TENSOR FIELDS

We prove a result regarding characterisations for smoothness of a tensor field. In what follows, let M be a smooth manifold of dimension n.

Definition. We define the (k,l)-*tensor bundle* to be the set

$$T^{(k,l)}TM := \coprod_{p\in M} T^{(k,l)}(T_pM) = \Big\{(p, A_p) \,\Big|\, p \in M, A_p \in T^{(k,l)}(T_pM)\Big\}.$$

Let $\pi : T^{(k,l)}TM \to M$ denote the projection map $(p, A_p) \mapsto p$.

The charts on $T^{(k,l)}TM$ are constructed in the following way: Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ be the smooth structure for M. For each α, we define $\widetilde{U}_\alpha := \pi^{-1}(U_\alpha)$, and we define $\widetilde{\pi}_\alpha : \widetilde{U}_\alpha \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^{n^{k+l}}$ by

$$(p, A^{i_1\cdots i_k}_{j_1\cdots j_l}\partial_{i_1} \otimes \cdots \partial_{i_k} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l}|_p) \mapsto \left(\varphi_\alpha(p), (A^{i_1\cdots i_k}_{j_1\cdots j_l})\right).$$

Proposition. *There exists a unique topology for $T^{(k,l)}TM$ which makes $T^{(k,l)}TM$ an $(n + n^{k+l})$-dimensional topological manifold, with $\widetilde{\mathcal{A}} = \{(\widetilde{U}_\alpha, \widetilde{\varphi}_\alpha)\}$ as a smooth atlas.*

We have already proven the result above for the tangent bundle. The proof for the general case is analogous.

Definition. A *rough (k,l)-tensor field* is a map $A : M \to T^{(k,l)}TM$ such that $\pi \circ A = \mathrm{Id}_M$.

We denote the space of smooth (k,l)-tensor fields by $\Gamma(T^{(k,l)}TM)$. We denote the spaces of smooth vector and covector fields by $\mathfrak{X}(M)$ and $\mathfrak{X}^*(M)$, respectively.

Proposition. *Let $A : M \to T^{(k,l)}TM$ be a rough tensor field. Then the following are equivalent:*

(a) A is smooth.
(b) In every chart, the component functions of A are smooth.
(c) Each point in M is contained in a chart where the component functions of A are smooth.
(d) For every $\omega^1, \dots, \omega^k \in \mathfrak{X}^*(M)$ and $X_1, \dots, X_l \in \mathfrak{X}(M)$, the map

$$f := A(\omega^1, \dots, \omega^k, X_1, \dots X_l) : M \to \mathbb{R}$$

is smooth.
(e) For each open $U \subseteq M$, if $\omega^1, \dots, \omega^k : U \to T^*M$ and $X_1, \dots, X_l : U \to TM$ are smooth covector and vector fields, respectively, then the map

$$g := A(\omega^1, \dots, \omega^k, X_1, \dots X_l) : U \to \mathbb{R}$$

is smooth.

Proof. We prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$, then $(a) \Rightarrow (d) \Rightarrow (e) \Rightarrow (b)$.

To prove $(a) \Rightarrow (b)$, suppose A is smooth. Let (U, φ) be a chart for M. Let $(\widetilde{U}, \widetilde{\varphi})$ denote the corresponding chart for $T^{(k,l)}TM$. Then observe that we can write

$$A^{i_1 \cdots i_k}_{j_1 \cdots j_l} = \Pi \circ \widetilde{\varphi} \circ A,$$

where $\Pi : \mathbb{R}^{n+n^{k+l}} \to \mathbb{R}$ is the projection map onto the corresponding coordinate. Thus, each component function is a composition of smooth maps.

It is immediate that $(b) \Rightarrow (c)$.

To prove $(c) \Rightarrow (a)$, suppose each point in M is contained in a chart where the component functions of A are smooth. Let $p \in M$. Then there exists a chart (U, φ) for M containing p such that the induced component functions $A^{i_1 \cdots i_k}_{j_1 \cdots j_l}$ are smooth. Let $(\widetilde{U}, \widetilde{\varphi})$ denote the corresponding chart for $T^{(k,l)}TM$. Then

$$\widetilde{\varphi} \circ A \circ \varphi^{-1} = \left(\mathrm{Id}_{\mathbb{R}^n}, (A^{i_1 \cdots i_k}_{j_1 \cdots j_l} \circ \varphi^{-1})\right),$$

which is smooth.

To prove $(b) \Rightarrow (d)$, suppose the component functions of A induced by any chart are smooth. Fix $\omega^1, \ldots, \omega^k \in \mathfrak{X}^*(M)$ and $X_1, \ldots, X_l \in \mathfrak{X}(M)$. Fix $p \in M$, and let (U, φ) be a chart containing p. Let us write $\omega^r = \omega^i_{i_r} dx^{i_r}$ and $X_s = X^{j_s}_s \partial_{j_s}$. Then

$$f = \omega^1_{i_1} \cdots \omega^k_{i_k} X^{j_1}_1 \cdots X^{j_l}_l A^{i_1 \cdots i_k}_{j_1 \cdots j_l}.$$

Thus, f is smooth on a neighbourhood around every point in its domain.

To prove $(d) \Rightarrow (e)$, suppose f in the proposition is smooth for any choice of smooth covector and vector fields on M. Let $U \subseteq M$ be open, and fix smooth covector and vector fields

$$\omega^1, \ldots, \omega^k : U \to T^*M \quad \text{and} \quad X_1, \ldots, X_l : U \to TM.$$

Fix $p \in U$. We can write $p \in B \subseteq \overline{B} \subseteq U$ for some open $B \subseteq M$. Let $\widetilde{\omega}^1, \ldots, \widetilde{\omega}^k \in \mathfrak{X}^*(M)$ and $\widetilde{X}_1, \ldots, \widetilde{X}_l \in \mathfrak{X}(M)$ be fields such that $\widetilde{\omega}^i|_B = \omega^i|_B$ and $\widetilde{X}_j|_B = X_j|_B$. Then

$$g = A(\widetilde{\omega}^1, \ldots, \widetilde{\omega}^k, \widetilde{X}_1, \ldots, \widetilde{X}_l) \quad \text{on } B,$$

which means that g is smooth on a neighbourhood around every point in its domain.

Finally, to prove $(e) \Rightarrow (b)$, suppose that g is smooth on any open V and for any choice of covector and vector fields on V. Let (U, φ) be a chart on M. Then we can write $A^{i_1 \cdots i_k}_{j_1 \cdots j_l} = A(dx^{i_1}, \ldots, dx^{i_k}, \partial_{j_1}, \ldots, \partial_{j_l})$. □

Proposition. *The space of smooth (k,l)-tensor fields is a $C^\infty(M)$-module.*

Proof. First, it is straightforward to check that rough (k,l)-tensor fields form a $C^\infty(M)$-module.

Now, given $A, B \in \Gamma(T^{(k,l)}TM)$ and $f \in C^\infty(M)$, we find that $(A+B)^{i_1 \cdots i_k}_{j_1 \cdots j_l} = A^{i_1 \cdots i_k}_{j_1 \cdots j_l} + B^{i_1 \cdots i_k}_{j_1 \cdots j_l}$ and $(fA)^{i_1 \cdots i_k}_{j_1 \cdots j_l} = f A^{i_1 \cdots i_k}_{j_1 \cdots j_l}$ in any chart in M. The previous proposition then tells us that $A + B$ and fA are also smooth. □