Coding of bounded solutions of equation $u_{xx} - u + \eta(x)u^3 = 0$ with periodic piecewise constant function $\eta(x)$

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March, 2021

Objective & Motivation

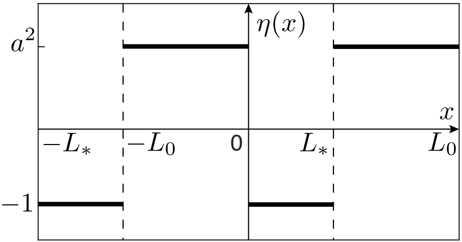
Our objective is an equation

$$u_{xx} - u + \eta(x)u^3 = 0, \qquad (1)$$

$\eta(x)$ is a periodic piecewise-constant function of period $L = L_* + L_0$,

$$\eta(x) = \begin{cases} -1, & x \in [0; L_*]; \\ a^2, & x \in [L_*; L_* + L_0], \end{cases}$$

where $a \in \mathbb{R}$.



Our motivation is a GPE equation:

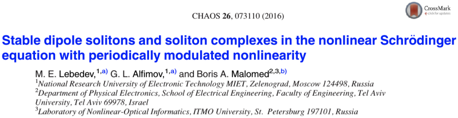
$$i\Psi_t + \Psi_{xx} + P(x)|\Psi|^2\Psi = 0, \qquad (2)$$

$P(x) \in \mathbb{R}$ is a periodic function that changes its sing on the period.

Stationary states equation:

$$u_{xx} + \omega u + P(x)u^3 = 0, \quad \omega < 0.$$

We wrote a paper!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Stable dipole solitons and soliton complexes in the nonlinear Schrödinger equation with periodically modulated nonlinearity

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Many localised stationary states was found (numerically).

Part I.
Common sense

Phase portraits

$\mathcal{P}_*$ mapping:

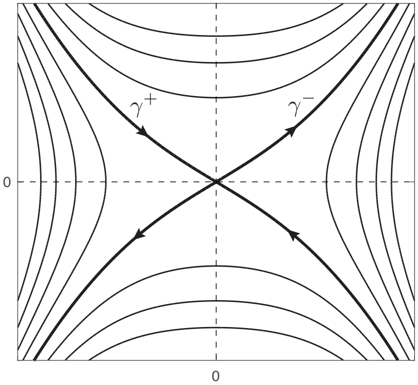


Figure 1: Phase portrait for the equation $u_{xx} - u - u^3 = 0$.

$\mathcal{P}_0$ mapping:

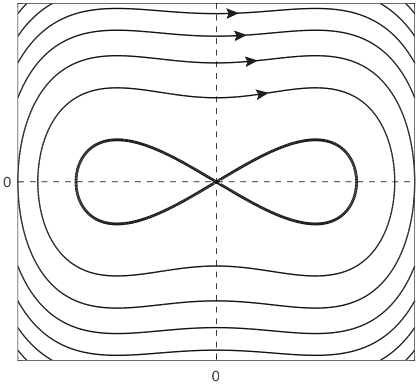


Figure 2: Phase portrait for the equation $u_{xx} - u + u^3 = 0$.

Poincaré map $\mathcal{P}(u_0, u_0') = (u(L), u'(L))$, $u(x)$ is a solution of (1) with initial conditions (u_0, u_0'); $\mathcal{P} = \mathcal{P}_0\mathcal{P}_*$.

Part II.
Poincaré map

$\mathrm{dom}(\mathcal{P}_*) = \mathscr{U}^+_{L_*}$

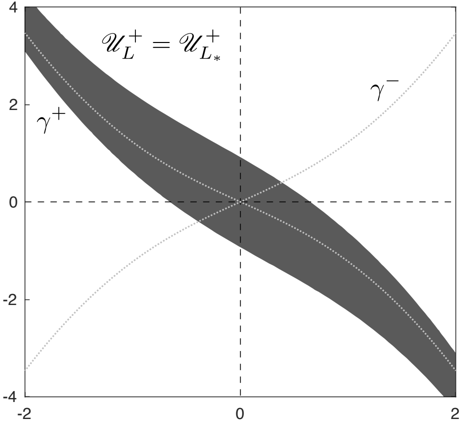


Figure 3: Domain of $\mathcal{P}$ for the parameters $(L_*, L_0, a) = (2, 1, 1)$; $\gamma^\pm$ are separatrices for the equation $u_{xx} - u - u^3 = 0$.

