TANGENT BUNDLE

Let M be a smooth manifold of dimension n. We define the *tangent bundle of* M to be the disjoint union of all the tangent spaces:

$$TM := \coprod_{p \in M} T_pM = \{(p, v) \mid p \in M,\ v \in T_pM\}.$$

We define the projection $\pi : TM \to M$ by $(p, v) \mapsto p$.

Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ be the smooth structure on M. For each α, let $\{\partial_i^\alpha|_p\}$ be the coordinate vectors at T_pM induced by φ_α. Define $\widetilde{U}_\alpha := \pi^{-1}(U_\alpha)$, and define

$$\widetilde{\varphi}_\alpha : \widetilde{U}_\alpha \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^n, \qquad (p, v^i \partial_i^\alpha|_p) \mapsto (\varphi_\alpha(p), v^1, \ldots, v^n).$$

Finally, set $\widetilde{\mathcal{A}} := \{(\widetilde{U}_\alpha, \widetilde{\varphi}_\alpha)\}$.

Proposition. *The tangent bundle TM has a unique topology which makes TM a $2n$-dimensional topological manifold, with $\widetilde{\mathcal{A}}$ as a smooth atlas. Moreover, π is smooth.*

Proof. We use the Smooth Manifold Chart Lemma.

(i) First, each $\widetilde{\varphi}_\alpha$ is invertible, with inverse given by

$$\widetilde{\varphi}_\alpha^{-1} : \varphi_\alpha(U_\alpha) \times \mathbb{R}^n \to \widetilde{U}_\alpha, \qquad (x, v^1, \ldots, v^n) \mapsto (\varphi_\alpha^{-1}(x), v^i \partial_i^\alpha|_{\varphi_\alpha^{-1}(x)}).$$

Moreover, each $\widetilde{\varphi}_\alpha(\widetilde{U}_\alpha) = \varphi_\alpha(U_\alpha) \times \mathbb{R}^n$ is open in $\mathbb{R}^{2n}$.

(ii) For each α and β, we find that $\widetilde{\varphi}_\alpha(\widetilde{U}_\alpha \cap \widetilde{U}_\beta) = \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n$, which is open in $\mathbb{R}^{2n}$.

(iii) Fix α and β, and consider the transition map $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}$. We find that

$$\begin{aligned} \widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}(x, v^1, \ldots, v^n) &= \widetilde{\varphi}_\beta(\varphi_\alpha^{-1}(x), v^i \partial_i^\alpha|_{\varphi_\alpha^{-1}(x)}) \\ &= \widetilde{\varphi}_\beta(\varphi_\alpha^{-1}(x), v^i \partial_i \tau_{\alpha\beta}^j(x)\, \partial_j^\beta|_{\varphi_\alpha^{-1}(x)}) = \left((\varphi_\beta \circ \varphi_\alpha^{-1})(x), v^i \partial_i \tau_{\alpha\beta}(x)\right), \end{aligned}$$

where $\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$. Thus, $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}$ is smooth.

(iv) Countably many $\widetilde{U}_\alpha$s cover TM because countably many U_αs cover M.

(v) The Hausdorff property is straightforward to check.

The projection π is smooth, because we can represent π in coordinates as a projection onto the first n coordinates. □

Remark. The topology for TM is generated by sets of the form $\widetilde{\varphi}_\alpha^{-1}(V)$, where $V \subseteq \mathbb{R}^{2n}$ is open.