Some for the Price of One: Vote Buying on a Network

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Abstract

Social structure is an important determinant of political outcomes, yet analyses of vote buying do not account for interdependence between actors. We provide a formal model in which candidates can improve their electoral performance by providing private transfers to policy-motivated voters connected on a social network at the expense of a public good. Advances in spectral graph theory allow us to analytically derive comparative statics explicitly in terms of deep parameters governing social structure. Contrary to much existing theory, equilibrium transfers are not determined by network density, but primarily by group fractionalization and homophily, and are driven by a disproportionate targeting of minorities. In addition, we extend the model to account for heterogeneous information structure, demonstrating density still does not affect candidate strategies on average.

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1 Introduction

A central task of political science is to understand the efficacy and distribution of private political gains, potentially at the expense of public resources. While it is frequently asserted in the empirical literature that social structure plays an important role in determining the efficacy of localized targeting, the exact mechanisms through which interdependence between actors influence equilibrium outcomes have not been systematically theorized. This is largely due to an immediate problem that frequently arises in the analysis of strategic interaction on complex networks: equilibrium strategies, even when unique, are frequently defined only implicitly and grow combinatorically in complexity with the size of the network. As a consequence, it is not typically possible to unambiguously characterize how systematic structural variation in social relations influences aggregate outcomes.

A growing literature in economics and graph theory, however, provides a technical framework for handling strategic interactions on dense social networks. Simultaneously, recent work in political science has emphasized the defining role of social network structure in both the form and efficacy of clientelist electoral linkages, especially in the form of vote buying (Holland and Palmer-Rubin 2015; Cruz, Labonne, and Querubin 2017; Cruz 2019). Due to the potential for highly complex interactions on large networks, however, it is difficult to derive reliable predictions from heuristic reasoning. In particular, while a long tradition has emphasized the importance of the "density" of in-group networks (Fearon and Laitin 1996; Putnam 2000; Miguel and Gugerty 2005; Chandra 2007; Habyarimana, Humphreys, Posner, and Weinstein 2009; Gubler and Selway 2012), recent work by Larson and Lewis (2017) finds the opposite tendency in their detailed study of two Ugandan villages, highlighting the need for theory that takes network structure seriously. Similarly, bilateral alignments in international relations have been found to constitute important determinants of the receipt of targeted inducements (Vreeland and Dreher 2014), but the role of additional interdependencies *between* recipients has been largely overlooked.

This article addresses this need, studying a networked model of a large election in which candidates compete to influence policy-motivated voters with private transfers (bribes) by combining the techniques of random graph analysis with insights from a series of papers that have observed the solution to a broad class of games on networks correspond to the vector of Katz-Bonacich centralities. In doing so, we obtain sharp results characterizing the equilibrium distribution of transfers while retaining much of the rich complexity of the setting.

For our analysis, we rely on a now well-established result from Ballester, Calvó-Armengol, and Zenou (2006), who show that a broad class of games can be studied as a network game where the Nash equilibrium strategies are proportional to the Katz-Bonacich centrality of agents (Katz 1953; Bonacich 1987). This approach facilitated the analysis of settings as diverse as peer effects in education (Calvó-Armengol, Patacchini, and Zenou 2009) and co-sponsorship networks in the U.S. Congress (Battaglini and Patacchini 2018; Battaglini, Sciabolazza, and Patacchini 2020), providing a good empirical fit when compared to alternative measures of centrality from the social networks literature.

Despite the appealing parsimony of this approach, however, the sensitivity of any agent's Katz-Bonacich centrality to small changes elsewhere on the network makes analysis of the role played by the underlying structural features of the network exceedingly difficult. In order to overcome this problem, we adopt an approach that has recently gained in popularity in the literature on learning and information diffusion (Board and Meyer-ter Vehn 2021), eschewing consideration of exact graphs in favor of a random generative model that shares the same features of interest, such as network density, homophily, or fractionalization. In this regard, important recent work by Dasaratha (2020) and Mostagir and Siderius (2021) provides a framework for studying the dependence of centrality measures on social structure based on the observation that, in large networks, these measures are close to their values in an appropriately-defined "average" network with high probability.

Having derived a novel measure of centrality as the unique equilibrium of the model presented in the next section, we prove analogous results that permit us to study how the efficacy of vote buying and equilibrium transfers depend on social structure. By extending several key results in the spectral theory of random graphs (Chung and Radcliffe 2011) to the case of weighted normalized graphs, in Section 3.1 we derive mild conditions on the behavior of social ties in the expected network that ensure the exact network can be replaced with its average counterpart without loss as the number of voters grows large. By doing so, we are able to derive closed-form algebraic expressions for the equilibrium transfers to each voter, generating sharp comparative statics.

In particular, our analytical framework allows us to overcome a major limitation of many network models: the need to begin by taking a highly complex discrete graph structure as granted. By focusing instead on the underlying generative process that gives rise to observed social structure, we can derive closed-form expressions for the dependence of equilibrium strategies on deep features of society (such as homophily, density, and fractionalization) that hold with probability approaching unity in large societies.

1.1 Vote Buying and Social Influence

Vote buying, although widely considered illegitimate, is a prevalent feature of younger democracies and is associated with a variety of negative economic, social, and institutional outcomes (Keefer 2007; Hicken 2011). While the use of private inducements as an electoral strategy in the absence of strong partisan attachments may be effective, prospective vote buyers face several key challenges in implementing these strategies in large elections. The first and most extensively studied problem is that of commitment—since the exchange is illegal and ballots are typically secret, politicians cannot be certain that voters will follow through on their promise, nor can voters be sure that politicians will deliver on any promise of goods if elected (Brusco, Nazareno, and Stokes 2004; Nichter 2008; Finan and Schechter 2012; Keefer and Vlaicu 2017). This problem can be overcome in practice, principally through a combination of normative commitment to reciprocity and the integration of voters into dense social networks that facilitate information flows and two-way monitoring (Stokes 2005; Nichter 2008; Finan and Schechter 2012; Calvo and Murillo 2013; Frye, Reuter, and Szakonyi 2019). Stable connections to voters thus allow politicians to simultaneously cultivate social norms that are conducive to successful vote buying and to gather information on their actual vote choice. In addition, connections between voters may play a crucial role, as social pressure and peer-monitoring make highly-connected voters more likely to follow through on a vote buying agreement (Cruz 2019).

A second difficulty relates to the identification of voters and the effective delivery of benefits to them. In large elections, no candidate is likely to have enough information, resources, or bureaucratic capacity to reliably identify, approach, and provide private goods to swing voters in a timely and cost-efficient manner. The cultivation of partisan political networks may assist in relaying information to politicians and providing established distribution channels (Stokes 2005; Calvo and Murillo 2013). This also creates a general advantage in clientelist structures for those voters with pre-existing ties (e.g., family relationships) to candidates, as they are more likely to be targeted for high-value goods (Fafchamps and Labonne 2017; Cruz, Labonne, and Querubin 2017).

A final challenge, which increases in magnitude with the size of the election, is the imbalance between the resources available for patronage and the number of votes that needs to be won. While high-credibility goods such as public-sector jobs or political favors are most effective at solving the commitment problem (Robinson and Verdier 2013; Oliveros 2016), the supply of such goods is unlikely to be sufficient to win even local elections that are decided by voting margins of several thousand. Moreover, in many developing countries, candidates lack the financial and organizational resources necessary to establish the kind of monitoring machines highlighted above, exacerbating the problem by making monitoring essentially impossible (Kramon 2016*a*; Guardado and Wantchekon 2018). Hence, a growing literature has begun to recognize another dimension of vote buying as an electoral strategy: handouts may serve to provide voters with information about the likely behavior of the candidate once in office (Kramon 2016*a*,*b*; Auerbach and Thachil 2018; Auerbach 2019).

Thus, a consistent theme in studies of vote buying (and clientelism more generally) is that network linkages that connect elites to voters and voters to one another are crucial determinants of the efficacy of these electoral strategies. However, the specific features of social networks that are most conducive to vote buying, as well as the types of voters most likely to be targeted, remain largely unspecified. For instance, while a key conclusion of the model presented in Stokes (2005, p. 318) is that clientelist parties are "effective to the extent that they insert themselves into the social networks of constituents," the actual interaction between candidates and voters is modeled as an iterated Prisoners' Dilemma, yielding no clear predictions regarding the consequences of variation in "insertion."

Another important example of this difficulty can be found in the closely related literature on the political economy of ethnicity, where the "density" of ethnic groups is taken to be one of their defining features and acts as a key mediating variable between diversity and outcomes such as conflict, public goods provision, and the development of clientelism (Putnam 2000; Miguel and Gugerty 2005; Chandra 2007; Habyarimana et al. 2009; Gubler and Selway 2012). One of the most influential formal statements of this perspective comes from Fearon and Laitin (1996), who study a model explaining the prevalence of cooperative equilibria in diverse societies by the relatively high probability of interaction among members of the same group, rather than between groups. Although this high probability is interpreted as a consequence of the density of in-group networks, connections between individual agents are not explicitly modeled, making it difficult to derive precise predictions about how variations in social structure affect the likelihood of cooperation. Indeed, in an empirical analysis of the features of actual ethnic networks in two Ugandan villages, Larson and Lewis (2017) find the opposite: network density is positively correlated with diversity and, contrary to theoretical expectations, including conventional predictions from social network theory regarding the "strength of weak ties" (Granovetter 1973) has negative consequences for information spread. While data on politically relevant social networks at this level of detail remain sparse, this example highlights the need to engage explicitly with network structure in order to generate empirically valid comparative predictions.

Given the inherent complexity of social networks, this is a task that would seem well-suited to a formal modeling approach. Few models of vote buying, however, have considered the role of connections between players. Indeed, many of the most influential models of vote buying assume continuous distributions of voters (Groseclose and Snyder 1996; Lizzeri and Persico 2001), for which results do not necessarily generalize to finite populations (Banks 2000; Dekel, Jackson, and Wolinsky 2008). While this literature has generated important insights into the possibility of vote buying to induce inefficient supermajority coalitions (Groseclose and Snyder 1996; Banks 2000), the difficulty of overcoming private incentives by providing public goods (Lizzeri and Persico 2001), the role of varying commitment structures and institutions in mitigating the inefficiencies introduced by vote buying (Dal Bó 2007; Dekel, Jackson, and Wolinsky 2008), and the institutional factors driving the mix of strategies chosen by clientelist machines (Gans-Morse, Mazzuca, and Nichter 2014), variation is driven either by individual factors (preferences) or by macro-level variation in institutional environments. Despite the prominent role afforded to ties between actors in empirical accounts, however, this aspect of the strategic environment has gone largely unexplored.

An important exception, however, comes from Battaglini and Patacchini (2018), who study the problem of influencing members of a legislature through campaign contributions using a network model.[1] The major finding of this work is that the equilibrium transfers to voters (legislators) are proportional to their Katz-Bonacich centrality (Bonacich 1987), weighted by the equilibrium probability of pivotality. While this result is elegant, both the structure of the game and the equilibrium strategies (which require the inversion of a matrix of arbitrary size) remain highly complex objects, making it essentially impossible to derive useful comparative statics. Since, given an arbitrary graph on n vertices, there are $2^{-1}n(n-1)$ possible undirected edges, the number of possible alterations to be considered increases rapidly with the size of the voting population. When combined with the fact that the addition or removal of a single edge could dramatically alter the equilibrium to *all* agents, this

[1] See also Battaglini, Sciabolazza, and Patacchini (2020).

largely precludes any analysis of the dependence of strategies on the underlying social structure.

In order to resolve this difficulty, we draw on recent advances in the analysis of random graphs, which make it possible to draw sharp conclusions about the effects of social structure by shifting attention from realized networks to an underlying generative model.

2 Model

We begin by assuming that voters care about policy in a unidimensional space and the provision of a public good, but they can also be "bribed" with private transfers that influence their likelihood of voting for one candidate over another. Since we are primarily interested in elections where n is sufficiently large that the probability of pivotality is approximately zero, we assume expressive voting based on net preference after transfers. In order to retain our focus on the network-specific elements of the model, we additionally assume that both candidates are endowed with full commitment power, so that transfers can be treated as either up-front or as campaign promises without consequence (Dekel, Jackson, and Wolinsky 2008). These bribes, however, are not a binding contract—voters may receive transfers from both candidates and will ultimately vote in accordance with their own preferences. As such, neither candidate can ever be certain that a vote has been "bought," reflecting the commitment problem highlighted in much of the literature.

The key feature of this model is the nature of network dependence. In addition to being influenced by direct transfers and campaign promises, voters place some weight on the likelihood of their neighbors voting for a given candidate. Modeling network spillovers in this way allows us to capture the main features of the strategic situation identified in the literature while retaining a tractable model capable of generating clear predictions.

Substantively, this mechanism has two main interpretations. First, voters can be thought of as communicating with their acquaintances about their intent to vote, which provides information about the candidate's desirability (e.g., their ability to provide voters with employment opportunities). Providing a transfer to a given voter therefore also increases the likelihood of their connections supporting that candidate since they hold more favorable posteriors, consistent with empirical research highlighting the informational role of vote buying (Kramon 2016*b*). Importantly, spillovers are not limited to a voter's direct connections. Since these direct contacts will have updated their beliefs, they then communicate these to their connections, who then communicate to their connections, and so on. Thus, a bribe paid to any voter on the network will positively impact *all* voter's likelihood of voting for a candidate on a connected graph, albeit with diminishing returns in social distance.

A second interpretation of social spillovers is as a consequence of social pressure. Even if voters do not gain any relevant information from their neighbors, they may still be intrinsically motivated to take the same action as a majority of them. Common mechanisms underlying this are a desire for conformity—voters may simply derive utility from doing the same thing as those around them—and a fear of social sanctioning. This latter channel has been emphasized in the empirical literature as a solution to the commitment problem faced by prospective vote buyers, as voters police one another's decision to vote for the "right" candidate (Nichter 2008; Finan and Schechter 2012; Calvo and Murillo 2013), making it particularly pertinent.

An important caveat regarding network dependence is that we normalize the total social influence of any voter's connections to 1, so that all voters place equal weight on their neighbors' vote probabilities. Intuitively, this implies that the influence exerted on voter i by their network neighbor j is greater if j is i's only neighbor than it would have been if i had been connected to a hundred other voters. Equivalently, voters can be thought of as making their decisions based on a weighted average of their neighbors' actions, and not a sum. It is worth noting that this aspect of the model diverges somewhat from others in the literature, such as that of Battaglini and Patacchini (2018), who assume implicitly that the most connected voters are also the most easily influenced. We feel that our approach more accurately captures the strategic environment we seek to understand.

Finally, candidates care about both their electoral performance their "programmatic" commitment to provide a public good, but face a potential trade-off between these two goals. The funds for private transfers to voters, which can provide an electoral advantage, must be diverted from the provision of the public good. In order to maximize the clarity of the results, we assume that candidates attempt to maximize vote share rather than to win a majority, although simulations indicate that our main results are substantively unchanged if candidates only value winning. In addition, this assumption is empirically appropriate in many cases where vote buying is common, especially in authoritarian regimes where incumbents frequently seek to achieve overwhelming vote shares, not just a majority (Reuter and Robertson 2012).

2.1 Setup

Consider a game with n voters that need to make a choice between two candidates. All voters are located on a network $\mathcal{G}$, which is assumed to be completely connected. We use the terms network and graph interchangeably throughout the paper to refer to an undirected graph, which is an ordered pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a set of n vertices and $\mathcal{E}$ is a set of m edges such that $\mathcal{E} \subseteq \{\{x, x'\} : x, x' \in \mathcal{V} \wedge x \neq x'\}$. Each candidate $k = 1, 2$ is associated with a policy $y_k = k$, where the policy space is a subset of $\mathbb{R}$, and each voter $i \in \mathcal{V}$ is endowed with a group membership $\ell_i = 1, 2$, which corresponds to an ideal policy $x_i = \ell_i$. Substantively, groups may be interpreted as corresponding to any grouping that is both socially and politically meaningful, such as political parties and ethnic or religious groups.

