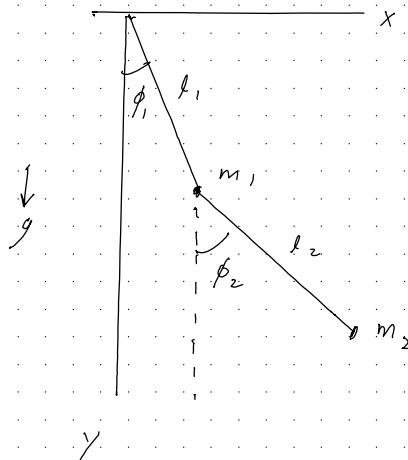


Soc 5, prob 1



$$x_1 = l_1 \sin \phi_1$$

$$y_1 = l_1 \cos \phi_1$$

$$x_2 = x_1 + l_2 \sin \phi_2 = l_1 \sin \phi_1 + l_2 \sin \phi_2$$

$$y_2 = y_1 + l_2 \cos \phi_2 = l_1 \cos \phi_1 + l_2 \cos \phi_2$$

$$U = -m_1 g y_1 - m_2 g y_2$$

$$= -m_1 g l_1 \cos \phi_1 - m_2 g (l_1 \cos \phi_1 + l_2 \cos \phi_2)$$

$$= -(m_1 + m_2) g l_1 \cos \phi_1 - m_2 g l_2 \cos \phi_2$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\dot{x}_1 = l_1 \cos \phi_1 \dot{\phi}_1 \quad \dot{y}_1 = -l_1 \sin \phi_1 \dot{\phi}_1$$

$$\dot{x}_1^2 = l_1^2 \cos^2 \phi_1 \dot{\phi}_1^2 \quad \dot{y}_1^2 = l_1^2 \sin^2 \phi_1 \dot{\phi}_1^2$$

$$\text{Thus, } \dot{x}_1^2 + \dot{y}_1^2 = l_1^2 (\sin^2 \phi_1 + \cos^2 \phi_1) \dot{\phi}_1^2$$

$$= l_1^2 \dot{\phi}_1^2$$

$$\dot{x}_2 = l_1 \cos \phi_1 \dot{\phi}_1 + l_2 \cos \phi_2 \dot{\phi}_2$$

$$\rightarrow \dot{x}_2^2 = l_1^2 \cos^2 \phi_1 \dot{\phi}_1^2 + l_2^2 \cos^2 \phi_2 \dot{\phi}_2^2 + 2l_1 l_2 \cos \phi_1 \cos \phi_2 \dot{\phi}_1 \dot{\phi}_2$$

$$\dot{y}_2 = -l_1 \sin \phi_1 \dot{\phi}_1 - l_2 \sin \phi_2 \dot{\phi}_2$$

$$\rightarrow \dot{y}_2^2 = l_1^2 \sin^2 \phi_1 \dot{\phi}_1^2 + l_2^2 \sin^2 \phi_2 \dot{\phi}_2^2 + 2l_1 l_2 \sin \phi_1 \sin \phi_2 \dot{\phi}_1 \dot{\phi}_2$$

From,

$$\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 (\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

$$= l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

$$\text{so, } T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2$$

$$+ m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

$$= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

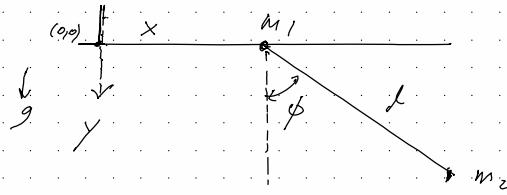
$$U = -(m_1 + m_2) g l_1 \cos \phi_1 - m_2 g l_2 \cos \phi_2$$

$$L = T - U$$

$$= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

$$+ (m_1 + m_2) g l_1 \cos \phi_1 + m_2 g l_2 \cos \phi_2$$

### Sec 5 Prob. 2



Generalised coords:  $x, \phi$

$$(x_1, y_1) = (x, 0)$$

$$(x_2, y_2) = (x + l \sin \phi, l \cos \phi)$$

$$U = -m_1 g y_1 - m_2 g y_2$$

$$= -m_2 g l \cos \phi$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\text{Now: } \dot{x}_1^2 + \dot{y}_1^2 = \dot{x}^2$$

$$\begin{aligned}\dot{x}_2^2 + \dot{y}_2^2 &= (\dot{x} + l \cos \phi \dot{\phi})^2 + (-l \sin \phi \dot{\phi})^2 \\ &= \dot{x}^2 + l^2 \cos^2 \phi \dot{\phi}^2 + 2 l \cos \phi \dot{x} \dot{\phi}\end{aligned}$$

$$= \dot{x}^2 + l^2 \dot{\phi}^2 + 2 l \cos \phi \dot{x} \dot{\phi}$$

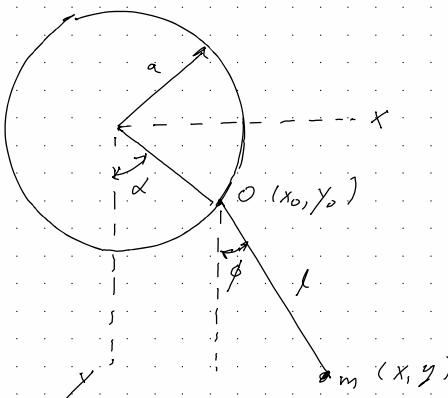
$$\rightarrow T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}^2 + l^2 \dot{\phi}^2 + 2 l \cos \phi \dot{x} \dot{\phi})$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \cos \phi \dot{x} \dot{\phi}$$

$$\begin{aligned}L &= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \cos \phi \dot{x} \dot{\phi} \\ &\quad + m_2 g l \cos \phi\end{aligned}$$

### Sec 5, Prob 3

(a)



point of support O moves along circle.

$$x_0 = a \sin \alpha \quad , \quad y_0 = a \cos \alpha$$

where  $\alpha = \omega t$ ,  $\omega = \text{const}$

Pendulum bob:

$$\begin{aligned}(x, y) : \quad x &= x_0 + l \sin \phi \\ y &= y_0 + l \cos \phi\end{aligned}$$

$$U = -m g y = -m g y_0 - m g l \cos \phi$$

Specified function of time.

[can ignore in L]

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{x}_0 + l \cos \phi \dot{\phi}$$

$$\dot{x}^2 = \dot{x}_0^2 + l^2 \cos^2 \phi \dot{\phi}^2 + 2 l \cos \phi \dot{x}_0 \dot{\phi}$$

$$\dot{y} = \dot{x}_0 - l \sin \phi$$

$$\dot{y}^2 = \dot{x}_0^2 + l^2 \sin^2 \phi \dot{\phi}^2 - 2l \sin \phi \dot{x}_0 \dot{\phi}$$

thus,

$$T = \frac{1}{2} m(\dot{x}_0^2 + \dot{y}^2)$$

$$= \frac{1}{2} m (\dot{x}_0^2 + l^2 \cos^2 \phi \dot{\phi}^2 + 2l \cos \phi \dot{x}_0 \dot{\phi})$$

$$+ \dot{y}^2 + l^2 \sin^2 \phi \dot{\phi}^2 - 2l \sin \phi \dot{x}_0 \dot{\phi})$$

$$= \frac{1}{2} m(\dot{x}_0^2 + \dot{y}^2) + \frac{1}{2} m l^2 \dot{\phi}^2 + m l \dot{\phi} (\dot{x}_0 \cos \phi - \dot{y}_0 \sin \phi)$$

$$\text{NOTE: } \dot{x}_0^2 + \dot{y}^2 = a^2 \dot{x}^2 = a^2 \gamma^2$$

since this is a specified function of time, we can ignore it in the Lagrangian;

$$\text{thus, } L = \frac{1}{2} m l^2 \dot{\phi}^2 + m l \dot{\phi} (\dot{x}_0 \cos \phi - \dot{y}_0 \sin \phi) + m g l \cos \phi$$

We can rewrite the second term:

$$x_0 = a \sin \alpha \rightarrow \dot{x}_0 = a \cos \alpha \dot{\alpha} \quad (\alpha = \gamma)$$

$$y_0 = a \cos \alpha \rightarrow \dot{y}_0 = -a \sin \alpha \dot{\alpha}$$

thus,

$$m l \dot{\phi} (\dot{x}_0 \cos \phi - \dot{y}_0 \sin \phi) = m l \dot{\phi} a \gamma (\cos \alpha \cos \phi + \sin \alpha \sin \phi) \\ = m l \dot{\phi} a \gamma \cos(\phi - \alpha) \\ = m l \dot{\phi} a \gamma \cos(\phi - \gamma t)$$

$$\text{Now: } \frac{d}{dt} [m l a \gamma \sin(\phi - \gamma t)]$$

$$= m l a \gamma \cos(\phi - \gamma t) (\dot{\phi} - \gamma)$$

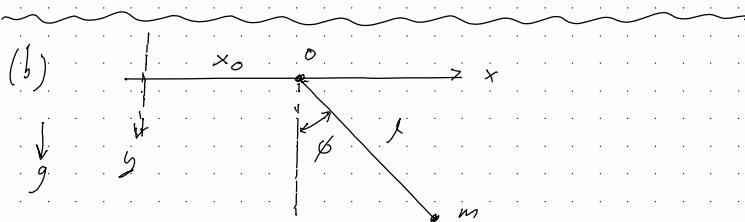
$$= m l a \dot{\phi} \gamma \cos(\phi - \gamma t) - m l a \gamma^2 \cos(\phi - \gamma t)$$

thus,

$$m l a \dot{\phi} \gamma \cos(\phi - \gamma t) = \frac{d}{dt} [m l a \gamma \sin(\phi - \gamma t)] + m l a \gamma^2 \cos(\phi - \gamma t)$$

(and we can ignore the total time derivative in the Lagrangian)

$$\rightarrow L = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \cos \phi + m l a \gamma^2 \cos(\phi - \gamma t)$$



point O moving according to  $x_0 = a \cos \gamma t$

$$x = x_0 + a \sin \gamma t$$

$$y = a \cos \gamma t$$

$$v = -a \gamma \sin \gamma t = -a \gamma l \cos \phi$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{x}_0 + l \cos \phi \dot{\phi}, \quad x_0 = a \cos \gamma t$$

$$= -a \sin(\gamma t) \dot{\gamma} + l \cos \phi \dot{\phi}$$

$$\rightarrow \dot{x}^2 = a^2 \dot{\gamma}^2 \sin^2 \gamma t + l^2 \cos^2 \phi \dot{\phi}^2$$

$$- 2al\dot{\gamma} \dot{\phi} \sin(\gamma t) \cos \phi$$

$$\dot{y} = -l \sin \phi \dot{\phi}$$

$$\rightarrow \dot{y}^2 = l^2 \sin^2 \phi \dot{\phi}^2$$

Thus,  $T = \frac{1}{2} m (a^2 \dot{\gamma}^2 \sin^2 \gamma t + l^2 \cos^2 \phi \dot{\phi}^2)$

$$- 2al\dot{\gamma} \dot{\phi} \sin(\gamma t) \cos \phi + l^2 \sin^2 \phi \dot{\phi}^2)$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + \frac{1}{2} m a^2 \dot{\gamma}^2 \sin^2 \gamma t$$

specified  
function  
of  
time  
(ignore)

$$- mal\dot{\gamma} \dot{\phi} \sin(\gamma t) \cos \phi$$

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 - mal\dot{\gamma} \dot{\phi} \sin(\gamma t) \cos \phi + mgl \cos \phi$$

2<sup>nd</sup> term:

$$- \frac{d}{dt} [ mal\dot{\gamma} \sin(\gamma t) \sin \phi ]$$

$$= - mal\dot{\gamma}^2 \cos(\gamma t) \sin \phi - mal\dot{\gamma} \dot{\phi} \sin(\gamma t) \cos \phi$$

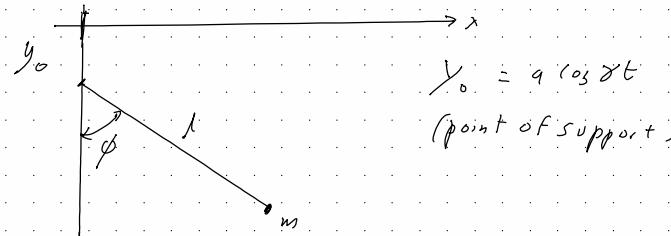
$$- mal\dot{\gamma} \dot{\phi} \sin(\gamma t) \cos \phi = - \frac{d}{dt} [ ] + mal\dot{\gamma}^2 \cos(\gamma t) \sin \phi$$

is no se

Thus, ignoring total time derivatives,

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl \cos \phi + mal\dot{\gamma}^2 \cos(\gamma t) \sin \phi$$

(c)



$$y_0 = a \cos \gamma t$$

(point of support)

$$x = l \sin \phi$$

$$y = y_0 + l \cos \phi$$

$$= a \cos \gamma t + l \cos \phi$$

$$U = -mgy$$

$$= -mg(a \cos \gamma t + l \cos \phi)$$

specified function of time [can ignore]

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = l \cos \phi \dot{\phi}$$

$$\dot{x}^2 = l^2 \cos^2 \phi \dot{\phi}^2$$

$$\dot{y} = -a \dot{\gamma} \sin(\gamma t) - l \sin \phi \dot{\phi}$$

$$\dot{y}^2 = a^2 \dot{\gamma}^2 \sin^2(\gamma t) + l^2 \sin^2 \phi \dot{\phi}^2 + 2al\dot{\gamma} \sin(\gamma t) \sin \phi \dot{\phi}$$

specified function of time [can ignore]

thus, ignoring this function of time

$$T = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \dot{\phi} \sin(\gamma t) \sin \phi$$

$$\rightarrow L = T - U$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \dot{\phi} \sin(\gamma t) \sin \phi + m g l \cos \phi$$

Rewrite 2<sup>nd</sup> term:

$$-\frac{d}{dt} [m a l \dot{\phi} \sin(\gamma t) \cos \phi] = -m a l \gamma^2 \cos(\gamma t) \cos \phi$$

$$+ m a l \dot{\gamma} \sin(\gamma t) \sin \phi \dot{\phi}$$

$$\text{so } m a l \dot{\phi} \sin(\gamma t) \sin \phi = -\frac{d}{dt} [ ] + m a l \gamma^2 \cos(\gamma t) \cos \phi$$

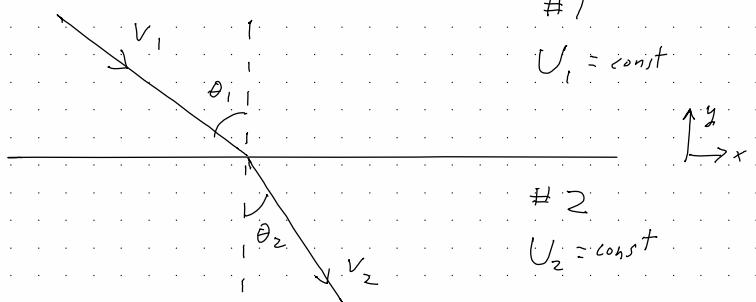
thus, ignoring total time derivative

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \cos \phi + m a l \gamma^2 \cos(\gamma t) \cos \phi$$

Sec 7, Prob 1

#1

$$U_1 = \text{const}$$



#2

$$U_2 = \text{const}$$

- Energy conserved, since no time dependence.
- Also momentum in x-direction ( $\parallel$  to interface) is conserved, since no x-dependence of the potential

$$U(x, y) = \begin{cases} U_1 & y \geq 0 \\ U_2 & y < 0 \end{cases}$$

$V_1$  : given

$$E = \frac{1}{2} m V_1^2 + U_1 = \frac{1}{2} m V_2^2 + U_2$$

$$\rightarrow \frac{1}{2} m V_2^2 = \frac{1}{2} m V_1^2 + (U_1 - U_2)$$

$$V_2^2 = V_1^2 + \frac{2(U_1 - U_2)}{m}$$

$$\text{so, } V_2 = V_1 \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m V_1^2}}$$

The angles  $\theta_1, \theta_2$  are related by

$$p_{1x} = p_{2x}$$

$$\mu v_1 \sin \theta_1 = \mu v_2 \sin \theta_2$$

$$\text{thus, } \frac{\sin \theta_1}{\sin \theta_2} = \frac{v_2}{v_1} = \sqrt{1 + \frac{(v_1 - v_2)^2}{2m v_1^2}}$$

### Sec 8; Prob 1

Transformations of action  $S = \int L dt$

$K, K'$ : two inertial frames

$K'$  moves with velocity  $\vec{V}$  wrt  $K$

Assume that  $K, K'$  coincide at  $t=0$  so  
 $\vec{r}_a = \vec{r}'_a$  wrt these two frames

$$\text{Now: } \vec{v}_a = \vec{V} + \vec{v}'_a$$

$$L = T - U$$

$$= \sum_a \frac{1}{2} m_a |\vec{v}_a|^2 - U(\vec{r}_1, \vec{r}_2, \dots, t)$$

$$|\vec{v}_a|^2 = |\vec{V} + \vec{v}'_a|^2$$

$$= |\vec{V}|^2 + |\vec{v}'_a|^2 + 2 \vec{V} \cdot \vec{v}'_a$$

so

$$L = \sum_a \frac{1}{2} m_a (|\vec{V}|^2 + |\vec{v}'_a|^2 + 2 \vec{V} \cdot \vec{v}'_a) - U$$

$$= \frac{1}{2} \mu V^2 + T' + \vec{V} \cdot \sum_a \vec{v}'_a - U$$

$$= T' - U + \frac{1}{2} \mu V^2 + \vec{P}' \cdot \vec{V}$$

$$= L' + \frac{1}{2} \mu V^2 + \vec{P}' \cdot \vec{V}$$

where  $\vec{P}'$  = total momentum wrt  $K'$

$$\mu = \sum_a m_a \leftarrow \text{total mass}$$

$$\begin{aligned}
 S &= \int_{t_1}^{t_2} \int \rho dt \\
 &= \int_{t_1}^{t_2} (\bar{L}' + \frac{1}{2}\mu V^2 + \bar{\rho}' \cdot \vec{V}) dt \\
 &= S' + \frac{1}{2}\mu V^2(t_2 - t_1) + \vec{V} \cdot \int_{t_1}^{t_2} \bar{\rho}' dt \\
 &\quad \text{does not change EOMs}
 \end{aligned}$$

$$\begin{aligned}
 \vec{V} \cdot \int_{t_1}^{t_2} \bar{\rho}' dt &= \vec{V} \cdot \int_{t_1}^{t_2} \sum_m \vec{v}_a' dt \\
 &= \vec{V} \cdot \sum_a \int_{t_1}^{t_2} \left( \frac{d\vec{r}_a}{dt} \right) dt \\
 &= \vec{V} \cdot \sum_a \vec{r}_a \Big|_{t_1}^{t_2} \\
 &= \vec{V} \cdot \left( \mu \vec{R}(t_2) - \mu \vec{R}(t_1) \right) \\
 &= \mu \vec{V} \cdot \left( \vec{R}(t_2) - \vec{R}(t_1) \right) \\
 &\quad \text{difference in com positions}
 \end{aligned}$$

$$\text{So: } S = S' + \frac{1}{2}\mu V^2(t_2 - t_1) + \mu \vec{V} \cdot \left( \vec{R}(t_2) - \vec{R}(t_1) \right)$$

