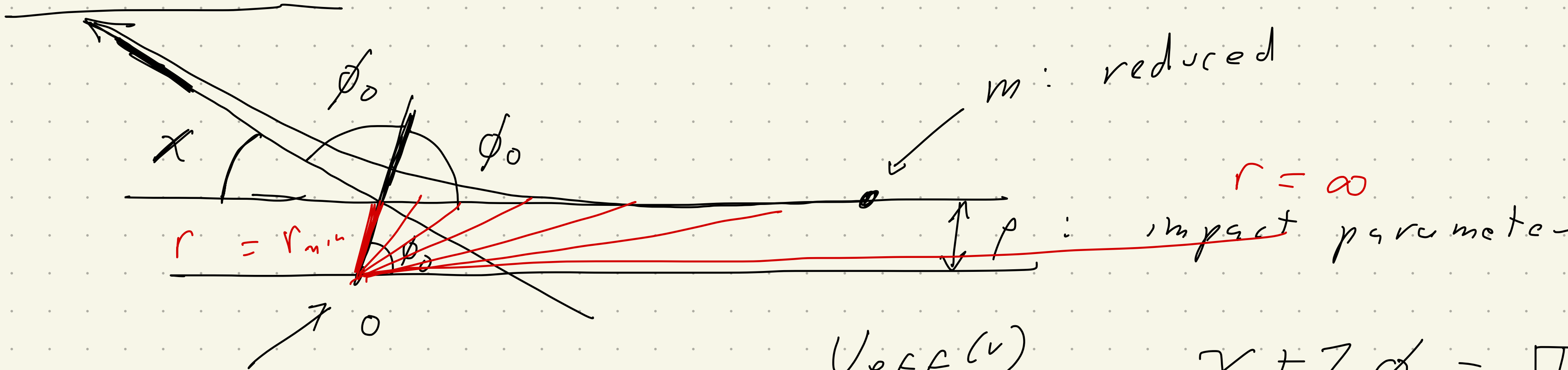
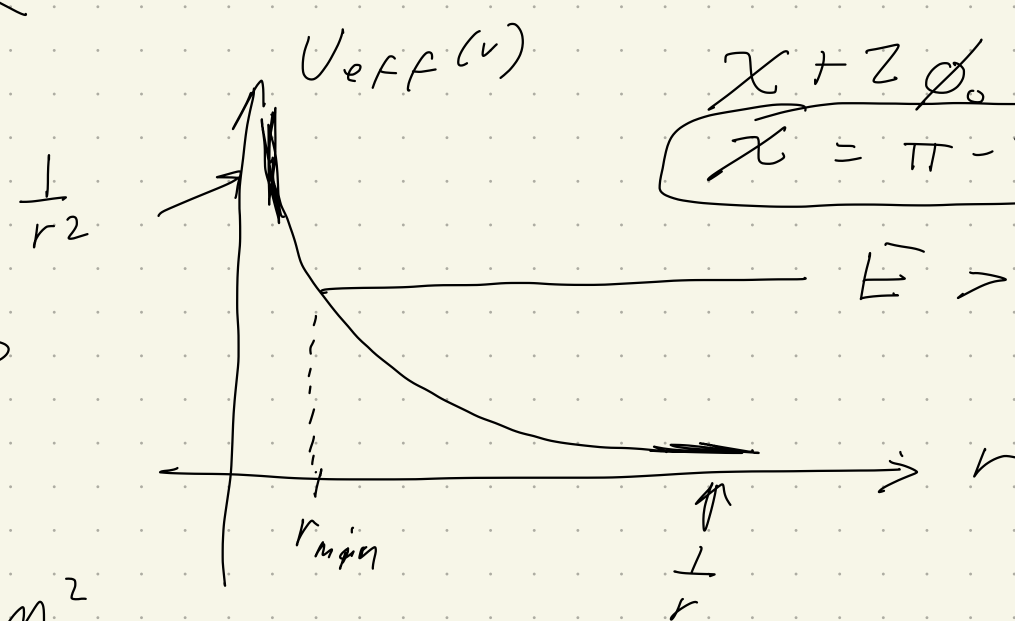
Lecture #16 Thurs 10/14



COM of original system

$$\chi + 2\phi_0 = \pi$$

$$\boxed{\chi = \pi - 2\phi_0}$$



$U(r) = \frac{\alpha}{r}$, $\alpha > 0$

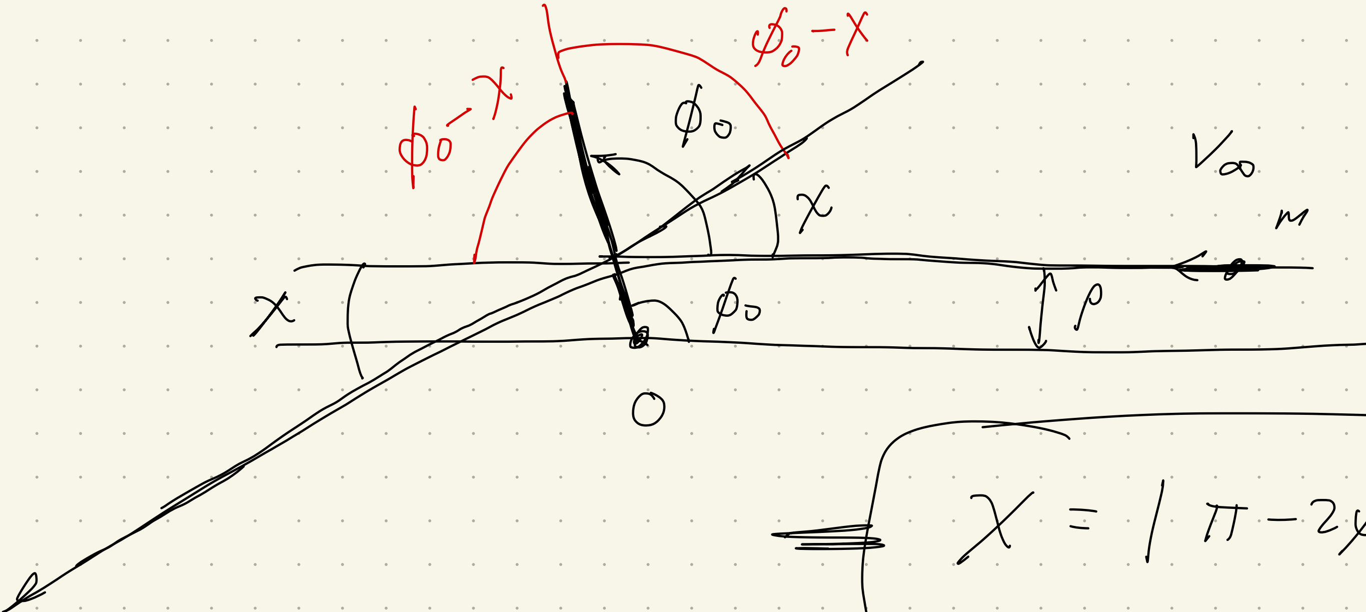
(repulsive)

$$U_{eff}(r) = U(r) + \frac{M_z^2}{2mr^2}$$

$$= \frac{\alpha}{r} + \frac{M_z^2}{2mr^2}$$

$E = U_{eff}(r_{min})$

turning point



$2(\phi_0 - \chi) + \chi = \pi$

$2\phi_0 - \chi = \pi$

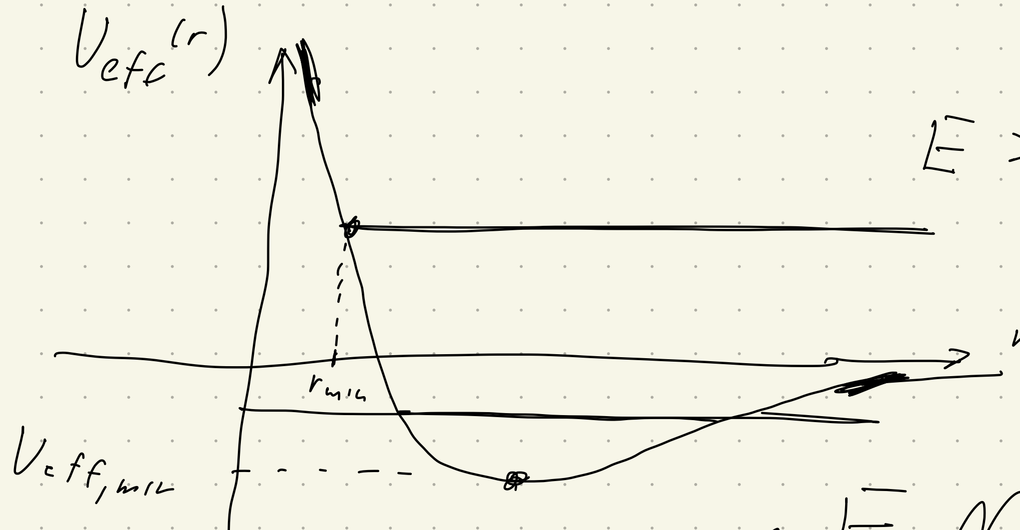
$$\boxed{\chi = 2\phi_0 - \pi}$$

$$\boxed{\chi = |\pi - 2\phi_0|}$$

for both repulsive & attractive

$U(r) = -\frac{\alpha}{r}$

$V_{eff}(r) = -\frac{\alpha}{r} + \frac{M_t^2}{2mr^2}$



$E > 0$

$$\boxed{E = \frac{1}{2} m v_\infty^2, \quad M = m\rho v_\infty}$$

E, M

ρ, v_∞

Sec 14:

$$t = \pm \int \frac{dr}{\sqrt{}} + \text{const}$$

$$\Rightarrow \phi = \pm \int \frac{M\, dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M^2}{r^2}}} + \text{const}$$

$$E = U_{eff}(r_{min})$$

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M\, dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M^2}{r^2}}}$$

$$M = m\rho v_\infty$$

$$E = \frac{1}{2} m v_\infty^2$$

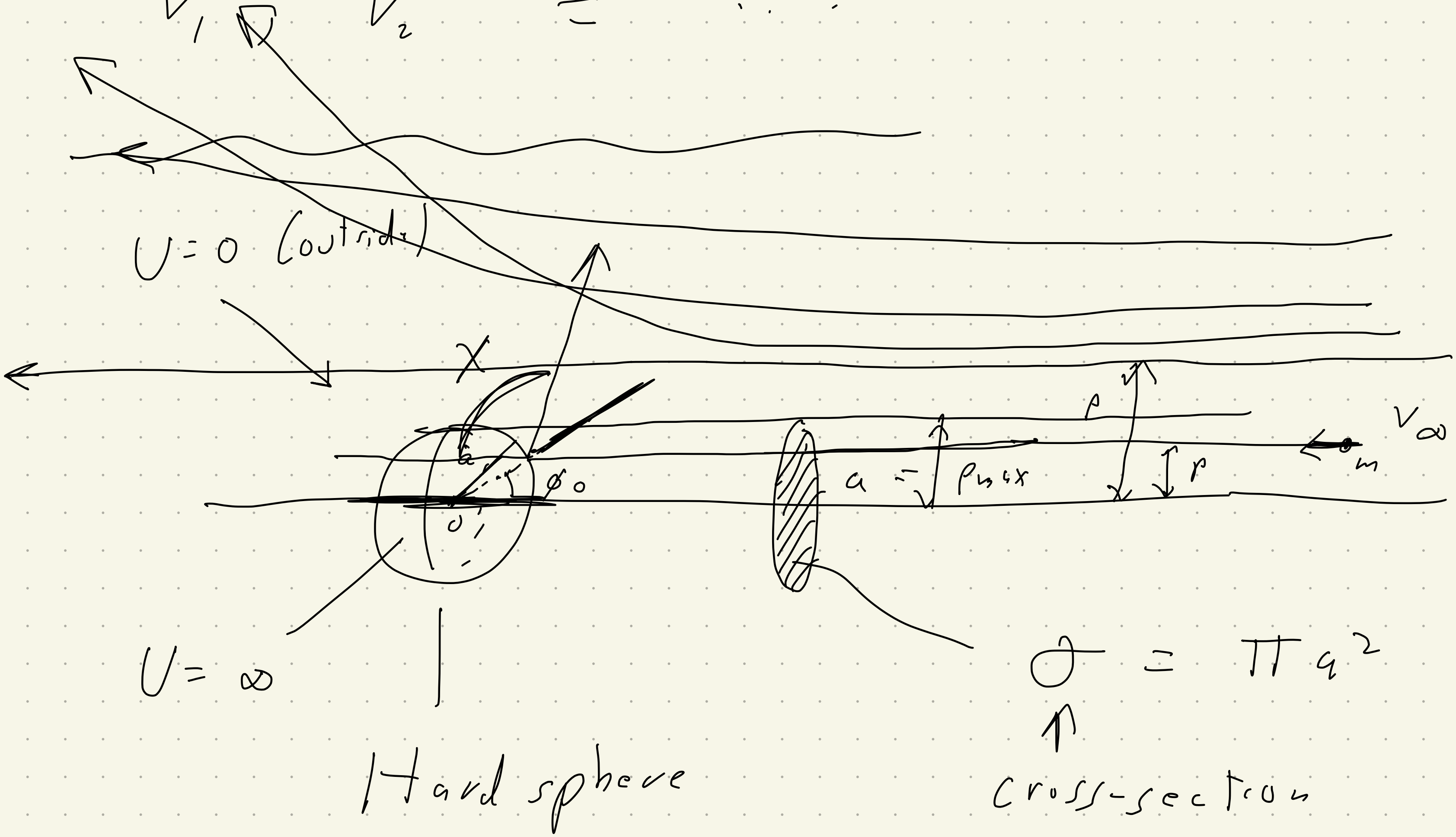
$$= \int_{r_{min}}^{\infty} \frac{m\rho v_\infty\, dr/r^2}{\sqrt{2m\left(\frac{1}{2}mv_\infty^2 - U(r)\right) - \frac{m^2\rho^2 v_\infty^2}{r^2}}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \frac{U(r)}{\frac{1}{2}mv_\infty^2} - \rho^2/r^2}}$$

$$\Theta_2 = \frac{1}{2}(\pi - \chi) \qquad \chi = |\pi - 2\phi_0|$$

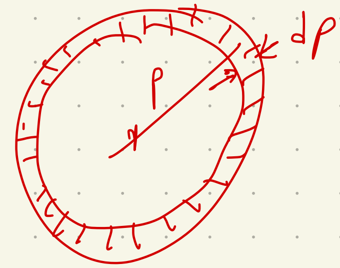
$$\tan\Theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2 \cos\chi}$$

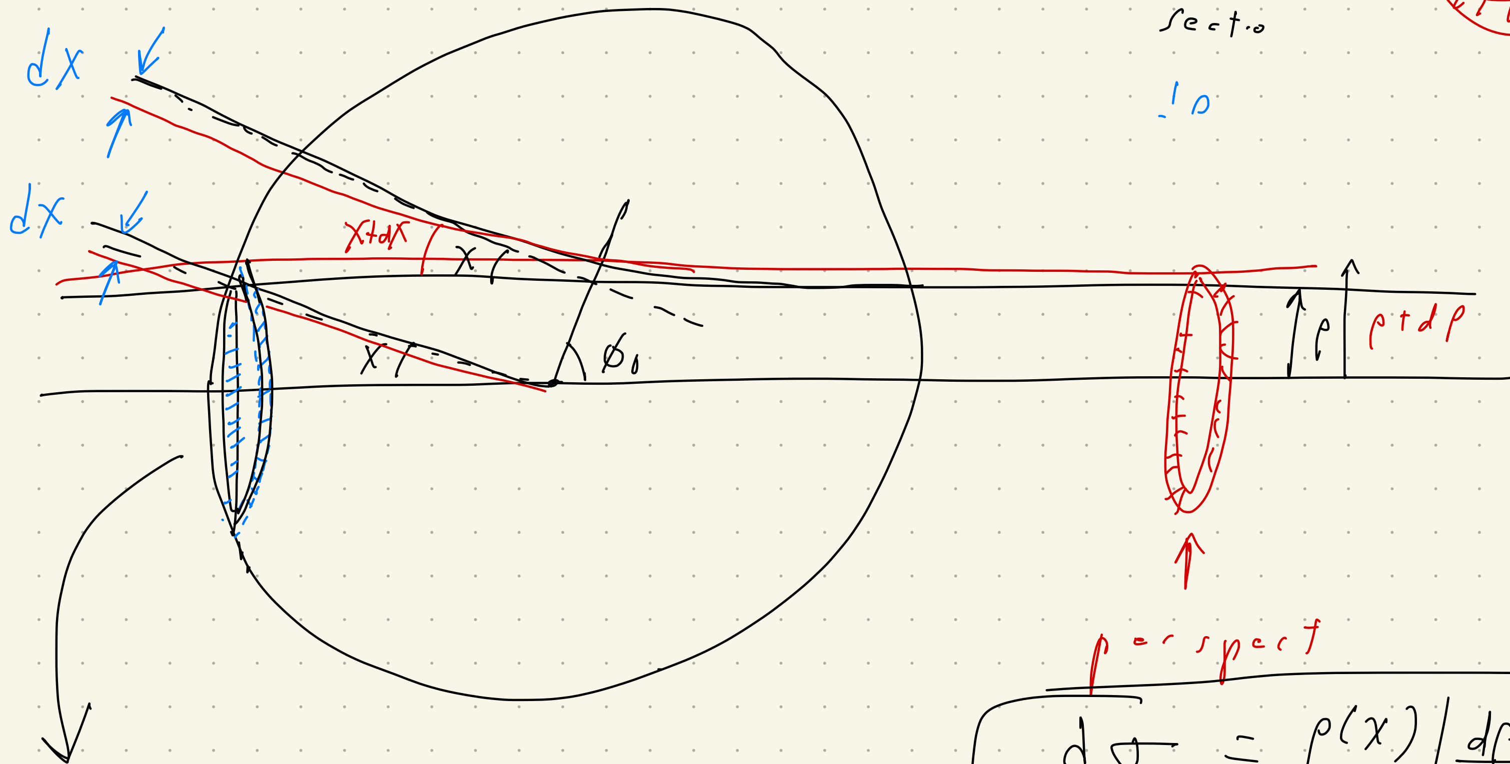
$v_1' \quad v_2' = \ldots$

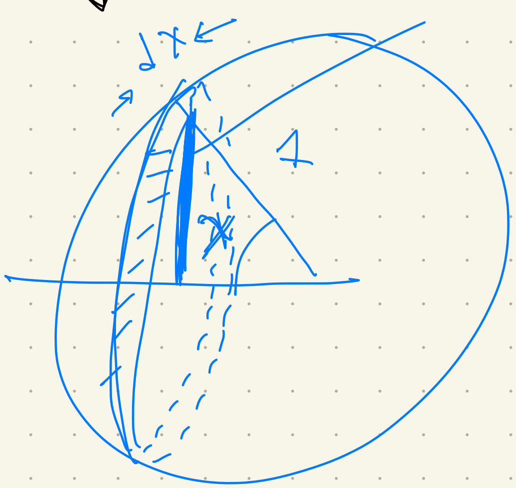


$$\boxed{d\sigma = 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi} \longleftarrow \boxed{d\sigma = 2\pi\rho\, d\rho}$$

differential cross section







perspect

$$\boxed{d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega}$$

$$\underbrace{d\Omega}_{\text{solid angle}} = 2\pi \sin\chi\, d\chi$$

$$\frac{d\sigma}{d\Omega} = \frac{\cancel{2\pi}\rho\, d\rho}{\cancel{2\pi}\sin\chi\, d\chi} = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right|$$

$$\frac{d\sigma}{d\Omega} = \frac{\rho}{\sin\chi}\left|\frac{d\rho}{d\chi}\right|$$ COM frame

$$\frac{d\sigma_1}{d\Omega_1} = \frac{\rho}{\sin\theta_1}\left|\frac{d\rho}{d\theta_1}\right|$$ Lab frame θ_1

$$\frac{d\sigma_2}{d\Omega_2} = \frac{\rho}{\sin\theta_2}\left|\frac{d\rho}{d\theta_2}\right|$$ Lab frame θ_2

$$\frac{d\sigma_1}{d\Omega_1} = \frac{\sin\chi}{\sin\theta_1}\left|\frac{d\chi}{d\theta_1}\right|\frac{d\sigma}{d\Omega}$$

$$= \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|\frac{d\sigma}{d\Omega}$$

calculated for the COM

Lecture #17: Tues Oct 19th

Next week: start small oscillations

Today/Thurs: Q & A (collisions and scatterings)

Lecture #18: Thur Oct 21st

Next week: small oscillations

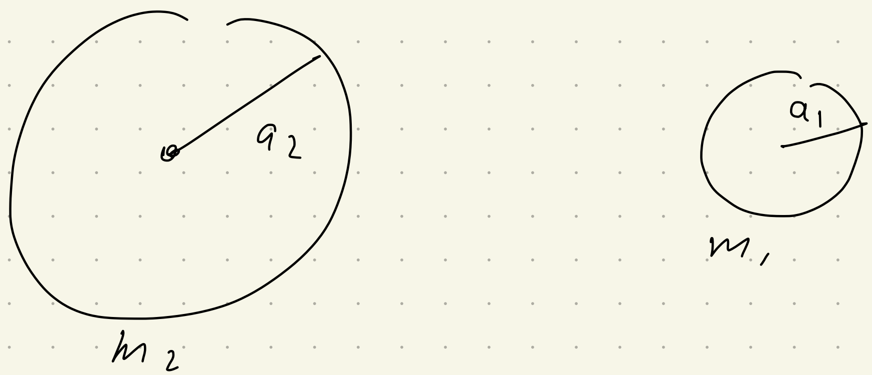
Today: Q & A, Quiz #3

(last 20 minutes)

$\boxed{\text{Q3\_name.pdf}}$

Quiz #3:

Two hard spheres with masses m_1, m_2 and radii a_1, a_2:



$$\rho(\chi) = (a_1 + a_2)\cos\left(\frac{\chi}{2}\right)$$

a) Find $\rho = \rho(\chi)$ where χ = scattering angle wrt CoM frame

b) What value of ρ will give $\theta_2 = 60°$?

