Calculus: A Comprehensive Guide

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January 25, 2024

1 Limits by Factoring

1.1 Problem 1

Consider the function:

$$f(x) = \frac{x^2 + x - 6}{x - 2}$$

Find $\lim_{x \to 2} f(x)$.
Solution:
The function is initially undefined at $x = 2$. However, by factoring, we get:

$$f(x) = \frac{(x+3)(x-2)}{x-2} = x + 3$$

We factor it the following way because:
$3\dot{*} - 2 = -6$
$3 + -2 = 1$

Therefore, $f(x) = \begin{cases} x + 3 & \text{for all } x \text{ except } x = 2 \\ \text{undefined} & \text{when } x = 2 \end{cases}$

Intermediate Step:

$$f(x) = \frac{x^2 + x - 6}{x - 2} = \frac{(x+3)(x-2)}{x-2} = x + 3$$

1.2 Problem 2

Consider the function $f(x) = \dfrac{x^4 + 3x^2 - 10x^2}{x^2 - 2x}$.

After factorization, we get:

$$f(x) = \frac{x^2(x^2 + 3 - 10)}{x(x-2)}$$

$$f(x) = \frac{x^2(x+5)(x-2)}{x(x-2)}$$

Which is the same as:

$$f(x) = \frac{x \cdot x(x+5)(x-2)}{x(x-2)}$$

After cancellation, we get:

$$f(x) = x(x+5), \text{ when } x \neq 2$$

2 Limits using Conjugates

2.1 Problem 1

Using these algebraic rules:

$$(a+b)(a-b) = a^2 - b^2$$
$$(\sqrt{a}+b)(\sqrt{a}-b) = a - b^2$$

Remember that these rules apply regardless of whether we use them on terms in the denominator or numerator. Just use the same steps, just accordingly.

Consider the function:

$$f(x) = \frac{x-3}{2-\sqrt{x+1}}$$

The problem is: $\lim_{x\to 3} f(x)$.

Using conjugates, we can rewrite our function:

$$= \frac{(x-3)}{2-\sqrt{x+1}} \cdot \frac{2+\sqrt{x+1}}{2+\sqrt{x+1}}$$

$$= \frac{(x-3)(2+\sqrt{x+1})}{2^2-(x+1)}$$

We use the rule: $1-(b+c) = 1-b-c$

$$\frac{(x-3)(2+\sqrt{x+1})}{-x+3}$$

Which can be rewritten as

$$\frac{(x-3)(2+\sqrt{x+1})}{-1(x-3)}$$

After cancellation, we get

$$f(x) = \frac{2+\sqrt{x+1}}{-1}, \text{ when } x \neq 3$$

Now we can find the limit:

$$\lim_{x\to 3} f(x) = -3$$

2.2 Problem 2

Consider the function:

$$f(x) = \frac{x-3}{\sqrt{4x+4}-4}$$

Find $\lim_{x\to 3} f(x)$.
Solution:
We use the following rules:

$$(a+b)(a-b) = a^2 - b^2$$
$$(\sqrt{a}+b)(\sqrt{a}-b) = a - b^2$$

And we get:

$$f(x) = \frac{x-3}{\sqrt{4x+4}-4}$$
$$= \frac{(x-3)}{\sqrt{4x+4}-4} \cdot \frac{\sqrt{4x+4}+4}{\sqrt{4x+4}+4}$$
$$= \frac{(x-3)(\sqrt{4x+4}-4)}{(4x+4)-4^2}$$
$$= \frac{(x-3)(\sqrt{4x+4}-4)}{4(x-3)}$$
$$= \frac{\sqrt{4x+4}+4}{4}$$

Therefore, $f(x) = \begin{cases} \frac{\sqrt{4x+4}+4}{4} & \text{for all } x \text{ except } x = 3 \\ \text{undefined} & \text{when } x = 3 \end{cases}$

We now use the theorem that states: If $f(x) = g(x)$ for all x-values in a given interval except for $x = c$, then

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$$

.

So,

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{\sqrt{4x+4}+4}{4} = \frac{2+4}{4} = 2$$

2.3 Example 3

Consider the function:

$$f(x) = \frac{x-2}{1-\sqrt{3x-5}}$$

Find $\lim_{x\to 2} f(x)$.
Solution:

And we get:

$$f(x) = \frac{x-2}{1-\sqrt{3x-5}}$$

$$= \frac{(x-2)}{1-\sqrt{3x-5}} \cdot \frac{(1+\sqrt{3x-5})}{(1+\sqrt{3x-5})}$$

$$= \frac{(x-2)(1+\sqrt{3x-5})}{1^2-(3x-5)}$$

$$= \frac{(x-2)(1+\sqrt{3x-5})}{1-(3x-5)}$$

Using the rule $a-(b+c) = a-b-c$, we get:

$$\frac{(x-2)(1+\sqrt{3x-5})}{1-3x+5} = \frac{(x-2)(1+\sqrt{3x-5})}{-3(x-2)}$$

After cancelation, we get:

$$\frac{1+\sqrt{3x-5}}{-3}$$

So, $f(x) = \begin{cases} \frac{1+\sqrt{3x-5}}{-3} & \text{for all } x \text{ except } x=2 \\ \text{undefined} & \text{when } x=2 \end{cases}$

Now, we find the limit using the theorem that states: If $f(x) = g(x)$ for all x-values in a given interval except for $x = c$, then

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$$

.

So,

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{1+\sqrt{3x-5}}{-3} = -\frac{2}{3}$$

3 Proof with Epsilon-Delta

3.1 Example Problem

Given $\varepsilon > 0$, we can find $\delta > 0$ such that $|x-c| < \delta \implies |f(x)-L| < \varepsilon$.

Consider the function:

$$f(x) = \begin{cases} 2x & \text{for } x \neq 5 \\ x & \text{for } x = 5 \end{cases}$$

Find $\lim_{x\to 5} f(x)$.

Solution:

For $x \neq 5$:

$$|f(x) - L| < 2\delta \quad \text{(where } L = 10\text{)}$$

Choose $\delta = \varepsilon/2$.
Therefore, $|f(x) - L| < \varepsilon$.
Intermediate Step:

$$|f(x) - L| < 2\delta \quad \text{(where } L = 10\text{)}$$

4 Trig limit using Pythagorean identity

First, we need to be aware of the different trigonometric identities:
Basic Definitions:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Pythagorean Identities:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\tan^2(\theta) + 1 = \sec^2(\theta)$$
$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Sum and Difference Identities:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

Double Angle Identities:

$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = \cos^2(A) - \sin^2(A) = 2\cos^2(A) - 1 = 1 - 2\sin^2(A)$$

$$\cos^2(x) = 1 - \sin^2(x)$$

$$\sin^2(x) = 1 - \cos^2(x)$$

Half-Angle Identities:

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}$$
$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos(A)}{2}}$$

Product-to-Sum Identities:

$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$
$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

1. **Definition of Cotangent:**

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

Therefore,

$$\cot^2(x) = \frac{\cos^2(x)}{\sin^2(x)}$$

2. **Pythagorean Identity:**

$$\sin^2(x) + \cos^2(x) = 1$$

4.1 Problem 1

Consider the problem: We have the function:

$$f(x) = \frac{1-\cos(\theta)}{2\sin^2(\theta)}$$

$$\lim_{\theta\to 0} f(x)$$

The function will be undefined when $\theta = 0$, but we can use the following rules to find the limit:

Rule 1: $\sin^2(\theta) + \cos^2(\theta) = 1$

Rule 2: $\sin^2(\theta) = 1 - \cos^2(\theta)$

So, after applying these rules to our function, we get (with intermediate steps):

$$f(x) = \frac{1-\cos(\theta)}{2\sin^2(\theta)} = \frac{1-\cos(\theta)}{2(1-\cos^2(\theta))}$$

Now, we can use the following algebraic rule:

$$(a+b)(a-b) = a^2 - b^2$$

where $a = 2$ and $b = (1 - \cos^2(\theta))$. We get:

$$f(x) = \frac{1 - \cos(\theta)}{2(1 + \cos(\theta))(1 - \cos(\theta))}$$

Now, after cancellation, we get the following:

$$f(x) = \frac{1}{2 + 2\cos(\theta)}$$

So now we can write our function as follows:

$$f(x) = \frac{1}{2 + 2\cos(\theta)}, \text{ when } \theta \neq 0$$

Now, we can find the limit:

$$\lim_{\theta \to 0} f(x) = \frac{1}{4}$$

4.2 Problem 2

We have the function:

$$f(\theta) = \frac{1 + \sqrt{2}\sin(\theta)}{\cos^2(\theta)}$$

$$\lim_{\theta \to -\frac{\pi}{4}} f(\theta)$$

The function is undefined when $\theta = -\frac{\pi}{4}$.
In this case, we use the rules:

$$\cos(2A) = \cos^2(A) - \sin^2(A) = 2\cos^2(A) - 1 = 1 - 2\sin^2(A)$$

And:

$$(a - b)(a + b) = a^2 - b^2$$

Specifically, we use

$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$

where $a = 1$ and $b = 2\sin^2(\theta)$, using $(a - b)(a + b) = a^2 - b^2$ to get $(1 + \sqrt{2}\sin(\theta))(1 - \sqrt{2}\sin(\theta))$.
We manipulate our expression to be:

$$f(\theta) = \frac{1 + \sqrt{2}\sin(\theta)}{(1 + \sqrt{2}\sin(\theta))(1 - \sqrt{2}\sin(\theta))}$$

After cancellation, we get:

$$f(\theta) = \frac{1}{1 - \sqrt{2}\sin(\theta)}$$

Now, we can find:

$$\lim_{\theta \to -\frac{\pi}{4}} f(\theta) = \frac{1}{2}$$

4.3 Problem 3

Given expression:

$$\frac{\cot^2(x)}{1-\sin(x)}$$

We are evaluating the limit as x approaches $\frac{\pi}{2}$, and we encounter an indeterminate form $\frac{0}{0}$. To resolve this, we use factorization and trigonometric identities.