Theorem 1

$\forall L_*$, $0 < L_* < +\infty$, set $\mathscr{U}^+_{L_*}$ is an infinite curvilinear open strip, that

(a) *$\mathscr{U}^+_{L_*}$ is symmetric with respect to the origin and contains γ^+;*

(b) *$\mathscr{U}^+_{L_*}$ is bounded by two symmetric monotonically decreasing curves (which are C^1 functions);*

(c) *vertical dimension of the $\mathscr{U}^+_{L_*}$ tends to zero exponentially when $L_* \to +\infty$.*

$$\mathscr{U}^+_L \equiv \mathrm{dom}(\mathcal{P}) = \mathrm{dom}(\mathcal{P}_0\mathcal{P}_*) = \mathrm{dom}(\mathcal{P}_*) \equiv \mathscr{U}^+_{L_*}.$$

$\mathcal{P}_*(\mathscr{U}_L^+) = \mathscr{U}_{L_*}^-$

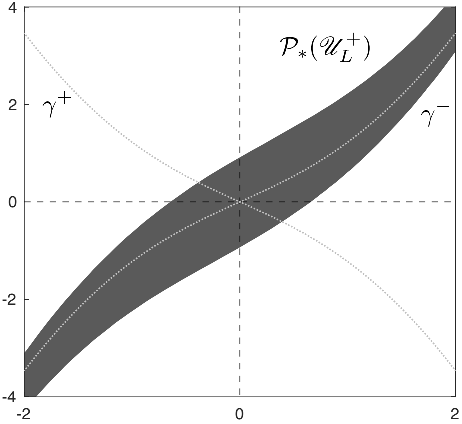


Figure 4: $\mathcal{P}_*$–image of $\mathscr{U}_L^+$ for the parameters $(L_*, L_0, a) = (2, 1, 1)$.

Theorem 2

$\forall L_*$, $0 < L_* < +\infty$, set $\mathscr{U}_{L_*}^-$ is an infinite curvilinear open strip, that

(a) *$\mathscr{U}_{L_*}^-$ is symmetric with respect to the origin and contains γ^-;*

(b) *$\mathscr{U}_{L_*}^-$ is bounded by two symmetric monotonically increasing curves (which are C^1 functions);*

(c) *vertical dimension of the $\mathscr{U}_{L_*}^-$ tends to zero exponentially when $L_* \to +\infty$.*

$$\mathcal{P}_*(\mathscr{U}_L^+) = I\mathscr{U}_L^+,$$

where I is a reflection with respect to the u axis on the phase plane (u, u').

$\mathcal{P}_0(\gamma_-)$

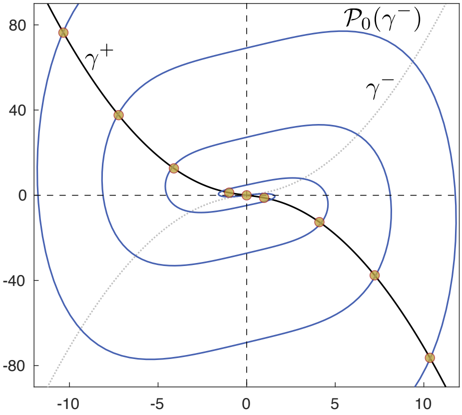


Figure 5: $\mathcal{P}_0(\gamma^-)$ (solid blue line) is an infinite spiral; yellow dots are the points of intersections $\mathcal{P}_0(\gamma^-) \cap \gamma^+$ predicted by the equation (3).

