RIEMANNIAN DISTANCE FUNCTION

We are going to prove that the Riemannian distance function satisfies the axioms of a distance metric, and that it generates the original manifold topology.

Let (M, g) be a Riemannian manifold, and let $\gamma : [a, b] \to M$ be a piecewise-smooth curve. We define the *length of* γ *with respect to* g by

$$L_g(\gamma) := \int_a^b |\gamma'(t)|dt.$$

An immediate consequence of this definition is that, if $a < c < b$, then

$$L_g(\gamma) = L_g(\gamma|_{[a,c]}) + L_g(\gamma|_{[c,b]}).$$

An *isometry* is a diffeomorphism $F : (M, g) \to (\widetilde{M}, \widetilde{g})$ satisfying $g = F^*\widetilde{g}$. The lengths of curves are invariant under isometries, as one would expect.

Proposition. *Let $F : (M, g) \to (\widetilde{M}, \widetilde{g})$ be an isometry. Let $\gamma : [a, b] \to M$ be a piecewise-smooth curve. Then $L_{\widetilde{g}}(F \circ \gamma) = L_g(\gamma)$.*

Proof. Given $t \in [a, b]$ where γ is smooth, unraveling $g = F^*\widetilde{g}$ gives us

$$|\gamma'(t)|^2 = |(F \circ \gamma)'(t)|^2.$$ □

Let $\gamma : [a, b] \to M$ and $\widetilde{\gamma} : [c, d] \to M$ be a piecewise-smooth curves. We say $\widetilde{\gamma}$ is a *reparameterisation of* γ if there is some diffeomorphism $\varphi : [c, d] \to [a, b]$ such that $\widetilde{\gamma} = \gamma \circ \varphi$. The following proposition shows that lengths are invariant under reparameterisations.

Proposition. *Let (M, g) be a Riemannian manifold, and let γ, $\widetilde{\gamma}$ and φ be as above. Then $L_g(\widetilde{\gamma}) = L_g(\gamma)$.*

Proof. We know that φ' never vanishes. Without loss of generality, assume φ' is always positive. Then

$$L_g(\widetilde{\gamma}) = \int_c^d |\widetilde{\gamma}'(t)|dt = \int_c^d |\gamma'(\varphi(t))|\varphi'(t)dt = \int_a^b |\gamma'(s)|ds = L_g(\gamma).$$ □

Before we define the Riemannian distance function, we need to verify that there is always some piecewise-smooth curve connecting any two points on a Riemannian manifold.

Proposition. *Let (M, g) be a connected Riemannian manifold. Then there exists a piecewise-smooth curve joining any two points in M.*

Proof. Fix $p_0 \in M$, and define C to be the set of all points in $p \in M$ such that there is a piecewise-smooth curve joining p_0 and p. Since C is non-empty, we are done if we show C is both open and closed. For any $p \in C$, let B_p be a coordinate ball around p.

Observe that each B_p is contained in C: given $q \in B_p$, we can construct a piecewise-smooth curve from p_0 to q by joining p_0 to p, then joining p to q. Thus, C is open.

Suppose $p \in \overline{C}$. Then $C \cap B_p$ is non-empty, so an argument similar to the previous paragraph shows that $p \in C$. Therefore, C is closed. $\square$

Definition. Let (M, g) be a connected Riemannian manifold. We define the *Riemannian distance function* $d : M \times M \to \mathbb{R}$ by

$$d_g(p, q) := \inf \left\{ L_g(\gamma) \;\middle|\; \gamma \text{ is piecewise-smooth curve joining } p \text{ to } q \right\}.$$

Since isometries preserve lengths of curves, it follows immediately that the distance function is invariant under isometries. Thus, a Riemannian isometry is also an isometry in the metric space sense.

Proposition. *Let $F : (M, g) \to (\widetilde{M}, \widetilde{g})$ be an isometry between two connected Riemannian manifolds. Then*

$$d_{\widetilde{g}}(F(p), F(q)) = d_g(p, q).$$

The following proposition shows that the Riemannian distance function on $\mathbb{R}^n$ is what we would expect.

Example. Consider $M = \mathbb{R}^n$. Then $d_{\overline{g}}(x, y) = |x - y|$.

Proof. Suppose $x \neq y$. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a curve joining x and y. Let $v = (y - x)/|y - x|$, which is a unit vector. Then

$$|y - x| = \langle y - x, v \rangle = \int_a^b \langle \gamma'(t), v \rangle dt \leq \int_a^b |\gamma'(t)| dt = L_{\overline{g}}(\gamma),$$

where we have used the Fundamental Theorem of Calculus in the second relation, and the Cauchy-Schwartz inequality in the third. The result follows because the length of the straight line from x to y is $|x - y|$. $\square$

The following lemma shows that we can estimate the norm induced by any g on a Euclidean space by the standard norm. The key is that c and C are independent of x, even though $|\cdot|_g$ may be different at each x.