1. **Definition of Cotangent:**

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

Therefore,

$$\cot^2(x) = \frac{\cos^2(x)}{\sin^2(x)}$$

2. **Pythagorean Identity:**

$$\sin^2(x) + \cos^2(x) = 1$$

Rewrite $\cot^2(x)$ using the Pythagorean identity:

$$\frac{\cot^2(x)}{1-\sin(x)} = \frac{\cos^2(x)}{\sin^2(x)[1-\sin(x)]}$$

3. **Factorization (Difference of Squares):**

$$1-\sin^2(x) = (1-\sin(x))(1+\sin(x))$$

Substitute this into the expression:

$$\frac{(1-\sin(x))(1+\sin(x))}{\sin^2(x)[1-\sin(x)]}$$

4. **Cancellation of Common Factors:**

$$\frac{(1+\sin(x))}{\sin^2(x)}$$

This expression is equivalent to the original one, except at points where $\sin(x) = 1$, particularly at $x = \frac{\pi}{2}$.

5. **Limit Evaluation:**

$$\lim_{x\to\frac{\pi}{2}} \frac{\cot^2(x)}{1-\sin(x)} = \lim_{x\to\frac{\pi}{2}} \frac{(1+\sin(x))}{\sin^2(x)}$$

Now, you can find the limit by direct substitution since the indeterminate form is resolved. Substitute $x = \frac{\pi}{2}$ into the expression to get the limit.

4.4 Problem 4

Consider the function $f(x) = \frac{3\cos^2(x)}{2-2\sin(x)}$.

We want to find $\lim_{x\to\frac{\pi}{2}} f(x)$.

Using the Pythagorean identity, we can simplify $f(x)$ to:

$$f(x) = \frac{3-(1-\sin^2(x))}{2-2\sin(x)}$$

Further simplifying using the difference of squares, we get:

$$f(x) = \frac{3(1+\sin(x))(1-\sin(x))}{2(1-\sin(x))}$$

After cancellation, the expression becomes:

$$f(x) = \frac{3\sin(x)}{2(1-\sin(x))}, \quad x \neq \frac{\pi}{2}$$

Now, let's evaluate the limit as x approaches $\frac{\pi}{2}$:

$$\lim_{x\to\frac{\pi}{2}} f(x) = 3$$

5 Limits at Infinity Quotients

Consider the following examples, where x approaches infinity:

5.1 Example 1: Fraction Approaching a Constant Above 1

$$\lim_{x\to\infty} \frac{3x^2+2x+1}{x^2+1} = 3$$

After dividing each term by the highest order term (which is x^2), we get:

$$f(x) = \frac{3+\frac{2}{x}+\frac{1}{x^2}}{1+\frac{1}{x^2}}$$

As $x \to \infty$, the terms with $\frac{1}{x}$ and $\frac{1}{x^2}$ become negligible, and the function approaches the constant 3.

5.2 Example 2: Fraction Approaching 0

$$\lim_{x\to\infty} \frac{2x+5}{x^3+3x} = 0$$

After simplifying, we have:

$$g(x) = \frac{\frac{2}{x^2}+\frac{5}{x}}{x}$$

As $x \to \infty$, the terms with $\frac{1}{x^2}$ and $\frac{1}{x}$ tend towards zero, and the function approaches 0.

5.3 Example 3: Unbounded Fraction

$$\lim_{x \to \infty} \frac{4x^3 + 2x^2 + 1}{x^2 + 3} = \infty$$

After simplification:

$$h(x) = \frac{4 + \frac{2}{x} + \frac{1}{x^2}}{1 + \frac{3}{x^2}}$$

In this case, as $x \to \infty$, the terms with $\frac{1}{x}$ and $\frac{1}{x^2}$ in the numerator dominate, leading to an unbounded limit.

Summary:

- (1): If the highest order term in the denominator grows faster, the limit approaches 0.
- (2) If both grow at the same rate, the limit approaches a constant.
- (3) If the highest order term in the numerator grows faster, the limit is unbounded.

5.4 Example 4: Square Root Difference

Consider the function:

$$f(x) = \sqrt{100 + x} - \sqrt{x}$$

Find $\lim_{x \to \infty} f(x)$.
Start with the expression:

$$f(x) = \sqrt{100 + x} - \sqrt{x}$$

We multiply with something that's equal to 1

$$f(x) = \sqrt{100 + x} - \sqrt{x} \times \frac{(\sqrt{100 + x} + \sqrt{x})}{(\sqrt{100 + x} + \sqrt{x})}$$

Recall $(a - b)(a + b) = a^2 - b^2$
Multiply the numerator and denominator by the conjugate:

$$f(x) = \frac{(\sqrt{100 + x} - \sqrt{x})(\sqrt{100 + x} + \sqrt{x})}{(\sqrt{100 + x} + \sqrt{x})}$$

Now continue with the simplification:

$$f(x) = \frac{(100 + x - x)}{\sqrt{100 + x} + \sqrt{x}}$$

Simplify further:

$$f(x) = \frac{100}{\sqrt{100 + x} + \sqrt{x}}$$

As $x \to \infty$, the terms with $\sqrt{100 + x}$ and $\sqrt{x}$ become negligible, and the function approaches 0.

Therefore, the limit is:

$$\lim_{x \to \infty} f(x) = 0$$

6 Derivatives

The slope of a secant line can be found using the formula $\frac{\Delta y}{\Delta x}$.

Let's consider the function

$$f(x) = x^2 - 2x + 5.$$

Suppose the secant lines intercept our function at $x = -1$ and $x = t$. We can input the values for x and y into a table as follows:

x	$f(x)$
-1	8
t	$t^2 - 2t + 5$

Now, we can use the formula to find the slope coefficient:

$$\frac{(t^2 - 2t + 5 - 8)}{t - (-1)} = \frac{(t - 3)(t + 1)}{t + 1} = t - 1.$$

6.1 f'(x)

To find $f'(x)$, we need to find the secant line as it approaches the point:

$$\frac{dy}{dx}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

or

$$f'(t) = \lim_{x \to t} \frac{f(x) - f(t)}{x - t}$$

An example could be $f(x) = \ln(x)$. For this case, let's find $f'(e)$:

$$f'(e) = \lim_{h \to 0} \frac{\ln(e+h) - \ln(e)}{h}$$

and

$$f'(e) = \lim_{x \to e} \frac{\ln(x) - \ln(e)}{x - e}$$

7 Power Rule

For a function $f(x) = x^n$, where $n \neq 0$, the derivative $f'(x)$ is given by:

$$f'(x) = nx^{(n-1)}$$

Examples

$$\frac{1}{x^n} = x^{-n}$$

$$\sqrt[n]{x} = x^{1/n}$$

$$\frac{n}{x} = n \cdot x^{-1}$$

$$\frac{2}{x^3} = 2\dot{x}^{-3}$$

$$f(x) = \frac{1}{x} = x^{-1} = \frac{d}{dx}(-x^{-2})$$

$$f(x) = \frac{1}{x^{10}} = x^{-10} = \frac{d}{dx}(-10x^{-11})$$

$$f(x) = \frac{5}{x^{10}} = 5x^{-10} = \frac{d}{dx}(-50x^{-11}) = \frac{d}{dx}\left(\frac{-50}{x^{11}}\right)$$

$$f(x) = \sqrt[3]{x} = x^{1/3} = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right)$$

$$f(x) = \sqrt[3]{x^2} = (x^2)^{1/3} = x^{2/3} = \frac{d}{dx}\left(\frac{2}{3}x^{-1/3}\right), \text{ where } f'(8) = \frac{1}{3}$$

8 Basic Derivative Rules

Power Rule

$$\frac{d}{dx}(x^0) = \frac{d}{dx}(1) = 0$$

Constant Rule

$$\frac{d}{dx}(c) = 0$$

Constant Multiple Rule

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}(f(x)) = c \cdot f'(x)$$

Sum and difference Rule (+/-)

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)) = f'(x) + g'(x)$$

8.0.1 Example

$$\frac{d}{dx}(2x \cdot x^5) = 2 \cdot \frac{d}{dx}(x^5) = 2 \cdot 5x^4 = 10x^4$$
$$\frac{d}{dx}(x^3 + x^{-4}) = 3x^2 + (-4)x^{-5} = 3x^3 - 4x^{-5}$$

$$f(x) = 2x^3 - 7x^2 + 3x - 100$$
$$f'(x) = 2 \cdot 3x^2 - 7 \cdot 2x + 3 + 0 = 6x^2 - 14x + 3$$
$$f'(8) = 6(8)^2 - 14(8) + 3$$

9 Tangents of Polynomials

To find the tangents of polynomials, we use the formula $y - y_1 = m(x - x_1)$.

