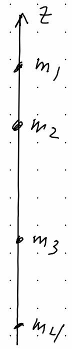
Sec 32, Prob 1:

(a)



$I_3 = 0$ since $x_a = y_a = 0$ for all masses

$$I_1 = \sum_a m_a (r_a^2 - \cancel{x_a^2}^{\,0}) = \sum_a m_a z_a^2$$

$$I_2 = \sum_a m_a (r_a^2 - \cancel{y_a^2}^{\,0}) = \sum_a m_a z_a^2$$

$$\rightarrow I_1 = I_2 \equiv I$$
$$= \sum_a m_a z_a^2$$

...

(assuming COM at $z=0$)

If COM is not at $z=0$, but at z_{com}, then:

$$I = \sum_a m_a (z_a - z_{com})^2, \quad z_{com} = \frac{1}{\mu}\sum_b m_b z_b$$

$$= \sum_a m_a (z_a^2 + z_{com}^2 - 2 z_{com} z_a)$$

$$= \sum_a m_a z_a^2 + \mu z_{com}^2 - 2 z_{com} \underbrace{\sum_a m_a z_a}_{\mu z_{com}}$$

$$= \sum_a m_a z_a^2 - \mu z_{com}^2$$

This last expression can be rewritten in terms of $l_{ab} \equiv |z_a - z_b|$ as follows:

$$I = \frac{1}{2}\sum_a m_a z_a^2 + \frac{1}{2}\sum_b m_b z_b^2 - \frac{1}{\mu}\Big(\sum_a m_a z_a\Big)\Big(\sum_b m_b z_b\Big)$$

$$= \frac{1}{2\mu}\sum_a\sum_b m_b m_a z_a^2 + \frac{1}{2\mu}\sum_a\sum_b m_a m_b z_b^2 - \frac{1}{\mu}\sum_a\sum_b m_a m_b z_a z_b$$

Thus

$$I = \frac{1}{2\mu}\sum_a\sum_b m_a m_b (z_a^2 + z_b^2 - 2 z_a z_b)$$

$$= \frac{1}{2\mu}\sum_a\sum_b m_a m_b (z_a - z_b)^2$$

$$= \frac{1}{2\mu}\sum_a\sum_b m_a m_b l_{ab}^2$$

NOTE: For just two masses:

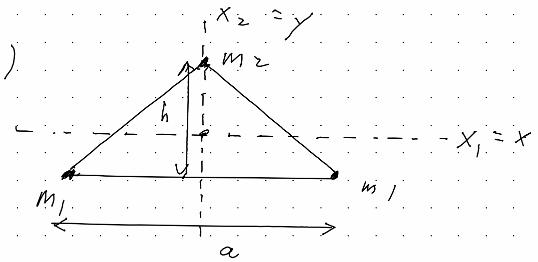
$$I = \frac{1}{2\mu}(m_1 m_2 l^2 + m_2 m_1 l^2)$$

$$= \frac{m_1 m_2}{\mu} l^2$$

$$= m l^2 \quad \text{where} \quad m \equiv \frac{m_1 m_2}{m_1 + m_2}$$

$$l = |z_1 - z_2|$$

(b)



Assume COM at $(x_1, x_2) = (x, y) = (0, 0)$

Then $2m_1 y_1 + m_2 y_2 = 0$

where $y_2 - y_1 = h$

thus, $2m_1 y_1 + m_2(h + y_1) = 0$

$$(2m_1 + m_2) y_1 + m_2 h = 0$$

$$y_1 = \frac{-m_2 h}{\mu} \quad , \quad \mu = 2m_1 + m_2$$

= total mass

and
$$y_2 = y_1 + h$$
$$= \frac{-m_2 h}{\mu} + h$$
$$= \frac{(\mu - m_2) h}{\mu}$$
$$= \frac{2m_1 h}{\mu}$$

All masses have $z_a = 0$

Thus, $I_3 = \sum_a m_a (r_a^2 - z_a^2) = \sum_a m_a (x_a^2 + y_a^2)$

$$I_1 = \sum_a m_a (r_a^2 - x_a^2) = \sum_a m_a y_a^2$$

$$I_2 = \sum_a m_a (r_a^2 - y_a^2) = \sum_a m_a x_a^2$$

Thus, $I_3 = I_1 + I_2$

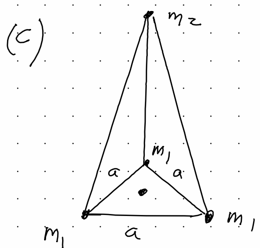
so need to calculate I_1, I_2.

$$I_1 = \sum_a m_a y_a^2$$
$$= 2m_1 y_1^2 + m_2 y_2^2$$
$$= 2m_1 \frac{m_2^2 h^2}{\mu^2} + m_2 \frac{4m_1^2 h^2}{\mu^2}$$
$$= \frac{2m_1 m_2 h^2}{\mu^2} (m_2 + 2m_1)$$
$$= \boxed{\frac{2m_1 m_2}{\mu} h^2}$$

