1. Consider the income process:

$$y_{it} = \alpha_i + \zeta_{it} + \varepsilon_{it}$$

where $\zeta_{it} = \rho\,\zeta_{it-1} + \eta_{it-1}$ and

$\eta_{it}, \varepsilon_{it}$ are iid w/

$$\mathbb{V}[\alpha_i] = \sigma^2_\alpha,\quad \mathbb{V}[\eta_{it}] = \sigma^2_\eta,\quad \mathbb{V}[\varepsilon_{it}] = \sigma^2_{\varepsilon,t}$$

Use population moments to argue that the parameters $\sigma^2_\alpha, \rho, \sigma^2_\eta, \{\sigma^2_{\varepsilon,t}\}_{t=1}^{T}$ can be identified w/ panel data.

Solution:

$$\mathbb{V}[y_{it}] = \sigma^2_\alpha + \mathbb{V}(\zeta_{it}) + \sigma^2_{\varepsilon,t}$$

$$\mathbb{V}[y_{it+1}] = \sigma^2_\alpha + \rho^2\,\mathbb{V}(\zeta_{it}) + \sigma^2_\eta + \sigma^2_{\varepsilon,t}$$

$$\mathbb{C}(y_{it}, y_{it+1}) = \sigma^2_\alpha + \rho\,\mathbb{V}(\zeta_{it})$$

$$\mathbb{C}(y_{it}, y_{it+2}) = \sigma^2_\alpha + \rho^2\,\mathbb{V}(\zeta_{it})$$

$$\mathbb{C}(y_{it}, y_{it+3}) = \sigma^2_\alpha + \rho^3\,\mathbb{V}(\zeta_{it}).$$

now:

$$\rho = \frac{\mathbb{C}(y_{it}, y_{it+3}) - \mathbb{C}(y_{it}, y_{it+2})}{\mathbb{C}(y_{it}, y_{it+2}) - \mathbb{C}(y_{it}, y_{it+1})} = \frac{(\rho^3 - \rho^2)\,\mathbb{V}(\zeta_{it})}{(\rho^2 - \rho)\,\mathbb{V}(\zeta_{it})}$$

$$\mathbb{V}(\zeta_{it}) = \frac{\mathbb{C}(y_{it}, y_{it+2}) - \mathbb{C}(y_{it}, y_{it+1})}{\rho^2 - \rho}.$$

$$\sigma_\alpha^2 = \mathbb{C}(y_{it}, y_{it+1}) - \rho \mathbb{V}(\zeta_{it}).$$

$$\sigma_{\varepsilon,t}^2 = \mathbb{V}[y_{it}] - \sigma_\alpha^2 - \mathbb{V}(\zeta_{it}) \quad \text{for each } t.$$

$$\sigma_\eta^2 = \mathbb{V}[y_{it+1}] - \sigma_\alpha^2 - \rho^2 \mathbb{V}(\zeta_{it}) - \sigma_{\varepsilon,t}^2.$$

Done!

---

2. Now suppose you are using the above identification arguments to get estimates of the income process from Question 1. As part of this you would like to estimate $\mathbb{V}[y_{it}]$ for each age t in the data (variable name AGE). Write the missing line of code below.

```
var_ests <- data %>%
        # missing line!
        summarise(var_y_t = var(log_income))
```

Solution: group_by(AGE).

---

2. Consider the income process:

$$y_{it} = \alpha_i + \zeta_{it} + \varepsilon_{it} + \theta \varepsilon_{it-1}$$

where $\zeta_{it} = \rho \zeta_{it-1} + \eta_{it}$, $(\eta_{it}, \varepsilon_{it})$ are iid

with $\mathbb{V}[\alpha_i] = \sigma^2_\alpha$, $\mathbb{V}(\varepsilon_{it}) = \sigma^2_\varepsilon$, $\mathbb{V}(\eta_{it}) = \sigma^2_\eta$.

Use population moments to identify the parameters of the income process.

Solution:

$$\mathbb{V}(y_{it}) = \sigma^2_\alpha + \mathbb{V}(\zeta_{it}) + (1+\theta^2)\sigma^2_\varepsilon .$$

$$C(y_{it}, y_{it+1}) = C_1 = \sigma^2_\alpha + \rho \mathbb{V}(\zeta_{it}) + \theta \sigma^2_\varepsilon$$

$$C(y_{it}, y_{it+2}) = C_2 = \sigma^2_\alpha + \rho^2 \mathbb{V}(\zeta_{it})$$

$$C(y_{it}, y_{it+s}) = C_s = \sigma^2_\alpha + \rho^s \mathbb{V}(\zeta_{it}) \quad \forall\ s > 1.$$

$$\rho = \frac{C_4 - C_3}{C_3 - C_2} = \frac{\rho^4 - \rho^3}{\rho^3 - \rho^2} \checkmark.$$

$$\mathbb{V}(\zeta_{it}) = \frac{C_3 - C_2}{\rho^3 - \rho^2}, \quad \sigma^2_\alpha = C_2 - \rho^2 \mathbb{V}(\zeta_{it}),$$

$$\frac{(1+\theta^2)}{\theta} = \frac{\mathbb{V}(y_{it}) - \sigma^2_\alpha - \mathbb{V}(\zeta_{it})}{C_1 - \sigma^2_\alpha - \rho \mathbb{V}(\zeta_{it})}$$

→ θ is solution to quadratic.

