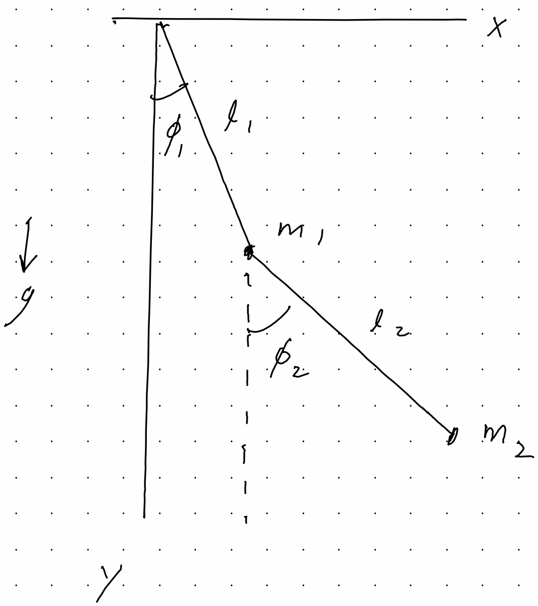
sec5, prob1



$x_1 = l_1 \sin\phi_1$

$y_1 = l_1 \cos\phi_1$

$x_2 = x_1 + l_2 \sin\phi_2 = l_1 \sin\phi_1 + l_2 \sin\phi_2$

$y_2 = y_1 + l_2 \cos\phi_2 = l_1 \cos\phi_1 + l_2 \cos\phi_2$

$U = -m_1 g y_1 - m_2 g y_2$

$= -m_1 g l_1 \cos\phi_1 - m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$

$= -(m_1 + m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$

$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$

$\dot{x}_1 = l_1 \cos\phi_1 \dot{\phi}_1$ $\qquad$ $\dot{y}_1 = -l_1 \sin\phi_1 \dot{\phi}_1$

$\dot{x}_1^2 = l_1^2 \cos^2\phi_1 \dot{\phi}_1^2$ $\qquad$ $\dot{y}_1^2 = l_1^2 \sin^2\phi_1 \dot{\phi}_1^2$

Thus, $\dot{x}_1^2+\dot{y}_1^2 = l_1^2(\sin^2\phi_1+\cos^2\phi_1)\dot{\phi}_1^2$

$$= l_1^2\dot{\phi}_1^2$$

$$\dot{x}_2 = l_1\cos\phi_1\dot{\phi}_1 + l_2\cos\phi_2\dot{\phi}_2$$

$$\rightarrow \dot{x}_2^2 = l_1^2\cos^2\phi_1\dot{\phi}_1^2 + l_2^2\cos^2\phi_2\dot{\phi}_2^2 + 2l_1l_2\cos\phi_1\cos\phi_2\dot{\phi}_1\dot{\phi}_2$$

$$\dot{y}_2 = -l_1\sin\phi_1\dot{\phi}_1 - l_2\sin\phi_2\dot{\phi}_2$$

$$\rightarrow \dot{y}_2^2 = l_1^2\sin^2\phi_1\dot{\phi}_1^2 + l_2^2\sin^2\phi_2\dot{\phi}_2^2 + 2l_1l_2\sin\phi_1\sin\phi_2\dot{\phi}_1\dot{\phi}_2$$

Thus,

$$\dot{x}_2^2+\dot{y}_2^2 = l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2l_1l_2(\cos\phi_1\cos\phi_2+\sin\phi_1\sin\phi_2)\dot{\phi}_1\dot{\phi}_2$$

$$= l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$

so $T = \frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2)$

$$= \frac{1}{2}m_1l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2$$

$$+ m_2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$

$$= \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2 + m_2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$

