Modelling Vibrations of Suspension Bridges

Abstract

We measure the relationship between the normal modes of a bridge and the effect on displacement, of a man jumping on the bridge. We begin with a simple model, making many approximations. We gradually improve our model, with the aim of a realistic simulation. We then compute the bridges normal modes. After modelling a man jumping on the bridge, we measure the maximum vertical displacement as a function of jumping frequency. We deduce maximum vertical displacement occurs when the man jumps at a frequency of a normal mode. We also discover the effects of destructive and constructive wave interference.

1 Introduction

On July 1st 1940, Tacoma Bridge, the third largest suspension bridge in the world, collapsed due to an underestimated effect called aeroelastic flutter [1]. Aeroelastic Flutter is the vibration of an elastic structure, caused by the positive damping exerted by a fluid flow. In the case of Tacoma and many Bridges across the world, the fluid flow was wind. We are exposed to countless elastic structures in our lives, so what causes this effect? Why doesn't this happen more often? How can we prevent it?

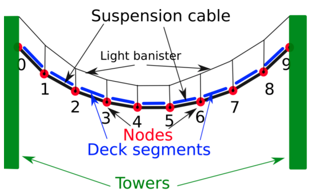
It seems the force has a greatest effect, when it resonates with the bridge, that is when the frequency of oscillations and normal modes are the same. In this investigation, much like in the Mythbusters Break Step Bridge episode [2], the force exerted will be a person jumping on the bridge at a set frequency. We will measure the relationship between the frequency of jumping and the maximum displacement reached by the bridge, comparing that with the normal modes of the bridge. We should then be able to advise engineers on the future construction of suspension bridges.

Initially, to calculate the normal modes of the bridge, we model the bridge as a system of nodes connected by rods. Throughout the investigation, we progressively improve our model until it is sufficiently accurate to simulate motion. This includes taking into account the bending and extension of the cable. Using some parameters, we compute the bridges normal modes. We then simulate a man jumping on the bridge with different frequencies and positions, measuring the maximum vertical displacement reached. We find greatest vertical displacement is reached when the man jumps at the frequency of a normal mode.

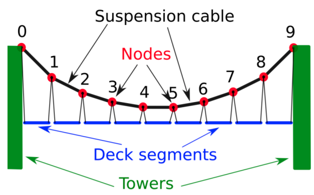
2 Bridge Model

2.1 Bridge Design

We will base our model of the bridge on the simple curved deck design 9a over the flat deck design 9b. This is because it's easier to model nodes that are equally spread and we won't have to model the support cables.



(a) Simple Curved Deck



(b) Flat Deck

Figure 1

We will include two towers on either side, with one cable in between. This means we will not take into account any twisting of the cable, limiting us to only 2 dimensions. We can also neglect the weight of the banisters and cable, allowing us to model oscillations with greater ease, without compromising accuracy.

2.2 Modelling the Bridge (as a Catenary)

One way to model the bridge, would be to view it as a continuous piece of string with linear density $w_0 = \frac{M}{L}$ and length $L = 2s$ as show in Figure 2. This would mean the bridge decking has its weight continuously spread along the cable.

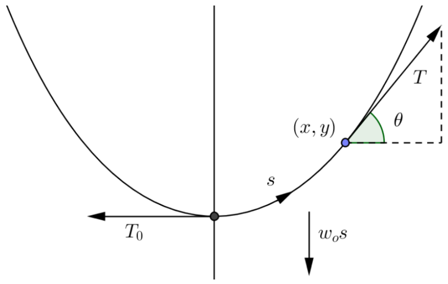


Figure 2: Continuous Segment of Bridge

At the point of static equilibrium, by considering vertical and horizontal components separately:

$$T_0 = T\cos(\theta).$$
$$w_0 s = T\sin(\theta).$$

By setting $c = \frac{w_0}{T_0}$, combining the two equations and differentiating with respect to x:

$$\frac{d^2y}{dx^2} = c\frac{dy}{ds}.$$

Using the derivative of an arc length [?], and the substitution $p = \dfrac{dy}{dx}$:

$$\frac{dp}{dx} = c\sqrt{1+p^2}.$$

This can then be solved using a separation of variables:

$$\ln(\sqrt{1+p^2} + p) = cx + B.$$

Using the condition that at the point $x = 0$, $\dfrac{dy}{dx} = p = 0$, we find $B = 0$, thus after some manipulation:

$$p = \frac{dy}{dx} = \frac{e^{cx} - e^{-cx}}{2} \qquad \Longrightarrow \qquad y = \frac{e^{cx} - e^{-cx}}{2c} + C.$$

Finally, as a profile for the bridge at rest, we get:

$$\boxed{y = \frac{T_0 L}{M}\cosh\left(\frac{M}{LT_0}x\right) + C.} \tag{1}$$

The integration constant can be used to set the x coordinate of the towers.