→ $$\sigma^2_\varepsilon = \frac{\mathbb{V}(y_{it}) - \sigma^2_\alpha - \mathbb{V}(\zeta_{it})}{(1+\theta^2)}$$

Done!

3. Consider the following one period model:

$$C_i = y_i = \exp(\alpha_i + \varepsilon_i).$$

where $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$, $\alpha_i \sim N(\mu, \sigma_\alpha^2)$. Suppose that α_i is known to each individual and ε_i is income risk. Assume log utility s.t. $V(\alpha_i) = \mathbb{E}[\log(C_i) | \alpha_i]$.

(a) Show that $V(\alpha_i) = \alpha_i$.

(b) Suppose a planner solves:

$$\max_{\{C_i\}} \int \log(C_i)\,di \quad \text{s.t.} \quad \int C_i\,di = \int \exp(\alpha_i + \varepsilon_i)\,di.$$

Show that the solution is

$$C_i^* = C^* = \exp\left(\mu + \tfrac{1}{2}(\sigma_\varepsilon^2 + \sigma_\alpha^2)\right)$$

Hint: if $z \sim N(\mu, \sigma_\varepsilon^2)$, $\mathbb{E}[\exp(z)] = \exp(\mu + 1/2\sigma^2)$.

(c) Show that the certainty equivalent $\bar{C}(\alpha_i)$ that delivers $V(\alpha_i)$ is $\bar{C}(\alpha_i) = \exp(\alpha_i)$.

(d) Show that $\bar{C} = \int \bar{C}(\alpha_i) = \exp(\mu + 1/2\,\sigma_\alpha^2)$

(e) Show that the planner's objective is μ in the baseline and $\mu + 1/2(\sigma_\varepsilon^2 + \sigma_\alpha^2)$ at the optimum.

(f) Decompose $w^* - w$ into two components:

(1) $w^* - \log(\bar{C})$

(2) $\log(\bar{C}) - w$.

What is (1)? What is (2)? Which term represents the gains from insurance?

(g) Interpret this result.

Solution

(a) $V(\alpha_i) = \mathbb{E}[\log(c_i) | \alpha_i] = \mathbb{E}[\alpha_i + \varepsilon_i | \alpha_i] = \alpha_i$.

(b) $\int \exp(\alpha_i + \varepsilon_i) di = \exp(\mu + 1/2(\sigma_\varepsilon^2 + \sigma_\alpha^2))$

since $\alpha_i + \varepsilon_i \sim N(\mu, \sigma_\alpha^2 + \sigma_\varepsilon^2)$.

Planner's FOC: $1/c_i^* = \lambda \quad \forall i$

where λ is multiplier on resource constraint. Hence $c_i = c^* \quad \forall i$.

Plugging into resource constraint gives

$$c^* = \exp(\mu + 1/2(\sigma_\alpha^2 + \sigma_\varepsilon^2)).$$

(c) $\bar{c}(\alpha_i)$ solves

$$\log(\bar{c}(\alpha_i)) = V(\alpha_i) = \alpha_i$$

$$\Leftrightarrow \quad \bar{c}(\alpha_i) = \exp(\alpha_i).$$

(d) Since $\alpha_i \sim N(\mu, \sigma_\alpha^2)$,

$$\bar{c} = \int \bar{c}(\alpha_i)\, di = \exp(\mu + 1/2\, \sigma_\alpha^2).$$

(e) <u>Baseline</u>: $W = \int \log(c_i) = \int \log(\exp(\alpha_i + \varepsilon_i))\, di$

$$= \mathbb{E}[\alpha_i + \varepsilon_i] = \mu.$$

$$W^* = \log(c^*) = \mu + 1/2(\sigma_\alpha^2 + \sigma_\varepsilon^2).$$

(f) (1) $W^* - \log(\bar{c}) = 1/2\, \sigma_\varepsilon^2 \leftarrow$ Insurance

(2) $\log(\bar{c}) - W = 1/2\, \sigma_\alpha^2 \leftarrow$ Redistribution.

(g) Makes sense! σ_ε^2 is variance component in y coming from risk, σ_α^2 is the component coming from ex-ante inequality.

