TENSOR CHARACTERISATION LEMMA

In what follows, let M be a smooth manifold of dimension n.

Let $\mathcal{T}^{(k,l)}(M)$ denote the $C^\infty(M)$-module of all smooth (k,l)-tensor fields. Let $\mathcal{M}^{(k,l)}(M)$ denote the $C^\infty(M)$-module of all $C^\infty(M)$-multilinear maps of the form

$$\underbrace{\mathfrak{X}^*(M)\times\cdots\times\mathfrak{X}^*(M)}_{k \text{ copies}}\times\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{l \text{ copies}}\to C^\infty(M).$$

A tensor field defines an element of $\mathcal{M}^{(k,l)}(M)$.

Proposition. *Let $A\in\mathcal{T}^{(k,l)}(M)$. Let*

$$\mathcal{M}(A):\underbrace{\mathfrak{X}^*(M)\times\cdots\times\mathfrak{X}^*(M)}_{k \textit{ copies}}\times\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{l \textit{ copies}}\to C^\infty(M)$$

be defined by

$$\mathcal{M}(A)(\omega^1,\dots,\omega^k,X_1,\dots X_l)(p):=A_p(\omega^1|_p,,\dots,\omega^k|_p,X_1|_p,\dots,X_l|_p).$$

Then $\mathcal{M}(A)$ belongs to $\mathcal{M}^{(k,l)}(M)$.

Proof. The codomain makes sense because a previous result tells us

$$\mathcal{M}(A)(\omega^1,\dots,\omega^k,X_1,\dots X_l):M\to\mathbb{R}$$

is smooth. Multilinearity is straightforward to verify. □

For the following two lemmas, let $\mathcal{A}\in\mathcal{M}^{(k,l)}(M)$, let $\omega^1,\dots,\omega^k\in\mathfrak{X}^*(M)$, and let $X_1,\dots,X_l\in\mathfrak{X}(M)$. Let $p\in M$ be fixed.

Lemma. *Suppose $X_{j_0}\equiv 0$ on a neighbourhood B of p for some j_0. Then*

$$\mathcal{A}(\omega^1,\dots,\omega^k,X_1,\dots,X_k)(p)=0.$$

Proof. Let ψ be a bump function supported in B such that $\psi(p)=1$. Then $\psi X_{j_0}\equiv 0$, so $0=\mathcal{A}(\omega^1,\dots,\psi X_i,\dots,X_k)(p)=\psi(p)\mathcal{A}(\omega^1,\dots,X_k)(p)$. □

Lemma. *Suppose $X_{j_0}|_p=0$ for some j_0. Then*

$$\mathcal{A}(\omega^1,\dots,\omega^k,X_1,\dots,X_k)(p)=0.$$

Proof. Let (U,φ) be a chart around p. Then we can write $X_{j_0}=X^m\partial_m:U\to TM$, where $X^m:U\to\mathbb{R}$ are smooth. We can also write $p\in B\subseteq\overline{B}\subseteq U$ for some open B in M. Let $E_m\in\mathfrak{X}(M)$ satisfy $E_m|_B=\partial_m|_B$, and let $f^m\in C^\infty(M)$ satisfy $f^m|_B=X^m|_B$. Then $f^mE_m\in\mathfrak{X}(M)$ satisfies $X=f^mE_m$ on B. Furthermore, we have $f^m(p)=X^m(p)=0$. Thus, the previous lemma implies

$$\mathcal{A}(\omega^1,\dots,X_i,\dots,X_k)(p)=f^m(p)\mathcal{A}(\omega^1,\dots,E_m,\dots,X_k)(p)=0.\quad\square$$

The results analogous to the previous two lemmas hold for covector fields. Thus, the value of an element of $\mathcal{M}^{(k,l)}(M)$ at p depends only on the values of the covector and vector fields at p.

Proposition. $\mathcal{A} \in \mathcal{M}^{(k,l)}(M)$ *Define* $\mathcal{T}(\mathcal{A}) : M \to T^{(k,l)}TM$ *by*

$$\mathcal{T}(\mathcal{A})_p(\alpha^k, \ldots, \alpha^k, v_1, \ldots, v_l) := \mathcal{A}(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l)(p),$$

where $\omega^1, \ldots, \omega^k \in \mathfrak{X}^*(M)$ *and* $X_1, \ldots, X_l \in \mathfrak{X}(M)$ *satisfy* $\omega^i|_p = \alpha^p$ *and* $X_j|_p = v_j$*. Then* $\mathcal{T}(\mathcal{A})$ *is well-defined and belongs to* $\mathcal{T}^{(k,l)}(M)$*.*

Proof. The previous lemma shows that $\mathcal{T}(\mathcal{A})$ is well-defined. Given $\omega^1, \ldots, \omega^k \in \mathfrak{X}^*(M)$ and $X_1, \ldots, X_l \in \mathfrak{X}(M)$, notice that the map $p \mapsto \mathcal{T}(\mathcal{A})_p(\omega^1|_p, \ldots, X_l|_p)$ is nothing more than $\mathcal{A}(\omega^1, \ldots, X_l)$, so $\mathcal{T}(\mathcal{A})$ is smooth by a previous result. $\square$

Proposition. *The maps*

$$\mathcal{M} : \mathcal{T}^{(k,l)}(M) \longrightarrow \mathcal{M}^{(k,l)}(M), \quad A \mapsto \mathcal{M}(A),$$
$$\mathcal{T} : \mathcal{M}^{(k,l)}(M) \longrightarrow \mathcal{T}^{(k,l)}(M), \quad \mathcal{A} \mapsto \mathcal{T}(\mathcal{A})$$

are $C^\infty(M)$*-linear isomorphisms and inverses of each other.*

Proof. It is straightforward to check that $\mathcal{M}$ is $C^\infty(M)$-linear. Unraveling the definitions shows that $\mathcal{M} \circ \mathcal{T}$ and $\mathcal{T} \circ \mathcal{M}$ are identity maps. $\square$