Theorem 3

$\mathcal{P}_0$–image of the curve γ^- is an infinite spiral, it intersects γ^+ infinitely many times at the points $\{0\} \cup \{u_{\pm n}\}$, where

$$u_{\pm n} = \pm \frac{2a^{3/2} x_{n-1}}{\sqrt[4]{a^2+1}} L_0^{-1} + O(L_0); \quad (3)$$

$L_0 \to 0$, and x_n are determined as

$$x_n = cn^{-1}\left(\frac{\sqrt{a}}{\sqrt[4]{a^2+1}}, k\right) + K(k)n,$$

where $k = 1/\sqrt{2}$, $n \in \mathbb{N}$.

Here $\mathrm{K}(\cdot)$ is the complete elliptic integral of the 1st kind, cn^{-1} is an inverse elliptic cosine.

$\mathcal{P}_0(\mathscr{U}_{L_*}^-) = \mathscr{U}_L^-$

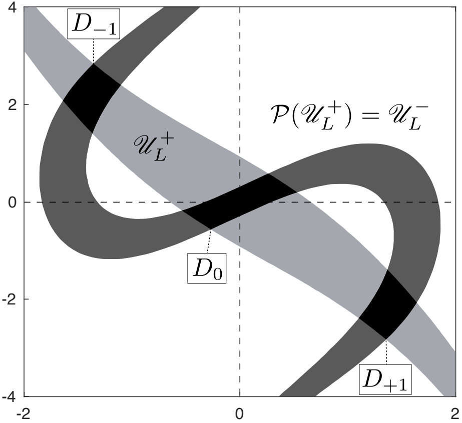


Figure 6: Three components D_{-1}, D_0, D_{+1} (black) of the set $\mathscr{U}_L = \mathscr{U}_L^+ \cap \mathscr{U}_L^-$.

$$\mathscr{U}_L^- \equiv \mathrm{dom}(\mathcal{P}^{-1}) = \mathcal{P}_0(\mathscr{U}_{L_*}^-) = \mathcal{P}(\mathscr{U}_L^-).$$

- Both $\mathcal{P}$ and $\mathcal{P}^{-1}$ are defined on $\mathscr{U}_L = \mathscr{U}_L^+ \cap \mathscr{U}_L^-$.
- $\mathscr{U}_L$ consists of infinite number of components $\mathscr{U}_L = \bigcup_{i \in S} D_i$.
- Each component except of the central one (D_0) is a curvilinear quadrangle with monotonic boarders (*island*).
- D_0 can be made an *island* by varying parameters (L_*, L_0, a).

$\mathcal{P}(\mathscr{U}_L) = \bigcup_{i \in S} \mathcal{P}(D_i)$

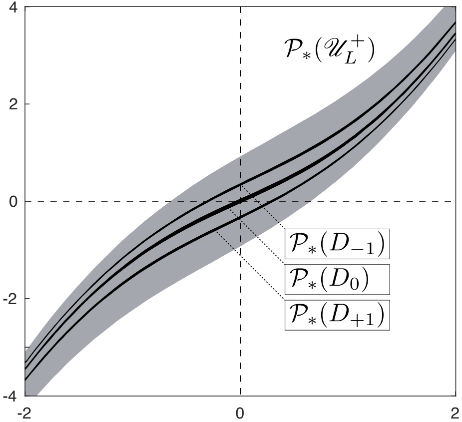


Figure 7: $\mathcal{P}_*$–image of the components D_{-1}, D_0, D_{+1} of $\mathscr{U}_L$.

