1 Problem 1

We are given the following information:
The interarrival times of the two classes are λ_1 and λ_2.
The service time, if given the full capacity C, is μ_1 and μ_2.
Using this information, we can compute the service requirement:

$$ServiceTime = \frac{ServiceRequirement}{ServerCapacity}$$

$$\mu_i = \frac{ServiceRequirement_i}{C}$$

$$ServiceRequirement_i = \mu_i C$$

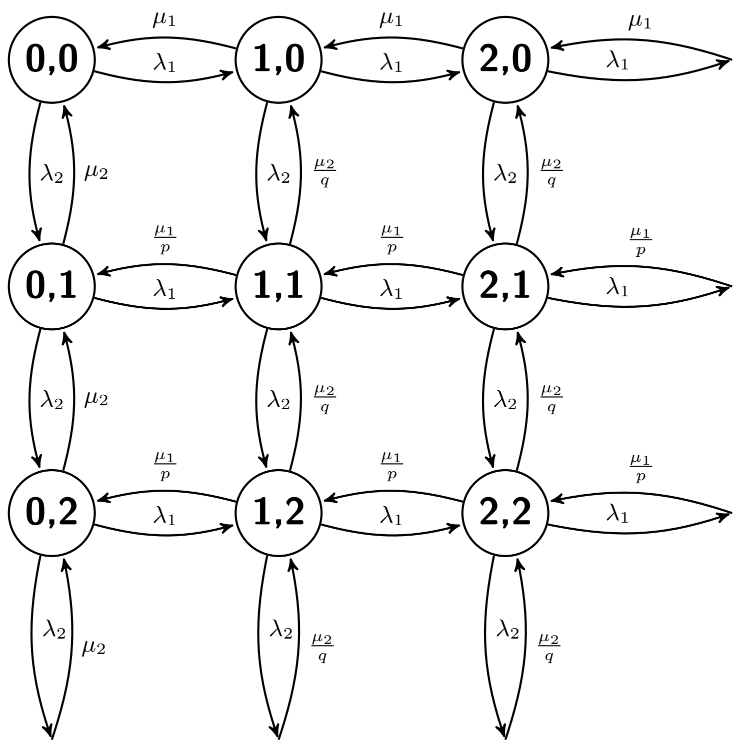
Using the service requirement, we can compute the service time under shared capacity.

$$ServiceTime_1 = \frac{\mu_1 \cancel{C}}{p\cancel{C}}$$

$$ServiceTime_2 = \frac{\mu_2 \cancel{C}}{q\cancel{C}}$$

$$ServiceTime_1 = \frac{\mu_1}{p}$$

$$ServiceTime_2 = \frac{\mu_2}{q}$$



2 Problem 2

2.1 Part a: Determine G

We know that all the probabilities must sum two zero. So:

$$\sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty} P(n_0, n_1) = 1$$

$$\iff \sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty} \frac{1}{G}\frac{p_0^{n_0}}{n_0!}p_1^{n_1} = 1$$

Split up the sum of product into a product of sums:

$$\iff \frac{1}{G}\left(\sum_{n_0=0}^{\infty}\frac{p_0^{n_0}}{n_0!}\right)\left(\sum_{n_1=0}^{\infty} p_1^{n_1}\right) = 1$$

Applying $\sum_k \frac{z^k}{k!} = e^z$ to the first sum
Applying $\sum_{n=k} x_n = \frac{x^k}{1-x}$ to the second sum
Multiplying both sides by G

$$G = \frac{e^{p_0}}{1-p_1} \text{ where } 0 \leq p_1 < 1$$

2.2 Part b: Determine $\mathbf{E}[K = N_0 + N_1]$

From the previous problem we know that:

$$\frac{1}{G} = \frac{1-p_1}{e^{p_0}}$$

Before solving the given problem, we compute the marginal probabilities:

$$P(N_0 = n_0) = \sum_{n_1=0}^{\infty} P(n_0, n_1)$$

$$P(N_0 = n_0) = \frac{1-p_1}{e^{p_0}} \sum_{n_1=0}^{\infty} \frac{p_0^{n_0}}{n_0!} p_1^{n_1}$$

Bringing the constants out

$$P(N_0 = n_0) = \frac{1-p_1}{e^{p_0}} \frac{p_0^{n_0}}{n_0!} \sum_{n_1=0}^{\infty} p_1^{n_1}$$