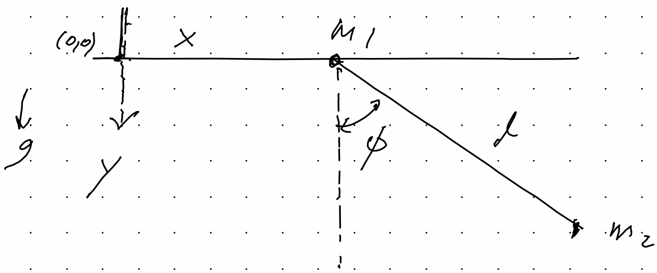
$$U = -(m_1+m_2)gl_1\cos\phi_1 - m_2gl_2\cos\phi_2$$

$$L = T-U$$

$$= \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2 + m_2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$

$$+(m_1+m_2)gl_1\cos\phi_1 + m_2gl_2\cos\phi_2$$

Sec5 Prob. 2



Generalised coords: x, ϕ

$(x_1, y_1) = (x, 0)$

$(x_2, y_2) = (x + l\sin\phi,\ l\cos\phi)$

$U = -m_1 g y_1 - m_2 g y_2$

$\quad = -m_2 g l \cos\phi$

$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$

Now: $\dot{x}_1^2 + \dot{y}_1^2 = \dot{x}^2$

$\dot{x}_2^2 + \dot{y}_2^2 = (\dot{x} + l\cos\phi\,\dot{\phi})^2 + (-l\sin\phi\,\dot{\phi})^2$

$\quad = \dot{x}^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi}$

$\quad\quad + l^2\sin^2\phi\,\dot{\phi}^2$

$\quad = \dot{x}^2 + l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi}$

$\mapsto T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}^2 + l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi})$

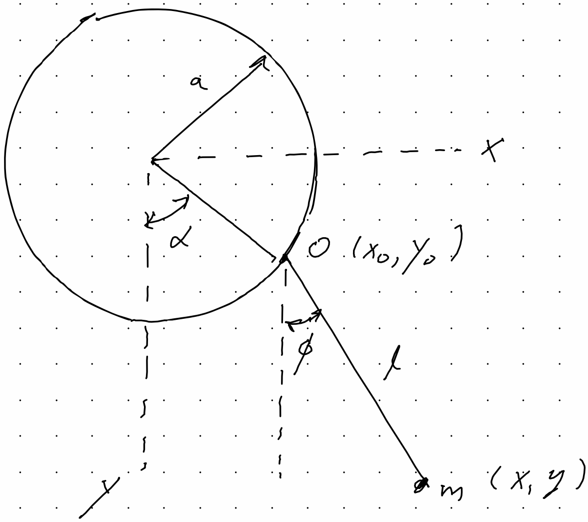
$\quad = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \cos\phi\,\dot{x}\dot{\phi}$

$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l\cos\phi\,\dot{x}\dot{\phi}$

$\quad + m_2 g l \cos\phi$

Sec 5, Prob 3

(a)



point of support O moves along circle:

$x_0 = a \sin\alpha \quad , \quad y_0 = a \cos\alpha$

where $\alpha = \gamma t \quad , \quad \gamma = \text{const}$

Pendulum bob:

$(x,y): \quad x = x_0 + l \sin\phi$

$\qquad\qquad y = y_0 + l \cos\phi$

$$U = -mgy = \underbrace{-mgy_0}_{\substack{\text{specified function of time} \\ [\text{can ignore in } L]}} - mgl\cos\phi$$

$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

$\dot{x} = \dot{x}_0 + l \cos\phi \, \dot{\phi}$

$\dot{x}^2 = \dot{x}_0^2 + l^2 \cos^2\phi \, \dot{\phi}^2 + 2 l \cos\phi \, \dot{x}_0 \dot{\phi}$

$$\dot{y} = \dot{y}_0 - l\sin\phi\,\dot{\phi}$$

$$\dot{y}^2 = \dot{y}_0^2 + l^2\sin^2\phi\,\dot{\phi}^2 - 2l\sin\phi\,\dot{y}_0\dot{\phi}$$

Thus,

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) \\ &= \frac{1}{2}m\left(\dot{x}_0^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}_0\dot{\phi} \right. \\ &\qquad \left. + \dot{y}_0^2 + l^2\sin^2\phi\,\dot{\phi}^2 - 2l\sin\phi\,\dot{y}_0\dot{\phi}\right) \\ &= \frac{1}{2}m\left(\dot{x}_0^2+\dot{y}_0^2\right) + \frac{1}{2}ml^2\dot{\phi}^2 + ml\dot{\phi}\left(\dot{x}_0\cos\phi - \dot{y}_0\sin\phi\right) \end{aligned}$$