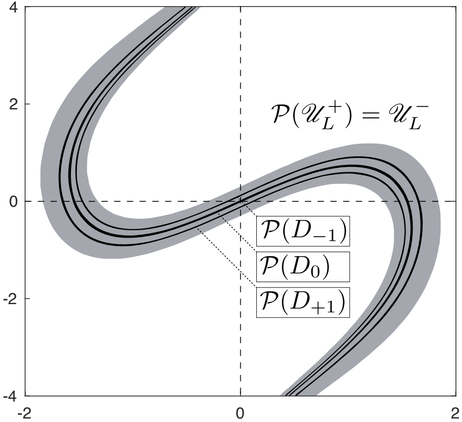


Figure 8: $\mathcal{P}$–image of the components D_{-1}, D_0, D_{+1} of $\mathscr{U}_L$.

$\bigcup_{i\in S}\mathcal{P}(D_i)\cap\mathscr{U}_L$

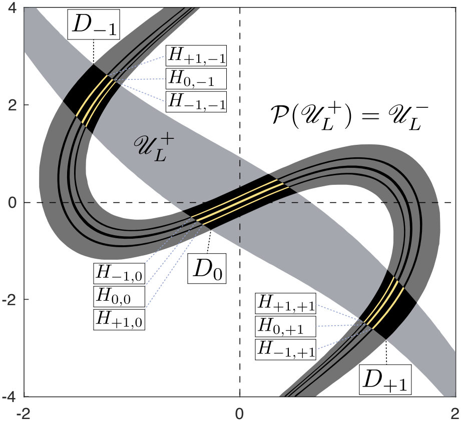


Figure 9: h-strips (yellow) as a result of intersections of $\mathcal{P}(D_i)$ and $\mathscr{U}_L$ sets for the components D_{-1}, D_0, D_{+1}.

$$\forall i, j,\ H_{i,j} = \mathcal{P}(D_i)\cap D_j \neq \varnothing.$$

- We call such sets as h-strips.
- Here h-strips consist of points where both $\mathcal{P}$ and $\mathcal{P}^{-2}$ are defined.
- This process can be continued, one can get points of initial condition where higher order of $\mathcal{P}^{-k}$ are defined.
- Continuation of the process results in sets of nested h-strips.

$\bigcup_{i\in S} \mathcal{P}^{-1}(D_i) \cap \mathscr{U}_L$

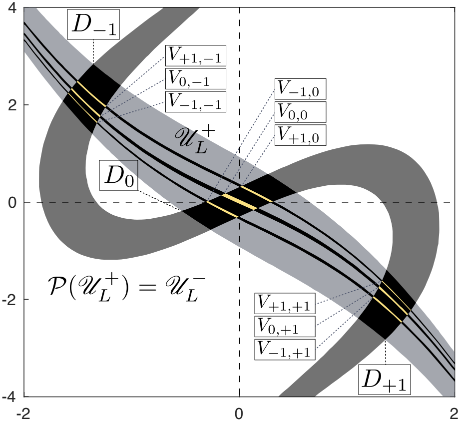


Figure 10: v-strips (yellow) as a result of intersections of $\mathcal{P}^{-1}(D_i)$ and $\mathscr{U}_L$ sets for the components D_{-1}, D_0, D_{+1}.

$$\forall i, j,\ V_{i,j} = \mathcal{P}^{-1}(D_i) \cap D_j \neq \varnothing.$$

- We call such sets as v-strips.
- Here v-strips consist of points where both $\mathcal{P}^2$ and $\mathcal{P}^{-1}$ are defined.
- This process can be continued, one can get points of initial condition where higher order of $\mathcal{P}^k$ are defined.
- Continuation of the process results in sets of nested v-strips.

All together

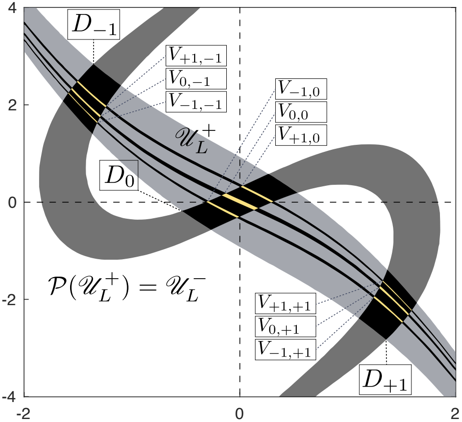


Figure 11: v-strips (yellow) as a result of intersections of $\mathcal{P}^{-1}(D_i)$ and $\mathscr{U}_L$ sets for the components D_{-1}, D_0, D_{+1}.

