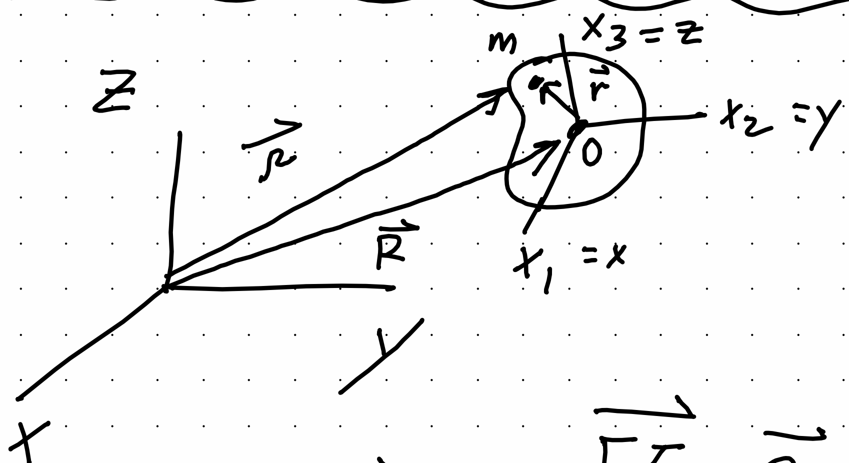
Lec #23: Tuesday Nov 10th

- Quiz #5: Thursday
- Midterm #2: Next Thursday 11/19 (Scattering, small oscillations, some rigid body)
- Today: Rigid body motion
(Sec 31–36, 38, 39)

38 — static equilibrium

39 — non-inertial



O: at COM (usually)

x_1, x_2, x_3: <u>fixed</u> in RB

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}$$

$\vec{\Omega}$: angular velocity vector

$$\left.\begin{aligned} \vec{V} &= \frac{d\vec{R}}{dt} \\ \vec{\Omega} &= \frac{d\vec{\phi}}{dt} \end{aligned}\right\} \dot{q}_i$$

$(\vec{R}, \vec{\phi})$: 6 DOF $= q_i$

$$T = \frac{1}{2} m v^2$$

$$\vec{p} = m \vec{v}$$

,

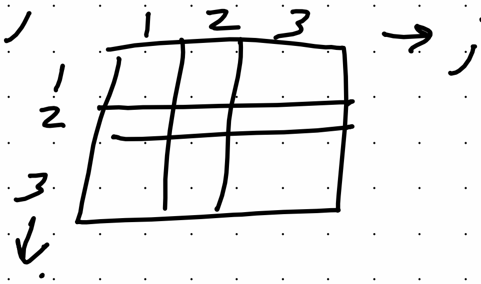
rotational quantities

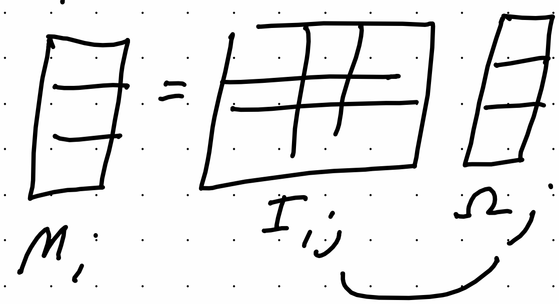
$$T_{rot} = \frac{1}{2} I \Omega^2$$

$$\vec{M} = I \vec{\Omega} \rightarrow M_i = \sum_j I_{ij} \Omega_j$$

I : moment of inertia

$\longrightarrow$ I_{ij} : inertia tensor





KE:

$$\vec{v}_a = \vec{V} + \vec{\Omega} \times \vec{r}_a$$

$$T = \frac{1}{2} \sum_a m_a |\vec{v}_a|^2$$

$$= \frac{1}{2} \sum_a m_a |\vec{V} + \vec{\Omega} \times \vec{r}_a|^2$$

$$|\vec{A} + \vec{B}|^2 = A^2 + B^2 + 2\vec{A}\cdot\vec{B}$$

$$= \frac{1}{2} \sum_a m_a \left(|\vec{V}|^2 + |\vec{\Omega} \times \vec{r}_a|^2 + 2\vec{V}\cdot(\vec{\Omega} \times \vec{r}_a) \right)$$

$$① = \frac{1}{2} \sum_a m_a |\vec{V}|^2 = \boxed{\frac{1}{2} \mu V^2}$$

total mass

$$③ = \sum_a m_a \vec{V} \cdot (\vec{\Omega} \times \vec{r}_a)$$

$$= \left(\sum_a m_a \vec{r}_a\right) \cdot (\vec{V} \times \vec{\Omega})$$

$$= \underbrace{\mu \vec{R}_{com}}_{=0 \text{ for } \boxed{O \text{ at COM}}} \cdot (\vec{V} \times \vec{\Omega})$$

$$\vec{A}\cdot(\vec{B} \times \vec{C}) = \vec{B}\cdot(\vec{C} \times \vec{A}) = \vec{C}\cdot(\vec{A} \times \vec{B})$$

(2) $= \frac{1}{2} \sum_a m_a \left| \vec{\Omega} \times \vec{r_a} \right|^2$

$$= \frac{1}{2} \sum_a m_a \left(\vec{\Omega} \times \vec{r_a}\right) \cdot \left(\vec{\Omega} \times \vec{r_a}\right)$$