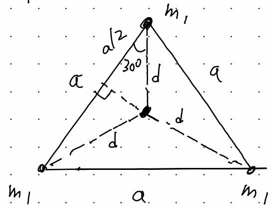
$$I_2 = \sum_a m_a x_a^2$$
$$= m_1 \left(\frac{a}{2}\right)^2 + m_1 \left(\frac{-a}{2}\right)^2$$
$$= \boxed{\frac{m_1 a^2}{2}}$$

$$I_3 = I_1 + I_2$$
$$= \boxed{\frac{2m_1 m_2}{\mu} h^2 + \frac{m_1 a^2}{2}}$$

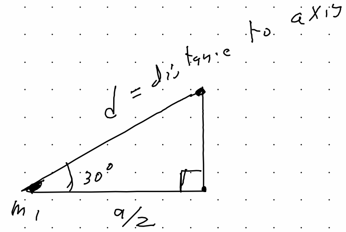
(c)



tetrahedron, height h
base: equilateral Δ with side length a



base



$$\cos 30^\circ = \frac{a}{2d} = \frac{\sqrt{3}}{2}$$

$$\rightarrow \quad d = \frac{a}{\sqrt{3}}$$

COM lies on axis of symmetry (z-axis)
Assume COM has $z=0$

Then $0 = m_2 z_2 + 3m_1 z_1$, $\quad z_2 - z_1 = h$

$$= m_2(z_1 + h) + 3m_1 z_1$$
$$= (3m_1 + m_2) z_1 + m_2 h$$

$$\rightarrow \quad z_1 = \frac{-m_2 h}{3m_1 + m_2} = -\frac{m_2 h}{\mu}$$

$$z_2 = h + z_1 = h - \frac{m_2 h}{\mu} = \frac{3m_1 h}{\mu}$$

Since a tetrahedron has 3-fold rotational symmetry, the x_1 principal axes can be chosen arbitrarily in the plane $\perp$ to the symmetry axis ($x_3 \equiv z$). [x_2 is $\perp$ to x_1, x_3]

Thus, $I_1 = I_2 \equiv I$

$$I_3 = \sum_a m_a (r_a^2 - z_a^2) = \sum_a m_a s_a^2 \quad \text{where } s^2 = r^2 - z^2$$

$$\left.\begin{aligned} I_1 &= \sum_a m_a (r_a^2 - x_a^2) \\ I_2 &= \sum_a m_a (r_a^2 - y_a^2) \end{aligned}\right\} \text{equal } (I_1 = I_2 \equiv I)$$

$$\begin{aligned} 2I &= I_1 + I_2 \\ &= \sum_a m_a (2r_a^2 - x_a^2 - y_a^2) \\ &= \sum_a m_a (2(s_a^2 + z_a^2) - s_a^2) \\ &= \sum_a m_a s_a^2 + 2\sum_a m_a z_a^2 \\ &= I_3 + 2\sum_a m_a z_a^2 \end{aligned}$$

Thus,

$$I = \frac{1}{2} I_3 + \sum_a m_a z_a^2$$

Now

$$I_3 = \sum_a m_a s_a^2 = 3 m_1 d^2 = \cancel{3} m_1 \frac{a^2}{\cancel{3}} = \boxed{m_1 a^2}$$

Also,

$$\sum_a m_a z_a^2 = 3m_1 z_1^2 + m_2 z_2^2$$

$$= 3m_1\left(-\frac{m_2 h}{\mu}\right)^2 + m_2\left(\frac{3m_1 h}{\mu}\right)^2$$

$$= \frac{3m_1 m_2^2 h^2}{\mu^2} + 9\frac{m_1^2 m_2 h^2}{\mu}$$

$$= \frac{3m_1 m_2 h^2}{\mu^2}\underbrace{(m_2 + 3m_1)}_{\mu}$$

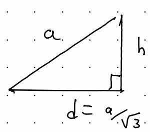
$$= \frac{3m_1 m_2 h^2}{\mu}$$

Thus,

$$I = \frac{1}{2}I_3 + \sum_a m_a z_a^2$$

$$= \boxed{\frac{1}{2}m_1 a^2 + \frac{3m_1 m_2 h^2}{\mu}}$$

Regular tetrahedron: $m_1 = m_2$



$$h^2 + \frac{a^2}{3} = a^2 \rightarrow h = \sqrt{\frac{2}{3}}\,a$$

$$\mu = 4m_1$$

$$I_3 = m_1 a^2$$

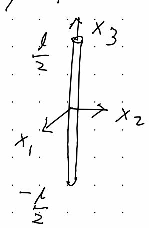
$$I = \frac{1}{2}m_1 a^2 + \frac{\cancel{3}\cancel{m_1} m_1}{4\cancel{m_1}}\left(\frac{2}{\cancel{3}}\right)a^2$$

$$= m_1 a^2 \;(= I_1 = I_2)$$

so $I_1 = I_2 = I_3 = m_1 a^2$

Sec 32, Prob 2:

(a) Thin rod of length ℓ:



$I_3 = \boxed{0}$

and $I_1 = I_2 \equiv I$

$$I = \int \rho\, dV\,(r^2 - x^2)$$

$$= \int \rho\, dV\, z^2$$

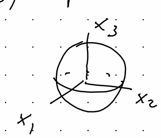
$$= \int_{-\ell/2}^{\ell/2} dz\left(\frac{\mu}{\ell}\right) z^2$$

$$= \frac{\mu}{\ell}\frac{z^3}{3}\Big|_{-\ell/2}^{\ell/2}$$

$$= \frac{\mu}{\ell}\frac{2}{3}\frac{\ell^3}{8}$$

$$= \boxed{\frac{1}{12}\mu\ell^2}$$

(b) Sphere of radius R:



$I_1 = I_2 = I_3 = I$

$$I = \frac{1}{3}(I_1 + I_2 + I_3)$$

$$= \frac{1}{3}\left[\int \rho\, dV\,(r^2 - x^2) + \int \rho\, dV\,(r^2 - y^2) + \int \rho\, dV\,(r^2 - z^2)\right]$$

$$= \frac{1}{3}\int \rho\, dV\left[3r^2 - x^2 - y^2 - z^2\right]$$

$$= \frac{2}{3}\int \rho\, dV\, r^2$$

$$I = \frac{2}{3}\int \rho\, dV\, r^2 \qquad \rho = \frac{\mu}{\frac{4}{3}\pi R^3}$$

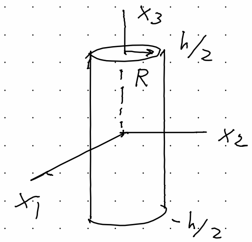
$$= \frac{2}{\cancel{3}} \frac{\mu}{\frac{4}{\cancel{3}}\pi R^3} \int_0^R r^4 dr \underbrace{\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi}_{4\pi}$$

$$= \frac{\mu}{2\pi R^3} \cdot 4\pi \int_0^R r^4 dr$$

$$= \frac{2\mu}{R^3} \frac{R^5}{5}$$

$$= \boxed{\frac{2}{5}\mu R^2}$$

(c)



$$\rho = \frac{\mu}{\pi R^2 \cdot h}$$

$$dV = ds\, s\, d\phi\, dz$$

where $s^2 = x^2 + y^2$

$$I_1 = I_2 \equiv I$$

$$2I = I_1 + I_2$$

$$= \int \rho\, dV (r^2 - x^2) + \int \rho\, dV (r^2 - y^2)$$

$$= \int \rho\, dV (2r^2 - s^2)$$

$$= \int \rho\, dV (2(s^2 + z^2) - s^2)$$

$$= \int \rho\, dV\, s^2 + 2\int \rho\, dV\, z^2$$

$$\rightarrow I = \frac{1}{2} I_3 + \int \rho\, dV \cdot z^2$$

$$I_3 = \int \rho\, dV\, s^2$$

$$= \frac{\mu}{\pi R^2 h} \int_0^R s^3 ds \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz$$

$$= \frac{\mu}{\pi R^2 \cancel{h}} \frac{R^4}{4} \cdot 2\pi \cdot \cancel{h}$$

$$= \boxed{\frac{1}{2}\mu R^2}$$

$$I = \frac{1}{2} I_3 + \int \rho\, dV\, z^2$$

$$\int \rho\, dV\, z^2 = \frac{\mu}{\pi R^2 h} \int_0^R s\, ds \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} z^2 dz$$

$$= \frac{\mu}{\cancel{\pi R^2} h} \frac{\cancel{R^2}}{\cancel{2}} \cdot \cancel{2}\cancel{\pi} \cdot \frac{z^3}{3}\Big|_{-h/2}^{h/2}$$

$$= \frac{\mu}{h} \frac{2}{3} \frac{h^3}{8}$$

$$= \frac{1}{12}\mu h^2$$

Thus,

$$I = \frac{1}{2}\left(\frac{1}{2}\mu R^2\right) + \frac{1}{12}\mu h^2$$

$$= \frac{1}{4}\mu R^2 + \frac{1}{12}\mu h^2$$

$$= \boxed{\frac{1}{4}\mu\left(R^2 + \frac{1}{3}h^2\right)}$$

NOTE: special limiting case:

(i) Thin rod $(R \to 0)$

$$I_3 = 0$$

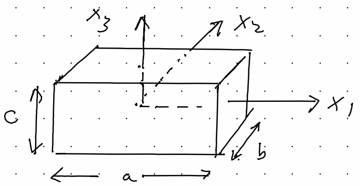
$$I_1 = I_2 = \frac{1}{12} M h^2$$

(ii) Thin disk $(h \to 0)$

$$I_3 = \frac{1}{2} M R^2$$

$$I_1 = I_2 = \frac{1}{4} M R^2$$

(d)



$$\rho = \frac{M}{abc}$$

$$dV = dx\,dy\,dz$$

$$I_1 = \int \rho\, dV\,(r^2 - x^2)$$

$$= \int \rho\, dV\,(y^2+z^2)$$

$$= \frac{M}{abc} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz\,(y^2+z^2)$$

$$= \frac{M}{\cancel{a}bc}\,\cancel{a} \int_{-b/2}^{b/2} dy \left(y^2 z + \frac{z^3}{3}\right)\Big|_{-c/2}^{c/2}$$

$$= \frac{M}{bc} \int_{-b/2}^{b/2} dy \left(c y^2 + \frac{2}{3}\frac{c^3}{8}\right)$$

$$= \frac{M}{bc}\left[c\frac{y^3}{3}\Big|_{-b/2}^{b/2} + \frac{1}{12} c^3 y \Big|_{-b/2}^{b/2} \right]$$

$$I_1 = \frac{M}{bc}\left[c\,\frac{2}{3}\frac{b^3}{8} + \frac{1}{12} b c^3 \right]$$

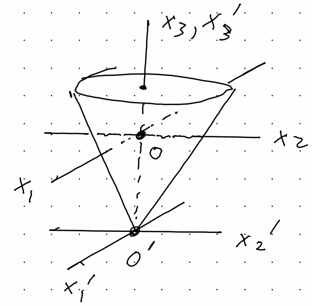
$$= \frac{M}{12bc}\left[cb^3 + bc^3 \right]$$

$$= \frac{M}{12}\left(b^2 + c^2\right)$$

$$I_2 = \frac{M}{12}\left(c^2 + a^2\right)$$

$$I_3 = \frac{M}{12}\left(a^2 + b^2\right)$$

(e)



First calculate I'_{ij} (w.r.t. x_1', x_2', x_3').

Then calculate I_{ij} via

$$I_{ij} = I'_{ij} - M(a^2 \delta_{ij} - a_i a_j)$$

where $\vec{a} = (0, 0, -d)$

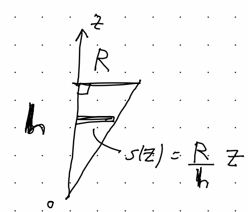
d: height of com above O'

Volume of cone:

$$V = \int dV$$

$$= \int_0^h dz \int_0^{2\pi} d\phi \int_0^{\frac{R}{h}z} s\,ds$$

$$= \cancel{2}\pi \int_0^h dz\, \frac{s^2}{\cancel{2}}\Big|_0^{Rz/h}$$



$$V = \pi \int_0^h dz \; \frac{R^2 z^2}{h^2}$$

$$= \frac{\pi R^2}{h^2} \; \frac{z^3}{3}\Big|_0^h$$

$$= \frac{\pi R^2}{h^2} \; \frac{h^3}{3}$$

$$= \boxed{\frac{1}{3}\pi R^2 h}$$

Thus, $\rho = \frac{M}{V} = \boxed{\frac{M}{\frac{1}{3}\pi R^2 h}}$ (mass density)

$$I_3' = \int \rho \, dV \, (r^2 - z^2)$$

$$= \int \rho \, dV \; s^2$$

$$= \rho \int_0^h dz \int_0^{2\pi} d\phi \int_0^{Rz/h} s^3 \, ds$$

$$= \frac{\rho}{4} \cdot 2\pi \int_0^h dz \; s^4 \Big|_0^{\frac{Rz}{h}}$$

$$= \frac{\rho \pi}{2} \int_0^h dz \left(\frac{R^4}{h^4}\right) z^4$$

$$= \frac{\rho \pi R^4}{2 h^4} \; \frac{h^5}{5}$$

$$= \frac{\rho \pi}{10} R^4 h$$

Thus,

$$I_3' = \frac{M}{\frac{1}{3}\cancel{\pi} R^2 \cancel{h}} \; \frac{\pi}{10} R^4 \cancel{h}$$

$$= \boxed{\frac{3}{10} M R^2}$$

Similar to the cylinder, we have $I_1' = I_2' \equiv I'$ where

$$I' = \frac{1}{2} I_3' + \int \rho \, dV \; z^2$$

Now:

$$\int \rho \, dV z^2 = \rho \int_0^h dz \; z^2 \int_0^{2\pi} d\phi \int_0^{Rz/h} s \, ds$$

$$= \cancel{2}\pi \rho \int_0^h dz \; z^2 \; \frac{s^2}{\cancel{2}}\Big|_0^{Rz/h}$$

$$= \pi \rho \frac{R^2}{h^2} \int_0^h dz \; z^4$$

$$= \pi \rho \frac{R^2}{h^2} \cdot \frac{h^5}{5}$$

$$= \frac{\pi}{5} \rho R^2 h^3$$

$$= \frac{\pi}{5}\left(\frac{M}{\frac{1}{3}\pi \cancel{R^2} h}\right) \cancel{R^2} h^3$$