Sec 9, Prob 1.

cylindrical coordinates,  $(s, \phi, z)$

$$s^2 = x^2 + y^2$$



$$x = s \cos \phi$$

$$y = s \sin \phi$$

$$z = z$$

$$\vec{M} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{r}$$

$$\text{Now, } M_x = m(y \dot{z} - z \dot{y})$$

$$M_y = m(z \dot{x} - x \dot{z})$$

$$M_z = m(x \dot{y} - y \dot{x})$$

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}$$

$$\dot{z} = z$$

$$\dot{y} = s \sin \phi + s \cos \phi \dot{\phi}$$

$$\dot{x} = s \cos \phi - s \sin \phi \dot{\phi}$$

$$\text{Thus, } M_x = m(s \sin \phi \dot{z} - z(s \sin \phi + s \cos \phi \dot{\phi}))$$

$$= m(s \sin \phi \dot{z} - z s \sin \phi - z s \cos \phi \dot{\phi})$$

$$M_y = m(z(s \cos \phi - s \sin \phi \dot{\phi}) - s \cos \phi \dot{z})$$

$$= m(z \cos \phi \dot{s} - z s \sin \phi \dot{\phi} - s \cos \phi \dot{z})$$

$$M_z = m(s \cos \phi (s \sin \phi + s \cos \phi \dot{\phi})$$

$$- s \sin \phi (s \cos \phi - s \sin \phi \dot{\phi})]$$

$$= m s^2 \dot{\phi}$$

$$\bar{M}^2 = M_x^2 + M_y^2 + M_z^2$$

$$= m^2 \left\{ (\sin \phi (sz - z's) - z s \cos \phi \dot{\phi})^2 + (\cos \phi (sz - z's) - z s \sin \phi \dot{\phi})^2 + (s^2 \dot{\phi})^2 \right\}$$

$$= m^2 \left\{ \sin^2 \phi (sz - z's)^2 + z^2 s^2 \cos^2 \phi \dot{\phi}^2 - 2 z s \sin \phi \cos \phi (sz - z's) + \cos^2 \phi (sz - z's)^2 + z^2 s^2 \sin^2 \phi \dot{\phi}^2 + 2 z s \sin \phi \cos \phi (sz - z's) + s^4 \dot{\phi}^2 \right\}$$

$$= m^2 \left\{ (sz - z's)^2 + z^2 s^2 \dot{\phi}^2 + s^4 \dot{\phi}^2 \right\}$$

$$= m^2 [ (sz - z's)^2 + s^2 \dot{\phi}^2 (z^2 + s^2) ]$$

### Sec 9, Prob 2

repeat for spherical polar coords.

$$M_x = m(yz - zy)$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$\text{Now: } x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\rightarrow \dot{x} = r \sin \theta \cos \phi + r \cos \theta \cos \phi \dot{\theta} - r \sin \theta \sin \phi \dot{\phi}$$

$$\dot{y} = r \sin \theta \sin \phi + r \cos \theta \sin \phi \dot{\theta} + r \sin \theta \cos \phi \dot{\phi}$$

$$\dot{z} = r \cos \theta - r \sin \theta \dot{\phi}$$

Then,

$$M_x = m(yz - zy)$$

$$= m \left\{ r \sin \theta \sin \phi (r \cos \theta - r \sin \theta \dot{\phi}) - r \cos \theta (r \sin \theta \sin \phi + r \cos \theta \sin \phi \dot{\theta} + r \sin \theta \cos \phi \dot{\phi}) \right\}$$

$$= m \left\{ -r^2 \sin^2 \theta \sin \phi \dot{\theta} - r^2 \cos^2 \theta \sin \phi \dot{\theta} - r^2 \sin \theta \cos \theta \cos \phi \dot{\phi} \right\}$$

$$= m \left\{ -r^2 \sin \phi \dot{\theta} - r^2 \sin \theta \cos \theta \cos \phi \dot{\phi} \right\}$$

$$= -mr^2 [\sin \phi \dot{\theta} + \sin \theta \cos \theta \cos \phi \dot{\phi}]$$

$$\begin{aligned}
 M_y &= m(z\dot{x} - x\dot{z}) \\
 &= m \{ r\cos\theta (r\sin\theta \cos\phi \dot{\theta} + r\cos\theta \cos\phi \dot{\phi} - r\sin\theta \sin\phi \dot{\phi}) \\
 &\quad - r\sin\theta \cos\phi (r\cos\theta \dot{\theta} - r\sin\theta \dot{\phi}) \} \\
 &= m \{ r^2 \cos^2\theta \cos\phi \dot{\theta} - r^2 \sin\theta \cos\theta \sin\phi \dot{\phi} \\
 &\quad + r^2 \sin^2\theta \cos\phi \dot{\theta} \} \\
 &= m \{ r^2 \cos\phi \dot{\theta} - r^2 \sin\theta \cos\theta \sin\phi \dot{\phi} \} \\
 &= mr^2 [\cos\phi \dot{\theta} - \sin\theta \cos\theta \sin\phi \dot{\phi}]
 \end{aligned}$$

$$\begin{aligned}
 M_z &= m(xy' - yx') \\
 &= m \{ r\sin\theta \cos\phi (r\sin\theta \sin\phi + r\cos\theta \cos\phi \dot{\theta} \\
 &\quad + r\sin\theta \cos\phi \dot{\phi}) \\
 &\quad - r\sin\theta \sin\phi (r\sin\theta \cos\phi + r\cos\theta \cos\phi \dot{\theta} \\
 &\quad - r\sin\theta \sin\phi \dot{\phi}) \} \\
 &= m [r^2 \sin^2\theta \cos^2\phi \dot{\theta} + r^2 \sin^2\theta \sin^2\phi \dot{\phi}] \\
 &= mr^2 \sin^2\theta \dot{\phi}
 \end{aligned}$$

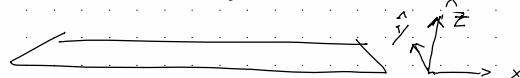
$$\begin{aligned}
 M^2 &= M_x^2 + M_y^2 + M_z^2 \\
 &= m^2 r^4 [\sin^2\phi \dot{\theta}^2 + \sin^2\theta \cos^2\theta \cos^2\phi \dot{\phi}^2] \\
 &\quad + m^2 r^4 [\cos^2\phi \dot{\theta}^2 - \sin^2\theta \cos^2\theta \sin^2\phi \dot{\phi}^2] \\
 &\quad + m^2 r^4 \sin^4\theta \dot{\phi}^2
 \end{aligned}$$

(cross terms will cancel)

$$\begin{aligned}
 M^2 &= m^2 r^4 \{ \sin^2\phi \dot{\theta}^2 + \sin^2\theta \cos^2\theta \cos^2\phi \dot{\phi}^2 \\
 &\quad + \cos^2\phi \dot{\theta}^2 + \sin^2\theta \cos^2\theta \sin^2\phi \dot{\phi}^2 \\
 &\quad + \sin^4\theta \dot{\phi}^2 \} \\
 &= m^2 r^4 [\dot{\theta}^2 + \sin^2\theta \cos^2\theta \dot{\phi}^2 + \sin^4\theta \dot{\phi}^2] \\
 &= m^2 r^4 [\dot{\theta}^2 + \sin^2\theta \phi^2 / (\cos^2\theta + \sin^2\theta)] \\
 &= m^2 r^4 [\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2]
 \end{aligned}$$

Sec 9, Prob 3

a) Infinite homogeneous plane



$P_x, P_y$  conserved

$M_z$  conserved where origin is anywhere in  $(x, y)$  plane

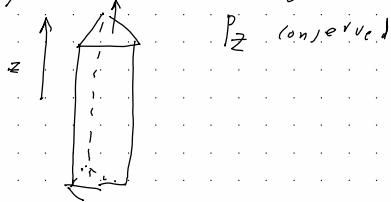
b) Infinite homogeneous cylinder



$P_z$  conserved

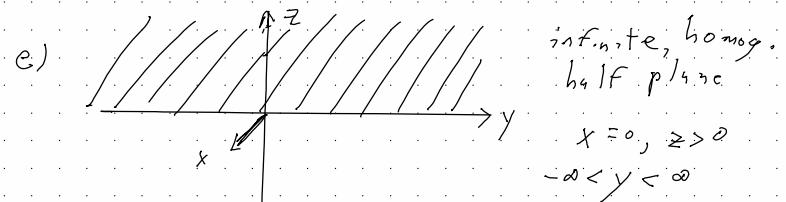
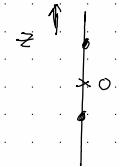
$M_z$  conserved, origin anywhere on  $Z$ -axis

c) Infinite homog prism



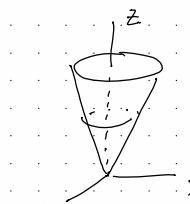
$P_z$  conserved

d) two points :  $M_z$  conserved, origin at midpoint of line connecting the two points



$P_y$  conserved

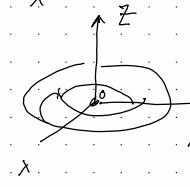
f)



homogeneous cone

$M_z$  conserved, origin anywhere on  $Z$ -axis.

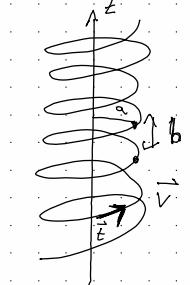
g)



homogeneous circular torus

$M_z$  conserved with  $\hat{z}$  directed as shown and origin at O.

b)

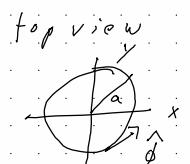


Field constant along helix:

$\Delta Z = b$  when  $\Delta\phi = 2\pi$   
at  $s^2 = x^2 + y^2 = a^2$

$$\frac{\Delta\phi}{2\pi} = \frac{\Delta Z}{b}$$

$$\rightarrow \Delta Z = \left(\frac{b}{2\pi}\right) \Delta\phi$$



$$\vec{F} = a\Delta\phi \hat{x} + \Delta Z \hat{z}$$

$$= \Delta\phi \underbrace{(a\hat{x} - \hat{x})}_{\vec{s}} + \Delta Z \hat{z}$$

$$\vec{E} = \Delta\phi (x\hat{y} - y\hat{x}) + \left(\frac{b}{2\pi}\right) \Delta\phi \hat{z}$$

$$= \Delta\phi \left[ x\hat{y} - y\hat{x} + \left(\frac{b}{2\pi}\right) \hat{z} \right]$$

Field unchanged if you move along  $\vec{t}$   
thus,  $\vec{P} \cdot \vec{t} = \text{const}$

$$\vec{P} \cdot \vec{t} = \Delta\phi [xP_y - yP_x + \left(\frac{b}{2\pi}\right) P_z]$$

$$= \Delta\phi \left[ M_z + \frac{b}{2\pi} P_z \right]$$

$$\text{so } M_z + \frac{b}{2\pi} P_z = \text{const}$$

where  $z = \text{axis of helix}$

$$b = 4z \text{ for } \Delta\phi = 2\pi \text{ at } s = a$$

### Sec 10, prob 1

Different masses, same path, same potential energy

$$L_1 = \frac{1}{2} m_1 v_1^2 - U$$

$$L_2 = \frac{1}{2} m_2 v_2^2 - U$$

$$\text{Thus, } m_1 v_1^2 = m_2 v_2^2$$

$$\frac{m_1}{t_1^2} = \frac{m_2}{t_2^2}$$

$$\rightarrow \left( \frac{t_2}{t_1} \right)^2 = \frac{m_2}{m_1}$$

$$\boxed{\text{or } \frac{t_2}{t_1} = \sqrt{\frac{m_2}{m_1}}}$$

Sec 10, Prob 2:

Same path, mass but potential energies differing by a constant

$$L_1 = \frac{1}{2}mV_1^2 - U_1$$

$$L_2 = \frac{1}{2}mV_2^2 - U_2$$

$$\frac{T_{b1}}{T_{b2}} = \frac{V_1^2}{V_2^2} = \frac{U_1}{U_2}$$

$$\rightarrow \frac{(1/t_1)^2}{(t/t_2)^2} = \frac{U_1}{U_2}$$

$$\text{so } \frac{t_2}{t_1} = \sqrt{\frac{U_1}{U_2}}$$

Sec 40 - Prob 1

single particle in a constant external field

$$L = \frac{1}{2}m\vec{v}^2 - U(\vec{r})$$

a) Cartesian coords  $(x, y, z)$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

$$\rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = p_x/m$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \rightarrow \dot{y} = p_y/m$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$H = \left( \frac{1}{2m} \vec{p}^2 - L \right) / \left. \vec{e} = \vec{e}(e_ip) \right|$$

$$= \left( p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z) \right) / \left. \vec{e} = \vec{e}(e_ip) \right|$$

$$= p_x \left( \frac{p_x}{m} \right) + p_y \left( \frac{p_y}{m} \right) + p_z \left( \frac{p_z}{m} \right) - \frac{1}{2}m \left( \left( \frac{p_x}{m} \right)^2 + \left( \frac{p_y}{m} \right)^2 + \left( \frac{p_z}{m} \right)^2 \right) + U(x, y, z)$$

$$= \frac{1}{2m} (\vec{p}^2) + U(x, y, z)$$

b) cylindrical coords  $(s, \phi, z)$ ,  $s^2 = x^2 + y^2$

$$L = \frac{1}{2}m(s^2\dot{s}^2 + \dot{s}^2\phi^2 + \dot{z}^2) - U(s, \phi, z)$$

$$\rightarrow p_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s} \rightarrow \dot{s} = p_s/m$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m s^2 \dot{\phi} \rightarrow \dot{\phi} = p_\phi / m s^2$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$\begin{aligned}
 H &= \left( p_r \dot{r} + p_\theta \dot{\theta} + p_z \dot{z} - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) + U(r, \theta, z) \right) \\
 &= p_r \left( \frac{p_r}{m} \right) + p_\theta \left( \frac{p_\theta}{mr^2} \right) + p_z \left( \frac{p_z}{m} \right) \\
 &\quad - \frac{1}{2} m \left( \left( \frac{p_r}{m} \right)^2 + r^2 \left( \frac{p_\theta}{mr^2} \right)^2 + \left( \frac{p_z}{m} \right)^2 \right) + U(r, \theta, z) \\
 &= \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2 \right) + U(r, \theta, z)
 \end{aligned}$$

c) spherical polar coords.  $(r, \theta, \phi)$

$$\begin{aligned}
 L &= \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 + \dot{r}^2) - U(r, \theta, \phi) \\
 \rightarrow p_r &= \frac{\partial L}{\partial \dot{r}} = m \dot{r} \rightarrow \dot{r} = p_r/m \\
 p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \rightarrow \dot{\theta} = p_\theta / mr^2 \\
 p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2 \theta \dot{\phi} \rightarrow \dot{\phi} = p_\phi / m r^2 \sin^2 \theta \\
 H &= \left( p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) + U(r, \theta, \phi) \right) \\
 &= p_r \left( \frac{p_r}{m} \right) + p_\theta \left( \frac{p_\theta}{mr^2} \right) + p_\phi \left( \frac{p_\phi}{mr^2 \sin^2 \theta} \right) \\
 &\quad - \frac{1}{2} m \left( \left( \frac{p_r}{m} \right)^2 + r^2 \left( \frac{p_\theta}{mr^2} \right)^2 + r^2 \sin^2 \theta \left( \frac{p_\phi}{mr^2 \sin^2 \theta} \right)^2 \right) + U(r, \theta, \phi) \\
 &= \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + U(r, \theta, \phi)
 \end{aligned}$$

sec 40 - prob 2

$$L = \frac{1}{2} m v^2 + m \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + \frac{1}{2} m |\vec{\Omega} \times \vec{r}|^2 - m \vec{W} \cdot \vec{r} - U$$

restrict to uniformly rotating frame of reference  $\vec{W} = 0$ ,  $\vec{\Omega} = \vec{\omega}$

$$\begin{aligned}
 \rightarrow L &= \frac{1}{2} m v^2 + m \vec{v} \cdot (\vec{\Omega} \times \vec{r}) \\
 &\quad + \frac{1}{2} m |\vec{\Omega} \times \vec{r}|^2 - U(r)
 \end{aligned}$$

$$\text{Now: } H = \vec{p} \cdot \vec{v} - L$$

$$\begin{aligned}
 \vec{p} &= \frac{\partial L}{\partial \vec{v}} = m \vec{v} + m \vec{\Omega} \times \vec{r} = m (\vec{v} + \vec{\Omega} \times \vec{r}) \\
 \rightarrow \vec{v} &= \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r}
 \end{aligned}$$

thus,

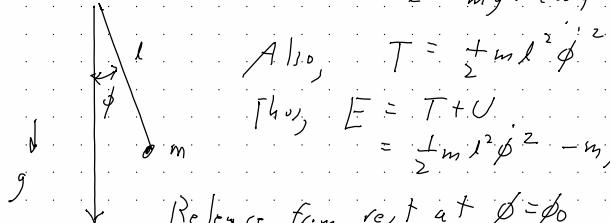
$$\begin{aligned}
 H &= \vec{p} \cdot \left( \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r} \right) - \frac{1}{2} m \left| \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r} \right|^2 \\
 &\quad - m \left( \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r} \right) \cdot \left( \vec{\Omega} \times \vec{r} \right) \\
 &\quad - \frac{1}{2} m |\vec{\Omega} \times \vec{r}|^2 + U(r)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{|\vec{p}|^2}{m} - \vec{p} \cdot (\vec{\Omega} \times \vec{r}) - \frac{1}{2} m \left( \frac{|\vec{p}|^2}{m^2} + |\vec{\Omega} \times \vec{r}|^2 \right) - \cancel{2 \vec{p} \cdot (\vec{\Omega} \times \vec{r})} \\
 &\quad - \vec{p} \cdot (\vec{\Omega} \times \vec{r}) + m |\vec{\Omega} \times \vec{r}|^2 - \frac{1}{2} m |\vec{\Omega} \times \vec{r}|^2 + U(r) \\
 &= \frac{|\vec{p}|^2}{2m} - \vec{p} \cdot (\vec{\Omega} \times \vec{r}) + U(r) \\
 &= \frac{|\vec{p}|^2}{2m} - \vec{\Omega} \cdot (\vec{r} \times \vec{p}) + U(r)
 \end{aligned}$$