In attempt to gain vote share, candidate k can extend n private bribes $b_{ik} \geq 0$. These bribes, however, come at the expense of a public good, which the candidate may also care about. A candidate k's problem is to choose $\boldsymbol{b}_k$ that solves

$$\begin{aligned} \max_{\boldsymbol{b}_k} \quad & \alpha \sum_{i \in \mathcal{V}} \phi_{ik}(\boldsymbol{b}_k, \boldsymbol{b}_{-k}) - \boldsymbol{b}_k \cdot \mathbf{1} \\ \text{subject to} \quad & b_{ik} \geq 0 \text{ for all } i \end{aligned}$$

where $\phi_{ik}(\cdot)$ is the probability voter i votes for candidate k and 1 dollar in bribes is valued at α in vote probability by the candidate. We thus normalize the value placed on a unit of public good by the candidate to one, so that α can be interpreted as the candidate's relative degree of office motivation. In particular, an α of 0 corresponds to a fully programmatic candidate, who trivially prefers to offer no bribes and promise the full amount of the public good.

Voters support the candidate that offers them a total higher payoff.[2] All voters care about policy according to a standard quadratic loss function and have $\gamma \geq 0$ value for a unit of public good, so that they incur a loss of γ for every dollar offered by a candidate to any voter. Additionally, voters have private information unknown to the candidates and other voters in the form of a private valence shock for each candidate, $\varepsilon_{ik} \in \mathbb{R}$. Without loss of generality, we can normalize $\varepsilon_{i2} = 0$ and define $\varepsilon_i := \varepsilon_{i1}$, which we assume is an independent, uniformly distributed random variable with mean zero and density $\theta > 0$. We interpret θ as the candidates' information, with smaller θ indicating less informed candidates. θ can also be taken as reflecting the intensity of the commitment problem facing candidates, as candidates with higher values can be more certain that a transfer to a voter will actually secure their vote.

Social structure also matters. In particular, voters prefer to vote for the same candidate as their neighbors as defined by the network. Denote by $\phi_{ik}(\cdot)$ the probability voter i votes for candidate k given all bribes, but before the realization of the valence shock ε_i. Then, each voter i places weight $w_{ij} \geq 0$ on voter j's probability of voting for candidate k if i and j are connected, and 0 otherwise. In the realized graph $\mathcal{G}$, the set of a voter i's social ties is denoted by $\mathcal{T}_i(\mathcal{G}) \subseteq \mathcal{V}$. The total social influence on each voter is normalized to 1, so that the actual influence of each neighbor j on i's utility is equal to $\left(\sum_{h \in \mathcal{T}_i(\mathcal{G})} w_{ih}\right)^{-1} w_{ij}$, implying that more highly connected voters are less influenced by each individual neighbor.

The expected payoff voter i receives from candidate k can thus be expressed as

$$U_i(k) = -(x_i - y_k)^2 + u(b_{ik}) + \Big(\sum_{h \in \mathcal{T}_i(\mathcal{G})} w_{ih}\Big)^{-1} \sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij}\phi_{jk}(\boldsymbol{b}) - \gamma \sum_{m \in \mathcal{V}} b_{mk} + \varepsilon_{ik} \quad (1)$$

[2] There is no obligation to vote for a candidate who offered them a bribe.

where $u(\cdot)$ is voter utility over bribes, which we assume is strictly increasing with diminishing marginal returns and that the rate of diminution is decreasing. That is, we require that the utility over bribes satisfy $u'(\cdot) > 0$, $u''(\cdot) < 0$, $u'''(\cdot) \geq 0$, $\lim_{b\to 0} u'(b) = \infty$, and $\lim_{b\to\infty} u'(b) = 0$. These assumptions include a wide range of plausible utilities, particularly logarithmic utility. Together, they ensure that all solution objects are well-defined and rule out the combinatoric problem of corner solutions.

Timing

The timing of the game is as follows.

1. Nature randomly chooses a private valence shock for each voter, $\varepsilon_i \sim \mathcal{U}\left[\frac{-1}{2\theta}, \frac{1}{2\theta}\right]$
2. For all voters $i \in \mathcal{V}$, each candidate $k = 0, 1$ offers a bribe $b_{ik} \geq 0$, which determines the residual public good offered
3. Each voter $i \in \mathcal{V}$ votes for candidate 1, $v_i = 1$, or candidate 2, $v_i = 2$

2.2 Equilibrium

A voter will cast a ballot for candidate 1 if and only if $U_i(1) \geq U_i(2)$. Here, candidates will not be able to perfectly anticipate voting behavior due to their imperfect information over voter preferences. Using equation (1), we can rewrite this as a condition on the size of the valence shock,

$$\varepsilon_i \leq (-1)^{x_i - 1} + u(b_{i1}) - u(b_{i2}) + \frac{\sum_{j\in\mathcal{T}_i(\mathcal{G})} w_{ij}(2\phi_j - 1)}{\sum_{j\in\mathcal{T}_i(\mathcal{G})} w_{ij}} + \gamma \sum_{m\in\mathcal{V}} (b_{m2} - b_{m1}),$$

where we have denoted $\phi_i := \phi_{i1}(\boldsymbol{b}) = 1 - \phi_{i2}(\boldsymbol{b})$ the probability a voter i votes for candidate 1. Noting that $\varepsilon_i \sim \mathcal{U}\left[\frac{-1}{2\theta}, \frac{1}{2\theta}\right]$ implies $\Pr(\varepsilon_i \leq \varepsilon) = \frac{1}{2} + \theta\varepsilon$, we can correspondingly write each voter's probability for voting for candidate 1 as

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \theta\left((-1)^{x_1-1} + u(b_{11}) - u(b_{12}) + \frac{\sum_{j\in\mathcal{T}_1(\mathcal{G})} w_{1j}(2\phi_j-1)}{\sum_{j\in\mathcal{T}_1(\mathcal{G})} w_{1j}} + \gamma\sum_{m\in\mathcal{V}}(b_{m2}-b_{m1})\right) \\ \vdots \\ \frac{1}{2} + \theta\left((-1)^{x_n-1} + u(b_{n1}) - u(b_{n2}) + \frac{\sum_{j\in\mathcal{T}_n(\mathcal{G})} w_{nj}(2\phi_j-1)}{\sum_{j\in\mathcal{T}_n(\mathcal{G})} w_{nj}} + \gamma\sum_{m\in\mathcal{V}}(b_{m2}-b_{m1})\right) \end{pmatrix}.$$

Here, $\boldsymbol{\phi}$ gives the unique vector of equilibrium vote probabilities. While each voter's utility is subject only to their neighbor's vote probabilities, this system of equations necessarily implies that a single voter's probability of supporting candidate 1 is a function of all other voter's probability of supporting 1. This occurs because, for example, a voter i's probability ϕ_i is affected by i's neighbor j's probability ϕ_j, which in turn is affected by j's neighbor m's probability ϕ_m. Since we rule out disconnected components, ϕ_i will both affect and be affected by all other voting probabilities throughout the entire network.

In equilibrium, each candidate chooses a vector of bribes that maximizes their utility, taking the other candidate's bribes as given. This gives rise to the n first-order conditions,

$$\sum_{j=1}^{n} \frac{\partial \phi_j}{\partial b_{ik}} = \frac{1-\lambda_k}{\alpha}$$

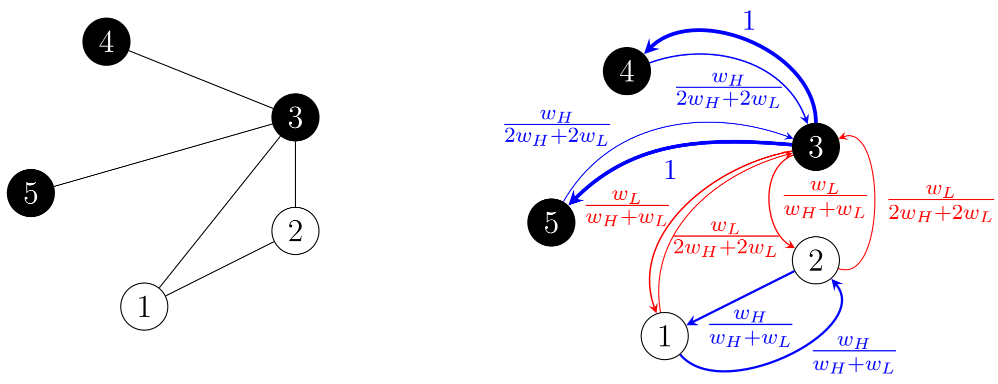
where λ_k is the Lagrangian multiplier associated with k's non-negativity constraint. Differentiating ϕ_i with respect to a bribe from candidate 1 to another voter h, we have

$$\frac{\partial \phi_i}{\partial b_{h1}} = \theta\left(u'(b_{h1})\mathbb{1}(i=h) + \frac{2\sum_{j\in\mathcal{T}_i(\mathcal{G})} w_{ij}\phi_j'}{\sum_{j\in\mathcal{T}_i(\mathcal{G})} w_{ij}} - \gamma\right).$$

Using this and the first-order conditions, we can rewrite the candidates' problem as

$$\left(\boldsymbol{J}\left[\boldsymbol{u}\right] - \boldsymbol{\Gamma}\right)^{\top} \cdot \left(\boldsymbol{I} - 2\theta\tilde{\boldsymbol{A}}\right)^{-1} \cdot \mathbf{1} = \frac{(1-\boldsymbol{\lambda})}{\alpha\theta} \tag{2}$$

where $\boldsymbol{J}[\cdot]$ is a diagonal matrix with $u'(b_i)$ as the non-zero entries, $\boldsymbol{\Gamma}$ is an $n \times n$ square



(a) Example of a realized graph (b) Induced weighted directed graph

Figure 1: An example of a realized (undirected) network with $n = 5$ and $\ell_1 = \ell_2 = 1$ and $\ell_3 = \ell_4 = \ell_5 = 2$ and the corresponding induced weighted directed network. In the weighted directed graph, thicker arrows indicate stronger influence, with $w_{ij} = w_H$ if $\ell_i = \ell_j$, $w_{ij} = w_L$ otherwise, and naturally $w_H > w_L$.

matrix such that every element of $\mathbf{\Gamma}$ is γ, $\boldsymbol{I}$ denotes the identity matrix, $\mathbf{1}$ denotes an n-vector of 1s, $\boldsymbol{\lambda}$ is an n-vector of Lagrange multipliers, and $\tilde{\boldsymbol{A}}$ is a normalized weighted adjacency matrix.

Definition 1. *Consider a realized graph $\mathcal{G}$ and a corresponding adjacency matrix $\boldsymbol{A}$ such that for all $i, j \in \mathcal{V}$, $A_{ij} = 1$ if $j \in \mathcal{T}_i(\mathcal{G})$ and $A_{ij} = 0$ otherwise. Then, the* ***normalized weighted adjacency matrix*** $\tilde{\boldsymbol{A}}$ *is given by, for all $i, j \in \mathcal{V}$,*

$$\tilde{A}_{ij} = \frac{w_{ij} A_{ij}}{\sum_{m \in \mathcal{V}} w_{im} A_{im}}.$$

By employing the normalized weighted adjacency matrix, we can account for several important features of social interaction. First, a voter i may be more influenced by one social tie than another. Second, it will be more difficult to influence a highly connected voter than a relatively disconnected one, e.g. an incremental change in the probability that i's neighbor j votes for candidate 1 will have less of an effect on i's vote probability if i has hundreds of neighbors than if j is i's only neighbor.

From the candidates' problem in equation (2), we can recover the equilibrium bribe that voter i receives from candidate k,

$$b_{ik} = [u']^{-1}\left(\gamma n + \frac{1}{\alpha\theta c_i}\right), \tag{3}$$

where c_i is the ith element of $\boldsymbol{c}$ and $\boldsymbol{c} = (\boldsymbol{I} - \theta\tilde{\boldsymbol{A}})^{-1}\mathbf{1}$ is our measure of centrality. This measure is analogous to each agent's Katz-Bonacich centrality on the weighted directed network corresponding to $\tilde{\boldsymbol{A}}$ with attenuation parameter θ. The nature of the strategic environment—specifically, the structure of social influence—can therefore be thought of as inducing a latent directed network with connections corresponding to the influence of i on j, which is decreasing in j's weighted degree and greatest when $\ell_i = \ell_j$. The value of a voter to a candidate is thus proportional to their centrality on this latent network, which captures the weighted sum of directed walks of any length that include that voter.

As the characterization makes clear, b_{ik} does not rely on policy considerations and hence, in equilibrium, both candidates bribe a given voter the same amount according to their centrality on the network.

Proposition 1 (Equality of bribes)**.** *In any equilibrium, $b_{i1} = b_{i2}$ for all $i \in \mathcal{V}$.*

To understand the intuition behind this result, recall that a voter's direct utility in bribes is increasing with diminishing marginal returns. Then, a candidate k will want to continue extending bribes to a voter i until the marginal gain in vote probability is equal to the marginal loss in public good. Since candidates want to maximize their total expected vote share, the point at which this occurs for voter i is the same for candidate 1 as it is for candidate 2. Voters will not receive more private inducements from candidates they are politically aligned with than from those they are not. Proposition 1 implies that the net direct gain from bribes is always zero, i.e. $u(b_{i1}) - u(b_{i2}) = 0$. Note also that $\gamma \sum_{i\in\mathcal{V}}(b_{i2} - b_{i1}) = 0$. Together, these facts lead to the next result.

Proposition 2 (Electoral outcomes)**.** *Vote buying does not affect electoral outcomes.*

In particular, the probability a voter i votes for candidate 1 *is*

$$\phi_i = \frac{1}{2} + \theta \left((-1)^{x_i - 1} + \sum_{j \in \mathcal{T}_i(\mathcal{G})} 2w_{ij}(\phi_j - 1) \right)$$

for all $i \in \mathcal{V}$, which is independent of candidate strategies.

Since an individual voter receives the same bribe from each candidate, neither candidate is successful in affecting change in the voter's probability of supporting them in the election. As no voter is influenced by the bribes (or, arguably, as all voters are influenced by the bribes equally in both directions), the equilibrium expected vote share is not altered by the candidates' bribes.[3]

Finally, it is straightforward from the assumptions on the derivatives of $u(\cdot)$ stated in the previous section to derive the following comparative statics by taking derivatives of the equilibrium bribes defined by equation (3).

Proposition 3 (Comparative Statics)**.** *For any voter i, the equilibrium transfers offered by both candidates are*

1. *Weakly decreasing in γ*
2. *Weakly decreasing in n*
3. *Weakly increasing in α*
4. *Weakly increasing in θ*

These results are consistent with the intuition of the basic strategic environment: the socially optimal bribes to voters would correspond to a transfer scheme such that the marginal value is equated with γ, but candidates provide additional transfers as a result of the network spillovers. The value of these spillovers is moderated by α—that

[3] Importantly, however, this is a feature of the two-candidate competitive environment. If, for instance, candidate 2 lacks the resources to distribute bribes or is ideologically committed to programmatic politics, then candidate 1 could achieve a large expected vote share, even if a majority of voters prefer candidate 2 on ideological grounds.

is, more office-motivated candidates place higher value on the additional increase to expected vote share—and by θ, which determines the likelihood of their realization.

Thus far, we have taken the exact realized network as given. In order to study the dependence of equilibrium strategy on social structure, we must transition to considering the underlying generating model that gave rise to the observed network. We now take up this problem in the next section.

3 Analysis

This section proceeds as follows. First, we present results related to graph theory that justify analysis of an average graph. This is needed for two reasons: since centrality is a highly nonlinear function of the realized network, it is convenient to replace the expected centrality of each individual voter—which is defined only implicitly—with that voter's centrality *on the expected network*, which is far more tractable. In addition, even simple generative models may result in realized networks with arbitrarily high variance in the emergent characteristics of individual vertices, such as centrality. We therefore require bounds on the spectrum of the induced normalized weighted network in order facilitate making claims that can be made to hold with arbitrarily high probability. These results are technical in nature and therefore detailed proofs are relegated to the Appendix.

Second, we employ these tools to study the implications of social structure on vote buying. In particular, we are able to derive closed-form expressions for the centrality of voters in each party, yielding sharp comparative statics in terms of the main features of social structure—namely, group fractionalization, density, and homophily.