$2\theta_2 + \chi = \pi \rightarrow \chi = \pi - 2\theta_2$
$= 60°$

$$\rho = (a_1 + a_2)\cos(30°) = \frac{\sqrt{3}}{2}(a_1 + a_2)$$

Lec #19: 10/26

Small oscillation: $\boxed{\text{Sec } 21}$, 22, 23

more than 1d

free oscillation in 1-d

forced oscillation in 1d

Damping: Sec 25 (not covered)

SHM: (simple harmonic motion) is sinusoidal

$$\omega = \frac{2\pi}{T}$$

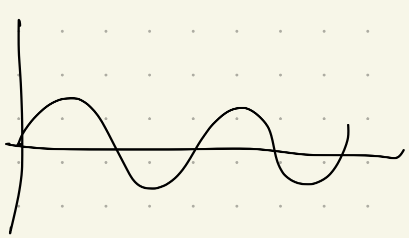
$$x(t) = a \cos(\omega t + \alpha)$$

amplitude

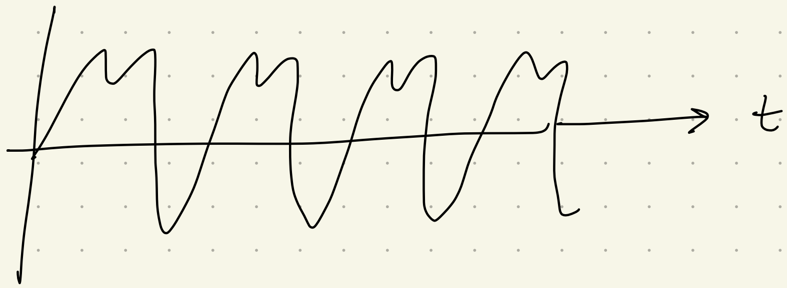
phase

angular freq

SHM

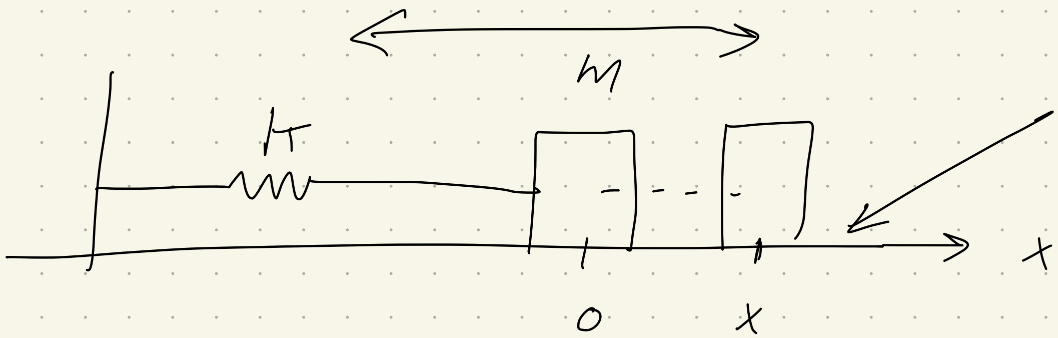


t



periodic

Example:



no friction

$$U = \frac{1}{2}kx^2$$

$$F = -\frac{dU}{dx} = -kx$$

$$F = -kx$$

linear

restoring

$$F = m\ddot{x} = -kx \rightarrow \ddot{x} = -\frac{k}{m}x$$

soln: $$x = c_1 \cos \omega t + c_2 \sin \omega t$$

$$\omega = \sqrt{\frac{k}{m}}$$

Alternative:

$$x = a\cos(\omega t + \alpha)$$

$$x = \mathrm{Re}\left[A e^{i\omega t}\right], \quad A = a e^{i\alpha}$$

$$T = \frac{1}{2} m \dot{x}^2, \quad U = \frac{1}{2} k x^2 \quad \Bigg\} \quad L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

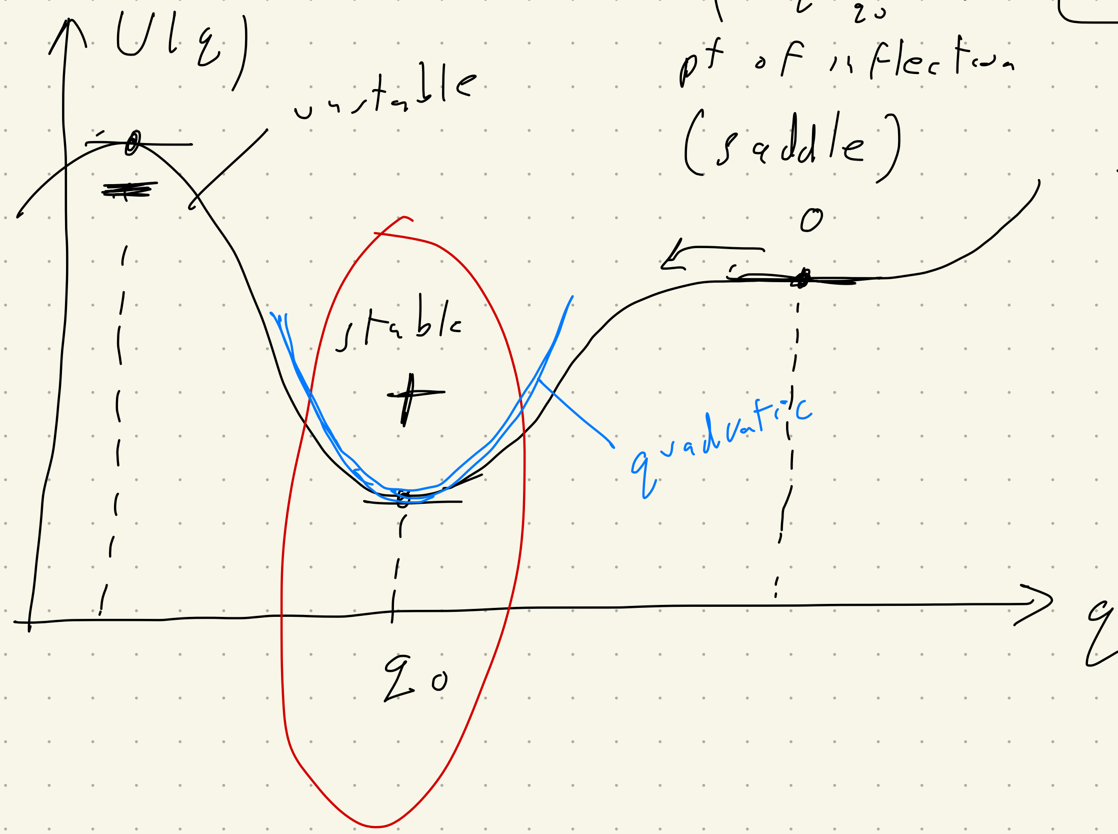
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow m\ddot{x} = -kx$$

$U(q) = q^3$

$q_0 = 0$

More generally:

$$\boxed{L = \frac{1}{2} a(q)\, \dot{q}^2 - U(q)}$$



$\left(\left.\frac{d^2U}{dq^2}\right|_{q_0} = 0\right)$

pt of inflection

(saddle)

Equilibrium: $(q = q_0)$

$$0 = F = -\left.\frac{dU}{dq}\right|_{q_0}$$

stable:

$$\left.\frac{d^2U}{dq^2}\right|_{q_0} > 0$$

$\longrightarrow U = \text{const}$

Expansion: (about q_0)

$$U(q) = \underbrace{U(q_0)}_{\substack{\text{const} \\ (\text{ignore in } L)}} + \overset{0 \ \text{equilibrium}}{\cancel{\frac{dU}{dq}}}\Big|_{q_0}\cdot(q-q_0) + \frac{1}{2}\overbrace{\frac{d^2U}{dq^2}\Big|_{q_0}}^{k>0}(q-q_0)^2$$

$$\underbrace{+\ \frac{1}{3!}\frac{d^3U}{dq^3}\Big|_{q_0}(q-q_0)^3 + \dots}_{\text{ignore for } q-q_0 \text{ small}}$$

$$\boxed{U(q) \approx \frac{1}{2}kx^2}$$

$$x \equiv q-q_0$$

$$k = \frac{d^2U}{dq^2}\Big|_{q_0}$$

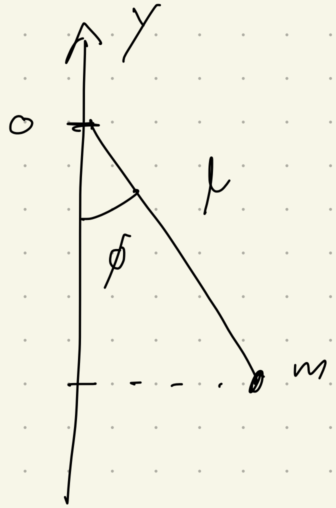
$$T = \frac{1}{2}a(q)\dot{q}^2$$

$$= \frac{1}{2}a(q_0)\dot{x}^2$$

$$\approx \frac{1}{2}m\dot{x}^2$$

$$\boxed{L \approx \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2}$$

Simple pendulum:



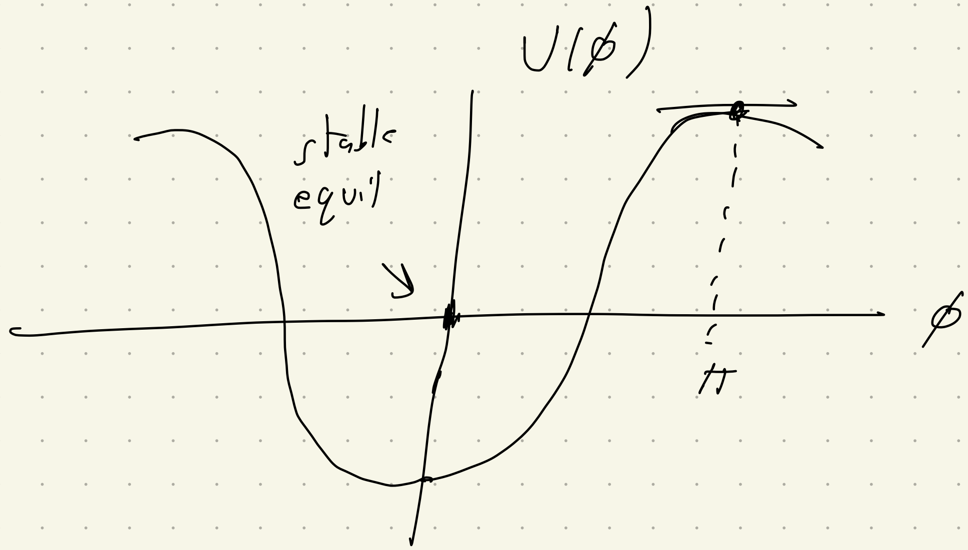
$$T = \frac{1}{2} m l^2 \dot{\phi}^2$$

$$U = -mgl\cos\phi$$

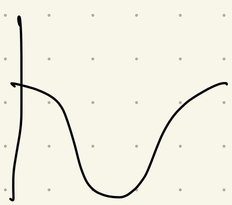
$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi$$

$U = mgy$

$= -mgl\cos\phi$



$\phi_0 = 0$ (equilibrium)



$$\cos\phi = 1 - \frac{1}{2}\phi^2 + \dots$$

$$\cos\phi = \underbrace{\cos 0}_{1} + \cancel{\underbrace{\frac{d(\cos\phi)}{d\phi}}_{\sin\phi}}\Bigg|_{\phi=0} \cdot \phi + \frac{1}{2}\frac{d^2(\cos\phi)}{d\phi^2}\Bigg|_{\phi=0}\phi^2$$

$\rightarrow 0$

$$+ \cdots$$

$$= 1 - \frac{1}{2}\phi^2 + \cdots$$

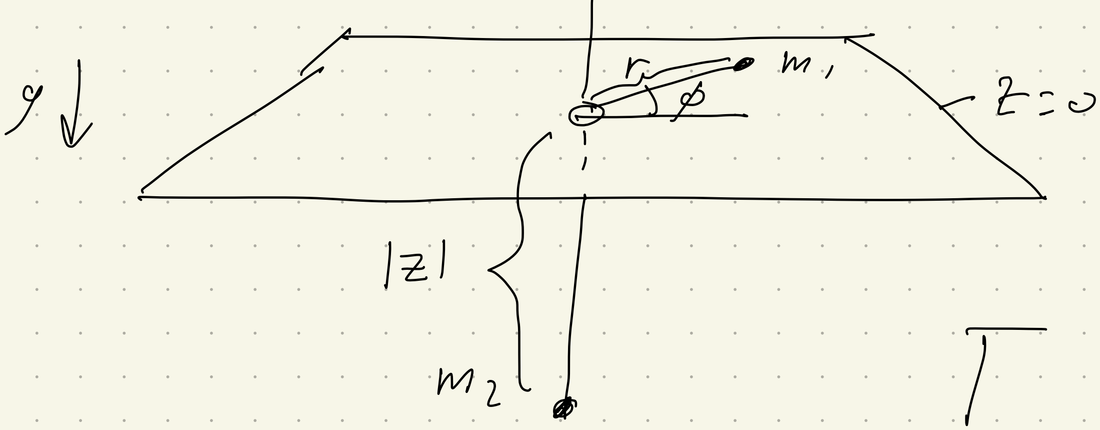
$$L = \frac{1}{2} m l^2 \dot\phi^2 + mgl\left(1 - \frac{1}{2}\phi^2\right)$$

$$= \frac{1}{2} \underbrace{m l^2}_{= M} \dot\phi^2 - \frac{1}{2}\underbrace{mgl}_{= K}\phi^2 + \underbrace{mgl}_{\text{ignore}}$$

$$L = \frac{1}{2} m \dot x^2 - \frac{1}{2} K x^2$$

$$\omega = \sqrt{\frac{K}{M}} = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}$$

Example:



length of string — $l = r + |z| = r - z$

z: location of m_2

r: for m_1

$$z = r - l \to \dot{z} = \dot{r}$$

$$T = \frac{1}{2} m_1 (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{1}{2} m_2 \dot{z}^2$$

$$U = m_2 g z = m_2 g (r - l)$$
$$= m_2 g r - m_2 g l$$

(ignore) const

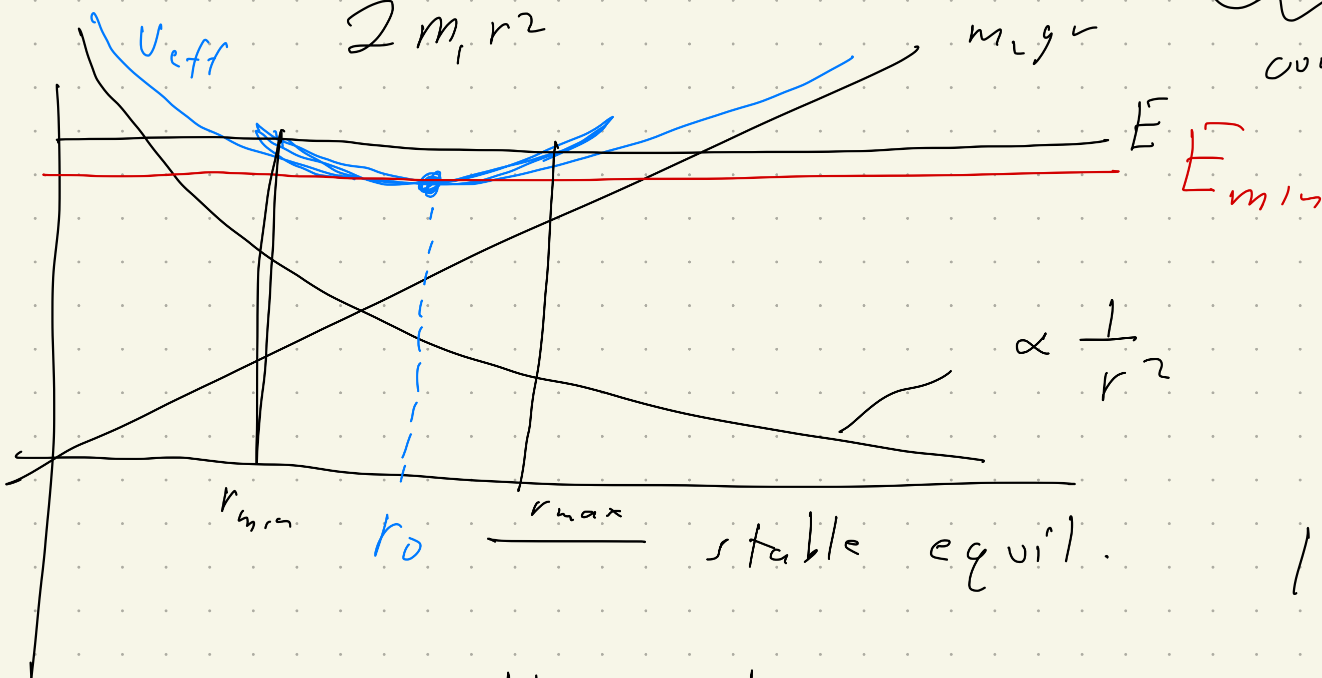
$$\boxed{L = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r}$$

i) $M_z = \frac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} \longrightarrow \dot{\phi} = \frac{M_z}{m_1 r^2}$

ii) $E = T + U = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \underbrace{\frac{1}{2} m_1 r^2 \dot{\phi}^2}_{\frac{M_z^2}{2 m_1 r^2}} + m_2 g r$

$$E = \frac{1}{2}\underbrace{(m_1+m_2)}_{M}\dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = \frac{M_z^2}{2m_1 r^2} + m_2 g r = \underbrace{U_{eff}(r_0)}_{\text{const}} + \frac{1}{2}K(r-r_0)^2$$



$$K = \left.\frac{d^2U_{eff}}{dr^2}\right|_{r=r_0}$$

$$|r-r_0| \ll r_0$$

$r_0 = ?$

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0}$$

$$= -\frac{M_z^2}{m_1 r_0^3} + m_2 g \rightarrow \boxed{M_z^2 = m_1 m_2 g r_0^3}$$

