$. In Figures 3, 4, and 5, we show the regions over which the main quantities of interest take positive and negative values assuming a moderate value $\hat{\theta}_2 = \frac{3}{4}$, while corresponding figures for other values of $\hat{\theta}_2$ can be found in Appendix A.

Unlike in the constant-information case, changes in parameters no longer have uniform effects. In particular, it is now possible for members of majority groups to

[10] See Supplemental Materials for details and code.

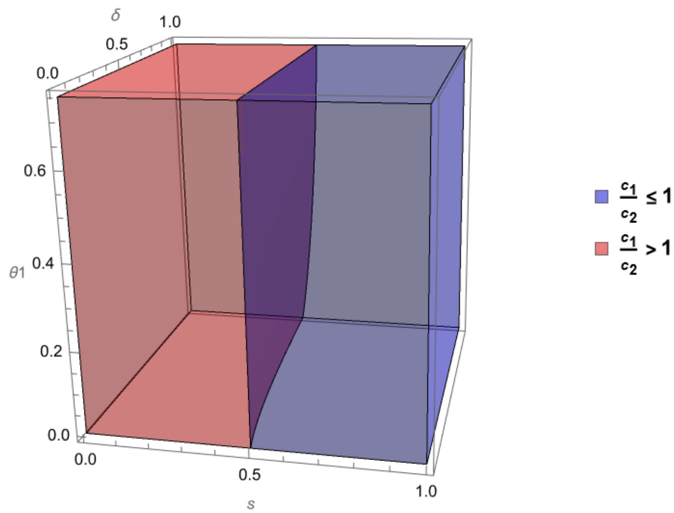
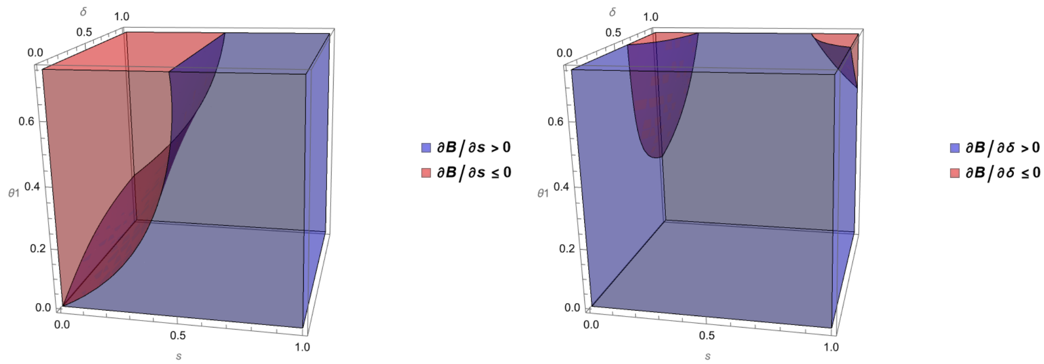


Figure 3: Relative size of expected centralities c_1 and c_2 as a function of s, δ, and $\hat{\theta}_1$ with $\hat{\theta}_2 = 0.75$. Note that the boundary at $s \approx 0.5$ is not a flat plane: for s sufficiently close to 0.5, there exist parameter values such that $c_1 \lesseqgtr c_2$.

have greater centrality on average, as shown in Figure 3. This kind of "flipping" only occurs when the groups are of approximately equal size—even with values of $\hat{\theta}_1$ and $\hat{\theta}_2$ arbitrarily close to 0 and 1, respectively, members of the minority group always receive more whenever $|s - 0.5| > 0.04$. However, it does serve to highlight the key difference from the baseline model: the greater marginal value of transfers to one group compared to the other, driven by prior type distribution, creates a countervailing influence whereby connections to the more predictable group become more valuable, regardless of group size. This effect is exactly reflected in Figure 5a, as changes in group size towards (away from) equality may now increase (decrease) inequality when the level of fractionalization is sufficiently close to 0.

Another clear illustration of this change from the baseline result can be seen in Figure 4a, which shows the effect of an increase in the size of group 1 on total expenditure. Whereas under homogeneous information an increase in the size of the majority group (greater fractionalization) always increases expenditures (see Proposition 5), this need not necessarily be the case when information varies by group. In



(a) Change in share of group 1 (b) Change in homophily

Figure 4: Effects of changes in social structure on total transfers given heterogeneous information by group with $\hat{\theta}_2 = 0.75$. A positive change in δ corresponds to a reduction in homophily, so that Panel (b) should be interpreted as showing that, for most parameter combinations, greater homophily leads to a reduction in transfers.

particular, when the less predictable (lower $\hat{\theta}_\ell$) group is in the minority, an increase in its size (reduction in fractionalization) can now lead to an increase in total expenditure, provided that its average posterior is sufficiently low. This is driven by a relatively higher rate of substitution into transfers to the majority group: as group sizes approach equality, members of the majority become relatively more valuable, while the increased number of cross-group ties offsets the reduction in transfers to the minority.

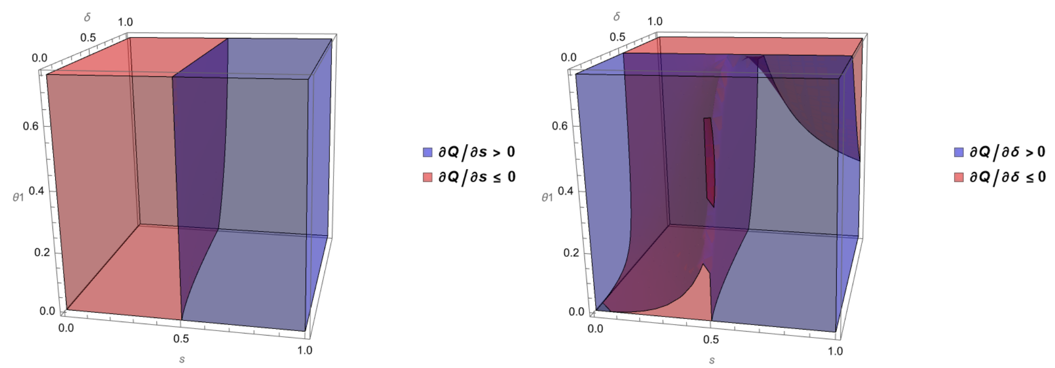
Finally, we consider the effect of a decrease in the level of homophily in the network (increase in δ), shown in Figures 4b and 5b. As in the baseline model, decreasing levels of homophily are uniformly associated with increases in expenditure unless at least one $\hat{\theta}_\ell$ is close to 1 and fractionalization is high. In other words, for an increase in homophily to be associated with an increase in spending, it is necessary that the corresponding relative decrease in cross-group ties actually be associated with an *increase* in the value of transfers to at least one group. Intuitively, this relies on two conditions: (1) the proportion of possible cross-group ties is sufficiently low that a marginal reduction in their probability does not have too large an effect and (2) the members of at least one group have sufficiently high $\hat{\theta}_\ell$ and are sufficiently numerous that, on average, ties within that group are more valuable than cross-group ties.

The relationship between homophily and inequality, shown in Figure 5b, is highly contingent. Similarly to the baseline model, when the less-predictable group (here, group 1) is in the majority, increases in homophily are generally associated with decreases in inequality, except when δ and $\hat{\theta}_1$ are both close to 1, corresponding to high predictability and low homophily. In this region, low-level increases in homophily have the effect of further increasing c_2 and decreasing c_1, exacerbating the existing inequality by weakening the equalizing effect of cross-group ties.

By contrast, when group 1 is the minority (and receiving higher average transfers), it is now possible for increases in homophily to cause increases in inequality even at low values of δ and $\hat{\theta}_1$. While increased homophily may increase c_1 if s is sufficiently close to 0 and δ to 1, for most parameter combinations increases in homophily decrease both c_1 and c_2 due to the loss of cross-group ties. However, when group 1 members are unpredictable compared to group 2 members, the net effect is to decrease c_2 by more than c_1, as members of the majority lose more total influence than do members of the minority. There are many parameter combinations that generate this effect, however, indicating that when the smaller group is also the less predictable electorally, the effects of homophily on inequality of transfers are generally quite ambiguous.

6 Discussion

In this article, we provide a formal model of vote buying on a network of policy-motivated voters who care about both private and public goods. We study the game under a given network structure and show that, in equilibrium, each individual voter receives the same bribe from both candidates. Since no voter receives more bribes from one candidate than the other, equilibrium expected vote share remains un-



(a) Change in share of group 1 (b) Change in homophily

Figure 5: Effects of changes in social structure on group inequality given heterogeneous information by group with $\hat{\theta}_2 = 0.75$. A positive change in δ corresponds to a reduction in homophily. Also note that the boundary at $s \approx 0.5$ in Panel (a) is not a flat plane: for s sufficiently close to 0.5, there exist parameter values such that $\frac{\partial Q}{\partial s} \lesseqgtr 0$.

changed and hence vote buying does not affect the electoral outcome. At the same time, the incorporation of network effects raises the marginal value of transfers to each voter, resulting in diversion of public resources that exceeds each voter's individual indifference point. In the absence of a vote-buying opponent with equal resources, moreover, these spillovers allow candidates to gain a large advantage from low spending.

