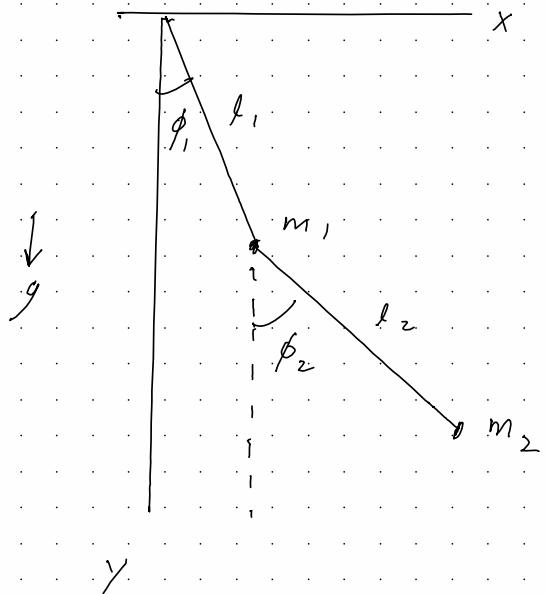


Soc 5, prob 1



$$x_1 = l_1 \sin \phi_1$$

$$y_1 = l_1 \cos \phi_1$$

$$x_2 = x_1 + l_2 \sin \phi_2 = l_1 \sin \phi_1 + l_2 \sin \phi_2$$

$$y_2 = y_1 + l_2 \cos \phi_2 = l_1 \cos \phi_1 + l_2 \cos \phi_2$$

$$U = -m_1 g y_1 - m_2 g y_2$$

$$= -m_1 g l_1 \cos \phi_1 - m_2 g (l_1 \cos \phi_1 + l_2 \cos \phi_2)$$

$$= -(m_1 + m_2) g l_1 \cos \phi_1 - m_2 g l_2 \cos \phi_2$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\dot{x}_1 = l_1 \cos \phi_1 \dot{\phi}_1$$

$$\dot{x}_1^2 = l_1^2 \cos^2 \phi_1 \dot{\phi}_1^2$$

$$\dot{y}_1 = -l_1 \sin \phi_1 \dot{\phi}_1$$

$$\dot{y}_1^2 = l_1^2 \sin^2 \phi_1 \dot{\phi}_1^2$$

$$\text{Thus, } \dot{x}_1^2 + \dot{y}_1^2 = l_1^2 (\sin^2 \phi_1 + \cos^2 \phi_1) \dot{\phi}_1^2 \\ = l_1^2 \dot{\phi}_1^2$$

$$x_2 = l_1 \cos \phi_1 \phi_1 + l_2 \cos \phi_2 \phi_2$$

$$\rightarrow \dot{x}_2^2 = l_1^2 \cos^2 \phi_1 \dot{\phi}_1^2 + l_2^2 \cos^2 \phi_2 \dot{\phi}_2^2 + 2l_1 l_2 \cos \phi_1 \cos \phi_2 \phi_1 \phi_2$$

$$\dot{y}_2 = -l_1 \sin \phi_1 \dot{\phi}_1 - l_2 \sin \phi_2 \dot{\phi}_2$$

$$\rightarrow \dot{y}_2^2 = l_1^2 \sin^2 \phi_1 \dot{\phi}_1^2 + l_2^2 \sin^2 \phi_2 \dot{\phi}_2^2 + 2l_1 l_2 \sin \phi_1 \sin \phi_2 \dot{\phi}_1 \dot{\phi}_2$$

Now,

$$\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 (\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2) \dot{\phi}_1 \dot{\phi}_2 \\ = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

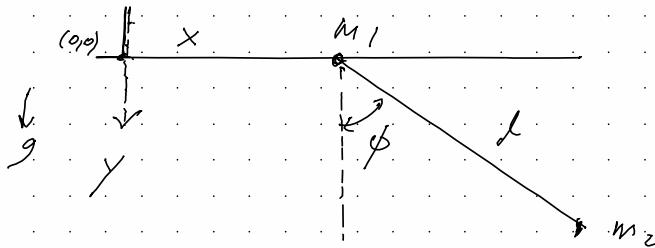
$$\text{so. } T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\ = \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 \\ + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 \\ = \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

$$U = -(m_1 + m_2) g l_1 \cos \phi_1 - m_2 g l_2 \cos \phi_2$$

$$L = T - U$$

$$= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 \\ + (m_1 + m_2) g l_1 \cos \phi_1 + m_2 g l_2 \cos \phi_2$$

Sec 5 Prob. 2



Generalised coords:  $x, \phi$

$$(x_1, y_1) = (x, 0)$$

$$(x_2, y_2) = (x + l \sin \phi, l \cos \phi)$$

$$U = -m_1 g y_1 - m_2 g y_2$$

$$= -m_2 g l \cos \phi$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\text{Now: } \dot{x}_1^2 + \dot{y}_1^2 = \dot{x}^2$$

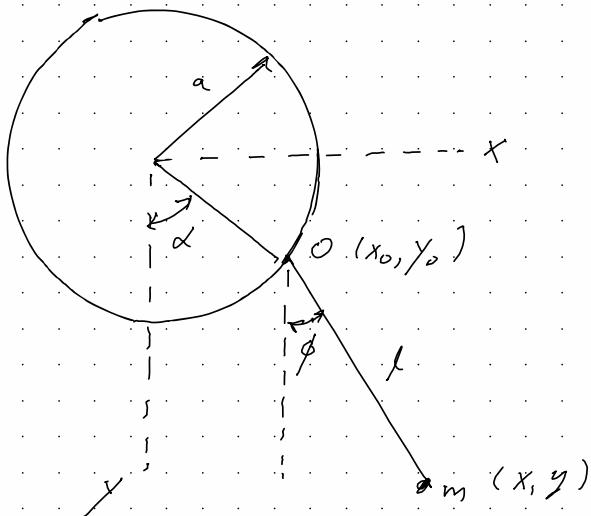
$$\begin{aligned} \dot{x}_2^2 + \dot{y}_2^2 &= (\dot{x} + l \cos \phi \dot{\phi})^2 + (-l \sin \phi \dot{\phi})^2 \\ &= \dot{x}^2 + l^2 \cos^2 \phi \dot{\phi}^2 + 2 l \cos \phi \dot{x} \dot{\phi} \\ &\quad + l^2 \sin^2 \phi \dot{\phi}^2 \\ &= \dot{x}^2 + l^2 \dot{\phi}^2 + 2 l \cos \phi \dot{x} \dot{\phi} \end{aligned}$$

$$\begin{aligned} \rightarrow T &= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}^2 + l^2 \dot{\phi}^2 + 2 l \cos \phi \dot{x} \dot{\phi}) \\ &= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \cos \phi \dot{x} \dot{\phi} \end{aligned}$$

$$\begin{aligned} L &= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \cos \phi \dot{x} \dot{\phi} \\ &\quad + m_2 g l \cos \phi \end{aligned}$$

Sec 5, Prob 3

(a)



point of support O moves along circle:

$$x_0 = a \sin \alpha \quad , \quad y_0 = a \cos \alpha$$

where  $\alpha = \gamma t$ ,  $\gamma = \text{const}$

pendulum bob:

$$(x, y) : \quad x = x_0 + l \sin \phi \\ y = y_0 + l \cos \phi$$

$$U = -mgy = -mg y_0 - mgl \cos \phi$$

$\downarrow$   
specified function of time.

