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RESEARCH ARTICLE

Inflation with $f(R, \phi)$ in Jordan frame

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Abstract
We consider an $f(R)$ action that is non-minimally coupled to a massive scalar field. The model closely resembles scalar–tensor theory and by conformal transformation can be transformed to Einstein frame. To avoid the ambiguity of the frame dependence, we obtain an exact analytical solution in Jordan frame and show that the model leads to a period of accelerated expansion with an exit. Further, we compute the scalar and tensor power spectrum for the model and compare them with observations.

Keywords Inflation · Model building · Generalized $f(R, \phi)$ · Power spectrum

1 Introduction

Cosmological Inflation [3,16,23,24,34] was originally introduced in the early 1980s to solve the cosmological puzzles like the horizon problem, flatness problem of the FRW model. It is now considered as the best paradigm for describing the early stages of the universe as it can explain the origin of structures in the universe and anisotropies in the Cosmic Microwave Background (CMB). The current PLANCK measurements indicate that the temperature fluctuations of CMB are nearly scale invariant [1]. Hence, the success of an inflationary model not only rests on providing a minimum of 50 e-foldings of inflation required to solve the cosmological puzzles, but a highly demanding requirement that the predicted power-spectrum being close to scale invariance [17,19, 22,36].

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While inflation is the successful paradigm for the early universe, we do not yet have a fundamental understanding of the mechanism that drives inflation. Inflationary models based on canonical scalar field built within General Relativity, need to satisfy slow-roll conditions [5,15]. In other words, the observations require the scalar field potentials to be nearly flat [1]. For the quantum fluctuations that exit the horizon during inflation to be nearly scale-invariant, the energy scale of the universe must remain almost a constant which require the canonical scalar field potential to be almost flat—almost like cosmological constant. While these flat potentials are phenomenologically successful, the scalar field sector of action deviates much from the standard model of particle physics. For instance, the renormalizability puts a constraint on the scalar field potential to be quartic i.e. $V(\phi) = m^2\phi^2 + \lambda\phi^4$, where λ is the coupling parameter and m is the mass. The inflationary models, based on canonical scalar field, require potentials of the form $V(\phi) = \sum_{n=0}^{N} c_{2n}\phi^{2n}$ where c_{2n}'s are real numbers and $N > 2$. However, in the standard model of particle physics and its minimal extensions, there is no candidate for inflaton with such flat potentials that could sustain inflation [17,19,22,36]. In this work, we construct an inflationary model where the inflaton is a massive scalar field which is theoretically well placed in the standard model of particle physics, however, in achieving this, we go beyond General Relativity.

The Einstein–Hilbert action can be treated as a limiting case of a more general action containing higher order invariants, and $f(R)$ gravity assumes such an action to be a function of Ricci scalar [7,9,27,31]. General Relativity is not renormalizable and, therefore, can not be quantized conventionally. However, the Einstein–Hilbert action supplemented with higher order curvature terms is renormalizable [6,30] which makes $f(R)$ gravity an interesting alternative to General Relativity. As usual, this comes with a prize, unlike General Relativity the resulting field equations of $f(R)$ gravity are not second order, but fourth order, which makes it non-trivial. $f(R)$ theories do not suffer from Oströgradsky instability [35].

In this work, we consider $f(R, \phi)$ [8,28] action, where $f(R)$ is non-minimally coupled to a massive scalar field, closely resembling scalar–tensor theories of gravity [11]. As mentioned above, the potential is fixed to be $m^2\phi^2$ and we choose the coupling function such that the modification to the Einstein–Hilbert action is dominant only at the initial phase of inflation. It is possible to perform a conformal transformation to the $f(R, \phi)$ action and transform it to Einstein–Hilbert action with an extra scalar field [24].

Scalar–tensor theories of gravity have a long-standing controversy about which frame (Einstein or Jordan) is the physical one [10,20,25]. To avoid these controversies and the ambiguity of the frame dependence, in this work, we consider the action without performing any identification or transformation with any other theory, frame or variables [25]. More specifically, we obtain an exact analytical solution in Jordan frame and show that the model leads to a period of accelerated expansion with an exit. We obtain the power spectrum for this model and show that it is nearly scale invariant.

The paper is organised as follows: In the next section, we discuss the model and obtain the exact analytical solution. To physically understand the dynamical equations, we obtain the exact de Sitter solution. In Sect. 2.2, we discuss our model and show that the constant H solution is a saddle point solution. We use this feature to highlight that the model has a graceful exit. In Sect. 3, we compute the power-spectrum of our model. Further, we discuss the key results and possible implications of our model in Sect. 4.

In this work, we consider $(-, +, +, +)$ metric signature. We use lower Latin alphabets for the 4-dimensional space-time and lower Greek alphabets for the 3-dimensional space. We use natural units $c = \hbar = 1$, $\kappa = 1/M_{\mathrm{P}}^2$, and M_{P} is the reduced Planck mass. We denote dot as derivative with respect to cosmic time t and $H(t) \equiv \dot{a}(t)/a(t)$. Various physical quantities with the overline refers to the values evaluated for the homogeneous and isotropic FRW background.

