EXISTENCE OF RIEMANNIAN METRICS

Let M be a smooth manifold. A *A Riemannian metric* is a smooth $(0, 2)$-tensor field $g \in \mathcal{T}^2(M)$ such that each g_p is an inner product.

Proposition. *Let M be a smooth manifold. Then M has a Riemannian metric.*

Proof. Let $\mathcal{A} = \{(U_\alpha, (x^i_\alpha))\}$ be the smooth structure for M. For each α, define

$$g_\alpha := \delta_{ij} dx^i_\alpha \otimes dx^j_\alpha,$$

which is a metric on U. Next, let $\{\psi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, and define $g : M \to T^{(0,2)}TM$ by

$$g := \sum \psi_\alpha g_\alpha.$$

Around every point in M, there is a neighbourhood where the sum above is finite. Thus, g is smooth. Now, fix $p \in M$. It is easy to see that each g_p is bilinear and symmetric. There exists some α_0 such that $\psi_{\alpha_0}(p) > 0$. Therefore, given non-zero $v \in T_pM$, we have

$$g_p(v, v) = \sum \psi_\alpha(p) g_\alpha|_p(v, v) \geq \psi_{\alpha_0}(p) g_{\alpha_0}|_p(v, v) > 0. \qquad \square$$