This profile is called a Catenary and is a very interesting function. Its equation was found by Leibniz, Huygens and Johann Bernoulli in 1691 and is the locus of the focus of a parabola rolling along a straight line [3]. If instead, we had opted for the flat deck model, this profile equation would be parabolic as the weight would be distributed differently. Unfortunately this equation will not be used later on, as another derivation is easier to model.

2.3 Modelling the Bridge (as derived in the assignment)

Another way to model the behaviour of the bridge is by representing it as a system of N nodes with positions (X_i, Z_i) and length L, each connected to adjacent nodes by a light rods. The end nodes are connected to fixed points at the towers with index 0 and N.

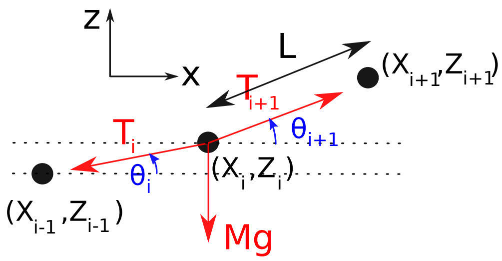


Figure 3: Force diagram for single node

If each deck segment has mass m, we know the load on each node is $m/2$ which we will denote M. By using the figure 3, much like the previous method by considering the static equilibrium:

$$T_i \cos(\theta_i) - T_i \cos(\theta_{i+1}) = 0.$$

$$T_{i+1} \sin(\theta_{i+1}) - T_i \sin(\theta_i) = Mg. \tag{2}$$

$$T_i \cos(\theta_i) = T.$$

$$T(\tan(\theta_{i+1}) - \tan(\theta_i)) = Mg. \tag{3}$$

By approximating $\tan(\theta_i) \approx \sin(\theta_i) \approx \frac{(Z_i - Z_{i-1})}{L}$, and through some manipulation of (3):

$$\frac{d^2 Z_i}{dx^2} \approx \frac{Mg}{TL}, \qquad \text{then as } N \to \infty, \qquad \boxed{Z(x) = \frac{Mg}{2TL}x^2}. \tag{4}$$

2.4 Equations of Motion for The Nodes

In order to build our dynamic equations, we will still base our model off Figure 3, but make some further assumptions. We set connecting strings between nodes to behave like light rigid springs, with spring constant K_{sc}, and also set N to be even and finite. The summary of the derivation of the system will be explained below, referring to the assignments appendix.

The derivation begins by assuming N to be even and thus $\theta_{N/2} = 0$, we can then form an equation for the successive angles based on (3). By using this equation, we can calculate the distance between two adjacent nodes in Z and X directions (assuming L constant). This equation is complicated, but can be simplified using the approximation $\sqrt{1+x} = 1 + \frac{x}{2}$.

With this simplified equation and the assumption that the cable is elastic, we use Hooks law to derive an equation for the vertical components of the forces acting on the nodes, combining the two gives us a set of dynamic equations as given below:

$$\begin{aligned} M\frac{d^2 X_i}{dt^2} = K_{sc}\Big(& (X_{i-1} - X_i)\cos^2(\theta_i) + (X_{i+1} - X_i)\cos^2(\theta_{i+1}) \\ & + (Z_{i-1} - Z_i)\cos(\theta_i)\sin(\theta_i) + (Z_{i+1} - Z_i)\cos(\theta_{i+1})\sin(\theta_{i+1})\Big). \end{aligned} \tag{5}$$

$$\begin{aligned} M\frac{d^2 Z_i}{dt^2} = K_{sc}\Big(& (Z_{i-1} - Z_i)\cos^2(\theta_i) + (Z_{i+1} - Z_i)\cos^2(\theta_{i+1}) \\ & + (X_{i-1} - X_i)\cos(\theta_i)\sin(\theta_i) + (X_{i+1} - X_i)\cos(\theta_{i+1})\sin(\theta_{i+1})\Big). \end{aligned} \tag{6}$$

3 Calculating Normal Modes

3.1 Solving for Base Case

Now we have derived a set of equations for the system of nodes, we will seek to find the normal modes (Standing waves) of the bridge. A normal mode is when all modes move sinusoidally with the same frequency and with a fixed phase relation, like Figure 4. Thus a general solution would be of the form $X_i(t) = Y_i \sin(\omega t)$ with ω as the frequency.