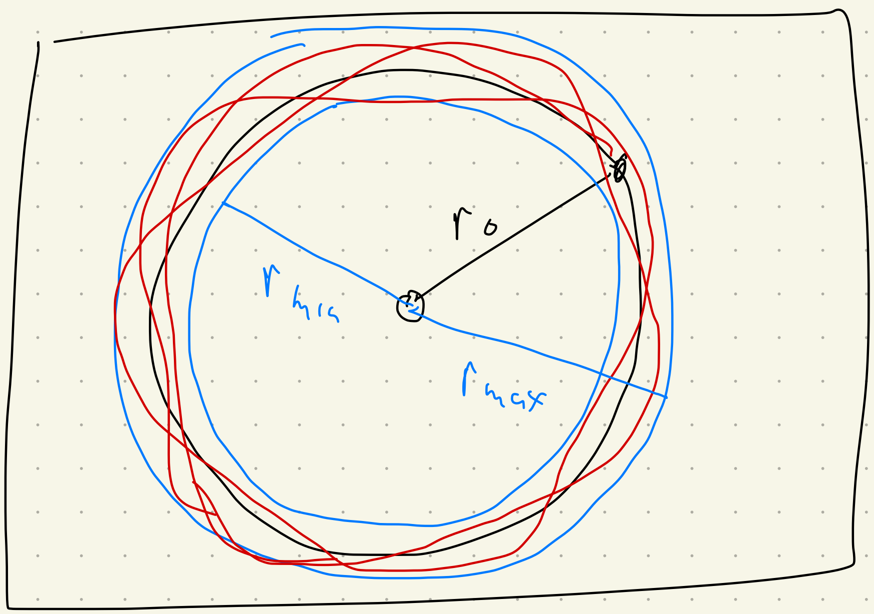
$$\left.\frac{d^2 U_{eff}}{dr^2}\right|_{r=r_0} = \frac{3 M_z^2}{m_1 r_0^4}$$

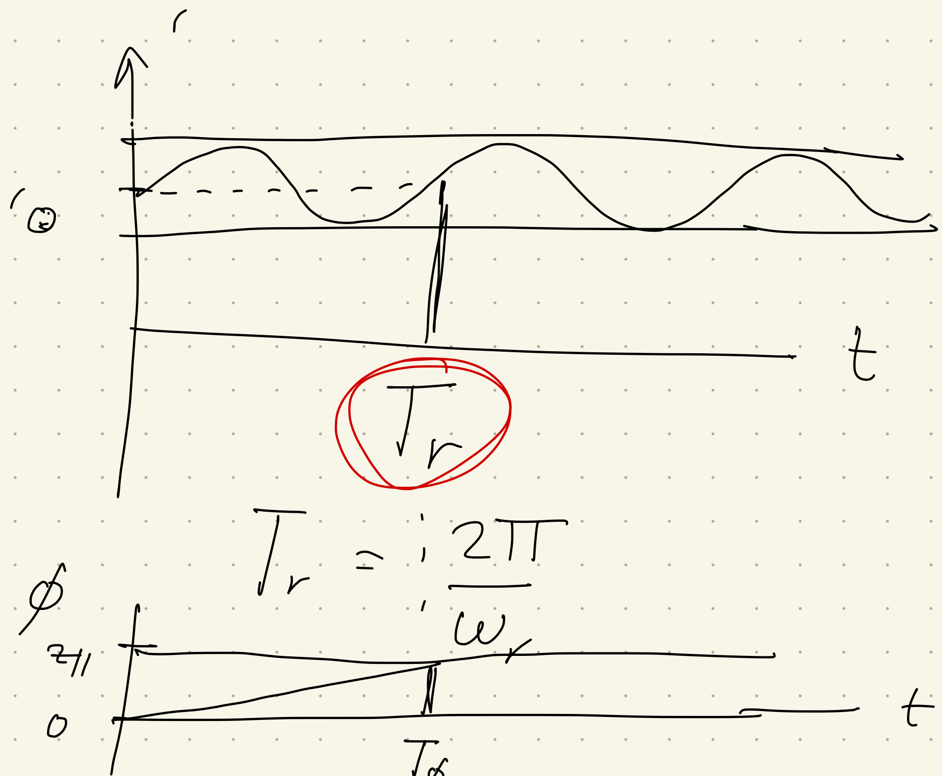
$$= \frac{3 \cancel{m_1} m_2 g r_0^3}{\cancel{m_1} r_0^4}$$

$$= \frac{3 m_2 g}{r_0} = K$$

$$\omega_r = \sqrt{\frac{K}{\underbrace{m_1+m_2}_{M}}}$$

$$= \sqrt{\frac{3 m_2 g}{(m_1+m_2) r_0}}$$





$$T_r = \frac{2\pi}{\omega_r}$$

$$T_\phi = \frac{2\pi}{\omega_\phi} = \frac{2\pi}{\dot\phi\big|_{r_0}}$$

$$\dot\phi = \frac{M_z}{m_1 r^2}$$

$$\omega_\phi = \dot\phi\Big|_{r=r_0} = \frac{M_z}{m_1 r_0^2} = \frac{\sqrt{m_1 m_2 g r_0^3}}{m_1 r_0^2}$$

$$\boxed{\omega_\phi = \sqrt{\frac{m_2 g}{m_1 r_0}}}$$

compare to

$$\boxed{\omega_r = \sqrt{\frac{3 m_2 g}{(m_1 + m_2) r_0}}}$$

If $\frac{3}{m_1+m_2} = \frac{1}{m_1}$ then $\omega_r = \omega_\phi$

$$3m_1 = m_1 + m_2$$

$$\boxed{2m_1 = m_2}$$

Lec #20: 10/28

Forced oscillation:

$$m\ddot{x} = -kx + F(t)$$

$$\ddot{x} + \frac{k}{m}x = \frac{F(t)}{m}$$

$$\boxed{\ddot{x} + \omega^2 x = \frac{F(t)}{m}}, \quad \omega \equiv \sqrt{\frac{k}{m}}$$

$$U(x) = \frac{1}{2}kx^2 - xF(t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$L = T - U$$

$$\frac{\partial L}{\partial x} = -kx + F(t)$$

general sol'n:

$$x(t) = x_h(t) + x_p(t)$$

homogeneous ($F(t)=0$)

particular (any solution for $F(t)$)

$$\boxed{x_h(t) = a\cos(\omega t + \alpha)}$$

a, α: two constant (Initial condition)

Suppose: $F(t) = f\cos(\gamma t + \beta)$

$$\ddot{x}_p + \omega^2 x_p = \frac{f}{m}\cos(\gamma t + \beta)$$

Guess!! $\quad x_p(t) = b\cos(\gamma t + \beta)$

$$\rightarrow -b\gamma^2 \cancel{\cos(\gamma t + \beta)} + \omega^2 b \cancel{\cos(\gamma t + \beta)} = \frac{f}{m}\cancel{\cos(\gamma t + \beta)}$$

$$b(\omega^2 - \gamma^2) = \frac{f}{m}$$

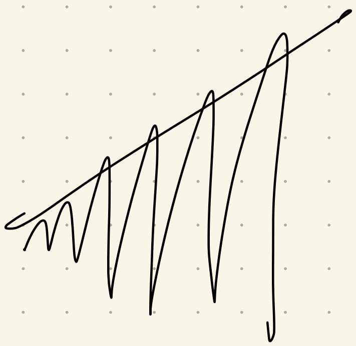
$$\rightarrow \boxed{b = \frac{f}{m(\omega^2 - \gamma^2)}}$$

$$\boxed{x_p(t) = \frac{f}{m(\omega^2 - \gamma^2)}\cos(\gamma t + \beta)}$$

$+ \; 10\cos(\omega t)$

$$\boxed{x(t) = a\cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)}\cos(\gamma t + \beta)}$$

$$x(t) = a\cos(\omega t+\alpha) + \underbrace{\frac{f}{m(\omega^2-\gamma^2)}\left[\cos(\gamma t+\beta) - \cos(\omega t+\beta)\right]}_{\frac{0}{0}}$$



L'hopital's

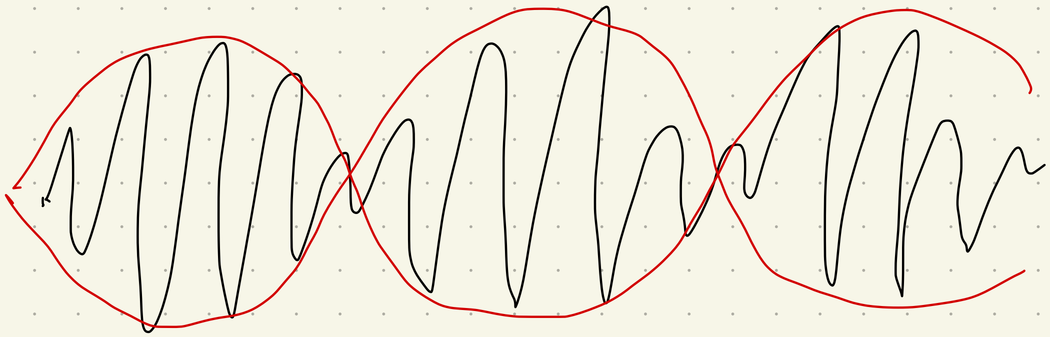
$$= \frac{\frac{d}{d\gamma}(\text{num})\Big|_{\gamma\to\omega}}{\frac{d}{d\gamma}(\text{den})\Big|_{\gamma\to\omega}}$$

$$= \frac{+ft\sin(\omega t+\beta)}{+2m\omega}$$

$$= \frac{ft}{2m\omega}\sin(\omega t+\beta)$$

at resonance ($\gamma=\omega$)

$$\boxed{x(t) = a\cos(\omega t+\alpha) + \frac{ft}{2m\omega}\sin(\omega t+\beta)}$$



$$\frac{\omega + \gamma}{2}$$

$$|\omega - \gamma| = \omega_{beat}$$

General: for arbitrary $F(t)$

$$F(t) = \mathrm{Re} \underbrace{\int_{-\infty}^{\infty} d\gamma \, \tilde{F}(\gamma) e^{i\gamma t}}_{\text{Fourier transform}}$$

$$\boxed{\ddot{x} + w^2 x = \frac{F(t)}{m}}$$

$$\boxed{y' + P(x)y = Q(x)}$$

Math methods

$y = y(x)$

Let: $\xi = \dot{x} + iwx$

complex

$$\dot{\xi} = \ddot{x} + iw\dot{x}$$
$$= \ddot{x} + iw(\xi - iwx)$$
$$= \ddot{x} + iw\xi + w^2 x$$

$$\rightarrow \dot{\xi} - iw\xi = \ddot{x} + w^2 x = \frac{F(t)}{m}$$

Homog: $\dot{\xi}_h - iw\xi_h = 0 \rightarrow \xi_h(t) = Ae^{iwt}$

complex constant

replace

Guess: $\boxed{\xi_p(t) = A(t)e^{iwt}}$

$$\boxed{\dot{\xi}_p - i\omega \xi_p = \frac{F(t)}{m}}, \quad \xi_p = A(t)e^{i\omega t}$$

$$\rightarrow \dot{A}e^{i\omega t} + \cancel{iA\omega e^{i\omega t}} \cancel{-i\omega A e^{i\omega t}} = \frac{F(t)}{m}$$

$$\dot{A} = \frac{e^{-i\omega t}F(t)}{m}$$

$$A(t) = \int dt \frac{F(t)}{m} e^{-i\omega t} + \text{const}$$

$$\boxed{\xi(t) = e^{i\omega t}\left[\int_0^t d\bar{t}\, \frac{F(\bar{t})}{m} e^{-i\omega \bar{t}} + \xi_0\right]}$$

$$\xi = \dot{x} + i\omega x \rightarrow \boxed{x(t) = \frac{1}{\omega}\,\mathrm{Im}(\xi(t))}$$

Complex constant (I.C.'s)

$$y' + P(x)\,y = Q(x) \qquad f(x)\,dx = g(y)\,dy$$

$$\frac{dy}{dx} + P(x)\,y - Q(x) = 0$$

$$\boxed{1\,dy + \big(P(x)\,y - Q(x)\big)\,dx = 0} \neq dU$$

$$dU(x,y) = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy$$

$$\frac{\partial^2 U}{\partial y\,\partial x} = \frac{\partial^2 U}{\partial x\,\partial y}$$

M. Boas

~~Mathemat~~

$$\boxed{\mu(x)\left[dy + \big(P(x)\,y - Q(x)\big)\,dx\right] = dU}$$

$$\frac{d\mu}{dx} = \frac{\partial}{\cancel{\partial y}} \left(P(x)\,\cancel{y} - Q(x) \right) \mu(x)$$

$$\frac{d\mu}{dx} = P(x)\,\mu(x)$$

$$\int \frac{d\mu}{\mu} = \int P(x)\,dx$$

$$\ln \mu = \int P(x)\,dx$$

$$\mu(x) = e^{\int p(x)\,dx}$$

Lec #21: Nov. 2nd

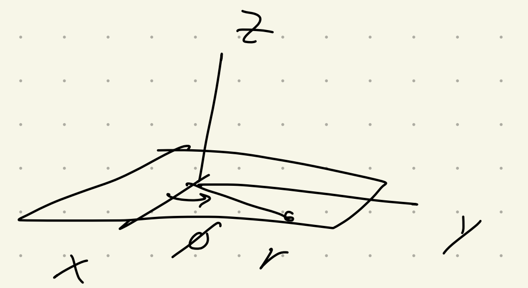
Today: Sec 23 Free oscillation in 2 or more dimensions.

Thurs: Rigid body motion

EXAM 2: Nov 18th $\rightarrow$ Nov 23rd

QUIZ #4: Today (at end of class)

Example: space oscillator



$$U = \frac{1}{2} k r^2 = \frac{1}{2} k (x^2 + y^2)$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$L = T - U$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k (x^2 + y^2)$$

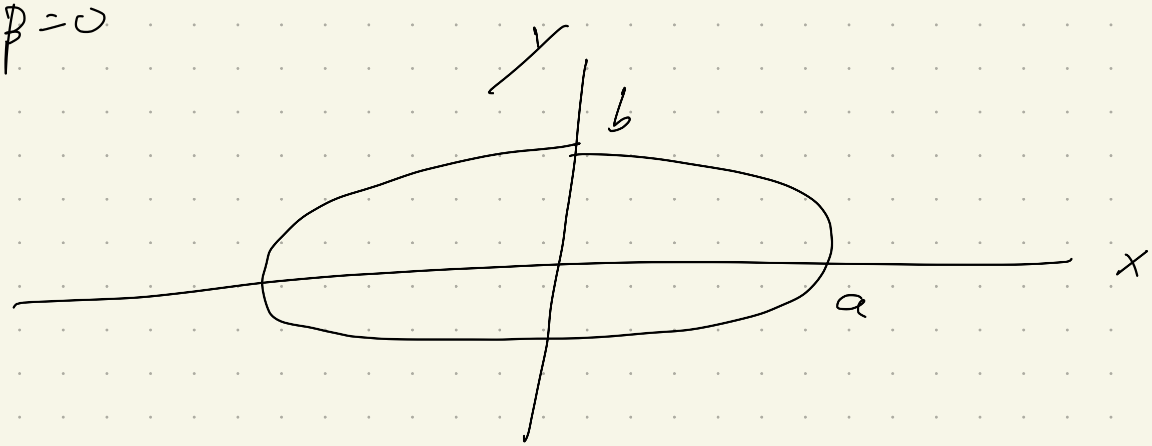
$$= \underbrace{\left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2\right)}_{\omega_x = \sqrt{\frac{k}{m}}} + \underbrace{\left(\frac{1}{2} m \dot{y}^2 - \frac{1}{2} k y^2\right)}_{\omega_y = \sqrt{\frac{k}{m}}}$$

$$\omega = \sqrt{\frac{k}{m}}$$

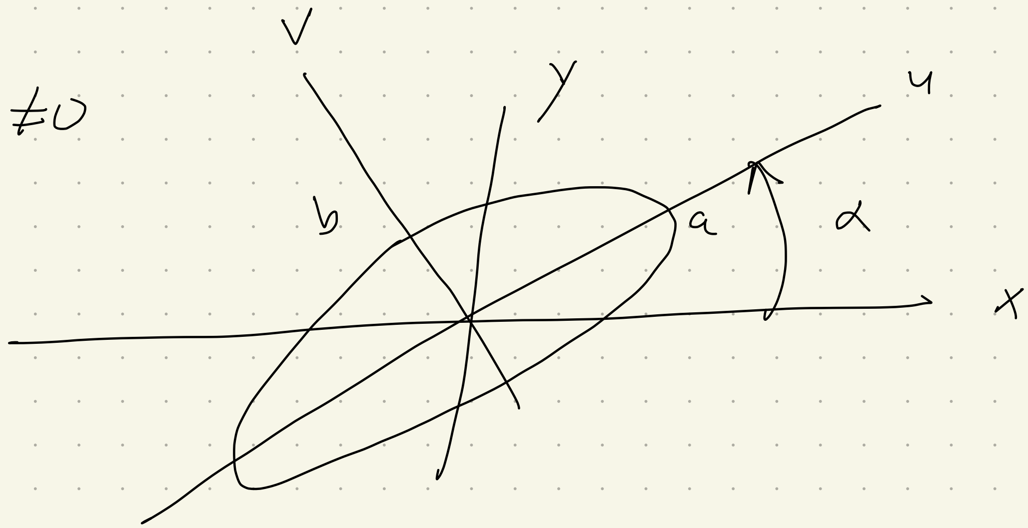
$$\boxed{\begin{aligned} x &= a \cos(\omega t + \alpha) \\ y &= b \sin(\omega t + \beta) \end{aligned}}$$

general solution

$\alpha = \beta = 0$



$\alpha = \beta \neq 0$

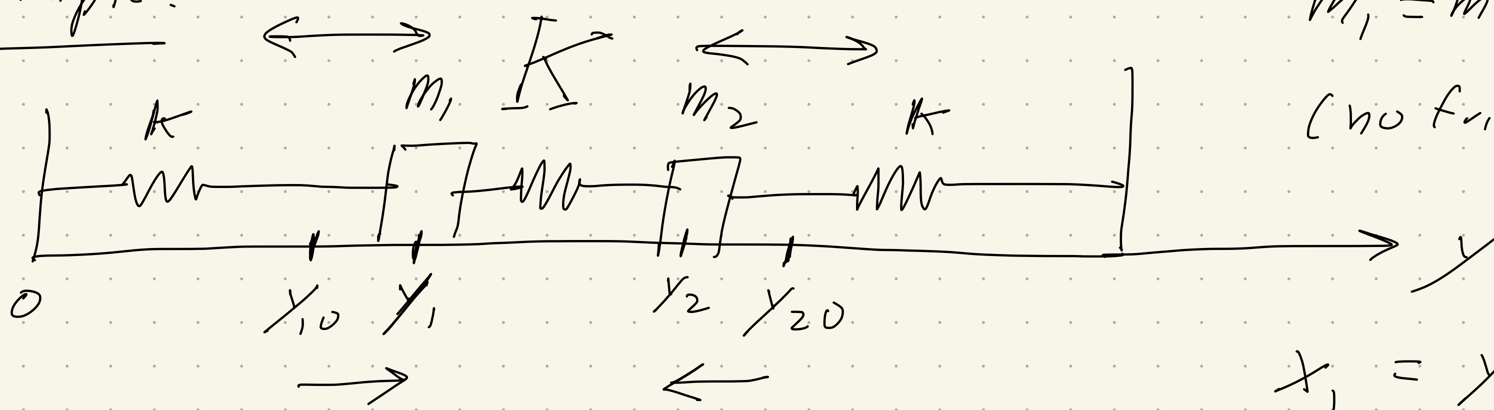


$\alpha \neq \beta$ —— ellipse

Example:

$m_1 = m_2 \equiv m$

(no friction)



$x_1 = y_1 - y_{10}$

$x_2 = y_2 - y_{20}$

$$L = T - U$$

$$T = \frac{1}{2} m \dot{y}_1^2 + \frac{1}{2} m \dot{y}_2^2 = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} K (x_2 - x_1)^2$$

$$= \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} K (x_2^2 + x_1^2 - 2 x_1 x_2)$$

$$= \frac{1}{2} (k + K) x_1^2 + \frac{1}{2} (k + K) x_2^2 - K x_1 x_2$$

$-\frac{1}{2} K x_1 x_2$
$-\frac{1}{2} K x_2 x_1$

$$T = \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k, \quad m_{ik} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$U = \frac{1}{2} \sum_{i,k} K_{ik} x_i x_k, \quad k_{ik} = \begin{pmatrix} k+K & -K \\ -K & k+K \end{pmatrix}$$

$$\boxed{L = \frac{1}{2}\sum_{i,k} m_{ik}\dot{x}_i\dot{x}_k - \frac{1}{2}\sum_{i,k} k_{ik} x_i x_k}$$ general