NOTE: $\dot{x}_0^2+\dot{y}_0^2 = a^2\dot{\alpha}^2 = a^2\gamma^2$

since this is a specified function of time, we can ignore it in the Lagrangian;

Thus,
$$L = \frac{1}{2}ml^2\dot{\phi}^2 + ml\dot{\phi}\left(\dot{x}_0\cos\phi - \dot{y}_0\sin\phi\right) + mgl\cos\phi$$

We can rewrite the second term:

$$x_0 = a\sin\alpha \rightarrow \dot{x}_0 = a\cos\alpha\,\dot{\alpha} \qquad (\dot{\alpha} \equiv \gamma)$$
$$y_0 = a\cos\alpha \rightarrow \dot{y}_0 = -a\sin\alpha\,\dot{\alpha}$$

Thus,

$$\begin{aligned} ml\dot{\phi}(\dot{x}_0\cos\phi - \dot{y}_0\sin\phi) &= ml\dot{\phi}a\gamma(\cos\alpha\cos\phi + \sin\alpha\sin\phi) \\ &= ml\dot{\phi}a\gamma\cos(\phi-\alpha) \\ &= ml\dot{\phi}a\gamma\cos(\phi-\gamma t) \end{aligned}$$

Now; $\frac{d}{dt}\left[m l a \gamma \sin(\phi - \gamma t)\right]$

$$= m l a \gamma \cos(\phi - \gamma t)(\dot{\phi} - \gamma)$$

$$= m l a \dot{\phi} \gamma \cos(\phi - \gamma t) - m l a \gamma^2 \cos(\phi - \gamma t)$$

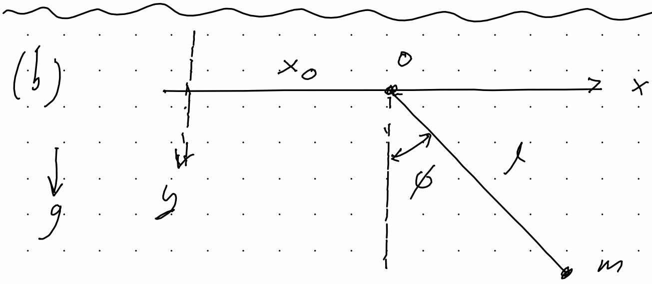
Thus,

$$m l a \dot{\phi} \gamma \cos(\phi - \gamma t) = \frac{d}{dt}\left[m l a \gamma \sin(\phi - \gamma t)\right] + m l a \gamma^2 \cos(\phi - \gamma t)$$

(and we can ignore the total time derivative in the Lagrangian)

$$\rightarrow L = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \cos\phi + m l a \gamma^2 \cos(\phi - \gamma t)$$

(b)



point O moving according to $x_0 = a \cos \gamma t$

$$x = x_0 + l \sin\phi$$

$$y = l \cos\phi$$

$$U = -mgy = -mgl\cos\phi$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{x}_0 + l \cos\phi \, \dot{\phi} \qquad , \qquad x_0 = a\cos \gamma t$$

$$= -a \sin(\gamma t)\, \gamma + l \cos\phi \, \dot{\phi}$$

$$\rightarrow \dot{x}^2 = a^2 \gamma^2 \sin^2 \gamma t + l^2 \cos^2\phi \, \dot{\phi}^2$$

$$- 2 a l \gamma \dot{\phi} \sin(\gamma t) \cos\phi$$

$$\dot{y} = -l \sin\phi \, \dot{\phi}$$

$$\rightarrow \dot{y}^2 = l^2 \sin^2\phi \, \dot{\phi}^2$$

Thus, $T = \frac{1}{2} m \Big(a^2 \gamma^2 \sin^2 \gamma t + l^2 \cos^2\phi \, \dot{\phi}^2$

$$- 2 a l \gamma \dot{\phi} \sin(\gamma t) \cos\phi + l^2 \sin^2\phi \, \dot{\phi}^2 \Big)$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + \frac{1}{2} m a^2 \gamma^2 \sin^2 \gamma t$$

specified function of time (ignore)

$$- m a l \gamma \dot{\phi} \sin(\gamma t) \cos\phi$$

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 - m a l \gamma \dot{\phi} \sin(\gamma t) \cos\phi + m g l \cos\phi$$

