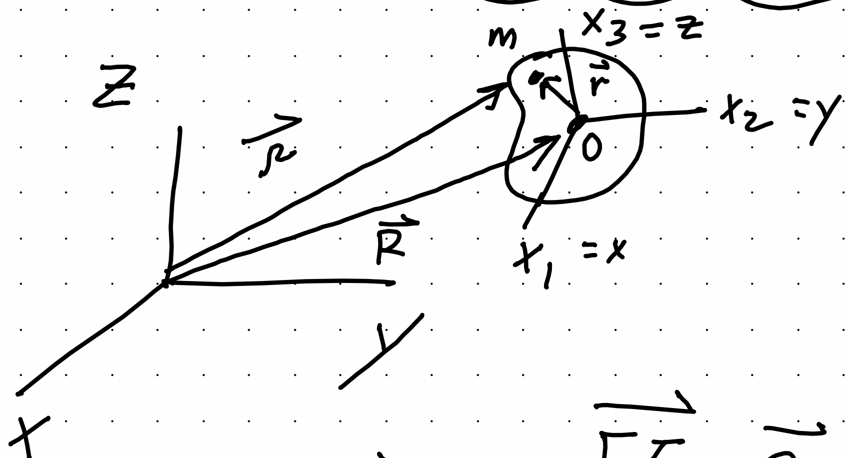
Lec #23: Tuesday Nov 10th

- Quiz #5: Thursday
- Midterm #2: Next Thursday 11/19 (Scattering, small oscillations, some rigid body)
- Today: Rigid body motion (Sec 31–36, 38, 39)
  - 39 — non-inertial
  - 38 — static equilibrium



O: at COM (usually)

x_1, x_2, x_3: fixed in RB

$$\vec{v} = \frac{d\vec{\rho}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}$$

$\vec{\Omega}$: angular velocity vector

$$\left.\begin{aligned} \vec{V} &= \frac{d\vec{R}}{dt} \\ \vec{\Omega} &= \frac{d\vec{\phi}}{dt} \end{aligned}\right\} \dot{q}_i$$

$(\vec{R}, \vec{\phi})$: 6 DOF $= q_i$

$$T = \frac{1}{2} m v^2$$
$$\vec{p} = m \vec{v}$$

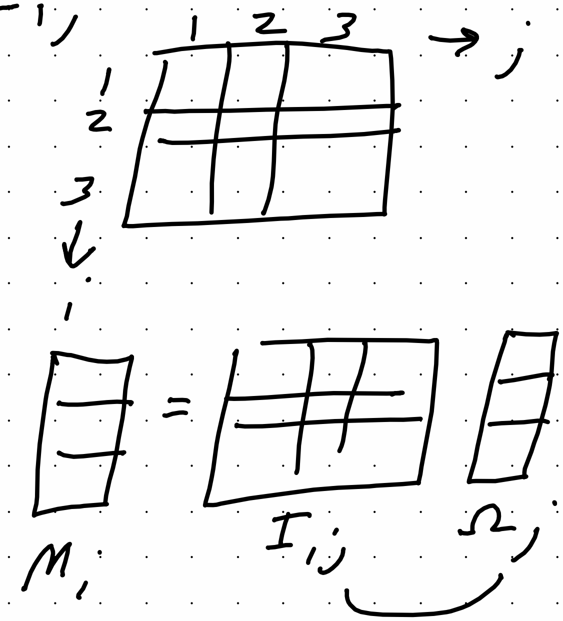
,

rotational quantities

$$T_{rot} = \frac{1}{2} I \Omega^2$$
$$\vec{M} = I \vec{\Omega} \rightarrow M_i = \sum_j I_{ij} \Omega_j$$

I : moment of inertia

$\longrightarrow$ I_{ij} : inertia tensor



KE:

$$T = \frac{1}{2}\sum_a m_a |\vec{v}_a|^2 , \qquad \vec{v}_a = \vec{V} + \vec{\Omega}\times\vec{r}_a$$

$$= \frac{1}{2}\sum_a m_a |\vec{V} + \vec{\Omega}\times\vec{r}_a|^2$$

$$|\vec{A}+\vec{B}|^2 = A^2 + B^2 + 2\vec{A}\cdot\vec{B}$$

$$= \frac{1}{2}\sum_a m_a \left(|\vec{V}|^2 + |\vec{\Omega}\times\vec{r}_a|^2 + 2\vec{V}\cdot(\vec{\Omega}\times\vec{r}_a) \right)$$