$\vec{p}$  is angular momentum

Sec. 11, Prob 1

Simple pendulum:  $U = -mg\gamma$   
 $= -mgl \cos\phi$



Also,  $T = \frac{1}{2}ml^2\dot{\phi}^2$

Thus,  $E = T + U$   
 $= \frac{1}{2}ml^2\dot{\phi}^2 - mgl \cos\phi$

Release from rest at  $\phi = \phi_0$

$$E = -mgl \cos\phi_0$$

$$\rightarrow \frac{1}{2}ml^2\dot{\phi}^2 - mgl \cos\phi = -mgl \cos\phi_0$$

$$T(E) = 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{E - U(\phi)}}$$

$$= 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{-mgl \cos\phi_0 + mgl \cos\phi}}$$

$$= 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$

$\phi < \phi_0$

Now:  $\cos\phi = \cos(2\frac{\phi}{2})$   
 $= \cos^2(\frac{\phi}{2}) - \sin^2(\frac{\phi}{2})$   
 $= 1 - 2\sin^2(\frac{\phi}{2})$

Also,  $\cos\phi_0 = 1 - 2\sin^2(\frac{\phi_0}{2})$

$$T(E) = 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{2 \left( \sin^2(\frac{\phi_0}{2}) - \sin^2(\frac{\phi}{2}) \right)}}$$

$$= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2(\frac{\phi_0}{2}) \left( 1 - \frac{\sin^2(\frac{\phi}{2})}{\sin^2(\frac{\phi_0}{2})} \right)}}$$

Let  $x = \frac{\sin(\frac{\phi}{2})}{\sin(\frac{\phi_0}{2})} \rightarrow dx = \frac{1}{2} \frac{\cos(\frac{\phi}{2})}{\sin(\frac{\phi_0}{2})}$

$$= \frac{d\phi}{2} \frac{\sqrt{1 - \sin^2(\frac{\phi}{2})}}{\sin(\frac{\phi_0}{2})}$$

$$= \frac{d\phi}{2} \frac{\sqrt{1 - \sin^2(\frac{\phi_0}{2})x^2}}{\sin(\frac{\phi_0}{2})}$$

Thus,  
 $T(E) = 2\sqrt{\frac{l}{g}} \int_0^1 \frac{2dx \sin(\frac{\phi_0}{2})}{\sqrt{1 - \sin^2(\frac{\phi_0}{2})x^2} \sin(\frac{\phi_0}{2}) \sqrt{1 - x^2}}$   
 $= 4\sqrt{\frac{l}{g}} \int_0^1 \frac{dx}{\sqrt{1 - x^2} \sqrt{1 - k^2 x^2}} \quad , \quad K = \sin(\frac{\phi_0}{2})$   
 $= 4\int_{\frac{1}{2}}^1 K(k)$

where  $K(k) = \text{complete elliptic integral of the 1st kind.}$

A problem from:

$$T(E) = 4\sqrt{\frac{E}{J}} \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-H^2 x^2}}$$

For  $H \ll 1$ :  $\frac{1}{\sqrt{1-H^2 x^2}} \approx 1 + \frac{1}{2} H^2 x^2$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-H^2 x^2}} \approx \int_0^1 \frac{dx}{\sqrt{1-x^2}} \left( 1 + \frac{1}{2} H^2 x^2 \right)$$

Now:  $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin(1) = \left[ \frac{\pi}{2} \right]$

$$\frac{1}{2} H^2 \int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1}{2} H^2 \int_0^{\pi/2} \frac{\sin^2 \theta \cos \theta d\theta}{\sqrt{1-\sin^2 \theta}}$$

Let:  $x = \sin \theta$        $\cos 2\theta = 1 - 2\sin^2 \theta$   
 $dx = \cos \theta d\theta$        $\rightarrow \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$   
 $x^2 = \sin^2 \theta$

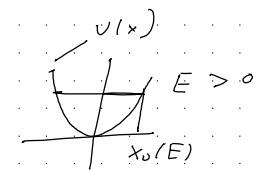
$$\begin{aligned} \rightarrow \frac{1}{2} H^2 \int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}} &= \frac{1}{2} H^2 \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2\theta) d\theta \\ &= \frac{1}{4} H^2 \left[ \frac{\pi}{2} - \frac{\sin 2\theta}{2} \Big|_0^{\pi/2} \right] \\ &= \boxed{\frac{1}{8} H^2 \pi} \quad |T = \sin \left( \frac{\phi_0}{2} \right) \approx \frac{\phi_0}{2}} \end{aligned}$$

$$Therefore, T(E) = \frac{4\sqrt{\frac{E}{J}}}{\sqrt{2}} \left( \frac{\pi}{2} + \frac{1}{8} H^2 \pi + \dots \right) = \boxed{2\sqrt{\frac{E}{J}} \left( 1 + \frac{1}{16} \phi_0^2 + \dots \right)}$$

Sec II, Prob 2:

(a)  $U = A|x|^n \quad x_0(E)$

$$T(E) = 2\sqrt{2m} \int_0^{\infty} \frac{dx}{\sqrt{E - Ax^n}}$$



where  $U(x_0) = E$

$$Ax_0^n = E$$

$$\rightarrow x_0 = \left( \frac{E}{A} \right)^{\frac{1}{n}}$$

$$\sqrt{E - Ax^n} = \sqrt{E} \sqrt{1 - \left( \frac{A}{E} \right)x^n}$$

$$= \sqrt{E} \sqrt{1 - t^n} \quad \text{where } t = \left( \frac{A}{E} \right)^{\frac{1}{n}} x$$

$$dt = \left( \frac{A}{E} \right)^{\frac{1}{n}} dx$$

$$t_0 = \left( \frac{A}{E} \right)^{\frac{1}{n}} x_0 = \left( \frac{A}{E} \right)^{\frac{1}{n}} \left( \frac{E}{A} \right)^{\frac{1}{n}} = 1$$

Thus,

$$T(E) = 2\sqrt{2m} \int_0^1 \left( \frac{E}{A} \right)^{\frac{1}{n}} \frac{dt}{\sqrt{1-t^n}}$$

Now let  $u = t^n$

$$\begin{aligned} du &= n t^{n-1} dt = n u^{\frac{n-1}{n}} dt \\ &= n u^{1-\frac{1}{n}} dt \end{aligned}$$

$$dt = \frac{1}{n} u^{\frac{1}{n}-1} du, \quad t=0, 1 \rightarrow u=0, 1$$

Thus

$$T(E) = 2 \sqrt{\frac{2m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 \frac{1}{n} \frac{u^{\frac{1}{n}-1}}{\sqrt{1-u}} du$$
$$= \frac{2}{n} \sqrt{\frac{2m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 u^{\frac{1}{n}-1} (1-u)^{-\frac{1}{2}} du$$

Recall: Beta Function

$$B(x, y) = \int_0^1 dt t^{x-1} (1-t)^{y-1} = \frac{1}{\Gamma(x)\Gamma(y)}$$

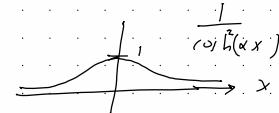
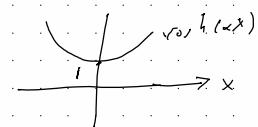
(where  $\operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0$ )

Thus,

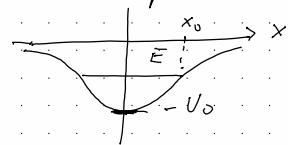
$$\boxed{T(E)} = \frac{2}{n} \sqrt{\frac{2m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \frac{\Gamma(\frac{1}{n}) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{n} + \frac{1}{2})} \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}$$
$$= \frac{2}{n} \sqrt{\frac{2\pi m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \frac{\Gamma(\frac{1}{n})}{\Gamma(\frac{1}{n} + \frac{1}{2})}$$

Gamma Function

$$(b) U = -U_0 / \cosh^2 \alpha x$$



$$-U_0 / \cosh^2$$



$$-U_0 \leq E < 0$$

Thus,  $E = -|E|$

$$|E| < U_0$$

$$T(E) = 2 \sqrt{2m} \int_0^{\infty} \frac{dx}{\sqrt{E - U(x)}}$$

where  $x_0$  given by

$$E = U(x_0)$$

$$= \frac{-U_0}{\cosh^2(\alpha x_0)}$$

$$\rightarrow \cosh^2(\alpha x_0) = -\frac{U_0}{E}$$

$$\cosh(\alpha x_0) = \sqrt{\frac{U_0}{|E|}}$$

$$\alpha x_0 = \cosh^{-1} \left( \sqrt{\frac{U_0}{|E|}} \right)$$

$$\boxed{x_0 = \frac{1}{\alpha} \cosh^{-1} \left( \sqrt{\frac{U_0}{|E|}} \right)}$$

$$\begin{aligned}
 \sqrt{E - U(x)} &= \sqrt{E + \frac{U_0}{\cosh^2(\alpha x)}} \quad \cosh^2 - \sinh^2 = 1 \\
 &= \frac{1}{\cosh(\alpha x)} \sqrt{E \cosh^2(\alpha x) + U_0} \\
 &= \frac{1}{\cosh(\alpha x)} \sqrt{E(1 + \sinh^2(\alpha x)) + U_0} \\
 &= \frac{1}{\cosh(\alpha x)} \sqrt{(E+U_0) + E \sinh^2(\alpha x)} \\
 &= \frac{1}{\cosh(\alpha x)} \sqrt{(U_0 - |E|) - |E| \sinh^2(\alpha x)}
 \end{aligned}$$

thus

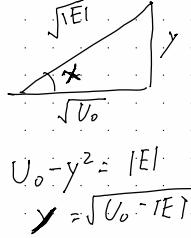
$$\begin{aligned}
 T(E) &= 2 \sqrt{2m} \int_0^{x_0} \frac{dx}{\sqrt{E - U(x)}} \\
 &= 2 \sqrt{2m} \int_0^{x_0} \frac{\cosh(\alpha x) dx}{\sqrt{(U_0 - |E|) - |E| \sinh^2(\alpha x)}}
 \end{aligned}$$

$$\text{Let: } t = \sinh(\alpha x) \rightarrow t_0 = \sinh(\alpha x_0)$$

$$dt = \alpha \cosh(\alpha x) dx$$

$$so \quad t_0 = \sinh^{-1} \left( \frac{U_0}{|E|} \right)$$

$$= \frac{U_0 - |E|}{|E|}$$



$$U_0 - y^2 = |E| \\ y = \sqrt{U_0 - |E|}$$

$$T(E) = 2 \sqrt{2m} \int_0^{\sqrt{\frac{U_0 - |E|}{|E|}}} dt$$

$$= 2 \sqrt{2m} \int_0^{\sqrt{\frac{U_0 - |E|}{|E|}}} \frac{dt}{\sqrt{U_0 - |E|} \sqrt{1 - \frac{|E|}{U_0 - |E|} t^2}}$$

$$\text{Def: } \sin u = \sqrt{\frac{|E|}{U_0 - |E|}} t$$

$$du \cos u = \sqrt{\frac{|E|}{U_0 - |E|}} dt$$

$$\begin{aligned}
 T(E) &= \frac{2 \sqrt{2m}}{\sqrt{U_0 - |E|}} \int_0^{\pi/2} \frac{\cos u du}{\sqrt{\frac{U_0 - |E|}{|E|}} \sqrt{1 - \frac{|E|}{U_0 - |E|} \sin^2 u}} \\
 &= \frac{\pi}{\alpha} \sqrt{\frac{2m}{|E|}} \frac{\pi}{2}
 \end{aligned}$$

$$\boxed{\frac{\pi}{\alpha} \sqrt{\frac{2m}{|E|}}}$$

$$(c) U = U_0 + \frac{1}{2} m \alpha x^2$$

$$T = 2\sqrt{2m} \int_0^{x_0} \frac{dx}{\sqrt{E - U(x)}}$$

where  $x_0$  determined by

$$E = U(x_0)$$

$$= U_0 + \frac{1}{2} m \alpha x_0^2$$

$$\Rightarrow x_0 = \tan^{-1} \left( \sqrt{\frac{E}{U_0}} \right)$$

$$\boxed{x_0 = \pm \tan^{-1} \left( \sqrt{\frac{E}{U_0}} \right)}$$

$$\sqrt{E - U(x)} = \sqrt{E - U_0 - \frac{1}{2} m \alpha x^2}$$

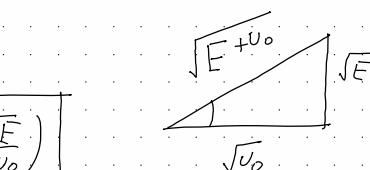
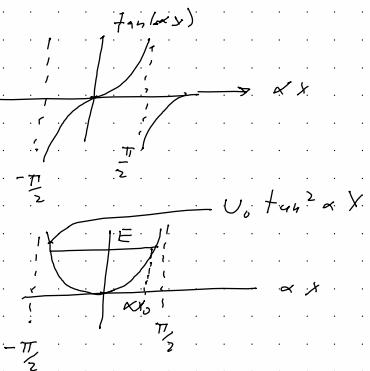
$$= \sqrt{E - U_0 - \frac{1}{2} m \sin^2(\alpha x)}$$

$$= \pm \sqrt{E \cos^2(\alpha x) - U_0 \sin^2(\alpha x)}$$

$$= \pm \sqrt{E (1 - \sin^2(\alpha x)) - U_0 \sin^2(\alpha x)}$$

$$= \pm \sqrt{E - (U_0 + E) \sin^2(\alpha x)}$$

$$= \frac{\sqrt{E}}{\cos \alpha x} \sqrt{1 - \frac{(U_0 + E)}{E} \sin^2(\alpha x)}$$



Thus

$$T(E) = 2\sqrt{2m} \int_0^{x_0} \frac{\cos \alpha x \, dx}{\sqrt{E - \frac{U_0 + E}{E} \sin^2 \alpha x}}$$

$$\text{Let } u = \sin \alpha x$$

$$du = \alpha \cos \alpha x \, dx$$

$$x=0, x_0 \rightarrow u=0$$

$$u_0 = \sin(\alpha x_0)$$

$$= \sin \left( \tan^{-1} \sqrt{\frac{E}{U_0}} \right)$$

$$= \sqrt{\frac{E}{E+U_0}}$$

$$u_0$$

$$\rightarrow T(E) = 2\sqrt{2m} \frac{1}{\sqrt{E}} \int_0^{u_0} \frac{du}{\sqrt{1 - \frac{U_0 + E}{E} u^2}}$$

$$\text{Let } \sqrt{\frac{U_0 + E}{E}} u = \sin \theta \rightarrow \sqrt{\frac{U_0 + E}{E}} du = \cos \theta \, d\theta$$

$$u=0 \rightarrow \theta=0$$

$$u=u_0 \rightarrow \sin \theta_0 = \sqrt{\frac{U_0 + E}{E}} \sqrt{\frac{E}{U_0 + E}} = 1 \Rightarrow \theta_0 = \frac{\pi}{2}$$

$$\rightarrow T(E) = 2\sqrt{2m} \frac{1}{\sqrt{E}} \int_0^{\pi/2} \frac{\cos \theta \, d\theta}{\sqrt{\frac{U_0 + E}{E}} \sqrt{1 - \sin^2 \theta}}$$

$$= \frac{2\sqrt{2m}}{\alpha \sqrt{E}} \sqrt{\frac{E}{U_0 + E}} \frac{\pi}{2} = \boxed{\frac{\pi \sqrt{2m}}{\alpha \sqrt{E}} \sqrt{\frac{E}{U_0 + E}}}$$

Sec 13, Prob 1

$\vec{X}$ : position vector of  $M$

$\vec{x}_a$ :  $a=1, 2, \dots, n$  position vector of  $n$  masses  
all with mass  $m$ .

