GEODESICS

Let M be a smooth manifold, and let ∇ be a connection.

Definition. We say a curve $\gamma : I \to M$ is a *geodesic* if $D_t\gamma' \equiv 0$. In other words, geodesics are precisely the curves with zero acceleration.

If $(x^1(t), \ldots, x^n(t))$ is a coordinate representation of γ, then

$$\ddot{x}^k(t) + \dot{\gamma}^i(t)\dot{\gamma}^j(t)\Gamma^k_{ij}(\gamma(t)) = 0.$$

The equation above is called the *geodesic equation*. If the image of a curve γ is contained in a single chart, then γ is a geodesic if and only if the geodesic equation holds.

The following proposition tells us that there are geodesics starting at any point in any direction, but they are only defined locally.

Proposition. *For any $p \in M$ and $v \in T_pM$, there exists a geodesic $\gamma : I \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. Moreover, any two such geodesics agree in their common domain.*

Proof. Let $(U, (x^i))$ be a chart around p. Standard ODE theory implies that there exist $(x^1(t), \ldots, x^n(t))$ which satisfy the geodesic equation and have the correct initial conditions. Sending $(x^1(t), \ldots, x^n(t))$ up onto the manifold gives the desired geodesic.

For uniqueness, suppose $\gamma, \widetilde{\gamma} : I \to M$ are two geodesics starting at p with initial velocity v. Using coordinates around p, uniqueness of ODE solutions in Euclidean space implies that γ and $\widetilde{\gamma}$ are equal on a neighbourhood of zero.

For the sake of contradiction, suppose $\gamma(t) \neq \widetilde{\gamma}(t)$ for some $t > 0$ (the case for $t < 0$ is analogous). Define

$$t_0 := \inf\{t > 0 \mid \gamma(t) \neq \widetilde{\gamma}(t)\}.$$

Continuity implies that $\gamma(t_0) = \widetilde{\gamma}(t_0)$ and $\gamma'(t_0) = \widetilde{\gamma}'(t_0)$. Applying a local uniqueness argument again gives us the contradiction. □

For any $p \in M$ and $v \in T_pM$, there exists a *maximal* geodesic $\gamma : I \to M$ starting at p with initial velocity v. *Maximal* here means that we cannot extend γ to a geodesic defined on a larger interval.