Additionally, our analytical framework allows us to overcome a major limitation of many network models: the need to begin by taking a highly complex discrete graph structure as granted. By employing techniques from spectral random graph theory to study the role of social structure, we are able to explicitly characterize the equilibrium transfers received by each voter in large networks and derive sharp comparative statics.

Particularly noteworthy are our findings regarding network density and homophily. Contrary to arguments frequently found in the literature, density does not affect either the level of group inequality or total spending except through its association with candidate information. Homophily, meanwhile, actually decreases both for most plausible parameter values. Similarly, in line with recent reevaluations of the ethnic diversity-public good provision connection (Singh and Vom Hau 2016), we find that candidates actually face the strongest incentive to siphon public resources for targeted inducements under low fractionalization with high levels of social integration. Another implication of the model is that candidates will tend to disproportionately target minority group members regardless of their own group affiliation, although this incentive can be offset by a systematic informational advantage with respect to in-group members.

When candidates have heterogeneous information about voters, fractionalization can promote vote buying strategies when not much is known about minorities but reduces inequality in almost all contexts. Homophily, on the other hand, typically reduces the extent of vote buying but has a highly nonlinear relationship with the degree of inequality among groups. A more general contribution of the varied information section is its implications for the dilemma of targeting "swing" versus "core" voters (Calvo and Murillo 2004; Stokes 2005; Nichter 2008). Here, swing voters are not necessarily those who are close to indifference, but instead those who have high variance in their random preference shock (i.e., $\theta_i = \overline{\theta}$). From the candidate's perspective, in other words, a strategically significant feature of swing voters is that their behavior on election day is difficult to predict, whether due to inherent features of the voters themselves (such as limited interest in politics) or a lack of knowledge by the candidate. This perspective becomes particularly relevant when considering the role of the network, as when the type distribution correlates with observable group characteristics, swing voters may not only be high types themselves, but also connected to many other high types. Our model therefore indicates that a previously overlooked tradeoff for candidates engaging in vote buying is the extent to which they can be certain of positive spillovers from targeting specific social groups. These findings suggest that the social position of swing voters should also be considered;

targeted redistribution may not only rely on individual characteristics, but also the way voters are embedded in society.

The somewhat surprising result that equilibrium transfers are entirely independent of network density comes directly from the tradeoff candidates face between seeking out the best-connected voters and those that have the greatest influence over others, i.e., those that are connected to many isolated voters. A counterfactual increase in density in a given society—that is, a symmetric increase in the connection probabilities of all voters—will not affect centrality (and hence transfers) since the increased number of connections is exactly offset by the reduction in the influence of each individual tie. This remains true even when accounting for diverging quality of candidate information, unless information structures are directly reliant on the underlying social network. The significance of this finding is that density in itself does not necessarily matter for the reasons it is frequently assumed to. Simply adding more ties to a network does not in itself have an effect on strategic behavior unless accompanied by a change in the strength of those ties.

These results have direct implications for vote buying in a variety of political settings. For instance, we predict the most intense vote buying to occur in social contexts with a large majority and small minority but with relatively low levels of social segregation. At the same time, members of the minority group are likely to benefit disproportionately from vote buying, especially when candidates are well-informed about voter preferences. In the absence of strong in-group preferences, this will tend to occur regardless of the group affiliation of those dispensing resources and may lead to targeting voters that *ex ante* prefer the opposition.

While a lack of fine-grained comparative data on social network structure in locations where vote buying occurs makes it difficult to evaluate many of these predictions directly, the main observable implications of our model are borne out by the empirical literature. In Romania, for instance, vote buying (both legal and illegal) tends to be particularly prevalent in areas with a large Romanian majority and small minority of moderately integrated Roma, but to a much lesser degree in areas with concentrated—and far more socially isolated—ethnically Hungarian minorities (Pop-Eleches, Pop-Eleches et al. 2012; Mares, Muntean, and Petrova 2017; Mares and Young 2019). Similarly, vote buying in contexts as diverse as Ukraine (Schlegel 2021) and Kenya (Gutiérrez-Romero 2014) tends to be organized around identifiable ethnic groups, with candidates targeting minorities, including those that are naturally predisposed to support their opponents.

Perhaps the most important implication of these findings is that empirical work on vote buying—and clientelism more broadly—ignores the role of social structure at its own peril. While it is now well-established that network position affects individual targeting, variations in meso- and macro-level features of social networks, especially homophily and information structures, strongly shape the strategic environment for would-be vote buyers. Moreover, we suggest that density of ties should neither be assumed a defining feature of ethnic groups nor a strategically relevant variable, further demonstrating the value of explicitly modeling network dependence.

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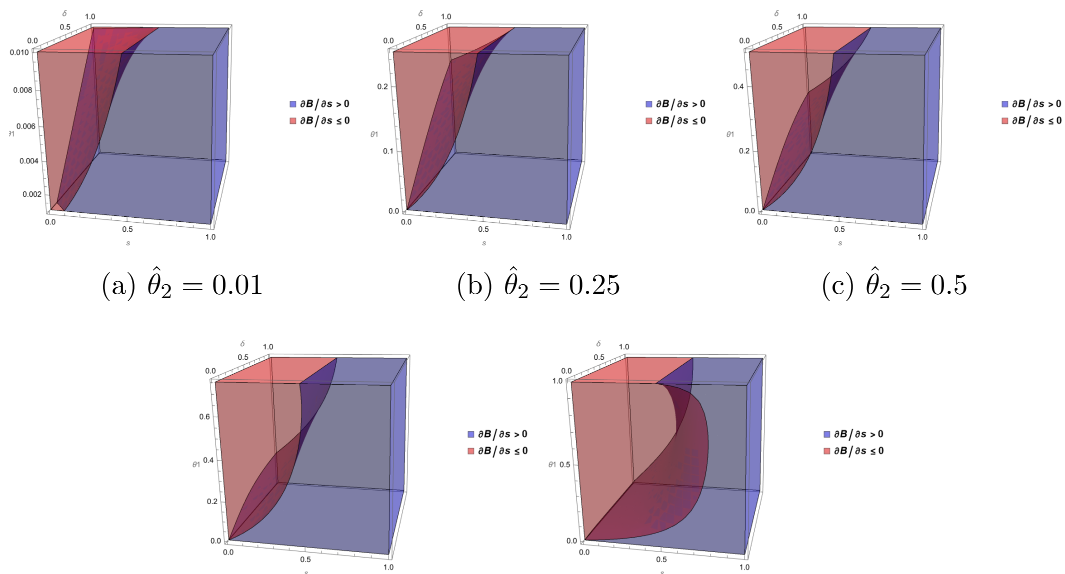
Online Appendix for "Some for the Price of One: Vote Buying on a Network"

Contents

A Additional Tables and Figures



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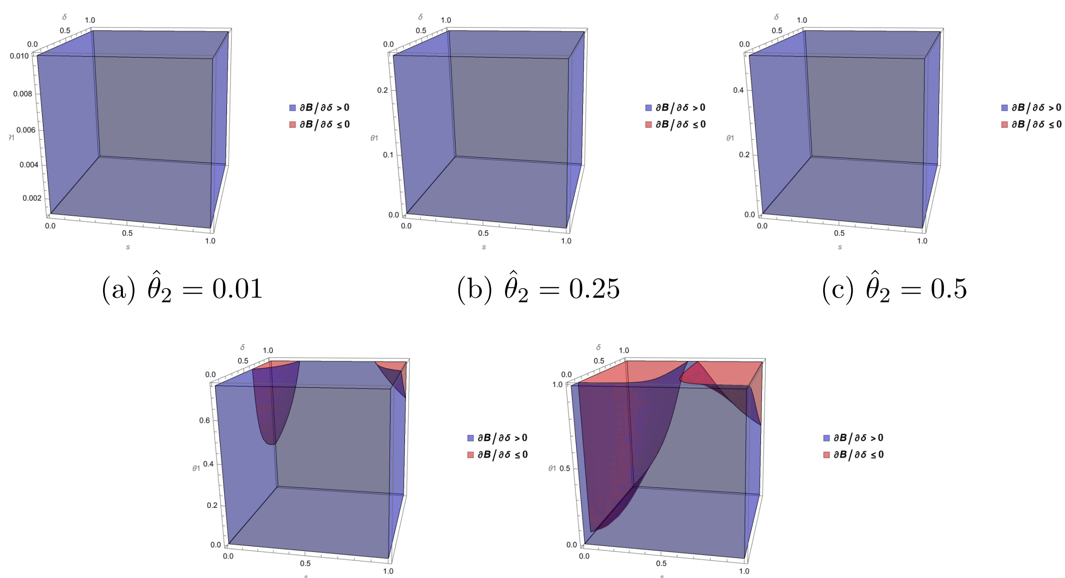
(d) $\hat{\theta}_2 = 0.75$ (e) $\hat{\theta}_2 = 0.99$