2nd term:

$$- \frac{d}{dt} \left[m a l \gamma \sin(\gamma t) \sin\phi \right]$$

$$= - m a l \gamma^2 \cos(\gamma t) \sin\phi - m a l \gamma \dot{\phi} \sin(\gamma t) \cos\phi$$

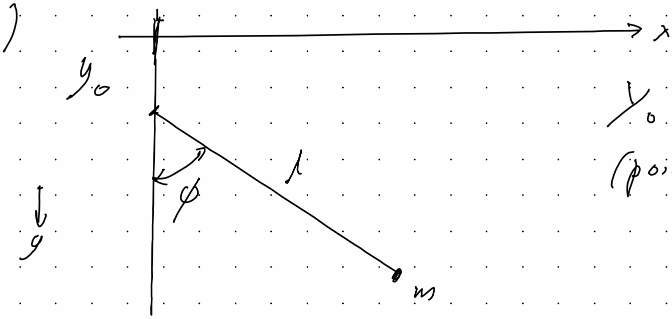
so

$$- m a l \gamma \dot{\phi} \sin(\gamma t) \cos\phi = \underbrace{- \frac{d}{dt} [\;]}_{\text{ignore}} + m a l \gamma^2 \cos(\gamma t) \sin\phi$$

Thus, ignoring total time derivatives,

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi + mal\gamma^2 \cos(\gamma t)\sin\phi$$

(c)



$y_0 = a\cos\gamma t$

(point of support)

$$x = l\sin\phi$$

$$y = y_0 + l\cos\phi$$

$$= a\cos\gamma t + l\cos\phi$$

$$U = -mgy$$

$$= \underbrace{-mga\cos\gamma t}_{} - mgl\cos\phi$$

specified function of time [can ignore]

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = l\cos\phi\,\dot{\phi}$$

$$\dot{x}^2 = l^2\cos^2\phi\,\dot{\phi}^2$$

$$\dot{y} = -a\gamma\sin(\gamma t) - l\sin\phi\,\dot{\phi}$$

$$\dot{y}^2 = \underbrace{a^2\gamma^2\sin^2(\gamma t)}_{} + l^2\sin^2\phi\,\dot{\phi}^2 + 2al\gamma\sin(\gamma t)\sin\phi\,\dot{\phi}$$

specified function of time [can ignore]

Thus, ignoring this function of time

$$T = \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \dot{\phi} \sin(\gamma t) \sin\phi$$

$$\rightarrow L = T - U$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \gamma \dot{\phi} \sin(\gamma t) \sin\phi + m g l \cos\phi$$

Rewrite 2nd term:

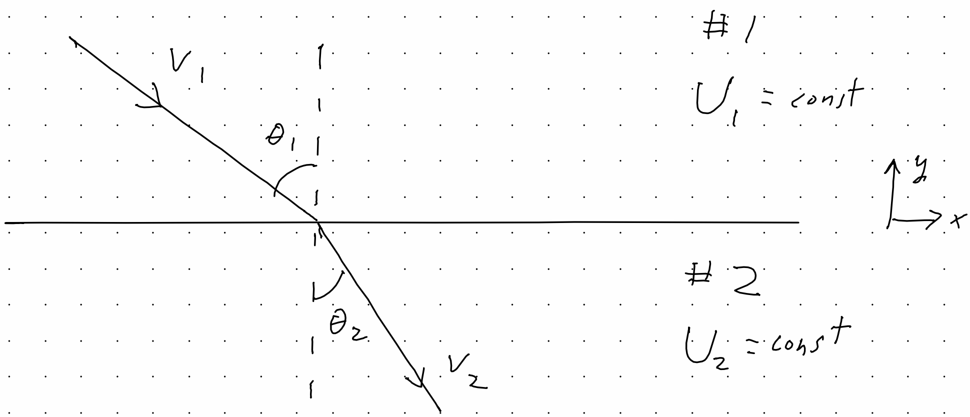
$$-\frac{d}{dt}[m a l \gamma \sin(\gamma t) \cos\phi] = -m a l \gamma^2 \cos(\gamma t) \cos\phi + m a l \gamma \sin(\gamma t) \sin\phi \, \dot{\phi}$$

so $m a l \gamma \dot{\phi} \sin(\gamma t) \sin\phi = -\frac{d}{dt}[\;] + m a l \gamma^2 \cos(\gamma t) \cos\phi$

thus, ignoring total time derivative

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \cos\phi + m a l \gamma^2 \cos(\gamma t) \cos\phi$$

