EMBEDDED SUBMANIFOLDS

We show that each embedded submanifold has a unique smooth structure, which is induced by the slice charts.

In what follows, let M and N be smooth manifolds of dimension m and n, respectively.

We say a smooth map $F : M \to N$ is a *smooth embedding* if

(i) the restriction $F : M \to F(M)$ is a homeomorphism, where $F(M)$ is equipped with the subspace topology.

(ii) The differential dF_p is injective at each $p \in M$.

It is straightforward to check that the composition of smooth embeddings is again a smooth embedding.

Definition. Let $S \subseteq M$, and suppose S is a smooth manifold. We say that S is an *embedded submanifold of* M if the inclusion map $\iota : S \hookrightarrow M$ is a smooth embedding.

An immediate consequence of this definition is that the topology on S must be the subspace topology induced from M.

The following proposition shows that embedded submanifolds are precisely the images of embeddings.

Proposition. *Let $F : M \to N$ be a smooth embedding, and let $S := F(M)$. Then S is a topological manifold when equipped with the subspace topology. Further, there is a unique smooth structure which makes S into an embedded submanifold and the following map into a diffeomorphism:*

$$\widetilde{F} : M \to S, \quad p \mapsto F(p).$$

Proof. Since $\widetilde{F}$ is a homeomorphism, we know S is a topological manifold.

Next, let $\mathcal{A}$ be the smooth structure for M. Given $(U, \varphi) \in \mathcal{A}$, define $\widetilde{U} := F(U)$, and define $\widetilde{\varphi} : \widetilde{U} \to \varphi(U)$. by the following diagram:

$$\begin{tikzcd} U \arrow[r, "\widetilde{F}"] \arrow[dr, "\varphi"'] & \widetilde{U} \arrow[d, "\widetilde{\varphi}"] \\ & \varphi(U) \end{tikzcd}$$

It is straightforward to show that $\widetilde{\mathcal{A}} := \{(\widetilde{U}, \widetilde{\varphi}) \mid (U, \varphi) \in \mathcal{A}\}$ is a smooth atlas for S. Furthermore, $\widetilde{F}$ is a diffeomorphism, because $\widetilde{\varphi} \circ \widetilde{F} \circ \varphi^{-1}$ and $\varphi \circ \widetilde{F}^{-1} \circ \widetilde{\varphi}$ are identity maps.

Finally, the following diagram tells us that S is an embedded submanifold, because the inclusion map is a composition of smooth embeddings:

$$\begin{tikzcd} M \arrow[r, "\widetilde{F}"] \arrow[dr, "F"'] & S \arrow[d, hook, "\iota"] \\ & N \end{tikzcd}$$

Uniqueness is straightforward to verify. $\square$

Let (U, φ) be a smooth chart for M. Let A be a subset of U. We say that A is a *k-slice of* (U, φ) if we can write

$$\varphi(A) = \Big\{(x^1, \ldots, x^k, 0, \ldots, 0) \in \varphi(U)\Big\}.$$

Let S be a subset of M. We say that a chart (U, φ) for M is a *k-slice chart for S in M* if $S \cap U$ is a k-slice of (U, φ). We say S satisfies the *k-slice condition* if every point in S is contained in a slice chart.

The following two propositions show that embedded submanifolds are precisely the subsets which satisfy the slice condition.

Proposition. *Let $S \subseteq M$ be an embedded submanifold of dimension k. Then S satisfies the k-slice condition.*

Proof. Fix $p \in S$. Since $\iota : S \to M$ is an embedding, the Rank Theorem tells us that there exist charts (V, ψ) in S centred at p and (U_0, φ) in M centred at p with $V \subseteq U_0$ such that $\widehat{\iota}$ is given by

$$(x^1, \ldots, x^k) \mapsto (x^1, \ldots, x^k, 0, \ldots, 0),$$

where $\widehat{\iota}$ is defined by the following diagram:

$$\begin{tikzcd} V \arrow[r, hook, "\iota"] \arrow[d, "\varphi"'] & U_0 \arrow[d, "\psi"] \\ \psi(V) \arrow[r, "\widehat{\iota}"'] & \varphi(U_0) \end{tikzcd}$$

Without loss of generality, assume that there exists $\varepsilon > 0$ such that $\psi(V)$ and $\varphi(U_0)$ are balls of radius ε centred at the origin in their respective spaces. (If not, we can restrict the charts so that these conditions are met.)

We can write $V = S \cap W$, where W is open in M. Define $U := W \cap U_0$. It is straighforward to check that $S \cap U = V$ is a k-slice of $(U, \varphi|_U)$. $\square$

Proposition. *Let S be a subset of M satisfying the k-slice condition. Then S is a topological manifold of dimension k when equipped with the subspace topology. Furthermore, there exists a smooth structure which makes S into an embedded submanifold.*

Proof. We know that S is Hausdorff and second-countable.