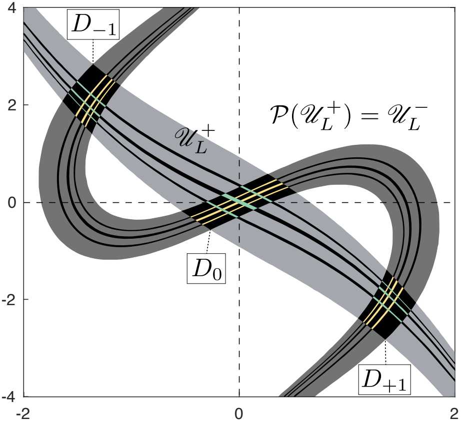


Figure 12: Intersections of h and v-strips (yellow and green) consist of points where both $\mathcal{P}^2$ and $\mathcal{P}^{-2}$ are defined.

Part III.
Solutions coding

$\mathcal{O} \to \mathcal{S}_\infty$

Orbit is a sequence of points $\{p_n\}$, that

$$\mathcal{P}(p_n) = p_{n+1}.$$

$\mathcal{O}$ – set of orbits of *regular* solution for the equation (1).

$\mathcal{S}$ – set of bi-infinite sequences over the infinite alphabet

$$\{\dots, i_{-1}, i_0, i_1, \dots\},$$

where each symbol corresponds to the connected component $D_i \in \mathscr{U}_L$.

It's easy to assign a *code* to the solution!

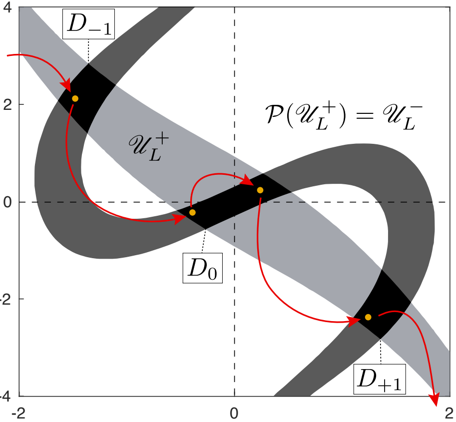


Figure 13: Sketch of an orbit of a *regular* solution (yellow dots) that corresponds to the code sequence $\{\dots, -1, 0, 0, +1, \dots\}$.

Solutions and their codes

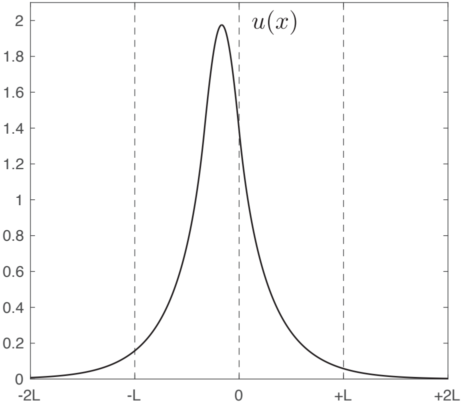


Figure 14: Solution of the code $\{\dots, 0, 0, +1, 0, 0, \dots\}$.

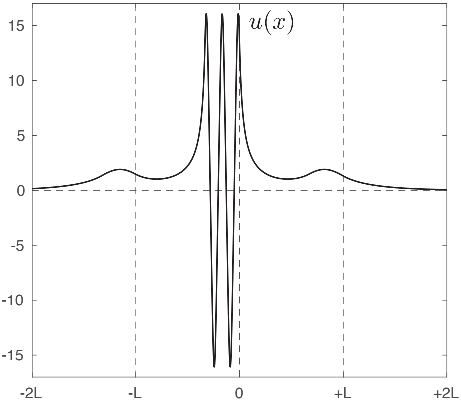


Figure 15: Solution of the code $\{\dots, 0, +1, +3, +1, 0, \dots\}$.