9.1 Example

Consider the function:

$$f(x) = 3x^2 + 8x^2$$

We want the slope when $x = -2$:

$$\frac{d}{dx}(9x^2 + 16x)$$
$$f'(-2) = 4$$

And when $x = -2$:

$$f(-2) = 8$$

So, the equation of the tangent line is given by:

$$y - 8 = 4(x + 2)$$
$$y - 8 = 4x + 8$$
$$y = 4x + 16$$

10 Derivatives of $\sin(x)$ and $\cos(x)$

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

Consider the problem:

$$\frac{d}{dx}\left(7\sin(x) - 3\cos(x) - \left(\frac{\pi}{\sqrt[3]{x}}\right)^2\right)$$

First, simplify $\left(\frac{\pi}{\sqrt[3]{x}}\right)^2$ as follows:

$$\frac{\pi^2}{(\sqrt[3]{x})^2} = \frac{\pi^2}{x^{1/3}} = \frac{\pi^2}{x^{2/3}} = \pi^2 x^{-2/3}$$

Now, to find the derivative, use the power rule, sine, and cosine rules:

$$f'(x) = 7\cos(x) - 3(-\sin(x)) - \left(\frac{2\pi^2}{3}\right)x^{-5/3}$$

11 Derivative of e^x and $\ln(x)$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

12 Product Rule

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

12.1 Examples

$$\frac{d}{dx}(x^2\sin(x)) = 2x\sin(x) + x^2\cos(x)$$

$$\begin{aligned}\frac{d}{dx}(e^x\cos(x)) &= e^x\cos(x) + e^x(-\sin(x))\\ &= e^x\cos(x) - e^x\sin(x)\\ &= e^x(\cos(x) - \sin(x))\end{aligned}$$

13 Quotient Rule

If $f(x) = \frac{u(x)}{v(x)}$, then the derivative $f'(x)$ using the quotient rule is given by:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}$$

If we have the expression

$$\frac{d}{dx}\left(\frac{x^2 - x + 5}{4x - 1}\right),$$

We can use the quotient rule by saying

$$u(x) = x^2 - x + 5 \quad \text{and} \quad v(x) = 4x - 1,$$

so

$$f'(x) = \frac{4x^2 - 2x - 19}{16x^2 - 8x + 1}.$$

13.1 Example 1

Let's consider the function $f(x) = \frac{x^2}{\cos(x)}$. The derivative $f'(x)$ is calculated as follows:

$$f'(x) = \frac{2x\cos(x) - x^2(-\sin(x))}{\cos^2(x)}$$

Simplifying:

$$f'(x) = \frac{2x\cos(x) + x^2\sin(x)}{\cos^2(x)}$$

13.2 Example 2

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sin(x)}{\sqrt{x}}\right) &= \frac{\cos(x)\sqrt{x} - \sin(x)\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x})^2} \\ &= \frac{\cos(x)\sqrt{x} - \sin(x)\left(\frac{1}{2\sqrt{x}}\right)}{x} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} \\ &= \frac{2x\cos(x) - \sin(x)}{2x\sqrt{x}}\end{aligned}$$

13.3 Example 3

$$\begin{aligned}\frac{d}{dx}\left(\frac{e^x}{\cos(x)}\right) &= \frac{e^x\cos(x) - e^x(-\sin(x))}{\cos^2(x)}\\ &= \frac{e^x\cos(x) + e^x\sin(x)}{\cos^2(x)}\\ &= \frac{e^x(\cos(x)+\sin(x))}{\cos^2(x)}\end{aligned}$$

13.4 Example 4

$$\begin{aligned}\frac{dy}{dx}\left(\frac{\sqrt{x}}{\sin(x)}\right) &= \frac{\frac{1}{2\sqrt{x}}\sin(x) - \sqrt{x}\cos(x)}{\sin^2(x)}\\ &= \frac{\left(\frac{1}{2\sqrt{x}}\sin(x) - \sqrt{x}\cos(x)\right)}{\sin^2(x)}\cdot\frac{2\sqrt{x}}{2\sqrt{x}}\\ &= \frac{\sin(x) - 2x\cos(x)}{2\sqrt{x}\sin^2(x)}\end{aligned}$$

13.5 Example 5

$$\begin{aligned}\frac{dy}{dx}\left(\frac{\ln(x)}{x^2}\right) &= \frac{\frac{1}{2}x^2 - \ln(x)\cdot 2x}{(x^2)^2}\\ &= \frac{x - \ln(x)\cdot 2x}{x^4}\\ &= \frac{x(1-2\ln(x))}{x^4}\\ &= \frac{1-2\ln(x)}{x^3}\end{aligned}$$

14 Derivatives of Trig Identities

If we need to find the derivative of trigonometric identities, we can use the quotient rule. The following expressions are useful:

$$\sin^2 x + \cos^2 x = 1$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\csc x = \frac{1}{\sin x}$$

15 Chain Rule

The chain rule is expressed as:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

15.1 Example 1

Let's find the derivative of $\ln(\sin(x))$:

$$\frac{d}{dx}(\ln(\sin(x))) = \frac{1}{\sin(x)} \cdot \cos(x)$$

15.2 Example 2

Consider the function $f(x) = (\cos(x))^3$. The derivative is:

$$f'(x) = 3(\cos(x))^2 \cdot (-\sin(x))$$

15.3 Example 3

For the function $f(x) = \sqrt{3x^2 - x}$, the derivative is:

$$f'(x) = \frac{1}{2}\sqrt{3x^2 - x} \cdot (6x - 1)$$

15.4 Example 4

Now, let's find the derivative of $f(x) = \ln(\sqrt{x})$:

$$f'(x) = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}$$

16 Derivatives of a^x

We know that some constant $a = e^{\ln(a)}$. Knowing this, we can use the chain rule to find derivatives of polynomials:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{\ln(a)}) = (\ln(a))(e^{\ln(a)})^x = (\ln(a))a^x$$

16.1 Example 1:

$$\frac{d}{dx}(8 \cdot 3^x) = 8\ln(3)3^x$$

16.2 Example 2:

$$\frac{d}{dx}(7^{x^2-x}) = \ln(7) \cdot 7^{x^2-x} \cdot (2x-1)$$

17 Derivatives of logarithmic functions

We know that $\log_a(b) = \frac{\log(b)}{\log(a)}$.

$$\frac{d}{dx}(\log_a(x)) = \frac{1}{\ln(a) \cdot x}$$

17.1 Example 1:

$$\frac{d}{dx}(\log_7(x)) = \frac{1}{\ln(7) \cdot x}$$

17.2 Example 2:

$$\frac{d}{dx}(-3\log_\pi(x)) = -\frac{3}{\ln(\pi) \cdot x}$$

17.3 Example 3:

$$\frac{d}{dx}(\log_4(x^2+x)) = \frac{1}{\ln(4)(x^2-x)} \cdot (2x+1) = \frac{2x+1}{\ln(4)(x^2-x)}$$

17.4 Example 4:

$$\frac{d}{dx}(\sec\left(\frac{3\pi}{2} - x\right)) = -\frac{\sin\left(\frac{3\pi}{2} - x\right)}{\cos^2\left(\frac{3\pi}{2} - x\right)}$$

17.5 Example 5:

$$\frac{d}{dx}\left(\sqrt[4]{x^3+4x^2+7}\right) = \frac{d}{dy}\left((x^3+4x^2+7)^{\frac{1}{4}}\right) = \frac{1}{4(x^3+4x^2+7)^{\frac{3}{4}}} \cdot (3x^2+8)$$

18 Implicit Differentiation

When we have two terms, we use implicit differentiation. We take the derivative of the second term with respect to y. We are using the chain rule in combination with other methods.