3.1 Graph Theory Results

The main result—an analogue of the spectral theorems of Chung and Radcliffe (2011) applied to the case of weighted, directed networks—allows us to place tight bounds on the deviation of the realized normalized weighted adjacency matrix from its expected counterpart.

Theorem 1. *Let $\boldsymbol{L}_W$ denote the normalized weighted Laplacian of $\mathcal{G}$, ω the largest total weight, and $\underline{d}$ the smallest expected degree. For any $\epsilon > 0$, there exists a $k > 0$ such that, for all i,*

$$\Pr\left(||\boldsymbol{L}_W - \bar{\boldsymbol{L}}_W|| \leq 4\sqrt{\frac{3\omega \ln(4n/\epsilon)}{\underline{d}}}\right) \geq 1 - \epsilon$$

if $\underline{d} > k\ln(n)$ and $\alpha\omega \leq \sqrt{\frac{\underline{d}}{3\ln(4n/\epsilon)}}$, where α is the smallest total weight.[4]

A major limitation of this result is that the bound depends on n, and can thus be arbitrarily large in large societies for a fixed ϵ. Under the assumption that the minimum degree grows at a rate greater than $\ln(n)$, it is straightforward to show the following result, which allows us to make asymptotic statements that will hold with high probability.[5]

Lemma 1. *For any $\epsilon > 0$, $\lim_{n\to\infty} \Pr(\|\boldsymbol{c}^{(n)}(\tilde{\boldsymbol{A}}) - \boldsymbol{c}^{(n)}(\bar{\tilde{\boldsymbol{A}}})\| > \epsilon) = 0$.*

This allows us to consider centrality on the average graph only, permitting analysis of comparative statics in terms of social structure rather than of a single realized graph.

3.2 Social Structure

Definition 2. *The* ***average normalized weighted adjacency matrix*** $\bar{\tilde{\boldsymbol{A}}}$ *is given by, for all $i, j \in \mathcal{V}$,*

$$\bar{\tilde{A}}_{ij} = \frac{w_{ij}p_{ij}}{\sum_{m\in\mathcal{V}} w_{im}p_{im}}.$$

For the following results, we assume that the graph is drawn according to a two-group stochastic block model with share $s \geq \frac{1}{2}$ of group 1, a probability p_H of intra-group connection, and a probability p_L of inter-group connection.[6] Further, assume

[4] Not to be confused with α in candidate utility.

[5] In fact, empirical work suggests that it grows at approximately rate $\ln(n)$ in most societies. However, simulation studies indicate that our results still hold with high probability if this is the case, and it is likely that a tighter bound is achievable.

[6] Notably, the parameter s captures all of the information provided by the Herfindahl-Hirschman index, which is a widely used empirical measure of concentration.

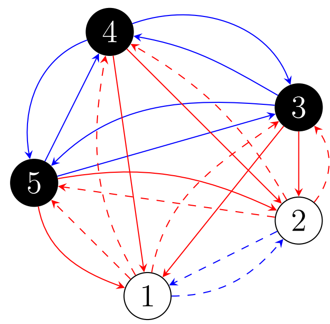


Figure 2: An example of an expected graph corresponding to the average normalized weighted adjacency matrix, from which the earlier example would have been generated. Blue lines indicate in-group influence, red lines indicate out-group influence, dashed lines indicate influence from a member of group 1, and solid lines indicate influence from a member of group 2.

for simplicity that $w_{ij} = w_H$ for in-group voters and $w_{ij} = w_L$ for out-group voters, with the natural assumption that $w_H \geq w_L$. Finally, we denote by $\delta \in (0,1)$ the ratio $\frac{w_L p_L}{w_H p_H}$, which thus captures the degree of homophily on the network (lower δ corresponds to more homophily).

The main result, which draws on the asymptotic bounds on the average adjacency matrix derived in the previous section, allows us to obtain closed-form expressions for each voter's centrality that hold with high probability given large n.

Proposition 4 (Expected Centrality)**.** *The centrality of a voter in party* 1 *and* 2 *is asymptotically equivalent to*

$$c_1 = \frac{-\delta + (\delta-1)s^2(\delta\theta+\delta+\theta-1) - (\delta-1)s(\delta\theta+\delta+\theta-1)}{sn((\delta-1)s-\delta)(-\theta+s(\delta+\theta-1)+1)}$$

and

$$c_2 = \frac{\delta - ((\delta-1)s^2(\delta\theta+\delta+\theta-1)) + (\delta-1)s(\delta\theta+\delta+\theta-1)}{n(s-1)((\delta-1)s+1)(s(\delta+\theta-1)-\delta)},$$

respectively, with probability approaching 1 *as* $n \to \infty$*.*

The proof of this result is left to the Appendix, but the key observation is that, unlike realized networks, which may be arbitrarily complex, the expected network is necessarily complete, since all voters have positive probability of being connected to all others. In particular, the adjacency matrix, while arbitrarily large, contains only four unique values, corresponding to directed connections within and between each group. With some algebra, it is therefore possible to derive an explicit formula for the inverse of this matrix, which in turn determines the value of each voter's centrality.

An immediate conclusion that follows from Proposition 4 is that, while centrality is a function of group sizes, information, and homophily, it is entirely *independent* of the actual density of the network.[7] Put differently, a uniform increase in connection probabilities between all voters would not influence equilibrium bribes in any way. Despite the prominence of density in many informal accounts of network effects, then, our model suggests this need not be a major determinant of the efficacy of vote buying, further highlighting the need for precisely formulated theory.

Also of note is that, under the assumption that $s \geq \frac{1}{2}$, it can easily be verified that $c_1 \leq c_2$, with equality only if the groups are of exactly equal size. In other words, each individual member of the minority group will always receive a higher equilibrium transfer than a member of the majority. Intuitively, this is driven by the fact that, assuming any positive degree of homophily, the influence of each member of the minority group increases as the group becomes smaller, making them more valuable to target.

Since these centralities are smoothly differentiable functions of all parameters, moreover, it is then straightforward to examine how the total spending of candidates, as well as the level of between-group inequality, depends on these parameters. In particular, let B denote the sum of bribes across all voters and Q denote the level of inequality, specifically $Q := (1 - b_1/b_2)^2$. Since $u(\cdot)$ is assumed to be concave with convex first derivative, it is sufficient to consider the total centrality rather than examining bribes directly. Then we have the following result.

Proposition 5 (Total bribes)**.** *For all $\theta \leq 2 - \sqrt{2}$, $\frac{\partial B}{\partial \delta} > 0$, $\frac{\partial B}{\partial s} > 0$, and $\frac{\partial B}{\partial \theta} > 0$.*

Since θ needs to be small for the solution to be well-defined, the condition for these results will always hold for large n. The effect of θ here is intuitive: better

[7] Note that while the centrality of each individual voter is decreasing in n, the total centrality is constant.

information increases the expected return on bribes. The other two results are non-obvious, however. In particular, the effect of δ is counterintuitive: an increase in δ (i.e., a decrease in homophily) increases total spending. While the mechanism for this is straightforward—higher δ corresponds to stronger "weak ties," raising the value of transfers to all voters—it is again at odds with many informal descriptions of dense networks that are argued to be particularly amenable to vote buying due to their high degree of homophily.

It is similarly straightforward to study the effect of network parameters on inequality by taking derivatives of the negative of the ratio of the two centralities:

Proposition 6 (Group size and inequality)**.** *Let Q denote the total inequality, or $Q = (1 - b_1/b_2)^2$. Then $\frac{\partial Q}{\partial \theta} \geq 0$, with equality if $s = \frac{1}{2}$ or $s = 1$, and $\frac{\partial Q}{\partial s} > 0$ for all parameter values. Moreover, $\frac{\partial Q}{\partial \delta} < 0$ only if $\delta \geq \sqrt{1-\theta}$, and is positive otherwise.*

Once again, the effects of information and group size are intuitive, suggesting that better informed candidates in more demographically uneven societies will concentrate their resources more intensely in the groups that provide the highest return. As with total spending, however, the effects of homophily are surprising. While higher homophily (lower δ) can increase inequality, this only holds for networks that exhibit extremely low degrees of homophily. In contrast, at all empirically plausible levels of homophily, increasing the relative influence of voters on members of their own group actually decreases the overall inequality of transfers from candidates. This result is especially surprising given that Dasaratha (2020) arrives at the opposite conclusion regarding Katz centrality on an undirected and unweighted network.

In fact, the key to understanding this result is the tradeoff faced by candidates between targeting highly-connected voters, who influence many others, and voters whose neighbors are *not* highly connected, since they are more easily influenced. In the extreme, as δ approaches 0, the greater value of transfers to members of the minority is completely offset by their disconnectedness from the majority, such that the equilibrium bribes approach equality.

4 Heterogeneous Information

A key feature of the baseline model is that both candidates have identical and completely homogeneous information about the preferences of all voters, modeled as a single commonly known value of θ. Among the competing candidates, homogeneous information results in homogeneous behavior. In practice, however, this is unlikely to hold true. In fact, a large body of research has emphasized the crucial intermediary role of brokers who possess superior knowledge about particular groups of voters, and in competitive settings informational asymmetries across candidates may account for divergent strategies (Stokes, Dunning, Nazareno, and Brusco 2013).

In this section, we study the consequences of relaxing this assumption, allowing the precision of candidates' information to vary arbitrarily across candidates and voters.

4.1 Equilibrium

We pick up from the setup of the baseline model, but now the information held by candidate k about voter i's preferences is allowed to vary. In particular, voter i's net preference for candidate 1, ε_i, is now drawn from one of two uniform distributions with density $\theta_i \in \{\underline{\theta}, \overline{\theta}\}$ with $\overline{\theta} > \underline{\theta}$. We can think about θ_i as voter i's private type, which is unknown to candidates.

While the candidates do not know which distribution voter i's net preference was drawn from, they have common priors and receive signals about each voter's type $m_i \in \{\underline{\theta}, \overline{\theta}\}$ such that $m_i = \theta_i$ with a probability (assumed greater than half) that relies on the voter-candidate pair. In other words, candidates receive informative signals about the preferences of voters and those signals may be more precise for some voters than for others. After receiving $\boldsymbol{m} = (m_1, \ldots, m_n)$, candidates form posterior beliefs $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ where $\mu_i = \Pr(\theta_i = \underline{\theta} | m_i)$ and distribute bribes accordingly.

Incorporating uncertainty over voter types, we can rewrite the new candidates' problem as

$$\max_{\boldsymbol{b}_k} \;\; \alpha \sum_{i\in\mathcal{V}} \mathbb{E}_\mu[\phi_{ik}(\boldsymbol{b}_k, \boldsymbol{b}_{-k})] - \boldsymbol{b}_k \cdot \mathbf{1} \tag{4}$$

$$\text{subject to} \;\; b_{ik} \geq 0 \text{ for all } i. \tag{5}$$

Then, exactly as before, we can write each voter's probability for voting for candidate 1 as a function of all other voter's probability for voting for candidate 1,

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \theta_1 \left(V_1(\boldsymbol{b}) + \frac{\sum_{j\in\mathcal{T}_1(\mathcal{G})} w_{1j}(2\phi_j - 1)}{\sum_{j\in\mathcal{T}_1(\mathcal{G})} w_{1j}} \right) \\ \vdots \\ \frac{1}{2} + \theta_n \left(V_n(\boldsymbol{b}) + \frac{\sum_{j\in\mathcal{T}_n(\mathcal{G})} w_{nj}(2\phi_j - 1)}{\sum_{j\in\mathcal{T}_n(\mathcal{G})} w_{nj}} \right), \end{pmatrix}$$

where we denote i's net preference for candidate 1 short of network effects by $V_i(\boldsymbol{b})$ for notational convenience so that

$$V_i(\boldsymbol{b}) := (-1)^{x_i - 1} + u(b_{i1}) - u(b_{i2}) + \gamma \sum_{m\in\mathcal{V}} (b_{m2} - b_{m1}).$$

Unlike the baseline case, however, candidates maximize an expected utility that now relies on their posterior beliefs of voter types. In particular, we need to characterize the candidates' expected vote share conditional on their signals. Using the above equation, we know this can be expressed

$$\begin{aligned} \mathbb{E}_\mu[\phi_i] &= \frac{1}{2} + \mathbb{E}_\mu\left[\theta_i \left(V_i(\boldsymbol{b}) + \frac{\sum_{j\in\mathcal{T}_i(\mathcal{G})} w_{ij}(2\phi_j - 1)}{\sum_{j\in\mathcal{T}_i(\mathcal{G})} w_{ij}} \right)\right] \\ &= \frac{1}{2} + \mathbb{E}_\mu[\theta_i] V_i(\boldsymbol{b}) + \frac{\sum_{j\in\mathcal{T}_i(\mathcal{G})} w_{ij}(2\mathbb{E}_\mu[\theta_i\phi_j] - \mathbb{E}_\mu[\theta_i])}{\sum_{j\in\mathcal{T}_i(\mathcal{G})} w_{ij}}. \end{aligned}$$

For the remainder of this section, we restrict all edge weights to $w_{ij} = 1$ for all i, j for simplicity. This assumption is without loss of generality as Proposition 4 establishes that, on average for large n, it is only the ratio $\frac{w_L p_L}{w_H p_H}$ that determines outcomes. Hence, as long as tie formation probabilities can vary freely, edge weights have no independent effect in the expected network, which is again our focus in this section.

First, note that $\mathbb{E}_\mu[\theta_i] = \overline{\theta} - \mu_i(\overline{\theta} - \underline{\theta})$. This can be thought of as the candidate's net "information" about voter i, taking into account both first-order uncertainty about i's vote choice and second-order uncertainty over her type. Second, by Lemmas 3 and 4 in the Appendix, we know that unilateral changes in a particular voter's type has a negligible influence on changes in other voter's vote probabilities as the network grows sufficiently large, i.e. $\phi_j(\cdot|\boldsymbol{\theta} : \theta_i = \underline{\theta}) \approx \phi_j(\cdot|\boldsymbol{\theta} : \theta_i = \overline{\theta})$ as $n \to \infty$. Hence we can conclude that, asymptotically, $\mathbb{E}_\mu[\theta_i \phi_j] = \left(\overline{\theta} - \mu_i(\overline{\theta} - \underline{\theta})\right) \mathbb{E}_\mu[\phi_j]$. This allows us to recover n first-order conditions,

$$\frac{\partial \mathbb{E}_\mu[\phi_i]}{\partial b_{h1}} = \left(\overline{\theta} - \mu_i(\overline{\theta} - \underline{\theta})\right) \left(u'(b_{h1}) \mathbb{1}(i = h) + \frac{2 \sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij} [\mathbb{E}[\phi_j]]'}{\sum_{j \in \mathcal{T}_i(\mathcal{G})} w_{ij}} - \gamma \right).$$

As before, the above equations allow us to rewrite the candidates' problem as

$$(\boldsymbol{J}[\boldsymbol{u}] - \boldsymbol{\Gamma})^\top \cdot \left(\boldsymbol{I} - 2\Theta \tilde{\boldsymbol{A}}\right)^{-1} \cdot \boldsymbol{1} = \Theta^{-1} \cdot \frac{(1 - \boldsymbol{\lambda})}{\alpha}, \tag{6}$$

where $\Theta := \overline{\theta} \boldsymbol{I} - (\overline{\theta} - \underline{\theta}) \boldsymbol{M}$ and $\boldsymbol{M}$ is an $n \times n$ diagonal matrix with posteriors μ_i as non-zero elements, with the rest defined as in the baseline model. Equilibrium bribes can therefore be explicitly expressed,

$$b_{ik} = [u']^{-1} \left(\gamma n + \frac{1}{\alpha \hat{\theta}_{ik} c_i} \right),$$

where the only difference from the previous section is that equilibrium bribes to a voter i from a candidate k rely not only on their centrality measure c_i, now the ith element of $\boldsymbol{c} = (\boldsymbol{I} - \Theta \tilde{\boldsymbol{A}})^{-1} \boldsymbol{1}$, but also on candidate k's belief about voter i's type, $\hat{\theta}_{ik} := \mathbb{E}_\mu[\theta_i]$. By the law of iterated expectations, since each candidate's posteriors are equal to their priors in expectation, they will act according to their common priors on average. Therefore, it is necessarily true that $\mathbb{E}[\hat{\theta}_{i1}] = \mathbb{E}[\hat{\theta}_{i2}] = \hat{\theta}_i$ for each voter i.