[can ignore in L]

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{x}_0 + l \cos \phi \quad \dot{\phi}$$

$$\dot{x}^2 = \dot{x}_0^2 + l^2 \cos^2 \phi \dot{\phi}^2 + 2l \cos \phi \dot{x}_0 \dot{\phi}$$

$$\dot{y} = \dot{y}_0 - l \sin \phi$$

$$\dot{y}^2 = \dot{x}_0^2 + l^2 \sin^2 \phi \dot{\phi}^2 - 2l \sin \phi \dot{x}_0 \dot{\phi}$$

thus,

$$T = \frac{1}{2} m (\dot{x}_0^2 + \dot{y}^2)$$

$$= \frac{1}{2} m (\dot{x}_0^2 + l^2 \cos^2 \phi \dot{\phi}^2 + 2l \cos \phi \dot{x}_0 \dot{\phi} + \dot{y}_0^2 + l^2 \sin^2 \phi \dot{\phi}^2 - 2l \sin \phi \dot{x}_0 \dot{\phi})$$

$$= \frac{1}{2} m (\dot{x}_0^2 + \dot{y}_0^2) + \frac{1}{2} m l^2 \dot{\phi}^2 + m l \dot{\phi} (\dot{x}_0 \cos \phi - \dot{y}_0 \sin \phi)$$

$$\text{Note: } \dot{x}_0^2 + \dot{y}_0^2 = a^2 \dot{\alpha}^2 = a^2 \gamma^2$$

since this is a specified function of time, we can ignore it in the Lagrangian;

$$\text{thus, } L = \frac{1}{2} m l^2 \dot{\phi}^2 + m l \dot{\phi} (\dot{x}_0 \cos \phi - \dot{y}_0 \sin \phi) + m g l \cos \phi$$

We can rewrite the second term:

$$\begin{aligned} x_0 &= a \sin \alpha & \rightarrow \dot{x}_0 = a \cos \alpha \dot{\alpha} & (\alpha = \gamma) \\ y_0 &= a \cos \alpha & \rightarrow \dot{y}_0 = -a \sin \alpha \dot{\alpha} \end{aligned}$$

thus,

$$\begin{aligned} m l \dot{\phi} (\dot{x}_0 \cos \phi - \dot{y}_0 \sin \phi) &= m l \dot{\phi} a \gamma (\cos \alpha \cos \phi + \sin \alpha \sin \phi) \\ &= m l \dot{\phi} a \gamma \cos(\phi - \alpha) \\ &= m l \dot{\phi} a \gamma \cos(\phi - \gamma t) \end{aligned}$$

$$\text{Now, } \frac{d}{dt} [m\lambda \gamma \sin(\phi - \gamma t)]$$

$$= m\lambda \gamma \cos(\phi - \gamma t) (\dot{\phi} - \gamma)$$

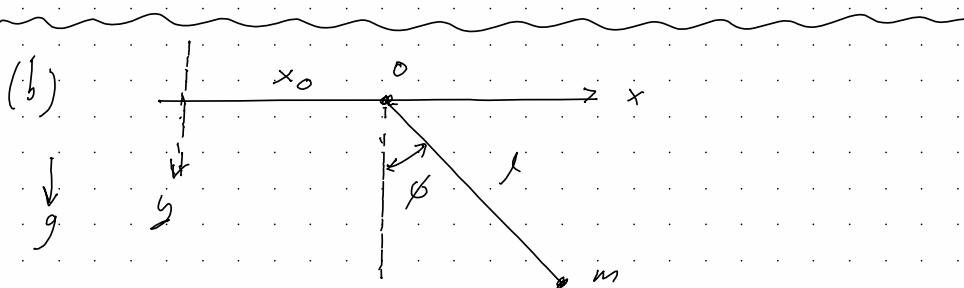
$$= m\lambda \dot{\phi} \gamma \cos(\phi - \gamma t) - m\lambda \gamma^2 \cos(\phi - \gamma t)$$

Thus,

$$m\lambda \dot{\phi} \gamma \cos(\phi - \gamma t) = \frac{d}{dt} [m\lambda \gamma \sin(\phi - \gamma t)] + m\lambda \gamma^2 \cos(\phi - \gamma t)$$

(and we can ignore the total time derivative in the Lagrangian)

$$\rightarrow L = \frac{1}{2} m \lambda^2 \dot{\phi}^2 + mg \lambda \cos \phi + m\lambda \gamma^2 \cos(\phi - \gamma t)$$



point O moving according to  $x_0 = a \cos \gamma t$

$$x = x_0 + l \sin \phi$$

$$y = l \cos \phi$$

$$U = -mgy = -mg l \cos \phi$$

$$T = \frac{1}{2} m(x^2 + y^2)$$

$$\dot{x} = \dot{x}_0 + l \cos \phi \dot{\phi}, \quad x_0 = a \cos \gamma t$$

$$= -a \sin(\gamma t) \dot{\phi} + l \cos \phi \dot{\phi}$$

$$\rightarrow \dot{x}^2 = a^2 \dot{\gamma}^2 \sin^2 \gamma t + l^2 \cos^2 \phi \dot{\phi}^2$$

$$- 2al\gamma \dot{\phi} \sin(\gamma t) \cos \phi$$

$$\dot{y} = -a \sin \phi \dot{\phi}$$

$$\rightarrow \dot{y}^2 = l^2 \sin^2 \phi \dot{\phi}^2$$

Thus,  $T = \frac{1}{2} m (a^2 \dot{\gamma}^2 \sin^2 \gamma t + l^2 \cos^2 \phi \dot{\phi}^2)$

$$- 2al\gamma \dot{\phi} \sin(\gamma t) \cos \phi + l^2 \sin^2 \phi \dot{\phi}^2$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + \underbrace{\frac{1}{2} m a^2 \dot{\gamma}^2 \sin^2 \gamma t}_{\text{specified function of time}} - mal\gamma \dot{\phi} \sin(\gamma t) \cos \phi$$

(ignore)

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 - mal\gamma \dot{\phi} \sin(\gamma t) \cos \phi + mgh \cos \phi$$

2nd term:

$$- \frac{d}{dt} [ mal\gamma \sin(\gamma t) \sin \phi ]$$

$$= -mal\gamma^2 \cos(\gamma t) \sin \phi - mal\gamma \dot{\phi} \sin(\gamma t) \cos \phi$$

so

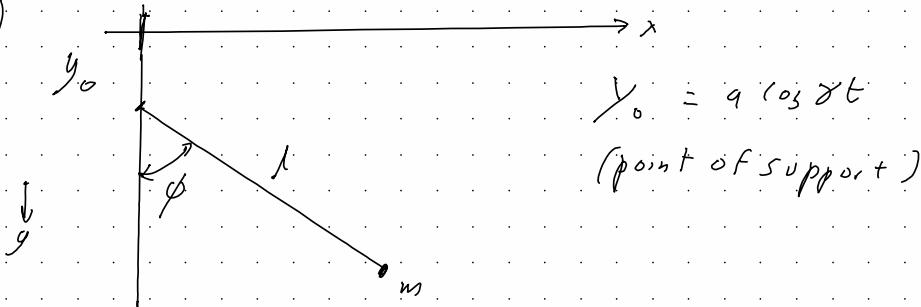
$$- mal\gamma \dot{\phi} \sin(\gamma t) \cos \phi = - \underbrace{\frac{d}{dt} [ ]}_{\text{is no se}} + mal\gamma^2 \cos(\gamma t) \sin \phi$$

Thus, ignoring total time derivative,

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \cos \phi + m a l \gamma^2 \cos(\gamma t) \sin \phi$$



(c)



$$x = l \cos \phi$$

$$y = y_0 + l \cos \phi$$

$$= a \cos \gamma t + l \cos \phi$$

$$U = -mgy$$

$$= -mg a \cos \gamma t - mg l \cos \phi$$

specified function of time [can ignore]

$$T = \frac{1}{2} m (x^2 + y^2)$$

$$\dot{x} = l \cos \phi \dot{\phi}$$

$$\dot{x}^2 = l^2 \cos^2 \phi \dot{\phi}^2$$

$$\dot{y} = -a \gamma \sin(\gamma t) - l \sin \phi \dot{\phi}$$

$$\dot{y}^2 = a^2 \gamma^2 \sin^2(\gamma t) + l^2 \sin^2 \phi \dot{\phi}^2 + 2al\gamma \sin(\gamma t) \sin \phi \dot{\phi}$$

specified function of time [can ignore]

thus, ignoring this function of time

$$T = \frac{1}{2} ml^2 \dot{\phi}^2 + mgl \dot{\phi} \sin(\gamma t) \sin \phi$$

$$\rightarrow L = T - U$$

$$= \frac{1}{2} ml^2 \dot{\phi}^2 + mgl \dot{\phi} \sin(\gamma t) \sin \phi + mgl \cos \phi$$

Rewrite 2<sup>nd</sup> term:

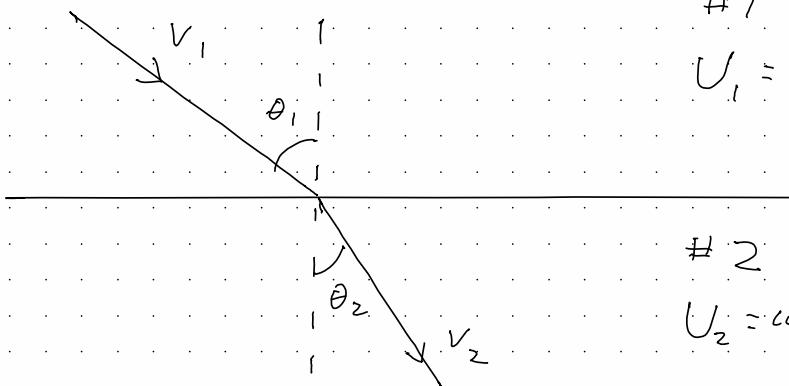
$$-\frac{d}{dt} [mgl \dot{\phi} \sin(\gamma t) \cos \phi] = -mgl \gamma^2 \cos(\gamma t) \cos \phi \\ + mgl \gamma \sin(\gamma t) \sin \phi$$

$$\text{so } mgl \dot{\phi} \sin(\gamma t) \sin \phi = -\frac{d}{dt} [-] + mgl \gamma^2 \cos(\gamma t) \cos \phi$$

thus, ignoring total time derivative

$$L = \frac{1}{2} ml^2 \dot{\phi}^2 + mgl \cos \phi + mgl \gamma^2 \cos(\gamma t) \cos \phi$$

Sect 7, prob 1



#1

$$U_1 = \text{const}$$

#2

$$U_2 = \text{const}$$

- Energy conserved, since no time dependence.
- Also momentum in  $x$ -direction ( $\parallel$  to interface) is conserved, since no  $x$ -dependence of the potential

$$U(x, y) = \begin{cases} U_1 & y \geq 0 \\ U_2 & y < 0 \end{cases}$$

$V_1$  : given

$$E = \frac{1}{2} m v_1^2 + U_1 = \frac{1}{2} m v_2^2 + U_2$$

$$\rightarrow \frac{1}{2} m v_2^2 = \frac{1}{2} m v_1^2 + (U_1 - U_2)$$

$$v_2^2 = v_1^2 + \frac{2(U_1 - U_2)}{m}$$

$$\text{so, } v_2 = v_1 \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

The angles  $\theta_1, \theta_2$  are related by

$$p_{1x} = p_{2x}$$

$$\mu V_1 \sin \theta_1 = \mu V_2 \sin \theta_2$$

$$\tan \theta_1 = \frac{\sin \theta_1}{\cos \theta_2} = \frac{V_2}{V_1} = \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

Sec 8, Prob 1

Transformation of action  $S = \int L dt$

$F, F'$ : two inertial frames

$F'$  moves with velocity  $\vec{V}$  wrt  $F$

Assume that  $F, F'$  coincide at  $t=0$  so  
 $\vec{r}_a = \vec{r}'_a$  wrt these two frames

$$\text{Now: } \vec{v}_a = \vec{V} + \vec{v}'_a$$

$$L = T - U$$

$$= \sum_a \frac{1}{2} m_a |\vec{v}_a|^2 - U(\vec{r}_1, \vec{r}_2, \dots, t)$$

$$|\vec{v}_a|^2 = |\vec{V} + \vec{v}'_a|^2$$

$$= |\vec{V}|^2 + |\vec{v}'_a|^2 + 2 \vec{V} \cdot \vec{v}'_a$$

so

$$L = \sum_a \frac{1}{2} m_a (|\vec{V}|^2 + |\vec{v}'_a|^2 + 2 \vec{V} \cdot \vec{v}'_a) - U$$

$$= \frac{1}{2} \mu V^2 + T' + \vec{V} \cdot \sum_a \vec{v}'_a - U$$

$$= T' - U + \frac{1}{2} \mu V^2 + \vec{P}' \cdot \vec{V}$$

$$= L' + \frac{1}{2} \mu V^2 + \vec{P}' \cdot \vec{V}$$

where  $\vec{P}'$  = total momentum wrt  $F'$

$$\mu = \sum_a m_a \leftarrow \text{total mass}$$

$$\begin{aligned}
 S &= \int_{t_1}^{t_2} L dt \\
 &= \int_{t_1}^{t_2} (\overline{L}' + \frac{1}{2}\mu V^2 + \vec{P}' \cdot \vec{V}) dt \\
 &= S' + \underbrace{\frac{1}{2}\mu V^2(t_2 - t_1)}_{\text{doesn't change}} + \vec{V} \cdot \int_{t_1}^{t_2} \vec{P}' dt
 \end{aligned}$$

EOMs

Now,

$$\begin{aligned}
 \overline{\vec{V}} \cdot \int_{t_1}^{t_2} \vec{P}' dt &= \overline{V} \cdot \int_{t_1}^{t_2} \sum_a m_a \vec{v}_a' dt \\
 &= \overline{V} \cdot \sum_a m_a \int_{t_1}^{t_2} \left( \frac{d\vec{r}_a}{dt} \right) dt \\
 &= \overline{V} \cdot \sum_a m_a \vec{r}_a \Big|_{t_1}^{t_2} \\
 &= \overline{V} \cdot \left( \mu \vec{R}(t_2) - \mu \vec{R}(t_1) \right) \\
 &= \mu \overline{V} \cdot \underbrace{\left( \vec{R}(t_2) - \vec{R}(t_1) \right)}_{\text{difference in com position}}
 \end{aligned}$$

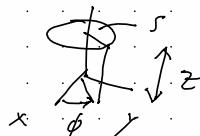
So:

$$S = S' + \frac{1}{2}\mu V^2(t_2 - t_1) + \mu \overline{V} \cdot (\vec{R}(t_2) - \vec{R}(t_1))$$

Sec 9, Prob 1

cylindrical coordinates,  $(s, \phi, z)$

$$s^2 = x^2 + y^2$$



$$x = s \cos \phi$$

$$y = s \sin \phi$$

$$z = z$$

$$\vec{M} = \vec{r} \times \vec{\phi} = m \vec{r} \times \vec{r}$$

$$\text{Now, } M_x = m(yz - zy)$$

$$M_y = m(zx - xz)$$

$$M_z = m(xy - yx)$$

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}$$

$$\vec{z} = z$$

$$\vec{y} = s \sin \phi \hat{i} + s \cos \phi \hat{j}$$

$$\vec{x} = s \cos \phi \hat{i} - s \sin \phi \hat{j}$$

$$\begin{aligned} \text{Thus, } M_x &= m(s \sin \phi z - z(s \sin \phi + s \cos \phi \dot{\phi})) \\ &= m(s \sin \phi z - z s \sin \phi \dot{s} - z s \cos \phi \dot{\phi}) \\ M_y &= m(z(s \cos \phi - s \sin \phi \dot{\phi}) - s \cos \phi z) \\ &= m(z \cos \phi \dot{s} - z s \sin \phi \dot{\phi} - s \cos \phi z) \end{aligned}$$

$$\begin{aligned} M_z &= m(s \cos \phi (s \sin \phi + s \cos \phi \dot{\phi}) \\ &\quad - s \sin \phi (s \cos \phi - s \sin \phi \dot{\phi})) \\ &= m s^2 \dot{\phi} \end{aligned}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$\begin{aligned}
 &= m^2 \left\{ \begin{aligned}
 &(\sin \phi (sz - z's) - z's \cos \phi \dot{\phi})^2 \\
 &+ (\cos \phi (sz - z's) - z's \sin \phi \dot{\phi})^2 \\
 &+ (s^2 \dot{\phi})^2 \end{aligned} \right\} \\
 &= m^2 \left\{ \begin{aligned}
 &\sin^2 \phi (sz - z's)^2 + z'^2 s^2 \cos^2 \phi \dot{\phi}^2 \\
 &- \cancel{2z's \sin \phi \cos \phi \dot{\phi} (sz - z's)} \\
 &+ \cancel{2z's \sin \phi \cos \phi \dot{\phi} (sz - z's)} \\
 &+ s^4 \dot{\phi}^2 \end{aligned} \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= m^2 \left\{ (sz - z')^2 + z'^2 s^2 \dot{\phi}^2 + s^4 \dot{\phi}^2 \right\} \\
 &= m^2 \left[ (sz - z')^2 + s^2 \dot{\phi}^2 (z'^2 + s^2) \right]
 \end{aligned}$$

