

SMOOTH MANIFOLD CHART LEMMA

This is a result that I am having difficulty in understanding.

Proposition. *Let M be a set. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be a collection of pairs, where each U_α is a subset of M , and each $\varphi_\alpha : U_\alpha \rightarrow \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ is a map. Suppose the following is true:*

- (i) *Each $\varphi_\alpha : U_\alpha \rightarrow \varphi_\alpha(U_\alpha)$ is a bijection, where $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ is open.*
- (ii) *For each $\alpha, \beta \in I$, $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in \mathbb{R}^n .*
- (iii) *Each transition map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth.*
- (iv) *Countably many U_α s cover M .*
- (v) *If p and q are distinct points in M , then either $p, q \in U_\alpha$ for some $\alpha \in I$, or p and q are separated by disjoint sets in \mathcal{A} .*

Then M has a unique topology \mathcal{T} which makes (M, \mathcal{T}) a topological manifold with \mathcal{A} as a smooth atlas.

Proof. Define $\mathcal{B} := \{\varphi_\alpha^{-1}(V) \mid \alpha \in I, V \subseteq \mathbb{R}^n \text{ is open}\}$. Let us show that (a) \mathcal{B} forms a basis, (b) M is a topological manifold with the topology \mathcal{T} generated by \mathcal{B} , and (c) \mathcal{T} is the unique topology such that \mathcal{A} is a smooth atlas.

- (a) Observe that $U_\alpha = \varphi_\alpha^{-1}(\mathbb{R}^n)$, so (iv) implies that \mathcal{B} covers M . Given two elements $\varphi_\alpha^{-1}(V)$ and $\varphi_\beta^{-1}(W)$ in \mathcal{B} , we find their intersection is again in \mathcal{B} :

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}(V \cap (\varphi_\beta \circ \varphi_\alpha^{-1})^{-1}(W)).$$

Thus, \mathcal{B} is a basis. Let \mathcal{T} be the topology generated by \mathcal{B} .

- (b) It is straightforward to show that each $\varphi_\alpha : U_\alpha \rightarrow \varphi_\alpha(U_\alpha)$ is a homeomorphism. Thus, M is locally-Euclidean. Hausdorffness follows from (v). Finally, second-countability follows because each U_α is second-countable, and M is covered by countably many U_α s.
- (c) It follows from (iv) and (iii) that \mathcal{A} is a smooth atlas for (M, \mathcal{T}) .

Now, suppose $\tilde{\mathcal{T}}$ is another topology for M such that \mathcal{A} is an atlas for $(M, \tilde{\mathcal{T}})$. This means that each U_α belongs to $\tilde{\mathcal{T}}$, and each $\varphi_\alpha : U_\alpha \rightarrow \varphi_\alpha(U_\alpha)$ is a homeomorphism with respect to $\tilde{\mathcal{T}}$. It follows that \mathcal{B} is contained in $\tilde{\mathcal{T}}$, so $\tilde{\mathcal{T}} = \mathcal{T}$. \square