2 Model and exact background solution

We consider the following action:

$$S = \int d^4x\,\sqrt{-g}\left[\frac{1}{2}f(R,\phi) - \frac{\omega}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right], \tag{1}$$

where ϕ is the scalar field, $V(\phi)$ is the scalar field potential and, we assume

$$f(R,\phi) = h(\phi)\left(R + \alpha R^2\right) \tag{2}$$

$h(\phi)$ is non-minimal coupling function. It is important to note that $\omega = 1$ corresponds to canonical scalar field while $\omega = -1$ corresponds to ghost [4]. Varying action (1) w.r.t. the field ϕ and the metric leads to the following equations of motion [13]:

$$\Box\phi + \frac{1}{2\omega}\left(\omega_{,\phi}\phi^{;a}\phi_{,a} + f_{,\phi} - 2V_{,\phi}\right) = 0 \tag{3}$$

$$F\,G^p_q = \omega\left(\phi^{;p}\phi_{;q} - \frac{1}{2}\delta^p_q\phi^{;c}\phi_{;c}\right) - \frac{1}{2}\delta^p_q\left(RF - f + 2V\right) + F^{;p}_{\ \ q} - \delta^p_q\Box F \tag{4}$$

where $F = \partial f(\phi, R)/\partial R$. The stress-tensor of the scalar field and modified gravity are given by

$$T^{\phi}_{pq} = \omega\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{pq}\partial_c\phi\partial^c\phi\right) - g_{pq}V(\phi) \tag{5a}$$

$$T^{MG}_{pq} = \frac{g_{pq}}{2}\left(f - F\,R\right) + \nabla_p\nabla_p F - g_{pq}\Box F \tag{5b}$$

In the rest of this section, we obtain exact solution for the FRW background in the Jordan frame (without performing conformal transformation).

2.1 Exact background solution in Jordan frame

As mentioned earlier, to avoid ambiguity, we obtain the exact solution for the spatially flat Friedmann–Robertson–Walker (FRW) background:

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) \tag{6}$$

in the Jordan frame. Note that $a(t)$ is the scale factor. From Eqs. (3, 4), equation of motion for $\phi(t)$ and the Hubble parameter, $H(t)$, are:

$$6\dot{h}H^2 + 72\dot{h}H^4\alpha + 72\dot{h}H^2\alpha\dot{H} + 3\dot{h}\dot{H} + 18\dot{h}\dot{H}^2\alpha - \dot{V} - \omega\dot{\phi}\ddot{\phi} - 3\omega H\dot{\phi}^2 = 0 \tag{7a}$$

$$\begin{aligned}&-\frac{1}{2}\omega\dot{\phi}^2 + 3hH^2 + 108\alpha hH^2\dot{H} - 18h\dot{H}^2\alpha - V + 3H\dot{h}\\&\quad + 72\dot{h}\alpha H^3 + 36H\dot{h}\alpha\dot{H} + 36Hh\alpha\ddot{H} = 0\end{aligned} \tag{7b}$$

$$\begin{aligned}&2h\dot{H} + 108\alpha hH^2\dot{H} + 48\dot{h}H^3\alpha + 54h\dot{H}^2\alpha + 3hH^2 + \ddot{h}\\&\quad + \frac{1}{2}\omega\dot{\phi}^2 120H\dot{h}\dot{H}\alpha + 72h\ddot{H}\alpha + 2H\dot{h}\\&\quad - V + 24\alpha H^2\ddot{h} + 12\alpha\dot{H}\ddot{h} + 24\alpha\ddot{H}\dot{h} + 12h\alpha\dddot{H} = 0\end{aligned} \tag{7c}$$

Rewriting Eqs. (7a, 7b), we get

$$\begin{aligned}&-2h\dot{H} - 72h\dot{H}^2\alpha - \omega\dot{\phi}^2 - \ddot{h} - 84H\dot{h}\dot{H}\alpha - 36Hh\ddot{H}\alpha + 24\alpha\dot{h}H^3\\&\quad + H\dot{h} - 24\ddot{h}H^2\alpha - 12\alpha\ddot{h}\dot{H} - 24\alpha\dot{h}\ddot{H} - 12\alpha h\dddot{H} = 0\end{aligned} \tag{8}$$