$$= \frac{3}{5} M h^2$$

So

$$I' = \frac{1}{2}\left(\frac{3}{10}\mu R^2\right) + \frac{3}{5}\mu h^2$$

$$= \boxed{\frac{3}{5}\mu\left(\frac{R^2}{4} + h^2\right)} = I_1' = I_2'$$

Need to find location of COM:

$$d = \frac{1}{\mu}\int \rho\, dV\, z$$

$$= \frac{1}{\mu}\rho \int_0^h z\,dz \int_0^{2\pi} d\phi \int_0^{Rz/h} s\,ds$$

$$= \frac{1}{\mu}\rho \cdot \cancel{2}\pi \int_0^h z\,dz\ \frac{1}{\cancel{2}}\left(\frac{R}{h}\right)^2 z^2$$

$$= \frac{\pi\rho}{\mu}\frac{R^2}{h^2}\frac{z^4}{4}\Big|_0^h$$

$$= \frac{\pi\rho}{4\mu} R^2 h^2$$

$$= \frac{\cancel{\pi}}{4\cancel{\mu}} \cdot \frac{\cancel{\mu}}{\frac{1}{3}\cancel{\pi}\cancel{R^2}h}\cancel{R^2}h^2$$

$$= \boxed{\frac{3}{4}h}$$

Thus,

$$I_{ij} = I_{ij}' - \mu(a^2\delta_{ij} - a_i a_j)$$

where $\vec{a} = (0, 0, -\frac{3}{4}h) \rightarrow a^2 = \frac{9}{16}h^2$

$$\rightarrow I_1 = I_1' - \mu a^2$$

$$= \frac{3}{5}\mu\left(\frac{R^2}{4} + h^2\right) - \mu\frac{9}{16}h^2$$

$$= \frac{3}{20}\mu R^2 + \mu h^2\underbrace{\left(\frac{3}{5} - \frac{9}{16}\right)}_{\frac{48-45}{80} = \frac{3}{80}}$$

$$= \boxed{\frac{3}{20}\mu\left(R^2 + \frac{h^2}{4}\right)}$$

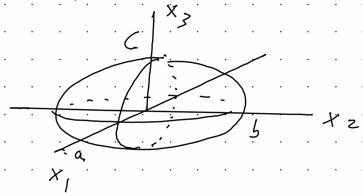
Also, $I_2 = I_1$

Finally, $I_3 = I_3' - \mu(\cancel{a^2 - a^2})^{\to 0}$

$$= I_3'$$

$$= \boxed{\frac{3}{10}\mu R^2}$$

(f) Ellipsoid with semi-axes a, b, c



$(a, b, c) \leftrightarrow (x_1, x_2, x_3)$

Define rescaled coordinates:

$$(u, v, w) \equiv \left(\frac{x_1}{a}, \frac{x_2}{b}, \frac{x_3}{c}\right)$$

So that boundary of ellipsoid

$$1 = \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 + \left(\frac{x_3}{c}\right)^2 = \underbrace{u^2+v^2+w^2}_{\text{unit 2-sphere}}$$

Volume:

$$V = \int dx_1 \int dx_2 \int dx_3$$

$$= abc \int du \int dv \int dw$$

$$= abc \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{1} r^2 dr$$

$$= abc \; 2\pi \cdot 2 \cdot \frac{r^3}{3}\Big|_0^1$$

$$= \boxed{\frac{4}{3}\pi abc}$$

$$\longrightarrow \rho = \frac{M}{\frac{4}{3}\pi abc}$$

$$I_3 = \int \rho \, dV \, (r^2 - z^2)$$

$$= \int \rho \, dV \, (x^2 + y^2)$$

$$= \frac{M}{\frac{4}{3}\pi abc} \iiint dx \, dy \, dz \, (x^2 + y^2)$$

$$= \frac{M}{\frac{4}{3}\pi \cancel{abc}} \cancel{abc} \iiint du \, dv \, dw \, (a^2u^2 + b^2v^2)$$

$$= \frac{M}{\frac{4}{3}\pi} \int_0^1 r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \big(a^2 r^2 \sin^2\theta \cos^2\phi$$

$$+ b^2 r^2 \sin^2\theta \sin^2\phi\big)$$

<u>Now:</u>

$$\int_0^{\pi} \sin\theta \, d\theta \, \sin^2\theta = \int_{-1}^{1} d(\cos\theta)(1 - \cos^2\theta)$$

$$= \int_{-1}^{1} dx \, (1 - x^2)$$

$$= \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$= 2 \cdot \frac{2}{3} = \boxed{\frac{4}{3}}$$

$$\int_0^1 r^4 dr = \frac{r^5}{5}\Big|_0^1 = \boxed{\frac{1}{5}}$$

$$\int_0^{2\pi} d\phi \begin{Bmatrix} \sin^2\phi \\ \cos^2\phi \end{Bmatrix} = 2\pi \cdot \frac{1}{2} = \boxed{\pi}$$