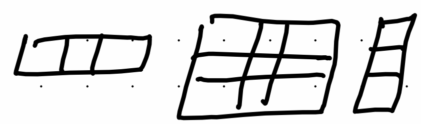
$$= \frac{1}{2} \sum_a m_a \; \vec{\Omega} \cdot \left(\vec{r_a} \times \left(\vec{\Omega} \times \vec{r_a}\right)\right)$$

$$= \frac{1}{2} \sum_a m_a \; \vec{\Omega} \cdot \left(\vec{\Omega}\, r_a^2 - \vec{r_a} \left(\vec{r_a} \cdot \vec{\Omega}\right)\right)$$

$$= \frac{1}{2} \sum_a m_a \left(\Omega^2 r_a^2 - \left(\vec{\Omega} \cdot \vec{r_a}\right)^2 \right)$$

$$= \frac{1}{2} \sum_a m_a \left(\sum_{i,j} \Omega_i \Omega_j \delta_{ij}\, r_a^2 - \sum_{i,j} \Omega_i r_{ai} \Omega_j r_{aj} \right)$$

$$= \frac{1}{2} \sum_{i,j} \left(\sum_a m_a \left(r_a^2 \delta_{ij} - r_{ai} r_{aj} \right)\right) \Omega_i \Omega_j$$

$$= \boxed{\frac{1}{2} \sum_{i,j} I_{ij}\, \Omega_i \Omega_j} \quad \left(= \frac{1}{2} \right.$$  $\left. \right)$

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B} \left(\vec{A} \cdot \vec{C}\right) - \vec{C} \left(\vec{A} \cdot \vec{B}\right)$$

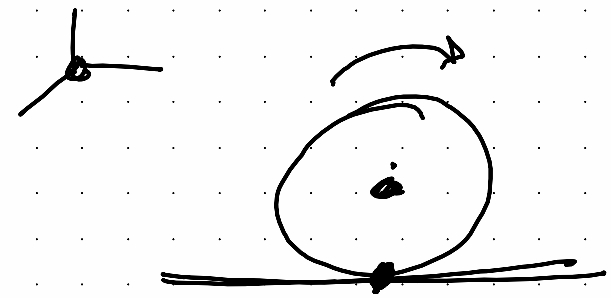
$$\boxed{T = \underbrace{\frac{1}{2}\mu V^2}_{\text{trans}} + \underbrace{\frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\Omega_j}_{\text{rotational}}}$$

$+$ [illegible]

$\left(\frac{1}{2} I \Omega^2\right)$ Freshman physics

for COM at origin
OF RB Frame

$\vec{M}$: wrt COM of body



$$\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a$$

$$= \sum_a m_a\, \vec{r}_a \times \vec{v}_a$$

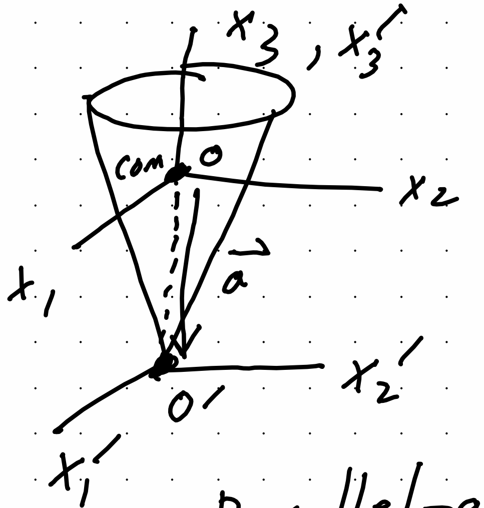
$$= \sum_a m_a\, \vec{r}_a \times \left(\vec{V} + \vec{\Omega} \times \vec{r}_a\right)$$

$$= \sum_a m_a\, \vec{r}_a \times \left(\vec{\Omega} \times \vec{r}_a\right)$$

$$= \sum_a m_a \left(\vec{\Omega}\, r_a^2 - \vec{r}_a\left(\vec{r}_a \cdot \vec{\Omega}\right)\right)$$

$$\boxed{M_i = \sum_j I_{ij}\,\Omega_j}$$

$\vec{M} = I\vec{\Omega}$ (Fresh. physics)



parallel-axis theorem

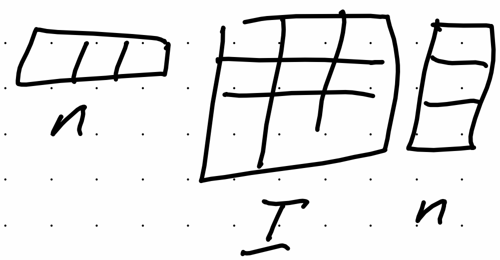
$$\underbrace{I'_{ij}}_{\text{wrt } O'} \, , \quad \underbrace{I_{ij}}_{\text{wrt } O \text{ (com)}}$$

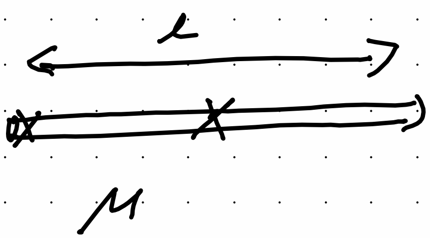
$$I'_{ij} = I_{ij} + \mu(a^2\delta_{ij} - a_i a_j)$$

$\vec{a}$: vector from O to O'

$$\underbrace{I(\hat{n})}_{\text{moment of inertia}} = \sum_{i,j} I_{ij} \underbrace{n_i n_j}$$