Solutions and their codes

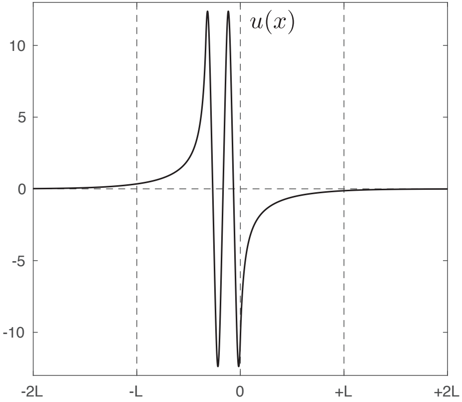


Figure 16: Solution of the code $\{\dots, 0, 0, -3, 0, 0, \dots\}$.

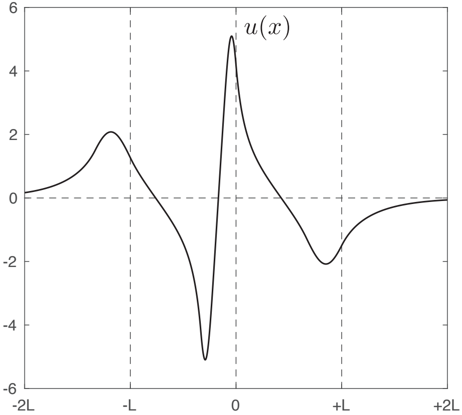


Figure 17: Solution of the code $\{\dots, 0, +1, +2, -1, 0, \dots\}$.

$\mathcal{S}_\infty \to \mathcal{O}$?

By code $\{\dots, i_{-1}, i_0, i_1, \dots\}$ find initial conditions for the equation (1) whose orbit visit components of $\mathscr{U}_L$ in a right order ($X \xrightarrow{\mathcal{P}} Y = \mathcal{P}(X) \cap Y$).

$D_{i_0} \supseteq H_{i_0} = D_{i_0}$;

$D_{i_0} \supseteq H_{i_{-1},i_0} = D_{i_{-1}} \xrightarrow{\mathcal{P}} D_{i_0}$;

$D_{i_0} \supseteq H_{i_{-2},i_{-1},i_0} = D_{i_{-2}} \xrightarrow{\mathcal{P}} D_{i_{-1}} \xrightarrow{\mathcal{P}} D_{i_0}$;

$\dots.$

Nested h-strips $\{H_n\}$:

$$\dots \subseteq H_{i_{-2},i_{-1},i_0} \subseteq H_{i_{-1},i_0} \subseteq H_{i_0} = D_{i_0}.$$

$$\{H_n\} \xrightarrow{n\to\infty} h?$$

$D_{i_0} \supseteq V_{i_0} = D_{i_0}$;

$D_{i_0} \supseteq V_{i_1,i_0} = D_{i_1} \xrightarrow{\mathcal{P}^{-1}} D_{i_0}$;

$D_{i_0} \supseteq V_{i_2,i_1,i_0} = D_{i_2} \xrightarrow{\mathcal{P}^{-1}} D_{i_1} \xrightarrow{\mathcal{P}^{-1}} D_{i_0}$;

$\dots.$

Nested v-strips $\{V_n\}$:

$$\dots \subseteq V_{i_2,i_1,i_0} \subseteq V_{i_1,i_0} \subseteq V_{i_0} = D_{i_0}$$

$$\{V_n\} \xrightarrow{n\to\infty} v?$$

Initial conditions in D_{i_0} should belongs to the intersection $h \cap v$.

Part IV.
Uniqueness

What is $h \cap v$?