18.1 Example 1

Consider the equation $x^3 + y^2 - x^2 = 5$. Applying implicit differentiation:

$$\frac{d}{dx}(x^3 y) + \frac{d}{dx}(y^2) - \frac{d}{dx}(x^2) = \frac{d}{dx}(5)$$

This leads to:

$$3x^2 y + x^3 \frac{dy}{dx} + 2y\frac{dy}{dx} - 2x = 0$$

Isolating $\frac{dy}{dx}$:

$$\frac{dy}{dx}(x^3 + 2y) = 2x - 3x^2 y$$

So,

$$\frac{dy}{dx} = \frac{2x - 3x^2 y}{x^3 + 2y}$$

Now, to find the slope of the function at the point (2, 3), substitute $x = 2$ and $y = 3$ into the formula:

$$\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{2 \cdot 2 - 3 \cdot 2^2 \cdot 3}{2^3 + 2 \cdot 3}$$

Simplifying, we find:

$$\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{-20}{14} = -\frac{10}{7}$$

18.2 Example 2

Consider the problem:

$$\frac{d}{dx}(2xy + x^3 - 3y^2) = 5$$

Now, taking the implicit derivative involving the product rule and sum rule:

$$2y + 2x\frac{dy}{dx} + 3x^2 - 6y\frac{dy}{dx} = 0$$

Isolating $\frac{dy}{dx}$:

$$\frac{dy}{dx}(2x - 6y) = -2y - 3x^2$$

So,

$$\frac{dy}{dx} = -\frac{2y + 3x^2}{2x - 6y}$$

19 Derivatives of inverse functions

If two functions are inverses of each other, let's say $g(x)$ and $h(x)$ are the inverses, we can express their relationship as follows:

$$h'(x) = \frac{1}{g'(h(x))} \tag{1}$$

20 Derivatives of inverse trigonometric functions

For $y = \sin^{-1}(x)$:

$$\begin{aligned}
&y = \sin^{-1}(x) \quad \leftrightarrow \quad \sin(y) = x \\
&\frac{d}{dx}(\sin(y)) = \frac{d}{dx}(x) \\
&\cos(y)\frac{dy}{dx} = 1 \\
&\frac{dy}{dx} = \frac{1}{\cos(y)}
\end{aligned}$$

From the Pythagorean theorem: $\cos^2(y) + \sin^2(y) = 1$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2(y)}} \quad \text{where} \quad \sin(y) = x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Similarly:

$$\frac{dy}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{dy}{dx}(\tan(x)) = \sec^2(x) = \frac{1}{\cos^2(x)}$$

For $y = \tan^{-1}(x) \quad \leftrightarrow \quad \tan(y) = x$:

$$
\begin{aligned}
&\frac{d}{dx}(\tan(y)) = \frac{d}{dx}(x) \\
&\frac{1}{\cos^2(y)}\frac{dy}{dx} = 1 \\
&\frac{dy}{dx} = \cos^2(y) \\
&\frac{dy}{dx} = \frac{\cos^2(y)}{\sin^2(y) + \cos^2(y)} \\
&\frac{dy}{dx} = \frac{\cos^2(y)}{\sin^2(y) + \cos^2(y)}\left(\frac{1/\cos^2(y)}{1/\cos^2(y)}\right) \\
&\frac{dy}{dx} = \frac{1}{1 + (\sin(y)/\cos(y))^2} \\
&\frac{dy}{dx} = \frac{1}{1 + \tan^2(y)} \\
&\frac{dy}{dx} = \frac{1}{1 + x^2}
\end{aligned}
$$

20.1 Example 1

Evaluate $\frac{d}{dx}(\tan^{-1}(-4x))$ at $x = 3$:

$$
\begin{aligned}
&\frac{d}{dx}(\tan^{-1}(x)) \cdot \frac{d}{dx}(-4x) \\
&= \frac{1}{1 + x^2} \cdot (-4) \\
&= -\frac{4}{1 + (-4x)^2} \\
&= -\frac{4}{1 + 16x^2}
\end{aligned}
$$

Now evaluating at $x = 3$:

$$
-\frac{4}{1 + 16(3)^2} = -\frac{4}{145}
$$

20.2 Example 2

Evaluate $\frac{d}{dx}(\cos^{-1}(-2x))$ at $x = \frac{1}{4}$:

$$\frac{d}{dx}(\cos^{-1}(x)) \cdot \frac{d}{dx}(-2x) = \left(-\frac{1}{\sqrt{1-x^2}}\right) \cdot (-2)$$
$$= \frac{2}{\sqrt{1 - 4\left(\frac{1}{4}\right)^2}}$$
$$= \frac{2}{\sqrt{\frac{3}{4}}}$$

20.3 Example 3:

Let $y = \sin^{-1}\left(\frac{x}{4}\right)$. Find $\frac{dy}{dx}$.

We use the chain rule:

$$\frac{d}{dx}\left(\sin^{-1}\left(g(x)\right)\right) = \frac{d}{dx}\left(\frac{x}{4}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{4}\right)^2}} \cdot \frac{1}{4} = \frac{1}{\sqrt{1-\frac{x^2}{16}}} \cdot \frac{1}{4} = \frac{1}{4\sqrt{\frac{x^2}{16}}}$$

20.4 Example 4

Evaluate $\frac{dy}{dx}$ of:

$$y = \sin^{-1}(-4x) \text{ at } x = -\frac{1}{6}$$

$$\frac{d}{dx}(\sin^{-1}(x)) \cdot \frac{d}{dx}(-4x) = \frac{1}{\sqrt{1-x^2}} \cdot (-4) = -\frac{4}{\sqrt{1-(-4x)^2}} = -\frac{4}{\sqrt{1-16x^2}}$$

Now, evaluating at $x = -\frac{1}{6}$, we get:

$$= -\frac{4}{\sqrt{1-(-4)^2}} \cdot (1 - 16(-1/6)^2) = -\frac{12}{\sqrt{5}}$$

21 Using multiple derivative rules

$$\frac{d}{dx}(uv) = u'v + uv'$$

$$\frac{d}{dx}(uvw) = u'vw + uv'w + uvw'$$

$$(f(g(h(x))))' = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

21.1 Example 1

$$y = \sin^3(x^2) = (\sin(x^2))^3$$
$$\frac{dy}{dx} = 3(\sin(x^2)^2 \cdot \frac{d}{dx}(\sin(x^2))$$
$$= 3(\sin(x^2)^2 \cdot \cos(x^2) \cdot \frac{d}{dx}(x^2))$$
$$= 3(\sin(x^2)^2 \cdot \cos(x^2) \cdot 2x)$$
$$= 6x\sin^2(x^2)\cos(x^2)$$

21.2 Example 2

$$\frac{d}{dx}(e^{\cos x}\cos(e^x)) = \frac{d}{dx}(e^{\cos x}) \cdot \cos(e^x) + e^{\cos x} \cdot \frac{d}{dx}(\cos(e^x))$$
$$= e^{\cos x}(-\sin x)\cos(e^x) + e^{\cos x} - \sin(e^x)$$
$$= -e^{\cos x}\sin(x)\cos(e^x) - e^x e^{\cos x}\sin(e^x)$$

21.3 Example 3

$$\frac{d}{dx}(\sin(\ln(x^2))) = \frac{d}{dx}(f(g(h(x))))$$
$$= f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$
$$= \cos(\ln x^2) \cdot \frac{1}{x^2} \cdot 2x$$
$$= \frac{2}{x}\cos(\ln x^2)$$

22 Second derivatives implicit equation

22.1 Example

Given the implicit equation $y^2 - x^2 = 4$, we want to find $\frac{d^2y}{dx^2}$.

Starting with the implicit equation:

$$2y\frac{dy}{dx} - 2x = 0$$

Isolating $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} = 2x$$

Dividing by $2y$:

$$\frac{dy}{dx} = \frac{x}{y}$$

Taking the derivative with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{x}{y}\right)$$

Using the product rule $(u'v + uv')$:

$$\frac{1}{y} + x\left(-\frac{1}{y^2}\frac{dy}{dx}\right)$$

Substituting $\frac{dy}{dx} = \frac{x}{y}$:

$$\frac{1}{y} - \frac{x^2}{y^3}$$

Therefore, the second derivative is:

$$\frac{d^2y}{dx^2} = \frac{1}{y} - \frac{x^2}{y^3}$$

23 Composite Exponential Function Differentiation

$$\frac{\partial}{\partial x}(y) = \frac{\partial}{\partial x}(x^x)$$

$$\frac{\partial}{\partial x}(\ln(y)) = \frac{\partial}{\partial x}(\ln(x^x))$$

$$= y(\ln(x) + 1) \quad \leftrightarrow \quad x^x(\ln(x) + 1)$$

23.1 Generic example

The generic formula for finding the partial derivative of $y = f(x)^{g(x)}$ using logarithmic differentiation is as follows:

1. Original Expression:
$$y = f(x)^{g(x)}$$

2. Natural Logarithm:
$$\ln(y) = g(x)\ln(f(x))$$

3. Differentiation:

$$\frac{1}{y}\frac{\partial y}{\partial x} = g'(x)\ln(f(x)) + g(x)\cdot\frac{f'(x)}{f(x)}$$

4. Solve for $\frac{\partial y}{\partial x}$:

$$\frac{\partial y}{\partial x} = y\left(g'(x)\ln(f(x)) + g(x)\cdot\frac{f'(x)}{f(x)}\right)$$

Substitute back the original expression for y:

$$\frac{\partial y}{\partial x} = f(x)^{g(x)}\left(g'(x)\ln(f(x)) + g(x)\cdot\frac{f'(x)}{f(x)}\right)$$

24 L'Hôpital's Rule

L'Hôpital's Rule is a powerful tool for evaluating limits of indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$. It states that if $\lim_{x\to a}\frac{f(x)}{g(x)}$ is an indeterminate form, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the limit on the right-hand side exists.

