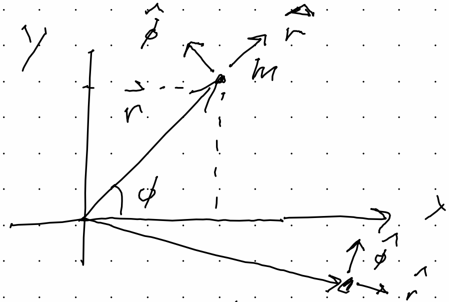
PHYS 5306



2-d motion:

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$$\vec r = x\,\hat x + y\,\hat y$$

$$\vec r = \vec r(t)$$

$$\vec r = r\,\hat r$$

$$\vec v = \dot{\vec r} = \dot x\,\hat x + \dot y\,\hat y$$

$$= \dot r\,\hat r + r\,\dot{\hat r}$$

$$\hat r = \cos\phi\,\hat x + \sin\phi\,\hat y$$

$$\hat\phi = \underline{\quad}\,\hat x \;\underline{\quad}\,\hat y$$

$$\dot{\hat r}$$

$$\vec a = \ddot{\vec r} = \bigcirc\,\hat r + \bigcirc\,\hat\phi$$

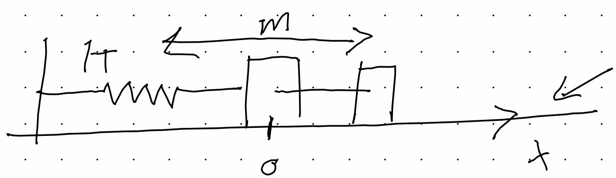
- Kinetic energy for a single mass m in Cartesian, sph. polar, and polar coordinates.

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

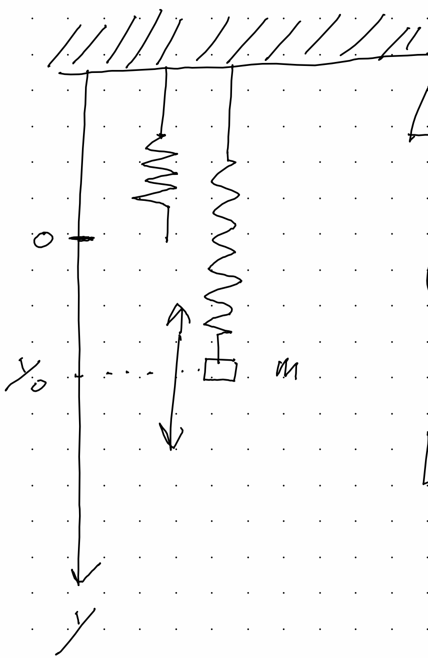
$$= \frac{1}{2} m \left(\qquad \right)$$ (r, θ, ϕ) sph. polar

$$= \frac{1}{2} m \left(\qquad \right)$$ (ρ, ϕ, z) cylindrical

$x = r \sin\theta \cos\phi$, etc



$$U = \frac{1}{2} k x^2$$



$$L = T - U$$

$$T = \frac{1}{2} m \dot{y}^2$$

$$U = \underbrace{\frac{1}{2} k y^2}_{U_s} \underbrace{- m g y}_{+\ U_g}$$

$$E = T + U$$

$$= \frac{1}{2} m \dot{y}^2 + \underbrace{\frac{1}{2} k y^2 - m g y}_{U_{eff}(y)}$$

plot this

$$\dot{x}^2 = \left(\dot{X} - l\dot{\phi}\cos\phi\right)^2$$

$$= \dot{X}^2 + l^2\dot{\phi}^2\cos^2\phi - 2l\dot{X}\dot{\phi}\cos\phi$$

$$\dot{y}^2 = l^2\dot{\phi}^2\sin^2\phi$$

$$\rightarrow \dot{x}^2 + \dot{y}^2 = \dot{X}^2 + l^2\dot{\phi}^2 - 2l\dot{X}\dot{\phi}\cos\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\dot{X}^2 + \frac{1}{2}ml^2\dot{\phi}^2 - ml\dot{X}\dot{\phi}\cos\phi$$

