1. I went over the Lane-Emden equation pretty quickly in class, so here's our chance to explore it further. None of these questions require you to code.

   (a) Starting from the hydrostatic equilibrium equation (top of page 2 in Lecture 2) and the polytropic model, derive the Lane-Emden eqaution (top of page 3). Explain your steps carefully.

   (b) Write the Lane-Emden equation as two coupled first order ODEs (the answer is on Worksheet 3).

   (c) Next we want to derive the third boundary condition, that $d\varrho/d\eta = 0$ at $\eta = 0$ (I think the first two are self-explanatory; let me know if they're not). To do that, consider a small volume at the centre of the star with a radius δ. Show that the pressure gradient $dp/dr \to 0$ as $\delta \to 0$, and hence that the density gradient should as well. (*Hint:* Use the hydrostatic equilibrium equation again.)

   (d) Finally, show that, for $n = 0$, the solution to the Lane-Emden equation is

$$\varrho(\eta) = 1 - \frac{\eta^2}{6}.$$

2. In class we discussed simple toy models for white dwarf stars. Before we handle a more complicated model, I want to return to these simple ones first. Recall that $M = 4\pi\rho_c\lambda^3 m$ and $r = \lambda\eta$.

   (a) For the non-relativistic model ($n = 1.5$), show that $M \propto \rho_c^{1/2}$, that $r \propto \rho_c^{-1/3}$, and hence that

$$M \propto R^{-2/3},$$

   where M is the total mass of the star and R is the radius of it.

   (b) For the relativistic model ($n = 3$), show that M is constant regardless of central density ρ_c. What is the value of M, in solar masses?

3. In reality, white dwarfs can be both nonrelativistic and relativistic. We need an equation of state that combines the two above. One simple way to do this (there are more complicated ways, but this gives us a good approximation) is to define a characteristic density ρ_0 where the two pressures are equal. This leads to

$$\rho_0 = \left(\frac{k_\mathrm{r}}{k_\mathrm{nr}}\right) = 3.789 \times 10^6 \text{ g cm}^{-3}.$$

   We'll then combine the equations of state in the following way:

$$p = \frac{p_\mathrm{nr}p_\mathrm{r}}{\sqrt{p_\mathrm{nr}^2 + p_\mathrm{r}^2}} = \frac{k_\mathrm{nr}\rho^{5/3}}{\sqrt{1 + (\rho/\rho_0)^{2/3}}}.$$

   Basically, this equation of state allows us to vary between a relativistic regime (where $p_\mathrm{r} > p_\mathrm{nr}$) and a non-relativistic regime (where $p_\mathrm{nr} > p_\mathrm{r}$). However, this equation of state is *not* a polytrope, and things get more complicated.

   (a) Combine equations 1 and 3 in the notes (bottom half of page 1) to get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho.$$

   Now let $\rho(r) = \rho_0\varrho(r)$ (note that this is *not* the central density ρ_c as before) and $r = \lambda\eta$ and show that the equation becomes

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left[\frac{\eta^2}{\varrho}\frac{d(\varrho^{5/3}(1+\varrho^{2/3})^{-1/2}}{d\eta}\right] = -\varrho.$$

(b) This is ...not a nice looking equation. But we're going to solve it numerically, anyway, so it doesn't really matter how complicated it is. First, though, we have to write it as two couple first order ODEs:

$$\frac{d\varrho}{d\eta} = \sigma$$
$$\frac{d\sigma}{d\eta} = -\frac{2}{\eta}\sigma - \frac{A(\varrho)}{B(\varrho)}\sigma^2 - \frac{1}{B(\varrho)}\varrho,$$

where $A(\varrho)$ and $B(\varrho)$ are the rather complicated equations

$$A(x) = -(5/9)x^{-4/3}(1+x^{2/3})^{-1/2} - (2/3)x^{-2/3}(1+x^{2/3})^{-3/2} + (1/3)(1+x^{2/3})^{-5/2},$$
$$B(x) = (5/3)x^{-1/3}(1+x^{2/3})^{-1/2} - (1/3)x^{1/3}(1+x^{2/3})^{-3/2}.$$

(these are not easy to derive, but feel free to try if you like. You don't *have* to, though). The boundary conditions are the usual:

$$\varrho(0) = \varrho_c, \quad \frac{d\varrho}{d\eta} = 0 \text{ at } \eta = 0.$$

Note, though, that ϱ_c isn't equal to one, but rather the value of the central density in units of ρ_0.

As a first test case, try solving the equations with $\varrho_c = 1$ (this just means that the density at the centre of the star is $\rho_0 = .3789 \times 10^6$ g/cm^3). Stop the integration when the density drops below 0.1% or so of ϱ_c (rather than waiting for it to go negative, which might lead to some numerical difficulties). Plot $\rho(r)$, identify the radius of the surface, and calculate the total mass of your white dwarf. Compare your plot with the nonrelativistic and relativistic models.

(c) All of the above leads to this part – now the real work can actually start. We're going to generate a bunch of models of different central densities. Do about 25 models, with ρ_cs that range from 10^4 g/cm^3 to 10^{12} g cm^3. Then:

i. Make a plot of the masses of your white dwarf models (in $M_\odot$) versus the logarithm of their central density. Identify the maximum stable mass of a white dwarf.

ii. Make a plot of the radii of your models (in units of $R_\odot/100$) versus the logarithm of their central density.

iii. Make a plot of the mass of your models versus their radii. Be sure to compare your results with the results from parts A and B above.

iv. This last plot, of the radii versus mass, is a famous result in white dwarf physics. Is our model any good? Do some research and see if you can find a paper that shows actual observed data for white dwarf mass and radii. Write out the reference you use, and comment on how good your model is.