PHY4410 Lecture 1 - Solving ODEs

Consider a general 2nd order ODEs:

$$\frac{d^2y}{dx^2} + q(x,y)\frac{dy}{dx} + r(x,y) = 0$$

It's always possible to write this as 2 1st order ODEs:

$$\begin{cases} \dfrac{dy}{dx} = z \\ \dfrac{dz}{dx} = -q(x,y)\,z - r(x,y) \end{cases}$$

So we need to solve

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \dots y_N) \qquad i = 1 \dots N$$

We'll also need BCs.

$\Rightarrow$ Initial value problem: we know the y_i's at some initial point and we want them at some later point.

Euler's Method

Consider just one ODE

$$\frac{dy}{dx} = f(x,y)$$

Write $dx \rightarrow \Delta x$ and $dy \rightarrow \Delta y$

$$\Rightarrow \frac{\Delta y}{\Delta x} = f(x,y) \Rightarrow \Delta y = f(x,y)\,\Delta x$$

Then

$$y_{n+1} = y_n + \Delta y = y_n + \Delta x\, f(x_n, y_n)$$

Notation:

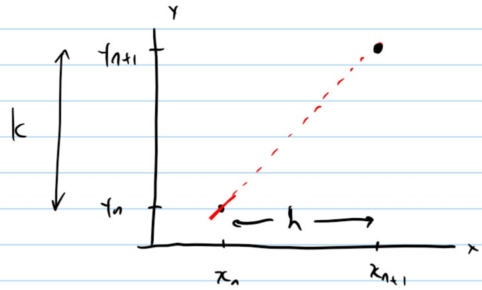
$$h = \Delta x$$

$$k_n = h f(x_n, y_n)$$

$$\boxed{y_{n+1} = y_n + k_n}$$

Euler's method.

What exactly is going on here?



Example : $\frac{dy^2}{dx^2} = -y$

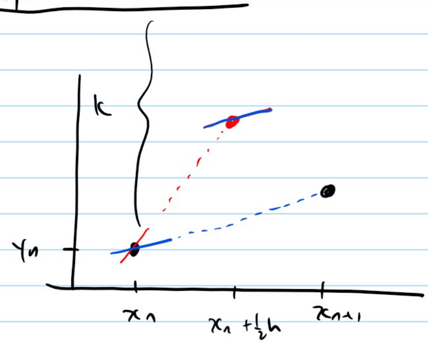
Define $z \equiv \frac{dy}{dx} \Rightarrow \begin{cases} \frac{dy}{dx} = z = f(x,y,z) \\ \frac{dz}{dx} = -y = g(x,y,z) \end{cases}$

Suppose $y_0 = 1$, $z_0 = 0$ at $x_0 = 0$.

$$y_1 = y_0 + k_0 = y_0 + hz_0 = 1$$

$$z_1 = z_0 + l_0 = z_0 + h(-y_0) = -h$$

Midpoint Method



1. Use derivative at x_n to estimate of y at $x_n + \frac{1}{2}h$
2. measure derivative there: $dy = hf(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}k_n)$
3. Use that derivative to estimate y at x_{n+1}

Then $k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}k_1)$

$\Rightarrow$ $\boxed{y_{n+1} = y_n + k_2}$ ▫ Midpoint method.

4th order Runge-Kutta Method RK4

Use derivatives at :

- one at the start
- two at the midpoint
- one at the end

$\Rightarrow$

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_n + h/2, \; y_n + k_1/2)$$

$$k_3 = h f(x_n + h/2, \; y_n + k_2/2)$$

$$k_4 = h f(x_n + h, \; y_n + k_3)$$

$$\boxed{y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}}$$

□ RK4.