Sec 9, Prob 2

repeat for spherical polar coords.

$$M_x = m(yz - zy), \text{ cyclic}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$\text{Now: } x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\rightarrow \dot{x} = r \sin \theta \cos \phi + r \cos \theta \cos \phi \dot{\theta} - r \sin \theta \sin \phi \dot{\phi}$$

$$\dot{y} = r \sin \theta \sin \phi + r \cos \theta \sin \phi \dot{\theta} + r \sin \theta \cos \phi \dot{\phi}$$

$$\dot{z} = r \cos \theta \dot{\theta} - r \sin \theta \dot{\phi}$$

Thus,

$$M_x = m(yz - zy)$$

$$= m \{ r \sin \theta \sin \phi (r \cos \theta - r \sin \theta \dot{\theta})$$

$$- r \cos \theta (r \sin \theta \sin \phi + r \cos \theta \sin \phi \dot{\theta} + r \sin \theta \cos \phi \dot{\phi}) \}$$

$$= m \{ -r^2 \sin^2 \theta \sin \phi \dot{\theta} - r^2 \cos^2 \theta \sin \phi \dot{\theta} - r^2 \sin \theta \cos \theta \cos \phi \dot{\phi} \}$$

$$= m \{ -r^2 \sin \phi \dot{\theta} - r^2 \sin \theta \cos \theta \cos \phi \dot{\phi} \}$$

$$= -mr^2 [\sin \phi \dot{\theta} + \sin \theta \cos \theta \cos \phi \dot{\phi}]$$

$$\begin{aligned}
 M_x &= m(z\dot{x} - x\dot{z}) \\
 &= m \left\{ r \cos \theta (r \sin \theta \cos \phi \dot{\theta} + r \cos \theta \cos \phi \dot{\phi} - r \sin \theta \sin \phi \dot{\phi}) \right. \\
 &\quad \left. - r \sin \theta \cos \phi (r \cos \theta \dot{\theta} - r \sin \theta \dot{\phi}) \right\} \\
 &= m \left\{ r^2 \cos^2 \theta \cos \phi \dot{\theta} - r^2 \sin \theta \cos \theta \sin \phi \dot{\phi} \right. \\
 &\quad \left. + r^2 \sin^2 \theta \cos \phi \dot{\theta} \right\} \\
 &= m r^2 [\cos \phi \dot{\theta} - \sin \theta \cos \theta \sin \phi \dot{\phi}] \\
 M_z &= m(x\dot{y} - y\dot{x}) \\
 &= m \left\{ r \sin \theta \cos \phi (r \sin \theta \sin \phi \dot{\theta} + r \cos \theta \sin \phi \dot{\phi}) \right. \\
 &\quad \left. + r \sin \theta \sin \phi (r \sin \theta \cos \phi \dot{\theta} + r \cos \theta \cos \phi \dot{\theta} \right. \\
 &\quad \left. - r \sin \theta \sin \phi \dot{\phi}) \right\} \\
 &= m [r^2 \sin^2 \theta \cos^2 \phi \dot{\theta} + r^2 \sin^2 \theta \sin^2 \phi \dot{\phi}] \\
 &= m r^2 \sin^2 \theta \dot{\phi}
 \end{aligned}$$

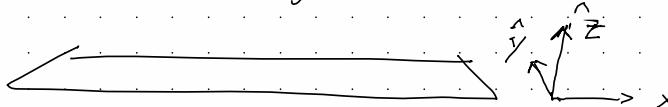
$$\begin{aligned}
 M^2 &= M_x^2 + M_y^2 + M_z^2 \\
 &= m^2 r^4 [\sin \phi \dot{\theta} + \sin \theta \cos \theta \cos \phi \dot{\phi}]^2 \\
 &\quad + m^2 r^4 [\cos \phi \dot{\theta} - \sin \theta \cos \theta \sin \phi \dot{\phi}]^2 \\
 &\quad + m^2 r^4 \sin^4 \theta \dot{\phi}^2
 \end{aligned}$$

(cross terms will cancel)

$$\begin{aligned}
 M^2 &= m^2 r^4 \left\{ \sin^2 \theta \dot{\theta}^2 + \sin^2 \theta \cos^2 \theta \cos^2 \phi \dot{\phi}^2 \right. \\
 &\quad + \cos^2 \theta \dot{\theta}^2 + \sin^2 \theta \cos^2 \theta \sin^2 \phi \dot{\phi}^2 \\
 &\quad \left. + \sin^4 \theta \dot{\phi}^2 \right\} \\
 &= m^2 r^4 \left[ \dot{\theta}^2 + \sin^2 \theta \cos^2 \theta \dot{\phi}^2 + \sin^4 \theta \dot{\phi}^2 \right] \\
 &= m^2 r^4 \left[ \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 / (\cos^2 \theta + \sin^2 \theta) \right] \\
 &= m^2 r^4 \left[ \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right]
 \end{aligned}$$

Sec 9, Prob 3

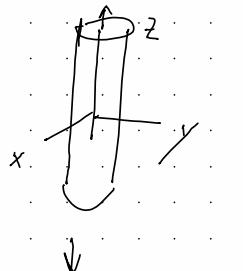
a) Infinite homogeneous plane



$P_x, P_y$  conserved

$M_z$  conserved where origin is anywhere in  $(x, y)$  plane

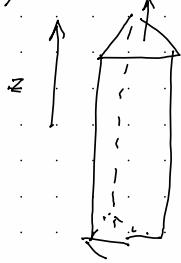
b) Infinite homogeneous cylinder



$P_z$  conserved

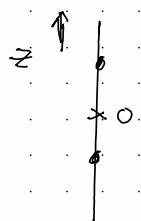
$M_z$  conserved, origin anywhere on z-axis

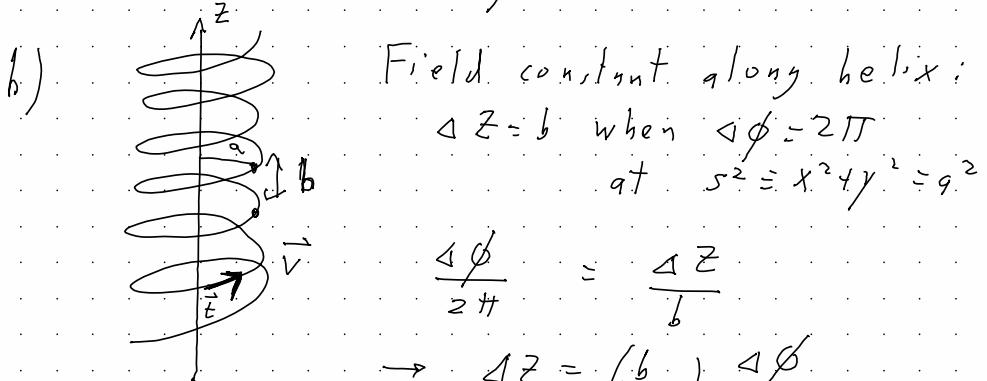
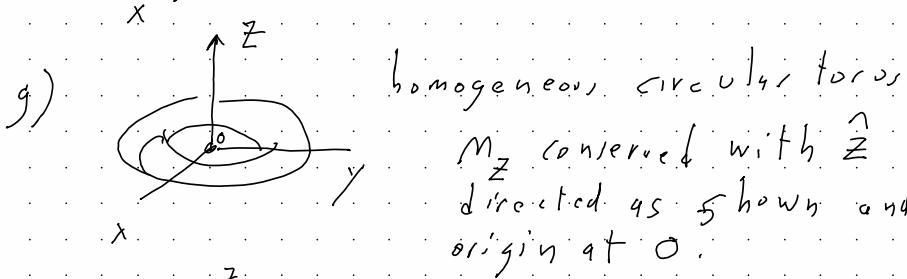
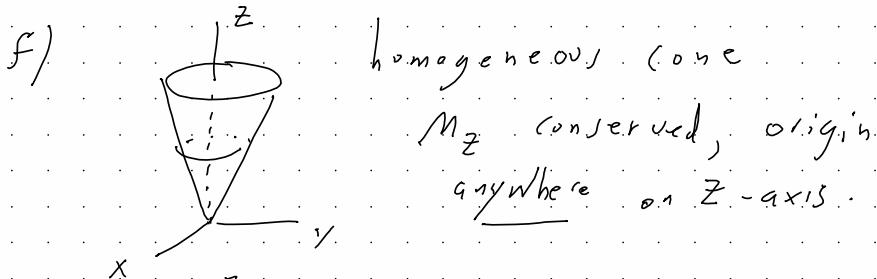
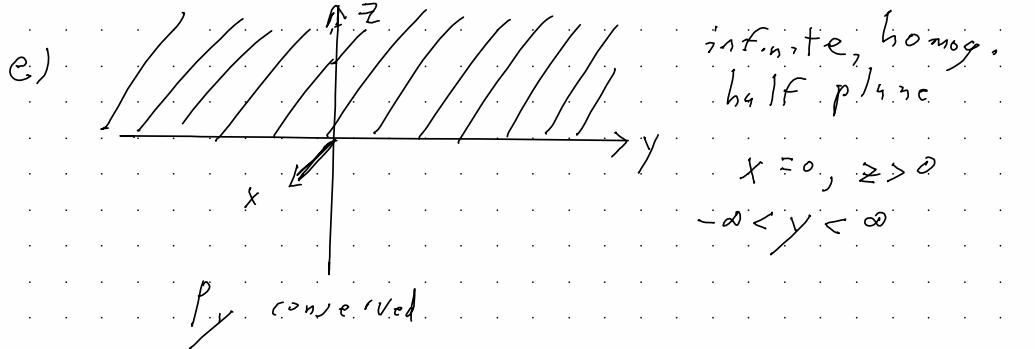
c) Infinite homogeneous prism



