# Multivariable Calculus

Joseph Margaryan

March 11, 2024

## Partial Derivatives

### First-Order Partial Derivatives

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to \infty} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

### Examples

$$f(x,y) = \tan(xy^{2.5})$$

$$\frac{\partial^3 f}{\partial y^3} = 2.5y^{1.5}\tan(x) \to 3.75y^{0.5}\tan(x) \to 1.875y^{-0.5}\tan(x)$$

$$\frac{\partial^2 f}{\partial y^2}(2e^y x^2) = 2e^y x^2$$

$$\frac{\partial^2 f}{\partial x \partial y}(4yx - x + e^y) = 4y - 1 \to 4$$

$$\frac{\partial^2 f}{\partial x^2}(x^6 y) = 6x^5 y \to 30x^4 y$$

## Gradients

$$\nabla f = \text{grad } f$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \ldots + \frac{\partial f}{\partial z}\hat{n}$$

$$f(x,y) = x^2 \sin(y)$$

$$\frac{\partial f}{\partial x} = 2x\sin(y)$$

$$\frac{\partial f}{\partial y} = x^2\cos(y)$$

$$\nabla f(x,y) = \begin{bmatrix} 2x\sin(y) \\ \\ x^2\cos(y) \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

# 1 Finding Gradients

## 1.1 Example 1

$$f(x,y,z) = xy + yz + zx$$
$$\frac{\partial f}{\partial x} = y + z$$
$$\frac{\partial f}{\partial y} = x + z$$
$$\frac{\partial f}{\partial z} = y + x$$
$$\nabla f(-4,3,-1) = (-4,3,1)$$

## 1.2 Example 2

$$f(x,y) = xe^y$$
$$\frac{\partial f}{\partial x} = e^y$$
$$\frac{\partial f}{\partial y} = xe^y$$
$$\nabla f(4,0) = (1,4)$$

## 1.3 Example 3

$$f(x,y,z) = \tan(xyz)$$
$$\frac{\partial f}{\partial x} = yz\sec^2(xyz)$$
$$\frac{\partial f}{\partial y} = xz\sec^2(xyz)$$
$$\frac{\partial f}{\partial z} = xy\sec^2(xyz)$$
$$\nabla f = (yz\sec^2(xyz), xz\sec^2(xyz), xy\sec^2(xyz))$$

### 1.4 Example 4

Consider the function $f(x, y) = \frac{1}{x^2+y^2} = (x^2 + y^2)^{-1}$.

Partial derivatives:

$$\frac{\partial f}{\partial x} = -\frac{2x}{(x^2+y^2)^2} = -\frac{2x}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = -\frac{2y}{(x^2+y^2)^2} = -\frac{2y}{(x^2+y^2)^2}$$

$$\nabla f(\mathbf{a}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

## 2 Directional Derivatives

Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a vector in the plane. The directional derivative of a function $f(x, y)$ in the direction of $\vec{v}$ is denoted by $\nabla\vec{v}f(a, b)$ and is given by:

$$\nabla\vec{v}f(a, b) = a\left(\frac{\partial f}{\partial x}\right) + b\left(\frac{\partial f}{\partial y}\right)$$

Here, $\nabla\vec{v}f(a, b)$ represents the directional derivative of $f$ at the point $(a, b)$ in the direction of the vector $\vec{v}$.

The directional derivative along the vector $\vec{v}$ with respect to $x$ is expressed as:

$$\left(\frac{\partial f}{\partial x}\right)(\vec{v}) = \lim_{h\to 0} \frac{f(\vec{v} + h\uparrow) - f(\vec{v})}{h}$$

Similarly, the directional derivative along the vector $\vec{v}$ with respect to $y$ is:

$$\left(\frac{\partial f}{\partial y}\right)(\vec{v}) = \lim_{h\to 0} \frac{f(\vec{v} + h\downarrow) - f(\vec{v})}{h}$$

Finally, the total directional derivative of $f$ at the point $(a, b)$ in the direction of $\vec{v}$ is given by:

$$\nabla\vec{v}f(\vec{v}) = \lim_{h\to 0} \frac{f(a + ha, b + hb) - f(a, b)}{h}$$

These formulas provide a way to compute the rate of change of the function $f$ in a specific direction.

# 3 Example 1

Consider the function $f(x, y, z) = x^2 - 6y + 2z$.

Let $\mathbf{a} = (1, -1, 1)$ and $\mathbf{v} = (-1, 2, 2)$.

The directional derivative is given by

$$\lim_{h\to 0} \frac{f(\mathbf{a} + h\mathbf{v}) - f(\mathbf{a})}{h} = \lim_{h\to 0} \frac{f(1-h, -1+2h, 1+2h) - f(1, -1, 1)}{h}.$$

Evaluating this expression, we get

$$\lim_{h\to 0} \frac{(1 - 2h + h^2 - 6 - 12h + 2 + 4h - 1 - 6 - 2)}{h} = \lim_{h\to 0} \frac{(h^2 - 10h)}{h} = \lim_{h\to 0}(h-10) = -10.$$

# 4 Example 2

Consider the function $f(x, y, z) = \frac{xy}{z}$.