Sec 7, Prob 1



- Energy conserved, since no time dependence.
- Also momentum in x-direction (// to interface) is conserved, since no x-dependence of the potential

$$U(x,y) = \begin{cases} U_1 & y \geq 0 \\ U_2 & y < 0 \end{cases}$$

V_1 : given

$$E = \frac{1}{2} m v_1^2 + U_1 = \frac{1}{2} m v_2^2 + U_2$$

$$\rightarrow \quad \frac{1}{2} m v_2^2 = \frac{1}{2} m v_1^2 + (U_1 - U_2)$$

$$v_2^2 = v_1^2 + \frac{2(U_1 - U_2)}{m}$$

so, $$v_2 = v_1 \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

The angles θ_1, θ_2 are related by

$$p_{1x} = p_{2x}$$

$$\cancel{m} v_1 \sin\theta_1 = \cancel{m} v_2 \sin\theta_2$$

thus, $$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

Sec 8, Prob 1

Transformation of action $S = \int L \, dt$

K, K': two inertial frames

K' moves with velocity $\vec{V}$ wrt K

Assume that K, K' coincide at $t=0$ so $\vec{r}_a = \vec{r}_a{}'$ wrt these two frames

Now: $\vec{v}_a = \vec{V} + \vec{v}_a{}'$

$$L = T - U$$
$$= \sum_a \frac{1}{2} m_a |\vec{v}_a|^2 - U(\vec{r}_1, \vec{r}_2, \ldots, t)$$

$$|\vec{v}_a|^2 = |\vec{V} + \vec{v}_a{}'|^2$$
$$= |\vec{V}|^2 + |\vec{v}_a{}'|^2 + 2\vec{V} \cdot \vec{v}_a{}'$$

So

$$L = \sum_a \frac{1}{2} m_a \left(|\vec{V}|^2 + |\vec{v}_a{}'|^2 + 2\vec{V} \cdot \vec{v}_a{}'\right) - U$$
$$= \frac{1}{2}\mu V^2 + T' + \vec{V} \cdot \sum_a m_a \vec{v}_a{}' - U$$

$$= T' - U + \frac{1}{2}\mu V^2 + \vec{P}' \cdot \vec{V}$$
$$= L' + \frac{1}{2}\mu V^2 + \vec{P}' \cdot \vec{V}$$

where $\vec{P}'$ = total momentum wrt K'

$\mu = \sum_a m_a$ = total mass

$$S = \int_{t_1}^{t_2} L\,dt$$

$$= \int_{t_1}^{t_2} \left(L' + \tfrac{1}{2}\mu V^2 + \vec{p}\,' \cdot \vec{V} \right) dt$$

$$= S' + \underbrace{\tfrac{1}{2}\mu V^2 (t_2 - t_1)}_{\text{doesn't change EOMs}} + \vec{V} \cdot \int_{t_1}^{t_2} \vec{p}\,' dt$$

Now:

$$\vec{V} \cdot \int_{t_1}^{t_2} \vec{p}\,' dt = \vec{V} \cdot \int_{t_1}^{t_2} \sum_a m_a \vec{v}_a\,' dt$$

$$= \vec{V} \cdot \sum_a m_a \int_{t_1}^{t_2} \left(\frac{d\vec{r}_a}{dt} \right) dt$$

$$= \vec{V} \cdot \sum_a m_a \vec{r}_a \Big|_{t_1}^{t_2}$$

$$= \vec{V} \cdot \left(\mu \vec{R}(t_2) - \mu \vec{R}(t_1) \right)$$

$$= \mu \vec{V} \cdot \left(\underbrace{\vec{R}(t_2) - \vec{R}(t_1)}_{\text{difference in COM positions}} \right)$$

