RESTRICTING MAPS TO SUBMANIFOLDS

It is straightforward to see that smoothness of maps is preserved when we strict either the domain or codomain to open submanifolds. The following two propositions show that the same is true for general embedded submanifolds.

Proposition. *Let $F : M \to N$ be smooth, and let $S \subseteq M$ be an embedded submanifold. Then $F|_S : S \to N$ is smooth.*

Proof. The result follows because $F|_S = F \circ \iota$. □

Proposition. *Let $F : M \to N$ be smooth, and let $S \subseteq N$ be an embedded submanifold containing $F(M)$. Then the map $\widetilde{F} : M \to S$, $p \mapsto F(p)$ is smooth.*

Proof. Fix $p \in M$. Let (V, ψ) be a slice chart for S in N around $F(p)$. Let $(\widetilde{V}, \widetilde{\psi})$ be the induced smooth chart for S. Let (U, φ) be a chart such that $F(U) \subseteq \widetilde{V}$ (we can choose such a chart because F is continuous).

Now, observe that the following diagram commutes:

$$\begin{tikzcd}
\varphi(U) \arrow[r, "\varphi^{-1}"] & U \arrow[r, "\widetilde{F}"] \arrow[dr, "F"'] & \widetilde{V} \arrow[rr, "\widetilde{\psi}"] & & \widetilde{\psi}(\widetilde{V}) \\
& & V \arrow[r, "\psi"] & \psi(V) \arrow[ur, "\pi"] &
\end{tikzcd}$$

We know that $\pi \circ \psi \circ F \circ \varphi^{-1}$ is a composition of smooth maps. Thus, $\widetilde{\psi} \circ \widetilde{F} \circ \varphi^{-1}$ is smooth. □