Figure 6: $\frac{\partial B}{\partial s}$ as a function of parameters for varying values of $\hat{\theta}_2$



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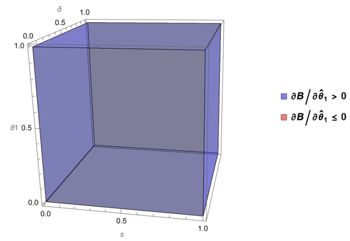
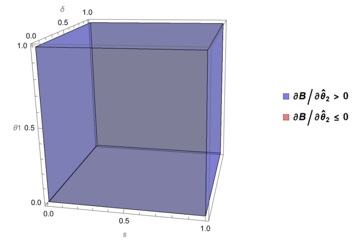
(d) $\hat{\theta}_2 = 0.75$ (e) $\hat{\theta}_2 = 0.99$

Figure 7: $\frac{\partial B}{\partial \delta}$ as a function of parameters for varying values of $\hat{\theta}_2$

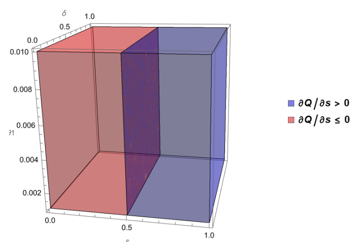


(a) $\frac{\partial B}{\partial \hat{\theta}_1}$



(b) $\frac{\partial B}{\partial \hat{\theta}_2}$

Figure 8: Derivatives of total transfers with respect to information as a function of parameters. These derivatives are always positive and do not depend on parameter choice.



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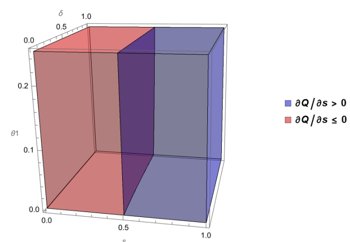

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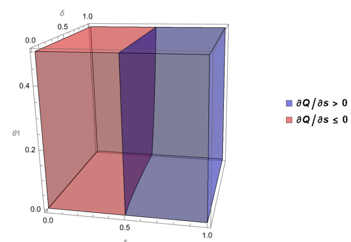

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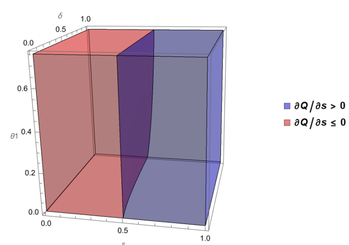

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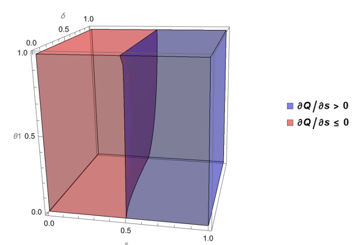
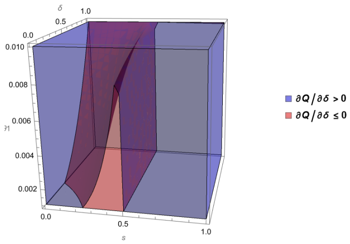


(e) $\hat{\theta}_2 = 0.99$

Figure 9: $\frac{\partial Q}{\partial s}$ as a function of parameters for varying values of $\hat{\theta}_2$



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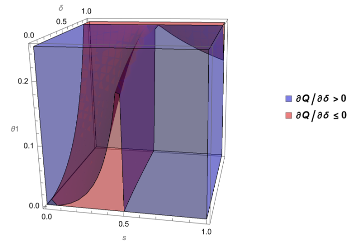

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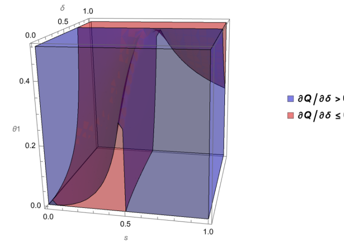

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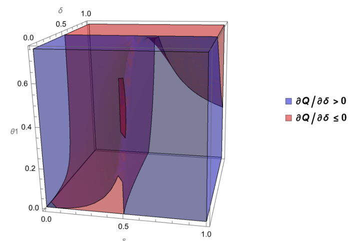

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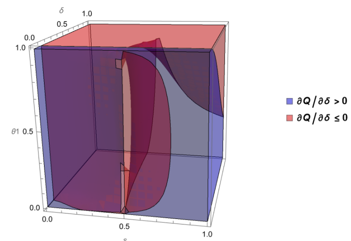
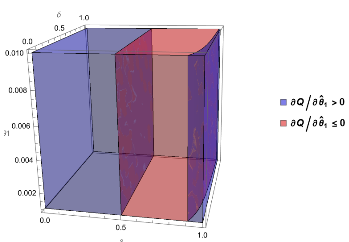


(e) $\hat{\theta}_2 = 0.99$

Figure 10: $\frac{\partial Q}{\partial \delta}$ as a function of parameters for varying values of $\hat{\theta}_2$



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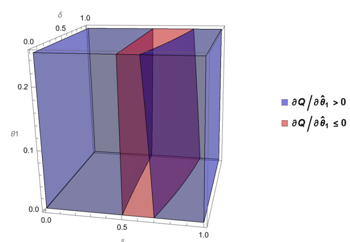

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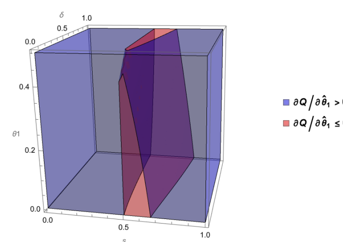

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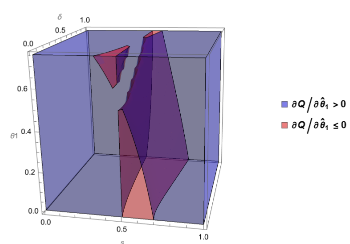

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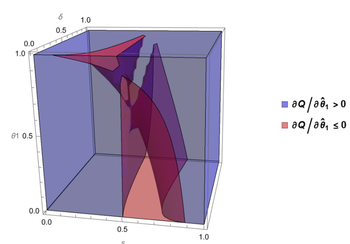
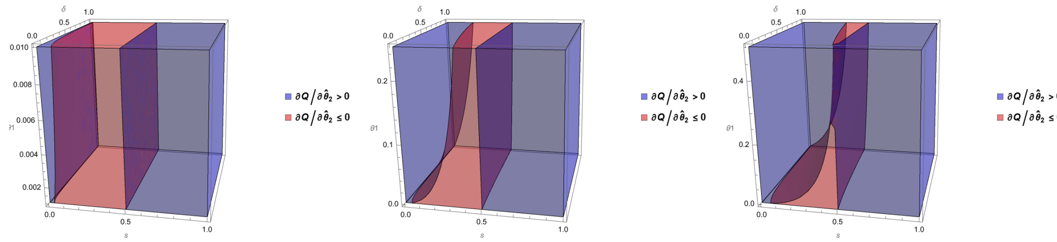


(e) $\hat{\theta}_2 = 0.99$

Figure 11: $\frac{\partial Q}{\partial \hat{\theta}_1}$ as a function of parameters for varying values of $\hat{\theta}_2$



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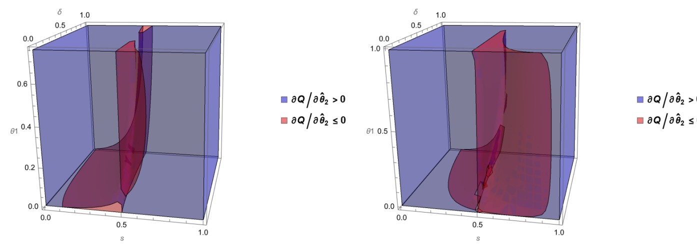

(d) $\hat{\theta}_2 = 0.75$ (e) $\hat{\theta}_2 = 0.99$

Figure 12: $\frac{\partial Q}{\partial \hat{\theta}_2}$ as a function of parameters for varying values of $\hat{\theta}_2$

B Network-Dependent Information

It is natural to think network dependence and candidate information are mutually reinforcing: social connections between brokers and voters are important precisely because they provide information about voter preferences, facilitating more accurate targeting (Finan and Schechter 2012; Stokes et al. 2013). In this section, we therefore consider a special case of the heterogeneous information model in which candidates' posterior beliefs about voter behavior depend on their position in the network.

We now place the candidates on the network and assume that a voter's type θ_i depends on their position on the network. If voters whose shortest-path distance from a candidate is lower have a higher probability that $\theta_i = \overline{\theta}$, then a systematic relationship will exist between i's probability of connecting with candidates p_{ik} and their average posterior information $\hat{\theta}_i$.[11]

In contrast to previous results, density may in fact play an important role in this setting, as increases in edge density will increase the likelihood of all voters being type $\overline{\theta}$ through a decrease in their expected distance from any candidate, which will directly affect candidates' posterior beliefs and the corresponding equilibrium bribes. Consequently, changes in density will now affect expected equilibrium behavior by changing the prior distribution of types. Although $\theta_i = \overline{\theta}$ symmetrically increases a voter's value to both candidates, equilibrium electoral outcomes may be affected if candidates enjoy systematically more precise information about socially proximate voters. In the two-candidate case, candidates will not only target their own close neighbors but also their opponent's, since both are associated with higher posteriors.