Let $\mathbf{a} = (-3, 2, 1)$ and $\mathbf{v} = (1, 1, 1)$.

The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{y}{z},$$
$$\frac{\partial f}{\partial y} = \frac{x}{z},$$
$$\frac{\partial f}{\partial z} = -\frac{xy}{z^2}.$$

The gradient is given by $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$.

At $\mathbf{a} = (-3, 2, 1)$, the gradient is $\nabla f(\mathbf{a}) = (2, -3, 6)$.

The dot product $(\nabla f(\mathbf{a})) \cdot \mathbf{v} = (2, -3, 6) \cdot (1, 1, 1) = 5$.

# Example 3

Consider the function $f(x, y) = \tan(y^2) - 3xy$.

Given point $\mathbf{a} = (-1, 1)$ and vector $\mathbf{v} = (1, 0)$.

Partial derivatives:

$$\frac{\partial f}{\partial x} = -3y$$

$$\frac{\partial f}{\partial y} = 2y\tan(y)\sec^2(y)$$

Gradient:

$$\nabla f(\mathbf{a}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (-3, 2\tan(1)\sec^2(1))$$

Dot product:

$$\nabla f(\mathbf{a}) \cdot \mathbf{v} = -3$$

## Example 4

Consider the function $f(x, y, z) = y^2 \sin(x) + xz$.
Given point $\mathbf{a} = (0, 2, 1)$ and vector $\mathbf{v} = (1, 0, -2)$.
Partial derivatives:

$$\frac{\partial f}{\partial x} = y^2 \cos(x) + z$$

$$\frac{\partial f}{\partial y} = 2y \sin(x)$$

$$\frac{\partial f}{\partial z} = x$$

Gradient:

$$\nabla f(\mathbf{a}) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (5, 0, 0)$$

Dot product:

$$\nabla f(\mathbf{a}) \cdot \mathbf{v} = 5$$

## Example 5

Consider the function $f(x, y, z) = x^2 \ln(z)$.
Given point $\mathbf{a} = (-1, 0, e)$ and vector $\mathbf{v} = (1, 0, 0)$.
Limit expression:

$$\lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{v}) - f(\mathbf{a})}{h}$$

$$\lim_{h \to 0} \frac{f(-1 + h, 0, e) - f(-1, 0, e)}{h}$$

$$\lim_{h \to 0} \frac{(-1 + h)^2 \ln(e) - (-1)^2 \ln(e)}{h} = -4 = \frac{\partial f}{\partial x} \mathbf{v}$$

## Example 6

Consider the function $f(x, y) = \sin(xy)$.
Given point $\mathbf{a} = (4, 0)$ and vector $\mathbf{v} = (0, 1)$.
Partial derivatives:

$$\frac{\partial f}{\partial x} = y \cos(xy)$$

$$\frac{\partial f}{\partial y} = x \cos(xy)$$

Gradient:

$$\nabla f(\mathbf{a}) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

Evaluate dot product:

$$\nabla f(\mathbf{a}) \cdot \mathbf{v} = 4$$

# 5 Multivariable Chain Rule

The multivariable chain rule expresses the derivative of a composite function. If $f(x(t), y(t))$ is a function of two variables and $x$ and $y$ are both functions of $t$, then:

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

For a composition of functions $h(t) = f(g(t))$, the chain rule can be expressed as:

$$\frac{dh}{dt} = \nabla f(g(t)) \cdot \mathbf{g}'(t)$$

## 5.1 Example

Let $f(x, y, z) = xy - x\sqrt{z}$ and $g(t) = (2, -t, 4t^2)$.

The gradient of $f$ is given by:

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

Where:

$$\frac{\partial f}{\partial x} = y - \frac{x}{\sqrt{z}}, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = -\frac{x}{2\sqrt{z}}$$

So,

$$\nabla f = \left( y - \frac{x}{\sqrt{z}}, x, -\frac{x}{2\sqrt{z}} \right)$$

The derivative of $g(t)$ is:

$$\mathbf{g}'(t) = (0, -1, 8t)$$

The chain rule for this example becomes:

$$\frac{d}{dt} f(g(t)) = \nabla f(g(t)) \cdot \mathbf{g}'(t)$$

$$\begin{aligned} \frac{d}{dt} f(g(t)) &= \left( -t - \sqrt[4t^2]{}, 2, -\frac{1}{\sqrt{4t^2}} \right) \cdot (0, -1, 8t) \\ &= -t - \sqrt{4t^2} - 16t^2 \end{aligned}$$

## 6 Divergence

In multivariable calculus, divergence is a vector operator that operates on a vector field, producing a scalar field as a result. It is denoted by the symbol $\nabla\cdot$ (read as "nabla dot"). The divergence of a vector field $\mathbf{F}$ is defined as follows:

If $\mathbf{F} = \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix}$ is a vector field in three-dimensional space $R^3$, then the divergence $\nabla \cdot \mathbf{F}$ is given by:

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

In other words, the divergence of a vector field measures the rate at which the vectors of the field are spreading out from or converging toward a given point in space. If the divergence is positive at a point, it indicates that the vectors are spreading out from that point. If it's negative, it indicates convergence, and if it's zero, it means there is no spreading or converging.