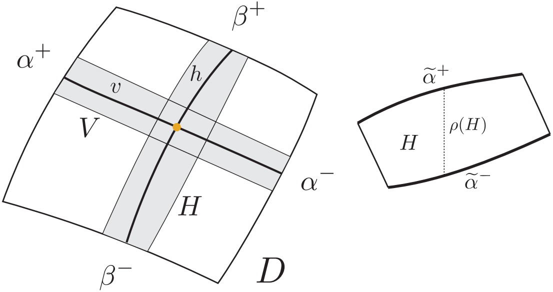
Theorem 4

Let

- *all* $\{H_n\}$ *have monotone increasing / decreasing borders which are graphs of* γ*-Lipschitz functions, and* $\rho(H_{n+1}) \le (1/\mu)\rho(H_n)$, $\mu > 1$;
- *all* $\{V_n\}$ *have monotone decreasing / increasing borders which are graphs of* γ*-Lipschitz functions, and* $\rho(V_{n+1}) \le (1/\nu)\rho(V_n)$, $\nu > 1$;

then the intersection $h \cap v$ *consists of just one point!*

Here $\rho(\cdot)$ is a vertical (for H_n) or horizontal (for V_n) width of the strips.



$D\mathcal{P}_p$, $D\mathcal{P}_p^{-1}$

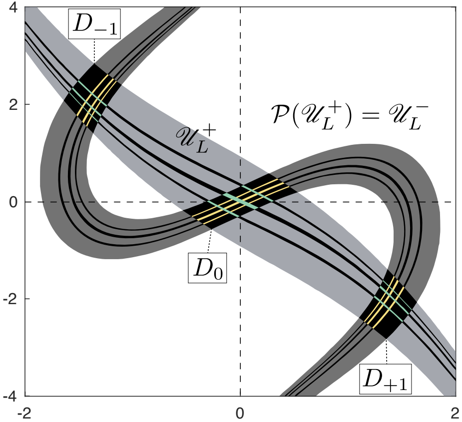


Figure 18: $H_{i,j}$ (yellow) and $V_{i,j}$ (green) are the points of interest.

$D\mathcal{P}_p$ – linearisation of the map $\mathcal{P}$ at the point p; $D\mathcal{P}_p^{-1}$ – its inverse.

If linear operators $D\mathcal{P}_p$, $D\mathcal{P}_p^{-1}$ satisfy some restrictions then the conditions of the above mentioned theorem are valid!

Theorem about h-strips mapping

Theorem 5 (About h-strips mapping)

Let Poincaré map $\mathcal{P}$ and its inverse $\mathcal{P}^{-1}$ are defined on an island set $\bigcup_{i\in S} D_i$, where S – set of indices, $\forall i,j \in S$ set $V_{j,i} = \mathcal{P}^{-1}D_j \cap D_i$ is non-empty, $\mathcal{P}$ is defined on the closure $\overline{V_{j,i}}$, and one the following conditions held:

(1) *borders $\alpha_i^{\pm}$ of an island D_i are increasing curves, $\forall p \in \overline{V_{j,i}}$ signs of the operator $D\mathcal{P}_p = (a_{mn})$ have exactly one of the following configurations:*

$$(a)\ \begin{pmatrix} + & + \\ + & + \end{pmatrix},\quad (b)\ \begin{pmatrix} - & - \\ - & - \end{pmatrix},\quad (c)\ \begin{pmatrix} + & + \\ - & - \end{pmatrix},\quad (d)\ \begin{pmatrix} - & - \\ + & + \end{pmatrix};$$

borders $\alpha_j^{\pm}$ of D_j are increasing for (a), (b), and decreasing for (c), (d);