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) = \frac{\partial L}{\partial x_j} \qquad j = 1, 2, \cdots$$

$$\boxed{\frac{d}{dt}\left(\sum_k m_{jk}\dot{x}_k\right) = -\sum_k k_{jk} x_k}$$

$$\underbrace{\frac{1}{2}\sum_{i,k} m_{ik}\dot{x}_i\dot{x}_k}_{} = \frac{1}{2}\Big(m_{11}\dot{x}_1^2 + m_{12}\dot{x}_1\dot{x}_2 + m_{13}\dot{x}_1\dot{x}_3$$
$$+ m_{21}\dot{x}_2\dot{x}_1 + m_{22}\dot{x}_2^2 + m_{23}\dot{x}_2\dot{x}_3$$
$$+ m_{31}\dot{x}_3\dot{x}_1 + m_{32}\dot{x}_3\dot{x}_2 + m_{33}\dot{x}_3^2\Big)$$

$$\frac{\partial L}{\partial \dot{x}_2} = \frac{1}{2}\left(m_{12}\dot{x}_1 + m_{21}\dot{x}_1 + 2m_{22}\dot{x}_2 + m_{23}\dot{x}_3 + m_{32}\dot{x}_3\right)$$
$$= \cancel{\frac{1}{2}}\left(\cancel{2}m_{21}\dot{x}_1 + \cancel{2}m_{22}\dot{x}_2 + \cancel{2}m_{23}\dot{x}_3\right)$$
$$= \sum_k m_{2k}\dot{x}_k$$

$$\sum_k m_{jk}\,\ddot{x}_k = -\sum_k k_{jk}\, x_k$$

Guess: $x_k = A_k e^{i\omega t}$

$\rightarrow \ddot{x}_k = -\omega^2 A_k e^{i\omega t}$

$$\sum_k -\omega^2 m_{jk} A_k \cancel{e^{i\omega t}} = \underbrace{-\sum_k k_{jk} A_k}_{\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} \\ \end{bmatrix}} \cancel{e^{i\omega t}}$$

$$\boxed{\sum_k \underbrace{\left(k_{jk} - \omega^2 m_{jk}\right)}_{\begin{bmatrix} & \\ & \end{bmatrix}} A_k = 0}$$ — 0 vector

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If invertible

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{\underline{M}} \cdot \vec{v} = \lambda \vec{v}$$

$$\underbrace{\left(\underline{\underline{M}} - \lambda \underline{\underline{\mathbb{1}}} \right)}_{\det = 0} \cdot \vec{v} = 0$$

$$\underbrace{0 = \det\left(K_{j k} - \omega^2 m_{j k}\right)}$$

characteristic equation

polynomial equation $\rightarrow (\omega^2)^N$

Normal mode freqs,

characteristic freqs,

eigen freqs.

$$0 = \det\left(\begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\right)$$

ω_α^2

α label the N different eigen freqs.

$$= \det\begin{bmatrix} k+K-\omega^2 m & -K \\ -K & k+K-\omega^2 m \end{bmatrix}$$

$$= \left((k+K)-\omega^2 m\right)^2 - K^2$$

$$\left((k+K) - \omega^2 m \right)^2 = K^2$$

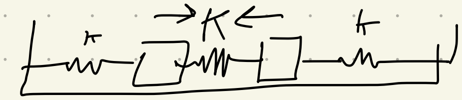
$$(k+K) - \omega^2 m = \pm K$$

$$\rightarrow \quad \omega^2 = \frac{(k+K) \pm K}{m}$$

$$\boxed{\omega_+^2 = \frac{k+2K}{m} , \quad \omega_-^2 = \frac{k}{m}} \leftarrow \text{eigen freqs.}$$

solve for eigenvectors:

$$\omega_+^2 = \frac{k+2K}{m} \qquad V_+ = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

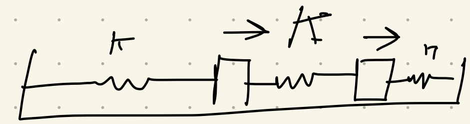


$$\omega_-^2 = \frac{k}{m} \qquad V_- = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

normal mode oscillation

$$\Delta_{k\alpha} = \frac{1}{2} \begin{array}{c} + \quad - \to \alpha \\ \begin{bmatrix} V_+ & V_- \end{bmatrix} \end{array}$$

$\downarrow k$ matrix of eigenvector



$$\boxed{\begin{aligned} x_k &= \mathrm{Re}\left[\sum_\alpha \Delta_{k\alpha} C_\alpha e^{i\omega_\alpha t}\right] \\ &= \sum_\alpha \Delta_{k\alpha}\Theta_\alpha \\ \Theta_\alpha &= \mathrm{Re}\left[C_\alpha e^{i\omega_\alpha t}\right] \end{aligned}}$$

complex const determined by ICs

normal coords

$$L = T - U$$

$$= \frac{1}{2}\sum_{i,k} m_{ik}\,\dot{x}_i\,\dot{x}_k - \frac{1}{2}\sum_{i,k} k_{ik}\,x_i\,x_k$$

$$= \frac{1}{2}\,\dot{x}^T m\,\dot{x} - \frac{1}{2}\,x^T k\,x$$

$$= \frac{1}{2}\,\dot{\theta}^T(\Delta^T m\,\Delta)\,\dot{\theta} - \frac{1}{2}\,\theta^T(\Delta^T k\,\Delta)\,\theta$$

$$= \frac{1}{2}\sum_{\alpha} \mathcal{M}_\alpha\,\dot{\theta}_\alpha^2 - \frac{1}{2}\sum_{\alpha} \mathcal{K}_\alpha\,\theta_\alpha^2$$

☆☆

where $\mathcal{M}$ = diagonal matrix $= \Delta^T m\,\Delta$

$\mathcal{K}$ = " $= \Delta^T k\,\Delta$

eigenvectors diagonalize both m_{ik} and k_{ik} with

$$\boxed{\mathcal{K}_\alpha = \mathcal{M}_\alpha\,\omega_\alpha^2}$$

CHECK:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$

$$= \frac{m}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$

$$= \frac{m}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \rightarrow \mathcal{M}_+ = m,\ \mathcal{M}_- = m$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} k+2K & k \\ -k-2K & k \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2k+4K & 0 \\ 0 & 2k \end{bmatrix}$$

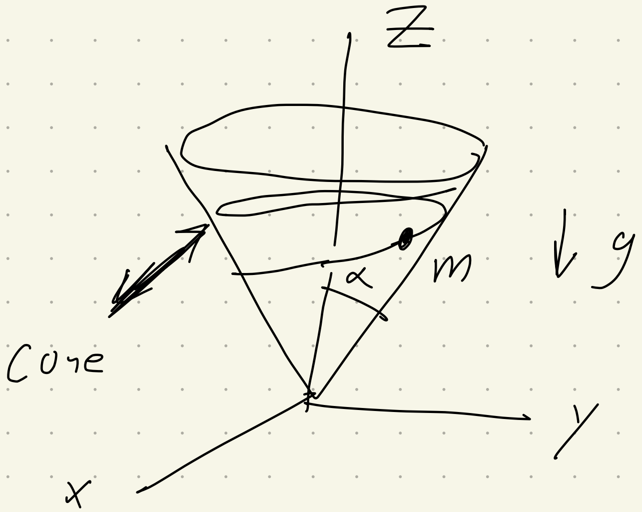
$$= \begin{bmatrix} k+2K & 0 \\ 0 & k \end{bmatrix}$$

$$= \begin{bmatrix} \omega_+^2 m & 0 \\ 0 & \omega_-^2 m \end{bmatrix} \longrightarrow \begin{aligned} \mathcal{K}_+ &= \omega_+^2 m \\ \mathcal{K}_- &= \omega_-^2 m \end{aligned}$$

Thus,
$$L = \frac{1}{2} m\left(\dot{\theta}_+^2 + \dot{\theta}_-^2\right) - \frac{1}{2}\left(m\omega_+^2\theta_+^2 + m\omega_-^2\theta_-^2\right)$$
$$= \left(\frac{1}{2} m \dot{\theta}_+^2 - \frac{1}{2} m \omega_+^2 \theta_+^2\right) + \left(\frac{1}{2} m \dot{\theta}_-^2 - \frac{1}{2} m \omega_-^2 \theta_-^2\right)$$

Quiz #4:

Calculate freq of small oscillations around the circular orbit solution $r = r_0$



α fixed

(Spherical coords)

a) Lagrangian?

b) Determine r_0

(relationship between r_0 and M_z)

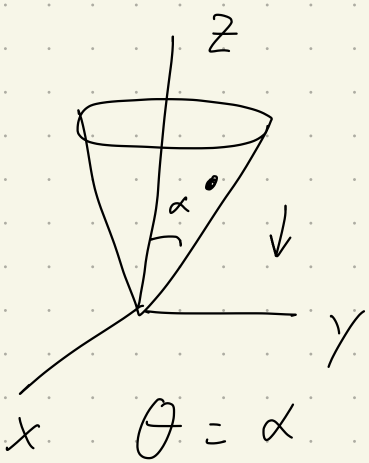
c) Determine ω_r

q4_name.pdf

joseph.d.romano@ttu.edu

Lec #22: Thurs Nov 4th

- Next 5 lectures rigid body motion 31-36, 38
- EXAM 2 - Tues 11/23 (not Thur 11/18)



$\theta = \alpha$

(sph. polar)

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \sin^2\alpha \, \dot{\phi}^2)$$

$$U = mgZ = mgr\cos\alpha$$

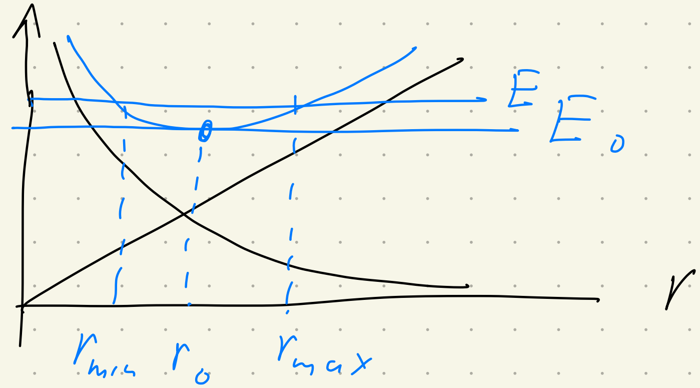
$$L = T - U$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \sin^2\alpha \, \dot{\phi}^2) - mgr\cos\alpha$$

$$M_z = mr^2\sin^2\alpha \, \dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{mr^2\sin^2\alpha}$$

$$E = T + U = \sum_i \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right)$$

$$= \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{M_z^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha}_{U_{eff}(r)}$$



$$\omega_r = \sqrt{\frac{K}{m}}$$

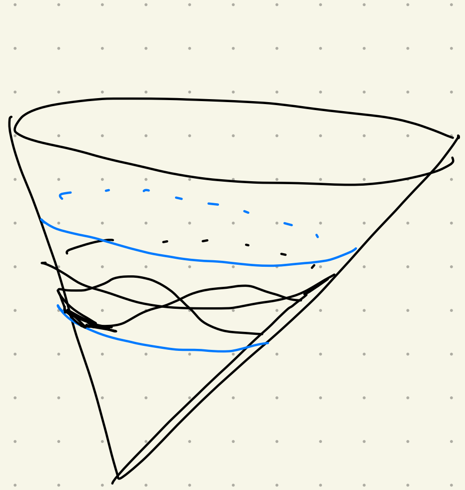
$$K = \left.\frac{d^2 U_{eff}}{dr^2}\right|_{r=r_0} = \frac{3\,mg\cos\alpha}{r_0}$$

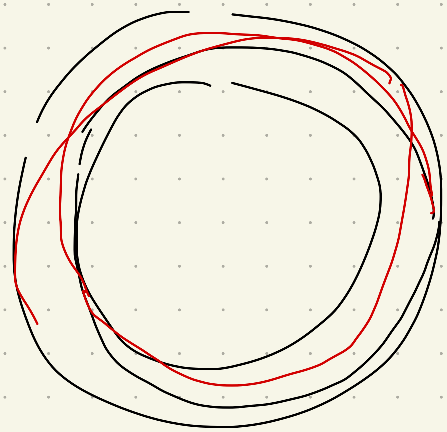
$$\underbrace{0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0}}_{M_z^2 = m^2 g r_0^3 \sin^2\alpha\cos\alpha}$$

$$\rightarrow \quad \omega_r = \sqrt{\frac{K}{m}} = \sqrt{\frac{3g\cos\alpha}{r_0}}$$

$$\omega_\phi = \left.\dot\phi\right|_{r=r_0} = \sqrt{\frac{g\cos\alpha}{r_0\sin^2\alpha}}$$

don't agree (not closed)

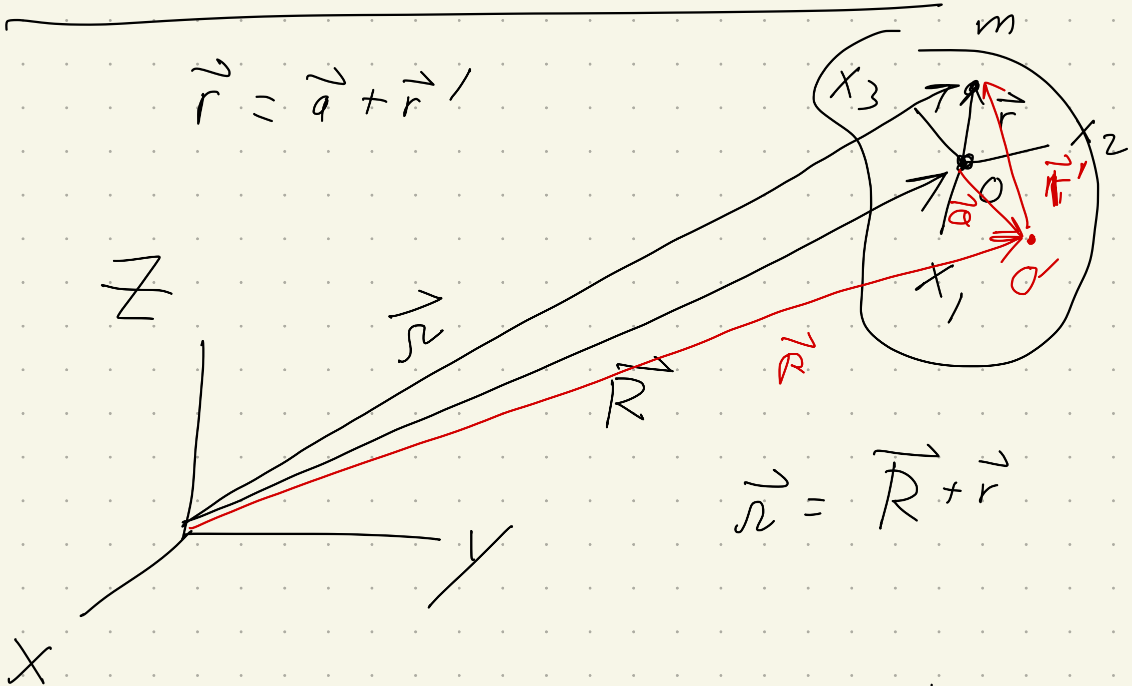




Rigid body motion:

$x_i = (x_1, x_2, x_3)$
$\quad = (x, y, z)$

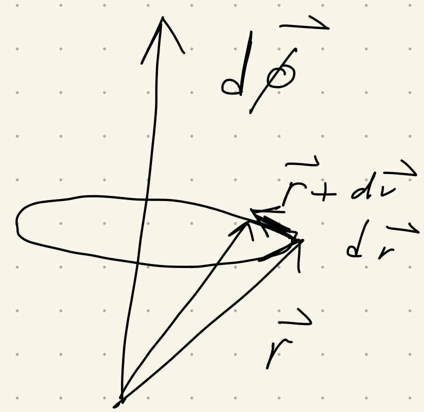
$\vec r = \vec a + \vec r\,'$



$\vec\rho = \vec R + \vec r$

$$\boxed{\vec v = \frac{d\vec\rho}{dt} = \frac{d\vec R}{dt} + \frac{d\vec r}{dt} = \vec V + \vec\Omega \times \vec r}$$

axis of rotation



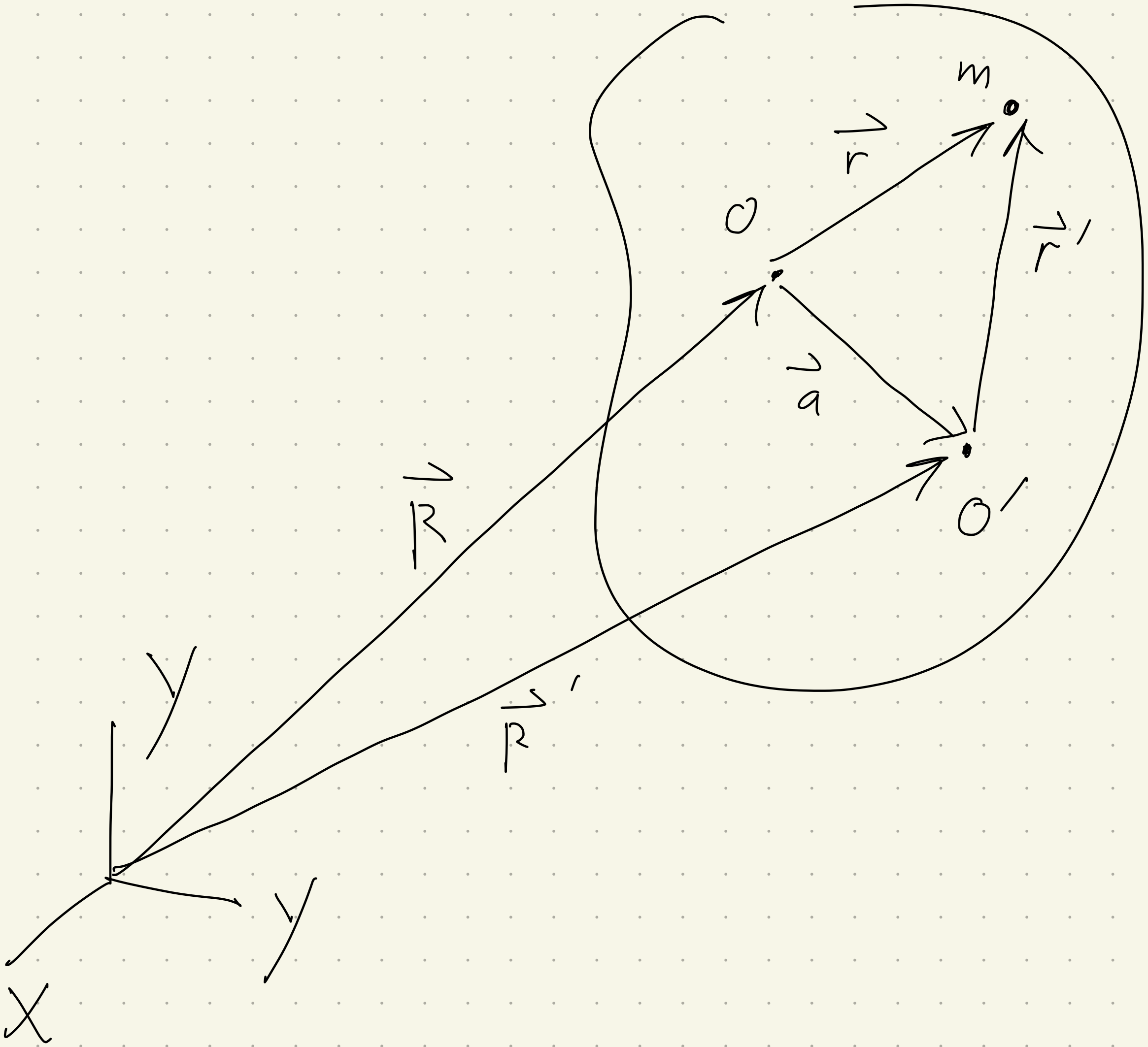
$|d\vec\phi| = d\phi$

$d\vec r = d\vec\phi \times \vec r$

most convenient if O is at com of RB.