**Example:**

Let $\mathbf{F} = \begin{bmatrix} (xz)^2 \\ \cos(xy) \\ e^{7z} \end{bmatrix}$.

The divergence $\nabla \cdot \mathbf{F}$ is calculated as follows:

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}[(xz)^2] + \frac{\partial}{\partial y}[\cos(xy)] + \frac{\partial}{\partial z}[e^{7z}] \\ &= 2xz^2 - x\sin(xy) + 7e^{7z} \end{aligned}$$

## 2D Curl in Vector Calculus

In vector calculus, the curl is a vector operator that describes the rotation of a vector field. For a two-dimensional vector field $\mathbf{F} = \langle P, Q \rangle$, the 2D curl $\mathrm{curl}_{2D}(\mathbf{F})$ is given by:

$$\mathrm{curl}_{2D}(\mathbf{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Here, $P$ and $Q$ are the components of the vector field.

## Example

Let $\mathbf{F} = \begin{bmatrix} y^3 - 9y \\ x^3 - 9x \end{bmatrix}$. The 2D curl of $\mathbf{F}$ is:

$$\mathrm{curl}_{2D}(\mathbf{F}) = \frac{\partial}{\partial x}(x^3 - 9x) - \frac{\partial}{\partial y}(y^3 - 9y) = 3x^2 - 9 - (3y^2 - 9)$$

When $(x, y) = (3, 0)$, there is a counter-clockwise rotation, so its result will be positive:

$$\text{curl}_{2D}(\mathbf{F})\Big|_{(3,0)} = 3(3)^2 - 9 - (3(0)^2 - 9) = 18$$

### 6.1 Example 2

$$\vec{V}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} = \begin{bmatrix} 1 \\ 2xln(xy) \end{bmatrix}$$

$$\text{curl}_{2D}(\vec{V})\Big|_{(3,0)} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big|_{(3,0)}$$

**Use Product rule and chain rule**

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

**Answer**

$$2ln(xy) + 2\Big|_{(3,0)}$$

## 7 3D Curl

The curl of a vector field $\vec{V}$ in three dimensions is given by:

$$\vec{V} = \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix}$$

The curl of $\vec{V}$, denoted as $\nabla \times \vec{V}$, is defined as:

$$\nabla \times \vec{V} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

Which expands to:

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$$

### 7.1 Example

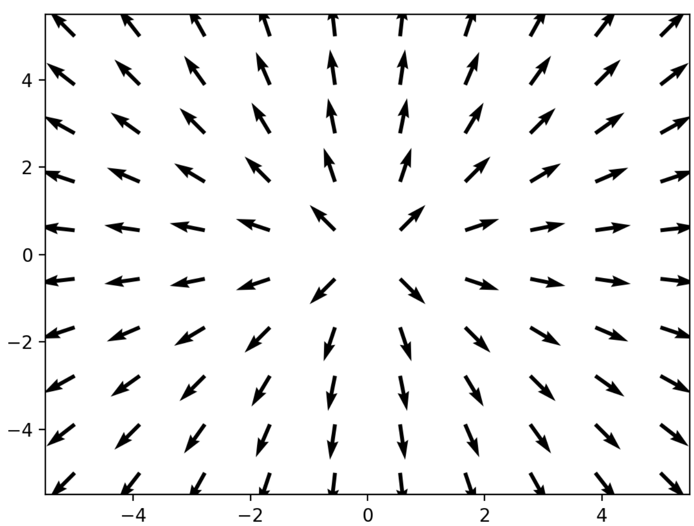
Consider the vector field $\mathbf{F}(x,y,z) = (4, \sin(xz), \cos(y) + \sin(z))$. The curl of $\mathbf{F}$ is given by:

$$\nabla \times \mathbf{F} = (-\sin(y) - xcos(xz))\hat{\mathbf{i}} + (0-0)\hat{\mathbf{j}} + (z\cos(xz) - 0)\hat{\mathbf{k}}$$

# 8 Vector and Scalar Fields

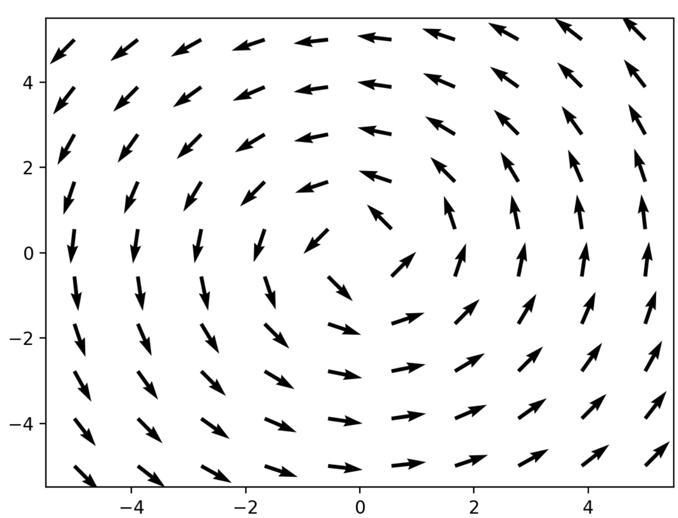
Let's consider two examples of vector fields in 2D:

$$\mathbf{F}(x,y) = \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{i}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{j}}$$

