The Generalized Roy Model

Lecture Notes

Selection in General

The Selection Equation

- Start with the selection equation without functional form restrictions
- Let $P(X, Z) = P[D = 1|X, Z]$ be the propensity score

Normalization

- For any distribution of V, since F_V is monotonically increasing (under support conditions), we can write:

$$D = \mathbf{1}\{\mu_d(X, Z) \geq V\} = \mathbf{1}\{F_V(\mu_d(X, Z)) \geq F_V(V)\}$$

- Since $F_V(V)$ is a uniform random variable in $[0, 1]$, we can write without loss of generality:

$$D = \mathbf{1}\{P(X, Z) - V \geq 0\}, \qquad V \sim U[0, 1]$$

- This is a normalization: we're expressing selection in terms of the propensity score

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Conditional Expectations and Selection

- Consider the conditional expectations of outcomes:

$$\mathbb{E}[Y|X, Z, D = 1] = \mu_1(X) + \underbrace{\mathbb{E}[U_1|V \leq P(X, Z)]}_{h_1(P(X,Z))}$$

$$\mathbb{E}[Y|X, Z, D = 0] = \mu_0(X) + \underbrace{\mathbb{E}[U_0|V > P(X, Z)]}_{h_0(P(X,Z))}$$

Two Key Observations

1. **The Classic Selection Problem**
   - If unobservable V determining D is related to unobservables (U_0, U_1) determining potential outcomes
   - Then difference in conditional means is contaminated by the *selection effect*
   - Cannot simply compare treated vs untreated means
2. **Dimension Reduction**

- Selection model implies dimension reduction in $\mathbb{E}[U_D|X, Z]$
- The combined propensity $P(X, Z)$ encodes all relevant information to control for selection
- This is a useful property
- But the underlying index model is *not* without loss of generality

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Identification by Functional Form

Setup: Joint Normality

- Assume (V, U_0, U_1) are jointly normally distributed:

$$\begin{bmatrix} V \\ U_0 \\ U_1 \end{bmatrix} = \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} 1 & \sigma_{V0} & \sigma_{V1} \\ \sigma_{V0} & \sigma_0^2 & \sigma_{01} \\ \sigma_{V1} & \sigma_{01} & \sigma_1^2 \end{bmatrix}\right)$$

Normalizations

- Location normalized: mean set to zero
- Scale of V normalized: unit variance

Additional Assumptions

- Each μ_D is linear in X: $\mu_D(X) = X\beta_D$
- Data: single cross-section (Y_D, D, X)
- Identification is about population values, so take joint distribution $\mathbb{P}_{Y_D,D,X}$ as given

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Step 1: Identifying the Selection Equation

- Distribution of D given X is a **probit model**:

$$P[D = 1|X] = \Phi(\mu_d(X))$$

  where Φ is the standard normal CDF

- Thus $\mu_d(X)$ is identified:

$$\mu_d(X) = \Phi^{-1}(P[D = 1|X])$$

  for any X

- If we impose $\mu_d(X) = X\gamma$:
  - Identification of each γ follows from usual rank condition
  - X must be full-rank with positive probability (like OLS)

Step 2: Identifying the Outcome Equations

For the Treated

$$\mathbb{E}[Y_1|X, D = 1] = X\beta_1 + \mathbb{E}[U_1|V < \mu_d(X)]$$

- Under joint normality, the conditional expectation has a closed form:

$$\mathbb{E}[Y_1|X, D = 1] = X\beta_1 - \sigma_{V1}\frac{\phi(\mu_d(X))}{\Phi(\mu_d(X))}$$

- The term $\frac{\phi(\mu_d(X))}{\Phi(\mu_d(X))}$ is the **inverse Mills ratio**

For the Untreated

$$\mathbb{E}[Y_0|X, D = 0] = X\beta_0 + \sigma_{V0}\frac{\phi(\mu_d(X))}{1 - \Phi(\mu_d(X))}$$

Identification Result

- Under rank conditions for X, both β_0 and β_1 are identified
- Therefore $ATE(X) = X(\beta_1 - \beta_0)$ is identified for all X

Treatment Effect Heterogeneity

- Cannot identify the full distribution of treatment effects
- But can identify the ATE among individuals with treatment propensity V:

$$\mathbb{E}[Y_1 - Y_0|X, V] = X(\beta_1 - \beta_0) + \underbrace{(\sigma_{V1} - \sigma_{V0})V}_{\mathbb{E}[U_1 - U_0|V]}$$

- This shows how treatment effects vary with unobserved selection propensity
- We'll return to this object (the MTE) later

No Exclusion Restriction Needed?

