# CONNECTIONS ON TENSOR BUNDLES

These are notes I have taken from Lee RM. In what follows, Let $M$ be a smooth manifold, and let $\nabla$ be a connection on $TM$. There is a canonical way we extend $\nabla$ to every tensor bundle:

**Definition.** On each tensor bundle $T^{(k,l)}TM$, we define

$$\nabla^{(k,l)} : \mathfrak{X}(M) \times \Gamma(T^{(k,l)}TM) \to \Gamma(T^{(k,l)}TM)$$

by the following. On $T^*M$, we set

$$\left(\nabla_X^{(0,1)}\omega\right)(Y) := X(\omega(Y)) - \omega(\nabla_X Y),$$

and on $T^{(k,l)}TM$, we set

$$\begin{aligned}\left(\nabla_X^{(k,l)}F\right)(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l) = X(F(\omega^1,\dots,\omega^k,Y_1\dots,Y_l))\\ - \sum_{i=1}^k F(\omega^1,\dots,\nabla_X\omega^i,\dots,\omega^k,Y_1,\dots,Y_l)\\ - \sum_{j=1}^l F(\omega^1,\dots,\dots,\omega^k,Y_1,\dots,\nabla_X Y_j,\dots,Y_l).\end{aligned}$$

It is straightforward to verify that each $\nabla_X^{(k,l)}F$ is indeed a $(k,l)$-tensor field, and that each $\nabla^{(k,l)}$ is a connection.

The following proposition shows how to compute $\nabla^{(k,l)}$ in coordinates.

**Proposition.** *Let $(U,(x^i))$ be a chart. Then*

$$\nabla_X^{(0,1)}\omega = X\omega_k dx^k - X^j\omega_i\Gamma_{jk}^i dx^k,$$

*and*

$$\begin{aligned}\nabla_X^{(k,l)}F = \left(XF_{j_1\cdots j_l}^{i_1\cdots i_k} + \sum_{s=1}^k X^m F_{j_1\cdots j_l}^{i_1\cdots p\cdots i_k}\Gamma_{mp}^{i_s} - \sum_{s=1}^l X^m F_{j_1\cdots p\cdots j_l}^{i_1\cdots i_k}\Gamma_{mj_s}^p\right)\times\\ \partial_{i_1}\otimes\cdots\otimes\partial_{i_k}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_l}.\end{aligned}$$

*Proof.* For the first formula, evaluating both sides at $Y$ gives $Y^iX\omega_i - \omega_iX^jY^l\Gamma_{jl}^i$. I didn't have time to prove the second formula, but I think the approach is the same. □

The following proposition shows why the connections defined above are "natural".

**Proposition.** *The connections $\nabla^{(k,l)}$ satisfy the following properties:*

(i) $\nabla^{(1,0)}$ *is equal to the original connection* $\nabla$.
(ii) $\nabla^{(0,0)}$ *is just differentiation of functions:* $\nabla_X^{(0,0)}f = Xf$.
(iii) *We have a product rule for tensor products:*

$$\nabla_X(F\otimes G) = \nabla_X F\otimes G + F\otimes\nabla_X G.$$

(iv) $\nabla$ *commutes with contraction:* $\nabla_X(\operatorname{tr} F) = \operatorname{tr}(\nabla_X F)$.

(v) *We have a product rule for the natural pairing:*

$$\nabla_X \langle \omega, Y \rangle = \langle \nabla_X \omega, Y \rangle + \langle \omega, \nabla_X Y \rangle.$$

*Proof.* The only property that I found difficult to prove is (iv). I worked it out by writing $\nabla_X(\operatorname{tr} F)$ and $\operatorname{tr}(\nabla_X F)$ in coordinates using the formula given in the previous proposition. $\square$

Given a $(k,l)$-tensor field $F \in \Gamma(T^{(k,l)}TM)$, we define the *total covariant derivative of* $F$ as the $(k, l+1)$-tensor field given by

$$\nabla F : \underbrace{\mathfrak{X}^*(M) \times \cdots \times \mathfrak{X}^*(M)}_{k\text{-copies}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l+1\text{-copies}} \to C^\infty(M),$$

$$(\nabla F)(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l, X) := (\nabla_X F)(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l).$$

The map $\nabla F$ is a $(k, l+1)$-tensor field because it is $C^\infty(M)$-multilinear.