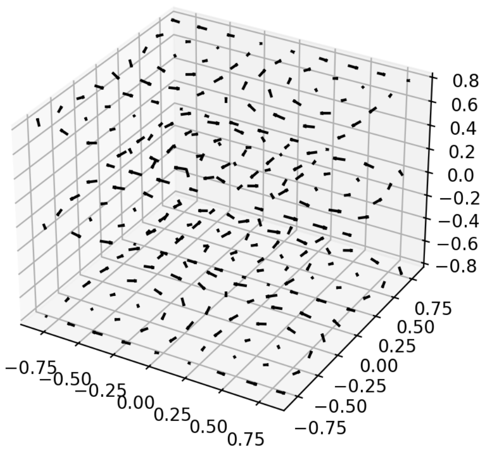



And another vector field:

$$\mathbf{G}(x,y) = \frac{-y}{\sqrt{x^2+y^2}}\hat{\mathbf{i}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{j}}$$



## 8.1 Example 3D Vector Field




# 9 Laplacian

The Laplacian of a scalar function $f$, denoted as $\nabla^2 f$ or $\triangle f$, is defined as the divergence of the gradient of $f$:

$$\nabla^2 f = \nabla \cdot \nabla f = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_k} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_k} \end{bmatrix}$$

This can be further expressed as:

$$= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_k^2}$$

And in summation notation:

$$\nabla^2 f = \sum_{i=1}^{k} \frac{\partial^2 f}{\partial x_i^2}$$

## 9.1 Example

Consider the function $f(x, y) = 3 + \cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)$. The Laplacian of $f$ is given by:

$$\nabla^2 f = \frac{1}{2} - \sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\hat{\mathbf{i}} + \frac{1}{2}\cos\left(\frac{y}{2}\right)\cos\left(\frac{x}{2}\right)\hat{\mathbf{j}}$$

**$\nabla$ is the divergence operator and $\nabla f$ represents the gradient of $f$.**

## 9.2 Example 2

$$f(x, y) = x^3 + x^2y + xy^2 + y^3$$
$$\nabla f = \nabla \cdot \nabla f$$
$$\frac{\partial f}{\partial x} = 3x^2 + 2xy + y^2$$
$$\frac{\partial f}{\partial y} = x^2 + 2xy + 3y^2$$
$$\frac{\partial^2 f}{\partial x^2} = 6x + 2y$$
$$\frac{\partial^2 f}{\partial y^2} = 2x + 6y$$
$$\nabla^2 f = 8x + 8y$$

## 9.3 Example 3

$$f(x, y, z) = x^2z^4 - z^2y^3 + x$$
$$\frac{\partial f}{\partial x} = 2xz^4 + 1$$
$$\frac{\partial f}{\partial y} = -3z^2y^2$$
$$\frac{\partial f}{\partial z} = 4x^2z^3 - 2zy^3$$
$$\frac{\partial^2 f}{\partial x^2} = 2z^4$$
$$\frac{\partial^2 f}{\partial y^2} = -6z^2y$$
$$\frac{\partial^2 f}{\partial z^2} = 12x^2z^2 - 2y^3$$
$$\nabla^2 f = 2z^4 - 6z^2y - 2y^3 + 12x^2z^2 - 2y^3$$

### 9.4 Example 4

$$f(x,y) = xy - e^{xy}$$

$$\triangle f \quad \text{(Laplace operator)}$$

From the chain rule:

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

Partial derivatives:

$$\frac{\partial f}{\partial x} = y - e^{xy}y$$

$$\frac{\partial f}{\partial y} = x - e^{xy}x$$

Second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = -e^{xy}y^2$$

$$\frac{\partial^2 f}{\partial y^2} = -e^{xy}x^2$$

Laplace operator of $f$:

$$\triangle f = -e^{xy}y^2 - e^{xy}x^2$$

### 9.5 Example 5

$$f(x,y,z) = xy - x^2z + y^2z$$

$$\nabla^2 f = \nabla \cdot \nabla f = 0$$

So, in this case, our multivariable function $f(x,y,z)$ is what we refer to as harmonic.

## 10 Jacobian

Suppose $f : R^n \to R^m$ is a function such that each of its first-order partial derivatives exists on $R^n$. This function takes a point $\mathbf{x} \in R^n$ as input and produces the vector $f(\mathbf{x}) \in R^m$ as output. Then the Jacobian matrix of $f$ is defined to be an $m \times n$ matrix, denoted by $\mathbf{J}$, whose $(i,j)$-th entry is $J_{ij} = \frac{\partial f_i}{\partial x_j}$.