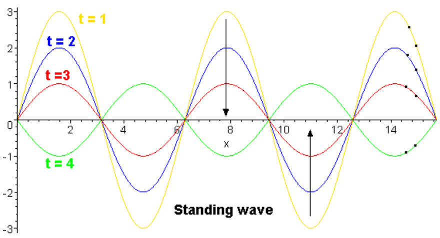


Figure 4: Normal Modes

Substituting into (5) and (6), we get equations of the form $AY = -\omega^2 Y$ for displacement in X and Z directions respectively. As an initial approximation, we can disregard the $sin\theta cos\theta$ components.

$$X: \qquad A_{i,j} = \frac{K_{sc}}{M}(-\delta_{i,j}(\cos^2\theta_i + \cos^2\theta_{i+1}) + \delta_{i+1,j}\cos^2\theta_{i+1} + \delta_{i-1,j}\cos^2\theta_i). \tag{7}$$

$$Z: \qquad A_{i,j} = \frac{K_{sc}}{M}(-\delta_{i,j}(\sin^2\theta_i + \sin^2\theta_{i+1}) + \delta_{i+1,j}\sin^2\theta_{i+1} + \delta_{i-1,j}\sin^2\theta_i). \tag{8}$$

Q1

We notice (7) is difficult to solve, so initially we take the tension to be very large, which means $\theta \to 0$, then using (3):

$$-\omega^2 Y_i = \frac{K_{sc}}{M}(Y_{i-1} - 2Y_i + Y_{i+1}).$$

Then as $N \to \infty$, i becomes continuous and Y a function of x in the horizontal direction:

$$Y_i = Y(iL).$$

We can use (10) to find Y_{i+1} and Y_{i-1} through a Taylor Expansion:

$$Y_{i\pm1} = Y(iL \pm L) \approx Y(iL) \pm L\frac{d}{dx}\Big(Y(iL)\Big) + \frac{L^2}{2}\frac{d^2}{dx^2}\Big(Y(iL)\Big).$$

Which we substitute into (9):

$$Y_{i-1} - 2Y_i + Y_{i+1} = L^2\frac{d^2}{dx^2}\Big(Y(iL)\Big).$$

Finally

$$\boxed{-\omega^2 Y(x) = \frac{L^2 K_{sc}}{M}\frac{d^2 Y(x)}{dx^2}.} \tag{9}$$

Q2

To solve this equation, we make the substitution $Y(x) = e^{\lambda x}$, such that:

$$-\omega^2 e^{\lambda x} = \frac{L^2 K_{sc}}{M} \lambda^2 e^{\lambda x}.$$

Which we arrange to find λ:

$$\lambda = \pm\sqrt{-\frac{\omega^2 M}{L^2 K_{sc}}} = i\frac{\omega}{L}\sqrt{\frac{M}{K_{sc}}}.$$

The general solution is hence:

$$Y(x) = A\cos\left(\frac{x\omega}{L}\sqrt{\frac{M}{K_{sc}}}\right) + B\sin\left(\frac{x\omega}{L}\sqrt{\frac{M}{K_{sc}}}\right).$$

And using boundary conditions:

$$Y(0) = A = 0.$$
$$Y(D) = B\sin\left(\frac{\omega D}{L}\sqrt{\frac{M}{K_{sc}}}\right) = 0.$$

This means:

$$\frac{\omega M}{L}\sqrt{\frac{M}{K_{sc}}} = n\pi, \qquad \text{for } n \in \mathbb{Z}.$$

As all components are positive, the possible values of ω are:

$$\boxed{\omega = \frac{L\pi n}{D}\sqrt{\frac{K_{sc}}{M}}}, \qquad \text{for } n \in \mathbb{N}. \tag{10}$$

3.2 Parameters

Another way to solve equations (7) and (8), is numerically using Python. For the remainder of this investigation, we define the parameters of the bridge by the following:

- $N = 19$.
- $M = 15kg$.
- $K_{sc} = 5x10^7 N/m$.
- Tower angle= 30 deg.
- $K_{bend} = 10^4 N/m$ (Used later).
- Weight of man= $80kg$.