For this reason, we focus on the one-candidate case, which arises naturally in many applications. For example, vote brokers typically rely heavily on personal connections with voters, as the preferences of more socially distant voters are less legible, and only

[11] Since θ_i measures the predictability of a voter's behavior given their observable characteristics, it is plausible that voters who are more connected to political candidates will vote more consistently.

one agent typically makes offers in a given locality (Stokes et al. 2013; Holland and Palmer-Rubin 2015). Similarly, targeted inducements in the international arena are frequently offered by a single hegemonic power (Vreeland and Dreher 2014).

Figures 13 and 14 show the results of model simulations. Since the asymptotic results in the preceding section require independence between θ_i and p_{ij}, we take the approach of estimating $\mathbb{E}[b_i]$ on a finite network directly through 1,000 repeated draws of networks from the stochastic block model. For each combination of parameters, we calculate the average transfers to members of each group, $\bar{b}_\ell = \mathbb{E}[b_i|\ell_i]$, the overall inequality, $Q = \left|1 - \bar{b}_1/\bar{b}_2\right|$, and the total transfers $B = \sum_i b_i$ at each draw. Estimates are then calculated as the average across all draws, along with bootstrapped 95% percentile confidence intervals.

Since our focus is primarily on social structure, we repeat the process over a grid of values between 0 and 1 for each of $\delta = \frac{p_L}{p_H}$ and $s = \frac{n_1}{n}$, which govern homophily and fractionalization. To reduce the computational burden, the remaining parameters are held constant at moderate values of $n = 200, \alpha = 1, \gamma = \frac{1}{400}, u(b) = 200\ln(b)$, while we assume for simplicity that candidate k's information decays in social distance according to a power law, $\hat{\theta}_i = 2^{-d(i,k)}$.

As Figure 13 demonstrates, the introduction of network dependence in a one-candidate environment introduces a distinct element of in-group favoritism, consistent with empirical observations across a variety of settings. As might be expected, this favoritism is entirely driven by homophily: when δ is close to 1 (low homophily), the two groups receive almost identical amounts on average, with consistently large differences emerging only when homophily is quite extreme. Intuitively, preference for the candidate's own group members is driven by lower social distance on average, amplified by greater separation between the groups, which leads to higher confidence that bribes to them will lead to an increase in vote share. In contrast to the independent case, this tendency to favor in-groups completely outweighs the preference

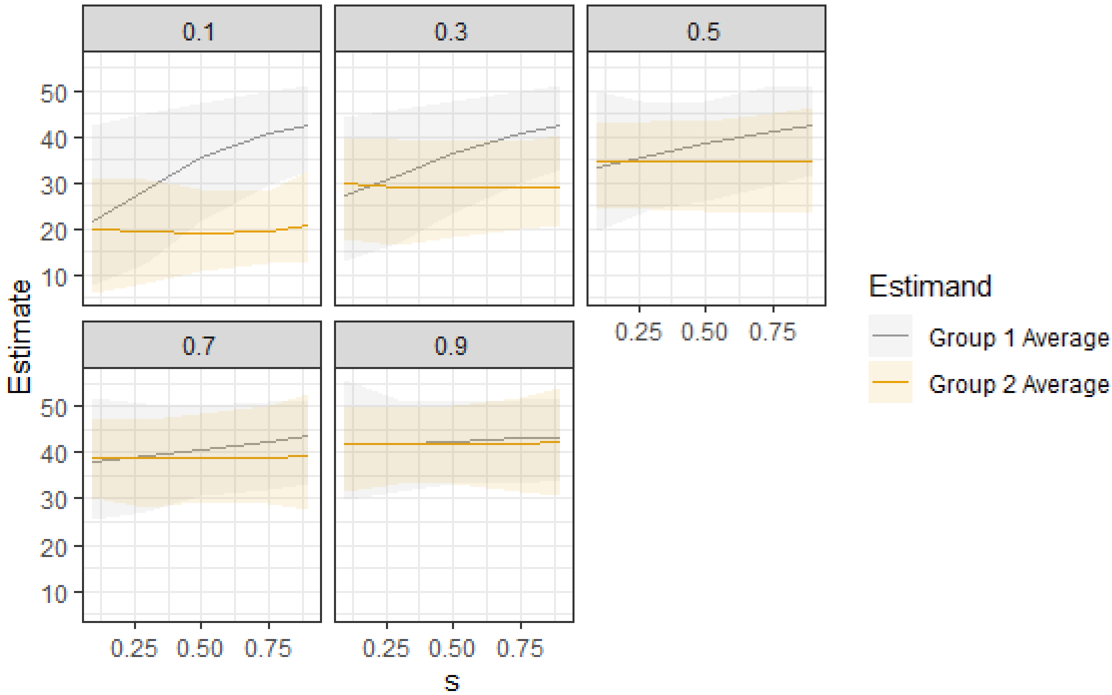

Figure 13: Average transfers from candidate 1 by group as a function of group 1's share (s), by homophily (δ).

for minorities. While the relative size of the groups has little effect on average on transfers to out-group members, when homophily is large transfers to the in-group are sharply increasing in group size. Under homophily, an increase in the size of the in-group further decreases the expected shortest path distance, thus raising the value of transfers to all group members. As a consequence, this also tends to increase the level of inequality (Figure 15) as more funds are diverted to the candidate's in-group members.

As can be seen in Figure 14, however, this effect is insufficient to completely offset the overall loss in network spillovers induced by an increase in homophily. While, for a given value of δ, expenditure on private transfers is highest when $s \to 1$, driven by increases in in-group spending, it is also increasing in δ for all values of s. In fact, as s approaches 1, the probabilities converge to the same values as when $\delta = 1$, so that in either case the candidate is, on average, as close as possible to all voters.

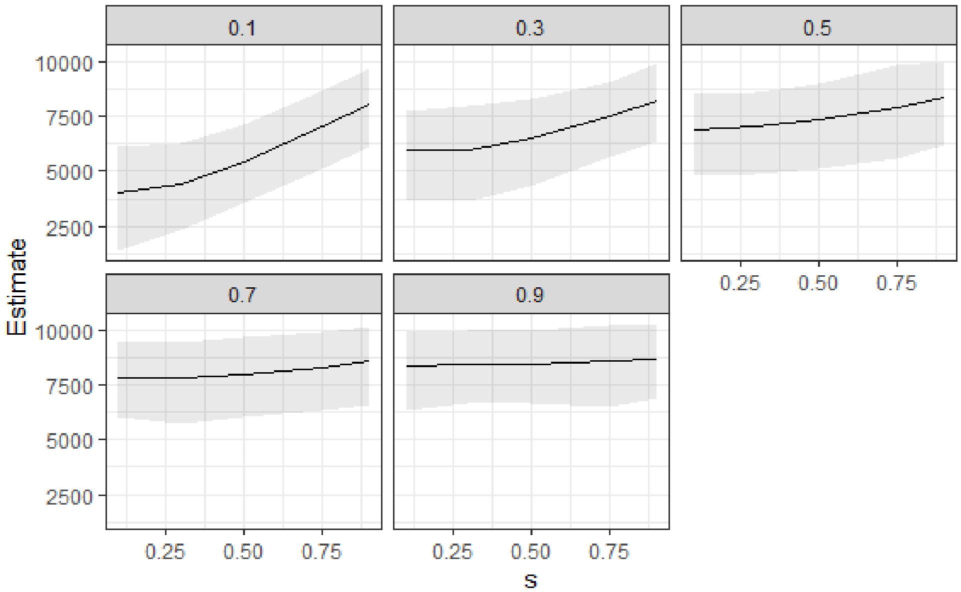

Figure 14: Total transfers from candidate 1 as a function of group 1's share (s), by homophily (δ)

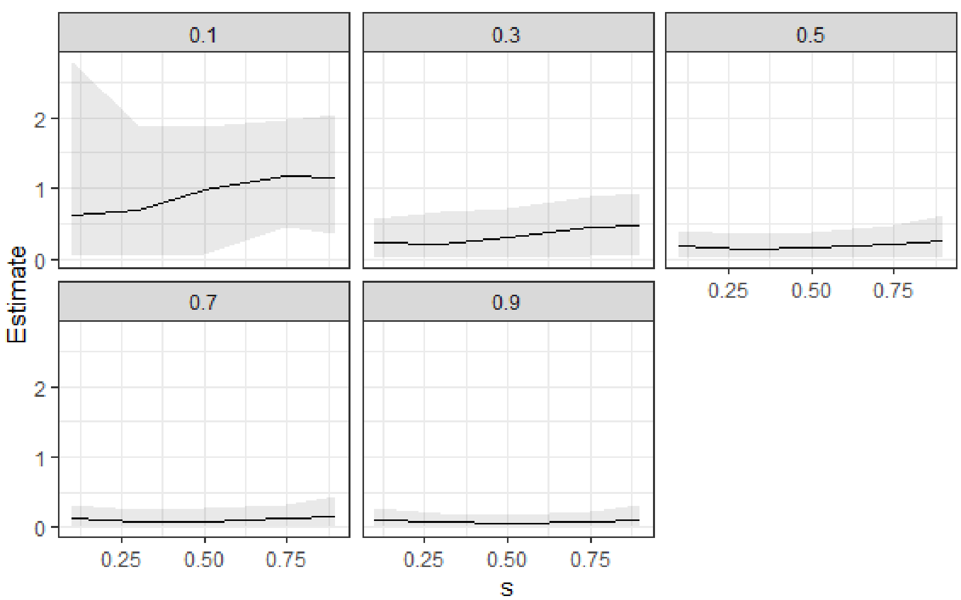

Figure 15: Inequality of average transfers from candidate 1 as a function of group 1's share (s), by homophily (δ)

When information depends on network structure in this way, therefore, we would expect the greatest diversion of funds from public to private goods in societies that are either perfectly homogeneous or have minimal levels of social segregation.

