SMOOTH MANIFOLD CHART LEMMA

To make a set M into a smooth manifold, it suffices to endow it with a candidate for a smooth atlas. There is a unique topology which makes everything work.

Proposition. *Let M be a set. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be a collection of pairs, where each U_α is a subset of M, and each $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ is a map. Suppose the following is true:*

(i) Each $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a bijection, where $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ is open.
(ii) For each $\alpha, \beta \in I$, $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.
(iii) Each transition map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth.
(iv) Countably many U_αs cover M.
(v) If p and q are distinct points in M, then either $p, q \in U_\alpha$ for some $\alpha \in I$, or p and q are separated by disjoint sets in $\mathcal{A}$.

Then M has a unique topology $\mathcal{T}$ which makes $(M, \mathcal{T})$ a topological manifold with $\mathcal{A}$ as a smooth atlas.

Proof. Define $\mathcal{B} := \{\varphi_\alpha^{-1}(V) \mid \alpha \in I,\ V \subseteq \mathbb{R}^n \text{ is open}\}$. Let us show that (a) $\mathcal{B}$ forms a basis, (b) M is a topological manifold with the topology $\mathcal{T}$ generated by $\mathcal{B}$, and (c) $\mathcal{T}$ is the unique topology such that $\mathcal{A}$ is a smooth atlas.

(a) Observe that $U_\alpha = \varphi_\alpha^{-1}(\mathbb{R}^n)$, so (iv) implies that $\mathcal{B}$ covers M. Given two elements $\varphi_\alpha^{-1}(V)$ and $\varphi_\beta^{-1}(W)$ in $\mathcal{B}$, we find their intersection is again in $\mathcal{B}$:

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}(V \cap (\varphi_\beta \circ \varphi_\alpha^{-1})^{-1}(W)).$$

Thus, $\mathcal{B}$ is a basis. Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$.

(b) It is straightforward to show that each $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a homeomorphism. Thus, M is locally-Euclidean. Hausdorffness follows from (v). Finally, second-countability follows because each U_α is second-countable, and M is covered by countably many U_αs.

(c) It follows from (iv) and (iii) that $\mathcal{A}$ is a smooth atlas for $(M, \mathcal{T})$.

Now, suppose $\widetilde{\mathcal{T}}$ is another topology for M such that $\mathcal{A}$ is an atlas for $(M, \widetilde{\mathcal{T}})$. This means that each U_α belongs to $\widetilde{\mathcal{T}}$, and each $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a homeomorphism with respect to $\widetilde{\mathcal{T}}$. It follows that $\mathcal{B}$ is contained in $\widetilde{\mathcal{T}}$, so $\widetilde{\mathcal{T}} = \mathcal{T}$. $\square$

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