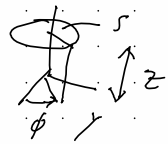
So:

$$S = S' + \tfrac{1}{2}\mu V^2 (t_2 - t_1) + \mu \vec{V} \cdot \left(\vec{R}(t_2) - \vec{R}(t_1) \right)$$

Sec 9, Prob 1

cylindrical coordinates (s, ϕ, z)

$s^2 = x^2 + y^2$



$x = s\cos\phi$

$y = s\sin\phi$

$z = z$

$\vec{M} = \vec{r} \times \vec{p} = m\vec{r} \times \dot{\vec{r}}$

Now: $M_x = m(y\dot{z} - z\dot{y})$

$M_y = m(z\dot{x} - x\dot{z})$

$M_z = m(x\dot{y} - y\dot{x})$

$M = \sqrt{M_x^2 + M_y^2 + M_z^2}$

$\dot{z} = \dot{z}$

$\dot{y} = \dot{s}\sin\phi + s\cos\phi\,\dot{\phi}$

$\dot{x} = \dot{s}\cos\phi - s\sin\phi\,\dot{\phi}$

Thus, $M_x = m\left(s\sin\phi\,\dot{z} - z(\dot{s}\sin\phi + s\cos\phi\,\dot{\phi})\right)$

$= m(s\sin\phi\,\dot{z} - z\sin\phi\,\dot{s} - zs\cos\phi\,\dot{\phi})$

$M_y = m\left(z(\dot{s}\cos\phi - s\sin\phi\,\dot{\phi}) - s\cos\phi\,\dot{z}\right)$

$= m(z\cos\phi\,\dot{s} - zs\sin\phi\,\dot{\phi} - s\cos\phi\,\dot{z})$

$M_z = m[\,s\cos\phi\,(\cancel{\dot{s}\sin\phi} + s\cos\phi\,\dot{\phi})$
$- s\sin\phi\,(\cancel{\dot{s}\cos\phi} - s\sin\phi\,\dot{\phi})\,]$

$= ms^2\dot{\phi}$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$= m^2 \Big\{ \big(\sin\phi\,(s\dot z - z\dot s) - zs\cos\phi\,\dot\phi\big)^2$$

$$+ \big(-\cos\phi\,(s\dot z - z\dot s) - zs\sin\phi\,\dot\phi\big)^2$$

$$+ \big(s^2\dot\phi\big)^2 \Big\}$$

$$= m^2 \Big\{ \sin^2\phi\,(s\dot z - z\dot s)^2 + z^2 s^2 \cos^2\phi\,\dot\phi^2$$

$$\cancel{- 2zs\sin\phi\cos\phi\,\dot\phi\,(s\dot z - z\dot s)}$$

$$+ \cos^2\phi\,(s\dot z - z\dot s)^2 + z^2 s^2 \sin^2\phi\,\dot\phi^2$$

$$\cancel{+ 2zs\sin\phi\cos\phi\,\dot\phi\,(s\dot z - z\dot s)}$$

$$+ s^4\dot\phi^2 \Big\}$$

$$= m^2 \Big\{ (s\dot z - z\dot s)^2 + z^2 s^2 \dot\phi^2 + s^4\dot\phi^2 \Big\}$$

$$= m^2 \Big[(s\dot z - z\dot s)^2 + s^2\dot\phi^2 (z^2 + s^2) \Big]$$

Sec 9, Prob 2

repeat for spherical polar coords.

$M_x = m(y\dot{z} - z\dot{y})$, cyclic

$M^2 = M_x^2 + M_y^2 + M_z^2$

Now: $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$\rightarrow \dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi}$

$\dot{y} = \dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi}$

$\dot{z} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$

Thus,

$M_x = m(y\dot{z} - z\dot{y})$

$= m\{ r\sin\theta\sin\phi(\cancel{\dot{r}\cos\theta} - r\sin\theta\,\dot{\theta})$

$- r\cos\theta(\cancel{\dot{r}\sin\theta\sin\phi} + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi})\}$

$= m\{ -r^2\sin^2\theta\sin\phi\,\dot{\theta} - r^2\cos^2\theta\sin\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\cos\phi\,\dot{\phi}\}$

$= m\{ -r^2\sin\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\cos\phi\,\dot{\phi}\}$

