Physonomicon

Brasides

March 1, 2024

Contents

1 Derivations for One Dimensional Motion

1.1 Average velocity $\bar{v}$ vs Instantaneous velocity v

1.1.1 tl;dr

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} \qquad avg\ velocity$$

$$x(t) = At^n \qquad position\ as\ function\ of\ time$$

$$\bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

$$Definition\ of\ v(t): \quad v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \iff v(t) = \frac{dx}{dt}$$

Where A is a constant and n is an integer.

1.1.2 Defining $\bar{v}$

Average velocity $\bar{v}$ is defined for position x_1 at time t_1 and position x_2 at time t_2:

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1}$$

Because we may consider the position x to be a function of time, we may instead choose to rewrite this as:

$$\bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

Where $x(t) = At^n$

In order to go from $\bar{v}$ to instantaneous velocity v, we swap in values that are more appropriate for calculus:

Let $t_1 = t$ *and* $t_2 = t + \Delta t$

Now we may consider the limit as $\Delta t \to 0$, which we shall define as instantaneous velocity $v(t)$:

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \iff v(t) = \frac{dx}{dt} = \frac{d}{dt}x(t) \tag{1.1}$$

Where the function $x(t)$ has the form: Where A is a constant and n is an integer.

1.2 Average acceleration $\bar{a}$ vs Instantaneous acceleration a

1.2.1 tl;dr

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} \qquad avg\ velocity$$

$$v(t) = \frac{dx}{dt} \qquad inst.\ velocity\ as\ function\ of\ time$$

$$\bar{a} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

$$Definition\ of\ a(t): \quad a(t) = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} \iff a(t) = \frac{d}{dt} v(t) = \frac{d^2 x}{dt^2}$$

1.2.2 Defining average velocity $\bar{a}$

The definition of acceleration has a similar form to the definition of velocity above, and is written in its most simple terms as:

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1}$$

Because we may consider instantaneous velocity as a function of time $v(t)$, we may instead choose to write this as:

$$\bar{a} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

Now we define the terms differently to more clearly apply the limit to arrive at the derivative:

$$t_1 = t, \quad t_2 = t + \Delta t$$

Now we may define the instantaneous acceleration a to be the limit of the instantaneous velocity when the $\Delta t \to 0$:

$$a = \lim_{x \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} \iff a = \frac{d}{dt} v(t) \tag{1.2}$$

1.3 Notation

As a simplification, we always take the initial time $t_0 = 0$ and define our equations with the supposition that acceleration is constant. Thus we write:

$$t = t_f$$

$$t = t + 0 = t - t_0 = \Delta t$$

$$\boxed{\Delta t = t} \tag{1.3}$$

$$\Delta x = \Delta x - x_0$$

$$\Delta v = v - v_0$$

$$\bar{a} = a = constant$$

1.4 Deriving the first kinematic equation

1.4.1 Position x as a function of $\bar{v}$

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

$$\bar{v} = \frac{x - x_0}{t}$$

$$x = x_0 + \bar{v}t$$

1.4.2 Final Velocity v as a function of time t and acceleration $a = \bar{a}$

$$a = \frac{\Delta v}{\Delta t} \qquad t = \Delta t$$

$$a = \frac{v - v_0}{t}$$

$$v = v_0 + at \tag{1.4}$$

1.4.3 Final Velocity v as a function of distance x and acceleration $a = \bar{a}$

Starting with

$$v = v_0 + at$$

Rearranging for t

$$t = \frac{v - v_0}{a}$$

By definition $\bar{v}$

$$\bar{v} = \frac{v_0 + v}{2}$$

Now substitute into equation for position

$$x = x_0 + \bar{v}t$$

$$x = x_0 + \frac{v_0 + v}{2} * \frac{v - v_0}{a}$$

$$x = x_0 + \frac{v^2 - v_0^2}{2a}$$

Solving for v^2

$$v^2 = v_0^2 + 2a(x - x_0)$$

Or

$$\boxed{v^2 = v_0^2 + 2a\Delta x}$$

1.4.4 Deriving position x as a function of velocity v, time t, and acceleration $a = \bar{a}$

We begin with position as a function of $\bar{v}$:

$$x = x_0 + \bar{v}t$$

We know we want v, t, and a, so we start with the equation for v and rearrange to substitute $\bar{v}$:

$$v = v_0 + at$$

$$\textit{add } v_0 \textit{ to both sides}$$

$$v + v_0 = 2v_0 + at$$

$$\frac{v + v_0}{2} = v_0 + \frac{at}{2}$$

$$\textit{substitute } \bar{v} = \frac{v + v_0}{2}$$

$$\bar{v} = v_0 + \frac{1}{2}at$$

$$\textit{substite}$$

$$x = x_0 + (v_0 + \frac{1}{2}at)t$$

$$\boxed{x = x_0 + v_0 t + \frac{1}{2}at^2}$$

1.5 Kinematic Equations from Integral Calculus

1.5.1 $v(t) = v_0 + at$ by Integration

We defined the relationship between acceleration and velocity in (1.2) as:

$$a = \frac{d}{dt}v(t)$$

Switching sides, as we are more interested in $v(t)$, and rewriting a as a function of time $a(t)$:

$$\frac{d}{dt}v(t) = a(t)$$

$$\int \frac{d}{dt}v(t)\,dt = \int a(t)\,dt + C_1$$

$$v(t) = \int a(t)\,dt + C_1 \tag{1.5}$$

For constant acceleration in (1.5):