C Proofs

Lemma 2 (Chung and Radcliffe (2011))**.** *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ be bounded independent random Hermitian matrices and set $M > 0 : ||\boldsymbol{X}_i - \mathbb{E}(\boldsymbol{X}_i)||_2 \leq M \ \forall i = 1, \ldots, m$. Then for any $a > 0$,*

$$\Pr(||\boldsymbol{X} - \mathbb{E}(\boldsymbol{X})||_2 > a) \leq 2n \exp\left(-\frac{a^2}{2v^2 + 2Ma/3}\right)$$

where $\boldsymbol{X} = \sum_{i=1}^m \boldsymbol{X}_i$ and $v^2 = ||\sum_{i=1}^m \mathbb{V}(\boldsymbol{X}_i)||$.[12]

Theorem 1. *Let $\boldsymbol{L}_W$ denote the normalized weighted Laplacian of $\mathcal{G}$, ω the largest total weight, and $\underline{d}$ the smallest expected degree. For any $\epsilon > 0$, there exists a $k > 0$ such that, for all i,*

$$\Pr\left(||\boldsymbol{L}_W - \bar{\boldsymbol{L}}_W|| \leq 4\sqrt{\frac{3\omega\ln(4n/\epsilon)}{\underline{d}}}\right) \geq 1 - \epsilon$$

if $\underline{d} > k\ln(n)$ and $\alpha\omega \leq \sqrt{\frac{\underline{d}}{3\ln(4n/\epsilon)}}$, where α is the smallest total weight.[13]

Proof of Theorem 1. Let $\mathcal{G}$ be an undirected random graph such that all edge formation probabilities are jointly independent. Denote by $\boldsymbol{A}$ the adjacency matrix, $\boldsymbol{W}$ a matrix of weights, and $\boldsymbol{A}_W$ the weighted adjacency matrix, such that $\boldsymbol{A}_W = \boldsymbol{W} \odot \boldsymbol{A}$.[14] Let $\boldsymbol{D}_W$ be the diagonal degree matrix such that $\{\bar{\boldsymbol{D}}_W\}_{ii} = \sum_j w_{ij}a_{ij}$, and denote by $\bar{\boldsymbol{A}}, \bar{\boldsymbol{D}}_W$ the expected equivalents. Finally, let $\boldsymbol{L}_W = \boldsymbol{I} - \boldsymbol{D}_W^{-1/2}\boldsymbol{A}_W\boldsymbol{D}_W^{-1/2}$ denote the normalized weighted Laplacian of $\mathcal{G}$, $\omega = \max_{i,j} w_{ij}$ be the largest total weight, $\alpha = \min_{i,j} w_{ij}$ the smallest, and $\delta = \min_i\{\bar{\boldsymbol{D}}_W\}_{ii}$ the smallest expected degree.

Denote $\bar{d}_i$ as the expected (weighted) degree of node i. By the triangle inequality, for any matrix $\boldsymbol{C}$,

[12] See Theorem 5 in Chung and Radcliffe (2011) for the proof.
[13] Note that this is distinct from α in candidate utility.
[14] Note that $\odot$ indicates the Hadamard (element-wise) product.

$$||\boldsymbol{L}_W - \bar{\boldsymbol{L}}_W|| \leq ||\boldsymbol{C} - \bar{\boldsymbol{L}}_W|| + ||\boldsymbol{L}_W - \boldsymbol{C}||$$

In particular, let $\boldsymbol{C} = \boldsymbol{I} - \bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{A}_W\bar{\boldsymbol{D}}_W^{-1/2}$. Then since the degree matrices are diagonal, we have $\boldsymbol{C} - \bar{\boldsymbol{L}}_W = \bar{\boldsymbol{D}}_W^{-1/2}(\boldsymbol{A}_W - \bar{\boldsymbol{A}}_W)\bar{\boldsymbol{D}}_W^{-1/2}$. Denoting by $\boldsymbol{A}^{ij}$ the matrix that is equal to 1 in the i,jth and j,ith positions and 0 elsewhere, we can write i,jth entry of $\boldsymbol{C} - \bar{\boldsymbol{L}}$ as

$$\boldsymbol{X}_{ij} = \bar{\boldsymbol{D}}_W^{-1/2}(w_{ij}(a_{ij} - p_{ij})\boldsymbol{A}^{ij})\bar{\boldsymbol{D}}_W^{-1/2} = \frac{w_{ij}(a_{ij} - p_{ij})}{\sqrt{\bar{d}_i\bar{d}_j}}\boldsymbol{A}^{ij}$$

Then clearly $\boldsymbol{C} - \bar{\boldsymbol{L}} = \sum \boldsymbol{X}_{ij}$, so Lemma 2 applies. Since $\mathbb{E}(a_{ij}) = p_{ij}$, we have that $\mathbb{E}(\boldsymbol{X}_{ij}) = \boldsymbol{0}$, so that $v^2 = ||\sum \mathbb{E}(\boldsymbol{X}_{ij}^2)||$. Also, each $\boldsymbol{X}_{ij}$ is bounded above by $\|\boldsymbol{X}_{ij}\| \leq \frac{\omega}{\delta}$ Now clearly

$$\mathbb{E}(\boldsymbol{X}_{ij}^2) = \begin{cases} \frac{w_{ij}^2}{\bar{d}_i\bar{d}_j}(p_{ij})(1-p_{ij})(A^{ii} + A^{jj}) & i \neq j \\ \frac{w_{ii}^2}{\bar{d}_i^2}(p_{ij})(1-p_{ij})A^{ii} & i = j \end{cases}$$

Now we can write

$$\begin{aligned} v^2 &= \left\| \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{w_{ij}^2}{\bar{d}_i\bar{d}_j}(p_{ij})(1-p_{ij})A^{ii} \right\| \\ &= \max_i \left(\sum_{j=1}^{n} \frac{w_{ij}^2}{\bar{d}_i\bar{d}_j}(p_{ij})(1-p_{ij}) \right) \\ &\leq \max_i \left(\frac{\omega}{\delta} \sum_{j=1}^{n} \frac{w_{ij}}{\bar{d}_i}(p_{ij}) \right) \\ &= \frac{\omega}{\delta} \end{aligned}$$

For notational convenience denote $a = \sqrt{\frac{3\omega\ln(4n/\epsilon)}{\delta}}$ and δ so that $a < 1$. In particular, we must have $\delta > 3\omega(\ln(4) + \ln(n) - \ln(\epsilon))$, so that if $k \geq 3\omega(1 + \ln(4/\epsilon))$, $\delta \geq k\ln(n)$

guarantees the result. Then from Lemma 2

$$\begin{aligned}
\Pr(\|\boldsymbol{C} - \bar{\boldsymbol{L}}_W\| > a) &\leq 2n \exp\left(-\frac{\frac{3n\omega^2 \ln(4n/\epsilon)}{n\delta}}{2n\omega^2/\delta + 2an\omega^2/3\delta}\right) \\
&= 2n \exp\left(-\frac{\frac{3n\omega^2 \ln(4n/\epsilon)}{\delta}}{2n\omega^2(3+a)/3\delta}\right) \\
&= 2n \exp\left(-\frac{9\ln(4n/\epsilon)}{6+2a}\right) \\
&\leq 2n \exp\left(-\frac{9\ln(4n/\epsilon)}{9}\right) \\
&= \frac{\epsilon}{2}
\end{aligned}$$

Now for the second term, note that d_i is a sum of random variables that are bounded between 0 and ω. Then by Hoeffding's Inequality, we have that, for any t,

$$\Pr(|d_i - \bar{d}_i| > t\bar{d}_i) \leq 2\exp\left(-\frac{t^2\bar{d}_i^2}{n\omega^2}\right) \leq 2\exp\left(-\frac{t^2\delta^2}{n\omega^2}\right)$$