Now, because one candidate may have more informative signals than the other, equality of bribes as in Proposition 1 may no longer holds in this setting for a particular realization; however, equality of bribes will continue to hold in expectation. This remains true even when there is diverging quality of information among the candidates. For a given network, the candidate with more accurate signals will be more responsive to the realized voter types, which means they will have an advantage over their opponent in the sense that they can better anticipate whether they should spend more or less on specific voters. However, as long as each candidate has well-specified prior beliefs about voter types, then both candidates will spend the same amount on each voter on average.

Further, the baseline model's result on electoral outcomes (Proposition 2) continues to hold for any realization if and only if candidates have the same quality of information—by introducing an informational advantage to one candidate, the better-informed candidate will be able to improve their electoral performance in expectation by more precisely allocating bribes to the voters with the greatest marginal return. Nonetheless, gains in electoral performance via vote buying remain orthogonal to group membership to the extent the informational structure is orthogonal to group membership.

4.2 Comparative Statics

A natural question that arises from this extension is how the information available to candidates about voter preferences relates to the structure of society. There are two main sources of variation in information we analyse in this section: cross-group differences in the prior distribution of types and network-dependent variations in posterior information. In this section, we study the first type of variation analytically, while the following section considers the impact of network dependencies with the aid of simulations.

Differences in information may arise due to systematic differences between the two groups, and thus affect both candidates symmetrically. For instance, if partisanship is stronger in one party than another, then candidates may view those associated with the "weaker" party as more likely to be swing voters. Intuitively, the first-order effect of this variation is to reduce the value of transfers to members of the less predictable group, as they are associated with a lower marginal value in expectation. It is unclear *a priori*, however, how this affects the comparative statics derived in the homogeneous case, as the reduced value of members of this groups also reduces the significance of all flow-on effects in the network in a complex fashion.

We therefore begin by studying the case where posteriors (in expectation) differ symmetrically for both candidates, assuming without loss of generality that $\hat{\theta}_1 < \hat{\theta}_2$. Note that this may be interpreted as reflecting a difference in prior distributions across groups, affecting both candidates symmetrically. Applying the same approach as for the baseline model,[8] we find the following

Proposition 7. *for n sufficiently large, with probability approaching one the centrality of a voter in group* 1 *is asymptotically equivalent to*

$$c_1 = \frac{\delta\left(-\hat{\theta}_1 + \hat{\theta}_2 - 1\right) + (\delta-1)s^2\left(\delta\hat{\theta}_1 + \delta + \hat{\theta}_2 - 1\right) - s\left(\delta^2(\hat{\theta}_1+1) - 2\delta(\hat{\theta}_1 - \hat{\theta}_2 + 1) - \hat{\theta}_2 + 1\right)}{ns((\delta-1)s-\delta)(-\hat{\theta}_2 + s(\delta + \hat{\theta}_2 - 1) + 1)}$$

and for a voter in group 2*,*

$$c_2 = \frac{\delta - \left((\delta-1)s^2(\delta\hat{\theta}_2 + \delta + \hat{\theta}_1 - 1)\right) + s\left(\delta^2(\hat{\theta}_2+1) - 2\delta - \hat{\theta}_1 + 1\right)}{n(s-1)((\delta-1)s+1)(s(\delta+\hat{\theta}_1 - 1) - \delta)}.$$

Due to the greater complexity of these expressions compared to the baseline model, it is no longer feasible to provide explicit characterizations for the main comparative statics—in most cases, the sign of the relevant derivatives depends on the four parameters s, δ, $\hat{\theta}_1$, and $\hat{\theta}_2$ in a highly non-linear fashion. In this section, we therefore adopt the approach of evaluating each derivative at a fine grid of points in $(s, \delta, \hat{\theta}_1)$ space, holding $\hat{\theta}_2$ constant at a range of values, assuming without loss of generality that $\hat{\theta}_1 < \hat{\theta}_2$. In Figures 3, 4, and 5, we show the regions over which the main quantities of interest take positive and negative values assuming a moderate value $\hat{\theta}_2 = \frac{3}{4}$, while corresponding figures for other values of $\hat{\theta}_2$ can be found in Appendix A.

Unlike in the constant-information case, changes in parameters no longer have uniform effects. In particular, it is now possible for members of majority groups to have greater centrality on average, as shown in Figure 3. This kind of "flipping" only occurs when the groups are of approximately equal size—even with values of

[8] See Supplemental Materials for details and code.

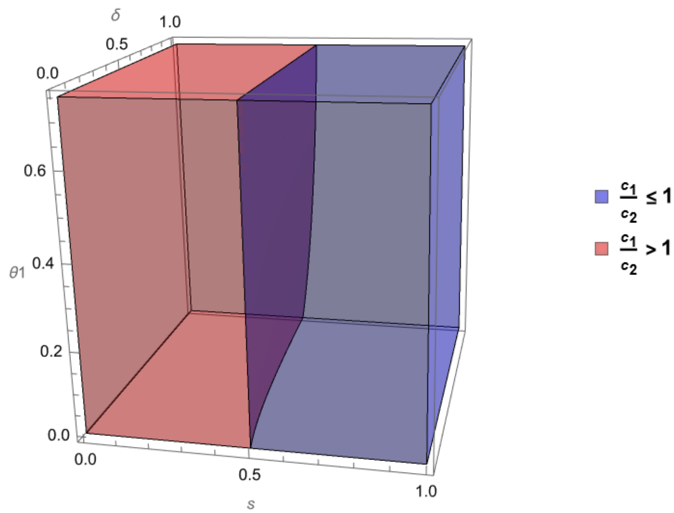
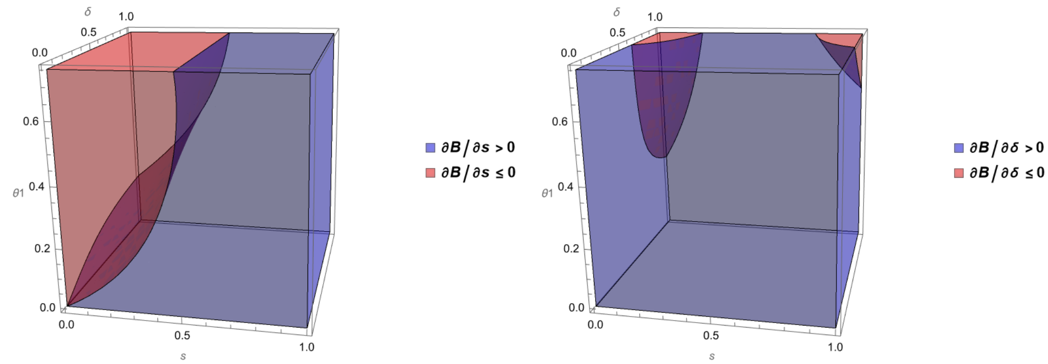


Figure 3: Relative size of expected centralities c_1 and c_2 as a function of s, δ, and $\hat{\theta}_1$ with $\hat{\theta}_2 = 0.75$. Note that the boundary at $s \approx 0.5$ is not a flat plane: for s sufficiently close to 0.5, there exist parameter values such that $c_1 \lesseqgtr c_2$.

$\hat{\theta}_1$ and $\hat{\theta}_2$ arbitrarily close to 0 and 1, respectively, members of the minority group always receive more whenever $|s - 0.5| > 0.04$. However, it does serve to highlight the key difference from the baseline model: the greater marginal value of transfers to one group compared to the other, driven by prior type distribution, creates a countervailing influence whereby connections to the more predictable group become more valuable, regardless of group size. This effect is exactly reflected in Figure 5a, as changes in group size towards (away from) equality may now increase (decrease) inequality when the level of fractionalization is sufficiently close to 0.

Another clear illustration of this change from the baseline result can be seen in Figure 4a, which shows the effect of an increase in the size of group 1 on total expenditure. Whereas under homogeneous information an increase in the size of the majority group (greater fractionalization) always increases expenditures (see Proposition 5), this need not necessarily be the case when information varies by group. In particular, when the less predictable (lower $\hat{\theta}_\ell$) group is in the minority, an increase in its size (reduction in fractionalization) can now lead to an increase in total expenditure, provided that its average posterior is sufficiently low. This is driven by a



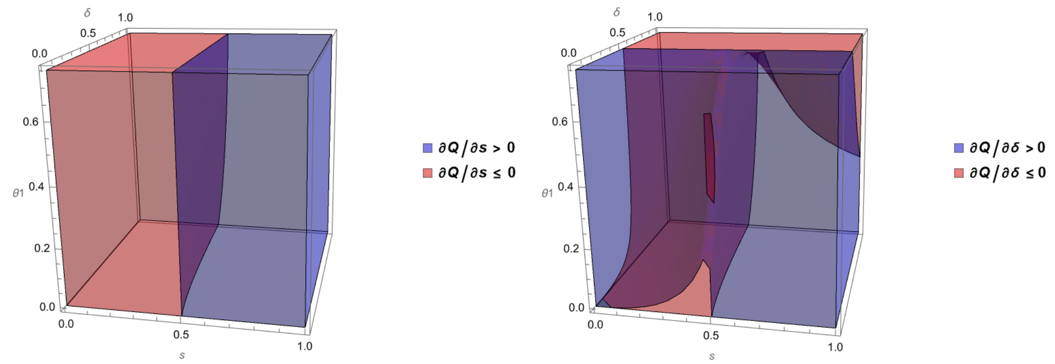
(a) Change in share of group 1 (b) Change in homophily

Figure 4: Effects of changes in social structure on total transfers given heterogeneous information by group with $\hat{\theta}_2 = 0.75$. Note that a positive change in δ corresponds to a reduction in homophily, so that Panel (b) should be interpreted as showing that, for most parameter combinations, greater homophily leads to a reduction in transfers.

relatively higher rate of substitution into transfers to the majority group: as group sizes approach equality, members of the majority become relatively more valuable, while the increased number of cross-group ties offsets the reduction in transfers to the minority.

Finally, we consider the effect of a decrease in the level of homophily in the network (increase in δ), shown in Figures 4b and 5b. As in the baseline model, decreasing levels of homophily are uniformly associated with increases in expenditure unless at least one $\hat{\theta}_\ell$ is close to one and fractionalization is high. In other words, for an increase in homophily to be associated with an increase in spending, it is necessary that the corresponding relative decrease in cross-group ties actually be associated with an *increase* in the value of transfers to at least one group. Intuitively, this generally relies on two conditions holding: (1) the proportion of possible cross-group ties is sufficiently low that a marginal reduction in their probability does not have too large of an effect and (2) the members of at least one group have sufficiently high $\hat{\theta}_\ell$ and are sufficiently numerous that, on average, ties within that group are more valuable than cross-group ties.

The relationship between homophily and inequality, shown in Figure 5b, is highly contingent. Similarly to the baseline model, when the less-predictable group (here,



(a) Change in share of group 1

(b) Change in homophily

Figure 5: Effects of changes in social structure on group inequality given heterogeneous information by group with $\hat{\theta}_2 = 0.75$. Note that a positive change in δ corresponds to a reduction in homophily. Also note that the boundary at $s \approx 0.5$ in Panel (a) is not a flat plane: for s sufficiently close to 0.5, there exist parameter values such that $\frac{\partial Q}{\partial s} \lessgtr 0$.

group 1), is in the majority, increases in homophily are generally associated with decreases in inequality, except when δ and $\hat{\theta}_1$ are both close to 1, corresponding to high predictability and low homophily. In this region, low-level increases in homophily have the effect of further increasing c_2 and decreasing c_1, exacerbating the existing inequality by weakening the equalizing effect of cross-group ties.

By contrast, when group 1 members in the minority (and thus receiving higher average transfers than their counterparts in group 2), it is now possible for increases in homophily to cause increases in inequality even at low values of δ and $\hat{\theta}_1$. While increased homophily may increase c_1 if s is sufficiently close to 0 and δ to 1, for most parameter combinations increases in homophily decrease both c_1 and c_2 due to the loss of cross-group ties. However, when group 1 members are unpredictable compared to group 2 members, the net effect is to decrease c_2 by more than c_1, as members of the majority lose more total influence than do members of the minority. There are many parameter combinations that generate this effect, however, indicating that when the smaller group is also the less predictable electorally, the effects of homophily on inequality of transfers are generally quite ambiguous.

4.3 Network-Dependent Information

In the preceding sections, density was shown to play a minor role in equilibrium behavior of particular graph realizations and no role whatsoever on average, even when signal accuracy is allowed to depend arbitrarily on network distance. It is straightforward to see that we can break this result by establishing a systematic relationship between density and one of the variables that do matter.

While there are several ways to accomplish this,[9] there is only one that is natural and consistent with the existing literature. Consider placing the candidates on the network (with corresponding group membership) and assuming that a voter's type θ_i is not independent from their position on the network. In particular, if voters whose shortest-path distance from a candidate is lower have a higher probability that $\theta_i = \overline{\theta}$, then a systematic relationship will exist between i's probability of connecting with candidates p_{ik} and their average posterior information $\hat{\theta}_i$.[10]

In this setting, density may in fact play an important role, as increases in density will increase the likelihood of all voters being type $\overline{\theta}$ through a decrease in their expected distance from any candidate, which will directly affect candidates' posterior beliefs and the corresponding equilibrium bribes. Therefore, unlike before, changes in density will now affect expected equilibrium behavior by changing the prior distribution of types. Importantly, however, because the event that $\theta_i = \overline{\theta}$ symmetrically increases their value to both candidates, network dependency of this kind will not induce strategic divergence between candidates on average, although equilibrium electoral outcomes may nevertheless be affected if candidates enjoy systematically better information about socially proximate voters in the form of more precise signals. Moreover, in the two-candidate case, candidates will not only target their own close neighbors but also their opponent's close neighbors with higher transfers, since both groups of voters are associated with higher posteriors.

[9] For example, if candidates' degree of office motivation were inversely proportional to their social integration, then more dense networks would tend to see lower levels of vote buying. However, not only is the empirical plausibility of this assumption unclear, it would yield the opposite relationship from that usually argued.

[10] Since θ_i measures the predictability of a voter's behavior given their observable characteristics, it is plausible that voters who are more connected to political candidates will vote more consistently.

For this reason, we focus on this section on the non-competitive one-candidate case, which arises naturally in a wide variety of applications. For example, vote brokers typically rely heavily on personal connections with voters to inform their targeting of benefits, as the preferences of more socially distant voters are less legible (Stokes et al. 2013). In many cases, such as under electoral authoritarianism, only one party actively engages in vote buying, while its opponents either lack resources or are ideologically committed to programmatic linkages. Even when multiple clientelist parties compete to buy votes, moreover, competition occurs at the stage of contracting between parties and brokers across localities, rather than between parties in a given locality (Holland and Palmer-Rubin 2015). Similarly, targeted inducements in the international arena are frequently determined by the strategy of a single hegemonic power, rather than strategic competition on equal terms (Vreeland and Dreher 2014).

Figures 6 and 7 show the results of a series of simulations of the model for a range of plausible parameter combinations. Since the asymptotic results upon which the preceding section relies require independence between θ_i and $p_{i,j}$ we instead take the approach of estimating $\mathbb{E}[b_i]$ on a finite network directly through 1,000 repeated draws of networks from the stochastic block model governed by the given network parameters. For each combination of parameters, we calculate the average transfers to members of each group, $\bar{b}_\ell = \mathbb{E}[b_i|\ell_i]$, the overall inequality, $Q = \left|1 - \bar{b}_1/\bar{b}_2\right|$, and the total transfers $B = \sum_i b_i$ at each draw. Estimates are then calculated as the average across all draws, along with bootstrapped 95% confidence intervals based on the corresponding percentiles.