Example 1: Indeterminate Form $\frac{0}{0}$

Consider the limit:

$$\lim_{x\to c}\frac{\sin(x)}{x}$$

This limit is of the form $\frac{0}{0}$ as x approaches c. Applying L'Hôpital's Rule:

$$\lim_{x\to c}\frac{\sin(x)}{x} = \lim_{x\to c}\frac{\cos(x)}{1} = \cos(c)$$

So, the limit is $\cos(c)$.

Example 2: Limit as $x \to \infty$

Now, consider the limit:

$$\lim_{x\to\infty}\frac{2x^2+3x+1}{3x^2-2x+5}$$

This limit is of the form $\frac{\infty}{\infty}$ as x approaches infinity. Applying L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{2x^2+3x+1}{3x^2-2x+5} = \lim_{x\to\infty}\frac{4x+3}{6x-2} = \lim_{x\to\infty}\frac{4}{6} = \frac{2}{3}$$

Alternatively, if we factor out the highest term in the numerator and denominator:

$$\lim_{x\to\infty}\frac{2x^2+3x+1}{3x^2-2x+5}=\lim_{x\to\infty}\frac{x^2(2+\frac{3}{x}+\frac{1}{x^2})}{x^2(3-\frac{2}{x}+\frac{5}{x^2})}=\lim_{x\to\infty}\frac{2+\frac{3}{x}+\frac{1}{x^2}}{3-\frac{2}{x}+\frac{5}{x^2}}=\frac{2}{3}$$

So, we see that both methods lead to the same constant limit.

25 L'Hôpital's Rule (Composite Exponential Functions)

Consider the limit:

$$\lim_{x\to 0}\cos(2\pi x)^{\frac{1}{x}}$$

Direct substitution leads to an indeterminate form 1^{∞}.
Let $y=\cos(2\pi x)^{\frac{1}{x}}$. Taking the natural logarithm of both sides:

$$\ln(y)=\frac{1}{x}\ln(\cos(2\pi x))=\frac{\ln(\cos(2\pi x))}{x}$$

Now, we can use L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{\frac{\partial}{\partial x}(\ln(\cos(2\pi x)))}{\frac{\partial}{\partial x}(x)}=\lim_{x\to 0}\frac{\left(\frac{1}{\cos(2\pi x)}\cdot -2\pi\sin(2\pi x)\right)}{1}=\frac{-2\pi\sin(2\pi x)}{\cos(2\pi x)}$$

Now, find:

$$\lim_{x\to 0}\frac{-2\pi\sin(2\pi x)}{\cos(2\pi x)}=\frac{0}{1}=0$$

$$\ln\left(\lim_{x\to\infty}y\right)=0$$

To find the limit of y, we use:

$$e^{\ln(\lim_{x\to\infty}y)}=e^0=1$$

So

$$\lim_{x\to\infty}y=1$$

25.1 Example 2

Consider the limit:

$$\lim_{x\to\infty}(1+5e^x)^{\frac{1}{x}}$$

Let $y=(1+5e^x)^{\frac{1}{x}}$. Taking the natural logarithm of both sides:

$$\ln(y)=\ln\left((1+5e^x)^{\frac{1}{x}}\right)=\frac{1}{x}\ln(1+5e^x)=\frac{\ln(1+5e^x)}{x}$$

Now, find the derivative of $\ln(1+5e^x)$:

$$\lim_{x\to\infty}\frac{\frac{\partial}{\partial x}(\ln(1+5e^x))}{\frac{\partial}{\partial x}(x)}=\lim_{x\to\infty}\frac{\frac{5e^x}{1+5e^x}}{1}=1$$

So, as x approaches infinity, $\ln(y)$ approaches 1. Therefore,

$$e^1=e$$

$$\lim_{x\to\infty}y=e$$

25.2 Example 3

$$\lim_{x\to 0}(2e^x-1)^{\frac{1}{x^2+2x}}$$

$$\ln(y)=\frac{\ln(2e^x-1)}{x^2+2x}$$

Now we use L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{\frac{d}{dx}(\ln(2e^x-1))}{\frac{d}{dx}(x^2+2x)}=\frac{\frac{2e^x}{2e^x-1}}{2x+1}=1$$

$$\lim_{x\to 0}\ln(y)=1$$

$$e^1=e$$

$$\lim_{x\to 0}y=e$$

25.3 Example 4

$$\lim_{x\to\infty}(3x+1)^{\frac{4}{x}}$$

$$\ln(y)=\frac{4\ln(3x+1)}{x}$$

Now we use L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{\frac{d}{dx}(4\ln(3x+1))}{\frac{d}{dx}(x)}=\frac{12}{3x+1}$$

This means

$$\lim_{x\to\infty}\ln(y)=0$$

This is the same as

$$\ln\left(\lim_{x\to\infty}y\right)=0$$

To find the limit of y, we use:

$$e^{\ln(\lim_{x\to\infty}y)}=e^0=1$$

So

$$\lim_{x\to\infty}y=1$$

25.4 Example 5

$$y=\left(1+\frac{2}{x}\right)^{3x}$$

$$\ln(y)=\ln\left(\left(1+\frac{2}{x}\right)^{3x}\right)$$

$$=3x\ln\left(1+\frac{2}{x}\right)$$

$$=\frac{3\ln\left(1+\frac{2}{x}\right)}{x^{-1}}$$

26 Mean Value Theorem

If $f(x)$ is both differentiable and continuous on the interval $[a,b]$, $\exists\ c\in(a,b)$ such that:

$$\frac{\Delta y}{\Delta x}=\frac{f(b)-f(a)}{b-a}=f'(c)$$

26.1 Example

Let's consider the function $f(x) = x^3 + 6x^2 + 6x$, and let c be the number that satisfies the Mean Value Theorem in the interval $[-6, 0]$.

$$\frac{f(b) - f(a)}{b - a} = \frac{f(0) - f(-6)}{0 - (-6)} = \frac{36}{6} = 6$$

The derivative of $f(x)$ is given by $\frac{d}{dx}f(x) = 3x^2 + 12x + 6$.
Setting $3x^2 + 12x + 6 = 6$, we simplify:

$$3x(x + 4) = 6$$

The solutions are $x = \{0, -4\}$. However, since c must be in the open interval $(-6, 0)$, the solution is $c = -4$.

27 Analyzing functions

27.1 Critical points

A critical point on a function can be either the local or global minimum or maximum point on the graph. To find such points, we find where $f'(x) = 0$. Let's say we want to determine when a function is decreasing or increasing.

Let's take the function $f(x) = -x^3 + 4x^2 - 5x$. We take the derivative $\frac{d}{dx}(f(x)) = f'(x) = -3x^2 + 8x - 5$. To find the critical points, we set $f'(x) = 0$:

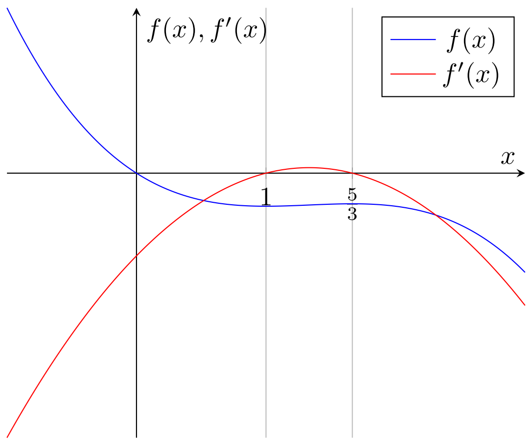
$$-3x^2 + 8x - 5 = 0$$

Using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we get $x = 1$ and $x = \frac{5}{3}$. Now, we have our number line and know that there is a minimum or maximum point at $x = 1$ and $x = \frac{5}{3}$.

Next, we find $f'(0)$ and $f'(2)$ to see if the function is decreasing or increasing beyond the minimum or maximum points. We find $f'(0)$ to be a negative number, indicating that the function is decreasing below 1. Similarly, $f'(2)$ is also negative. When we evaluate $f'\left(\frac{4}{3}\right)$, which is in the middle, we see that the function is increasing; therefore, by our analysis, we can say that the function $f(x)$ is increasing if and only if $1 < x < \frac{5}{3}$.