Q4

With these parameters, we are able to calculate the distance between the two towers '$\mathcal{L}$' and the depth of the lowest point of the bridge at rest 'h'. We form these equations, paying close attention to the indexing:

$$\mathcal{L} = Lsum_{n=1}^{N-1} L\cos(\theta_n). \qquad h = \sum_{n=1}^{\frac{N}{2}-1} L\sin(\theta_n).$$

Using the 'get_distance_and_depth()' function, we find that $\mathcal{L} = 17.988m$ (3d.p.) and $h = 2.660m$ (3d.p.).

3.3 Solving Using Python

Our program inputs these parameters and calculates the eigenvalues and corresponding eigenvectors of the equation $AY = -w^2 Y$, which returns the eigenvalues with values $\lambda = -\omega^2$. These frequencies correspond to the normal modes of the bridge for the X and Z directions seperately:

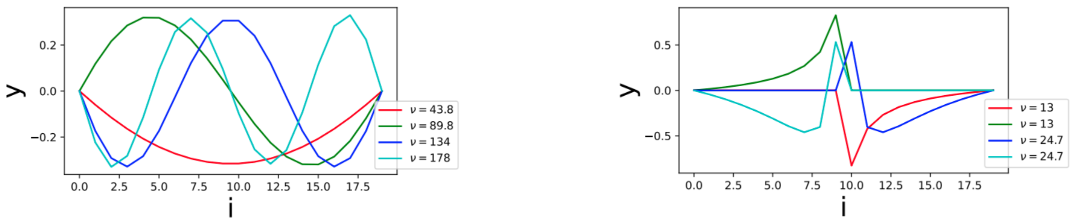


Figure 5: 4 Lowest Normal Modes in X and Z

These calculations are a good start, but they don't look quite right and we can improve our model further. We will now take into account the force it takes to bend the cable at the nodes and combine the two equations (7) and (8). Again, we will assume solutions of the form $X_i(t) = x_i \sin(\omega t)$ and $Z_i(t) = z_i \sin(\omega t)$.

$$\begin{aligned} -\omega^2 z_i =& \frac{K_{sc}}{M}\Big((z_{i-1} - z_i)\sin^2(\theta_i) + (z_{i+1} - z_i)\sin^2(\theta i + 1) + (x_{i-1} - x_i)\sin(\theta_i)\cos(\theta_i) \\ &+ (x_{i+1} - x_i)\sin(\theta_{i+1})\cos(\theta_{i+1})\Big) + \frac{K_{bend}}{M}\left(\frac{z_{i-1} + z_{i+1}}{2} - z_i\right). \end{aligned} \tag{11}$$

$$\begin{aligned} -\omega^2 x_i =& \frac{K_{sc}}{M}\Big((x_{i-1} - x_i)\cos^2(\theta_i) + (x_{i+1} - x_i)\cos^2(\theta_{i+1}) + (z_{i-1} - z_i)\sin(\theta_i)\cos(\theta_i) \\ &+ (z_{i+1} - z_i)\sin(\theta_{i+1})\cos(\theta_{i+1})\Big). \end{aligned} \tag{12}$$

By rearranging equations (11) and (12) much like we did for equations (7) and (8) into Matrix form, we will again use our Python program to calculate the Normal Modes with the improved model. The first 5 Updated Normal mode profiles are returned:

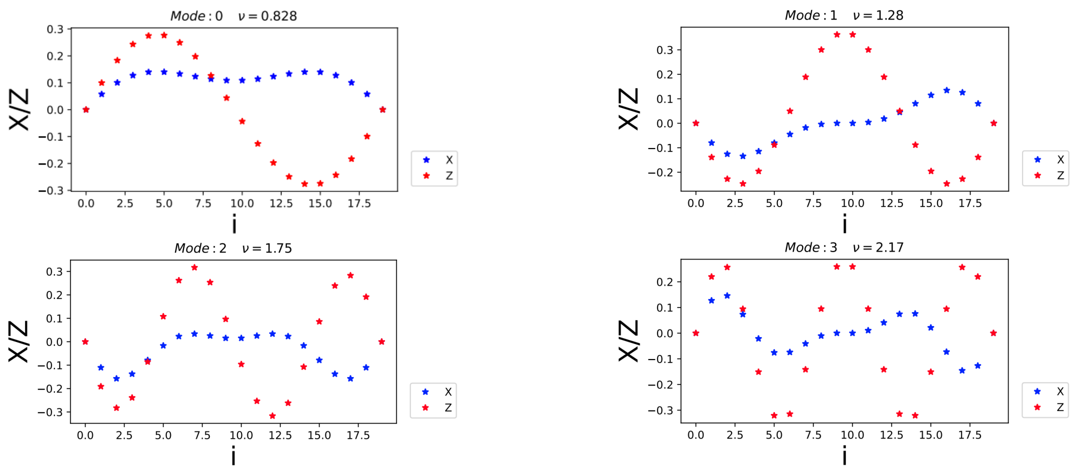


Figure 6: New 4 Lowest Normal Modes in X and Z

Q3

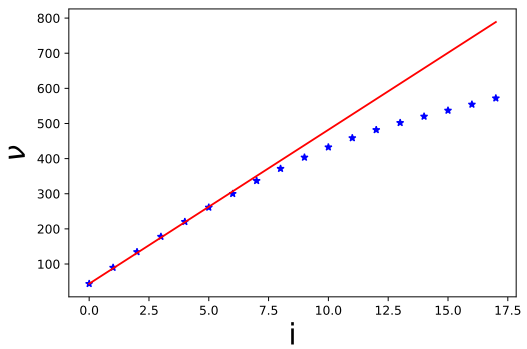
We notice that the normal mode profiles in 5 for the horizontal displacements look unnatural. This is because without a bending coefficient, bridge segments are articulated causing unrealistic bridge deformations. with the addition of our K_{bend} coefficient, the horizontal displacements look more natural.

Because we know the relative amplitudes produced in Figure 5 are correct and the amplitudes produced in 6 are set arbitrarily by our code, we can use this data to justify neglecting the initial approximations in equations chapter 3.1

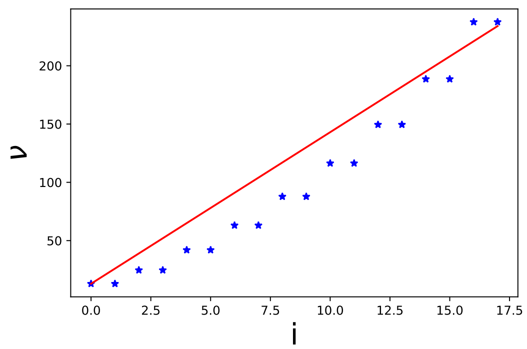
First we compare vertical displacements between Figure 5 and 5. We notice that the profile of the normal modes are barely effected by the inclusion of horizontal displacements in the calculations. For example the similarity between the green mode and mode 0. This makes practical sense, as horizontal displacements are relatively small compared to vertical displacements and would have little effect on the overall profile.

Next we compare horizontal displacements between Figure 5 and 5. We notice there is a large difference between the profile and amplitude of the nodes. For example between the red mode and mode 0, which have similar frequencies. This also makes sense, as vertical displacements are much larger than horizontal displacements and would have a large effect of the equations of motion.

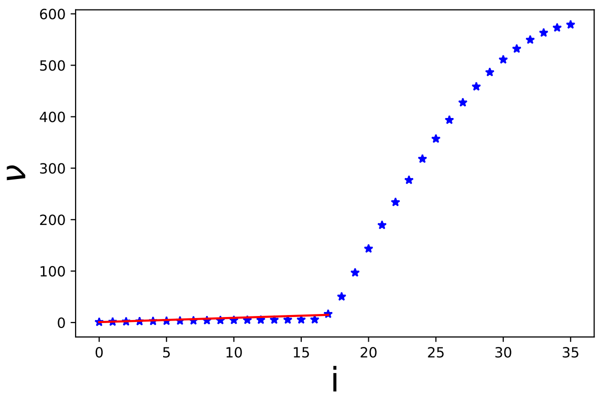
We can also compare the accuracy of normal mode frequencies between the two approaches, which are represented as graphs below, where the red line is the frequency we calculated in Q2.



