VECTOR FIELDS ALONG CURVES

In what follows, let M be a smooth manifold, let ∇ be a connection, and let $\gamma : I \to M$ be a smooth curve.

Definition. A *smooth vector field along* γ is a smooth map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for each $t \in I$. We denote the $C^\infty(I)$-module of all smooth vector fields along γ by $\mathfrak{X}(\gamma)$.

Given $V \in \mathfrak{X}(\gamma)$, we say that $\widetilde{V} \in \mathfrak{X}(M)$ is an *extension* of V if $V(t) = \widetilde{V}_{\gamma(t)}$ for each $t \in I$. On the other hand, given $\widetilde{V} \in \mathfrak{X}(M)$, we can induce a vector field along γ by $V := \widetilde{V} \circ \gamma$. In particular, $\partial_i \circ \gamma$ is a vector field along γ.

Proposition. *There exists a unique operator*

$$D_t : \mathfrak{X}(M) \to \mathfrak{X}(M)$$

satisfying the following properties:

(i) D_t *is linear.*
(ii) D_t *satisfies the product rule:* $D_t(fV) = fD_tV + f'V$.
(iii) *We can write* $D_tV(t) = \nabla_{\gamma'(t)}\widetilde{V}$ *for any extension* $\widetilde{V}$ *of* V.

Proof. Let us first show uniqueness. Suppose D_t is such an operator. It can be checked that D_t depends only on local data, which means that we can work in coordinates (Why?). Now, fix $t_0 \in I$, and let $(U, (x^i))$ be a chart around $\gamma(t_0)$. We find

$$D_tV(t) = \dot{V}^j(t)\partial_j|_{\gamma(t)} + V^j(t)\dot{\gamma}^i(t)\Gamma_{ij}^k(\gamma(t))\partial_k|_{\gamma(t)}$$

for any t sufficiently close to t_0. This shows uniqueness, since D_t depends only on things we already know.

For existence, we can define D_t by the formula given above, which satisfies (i), (ii) and (iii). Uniqueness shows that the definition is independent of the choice of coordinates (Why?). $\square$

We say D_tV is the *covariant derivative of* V *along* γ.