$\hat{n}$: axis of rotation





$$\boxed{\begin{aligned} I_{com} &= \tfrac{1}{12}\mu l^2 \\ I_{end} &= \tfrac{1}{3}\mu l^2 \end{aligned}}$$

$$\boxed{\underbrace{I_{ij}}_{\substack{3\times 3 \text{ real} \\ \text{symmetric}}} = \sum_a m_a \left(r_a^2 \delta_{ij} - r_{ai} r_{aj} \right) = \int \underbrace{\rho\, dV}_{dm} \left(r^2 \delta_{ij} - r_i r_j \right)}$$

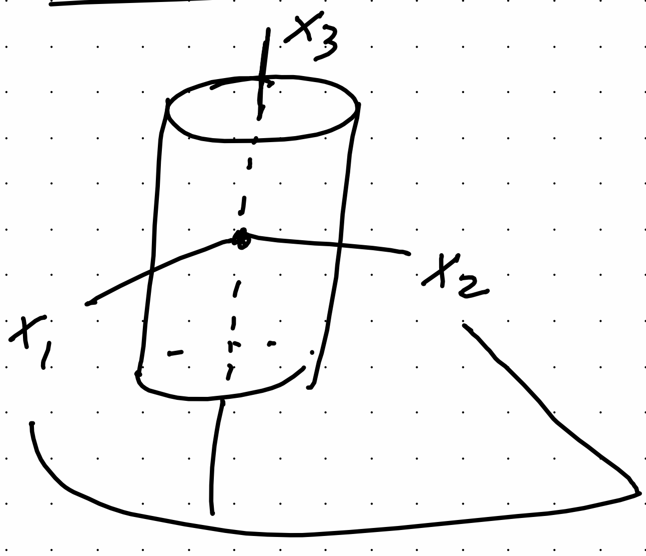
can always be diagonalized

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

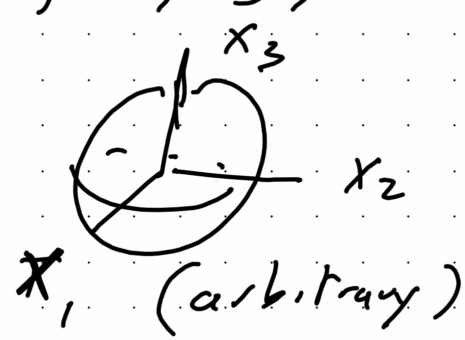
$$I_{ij} = I_i \delta_{ij}$$

$$T_{rot} = \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j = \frac{1}{2}\left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 \right)$$

<u>principle axes:</u> (x_1, x_2, x_3)

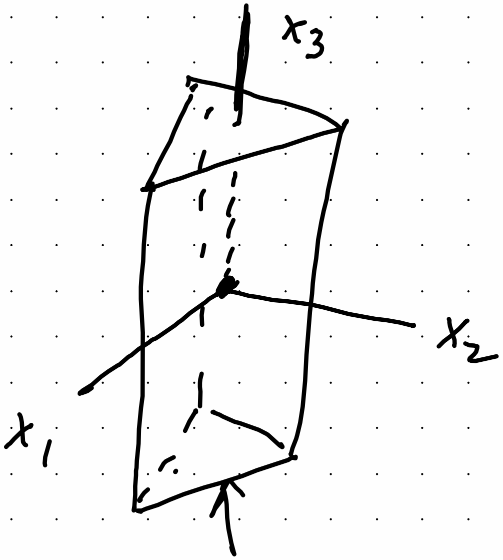


arbitrary

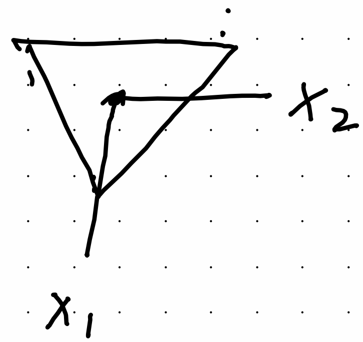


$$M_i = \sum_j I_{ij} \Omega_j$$

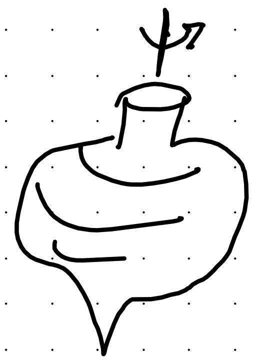
$$M_1 = I_1 \Omega_1, \quad M_2 = I_2 \Omega_2, \quad M_3 = I_3 \Omega_3$$