Closed system  $\rightarrow$  linear momentum conserved  
 $\rightarrow$  COM frame

$$M\vec{X} + m(\vec{x}_1 + \vec{x}_2 + \dots + \vec{x}_n) = 0$$

$$\text{or } M\vec{X} + m \sum_{a=1}^n \vec{x}_a = 0 \quad (1)$$

$$\text{Taking time derivative } \rightarrow M\dot{\vec{X}} + \sum_a \dot{\vec{x}}_a = 0. \quad (2)$$

Define relative position vectors:

$$\vec{r}_1 \equiv \vec{x}_1 - \vec{X}$$

$$\vec{r}_2 \equiv \vec{x}_2 - \vec{X}$$

etc.

$$\text{or } \vec{r}_a \equiv \vec{x}_a - \vec{X}, \quad a=1, 2, \dots, n. \quad (3)$$

Summing up (3):

$$\begin{aligned} \sum \vec{r}_a &= \sum (\vec{x}_a - \vec{X}) \\ &= \sum \vec{x}_a - n \vec{X} \quad \text{total mass} \\ &= -\frac{M}{m} \vec{X} - n \vec{X} \\ &= -\frac{(M+n)m}{m} \vec{X} = -\mu \vec{X} \end{aligned}$$

$$\text{Thus, } \vec{X} = -\frac{m}{\mu} \sum \vec{r}_a$$

$$\text{and } \vec{x}_a = \vec{r}_a + \vec{X}$$

give you  $\vec{x}_a, \vec{X}$  in terms of  $\vec{r}_a$ :

$$\begin{aligned} \text{Hence: } T &= \frac{1}{2} \sum_a m |\vec{x}_a|^2 + \frac{1}{2} M |\vec{X}|^2 \\ &= \frac{1}{2} m \sum_a |\vec{r}_a + \vec{X}|^2 + \frac{1}{2} M |\vec{X}|^2 \\ &= \frac{1}{2} m \sum_a |\vec{r}_a|^2 + \frac{1}{2} \sum_a m 2\vec{r}_a \cdot \vec{X} \\ &\quad + \frac{1}{2} \sum_a m |\vec{X}|^2 + \frac{1}{2} M |\vec{X}|^2 \\ &= \frac{1}{2} m \sum_a |\vec{r}_a|^2 + m \left( \sum_a \vec{r}_a \right) \cdot \vec{X} \\ &\quad + \frac{1}{2} nm |\vec{X}|^2 + \frac{1}{2} M |\vec{X}|^2 \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} m \sum_a |\vec{r}_a|^2 - \mu |\vec{X}|^2 + \frac{1}{2} (nm + M) |\vec{X}|^2 \\ &= \frac{1}{2} m \sum_a |\vec{r}_a|^2 - \frac{1}{2} \mu |\vec{X}|^2 \end{aligned}$$

Now rewrite in  $t$  form:

$$-\frac{1}{2} \mu |\vec{X}|^2 = -\frac{1}{2} \mu \frac{m^2}{\mu^2} \sum_a |\vec{r}_a|^2 = -\frac{1}{2} \frac{m^2}{\mu} \sum_a |\vec{r}_a|^2$$

$$\text{Thus, } T = \frac{1}{2} m \sum_a |\vec{r}_a|^2 - \frac{1}{2} \frac{m^2}{\mu} \sum_a |\vec{r}_a|^2$$

## Potential energy

$$U = U(|\vec{x}_1 - \vec{x}_2|, |\vec{x}_1 - \vec{x}_3|, \dots, |\vec{x}_1 - \vec{x}_n|)$$

$$|\vec{x}_1 - \vec{x}|, |\vec{x}_2 - \vec{x}|, \dots, |\vec{x}_n - \vec{x}|$$

$$= \cup \left( |\vec{r}_1 - \vec{r}_2|, |\vec{r}_1 - \vec{r}_3|, \dots, |\vec{r}_1 - \vec{r}_n|, \right. \\ \left. |\vec{r}_2|, |\vec{r}_3|, \dots, |\vec{r}_n| \right)$$

which depends only on the relative position vectors  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ .

Thus,

$$L = \frac{1}{2} m \sum_a |\vec{r}_a|^2 - \frac{1}{2} \frac{m^2}{M} \left| \sum_a \vec{r}_a \right|^2 - U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n)$$

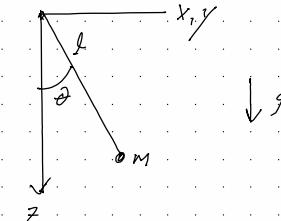
Sec 14, Prob. 1

## Spherical pendulum

$$T = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)$$

$$U = -mgz$$

$$= -mg\ell \cos \theta$$



Thur

$$f = T \cup$$

$$= \frac{1}{2} m l^2 (\dot{\theta}^2 + m^2 \dot{\theta} \dot{\phi}^2) + m g l \cos \theta$$

$$\text{No } t\text{-dependence} \rightarrow E = T + U = \text{const}$$

$$\text{No } \phi\text{-dependence} \rightarrow \frac{\partial L}{\partial \dot{\phi}} = m l^2 \sin^2 \theta \dot{\phi} \equiv M_2 = \cos \theta$$

$$E = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - mg l \cos \theta$$

$$= \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2 \theta \frac{M_2^2}{m^2 l^4 \sin^2 \theta}) - mgl \cos \theta$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{M_2^2}{2m l^2 \sin^2 \theta} - m g l \cos \theta$$

$$U_{eff}(\theta)$$

$$E = \frac{1}{2} m \lambda^2 \dot{\theta}^2 + \frac{M_2^2}{2 m \lambda^2 \sin^2 \theta} + U(\theta)$$

$$\rightarrow \frac{1}{2} m_1^2 \dot{\theta}^2 = E - U(\theta) - \frac{M^2}{2 m_1^2 \sin^2 \theta}$$

$$\textcircled{2} = \sqrt{\frac{2}{m_e^2} (E - U(\theta))} - \frac{M_e^2}{m_e^2 e^2 q_{31} m^2}$$

$$\frac{d\theta}{dt} = \dot{\theta} = \sqrt{\frac{2}{ml^2} (E + mgl\cos\theta) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}$$

$$\rightarrow dt = \frac{d\theta}{\sqrt{\dots}}$$

$$t = \int \frac{d\theta}{\sqrt{\frac{2}{ml^2} (E + mgl\cos\theta) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}} + C_1 t$$

Path: use  $M_z = ml^2 \sin^2\theta \dot{\phi}$

$$\text{Thus, } \frac{d\theta}{dt} = \frac{d\theta}{d\phi} \frac{d\phi}{dt} \\ = \frac{d\theta}{d\phi} \frac{M_z}{ml^2 \sin^2\theta}$$

$$\text{Thus, } \frac{d\theta}{d\phi} = \frac{d\theta}{dt} \frac{ml^2 \sin^2\theta}{M_z} = \sqrt{\frac{ml^2 \sin^2\theta}{M_z}}$$

$$d\phi = \frac{d\theta M_z}{\sqrt{ml^2 \sin^2\theta}}$$

$$\phi = \int \frac{M_z d\theta / ml^2 \sin^2\theta}{\sqrt{\frac{2}{ml^2} (E + mgl\cos\theta) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}} + C_2 \phi$$

Turning points: (where  $\dot{\theta} = 0$ )

$$E = V_{\text{eff}}(r)$$

$$= \frac{M_z^2}{2ml^2 \sin^2\theta} - mgl\cos\theta$$

$$\rightarrow 2Eml^2 \sin^2\theta = M_z^2 - 2mgl^3 \sin^2\theta \cos\theta$$

$$2Eml^2 (1 - \cos^2\theta) = M_z^2 - 2mgl^3 (1 - \cos^2\theta) \cos\theta$$

$$2Eml^2 - 2Eml^2 \cos^2\theta$$

$$= M_z^2 - 2mgl^3 \cos\theta + 2mgl^3 \cos^3\theta$$

$$\text{Thus, } 2mgl^3 \cos^3\theta + 2Eml^2 \cos^2\theta - 2mgl^3 \cos\theta$$

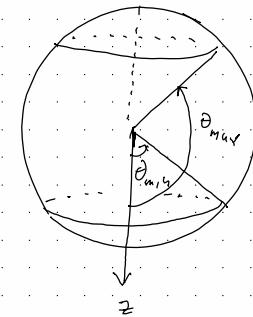
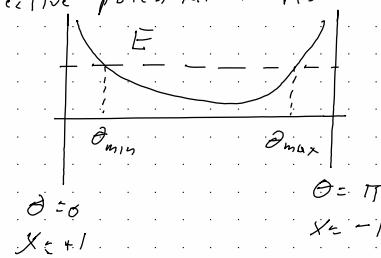
$$+ (M_z^2 - 2Eml^2) = 0$$

Divide by  $2mgl^3$ :

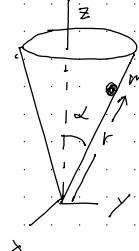
$$\rightarrow \cos^3\theta + \frac{E}{mgl} \cos^2\theta - \cos\theta + \left( \frac{M_z^2}{2mgl^3} - \frac{E}{mgl} \right) = 0$$

Cubic equation for  $\cos\theta$ :

Effective potential looks like:



Sec 14, Prob. 2



spherical polar coords  $(r, \theta, \phi)$

Constraint  $\theta = \alpha$

Generalized coords:  $(r, \phi)$

$$T = \frac{1}{2}m(r^2 + r^2\sin^2\alpha\dot{\phi}^2)$$

$$= \frac{1}{2}m(r^2 + r^2\sin^2\alpha\dot{\phi}^2)$$

$$U = mgz$$

$$= mg r \cos \alpha$$

$$L = T - U$$

$$= \frac{1}{2}m(r^2 + r^2\sin^2\alpha\dot{\phi}^2) - mg r \cos \alpha$$

$E = \text{const}$  (since no explicit t dependence)

$$\frac{dL}{d\phi} = mr^2\sin^2\alpha\dot{\phi} = M_Z = \text{const}$$

(since no explicit  $\phi$  dependence)

$$E = T + U$$

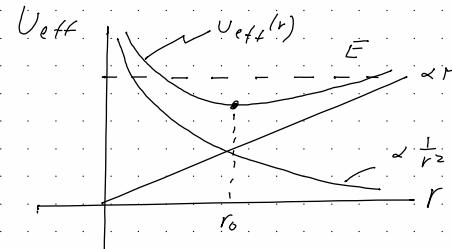
$$= \frac{1}{2}m(r^2 + r^2\sin^2\alpha\dot{\phi}^2) + mg r \cos \alpha$$

$$= \frac{1}{2}mr^2 + \frac{1}{2}mr^2\sin^2\alpha\left(\frac{M_Z^2}{m^2r^4\sin^4\alpha}\right) + mg r \cos \alpha$$

$$= \frac{1}{2}mr^2 + \frac{M_Z^2}{2mr^2\sin^2\alpha} + mg r \cos \alpha$$

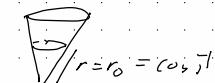
$$= \frac{1}{2}mr^2 + U_{\text{eff}}(r)$$

$$U_{\text{eff}}(r) = \frac{M_Z^2}{2mr^2\sin^2\alpha} + mg r \cos \alpha$$



Bound orbits for  $E > U_{\text{eff}, \min} = U_{\text{eff}}(r_0)$

$r_0$ : stable circular orbit



t-equation:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}(E - mg r \cos \alpha) - \frac{M_Z^2}{m^2 r^2 \sin^2 \alpha}}} + \text{const}$$

Using  $M_Z = mr^2 \sin^2 \alpha \dot{\phi}$

$$\rightarrow \frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{M_Z}{mr^2 \sin^2 \alpha}$$

$\phi$ -equation:

$$\phi = \int \frac{\left(\frac{M_Z}{\sin^2 \alpha}\right) dr}{\sqrt{\frac{2m(E - mg r \cos \alpha)}{r^2 \sin^2 \alpha} - \frac{M_Z^2}{r^2 \sin^2 \alpha}}} + \text{const}$$

Turning points:  $r = r_{\min}, r_{\max}$   
Determined by effective potential

$$E = U_{\text{eff}}(r)$$

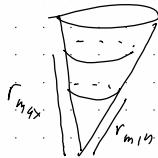
$$= \frac{M_2^2}{2m r^2 \sin^2 \alpha} + m g r \cos \alpha$$

$$\rightarrow 2m E r^2 \sin^2 \alpha = M_2^2 + 2m^2 g r^3 \sin^2 \alpha \cos \alpha$$

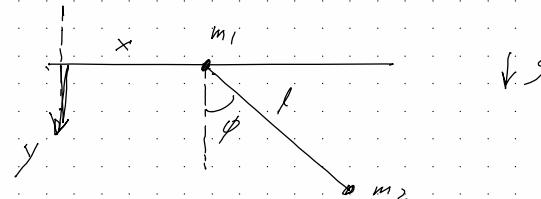
$$0 = 2m^2 g r^3 \sin^2 \alpha \cos \alpha - 2m E r^2 \sin^2 \alpha + M_2^2$$

$$= r^3 - \left( \frac{E}{m g \cos \alpha} \right) r^2 + \frac{M_2^2}{2m^2 g \sin^2 \alpha \cos \alpha}$$

cubic equation again



### Sec 14, Prob 3



From Sec 5, Prob 2 we have

$$L = \frac{1}{2}(m_1 + m_2)x^2 + \frac{1}{2}m_2 l^2 \dot{\phi}^2 + 2l \dot{x} \dot{\phi} \cos \phi + m_2 g l \cos \phi$$

No dependence on  $x$ :

$$\rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)x + m_2 l \dot{\phi} \cos \phi = \text{const}$$

( $x$ -component of total momentum)

No explicit  $t$ -dependence

$$\begin{aligned} \rightarrow E &= T + U = \text{const} \\ &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2 \dot{\phi}^2 + 2l \dot{x} \dot{\phi} \cos \phi) - m_2 g l \cos \phi \end{aligned}$$

Want in frame where  $\text{com}_x = 0$ :

$$\begin{aligned} \text{com}_x &= m_1 x + m_2(x + l \sin \phi) \\ &= (m_1 + m_2)x + m_2 l \sin \phi \end{aligned}$$

$$\text{com}_x = 0 \rightarrow x = -\left(\frac{m_2}{m_1 + m_2}\right) l \sin \phi$$

$$\dot{x} = -\left(\frac{m_2}{m_1+m_2}\right) l \cos\phi \dot{\phi}$$

$t^{\text{loss}}$

$$E = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi)$$

$$- m_2 g l \cos\phi$$

$$= \frac{1}{2}(m_1+m_2) \frac{m_2^2}{(m_1+m_2)^2} l^2 \cos^2\phi \dot{\phi}^2$$

$$+ \frac{1}{2}m_2 l^2 \dot{\phi}^2 - m_2 \left(\frac{m_2}{m_1+m_2}\right) l^2 \cos^2\phi \dot{x}^2$$

$$- m_2 g l \cos\phi$$

$$= \frac{1}{2}m_2 l^2 \dot{\phi}^2 \left[ 1 - \left(\frac{m_2}{m_1+m_2}\right) \cos^2\phi \right]$$

$$- m_2 g l \cos\phi$$

1-d problem:

$$\underline{E + m_2 g l \cos\phi} = \frac{1}{2}m_2 l^2 \dot{\phi}^2$$

$$\left[ 1 - \left(\frac{m_2}{m_1+m_2}\right) \cos^2\phi \right]$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{\frac{2}{m_2 l^2} \left( E + m_2 g l \cos\phi \right)} \overline{1 - \left(\frac{m_2}{m_1+m_2}\right) \cos^2\phi}$$

$$\rightarrow dt = \frac{d\phi}{\sqrt{}}$$

$$= d\phi \sqrt{1 - \left(\frac{m_2}{m_1+m_2}\right) \cos^2\phi}$$

$$= d\phi \sqrt{\frac{m_2 l^2}{2} \frac{1}{m_1+m_2}} \sqrt{\frac{(m_1+m_2) - m_2 \cos^2\phi}{E + m_2 g l \cos\phi}}$$

$$= d\phi \sqrt{\frac{m_2}{m_1+m_2}} \sqrt{\frac{l^2}{2}} \sqrt{\frac{m_1+m_2 \sin^2\phi}{E + m_2 g l \cos\phi}}$$

so

$$t = \sqrt{\left(\frac{m_2}{m_1+m_2}\right) \frac{l^2}{2}} \int d\phi \sqrt{\frac{m_1+m_2 \sin^2\phi}{E + m_2 g l \cos\phi}} + \text{const}$$

$$\text{Now: } x_2 = x + l \sin\phi$$

$$y_2 = l \cos\phi$$

$$\text{using } x = -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi$$

$$\rightarrow x_2 = \left[ -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi + l \cos\phi \right] = \left(\frac{m_1}{m_1+m_2}\right) l \sin\phi$$

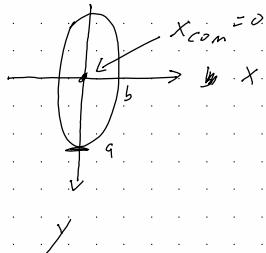
$$x_2 = \left( \frac{m_1}{m_1 + m_2} \right) l \sin \phi = b \sin \phi$$

$$y_2 = l \cos \phi = a \cos \phi$$

$$\left(\frac{y_2}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = \cos^2 \phi + \sin^2 \phi = 1$$

which is an ellipse with semi-major and semi-minor axes:

$$a = l, \quad b = l \left( \frac{m_1}{m_1 + m_2} \right)$$



If  $m_1 \gg m_2$   
then  $a = b \approx l$   
so that  $m_2$  moves  
along a circular  
arc of radius  $l$ :

Sec 15, Prob 1

$$U = -\frac{\infty}{r}, \quad E = 0 \rightarrow e = 1$$

$$\frac{p}{r} = 1 + \cos \phi$$

when  $\phi = 0, \quad p = 2 \cdot r$

$$\text{so } r_{min} = \frac{p}{2}$$

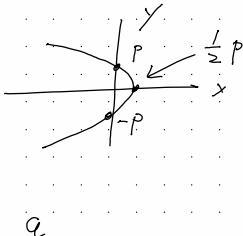
NOTE:  $p = r + r \cos \phi$   
 $= \sqrt{r^2 + y^2} + x$

$$\rightarrow (p-x)^2 = x^2 + y^2$$

$$p^2 + x^2 - 2px = x^2 + y^2$$

$$p^2 - y^2 = 2px$$

$$\rightarrow \boxed{x = \frac{p^2 - y^2}{2p}} \quad \text{parabola}$$



when  $y = 0, \quad x = \frac{1}{2}P$

$$x = 0, \quad y = \pm P$$

Time equation:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m} [E - U(r)] - \frac{m^2}{r^2}}} \quad t \text{ vs } r$$

$$U(r) = -\frac{\alpha}{r}, \quad p = \frac{mv^2}{m\alpha}, \quad e=1, \quad E=0$$

$$\rightarrow t = \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{mv^2}{m^2 r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{mv^2}{m^2 r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{dr}{\sqrt{\frac{2}{r} - \frac{p^2}{r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{r dr}{\sqrt{2r-p^2}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{r dr}{\sqrt{\frac{2r-1}{p}}} + \text{const}$$

Defn:  $\frac{2r-1}{p} = \xi^2 > 0 \quad (-\infty < \xi < \infty)$

$$2r = p(1+\xi^2)$$

$$\boxed{r = \frac{p}{2}(1+\xi^2)} \rightarrow dr = p\xi d\xi$$

$$r dr = \frac{p^2}{2}(\xi + \xi^3) d\xi$$

Thus,

$$t = \sqrt{\frac{m}{\alpha p}} \int \frac{\frac{p^2}{2}(\xi + \xi^3) d\xi}{\sqrt{\xi^2}} + \text{const}$$

$$= \sqrt{\frac{mp^3}{\alpha}} \frac{1}{2} \int (1+\xi^2) d\xi + \text{const}$$

$$= \frac{1}{2} \sqrt{\frac{mp^3}{\alpha}} \left( \xi + \frac{1}{3}\xi^3 \right) + \text{const}$$

choose const so that  $t=0 \Leftrightarrow \xi = 0$  (const=0)

$$\text{so, } \boxed{t = \frac{1}{2} \sqrt{\frac{mp^3}{\alpha}} \left( \xi + \frac{1}{3}\xi^3 \right)}$$

Now:  $\frac{p}{r} = 1 + \cos\phi$

$$p = r(1 + \cos\phi)$$

$$\phi = \frac{p}{2}(1+\xi^2)(1+\cos\phi)$$

$$Z = 1 + \xi^2 + \cos\phi + \xi^2 \cos\phi$$

$$1 - \xi^2 = (1+\xi^2) \cos\phi$$

$$\rightarrow \boxed{\cos\phi = \frac{1-\xi^2}{1+\xi^2}}$$

$$\begin{aligned} X &= r \cos \phi \\ &= \frac{p}{2} (1 + \xi^2) \left( \frac{1 - \xi^2}{1 + \xi^2} \right) \\ &= \frac{p}{2} (1 - \xi^2) \end{aligned}$$

Also,

$$\begin{aligned} x^2 + y^2 &= r^2 \\ \rightarrow y^2 &= r^2 - x^2 \\ &= \frac{p^2}{4} (1 + \xi^2)^2 - \frac{p^2}{4} (1 - \xi^2)^2 \\ &= \frac{p^2}{4} (x + \xi)^2 + 2\xi^2 - (x - \xi)^2 + 2\xi^2 \\ &= p^2 \xi^2 \end{aligned}$$

so  $y = p \xi$

Sec 15, Prob 3:

$$\Delta\phi = 2 \int_{r_{\min}}^{r_{\max}} \frac{M dr / r^2}{\sqrt{2m(E-U) - M^2/r^2}}$$

This is the change in  $\phi$  as  $r$  goes from  $r_{\min}$  to  $r_{\max}$  and then back to  $r_{\min}$ .

