TANGENT SPACES AND DIFFERENTIALS

We are going to show that T_pM has the same dimension as M. The idea is to first show $T_p\mathbb{R}^n \cong \mathbb{R}^n$, then use a chart to create a chain of isomorphisms.

Definition. Let M be a smooth manifold, and let $p \in M$. A *derivation at p* is a linear map $v : C^\infty(M) \to \mathbb{R}$ satisfying the product rule:

$$v(fg) = f(p)vg + g(p)vf \qquad \forall f, g \in C^\infty(M).$$

We define T_pM to be the collection of all derivations at p.

It is straightforward to check that T_pM is a linear subspace of $C^\infty(M)^*$.

Definition. Let $F : M \to N$ be a smooth map between smooth manifolds. Let $p \in M$. We define the *differential of F at p*, denoted $dF_p : T_pM \to T_{F(p)}N$, by

$$dF_p(v)f := v(f \circ F).$$

It is easy to check that given $v \in T_pM$, $dF_p(v)$ is indeed a derivation at $F(p)$.

Here are some properties of the differential:

Proposition. *Let $F : M \to N$ and $G : N \to P$ be smooth maps between smooth manifolds. Let $p \in M$.*

(i) The differential dF_p is linear.
(ii) The chain rule is satisfied: $d(G \circ F)_p = dG_{F(p)} \circ dF_p$.
(iii) The differential of the identity is the identity: $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$.
(iv) If F is a diffeomorphism, then dF_p is an isomorphism with $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Proof. We show (iii); the proofs for the others are routine. Let $v \in T_pM$, and let $f \in C^\infty(P)$. Then

$$dG_{F(p)}(dF_p(v))f = dF_p(v)(f \circ G) = v(f \circ G \circ F) = d(G \circ F)_p f. \quad \square$$

The following lemma shows that derivations only need local data.

Lemma. *Let M be a smooth manifold. Let $p \in M$, and let $v \in T_pM$. Suppose $f, g \in C^\infty(M)$, and $f = g$ on a neighbourhood around p. Then $vf = vg$.*

Proof. Define $h := f - g$. Observe that $\operatorname{supp} h \subseteq M \backslash \{p\}$, because $f = g$ on a neighbourhood of p. Let $\psi \in C^\infty(M)$ be a smooth bump function for $\operatorname{supp} h$ supported in $M\backslash\{p\}$.Then $h = h\psi$, so $v(h) = v(h\psi) = 0$. $\square$

The following proposition shows that we can identify T_pU with T_pM, where $U \subseteq M$ is an open submanifold.

Proposition. *Let M be a smooth manifold, and let $U \subseteq M$ be open with inclusion map $\iota : U \hookrightarrow M$. Let $p \in U$. Then $d\iota_p : T_pU \to T_pM$ is a linear isomorphism.*

Proof. Let B be a neighbourhood of p such that $\overline{B} \subseteq U$.

Let us show injectivity. Suppose $v \in T_pU$, and $d\iota_p(v) = 0$. Let $f \in C^\infty(U)$. Let $\widetilde{f} \in C^\infty(M)$ be an extension of f that agrees on $\overline{B}$. Then the lemma implies that $v(f) = v(\widetilde{f}|_U) = v(\widetilde{f} \circ \iota) = d\iota_p(v)\widetilde{f} = 0$.

Next, let us show surjectivity. Let $w \in T_pM$. Define $v : C^\infty(U) \to \mathbb{R}$ by $vf := w\widetilde{f}$, where $\widetilde{f} : M \to \mathbb{R}$ is any smooth map such that $\widetilde{f}$ and f agree on $\overline{B}$. The lemma guarantees that this map is well-defined. It is straightforward to check that v is indeed an element of T_pU. Finally, $d\iota_p(v)g = v(g \circ \iota) = wg$, as desired. $\square$

Let $p, v \in \mathbb{R}^n$, and let $f \in C^\infty(\mathbb{R}^n)$. Recall that we define the *directional derivative of f at p in the direction v* by

$$D_v|_p f := \left.\frac{d}{dt}\right|_{t=0} f(p + tv).$$

From calculus, we know that $D_v|_p$ is a derivation at p, so $D_v|_p \in T_p\mathbb{R}^n$.

Proposition. *Let $p \in \mathbb{R}^n$. Then the map $\Phi : \mathbb{R}^n \to T_p\mathbb{R}^n$, $v \mapsto D_v|_p$ is a linear isomorphism.*

Proof. Linearity follows because $D_v|_p f = \operatorname{grad} f_p \cdot v$. Further, if $v = (v^i)$, then we can write $D_v|_p f = v^i \partial_i|_p f$.

Let us show injectivity. Let $v = (v^i)$, and suppose $D_v|_p = 0$. Let π^j denote the projection onto the jth coordinate. Then $0 = D_v|_p \pi^j = v^i \partial_i|_p \pi^j = v^j$.

Next, let us show surjectivity. Let $w \in T_p\mathbb{R}^n$, and define $v := w(\pi^i)e_i$. Let $f \in C^\infty(\mathbb{R}^n)$. Taylor's Theorem implies that we can write

$$f = f(p) + (\pi^i - p^i)\partial_i|_p f + R,$$

where $R \in C^\infty(\mathbb{R}^n)$ satisfies $w(R) = 0$. Then $wf = w(\pi^i)\partial_i|_p f = D_v|_p f$. $\square$

Proposition. *Let M be a smooth manifold of dimension n. Let $p \in M$. Then T_pM is an n-dimensional vector space.*

Proof. Let (U, φ) be a chart around p. Then we can write

$$T_pM \cong T_pU \cong T_{\varphi(p)}(\varphi(U)) \cong T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n,$$

where the second equivalence follows because φ is a diffeomorphism. $\square$

Definition. Let M be a smooth manifold, and let (U, φ) be a chart containing $p \in M$. We define the *coordinate vectors at p induced by φ* by

$$\partial_i|_p := d(\varphi^{-1})_{\varphi(p)}(\partial_i|_{\varphi(p)}).$$

The previous proposition implies that $\{\partial_i|_p\}$ forms a basis for T_pM.