Applying $\sum_{n=k} x_n = \frac{x^k}{1-x}$ to the sum

$$P(N_0 = n_0) = \frac{1-p_1}{e^{p_0}} \frac{p_0^{n_0}}{n_0!} \frac{1}{1-p_1}$$

$$P(N_0 = n_0) = \frac{1-p_1}{e^{p_0}} \frac{p_0^{n_0}}{n_0!} \frac{1}{1-p_1}$$

$$P(N_0 = n_0) = \frac{p_0^{n_0}}{e^{p_0} n_0!} \tag{1}$$

Similarly we derive the marginal probability for $P(N_1 = n_1)$:

$$P(N_1 = n_1) = \sum_{n_0=0}^{\infty} P(n_0, n_1)$$

$$P(N_1 = n_1) = \frac{1-p_1}{e^{p_0}} \sum_{n_0=0}^{\infty} \frac{p_0^{n_0}}{n_0!} p_1^{n_1}$$

Bringing out the constant

$$P(N_1 = n_1) = \frac{1-p_1}{e^{p_0}} p_1^{n_1} \sum_{n_0=0}^{\infty} \frac{p_0^{n_0}}{n_0!}$$

Applying $\sum_k \frac{z^k}{k!} = e^z$ to the sum

$$P(N_1 = n_1) = \frac{1-p_1}{e^{p_0}} p_1^{n_1} e_{p_0}$$

$$P(N_1 = n_1) = (1-p_1) p_1^{n_1} \tag{2}$$

With (1) and (2), we now have enough information to solve for $E[K]$. From lectures we know that the expecation of a sum of random variables is the same as the sum of their expecations (without having to make any assumptions):

$$E[K] = E[N_0 + N_1] = E[N_0] + E[N_1]$$

Applying the definition of expecation to both expectations:

$$E[K] = \sum_{n_0=0}^{\infty} n_0 P(N_0 = n_0) + \sum_{n_1=0}^{\infty} n_1 P(N_1 = n_1)$$

Applying formulas from (1) and (2):

$$E[K] = \sum_{n_0=0}^{\infty} n_0 \frac{p_0^{n_0}}{e^{p_0} n_0!} + \sum_{n_1=0}^{\infty} n_1 (1-p_1) p_1^{n_1}$$

Bringing out the constants from the sums:

$$E[K] = \frac{1}{e^{p_0}} \sum_{n_0=0}^{\infty} n_0 \frac{p_0^{n_0}}{n_0!} + (1-p_1) \sum_{n_1=0}^{\infty} n_1 p_1^{n_1}$$

Applying $\sum_{k=0}^{\infty} k \frac{z^k}{k!} = ze^z$ to the first sum
Applying $\sum_{n=k}^{\infty} nx^n = \frac{kx^k}{1-x} + \frac{x^{k+1}}{(1-x)^2}$ to the second sum

$$E[K] = \frac{1}{e^{p_0}} p_0 e^{p_0} + (1-p_1) \frac{p_1}{(1-p_1)^2}$$

A little simplification leaves us with:

$$E[K] = p_0 + \frac{p_1}{(1-p_1)}$$

3 Problem 3

We are given:

$$MeanThinkTime = Z = \frac{1}{\mu_0} = 5sec$$

$$MeanServerServiceTime = \frac{1}{\mu_1} = 0.05sec$$

$$MeanDatabaseServiceTime = \frac{1}{\mu_2} = 0.1sec$$

Note that this is the same setup as in Assignment 1, Problem 1. As a result, we re-use the results from that question:

$$V_1 = \frac{\lambda_1}{\lambda_0} = 5$$

$$V_2 = \frac{\lambda_2}{\lambda_0} = 4$$

$$D_1 = V_1 S_1 = 5 * .05 = 0.25$$

$$D_b = D_2 = V_2 S_2 = 4 * .1 = 0.4$$

$$D = D_1 + D_2 = 0.25 + 0.4 = 0.65$$

$$N^* = \frac{D+Z}{D_b} = 14.125 \approx 14$$

$$R_{lower}(N) = \begin{cases} 0.65 \text{ if } N \leq 14 \\ .4N - 5 \text{ otherwise} \end{cases}$$