As it is evident, the equations of motion are higher order. To obtain the exact solution to the above Eqs. (7, 8), we use the following ansatz:

$$a(t) = a_0 e^{H_D t} \quad \text{and} \quad \phi = \phi_0 e^{-p\,H_D t} \tag{9}$$

where a_0, $H_D > 0$ and n are constants. ϕ_0 corresponds to the value of the scalar field at the start of inflation which is set to 1. Substituting the above ansatz (9) in Eq. (8) and solving for $h(\phi)$ [21], we get the following *exact relations*:

$$V(\phi) = \lambda_0 + m^2\,\phi^2 + \lambda_p\,\phi^{-p} \tag{10a}$$

$$h(\phi) = \mu_0 + \mu_2\,\phi^2 + \mu_p\,\phi^{-p} \tag{10b}$$

where

$$\mu_0 = \frac{1}{3\,H_D^2}\lambda_0 \tag{11}$$

$$\mu_2 = -\frac{\omega\,p}{\left(1 + 24\,\alpha\,H_D^2\right)(2 + 4\,p)} \tag{12}$$

$$m^2 = \left(3 + p(2p - 5)\left(1 + 24\,\alpha\,H_D^2\right)\right)\,H_D^2\,\mu_2 \tag{13}$$

$$\mu_p = \frac{1}{6\,H_D^2\left(12\,\alpha\,H_D^2 + 1\right)}\lambda_p\,, \tag{14}$$

λ_0, λ_p are arbitrary (integration) constants. It is important to note that this is an exact analytical expression for the background evolution. We like to stress the following points: First, as mentioned earlier, the solution is obtained without performing any conformal transformation. To our knowledge, there has been no exact inflationary solution in the Jordan frame. Second, for the exact de Sitter, the scalar field decays with time. This has been contrasted with general relativity where the scalar field is a constant. Third, the coupling function is strongly related to the potential. From the above expressions, it is clear that μ_0 depends on λ_0, similarly, μ_p depends on λ_p and m is related to μ_2. If λ_0, λ_p vanish, then automatically, corresponding μ_0, μ_p also vanish. This is important because, while the scalar field potential can be obtained from the standard model of particle physics, the non-minimal coupling term is completely independent. Since each of the terms in the non-minimal coupling term is related to the scalar field potential, for our exact model, the non-minimal coupling term is completely fixed by the standard model of particle physics. Lastly, the anatz (9) is for exact de Sitter leading to an accelerated expansion. Since the field decays exponentially (9), the non-minimal coupling function and the scalar field potential are dominated by ϕ^{-p}. This implies that once the inflation sets in, there is no way to stop the inflation if $\lambda_p \neq 0$. As mentioned in the introduction, our focus is to have less deviation from the standard model of particle physics, hence, we set $\lambda_0 = \lambda_p = 0$. Although $\lambda_0 = 0$ is not a requirement, this assumption simplifies our analysis as we will show in the next subsection.

2.2 Special case: $\lambda_0 = \lambda_p = 0$

As mentioned above, let us set $\lambda_0 = \lambda_p = 0$ and $\omega = 1$. We have:

$$V(\phi) = m^2 \phi^2 \; ; \; h(\phi) = \mu_2 \phi^2 \tag{15}$$

where

$$\mu_2 = -\frac{p}{\left(1 + 24\,\alpha\, H_D^2\right)(2 + 4\,p)}; m^2 = \left(3 + p(2p-5)(1 + 24\,\alpha\, H_D^2)\right) H_D^2\, \mu_2 \tag{16}$$

This is one of the main results of this work regarding which we would like to stress the following points: First, while λ_p is set to zero, p is an arbitrary number and can take any positive value. Second, μ_2 is directly related to m^2. Since, m^2 is positive definite, this leads to the condition that $24\alpha H_D^2 + 1 < 0$ or $\alpha < -1/(24 H_D^2)$. Since the Hubble parameter during inflation is large, this implies that α is small negative value. Third, to understand the exact analytical solution, from the stress-tensor (5), let us calculate $\rho + 3P$

$$\rho + 3P \equiv -T_0^0 + T_\alpha^\alpha = \left[p^2 - \frac{m^2}{H_D^2} + \mu_2\left[180\,\alpha H_D^2 + 3\,p\,(2p-1)(1 + 24\alpha H_D^2)\right]\right] 2\phi_0^2 H_D^2 e^{-2pH_Dt} \tag{17}$$

The first two terms correspond to the canonical scalar field while the last term correspond to the modifications to the gravity. For $p > 1/2$, the third term is always negative, while, for $p < 1/2$ the third term can be positive, negative or zero. Large value of $p(\gg 1)$ does not lead to inflation as $\rho + 3P = 0$. However, $p \ll 1$ leads to $\rho + 3P < 0$. In the analysis we take $p < 1$. It s important to note that $p \ll 1$ corresponds to constant scalar field. Lastly, the solution we have obtained is exact de Sitter which does not have an exit. The above analysis provides the possibility that if p changes from small value to large value this will lead to an inflation with exit. Other way of looking at this is to change the initial value of $\dot{\phi}$. Using the fact that, $\dot{\phi} \propto p$, it is possible to check what kind of inflationary solutions occur if $\dot{\phi}_{t=0} \neq \dot{\phi}^{dS}_{t=0}$. We will study this in the next subsection.

2.3 Saddle point and exit from inflation

In this section we show—both analytically and numerically—that the exact de Sitter solution obtained is a saddle point. Specifically, we show that the deviation of the initial values from the de Sitter value leads to either a smooth exit from the inflationary phase or to super inflation. Hence there exist a range of initial values for which we have a viable inflationary solution. Models of inflation based on the saddle point solutions have been considered in past, see Refs. [14,29,32].