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Here: - $f_i$ represents the $i$-th component function of the vector-valued function $\mathbf{f}$. - $x_j$ represents the $j$-th input variable. The elements $\frac{\partial f_i}{\partial x_j}$ of the Jacobian matrix represent the partial derivatives of the $i$-th component function with respect to the $j$-th input variable.

The Jacobian matrix provides valuable information about the local behavior of a multivariable function. It helps in understanding how small changes in each input variable contribute to changes in each component of the output vector. Additionally, the Jacobian is used in the linear approximation of a differentiable function near a specific point. If the function is differentiable at a point $\mathbf{a}$, then the Jacobian matrix $\mathbf{J}(\mathbf{f})$ evaluated at $\mathbf{a}$ is a linear transformation that approximates the change in $\mathbf{f}(\mathbf{x})$ near $\mathbf{a}$.

## 10.1 Example

Consider the vector-valued function $\mathbf{f} : R^2 \to R^2$ defined by:

$$\begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = \begin{bmatrix} x + \sin(y) \\ y + \sin(x) \end{bmatrix}$$

We want to find the Jacobian matrix $\mathbf{J}$ of $\mathbf{f}$ at a certain point $(-2, 1)$. The Jacobian matrix $\mathbf{J}$ is given by:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

Now, let's compute the partial derivatives:

$$\frac{\partial f_1}{\partial x} = 1, \quad \frac{\partial f_1}{\partial y} = \cos(y), \quad \frac{\partial f_2}{\partial x} = \cos(x), \quad \frac{\partial f_2}{\partial y} = 1$$

Substituting these values into $\mathbf{J}$:

$$\mathbf{J} = \begin{bmatrix} 1 & \cos(y) \\ \cos(x) & 1 \end{bmatrix}$$

Now, evaluating $\mathbf{J}$ at the point $(-2, 1)$:

$$\mathbf{J}(-2, 1) = \begin{bmatrix} 1 & \cos(1) \\ \cos(-2) & 1 \end{bmatrix}$$

This represents the Jacobian matrix of $\mathbf{f}$ at the specified point $(-2, 1)$, providing information about the local behavior of $\mathbf{f}$ near that point.

### 10.2 Example 2

Consider the vector-valued function $\mathbf{f} : R^3 \to R^3$ defined by:

$$\mathbf{f}(x, y, z) = \begin{bmatrix} ye^z \\ x - y \\ 3y + z^2 \end{bmatrix}$$

The Jacobian matrix $\mathbf{J}$ is given by:

$$\mathbf{J} = \begin{bmatrix} 0 & e^z & ye^z \\ 1 & -1 & 0 \\ 0 & 3 & 2z \end{bmatrix}$$

### 10.3 Example 3

Consider the vector-valued function $\mathbf{f} : R^2 \to R^3$ defined by:

$$\mathbf{f}(u, v) = \begin{bmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \sin(v) \end{bmatrix}$$

The Jacobian matrix $\mathbf{J}$ is given by:

$$\mathbf{J} = \begin{bmatrix} -\sin(u)\cos(v) & \cos(u) - \sin(v) \\ \cos(u)\cos(v) & \sin(u) - \sin(v) \\ 0 & \cos(v) \end{bmatrix}$$

### 10.4 Example 4

Consider the vector-valued function $\mathbf{f} : R^3 \to R^3$ defined by:

$$\mathbf{f}(x, y, z) = \begin{bmatrix} e^x e^y \\ y - x^2 \\ -x^2 + z \end{bmatrix}$$

The Jacobian matrix $\mathbf{J}$ is given by:

$$\mathbf{J} = \begin{bmatrix} e^x e^y & e^x e^y & 0 \\ -2x & 1 & 0 \\ -2x & 0 & 1 \end{bmatrix}$$

## 11 Jacobian determinant

- If $|x| > 1$, then the absolute value of the Jacobian determinant is greater than 1. This implies that the transformation described by the matrix is an expansion.

- If $x = 1$, then the Jacobian determinant is equal to 1. This implies that the transformation described by the matrix results in no change in volume (neither expansion nor contraction).
- If $0 < x < 1$, then the absolute value of the Jacobian determinant is between 0 and 1. This implies that the transformation described by the matrix is a contraction.
- If $x = 0$, then the Jacobian determinant is zero. This implies that the transformation described by the matrix collapses the space (contracting infinitely).

Let $f$ be a transformation from $R^2$ to $R^2$. Its Jacobian matrix is given below.

$$J(f) = \begin{bmatrix} -\sin(x) & -\cos(y) \\ \\ \cos(x) & \sin(y) \end{bmatrix}$$

The Jacobian determinant of $f$:

$$|J(f)| = -\sin(x)\sin(y) + \cos(y)\cos(x)$$

How will $f$ expand or contract space around the point $(\pi, \pi)$?
It's 1, so it will remain the same.