- Notice: identification holds *without* an excluded variable Z
- Follows entirely from assumptions of:
  - Linearity in μ_d
  - Normality of error terms
- These yield a particular parametric decomposition of conditional expectations

The Problem

- This is **identification by functional form**
- Identification depends crucially on these functional form assumptions
- Without linearity in μ_d: cannot separately identify μ_d from selection correction

- Is this credible for identifying key causal parameters?
- See Lewbel for broad discussion of these issues

Identification with Exclusion Restrictions

Formal Assumptions

1. **Exclusion:** $\mu_D(X, Z) = \mu_D(X)$ almost everywhere
   - Z does not directly affect potential outcomes
2. **Independence:** $Z \perp (V, U_0, U_1)|X$
   - Z is independent of unobservables conditional on X

Implications of the Exclusion Restriction

- We can write:
$$\mathbb{E}[U_1|X, Z, D = 1] = \mathbb{E}[U_1|V \leq P(X, Z)]$$
and similarly for $D = 0$
- The expectation of Y (unconditional on D):

$$\begin{aligned}\mathbb{E}[Y|X, P(X, Z) = p] &= \mu_0(X) + p[\mu_1(X) - \mu_0(X)] && (1)\\ &\quad + p\mathbb{E}[(U_1 - U_0)|V \leq P(X, Z)] && (2)\\ &= \mu_0(X) + p[\mu_1(X) - \mu_0(X)] + \int_0^p \mathbb{E}[(U_1 - U_0)|V = u]\, du && (3)\end{aligned}$$

The Marginal Treatment Effect (MTE)

Derivation

- Take derivative with respect to p:

$$\begin{aligned}\frac{\partial \mathbb{E}[Y|X, P(X, Z) = p]}{\partial p} &= \mu_1(X) - \mu_0(X) + \mathbb{E}[U_1 - U_0|V = p] && (4)\\ &= \mathbb{E}[Y_1 - Y_0|V = p] && (5)\\ &= MTE(p) && (6)\end{aligned}$$

Interpretation

- Heckman and Vytlacil (2005) call this the **Local Instrumental Variables** approach
- Define $MTE(p)$ as the **Marginal Treatment Effect**
- It is the average treatment effect among individuals with propensity $1 - p$ to take treatment
- Individuals with $V = p$ are exactly indifferent when $P(X, Z) = p$

Key Result

- Commonly used estimators (IV, DD) are weighted averages of MTE
- MTE is a useful building block for understanding what different estimators identify

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Identifying the ATE

- Under support conditions on Z:
- As support of $P(X, Z)$ approaches $[0, 1]$:
- The average treatment effect is identified:

$$ATE(X) = \mu_1(X) - \mu_0(X)$$

- Requires variation in Z that pushes propensity to both extremes

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Credibility: Functional Form vs Exclusion Restrictions

- Which is more credible?
  - Functional form restrictions (normality, linearity)
  - Exclusion and independence restrictions
- Assuming a "good" instrument exists, exclusion approach is preferable
- Functional form identification is fragile: results can change with different assumptions

Practical Compromise

- Even with an instrument, fully nonparametric estimation can be demanding
- Reasonable approach:
  1. Show conditions under which instrument provides nonparametric identification
  2. Introduce parametric assumptions that interpolate this plausible variation
  3. Establishes that identification is *not purely* driven by functional form
- See Cunha et al. (2010) and Carneiro et al. (2011) for examples

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Monotonicity and Potential Outcomes

The Imbens-Angrist Framework

- Generalized Roy model embeds the **potential outcomes framework**
- Imbens and Angrist (1994) consider what 2SLS estimates with heterogeneous treatment effects

Setup

- Index individuals by $\omega \in \Omega$
- Model as a triple: $(Y_1(\omega), Y_0(\omega), D_Z(\omega))$
- $D_Z(\omega) \in \{0, 1\}$: choice of individual ω when instrument takes value Z