---

4. Consider the 2 period Becker-Tomes model:

$$\max_{c_1, c_2, X, A} \; u(c_1) + \beta u(c_2)$$

$$\text{s.t.} \quad c_1 + \gamma X + A/1+r \leq \Omega + T/1+r$$

$$c_2 = A + wz X^\delta - T, \quad A \geq 0.$$

where T is a lump-sum tax paid by generation 2 to generation 1.

(a) Show that if a household is

not constrained, the value of T does not affect any allocations of C_1, C_2, X at the optimum. (Hint: try combining budget constraints).

(b) Show that for constrained households:

$$u'(C_1) = \delta w z X^{\delta-1} u'(C_2) \quad \text{and} \quad (1+r) \leq \frac{\delta w z X^{\delta-1}}{\tau}.$$

(c) Consider a marginal increase in T. What happens to the welfare of constrained & unconstrained households?

(d) Interpret.

Solution:

(a) If $A > 0$ at the optimum,

can combine to get:

$$C_1 + C_2/_{1+r} \leq \Omega + wzX^{\delta}/_{1+r} \qquad (*)$$

s.t. household solves:

$$\max_{C_1, C_2} \; u(C_1) + \beta u(C_2) \quad \text{s.t. } (*).$$

Hence T does not feature.

(b) When $A=0$, solve:

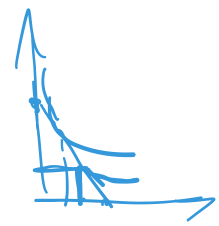
$$\max \; u(\Omega + T/_{1+r} - \psi X) + \beta u(wzX^{\delta} - T)$$

get
$$u'(\Omega + T/_{1+r} - \psi X) = \beta \delta w z X^{\delta-1} u'(wzX^{\delta} - T)$$
$$\iff u'(c_1)\psi = \beta \delta w z^{\delta-1} u'(wzX^{\delta} - T).$$

Note that the marginal return to assets is

$$-\frac{1}{1+r} u'(c_1) + \beta u'(c_2) \leq 0 \iff (1+r) \leq \frac{\delta w z X^{\delta-1}}{\psi}.$$

(if >0, then household should optimally increase A above 0).



(c) if $T \nearrow$ marginally change in welfare is

$$u'(c_1)\, 1/_{1+r} - \beta u'(c_2) \geq 0$$

for constrained households, where inequality follows from (b).

Since allocations for constrained households are unaffected by $T \Rightarrow$ no welfare effect.

Note: not clear if welfare of children $\nearrow$ for this intervention.

5. For same 2-period model,

suppose budget constraints are now:

$$c_1 + A/_{1+r} + \psi X \leq \lambda_1 \Omega^{1-\tau}$$
$$c_2 = A + \lambda_2 (wzX^{\delta})^{1-\tau}, \quad A \geq 0$$

where λ_1 and λ_2 are chosen to balance budget in each period.

(a) Write the FOC for unconstrained households.

(b) Recall that the planner sets

$$(1+r) = \frac{w z_i X^{\delta-1}}{\psi} \text{ for each household } i.$$

Can the equilibrium w/ taxation be efficient?

(C) Explain, in words, the efficiency costs of progressive taxation in this model. (Hint: compare the household's FOC to the planner's).

(d) Assuming log utility solve for X when households are constrained. Comment on solution relative to Baseline.

Solution:

(a) For unconstrained:

$$(1+r) = \lambda_2 \frac{(1-\tau)\delta (wz)^{1-\tau} X^{\delta(1-\tau)-1}}{\psi}$$

(b) Comparing

to the condition

$$(1+r) = \frac{\delta w z X^{\delta-1}}{\psi},$$

taxation distorts the optimal allocation of financial & human capital $\Rightarrow$ <u>not</u> <u>efficient</u>.

(c) Redistribution reduces the incentives for parents to invest.

$\Rightarrow$ An equity-efficiency trade-off

(d).

$$\max_{X} \log\left(\lambda_1 \Omega^{1-\tau} - \psi X\right) + \beta \log\left(\lambda_2 (w z X^{\delta})^{1-\tau}\right)$$

$$\rightarrow \quad \frac{\psi}{\lambda_1 \Omega^{1-\tau} - \psi X} = \beta \frac{(1-\tau)\delta}{X}$$

$$\Leftrightarrow \quad X = \frac{1}{\psi} \times \left[\frac{\beta(1-\tau)\delta}{1+\beta(1-\tau)\delta}\right] \lambda_1 \Omega^{1-\tau}.$$

$\rightarrow$ compared to

$$X = \frac{1}{\psi} \frac{\beta\delta}{1+\beta\delta} \Omega.$$

So taxation also distorts investment for unconstrained. This <u>could</u> offset distortion from incomplete markets, but depends on parameters $(\lambda_1, \lambda_2, \tau)$.