2.3.1 Analytical

Analytically, we look at the trajectories close to the de Sitter solution and evolve them in time. Rewriting the Eqs. (7, 8) in terms of the variable $\Delta = \dot{\phi}/\phi$:

$$\begin{aligned}\dot{\Delta} &= 144\alpha\mu_2 H^4 + 144\alpha\mu_2 \dot{H} H^2 + 36\alpha\mu_2 \dot{H}^2 - 3H\Delta - \Delta^2 + 12\mu_2 H^2 + 6\mu_2 \dot{H} - 2m^2 \\ \ddot{H} &= -4H^2\Delta - 3H\dot{H} - 2\Delta\dot{H} + \frac{1}{72}\frac{\Delta^2}{\alpha\mu_2 H} + \frac{1}{2}\frac{\dot{H}^2}{H} + \frac{1}{36}\frac{m^4}{\alpha\mu_2 H} - \frac{1}{12}\frac{H}{\alpha} - \frac{1}{6}\frac{\Delta}{\alpha}\end{aligned} \tag{18}$$

The fact that we can write the field equations only in terms of Δ implies that the dynamics (evolution of Hubble parameter, Number of e-foldings etc.) depends only on Δ and does not depend on ϕ or $\dot{\phi}$ independently. Let us define vector $\boldsymbol{v}$ as:

$$\boldsymbol{v} = \begin{pmatrix} H \\ \dot{H} \\ \Delta \end{pmatrix}$$

It is important to note that the de Sitter solution ($H = H_D$) is an equilibrium point ($\{\dot{\boldsymbol{v}}\}_{eq} = 0$) and

$$\{\boldsymbol{v}\}_{eq} = \begin{pmatrix} H_D \\ 0 \\ -pH_D \end{pmatrix}.$$

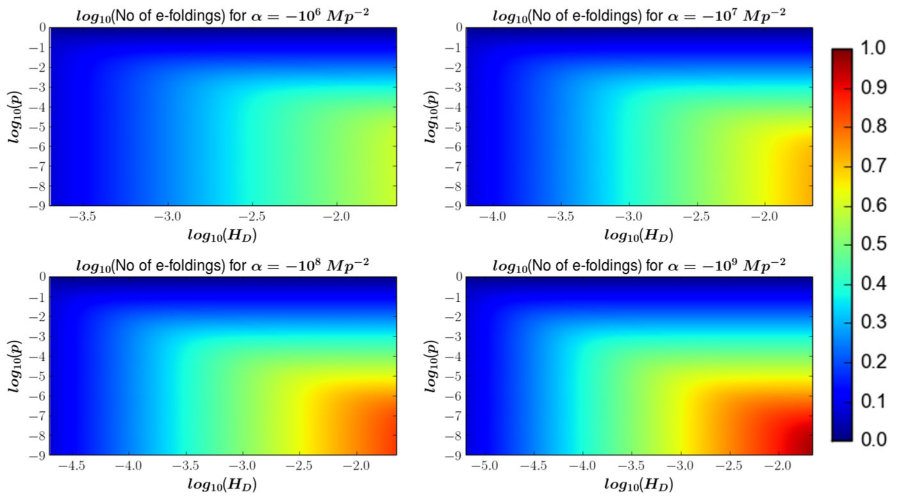

Fig. 1 Contour plot showing the dependence of the number of e-folding on H_D, p and α

The background equations for $\dot{\boldsymbol{v}}=f(\boldsymbol{v})$ can be written as

$$\dot{\boldsymbol{v}}=\begin{pmatrix}\dot{H}\\ \ddot{H}\\ \dot{\Delta}\end{pmatrix}=\left\{\begin{matrix}\dot{H}\\ -4H^2\Delta-3H\dot{H}-2\Delta\dot{H}+\frac{1}{72}\frac{\Delta^2}{\alpha\mu_2 H}+\frac{1}{2}\frac{\dot{H}^2}{H}+\frac{1}{36}\frac{m^4}{\alpha\mu_2 H}-\frac{1}{12}\frac{H}{\alpha}-\frac{1}{6}\frac{\Delta}{\alpha}\\ 144\alpha\mu_2 H^4+144\alpha\mu_2\dot{H}H^2+36\alpha\mu_2\dot{H}^2-3H\Delta-\Delta^2+12\mu_2 H^2+6\mu_2\dot{H}-2m^2\end{matrix}\right\}$$

As mentioned above, we perturb $\boldsymbol{v}=\boldsymbol{v}_{eq}+\delta\boldsymbol{v}$ and obtain the equation for $\delta\boldsymbol{v}$ by Taylor expanding $f(\boldsymbol{v})$ about the equilibrium point. Hence, we have:

$$\delta\dot{v}_i=\{\partial_j f_i\}_{eq}\delta v_j=J_{ij}\delta v_j\,. \tag{19}$$

See "Appendix A" for details. From the above analysis, it is clear that the de Sitter solution is a saddle point. Hence, for a range of initial conditions we have an inflationary phase eventually leading to exit. For the largest positive eigen value (λ), the number of e-foldings is given by:

$$N\approx\frac{H_D}{\lambda}ln\left(\frac{H_D^2}{\lambda(H_D-H_i)}\right).$$

Figure 1 contains the contour plot for the parameters p and H_D for different e-foldings by keeping $(H_D-H_i)/H_D$ constant.