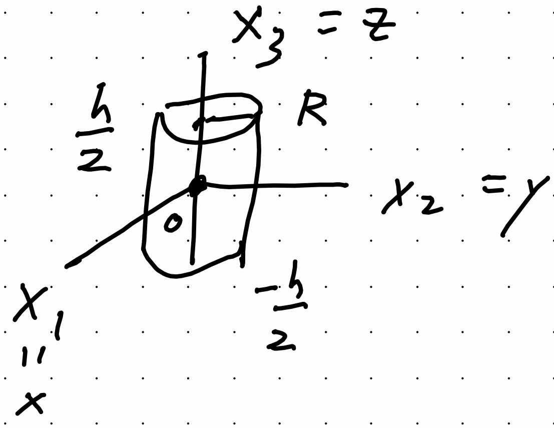


equilateral triang



$$\underbrace{I_1 = I_2 \neq I_3}_{\text{symmetrical top}}$$





total mass M

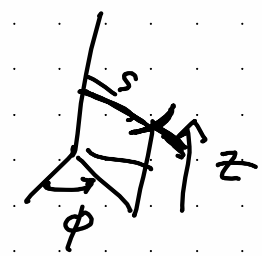
$$\rho = \frac{M}{\text{volume}} = \frac{M}{\pi R^2 h}$$

$$I_3 = I_{33} = \int \rho\, dV \left(r^2 \underbrace{\delta_{33}}_{1} - \underbrace{r_3 r_3}_{z^2} \right)$$

$$= \int \rho\, dV \left(r^2 - z^2 \right)$$

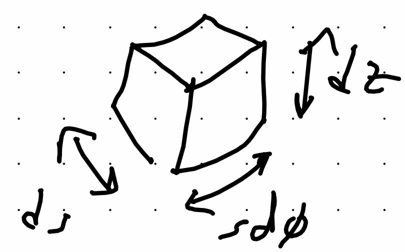
$$= \int \rho\, dV \left(x^2 + y^2 \right) = \int \rho\, dV s^2$$

<u>cylindrical:</u> s, ϕ, z $\qquad s^2 = x^2 + y^2$



$$dV = ds\; s\,d\phi\; dz$$

$$= s\,ds\,d\phi\,dz$$



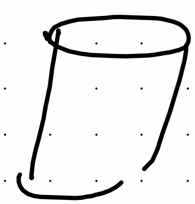
$$I_3 = \int \rho \, dV \, s^2$$

$$= \frac{M}{\pi R^2 h} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-h/2}^{h/2} dz}_{h} \underbrace{\int_0^{R} s^3 \, ds}_{\frac{R^4}{4}}$$

$\cancel{4\pi}$

$$= \frac{M}{\cancel{\pi} \cancel{R^2} \cancel{h}} \cdot 2 \cancel{\pi} \cancel{h} \frac{R^{\cancel{4}\,2}}{4}$$

$$= \boxed{\frac{1}{2} M R^2}$$



$$I_1 = I_2 \equiv I$$

$$I_1 = \int \rho \, dV (r^2 - x^2)$$

$$+ \; I_2 = \int \rho \, dV (r^2 - y^2)$$

$$2I = \int \rho \, dV \, (2r^2 - x^2 - y^2)$$

$$\left(\begin{array}{l} r^2 = s^2 + z^2 \\ x^2 + y^2 = s^2 \end{array} \right)$$

$$2I = \int \rho \, dV \, (s^2 + 2z^2)$$

$$\boxed{I = \frac{1}{2} \underbrace{\int \rho \, dV \, s^2}_{I_3} + \int \rho \, dV \, z^2 = \frac{1}{2} I_3 + \int \rho \, dV \, z^2}$$

easier to evaluate

$$\int \rho \, dV \, z^2 = \frac{M}{\pi R^2 h} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-h/2}^{h/2} dz \cdot z^2}_{\left.\frac{z^3}{3}\right|_{-h/2}^{h/2}} \underbrace{\int_0^R s \, ds}_{\frac{R^2}{2}}$$

$$\left.\frac{z^3}{3}\right|_{-h/2}^{h/2} = \frac{2}{3}\left(\frac{h}{2}\right)^3 = \frac{h^3}{12}$$

$$= \frac{M}{\cancel{\pi} \cancel{R^2} h} \cdot \cancel{2\pi} \cdot \frac{h^3}{12} \cdot \frac{\cancel{R^2}}{\cancel{2}}$$

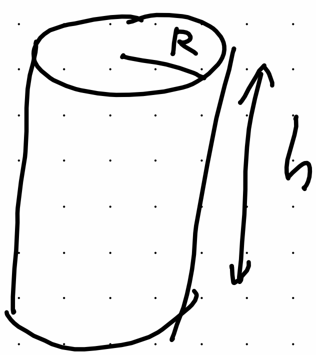
$$= \boxed{\frac{M h^2}{12}}$$

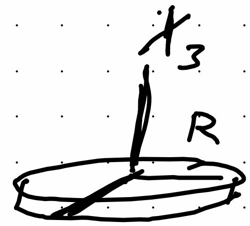
$$I = \frac{1}{2}\left(\frac{1}{2}\mu R^2\right) + \frac{\mu h^2}{12}$$

$$= \frac{1}{4}\mu R^2 + \frac{1}{12}\mu h^2$$

$$= \frac{1}{4}\mu\left(R^2 + \frac{1}{3}h^2\right) = I_1, I_2$$

$$I_3 = \frac{1}{2}\mu R^2$$



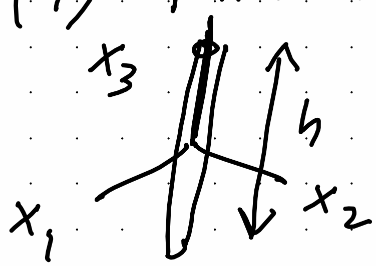


Limiting cases: (i) Disk ($h \to 0$)

$I_3 = \frac{1}{2}\mu R^2$

$I_1, I_2 = \frac{1}{4}\mu R^2$

(i) thin rod ($R \to 0$)