In assignment 1, we did not have to compute a bound on the throughput, so we do it here:

$$Y_{upper}(N) = \begin{cases} \frac{N}{D+Z} \text{ if } N \leq 14 \\ \frac{1}{D_b} \text{ otherwise} \end{cases} = \begin{cases} \frac{N}{5.65} \text{ if } N \leq 14 \\ 2.5 \text{ otherwise} \end{cases}$$

Last, we use the formulas from Mean Value Analysis:

$$Q_i(0) = 0 \; \forall i \text{ (base case)}$$

$$R_i(N) = \frac{1}{\mu_i}(1 + Q_i(N-1)),\ i = 1, 2, ..., M$$

$$R_1(N) = 0.05(1 + Q_i(N-1))$$

$$R_2(N) = 0.1(1 + Q_i(N-1))$$

$$R(N) = V_1R_1(N) + V_2R_2(N) = 5R_1(N) + 4R_2(N)$$

$$Y(N) = \frac{N}{Z + R(N)} = \frac{N}{5 + R(N)}$$

$$Q_i = Y(N)V_iR_i(N)$$

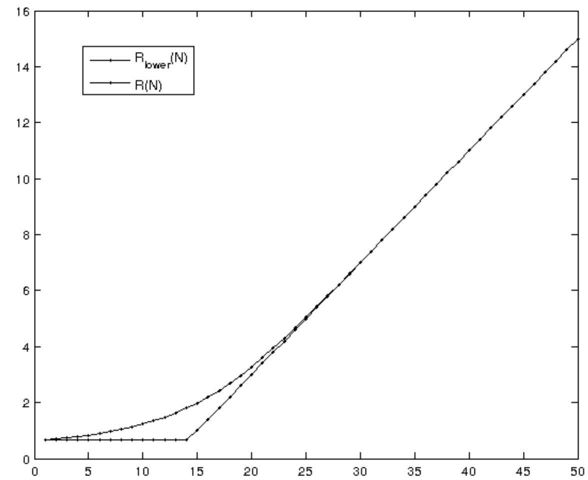
$$Q_1 = 5Y(N)R_1(N)$$

$$Q_2 = 4Y(N)R_2(N)$$

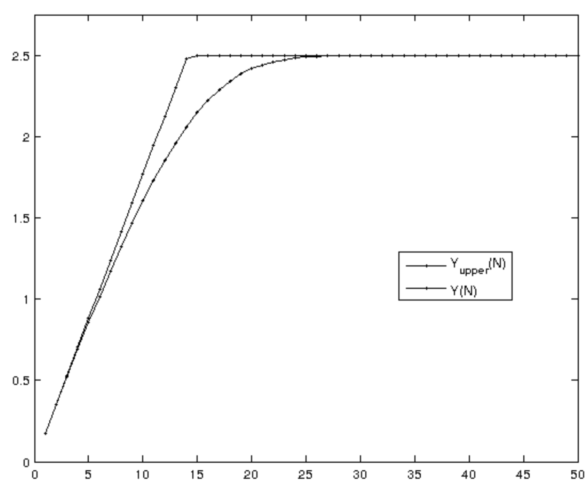
Now we have collected all the information we need for computing everything. I wrote the following program in Matlab to plot everything:

```
function problem3
close all; clear all;
  % N = 1 .. 50 (no zero)
  R_lower = zeros(1,50);
  Y_upper = zeros(1,50);
  R_1 = zeros(1,50);
  R_2 = zeros(1,50);
  R = zeros(1,50);
  Y = zeros(1,50);
  Q_1 = zeros(1,50);
  Q_2 = zeros(1,50);

  for N = 1:50
    % computing upper and lower bounds
    if N <= 14
      R_lower(N) = 0.65;
      Y_upper(N) = N/5.65;
    else
      R_lower(N) = .4*N - 5;
      Y_upper(N) = 2.5;
    end

    R_1(N) = 0.05*(1+getQN(Q_1,N-1));
    R_2(N) = 0.1*(1+getQN(Q_2,N-1));
    R(N) = 5*R_1(N) + 4*R_2(N);
    Y(N) = N/(5+R(N));
    Q_1(N) = 5 * Y(N) * R_1(N);
    Q_2(N) = 4 * Y(N) * R_2(N);
  end
R_lower
R
Y_upper
Y
  plot(1:50, R_lower(1,:), 'k-o', 'MarkerSize', 3)
hold on
plot(1:50, R(1,:), 'k-+' , 'MarkerSize', 3)
legend('R_{lower}(N)', 'R(N)');

figure
  plot(1:50, Y_upper(1,:), 'k-o', 'MarkerSize', 3)
hold on
plot(1:50, Y(1,:), 'k-+' , 'MarkerSize', 3)
legend('Y_{upper}(N)', 'Y(N)');
end

function [QN] = getQN(Q,N)
  if N == 0 QN = 0;
  else QN = Q(N);
  end
end
```