2.3.2 Numerical

We also studied the evolution of background Eq. (7) numerically for a time step of $10^{-4}M_{\mathrm{P}}^{-1}$ and for a precision of the field ϕ/ϕ_0, (in dimensionless units) 10^{-16}. Figure 2 contains the plot of slow-roll parameter $\epsilon=-\frac{\dot{H}}{H^2}$ and the scalar field ϕ as a function of the number of e-foldings for different initial values of $\dot{\phi}$. We have taken the values for the parameters to be $\mu_2 = 10^{-4}$, $\alpha = -10^8 M_p^{-2}$ and two different values for p, $p = 0.1$ and $p = 0.01$, that corresponds to $m = 6 \times 10^{-5} M_p$, $H_D = 4 \times 10^{-4} M_p$ and $m = 3.36 \times 10^{-6} M_p$, $H_D = 10^{-4} M_p$ respectively.

From the plots we infer that the number of e-foldings is larger for larger values of H_D and for smaller values of p. These are consistent with the analytical results as small value for p suggests that the Universe remain in the inflationary phase for a long period.

3 Scalar and tensor power spectra for the model

In this section, we compute the scalar and tensor power-spectrum for our model discussed in the previous section. For easy comparison, we use the same notation as in Ref. [12]. It is important to note that the analysis used in Ref. [12] is not applicable for our model and hence, the results obtained in Ref. [26] have to be interpreted cautiously.

3.1 Perturbations

The linear order perturbations about the FRW background is given by [12]

$$\begin{aligned} ds^2 &= -(1+2\theta)dt^2 - a(\beta_{,\alpha} + B_\alpha)dt dx^\alpha \\ &\quad + a^2 \left[g^{(3)}_{\alpha\beta}(1-2\psi) + 2\gamma_{,\alpha|\beta} + 2C_{\alpha|\beta} + 2C_{\alpha\beta} \right] \end{aligned} \tag{20}$$

where $\theta(x,t)$, $\beta(x,t)$, $\psi(x,t)$ and $\gamma(x,t)$ characterize the scalar-type perturbations, $B_\alpha(x,t)$ and $C_\alpha(x,t)$ are trace-free ($B^\alpha_{|\alpha} = C^\alpha_{|\alpha} = 0$) vector perturbations, and $C_{\alpha\beta}(x,t)$ is transverse, trace-free ($C^\beta_{\alpha|\beta} = 0 = C^\alpha_\alpha$) tensor perturbation. The scalar field is decomposed as $\phi(x,t) = \bar{\phi}(t) + \delta\phi(x,t)$.

The perturbed scalar equations in Newtonian gauge in the Fourier space are given by [12,13]

$$-F\psi + F\theta + \delta F = 0 \tag{21a}$$

$$-2F\dot{\psi} - 2FH\theta - \dot{F}\theta + \dot{\phi}\delta\phi + \dot{\delta F} - H\delta F = 0 \tag{21b}$$

$$\begin{aligned} &6FH\dot{\psi} + 6FH^2\theta + 2F\frac{k^2}{a^2}\psi - \dot{\phi}^2\theta + 3\dot{F}\dot{\psi} + 6\dot{F}H\theta + \dot{\phi}\dot{\delta\phi} \\ &\quad - \ddot{\phi}\delta\phi - 3H\dot{\phi}\delta\phi - 3H\dot{\delta F} + 3\dot{H}\delta F + 3H^2\delta F - \frac{k^2}{a^2}\delta F = 0 \end{aligned} \tag{21c}$$

$$\begin{aligned} &6F\ddot{\psi} + 12F\dot{H}\theta + 6FH\dot{\theta} + 12FH\dot{\psi} + 12FH^2\theta - 2F\frac{k^2}{a^2}\theta \\ &\quad + 3\dot{F}\dot{\psi} + 6\dot{F}H\theta + \dot{F}\dot{\theta} + 4\dot{\phi}^2\theta + 6\theta\ddot{F} \\ &\quad - 4\dot{\phi}\dot{\delta\phi} - 2\ddot{\phi}\delta\phi - 6H\dot{\phi}\delta\phi - 3\ddot{\delta F} - 3H\dot{\delta F} + 6H^2\delta F - \frac{k^2}{a^2}\delta F = 0 \end{aligned} \tag{21d}$$