$P_z$  conserved

d) two points :  $M_z$  conserved, origin at midpoint of line connecting the two points





$$\frac{\Delta\phi}{2\pi} = \frac{\Delta Z}{b}$$

$$\rightarrow \Delta Z = \left(\frac{b}{2\pi}\right) \Delta\phi$$

$$\vec{F} = a\Delta\phi \hat{x} + \Delta Z \hat{z}$$

$$= \underbrace{a\Delta\phi (xy \hat{y} - yx \hat{x})}_{\vec{s}} + \Delta Z \hat{z}$$

$$\vec{t} = \Delta\phi (x\hat{y} - y\hat{x}) + \left(\frac{b}{2\pi}\right) \Delta\phi \hat{z}$$

$$= \Delta\phi \left[ x\hat{y} - y\hat{x} + \left(\frac{b}{2\pi}\right) \hat{z} \right]$$

Field unchanged if you move along  $\vec{t}$ .  
Thus,  $\vec{P} \cdot \vec{t} = \text{const}$

$$\vec{P} \cdot \vec{t} = \Delta\phi [xP_y - yP_x + \left(\frac{b}{2\pi}\right) P_z]$$

$$= \Delta\phi \left[ M_z + \frac{b}{2\pi} P_z \right]$$

$$\text{so } M_z + \frac{b}{2\pi} P_z = \text{const}$$

where  $z$  = axis of helix

$$b = \Delta z \text{ for } \Delta\phi = 2\pi \text{ at } s = a$$

Sec 10, prob 1

Different masses, same path, same potential energy

$$L_1 = \frac{1}{2} m_1 v_1^2 - U$$

$$L_2 = \frac{1}{2} m_2 v_2^2 - U$$

Thus,  $m_1 v_1^2 = m_2 v_2^2$

$$\frac{m_1}{t_1^2} = \frac{m_2}{t_2^2}$$

$$\rightarrow \left( \frac{t_2}{t_1} \right)^2 = \frac{m_2}{m_1}$$

or 
$$\boxed{\frac{t_2}{t_1} = \sqrt{\frac{m_2}{m_1}}}$$

Sec 10, Prob 2:

Same path, mass but potential energies differ by a constant

$$L_1 = \frac{1}{2} m v_1^2 - U_1$$

$$L_2 = \frac{1}{2} m v_2^2 - U_2$$

$$\frac{T_{buj}}{V^2} = \frac{U_1}{U_2}$$

$$\rightarrow \frac{(1/t_1)^2}{(1/t_2)^2} = \frac{U_1}{U_2}$$

$$\text{so } \frac{t_2}{t_1} = \sqrt{\frac{U_1}{U_2}}$$

## Sec 40 - Prob 1

single particle in a constant external field

$$L = \frac{1}{2}m\vec{v}^2 - U(\vec{r})$$

a) Cartesian coords  $(x, y, z)$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

$$\rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = p_x/m$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \rightarrow \dot{y} = p_y/m$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$H = \left( \sum_i p_i \dot{q}_i - L \right) \Big|_{q_i = \dot{q}_i (\epsilon_{ip})}$$

$$= \left( p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z) \right)$$

$$= p_x \left( \frac{p_x}{m} \right) + p_y \left( \frac{p_y}{m} \right) + p_z \left( \frac{p_z}{m} \right) \quad \begin{matrix} \dot{x} = p_x/m \\ \dot{y} = p_y/m \\ \dot{z} = p_z/m \end{matrix}$$

$$- \frac{1}{2}m \left( \left( \frac{p_x}{m} \right)^2 + \left( \frac{p_y}{m} \right)^2 + \left( \frac{p_z}{m} \right)^2 \right) + U(x, y, z)$$

$$= \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + U(x, y, z)$$

b) cylindrical coords  $(s, \phi, z)$ ,  $s^2 = x^2 + y^2$

$$L = \frac{1}{2}m(s^2 \dot{s}^2 + \dot{s}^2 \phi^2 + \dot{z}^2) - U(s, \phi, z)$$

$$\rightarrow \bar{p}_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s} \rightarrow \dot{s} = \bar{p}_s/m$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m s^2 \dot{\phi} \rightarrow \dot{\phi} = p_\phi / m s^2$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$\begin{aligned}
 H &= \left( p_r \dot{r} + p_\theta \dot{\theta} + p_z \dot{z} - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) + U(r, \theta, z) \right) \\
 &= p_r \left( \frac{p_r}{m} \right) + p_\theta \left( \frac{p_\theta}{mr^2} \right) + p_z \left( \frac{p_z}{m} \right) \\
 &\quad - \frac{1}{2} m \left( \left( \frac{p_r}{m} \right)^2 + r^2 \left( \frac{p_\theta}{mr^2} \right)^2 + \left( \frac{p_z}{m} \right)^2 \right) + U(r, \theta, z) \\
 &= \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2 \right) + U(r, \theta, z)
 \end{aligned}$$

c) spherical polar coords  $(r, \theta, \phi)$

$$L = \frac{1}{2} m (r^2 \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - U(r, \theta, \phi)$$

$$\rightarrow p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r} \rightarrow \dot{r} = p_r/m$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \rightarrow \dot{\theta} = p_\theta / mr^2$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2 \theta \dot{\phi} \rightarrow \dot{\phi} = p_\phi / m r^2 \sin^2 \theta$$

$$H = \left( p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) + U(r, \theta, \phi) \right)$$

$$\begin{aligned}
 &= p_r \left( \frac{p_r}{m} \right) + p_\theta \left( \frac{p_\theta}{mr^2} \right) + p_\phi \left( \frac{p_\phi}{mr^2 \sin^2 \theta} \right) \\
 &\quad - \frac{1}{2} m \left( \left( \frac{p_r}{m} \right)^2 + r^2 \left( \frac{p_\theta}{mr^2} \right)^2 + r^2 \sin^2 \theta \left( \frac{p_\phi}{mr^2 \sin^2 \theta} \right)^2 \right) + U(r, \theta, \phi) \\
 &= \frac{1}{2m} \left( \frac{p_r^2}{r^2} + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + U(r, \theta, \phi)
 \end{aligned}$$

sec 40 - prob 2

$$L = \frac{1}{2} m v^2 + m \vec{v} \cdot (\vec{\omega} \times \vec{r}) + \frac{1}{2} m |\vec{\omega} \times \vec{r}|^2 + m \vec{r} \cdot \vec{\omega} - U$$

restrict to uniformly rotating frame of reference  $\vec{W} = \omega \hat{z}$ ,  $\vec{\Omega} = \omega \hat{z}$

$$\rightarrow L = \frac{1}{2} m v^2 + m \vec{v} \cdot (\vec{\omega} \times \vec{r}) + \frac{1}{2} m |\vec{\omega} \times \vec{r}|^2 + U(r)$$