Now in particular let $t = \sqrt{\frac{n\omega^2 \ln(4n/\epsilon)}{\delta^2}} = \sqrt{\frac{n\omega}{3\delta}}a$. We have $t < a < 1$ if $\delta > \frac{n\omega}{3}$. In our application, $\omega = \rho_H, \delta = n_0 p_L \rho_L + n_1 p_H \rho_H.$, so that for all i we obtain

$$\Pr(|d_i - \bar{d}_i| > t\bar{d}_i) \leq \frac{\epsilon}{2n}$$

Now note that

$$\left\|\bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{D}_W^{1/2} - \boldsymbol{I}\right\|_2 = \max_i \left|\sqrt{\frac{d_i}{\bar{d}_i}} - 1\right|$$

To bound this, note that from (C) we can conclude that $\Pr\left(\left|\frac{d_i}{\bar{d}_i} - 1\right| > t\right) \leq \frac{\epsilon}{2n}$ and hence with probability at least $1 - \frac{\epsilon}{2n}$,

$$\left\|\bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{D}_W^{1/2} - \boldsymbol{I}\right\|_2 < \sqrt{\frac{n\omega^2 \ln(4n/\epsilon)}{\delta^2}}$$

Finally, note that since $\|\boldsymbol{L}\|_2 \leq 2$ (Chung and Graham 1997), we have $\|\boldsymbol{I}-\boldsymbol{L}\|_2 \leq 1$. Now consider

$$\begin{aligned}
\|\boldsymbol{L}_W - \boldsymbol{C}\| &= \|\boldsymbol{I} - \boldsymbol{D}_W^{-1/2}\boldsymbol{A}_W\boldsymbol{D}_W^{-1/2} - \boldsymbol{I} + \bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{A}_W\bar{\boldsymbol{D}}_W^{-1/2}\| \\
&= \|(\boldsymbol{I} - \boldsymbol{L}_W)\bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{D}_W^{1/2}\boldsymbol{D}_W^{-1/2}\boldsymbol{A}_W\boldsymbol{D}_W^{-1/2}\boldsymbol{D}_W^{1/2}\bar{\boldsymbol{D}}_W^{-1/2}\| \\
&= \|(\boldsymbol{I} - \boldsymbol{L}_W)\bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{D}_W^{1/2}(\boldsymbol{I} - \boldsymbol{L})\boldsymbol{D}_W^{1/2}\bar{\boldsymbol{D}}_W^{-1/2}\| \\
&= \|(\bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{D}_W^{1/2} - \boldsymbol{I})(\boldsymbol{I} - \boldsymbol{L}_W)\boldsymbol{D}_W^{1/2}\bar{\boldsymbol{D}}_W^{-1/2} + (\boldsymbol{I} - \boldsymbol{L})(\boldsymbol{I} - \boldsymbol{D}_W^{1/2}\bar{\boldsymbol{D}}_W^{-1/2})\| \\
&\leq \|\bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{D}_W^{1/2} - \boldsymbol{I}\|\|\boldsymbol{D}_W^{1/2}\bar{\boldsymbol{D}}_W^{-1/2}\| + \|\boldsymbol{I} - \boldsymbol{D}_W^{1/2}\bar{\boldsymbol{D}}_W^{-1/2}\| \\
&\leq t^2 + 2t
\end{aligned}$$

Hence, finally,

$$\begin{aligned}
||\boldsymbol{L}_W - \bar{\boldsymbol{L}}_W|| &\leq ||\boldsymbol{C} - \bar{\boldsymbol{L}}_W|| + ||\boldsymbol{L}_W - \boldsymbol{C}|| \\
&\leq a + \frac{n\omega}{3\delta}a^2 + \sqrt{\frac{4n\omega}{3\delta}}a \\
&= a\left(\frac{\sqrt{3\delta} + 2\sqrt{n\omega}}{\sqrt{3\delta}} + \frac{n\omega}{3\delta}a\right) \\
&= a\left(1 + \frac{2\sqrt{3n\omega\delta} + n\omega a}{3\delta}\right)
\end{aligned}$$

Now, choose $k > 1$ such that

$$\delta \geq \frac{1}{3}\left(2n\omega\frac{\sqrt{k} + k + 1}{(k-1)^2}\right).$$

□

Proof of Proposition 4. By the result established in Lemma 1, it is sufficient to consider centrality on the average network. Under the stochastic block model, letting s_i denote the share of i's group without loss of generality, we have that the expected degree of i can be written as

$$\sum_{j=1}^{n} w_{ij}p_{ij} = s_i n w_H p_H + (1-s_i) n w_L p_L = n\left(w_L p_L + s_i(w_H p_H - w_L p_L)\right),^{15}$$

For notational convenience, we denote $w_H p_H = \rho, w_L p_L = \delta\rho$ for some $0 < \delta < 1$. The key observation is that the actual value of ρ is irrelevant, since it appears in both the denominator and numerator of each entry of the expected adjacency matrix. Thus, all results depend only on δ, the *relative* expected weight placed on out-group connections.

Note now that we can write the matrix $\boldsymbol{I} - \theta\bar{\tilde{\boldsymbol{A}}}$ as a 2×2 block matrix with blocks $\bar{\tilde{\boldsymbol{A}}}_{11} = \boldsymbol{I} - \frac{\theta}{n(\delta+s_1(1-\delta))}\left(\mathbf{1}_{s_1 n \times s_1 n} - \boldsymbol{I}\right)$, $\bar{\tilde{\boldsymbol{A}}}_{12} = -\frac{\theta\delta}{n(\delta+s_1(1-\delta))}\mathbf{1}_{s_1 n \times s_2 n}$, $\bar{\tilde{\boldsymbol{A}}}_{21} = -\frac{\theta\delta}{n(\delta+s_2(1-\delta))}\mathbf{1}_{s_2 n \times s_1 n}$, and $\bar{\tilde{\boldsymbol{A}}}_{22} = \boldsymbol{I} - \frac{\theta}{n(\delta+s_2(1-\delta))}\left(\mathbf{1}_{s_2 n \times s_2 n} - \boldsymbol{I}\right)$. To apply the formula for block inversion, we first want to identify $\bar{\tilde{\boldsymbol{A}}}_{11}^{-1}$. We conjecture that

$$P = \bar{\tilde{\boldsymbol{A}}}_{11}^{-1} = \begin{bmatrix} a_1 & b_1 & \cdots & b \\ b & a & \cdots & b \\ \vdots & \cdots & \ddots & \vdots \\ b & \cdots & \cdots & a \end{bmatrix}$$

Then we have that

$$\begin{bmatrix} 1 & -(n_1-1)\frac{\theta}{n(\delta+s_1(1-\delta))} \\ -\frac{\theta}{n(\delta+s_1(1-\delta))} & \left(1-(n_1-2)\frac{\theta}{n(\delta+s_1(1-\delta))}\right) \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which indeed has a unique solution. The inverse of the bottom-right block is identical, swapping group indices. Hence, we can construct the centrality vector according to

[15] Technically this is an approximation since $p_{ii} = 0$, but the loss is insignificant for large n, which is assumed here.

the formula:

$$\boldsymbol{c} = \begin{bmatrix} \left(\bar{\tilde{\boldsymbol{A}}}_{11} - \bar{\tilde{\boldsymbol{A}}}_{12}\bar{\tilde{\boldsymbol{A}}}_{22}^{-1}\bar{\tilde{\boldsymbol{A}}}_{21}\right)^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \left(\bar{\tilde{\boldsymbol{A}}}_{22} - \bar{\tilde{\boldsymbol{A}}}_{21}\bar{\tilde{\boldsymbol{A}}}_{11}^{-1}\bar{\tilde{\boldsymbol{A}}}_{12}\right)^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\bar{\tilde{\boldsymbol{A}}}_{12}\bar{\tilde{\boldsymbol{A}}}_{22}^{-1} \\ -\bar{\tilde{\boldsymbol{A}}}_{21}\bar{\tilde{\boldsymbol{A}}}_{11}^{-1} & \boldsymbol{I} \end{bmatrix} \tag{A1}$$

The main-diagonal blocks in the first matrix again have the same structure, with a single value on the main diagonal and another value on the off-diagonal. This has a similar structure to the previous matrix, and the inverse can thus be calculated analogously by solving for main and off-diagonal elements a_i', b_i'[16].

Remark. There is a substantive interpretation of a_i' and b_i': a_i' is the weighted average of the number of paths back to a voter in group i through the network, while b_i' is the weighted average of the number of paths to someone else in your group through the network. Because in the expected network, all voters are connected to all others, i's centrality does not depend on paths to the other group, because a linear dependence is induced (all paths within group essentially correspond to an equivalent cross-party path).