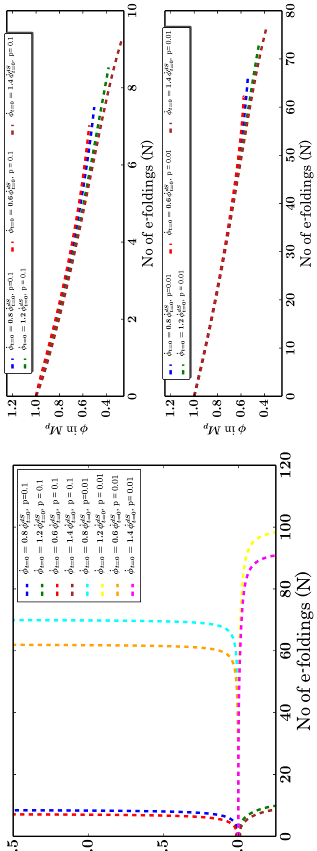

Fig. 2 Slow roll parameter ϵ and scalar field ϕ vs number of e-foldings (i) slow-roll parameter, ϵ and (ii) scalar field ϕ, for different initial values of $\frac{\dot{\phi}}{\phi}$

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{1}{2}f_{\phi\phi} + V_{\phi\phi}\delta\phi + \frac{k^2}{a^2}\delta\phi - 3\dot{\phi}\dot{\psi} - 6H\dot{\phi}\theta - \dot{\phi}\dot{\theta}$$
$$- 2\ddot{\phi}\theta + 3F_\phi\ddot{\psi} + 6F_\phi\dot{H}\theta + 3HF_\phi\dot{\theta} + 12F_\phi H\dot{\psi} + 12F_\phi H^2\theta$$
$$+ 2F_\phi\frac{k^2}{a^2}\psi - F_\phi\frac{k^2}{a^2}\theta = 0 \tag{21e}$$
$$\delta F - F_\phi\delta\phi + F_R\delta R = 0 \tag{21f}$$

where

$$\delta R = -6\ddot{\psi} - 12\dot{H}\theta - 6H\dot{\theta} - 24H\dot{\psi} - 24H^2\theta - 4\frac{k^2}{a^2}\psi + 2\frac{k^2}{a^2}\theta$$

It is important to note that only three of the above six equations are independent and we have rewritten the perturbation of F in Eq. (21f) as an independent equation.

The tensor perturbations in the Fourier space are given by:

$$\ddot{C}^\alpha_\beta + \left(\frac{\dot{F}}{F} + 3H\right)\dot{C}^\alpha_\beta + \frac{k^2}{a^2}C^\alpha_\beta = 0 \tag{22}$$

In the rest of this section, we obtain the scalar and tensor power spectrum for the exact de Sitter model. The reason is two fold: First, for an inflationary model with an exit, the scale factor evolution can only be obtained numerically and hence, the power-spectrum can not be evaluated exactly. Second, the perturbations equations are complicated and for the exact de sitter, the perturbation equations can be simplified and we can obtain analytical solutions in the super-Hubble scales.

We obtain the scalar power-spectrum in the limit $p \ll 1$, however, the tensor power-spectrum is obtained for any p that leads to inflation.

3.2 Scalar power spectrum

Since the analysis is in Jordan frame, the quantity we need to evaluate inorder to compare with the observations is 3-Curvature perturbation ($\mathcal{R}$) which is given by:

$$\mathcal{R} = \psi + \frac{H}{\dot{\phi}}\delta\phi \tag{23}$$

Before we proceed with the evaluation, we would like to point that $\mathcal{R}$ is conserved at large scales. The constancy of $\mathcal{R}$ is a consequence of the local energy conservation and is valid for any relativistic theory of gravity [18,33]. In the rest of the analysis, we will not be including the entropy perturbations as these will vanish at the super-Hubble scales.

As the Eq. (21) are highly coupled, non-linear and higher-order, we need to follow different strategy to obtain the differential equation for the 3-curvature perturbation.

First, is to obtain a solution to the differential equation of the combined variable $\theta + \psi \equiv \Theta$. Physically, Θ is the Bardeen potential in the Einstein frame. From Eqs.(21a, 21b, 21c), the differential equation for Θ is given by:

$$F\ddot{\Theta} + \left(3\dot{F} + H\,F - -\frac{2F\ddot{\phi}}{\dot{\phi}}\right)\dot{\Theta} + \left(\frac{k^2}{a^2}F - \ddot{F} - \frac{2FH\ddot{\phi}}{\dot{\phi}} + \frac{2\dot{F}\ddot{\phi}}{\dot{\phi}} + H\dot{F} + 4\dot{H}F\right)\Theta$$
$$= \left(\dot{\phi}^2 + 6F\dot{H} - 3\dot{F}H - 3\ddot{F} + \frac{6\dot{F}\ddot{\phi}}{\dot{\phi}}\right)\theta \tag{24}$$

For the exact analytical solution in the previous section (H is a constant), Eq. (24) becomes:

$$\ddot{\Theta} + H_D(1-4p)\dot{\Theta} + \frac{k^2}{a^2}\Theta - 4\,p\,H_D^2\,(1-p)\theta = 0 \tag{25}$$

In the small wavelength limit $\frac{k}{a} \gg 1$, the last two terms in the LHS can be approximated as

$$\left(\frac{k^2}{a^2} - 4\,p\,H_D^2\,(1-p)\right)\theta + \frac{k^2}{a^2}\psi \simeq \frac{k^2}{a^2}\Theta\,.$$

we then have,

$$\ddot{\Theta} + H_D(1-4p)\dot{\Theta} + \frac{k^2}{a^2}\Theta \simeq 0 \tag{26}$$