$$U = -mgl\cos\phi$$

$$\boxed{L = \frac{1}{2}m\dot{X}^2 + \frac{1}{2}ml^2\dot{\phi}^2 - ml\dot{X}\dot{\phi}\cos\phi + mgl\cos\phi}$$

$$L = L(\phi, \dot{\phi}, t) \longrightarrow L' = L + \frac{d}{dt}f(\phi, t)$$

$$L = \cancel{\tfrac{1}{2} m \dot{X}^2} + \tfrac{1}{2} m l^2 \dot{\phi}^2 - m l \dot{X} \dot{\phi} \cos\phi + m g l \cos\phi$$

prescribed function of time $\longrightarrow$ ignore

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$$\dot{X} \dot{\phi} \cos\phi = \underbrace{\frac{d}{dt}\left(\dot{X} \sin\phi\right)}_{\ddot{X} \sin\phi + \dot{X} \dot{\phi} \cos\phi} - \ddot{X} \sin\phi$$

$$-m l \dot{X} \dot{\phi} \cos\phi = \underbrace{\frac{d}{dt}\left(-m l \dot{X} \dot{\phi} \cos\phi\right)} + m l \ddot{X} \sin\phi$$

ignore this in Lagrangian

$$\boxed{L = \tfrac{1}{2} m l^2 \dot{\phi}^2 + m l a \sin\phi + m g l \cos\phi}$$

i) show that both Lagrangians give the same EOMs

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi} \iff \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot\phi}\right) = \frac{\partial L'}{\partial \phi}$$

ii) $L' = \frac{1}{2} m l^2 \dot\phi^2 + mal \sin\phi + mgl\cos\phi$

does not depend explicitly on time

$$\rightarrow \quad E = h = H = \underbrace{\dot\phi \frac{\partial L}{\partial \dot\phi}}_{} - L = \text{const}$$

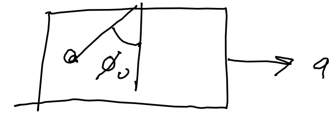
$$= \sum_i \dot q_i \underbrace{\frac{\partial L}{\partial \dot q_i}}_{p_i} - L \quad \text{(more generally)}$$

p_i = momentum conjugate to q_i

in general $\rightarrow$ $\neq T + U \leftarrow$ total mechanical energy

iii) Equil solution: $\dot\phi = 0$

$$\tan\phi_0 = \frac{a}{g}$$



Use EOMs from Lagrangian to show this

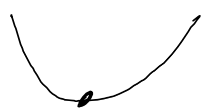
$$\boxed{E = \tfrac{1}{2} m l^2 \dot\phi^2 - m l a \sin\phi - m g l \cos\phi}$$

$$= \dot\phi \frac{\partial L}{\partial \dot\phi} - L$$

$$E = \tfrac{1}{2} m l^2 \dot\phi^2 + U_{eff}(\phi)$$

$$U_{eff}(\phi) = -ml(a\sin\phi + g\cos\phi)$$

☆ graph it



$$0 = \left.\frac{dU_{eff}}{d\phi}\right|_{\phi=\phi_0}$$

$$U_{eff}(\phi) = U_{eff}(\phi_0) + \cancelto{0}{\left.\frac{dU_{eff}}{d\phi}\right|_{\phi_0}}(\phi-\phi_0) + \frac{1}{2}\left.\frac{d^2U_{eff}}{d\phi}\right|_{\phi_0}(\phi-\phi_0)^2 + \cdots$$