(a) X Normal Mode frequencies



(b) Z Normal Mode frequencies



(c) XZ Normal Mode frequencies

These support our initial statements, as the combined XZ calculations are very accurate (up to a reasonable high threshold).

Therefore, it would be reasonable to neglect the horizontal displacements in the equation for vertical displacements. But it would not be reasonable to neglect vertical displacements in the equation for horizontal displacements. For the remainder of this investigation we will use the more complicated, but more accurate system of equations (5) and (6).

4 Introducing External Forces

In this section we will now model a man jumping on the bridge with frequency v and measure the maximum vertical displacement with a range of frequencies.

4.1 Friction and Applied Load

As a starting point, we will also take into account any friction within the system, which we denote by $-\Gamma\frac{dX_i}{dt}$. Using (5) and (6), and the addition of a man at rest ($Load_i$), our improved model is now:

$$
\begin{aligned}
M\frac{d^2Z_i}{dt^2} =& K_{sc}\Big((Z_{i-1}-Z_i)\sin^2(\theta_i) + (Z_{i+1}-Z_i)\sin^2(\theta i+1) + (X_{i-1}-X_i)\sin(\theta_i)\cos(\theta_i). \\
&+ (X_{i+1}-X_i)\sin(\theta_{i+1})\cos(\theta_{i+1})\Big) + K_{bend}\left(\frac{Zi-1+Z_{i+1}}{2} - Z_i\right) \\
&- \Gamma\frac{dZ_i}{dt} + Load_i(t).
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
M\frac{d^2X_i}{dt^2} =& K_{sc}\Big((X_{i-1}-X_i)\cos^2(\theta_i) + (X_{i+1}-X_i)\cos^2(\theta_{i+1}) + (Z_{i-1}-Z_i)\sin(\theta_i)\cos(\theta_i). \\
&+ (Z_{i+1}-Z_i)\sin(\theta_{i+1})\cos(\theta_{i+1})\Big) - \Gamma\frac{dX_i}{dt}.
\end{aligned} \tag{14}
$$

We notice that (13) and (14) are systems of 2(N-2) second ordinary differential equations which we can write as 4(N-2) first order differential equations. After ignoring $Load_i$, we begin with a simple rearrangement: **Q5**

$$
\begin{aligned}
X_i'\prime &= \frac{F_{x,i}}{M} - \frac{\Gamma X_i'}{M}. \\
Z_i'\prime &= \frac{F_{z,i}}{M} + \frac{K_{bend}}{M}\left(\frac{Z_{i-1}+Z_{i+1}}{2} - Z_i\right) - \frac{\Gamma Z_i'}{M}.
\end{aligned}
$$

We make the substitutions:

$$
\begin{aligned}
u_i &= X_i & p_i &= Z_i. \\
v_i &= X_i' & q_i &= Z_i'.
\end{aligned}
$$

After the substitutions, we have our $4(N-2)$ first order ordinary differential equations:

$$
\begin{aligned}
u_i' &= v_i, \\
v_i' &= \frac{F_{x,i}}{M} - \frac{\Gamma v_i}{M}. \\
p_i' &= q_i. \\
q_i' &= \frac{F_{z,i}}{M} + \frac{K_{bend}}{M}\left(\frac{p_{i-1}+p_{i+1}}{2} - p_i\right) - \frac{\Gamma q_i}{M}.
\end{aligned}
$$

Solving first order ODE's using python is much easier than solving second order ODE's.

Initially we will take $Load_i$ to be constant, which represents a man standing on the bridge with no movement. By running our program, we compute the maximum displacement of the bridge as a function of time. Included are snippets of the position of the nodes at times $0.1, 1, 10$ and 100 seconds.