3.1 Part a: Response Time



3.2 Part b: Throughput



4 Problem 4

For large n, we apply the Central Limit Theorm

$$\frac{\sum_{i=0}^{n} X_i - \sum_{i=0}^{n} \mu_i}{\sqrt{\sum_{i=0}^{n} \sigma_i^2}} \sim N(0,1)$$

All μ_i's and σ_i's are identical:

$$\iff \frac{\sum_{i=0}^{n} X_i - n\mu}{\sqrt{n}\sigma} \sim N(0,1)$$

There the expecation is zero. We can use this fact to determine the expecation of Y.

$$E[\frac{\sum_{i=0}^{n} X_i - n\mu}{\sqrt{n}\sigma}] = 0$$

$$\iff E[\sum_{i=0}^{n} X_i - n\mu] = 0$$

The expecation of a sum, is the sum of the expecations. Additionally, the expecation of a constant is just the constant.

$$\iff E[\sum_{i=0}^{n} X_i] = n\mu$$

We can put the constant into the expecation, and we get E[Y].

$$\iff E[\frac{\sum_{i=0}^{n} X_i}{n}] = E[Y] = \mu$$

Now for the variance:

$$var(\frac{\sum_{i=0}^{n} X_i - n\mu}{\sqrt{n}\sigma}) = 1$$

The variance of a random variable plus a constant is just the variance of the random variable ($var(X + a) = var(X)$).

$$\iff var(\frac{\sum_{i=0}^{n} X_i}{\sqrt{n}\sigma}) = 1$$

Multiplying both sides by $\frac{1}{n}$ and bringing out the constant from the denominator of the variance ($var(aX) = a^2 var(X)$).

$$\iff \frac{1}{n\sigma^2} var(\frac{\sum_{i=0}^{n} X_i}{\sqrt{n}}) = \frac{1}{n}$$

Bring the σ over and placing the constant n, inside the variance leaves us with var(Y).

$$\iff var(\frac{\sum_{i=0}^{n} X_i}{n}) = var(Y) = \frac{\sigma^2}{n}$$

Now we can apply Chebychev's Equality. Suppose X is randomly distributed with finite mean μ and variance σ^2. then for all a > 0

$$P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$$

In our case, the random variable is Y, and the mean stays the same. Secondly instead of a, we have ϵ. Lastly, the variance is $\frac{\sigma^2}{n}$ instead. This leaves us with expected result:

$$P(|Y - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$$

5 Problem 5

Let D be a discrete random variable with the following distribution: $P(D = d) = 1$ and $P(D = x) = 0$ for all $x \neq d$.

5.1 Part a: Determine the MGF of D

$$M_D(t) = E[e^{tD}]$$

Subbing in the definition of expecation

$$M_D(t) = \sum_{i=0}^{\infty} e^{ti} P(D = i)$$

However, i=d is the only value in the sum that has a non-zero probability.

$$M_D(t) = e^{td} P(D = d) = e^{td}(1)$$

$$M_D(t) = e^{td}$$

Taking the r^{th} derivative with respect to t of the MFG, gives us that $E[X^r] = d^r e^{d0} = d^r$ for all $r \geq 1$

5.2 Part b

$$M_Y(t) = E[e^{tY}] = E[\prod_{i=1}^{n} e^{tX_i}]$$

Since they are all independantly distributed we can bring the product out of the expecation:

$$= \prod_{i=1}^{n} E[e^{tX_i}] = \prod_{i=1}^{n} M_{X_i}(t)$$

From class, we already know the MGF of the exponential distribution with parameter λ is $M_X = \frac{\lambda}{\lambda - t}$ Replacing λ with $\frac{n}{d}$ gives us the MGF of each of the random variables:

$$= \prod_{i=1}^{n} \frac{\frac{n}{d}}{\frac{n}{d} - t}$$

$$M_Y(t) = \frac{n}{d}^n (\frac{n}{d} - t)^{-n}$$

Taking the first derivative:

$$\frac{dM_Y(t)}{dt} = \frac{n}{d}^n (-n)(\frac{n}{d} - t)^{-n-1}(-1)$$

Now the second derivative.