$I_3 = 0$

$I_1 = I_2 = \frac{1}{12}\mu h^2$

$$L = T - U$$

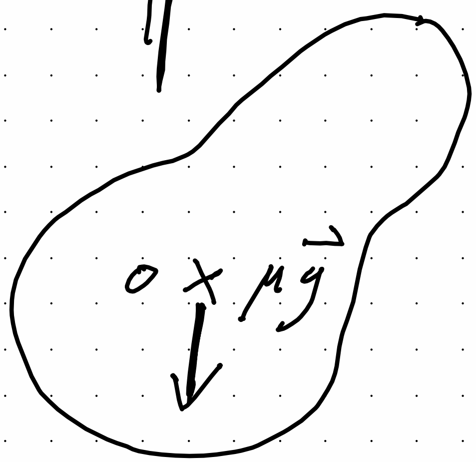
$$= \frac{1}{2}\mu V^2 + \frac{1}{2}\sum_{i,j} I_{ij}\Omega_i\Omega_j - U$$

$$\vec{V} = \dot{\vec{R}} \qquad L(\vec{R}, \vec{\phi}, \dot{\vec{R}}, \dot{\vec{\phi}})$$

$$\vec{\Omega} = \dot{\vec{\phi}}$$

$\vec{g}$ 

uniform
Field

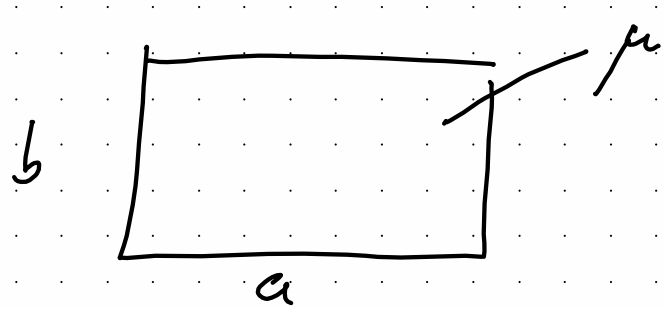


Lecture #24: Thursday 11/12

- Quiz #5 (today)
- Midterm #2 (next Thursday) (scattering, small oscillations, RB motion)
- Today's topics:
  (1) RB EOMs
  (2) Euler's equations
  (3) Euler angles

Q5: Calculate the principal moments of inertia for a 2-d rectangle with sidelengths a, b.

uniform

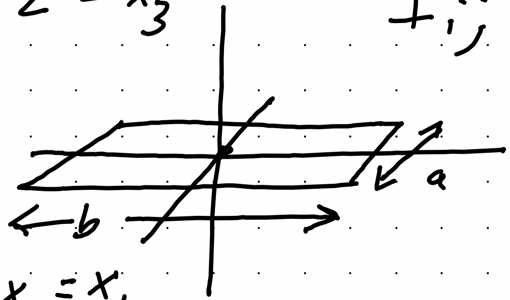


name_q5.pdf

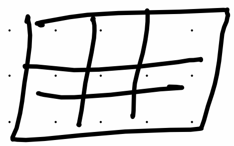
$z = x_3$

$I_{ij} = \int \rho\, dV\,(r^2 \delta_{ij} - r_i r_j)$

$r^2 = x^2 + y^2 + z^2$



$x_2 = y$



$x = x_1$

$dm = \rho\, dV$

$= \sigma\, dx\, dy$

$= \frac{M}{ab}\, dx\, dy$

$I_1 = \int \rho\, dV\,(r^2 - x^2)$

$\parallel$

I_{11}

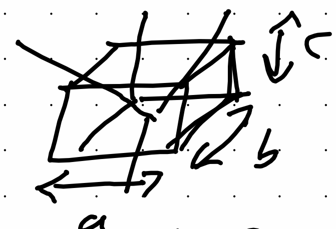
$= \int \rho\, dV\, y^2$

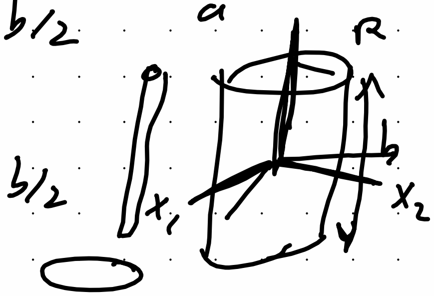
$= \frac{M}{ab} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\, y^2$

$= \frac{M}{\cancel{a}b}\, \cancel{a}\, \left.\frac{y^3}{3}\right|_{-b/2}^{b/2}$

$= \frac{M}{b}\, \frac{2}{3} \left(\frac{b}{2}\right)^3$

$= \frac{M}{b}\, \frac{2}{3}\, \frac{b^3}{8} = \boxed{\frac{1}{12} M b^2}$





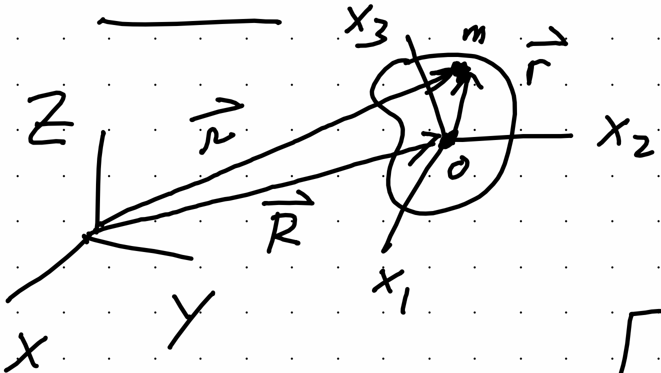
$\boxed{I_2 = \frac{1}{12} M a^2}$

$I_3 = \int \sigma\, dx\, dy\, (x^2 + y^2)$

$= I_1 + I_2$

$\boxed{I_3 = \frac{1}{12} M (a^2 + b^2)}$

EOMs:



Final result:

$$\frac{d\vec P}{dt} = \vec F = \sum \vec f$$

$$\frac{d\vec M}{dt} = \vec K = \sum \vec r \times \vec f$$

$$\vec\rho = \vec R + \vec r\,, \quad \boxed{\delta\vec\rho = \delta\vec R + \delta\vec\phi \times \vec r}$$

$\delta\Omega_K$

$$L = T - U$$

$$= \frac{1}{2}\mu|\vec V|^2 + \frac{1}{2}\sum_{i,j} I_{ij}\Omega_i\Omega_j - U(\vec\rho)$$

$$\delta L = \mu \vec V\cdot\delta\vec V + \sum_{i,j} I_{ij}\Omega_j\,\delta\Omega_i \boxed{- \sum \frac{\partial U}{\partial\vec\rho}\cdot\delta\vec\rho}$$

$$- \sum \frac{\partial U}{\partial\vec\rho}\cdot\left(\delta\vec R + \delta\vec\phi\times\vec r\right)$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial\vec V}\right) = \frac{\partial L}{\partial\vec R}}\,, \quad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial\vec\Omega}\right) = \frac{\partial L}{\partial\vec\phi}}$$

$$\frac{1}{2}\sum_{i,j} I_{ij}\,\underline{\underline{\delta\Omega_i}}\,\Omega_j + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\delta\Omega_j$$

$$= \qquad {''} \qquad + \frac{1}{2}\sum_{j,i} I_{ji}\,\Omega_j\,\underline{\underline{\delta\Omega_i}}$$

$$= \qquad {''} \qquad + \frac{1}{2}\underbrace{\sum_{j,i}}_{\text{swap}} I_{ij}\,\Omega_j\,\delta\Omega_i$$

$$= \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i$$

$$= \boxed{\sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i}$$

$$\delta L = m\vec{V}\cdot\underline{\delta\vec{V}} + \sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i \qquad \vec{A}\cdot(\vec{B}\times\vec{C}) \;/\; \vec{B}\cdot(\vec{C}\times\vec{A})$$

$$- \left(\sum \frac{\partial U}{\partial \vec{r}}\right)\cdot\delta\vec{R} \; \underbrace{- \sum \frac{\partial U}{\partial \vec{r}}\cdot(\delta\vec{\phi}\times\vec{r})}$$

$$= - \sum \delta\vec{\phi}\cdot\left(\vec{r}\times\frac{\partial U}{\partial \vec{r}}\right)$$

$$= - \delta\vec{\phi}\cdot\left(\sum \vec{r}\times\frac{\partial U}{\partial \vec{r}}\right)$$

$$\sum_i M_i\,\delta\Omega_i = \vec{M}\cdot\delta\vec{\Omega}$$

$$dF = \cancel{\bigcirc}\,dx + \cancel{\bigcirc}\,dy$$

$$\boxed{\delta L = \vec{P}\cdot\delta\vec{V} + \vec{M}\cdot\delta\vec{\Omega} + \left(\sum\vec{f}\right)\cdot\delta\vec{R} + \delta\vec{\phi}\cdot\left(\sum\vec{r}\times\vec{f}\right)}$$

$$\vec{p} = \frac{\partial L}{\partial \vec{V}}, \quad \vec{M} = \frac{\partial L}{\partial \vec{\Omega}}, \quad \sum\vec{f} = \frac{\partial L}{\partial \vec{R}}, \quad \sum\vec{r}\times\vec{f} = \frac{\partial L}{\partial \vec{\phi}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{V}}\right) = \frac{\partial L}{\partial \vec{R}} \rightarrow \boxed{\frac{d\vec{p}}{dt} = \sum\vec{f} \equiv \vec{F}}$$

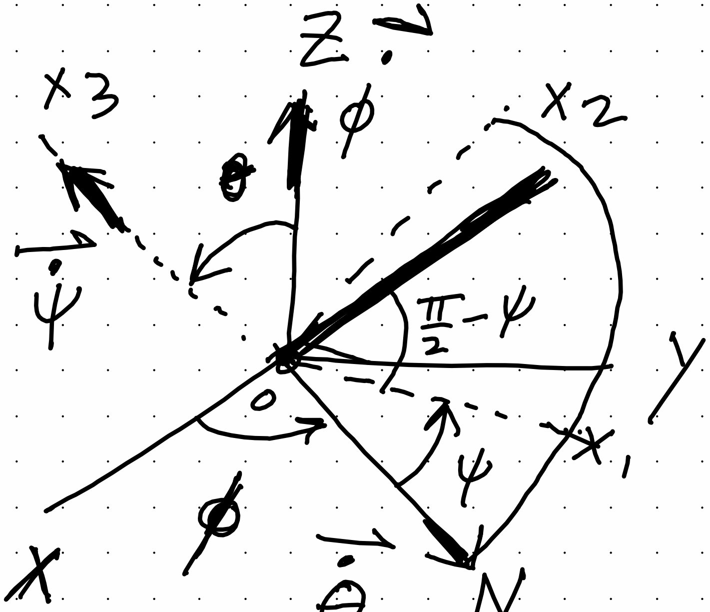
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{\Omega}}\right) = \frac{\partial L}{\partial \vec{\phi}} \rightarrow \boxed{\frac{d\vec{M}}{dt} = \sum\vec{r}\times\vec{f} = \vec{K}}$$