A closed bound orbit would have  $\Delta\phi = 2\pi m/n$  for  $m, n$  integers.

Consider:  $U = -\frac{\alpha}{r} + \delta U$  where  $|\delta U| \ll |\frac{\alpha}{r}|$

$$\text{For } \delta U = 0, \Delta\phi = 2\pi$$

$$\Delta\phi = 2 \int_{r_{\min}}^{r_{\max}} \frac{M dr / r^2}{\sqrt{2m(E - (U + \delta U)) - M^2/r^2}}$$

$$= 2 \int_{r_{\min}}^{r_{\max}} \frac{M dr / r^2}{\sqrt{2m(E-U) - 2m\delta U - M^2/r^2}}$$

$$= 2 \int_{r_{\min}}^{r_{\max}} \frac{M dr / r^2}{\sqrt{(2m(E-U) - M^2/r^2)} \left( 1 - \frac{2m\delta U}{2m(E-U) - M^2/r^2} \right)}$$

$$\approx 2 \int_{r_{\min}}^{r_{\max}} \frac{M dr / r^2}{\sqrt{2m(E-U) - M^2/r^2}} \left[ 1 + \frac{m\delta U}{2m(E-U) - M^2/r^2} \right]$$

For  $V = -\alpha/r$ :

$$\Delta\phi \approx 2\pi + 2 \int_{r_{min}}^{r_{max}} \frac{Md\tau/r^2}{\sqrt{2m(E-U) - M^2/r^2}} d\delta U$$

$$\approx 2\pi + \delta\phi$$

where

$$r_{max}$$

$$\delta\phi \equiv \int_{r_{min}}^{r_{max}} \frac{2m\delta U M dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}}^{3/2}$$

$$= \frac{2}{2M} \left[ \int_{r_{min}}^{r_{max}} \frac{2m\delta U dr}{\sqrt{2m(E-U) - M^2/r^2}} \right]$$

Evaluate terms in integrand along unperturbed path since  $\delta U$  is already small

$$\frac{p}{r} = 1 + e\cos\phi \quad , \quad p = q(1-e^2)$$

$$\rightarrow -\frac{p}{r^2} dr = -e\sin\phi d\phi \rightarrow \boxed{dr = \frac{r^2 e \sin\phi}{p} d\phi}$$

$$\sqrt{ } = \sqrt{2mE + 2m\alpha - \frac{M^2}{r^2}}$$

$$= \sqrt{-2m|E| + \frac{4m|E|\alpha}{r} - \frac{2m|E|p^2}{(1-e^2)^2}}$$

$$\begin{aligned} \sqrt{ } &= \sqrt{2m|E|} \sqrt{-1 + \frac{2}{(1-e^2)}(1+e\cos\phi) - \frac{1}{(1-e^2)}(1+e\cos\phi)^2} \\ &= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}} \sqrt{-1/(1-e^2) + 2/(1+e\cos\phi) - (1+2e\cos\phi + e^2\cos^2\phi)} \\ &= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}} \sqrt{e^2(1-\cos^2\phi)} \\ &= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}} e \sin\phi \end{aligned}$$

Thus,

$$\begin{aligned} |\delta\phi| &= \frac{2}{2M} \left[ 2m \int_0^\pi \frac{\delta U r^2 e \sin\phi d\phi}{\sqrt{2m|E|} e \sin\phi} \right] \\ &= \frac{2}{2M} \left[ \frac{2m}{p \sqrt{\frac{2m|E|}{1-e^2}}} \int_0^\pi d\phi r^2 \delta U \right] \\ &= \frac{2}{2M} \left[ \frac{2m}{M} \int_0^\pi d\phi r^2 \delta U \right] \end{aligned}$$

$$\text{using (15.6): } \frac{M}{\sqrt{2m|E|}} = \frac{p}{\sqrt{1-e^2}}$$

Evaluate:

$$\delta\phi = \frac{\partial}{\partial M} \left[ \frac{2m}{M} \int_0^\pi d\phi r^2 \delta U \right]$$

For (a)  $\delta U = \beta/r^2$ , (b)  $\delta U = \gamma/r^3$

(a)  $\delta\phi = \frac{\partial}{\partial M} \left[ \frac{2m}{M} \int_0^\pi d\phi \beta \right]$

$$= 2\pi\beta m \frac{\partial}{\partial M} \left( \frac{1}{M} \right)$$

$$= -\frac{2\pi\beta m}{M^2}$$

$$= \boxed{-\frac{2\pi\beta}{\alpha p}}$$

Recall:  $j^0 = \frac{M^2}{m\alpha}$

$$\leftarrow \quad \alpha p = \frac{M^2}{m}$$

(b)  $\delta\phi = \frac{\partial}{\partial M} \left[ \frac{2m}{M} \int_0^\pi d\phi \frac{\gamma}{r} \right]$

$$= \frac{\partial}{\partial M} \left[ \frac{2m\gamma}{M} \int_0^\pi d\phi \left( \frac{1 + e^{i\phi}}{r} \right) \right]$$

$$= 2m\gamma \frac{\partial}{\partial M} \left[ \frac{1}{mp} \left( \pi + e^{i\phi} \right) \right]$$

Now:  $\frac{1}{mp} = \frac{m\alpha}{M^3} \rightarrow \frac{\partial}{\partial M} \left( \frac{1}{mp} \right) = -\frac{3m\alpha}{M^4}$

so  $\delta\phi = -6\pi\gamma \frac{m^2\alpha}{M^4}$

$$= -6\pi\gamma\alpha \left( \frac{1}{p\alpha} \right)^2$$

$$= \boxed{-\frac{6\pi\gamma}{p^2\alpha}}$$

Sec 16, Prob 2

From (16.7) we have

$$p(\theta_0)d\theta_0 = \frac{1}{2} \sin\theta_0 d\theta_0 \\ = -\frac{1}{2} d(\cos\theta_0)$$

where  $\theta_0$  is the angle of one of the emitted particles in the com frame.

We would like to find  $p(\theta)d\theta$ , where  $\theta$  is the angle of one of the emitted particles in the lab frame.

$$\sin\theta p(\theta)d\theta = p(\theta_0)d\theta_0$$

we just need to find  $\theta_0$  as a function of  $\theta$ .

This is given by (16.6) which we first derive.

Proof: Given  $\tan\theta = \frac{v_0 \sin\theta_0}{v_0 \cos\theta_0 + V}$

we have

$$\tan\theta(v_0 \cos\theta_0 + V) = v_0 \sqrt{1 - \cos^2\theta_0}$$

Square both sides

$$\tan^2\theta(v_0^2 \cos^2\theta_0 + V^2 + 2v_0 V \cos\theta_0) = v_0^2(1 - \cos^2\theta_0)$$

$$\underbrace{(\tan^2\theta)v_0^2 \cos^2\theta_0}_{\sec^2\theta} + 2v_0 V \tan^2\theta \cos\theta_0 + (V^2 + \tan^2\theta - v_0^2) = 0$$

Quadratic equation for  $\cos\theta_0$ .

$$\rightarrow \cos\theta_0 = -2v_0 V \tan^2\theta \pm \sqrt{4v_0^2 V^2 \tan^4\theta - 4v_0^2 \sec^2\theta (V^2 \tan^2\theta - v_0^2)} \\ \geq v_0^2 \sec^2\theta$$

$$= -\frac{V}{v_0} \sin^2\theta \pm \frac{1}{\sec\theta} \sqrt{\left(\frac{V}{v_0}\right)^2 \tan^4\theta - \sec^2\theta \left(\left(\frac{V}{v_0}\right)^2 \tan^2\theta - 1\right)}$$

$$= -\frac{V}{v_0} \sin^2\theta \pm \frac{1}{\sec\theta} \sqrt{\left(\frac{V}{v_0}\right)^2 \left(\frac{\sin^4\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}\right) + 1}$$

$$= -\frac{V}{v_0} \sin^2\theta \pm \cos\theta \sqrt{\left(\frac{V}{v_0}\right)^2 \frac{\sin^2\theta(\sin^2\theta - 1)}{\cos^2\theta} + 1}$$

$$= -\frac{V}{v_0} \sin^2\theta \pm \cos\theta \sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}$$

(for  $v_0 > V$  take +, for  $v_0 < V$  take both signs.)

Now differentiate both sides:

$$d(\cos\theta_0) = -\frac{V}{v_0} 2 \sin\theta \cos\theta d\theta \mp \sin\theta d\theta \sqrt{\cos^2\theta \left(\frac{1}{v_0^2}\right) + \left(\frac{V}{v_0}\right)^2 \sin^2\theta \cos\theta d\theta}$$

$$= \sin\theta d\theta \left[ -2 \frac{V \cos\theta}{v_0} \mp \sqrt{1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta} \frac{1}{v_0} \right]$$

Now:

$$\begin{aligned} & \sqrt{1 + \left(\frac{V}{v_0}\right)^2 \cos^2 \theta} \sqrt{\frac{1}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}}} \\ &= \frac{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta + \left(\frac{V}{v_0}\right)^2 \cos^2 \theta}{\sqrt{1 + \left(\frac{V}{v_0}\right)^2 (\cos^2 \theta - \sin^2 \theta)}} \\ &= \frac{1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta}{\sqrt{1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta}} \end{aligned}$$

Thus,

$$d(1/\omega_0) = \underbrace{-\sin \theta d\theta}_{d(\cos \theta)} \left[ 2 \frac{V}{v_0} \cos \theta \pm \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}} \right]$$

So:

$$p(\theta) d\theta = \frac{1}{2} \sin \theta d\theta \left[ 2 \frac{V}{v_0} \cos \theta \pm \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}} \right]$$

Now: for  $v_0 > V$ , take + sign ( $\theta \in [0, \pi]$ )

for  $v_0 < V$ , as  $\theta_0$  increases from 0 to  $\pi$

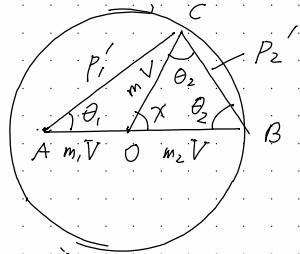
$\theta$  increases from  $0 \rightarrow \theta_{\max}$   
 $\theta$  decreases from  $\theta_{\max} \rightarrow 0$

Thus, for  $v_0 < V$  need to take the difference of the + and - expressions:

$$\begin{aligned} p(\theta) d\theta &= \frac{1}{2} \sin \theta d\theta \left[ 2 \frac{V}{v_0} \cos \theta + \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2 \theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}} \right] \\ &\quad - \frac{1}{2} \sin \theta d\theta \left[ 2 \frac{V}{v_0} \cos \theta - \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2 \theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}} \right] \\ &= \frac{\sin \theta d\theta \left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2 \theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}} \end{aligned}$$

which is valid for  $0 \leq \theta \leq \theta_{\max} = \sin^{-1}\left(\frac{v_0}{V}\right)$

Sec 17, Prob 1:



$$\begin{aligned} x + 2\theta_2 &= \pi \\ x &= \pi - 2\theta_2 \end{aligned}$$

From the above figure:

$$\begin{aligned} (m_2 v_2')^2 &= 2(mv)^2 - 2(mv)^2 \cos x \\ &= 2m^2 v^2 (1 - \cos(\pi - 2\theta_2)) \\ &= 2m^2 v^2 \left[ 1 - (\cos(\pi) \cos(2\theta_2) + \sin(\pi) \sin(2\theta_2)) \right] \\ &= 2m^2 v^2 (1 + \cos(2\theta_2)) \end{aligned}$$

$$\rightarrow v_2' = \sqrt{2} \left( \frac{m}{m_2} \right) v \sqrt{1 + (\cos^2 \theta_2 - \sin^2 \theta_2)}$$

$$= \sqrt{2} \left( \frac{m_1}{m_1 + m_2} \right) v \sqrt{2 \cos^2 \theta_2}$$

$$= 2 \left( \frac{m_1}{m_1 + m_2} \right) v \cos \theta_2$$

$$\text{Thus, } \left( \frac{v_2'}{v} \right) = \left( \frac{2m_1}{m_1 + m_2} \right) \cos \theta_2$$

Also,

$$\begin{aligned} (mv)^2 &= (m_1 v_1')^2 + (m_2 v_2')^2 - 2m_1^2 v_1' v_2' \cos \theta_1 \\ \Rightarrow (m_1 v_1')^2 &- 2m_1 v_1' v_2' \cos \theta_1 (m_2 v_2') + m_1^2 v_1'^2 - m_1^2 v^2 = 0 \end{aligned}$$

$$\text{Now: } v = \frac{m_1 v_1' + m_2 v_2'}{m_1 + m_2} = \frac{m_1 v}{m_1 + m_2}$$

thus,

$$\begin{aligned} (m_1 v_1')^2 &- 2 \left( \frac{m_1^2 v}{m_1 + m_2} \right) \cos \theta_1 m_1 v_1' + \frac{m_1^2 m_2^2 v^2}{(m_1 + m_2)^2} \\ &- \frac{m_1^2 m_2^2}{(m_1 + m_2)^2} v^2 = 0 \end{aligned}$$

$$\rightarrow (v_1')^2 - 2 \left( \frac{m_1}{m_1 + m_2} \right) \cos \theta_1 v_1' + \frac{m_1^2 - m_2^2}{(m_1 + m_2)^2} v^2 = 0$$

$$(v_1')^2 - 2 \left( \frac{m_1 v}{m_1 + m_2} \right) \cos \theta_1 v_1' + \left( \frac{m_1 - m_2}{m_1 + m_2} \right) v^2 = 0$$

$$\left( \frac{v_1'}{v} \right)^2 - 2 \left( \frac{m_1}{m_1 + m_2} \right) \cos \theta_1 \left( \frac{v_1'}{v} \right) + \left( \frac{m_1 - m_2}{m_1 + m_2} \right) = 0$$

Quadratic equations

$$\frac{v'}{v} = \frac{2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \sqrt{4\left(\frac{m_1}{m_1+m_2}\right)^2\cos^2\theta_1 - 4\left(\frac{m_1-m_2}{m_1+m_2}\right)}}{2}$$

2

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right) \sqrt{m_1^2\cos^2\theta_1 - (m_1-m_2)(m_1+m_2)}$$

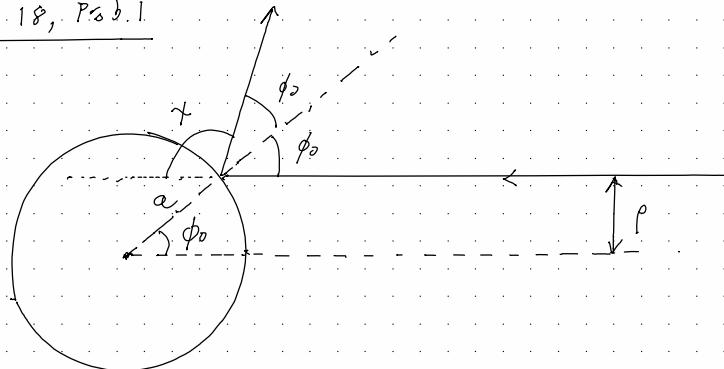
$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right) \sqrt{m_1^2(\cos^2\theta_1 - 1) + m_2^2}$$

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right) \sqrt{m_2^2 - m_1^2\sin^2\theta_1}$$