1 $$= \frac{1}{2}\sum_a m_a |\vec{V}|^2 = \boxed{\frac{1}{2}\mu V^2}$$

μ: total mass

3 $$= \sum_a m_a \vec{V}\cdot(\vec{\Omega}\times\vec{r}_a)$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$= \left(\sum_a m_a \vec{r}_a\right)\cdot(\vec{V}\times\vec{\Omega})$$

$$= \underbrace{\mu \vec{R}_{com}}_{=0 \text{ for}}\cdot(\vec{V}\times\vec{\Omega})$$

$\boxed{O \text{ at COM}}$

(2) $= \frac{1}{2} \sum_a m_a \left| \vec{\Omega} \times \vec{r_a} \right|^2$

$$= \frac{1}{2} \sum_a m_a \left(\vec{\Omega} \times \vec{r_a} \right) \cdot \left(\vec{\Omega} \times \vec{r_a} \right)$$

$$= \frac{1}{2} \sum_a m_a \, \vec{\Omega} \cdot \left(\vec{r_a} \times \left(\vec{\Omega} \times \vec{r_a} \right) \right)$$

$$= \frac{1}{2} \sum_a m_a \, \vec{\Omega} \cdot \left(\vec{\Omega} \, r_a^2 - \vec{r_a} \left(\vec{r_a} \cdot \vec{\Omega} \right) \right)$$

$$= \frac{1}{2} \sum_a m_a \left(\Omega^2 r_a^2 - \left(\vec{\Omega} \cdot \vec{r_a} \right)^2 \right)$$

$$= \frac{1}{2} \sum_a m_a \left(\sum_{i,j} \Omega_i \Omega_j \delta_{ij} r_a^2 - \sum_{i,j} \Omega_i r_{ai} \Omega_j r_{aj} \right)$$

$$= \frac{1}{2} \sum_{i,j} \left(\sum_a m_a \left(r_a^2 \delta_{ij} - r_{ai} r_{aj} \right) \right) \Omega_i \Omega_j$$

$$= \boxed{\frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j} \quad \left(= \frac{1}{2} \right)$$

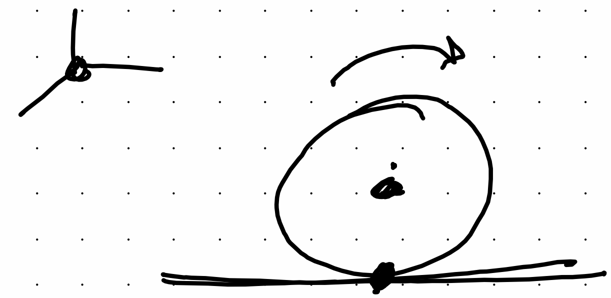
$$\vec{A} \times \left(\vec{B} \times \vec{C} \right) = \vec{B} \left(\vec{A} \cdot \vec{C} \right) - \vec{C} \left(\vec{A} \cdot \vec{B} \right)$$

$$\boxed{T = \underbrace{\tfrac{1}{2}\mu V^2}_{\text{trans}} + \underbrace{\tfrac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\Omega_j}_{\text{rotational}}} + \ldots$$

$\left(\tfrac{1}{2} I \Omega^2\right)$ Freshman physics

for COM at origin
of RB Frame

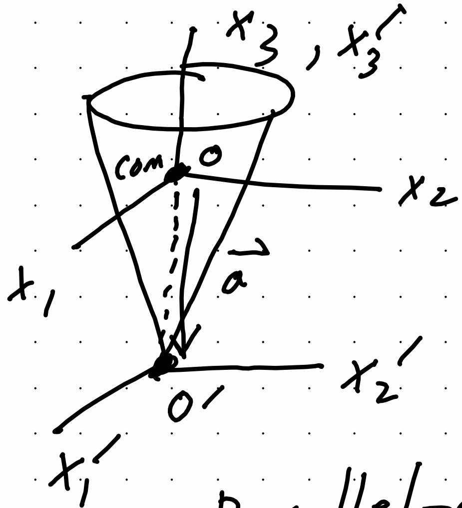
$\vec{M}$: wrt COM of body



$$\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a$$
$$= \sum_a m_a\, \vec{r}_a \times \vec{v}_a$$
$$= \sum_a m_a\, \vec{r}_a \times \left(\vec{V} + \vec{\Omega} \times \vec{r}_a\right)$$
$$= \sum_a m_a\, \vec{r}_a \times \left(\vec{\Omega} \times \vec{r}_a\right)$$
$$= \sum_a m_a \left(\vec{\Omega}\, r_a^2 - \vec{r}_a\left(\vec{r}_a \cdot \vec{\Omega}\right)\right)$$

