SMOOTH MAPS

Definition. We say that a map $F : M \to N$ between smooth manifolds is *smooth* if for every $p \in M$, there exists smooth charts (U, φ) and (V, ψ) containing p and $F(p)$ respectively, such that $F(U) \subseteq V$ and $\widehat{F} := \psi \circ F \circ \varphi^{-1}$ is smooth.

$$\begin{tikzcd} U \arrow[r, "F"] \arrow[d, "\varphi"'] & V \arrow[d, "\psi"] \\ \widehat{U} \arrow[r, "\widehat{F}"'] & \widehat{V} \end{tikzcd}$$

Remark. Suppose $F : M \to N$ is smooth. Then F is continuous on a neighbourhood around each point, because we can write $F = \psi^{-1} \circ \widehat{F} \circ \varphi$. Thus, F is continuous by the pasting lemma.

Any $\widehat{F}$ constructed by composing F with charts is called a *coordinate representation* of F. The smoothness of $\widehat{F}$ is independent of the choice of charts, as the next proposition shows.

Proposition. *Let $F : M \to N$ be smooth. Let (U, φ) and (V, ψ) be smooth charts for M and N respectively. Then $\widehat{F} := \psi \circ F \circ \varphi^{-1}$ is smooth on $\varphi(U \cap F^{-1}(V))$.*

Proof. Let $\varphi(p) \in \varphi(U \cap F^{-1}(V))$. Then there exists smooth charts (W, θ) and (X, η) containing p and $F(p)$ respectively, such that $F(W) \subseteq X$, and $\eta \circ F \circ \theta^{-1}$ is smooth. Observe that the following diagram commutes:

$$\begin{tikzcd} & U \cap F^{-1}(V) \cap W \arrow[dl, "\varphi"'] \arrow[d, "\theta"] \arrow[rr, "F"] & & V \cap X \arrow[d, "\eta"] \arrow[dr, "\psi"] & \\ \varphi(U \cap F^{-1}(V)) \arrow[r, "\theta\circ\varphi^{-1}"'] & \theta(U \cap W) \arrow[rr, "\eta\circ F\circ\theta^{-1}"'] & & \eta(V \cap X) \arrow[r, "\psi\circ\eta^{-1}"'] & \psi(V \cap X) \end{tikzcd}$$

Thus, $\widehat{F} = \psi \circ F \circ \varphi^{-1} = (\psi \circ \eta^{-1}) \circ (\eta \circ F \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1})$ is smooth on a neighbourhood around $\varphi(p)$. Therefore, $\widehat{F}$ is smooth, because it is smooth on a neighbourhood of every point in its domain. $\square$

Examples. Here are a few examples of smooth maps.

- The identity map $\mathrm{id} : M \to M$ is smooth.
- Any constant map is smooth.
- If (U, φ) is a smooth chart for M, then $\varphi : U \to \widehat{U}$ is a smooth map.

Proposition. *Let $F : K \to M$ and $G : M \to N$ be smooth maps. Then $G \circ F$ is smooth.*

Proof. Fix $p \in K$. Then there exist smooth charts (U, φ) and (V, ψ) around p and $f(p)$ respectively, such that $F(U) \subseteq V$, and $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is smooth. Also, there exist smooth charts (W, θ) and (X, η) around $F(p)$ and $G(F(p))$ respectively, such that $G(W) \subseteq X$, and $\widehat{G} = \eta \circ G \circ \theta^{-1}$ is smooth.

Now, observe that $(G \circ F)(F^{-1}(V \cap W)) \subseteq X$, and the following diagram commutes:

$$\begin{tikzcd}
 & F^{-1}(V \cap W) \arrow[r, "F"] \arrow[dl, "\varphi"'] & V \cap W \arrow[r, "G"] \arrow[dl, "\psi"'] \arrow[dr, "\theta"] & X \arrow[dr, "\eta"] & \\
\varphi(F^{-1}(V \cap W)) \arrow[r, "\widehat{F}"'] & \psi(V \cap W) \arrow[rr, "\theta \circ \psi^{-1}"'] & & \theta(V \cap W) \arrow[r, "\widehat{G}"'] & \eta(X)
\end{tikzcd}$$

Thus, $\eta \circ (G \circ F) \circ \varphi^{-1}$ is smooth. $\square$