## 11.1 Example 2

Let $f$ be a transformation from $R^3$ to $R^3$. Its Jacobian matrix is given below.

$$J(f) = \begin{bmatrix} 1 & 0 & 0 \\ \\ y & x & 0 \\ \\ yz & xz & xy \end{bmatrix}$$

The Jacobian determinant of $f$:

$$|J(f)| = 1 \begin{vmatrix} x & 0 \\ xz & xy \end{vmatrix} - 0 \begin{vmatrix} y & 0 \\ yz & xy \end{vmatrix} + 0 \begin{vmatrix} y & x \\ yz & xz \end{vmatrix}$$

$$= 1(x * xy - x * yz) - 0(y * xy - 0 * yz) + 0(y * xz - x * yz) = x^2 y$$

At the point (2,0,3) the determinant is 0 which means the matrix is contracting infinitely

## 11.2 Example 3

Let $f$ be a transformation from $R^3$ to $R^3$. Its Jacobian matrix is given below.

$$J(f) = \begin{bmatrix} 0 & -zsin(y) & cos(y) \\ cos(z) & 0 & -xsin(z) \\ -ysin(x) & cos(x) & 0 \end{bmatrix}$$

The Jacobian determinant of $f$:

$$|J(f)| = 0 \begin{vmatrix} 0 & -xsin(z) \\ cos(x) & 0 \end{vmatrix} - (-zsin(y) \begin{vmatrix} cos(z) & -xsin(z) \\ -ysin(x) & 0 \end{vmatrix} + cos(y) \begin{vmatrix} cos(z) & 0 \\ -ysin(x) & cos(x) \end{vmatrix}$$
$$= 0 + zsin(y)(xsin(z) - ysin(x)) + cos(y)(cos(z) * cos(x))$$
$$= -zsin(y)xsin(z)ysin(x) + cos(y)cos(z)cos(x)$$

At the point $(\frac{\pi}{3}, 0, \frac{\pi}{3})$ the determinant is $\frac{1}{4}$ which means the matrix is contracting

# 12 Tangent Plane

The tangent plane is a geometrical surface that shows the tangent line at a certain point in a multivariable function. It can be calculated as follows:

$$T(x, y, z) = f(x_0, y_0, z_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial f}{\partial z}(z - z_0)$$
$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

at a given point $(x_0, y_0, z_0)$.

## 12.1 Example

Consider the function:

$$f(x, y, z) = x^3 - yz + \sin(x)$$

with partial derivatives:

$$f_x = 3x^2 + \cos(x)$$
$$f_y = -z$$
$$f_z = -y$$

To find the tangent plane at a specific point, say $(x_0, y_0, z_0) = (\pi, 4, 5)$, we plug in the values of the function and its partial derivatives at that point into the equation of the tangent plane:

$$3\pi^2 - 1(x - \pi) - 5(y - 4) - 4(z - 5)$$

### 12.2 Example 2

$$z = ye^{2x} - y^2$$

We want the Tangent plane at $S(1, 5)$

$$f_x = 2ye^{2x}$$
$$f_y = e^{2x} - 2y$$
$$z = 5e^2 - 25 + 10e^2(x - 1) + (e^2 - 10)(y - 5)$$

## 13 Quadratic Approximation

$$\begin{aligned} z = f(x, y) &+ f_x(x_0, y_0)(x - x_0) \\ &+ f_y(x_0, y_0)(y - y_0) \\ &+ \frac{1}{2} f_{xx}(x_0, y_0)(x - x_0)^2 \\ &+ f_{xy}(x_0, y_0)(x - x_0)(y - y_0) \\ &+ \frac{1}{2} f_{yy}(x_0, y_0)(y - y_0)^2 \end{aligned}$$

## 14 Hessian Matrix

The Hessian matrix $\mathbf{H}$ is given by:

$$\mathbf{H_{x,y}} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

The Hessian matrix $\mathbf{H}$ for the function $f(x, y, z)$ is given by:

$$\mathbf{H_{x,y,z}} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

## 15 Critical Points of multivariable functions

If we have a multivariable function and we want to find potential maxima, minima, or saddle points, we can find it like this:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0_k \end{bmatrix}$$

So we need to solve the system of equations that satisfies the following property:

$$\frac{\partial f}{\partial x_1} = 0$$
$$\frac{\partial f}{\partial x_2} = 0$$
$$\vdots$$
$$\frac{\partial f}{\partial x_k} = 0$$

Consider the function: $f(x, y, z) = sin(z + x) - x^2 - y^2$

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla f = (cos(x + z) - 2x, -2y, cos(x + z))$$

We need to solve the system of linear equation

$$\begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} cos(x + z) - 2x \\ -2y \\ cos(x + z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using python we can:

```
    from sympy import symbols, sin, cos, solve

# Define the variables
x, y, z = symbols('x y z')
```

```
# Define the function
f = sin(z + x) - x**2 - y**2

# Compute the partial derivatives
df_dx = f.diff(x)
df_dy = f.diff(y)
df_dz = f.diff(z)

# Find the critical points
critical_points = solve((df_dx, df_dy, df_dz), (x, y, z))

# Print the critical points
print("Critical Points:", critical_points)
```