(2) *borders $\alpha_i^{\pm}$ of an island D_i are decreasing curves, $\forall p \in \overline{V_{j,i}}$ signs of the operator $D\mathcal{P}_p = (a_{mn})$ have exactly one of the following configurations:*

$$(a)\ \begin{pmatrix} + & - \\ - & + \end{pmatrix},\quad (b)\ \begin{pmatrix} - & + \\ + & - \end{pmatrix},\quad (c)\ \begin{pmatrix} + & - \\ + & - \end{pmatrix},\quad (d)\ \begin{pmatrix} - & + \\ - & + \end{pmatrix};$$

borders $\alpha_j^{\pm}$ of D_j are decreasing for (a), (b), and increasing for (c), (d);

and moreover $\exists \mu > 1$ such that $\forall p \in \overline{V_{j,i}}$, $|a_{11}| \geq \mu$, then for any monotone h-strip $H \in D_i$, $\forall j \in S$, $\mathcal{P}H \cap D_j = \widetilde{H}_j$ is also a monotone h-strip, and $\rho(\widetilde{H}_j) \leq (1/\mu)\rho(H)$.

Theorem about v-strips mapping

Theorem 6 (About v-strips mapping)

Let Poincaré map $\mathcal{P}$ and its inverse $\mathcal{P}^{-1}$ are defined on an island set $\bigcup_{i \in S} D_i$, where S – set of indices, $\forall i, j \in S$ set $H_{i,j} = \mathcal{P}D_i \cap D_j$ is non-empty, $\mathcal{P}^{-1}$ is defined on the closure $\overline{H_{i,j}}$, and one the following conditions held:

(1) *borders $\beta_i^{\pm}$ of an island D_i are increasing curves, $\forall p \in \overline{H_{i,j}}$ signs of the operator $D\mathcal{P}_0^{-1} = (b_{mn})$ have exactly one of the following configurations:*

$$(a)\ \begin{pmatrix} + & + \\ + & + \end{pmatrix}, \quad (b)\ \begin{pmatrix} - & - \\ - & - \end{pmatrix}, \quad (c)\ \begin{pmatrix} + & + \\ - & - \end{pmatrix}, \quad (d)\ \begin{pmatrix} - & - \\ + & + \end{pmatrix};$$

borders $\beta_j^{\pm}$ of D_j are increasing for (a), (b), and decreasing for (c), (d);

(2) *borders $\beta_i^{\pm}$ of an island D_i are decreasing curves, $\forall p \in \overline{H_{i,j}}$ signs of the operator $D\mathcal{P}_0^{-1} = (b_{mn})$ have exactly one of the following configurations:*

$$(a)\ \begin{pmatrix} + & - \\ - & + \end{pmatrix}, \quad (b)\ \begin{pmatrix} - & + \\ + & - \end{pmatrix}, \quad (c)\ \begin{pmatrix} + & - \\ + & - \end{pmatrix}, \quad (d)\ \begin{pmatrix} - & + \\ - & + \end{pmatrix};$$

borders $\beta_j^{\pm}$ of D_j are decreasing for (a), (b), and increasing for (c), (d);

and moreover $\exists \nu > 1$ such that $\forall p \in \overline{H_{i,j}}$, $|b_{22}| \geq \nu$, then for any monotone v-strip $V \in D_j$, $\forall i \in S$, $\mathcal{P}^{-1}V \cap D_i = \widetilde{V}_i$ is also a monotone v-полоса, and $\rho(\widetilde{V}_i) \leq (1/\nu)\rho(V)$.

Proof in an asymptotic limit

Denote by $\mathcal{S}(b)$, $b \in \mathbb{R}$, a set of solutions for equation (1) such that $|u(x)| < b$ on the whole real axis $\mathbb{R}$.

Denote by Ω_n the set of bi-infinite sequences $\{\ldots, i_{-1}, i_0, i_1, \ldots\}$ where i_k, $k = 0, \pm 1, \ldots$, is an integer, $-n \le i_k \le n$.

Theorem 7

$\forall N$ there exists a pair $(\widetilde{L}_*, \widetilde{L}_0)$ such that for any pair (L_*, L_0), $L_* > \widetilde{L}_*$ and $0 < L_0 < \widetilde{L}_0$, there exist a sequence $b_0 < b_1 < \ldots < b_N$, and a homeomorphism T such that $T\mathcal{S}(b_n) = \Omega_n$, $n = 0, 1, \ldots, N$.

Thanks for your attention!