Euler's equations / Euler angles ☆

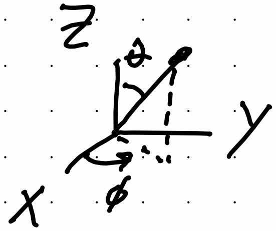
$\vec{R}, \vec{\phi}$

Ω_i : ith component of $\vec{\Omega}$

L wrt $\hat{x}_i$



(line of node)



$$\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta} + \vec{\dot\psi}$$

$$\vec{\dot\psi} = \dot\psi\,\hat{x}_3$$

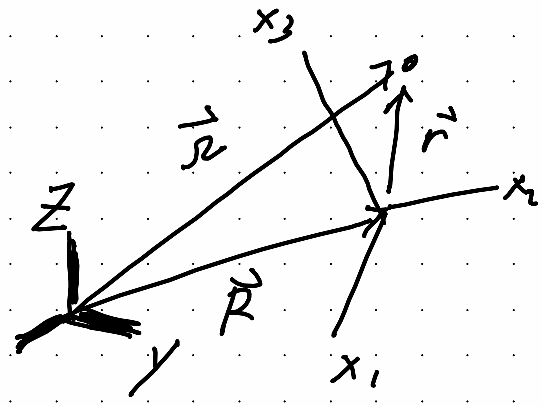
$$\vec{\dot\theta} = \dot\theta\cos\psi\,\hat{x}_1 - \dot\theta\sin\psi\,\hat{x}_2$$

$$\vec{\dot\phi} = \dot\phi\cos\theta\,\hat{x}_3 + \dot\phi\sin\theta\left(\underbrace{\sin\psi}_{\cos(\frac{\pi}{2}-\psi)}\hat{x}_1 + \underbrace{\cos\psi}_{\sin(\frac{\pi}{2}-\psi)}\hat{x}_2\right)$$

Announcements

- Midterm II is this thursday
- Today:
  - i) Euler angles
  - ii) Euler's equation for RB motion
  - iii) Free rotation with $\vec{\Omega} = \text{const}$
  - iv) " of a symmetric top ($I_1 = I_2$)
  - v) Heavy symmetrical top with one point fixed [prob 35.1]

$$\vec{\Omega} = (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\,\hat{x}_1 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\,\hat{x}_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{x}_3$$



$$\boxed{\begin{aligned} \Omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \Omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \Omega_3 &= \dot{\psi} + \dot{\phi}\cos\theta \end{aligned}}$$

$$\vec{v} = \frac{d\vec{\mathcal{R}}}{dt} = \vec{V} + \vec{\Omega}\times\vec{r}$$

$\frac{d'\vec{r}}{dt}$ wrt RB

Euler's equations:

$$\frac{d\vec{P}}{dt} = \sum \vec{f} = \vec{F}$$

$$\frac{d\vec{M}}{dt} = \sum \vec{r}\times\vec{f} = \vec{K}$$

(wrt RB axes) $\vec{A}$: any vector

$$\boxed{\underbrace{\frac{d\vec{A}}{dt}}_{\text{wrt inertial frame}} = \underbrace{\frac{d'\vec{A}}{dt}}_{\text{wrt rotating frame}} + \underbrace{\vec{\Omega}\times\vec{A}}_{\text{angular velocity of the rotating frame}}}$$

$$\underbrace{\left(\frac{d'\vec{A}}{dt}\right)_i}_{\text{cartesian components wrt rotating frame}} = \frac{dA_i}{dt} \equiv \dot{A}_i$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}$$

$$= \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$$

$$\cancel{= \ddot{r}\hat{r}}$$

$$\vec{A} = \sum_i A_i \hat{x}_i$$

$$\frac{d\vec{A}}{dt} = \underbrace{\sum_i \left(\frac{dA_i}{dt}\right)\hat{x}_i}_{\frac{d'\vec{A}}{dt}} + \underbrace{\sum_i A_i \frac{d\hat{x}_i}{dt}}_{\vec{\Omega}\times\vec{A}}$$

$$\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\Omega} \times \vec{A} \quad , \quad \left(\frac{d'\vec{A}}{dt}\right)_i = \dot{A}_i$$

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d'\vec{P}}{dt} + \vec{\Omega} \times \vec{P} \qquad \Big| \quad \vec{P} = \mu \vec{V}$$

$$\rightarrow F_1 = \dot{P}_1 + (\vec{\Omega} \times \vec{P})_1$$

$$= \dot{P}_1 + \Omega_2 P_3 - \Omega_3 P_2$$

$$= \mu(\dot{V}_1 + \Omega_2 V_3 - \Omega_3 V_2)$$

similar equations for F_2, F_3

$$\boxed{\vec{K} = \frac{d\vec{M}}{dt}} = \frac{d'\vec{M}}{dt} + \vec{\Omega} \times \vec{M} \qquad M_i = I_i \Omega_i$$

$$K_x = \dot{M}_x$$

$$K_1 = \dot{M}_1 + \Omega_2 M_3 - \Omega_3 M_2$$

$$= I_1 \dot{\Omega}_1 + \Omega_2 I_3 \Omega_3 - \Omega_3 I_2 \Omega_2$$

$$= I_1 \dot{\Omega}_1 + \Omega_2 \Omega_3 (I_3 - I_2)$$

*(similar equations for K_2, K_3)