$= -mr^2[\sin\phi\,\dot{\theta} + \sin\theta\cos\theta\cos\phi\,\dot{\phi}]$

$$M_y = m(z\dot{x} - x\dot{z})$$

$$= m\{ r\cos\theta\,(\cancel{\dot{r}\sin\theta\cos\phi} + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi})$$

$$- r\sin\theta\cos\phi\,(\cancel{\dot{r}\cos\theta} - r\sin\theta\,\dot{\theta})$$

$$= m\{ r^2\cos^2\theta\cos\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\sin\phi\,\dot{\phi}$$

$$+ r^2\sin^2\theta\cos\phi\,\dot{\theta}\,\}$$

$$= m[\, r^2\cos\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\sin\phi\,\dot{\phi}\,]$$

$$= mr^2[\,\cos\phi\,\dot{\theta} - \sin\theta\cos\theta\sin\phi\,\dot{\phi}\,]$$

$$M_z = m(x\dot{y} - y\dot{x})$$

$$= m\{ r\sin\theta\cos\phi\,(\cancel{\dot{r}\sin\theta\sin\phi} + \cancel{r\cos\theta\sin\phi\,\dot{\theta}} + r\sin\theta\cos\phi\,\dot{\phi})$$

$$- r\sin\theta\sin\phi\,(\cancel{\dot{r}\sin\theta\cos\phi} + \cancel{r\cos\theta\cos\phi\,\dot{\theta}} - r\sin\theta\sin\phi\,\dot{\phi})\,\}$$

$$= m[\, r^2\sin^2\theta\cos^2\phi\,\dot{\phi} + r^2\sin^2\theta\sin^2\phi\,\dot{\phi}\,]$$

$$= mr^2\sin^2\theta\,\dot{\phi}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$= m^2r^4[\,\sin\phi\,\dot{\theta} + \sin\theta\cos\theta\cos\phi\,\dot{\phi}\,]^2$$

$$+ m^2r^4[\,\cos\phi\,\dot{\theta} - \sin\theta\cos\theta\sin\phi\,\dot{\phi}\,]^2$$

$$+ m^2r^4\sin^4\theta\,\dot{\phi}^2$$

cross terms will cancel

$$M^2 = m^2 r^4 \Big\{ \sin^2\phi\, \dot{\theta}^2 + \sin^2\theta \cos^2\theta \cos^2\phi\, \dot{\phi}^2$$

$$+ \cos^2\phi\, \dot{\theta}^2 + \sin^2\theta \cos^2\theta \sin^2\phi\, \dot{\phi}^2$$

$$+ \sin^4\theta\, \dot{\phi}^2 \Big\}$$

$$= m^2 r^4 \left[\dot{\theta}^2 + \sin^2\theta \cos^2\theta\, \dot{\phi}^2 + \sin^4\theta\, \dot{\phi}^2 \right]$$

$$= m^2 r^4 \left[\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2 (\cos^2\theta + \sin^2\theta) \right]$$

$$= m^2 r^4 \left[\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2 \right]$$

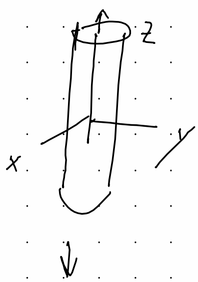
Sec 9, Prob 3

a) Infinite homogeneous plane

P_x, P_y conserved

M_z conserved, where origin is anywhere in (x, y) plane

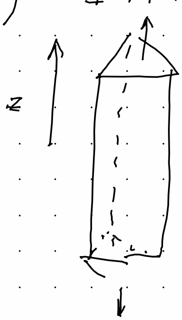
b) Infinite homogeneous cylinder



P_z conserved

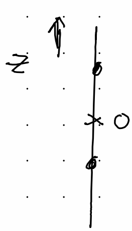
M_z conserved, origin anywhere on z-axis

c) Infinite homog. prism

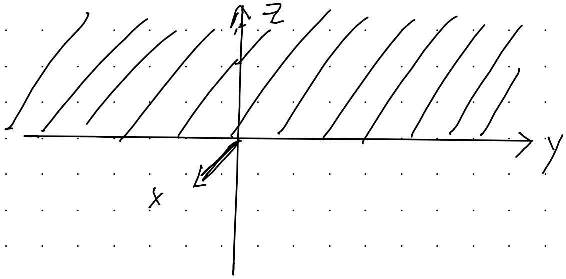


P_z conserved

d) two points : M_z conserved, origin at midpoint of line connecting the two points



e)