Since our focus is primarily on social structure, we repeat the process over a grid of values between 0 and 1 for each of $\rho = p_H$, $\delta = \frac{p_L}{p_H}$ and $s = \frac{n_1}{n}$, which govern density, homophily, and fractionalization, respectively. Due to the computational burden of calculating large matrix inverses at each iteration, the model's remaining parameters are held constant at moderate values[11] of $n = 200, \alpha = 1, \gamma = \frac{1}{400}, u(b) = 200\ln(b)$,

[11] Note that, of these, only n is likely to significantly affect results, as the other parameters simply scale the candidate's optimal tradeoff and do not affect centrality directly. This value of n was chosen both for the sake of computational feasibility and because it corresponds to the approximate size of many populations of interest, including the international society of states and the villages

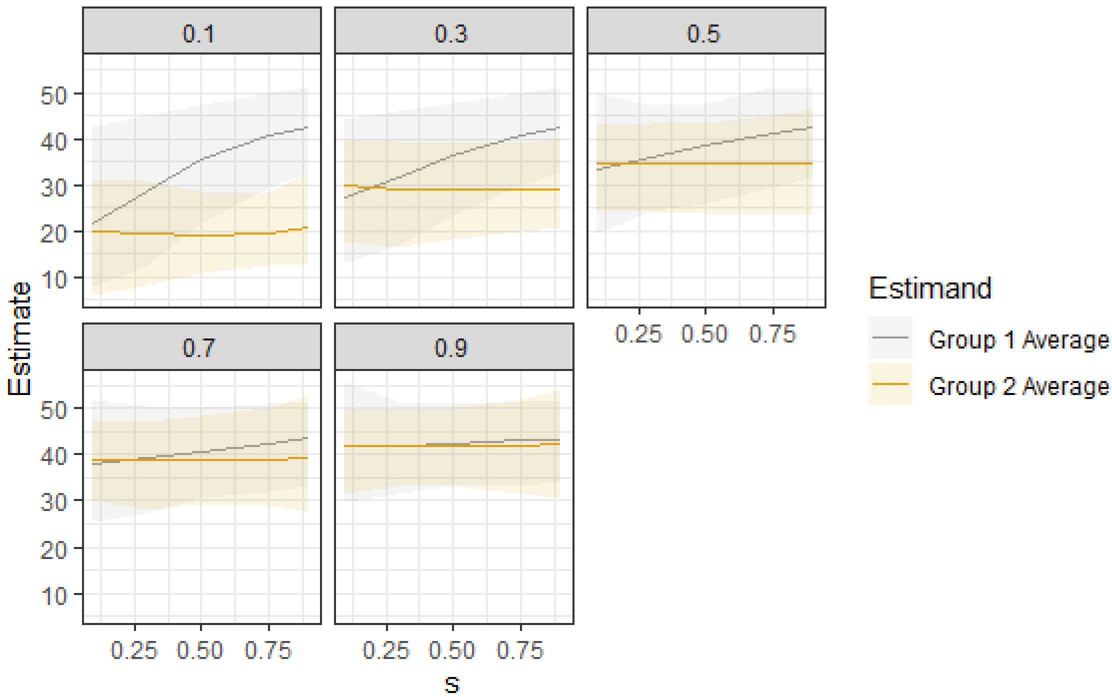


Figure 6: Average transfers from candidate 1 by group as a function of group 1's share (s), by homophily (δ).

while we assume for simplicity that candidate k's information decays in social distance according to a power law, $\hat{\theta}_i = 2^{-d(i,k)}$.

As Figure 6 clearly demonstrates, the introduction of network dependence in a one-candidate environment introduces a clear element of in-group favoritism, consistent with empirical observations across a variety of settings. As might be expected, this favoritism is entirely driven by homophily: when δ is close to 1 (low homophily), the two groups receive almost identical amounts on average, with consistently large differences emerging only when homophily is quite extreme. Intuitively, preference for the candidate's own group members is driven by lower social distance on average, amplified by greater separation between the groups, which leads to higher confidence that bribes to them will lead to an increase in vote share. In striking contrast to the independent case, this tendency to favor in-groups completely outweighs the prefer-

studied by Larson and Lewis (2017). Moreover, increases in n mainly affect the variability and scale of the estimates, and not their averages, consistent with the analytical results of the preceding sections.

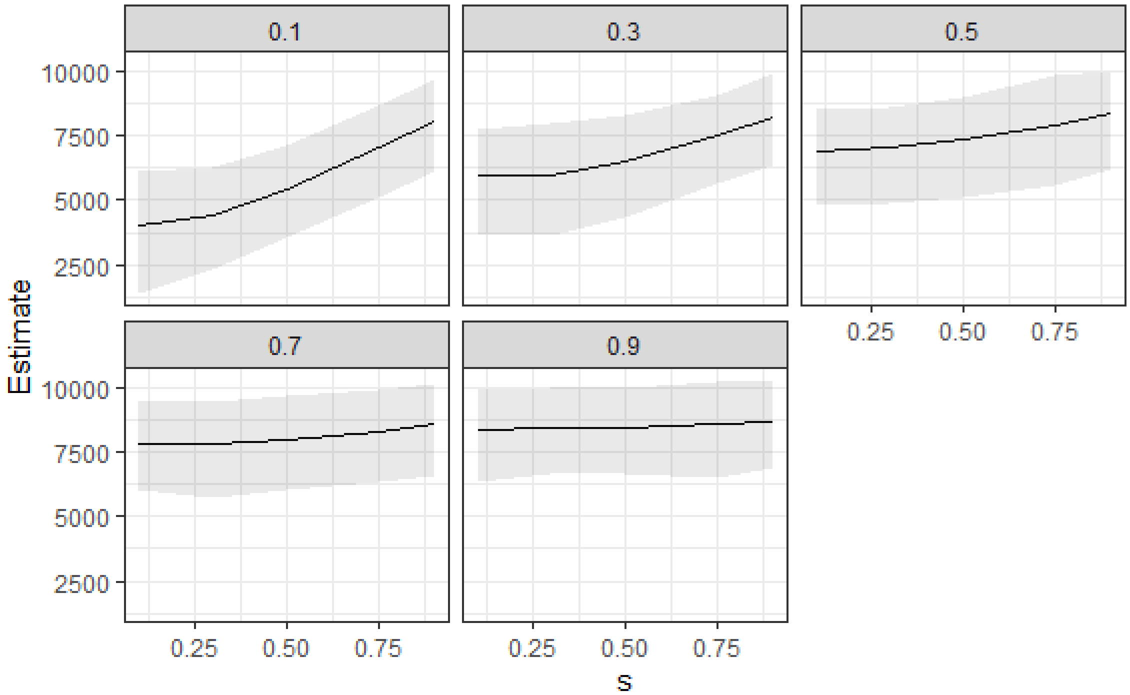


Figure 7: Total transfers from candidate 1 as a function of group 1's share (s), by homophily (δ)

ence for minorities. Indeed, while the relative size of the groups has little effect on average on transfers to out-group members, when homophily is large transfers to the in-group are sharply increasing in its size. Under homophily, an increase in the size of the in-group further decreases the expected shortest path distance, thus raising the value of transfers to all group members. As a consequence, this also tends to increase the level of inequality (Figure 8) as more funds are diverted to the candidate's in-group members.

As can be seen in Figure 7, however, this effect is insufficient to completely offset the overall loss in network spillovers induced by an increase in homophily. While, for a given value of δ, expenditure on private transfers is highest when $s \to 1$, driven by increases in in-group spending, it is also increasing in δ for all values of s. In fact, as s approaches 1, the probabilities converge to the same values as when $\delta = 1$, so that in either case the candidate is, on average, as close as possible to all voters. When information depends on network structure in this way, therefore, we again expect the greatest diversion of funds from public to private goods in societies that are either

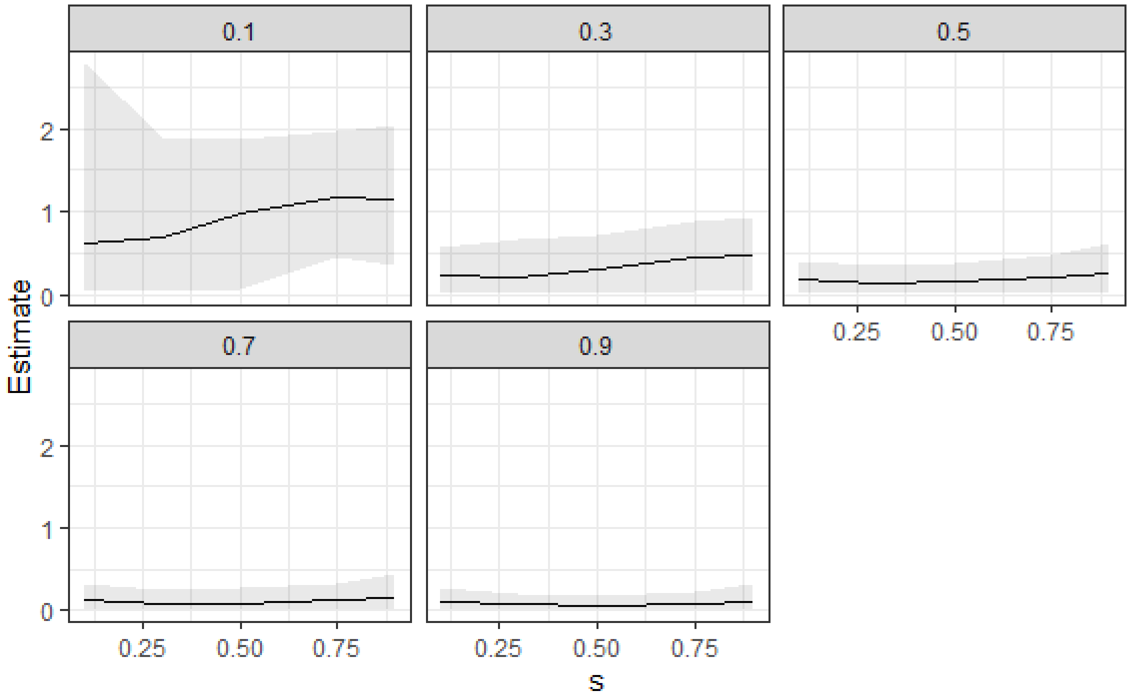


Figure 8: Inequality of average transfers from candidate 1 as a function of group 1's share (s), by homophily (δ)

perfectly homogeneous or have minimal levels of social segregation.

5 Discussion

In this article, we provide a formal model of vote buying on a network of policy-motivated voters who care about both private and public goods and candidates who can extend bribes to improve their electoral performance. We study the game under a given network structure and show that, in equilibrium, each individual voter receives the same bribe from both candidates. Since no candidate receives more bribes from one candidate than the other, equilibrium expected vote share remains unchanged and hence vote buying does not affect the electoral outcomes.

Additionally, we employ techniques from spectral random graph theory to research the role of social structure. In doing so, we are able to explicitly characterize the equilibrium transfers received by each voter in large networks and derive sharp comparative statics regarding the role of social structure. Particularly noteworthy are our findings regarding network density and homophily: contrary to arguments frequently found in the literature, density does not affect either the level of group inequality or total spending, while homophily actually decreases both for most plausible parameter combinations.

These results have direct implications on vote buying in a variety of political settings. For instance, we predict the most intense vote-buying to occur in social contexts with a large majority and small minority but with relatively low levels of social segregation. At the same time, members of the minority are likely to benefit disproportionately from vote buying, especially when candidates are well-informed about voter preferences.

There are a number of interesting questions our model provokes that we leave to future research. For example, the voters in our model do not strategically transmit information to their neighbors, but instead transparently share their vote probabilities. A natural extension is therefore to allow voters to attempt to persuade their neighbors by revealing information with the goal of maximizing their expected transfers. Such a model may be able to answer questions about how polarization in a society affects vote buying and how large, heterogeneous biases may interact with features of a social network studied in this paper, such as density and homophily.

The role of social connections between candidates and voters is also deserving of further consideration. In particular, consider an extension to the signaling model considered here in which candidates have limited information about the network structure, but can pay a cost that is proportional to their social distance to discover pertinent information about voters, such as party membership and policy preferences. This extension could further inform how candidate location in the existing social structure affects the extent of vote buying and the distribution of private gains, as well as accounting for the empirically observed tendency to favor members of the candidate's in-group.

References

Auerbach, Adam Michael. 2019. *Demanding development: The politics of public goods provision in India's urban slums.* Cambridge University Press.

Auerbach, Adam Michael, and Tariq Thachil. 2018. "How clients select brokers: competition and choice in India's slums." *American Political Science Review* 112(4):775–791.

Ballester, Coralio, Antoni Calvó-Armengol, and Yves Zenou. 2006. "Who's who in networks. Wanted: The key player." *Econometrica* 74(5):1403–1417.

Banks, Jeffrey S. 2000. "Buying supermajorities in finite legislatures." *American Political Science Review* 94(3):677–681.

Battaglini, Marco, and Eleonora Patacchini. 2018. "Influencing connected legislators." *Journal of Political Economy* 126(6):2277–2322.

Battaglini, Marco, Valerio Leone Sciabolazza, and Eleonora Patacchini. 2020. "Effectiveness of connected legislators." *American Journal of Political Science* 64(4):739–756.

Board, Simon, and Moritz Meyer-ter Vehn. 2021. "Learning dynamics in social networks." *Econometrica* 89(6):2601–2635.

Bonacich, Phillip. 1987. "Power and centrality: A family of measures." *American journal of sociology* 92(5):1170–1182.

Brusco, Valeria, Marcelo Nazareno, and Susan C Stokes. 2004. "Vote buying in Argentina." *Latin American research review* pp. 66–88.

Calvó-Armengol, Antoni, Eleonora Patacchini, and Yves Zenou. 2009. "Peer effects and social networks in education." *The review of economic studies* 76(4):1239–1267.

Calvo, Ernesto, and Maria Victoria Murillo. 2013. "When parties meet voters: Assessing political linkages through partisan networks and distributive expectations in Argentina and Chile." *Comparative Political Studies* 46(7):851–882.

Chandra, Kanchan. 2007. *Why ethnic parties succeed: Patronage and ethnic head counts in India*. Cambridge University Press.

Chung, Fan, and Mary Radcliffe. 2011. "On the spectra of general random graphs." *the electronic journal of combinatorics* pp. P215–P215.

Chung, Fan RK, and Fan Chung Graham. 1997. *Spectral graph theory.* Number 92 American Mathematical Soc.

Cruz, Cesi. 2019. "Social networks and the targeting of vote buying." *Comparative Political Studies* 52(3):382–411.

Cruz, Cesi, Julien Labonne, and Pablo Querubin. 2017. "Politician family networks and electoral outcomes: Evidence from the Philippines." *American Economic Review* 107(10):3006–37.

Dal Bó, Ernesto. 2007. "Bribing voters." *American Journal of Political Science* 51(4):789–803.

Dasaratha, Krishna. 2020. "Distributions of centrality on networks." *Games and Economic Behavior* 122:1–27.

Dekel, Eddie, Matthew O Jackson, and Asher Wolinsky. 2008. "Vote buying: General elections." *Journal of Political Economy* 116(2):351–380.

Fafchamps, Marcel, and Julien Labonne. 2017. "Do politicians' relatives get better jobs? evidence from municipal elections." *The Journal of Law, Economics, and Organization* 33(2):268–300.

Fearon, James D, and David D Laitin. 1996. "Explaining interethnic cooperation." *American political science review* 90(4):715–735.

Finan, Frederico, and Laura Schechter. 2012. "Vote-buying and reciprocity." *Econometrica* 80(2):863–881.

Frye, Timothy, Ora John Reuter, and David Szakonyi. 2019. "Vote brokers, clientelist appeals, and voter turnout: Evidence from Russia and Venezuela." *World politics* 71(4):710–746.

Gans-Morse, Jordan, Sebastian Mazzuca, and Simeon Nichter. 2014. "Varieties of clientelism: Machine politics during elections." *American Journal of Political Science* 58(2):415–432.

Granovetter, Mark S. 1973. "The strength of weak ties." *American journal of sociology* 78(6):1360–1380.

Groseclose, Tim, and James M Snyder. 1996. "Buying supermajorities." *American Political Science Review* 90(2):303–315.