Substituting these values into equation (A1), we then have that

$$c_i = (1 + n_{-i})a_i' + (1 + n_{-i})(n_i - 1)b_i'$$

[16] A unique solution again exists, but we suppress the exact expression as it is extremely complex. The mathematica file used to calculate these values is available upon request from the authors.

or, without loss of generality,

$$
\begin{aligned}
c_1 = n\Big(& -\delta\theta - \delta n + (\delta-1)ns^2(\delta\theta+\delta+\theta-1) \\
& -(\delta-1)ns(\delta\theta+\delta+\theta-1) + (\delta-1)\theta s\Big) \\
\cdot \Big(& -\theta^2 + n^3 s((\delta-1)s-\delta)(-\theta+s(\delta+\theta-1)+1) \\
& + \theta n^2 s(-\delta-\theta+s(2\delta+\theta-2)+1) + \theta n\left(\theta+\delta^2\theta s - \delta s + s - 1\right)\Big)^{-1}.
\end{aligned}
$$

Since n is by assumption large, this expression is asymptotically equivalent to its leading term. We can thus simplify further, writing

$$
c_1 \sim \frac{-\delta + (\delta-1)s^2(\delta\theta+\delta+\theta-1) - (\delta-1)s(\delta\theta+\delta+\theta-1)}{sn((\delta-1)s-\delta)(-\theta+s(\delta+\theta-1)+1)}.
$$

□

Proof of Proposition 7. It is straightforward to see that, replacing θ with Θ, the modified centrality of agenti i is now equal to their DeGroot centrality (Mostagir, Ozdaglar, and Siderius 2022) on the normalized network. It then follows from Theorem 1 and from the proof of Theorem 1 in Mostagir and Siderius (2021) that we can again consider the expected network only. The result then follows from an analogous argument to the proof of the preceding proposition, calculated using the accompanying Wolfram Mathematica code. □

Proof of Lemma 1. Let $\mathcal{G}^{(n)}$ be a sequence of random graphs over n vertices, and denote by $\delta_{(n)}$ the smallest expected weighted degree, i.e., $\delta_{(n)} = \min_i \sum_j w_{ij}^{(n)} p_{ij}^{(n)}$. Further, let $\bar{w}_{(n)} = \max_{i,j} w_{ij}^{(n)}$ and $\underline{w}_{(n)} = \min_{i,j} w_{ij}^{(n)}$ be the largest and smallest individual weights, satisfying $\frac{\bar{w}_{(n)}}{\underline{w}_{(n)}} \leq \omega$ for some $\omega > 0$ for all n. Then if there exists a non-decreasing sequence of $k_{(n)} > 0$ such that $\delta_{(n)} \geq k_{(n)} \ln(n)$ and $\underline{w}_{(n)} \cdot \bar{w}_{(n)} =$

$o\left(\sqrt{\frac{\delta_{(n)}}{\ln(n)}}\right)$, then the realized centrality vector centrality vector $\boldsymbol{c}^{(n)}(\tilde{\boldsymbol{A}})$ is with high probability close to the centrality of the average graph $\boldsymbol{c}^{(n)}(\bar{\tilde{\boldsymbol{A}}})$ for large n.

Under the stated assumptions, we can apply Theorem 1 to conclude that for any $\xi > 0$, for all n we have

$$\Pr\left(||\boldsymbol{L}_W - \bar{\boldsymbol{L}}_W|| \leq 4\sqrt{\frac{3\omega\ln(4n/\xi)}{\delta}}\right) \geq 1 - \xi$$

Furthermore, by assumption $\lim_{n\to\infty} 4\sqrt{\frac{3\omega\ln(4n/\xi)}{\delta}} = 0$ regardless of the ξ chosen, so that under the 2-norm,

$$\boldsymbol{L}_W \underset{p}{\rightarrow} \bar{\boldsymbol{L}}_W$$

Now, for convenience call $\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{L}_W$ and $\bar{\boldsymbol{B}}$ the expected equivalent. Now clearly also $\boldsymbol{B} \underset{p}{\rightarrow} \bar{\boldsymbol{B}}$, and furthermore we can write $\boldsymbol{B} = \boldsymbol{D}^{-1/2}\boldsymbol{A}_W\boldsymbol{D}^{-1/2} = \boldsymbol{D}^{-1/2}\tilde{\boldsymbol{A}}\boldsymbol{D}^{1/2}$. So we can write, using properties of matrix norms (abusing notation in the second step slightly so that the maximum is over the norm of the matrices) and the above result,

$$\begin{aligned}
\limsup_{n\to\infty} \|\tilde{\boldsymbol{A}} - \bar{\tilde{\boldsymbol{A}}}\| &= \limsup_{n\to\infty} \|\boldsymbol{D}^{1/2}\boldsymbol{B}\boldsymbol{D}^{-1/2} - \bar{\boldsymbol{D}}^{1/2}\bar{\boldsymbol{B}}\bar{\boldsymbol{D}}^{-1/2}\| \\
&\leq \limsup_{n\to\infty} \| \max\left\{\boldsymbol{D}^{1/2}, \bar{\boldsymbol{D}}^{1/2}\right\}(\boldsymbol{B} - \bar{\boldsymbol{B}}) \max\left\{\boldsymbol{D}^{-1/2}, \bar{\boldsymbol{D}}^{-1/2}\right\} \| \\
&\leq \limsup_{n\to\infty} \max\left\{\|\boldsymbol{D}^{1/2}\|, \|\bar{\boldsymbol{D}}^{1/2}\|\right\} \|\boldsymbol{B} - \bar{\boldsymbol{B}}\| \max\left\{\|\boldsymbol{D}^{-1/2}\|, \|\bar{\boldsymbol{D}}^{-1/2}\|\right\} \\
&\leq \limsup_{n\to\infty} \xi \max\left\{\|\boldsymbol{D}^{1/2}\|, \|\bar{\boldsymbol{D}}^{1/2}\|\right\} \max\left\{\|\boldsymbol{D}^{-1/2}\|, \|\bar{\boldsymbol{D}}^{-1/2}\|\right\}
\end{aligned}$$

Now since ξ can be chosen to be arbitrarily small, it is sufficient to establish that both $\|\boldsymbol{D}^{1/2}\|_2\|\bar{\boldsymbol{D}}^{-1/2}\|_2$ and $\|\boldsymbol{D}^{1/2}\|_2\|\bar{\boldsymbol{D}}^{-1/2}\|_2$ are bounded by a constant almost surely. To see that they are, observe that

$$\|\boldsymbol{D}^{1/2}\|_2\|\bar{\boldsymbol{D}}^{-1/2}\|_2 = \sqrt{\frac{\max_i \sum_j w_{ij}a_{ij}}{\max_i \sum_j w_{ij}p_{ij}}} \leq \sqrt{\frac{\bar{w}_{(n)}}{\underline{w}_{(n)}}} \max_i \sqrt{\frac{\sum_j a_{ij}}{\sum_j p_{ij}}} \leq \sqrt{\omega} \max_i \sqrt{\frac{\sum_j a_{ij}}{\sum_j p_{ij}}}$$

and similarly

$$\|\boldsymbol{D}^{1/2}\|_2\|\bar{\boldsymbol{D}}^{-1/2}\|_2 = \sqrt{\frac{\max_i \sum_j w_{ij}p_{ij}}{\max_i \sum_j w_{ij}a_{ij}}} \leq \sqrt{\frac{\bar{w}_{(n)}}{\underline{w}_{(n)}}} \max_i \sqrt{\frac{\sum_j p_{ij}}{\sum_j a_{ij}}} \leq \sqrt{\omega} \max_i \sqrt{\frac{\sum_j p_{ij}}{\sum_j a_{ij}}}$$

But since the a_{ij} are distributed Bernoulli(p_{ij}), (see, e.g., Mostagir and Siderius (2021)) both $\max_i \sqrt{\frac{\sum_j p_{ij}}{\sum_j a_{ij}}}$ and $\max_i \sqrt{\frac{\sum_j a_{ij}}{\sum_j p_{ij}}}$ converge in probability to 1, so that we have for any $\xi > 0$

$$\limsup_{n\to\infty} \|\tilde{\boldsymbol{A}} - \bar{\tilde{\boldsymbol{A}}}\| \leq \xi\sqrt{\omega}$$

That is, the weighted adjacency matrix can be made arbitrarily close to its expected counterpart.

