CONNECTIONS

A connection is an additional piece of data that we put on a smooth manifold. It gives us a way differentiate vector fields. On a Riemannian manifold, there exists a unique connection which is the "nicest", called the Levi-Civita connection. Let us first study connections in general.

Definition. Let M be a smooth manifold. A *connection on TM* is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (X, Y) \mapsto \nabla_X Y$$

which is

(i) $C^\infty(M)$-linear in the first argument,
(ii) $\mathbb{R}$-linear in the second argument, and
(iii) satisfies the product rule

$$\nabla_X(fY) = f\nabla_X Y + Xf\, Y.$$

We say $\nabla_X Y$ is the *covariant derivative of Y in the direction X.*

The definition extends to all tensor bundles. Note that connections are *not* tensor fields, because they are not $C^\infty(M)$-linear in the second argument.

For the rest of this document, let M be a smooth manifold, and let ∇ be a connection.

The following proposition shows that connections depend only on local data.

Proposition. *Let $X, \widetilde{X}, Y, \widetilde{Y} \in \mathfrak{X}(M)$, and suppose $X = \widetilde{X}$ and $Y = \widetilde{Y}$ on a neighbourhood B of p. Then $\nabla_X Y|_p = \nabla_{\widetilde{X}} \widetilde{Y}|_p$.*

Proof. Suppose $Y \equiv 0$ on B. Let ψ be a bump function for $M \backslash B$ supported in $M \backslash \{p\}$. Then $Y = \psi Y$, so $\nabla_X Y|_p = \nabla_X(\psi Y)|_p = 0$. A similar argument works for X. □

The following proposition shows that we can use the same connection on vector fields defined locally.

Proposition. *Let $U \subseteq M$ be open. Then there exists a unique connection on U*

$$\nabla^U : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$$

which satisfies $\nabla^U_{X|_U} Y|_U = (\nabla_X Y)|_U$.

Proof. For uniqueness, suppose ∇^U is such a connection. Let $p \in U$, and let $X, Y \in \mathfrak{X}(U)$. Let $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(M)$ be vector fields which satisfy $X|_B = \widetilde{X}|_B$ and $Y|_B = \widetilde{Y}|_B$ for some open B containing p. Then

$$\nabla^U_X Y|_p = \nabla^U_{\widetilde{X}|_U}(\widetilde{Y}|_U)|_p = \nabla_{\widetilde{X}} \widetilde{Y}|_p.$$

The previous proposition shows that the choices of extensions don't matter. Thus we have shown uniqueness, because ∇^U depends only on ∇.

For existence, define $\nabla^U_X Y|_p := \nabla_{\widetilde{X}} \widetilde{Y}|_p$, where $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(M)$ agree with X and Y on a neighbourhood of p. It is straightforward to check ∇^U is indeed a connection. □

In fact, the first argument of ∇ depends only on pointwise data, as the next proposition shows. Thus, given $p \in M$ and $v \in T_pM$, the expression $\nabla_v Y$ makes sense.

Proposition. *Let* $X, \widetilde{X}, Y \in \mathfrak{X}(M)$, *and let* $p \in M$. *Suppose* $X_p = \widetilde{X}_p$. *Then* $\nabla_X Y|_p = \nabla_{\widetilde{X}} Y|_p$.

Proof. Suppose $X_p = 0$. Let $(U, (x^i))$ be a chart around p, and write $X = X^i \partial_i$. Then $X^i(p) = 0$, so $\nabla_X Y|_p = x^i(p) \nabla_{\partial_i} Y|_p = 0$. $\square$

Let $(U, (x^i))$ be a chart. We can write

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k,$$

where the Γ_{ij}^k are n^3 smooth functions. If ∇ is the Levi-Civita connection, we call Γ_{ij}^k the *Christoffel symbols*. A direct computation shows that

$$\nabla_X Y = XY^k \partial_k + X^i Y^j \Gamma_{ij}^k \partial_k.$$