Guardado, Jenny, and Leonard Wantchekon. 2018. "Do electoral handouts affect voting behavior?" *Electoral Studies* 53:139–149.

Gubler, Joshua R, and Joel Sawat Selway. 2012. "Horizontal inequality, crosscutting cleavages, and civil war." *Journal of Conflict Resolution* 56(2):206–232.

Habyarimana, James, Macartan Humphreys, Daniel N Posner, and Jeremy M Weinstein. 2009. *Coethnicity: Diversity and the Dilemmas of Collective Action*. Russell Sage Foundation.

Hicken, Allen. 2011. "Clientelism." *Annual review of political science* 14:289–310.

Holland, Alisha C, and Brian Palmer-Rubin. 2015. "Beyond the machine: Clientelist brokers and interest organizations in Latin America." *Comparative Political Studies* 48(9):1186–1223.

Jackson, Matthew O. 2008. Average distance, diameter, and clustering in social networks with homophily. In *International Workshop on Internet and Network Economics*. Springer pp. 4–11.

Katz, Leo. 1953. "A new status index derived from sociometric analysis." *Psychometrika* 18(1):39–43.

Keefer, Philip. 2007. "Clientelism, credibility, and the policy choices of young democracies." *American journal of political science* 51(4):804–821.

Keefer, Philip, and Razvan Vlaicu. 2017. "Vote buying and campaign promises." *Journal of Comparative Economics* 45(4):773–792.

Kramon, Eric. 2016*a*. "Electoral handouts as information: Explaining unmonitored vote buying." *World Politics* 68(3):454–498.

Kramon, Eric. 2016*b*. "Where is vote buying effective? Evidence from a list experiment in Kenya." *Electoral Studies* 44:397–408.

Larson, Jennifer M, and Janet I Lewis. 2017. "Ethnic networks." *American Journal of Political Science* 61(2):350–364.

Lizzeri, Alessandro, and Nicola Persico. 2001. "The provision of public goods under alternative electoral incentives." *American Economic Review* 91(1):225–239.

Miguel, Edward, and Mary Kay Gugerty. 2005. "Ethnic diversity, social sanctions, and public goods in Kenya." *Journal of public Economics* 89(11-12):2325–2368.

Mostagir, Mohamed, Asuman Ozdaglar, and James Siderius. 2022. "When is society susceptible to manipulation?" *Management Science* .

Mostagir, Mohamed, and James Siderius. 2021. "Centrality in stochastic networks." *Working paper* .

Nichter, Simeon. 2008. "Vote buying or turnout buying? Machine politics and the secret ballot." *American political science review* 102(1):19–31.

Oliveros, Virginia. 2016. "Making it personal: Clientelism, favors, and the personalization of public administration in Argentina." *Comparative Politics* 48(3):373–391.

Putnam, Robert D. 2000. *Bowling alone: The collapse and revival of American community.* Simon and schuster.

Reuter, Ora John, and Graeme B Robertson. 2012. "Subnational appointments in authoritarian regimes: Evidence from Russian gubernatorial appointments." *The Journal of politics* 74(4):1023–1037.

Robinson, James A, and Thierry Verdier. 2013. "The political economy of clientelism." *The Scandinavian Journal of Economics* 115(2):260–291.

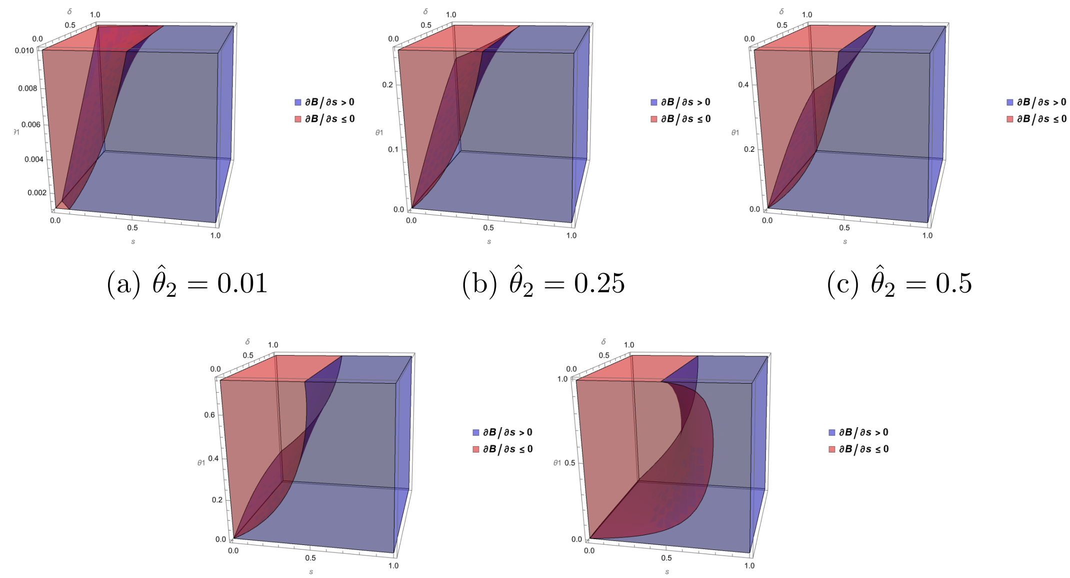
Stokes, Susan C. 2005. "Perverse accountability: A formal model of machine politics with evidence from Argentina." *American political science review* 99(3):315–325.

Stokes, Susan C, Thad Dunning, Marcelo Nazareno, and Valeria Brusco. 2013. *Brokers, voters, and clientelism: The puzzle of distributive politics*. Cambridge University Press.

Vreeland, James Raymond, and Axel Dreher. 2014. *The political economy of the United Nations Security Council: Money and influence*. Cambridge University Press.

For online publication.

A Additional Tables and Figures



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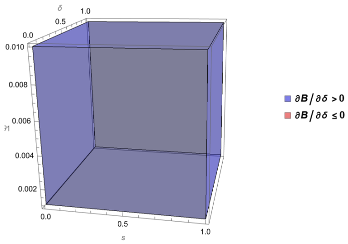
(d) $\hat{\theta}_2 = 0.75$ (e) $\hat{\theta}_2 = 0.99$

Figure 9: $\frac{\partial B}{\partial s}$ as a function of parameters for varying values of $\hat{\theta}_2$



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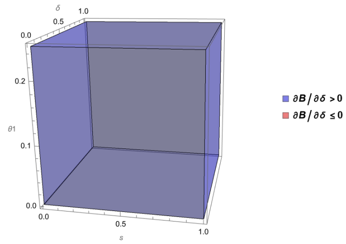

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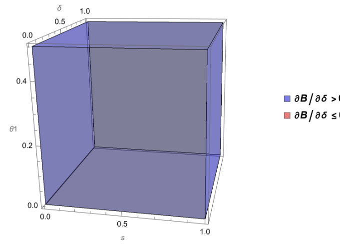

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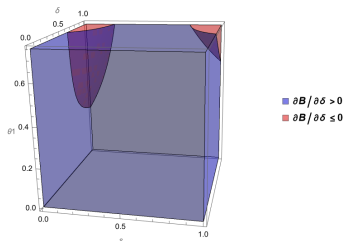

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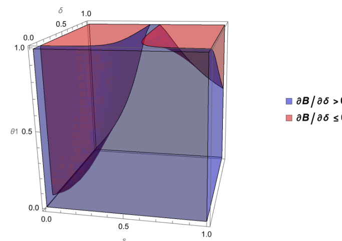

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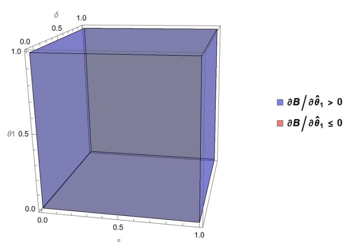
Figure 10: $\frac{\partial B}{\partial \delta}$ as a function of parameters for varying values of $\hat{\theta}_2$



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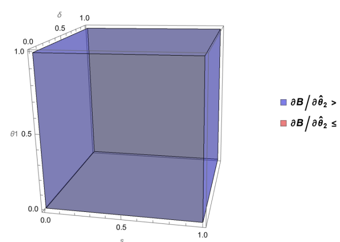
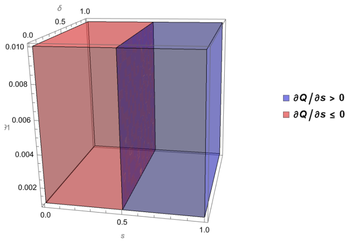


(b) $\frac{\partial B}{\partial \hat{\theta}_2}$

Figure 11: Derivatives of total transfers with respect to information as a function of parameters. Note that these derivatives are always positive and do not depend on parameter choice.



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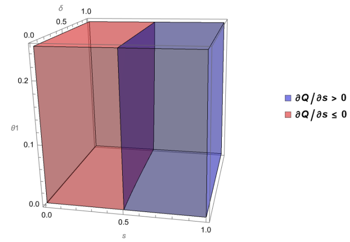

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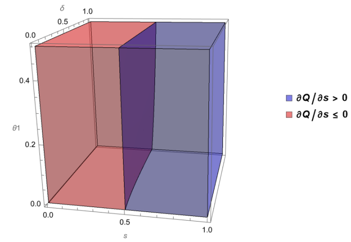

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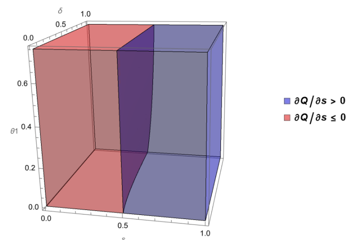

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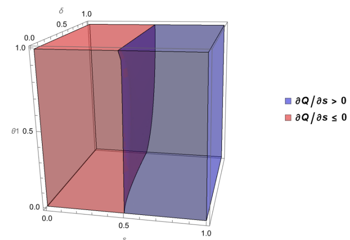

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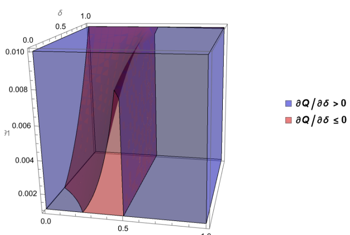
Figure 12: $\frac{\partial Q}{\partial s}$ as a function of parameters for varying values of $\hat{\theta}_2$



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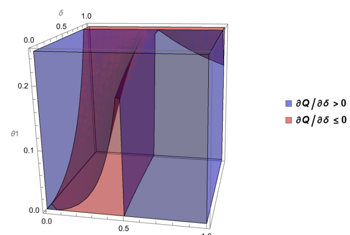

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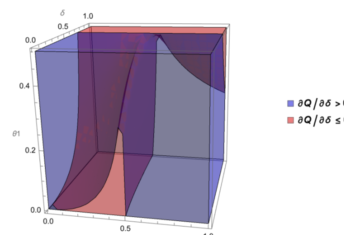

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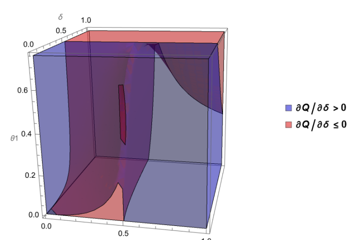

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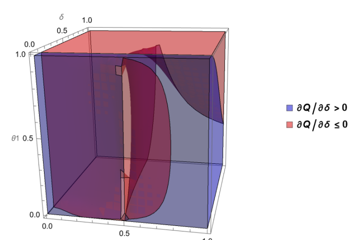

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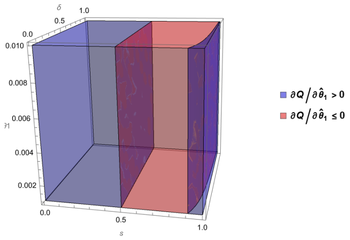
Figure 13: $\frac{\partial Q}{\partial \delta}$ as a function of parameters for varying values of $\hat{\theta}_2$



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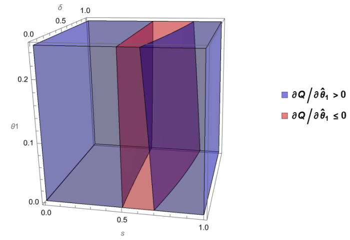

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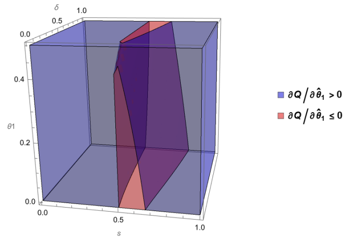

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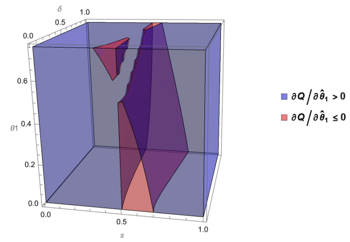

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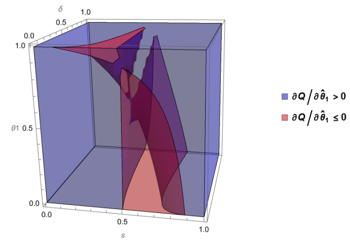

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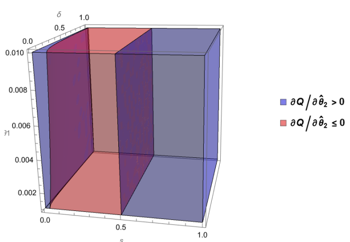
Figure 14: $\frac{\partial Q}{\partial \hat{\theta}_1}$ as a function of parameters for varying values of $\hat{\theta}_2$



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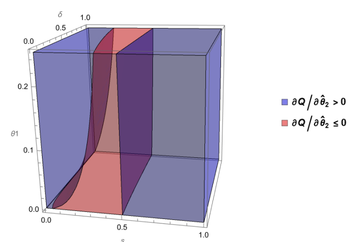

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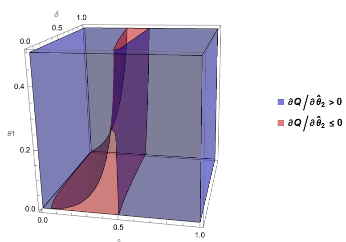

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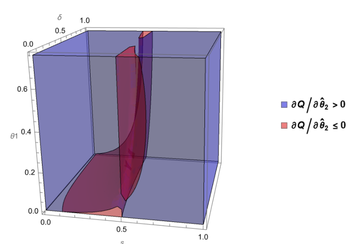

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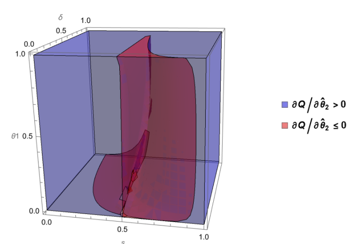

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Figure 15: $\frac{\partial Q}{\partial \hat{\theta}_2}$ as a function of parameters for varying values of $\hat{\theta}_2$

B Proofs

Lemma 2 (Chung and Radcliffe (2011)). *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ be bounded independent random Hermitian matrices and set $M > 0 : ||\boldsymbol{X}_i - \mathbb{E}(\boldsymbol{X}_i)||_2 \leq M \; \forall i = 1, \ldots, m$. Then for any $a > 0$,*

$$\Pr(||\boldsymbol{X} - \mathbb{E}(\boldsymbol{X})||_2 > a) \leq 2n \exp\left(-\frac{a^2}{2v^2 + 2Ma/3}\right)$$

where $\boldsymbol{X} = \sum_{i=1}^m \boldsymbol{X}_i$ and $v^2 = ||\sum_{i=1}^m \mathbb{V}(\boldsymbol{X}_i)||$.[12]

Proof of Theorem 1. Let $\mathcal{G}$ be an undirected random graph such that all edge formation probabilities are jointly independent. Denote by $\boldsymbol{A}$ the adjacency matrix, $\boldsymbol{W}$ a matrix of weights, and $\boldsymbol{A}_W$ the weighted adjacency matrix, such that $\boldsymbol{A}_W = \boldsymbol{W} \odot \boldsymbol{A}$.[13] Let $\boldsymbol{D}_W$ be the diagonal degree matrix such that $\{\bar{\boldsymbol{D}}_W\}_{ii} = \sum_j w_{ij} a_{ij}$, and denote by $\bar{\boldsymbol{A}}, \bar{\boldsymbol{D}}_W$ the expected equivalents. Finally, let $\boldsymbol{L}_W = \boldsymbol{I} - \boldsymbol{D}_W^{-1/2} \boldsymbol{A}_W \boldsymbol{D}_W^{-1/2}$ denote the normalized weighted Laplacian of $\mathcal{G}$, $\omega = \max_{i,j} w_{ij}$ be the largest total weight, $\alpha = \min_{i,j} w_{ij}$ the smallest, and $\delta = \min_i \{\bar{\boldsymbol{D}}_W\}_{ii}$ the smallest expected degree.