27.2 Relative Maximum and Minimum Points

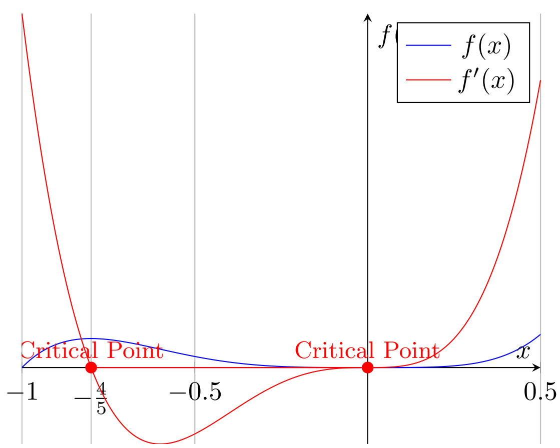
To find relative maximum and minimum points on a function, we start by identifying critical points where $f'(x) = 0$. Let's consider the function $f(x) = x^5 + x^4$, and its derivative $f'(x) = x^3(5x + 4)$. The critical points are at $x = 0$ and $x = -\frac{4}{5}$.

Next, we examine the behavior of $f'(x)$ around these critical points. We evaluate f' at points slightly smaller and larger than each critical point. For instance, we choose c slightly smaller than 0 and k slightly smaller than $-\frac{4}{5}$. We then determine whether $f'(c)$ and $f'(k)$ are negative or positive.

x	-1	$-\frac{4}{5}$	-0.5	0.5
$f'(x)$	$\uparrow$	$\rightarrow$	$\downarrow$	$\uparrow$

On the number line, we observe an ascending arrow over -1, a straight arrow over $-\frac{4}{5}$, a decreasing arrow over -0.5, and an ascending arrow over 0.5. These indicate the behavior of the derivative $f'(x)$ at these points.

Now, let's visualize this on a graph:



In the graph, we observe the function $f(x)$ and its derivative $f'(x)$. The critical points correspond to points where $f'(x) = 0$, and the behavior around these points helps identify whether there are relative maximum or minimum points on the function.

This analysis provides valuable insights into the behavior of the function and allows us to locate points of interest.

28 Inflection Points

Inflection points are points on a function that change its concavity. They occur where the rate of change transitions from ascending to descending. Algebraically, we can find inflection points by examining the second derivative of the function.

28.1 Example

Consider the function

$$h(x) = x^5 + 5x^4$$

The first derivative is given by

$$h'(x) = 5x^4 + 20x^3$$

And the second derivative is

$$h''(x) = 20x^3 + 60x^2$$

To find critical points, set the second derivative equal to zero:

$$20x^2(x + 3) = 0$$

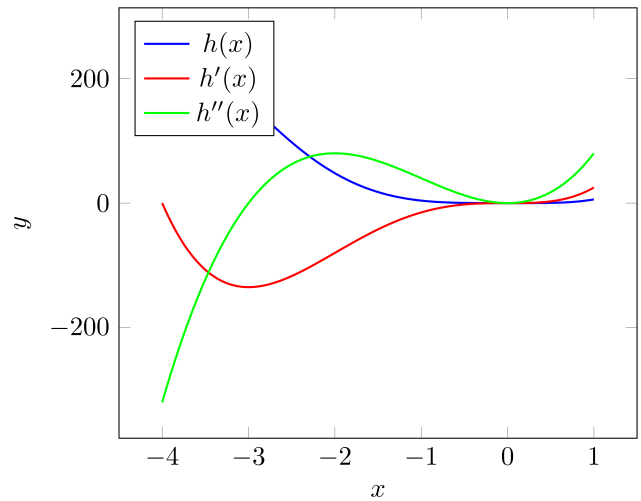
This gives critical points $x = -3$ and $x = 0$.

Now, let's analyze the sign of the second derivative at neighboring points:

x	$h''(x)$
-4	$\downarrow$
-3	$\rightarrow$
-2	$\uparrow$
-1	$\uparrow$
0	$\rightarrow$
1	$\uparrow$

The table shows that the rate of change is negative for $x = -4$, becomes positive at $x = -3$, and remains positive thereafter. Therefore, there is one inflection point at $x = -3$ where the rate of change changes from descending to ascending.

Now, let's visualize the function, its derivative, and the second derivative on a graph:



This graph visually illustrates how the function, its first derivative, and its second derivative relate to each other.

29 Integrals

The integral of a function is defined as follows:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\,dx \tag{2}$$

Where:

- n is the number of subintervals,
- x_i are the sample points in the i-th subinterval,
- Δx_i is the width of the i-th subinterval,
- $f(x_i)$ is the function evaluated at x_i,
- $\int_a^b f(x)\,dx$ represents the definite integral of $f(x)$ from a to b.

30 Riemann Sums

Let $\Delta x = \frac{b-a}{n}$ be the width of each subinterval, where $[a, b]$ is the interval of integration and n is the number of subintervals.

30.1 Right Riemann Sum

The right Riemann sum is given by:

$$\sum_{i=m}^{n} f(x_i)\Delta x_i$$

30.2 Left Riemann Sum

The left Riemann sum is given by:

$$\sum_{i=m}^{n} f(x_{i-1})\Delta x_i$$

30.3 Midpoint Riemann Sum

The midpoint Riemann sum is given by:

$$\sum_{i=m}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x_i$$

30.4 Trapezoidal Riemann Sum

The trapezoidal Riemann sum is given by:

$$\sum_{i=m}^{n} \frac{f(x_{i-1})+f(x_i)}{2}\Delta x_i$$

Where

$$x_{i-1} = a$$
$$x_i = b$$

30.5 Limit Definition of the Definite Integral

The definite integral of $f(x)$ over the interval $[a, b]$ is defined as the limit of Riemann sums as the number of subintervals approaches infinity:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=m}^{n} f(a+\Delta x_i)\Delta x$$

31 Fundamental Theorem in Calculus

31.1 Statement

The Fundamental Theorem of Calculus states that for a function $f(x)$ continuous on the closed interval $[a, b]$ and $F(x)$ defined by the definite integral:

$$F(x) = \int_a^b f(x)\, dx$$

Then, the antiderivative of $f(x)$, denoted by $F(x)$, is given by:

$$F(x) = F(b) - F(a)$$

Furthermore, the derivative of $F(x)$ is equal to the original function $f(x)$:

$$F'(x) = f(x)$$

31.2 Example

Consider the function $G(x) = \cos(3x)$ and let $g(x)$ be its derivative, $g(x) = G'(x)$. We want to find the definite integral:

$$\int_0^\pi g(x)\, dx$$

Using the Fundamental Theorem of Calculus:

$$\int_0^\pi g(x)\, dx = G(\pi) - G(0)$$

Substitute the values:

$$\int_0^\pi g(x)\, dx = \cos(3\pi) - \cos(0) = -2$$

32 Reverse Power Rule

The Reverse Power Rule is a useful rule for finding antiderivatives. It states the following:

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$$

where n is a constant and C is the constant of integration.
Additionally, the rule extends to the following cases:

$$\int c \cdot f(x)\, dx = c \int f(x)\, dx$$

for any constant c, and

$$\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$$

for functions $f(x)$ and $g(x)$.

32.1 Examples

32.1.1 Example 1

Evaluate the integral:

$$\int \frac{x^3 + 3x^2 - 5}{x^2}\,dx$$

Applying the Reverse Power Rule:

$$\int \frac{x^3 + 3x^2 - 5}{x^2}\,dx = \int (x + 3 - 5x^{-2})\,dx$$

Now apply the Reverse Power Rule to each term:

$$= \frac{x^2}{2} + 3x - 5x^{-1} + C$$

32.1.2 Example 2

Evaluate the integral:

$$\int \sqrt[3]{x^3}\,dx$$

Applying the Reverse Power Rule:

$$\int \sqrt[3]{x^3}\,dx = \int x^{5/3}\,dx$$

Now apply the Reverse Power Rule:

$$= \frac{x^{8/3}}{8/3} + C$$

Simplify the result:

$$= \frac{3x^{8/3}}{8} + C$$

33 Integral Rules

33.1 Sum/Difference Rule

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

33.2 Constant Multiple Rule

$$\int_a^b k \cdot f(x)\,dx = k \int_a^b f(x)\,dx$$

33.3 Reverse Interval Rule

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

33.4 Zero-length Interval Rule

$$\int_a^a f(x)\,dx = 0$$

33.5 Adding Intervals Rule

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

34 Integral Examples

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int (e^x + \frac{6}{x})\,dx = e^x + 6\ln|x| + C$$

$$\int (\sin x + \cos x)\,dx = -\cos x + \sin x + C$$

$$\begin{aligned}\int 3\csc x\cot x\,dx \quad &(\text{Let } u = \csc x,\ du = -\csc x\cot x\,dx)\\ &= -3\int du\\ &= -3u + C\\ &= -3\csc x + C\end{aligned}$$

$$\begin{aligned}\int 5\csc x\cot x\,dx &= 5\int (\csc x\cot x + \csc^2 x)\,dx\\ &= 5\int (du + \frac{du}{dx})\,dx \quad (\text{Let } u = \csc x,\ du = -\csc x\cot x\,dx)\\ &= 5\int du\\ &= 5u + C\\ &= 5\csc x + C\end{aligned}$$

35 u-substitution

Consider the integral

$$\int (3x^2 + 2x)e^{(x^3+x^2)}\, dx.$$

Let's make the substitution $u = x^3 + x^2$. Then, $du/dx = 3x^2 + 2x$, and $du = (3x^2 + 2x)\, dx$.

