Coding of bounded solutions of equation $u_{xx} - u + \eta(x)u^3 = 0$ with periodic piecewise constant function $\eta(x)$

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Consider a one-dimensional second-order differential equation

$$u_{xx} - u + \eta(x)u^3 = 0, \tag{1}$$

where $\eta(x)$ is a periodic piecewise-constant function of period $L + \ell$,

$$\eta(x) = \begin{cases} -1, & x \in [0; L]; \\ \xi, & x \in [L; L + \ell], \end{cases} \tag{2}$$

where $\xi > 0$. Let us define two topological spaces.

At first, denote by $\mathcal{S}(b)$, $b \in \mathbb{R}$, a set of solutions for equation (1) such that $|u(x)| < b$ on the whole real axis $\mathbb{R}$. Evidently, $b_1 < b_2$ implies $\mathcal{S}(b_1) \subseteq \mathcal{S}(b_2)$. One can define a metric ρ in $\mathcal{S}(b)$ as follows,

$$\rho(v, w) = \sqrt{(v(0) - w(0))^2 + (v_x(0) - w_x(0))^2}, \quad v(x),\, w(x) \in \mathcal{S}(b). \tag{3}$$

This implies that $\mathcal{S}(b)$ can be regarded as topological space where neighbourhood $U_\varepsilon(u)$ of an element $u \in \mathcal{S}(b)$ is defined as $U_\varepsilon(u) = \{v |\ \rho(u, v) < \varepsilon\}$.

At second, denote by Ω_n the set of bi-infinite sequences $\{\ldots, i_{-1}, i_0, i_1, \ldots\}$ where i_k, $k = 0, \pm 1, \ldots$, is an integer, $-n \le i_k \le n$. Evidently that for $n_1 < n_2$ one has $\Omega_{n_1} \subset \Omega_{n_2}$. The set Ω_n can be regarded as topological space where neighbourhood $W_k(\omega^*)$ of an element $\omega^* = \{\ldots, i^*_{-1}, i^*_0, i^*_1, \ldots\} \in \Omega_n$ is defined as $W_k(\omega^*) = \{\omega |\ i^*_s = i_s,\ |s| < k\}$.

The main result of our study is the following theorem.

Theorem. *For any N there exists a pair (L_0, ℓ_0) such that for any pair (L, ℓ), $L > L_0$ and $0 < \ell < \ell_0$, there exist a sequence*

$$b_0 < b_1 < \ldots < b_N,$$

and a homeomorphism T such that $T\mathcal{S}(b_n) = \Omega_n$, $n = 0, 1, \ldots, N$.

The theorem can be illustrated by the following diagram:

$$\begin{array}{ccccccc} \mathcal{S}(b_0) & \subset & \mathcal{S}(b_1) & \subset & \cdots & \subset & \mathcal{S}(b_N) \\ \downarrow T & & \downarrow T & & & & \downarrow T \\ \Omega_0 & \subset & \Omega_1 & \subset & \cdots & \subset & \Omega_N \end{array} \tag{4}$$

The theorem is proved for ξ below a threshold ξ_0 that is a root of some transcendent equation.