$\varkappa$

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \cdots$$

$$E = T + U_{eff}(\phi)$$

$$= T + U_{eff}(\phi_0) + \frac{1}{2}\varkappa(\phi-\phi_0)^2$$

$$= \frac{1}{2}ml^2\underline{\dot{\phi}}^2 + U_{eff}(\phi_0) + \frac{1}{2}\varkappa\,\underline{(\phi-\phi_0)}^2$$

const → ignore

$x = \phi - \phi_0$ $\qquad |x| \ll 1$

$\dot{x} = \dot{\phi}$

$$E = \frac{1}{2}\underbrace{ml^2}_{M}\dot{x}^2 + \frac{1}{2}\varkappa x^2, \quad \omega = \sqrt{\frac{\varkappa}{M}} = \sqrt{\frac{\varkappa}{ml^2}}$$

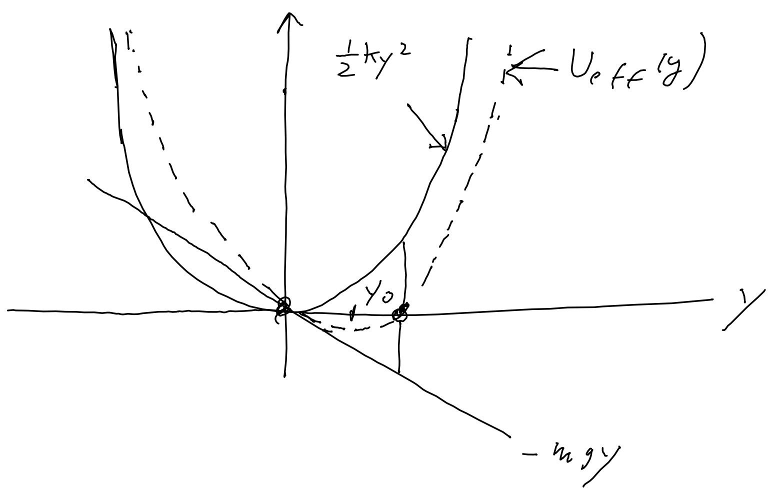
Answers to problems posed at the end of the last class

(1) $\vec{a} = (\underbrace{\ddot{r} - r\dot{\phi}^2}_{\text{centripetal acceleration}})\hat{r} + (2\dot{r}\dot{\phi} + \underbrace{r\ddot{\phi}}_{\text{tangential acceleration}})\hat{\phi}$

(2) $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$

$= \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$

(3) plot $U_{eff}(y) = \frac{1}{2}ky^2 - mgy$



$U_{eff}(y) = U_{eff}(y_0) + \frac{1}{2}\kappa(y-y_0)^2$

where y_0 is the solution to

$0 = \left.\frac{dU_{eff}}{dy}\right|_{y_0}$

and κ is given by

$\kappa \equiv \left.\frac{d^2U_{eff}}{dy^2}\right|_{y_0}$

$$0 = \left.\frac{dU_{eff}}{dy}\right|_{y_0} = k y_0 - mg \rightarrow y_0 = \frac{mg}{k}$$

$$\mathcal{K} = \left.\frac{d^2 U_{eff}}{dy^2}\right|_{y_0} = k$$ (so $\mathcal{K}$ is the same as k for this problem)

$$U_{eff}(y_0) = \frac{1}{2} k y_0^2 - mg y_0 = \frac{1}{2} k \left(\frac{mg}{k}\right)^2 - mg\left(\frac{mg}{k}\right) = -\frac{1}{2}\frac{m^2 g^2}{k}$$

$$\rightarrow \boxed{U_{eff}(y) = -\frac{1}{2}\frac{m^2 g^2}{k} + \frac{1}{2} k \left(y - \frac{mg}{k}\right)^2}$$

You can also obtain the same expression for $U_{eff}(y)$ by completing the square:

$$\begin{aligned} U_{eff}(y) &= \frac{1}{2} k y^2 - mgy \\ &= \frac{1}{2} k \left(y^2 - \frac{2mg}{k} y\right) \\ &= \frac{1}{2} k \left[\left(y - \frac{mg}{k}\right)^2 - \frac{m^2 g^2}{k^2}\right] \\ &= \frac{1}{2} k \left(y - \frac{mg}{k}\right)^2 - \frac{1}{2}\frac{m^2 g^2}{k} \end{aligned}$$