TENSORS

We review tensors and the contraction operator.

In what follows, let V be an n-dimensional real vector space. If (b_i) is a basis for V, we define the *dual basis* by $\beta^j(\lambda^i b_i) := \lambda^j$. In other words, β^j projects onto the jth coordinate.

Definition. A (k,l)-*tensor* is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}} \to \mathbb{R}.$$

We denote the vector space of (k,l)-tensors by

$$T^{(k,l)}(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ copies}}$$

If $F \in T^{(k,l)}(V)$ and $G \in T^{(p,q)}(V)$, we define $F \otimes G \in T^{(k+p,l+q)}(V)$ by

$$\begin{aligned}&(F \otimes G)(\omega^1, \ldots, \omega^{k+p}, v_1, \ldots, v_{l+q})\\ &:= F(\omega^1, \ldots, \omega^k, v_1, \ldots, v_l)G(\omega^{k+1}, \ldots, \omega^{k+p}, v_{l+1}, \ldots, v_{l+q}).\end{aligned}$$

Proposition. *Let (b_i) be a basis for V, and let (β^j) be the dual basis. Then*

$$b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_l}$$

form a basis for $T^{(k,l)}(V)$, where i_p and j_q run through $1, \ldots, n$.

Proof. Suppose $F = F^{i_1 \cdots i_k}_{j_1 \cdots j_l} b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_l} = 0$. Fix indices $r_1, \ldots, r_k$ and $s_1, \ldots, s_l$. Then $0 = F(\beta^{r_1}, \ldots, \beta^{r_k}, b_{s_1}, \ldots, b_{s_l}) = F^{r_1 \cdots r_k}_{s_1 \cdots s_l}$.This shows linear independence.

Suppose $F \in T^{(k,l)}(V)$. Define $F^{i_1 \cdots i_k}_{j_1 \cdots j_l} := F(\beta^{i_1}, \ldots, \beta^{i_k}, b_{j_1}, \ldots, b_{j_l})$. Let

$$(\lambda^1_{r_1}\beta^{r_1}, \ldots, \lambda^1_{r_k}\beta^{r_k}, \mu^{s_1}_1 b_{s_1}, \ldots, \mu^{s_l}_l b_{s_l}) \in \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}}$$

be an arbitrary element. Then

$$\begin{aligned}&F(\lambda^1_{r_1}\beta^{r_1}, \ldots, \lambda^1_{r_k}\beta^{r_k}, \mu^{s_1}_1 b_{s_1}, \ldots, \mu^{s_l}_l b_{s_l}) = \lambda^1_{r_1} \cdots \lambda^1_{r_k} \mu^{s_1}_1 \cdots \mu^{s_l}_l F^{r_1 \cdots r_k}_{s_1 \cdots s_l}\\ &= (F^{i_1 \cdots i_k}_{j_1 \cdots j_l} b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_l})(\lambda^1_{r_1}\beta^{r_1}, \ldots, \lambda^1_{r_k}\beta^{r_k}, \mu^{s_1}_1 b_{s_1}, \ldots, \mu^{s_l}_l b_{s_l}).\end{aligned}$$

Thus, we have shown the spanning property. □

Proposition. *The space of tensors $T^{(k+1,l)}(V)$ is canonically isomorphic to the space of multilinear maps of type*

$$\underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}} \to V.$$

Proof. Let us show $\mathrm{End}(V) \simeq T^{(1,1)}(V)$. The argument for the general case is analogous.

Define $\Phi : \mathrm{End}(V) \to T^{(1,1)}(V)$ by $\Phi(A)(\omega, v) := \omega(Av)$. It is straightforward to check that Φ is linear.

Suppose $\Phi(A) = 0$. Let (b_i) be a basis with dual basis (β^j). Let $Av = \lambda^i b_i$. Then $0 = \Phi(A)(\beta^j, v) = \lambda^j$, so $Av = 0$. Thus, $A = 0$. This shows injectivity.

Next, let $F \in T^{(1,1)}(V)$, and define $Av := F(\cdot, v)$. Then $\Phi(A)(\omega, v) = \omega(F(\cdot, v)) = F(\omega, v)$. Thus, Φ is surjective. $\square$

Definition. We define the *contraction operator* $\operatorname{tr} : T^{(k+1,l+1)}(V) \to T^{(k,l)}(V)$ by

$$(\operatorname{tr} F)(\omega^1, \ldots, \omega^k, v_1, \ldots, v_l) := \operatorname{tr} A, \quad \text{where} \quad A : v \mapsto F(\omega^1, \ldots, \omega^k, \bullet, v_1, \ldots, v_l, v).$$

The following proposition shows how to compute the trace once we have fixed a basis.

Proposition. *Let (b_i) be a basis for V, and let (β^j) be the dual basis. Let $F \in T^{(k+1,l+1)}(V)$. Write $F = F^{i_1 \cdots i_k}_{j_1 \cdots j_l} b_{i_1} \otimes \cdots \otimes b_{i_{k+1}} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_{l+1}}$. Then*

$$\operatorname{tr} F = F^{i_1 \cdots i_k m}_{j_1 \cdots j_l m} b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \cdots \otimes \beta^{j_l}.$$

Proof. Let us consider $\operatorname{tr} : T^{(1,1)}(V) \to \mathbb{R}$. The proof for the general case is analogous.

Let $F = F^i_j b_i \otimes \beta^j$. Consider $A : v \mapsto F(\bullet, v)$. It is straightforward to show that $[A]^{(b_j)}_{(b_i)} = (F^i_j)$, so $\operatorname{tr} F = \operatorname{tr} [A]^{(b_j)}_{(b_i)} = F^m_m$. $\square$