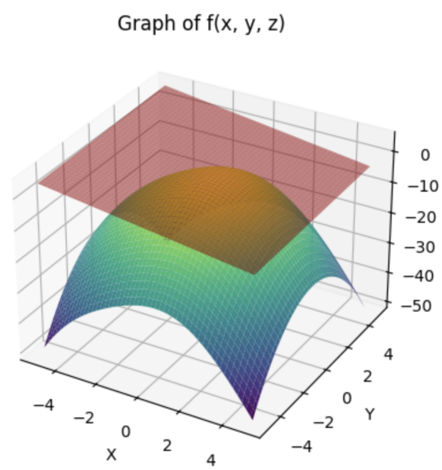



Figure 1: f(x, y, z) with a tangent plane at the critical point

## 15.1 Maxima, Minima, Saddle points

To determine whether a critical point is a local Maxima, Minima, or a saddle point, we need to find the determinant of the Hessian Matrix evaluated at the critical points:

$$|H_{x_0,y_0}| = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y}\frac{\partial^2 f}{\partial y \partial x}$$

$$|H| = f_{xx}f_{yy} - f_{xy}f_{yx}$$

In most cases, we can use the fact that:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$|H| > 0 \ \& \ f_{xx} > 0, implies- > \textbf{local minima}$$

$$|H| > 0 \ \& \ f_{xx} < 0, implies- > \textbf{local maxima}$$

$$|H| < 0, implies- > \textbf{saddle point}$$

```
    import sympy as sp

x, y = sp.symbols('x y')
f = x**3+y**3-x-y
dx = sp.diff(f, x)
dy = sp.diff(f, y)
dxx = sp.diff(dx, x)
dyy = sp.diff(dy, y)
dxdy = sp.diff(dx, y)
crit = sp.solve((dx, dy), (x, y))
h_det = dxx * dyy - dxdy ** 2

# Print critical points
print("Critical Points:", crit)

# Evaluate the determinant of the Hessian matrix at each critical point
for point in crit:
    h_det_value = h_det.subs({x: point[0], y: point[1]})
    print("Hessian Determinant at {}: {}".format(point, h_det_value))
```

# 16 Gradient Descent Algorithm

The Gradient Descent Algorithm is an iteration that iteratively comes closer to the system of equations that we find. $\nabla f = 0$
In certain instances we may have hundreds or even thousands of features and manipulating such large expressions becomes impossible to satisfy the system of linear equation:

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The Gradient Descent algorithm iteratively updates the parameters of a model by a learning rate $\alpha$ to minimize a loss function. At each iteration, the parameters are adjusted in the direction opposite to the gradient of the loss function with respect to the parameters, aiming to find the local minimum of the loss function.

$$x_{n-1} = x_n - \alpha \nabla f(x_n)$$

## 16.1 Example

Consider the function:

$$f(x, y) = x^2 - y$$

a learning rate:

$$\alpha = 0.1$$

We have the Gradient:

$$\nabla f = (2x, -1)$$

We choose a starting point from which our algorithm will iteratively optimize. Lets say:

$$x_n = (5, 3)$$

Our Gradient Descent algorithm becomes as follow:

$$x_{n-1} = (5, 3) - [0.1 * (10, -1)] = (4, 3.1)$$

Our algorithm successfully computed the next set of features to be (4, 3.1) after iteration 1. We would continue this as long as we wish to train our model and optimize the parameters to satisfy the system of equations derived from our Cost function. Using a small learning rate prevents us from ending up on the other side of the curve from which the slope of the tangent line = 0. We could theoretically come infinitely close to the perfect set of parameters:

$$\lim_{n \to \infty} x_n = x^*$$

Where $x^*$ is the optimal set of parameters that minimizes the loss function, X

To minimize the loss function effectively, it's crucial to identify the critical points corresponding to the appropriate local minima. However, challenges arise due to the presence of saddle points or multiple critical points in the loss landscape, which can hinder the convergence of optimization algorithms. Researchers have developed advanced optimization techniques, including Stochastic Gradient Descent, Adam, and others, to address these challenges and navigate the loss landscape more effectively.

# 17 Line Integrals

$$\int_C f(x, y) ds = \int_a^b f(h(t), g(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

## 17.1 Example

Suppose we have a scalar field $f(x, y) = x - y$ and a curve $C$ Thats parameterized by $\alpha(t) = (2cos(t), 2sin(t))$ for $0 < t < \pi$

$$f(h(t), g(t)) = 2cos(t) - 2sin(t)$$
$$ds = \sqrt{4(-sin(t)^2 + 4(cos(t)^2)}$$
$$ds = 2\sqrt{sin^2(t) + cos^2(t)}$$
$$ds = 2\sqrt(1) = 2$$
$$\int_0^\pi 2cos(t) - 2sin(t)(2) = -8$$

### 17.2 Example 2

$$f(x, y) = \frac{x^2}{r}$$
$$C :\rightarrow \alpha(t) = (rcos(t), rsin(t))$$
$$f(\alpha x, \alpha y) = \frac{(rcost)^2)}{r} = rcos^2(t)$$
$$ds = \sqrt{(-rsint)^2 + (rcost)^2} = \sqrt{-r^2 sin^2 t + r^2 cos^2 t}$$
$$= \sqrt{r^2(sin^2 + r^2 cos^2} = \sqrt{r^2} = r$$
$$\int_0^{2\pi} rcos^2(t)(r) = \pi r2$$