$$\frac{d^2 M_Y(t)}{dt^2} = \frac{n}{d}^n (n)(n+1)(\frac{n}{d} - t)^{-n-2}$$

We can see a pattern for the r^{th} derivative emerging:

$$\frac{d^r M_Y(t)}{dt^r} = \frac{n}{d}^n (n)(n+1)...(n+r-1)(\frac{n}{d} - t)^{-n-r}$$

If we assume n is large, we can replace the $(n)(n+1)...(n+r-1)$ with n^r:

$$\frac{d^r M_Y(t)}{dt^r} = \frac{n}{d}^n n^r (\frac{n}{d} - t)^{-n-r}$$

Setting t = 0 and a little cleaning up

$$\frac{d^r M_Y(0)}{dt^r} = \frac{n^n}{d^n} n^r (\frac{n}{d} - 0)^{-(n+r)}$$

$$\frac{d^r M_Y(0)}{dt^r} = \frac{n^n}{d^n} n^r \frac{d^{n+r}}{n^{n+r}}$$

The n^{n+r} cancels, and d^n cancels leaving

$$\frac{d^r M_Y(0)}{dt^r} = d^r$$

This matches the MGF of D. As a result, if n is large enough, Y can approximate D.

6 Problem 6

6.1 Part a: MGF of Y

$$M_{Y|X=x}(t) = E[e^{tY}|X = x] = \sum_{n=0}^{\infty} e^{tn} P(Y = n|X = x)$$

$$= \sum_{n=0}^{\infty} e^{tn} \frac{(\lambda x)^n}{n!} e^{-\lambda x}$$

Moving out everything without an n from the sum and moving the e^t under the same n as λ:

$$= e^{-\lambda x} \sum_{n=0}^{\infty} \frac{(\lambda e^t x)^n}{n!}$$

Applying the taylor sum for e^z:

$$= e^{-\lambda x} e^{(\lambda e^t x)}$$

$$M_{Y|X=x}(t) = e^{\lambda x(e^t - 1)}$$

We can use total expecation do obtain the MGF:

$$M_Y(t) = \int_{-\infty}^{\infty} E[e^{tY}|X = x]P(X = x)dx$$

$$= \int_{-\infty}^{\infty} e^{\lambda x(e^t-1)} P(X = x)dx$$

However, we use the conditioned form in part b.

6.2 Part b: Variance and Expectation of Y

6.2.1 Expectation

Taking the first derivative of the conditioned MFG:
Let $y = \lambda x(e^t - 1)$

$$\frac{dM_{Y|X=x}(t)}{dt} = e^y \frac{dy}{dt}$$

$$\frac{dM_{Y|X=x}(t)}{dt} = e^y \lambda x e^t$$

Setting t to zero gives us the conditioned expectation. Also note that when t=0, y also is 0.

$$E[Y|X = x] = \frac{dM_{Y|X=x}(0)}{dt} = e^0 \lambda x e^0 = \lambda x$$

Now applying total expectation:

$$E[Y] = \int_{-\infty}^{\infty} E[Y|X = x]P(X = x)$$

$$= \int_{-\infty}^{\infty} \lambda x P(X = x) = \lambda \int_{-\infty}^{\infty} x P(X = x)$$

Applying the definition of the expectation of X in reverse:

$$E[Y] = \lambda E[X]$$

6.2.2 Variance

Earlier we had the following first derivative:

$$\frac{dM_{Y|X=x}(t)}{dt} = \lambda x e^{\lambda x(e^t-1)+t}$$

Let $y = \lambda x(e^t - 1) + t$ then $\frac{dy}{dt} = xe^t + 1$

$$\frac{d^2 M_{Y|X=x}(t)}{dt^2} = e^y \lambda x \frac{dy}{dt}$$

$$\frac{d^2 M_{Y|X=x}(t)}{dt^2} = e^y \lambda x(xe^t + 1)$$

Setting t=0 also gives y=0. We set t=0 to obtain the conditioned expecatation $E[Y^2|X = x]$:

$$E[Y^2|X = x] = \frac{d^2 M_{Y|X=x}(0)}{dt^2} = e^0 \lambda x(xe^0 + 1) = \lambda x(x + 1)$$

$$E[Y^2|X = x] = \lambda x^2 + \lambda x$$

Applying total expectation:

$$E[Y^2] = \int_{-\infty}^{\infty} E[Y^2|X = x]P(X = x)$$

$$= \int_{-\infty}^{\infty} \lambda x^2 P(X = x) + \int_{-\infty}^{\infty} \lambda x P(X = x)$$