The + sign holds for  $m_1 < m_2$

+/- signs hold for  $m_1 > m_2$

Sec 18, Prob. 1



$$x + 2\phi_0 = \pi \rightarrow \phi_0 = \frac{\pi}{2} - \frac{x}{2}$$

$$r = a \sin \phi_0$$

$$= a \sin\left(\frac{\pi}{2} - \frac{x}{2}\right)$$

$$= a \left( \sin\left(\frac{\pi}{2}\right) \cos\left(\frac{x}{2}\right) - \cos\left(\frac{\pi}{2}\right) \sin\left(\frac{x}{2}\right) \right)$$

$$= a \cos\left(\frac{x}{2}\right)$$

$$d\sigma = 2\pi r d\rho$$

$$= 2\pi r(x) \left| \frac{dp}{dx} \right| dx$$

$$= \frac{p'(x)}{\sin x} \left| \frac{dp}{dx} \right| d\Omega$$

where  $d\Omega = \text{solid angle}$

$$= 2\pi \sin x dx$$

integral over  
 $d\phi$

$$\rho = a \cos\left(\frac{x}{2}\right)$$

$$d\rho = -a \frac{1}{2} \sin\left(\frac{x}{2}\right) dx$$

$$= -\frac{a}{2} \sin\left(\frac{x}{2}\right) dx$$

thus,

$$d\sigma = \frac{\rho(x)}{\sin x} \left| \frac{dp}{dx} \right| d\Omega$$

$$= \frac{a \cos(x/2)}{\sin x} \frac{a}{2} \sin\left(\frac{x}{2}\right) d\Omega$$

$$= \frac{a^2}{2} \frac{\sin(X_2) \cos(X/2)}{\sin x} d\Omega$$

$$= \boxed{\frac{a^2}{4} d\Omega} \quad (\text{since } \sin x = 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right))$$

Total cross section.

$$\Sigma = \int d\sigma = \frac{a^2}{4} \int d\Omega = \frac{a^2}{4} \cdot 4\pi = \boxed{4\pi a^2}$$

Now calculate differential cross section in

lab frame for both  $m_1$  and  $m_2$

Use the result that

$$d\sigma_1 = \frac{\rho(\theta_1)}{\sin \theta_1} \left| \frac{dp}{d\theta_1} \right| d\Omega_1 = \rho \left| \frac{dp}{d(\cos \theta_1)} \right| d\Omega_1$$

comprise to:

$$d\sigma = \rho \left| \frac{dp}{d(\cos x)} \right| d\Omega$$

$$\rightarrow \frac{d\sigma_1}{d\Omega_1} = \rho \left| \frac{dp}{d(\cos \theta_1)} \right|$$

$$= \boxed{\frac{d(\cos x)}{d(\cos \theta_1)}} \frac{d\sigma}{d\Omega}$$

So we need to evaluate:

$$\frac{d(\cos x)}{d(\cos \theta_1)} \quad \text{and} \quad \frac{d(\cos x)}{d(\cos \theta_2)}$$

start with  $\theta_2$ :  $(17.4)$

$$\theta_2 = \frac{1}{2}(\pi - x) \rightarrow \boxed{x = \pi - 2\theta_2}$$

$$\rightarrow \cos x = \cos(\pi - 2\theta_2)$$

$$= \cos \pi \cos(2\theta_2) + \sin \pi \sin(2\theta_2)$$

$$= -\cos(2\theta_2)$$

$$= -(\cos^2 \theta_2 - \sin^2 \theta_2)$$

$$= -(2\cos^2 \theta_2 - 1)$$

$$= -2\cos^2 \theta_2 + 1$$

$$\text{Thus, } \boxed{d(\cos x) = -4 \cos \theta_2 d(\cos \theta_2)}$$

Thus,

$$\begin{aligned}\frac{d\sigma_2}{d\Omega_2} &= \frac{d\sigma}{d\Omega} \left| \frac{d(\cos X)}{d(\cos \theta_2)} \right| \\ &= \frac{1}{4} a^2 \cdot |4 \cos \theta_2| \\ &= a^2 |\cos \theta_2|\end{aligned}$$

So  $d\sigma_2 = a^2 |\cos \theta_2| d\Omega_2$

Now consider  $\theta_1$ :

From (17.4):

$$t_{11} \theta_1 = \frac{m_2 \sin X}{m_1 + m_2 \cos X}$$

Compare with

$$t_{11} \theta = \frac{v_0 \sin \theta_0}{V + v_0 \cos \theta_0} \quad (16.5)$$

make identifications:  $\theta \rightarrow \theta_1$ ,  $v_0 \rightarrow m_2$ ,  $V \rightarrow m_1$

Then we can write down from (16.6).

$$\cos X = -\frac{m_1 \sin^2 \theta_1}{m_2} \pm \cos \theta_1 \sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2 \theta_1}$$

[See also sec 16, prob. 2 where we derived this for  $\theta$  and  $\theta_0$ .]

We also worked out the derivative:

$$d(\cos \theta_1) = d(\cos \theta) \left[ 2 \frac{V \cos \theta}{v_0} \pm \frac{1 + \left(\frac{V}{v_0}\right)^2 \cos^2 \theta}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}} \right]$$