Denote $\bar{d}_i$ as the expected (weighted) degree of node i. By the triangle inequality, for any matrix $\boldsymbol{C}$,

$$||\boldsymbol{L}_W - \bar{\boldsymbol{L}}_W|| \leq ||\boldsymbol{C} - \bar{\boldsymbol{L}}_W|| + ||\boldsymbol{L}_W - \boldsymbol{C}||$$

In particular, let $\boldsymbol{C} = I - \bar{\boldsymbol{D}}_W^{-1/2} \boldsymbol{A}_W \bar{\boldsymbol{D}}_W^{-1/2}$. Then since the degree matrices are diagonal, we have $\boldsymbol{C} - \bar{\boldsymbol{L}}_W = \bar{\boldsymbol{D}}_W^{-1/2} (\boldsymbol{A}_W - \bar{\boldsymbol{A}}_W) \bar{\boldsymbol{D}}_W^{-1/2}$. Denoting by $\boldsymbol{A}^{ij}$ the matrix that is equal to 1 in the i,jth and j,ith positions and 0 elsewhere, we can write i,jth entry of $\boldsymbol{C} - \bar{\boldsymbol{L}}$ as

$$\boldsymbol{X}_{ij} = \bar{\boldsymbol{D}}_W^{-1/2} (w_{ij}(a_{ij} - p_{ij}) \boldsymbol{A}^{ij}) \bar{\boldsymbol{D}}_W^{-1/2} = \frac{w_{ij}(a_{ij} - p_{ij})}{\sqrt{\bar{d}_i \bar{d}_j}} \boldsymbol{A}^{ij}$$

Then clearly $\boldsymbol{C} - \bar{\boldsymbol{L}} = \sum \boldsymbol{X}_{ij}$, so Lemma 2 applies. Note that since $\mathbb{E}(a_{ij}) = p_{ij}$, we have that $\mathbb{E}(\boldsymbol{X}_{ij}) = \boldsymbol{0}$, so that $v^2 = ||\sum \mathbb{E}(\boldsymbol{X}_{ij}^2)||$. Also, each $\boldsymbol{X}_{ij}$ is bounded

[12] See Theorem 5 in Chung and Radcliffe (2011) for the proof.
[13] Note that $\odot$ indicates the Hadamard (element-wise) product.

above by $\|\boldsymbol{X}_{ij}\| \leq \frac{\omega}{\delta}$ Now clearly

$$\mathbb{E}(\boldsymbol{X}_{ij}^2) = \begin{cases} \frac{w_{ij}^2}{\bar{d}_i \bar{d}_j}(p_{ij})(1-p_{ij})(A^{ii}+A^{jj}) & i \neq j \\ \frac{w_{ii}^2}{\bar{d}_i^2}(p_{ij})(1-p_{ij})A^{ii} & i = j \end{cases}$$

Now we can write

$$\begin{aligned} v^2 &= \left\| \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{w_{ij}^2}{\bar{d}_i \bar{d}_j}(p_{ij})(1-p_{ij})A^{ii} \right\| \\ &= \max_i \left(\sum_{j=1}^{n} \frac{w_{ij}^2}{\bar{d}_i \bar{d}_j}(p_{ij})(1-p_{ij}) \right) \\ &\leq \max_i \left(\frac{\omega}{\delta} \sum_{j=1}^{n} \frac{w_{ij}}{\bar{d}_i}(p_{ij}) \right) \\ &= \frac{\omega}{\delta} \end{aligned}$$

For notational convenience denote $a = \sqrt{\frac{3\omega \ln(4n/\epsilon)}{\delta}}$ and δ so that $a < 1$. In particular, we must have $\delta > 3\omega(\ln(4) + \ln(n) - \ln(\epsilon))$, so that if $k \geq 3\omega(1 + \ln(4/\epsilon))$, $\delta \geq k \ln(n)$ guarantees the result. Then from Lemma 2

$$\begin{aligned} \Pr(\|\boldsymbol{C} - \bar{\boldsymbol{L}}_W\| > a) &\leq 2n \exp\left(-\frac{\frac{3n\omega^2 \ln(4n/\epsilon)}{n\delta}}{2n\omega^2/\delta + 2an\omega^2/3\delta} \right) \\ &= 2n \exp\left(-\frac{\frac{3n\omega^2 \ln(4n/\epsilon)}{\delta}}{2n\omega^2(3+a)/3\delta} \right) \\ &= 2n \exp\left(-\frac{9\ln(4n/\epsilon)}{6+2a} \right) \\ &\leq 2n \exp\left(-\frac{9\ln(4n/\epsilon)}{9} \right) \\ &= \frac{\epsilon}{2} \end{aligned}$$

Now for the second term, note that d_i is a sum of random variables that are bounded between 0 and ω. Then by Hoeffding's Inequality, we have that, for any t,

$$\Pr(|d_i - \bar{d}_i| > t\bar{d}_i) \leq 2\exp\left(-\frac{t^2\bar{d}_i^2}{n\omega^2}\right) \leq 2\exp\left(-\frac{t^2\delta^2}{n\omega^2}\right)$$

Now in particular let $t = \sqrt{\frac{n\omega^2\ln(4n/\epsilon)}{\delta^2}} = \sqrt{\frac{n\omega}{3\delta}}a$. We have $t < a < 1$ if $\delta > \frac{n\omega}{3}$. In our application, $\omega = \rho_H, \delta = n_0 p_L \rho_L + n_1 p_H \rho_H.$, so that for all i we obtain

$$\Pr(|d_i - \bar{d}_i| > t\bar{d}_i) \leq \frac{\epsilon}{2n}$$

Now note that

$$\left\|\bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{D}_W^{1/2} - \boldsymbol{I}\right\|_2 = \max_i \left|\sqrt{\frac{d_i}{\bar{d}_i}} - 1\right|$$

To bound this, note that from (B) we can conclude that $\Pr\left(\left|\frac{d_i}{\bar{d}_i} - 1\right| > t\right) \leq \frac{\epsilon}{2n}$ and hence with probability at least $1 - \frac{\epsilon}{2n}$,

$$\left\|\bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{D}_W^{1/2} - \boldsymbol{I}\right\|_2 < \sqrt{\frac{n\omega^2\ln(4n/\epsilon)}{\delta^2}}$$

Finally, note that since $\|\boldsymbol{L}\|_2 \leq 2$ (Chung and Graham 1997), we have $\|\boldsymbol{I} - \boldsymbol{L}\|_2 \leq 1$. Now consider

$$\begin{aligned}
\|\boldsymbol{L}_W - \boldsymbol{C}\| &= \|\boldsymbol{I} - \boldsymbol{D}_W^{-1/2}\boldsymbol{A}_W\boldsymbol{D}_W^{-1/2} - I + \bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{A}_W\bar{\boldsymbol{D}}_W^{-1/2}\| \\
&= \|(\boldsymbol{I} - \boldsymbol{L}_W)\bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{D}_W^{1/2}\boldsymbol{D}_W^{-1/2}\boldsymbol{A}_W\boldsymbol{D}_W^{-1/2}\boldsymbol{D}_W^{1/2}\bar{\boldsymbol{D}}_W^{-1/2}\| \\
&= \|(\boldsymbol{I} - \boldsymbol{L}_W)\bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{D}_W^{1/2}(\boldsymbol{I} - \boldsymbol{L})\boldsymbol{D}_W^{1/2}\bar{\boldsymbol{D}}_W^{-1/2}\| \\
&= \|(\bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{D}_W^{1/2} - \boldsymbol{I})(\boldsymbol{I} - \boldsymbol{L}_W)\boldsymbol{D}_W^{1/2}\bar{\boldsymbol{D}}_W^{-1/2} + (\boldsymbol{I} - \boldsymbol{L})(\boldsymbol{I} - \boldsymbol{D}_W^{1/2}\bar{\boldsymbol{D}}_W^{-1/2})\| \\
&\leq \|\bar{\boldsymbol{D}}_W^{-1/2}\boldsymbol{D}_W^{1/2} - \boldsymbol{I}\|\|\boldsymbol{D}_W^{1/2}\bar{\boldsymbol{D}}_W^{-1/2}\| + \|\boldsymbol{I} - \boldsymbol{D}_W^{1/2}\bar{\boldsymbol{D}}_W^{-1/2}\| \\
&\leq t^2 + 2t
\end{aligned}$$

Hence, finally,

$$
\begin{aligned}
||\boldsymbol{L}_W - \bar{\boldsymbol{L}}_W|| &\leq ||\boldsymbol{C} - \bar{\boldsymbol{L}}_W|| + ||\boldsymbol{L}_W - \boldsymbol{C}|| \\
&\leq a + \frac{n\omega}{3\delta}a^2 + \sqrt{\frac{4n\omega}{3\delta}}a \\
&= a\left(\frac{\sqrt{3\delta} + 2\sqrt{n\omega}}{\sqrt{3\delta}} + \frac{n\omega}{3\delta}a\right) \\
&= a\left(1 + \frac{2\sqrt{3n\omega\delta} + n\omega a}{3\delta}\right)
\end{aligned}
$$

Now, choose $k > 1$ such that

$$
\delta \geq \frac{1}{3}\left(2n\omega\frac{\sqrt{k} + k + 1}{(k-1)^2}\right)
$$

□

Proof of Proposition 4. By the result established in Lemma 1, it is sufficient to consider centrality on the average network. Under the stochastic block model, letting s_i denote the share of i's group without loss of generality, we have that the expected degree of i can be written as

$$
\sum_{j=1}^{n} w_{ij}p_{ij} = s_i n w_H p_H + (1 - s_i)n w_L p_L = n\left(w_L p_L + s_i(w_H p_H - w_L p_L)\right), ^{14}
$$

For notational convenience, we denote $w_H p_H = \rho, w_L p_L = \delta\rho$ for some $0 < \delta < 1$. The key observation is that the actual value of ρ is irrelevant, since it appears in both the denominator and numerator of each entry of the expected adjacency matrix. Thus, all results depend only on δ, the *relative* expected weight placed on out-group connections.

Note now that we can write the matrix $\boldsymbol{I} - \theta\bar{\bar{\boldsymbol{A}}}$ as a 2×2 block matrix with blocks $\bar{\bar{\boldsymbol{A}}}_{11} = \boldsymbol{I} - \frac{\theta}{n(\delta+s_1(1-\delta))}\left(\mathbf{1}_{s_1 n \times s_1 n} - \boldsymbol{I}\right)$, $\bar{\bar{\boldsymbol{A}}}_{12} = -\frac{\theta\delta}{n(\delta+s_1(1-\delta))}\mathbf{1}_{s_1 n \times s_2 n}$, $\bar{\bar{\boldsymbol{A}}}_{21} =$

[14] Technically this is an approximation since $p_{ii} = 0$, but the loss is insignificant for large n, which is assumed here.

$-\frac{\theta\delta}{n(\delta+s_2(1-\delta))}\mathbf{1}_{s_2 n\times s_1 n}$, and $\bar{\tilde{\boldsymbol{A}}}_{22} = \boldsymbol{I} - \frac{\theta}{n(\delta+s_2(1-\delta))}(\mathbf{1}_{s_2 n\times s_2 n} - \boldsymbol{I})$. To apply the formula for block inversion, we first want to identify $\bar{\tilde{\boldsymbol{A}}}_{11}^{-1}$. We conjecture that

$$P = \bar{\tilde{\boldsymbol{A}}}_{11}^{-1} = \begin{bmatrix} a_1 & b_1 & \cdots & b \\ b & a & \cdots & b \\ \vdots & \cdots & \ddots & \vdots \\ b & \cdots & \cdots & a \end{bmatrix}$$

Then we have that

$$\begin{bmatrix} 1 & -(n_1-1)\frac{\theta}{n(\delta+s_1(1-\delta))} \\ -\frac{\theta}{n(\delta+s_1(1-\delta))} & \left(1-(n_1-2)\frac{\theta}{n(\delta+s_1(1-\delta))}\right) \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which indeed has a unique solution. The inverse of the bottom-right block is identical, swapping group indices. Hence, we can construct the centrality vector according to the formula:

$$\boldsymbol{c} = \begin{bmatrix} \left(\bar{\tilde{\boldsymbol{A}}}_{11} - \bar{\tilde{\boldsymbol{A}}}_{12}\bar{\tilde{\boldsymbol{A}}}_{22}^{-1}\bar{\tilde{\boldsymbol{A}}}_{21}\right)^{-1} & \mathbf{0} \\ \mathbf{0} & \left(\bar{\tilde{\boldsymbol{A}}}_{22} - \bar{\tilde{\boldsymbol{A}}}_{21}\bar{\tilde{\boldsymbol{A}}}_{11}^{-1}\bar{\tilde{\boldsymbol{A}}}_{12}\right)^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\bar{\tilde{\boldsymbol{A}}}_{12}\bar{\tilde{\boldsymbol{A}}}_{22}^{-1} \\ -\bar{\tilde{\boldsymbol{A}}}_{21}\bar{\tilde{\boldsymbol{A}}}_{11}^{-1} & \boldsymbol{I} \end{bmatrix} \tag{A1}$$

The main-diagonal blocks in the first matrix again have the same structure, with a single value on the main diagonal and another value on the off-diagonal. This has a similar structure to the previous matrix, and the inverse can thus be calculated analogously by solving for main and off-diagonal elements a_i', b_i'[15].

Remark. There is a substantive interpretation of a_i' and b_i': a_i' is the weighted average of the number of paths back to a voter in group i through the network, while b_i' is the weighted average of the number of paths to someone else in your group through the network. Because in the expected network, all voters are connected to all others, i's centrality does not depend on paths to the other group, because a linear dependence is induced (all paths within group essentially correspond to an equivalent cross-party path).

[15] A unique solution again exists, but we suppress the exact expression as it is extremely complex. The mathematica file used to calculate these values is available upon request from the authors.

Substituting these values into equation (A1), we then have that

$$c_i = (1 + n_{-i})a'_i + (1 + n_{-i})(n_i - 1)b'_i$$

or, without loss of generality,

$$\begin{aligned} c_1 = n\Big(&- \delta\theta - \delta n + (\delta - 1)ns^2(\delta\theta + \delta + \theta - 1) \\ &- (\delta - 1)ns(\delta\theta + \delta + \theta - 1) + (\delta - 1)\theta s\Big) \\ \cdot \Big(&- \theta^2 + n^3 s((\delta - 1)s - \delta)(-\theta + s(\delta + \theta - 1) + 1) \\ &+ \theta n^2 s(-\delta - \theta + s(2\delta + \theta - 2) + 1) + \theta n \left(\theta + \delta^2\theta s - \delta s + s - 1\right) \Big)^{-1}. \end{aligned}$$

Since n is by assumption large, this expression is asymptotically equivalent to its leading term. We can thus simplify further, writing

$$c_1 \sim \frac{-\delta + (\delta - 1)s^2(\delta\theta + \delta + \theta - 1) - (\delta - 1)s(\delta\theta + \delta + \theta - 1)}{sn((\delta - 1)s - \delta)(-\theta + s(\delta + \theta - 1) + 1)}.$$

□

Proof of Proposition 7. It is straightforward to see that, replacing θ with Θ, the modified centrality of agenti i is now equal to their DeGroot centrality (Mostagir, Ozdaglar, and Siderius 2022) on the normalized network. It then follows from Theorem 1 and from the proof of Theorem 1 in Mostagir and Siderius (2021) that we can again consider the expected network only. The result then follows from an analogous argument to the proof of the preceding proposition, calculated using the accompanying Wolfram Mathematica code. □

Proof of Lemma 1. Let $\mathcal{G}^{(n)}$ be a sequence of random graphs over n vertices, and denote by $\delta_{(n)}$ the smallest expected weighted degree, i.e. $\delta_{(n)} = \min_i \sum_j w_{ij}^{(n)} p_{ij}^{(n)}$. Further, let $\bar{w}_{(n)} = \max_{i,j} w_{ij}^{(n)}$ and $\underline{w}_{(n)} = \min_{i,j} w_{ij}^{(n)}$ be the largest and smallest individual weights, satisfying $\frac{\bar{w}_{(n)}}{\underline{w}_{(n)}} \leq \omega$ for some $\omega > 0$ for all n. Then if there exists a non-decreasing sequence of $k_{(n)} > 0$ such that $\delta_{(n)} \geq k_{(n)} \ln(n)$ and $\underline{w}_{(n)} \cdot \bar{w}_{(n)} =$

$o\left(\sqrt{\frac{\delta_{(n)}}{\ln(n)}}\right)$, then the realized centrality vector centrality vector $\boldsymbol{c}^{(n)}(\tilde{\boldsymbol{A}})$ is with high probability close to the centrality of the average graph $\boldsymbol{c}^{(n)}(\bar{\tilde{\boldsymbol{A}}})$ for large n.