Thus,

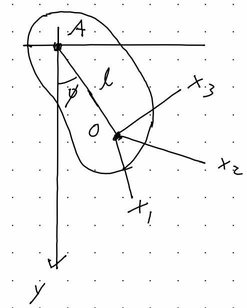
$$I_3 = \frac{M}{\cancel{\frac{4}{3}}\cancel{\pi}}\left(a^2 \cancel{\tfrac{4}{3}} \cdot \frac{1}{5} \cdot \cancel{\pi} + b^2 \cancel{\tfrac{4}{3}} \cdot \frac{1}{5} \cdot \cancel{\pi}\right)$$

$$= \boxed{\frac{M}{5}(a^2+b^2)}$$

cyclically permuting a, b, c $\rightarrow$

$$\boxed{\begin{aligned} I_1 &= \frac{M}{5}(b^2+c^2) \\ I_2 &= \frac{M}{5}(c^2+a^2) \end{aligned}}$$

Sec 32, Prob 3:



com at O

rotation axis at A, out of page

$\vec{\Omega} = \dot{\phi}\hat{n}$

$U = \mu g l(1-\cos\phi)$

$\approx \frac{1}{2}\mu g l \phi^2$ for $\phi << 1$

$$L = T - U$$

$$T = \frac{1}{2}\mu V^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$\vec{\Omega} = \dot{\phi}\hat{n} \rightarrow \begin{aligned} \Omega_1 &= \dot{\phi}\,\hat{n}\cdot\hat{x}_1 = \dot{\phi}\cos\alpha \\ \Omega_2 &= \dot{\phi}\,\hat{n}\cdot\hat{x}_2 = \dot{\phi}\cos\beta \\ \Omega_3 &= \dot{\phi}\,\hat{n}\cdot x_3 = \dot{\phi}\cos\gamma \end{aligned}$$

$$V = l\dot{\phi}$$

Thus,

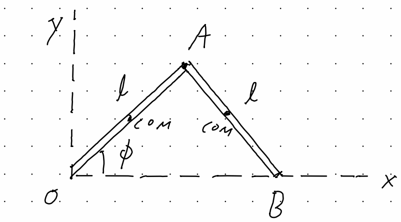
$$T = \frac{1}{2}\mu l^2 \dot{\phi}^2 + \frac{1}{2}\dot{\phi}^2\left(I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)$$

so

$$L = \frac{1}{2}\dot{\phi}^2\left(\mu l^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right) - \frac{1}{2}\mu g l \phi^2$$

$$\rightarrow \omega = \sqrt{\frac{\mu g l}{\mu l^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma}}$$

Sec. 32, Prob 4



two uniform rods

$$I_{com} = \frac{1}{12} M l^2$$

$$I_{end} = \frac{1}{3} M l^2$$

$$T = T_1 + T_2$$

$$T_1 = \frac{1}{2} M \left(\frac{l}{2}\right)^2 \dot{\phi}^2 + \frac{1}{2} I_{com} \dot{\phi}^2$$

$$= \frac{1}{8} M l^2 \dot{\phi}^2 + \frac{1}{24} M l^2 \dot{\phi}^2$$

$$= \left(\frac{1}{8} + \frac{1}{24}\right) M l^2 \dot{\phi}^2$$

$$= \frac{1}{6} M l^2 \dot{\phi}^2$$

$$T_2 = \frac{1}{2} M V^2 + \frac{1}{2} I_{com} \dot{\phi}^2$$

Now: $V^2 = \dot{x}^2 + \dot{y}^2$

$$x = \frac{3}{2} l \cos\phi \quad , \quad y = \frac{l}{2} \sin\phi$$

$$\dot{x} = -\frac{3}{2} l \sin\phi \, \dot{\phi} \quad , \quad \dot{y} = \frac{l}{2} \cos\phi \, \dot{\phi}$$

$$\rightarrow V^2 = \frac{9}{4} l^2 \sin^2\phi \, \dot{\phi}^2 + \frac{l^2}{4} \cos^2\phi \, \dot{\phi}^2$$

$$= 2 l^2 \sin^2\phi \, \dot{\phi}^2 + \frac{l^2}{4} \dot{\phi}^2$$

$$= 2 l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{8}\right)$$

so

$$T_2 = \frac{1}{\cancel{2}} M \cancel{2} l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{8}\right) + \frac{1}{24} M l^2 \dot{\phi}^2$$

$$= M l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{8} + \frac{1}{24}\right)$$

$$= M l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{6}\right)$$

Thus,

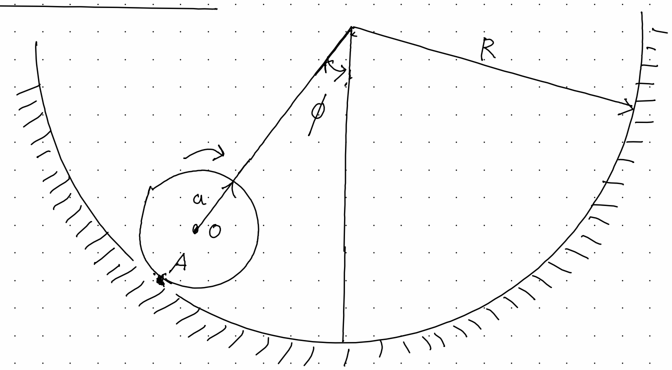
$$T = T_1 + T_2$$

$$= \frac{1}{6} M l^2 \dot{\phi}^2 + M l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{6}\right)$$

$$= M l^2 \dot{\phi}^2 \left(\frac{1}{3} + \sin^2\phi\right)$$

$$= \frac{1}{3} M l^2 \dot{\phi}^2 \left(1 + 3 \sin^2\phi\right)$$

Sec 32, Prob 6:



Homogeneous cylinder of radius a, mass μ:

$I_3 = \frac{1}{2}\mu a^2$ (about com)

$V = -(R-a)\dot\phi$ (velocity of com)

Instantaneous axis of rotation at A:

$0 = \vec V + \vec\Omega \times (-a\hat n)$, $\vec\Omega$ into page

so $V = \Omega a$

Thus, $\Omega a = (R-a)|\dot\phi|$

$$\Omega = \left(\frac{R-a}{a}\right)|\dot\phi|$$

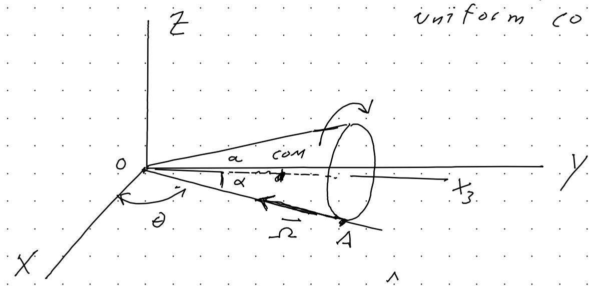
$$T = \frac{1}{2}\mu V^2 + \frac{1}{2} I_3 \Omega^2$$

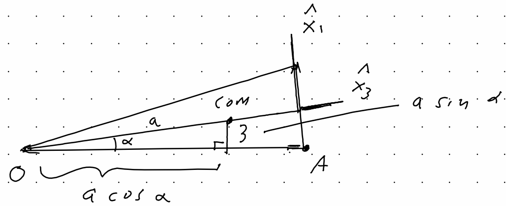
$$= \frac{1}{2}\mu\left(\frac{R-a}{\cancel{a}}\right)^2\cancel{a^2}\dot\phi^2 + \frac{1}{4}\mu\cancel{a^2}\left(\frac{R-a}{\cancel{a}}\right)^2\dot\phi^2$$

$$= \mu(R-a)^2\dot\phi^2\left(\frac{1}{2}+\frac{1}{4}\right) = \boxed{\frac{3}{4}\mu(R-a)^2\dot\phi^2}$$

Sec 32, Prob 7.

radius R, height h
uniform cone





V = velocity of COM
$= a\cos\alpha\,\dot\theta$

OA: instantaneous axis of rotation

$0 = \vec V - a\sin\alpha\,\vec\Omega \times \hat n$ $(\hat n = \hat z)$

$0 = V - a\sin\alpha\,\Omega$

Thus, $a\sin\alpha\,\Omega = V = a\cos\alpha\,\dot\theta$

$$\boxed{\Omega = \cot\alpha\,\dot\theta}$$

$\vec\Omega$: directed from A to O

$\Omega_3 = -\Omega\cos\alpha = -\frac{\cos^2\alpha}{\sin\alpha}\dot\theta$

$\Omega_1 = \Omega\sin\alpha = \cos\alpha\,\dot\theta$

Thus,

$(I_2 = I_1)$

$$T = \frac{1}{2}\mu v^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_1\Omega_2^2 + I_3\Omega_3^2\right)$$

$$= \frac{1}{2}\mu a^2\cos^2\alpha\,\dot\theta^2 + \frac{1}{2}\left(I_1\cos^2\alpha\,\dot\theta^2 + I_3\frac{\cos^4\alpha}{\sin^2\alpha}\dot\theta^2\right)$$

Now: $I_1 = I_2 = \frac{3}{20}\mu\left(R^2 + \frac{1}{4}h^2\right)$

$I_3 = \frac{3}{10}\mu R^2$

Also: $a = \frac{3}{4}h$

$\tan\alpha = \frac{R}{h} \rightarrow R = h\tan\alpha$

Thus,

$$T = \frac{1}{2}\mu\left(\frac{9}{16}\right)h^2\cos^2\alpha\,\dot\theta^2$$

$$+ \frac{1}{2}\Big[\frac{3}{20}\mu\left(h^2\tan^2\alpha + \frac{1}{4}h^2\right)\cos^2\alpha\,\dot\theta^2$$

$$+ \frac{3}{10}\mu h^2\tan^2\alpha\frac{\cos^4\alpha}{\sin^2\alpha}\dot\theta^2\Big]$$

$$= \mu h^2\dot\theta^2\Big[\frac{9}{32}\cos^2\alpha + \frac{3}{40}\overbrace{\sin^2\alpha}^{(1-\cos^2\alpha)} + \frac{3}{160}\cos^2\alpha$$

$$+ \frac{3}{20}\cos^2\alpha\Big]$$

$$= \mu h^2\dot\theta^2\left[\frac{3}{40} + \cos^2\alpha\left(\frac{9}{32} - \frac{3}{40} + \frac{3}{160} + \frac{3}{20}\right)\right]$$