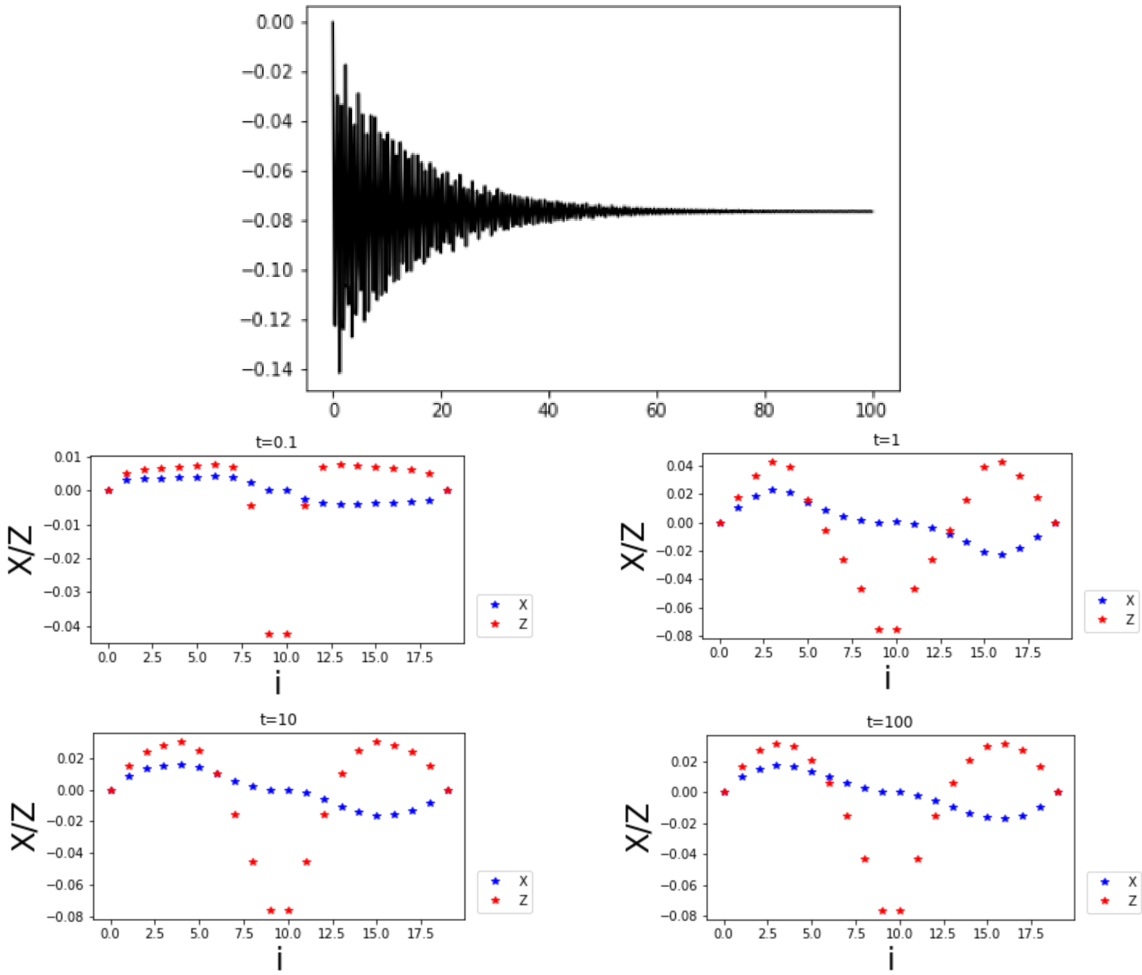


Figure 8: 4 Lowest Normal Modes in X and Z

4.2 Man Jumping on Bridge

Now we are finally able to model a man jumping on the bridge, we do this by viewing his jumping as a force applied sinusoidaly with frequency v, that only takes effect when he is not in the air. Where i is his position on the bridge, we model this using the equation.