Let (U, φ) be a slice chart for S. Let $\widetilde{U} := S \cap U$, which is a slice of (U, φ). Let $\pi : \mathbb{R}^m \to \mathbb{R}^k$ denote the projection onto the first k coordinates. Let $\widetilde{\varphi}(\widetilde{U}) := \pi(\varphi(U))$, and define $\widetilde{\varphi}$ by the following diagram:

$$\begin{tikzcd} \widetilde{U} \arrow[d, "\varphi"'] \arrow[dr, "\widetilde{\varphi}"] & \\ \varphi(\widetilde{U}) \arrow[r, "\pi"'] & \widetilde{\varphi}(\widetilde{U}) \end{tikzcd}$$

We know that $\widetilde{U}$ is open in S by definition, and $\widetilde{\varphi}(\widetilde{U})$ is open because φ and π are open maps. Observe that

$$i : (x^1, \ldots, x^k) \mapsto (x^1, \ldots, x^k, 0, \ldots, 0)$$

is the inverse of π. Therefore, $\widetilde{\varphi}$ and its inverse are compositions of continuous maps. This shows that S is locally Euclidean.

Let us show that

$$\left\{ (\widetilde{U}, \widetilde{\varphi}) \;\middle|\; (U, \varphi) \text{ is a slice chart for } S \right\}$$

is a smooth atlas for S. Let $(\widetilde{U}, \widetilde{\varphi})$ and $(\widetilde{V}, \widetilde{\psi})$ be two such charts. Then the following diagram commutes:

$$\begin{tikzcd}
 & & & \widetilde{U} \cap \widetilde{V} \arrow[dlll, "\widetilde{\varphi}"'] \arrow[dl, "\varphi"] \arrow[dr, "\psi"'] \arrow[drrr, "\widetilde{\psi}"] & & & \\
\widetilde{\varphi}(\widetilde{U} \cap \widetilde{V}) \arrow[rr, "i"'] & & \varphi(\widetilde{U} \cap \widetilde{V}) \arrow[rr, "\psi \circ \varphi^{-1}"'] & & \psi(\widetilde{U} \cap \widetilde{V}) \arrow[rr, "\pi"'] & & \widetilde{\psi}(\widetilde{U} \cap \widetilde{V})
\end{tikzcd}$$

Thus, $\widetilde{\psi} \circ \widetilde{\varphi}^{-1}$ is a composition of smooth maps.

Finally, let us show that $\iota : S \hookrightarrow M$ is a smooth embedding. Because S is equipped with the subspace topology, we know $\iota : S \to S$ is a homeomorphism. Now, let (U, φ) be a slice chart, and let $(\widetilde{U}, \widetilde{\varphi})$ be the corresponding chart for S. We find the following diagram commutes:

$$\begin{tikzcd}
\widetilde{U} \arrow[r, "\iota"] \arrow[d, "\widetilde{\varphi}"'] & U \arrow[d, "\varphi"] \\
\widetilde{\varphi}(\widetilde{U}) \arrow[r, "i"'] & \varphi(U)
\end{tikzcd}$$

Thus, ι is smooth, and its differential is injective at each point. $\square$

If $S \subseteq M$ is an embedded submanifold, then the set S satisfies the slice condition. The slice charts generate a smooth structure. The following proposition shows that this generated smooth structure is the same as the original smooth structure on S.

Proposition. *Let $S \subseteq M$ be an embedded submanifold with smooth structure $\mathcal{A}$. Let $\widetilde{\mathcal{A}}$ be the smooth structure generated by charts of the form $(\widetilde{U}, \widetilde{\varphi})$ defined in the proof of the previous proposition. Then $\mathcal{A} = \widetilde{\mathcal{A}}$.*

Proof. Fix $(\widetilde{U}, \widetilde{\varphi}) \in \widetilde{\mathcal{A}}$. It suffices to show that $(\widetilde{U}, \widetilde{\varphi})$ is smoothly compatible with $\mathcal{A}$. We can do this by showing that $\widetilde{\varphi}$ is diffeomorphism with respect to $\mathcal{A}$. First, $\widetilde{\varphi}$ is smooth respect to $\mathcal{A}$, because it is the composition of the following smooth maps:

$$\widetilde{U} \xrightarrow{\iota} U \xrightarrow{\varphi} \mathbb{R}^m \xrightarrow{\pi} \mathbb{R}^k.$$

Now, fix $p \in \widetilde{U}$. Let i be the map given in the previous proposition. Observe that the following diagram commutes:

$$\begin{tikzcd}
T_pS \arrow[r, "d\iota_p"] \arrow[d, "d\widetilde{\varphi}_p"'] & T_pM \arrow[d, "d\varphi_p"] \\
T_{\widetilde{\varphi}(p)}\mathbb{R}^k \arrow[r, "di_{\widetilde{\varphi}(p)}"'] & T_{\varphi(p)}\mathbb{R}^m
\end{tikzcd}$$

Now, $d\varphi_p \circ d\iota_p$ is injective, since ι is an embedding. It follows that $d\widetilde{\varphi}_p$ must also be injective, so it is an isomorphism. It follows by the Inverse Function Theorem that $\widetilde{\varphi}$ is a diffeomorphism with respect to $\mathcal{A}$. $\square$

Thus, the smooth structure of each embedded submanifold is unique, and is induced by the slice charts.