$$\frac{d\vec r}{dt} = \frac{d\vec\phi}{dt} \times \vec r = \vec\Omega \times \vec r$$

$$\vec{r} = \vec{a} + \vec{r}\,' \longrightarrow \vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$$



$$\vec{v} = \vec{V}\,' + \vec{\Omega}\,' \times \vec{r}\,'$$

$$\vec{v} = \vec{V} + \vec{\Omega} \times (\vec{a} + \vec{r}\,')$$

$$= \vec{V} + \vec{\Omega} \times \vec{a} + \vec{\Omega} \times \vec{r}\,'$$

$$\vec{V}\,' = \vec{V} + \vec{\Omega} \times \vec{a}$$

$$\vec{\Omega}\,' = \vec{\Omega}$$

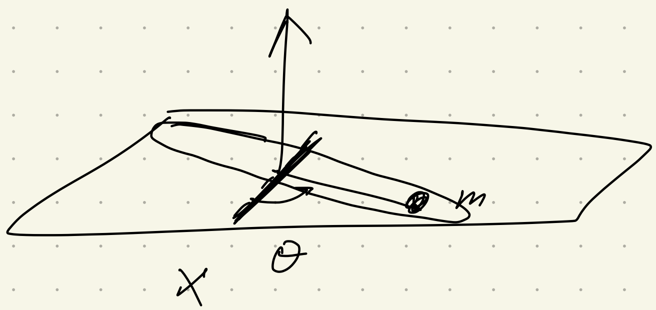
2-d rotational motion:

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt}} \qquad \vec{p} = m\vec{v}$$

$$\theta, \quad \frac{d\theta}{dt} = \omega, \quad T_{rot} = \frac{1}{2} I \omega^2$$

moment of inertia

axis of rotation (fixed



3-d

$$L = I\omega$$

I_{ij} : inertia tensor

$$M_i = \sum_k I_{ik} \Omega_k$$

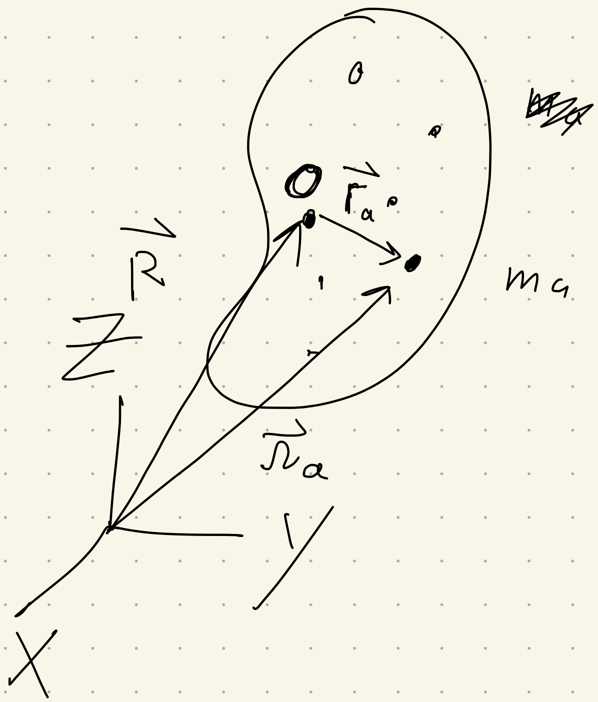
(3x3 symmetric matrix)

$$\vec{M} = \underline{\underline{I}}\, \vec{\Omega}$$

$$\boxed{\vec{N} = \frac{d\vec{M}}{dt}}$$

KE :

m_a —— a : labels the mass point in the rigid body



$$T = \sum_a \frac{1}{2} m_a |\vec{v}_a|^2$$

$$\vec{v}_a = \vec{V} + \vec{\Omega} \times \vec{r}_a$$

$$|\vec{v}_a|^2 = (\vec{V} + \vec{\Omega} \times \vec{r}_a) \cdot (\vec{V} + \vec{\Omega} \times \vec{r}_a)$$

$$= \underline{|\vec{V}|^2 + 2\vec{V} \cdot (\vec{\Omega} \times \vec{r}_a)}$$

$$+ \underline{(\vec{\Omega} \times \vec{r}_a) \cdot (\vec{\Omega} \times \vec{r}_a)}$$

1st term $\quad \sum_a \frac{1}{2} m_a |\vec{V}|^2 = \frac{1}{2} \mu |\vec{V}|^2$

$\mu = \sum_a m_a$ = total mass

$$2^{nd}\text{ term} = \sum_a \cancel{\tfrac{1}{2}}\,\cancel{2}\, m_a \vec{V}\cdot(\vec{\Omega}\times\vec{r}_a)$$

$$= \left(\sum_a m_a \vec{r}_a\right)\cdot(\vec{V}\times\vec{\Omega}) = 0 \left(\begin{array}{l}\text{if}\\ \text{origin}\\ \text{is at}\\ \text{COM}\end{array}\right)$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$3^{rd}\text{ term} = (\vec{\Omega}\times\vec{r}_a)\cdot(\vec{\Omega}\times\vec{r}_a)$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$= \vec{\Omega}\cdot(\vec{r}_a\times(\vec{\Omega}\times\vec{r}_a))$$

$$= \vec{\Omega}\cdot(\vec{\Omega}\,|\vec{r}_a|^2 - \vec{r}_a(\vec{r}_a\cdot\vec{\Omega}))$$

$$= \vec{\Omega}\cdot\vec{\Omega}\, r_a^2 - (\vec{\Omega}\cdot\vec{r}_a)(\vec{\Omega}\cdot\vec{r}_a)$$

$$T = \frac{1}{2} M \left|\vec{V}\right|^2 + \sum_a \frac{1}{2} m_a \left(\underbrace{\vec{\Omega}\cdot\vec{\Omega}}\, r_a^2 - \underbrace{\left(\vec{\Omega}\cdot\vec{r_a}\right)\left(\vec{\Omega}\cdot\vec{r_a}\right)} \right)$$

$$\sum_i \Omega_i^2 = \sum_{i,j} \Omega_i \Omega_j \delta_{ij}$$

$$\sum_i \Omega_i x_{ai} \sum_j \Omega_j x_{aj}$$

$$= \frac{1}{2} M \left|\vec{V}\right|^2 + \sum_a \frac{1}{2} m_a \sum_{i,j} \left(r_a^2 \delta_{ij} - x_{ai} x_{aj} \right) \Omega_i \Omega_j$$

$$= \frac{1}{2} M \left|\vec{V}\right|^2 + \frac{1}{2} \sum_{i,j} \underbrace{\left(\sum_a m_a \left(r_a^2 \delta_{ij} - x_{ai} x_{aj} \right) \right)}_{I_{ij}} \Omega_i \Omega_j$$

$$= \boxed{\frac{1}{2} M \left|\vec{V}\right|^2 + \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j}$$

Lec #23: Nov 9th

Quiz #5: next Tuesday

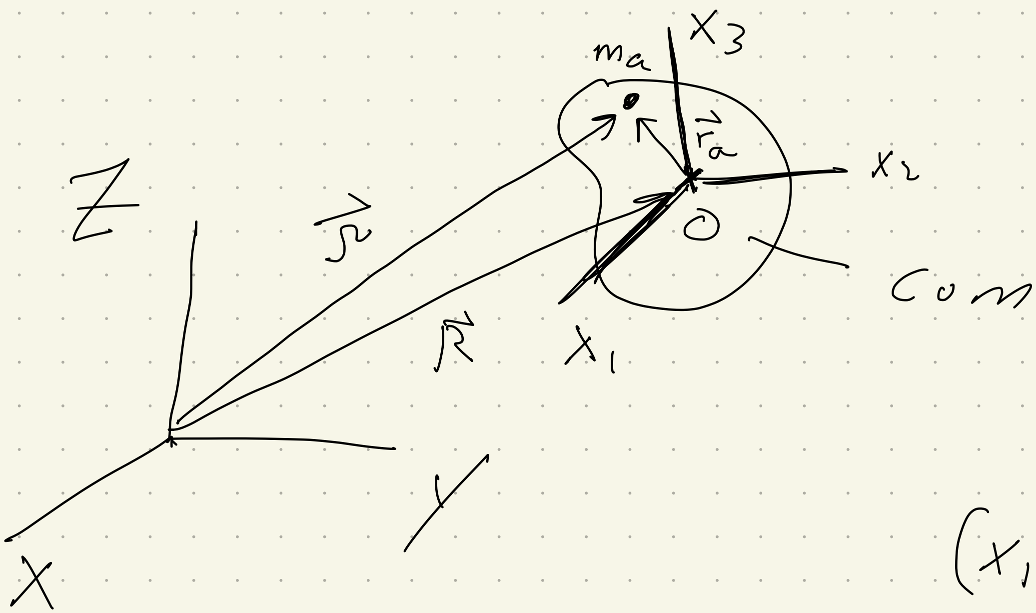
EXAM 2: Tuesday 11/23

Today: Inertia tensor

Thurs: Equation of motion

Last time:

$$T = \frac{1}{2}\mu|\vec{V}|^2 + \frac{1}{2}\underbrace{I_{ij}\Omega_i\Omega_j}_{\underbrace{\sum_{i=1}^{3}\sum_{j=1}^{3}}_{\text{"drop"}} I_{ij}\Omega_i\Omega_j}$$



$(x_1, x_2, x_3) = (x, y, z)$

$$I_{ij} = \sum_a m_a \left(r_a^2 \delta_{ij} - x_{ai} x_{aj} \right)$$

$$r_a^2 = \sum_i x_{ai}^2$$

$$dm = \rho \, dV$$

mass density

$$\boxed{I_{ij} = \int \rho \, dV \left(r^2 \delta_{ij} - x_i x_j \right)}$$

Angular momentum: (wrt COM)

$$\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a \qquad \vec{p}_a = m_a \vec{v}_a$$

$$= \sum_a m_a \vec{r}_a \times \vec{v}_a$$

$$= \sum_a m_a \vec{r}_a \times \left(\cancel{\vec{V}}^{\,0} + \vec{\Omega} \times \vec{r}_a \right)$$

$$= \sum_a m_a \vec{r}_a \times \left(\vec{\Omega} \times \vec{r}_a \right)$$

$$\vec{M} = \sum_a m_a \left(\vec{\Omega}\, r_a^2 - \vec{r}_a \left(\vec{\Omega} \cdot \vec{r}_a \right) \right)$$

$$M_i = \sum_a m_a \left(\underbrace{\Omega_i}_{\delta_{ij} \Omega_j} r_a^2 - x_{ai} \left(\Omega_j\, x_{aj} \right) \right)$$

$$= \sum_a m_a \left(\delta_{ij} r_a^2 - x_{ai} x_{aj} \right) \Omega_j$$

$$= I_{ij} \Omega_j$$

$$\boxed{\vec{M} = \underline{\underline{I}} \cdot \vec{\Omega}}$$

$$\boxed{M_i = I_{ij} \Omega_j \,, \quad T = \frac{1}{2} \mu V^2 + \frac{1}{2} I_{ij} \Omega_i \Omega_j}$$

COM Frame

Properties of I_{ij}:

inertia tensor
(3x3 matrix)

1) I_{ij} real, symmetric $(I_{ij} = I_{ji})$

can always diagonalize:

$$I_{ij} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

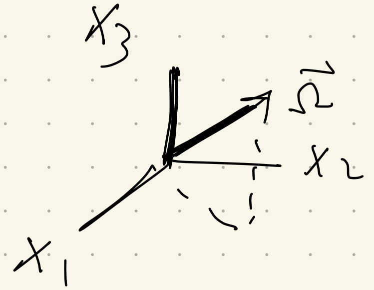
$$= \delta_{ij} I_i \quad \text{(no summation)}$$

(x_1, x_2, x_3) for such a basis are called "principle axes"

$$T = \frac{1}{2} \mu V^2 + \frac{1}{2} I_i \Omega_i^2$$

$$M_i = I_i \Omega_i$$

$$(M_1 = I_1 \Omega_1$$
$$M_2 = I_2 \Omega_2$$
$$M_3 = I_3 \Omega_3)$$

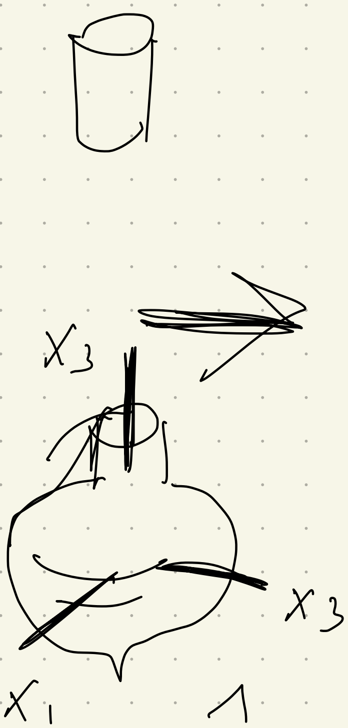


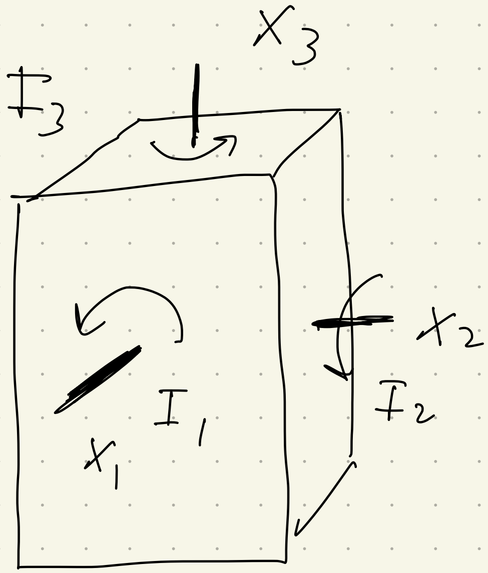
$\vec{\Omega} = (1, 1, 1)$

$I_1 = 1, \; I_2 = 1, \; I_3 = 2$

$\vec{M} = (1, 1, 2)$

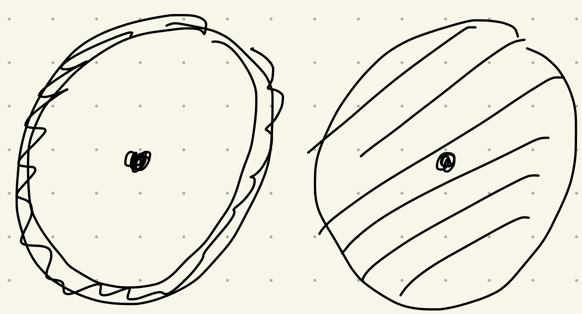
$I_1 \neq I_2 \neq I_3$ in general



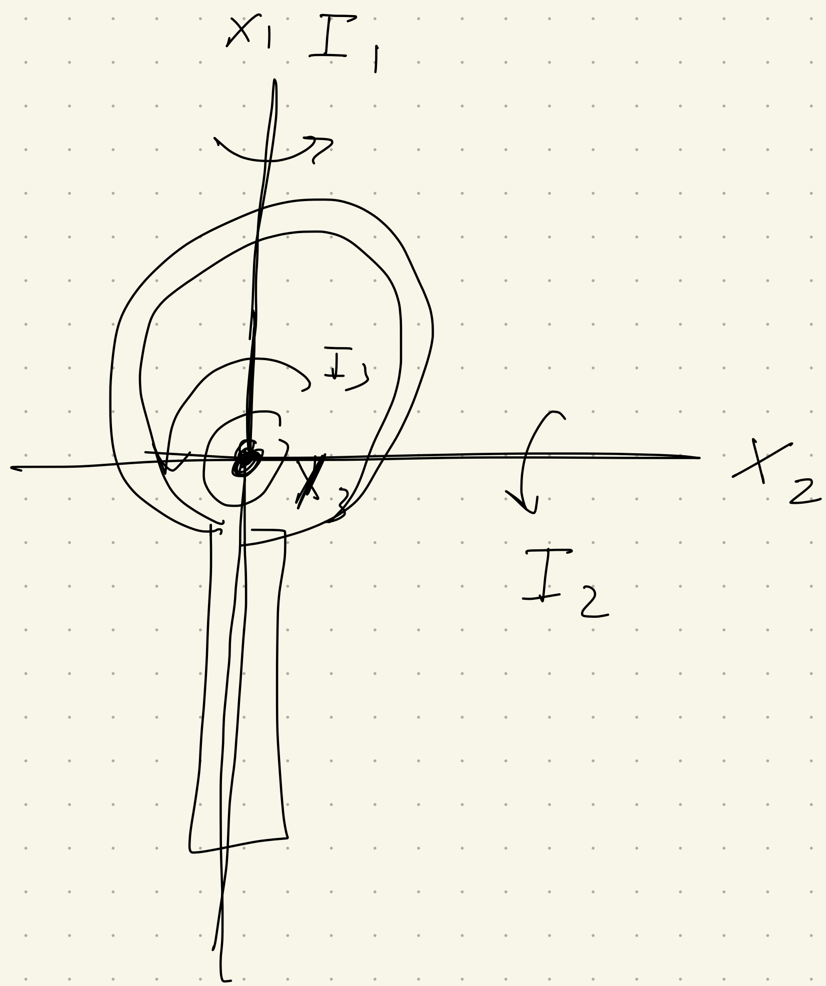


$$\underbrace{I_1 > I_2 > I_3}_{\text{moment of inertia about axis } x_1,\, x_2,\, x_3}$$

$\hat{n}$: $$\underbrace{I(\hat{n}) = I_{ij}\, n_i\, n_j}_{\text{moment of inertia about } \hat{n}}$$



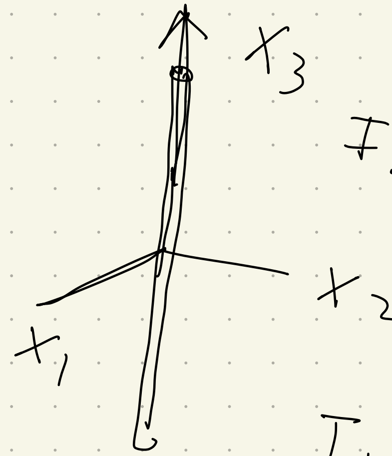
$I_1 < I_2 < I_3$



$I_3 = 0$ "rotator"

$I_1 = I_2 = I_3$ "sphere"

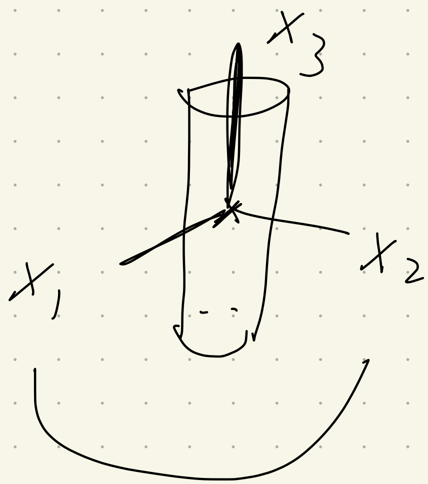
"cube"



$I_3 = 0$

$I_1 = I_2 \neq 0$

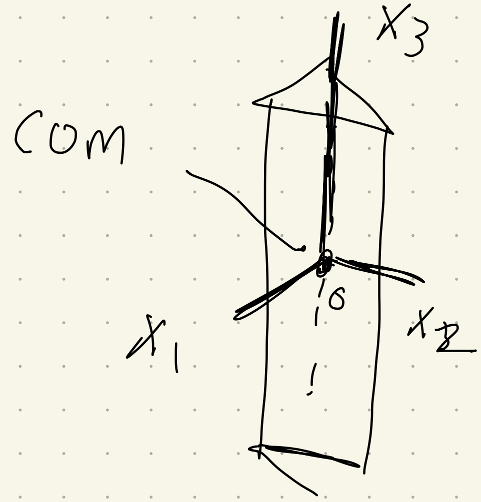
$I_1 = I_2$, "symmetric top"



arbitrarily chosen
in plane $\perp$ to x_3

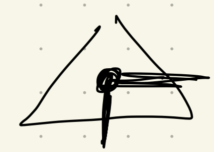
$I_1 \neq I_2 \neq I_3$

"asymmetric top"

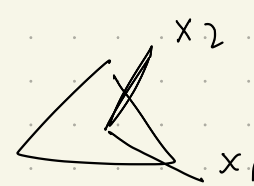


$I_1 \overset{??}{=} I_2$

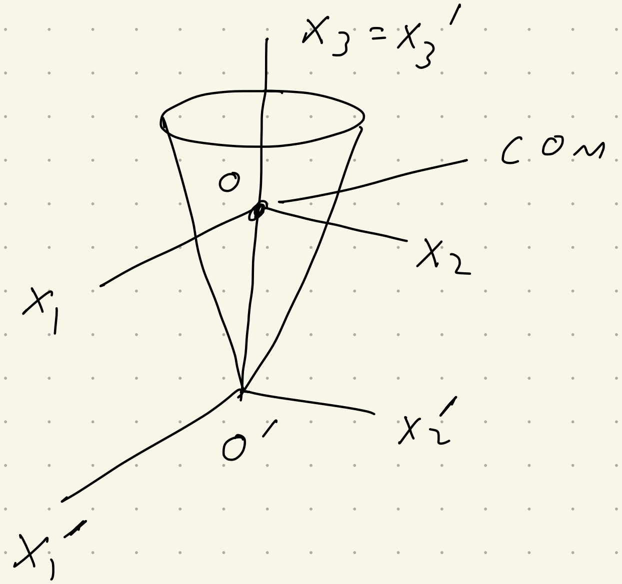
(top view)