Lemma. *Let g be a metric on $U \subseteq \mathbb{R}^n$, and let $K \subseteq U$ be compact. Then there exist $c, C > 0$ such that for every $x \in K$ and every $v_x \in T_x\mathbb{R}^n$, we have*

$$c|v_x|_{\overline{g}} \leq |v_x|_g \leq C|v_x|_{\overline{g}}.$$

Proof. Define

$$L := \left\{ (x, v_x) \in T\mathbb{R}^n \;\middle|\; x \in K,\ |v_x|_{\overline{g}} = 1 \right\}.$$

Then $L \simeq K \times \mathbb{S}^{n-1}$, so L is compact. Since $|\cdot|_g$ is continuous, there exist positive c and C such that $c \leq |v_x|_g \leq C$ for every $(x, v_x) \in L$. Homogeneity gives the result. $\square$

If (M, g) is a connected Riemannian manifold, we define

$$B_g(p, \varepsilon) := \{q \in M \mid d_g(p, q) < \varepsilon\}.$$

Lemma. *Let (M, g) be a connected Riemannian manifold, and let U be an open set (in the manifold topology) containing $p \in M$. Then there exists $\varepsilon > 0$ such that $B_g(p, \varepsilon)$ is a subset of U.*

Proof. Let (U_0, φ) be a chart containing p such that $U_0 \subseteq U$. Let $\varepsilon > 0$ be small enough so that $V := B_{\varphi^*\overline{g}}(p, \varepsilon)$ satisfies $\overline{V} \subseteq U_0$. The previous lemma implies the existence of $c > 0$ such that $L_{(\varphi^{-1})^*g}(\widehat{\gamma}) \geq cL_{\overline{g}}(\widehat{\gamma})$ for any curve $\widehat{\gamma}$ in $\overline{B_{\overline{g}}(\varphi(p), \varepsilon)}$.

Now, let $q \in M \backslash U$, and let $\gamma : [a, b] \to M$ be a piecewise-smooth curve joining p to q. Define

$$t_0 := \inf\{t \in [a, b] \mid \gamma(t) \notin \overline{V}\}.$$

By continuity from both sides, it follows that $\gamma(t_0) \in \partial V$, and $\gamma(t) \in \overline{V}$ for every $t \in [a, t_0]$. Thus, $\widehat{\gamma} := \varphi \circ \gamma|_{[a,t_0]}$ is a curve in $\overline{B_{\overline{g}}(\varphi(p), \varepsilon)}$, with $\widehat{\gamma}(t_0)$ on the boundary. We obtain

$$L_g(\gamma) \geq L_g(\gamma|_{[a,t_0]}) = L_{(\varphi^{-1})^*g}(\widehat{\gamma}) \geq cL_{\overline{g}}(\widehat{\gamma}) \geq c|\widehat{\gamma}(t_0) - \varphi(p)| = c\varepsilon.$$

Taking the infimum gives $d_g(p, q) \geq c\varepsilon$ whenever $q \notin U$. Taking the contrapositive gives the result. □

Proposition. *Let (M, g) be a connected Riemannian manifold. Then (M, d_g) is a metric space.*

Proof. It is clear that $d_g(p, p) = 0$. Symmetry of d_g follows from the fact that a curve joining p and q can be reparameterised to join q to p.

Let $p, q, r \in M$ be arbitrary. Let α and β be piecewise-smooth curves joining p to q and q to r, respectively. Let γ be the curve which travels along α, then travels along β. Then we have $d(p, r) \leq L_g(\gamma) = L_g(\alpha) + L_g(\beta)$. Taking infimums gives the triangle inequality.

Finally, suppose $p, q \in M$ and $p \neq q$. Let (U, φ) be a chart containing p but not q. The second lemma gives $d(p, q) > 0$. □

Proposition. *Let (M, g) be a connected Riemannian manifold. Let $\mathcal{T}$ denote the underlying manifold topology. Let $\mathcal{T}'$ denote the topology generated by the distance function d_g. Then $\mathcal{T} = \mathcal{T}'$.*

Proof. The second lemma shows that $\mathcal{T} \subseteq \mathcal{T}'$.

Conversely, suppose $W \in \mathcal{T}'$ Let (U, φ) be a chart around p. Let $r > 0$ be small enough so that $V = B_{\varphi^*\overline{g}}(\varphi(p), r)$ satisfies $\overline{V} \subseteq U$. The first lemma implies the existence of $C > 0$ such that $L_{(\varphi^{-1})^*g}(\widehat{\gamma}) \leq CL_{\overline{g}}(\widehat{\gamma})$ for any curve $\widehat{\gamma}$ in $\overline{B_{\overline{g}}(\varphi(p), r)}$.

Next, let $0 < \varepsilon < r$ be small enough so that $B_g(p, C\varepsilon) \subseteq W$. Observe that $B_{\varphi^*\overline{g}}(p, \varepsilon) = \varphi^{-1}(B_{\overline{g}}(\varphi(p), \varepsilon))$ belongs to $\mathcal{T}$.

Let $q \in B_{\varphi^*\overline{g}}(p, \varepsilon)$, and let $\widehat{\gamma}$ be the straight line segment from $\varphi(p)$ to $\varphi(q)$. Define $\gamma := \varphi^{-1} \circ \widehat{\gamma}$, which is a curve from p to q. Finally, we find

$$d_g(p, q) \leq L_g(\gamma) = L_{(\varphi^{-1})^*g}(\widehat{\gamma}) \leq CL_{\overline{g}}(\widehat{\gamma}) < C\varepsilon.$$

Thus, we have shown that $B_{\varphi^*\overline{g}}(p, \varepsilon) \subseteq B_g(p, C\varepsilon) \subseteq W$, which shows that W belongs to $\mathcal{T}$. □