Under the stated assumptions, we can apply Theorem 1 to conclude that for any $\xi > 0$, for all n we have

$$\Pr\left(||\boldsymbol{L}_W - \bar{\boldsymbol{L}}_W|| \leq 4\sqrt{\frac{3\omega\ln(4n/\xi)}{\delta}}\right) \geq 1-\xi$$

Furthermore, by assumption $\lim_{n\to\infty} 4\sqrt{\frac{3\omega\ln(4n/\xi)}{\delta}} = 0$ regardless of the ξ chosen, so that under the 2-norm,

$$\boldsymbol{L}_W \underset{p}{\to} \bar{\boldsymbol{L}}_W$$

Now, for convenience call $\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{L}_W$ and $\bar{\boldsymbol{B}}$ the expected equivalent. Now clearly also $\boldsymbol{B} \underset{p}{\to} \bar{\boldsymbol{B}}$, and furthermore we can write $\boldsymbol{B} = \boldsymbol{D}^{-1/2}\boldsymbol{A}_W\boldsymbol{D}^{-1/2} = \boldsymbol{D}^{-1/2}\tilde{\boldsymbol{A}}\boldsymbol{D}^{1/2}$. So we can write, using properties of matrix norms (abusing notation in the second step slightly so that the maximum is over the norm of the matrices) and the above result,

$$\begin{aligned}
\limsup_{n\to\infty} \|\tilde{\boldsymbol{A}} - \bar{\tilde{\boldsymbol{A}}}\| &= \limsup_{n\to\infty} \|\boldsymbol{D}^{1/2}\boldsymbol{B}\boldsymbol{D}^{-1/2} - \bar{\boldsymbol{D}}^{1/2}\bar{\boldsymbol{B}}\bar{\boldsymbol{D}}^{-1/2}\| \\
&\leq \limsup_{n\to\infty} \|\max\left\{\boldsymbol{D}^{1/2}, \bar{\boldsymbol{D}}^{1/2}\right\}(\boldsymbol{B} - \bar{\boldsymbol{B}})\max\left\{\boldsymbol{D}^{-1/2}, \bar{\boldsymbol{D}}^{-1/2}\right\}\| \\
&\leq \limsup_{n\to\infty} \max\left\{\|\boldsymbol{D}^{1/2}\|, \|\bar{\boldsymbol{D}}^{1/2}\|\right\}\|\boldsymbol{B} - \bar{\boldsymbol{B}}\|\max\left\{\|\boldsymbol{D}^{-1/2}\|, \|\bar{\boldsymbol{D}}^{-1/2}\|\right\} \\
&\leq \limsup_{n\to\infty} \xi \max\left\{\|\boldsymbol{D}^{1/2}\|, \|\bar{\boldsymbol{D}}^{1/2}\|\right\}\max\left\{\|\boldsymbol{D}^{-1/2}\|, \|\bar{\boldsymbol{D}}^{-1/2}\|\right\}
\end{aligned}$$

Now since ξ can be chosen to be arbitrarily small, it is sufficient to establish that both $\|\boldsymbol{D}^{1/2}\|_2\|\bar{\boldsymbol{D}}^{-1/2}\|_2$ and $\|\boldsymbol{D}^{1/2}\|_2\|\bar{\boldsymbol{D}}^{-1/2}\|_2$ are bounded by a constant almost surely. To see that they are, observe that

$$\|\boldsymbol{D}^{1/2}\|_2\|\bar{\boldsymbol{D}}^{-1/2}\|_2 = \sqrt{\frac{\max_i \sum_j w_{ij}a_{ij}}{\max_i \sum_j w_{ij}p_{ij}}} \leq \sqrt{\frac{\bar{w}_{(n)}}{\underline{w}_{(n)}}} \max_i \sqrt{\frac{\sum_j a_{ij}}{\sum_j p_{ij}}} \leq \sqrt{\omega} \max_i \sqrt{\frac{\sum_j a_{ij}}{\sum_j p_{ij}}}$$

and similarly

$$\|\boldsymbol{D}^{1/2}\|_2\|\bar{\boldsymbol{D}}^{-1/2}\|_2 = \sqrt{\frac{\max_i \sum_j w_{ij}p_{ij}}{\max_i \sum_j w_{ij}a_{ij}}} \leq \sqrt{\frac{\bar{w}_{(n)}}{\underline{w}_{(n)}}} \max_i \sqrt{\frac{\sum_j p_{ij}}{\sum_j a_{ij}}} \leq \sqrt{\omega} \max_i \sqrt{\frac{\sum_j p_{ij}}{\sum_j a_{ij}}}$$

But since the a_{ij} are distributed Bernoulli(p_{ij}), (see e.g. Mostagir and Siderius (2021)) both $\max_i \sqrt{\frac{\sum_j p_{ij}}{\sum_j a_{ij}}}$ and $\max_i \sqrt{\frac{\sum_j a_{ij}}{\sum_j p_{ij}}}$ converge in probability to 1, so that we have for any $\xi > 0$

$$\limsup_{n\to\infty} \|\tilde{\boldsymbol{A}} - \bar{\tilde{\boldsymbol{A}}}\| \leq \xi\sqrt{\omega}$$

That is, the weighted adjacency matrix can be made arbitrarily close to its expected counterpart.

We now wish to show that, for arbitrary $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(\|(\boldsymbol{I} - \theta\tilde{\boldsymbol{A}})^{-1} - (\boldsymbol{I} - \theta\bar{\tilde{\boldsymbol{A}}})^{-1}\| \geq \epsilon) = 0$$

The key observation is that the above result implies that for any $\mu > 0$, there exists sufficiently large n such that with probability approaching 1, $\|\tilde{\boldsymbol{A}}^k - \bar{\tilde{\boldsymbol{A}}}^k\| \leq \mu$ for all k. Then it is straightforward to note that (since by model assumptions we have $\theta < 1$, so the formula for infinite geometric series can be applied),

$$\begin{aligned}
\limsup_{n\to\infty} \|(\boldsymbol{I} - \theta\tilde{\boldsymbol{A}})^{-1} - (\boldsymbol{I} - \theta\bar{\tilde{\boldsymbol{A}}})^{-1}\| &= \limsup_{n\to\infty} \left\| \sum_{k=0}^{\infty} \theta^k \left(\tilde{\boldsymbol{A}}^k - \bar{\tilde{\boldsymbol{A}}}^k \right) \right\| \\
&\leq \limsup_{n\to\infty} \sum_{k=0}^{\infty} |\theta^k| \left\| \left(\tilde{\boldsymbol{A}}^k - \bar{\tilde{\boldsymbol{A}}}^k \right) \right\| \\
&\leq \sum_{k=0}^{\infty} \mu |\theta^k| \\
&= \frac{\mu}{1-\theta}
\end{aligned}$$

Since μ was chosen arbitrarily, this implies that for any $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(\|\boldsymbol{c}^{(n)}(\tilde{\boldsymbol{A}}) - \boldsymbol{c}^{(n)}(\bar{\tilde{\boldsymbol{A}}})\| > \epsilon) = 0$$

Finally, note that the assumption of non-vanishing spectral gap guarantees that the network is connected with high probability, so that the centrality is well-defined (Dasaratha 2020; Mostagir and Siderius 2021), completing the proof. □

Lemma 3. *Define the following terms.*

1. $\Delta_{ij} := \left| \phi_j(\cdot|\boldsymbol{\theta} : \theta_i = \underline{\theta}) - \phi_j(\cdot|\boldsymbol{\theta} : \theta_i = \overline{\theta}) \right|$
2. W_{ij}^k *the (weighted) sum of all length k walks beginning with i and ending with j*
3. $\mathcal{I}_{ij} := \frac{\sum_{k=1}^{\infty} \overline{\theta}^k W_{ij}^k}{\sum_h \sum_{k=1}^{\infty} \underline{\theta}^k W_{hj}^k}$

Then, for any sequence of graphs $\mathcal{G}^{(n)}$ such that $(\bar{d}_n/\overline{\theta})^{\mathrm{diam}(\mathcal{G})} = o(n)$, there exists a $z_n > 0$ such that $\Delta_{ij}(\mathcal{G}^{(n)}) < z_n \mathcal{I}_{ij}(\mathcal{G}^{(n)})$ which satisfies $z_n = o(n)$.[16]

Proof. Take a graph $\mathcal{G}$, a corresponding Δ_{ij}, and choose a constant $z_n \in \mathbb{R}$ where

$$z_n > z_n^* := \frac{\Delta_{ij}}{\mathcal{I}_{ij}}.$$

[16] This assumption is consistent with the actual characteristics of "well-behaved" social network graphs, both theoretically and empirically (Jackson 2008).

Note that the numerator is bounded by 1 and the denominator must be greater than 0, so it must be that z_n^* is finite. Then, z_n must satisfy $\Delta_{ij} < z_n \mathcal{I}_{ij}$. Moreover, it is clear that $z_n^* < \frac{1}{\mathcal{I}_{ij}}$.[17]

Consider now the denominator of $\mathcal{I}_{ij}$. Since this reflects the sum of all walks of length k ending in j, $\sum_h \sum_{k=1}^{\infty} \underline{\theta}^k W_{hj}^k$, it is straightforward to show by induction on the length of walks k that this must always equal $\underline{\theta}/(1-\underline{\theta})$.

To see this, note that first when $k = 1$, it is trivially true that there are d_j such walks (starting from each of j's neighbors), each with weight $\frac{1}{d_j}$, so that $\sum_h W_{hj}^1 = 1$.

Now assume that for arbitrary $k \geq 2$, $\sum_h W_{hj}^k = 1$. Then for any walk of length $k+1$ with start vertex h, the deletion of h from the walk produces a (not necessarily unique) walk of length k that begins from some h' adjacent to h. In particular, for each walk w_{hj}^k of length k beginning with vertex h there are d_h walks of length $k+1$, each with weight $\frac{1}{d_h} w_{hj}^k$.[18] The sum of all of these walks can then be expressed

$$\sum_{h' \in \mathcal{T}_h(\mathcal{G})} \frac{1}{d_h} w_{ij}^k = w_{ij}^k.$$

But note that this enumeration is exhaustive; that is, there is no walk of length $k+1$ that cannot be constructed in this fashion, and no two distinct w_{ij}^k can be extended to produce the same w_{ij}^{k+1}. Hence, we have that $W^{k+1} = \sum_i w_{ij}^k = 1$ by the induction assumption.

Moreover, this implies that, regardless of n,

$$\begin{aligned}
\sum_h \sum_{k=1}^{\infty} \underline{\theta}^k W_{hj}^k &= \sum_{k=1}^{\infty} \underline{\theta}^k \sum_h W_{hj}^k \\
&= \sum_{k=1}^{\infty} \underline{\theta}^k \\
&= \frac{\underline{\theta}}{1-\underline{\theta}},
\end{aligned}$$

[17] In fact, this inequality is necessarily extremely loose, since $\Delta_{ij} = 1$ would imply that a change in θ_i moves j from never voting for candidate 1 to voting for them with certainty, which cannot be true by construction.

[18] Note that here w_{hj}^k refers to a particular k-length walk from h to j, which should not be confused with w_{hj}, the weight of j's influence on h's voting behavior from other sections of the paper.

which implies

$$\mathcal{I}_{ij} = \frac{1-\underline{\theta}}{\underline{\theta}} \sum_{k=1}^{\infty} \overline{\theta}^k W_{ij}^k.$$

Now observe that, defining by $d(i,j)$ the unweighted shortest path distance from i to j, for any $k < d(i,j)$, there by definition exist no walks from i to j of length k, so that clearly $W_{ij}^k = 0$. Then a trivial lower bound on $\mathcal{I}_{ij}$ is $\overline{\theta}^{d(i,j)} W_{ij}^{d(i,j)}$. Moreover, by definition there exists at least one path of length $d(i,j)$, which has minimal weight if all vertices on the path have the maximum degree on the network. Hence, a uniform lower bound for any pair i,j on $\mathcal{G}$ is given by $\left(\overline{\theta}/\bar{d}_n\right)^{\mathrm{diam}(\mathcal{G})}$. Hence, we have that

$$z_n < \frac{\underline{\theta}}{1-\underline{\theta}} \left(\frac{\bar{d}_n}{\overline{\theta}}\right)^{\mathrm{diam}(\mathcal{G})} = o(n)$$

by assumption. □

Lemma 4. *Suppose that $\mathcal{G}$ is generated according to the stochastic block model and satisfies the assumptions of Theorem 1 and of Lemma 3 almost surely. Then Δ_{ij} converges in probability to* 0 *for all i,j as $n \to \infty$.*

Proof. We aim to bound Δ_{ij} using Lemma 3. Now fix some $\epsilon > 0$. Then it follows from the preceding proof that we want to show that for any i,j, there exists N such that for any $N > n$, the probability that

$$\max_{i,j} \sum_{k=1}^{\infty} \overline{\theta}^k W_{ij}^k > \frac{\underline{\theta}\epsilon}{1-\underline{\theta}} \tag{A2}$$

is arbitrarily small.

Now consider the left-hand side of (A2). Note that it follows from the proof of Lemma 1 that, under the given assumptions, we need only consider the average network, since the sum of walks from i to j is arbitrarily close with probability approaching unity for sufficiently large n. Now note that the expected network, defined as before, is simply a weighted complete graph K_n. Consequently, it is straightforward to enumerate all paths from i to j. We show that for all i,j,k, $\bar{W}_{ij}^K = O(n^{-1})$.

In general, the maximal weight for each connection i, j occurs when $\ell_i = \ell_j$ and for all other h, $\ell_h \neq \ell_i$, so that $w_{ij} < \frac{p_H}{(n-1)p_L}$. Since all walks are at their greatest when all weights are maximal, a (very loose) upper bound for $\bar{W}_{ij}^K$ is the corresponding entry of the k^{th} power of $\frac{p_H}{(n-1)p_L}\left(\mathbf{1}_{n\times n} - \boldsymbol{I}_n\right)$. Now note that

$$\begin{aligned}
\left[\left(\frac{p_H}{(n-1)p_L}\left(\mathbf{1}_{n\times n} - \boldsymbol{I}_n\right)\right)^k\right]_{ij} &= \left[\frac{p_H^k}{(n-1)^k p_L^k}\left(\mathbf{1}_{n\times n} - \boldsymbol{I}_n\right)^k\right]_{ij} \\
&\leq \left[\frac{p_H^k}{(n-1)^k p_L^k}\left(\mathbf{1}_{n\times n}\right)^k\right]_{ij} \\
&= \frac{n^{k-1} p_H^k}{(n-1)^k p_L^k} \\
&= O(n^{-1})
\end{aligned}$$

Hence, $p\text{-lim}_{n\to\infty} \mathcal{I}_{ij} = 0$, and so by Lemma 3, it follows also that $p\text{-lim}_{n\to\infty} \Delta_{ij} = 0$, completing the proof.

□