Second, rewrite $\delta\phi$ in terms of Θ. Using Eqs. (21a, 21b), for $p \ll 1$, we have

$$\delta\phi = \frac{\phi_0}{2H_D}\,e^{-p\,H_D\,t}\left(\dot{\Theta} + H_D\Theta\right) \tag{27}$$

Third, using the combination of Eqs. (21b, 21a) and (21f), we obtain ψ, θ and δF in terms of Θ, i. e.,

$$\theta = \frac{2}{3}\frac{1}{\frac{k^2}{a^2}}\ddot{\Theta} + \frac{1}{\frac{k^2}{a^2}}\dot{\Theta}\left(H_D - \frac{1}{12\alpha H_D}\right) + \frac{2}{3}\Theta \tag{28a}$$

$$\psi = \frac{1}{3}\Theta - \frac{2}{3}\frac{1}{\frac{k^2}{a^2}}\ddot{\Theta} - \frac{1}{\frac{k^2}{a^2}}\dot{\Theta}\left(H_D - \frac{1}{12\alpha H_D}\right) \tag{28b}$$

$$\delta F = n\phi_0^2 e^{-2pH_Dt}\left(\frac{1}{6}\Theta + \frac{2}{3}\frac{1}{\frac{k^2}{a^2}}\ddot{\Theta} + \frac{1}{\frac{k^2}{a^2}}\dot{\Theta}\left(H_D - \frac{1}{12\alpha H_D}\right)\right) \tag{28c}$$

Fourth, solve the differential equation (26) in the small wavelength limit to obtain Θ, i.e.

$$\Theta = e^{(4p-1)H_Dt/2}U_1 \tag{29}$$

where

$$U_1 = C_1 H^{(1)}_{\frac{1}{2}-2p}\left(\frac{ke^{-H_D t}}{a_0 H_D}\right) + C_2 H^{(2)}_{\frac{1}{2}-2p}\left(\frac{ke^{-H_D t}}{a_0 H_D}\right)$$

Fifth, we need to reduce the order of the differential equation of $\mathcal{R}$. From Eqs. (28), it is clear that the $\mathcal{R}$ contains higher derivatives of Θ. Interestingly, it can be shown that, $\ddot{\Theta}$ is linear in Θ and $\dot{\Theta}$, see Eq. 26. After a long calculation and setting the Bunch–Davies vacuum, in the limit $\frac{k}{a} \gg 1$ the Jordan frame curvature perturbation $\mathcal{R}_<$ is given by:

$$\mathcal{R}_< = \frac{H_D}{2a\sqrt{k}} e^{-ik\eta} \tag{30}$$

In the large scale limit, i.e. $\frac{k}{a} \to 0$, we can see that $\mathcal{R} = constant$ is a solution. Hence, we have

$$\mathcal{R}_> = C\,. \tag{31}$$

Matching the small wavelength and large wavelength solutions at $|k\eta| = 2\pi$ we have $C = \frac{\sqrt{2}H_D\pi}{k^{3/2}}$. For $p \ll 1$, the scalar power-spectrum is constant and is given by:

$$\mathcal{P}_{\mathcal{R}} = H_D^2\,. \tag{32}$$

3.3 Tensor power spectrum

Following Ref. [13], we can obtain the following equation of motion for the tensor perturbation for exact de Sitter solution:

$$\ddot{C}^\alpha_\beta + (-2pH_D + 3H_D)\,\dot{C}^\alpha_\beta + \frac{k^2}{a^2}C^\alpha_\beta = 0 \tag{33}$$

Defining $C^\alpha_\beta = v_g/z_g$ and $z_g = a\phi_0\sqrt{1 + 24\alpha H_D^2 e^{-pH_D t}}$, we have

$$v_g'' + \left(k^2 - \frac{z_g''}{z_g}\right)v_g = 0 \tag{34}$$

The solution to the above differential equation is again a sum of Hankel function:

$$v_g = \sqrt{-\eta}(\tilde{C}_1 H^{(1)}_{3/2-p}(-k\eta) + \tilde{C}_2 H^{(2)}_{3/2-p}(-k\eta)) \tag{35}$$

Setting the initial state to be Bunch–Davies vacuum, we have $\tilde{C}_2 = 0$ and $\tilde{C}_1 = \sqrt{\frac{\pi}{4}}$. Hence for tensor perturbation C^α_β, we have:

$$\nu_g = \sqrt{\frac{\pi}{4}}\sqrt{-\eta}H^{(1)}_{3/2-p}(-k\eta) \tag{36}$$

The Tensor power spectrum is given by $\mathcal{P}_g = 8\frac{k^3}{2\pi^2}|\mathcal{C}^{\oslash}_{\odot}|^2$ and is:

$$\mathcal{P}_g = 8\left(\frac{k}{k_*}\right)^{2p}\frac{2^{-2p}}{4\pi^2}H_D^2\left(\frac{\Gamma(3/2-p)}{\Gamma(3/2)}\right)^2 e^{2pH_D t_*}{\phi_0}^2\left(1+24\alpha H_D^2\right) \tag{37}$$

Tensor spectral index, $n_T = 2p$, which means that for a decaying scalar field the spectrum obtained is blue tilted.