Four Key Assumptions

1. **Exclusion:** $Y_D(\omega, Z) = Y_D(\omega)$ for $D \in \{0, 1\}$
   - Potential outcomes do not depend on Z
2. **Independence:** $Z \perp \omega$
   - Instrument is independent of individual type
3. **Monotonicity:** For any pair (z, z') in support of Z:
   - Either $D_Z(\omega) \geq D_{Z'}(\omega)$ for all ω
   - Or $D_Z(\omega) \leq D_{Z'}(\omega)$ for all ω
   - No "defiers" exist
4. **Relevance:** $P[\omega : D_1(\omega) \neq D_0(\omega)] > 0$
   - Instrument affects treatment for some individuals

Understanding Monotonicity with Binary Z

- When $Z \in \{0, 1\}$, partition Ω into four groups:

1. **Always takers:** $D_1(\omega) = D_0(\omega) = 1$
   - Always treated regardless of Z
2. **Never takers:** $D_1(\omega) = D_0(\omega) = 0$
   - Never treated regardless of Z
3. **Compliers:** $D_1(\omega) = 1$, $D_0(\omega) = 0$
   - Treated when $Z = 1$, untreated when $Z = 0$
   - "Comply" with the instrument
4. **Defiers:** $D_1(\omega) = 0$, $D_0(\omega) = 1$
   - Do the opposite of what instrument suggests

- **Monotonicity rules out defiers** (without loss of generality)

The Local Average Treatment Effect (LATE)

The 2SLS Estimand

$$\frac{\mathbb{E}[Y|Z = 1] - \mathbb{E}[Y|Z = 0]}{P[D = 1|Z = 1] - P[D = 1|Z = 0]}$$

Result (Imbens-Angrist)

- Under the four assumptions, this equals:

$$\mathbb{E}[Y_1(\omega) - Y_0(\omega)|\omega \in \text{Compliers}]$$

- The **Local Average Treatment Effect (LATE)**
- Average treatment effect *among compliers only*

Multi-valued Instruments

- For multi-valued Z: 2SLS produces a weighted average of LATEs
- Different complier groups for different values of Z

Importance

- This interpretation of IV is immensely widely used
- Know the definition and the assumptions
- LATE may differ from ATE if compliers are selected

Relationship to the Generalized Roy Model

Equivalence Result (Vytlacil 2002)

- The monotonicity assumption is **equivalent** to a latent index model

One Direction (Easy)

- The latent index model:
$$D = \mathbf{1}\{P(Z) - V \geq 0\}$$
obeys monotonicity by construction
- Higher $P(Z)$ means weakly more people select treatment

Other Direction (Intuitive)

- Can construct a mapping from Ω to $[0, 1]$ such that:
$$D_Z(\omega) = \mathbf{1}\{P(Z) - V(\omega) \geq 0\}$$
- The ranking $V(\omega)$ orders individuals by their resistance to treatment
- Monotonicity ensures this ranking is consistent across values of Z

Connecting LATE and MTE

Key Result

- For binary instrument $Z \in \{0, 1\}$, the 2SLS estimand equals:
$$\int_{P(Z=0)}^{P(Z=1)} MTE(u)\, du$$

Interpretation

- 2SLS averages the MTE over compliers
- Compliers are those with $V \in [P(Z = 0), P(Z = 1)]$
- These are the individuals induced to switch treatment status by the instrument

Implications

- LATE weights different parts of the MTE curve
- Different instruments weight different parts
- Understanding MTE helps understand what any particular IV estimate captures

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Summary

Key Takeaways

- **Selection Problem:** Comparing treated vs untreated confounded by selection on unobservables
- **Dimension Reduction:** Propensity score $P(X, Z)$ summarizes selection-relevant information
- **Identification by Functional Form:**
  - Joint normality + linearity identifies parameters without exclusion
  - But identification is fragile and driven by functional form
- **Identification with Exclusion:**
  - Exclusion + independence assumptions provide more credible identification
  - MTE is identified by derivative of $\mathbb{E}[Y|X, P]$ with respect to P
  - ATE identified if propensity has full support
- **Monotonicity and LATE:**
  - Imbens-Angrist: 2SLS identifies LATE under monotonicity
  - Vytlacil: Monotonicity equivalent to latent index model
  - LATE = integral of MTE over compliers
- **Practical Guidance:**
  - Show nonparametric identification conditions
  - Use parametric assumptions to interpolate
  - Establishes identification is not purely functional form