$$\boxed{M_i = \sum_j I_{ij}\,\Omega_j}$$

$\vec{M} = I\vec{\Omega}$ (Fresh. physics)



parallel-axis theorem

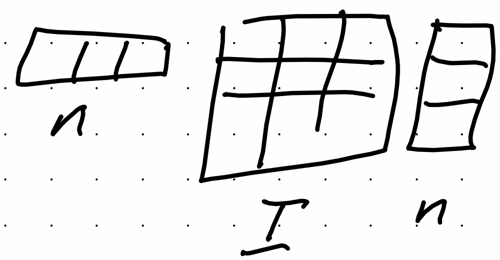
$$\underbrace{I'_{ij}}_{\text{wrt } O'} \; , \quad \underbrace{I_{ij}}_{\text{wrt } O \text{ (com)}}$$

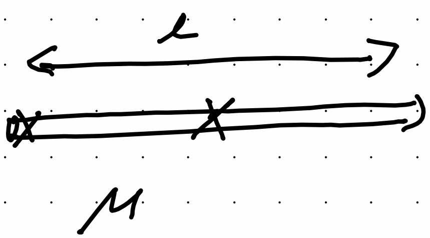
$$I'_{ij} = I_{ij} + \mu(a^2\delta_{ij} - a_i a_j)$$

$\vec{a}$: vector from O to O'

$$\underbrace{I(\hat{n})}_{\text{moment of inertia}} = \sum_{i,j} I_{ij} \underbrace{n_i n_j}$$

$\hat{n}$: axis of rotation





$$\boxed{\begin{aligned} I_{com} &= \frac{1}{12}\mu l^2 \\ I_{end} &= \frac{1}{3}\mu l^2 \end{aligned}}$$

$$\boxed{\underbrace{I_{ij}}_{\text{3x3 real symmetric}} = \sum_a m_a \left(r_a^2 \delta_{ij} - r_{ai} r_{aj} \right) = \int \underbrace{\rho\, dV}_{dm} \left(r^2 \delta_{ij} - r_i r_j \right)}$$

can always be diagonalized

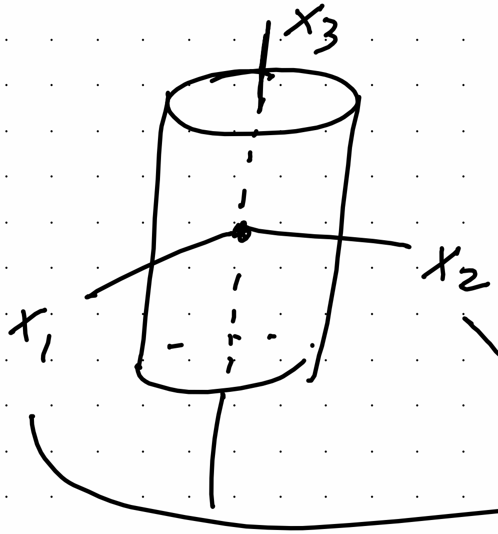
$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$I_{ij} = I_i \delta_{ij}$$

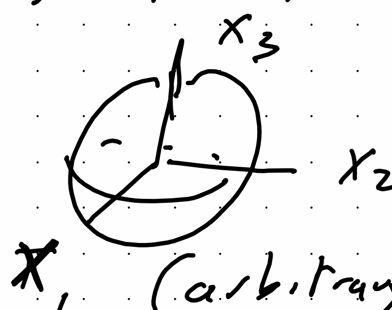
$$T_{rot} = \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j$$

$$= \frac{1}{2} \left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 \right)$$

<u>principle axes:</u> (x_1, x_2, x_3)