4 Discussions

In this work, in Jordan frame, we have obtained an exact inflationary model for an $f(R, \phi)$ model. The scalar field is massive, non-minimally coupled to $f(R)$ and does not have self-interacting potential. The scalar field potential is consistent with standard model of particle physics. We have shown analytically and numerically that the model has an inflationary solution with an exit. For large number of e-foldings, the inflationary model behaves close to de Sitter.

It is important to note that the scalar perturbations cannot be evaluated analytically for any value of p. To verify the validity of the model with the CMB observations, we have obtained the scalar power spectrum for $p \ll 1$. Under this limit, the scalar power spectrum is nearly scale-invariant. We need to numerically evaluate for general p to be constraint the model with the current PLANCK data.

For a more precise calculation which was possible in the case of tensor perturbation, we have shown that the obtained spectrum have a blue tilt—for the exact constant H solution. The requirement that the blue tilt must be very small [2] constrains the parameter $p \ll 1$. The running of spectral index is expected to be negative for our model. In order to show that we need to obtain the power-spectrum numerically, which is currently under investigation.

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5 Appendix A: Details of the analytical approach for the saddle point

Jacobi matrix, J_{ij}, is given by:

$$J_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 1/6\,\frac{72\,p\alpha\,{H_D}^2+p-1}{\alpha} & 2\,pH_D - 3\,H_D & 1/9\,\frac{-1-24\,\alpha\,{H_D}^2+p+24\,p\alpha\,{H_D}^2}{\alpha} \\ 3\,\frac{(-3+2\,p)pH_D}{1+2\,p} & -3\,\frac{p}{1+2\,p} & 2\,pH_D - 3\,H_D \end{bmatrix} \tag{A1}$$

Let the eigen value and eigen vector of J be λ_i and u_i. Then the phase space trajectory is given by:

$$\delta v_i = \sum_{i=1}^{i=3} c_i u_i e^{(\lambda_i t)} \tag{A2}$$

where

$$\lambda_i = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2\,pH_D - 3\,H_D \\ \frac{-9\,H_D\alpha - 12\,H_D\alpha\,p + 12\,H_D\alpha\,p^2 + \sqrt{81\,{H_D}^2\alpha^2 + 936\,{H_D}^2\alpha^2 p + 1944\,{H_D}^2\alpha^2 p^2 + 864\,{H_D}^2\alpha^2 p^3 + 144\,{H_D}^2\alpha^2 p^4 - 6\,p\alpha - 6\,\alpha + 12\,p^2\alpha}}{6(1+2\,p)\alpha} \\ -\frac{9\,H_D\alpha + 12\,H_D\alpha\,p - 12\,H_D\alpha\,p^2 + \sqrt{81\,{H_D}^2\alpha^2 + 936\,{H_D}^2\alpha^2 p + 1944\,{H_D}^2\alpha^2 p^2 + 864\,{H_D}^2\alpha^2 p^3 + 144\,{H_D}^2\alpha^2 p^4 - 6\,p\alpha - 6\,\alpha + 12\,p^2\alpha}}{6(1+2\,p)\alpha} \end{bmatrix}$$

and

$$u_1 = \begin{bmatrix} -\frac{2}{3}\frac{-1-24\alpha H_D^2+24\alpha p H_D^2+p}{72pH_D^2\alpha+p-1} \\ -\frac{2}{3}\frac{-1-24\alpha H_D^2+24\alpha p H_D^2+p}{72pH_D^2\alpha+p-1}\lambda_1 \\ 1 \end{bmatrix};$$

$$u_2 = \begin{bmatrix} \frac{1}{18}\frac{1-p-120\alpha p H_D^2-96\alpha p^2 H_D^2}{p\alpha\lambda_2(\lambda_2-\lambda_1)} \\ \frac{1}{18}\frac{1-p-120\alpha p H_D^2-96\alpha p^2 H_D^2}{p\alpha\lambda_2(\lambda_2-\lambda_1)}\lambda_2 \\ 1 \end{bmatrix};$$

$$u_3 = \begin{bmatrix} \frac{1}{18}\frac{1-p-120\alpha p H_D^2-96\alpha p^2 H_D^2}{p\alpha\lambda_3(\lambda_3-\lambda_1)} \\ \frac{1}{18}\frac{1-p-120\alpha p H_D^2-96\alpha p^2 H_D^2}{p\lambda_3(\lambda_3-\lambda_1)}\lambda_3 \\ 1 \end{bmatrix}$$

c_i's are constants whose values has to be fixed from initial values of Hubble parameter (H_i) and the initial value of $\dot{\phi}/\phi$ (Δ_i).

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