Now:  $H = \vec{p} \cdot \vec{v} - L$

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m \vec{v} + m \vec{\omega} \times \vec{r} = m(\vec{v} + \vec{\omega} \times \vec{r})$$

$$\rightarrow \vec{v} = \frac{1}{m} \vec{p} - \vec{\omega} \times \vec{r}$$

thus,

$$\begin{aligned} H &= \vec{p} \cdot \left( \frac{1}{m} \vec{p} - \vec{\omega} \times \vec{r} \right) - \frac{1}{2} m \left| \frac{1}{m} \vec{p} - \vec{\omega} \times \vec{r} \right|^2 \\ &\quad - m \left( \frac{1}{m} \vec{p} - \vec{\omega} \times \vec{r} \right) \cdot (\vec{\omega} \times \vec{r}) \\ &\quad - \frac{1}{2} m |\vec{\omega} \times \vec{r}|^2 + U(r) \end{aligned}$$

$$\begin{aligned} &= \frac{|\vec{p}|^2}{m} - \vec{p} \cdot (\vec{\omega} \times \vec{r}) - \frac{1}{2} m \left( \frac{|\vec{p}|^2}{m^2} + |\vec{\omega} \times \vec{r}|^2 \right) - \cancel{2 \vec{p} \cdot (\vec{\omega} \times \vec{r})} \\ &\quad - \vec{p} \cdot (\vec{\omega} \times \vec{r}) + m |\vec{\omega} \times \vec{r}|^2 - \frac{1}{2} m |\vec{\omega} \times \vec{r}|^2 + U(r) \end{aligned}$$

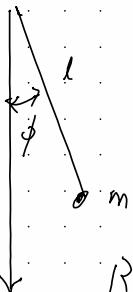
$$= \frac{|\vec{p}|^2}{2m} - \vec{p} \cdot (\vec{\omega} \times \vec{r}) + U(r)$$

$$= \frac{|\vec{p}|^2}{2m} - \vec{\omega} \cdot \underbrace{[\vec{r} \times \vec{p}]}_{\vec{m} : \text{angular momentum}} + U(r)$$

Sec 11, Prob 1.

Simple pendulum :  $U = -mgy$

$$= -mg l \cos\phi$$



$$\text{Also, } T = \frac{1}{2}ml^2\dot{\phi}^2$$

$$\text{Then, } E = T + U$$

$$= \frac{1}{2}ml^2\dot{\phi}^2 - mgyl \cos\phi$$

Release from rest at  $\phi = \phi_0$

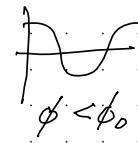
$$E = -mgl \cos\phi_0$$

$$\rightarrow \frac{1}{2}ml^2\dot{\phi}^2 - mgl \cos\phi = -mgl \cos\phi_0$$

$$T(E) = 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{E - U(\phi)}}$$

$$= 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{-mgl \cos\phi_0 + mgl \cos\phi}}$$

$$= 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$



$$\text{Now, } \cos\phi = \cos\left(2\frac{\phi}{2}\right)$$

$$\cos\phi > \cos\phi_0$$

$$= \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\text{Also, } \cos\phi_0 = 1 - 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$T(\tilde{E}) = 4 \sqrt{\frac{L}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{2 \left( \sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right) \right)}}$$

$$= 2 \sqrt{\frac{L}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2\left(\frac{\phi_0}{2}\right) \left( 1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)} \right)}}$$

Let  $x = \frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}$   $\rightarrow dx = \frac{1}{2} \frac{d\phi}{\frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}}$

$$= \frac{d\phi}{2} \frac{\sqrt{1 - \sin^2\left(\frac{\phi}{2}\right)}}{\sin\left(\frac{\phi_0}{2}\right)}$$

$$= \frac{d\phi}{2} \frac{\sqrt{1 - \sin^2\left(\frac{\phi_0}{2}\right)x^2}}{\sin\left(\frac{\phi_0}{2}\right)}$$

Thus,

$$T(\tilde{E}) = 2 \sqrt{\frac{L}{g}} \int_0^1 \frac{2 dx \sin\left(\frac{\phi_0}{2}\right)}{\sqrt{1 - \sin^2\left(\frac{\phi_0}{2}\right)x^2} \sin\left(\frac{\phi_0}{2}\right) \sqrt{1 - x^2}}$$

$$= 4 \sqrt{\frac{L}{g}} \int_0^1 \frac{dx}{\sqrt{1 - x^2} \sqrt{1 - K^2 x^2}} \quad , \quad K = \sin\left(\frac{\phi_0}{2}\right)$$

$$= 4 \sqrt{\frac{L}{g}} K(K)$$

where  $K(K) = \text{complete elliptic integral of the 1st kind.}$

## Approximation:

$$T(E) = 4\sqrt{\frac{E}{g}} \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-H^2 x^2}}$$

For  $H \ll 1$ :  $\frac{1}{\sqrt{1-H^2 x^2}} \approx 1 + \frac{1}{2} H^2 x^2$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-H^2 x^2}} \approx \int_0^1 \frac{dx}{\sqrt{1-x^2}} \left( 1 + \frac{1}{2} H^2 x^2 \right)$$

Now:  $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin(1) = \boxed{\frac{\pi}{2}}$

$$\frac{1}{2} R \int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1}{2} H^2 \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta \cos \theta d\theta}{\sqrt{1-\sin^2 \theta}}$$

Let:  $x = \sin \theta \quad \cos 2\theta = 1 - 2\sin^2 \theta$

$$dx = \cos \theta d\theta \quad \rightarrow \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$x^2 = \sin^2 \theta$$

$\frac{\pi}{2}$

$$\rightarrow \frac{1}{2} H^2 \int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1}{2} H^2 \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2\theta) d\theta$$

$$= \frac{1}{4} H^2 \left[ \frac{\pi}{2} - \frac{\sin 2\theta}{2} \Big|_0^{\frac{\pi}{2}} \right]$$

$$= \boxed{\frac{1}{8} H^2 \pi}$$

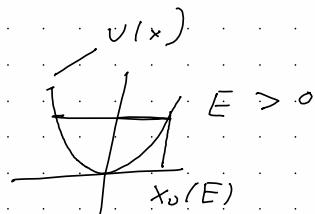
$$H = \sin \left( \frac{\phi_0}{2} \right) \approx \frac{\phi_0}{2}$$

$$T_0, T(E) = 4\sqrt{\frac{E}{g}} \left( \frac{\pi}{2} + \frac{1}{8} H^2 \pi + \dots \right) = \boxed{2\pi \sqrt{\frac{E}{g}} \left( 1 + \frac{1}{16} \phi_0^2 + \dots \right)}$$

Sec 11, Prob 2:

(a)  $U = A|x|^n \quad x_0(E)$

$$T(E) = 2\sqrt{2m} \int_0^{\infty} \frac{dx}{\sqrt{E - Ax^n}}$$



where  $U(x_0) = E$

$$\begin{aligned} A x_0^n &= E \\ \rightarrow x_0 &= (E/A)^{1/n} \end{aligned}$$

$$\begin{aligned} \sqrt{E - Ax^n} &= \sqrt{E} \sqrt{1 - (\frac{A}{E})x^n} \\ &= \sqrt{E} \sqrt{1 - t^n} \quad \text{where } t = \left(\frac{A}{E}\right)^{\frac{1}{n}} x \end{aligned}$$

$$dt = \left(\frac{A}{E}\right)^{\frac{1}{n}} dx$$

$$t_0 = \left(\frac{A}{E}\right)^{\frac{1}{n}} x_0 = \left(\frac{A}{E}\right)^{\frac{1}{n}} \left(\frac{E}{A}\right)^{\frac{1}{n}} = 1$$

Thus,

$$T(E) = 2\sqrt{2m} \int_0^1 \left(\frac{E}{A}\right)^{\frac{1}{n}} \frac{dt}{\sqrt{1-t^n}}$$