Bringing out the lambda:

$$= \lambda(\int_{-\infty}^{\infty} x^2 P(X = x) + \int_{-\infty}^{\infty} x P(X = x))$$

Applying the definition of expectation in reverse:

$$E[Y^2] = \lambda(E[X^2] + E[X])$$

Now solving for the var(Y):

$$var(Y) = E[Y^2] - (E[Y])^2 = \lambda(E[X^2] + E[X]) - (\lambda E[X])^2$$

$$= \lambda E[X^2] + \lambda E[X] - \lambda^2 E[X]^2$$

Applying $var(X) + E[X]^2 = E[X^2]$

$$= \lambda(var(X) + E[X]^2) + \lambda E[X] - \lambda^2 E[X]^2$$

$$= \lambda \left(var(X) + E[X] - \lambda E[X]^2\right)$$

7 Problem 7

We did not have to complete this problem because the material was not covered in class yet.

8 Problem 8

8.1 Part a

$$\bar{X} = \frac{1}{n}\sum_{n=0} X_i$$

$$= \frac{1}{5}(4.25 + 5.12 + 4.55 + 4.33 + 4.98) = 4.646$$

$$\begin{aligned} s^2 &= \frac{1}{n-1}\sum_{n=0}(X_i - \bar{X})^2 \\ &= \frac{1}{4}((4.25 - 4.646)^2 + (5.12 - 4.646)^2 \\ &+ (4.55 - 4.646)^2 + (4.33 - 4.646)^2 + (4.98 - 4.646)^2) \\ &\approx 0.60212 \end{aligned}$$

$$s \approx \sqrt{(}0.60212) \approx 0.77596$$

Since we have less than the recommended 30 replications, we need to apply the t-distribution here. As a result the 95% confidence interval is

$$\bar{X} \pm t_{0.975,4}\frac{s}{\sqrt{n}}$$

Consulting the t-table gives $t_{0.975,4} = 2.776$. Plugging in all the values gives:

$$4.646 \pm 2.776\frac{0.77596}{\sqrt{5}}$$

$$4.646 \pm 0.96332$$

Which gives the interval $(3.6827, 5.6093)$

8.2 Part b

$$\begin{aligned} \bar{X} &= \frac{1}{n}\sum_{n=0} X_i \\ &= \frac{1}{10}(5 * 4.646 + 5.23 \\ &+ 4.77 + 4.51 + 5.39 + 4.45) = 4.758 \end{aligned}$$

I ommit the calculations for the variance, as they are tedious.

$$s^2 = \frac{1}{n-1}\sum_{n=0}(X_i - \bar{X})^2 = \frac{1}{9}(1.44356) \approx 0.160396$$

$$s \approx \sqrt{(}0.160396) \approx 0.400494$$

Since we have less than the recommended 30 replications, we need to apply the t-distribution here. As a result the 95% confidence interval is

$$\bar{X} \pm t_{0.975,9}\frac{s}{\sqrt{n}}$$

Consulting the t-table gives $t_{0.975,9} = 2.262$. Plugging in all the values gives:

$$4.758 \pm 0.28648$$

Which gives the interval $(4.4715, 5.0444)$

8.3 Part c

Ten replicates is not enough so that the interval is not wider than $\bar{X} \pm 0.05\bar{X}$ because
$0.05 * 4.758 = 0.2379 < 0.28648$. As a result, we look for smallest n that is no wider than $\bar{X} \pm 0.05\bar{X}$.
Let $d = 0.05\bar{X} = 0.2379$
We assume that the mean, and variance to not change significantly when increasing the replicates (allowing us to re-use our mean, variance, and t-table values):

$$m = (t_{0.975,n-1}\frac{s}{d})^2$$

$$m \approx (2.262\frac{0.40049}{0.2379})^2 \approx 14.500$$

As a result, we need approximately 15 replications to acheived the desired width, which is 5 more replications.