TANGENT SPACE OF A SUBMANIFOLD

Proposition. *Let $S \subseteq M$ be an embedded submanifold and let $p \in S$. Then*

$$d\iota_p(T_pS) = \left\{ v \in T_pM \;\middle|\; \forall f \in C^\infty(M),\ f|_S \equiv 0 \text{ implies } vf = 0 \right\}.$$

Proof. Suppose $v = d\iota_p(w)$ belongs to $d\iota_p(T_pS)$. If $f \in C^\infty(M)$ satisfies $f|_S \equiv 0$, then $vf = w(f|_S) = 0$.

Conversely, suppose $v \in T_pM$, and $vf = 0$ whenever $f \in C^\infty(M)$ and $f|_S \equiv 0$. Let (U, φ) be a slice chart around p, and let $(\widetilde{U}, \widetilde{\varphi})$ be the induced chart for S. Let $\widetilde{\partial}_1|_p, \dots, \widetilde{\partial}_k|_p \in T_pS$ and $\partial_1|_p, \dots, \partial_n|_p \in T_pM$ denote the coordinate vectors induced by $(\widetilde{U}, \widetilde{\varphi})$ and (U, φ), respectively. The coordinate representation $\widehat{\iota}$ is given by $(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$, where $\widehat{\iota}$ is defined by the following diagram:

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\iota"] \arrow[d, "\widetilde{\varphi}"'] & U \arrow[d, "\varphi"] \\ \widetilde{\varphi}(\widetilde{U}) \arrow[r, "\widehat{\iota}"'] & \varphi(U) \end{tikzcd}$$

Now, write $v = \sum_{i=1}^n v^i \partial_i|_p$, and define $w := \sum_{i=1}^k v^i \widetilde{\partial}_i|_p$, which belongs to T_pS. By direct computation, we find $d\iota_p(w) = \sum_{i=1}^k \partial_i|_p$, so it remains to show that $v^{k+1} = \cdots = v^n = 0$.

Fix $j = k+1, \dots, n$, and let $\psi \in C^\infty(M)$ be a smooth bump function such that $\psi(p) = 1$ and $\operatorname{supp} \psi \subseteq U$. Define $f^i : M \to \mathbb{R}$ by $p \mapsto \psi(p)\pi^j(\varphi(p))$, where $\pi^j : \mathbb{R}^n \to \mathbb{R}$ is the jth coordinate projection. Then $f^j \in C^\infty(M)$ and $f^i|_S \equiv 0$. A direct computation shows $0 = vf^j = v^j$. $\square$