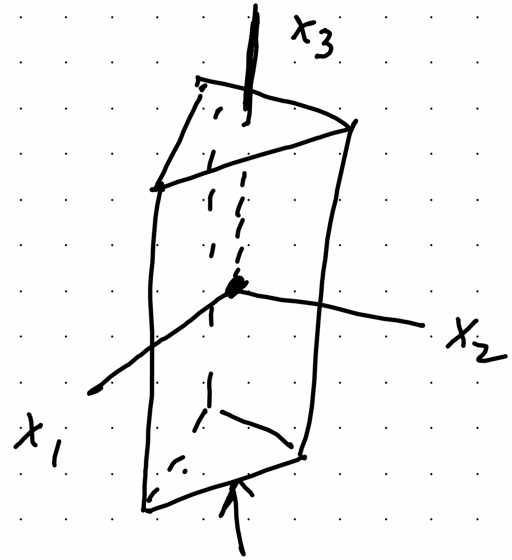
arbitrary



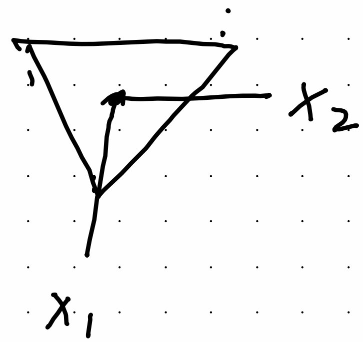
(arbitrary)

$$M_i = \sum_j I_{ij} \Omega_j$$

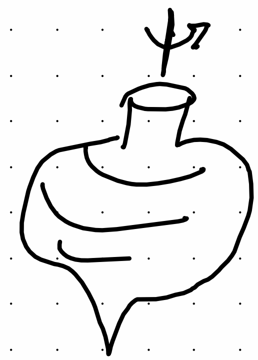
$$M_1 = I_1 \Omega_1, \quad M_2 = I_2 \Omega_2$$
$$M_3 = I_3 \Omega_3$$

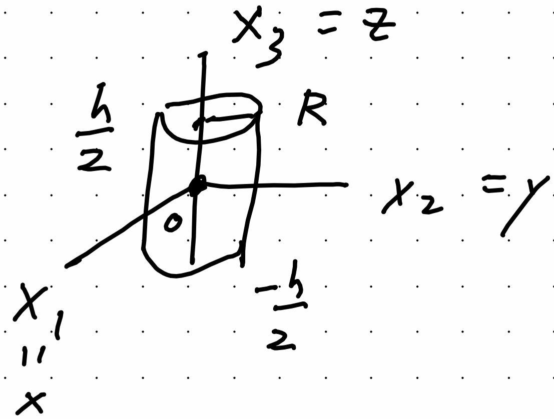


equilateral triang



$$\underbrace{I_1 = I_2 \neq I_3}_{\text{symmetrical top}}$$





total mass M

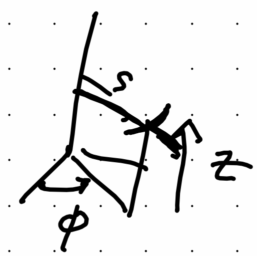
$$\rho = \frac{M}{\text{volume}} = \frac{M}{\pi R^2 h}$$

$$I_3 = I_{33} = \int \rho\, dV \left(r^2 \underbrace{\delta_{33}}_{1} - \underbrace{r_3 r_3}_{z^2}\right)$$

$$= \int \rho\, dV \left(r^2 - z^2\right)$$

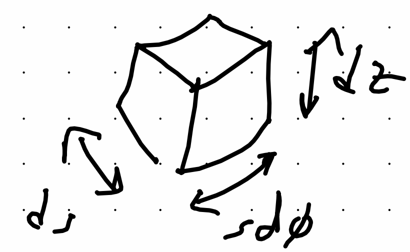
$$= \int \rho\, dV \left(x^2 + y^2\right) = \int \rho\, dV s^2$$

<u>cylindrical:</u> s, ϕ, z $\quad s^2 = x^2 + y^2$



$$dV = ds\; s\,d\phi\; dz$$

$$= s\,ds\,d\phi\,dz$$



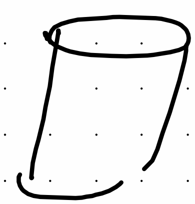
$$I_3 = \int \rho \, dV \, s^2$$

$$= \frac{M}{\pi R^2 h} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-h/2}^{h/2} dz}_{h} \underbrace{\int_0^{R} s^3 \, ds}_{\frac{R^4}{4}}$$

