SMOOTH MANIFOLDS

Definition. We say that a topological space M is an *n-dimensional topological manifold* if M is second-countable, Hausdorff, and locally-Euclidean.

Locally-Euclidean means that every point has an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$.

A *coordinate chart* is a pair $(U, \varphi : U \to \widehat{U})$, where $U \subseteq M$ and $\widehat{U} \subseteq \mathbb{R}^n$ are open, and φ is a homeomorphism.

We want to be able to differentiate maps between manifolds. *Compatibility* ensures that the smoothness of such maps are independent of the choice of charts.

Definition. Let M be a topological manifold. Let (U, φ) and (V, ψ) be charts. We say (U, φ) and (V, ψ) are *smoothly compatible* if $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism between Euclidean spaces.

A *smooth atlas* $\mathcal{A}$ is a collection of charts covering M such that any two are smoothly compatible. We say that a chart (U, φ) is compatible with $\mathcal{A}$ if (U, φ) is compatible with every chart in $\mathcal{A}$. A *smooth structure* is a maximal smooth atlas. (Maximal with respect to inclusion.)

A *smooth manifold* is a topological manifold equipped with a smooth structure.

In practice, it is impossible to explicitly define a smooth structure. However, the following proposition shows that we can generate a smooth structure from a smooth atlas.

Proposition. *Let M be a topological manifold, and let $\mathcal{A}$ be a smooth atlas. Then there exists a unique smooth structure $\overline{\mathcal{A}}$ containing $\mathcal{A}$.*

Proof. Define $\overline{\mathcal{A}} := \{(U, \varphi) \mid (U, \varphi) \text{ is a chart compatible with } \mathcal{A}\}$. Let us show that $\overline{\mathcal{A}}$ is (i) a smooth atlas, (ii) is maximal, and (iii) is the unique smooth structure containing $\mathcal{A}$.

(i) Let (U, φ) and (V, ψ) be two charts in $\overline{\mathcal{A}}$. Fix $\varphi(p) \in \varphi(U \cap V)$, and let (W, θ) be a chart in $\mathcal{A}$ containing p. Observe that the following diagram commutes:

$$\begin{tikzcd}
 & U \cap V \cap W \arrow[dl, "\varphi"'] \arrow[d, "\theta"] \arrow[dr, "\psi"] & \\
\varphi(U \cap V \cap W) \arrow[r, "\theta \circ \varphi^{-1}"'] & \theta(U \cap V \cap W) \arrow[r, "\psi \circ \theta^{-1}"'] & \psi(U \cap V \cap W)
\end{tikzcd}$$

Therefore, $\psi \circ \varphi^{-1} = (\varphi \circ \theta^{-1}) \circ (\theta \circ \psi^{-1})$ is smooth on a neighbourhood of $\varphi(p)$. Thus, $\psi \circ \varphi^{-1}$ is smooth, because it is smooth on a neighbourhood of every point in its domain. The argument to show smoothness of $\varphi \circ \psi^{-1}$ is identical.

(ii) Next, suppose $\mathcal{B}$ is a smooth atlas such that $\overline{\mathcal{A}} \subseteq \mathcal{B}$. Since $\mathcal{B}$ contains $\mathcal{A}$, every chart in $\mathcal{B}$ is compatible with $\mathcal{A}$, so $\mathcal{B} \subseteq \overline{\mathcal{A}}$.

(iii) Suppose $\mathcal{B}$ is another smooth structure containing $\mathcal{A}$. Since every chart in $\mathcal{B}$ is compatible with $\mathcal{A}$, we have $\mathcal{B} \subseteq \overline{\mathcal{A}}$, so $\mathcal{B} = \overline{\mathcal{A}}$ by maximality of $\mathcal{B}$. $\square$

We can put different smooth structures on the same underlying topological manifold. The following proposition gives a characterisation of when two different smooth atlases generate the same smooth structure.

Proposition. *Let M be a topological manifold, and let $\mathcal{A}$ and $\mathcal{B}$ be smooth atlases. Then $\overline{\mathcal{A}} = \overline{\mathcal{B}}$ if and only if $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas.*

Proof. Suppose $\mathcal{A} \cup \mathcal{B}$ is *not* a smooth atlas. Then there exists a chart in $\mathcal{B}$ not compatible with $\mathcal{A}$, so the definition of $\overline{\mathcal{A}}$ in the previous proposition shows that $\mathcal{B}$ is not a subset of $\overline{\mathcal{A}}$. In particular, $\overline{\mathcal{B}}$ is not a subset of $\overline{\mathcal{A}}$.

Conversely, suppose $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas. Observe that $\overline{\mathcal{A} \cup \mathcal{B}}$ is a smooth structure containing $\mathcal{A}$ and $\mathcal{B}$. Uniqueness implies $\overline{\mathcal{A}} = \overline{\mathcal{A} \cup \mathcal{B}} = \overline{\mathcal{B}}$. $\square$