so we can similarly write down

$$d(\cos X) = d(\cos \theta_1) \left[ 2 \frac{m_1 \cos \theta_1}{m_2} \pm \frac{1 + \left(\frac{m_1}{m_2}\right)^2 \cos(2\theta_1)}{\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2 \theta_1}} \right]$$

~~~~~

For  $m_1 < m_2$ : take + sign

For  $m_1 > m_2$ : As  $X$  increases from 0 to  $\pi$ ,

$\theta_1$  increases from 0 to  $\theta_{\max}$ ; then  $\theta_1$  decreases from  $\theta_{\max}$  to 0. In that case

$$\begin{aligned}d(\cos X) &= d(\cos \theta_1) [\phi + \theta] - d(\cos \theta_1) [\phi - \theta] \\ &= 2 d(\cos \theta_1) \frac{1 + \left(\frac{m_1}{m_2}\right)^2 \cos(2\theta_1)}{\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2 \theta_1}}\end{aligned}$$

Use: (for best)

$$\begin{aligned}d\sigma_1 &= \left( \frac{d\sigma}{d\Omega} \right) \left| \frac{d(\cos X)}{d(\cos \theta_1)} \right| d\Omega_1 \\ &\approx \frac{1}{4} a^2 \left| \frac{d(\cos X)}{d(\cos \theta_1)} \right| d\Omega_1\end{aligned}$$

16 v,

For  $m_1 < m_2$ :

$$d\sigma_1 = \frac{1}{4} a^2 \left[ 2 \left( \frac{m_1}{m_2} \right) \cos \theta_1 + \frac{1 + \left( \frac{m_1}{m_2} \right)^2 \cos(2\theta_1)}{\sqrt{1 - \left( \frac{m_1}{m_2} \right)^2 \sin^2 \theta_1}} \right] d\Omega_1$$

For  $m_1 > m_2$ :

$$\begin{aligned} d\sigma_1 &= \frac{1}{4} a^2 \cdot 2 \frac{1 + \left( \frac{m_1}{m_2} \right)^2 \cos(2\theta_1)}{\sqrt{1 - \left( \frac{m_1}{m_2} \right)^2 \sin^2 \theta_1}} d\Omega_1 \\ &= \frac{a^2}{2} \frac{1 + \left( \frac{m_1}{m_2} \right)^2 \cos(2\theta_1)}{\sqrt{1 - \left( \frac{m_1}{m_2} \right)^2 \sin^2 \theta_1}} d\Omega_1 \end{aligned}$$

Sec 18, Prob 2:

Hard sphere scattering again.

Calculate  $d\sigma$  in terms of  $dE$  where  
 $E$  = energy lost by scattered particle.

Now:  $E$  = energy lost by scattered particle

= energy gained by  $m_2$

$$= \frac{1}{2} m_2 (V_2')^2$$

From Fig. 16., we have (law of cosines):

$$\begin{aligned} (m_2 V_2')^2 &= (mV)^2 + (mV)^2 - 2(mV)^2 \cos X \\ &= 2(mV)^2 [1 - \cos X] \\ &= 2(mV)^2 2 \sin^2 \left( \frac{X}{2} \right) \end{aligned}$$

$$\text{so } m_2 V_2' = 2mV \sin \left( \frac{X}{2} \right)$$

$$\rightarrow V_2' = \left( \frac{m}{m_2} \right) V \sin \left( \frac{X}{2} \right)$$

$$= \left( \frac{m_1}{m_1 + m_2} \right) V \sin \left( \frac{X}{2} \right) \quad (17.5)$$

$$\text{NOTE: } d\sigma = \frac{1}{4} a^2 d\Omega$$

$$= \frac{1}{4} a^2 2\pi \sin X dx$$

$$= \frac{\pi a^2}{2} \int d(\cos X) /$$

So we would like to relate  $dE$  and  $d(\cos X)$ .

$$\text{Now: } E = \frac{1}{2} m_2 (v_z')^2$$

$$= \frac{1}{2} m_2 \frac{4 m_1^2 v^2}{(m_1 + m_2)^2} \sin^2\left(\frac{\chi}{2}\right)$$

$$= \frac{Z m_1^2 m_2}{(m_1 + m_2)^2} V_\infty^2 \sin^2\left(\frac{\chi}{2}\right) \quad (\text{since } V = V_\infty)$$

$$= E_{max} \sin^2\left(\frac{\chi}{2}\right)$$

$$\text{where } E_{max} = \frac{Z m_1^2 m_2}{(m_1 + m_2)^2} V_\infty^2$$

$$= 4 \left(\frac{m_1}{m_1 + m_2}\right) \frac{m_1 V_\infty^2}{2}$$

$$= 4 \left(\frac{m_1}{m_1 + m_2}\right) E$$

$$\text{Thus, } dE = E_{max} Z \sin\left(\frac{\chi}{2}\right) \cos\left(\frac{\chi}{2}\right) \frac{d\chi}{2}$$

$$= \frac{1}{2} E_{max} \sin X dX$$

$$= \frac{1}{2} E_{max} |d(\cos X)|$$

$$\text{So } d\sigma = \frac{\pi q^2}{2} |d(\cos X)|$$

$$= \frac{\pi q^2}{2} \frac{2}{E_{max}} dE = \boxed{\frac{\pi q^2}{E_{max}} dE}$$

which is a uniform distribution w.r.t.  $E$ .

### Sec 18, Prob. 4

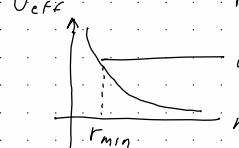
Effective cross section to "fall" to center of

$$U(r) = -\alpha/r^2 \quad (\alpha > 0)$$

$$U_{eff}(r) = U(r) + \frac{M^2}{2mr^2}$$

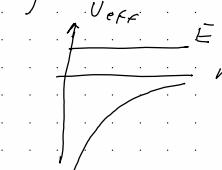
$$= -\frac{\alpha}{r^2} + \frac{M^2}{2mr^2}$$

$$= \frac{1}{r^2} \left( \frac{M^2}{2m} - \alpha \right)$$



$$\frac{M^2}{2m} - \alpha > 0$$

(don't fall to center)  
since  $r_{min} > 0$



$$\frac{M^2}{2m} - \alpha < 0$$

Fall to center ( $r=0$ )

For a given  $E = \frac{1}{2} m V_\infty^2 > 0$  need

$$\frac{M^2}{2m} - \alpha < 0$$

$$\frac{M^2}{2m} < \alpha$$

$$\rightarrow M_{max} \leq \sqrt{2m\alpha}$$

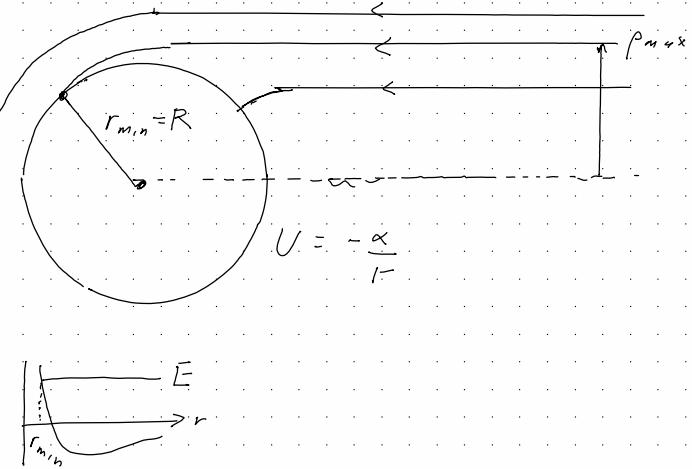
Cross section!  $\sigma = \pi M_{max}^2$ ,  $M = \rho m V_\infty$

Thus,

$$\begin{aligned}\sigma &= \pi p_{\max}^2 \\ &= \pi \frac{M_{\max}^2}{m^2 V_{\infty}^2} \\ &= \pi \frac{Z_m \alpha}{m^2 V_{\infty}^2} \\ &= \frac{\pi \alpha}{\frac{1}{2} m V_{\infty}^2} \\ &= \boxed{\frac{\pi \alpha}{E}}\end{aligned}$$

$$p_{\max} = \frac{M_{\max}}{m V_{\infty}}$$

Sec 18, Prob 6



turning point at  $r = R$

$$\begin{aligned}0 &= E - U_{eff}(R) \\ &= E - U(R) - \frac{M_{\max}^2}{2mR^2} \\ &= E + \frac{\alpha}{R} - \frac{M_{\max}^2}{2mR^2}\end{aligned}$$

$$\rightarrow \frac{M_{\max}^2}{2mR^2} = E + \frac{\alpha}{R}$$

$$M_{\max} = p_{\max} m V_{\infty}, \quad E = \frac{1}{2} m V_{\infty}^2$$

Thurj.

$$O = \pi \rho^2$$

$$= \pi \frac{M_{\max}^2}{m^2 V_\infty^2}$$

$$= \pi \frac{1}{m^2 V_\infty^2} 2mR^2 \left( E + \frac{\alpha}{R} \right)$$

$$= \pi R^2 \left( \frac{2}{m V_\infty^2} \right) \left( E + \frac{\alpha}{R} \right)$$

$\underbrace{\frac{1}{E}}$

$$= \boxed{\pi R^2 \left( 1 + \frac{\alpha}{ER} \right)}$$

where  $E = \frac{1}{2} m V_\infty^2 = \frac{1}{2} \left( \frac{m_1 m_2}{m_1 + m_2} \right) V_\infty^2$

and  $\alpha = G m_1 m_2$

Sec 19, Prob 1:

$$U = \frac{\alpha}{r^2}, \quad \alpha > 0$$

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{M dr / r^2}{\sqrt{2m [E - U(r)] - m^2 / r^2}}$$

Substitute:  $E = \frac{1}{2} m V_\infty^2$

$$M = \rho m V_\infty$$

$$\rightarrow \phi_0 = \int_{r_{\min}}^{\infty} \frac{\rho m V_\infty dr / r^2}{\sqrt{2m \left[ \frac{1}{2} m V_\infty^2 - U(r) \right] - \rho^2 m^2 V_\infty^2 / r^2}}$$

$$= \int_{r_{\min}}^{\infty} \frac{\rho dr / r^2}{\sqrt{1 - 2U(r)/m V_\infty^2 - \rho^2/r^2}}$$

$$= \int_{r_{\min}}^{\infty} \frac{\rho dr / r^2}{\sqrt{1 - \rho^2/r^2 - 2U(r)/m V_\infty^2}}$$

Substitute:  $U(r) = \frac{\alpha}{r^2}$

$$\sqrt{ } = \sqrt{1 - \rho^2/r^2 - \left( \frac{2\alpha}{m V_\infty^2} \right) \frac{1}{r^2}}$$

$$\therefore \sqrt{1 - \left( \rho^2 + \frac{2\alpha}{m V_\infty^2} \right) \frac{1}{r^2}} = \sqrt{1 - \frac{A^2}{r^2}}$$

$$A^2 = \rho^2 + \frac{2\alpha}{m V_\infty^2}$$

Thus,

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{\rho dr / r^2}{\sqrt{1 - \frac{A^2}{r^2}}}$$

$$= \int_A^{\infty} \frac{\rho dr / r^2}{\sqrt{1 - \frac{A^2}{r^2}}}$$

$$\text{Let } u = \frac{1}{r} \rightarrow du = -\frac{1}{r^2} dr$$

$$\frac{A^2}{r^2} = A^2 u^2$$

$$\phi_0 = - \int_{\frac{1}{A}}^{\frac{1}{A}} \frac{\rho du}{\sqrt{1 - A^2 u^2}}$$

$$= \int_0^{\frac{1}{A}} \frac{\rho du}{\sqrt{1 - A^2 u^2}}$$

$$\text{Let } \sin \theta = Au$$

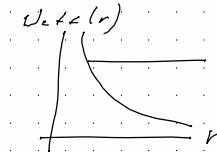
$$c \omega \theta d\theta = Adu$$

$$u=0, \frac{\pi}{A} \rightarrow \theta=0, \frac{\pi}{2}$$

$$\sqrt{1 - A^2 u^2} = \sqrt{1 - \sin^2 \theta} = \cos \theta$$

$$\rightarrow \phi_0 = \int_0^{\frac{\pi}{2}} \frac{\rho c \omega \theta d\theta / A}{\cos \theta} = \frac{\rho \frac{\pi}{2}}{A}$$

$$\boxed{\phi_0 = \frac{\pi}{2} \frac{\rho}{\sqrt{\rho^2 + \frac{2\alpha}{m V_\infty^2}}}}$$



$$\text{Repulsive scattering: } \chi + 2\phi_0 = \pi$$

$$\chi = \pi - 2\phi_0$$

$$\chi = \pi - \pi \frac{\rho}{\sqrt{\rho^2 + \frac{2\alpha}{m V_\infty^2}}} = \pi \left[ 1 - \frac{1}{\sqrt{1 + \frac{2\alpha}{\rho^2 m V_\infty^2}}} \right]$$

$$\left( \frac{\pi \rho}{\sqrt{\dots}} \right)^2 = (\pi - \chi)^2$$

$$\frac{\pi^2 \rho^2}{\rho^2 + \frac{2\alpha}{m V_\infty^2}} = (\pi - \chi)^2$$

$$\pi^2 \rho^2 = (\pi - \chi)^2 \rho^2 + (\pi - \chi)^2 \frac{2\alpha}{m V_\infty^2}$$

$$(\pi^2 - (\pi - \chi)^2) \rho^2 = (\pi - \chi)^2 \frac{2\alpha}{m V_\infty^2}$$

$$(\pi^2 - \pi^2 + 2\pi\chi - \chi^2) \rho^2 = (\pi - \chi)^2 \frac{2\alpha}{m V_\infty^2}$$

$$\rho^2 = \frac{(\pi - \chi)^2}{2\pi\chi - \chi^2} \frac{2\alpha}{m V_\infty^2}$$

$$\boxed{\rho = \frac{(\pi - \chi)}{\sqrt{2\pi\chi - \chi^2}} \sqrt{\frac{2\alpha}{m V_\infty^2}}}$$

Differential cross-section:

$$\begin{aligned} d\sigma &= 2\pi \rho d\rho \\ &= 2\pi \rho(x) \left| \frac{d\rho}{dx} \right| dx \\ &= \frac{\rho(x)}{\sin x} \left| \frac{d\rho}{dx} \right| d\Omega, \quad d\Omega = 2\pi \sin x dx \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d\rho}{dx} &= \frac{2\alpha}{\sqrt{mV_\infty^2}} \frac{-\sqrt{2\pi x - x^2} - 2\sqrt{(2\pi - 2x)(\pi - x)}}{2\pi x - x^2} \\ &= -\frac{\sqrt{2\alpha}}{mV_\infty^2} \frac{\sqrt{2\pi x - x^2} + \frac{(\pi - x)^2}{\sqrt{}}}{2\pi x - x^2} \\ &= -\frac{\sqrt{2\alpha}}{mV_\infty^2} \frac{2\pi x - x^2 + (\pi - x)^2}{(2\pi x - x^2)^{3/2}} \\ &= -\frac{\sqrt{2\alpha}}{mV_\infty^2} \frac{2\pi x - x^2 + \pi^2 + x^2 - 2\pi x}{(2\pi x - x^2)^{3/2}} \\ &= -\frac{\sqrt{2\alpha}}{mV_\infty^2} \frac{\pi^2}{(2\pi x - x^2)^{3/2}} \end{aligned}$$

So,

$$\begin{aligned} d\sigma &= \frac{(\pi - x)}{\sqrt{2\pi x - x^2}} \frac{\sqrt{2\alpha}}{mV_\infty^2} \frac{1}{\sin x} \frac{\sqrt{2\alpha}}{\sqrt{m\alpha}} \frac{\pi^2}{(2\pi x - x^2)^{3/2}} d\Omega \\ &\approx \boxed{\left[ \frac{(2\alpha)}{mV_\infty^2} \frac{d\Omega}{\sin x} \frac{\pi^2(\pi - x)}{(2\pi x - x^2)^{3/2}} \right]} \end{aligned}$$

Sec 20, Prob. 1 Small-angle scattering

start with (18.4):

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{\rho dr/r^2}{\sqrt{1 - \rho^2/r^2 - 2U/mV_\infty^2}}$$

Assume  $U$  is weak so that  $2U/mV_\infty^2 \ll 1$

$$\begin{aligned} \frac{1}{\sqrt{1 - \rho^2/r^2}} &= \frac{1}{\sqrt{(1 - \rho^2/r^2)} \left( 1 - \frac{2U/mV_\infty^2}{(1 - \rho^2/r^2)} \right)} \\ &\approx \frac{1}{\sqrt{1 - \rho^2/r^2}} \left( 1 + \frac{2U/mV_\infty^2}{1 - \rho^2/r^2} \right) \\ &= \frac{1}{\sqrt{1 - \rho^2/r^2}} + \frac{U/mV_\infty^2}{(1 - \rho^2/r^2)^{3/2}} \end{aligned}$$

can replace  $r_{\min}$  limit by  $\rho$ :

$$\int_{\rho}^{\infty} \frac{\rho dr/r^2}{\sqrt{1 - \rho^2/r^2}} = - \int_{\rho}^0 \frac{\rho dy}{\sqrt{1 - \rho^2 y^2}}$$

$$\begin{aligned} \text{let } u &= \frac{y}{r} \\ dy &= -\frac{1}{r^2} dr \\ \text{let } \rho u &= \sin \theta \\ \rho du &= \cos \theta d\theta \\ u = 1 &\rightarrow \theta = \frac{\pi}{2} \\ &= \boxed{\frac{\pi}{2}} \end{aligned}$$

Then,

$$\phi_0 \approx \frac{\pi}{2} + \frac{1}{m v_\infty^2} \int_p^\infty \frac{\rho dr / r^2 U(r)}{(1 - \rho^2/r^2)^{3/2}}$$
$$= \frac{\pi}{2} + \frac{1}{m v_\infty^2} \frac{\partial}{\partial p} \left[ \int_p^\infty \frac{U(r) dr}{\sqrt{1 - \rho^2/r^2}} \right]$$

Now:

$$\frac{\int_p^\infty U(r) dr}{r \sqrt{1 - \rho^2/r^2}} = u v \int_p^\infty - \int_p^\infty v dy$$

where  $u = U(r)$

$$dv = \frac{dr}{\sqrt{1 - \rho^2/r^2}} = \frac{r dr}{\sqrt{r^2 - \rho^2}} \quad x = r^2 - \rho^2$$
$$dx = 2r dr$$
$$= \frac{dx/2}{\sqrt{x}}$$

$$\rightarrow v = \frac{1}{2} \int \frac{dx}{\sqrt{x}} = \sqrt{x} + \text{const}$$
$$= \sqrt{r^2 - \rho^2} + \text{const}$$

so:

$$\frac{\int_p^\infty U(r) dr}{r \sqrt{1 - \rho^2/r^2}} = U(r) \cancel{\sqrt{r^2 - \rho^2}} \Big|_p^\infty - \int_p^\infty \left( \frac{dU}{dr} \right) dr \cancel{\sqrt{r^2 - \rho^2}}$$

assuming

$$U(r) \rightarrow 0 \text{ faster}$$

$$\text{than } \frac{1}{r} \text{ as } r \rightarrow \infty$$

so

$$\phi_0 = \frac{\pi}{2} + \frac{1}{m v_\infty^2} \frac{\partial}{\partial p} \left[ - \int_p^\infty \frac{dU}{dr} dr \sqrt{r^2 - \rho^2} \right]$$
$$= \frac{\pi}{2} + \frac{1}{m v_\infty^2} (-) \int_p^\infty \frac{dU}{dr} dr \frac{1}{2\sqrt{r^2 - \rho^2}} (-\cancel{\rho})$$
$$= \frac{\pi}{2} + \frac{p}{m v_\infty^2} \int_p^\infty dr \frac{dU/dr}{\sqrt{r^2 - \rho^2}}$$

Scattering angle  $X$ :

$$2\phi_0 + X = \pi$$

$$X = \pi - 2\phi_0$$

$$\rightarrow X = \pi - 2 \left( \frac{\pi}{2} + \frac{p}{m v_\infty^2} \int_p^\infty dr \frac{dU/dr}{\sqrt{r^2 - \rho^2}} \right)$$
$$= - \frac{2p}{m v_\infty^2} \int_p^\infty dr \frac{dU/dr}{\sqrt{r^2 - \rho^2}}$$

In terms of  $\Theta$ ,

$$\tan \Theta_1 = \frac{m_2 \sin X}{m_1 + m_2 \cos X} \rightarrow \Theta_1 \approx \frac{m_2 X}{m_1 + m_2}$$

Thur,

$$\begin{aligned}\theta' &\approx \left(\frac{m_2}{m_1+m_2}\right)x \\ &\approx \left(\frac{m_2}{m_1+m_2}\right) \left(\frac{-2\rho}{m_1 v_{\infty}^2}\right) \int_{\rho}^{\infty} dr \frac{dU/dr}{\sqrt{r^2 - \rho^2}} \\ &= \frac{-2\rho}{m_1 v_{\infty}^2} \int_{\rho}^{\infty} dr \frac{dU/dr}{\sqrt{r^2 - \rho^2}}\end{aligned}$$

which is Eq. (20,3)

Sec 21, Prob 1

$$x = a \cos(\omega t + \alpha)$$

Express  $a, \alpha$  in terms of  $x_0, v_0$

$$x_0 = a \cos \alpha$$

$$v = \dot{x} = -a \omega \sin(\omega t + \alpha)$$

$$\rightarrow v_0 = -a \omega \sin \alpha$$

$$\text{Thus, } \frac{v_0}{x_0} = -\omega \tan \alpha$$

$$\rightarrow \boxed{\tan \alpha = \frac{-v_0}{\omega x_0}}$$

$$\text{Also: } x_0 = a \cos \alpha$$

$$\frac{v_0}{\omega} = -a \sin \alpha$$

$$\rightarrow x_0^2 + \left(\frac{v_0}{\omega}\right)^2 = a^2 \cos^2 \alpha + a^2 \sin^2 \alpha \\ = a^2$$

$$\text{so } \boxed{a = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}}$$

Sec 21, Prob 2:

Diatomic molecules, different isotopes  
 $m_1, m_2$  and  $m'_1, m'_2$

$\xrightarrow{x}$       2-body  $\rightarrow$  1-body with com  
 $m_1 + m_2$        $T = \frac{1}{2} m \dot{x}^2, m = \frac{m_1 m_2}{m_1 + m_2}$   
 $U = \frac{1}{2} k x^2$

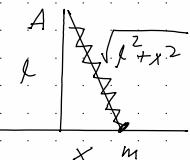
Theory,  $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}$

Similarly, for masses  $m'_1, m'_2$  with  $H' = H$ :

$$\omega' = \sqrt{\frac{k}{m'}} = \sqrt{\frac{k(m'_1 + m'_2)}{m'_1 m'_2}}$$

$$\rightarrow \frac{\omega'}{\omega} = \sqrt{\frac{k(m'_1 + m'_2)}{m'_1 m'_2}} \sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}}$$

$$= \sqrt{\left(\frac{m'_1 + m'_2}{m_1 + m_2}\right) \frac{m_1 m_2}{m'_1 m'_2}}$$

Sec 21, Prob 3:

$$U \approx F \delta t$$

$$\delta t = \sqrt{x^2 + x'^2} - t$$

$$= t \left( \sqrt{1 + \left(\frac{x}{t}\right)^2} - 1 \right)$$

$$= t \left( x + \frac{1}{2} \left(\frac{x}{t}\right)^2 + \dots - t \right)$$

$$\approx \frac{1}{2} \frac{x^2}{t}$$

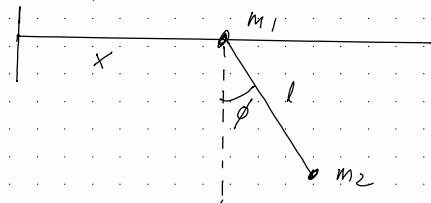
Theory,  $U \approx \frac{1}{2} \left(\frac{F}{\epsilon}\right) x^2 = \frac{1}{2} k x^2$  for  $H \equiv \frac{F}{\epsilon}$

Also,  $T = \frac{1}{2} m \dot{x}^2$

$$\rightarrow \omega = \sqrt{\frac{F}{\epsilon m}}$$

Sec 21, Prob 5.

Figure 2 from Sec 5:



Recall from Sec 14, Prob 3:

$$E = \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left( 1 - \left( \frac{m_2}{m_1 + m_2} \right) \cos^2 \phi \right) - m_2 g l \cos \phi$$

Now  $\phi = 0$  corresponds to stable equilibrium.

Small oscillations:  $\phi \ll 1$

Since  $\dot{\phi}^2$  is small can set  $\phi = 0$  in  $\left( 1 - \left( \frac{m_2}{m_1 + m_2} \right) \cos^2 \phi \right)$

$$\rightarrow 1 - \left( \frac{m_2}{m_1 + m_2} \right) \cos^2 \phi \rightarrow 1 - \frac{m_2}{m_1 + m_2} = \frac{m_1}{m_1 + m_2}$$

Then

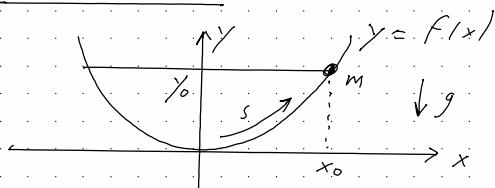
$$E = \frac{1}{2} m_1 l^2 \dot{\phi}^2 - m_2 g l \left( 1 - \frac{1}{2} \dot{\phi}^2 \right)$$

$$= \frac{1}{2} m_1 l^2 \dot{\phi}^2 + \frac{1}{2} m_2 g l \dot{\phi}^2 - \frac{m_2 g l}{m_1 + m_2}$$

Thus,

$$\omega = \sqrt{\frac{m_2 g l}{m_1 l^2}} = \sqrt{\left( \frac{m_1 + m_2}{m_1} \right) \frac{g}{l}} \quad (\rightarrow \sqrt{\frac{g}{l}} \text{ for } m_1 \gg m_2)$$

Sec 21, Prob 6



Find  $y = f(x)$  so that period  $P(y_0)$  is indep. of  $y_0$ .

$$\begin{aligned} \text{Need: } E &= \frac{1}{2} m v^2 + mgy \\ &= \frac{1}{2} m \left( \frac{ds}{dt} \right)^2 + \frac{1}{2} I s^2 \end{aligned}$$

where  $s$  = arc length along the curve

$$\begin{aligned} &= \int_0^s ds \\ &= \int_0^s \sqrt{dx^2 + dy^2} \\ &= \int_0^x dx \sqrt{1 + y'^2} \end{aligned}$$

$$\text{so } mgy = \frac{1}{2} I s^2$$

$$y = \frac{1}{2g} \frac{I}{m} s^2 \equiv A^2 s^2$$

