Lecture 2 - How to Model Stars

Equations of Stellar Structure

(1) $$\rho \cancel{\frac{D\underset{\sim}{u}}{Dt}} = -\underset{\sim}{\nabla} p - \rho \underset{\sim}{\nabla} \phi$$ (Euler) $\underset{\sim}{u} \rightarrow$ velocity

$\phi \rightarrow$ gravitational potential

"ma" = pressure gravity

(2) $$\cancel{\frac{D\rho}{Dt}} + \cancel{\rho \underset{\sim}{\nabla} \cdot \underset{\sim}{u}} = 0$$ (conservation of mass)

(3) $$\nabla^2 \phi = 4\pi G \rho$$ (gravity)

(4) $$\frac{DU}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = \epsilon - \frac{1}{\rho}\underset{\sim}{\nabla} \cdot \underset{\sim}{F}$$ (energy)

Some basic assumptions:

- time independence $\rightarrow$ static $\underset{\sim}{u} = 0$
- spherical symmetry

(1) $\rightarrow$ $$\frac{dp}{dr} = -\rho \frac{d\phi}{dr}$$

(2) $\rightarrow$ $$0 = 0$$

(3) $\rightarrow$ $$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = 4\pi G \rho$$

(4) $\rightarrow$ $$0 = \epsilon - \frac{1}{\rho}\underset{\sim}{\nabla} \cdot \underset{\sim}{F}$$

$\hookrightarrow$ Integrate: $$r^2 \frac{d\phi}{dr} = \underbrace{4\pi G \int_0^r \rho(r') r'^2 dr'}_{G M(r)}$$

$\hookrightarrow$ mass enclosed in radius r.

$$\frac{d\phi}{dr} = \frac{G M(r)}{r^2}$$

① $\rightarrow$ $$\boxed{\frac{dp}{dr} = -\frac{GM\rho}{r^2}}$$ Hydrostatic equilibrium

Without another relationship between p and ρ we can't solve this!

Polytropic Model

Assume $$\boxed{p = k\rho^{\gamma}}$$, k and γ are constant.

① $\rightarrow$ $$\frac{dp}{dr} = -\frac{GM\rho}{r^2} \Rightarrow \frac{r^2}{\rho}\frac{dp}{dr} = -GM$$

Take derivative of both sides:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -G\frac{dM}{dr}$$

From ③, $$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho$$

Use $p = k\rho^{\gamma} \Rightarrow \frac{dp}{dr} = k\gamma\rho^{\gamma-1}\frac{d\rho}{dr}$

$$\boxed{\frac{k\gamma}{r^2}\frac{d}{dr}\left(r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right) = -4\pi G\rho}$$

This is an ODE involving just density.

Clean it up:

- define a dimensionless parameter:

$$\rho(r) = \rho_c \, \theta^n(r)$$

$\hookrightarrow$ dimensionless density

- define n such that

$$\gamma = \frac{n+1}{n}$$

- define $\eta = \dfrac{r}{\lambda_n}$, $\lambda_n = \left[\dfrac{(n+1)\, k}{4\pi G} \rho_c^{\frac{(1-n)}{n}} \right]^{1/2}$

$\Rightarrow$ The ODE becomes

$$\boxed{\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\theta}{d\eta} \right) = -\theta^n}$$

Lane - Emden Equation

Boundary Conditions:

- At the surface, the density goes to zero:

$$\theta(\eta_s) = 0$$

↳ radius of surface η_s

- At the centre of the star, the density is ρ_c

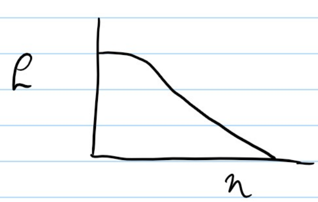
$$\Rightarrow \theta(0) = 1$$

- $\dfrac{d\theta}{d\eta} = 0$ at $\eta = 0$

Analytic Solutions:

- $n = 0$, $\theta(\eta) = 1 - \dfrac{\eta^2}{6}$

Surface is at $\theta(\eta_s) = 0 \Rightarrow \eta_s = \sqrt{6}$.



- $n = 1$, $\theta(\eta) = \dfrac{\sin\eta}{\eta}$, $\eta_s = \pi$

- $n = 5$, $\theta(\eta) = \dfrac{1}{\sqrt{1 + \frac{1}{3}\eta^2}}$, $\eta_s = \infty$. #

White Dwarfs

- He, O, C

→ nonrelativistic gas —
→ relativistic gas — } polytropes $p \propto \rho^\gamma$

Steps

- solve L-E eq. via RK4
- give ρ_c, calculate r_s, M
- look up the Chandrasekar limit and typical radii of WDs -

How good are the models?