Now let  $u = t^n$

$$\begin{aligned} du &= n t^{n-1} dt = n u^{\frac{n-1}{n}} dt \\ &= n u^{1-\frac{1}{n}} dt \end{aligned}$$

$$dt = \frac{1}{n} u^{\frac{1}{n}-1} du, \quad t=0, 1 \rightarrow u=0, 1$$

Thus

$$T(\bar{E}) = 2 \sqrt{\frac{2m}{\bar{E}}} \left(\frac{\bar{E}}{A}\right)^{\frac{1}{n}} \int_0^1 \frac{1}{n} \frac{u^{\frac{1}{n}-1}}{\sqrt{1-u}} du$$

$$= \frac{2}{n} \sqrt{\frac{2m}{\bar{E}}} \left(\frac{\bar{E}}{A}\right)^{\frac{1}{n}} \int_0^1 u^{\frac{1}{n}-1} (1-u)^{-\frac{1}{2}} du$$

Recall: Beta Function

Gamma Function

$$B(x, y) = \int_0^1 dt t^{x-1} (1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

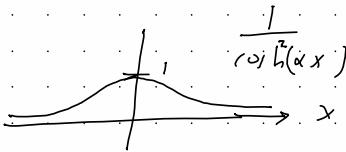
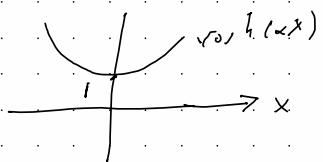
(where  $\operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0$ )

Thus,

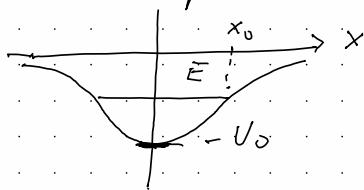
$$T(\bar{E}) = \frac{2}{n} \sqrt{\frac{2m}{\bar{E}}} \left(\frac{\bar{E}}{A}\right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)} \quad \leftarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{2}{n} \sqrt{\frac{2\pi m}{\bar{E}}} \left(\frac{\bar{E}}{A}\right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}$$

$$(b) \quad U = -U_0 / \cosh^2 \alpha x$$



$$-U_0 / \cosh^2$$



$$-U_0 \leq E < 0$$

$$\text{Thus, } E = -|E|$$

$$|E| < U_0$$

$$T(E) = 2\sqrt{2m} \int_0^{\infty} \frac{dx}{\sqrt{E - U(x)}}$$

where  $x_0$  given by

$$E = U(x_0)$$

$$= -\frac{U_0}{\cosh^2(\alpha x_0)}$$

$$\rightarrow \cosh^2(\alpha x_0) = -\frac{U_0}{E}$$

$$\cosh(\alpha x_0) = \sqrt{\frac{U_0}{|E|}}$$

$$\alpha x_0 = \coth^{-1}\left(\sqrt{\frac{U_0}{|E|}}\right)$$

$$x_0 = \frac{1}{\alpha} \coth^{-1}\left(\sqrt{\frac{U_0}{|E|}}\right)$$

$$\begin{aligned}
 \sqrt{E - U(x)} &= \sqrt{\frac{E + U_0}{\cosh^2(\alpha x)}} \quad c^2 - s^2 = 1 \\
 &= \frac{1}{\cosh(\alpha x)} \sqrt{E \cosh^2(\alpha x) + U_0} \\
 &= \frac{1}{\cosh(\alpha x)} \sqrt{E (1 + \sinh^2(\alpha x)) + U_0} \\
 &= \frac{1}{\cosh(\alpha x)} \sqrt{(E + U_0) + E \sinh^2(\alpha x)} \\
 &= \frac{1}{\cosh(\alpha x)} \sqrt{(U_0 - |E|) - |E| \sinh^2(\alpha x)}
 \end{aligned}$$

thus

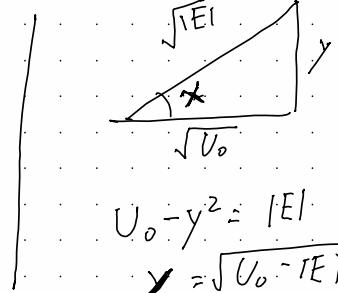
$$\begin{aligned}
 T(E) &= 2 \sqrt{2m} \int_{x_0}^{\infty} \frac{dx}{\sqrt{E - U(x)}} \\
 &= 2 \sqrt{2m} \int_0^{x_0} \frac{\cosh(\alpha x) dx}{\sqrt{(U_0 - |E|) - |E| \sinh^2(\alpha x)}}
 \end{aligned}$$

$$\text{Let: } t = \sinh(\alpha x) \rightarrow t_0 = \sinh(\alpha x_0)$$

$$dt = \alpha \cosh(\alpha x) dx$$

$$\text{so } t_0 = \sinh \left( \cosh^{-1} \sqrt{\frac{U_0}{|E|}} \right)$$

$$= \sqrt{\frac{U_0 - |E|}{|E|}}$$



$$T(E) \approx 2 \sqrt{2m} \int_0^{\infty} \frac{\sqrt{\frac{U_0 - |E|}{|E|}} dt}{\propto \sqrt{(U_0 - |E|)} - |E| t^2}$$

$$= 2 \sqrt{2m} \int_0^{\infty} \frac{\sqrt{\frac{U_0 - |E|}{|E|}} dt}{\propto \sqrt{U_0 - |E|} \sqrt{1 - \frac{|E|}{U_0 - |E|} t^2}}$$

Defn:  $\sin u = \sqrt{\frac{|E|}{U_0 - |E|}} t$

$$du \cos u = \sqrt{\frac{|E|}{U_0 - |E|}} dt$$

$$T(E) = \frac{2 \sqrt{2m}}{\propto \sqrt{U_0 - |E|}} \int_0^{\pi/2} \frac{\sqrt{\frac{U_0 - |E|}{|E|}} \cos u du}{\sqrt{1 - \frac{|E|}{U_0 - |E|} \sin^2 u}}$$

$$= \frac{\pi}{2} \sqrt{\frac{2m}{|E|}} \frac{\pi}{2}$$

$$= \boxed{\frac{\pi}{2} \sqrt{\frac{2m}{|E|}}}$$

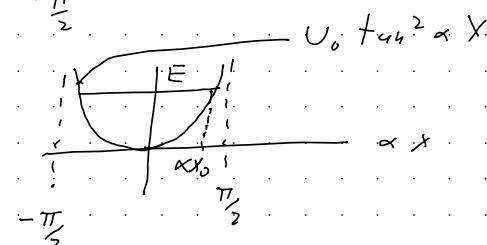
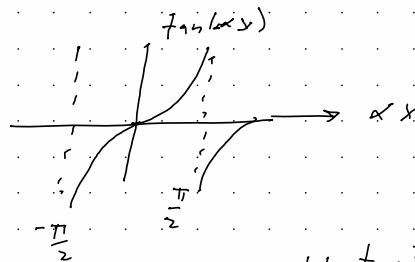
$$(c) U = U_0 \tan^2 \alpha x$$

$$T = 2\sqrt{2m} \int_0^{x_0} \frac{dx}{\sqrt{E - U(x)}}$$

where  $x_0$  determined by

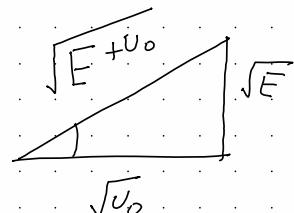
$$E = U(x_0)$$

$$= U_0 \tan^2(\alpha x_0)$$



$$\rightarrow \alpha x_0 = \tan^{-1} \left( \sqrt{\frac{E}{U_0}} \right)$$

$$\boxed{x_0 = \pm \tan^{-1} \left( \sqrt{\frac{E}{U_0}} \right)}$$



$$\sqrt{E - U(x)} = \sqrt{E - U_0 \tan^2(\alpha x)}$$

$$= \sqrt{E - U_0 \frac{\sin^2(\alpha x)}{\cos^2(\alpha x)}}$$

$$= \frac{1}{\cos(\alpha x)} \sqrt{E \cos^2(\alpha x) - U_0 \sin^2(\alpha x)}$$

$$= \frac{1}{\cos(\alpha x)} \sqrt{E (1 - \sin^2(\alpha x)) - U_0 \sin^2(\alpha x)}$$

$$= \frac{1}{\cos(\alpha x)} \sqrt{E - (U_0 + E) \sin^2(\alpha x)}$$

$$= \frac{\sqrt{E}}{\cos(\alpha x)} \sqrt{1 - \left( \frac{U_0 + E}{E} \right) \sin^2(\alpha x)}$$