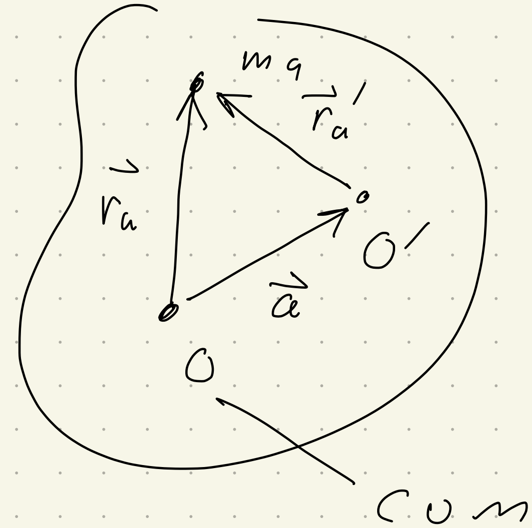


equilateral triangle



Shift the origin:





$$\vec r_a = \vec a + \vec r_a'$$

$$\vec r_a' = \vec r_a - \vec a$$

$$I'_{ij} = \sum_a m_a \Big(\delta_{ij} \underbrace{|\vec r_a'|^2}_{|\vec r_a|^2 + |\vec a|^2 + 2\vec r_a \cdot \vec a} - x'_{ai}\, x'_{aj} \Big)$$

$$(x_{ai} - a_i)(x_{aj} - a_j) = x_{ai}x_{aj} + a_i a_j - a_i x_{aj} - a_j x_{ai}$$

$$I'_{ij} = \underbrace{\sum_a m_a \left(\delta_{ij} |\vec{r}_a|^2 - x_{ai} x_{aj} \right)}_{I_{ij}}$$

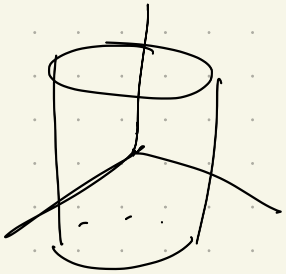
$$+ \underbrace{\sum_a m_a}_{\mu} \left(\delta_{ij} |\vec{a}|^2 - a_i a_j \right)$$

$$+ 2 \cancel{\sum_a m_a \vec{r}_a} \cdot \vec{a} - \cancel{\left(\sum_a m_a x_{aj} \right)} a_i - \cancel{\left(\sum_a m_a x_{ai} \right)} a_j$$

$$0 \qquad 0 \qquad 0$$

$$\mu \vec{R}_{COM} = 0$$

$$I'_{ij} = I_{ij} + \mu \left(\delta_{ij} a^2 - a_i a_j \right)$$



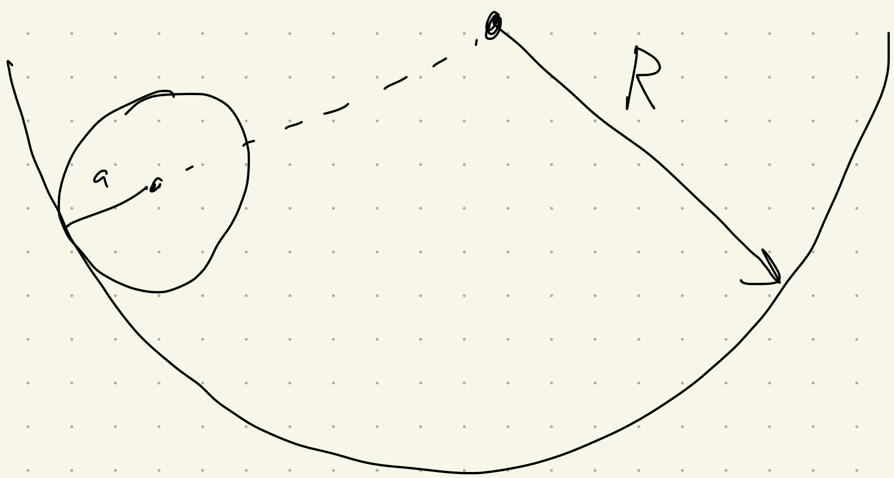
I_1, I_2, I_3

unif cylinder: μ, R, h

Sec 32, Prob 2c

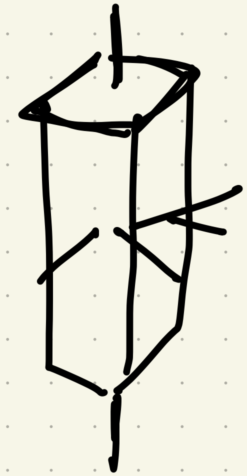
rolling without slipping

$T = ?$



$T = ?$

Lecture #24: Thur Nov 11th

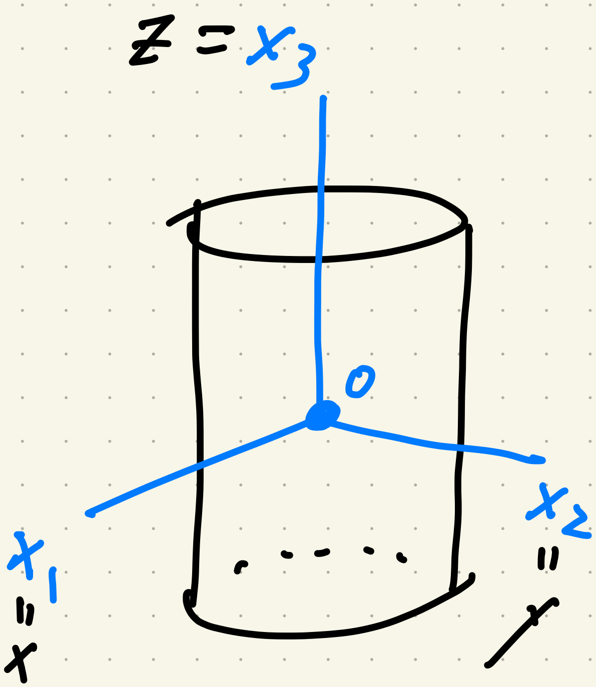


Quiz # 5 — Tuesday Nov 16th

Exam # 2 — Tuesday Nov 23rd

Inertia tensor:

$$I_{ij} = \int \rho \, dV \left(r^2 \delta_{ij} - x_i x_j \right)$$



Unif cylinder

mass: μ

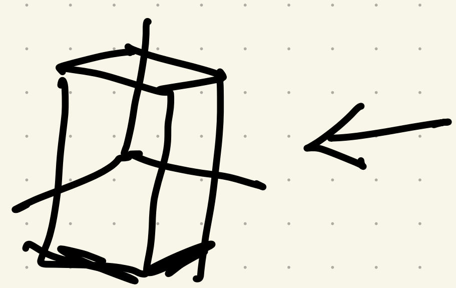
radius: R

height: h

$$\rho = \frac{\mu}{\pi R^2 \cdot h}$$

$$I_3 =$$

$$I = I_1 = I_2$$



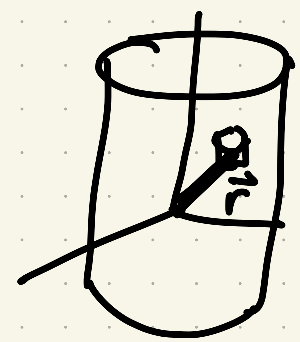
O : Com of the cylinder

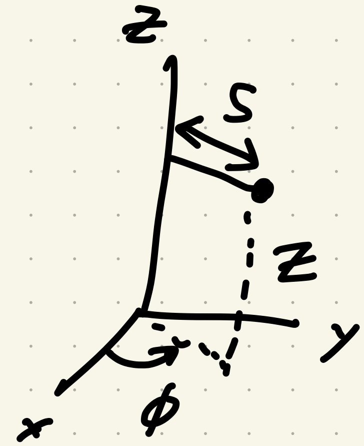
$$\sum_{i,j} I_{ij}\, n_i\, n_j = I(\hat{n})$$

$$I_3 = \sum_{i,j} I_{ij}\, (x_3)_i\, (x_3)_j = I_{33}$$

$$I_1 = I_{11}$$

$$I_2 = I_{22}$$





$$I_3 = \int \rho\, dV \left(r^2 \delta_{33} - x_3^2 \right)$$

$$= \int \rho\, dV \left(r^2 - z^2 \right)$$

sph. polar radius

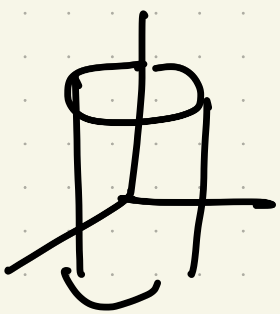
$$= \int \rho\, dV \left(\underbrace{x^2 + y^2}_{s^2} \right)$$

$$dV = s\, ds\, d\phi\, dz$$

$$I_3 = \rho \int_{s=0}^{R} \int_{\phi=0}^{2\pi} \int_{z=-\frac{h}{2}}^{h/2} s\,ds\,d\phi\,dz\; s^2$$

$$= \frac{M}{\pi R^2 h}\, 2\pi\, h \underbrace{\int_{s=0}^{R} s^3\,ds}_{\underbrace{\left.\frac{s^4}{4}\right|_0^R}_{\frac{R^4}{4}}}$$

$$= \boxed{\frac{1}{2} M R^2}$$



$$I = I_1 = \rho \int dV\,(\underbrace{r^2 - x_1^2}_{y^2+z^2})$$

$$= \rho \int dV\,(y^2+z^2)$$

(s, ϕ, z)

$y = s\sin\phi$

$x = s\cos\phi$

$$I = I_2 = \rho \int dV\,(x^2+z^2)$$

$$2I = I_1 + I_2$$

$$= \rho \int dV\,(x^2+y^2+2z^2)$$

$$= \rho \int dV\, s^2 + 2\rho \int dV\, z^2$$

$$\rightarrow \boxed{I = \frac{1}{2} I_3 + \rho \int dV\, z^2} \leftarrow$$

$$I_1 = I_2 = \frac{1}{2} I_3 + \rho \int dV\, z^2$$

$$= \frac{1}{4} M R^2 + \frac{M}{\pi R^2 h} \int_{s=0}^{R} \int_{z=-\frac{h}{2}}^{\frac{h}{2}} \int_{\phi=0}^{2\pi} s\, ds\, d\phi\, dz\, z^2$$

$$= \frac{1}{4} M R^2 + \frac{M}{\pi R^2 h}\, 2\pi\, \underbrace{\left(\frac{z^3}{3}\right)\Bigg|_{-h/2}^{h/2}}\; \frac{s^2}{2}\Bigg|_0^R$$

$$+ \frac{M}{\cancel{\pi}\cancel{R^2} h}\, \cancel{2}\cancel{\pi}\, \frac{2}{3}\left(\frac{h}{2}\right)^3 \frac{\cancel{R^2}}{\cancel{2}}$$

$$\frac{1}{12}$$

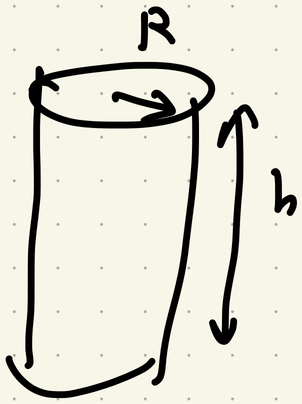
$$\frac{1}{12} M h^2$$

$I_1 = I_2$

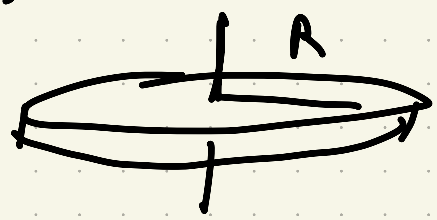
$$= \frac{1}{4} M R^2 + \frac{1}{12} M h^2 = \boxed{\frac{1}{4} M \left(R^2 + \frac{h^2}{3}\right)}$$

$$I_1 = I_2 = \frac{1}{4} M \left(R^2 + \frac{1}{3} h^2 \right)$$

$$I_3 = \frac{1}{2} M R^2$$



i) thin disk $(h \to 0)$



$$\begin{cases} I_3 = \frac{1}{2} M R^2 \\ I_1 = I_2 = \frac{1}{4} M R^2 \end{cases}$$

ii)  $(R \to 0)$

thin rod

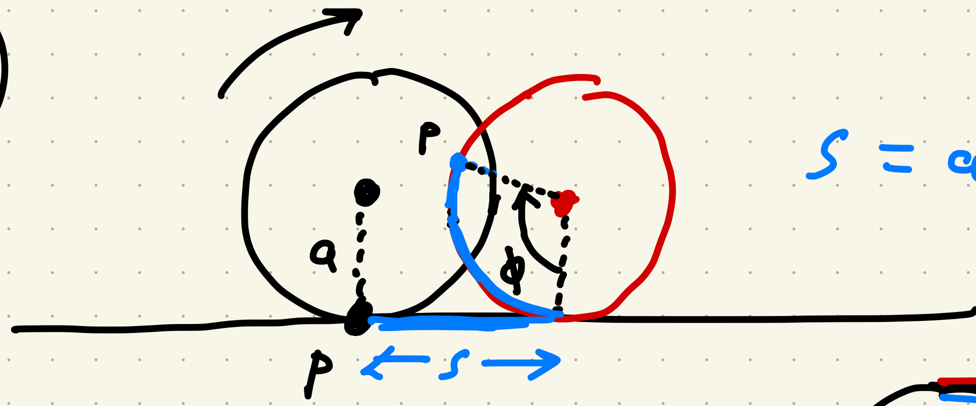
$$\begin{cases} I_3 = 0 \\ I_1 = I_2 = \frac{1}{12} M h^2 \end{cases}$$

Calculate: KE of a rolling cylinder (rolling w/out slipping) $(M, a, \text{height} = h)$

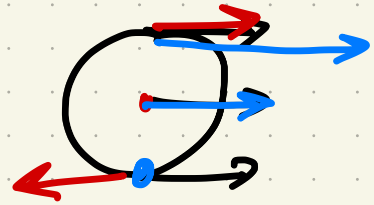
$$T = \frac{1}{2} M |\vec{V}|^2 + \frac{1}{2} I_{ij} \Omega_i \Omega_j$$

$$= \frac{1}{2} M |\vec{V}|^2 + \frac{1}{2}\left(\cancel{I_1 \Omega_1^2} + \cancel{I_2 \Omega_2^2} + I_3 \Omega_3^2\right)$$

i)



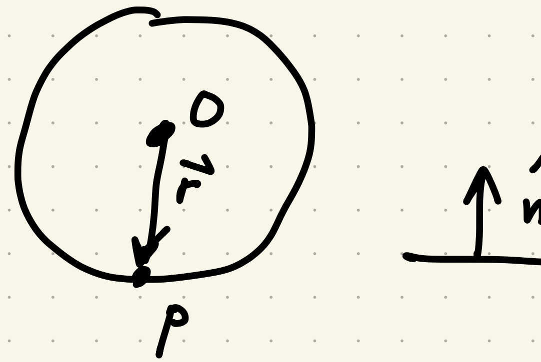
$$s = a\phi$$



$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$$

$$V = \dot{s} = a\dot{\phi}$$

$$\Omega = \frac{V}{a} = \frac{a\dot{\phi}}{a} = \dot{\phi}$$



$$T = \frac{1}{2} M a^2 \dot{\phi}^2 + \frac{1}{2}\left(\frac{1}{2} M a^2\right) \dot{\phi}^2$$

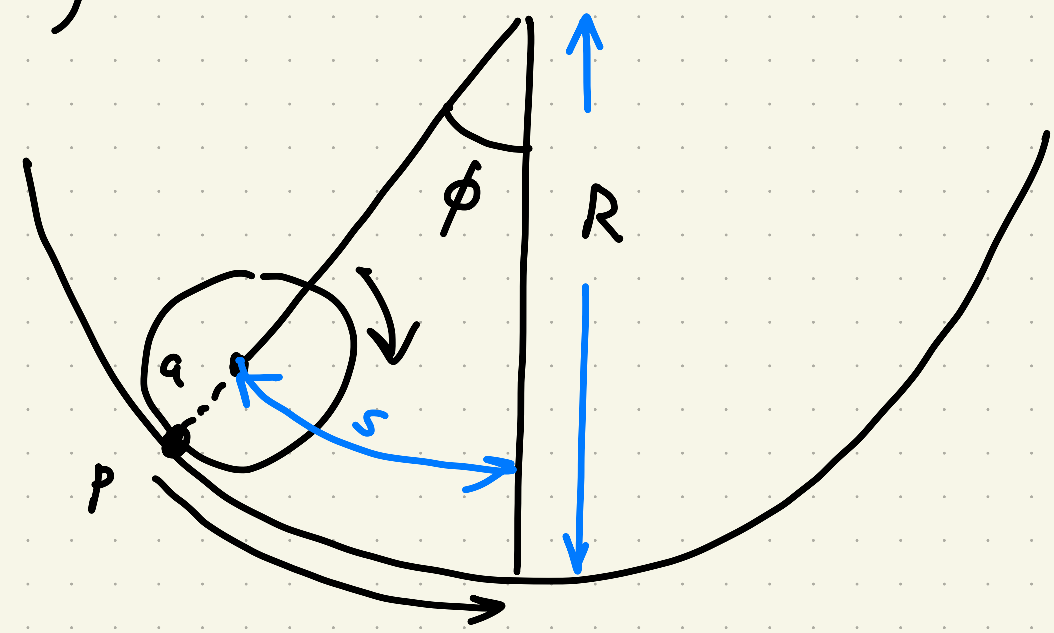
$$= \boxed{\frac{3}{4} M a^2 \dot{\phi}^2}$$

$$0 = \vec{V} + \vec{\Omega} \times (-a\hat{n})$$

$\vec{\Omega}$ into page

$$\boxed{V = \Omega a}$$

ii)



$$s = (R-a)\phi$$

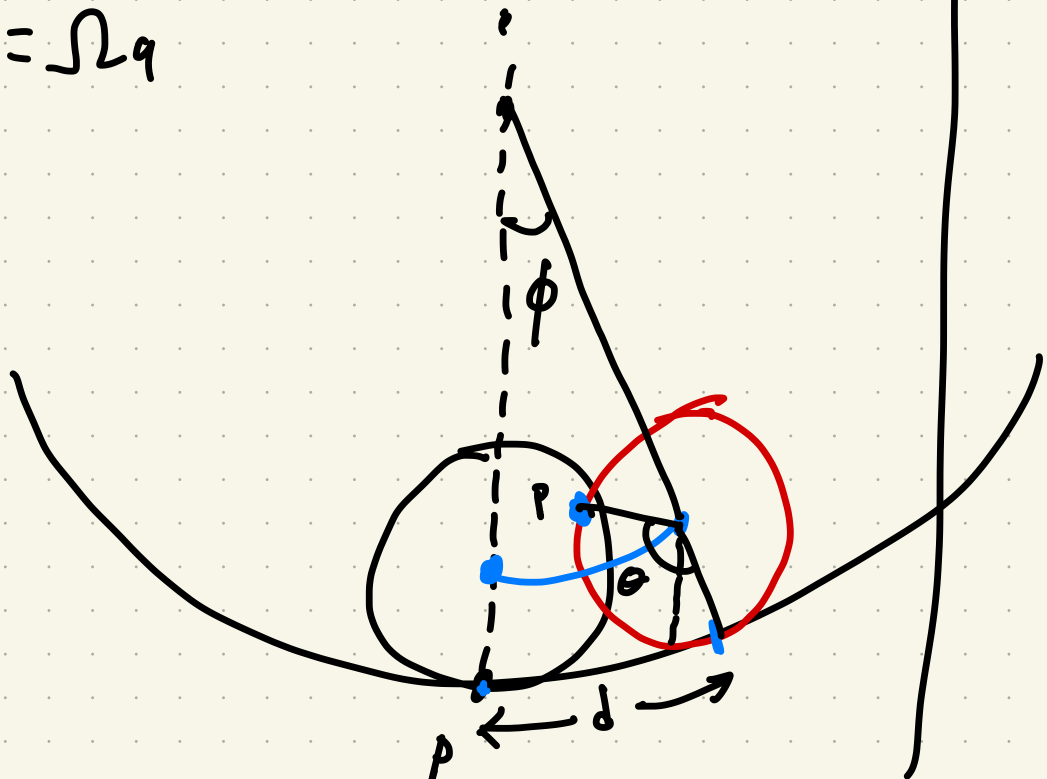
$$V = \Omega a$$

$$V = \dot{s}$$
$$= (R-a)\dot{\phi}$$

$$V = \Omega a$$

$$\Omega = \left(\frac{R-a}{a}\right)\dot{\phi} \rightarrow \boxed{T = \frac{3}{4} m (R-a)^2 \dot{\phi}^2}$$

$$V = \Omega a$$



$$\alpha = \theta - \phi$$

$$\dot{\alpha} = \dot{\theta} - \dot{\phi}$$

$$= \frac{R}{a}\dot{\phi} - \dot{\phi}$$

$$= \left(\frac{R}{a} - 1\right)\dot{\phi}$$

$$= \left(\frac{R-a}{a}\right)\dot{\phi}$$

$$= \Omega$$

$$a\theta = d = R\phi \rightarrow \theta = \frac{R}{a}\phi$$

$$s = (R-a)\phi$$

$$V = \dot{s} = (R-a)\dot{\phi} = \Omega a \rightarrow \boxed{\Omega = \left(\frac{R-a}{a}\right)\dot{\phi}}$$