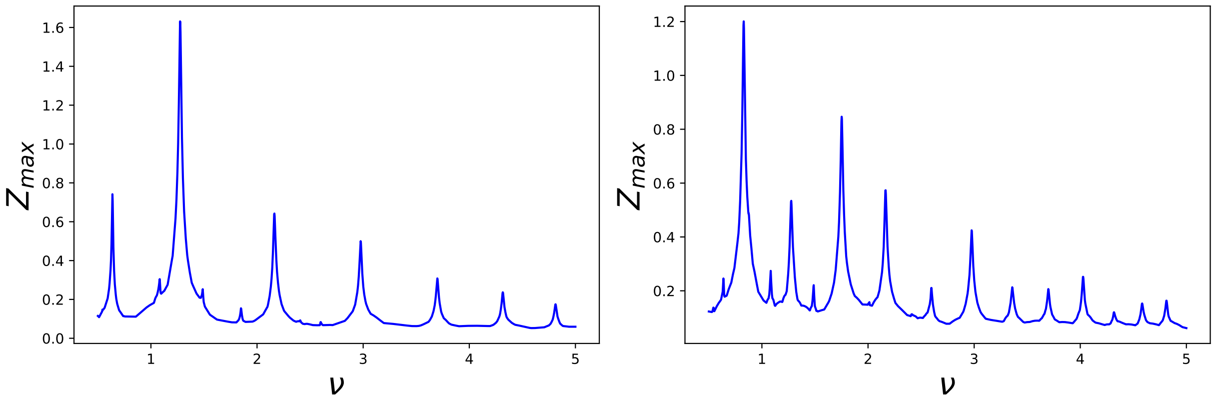
$$Load_i = -200H(\sin(2\pi vt) \tag{15}$$

Where $H(x) = x$ if $x > 0$ and 0 otherwise.

5 Conclusion

Q6

Including (13) and (14), we finally have the mathematical tools to compute the movement of the bridge when acted upon by a force, in this case a man. We use Python to simulate a man jumping on the bridge for $60s$, and record the maximum vertical displacement reached by the man when he is at position $i = 10$ and $i = 7$, after repeating this for a range of frequencies between $0.5Hz$ and $5Hz$, we generate a figure of the maximum vertical amplitude as a function of the mans frequency v. For these graphs, 2000 frequencies were tested per graph.



(a) $i = 10$

(b) $i = 7$

Figure 9

Frequencies at which maximum displacements occurs:

$i = 10$: [0.64, 1.25, 2.16, 2.97, 3.70, 4.31, 4.82].

$i = 7$: [0.88, 1.27, 1.76, 2.16, 2.59, 2.97, 3.31, 3.70, 4.03 4.31, 4.59, 4.82].

Normal Modes: [0.83, 1.28 1.75, 2.17, 2.59, 2.98, 3.36, 3.70, 4.03, 4.32, 4.58, 4.81].

By comparing the graphs for $i = 10$ and $i = 7$, we notice greatest displacement always occurs at the frequency of a normal mode. However at i=7, there are a greater number of resonating frequencies and the graph looks less regular.

This could be explained by the waves reflecting from the towers and constructively or destructively interfering with the mans jumping, depending on the wavelength/frequency and position on the bridge. When the man jumps at the centre of the bridge ($i = 10$), his jumping will destructively interfere with the reflection of his wave for an alternating number of normal mode frequencies, which perfectly coincides with what we observe.

This brings us perfectly back to out initial questions, what causes this effect? Why doesn't this happen more often? How can we prevent it?

This effect is clearly caused when positive damping occurs at a normal mode, as long as its not applied at a position on the bridge where the reflection could cause destructive interference. This resonance and extreme vertical displacement could put extreme pressure on the system and potentially cause disaster, much like that of Tacoma bridge.

From an engineers point of view, there are a number of ways to tackle this problem. One way would be to design the bridge so that the resonant frequencies of the bridge did not coincide with any natural periodic forces applied to the bridge. This could be people walking, cars stopping and starting, or wind blowing.

Another way to tackle this problem would be to install dampers onto the bridge, much like how engineers tackled Millennium Bridge's resonance in London. These dampers would absorb the energy and prevent any extreme movement.

Q7

For our investigation, we used a curved deck bridge design to base our equations. It might be practical to consider if this same outcome could be applied to flat deck bridges.

As I have already explained in Chapter 2.2, the most accurate way to model a curved deck bridge would be to model is as a Catenary, as in (1). However, we opted to use equations (3) and (4) which inaccurately model it as a parabola.

If a flat deck bridge were to be modelled by N nodes, we would notice that the weight of the bridge would be unevenly spread between these nodes. The profile of a flat deck is hence perfectly modelled by a parabola, and our set of equation would be even more accurate for this style of bridge. This is shown in this youtube video at 2:57 [5].

However, throughout this investigation, we have made many approximations of the bridges movement. For instance we do not take into account the rigidity between the end nodes and the path leading to the bridge, if a bridge was built this way, this could have a major impact on our results. We do also not reproduce the failure mechanism of the bridge, meaning we do not have a threshold to compare the success of various interventions.

References

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