$$\rightarrow \sqrt{y} = As, \quad A = \frac{1}{\sqrt{2g}} \sqrt{\frac{I}{m}}$$

$$\frac{1}{2\sqrt{y}} \frac{dy}{dx} = A \frac{ds}{dx} = A \sqrt{1+y'^2}$$

$$\text{Thus, } \frac{1}{2} \frac{y'}{\sqrt{y}} = A \sqrt{1+y'^2}$$

$$y'^2 = 4A^2 y (1+y'^2)$$

$$y'^2(1-4A^2y) = 4A^2y$$

$$\frac{dy}{dx} = y' = \sqrt{\frac{4A^2y}{1-4A^2y}}$$

$$\text{so } x = \int_0^y dy \sqrt{\frac{1-4A^2y}{4A^2y}}$$

multiple substitution

$$y = \frac{1}{8A^2} (1 - \cos \theta)$$

$$\frac{dy}{d\theta} = \frac{1}{8A^2} \sin \theta d\theta = \frac{1}{8A^2} \sqrt{1 - \cos^2 \theta} d\theta$$

$$\rightarrow \frac{1-4A^2y}{4A^2y} = \frac{1 - \frac{1}{2}(1 - \cos \theta)}{\frac{1}{2}(1 - \cos \theta)} = \frac{\frac{1}{2}(1 + \cos \theta)}{\frac{1}{2}(1 - \cos \theta)} = \frac{1 + \cos \theta}{1 - \cos \theta}$$

$$\text{Thus, } x = \int \frac{1}{8A^2} \sqrt{1 - \cos^2 \theta} d\theta \sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}}$$

$$= \frac{1}{8A^2} \int d\theta \sqrt{\frac{(1 - \cos \theta)(1 + \cos \theta)(1 + \cos \theta)}{1 - \cos \theta}}$$

$$x = \frac{1}{8A^2} \int d\theta (1 + \cos \theta)$$

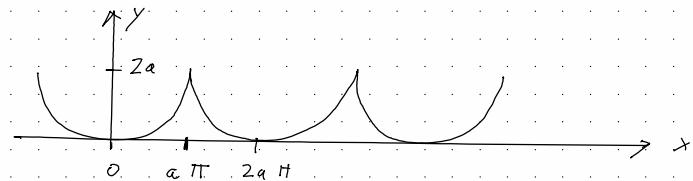
$$= \frac{1}{8A^2} (\theta + \sin \theta)$$

thus,

$$x = \frac{1}{8A^2} (\theta + \sin \theta) = a(\theta + \sin \theta)$$

$$y = \frac{1}{8A^2} (1 - \cos \theta) = a(1 - \cos \theta)$$

which are parametric equations for a cycloid



$$\text{NOTE: } 8A^2 = 8 \frac{1}{2g} \frac{\pi^2}{m} = \frac{4}{g} \frac{\pi^2}{m} = \frac{4}{g} w^2$$

Sec 22, Prob 1:

$$(a) F = F_0 = \cos \omega t$$

$$x = \dot{x} = 0 \text{ at } t=0 \Rightarrow \xi_0 = 0$$

$$\begin{aligned}\rightarrow \xi(t) &= e^{i\omega t} \left[ F_0 + \int_0^t dt' \frac{F(t')}{m} e^{-i\omega t'} \right] \\ &= e^{i\omega t} \frac{F_0}{m} \int_0^t dt' e^{-i\omega t'} \\ &= e^{i\omega t} \left( \frac{F_0}{-i\omega} \right) e^{-i\omega t} \Big|_0^t \\ &= e^{i\omega t} \left( \frac{F_0}{-i\omega} \right) (e^{-i\omega t} - 1) \\ &= \frac{iF_0}{m\omega} (1 - e^{i\omega t}) \\ &= \frac{iF_0}{m\omega} (1 - \cos \omega t - i \sin \omega t)\end{aligned}$$

$$\text{Now: } x(t) = \frac{\text{Im}(\xi(t))}{\omega}, \quad \dot{x}(t) = \text{Re}(\xi(t))$$

Thus,

$$\boxed{x(t) = \frac{F_0}{m\omega^2} (1 - \cos \omega t)}$$

$$(b) F(t) = a t$$

$$\xi(t) = e^{i\omega t} \frac{a}{m} \int_0^t dt' t' e^{-i\omega t'}$$

$$\text{Let } u = t' \rightarrow du = dt'$$

$$dv = e^{-i\omega t'} dt' \rightarrow v = \frac{1}{-i\omega} e^{-i\omega t'}$$

Thus,

$$\xi(t) = e^{i\omega t} \frac{a}{m} \left[ \frac{-t}{-i\omega} e^{-i\omega t} \right]_0^t + \frac{1}{-i\omega} \int_0^t dt' e^{-i\omega t'}$$

$$= e^{i\omega t} \frac{a}{m} \left[ \frac{t}{i\omega} e^{-i\omega t} + \frac{1}{\omega^2} (e^{-i\omega t} - 1) \right]$$

$$= \frac{a}{m} \left[ \left( \frac{it}{\omega} + \frac{1}{\omega^2} \right) - \frac{1}{\omega^2} e^{i\omega t} \right]$$

$$= \frac{a}{m} \left[ \left( \frac{it}{\omega} + \frac{1}{\omega^2} \right) - \frac{1}{\omega^2} (\cos \omega t + i \sin \omega t) \right]$$

$$= \frac{a}{m} \left[ \frac{1}{\omega^2} \left( 1 - \cos \omega t \right) + i \left( \frac{t}{\omega} - \frac{\sin \omega t}{\omega^2} \right) \right]$$

$$\rightarrow x(t) = \frac{\text{Im}(\xi(t))}{\omega}$$

$$= \boxed{\frac{a}{m\omega^3} (\omega t - \sin \omega t)}$$

$$(c) F(t) = F_0 \exp(-\alpha t)$$

$$\begin{aligned}\xi(t) &= e^{i\omega t} \frac{F_0}{m} \int_0^t d\bar{t} e^{-i\omega \bar{t}} e^{-i\omega t} \\ &= \frac{F_0}{m} e^{i\omega t} \frac{1}{-(i\omega + \alpha)} e^{-(i\omega + \alpha)t} \int_0^t \\ &= \frac{F_0}{m} e^{i\omega t} \frac{1}{-(i\omega + \alpha)} (e^{-(i\omega + \alpha)t} - 1)\end{aligned}$$

NOTE:  $\frac{1}{-(i\omega + \alpha)} \cdot \left(\frac{i\omega - \alpha}{i\omega - \alpha}\right) = \frac{i\omega - \alpha}{\alpha^2 + \omega^2}$

$$\begin{aligned}\rightarrow \xi(t) &= \frac{F_0}{m} \left( \frac{i\omega - \alpha}{\alpha^2 + \omega^2} \right) \left[ e^{-\alpha t} - e^{i\omega t} \right] \\ &= \frac{F_0}{m} \left( \frac{i\omega - \alpha}{\alpha^2 + \omega^2} \right) \left[ e^{-\alpha t} - \cos \omega t - i \sin \omega t \right]\end{aligned}$$

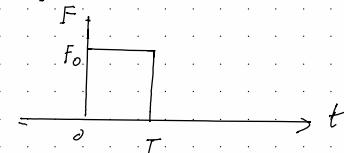
$$\begin{aligned}&= \frac{F_0}{m(\alpha^2 + \omega^2)} \left[ i(w e^{-\alpha t} - \omega \cos \omega t + \omega \sin \omega t) \right. \\ &\quad \left. - \alpha e^{-\alpha t} + \alpha \cos \omega t + \omega \sin \omega t \right]\end{aligned}$$

thus,

$$x(t) = F_m(\xi(t)) / \omega$$

$$\begin{aligned}&= \frac{F_0}{m\omega(\alpha^2 + \omega^2)} \left( w e^{-\alpha t} - \omega \cos \omega t + \omega \sin \omega t \right) \\ &= \boxed{\frac{F_0}{m(\alpha^2 + \omega^2)} \left( e^{-\alpha t} - \cos \omega t + \frac{\omega}{w} \sin \omega t \right)}$$

Sec 22, Prob 3:



Initial condition

$$\xi_0 = 0 \quad (\text{since } x_0 = \dot{x}_0 = 0)$$

For  $t > T$ :

$$\begin{aligned}\xi(t) &= e^{i\omega t} \int_0^t d\bar{t} \frac{F_0}{m} e^{-i\omega \bar{t}} \\ &= e^{i\omega t} \frac{F_0}{m} \left( \frac{1}{-i\omega} \right) (e^{-i\omega T} - 1) \\ &= \frac{iF_0}{m\omega} (e^{i\omega(t-T)} - e^{i\omega t}) \\ &= \frac{iF_0}{m\omega} \left[ \cos(\omega(t-T)) + i \sin(\omega(t-T)) \right. \\ &\quad \left. - \cos \omega t - i \sin \omega t \right] \\ &= \frac{iF_0}{m\omega} \left( \cos(\omega(t-T)) - \cos(\omega t) \right) \\ &\quad - \frac{F_0}{m\omega} (\sin(\omega t - T) - \sin(\omega t))\end{aligned}$$

$$\rightarrow x(t) = \frac{I_m(\xi(t))}{\omega}$$

$$= \frac{F_0}{m\omega^2} (\cos(\omega(t-T)) - \cos(\omega t))$$

We can also write for  $t > T$ :

$$x(t) = c_1 \cos(\omega(t-T)) + c_2 \sin(\omega(t-T))$$

$$\rightarrow \dot{x}(t) = -c_1 \omega \sin(\omega(t-T)) + c_2 \omega \cos(\omega(t-T))$$

$$\text{Thus, } x(T) = c_1, \quad \dot{x}(T) = c_2 \omega$$

Match with:

$$x(t) = \frac{F_0}{m\omega^2} (\cos(\omega(t-T)) - \cos(\omega t))$$

$$\dot{x}(t) = \frac{F_0}{m\omega} (-\sin(\omega(t-T)) + \sin(\omega t))$$

at  $t = T$ ,

$$\rightarrow c_1 = \frac{F_0}{m\omega^2} (1 - \cos(\omega T))$$

$$c_2 \omega = \frac{F_0}{m\omega} \sin(\omega T)$$

Thus,

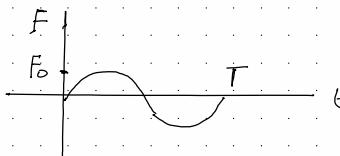
$$\alpha = \sqrt{c_1^2 + c_2^2}$$

$$= \frac{F_0}{m\omega^2} \sqrt{1 + \cos^2(\omega T) - 2 \cos(\omega T) + \sin^2(\omega T)}$$

$$= \frac{F_0}{m\omega^2} \sqrt{2(1 - \cos(\omega T))}$$

$$= \boxed{\frac{2F_0}{m\omega^2} \sin\left(\frac{\omega T}{2}\right)} \quad \text{using } 1 - \cos\theta = 2\sin^2(\frac{\theta}{2})$$

Sec 22, Prob 5:



$$T = \frac{2\pi}{\omega}$$

$$f(t) = F_0 \sin \omega t, \quad \xi_0 = 0$$

For  $t > T$ :

$$\xi(t) = e^{i\omega t} \frac{F_0}{m} \int_0^T dt' \sin \omega t' e^{-i\omega t'}$$

$$= \frac{F_0}{m} e^{i\omega t} \frac{1}{2i} \int_0^T dt' (e^{i\omega t'} - e^{-i\omega t'}) e^{-i\omega t'}$$

$$= \frac{F_0}{2mi} e^{i\omega t} \int_0^T dt' (1 - e^{-i2\omega t'})$$

$$= \frac{F_0}{2mi} e^{i\omega t} \left( T + \frac{1}{i2\omega} (e^{-i2\omega T} - 1) \right)$$

$$= \frac{F_0 T}{2mi} e^{i\omega t}$$

$$\text{Thus, } x(T) = \frac{\text{Im}(\xi(T))}{\omega} \quad \left| \begin{array}{l} \dot{x}(T) = \text{Re}(\xi(T)) \\ = 0 \end{array} \right.$$

$$= -\frac{F_0 T}{2mw}$$

We can also write for  $t > T$ :

$$x(t) = c_1 \cos(\omega(t-T)) + c_2 \sin(\omega(t-T))$$

$$\rightarrow \dot{x}(t) = -c_1 \omega \sin(\omega(t-T)) + c_2 \omega \cos(\omega(t-T))$$

$T^{\text{tors}}$

$$x(T) = c_1, \quad \dot{x}(T) = c_2 \omega$$

match w.t.

$$x(T) = \frac{-F_0 T}{2m\omega}, \quad \dot{x}(T) = 0$$

$$\rightarrow c_1 = \frac{-F_0 T}{2m\omega}, \quad c_2 = 0$$

$$\begin{aligned} \text{so } a &= \sqrt{c_1^2 + c_2^2} \\ &= \frac{F_0 T}{2m\omega} \end{aligned}$$

$$= \frac{F_0 \pi T / \omega}{2m\omega}$$

$$= \frac{F_0 \pi}{m\omega^2}$$

Sec 23, Prob 1:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}\omega_0^2(x^2 + y^2) + \alpha xy$$

$$= \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k - \frac{1}{2} \sum_{i,k} K_{ik} x_i x_k$$

where  $m_{ik} = \begin{array}{|c|c|} \hline 1 & 0 \\ 0 & 1 \\ \hline \end{array}$  and

$$\sum_{i,k} K_{ik} x_i x_k = \omega_0^2(x^2 + y^2) - 2\alpha xy$$

$$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\text{so } K_{ik} = \begin{pmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{pmatrix}$$

EOMs:

$$\frac{d}{dt} \sum_k m_{ik} \dot{x}_k = - \sum_k K_{ik} x_k$$

$$\sum_k m_{ik} \ddot{x}_k = - \sum_k K_{ik} x_k$$

$$\sum_k (m_{ik} \ddot{x}_k + K_{ik} x_k) = 0$$

Trial solution:  $x_k = A_k e^{i\omega t}$

$$\rightarrow \sum_k (-\omega^2 m_{ik} + K_{ik}) A_k e^{i\omega t} = 0$$

Thus,

$$\sum (k_{iH} - \omega^2 m_{iH}) A_H = 0$$

$$\rightarrow \det(k_i - \omega^2 m_{iH}) = 0$$

$$0 = \det \begin{vmatrix} \omega_0^2 - \omega^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega^2 \end{vmatrix}$$

$$= (\omega_0^2 - \omega^2)^2 - \alpha^2$$

$$= \omega^4 - 2\omega^2\omega_0^2 + (\omega_0^4 - \alpha^2)$$

Quadratic equation:

$$\omega_t^2 = \frac{2\omega_0^2 \pm \sqrt{4\omega_0^4 - 4(\omega_0^4 - \alpha^2)}}{2}$$

$$= \frac{2\omega_0^2 \pm 2\alpha}{2}$$

$$= [\omega_0^2 \pm \alpha] \text{ (normal mode freqs)}$$

/Normal mode vectors:

$$\underline{\omega_t^2 = \omega_0^2 + \alpha}$$

$$\sum_H (k_{iH} - \omega_t^2 m_{iH}) A_H = 0$$

$$\begin{vmatrix} \omega_0^2 - \omega_t^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega_t^2 \end{vmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} -\alpha & -\alpha \\ -\alpha & -\alpha \end{vmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Thus, } -\alpha(A_1 + A_2) = 0$$

$$A_2 = -A_1$$

$$\text{so Eigenvector is } \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = V_+$$

$$\underline{\omega_0^2 = \omega_-^2}$$

$$\begin{vmatrix} \omega_0^2 - \omega_-^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega_-^2 \end{vmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{vmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Thus, } A_2 = A_1$$

$$\rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = V_-$$

$V_+$  corresponds to out-of-phase motion:

$$V_+ e^{i\omega t} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\sqrt{\omega_0^2 + \alpha} t}$$

$V_-$  corresponds to in-phase motion:

$$V_- e^{i\omega t} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\sqrt{\omega_0^2 - \alpha} t}$$

NOTE:

For weak coupling,  $\alpha \ll \omega_0^2$ , we have

$$\begin{aligned}\omega_{\pm}^2 &= \omega_0^2 \pm \alpha \\ &= \omega_0^2 \left( 1 \pm \frac{\alpha}{\omega_0^2} \right)\end{aligned}$$

$$\begin{aligned}\omega_{\pm} &= \omega_0 \sqrt{1 \pm \frac{\alpha}{\omega_0^2}} \\ &\approx \omega_0 \left( 1 \pm \frac{\alpha}{2\omega_0^2} \right) \quad \text{nearly equal freq}\end{aligned}$$

Bent freq:

$$\begin{aligned}\omega_+ - \omega_- &\approx \omega_0 \left( 1 + \frac{\alpha}{2\omega_0^2} \right) - \omega_0 \left( 1 - \frac{\alpha}{2\omega_0^2} \right) \\ &= \boxed{\frac{\alpha}{\omega_0}}\end{aligned}$$

Sec 23, Prob 3:

$$U = \frac{1}{2} k r^2, \quad \text{Simple oscillator}$$

central potential  $\rightarrow$  motion in a plane ( $x, y$ )  
or ( $r, \phi$ )

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k (x^2 + y^2)$$

$$= \left( \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \right) + \left( \frac{1}{2} m \dot{y}^2 - \frac{1}{2} k y^2 \right)$$

EOMs:

$$m \ddot{x} = -kx \rightarrow x = a \cos(\omega t + \alpha)$$

$$m \ddot{y} = -ky \rightarrow y = b \cos(\omega t + \beta)$$

$$\text{where } \omega = \sqrt{\frac{k}{m}}$$

$$\begin{aligned}\text{Write } \omega t + \beta &= \omega t + \alpha + (\beta - \alpha) \\ &= \phi + \delta\end{aligned}$$

$$\text{where } \phi \equiv \omega t + \alpha, \quad \delta \equiv \beta - \alpha$$

$$\text{Then } x = a \cos \phi \rightarrow \boxed{\cos \phi = \frac{x}{a}}$$

$$\begin{aligned}y &= b \cos(\phi + \delta) \\ &= b (\cos \phi \cos \delta - \sin \phi \sin \delta)\end{aligned}$$

$$\begin{aligned}\rightarrow \frac{y}{b} &= \cos \phi \cos \delta - \sin \phi \sin \delta \\ &= \left( \frac{x}{a} \right) \cos \delta - \sin \phi \sin \delta\end{aligned}$$

$$\text{so } \boxed{\sin \phi = \frac{1}{\sin \delta} \left[ \left( \frac{x}{a} \right) \cos \delta - \frac{y}{b} \right]}$$

Square and add two boxed expressions

$$\cos^2 \phi = \left(\frac{x}{a}\right)^2$$

$$+ \sin^2 \phi = \frac{1}{\sin^2 \delta} \left[ \left(\frac{x}{a}\right)^2 \cos^2 \delta + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab} \cos \delta \right]$$

$$1 = \left(\frac{x}{a}\right)^2 \left(1 + \frac{\cos^2 \delta}{\sin^2 \delta}\right) + \left(\frac{y}{b}\right)^2 \frac{1}{\sin^2 \delta} - \frac{2xy}{ab} \frac{\cos \delta}{\sin^2 \delta}$$

$$\rightarrow \sin^2 \delta = \left(\frac{x}{a}\right)^2 \underbrace{\left(\sin^2 \delta + \cos^2 \delta\right)}_{=1} + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab} \cos \delta$$

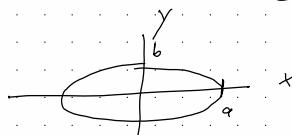
$$= \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab} \cos \delta$$

$$\text{Thus, } \boxed{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab} \cos \delta = \sin^2 \delta}$$

Suppose  $\delta = \frac{\pi}{2}$

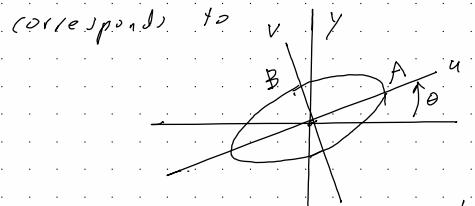
$$\text{Then } \cos \delta = 0, \sin \delta = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad (\text{ellipse centered at origin with semi-major axis } a, \text{ semi-minor axis } b)$$



In general

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab} \cos \delta = \sin^2 \delta$$



Find angle  $\theta$  such that the above expression maps to

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$$

$$\text{Now: } \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\rightarrow \sin^2 \delta = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab} \cos \delta$$

$$= \frac{1}{a^2} (\cos \theta u - \sin \theta v)^2 + \frac{1}{b^2} (\sin \theta u + \cos \theta v)^2 - \frac{2(\cos \theta u - \sin \theta v)(\sin \theta u + \cos \theta v)}{ab} \cos \delta$$

$$\begin{aligned}
&= \frac{1}{a^2} (u^2 \cos^2 \theta + v^2 \sin^2 \theta - 2uv \sin \theta \cos \theta) \\
&+ \frac{1}{b^2} (u^2 \sin^2 \theta + v^2 \cos^2 \theta + 2uv \sin \theta \cos \theta) \\
&- \frac{2}{ab} \cos \delta (u^2 \sin \theta \cos \theta - v^2 \sin \theta \cos \theta + uv \cos 2\theta) \\
&= u^2 \left( \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} - 2 \frac{\sin \theta \cos \theta \cos \delta}{ab} \right) \\
&+ v^2 \left( \frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{b^2} + 2 \frac{\sin \theta \cos \theta \cos \delta}{ab} \right) \\
&+ 2uv \left( -\frac{\sin \theta \cos \theta}{a^2} + \frac{\sin \theta \cos \theta}{b^2} - \frac{\cos 2\theta \cos \delta}{ab} \right)
\end{aligned}$$

We can make the factor multiplying  $4v$  equal to zero by choosing  $\theta$  appropriately:

$$-\sin \theta \cos \theta \left( \frac{1}{a^2} - \frac{1}{b^2} \right) = \frac{\cos 2\theta \cos \delta}{ab}$$

$$\frac{1}{2} \sin 2\theta \left( \frac{a^2 - b^2}{a^2 b^2} \right) = \frac{\cos 2\theta \cos \delta}{ab}$$

$$\boxed{\tan 2\theta = \left( \frac{2ab}{a^2 - b^2} \right) \cos \delta}$$

NOTE: if  $a=b$  then  $\theta = \cos 2\theta$

$$\rightarrow \theta = \frac{\pi}{4}$$

so choosing  $\theta$  as above, we have

$$\begin{aligned}
\sin^2 \delta &= u^2 \textcircled{1} + v^2 \textcircled{2} \\
1 &= u^2 \textcircled{1} + v^2 \frac{\textcircled{2}}{\sin^2 \delta}
\end{aligned}$$

$$= \left( \frac{u}{A} \right)^2 + \left( \frac{v}{B} \right)^2$$

$$\text{where } A^2 = \frac{\sin^2 \delta}{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} - 2 \frac{\sin \theta \cos \theta \cos \delta}{ab}}$$

$$B^2 = \frac{\sin^2 \delta}{\left( \frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{b^2} + 2 \frac{\sin \theta \cos \theta \cos \delta}{ab} \right)}$$