$\cancel{4\pi}$

$$= \frac{M}{\cancel{\pi} \cancel{R^2} \cancel{h}} \cdot 2 \cancel{\pi} \cancel{h} \frac{R^{\cancel{4}\,2}}{4}$$

$$= \boxed{\frac{1}{2} M R^2}$$



$$I_1 = I_2 \equiv I$$

$$I_1 = \int \rho \, dV (r^2 - x^2)$$

$$+ \; I_2 = \int \rho \, dV (r^2 - y^2)$$

$$2I = \int \rho \, dV \left(2r^2 - x^2 - y^2 \right)$$

$$\left(\begin{array}{l} r^2 = s^2 + z^2 \\ x^2 + y^2 = s^2 \end{array} \right)$$

$$2I = \int \rho \, dV \left(s^2 + 2z^2 \right)$$

$$\boxed{I} = \frac{1}{2} \underbrace{\int \rho \, dV \, s^2}_{I_3} + \int \rho \, dV \, z^2$$

$$= \frac{1}{2} I_3 + \int \rho \, dV \, z^2$$

easier to evaluate

$$\int \rho \, dV \, z^2 = \frac{M}{\pi R^2 h} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-h/2}^{h/2} dz \cdot z^2}_{\frac{z^3}{3}\Big|_{-h/2}^{h/2}} \underbrace{\int_0^R s \, ds}_{\frac{R^2}{2}}$$

$$\frac{z^3}{3}\Big|_{-h/2}^{h/2} = \frac{2}{3}\left(\frac{h}{2}\right)^3 = \frac{h^3}{12}$$

$$= \frac{M}{\cancel{\pi} \cancel{R^2} h} \cdot \cancel{2}\cancel{\pi} \cdot \frac{h^3}{12} \cdot \frac{\cancel{R^2}}{\cancel{2}}$$

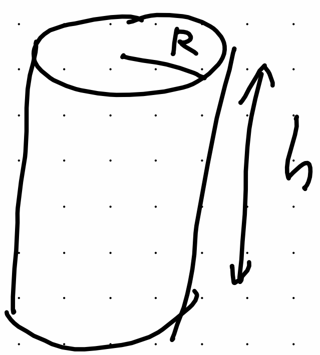
$$= \boxed{\frac{M h^2}{12}}$$

$$I = \frac{1}{2}\left(\frac{1}{2}\mu R^2\right) + \frac{\mu h^2}{12}$$

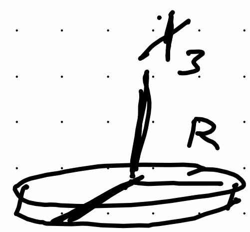
$$= \frac{1}{4}\mu R^2 + \frac{1}{12}\mu h^2$$

$$= \frac{1}{4}\mu\left(R^2 + \frac{1}{3}h^2\right) = I_1, I_2$$

$$I_3 = \frac{1}{2}\mu R^2$$



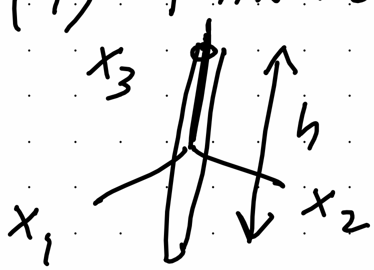
Limiting cases: (i) Disk ($h \to 0$)



$I_3 = \frac{1}{2}\mu R^2$

$I_1, I_2 = \frac{1}{4}\mu R^2$

(i) thin rod ($R \to 0$)



$I_3 = 0$

$I_1 = I_2 = \frac{1}{12}\mu h^2$

$$L = T - U$$

$$= \frac{1}{2}\mu V^2 + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\Omega_j - U$$

$$\vec{V} = \dot{\vec{R}}$$

$$\vec{\Omega} = \dot{\vec{\phi}}$$

$$L(\vec{R}, \vec{\phi}, \dot{\vec{R}}, \dot{\vec{\phi}})$$

$\vec{g} \downarrow$

uniform
Field