Now: $\frac{9}{32} - \frac{3}{40} + \frac{3}{160} + \frac{3}{20}$

$$= \frac{1}{160}\left[45 - 12 + 3 + 24\right]$$

$$= \frac{60}{160}$$

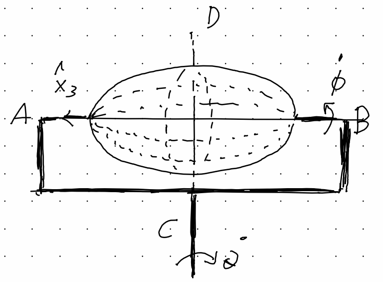
$$= \frac{15}{40}$$

Thus,

$$T = \mu h^2\dot\theta^2\left[\frac{3}{40} + \frac{15}{40}\cos^2\alpha\right]$$

$$= \boxed{\frac{3}{40}\mu h^2\dot\theta^2\left[1 + 5\cos^2\alpha\right]}$$

Sec 32, Prob 9.



homogeneous ellipsoid with principal moments of inertia I_1, I_2, I_3

$$\vec{\Omega} = \vec{\dot{\phi}} + \vec{\dot{\theta}}$$

Now: $\vec{\dot{\phi}} = \dot{\phi}\,\hat{x}_3$

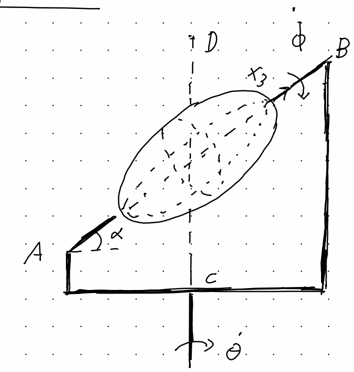
$$\vec{\dot{\theta}} = \dot{\theta}\left[\cos\phi\,\hat{x}_1 + \sin\phi\,\hat{x}_2\right]$$

so $\vec{\Omega} = \dot{\theta}\cos\phi\,\hat{x}_1 + \dot{\theta}\sin\phi\,\hat{x}_2 + \dot{\phi}\,\hat{x}_3$

$$T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$= \frac{1}{2}\left[\left(I_1\cos^2\phi + I_2\sin^2\phi\right)\dot{\theta}^2 + I_3\,\dot{\phi}^2\right]$$

Sec 32, Prob 10.



Uniform ellipsoid with $I_1 = I_2$ (circular cross section)

$$\vec{\Omega} = \vec{\dot{\phi}} + \vec{\dot{\theta}}$$

$$\vec{\dot{\phi}} = \dot{\phi}\,\hat{x}_3$$

$$\vec{\dot{\theta}} = \dot{\theta}\left[\cos\left(\tfrac{\pi}{2}-\alpha\right)\hat{x}_3 + \sin\left(\tfrac{\pi}{2}-\alpha\right)\left(\cos\phi\,\hat{x}_1 + \sin\phi\,\hat{x}_2\right)\right]$$

Now! $\cos\left(\frac{\pi}{2}-\alpha\right) = \cancel{\cos\left(\frac{\pi}{2}\right)}^{0}\cos\alpha + \sin\left(\frac{\pi}{2}\right)\sin\alpha$
$= \sin\alpha$

$\sin\left(\frac{\pi}{2}-\alpha\right) = \sin\frac{\pi}{2}\cos\alpha - \cancel{\cos\left(\frac{\pi}{2}\right)}^{0}\sin\alpha$
$= \cos\alpha$

so $\vec{\dot{\theta}} = \dot{\theta}\left[\sin\alpha\,\hat{x}_3 + \cos\alpha\cos\phi\,\hat{x}_1 + \cos\alpha\sin\phi\,\hat{x}_2\right]$

$\rightarrow \vec{\Omega} = \dot{\theta}\cos\alpha\left(\cos\phi\,\hat{x}_1 + \sin\phi\,\hat{x}_2\right) + \left(\dot{\phi} + \dot{\theta}\sin\alpha\right)\hat{x}_3$

$$T = \frac{1}{2}\left(I_1\Omega_1^2 + I_1\Omega_2^2 + I_3\Omega_3^2\right)$$

$$= \frac{1}{2}\left[I_1\,\dot{\theta}^2\cos^2\alpha\left(\cos^2\phi + \sin^2\phi\right) + I_3\left(\dot{\phi} + \dot{\theta}\sin\alpha\right)^2\right]$$

$$= \frac{1}{2}\left[I_1\cos^2\alpha\,\dot{\theta}^2 + I_3\left(\dot{\phi} + \dot{\theta}\sin\alpha\right)^2\right]$$