Free rotation: $K_i = 0$, $F_i = 0$

$$0 = I_1 \dot{\Omega}_1 + \Omega_2 \Omega_3 (I_3 - I_2), \cancel{\text{etc.}}$$
$$0 = I_2 \dot{\Omega}_2 + \Omega_3 \Omega_1 (I_1 - I_3)$$
$$0 = I_3 \dot{\Omega}_3 + \Omega_1 \Omega_2 (I_2 - I_1)$$

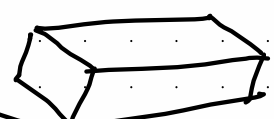
Free rotation with $\vec{\Omega} = \text{const}$:

$$\frac{d\vec{\Omega}}{dt} = 0$$
$$\frac{d'\vec{\Omega}}{dt} = 0$$
$$\frac{d\vec{\Omega}}{dt} = \frac{d'\vec{\Omega}}{dt} + \cancel{\vec{\Omega} \times \vec{\Omega}}$$
$$\frac{d\vec{M}}{dt} \neq \frac{d'\vec{M}}{dt}$$

$$0 = \Omega_2 \Omega_3 (I_3 - I_2)$$
$$0 = \Omega_3 \Omega_1 (I_1 - I_3)$$
$$0 = \Omega_1 \Omega_2 (I_2 - I_1)$$

$\Omega_1 = \text{const}$, $\Omega_2 = 0$, $\Omega_3 = 0$ ← stable

$I_1 < I_2 < I_3$



unstable ↓

$\Omega_2 = \text{const}$, $\Omega_1 = 0$, $\Omega_3 = 0$

$\Omega_3 = \text{const}$, $\Omega_1 = 0$, $\Omega_2 = 0$

$\Omega_1 = \text{const}, \quad \Omega_2 = 0, \quad \Omega_3 = 0$: exact

$$\boxed{\begin{aligned} \Omega_1 &= \text{const} + \epsilon_1 \\ \Omega_2 &= \epsilon_2 \\ \Omega_3 &= \epsilon_3 \end{aligned}}$$

$\epsilon_{1,2,3}$: small time dependent perturbations

Keep 0^{th} and 1^{st} order terms.
Ignore 2^{nd} order, e.g. $\epsilon_2 \epsilon_3$

$$0 = I_1 \frac{d}{dt}(\underset{\to 0}{\text{const}} + \epsilon_1) + \underbrace{\epsilon_2 \epsilon_3 (I_3 - I_2)}_{2^{nd} \text{ order} \to \text{ignore}}$$

$$\approx I \dot{\epsilon}_1$$

$$\to \epsilon_1 = \text{const} \to \Omega_1 = \text{const}$$

$$\left.\begin{aligned} 0 &= I_2 \dot{\epsilon}_2 + \epsilon_3 \Omega_1 (I_1 - I_3) \\ 0 &= I_3 \dot{\epsilon}_3 + \Omega_1 \epsilon_2 (I_2 - I_1) \end{aligned}\right\}$$ coupled 1^{st} order diff. equations

Differentiate

$$0 = I_2 \ddot{\epsilon}_2 + \dot{\epsilon}_3 \Omega_1 (I_1 - I_3)$$

$$= I_2 \ddot{\epsilon}_2 - \frac{\Omega_1 \epsilon_2 (I_2 - I_1)}{I_3} \Omega_1 (I_1 - I_3)$$

$I_1 < I_2 < I_3$

$$0 = I_2 \ddot{\epsilon}_2 + \dot{\epsilon}_3 \Omega_1 (I_1 - I_3)$$

$$= I_2 \ddot{\epsilon}_2 - \frac{\Omega_1 \epsilon_2 (I_2 - I_1)}{I_3} \Omega_1 (I_1 - I_3)$$

$$= \ddot{\epsilon}_2 + \underbrace{\frac{\Omega_1^2 (I_2 - I_1)(I_3 - I_1)}{I_2 I_3}}_{\omega^2} \epsilon_2$$

$$\boxed{\begin{matrix}(I_3 - I_2) \\ (I_1 - I_2)\end{matrix}}$$

$$= \ddot{\epsilon}_2 + \omega^2 \epsilon_2$$

$$\ddot{\epsilon}_2 = -\omega^2 \epsilon_2 \longrightarrow \text{SHM}$$

$$\epsilon_2 = A \cos \omega t + B \sin \omega t = C \cos(\omega t + \alpha)$$

Similarly,

$$\ddot{\epsilon}_3 = -\omega^2 \epsilon_3 \longrightarrow \epsilon_3 = D \cos(\omega t + \beta)$$

(ϵ_2, ϵ_3 are bound by their initial deviations away from 0)

For $\Omega_2 = \text{const}$ solution

perturbations

$$\ddot{\epsilon}_1 = +\omega^2 \epsilon_1$$

$$\ddot{\epsilon}_3 = +\omega^2 \epsilon_3$$

$$\rightarrow \quad \epsilon_1(t) = A \underbrace{e^{\omega t}}_{\text{grows exponentially}} + B \underbrace{e^{-i\omega t}}_{\text{damped exponentially}}$$

$\rightarrow$ instability