The integral becomes

$$\int e^u\, du = e^u + C = e^{x^3+x^2} + C,$$

where C is the constant of integration.

35.1 Example 2

Consider the integral

$$\int \sqrt{7x+9}\, dx.$$

We can simplify this by making the substitution $u = 7x + 9$, which gives $du = 7\, dx$. Thus,

$$\frac{1}{7}\int 7\sqrt{7x+9}\, dx = \frac{1}{7}\int \sqrt{u}\, du = \frac{1}{7} \cdot \frac{2}{3}u^{3/2} + C = \frac{2}{21}(7x+9)^{3/2} + C.$$

35.2 Example 3

Consider the integral

$$\int (2x+1)\sqrt{x^2+x}\, dx.$$

Let $u = x^2 + x$, then $du = (2x + 1)\, dx$. The integral becomes

$$\int \sqrt{u}\, du.$$

35.3 Example 4

Consider the integral

$$\int \frac{(\ln x)^{10}}{x}\, dx = \int (\ln x)^{10} \frac{1}{x}\, dx.$$

Let $u = \ln x$, then $du = \frac{1}{x}\, dx$. The integral becomes

$$\int u^{10}\, du.$$

35.4 Example 5

Consider the integral

$$\int \tan x \, dx = -\int \frac{\sin x}{\cos x} \, dx = -\int \frac{1}{\cos x}(-\sin x) \, dx.$$

Let $u = \cos x$, then $du = -\sin x \, dx$. The integral becomes

$$-\int \frac{1}{u} \, du.$$

35.5 Example 6

Consider the integral

$$\int \frac{4x^3}{x^4 + 7} \, dx.$$

Let $u = x^4$, then $du/dx = 4x^3$. The integral becomes

$$\int \frac{4x^3 \, dx}{x^4 + 7} = \int \frac{du}{u} = \int \frac{1}{u} \, dx = \ln |u| + C = \ln |x^4 + 7| + C.$$

35.6 Example 7

Consider the integral

$$\int \frac{\pi}{x \ln x} \, dx.$$

Let $u = \ln x$, then $du = \frac{1}{x} \, dx$ and $\frac{du}{dx} = \frac{1}{x}$. The integral becomes

$$\pi \int \frac{1}{\ln x} \cdot \frac{1}{x} \, dx = \pi \int \frac{1}{u} \, du = \pi \ln |u| + C = \pi \ln | \ln x| + C.$$

35.7 Example 8

Evaluate the following integral:

$$\int_0^1 x^2 \cdot 2^{x^3} \, dx$$

Using the fact that $2 = e^{\ln 2}$ and $2^{x^3} = (e^{\ln 2})^{x^3} = e^{\ln 2 \cdot x^3}$, let $u = \ln 2 \cdot x^3$. Then, $du = 3x^2 \ln 2 \, dx$, which can be expressed as $x^2 \cdot 3 \ln 2 \, dx = x^2 \ln 2^3 \, dx = x^2 \ln 8 \, dx$.

Now, the integral becomes:

$$\frac{1}{\ln 8} \int_0^1 \ln 8 \cdot x^2 \cdot e^{\ln 2 \cdot x^3} \, dx$$

Simplifying further:

$$-\frac{1}{\ln 8}\int_0^1 e^u\,du$$

The integral of e^u with respect to u is e^u, so the result is:

$$-\frac{1}{\ln 8}e^u + C$$

Now, substitute back $u = \ln 2 \cdot x^3$:

$$-\frac{1}{\ln 8}e^{\ln 2 \cdot x^3} + C$$

Finally, evaluate the expression from 0 to 1:

$$-\frac{1}{\ln 8}\left(e^{\ln 2} - 1\right) = \frac{1}{\ln 8}$$

35.8 Example 9

Evaluate the following integral:

$$\int (x+3)(x-1)^5\,dx$$

Let $u = x - 1$, then $du = dx$, and the integral becomes:

$$\int (u+4)u^5\,du = \frac{u^6}{6} + \frac{2}{3}u^5 + C = \frac{(x-1)^7}{7} + \frac{2}{3}(x-1)^6 + C$$

35.9 Example 10

Evaluate the following integral:

$$\int \frac{\cos 5x}{e^{\sin 5x}}\,dx$$

Let $u = \sin 5x$, then $du = 5\cos 5x\,dx$. After simplification:

$$\frac{1}{5}\int \frac{du}{e^u} = -\frac{1}{5}e^{-u} + C = -\frac{1}{5}e^{-\sin 5x} + C$$

35.10 Example 11

Evaluate the following integral:

$$\int \frac{2^{\ln x}}{x}\,dx$$

Let $u = \ln x$, then $du = \frac{1}{x}\,dx$. The integral becomes:

$$\int 2^u\,du = \frac{1}{\ln 2}2^u + C = \frac{1}{\ln 2}2^{\ln x} + C$$

35.11 Example 12

Let $u = 4x$, $du = 4\,dx$, then

$$\frac{1}{4}\int \frac{1}{2+\sin u}\cos u\,du$$

Let $s = \sin u$, $ds = \cos u\,du$, then

$$\frac{1}{4}\int \frac{1}{2+s}\,ds$$

$$= \frac{1}{4}\ln|2+s| + C$$

Substitute back $s = \sin u$:

$$= \frac{1}{4}\ln|2+\sin u| + C$$

Finally, substitute back $u = 4x$:

$$= \frac{1}{4}\ln|2+\sin 4x| + C$$

35.12 Example 13

Evaluate the following integral:

$$\int \frac{x-5}{-2x+2}\,dx = \int\left(-\frac{1}{2} - \frac{4}{-2x+2}\right)dx$$

After simplification:

$$-\frac{1}{2x} + 2\ln|x-1| + C$$

35.13 Integration using Long Division

Consider the integral:

$$\int \frac{2x^3 - 3x^2 - 3x + 2}{x-2}\,dx$$

To evaluate this integral, we can use long division.

$$\begin{array}{r|rrrr}
x-2 & 2x^3 & -3x^2 & -3x & +2 \\
\cline{2-3}
 & -(2x^3 & +4x^2) & & \\
\cline{2-5}
 & & -x^2 & -3x & \\
 & & +x^2 & -2x & \\
\cline{3-3}
 & & & -x & +2 \\
 & & & +x & -2 \\
 & & & & 0
\end{array}$$

Thus, we find that:

$$\int \frac{2x^3 - 3x^2 - 3x + 2}{x - 2}\, dx = \int (2x^2 + x - 1)\, dx$$

And now, we can easily evaluate the integral:

$$= \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + c$$

where c is the constant of integration.

35.14 Integration using completing the square

$$\int \frac{1}{\sqrt{k^2 - x^2}}\, dx = \arcsin\left(\frac{x}{k}\right) + C \qquad (3)$$

=

$$\int \frac{1}{\sqrt{k^2 - (x+h)^2}}\, dx$$

=

$$= \sin^{-1}\left(\frac{x+h}{k}\right)$$

$$\int \frac{a}{(x+h)^2 + k^2}\, dx = \frac{1}{k} \arctan\left(\frac{x+h}{k}\right) + C \qquad (4)$$

Consider the following integral:

$$\int \frac{1}{5x^2 - 30x + 65}\, dx$$

We can start by completing the square in the denominator:

$$= \frac{1}{5} \int \frac{1}{x^2 - 6x + 13}\, dx$$

Now, express the denominator as a perfect square:

$$= \frac{1}{5} \int \frac{1}{(x-3)^2 + 2^2}\, dx$$

Next, perform a substitution to simplify the integral further:

$$= \frac{1}{20} \int \frac{1}{\left(\frac{(x-3)^2}{2^2} + 1\right)}\, dx$$

Let's make the substitution $u = \frac{1}{2}x - \frac{3}{2}$. Then, $du = \frac{1}{2}dx$, and the integral becomes:

$$= \frac{1}{10} \int \frac{1}{u^2 + 2}\, du$$

Now, integrate with respect to u:

$$= \frac{1}{10}\arctan(u) + C$$

Finally, substitute back $u = \frac{1}{2}x - \frac{3}{2}$ to get the result in terms of x:

$$= \frac{1}{10}\tan^{-1}\left(\frac{1}{2}x - \frac{3}{2}\right) + C$$

where C is the constant of integration.