Lecture #25: Nov 16th

EXAM 2 - Next tuesday 11/23

(Zoom review session: 7pm - 8pm, Sunday)

Today: - Quiz #5

- RB motion

Thursday - RB motion / statics ??
(Sec 38)

Tues: 11/23 EXAM 2

Thurs: Thanksgiving

Tues: 11/30: non-inertial ref frames

Sat 12/4: 1:30 - 4:00pm $\begin{cases} \text{Sat} \\ \text{Sun} \\ \text{Mon} \end{cases}$

EOMs:

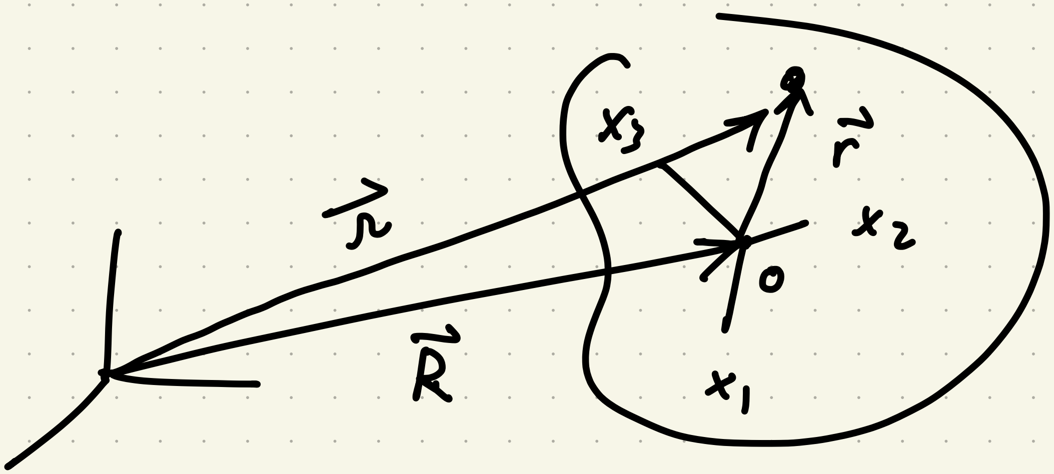
$$\frac{d\vec{M}}{dt} = \vec{K} = \sum \vec{r}\times\vec{f}$$

$$\sum_a \vec{r}_a \times \vec{f}_a$$

$$\frac{d\vec{P}}{dt} = \vec{F} = \sum \vec{f}$$

6 DOF's: $(\vec{R}, \vec{\phi}) = q_i$

$$(\vec{V}, \vec{\Omega}) = \dot{q}_i$$



inertial

$$\vec{v} = \frac{d\vec{\rho}}{dt}$$

$$= \frac{d}{dt}(\vec{R} + \vec{r})$$

$$= \frac{d\vec{R}}{dt} + \frac{d\vec{r}}{dt}$$

$$= \vec{V} + \vec{\Omega}\times\vec{r}$$

Derive:

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}\left(\sum \vec{p}\right) = \sum \frac{d\vec{p}}{dt} = \sum \vec{f}$$

$$\frac{d\vec{M}}{dt} = \frac{d}{dt}\left(\sum \vec{r}\times\vec{p}\right) = \sum\left(\cancelto{0}{\dot{\vec{r}}\times\vec{p}} + \vec{r}\times\underbrace{\dot{\vec{p}}}_{=\vec{f}}\right)$$

$$= \sum \vec{r}\times\vec{f}$$

$$\dot{\vec{r}} = m\vec{v} = m\frac{d\vec{\Omega}}{dt} = m\frac{d\vec{r}}{dt} = m\dot{\vec{r}}$$

$$\dot{\vec{r}} = \vec{\Omega}\times\vec{r}$$

Alternative derivation:

$$L = \underbrace{\frac{1}{2}\mu|\vec{V}|^2 + \frac{1}{2} I_{ij}\cdot\Omega_i\cdot\Omega_j}_{T} - \underbrace{U(\vec{r})}_{\text{"are"}}$$

$\sum_{i,j}$ implied

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{V}}\right) = \frac{\partial L}{\partial \vec{R}} \longleftrightarrow \frac{d\vec{P}}{dt} = \sum \vec{f}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{\Omega}}\right) = \frac{\partial L}{\partial \vec{\phi}} \longleftrightarrow \frac{d\vec{M}}{dt} = \sum \vec{r}\times\vec{f}$$

$$\frac{\partial L}{\partial \vec{V}} = \mu\vec{V} = \vec{P}$$

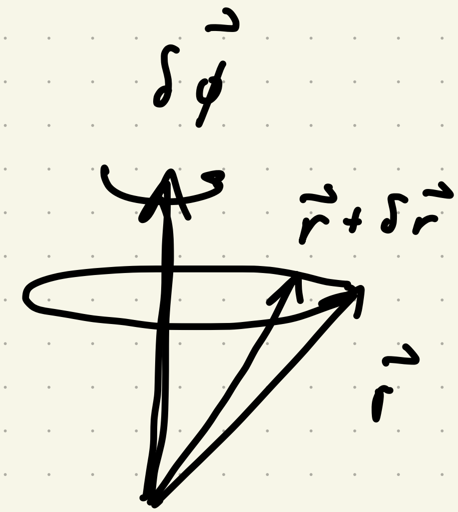
$$\frac{\partial L}{\partial \vec{\Omega}} = \vec{M}$$

$$\frac{\partial L}{\partial \Omega_i} = \sum_j I_{ij}\cdot\Omega_j = M_i$$

$$\frac{1}{2} I_{ij}\Omega_i\Omega_j = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$\frac{\partial L}{\partial \Omega_3} = I_3\Omega_3 = M_3$$

$$\delta U(\vec{r}) = \sum_a \frac{\partial U}{\partial \vec{r}_a} \cdot \delta\vec{r}_a$$



$$\vec{r} = \vec{R} + \vec{r}$$

$$\delta\vec{r} = \delta\vec{R} + \delta\vec{r}$$

$$= \delta\vec{R} + \delta\vec{\phi} \times \vec{r}$$

$$\delta U = \sum \underbrace{\frac{\partial U}{\partial \vec{r}}}_{-\vec{f}} \cdot \left(\delta\vec{R} + \delta\vec{\phi} \times \vec{r}\right)$$

$$\delta U = -\left(\sum \vec{f}\right)\cdot \delta\vec{R} - \sum \underbrace{\vec{f}\cdot\left(\delta\vec{\phi}\times\vec{r}\right)}_{\delta\vec{\phi}\cdot\sum\left(\vec{r}\times\vec{f}\right)}$$

$$\boxed{\frac{\partial U}{\partial \vec{R}} = -\sum \vec{f} = -\vec{F}}\ , \quad \boxed{\frac{\partial U}{\partial \vec{\phi}} = -\sum \vec{r}\times\vec{f} = -\vec{K}}$$

$$\frac{\partial L}{\partial \vec{R}} = -\frac{\partial U}{\partial \vec{R}} = \vec{F}\ , \qquad \frac{\partial L}{\partial \vec{\phi}} = -\frac{\partial U}{\partial \vec{\phi}} = \vec{K}$$

$U(x,y) \qquad dU = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy$

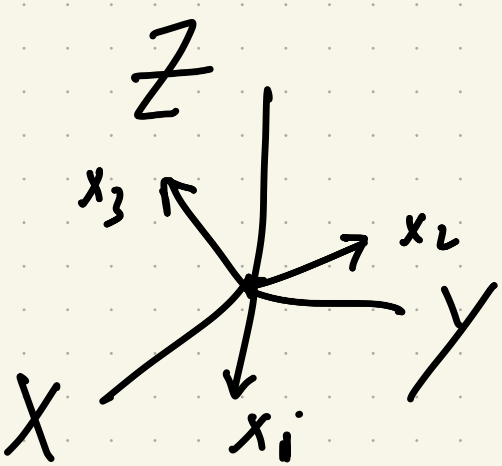
$$\delta U = \frac{\partial U}{\partial \vec{R}}\cdot\delta\vec{R} + \frac{\partial U}{\partial \vec{\phi}}\cdot\delta\vec{\phi}$$

Euler's equations for RB motion:

$$\underbrace{\frac{d\vec{A}}{dt}}_{\text{wrt inertial frame}} = \underbrace{\frac{d'\vec{A}}{dt}}_{\text{wrt rotating frame}} + \vec{\Omega}\times\vec{A}$$

$\vec{\Omega}$: ang. velocity of the rotating frame

$$\vec{A} = \sum_i A_i \hat{x}_i \qquad \left(= A_X \hat{X} + A_Y \hat{Y} + A_Z \hat{Z}\right)$$



$$\frac{d\vec{A}}{dt} = \underbrace{\sum_i \frac{dA_i}{dt}\hat{x}_i}_{\frac{d'\vec{A}}{dt}} + \underbrace{\sum_i A_i \frac{d\hat{x}_i}{dt}}_{\vec{\Omega}\times\vec{A}}$$

$$\left(\frac{d'\vec{A}}{dt}\right)_i = \frac{dA_i}{dt}$$

$$\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\Omega}\times\vec{A}$$

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d'\vec{P}}{dt} + \vec{\Omega}\times\vec{P}$$

$$P_i = \mu V_i$$

$$\boxed{F_1} = \left(\frac{d'\vec{P}}{dt}\right)_1 + \left(\vec{\Omega}\times\vec{P}\right)_1$$

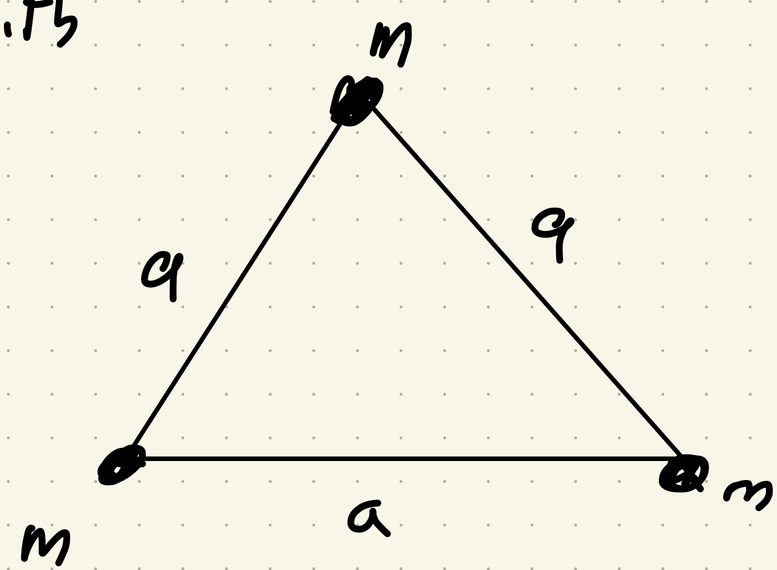
$$= \frac{dP_1}{dt} + \Omega_2 P_3 - \Omega_3 P_2$$

$$= \mu\left(\frac{dV_1}{dt} + \Omega_2 V_3 - \Omega_3 V_2\right)$$

+ cyclic

q5 – lastname !

Calculate the principal moments of inertia for a rigid body which consists of 3 equal masses at the corners of an equilateral triangle with side = a



Solution: take points in xy plane. Since masses are equal, and an equilateral Δ has $120°$ rotational symmetry $\rightarrow$ symmetric top (so $I_1 = I_2 \equiv I$)

Also, because points don't have a z-component:

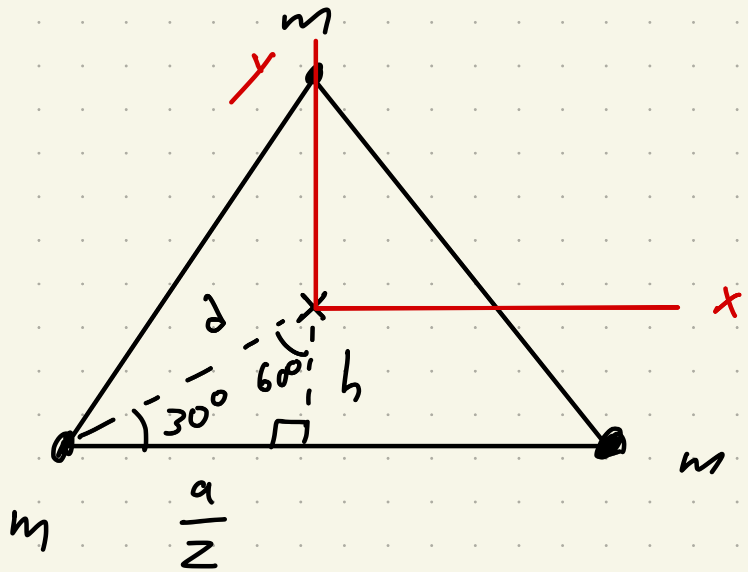
$$I_1 = \sum_a m_a (y_a^2 + \underbrace{z_a^2}_{0}) = \sum_a m_a y_a^2$$

$$I_2 = \sum_a m_a (\underbrace{z_a^2}_{0} + x_a^2) = \sum_a m_a x_a^2$$

$$\rightarrow \quad I_3 = \sum_a m_a (x_a^2 + y_a^2) = I_1 + I_2 = 2I$$

Thus, $I = \frac{1}{2} I_3$

To find I_3, note that $x_a^2 + y_a^2 = \perp$ distance of mass points from z-axis



$$\sin 30^\circ = \frac{h}{d} = \frac{1}{2}$$

$$\sin 60^\circ = \frac{a/2}{d} = \frac{\sqrt{3}}{2}$$

$$\rightarrow \boxed{d = \frac{a}{\sqrt{3}} \qquad h = \frac{d}{2} = \frac{a}{2\sqrt{3}}}$$

Thus,

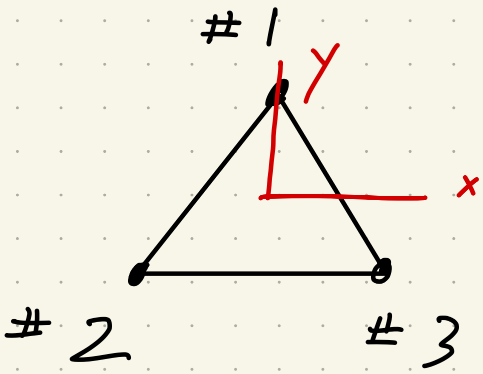
$$I_3 = \sum_a m_a (x_a^2 + y_a^2)$$

$$= \sum_a m_a d_a^2 = 3m\left(\frac{a}{\sqrt{3}}\right)^2 = \boxed{ma^2}$$

$$\rightarrow I_1 = I_2 = \frac{1}{2} I_3 = \boxed{\frac{1}{2} ma^2}$$

Alternative: (explicit calculation of I_1, I_2)

Take (x_1, x_2) to be (x, y)



Then #1: $(x, y) = \left(0, \frac{a}{\sqrt{3}}\right)$

#2: $(x, y) = \left(-\frac{a}{2}, -\frac{a}{2\sqrt{3}}\right)$

#3: $(x, y) = \left(\frac{a}{2}, -\frac{a}{2\sqrt{3}}\right)$

Then $I_1 = \sum_a m_a y_a^2 = m\left(\left(\frac{a}{\sqrt{3}}\right)^2 + \left(\frac{-a}{2\sqrt{3}}\right)^2 + \left(\frac{-a}{2\sqrt{3}}\right)^2\right)$

$= ma^2\left(\frac{1}{3} + \frac{1}{12} + \frac{1}{12}\right)$

$= \boxed{\frac{1}{2} ma^2}$

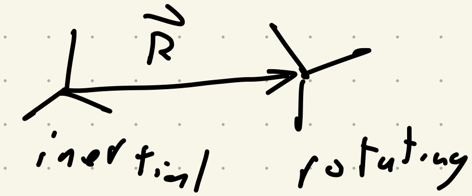
$I_2 = \sum_a m_a x_a^2 = m\left(0^2 + \left(\frac{-a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right)$

$= \boxed{\frac{1}{2} ma^2}$

$\rightarrow \quad I_3 = I_1 + I_2 = \boxed{ma^2}$

Recall: $$\underbrace{\frac{d\vec{A}}{dt}}_{\substack{\text{wrt}\\ \text{inertial}\\ \text{frame}}} = \underbrace{\frac{d'\vec{A}}{dt}}_{\substack{\text{wrt}\\ \text{rotating}\\ \text{frame}}} + \vec{\Omega} \times \vec{A}$$

$\vec{\Omega}$ — angular velocity of rotating frame



$$\vec{A} = \sum_i A_i \hat{x}_i$$

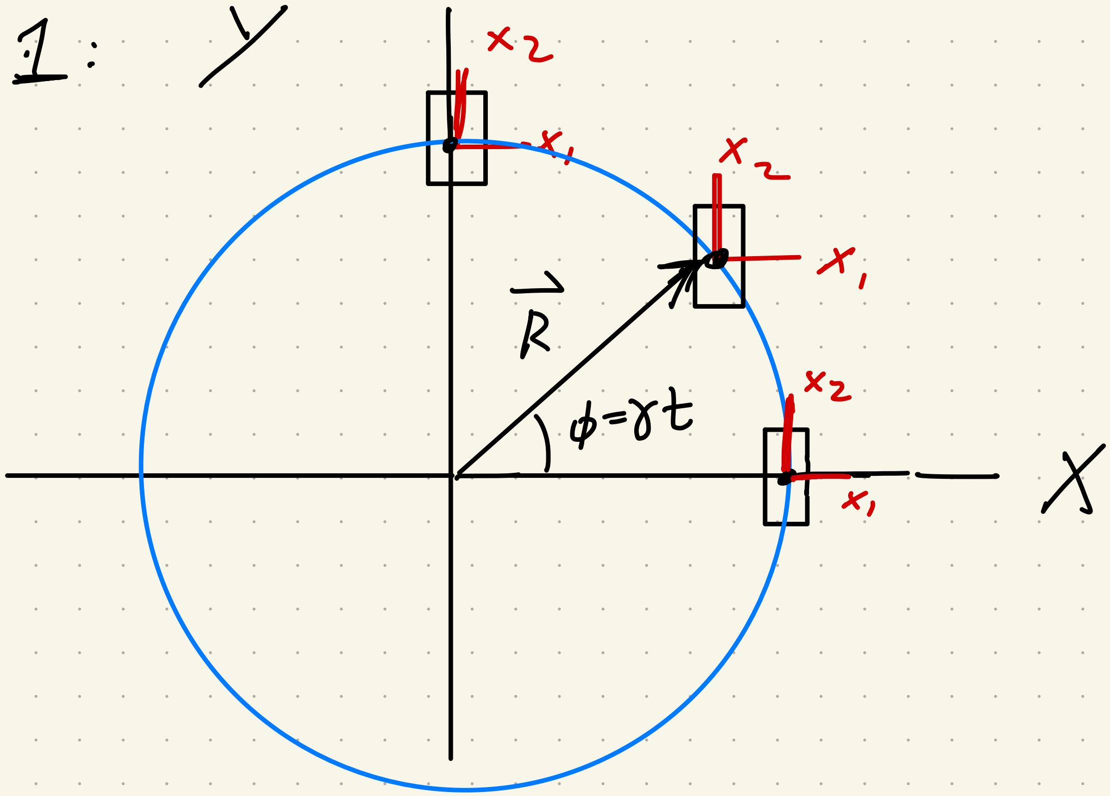
$\hat{x}_i$ — basis vectors (time-dependent) for rotating frame

$$\frac{d\vec{A}}{dt} = \underbrace{\sum_i \frac{dA_i}{dt} \hat{x}_i}_{\frac{d'\vec{A}}{dt}} + \underbrace{\sum_i A_i \frac{d\hat{x}_i}{dt}}_{\vec{\Omega} \times \vec{A}}$$

NOTE: i) $\frac{d\vec{A}}{dt}$, $\frac{d'\vec{A}}{dt}$ involve the same vector

ii) Inertial frame and rotating frame do not have to have the same origin.