## 18 Work

$$\int_C \overrightarrow{f} \cdot d\overrightarrow{r}$$
$$f(x, y) = y\hat{i} - x\hat{j}$$
$$C :\rightarrow x(t) = cos(t), y(t) = sin(t), for 0 < t < 2\pi$$
$$d\overrightarrow{r} = -sin(t)\hat{i} + cos(t)\hat{j}$$
$$\overrightarrow{f}(t) = sin(t)\hat{i} - cos(t)\hat{j}$$

$$-\int_0^{2\pi} dt = -2\pi$$

## 18.1 Example

Suppose we have a vector field $f(x, y) = (xcosy, ycosx)$ and a Curve C that is parameterized by $\alpha(t) = (2t, t)$ for $0 < t\pi$

$$\int_C f \cdot d\alpha = \int_{t_1}^{t_2} f(\alpha(t)) \cdot \alpha'(t)dt$$

$$f(x, y) = (xcosy, ycosx)$$

$$\alpha(t) = (2t, t)$$

$$f(\alpha(t)) = (2tcos(t), tcos(2t))$$

$$\alpha'(t) = (2, 1)$$

$$\rightarrow \int_0^{\pi} (2cos(t), tcos(2t)) \cdot (2, 1)dt$$

## 18.2 Example 2

$$f(x, y) = (e^x, y^{-1})$$

$$\alpha(t) = (3t, t^2)$$

$$f(\alpha(t)) = e^{3t}, \frac{1}{t^2}$$

$$\alpha'(t) = (3, 2t)$$

$$f(\alpha(t)) \cdot \alpha'(t) = (e^{3t}, \frac{1}{t^2}) \cdot (3t, t^2) = 3e^{3t} + \frac{2}{t}$$

$$\int_1^2 3e^{3t} + \frac{2}{t}dt$$

$$u = 3t, du = 3dt$$

$$\int \frac{2}{t} + e^u du$$

$$2log|2| + e^u du$$

$$2\log|2| + e^{3t}\Big|_1^2$$

## 19 Line integrals in conservative vector fields

Vector field:

$$\rho(x, y) = e^x + y$$

A curve parameterizes that.

$$C :\rightarrow (cos(t), sin(t))$$

Which is traversed through:

$$t = \frac{\pi}{2}, t = \frac{3\pi}{2}$$

$$C_{start} \rightarrow \left(\cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right)\right) = (0, 1)$$

$$C_{end} \rightarrow \left(\cos\left(\frac{3\pi}{2}\right), \sin\left(\frac{3\pi}{2}\right)\right) = (0, -1)$$

$$\int_C \nabla\rho \cdot ds = \rho(0, -1) - \rho(0, 1)$$

$$e^0 - 1 - (e^0 + 1) = -2$$

## 20 Closed line integral of conservative field

$$\oint_C (x^2 + y^2)dx + 2xydy$$

$$C :\rightarrow x = cost, y = sint, 0 < t < 2\pi$$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$dr = dx\hat{i} + dy\hat{j}$$

$$\vec{f} = \nabla F \Leftrightarrow \frac{\partial F}{\partial x} = x^2 + y^2 \wedge \frac{\partial F}{\partial y} = 2xy$$

$$F(x, y) = \frac{x^2}{3} + xy^2$$

$$\int_0^{2\pi} \vec{f} \cdot d\vec{r} = F(2\pi) - F(0)$$

$$x(0) = \cos(0) \quad x(2\pi) = \cos(2\pi)$$
$$y(0) = \sin(0) \quad y(2\pi) = \sin(2\pi)$$

$$\int_C \vec{f} \cdot d\vec{r} = F(x(2\pi), y(2\pi)) - F(x(0), y(0))$$

$$\int_C \vec{f} \cdot d\vec{r} = F(1, 0) - F(1, 0)$$

$$\left(\frac{1^3}{3} + 1 \cdot 0^2\right) - \left(\frac{1^3}{3} + 1 \cdot 0^2\right) = 0$$

# 21 Distinguishing between conservative vector fields

$$f(x, y) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}$$

$$P(x, y) = x^3 y^3 \wedge Q(x, y) = \frac{x^4 y}{2}$$

In order for a vector field to be considered conservative the following proposition must hold:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

In our case:

$$P_y = 2x^3 y \wedge Q_x = 2x^3 y$$

So, this is indeed a conservative vector field

# 22 Potential functions

$$f(x, y) = (2(x - y), 2(y - x))$$

find f such that:

$$f = \nabla F$$

$$F_x = 2(x - y) \wedge F_y = 2(y - x)$$

$$F = \int F_x dx \wedge F = \int F_y dy$$

Solve the equation for $H(y)$ and $G(x)$

$$F = \int F_x dx + H(y) = F = \int F_y dy + G(x)$$

We get

$$F(x, y) = x^2 - 2xy + y^2 + c$$