$$v(t) = \int a\,dt = at + C_1$$

$$v(0) = v_0 = a(0) + C_1$$

$$v_0 = C_1$$

$$\therefore \boxed{v(t) = v_0 + at} \tag{1.6}$$

1.5.2 $x(t)$ by Integration

We defined the function of position in (1.1) as:

$$\frac{d}{dt}x(t) = v(t)$$

$$\int \frac{d}{dt}x(t)\,dt = \int v(t)\,dt$$

$$x(t) = \int v(t)\,dt + C_2 \tag{1.7}$$

Substite equation for $v(t)$ from (1.6) in (1.7)

$$x(t) = \int v_0 + at\,dt$$

$$x(t) = v_0 t + \frac{1}{2}at^2 + C_2$$

$$x(0) = x_0 = v_0(0) + \frac{1}{2}a(0)^2 + C_2$$

$$x_0 = C_2$$

$$\therefore \boxed{x(t) = x_0 + v_0 t + \frac{1}{2}at^2} \tag{1.8}$$

Example on pg 136 of the textbook.

2 Projectile Motion Derivations

Motion of an object when:

$$a_x = 0$$

2.1 Defining vector quantities of motion for two dimensional vectors

Where $x(t)$ and $y(t)$ are defined for two dimensional position vector $\vec{r}(t)$ and similarly $v_x(t)$ and $v_y(t)$ are defined for velocity vector $\vec{v}(t)$, while $\vec{a}$ is similar but with $a_x = 0$:

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

$$\vec{v}(t) = v_x(t)\hat{i} + v_y(t)\hat{j}$$

$$\vec{a}(t) = 0\hat{i} + a_y(t)\hat{j}$$

2.2 Rewriting kinematic equations for x and y directions where $a_x = 0$

$$x(t) = x_0 + \bar{v}_x t$$

$$v_x(t) = v_{0,x}$$

$$x(t) = x_0 + v_{0,x} t$$

$$v_x^2(t) = v_{0,x}^2 t$$

$$y(t) = y_0 + \bar{v}_y t$$

$$v_y(t) = v_{0,y} + a_y t^2$$

$$y(t) = y_0 + v_{0,y} t + \frac{1}{2} a_y t^2$$

$$v_y^2(t) = v_{0,y}^2 + 2a_y(y - y_0)$$

2.3 Derivations of Projectile Motion Equations

2.3.1 Maximum height h

Requires knowing two of three: y, $v_{0,y}^2$, g
For $v_y = 0$, $y_0 = 0$, $a_y = -g$:

$$v_y^2 = v_{0,y}^2 + 2a_y(y - y_0)$$

$$0 = v_{0,y}^2 + 2(-g)(y - 0)$$

$$0 = v_{0,y}^2 - 2gy$$

$$\therefore \boxed{y = \frac{v_{0,y}^2}{2g}} \tag{2.1}$$

2.3.2 Direction of velocity θ_v

The direction of velocity does not need to be derived, as it relies on the triangle formed by v_x and v_y:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) \tag{2.2}$$

2.3.3 Time of Flight T_{tof}

For $y = y_0$, $a_y = -g$:

$$
\begin{aligned}
y - y_0 &= v_{0,y}t + \frac{1}{2}a_y t^2 \\
0 &= v_{0,y}t + \frac{1}{2}(-g)t^2 \\
0 &= v_{0,y}t - \frac{1}{2}gt^2 & v_{0,y} = \sin(\theta_0)v_0 \\
0 &= \sin(\theta_0)v_0 t - \frac{1}{2}gt^2 \\
0 &= t\left(\sin(\theta_0)v_0 - \frac{1}{2}gt\right) \\
0 &= \sin(\theta_0) - \frac{1}{2}gt \\
\frac{1}{2}gt &= \sin(\theta_0) \\
t &= \frac{2\sin(\theta_0)}{g}
\end{aligned}
$$

$$
\boxed{T_{tof} = \frac{2\sin(\theta_0)}{g}} \tag{2.3}
$$

2.3.4 Trajectory y as a function of x without time t

Need three of four: y, v_0, θ_0, x
For $a = -g$

$$x = v_{0,x}t$$

Solving for t:

$$
\begin{aligned}
t &= \frac{x}{v_{0,x}} \\
t &= \frac{x}{\cos(\theta_0)v_0}
\end{aligned}
$$

y-position:

$$
\begin{aligned}
y &= v_{0,y}t + \frac{1}{2}at^2 \\
y &= v_0\sin(\theta_0)t - \frac{1}{2}gt^2
\end{aligned}
$$

Substitute t:

$$y = v_0 \sin(\theta_0)\left(\frac{x}{\cos(\theta_0)v_0}\right) - \frac{1}{2}g\left(\frac{x}{\cos(\theta_0)v_0}\right)^2$$

$$\boxed{y = \tan(\theta_0)x - x^2\left(\frac{g}{2(v_0\cos\theta_0)^2}\right)} \tag{2.4}$$

2.3.5 Range

Starting with trajectory equation (2.4) where $y = 0$:

$$0 = \tan(\theta_0)x - x^2\left(\frac{g}{2(v_0\cos\theta_0)^2}\right)$$

$$x^2\left(\frac{g}{2(v_0\cos\theta_0)^2}\right) = \tan(\theta_0)x$$

$$x = \tan(\theta_0)\left(\frac{2(v_0\cos(\theta_0)^2)}{g}\right)$$

$$x = \frac{v_0^2 2\sin(\theta_0)\cos(\theta_0)}{g}$$

$$\boxed{x = R = \frac{v_0^2\sin(2\theta_0)}{g}} \tag{2.5}$$