We now wish to show that, for arbitrary $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(\|(\boldsymbol{I} - \theta\tilde{\boldsymbol{A}})^{-1} - (\boldsymbol{I} - \theta\bar{\tilde{\boldsymbol{A}}})^{-1}\| \geq \epsilon) = 0$$

The key observation is that the above result implies that for any $\mu > 0$, there exists sufficiently large n such that with probability approaching 1, $\|\tilde{\boldsymbol{A}}^k - \bar{\tilde{\boldsymbol{A}}}^k\| \leq \mu$ for all k. Then it is straightforward to note that (since by model assumptions we have $\theta < 1$, so the formula for infinite geometric series can be applied),

$$\begin{aligned}\limsup_{n\to\infty} \|(\boldsymbol{I} - \theta\tilde{\boldsymbol{A}})^{-1} - (\boldsymbol{I} - \theta\bar{\tilde{\boldsymbol{A}}})^{-1}\| &= \limsup_{n\to\infty} \left\| \sum_{k=0}^{\infty} \theta^k \left(\tilde{\boldsymbol{A}}^k - \bar{\tilde{\boldsymbol{A}}}^k \right) \right\| \\ &\leq \limsup_{n\to\infty} \sum_{k=0}^{\infty} |\theta^k| \left\| \left(\tilde{\boldsymbol{A}}^k - \bar{\tilde{\boldsymbol{A}}}^k \right) \right\| \\ &\leq \sum_{k=0}^{\infty} \mu|\theta^k| \\ &= \frac{\mu}{1-\theta}\end{aligned}$$

Since μ was chosen arbitrarily, this implies that for any $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(\|\boldsymbol{c}^{(n)}(\tilde{\boldsymbol{A}}) - \boldsymbol{c}^{(n)}(\bar{\tilde{\boldsymbol{A}}})\| > \epsilon) = 0$$

Finally, note that the assumption of non-vanishing spectral gap guarantees that the network is connected with high probability, so that the centrality is well-defined (Dasaratha 2020; Mostagir and Siderius 2021), completing the proof. $\square$

Lemma 3. *Define the following terms.*

1. $\Delta_{ij} := \left|\phi_j(\cdot|\boldsymbol{\theta} : \theta_i = \underline{\theta}) - \phi_j(\cdot|\boldsymbol{\theta} : \theta_i = \overline{\theta})\right|$
2. W_{ij}^k *the (weighted) sum of all length k walks beginning with i and ending with j*
3. $\mathcal{I}_{ij} := \frac{\sum_{k=1}^{\infty} \overline{\theta}^k W_{ij}^k}{\sum_h \sum_{k=1}^{\infty} \underline{\theta}^k W_{hj}^k}$

Then, for any sequence of graphs $\mathcal{G}^{(n)}$ such that $(\bar{d}_n/\overline{\theta})^{\mathrm{diam}(\mathcal{G})} = o(n)$, there exists a $z_n > 0$ such that $\Delta_{ij}(\mathcal{G}^{(n)}) < z_n\mathcal{I}_{ij}(\mathcal{G}^{(n)})$ which satisfies $z_n = o(n)$.[17]

Proof. Take a graph $\mathcal{G}$, a corresponding Δ_{ij}, and choose a constant $z_n \in \mathbb{R}$ where

$$z_n > z_n^* := \frac{\Delta_{ij}}{\mathcal{I}_{ij}}.$$

[17] This assumption is consistent with the actual characteristics of "well-behaved" social network graphs, both theoretically and empirically (Jackson 2008).

Note that the numerator is bounded by 1 and the denominator must be greater than 0, so it must be that z_n^* is finite. Then, z_n must satisfy $\Delta_{ij} < z_n \mathcal{I}_{ij}$. Moreover, it is clear that $z_n^* < \frac{1}{\mathcal{I}_{ij}}$.[18]

Consider now the denominator of $\mathcal{I}_{ij}$. Since this reflects the sum of all walks of length k ending in j, $\sum_h \sum_{k=1}^{\infty} \underline{\theta}^k W_{hj}^k$, it is straightforward to show by induction on the length of walks k that this must always equal $\underline{\theta}/(1-\underline{\theta})$.

To see this, note that first when $k = 1$, it is trivially true that there are d_j such walks (starting from each of j's neighbors), each with weight $\frac{1}{d_j}$, so that $\sum_h W_{hj}^1 = 1$.

Now assume that for arbitrary $k \geq 2$, $\sum_h W_{hj}^k = 1$. Then for any walk of length $k+1$ with start vertex h, the deletion of h from the walk produces a (not necessarily unique) walk of length k that begins from some h' adjacent to h. In particular, for each walk w_{hj}^k of length k beginning with vertex h there are d_h walks of length $k+1$, each with weight $\frac{1}{d_h} w_{hj}^k$.[19] The sum of all of these walks can then be expressed

$$\sum_{h' \in \mathcal{T}_h(\mathcal{G})} \frac{1}{d_h} w_{ij}^k = w_{ij}^k.$$

But note that this enumeration is exhaustive; that is, there is no walk of length $k+1$ that cannot be constructed in this fashion, and no two distinct w_{ij}^k can be extended to produce the same w_{ij}^{k+1}. Hence, we have that $W^{k+1} = \sum_i w_{ij}^k = 1$ by the induction assumption.

[18] In fact, this inequality is necessarily extremely loose, since $\Delta_{ij} = 1$ would imply that a change in θ_i moves j from never voting for candidate 1 to voting for them with certainty, which cannot be true by construction.

[19] Note that here w_{hj}^k refers to a particular k-length walk from h to j, which should not be confused with w_{hj}, the weight of j's influence on h's voting behavior from other sections of the paper.

Moreover, this implies that, regardless of n,

$$\begin{aligned}\sum_h \sum_{k=1}^{\infty} \underline{\theta}^k W_{hj}^k &= \sum_{k=1}^{\infty} \underline{\theta}^k \sum_h W_{hj}^k \\ &= \sum_{k=1}^{\infty} \underline{\theta}^k \\ &= \frac{\underline{\theta}}{1-\underline{\theta}},\end{aligned}$$

which implies

$$\mathcal{I}_{ij} = \frac{1-\underline{\theta}}{\underline{\theta}} \sum_{k=1}^{\infty} \overline{\theta}^k W_{ij}^k.$$

Now observe that, defining by $d(i,j)$ the unweighted shortest path distance from i to j, for any $k < d(i,j)$, there by definition exist no walks from i to j of length k, so that clearly $W_{ij}^k = 0$. Then a trivial lower bound on $\mathcal{I}_{ij}$ is $\overline{\theta}^{d(i,j)} W_{ij}^{d(i,j)}$. Moreover, by definition there exists at least one path of length $d(i,j)$, which has minimal weight if all vertices on the path have the maximum degree on the network. Hence, a uniform lower bound for any pair i,j on $\mathcal{G}$ is given by $\left(\overline{\theta}/\bar{d}_n\right)^{\operatorname{diam}(\mathcal{G})}$. Hence, we have that

$$z_n < \frac{\underline{\theta}}{1-\underline{\theta}} \left(\frac{\bar{d}_n}{\overline{\theta}}\right)^{\operatorname{diam}(\mathcal{G})} = o(n)$$

by assumption. $\square$

Lemma 4. *Suppose that $\mathcal{G}$ is generated according to the stochastic block model and satisfies the assumptions of Theorem 1 and of Lemma 3 almost surely. Then Δ_{ij} converges in probability to 0 for all i,j as $n \to \infty$.*

Proof. We aim to bound Δ_{ij} using Lemma 3. Now fix some $\epsilon > 0$. Then it follows from the preceding proof that we want to show that for any i,j, there exists N such that for any $N > n$, the probability that

$$\max_{i,j} \sum_{k=1}^{\infty} \overline{\theta}^k W_{ij}^k > \frac{\theta\epsilon}{1-\underline{\theta}} \tag{A2}$$

is arbitrarily small.

Now consider the left-hand side of (A2). Note that it follows from the proof of Lemma 1 that, under the given assumptions, we need only consider the average network, since the sum of walks from i to j is arbitrarily close with probability approaching unity for sufficiently large n. Now note that the expected network, defined as before, is simply a weighted complete graph K_n. Consequently, it is straightforward to enumerate all paths from i to j. We show that for all i, j, k, $\bar{W}_{ij}^K = O(n^{-1})$.

In general, the maximal weight for each connection i, j occurs when $\ell_i = \ell_j$ and for all other h, $\ell_h \neq \ell_i$, so that $w_{ij} < \frac{p_H}{(n-1)p_L}$. Since all walks are at their greatest when all weights are maximal, a (very loose) upper bound for $\bar{W}_{ij}^K$ is the corresponding entry of the k^{th} power of $\frac{p_H}{(n-1)p_L}(\mathbf{1}_{n\times n} - \boldsymbol{I}_n)$. Now note that

$$\begin{aligned}
\left[\left(\frac{p_H}{(n-1)p_L}(\mathbf{1}_{n\times n} - \boldsymbol{I}_n)\right)^k\right]_{ij} &= \left[\frac{p_H^k}{(n-1)^k p_L^k}(\mathbf{1}_{n\times n} - \boldsymbol{I}_n)^k\right]_{ij} \\
&\leq \left[\frac{p_H^k}{(n-1)^k p_L^k}(\mathbf{1}_{n\times n})^k\right]_{ij} \\
&= \frac{n^{k-1} p_H^k}{(n-1)^k p_L^k} \\
&= O(n^{-1})
\end{aligned}$$

Hence, $p\text{-}\lim_{n\to\infty} \mathcal{I}_{ij} = 0$, and so by Lemma 3, it follows also that $p\text{-}\lim_{n\to\infty} \Delta_{ij} = 0$, completing the proof.

□