36 Integration using trigonometric identities

$$\int \cos^3 x\,dx$$

We know from the Pythagorean identity that $\cos^2 x = 1 - \sin^2 x$, so now we have

$$\int \cos x \cdot (1 - \sin^2 x)\,dx$$

$$= \int \cos x\,dx - \int \cos x \cdot \sin^2 x\,dx$$

We let $u = \sin x$ and $du = \cos x\,dx$. So now we get

$$\sin x - \int u^2\,du$$

$$= \sin x - \frac{u^3}{3} + C$$

$$= \sin x - \frac{1}{3}\sin^3 x + C$$

36.1 Example 2

$$\int \sin^2 x \cdot \cos^3 x\,dx$$

$$= \int \sin^2 x \cdot \cos^2 x \cdot \cos x\,dx$$

$$= \int \sin^2(1 - \sin^2 x) \cdot \cos x\,dx$$

$$= \int (\sin^2 x - \sin^4 x) \cdot \cos x\,dx$$

Let $u = \sin x$, $du = \cos x\, dx$, and we get

$$\int (u^2 - u^4)\, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

36.2 Example 3

$$\int \sin^4 x\, dx$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 2x = \frac{1}{2}(1 - \cos 4x)$$

$$\int \left(\frac{1}{2}(1 - \cos 2x)\right)^2 dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(1 - \cos 4x)\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

37 Inverse Tangent and Integration

37.1 $\tan^{-1}(x)$

The equation is given by:

$$\frac{1}{x^2 + 1} = \tan^{-1}(x)$$

37.2 Integration with e^x

The integral expression:

$$\int \frac{e^x}{1+e^{2x}}\,dx$$

Manipulating the expression using substitution $u = e^x$:

$$= \int \frac{1}{1+e^{2x}} \cdot e^x\,dx$$

Let $u = e^x$, then $du = e^x\,dx$:

$$= \int \frac{1}{u^2}\,du$$

This integrates to:

$$= \tan^{-1}(u)$$

Substitute back $u = e^x$:

$$= \tan^{-1}(e^x)$$

38 Differential Equations

We want to see if a function is a solution to a differential equation:

$$f'(x) = \frac{3f(x)}{x \ln x}$$

Is $f(x) = 2(\ln x)^3$ a solution to the differential equation?

$$\begin{aligned} f(x) &= 2(\ln x)^3 \\ u &= x^3 \\ du &= 3x^2 \\ f'(x) &= \frac{6(\ln x)^2}{x} \\ \frac{6(\ln x)^2}{x} &= \frac{3f(x)}{x \ln x} \\ \frac{6(\ln x)^2}{x} &= \frac{6(\ln x)^2}{x} \end{aligned}$$

38.1 Example 2

$$\frac{dy}{dx} = y(2x-5)$$

Is $y = -e^{x^2-5x}$ a solution to the differential equation?

$$\begin{aligned} \frac{dy}{dx} &= -(2x-5)e^{x^2-5x} \\ -(2x-5)e^{x^2-5x} &= y(2x-5) \\ -(2x-5)e^{x^2-5x} &= -(2x-5)e^{x^2-5x} \end{aligned}$$

39 Solving Differential Equations

Given the differential equation:

$$\frac{dy}{dx} = -\frac{x}{ye^{x^2}}$$

Isolate $y\,dy$ on one side and $x\,dx$ on the other side:

$$y\,dy = -xe^{-x^2}\,dx$$

Integrate both sides:

$$\int y\,dy = \int -xe^{-x^2}\,dx$$

This leads to:

$$\frac{y^2}{2} + C_1 = -\frac{1}{2}e^{-x^2} + C_2$$

Subtract C_1 from both sides:

$$\frac{y^2}{2} = -\frac{1}{2}e^{-x^2} + C$$

Multiply both sides by 2:

$$y^2 = e^{-x^2} + 2C$$

Take the square root:

$$y = \pm\sqrt{e^{-x^2}}$$

39.1 Example 2

Given the differential equation:

$$\frac{dy}{dx} = \frac{1}{6x - 42}$$

We can rewrite it as:

$$\frac{dy}{dx} = \frac{1}{6(x-7)}$$

Now, integrate both sides:

$$\int 6\,dy = \int \frac{1}{x-7}\,dx$$

This leads to:

$$6y = \ln|x-7| + C_1$$

Solving for y:

$$y = \frac{\ln|x-7| + C_1}{6}$$

Alternatively, we can simplify the expression:

$$y = \frac{\ln|x-7|}{6} + C$$

39.2 Example 3

Given the differential equation:

$$\frac{dy}{dx} = -\frac{1}{4e^x y^2}$$

We can rewrite it as:

$$\frac{dy}{dx} = -\frac{1}{4e^x} y^{-2}$$

Further simplifying:

$$\frac{dy}{dx} = -\frac{e^x}{4y^2}$$

Now, integrate both sides:

$$\int -4y^2 \, dy = \int e^x \, dx$$

This leads to:

$$-\frac{4}{3} y^3 = e^x + C_1$$

Solving for y:

$$y^3 = -\frac{3e^x}{4} + C_1$$

Finally:

$$y = \sqrt[3]{-\frac{3e^x}{4}} + C$$

39.3 Differential Equations (Exponentials)

We consider the differential equation:

$$\frac{dy}{dx} = y$$

Separate variables:

$$\frac{1}{y} \, dy = dx$$

Integrate both sides:

$$\int \frac{1}{y} \, dy = \int dx$$

Evaluate the integral:

$$\ln |y| = x + C$$

Exponentiate both sides:

$$|y| = e^{x+C}$$

Simplify:

$$|y| = e^x \cdot e^C$$

Combine constants:

$$|y| = Ce^x$$

The solution to the differential equation is $|y| = Ce^x$.
Solve the equation when $f(4) = 1$:
When $y = 1$ and $x = 4$:

$$1 = Ce^4$$

Solve for C:

$$e^{-4} = C$$

Plug it back into our original solution:

$$y = e^{-4} \cdot e^x$$

Simplify:

$$y = e^{x-4}$$

Calculus Applications: Exponential Growth and Decay

In calculus, problems involving exponential growth and decay are common in various fields, such as finance, physics, and biology. These problems can be modeled using differential equations, providing a powerful tool for understanding and predicting changes over time.

General Form of Exponential Growth/Decay Model

The general form of an exponential growth or decay model is given by the differential equation:

$$\frac{dy}{dx} = ky$$

where y is the quantity of interest (e.g., value, population), x is time, and k is a constant that determines the rate of growth or decay.

Solving Exponential Growth/Decay Differential Equations

The solution to the differential equation can be found by separating variables and integrating:

$$\int \frac{1}{y}\, dy = \int k\, dx$$

This leads to the general solution:

$$y = Ce^{kx}$$

where C is an arbitrary constant determined by initial conditions.

Application to Real-World Problems

To apply these concepts to real-world problems, specific information about the initial conditions and growth/decay rates is needed. Once the constants are determined, the model can be used to make predictions for future values.

Now, let's explore a few examples to see how calculus can be applied to solve problems involving exponential growth and decay.

Problem 1: Computer Depreciation

The value of a computer is decreasing at a rate that is proportional to its current value. Let y be the value of the computer at time x. The differential equation is given by:

$$\frac{dy}{dx} = y$$

Solving this differential equation, we get:

$$\int \frac{1}{y}\, dy = \int dx$$

$$\ln|y| = x + C$$

$$|y| = e^{x+C} = Ce^{x}$$

Given that the computer is worth 850 initially ($y = 850), anditisworth306after2years(x = 2), wefindC$:

$$850e^2 = 306 \implies e^2 = \frac{306}{850} \implies e^2 = 0.36$$

$$2x = \ln(0.36) \implies x = \frac{\ln(0.36)}{2}$$

To find the value of the computer after 5 years ($x = 5$):

$$y(5) = 850e^{\frac{\ln(0.36)}{2}\cdot 5}$$

Problem 2: Rhodium Price Increase

The price of rhodium increases at a rate proportional to its current price. Let y be the price of rhodium at time x. The differential equation is given by:

$$\frac{dy}{dx} = ky$$

Given that the initial price is 475 and it quadruples every 25 months ($k = \frac{\ln(4)}{25}$), the price after 35 months ($x = 35$) is:

$$y(35) = 475e^{\frac{\ln(4)}{25} \cdot 35}$$

Problem 3: Decrease in Uninformed Students

The number of students who have not heard a controversial story decreases at a rate proportional to the number of students uninformed. Let y be the number of uninformed students at time x. The differential equation is given by:

$$\frac{dy}{dx} = -\frac{y}{3}$$

Given that there were initially 900 students uninformed ($y = 900$), the remaining after 7 days ($x = 7$) is:

$$y(7) = 900e^{-\frac{\ln(1/3)}{4} \cdot 7}$$