Example 1:



Com move in circle of radius R with unit angular velocity γ.

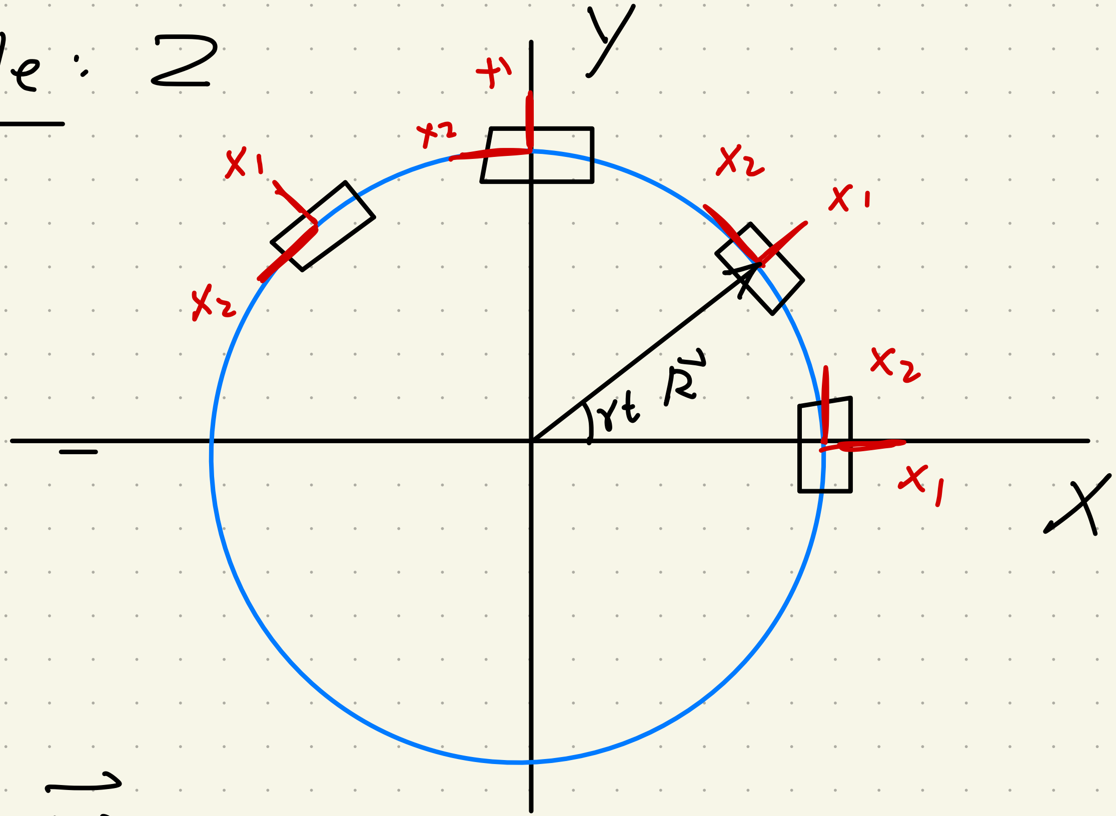
Thus, $\vec{R} = R\cos(\gamma t)\,\hat{X} + R\sin(\gamma t)\,\hat{Y}$

$$\vec{V} = \frac{d\vec{R}}{dt} = R\gamma\left(-\sin(\gamma t)\,\hat{X} + \cos(\gamma t)\,\hat{Y}\right)$$

Since $(\hat{x}_1, \hat{x}_2)$ point in same direction as $(\hat{X}, \hat{Y})$

then $\frac{d\vec{A}}{dt} = \frac{d\vec{A}'}{dt}$ for any $\vec{A}$, including $\vec{V}$

Example: 2



Now the rigid body frame axes are rotating relative to the fixed inertial frame

$\vec{R}$, $\vec{V}$ same as for example 1.

$$\hat{x}_1 = \cos(\gamma t)\,\hat{X} + \sin(\gamma t)\,\hat{Y}$$

$$\hat{x}_2 = -\sin(\gamma t)\,\hat{X} + \cos(\gamma t)\,\hat{Y}$$

$$\frac{d\hat{x}_1}{dt} = -\gamma \sin(\gamma t)\,\hat{X} + \gamma \cos(\gamma t)\,\hat{Y} = \gamma\,\hat{x}_2$$

$$\frac{d\hat{x}_2}{dt} = -\gamma \cos(\gamma t)\,\hat{X} - \gamma \sin(\gamma t)\,\hat{Y} = -\gamma\,\hat{x}_1$$

Thus, $\displaystyle\sum_i A_i \frac{d\hat{x}_i}{dt} = \gamma A_1 \hat{x}_2 - \gamma A_2 \hat{x}_1$

$$= \vec{\Omega} \times \vec{A}$$

where $\vec{\Omega} = \gamma \hat{Z} = \gamma \hat{x}_3$

check:

$$(\vec{\Omega} \times \vec{A})_1 = \cancel{\Omega_2}^{\,0} A_3 - \Omega_3 A_2 = -\gamma A_2$$

$$(\vec{\Omega} \times \vec{A})_2 = \Omega_3 A_1 - \cancel{\Omega_1}^{\,0} A_3 = \gamma A_1$$

$$(\vec{\Omega} \times \vec{A})_3 = \cancel{\Omega_1}^{\,0} A_2 - \cancel{\Omega_2}^{\,0} A_1 = 0$$

so $\dfrac{d\vec{A}}{dt} = \dfrac{d'\vec{A}}{dt} + \vec{\Omega} \times \vec{A}$ as claimed

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NOTE: In rotating frame $\vec{R} = R\hat{x}_1$, $\vec{V} = R\gamma \hat{x}_2$

so $\dfrac{d'\vec{R}}{dt} = 0$, $\dfrac{d'\vec{V}}{dt} = 0$ $\Longrightarrow \vec{V} = \dfrac{d\vec{R}}{dt} = 0 + \vec{\Omega} \times \vec{R}$

$$= -\gamma R \sin(\gamma t)\hat{x}_1 + \gamma R \cos(\gamma t)\hat{x}_2$$

Lecture #26:

EXAM 2 – next Tuesday

Review session (optional) – Sunday, 7pm
Q&A (via Zoom)

Last class: 11/30 (Tuesday) – non-inertial ref frame

scattering, small oscillations, RB motion

Euler's equations of RB motion:

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d'\vec{P}}{dt} + \vec{\Omega} \times \vec{P}$$

$$\vec{K} = \frac{d\vec{M}}{dt} = \frac{d'\vec{M}}{dt} + \vec{\Omega} \times \vec{M}$$

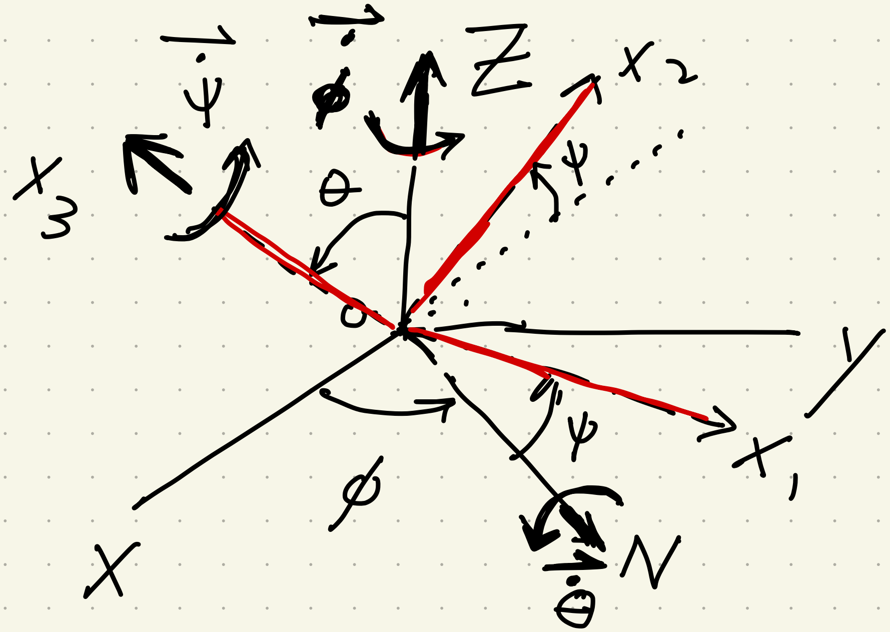
$$\boxed{F_1 = \mu\left(\frac{dV_1}{dt} + \Omega_2 V_3 - \Omega_3 V_2\right)}$$

similar for 2, 3

$$\boxed{K_1 = I_1 \frac{d\Omega_1}{dt} + \Omega_2 \Omega_3 (I_3 - I_2)}$$

similar for 2, 3

Euler angles: $\vec{\phi}$ (ϕ, θ, ψ)



ON: line of nodes

$$\boxed{\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta} + \vec{\dot\psi}}$$

$\Omega_1, \Omega_2, \Omega_3$

$\vec{\dot\psi} = \dot\psi \, \hat{x}_3$

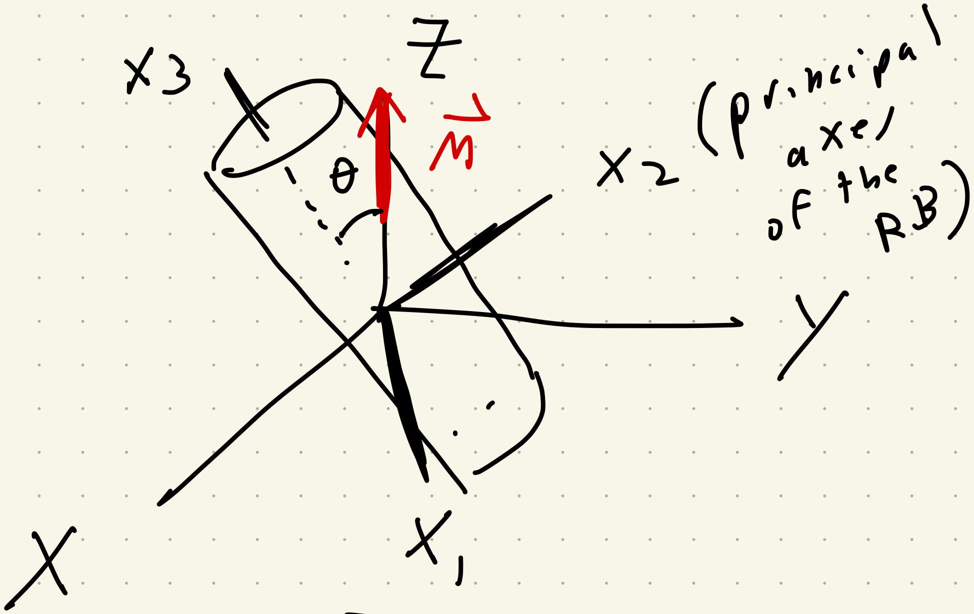
$\vec{\dot\theta} = \dot\theta \left(\cos\psi \, \hat{x}_1 - \sin\psi \, \hat{x}_2 \right)$

$\vec{\dot\phi} = \dot\phi \big(\cos\theta \, \hat{x}_3$

$+ \sin\theta \cos\psi \, \hat{x}_2$

$+ \sin\theta \sin\psi \, \hat{x}_1 \big)$

$$\boxed{\begin{aligned}\vec{\Omega} &= (\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi) \hat{x}_1 \\ &+ (\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi) \hat{x}_2 + (\dot\phi \cos\theta + \dot\psi) \hat{x}_3\end{aligned}}$$



X_1 in (X, Y) plane)

$\psi = 0$

Torque-free motion of a symmetric top

$\vec{K} = 0 = \frac{d\vec{M}}{dt}$

$\vec{M} = \text{const}$ (choose to point along $\hat{Z}$)

$$\boxed{\begin{aligned} \Omega_1 &= \dot{\theta} \\ \Omega_2 &= \dot{\phi} \sin\theta \\ \Omega_3 &= \dot{\phi}\cos\theta + \dot{\psi} \end{aligned}}$$

(when $\psi = 0$)

$M_1 = 0$

$M_2 = M \sin\theta$

$M_3 = M\cos\theta$

$M_1 = I_1 \Omega_1$, $M_2 = I_2 \Omega_2$,

$M_3 = I_3 \Omega_3$

$0 = I_1 \dot{\theta} \to \dot{\theta} = 0$

$\theta = \text{const}$

$M \cancel{\sin\theta} = I_2 \dot{\phi} \cancel{\sin\theta} \to \boxed{\dot{\phi} = \frac{M}{I_2}} = \text{const}$

$M_3 = I_3 \Omega_3$

$$\Omega_3 = \frac{M_3}{I_3} = \frac{M\cos\theta}{I_3}$$

$$\Omega_3 = \text{const} \longrightarrow \dot\psi = \text{const}$$

$$\dot\phi \cos\theta + \dot\psi = \frac{M\cos\theta}{I_3}$$

$\dot\phi = \text{const} = \dfrac{M}{I_2}$

$\theta = \theta_0$ $\qquad$ $\theta = \theta_0$

Spin freq $= \Omega_3$

wobble freq $= \dot\phi$

$$\frac{\text{spin}}{\text{wobble}} = \frac{\Omega_3}{\dot\phi} = \frac{\cancel{M}\cos\theta}{I_3}\,\frac{I_2}{\cancel{M}}$$

$$= \cos\theta \frac{I_2}{I_3} \simeq \frac{I_2}{I_3} = \boxed{\frac{1}{2}}$$

$\theta \approx 0$

$I_3 = \frac{1}{2} M R^2, \quad I_1 = I_2 = \frac{1}{4} M R^2$

Torque-free motion with $\vec{\Omega} = \text{const}$:

$\vec{M} = \text{const}$

$$0 = \Omega_2 \Omega_3 (I_3 - I_2)$$
$$0 = \Omega_3 \Omega_1 (I_1 - I_3)$$
$$0 = \Omega_1 \Omega_2 (I_2 - I_1)$$

$$\left.\begin{array}{l} \Omega_1 = \text{const}, \quad \Omega_2 = 0, \; \Omega_3 = 0 \\ \Omega_2 = \text{const}, \quad \Omega_3 = 0, \; \Omega_1 = 0 \\ \Omega_3 = \text{const}, \quad \Omega_1 = 0, \; \Omega_2 = 0 \end{array}\right\} \text{rotation about a single principal axis}$$

$$I_1 < I_2 < I_3$$

$\underline{I_1}$: $\quad \Omega_1 = \text{const}$, $\Omega_2 = 0$, $\Omega_3 = 0$

$\Rightarrow \boxed{\Omega_1 = \text{const} + \epsilon_1}$, $\Omega_2 = \epsilon_2$, $\Omega_3 = \epsilon_3$

$\epsilon_1, \epsilon_2, \epsilon_3$: functions of time

$|\epsilon_1| << 1$, etc.

$$M_1 = 0 = I_1 \dot{\Omega}_1 + \Omega_2 \Omega_3 (I_3 - I_2)$$

$$\dot{\Omega}_1 = -\Omega_2 \Omega_3 \left(\frac{I_3 - I_2}{I_1} \right)$$

$$\frac{d}{dt}\left(\cancelto{0}{\text{const}} + \epsilon_1 \right) = -\epsilon_2 \epsilon_3 \left(\frac{I_3 - I_2}{I_1} \right)$$

$$\dot{\epsilon}_1 = -\epsilon_2 \epsilon_3 \left(\frac{I_3 - I_2}{I_1} \right) = O(\epsilon^2) \approx 0$$

$\boxed{\epsilon_1 = \text{const}}$ $\rightarrow$ $\boxed{\Omega_1 = \text{const}}$

$$\rightarrow 0 = I_2 \dot{\Omega}_2 + \Omega_3 \Omega_1 (I_1 - I_3)$$

$$\dot{\Omega}_2 = -\Omega_3 \Omega_1 \frac{(I_1 - I_3)}{I_2}$$

$$\boxed{\dot{\epsilon}_2 = -\epsilon_3 \underbrace{\Omega_1}_{\text{const}} \frac{(I_1 - I_3)}{I_2}}$$

$$\rightarrow 0 = I_3 \dot{\Omega}_3 + \Omega_1 \Omega_2 (I_2 - I_1)$$

$$\dot{\Omega}_3 = -\underbrace{\Omega_1}_{\text{const}} \Omega_2 \frac{(I_2 - I_1)}{I_3}$$

$$\boxed{\dot{\epsilon}_3 = -\epsilon_2 \Omega_1 \frac{(I_2 - I_1)}{I_3}}$$

$$\ddot{\epsilon}_2 = -\Omega_1 \frac{(I_1 - I_3)}{I_2} \boxed{\dot{\epsilon}_3}$$

$$= +\Omega_1 \frac{(I_1 - I_3)}{I_2} \overbrace{\Omega_1 \frac{(I_2 - I_1)}{I_3} \epsilon_2}$$

$$= +\Omega_1^2 \underbrace{\frac{\overset{<0}{(I_1 - I_3)}\overset{>0}{(I_2 - I_1)}}{I_2 I_3}}_{<0} \epsilon_2$$

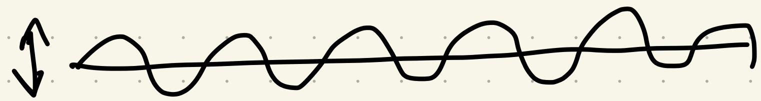
$(I_1 < I_2 < I_3)$

$$\boxed{\ddot{\epsilon}_2 = -\lambda^2 \epsilon_2}$$

similarly

$$\boxed{\ddot{\epsilon}_3 = -\lambda^2 \epsilon_3}$$

$$\overbrace{\epsilon_3, \epsilon_2 = a\cos(\lambda t) + b\sin(\lambda t)}^{\text{sinusoidal evolution}}$$



$\Rightarrow$ perturbations remain small

$\Downarrow$

$\Omega_1 =$ const solution

$\Omega_2 = 0$

$\Omega_3 = 0$

stable

Similarly:

$\boxed{I_3}$: $\Omega_3 = \text{const}$, $\Omega_1 = 0$, $\Omega_2 = 0$

$\rightarrow$ sinusoidal behavior for ϵ_1, ϵ_2

$\rightarrow$ stable

However: $I_1 < I_2 < I_3$

$\boxed{I_2}$: $\Omega_2 = \text{const}$, $\Omega_1 = 0$, $\Omega_3 = 0$

$\Omega_2 = \text{const} + \epsilon_2$, $\Omega_1 = \epsilon_1$, $\Omega_3 = \epsilon_3$

$$\ddot{\epsilon}_1 = \Omega_2^2 \frac{\overbrace{(I_2 - I_1)}^{>0}\overbrace{(I_3 - I_2)}^{>0}}{I_3 I_1} \epsilon_1 = A^2 \epsilon_1$$

$$\ddot{\epsilon}_1 = \lambda^2 \epsilon_1$$

$$\underbrace{\epsilon_1(t)} = a \underbrace{e^{\lambda t}}_{\substack{\text{grows} \\ \text{exponentially} \\ \text{with time.}}} + \underbrace{b e^{-\lambda t}}_{\text{decays to zero}}$$

$\Omega_1 = 0$

$$\Longrightarrow \boxed{\begin{gathered} \Omega_2 = \text{const}, \; \Omega_1 = 0, \; \Omega_3 = 0 \\ \text{is } \underline{\text{unstable}} \text{ to small perturbations.} \end{gathered}}$$