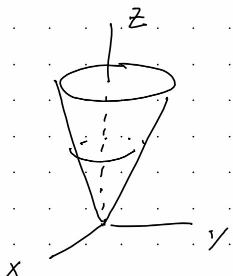
infinite, homog. half plane

$x = 0, \; z > 0$

$-\infty < y < \infty$

p_y conserved

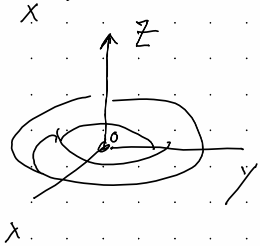
f)



homogeneous cone

M_z conserved, origin anywhere on z-axis.

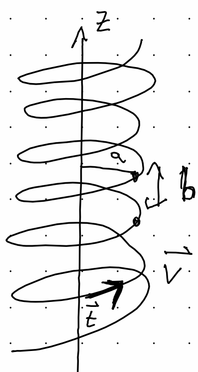
g)



homogeneous circular torus

M_z conserved with $\hat{z}$ directed as shown and origin at O.

h)



Field constant along helix:

$\Delta z = b$ when $\Delta\phi = 2\pi$

at $s^2 = x^2 + y^2 = a^2$

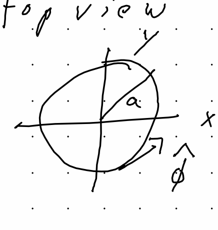
$$\frac{\Delta\phi}{2\pi} = \frac{\Delta z}{b}$$

$$\rightarrow \quad \Delta z = \left(\frac{b}{2\pi}\right)\Delta\phi$$

$$\vec{t} = a\Delta\phi\,\hat{\phi} + \Delta z\,\hat{z}$$

$$= \cancel{a}\Delta\phi\,\underbrace{\left(x\hat{y} - y\hat{x}\right)}_{\cancel{s}} + \Delta z\,\hat{z}$$

top view



$$\vec{t} = \Delta\phi\,(x\hat{y} - y\hat{x}) + \left(\frac{b}{2\pi}\right)\Delta\phi\,\hat{z}$$

$$= \Delta\phi\left[x\hat{y} - y\hat{x} + \left(\frac{b}{2\pi}\right)\hat{z}\right]$$

Field unchanged if you move along $\vec{t}$.

Thus, $\vec{P}\cdot\vec{t} = \text{const}$

$$\vec{P}\cdot\vec{t} = \Delta\phi\left[xP_y - yP_x + \left(\frac{b}{2\pi}\right)P_z\right]$$

$$= \Delta\phi\left[M_z + \frac{b}{2\pi}P_z\right]$$

so $M_z + \frac{b}{2\pi}P_z = \text{const}$

where z = axis of helix

$b = \Delta z$ for $\Delta\phi = 2\pi$ at $s = a$

Sec 10, prob 1

Different masses, same path, same potential energy

$L_1 = \frac{1}{2} m_1 v_1^2 - U$

$L_2 = \frac{1}{2} m_2 v_2^2 - U$

Thus, $m_1 v_1^2 = m_2 v_2^2$

$$\frac{m_1}{t_1^2} = \frac{m_2}{t_2^2}$$

$$\longrightarrow \left(\frac{t_2}{t_1}\right)^2 = \frac{m_2}{m_1}$$

or $\boxed{\frac{t_2}{t_1} = \sqrt{\frac{m_2}{m_1}}}$

Sec 10, Prob 2:

Same path, mass but potential energies differing by a constant

$$L_1 = \frac{1}{2} m v_1^2 - U_1$$

$$L_2 = \frac{1}{2} m v_2^2 - U_2$$

Thus, $$\frac{v_1^2}{v_2^2} = \frac{U_1}{U_2}$$

$$\rightarrow \frac{(l/t_1)^2}{(l/t_2)^2} = \frac{U_1}{U_2}$$

so $$\frac{t_2}{t_1} = \sqrt{\frac{U_1}{U_2}}$$