Thur

$$T(E) = 2\sqrt{2m} \int_0^{x_0} \frac{\cos \alpha x \, dx}{\sqrt{E} \sqrt{1 - \left(\frac{U_0+E}{E}\right) \sin^2 \alpha x}}$$

Let  $u = \sin \alpha x$

$$du = \alpha \cos \alpha x \, dx$$

$$\begin{aligned} x=0, x_0 &\rightarrow u=0 \\ u_0 &= \sin(\alpha x_0) \\ &= \sin\left(t_{u_0}^{-1} \sqrt{\frac{E}{U_0}}\right) \\ &= \sqrt{\frac{E}{E+U_0}} \end{aligned}$$

$$\rightarrow T(E) = 2\sqrt{2m} \frac{1}{\alpha} \int_0^{u_0} \frac{du}{\sqrt{1 - \frac{U_0+E}{E} u^2}}$$

$$\text{Let } \sqrt{\frac{U_0+E}{E}} u = \sin \theta \rightarrow \sqrt{\frac{U_0+E}{E}} du = \cos \theta d\theta$$

$$\begin{aligned} u=0 &\rightarrow \theta=0 \\ u=u_0 &\rightarrow \sin \theta_0 = \sqrt{\frac{U_0+E}{E}} \sqrt{\frac{E}{U_0+E}} = 1 \rightarrow \theta_0 = \frac{\pi}{2} \end{aligned}$$

$$\rightarrow T(E) = 2\sqrt{2m} \frac{1}{\alpha} \int_0^{\pi/2} \frac{\cos \theta \, d\theta}{\sqrt{\frac{U_0+E}{E}} \sqrt{1 - \sin^2 \theta}}$$

$$\begin{aligned} &= \frac{2}{\alpha} \sqrt{2m} \sqrt{\frac{E}{U_0+E}} \frac{\pi}{2} \\ &= \boxed{\frac{\pi}{\alpha} \sqrt{\frac{2m}{E}} \sqrt{\frac{E}{U_0+E}}} \end{aligned}$$

Sec 13, Prob 1

$\vec{X}$ : position vector of M

$\vec{x}_a$ :  $a = 1, 2, \dots, n$  position vector of n masses all with mass m

Closed system  $\rightarrow$  linear momentum conserved  
 $\rightarrow$  COM Frame

$$M\vec{\dot{X}} + m(\vec{\dot{x}}_1 + \vec{\dot{x}}_2 + \dots + \vec{\dot{x}}_n) = 0$$

$$\text{or } M\vec{\dot{X}} + m \sum_{a=1}^n \vec{\dot{x}}_a = 0 \quad (1)$$

$$\text{Taking time derivative } \rightarrow M\vec{\ddot{X}} + \sum_a \vec{\ddot{x}}_a = 0 \quad (2)$$

Define relative position vectors:

$$\vec{r}_1 \equiv \vec{x}_1 - \vec{X}$$

$$\vec{r}_2 \equiv \vec{x}_2 - \vec{X}$$

etc.

$$\text{or } \vec{r}_a \equiv \vec{x}_a - \vec{X}, \quad a = 1, 2, \dots, n \quad (3)$$

Summing up (3):

$$\begin{aligned} \sum \vec{r}_a &= \sum (\vec{x}_a - \vec{X}) \\ &= \sum \vec{x}_a - n \vec{X} \quad \boxed{\text{total mass}} \\ &= -\frac{M}{m} \vec{X} - n \vec{X} \\ &= -\frac{(M+n)m}{m} \vec{X} = -\mu \vec{X} \end{aligned}$$

$$\text{Thus, } \vec{\Sigma} = -\frac{m}{\mu} \sum_a \vec{r}_a$$

$$\text{and } \vec{x}_a = \vec{r}_a + \vec{\Sigma}$$

give you  $\vec{x}_a, \vec{\Sigma}$  in terms of  $\vec{r}_a$ :

$$\text{H.E.: } T = \frac{1}{2} \sum_a m |\vec{x}_a|^2 + \frac{1}{2} M |\vec{\Sigma}|^2$$

$$= \frac{1}{2} m \sum_a |\vec{r}_a + \vec{\Sigma}|^2 + \frac{1}{2} M |\vec{\Sigma}|^2$$

$$= \frac{1}{2} m \sum_a |\vec{r}_a|^2 + \frac{1}{2} m \sum_a \vec{r}_a \cdot \vec{\Sigma}$$

$$+ \frac{1}{2} \sum_a m |\vec{\Sigma}|^2 + \frac{1}{2} M |\vec{\Sigma}|^2$$

$$= \frac{1}{2} m \sum_a |\vec{r}_a|^2 + m \left( \sum_a \vec{r}_a \right) \cdot \vec{\Sigma}$$

$$+ \frac{1}{2} m |\vec{\Sigma}|^2 + \frac{1}{2} M |\vec{\Sigma}|^2$$

$$= \frac{1}{2} m \sum_a |\vec{r}_a|^2 - \mu |\vec{\Sigma}|^2 + \frac{1}{2} (hm + M) |\vec{\Sigma}|^2$$

$$= \frac{1}{2} m \sum_a |\vec{r}_a|^2 - \frac{1}{2} \mu |\vec{\Sigma}|^2$$

Now rewrite  $I_1, I_2$  term:

$$-\frac{1}{2} \mu |\vec{\Sigma}|^2 = -\frac{1}{2} \mu \frac{m^2}{\mu^2} \left| \sum_a \vec{r}_a \right|^2 = -\frac{1}{2} \frac{m^2}{\mu} \left| \sum_a \vec{r}_a \right|^2$$

$$\text{Thus, } T = \frac{1}{2} m \sum_a |\vec{r}_a|^2 - \frac{1}{2} \frac{m^2}{\mu} \left| \sum_a \vec{r}_a \right|^2$$

Potential energy

$$U = U(|\vec{x}_1 - \vec{x}_2|, |\vec{x}_1 - \vec{x}_3|, \dots, |\vec{x}_1 - \vec{x}_n|, \\ |\vec{x}_1 - \vec{x}|, |\vec{x}_2 - \vec{x}|, \dots, |\vec{x}_n - \vec{x}|) \\ = U(|\vec{r}_1 - \vec{r}_2|, |\vec{r}_1 - \vec{r}_3|, \dots, |\vec{r}_1 - \vec{r}_n|, \\ |\vec{r}_1|, |\vec{r}_2|, \dots, |\vec{r}_n|)$$

which depends only on the relative position vectors  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ .

Thus,

$$L = \frac{1}{2} m \sum_a |\vec{r}_a|^2 - \frac{1}{2} \frac{m^2}{M} \sum_a |\vec{r}_a|^2 - U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n)$$

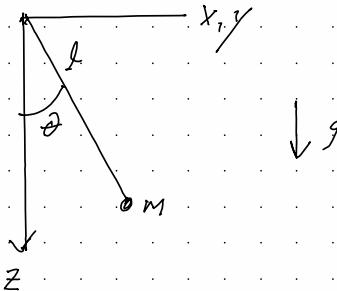
Sec 14, Prob 1

Spherical pendulum:

$$T = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)$$

$$U = -mgz$$

$$= -mgl \cos \theta$$



Thus,

$$L = T - U$$

$$= \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + mgl \cos \theta$$

$$\text{No } t\text{-dependence} \rightarrow E = T + U = \text{const}$$

$$\text{No } \phi\text{-dependence} \rightarrow \frac{\partial L}{\partial \dot{\phi}} = ml^2 \sin^2 \theta \dot{\phi} \equiv M_z = \text{const}$$

$$E = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - mgl \cos \theta$$

$$= \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2 \theta \frac{M_z^2}{m^2 l^4 \sin^4 \theta}) - mgl \cos \theta$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{M_z^2}{2 m l^2 \sin^2 \theta} - mgl \cos \theta$$

$$U_{\text{eff}}(\theta)$$

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{M_z^2}{2 m l^2 \sin^2 \theta} + U(\theta)$$

$$\rightarrow \frac{1}{2} m l^2 \dot{\theta}^2 = E - U(\theta) - \frac{M_z^2}{2 m l^2 \sin^2 \theta}$$

$$\dot{\theta} = \sqrt{\frac{2}{ml^2} (E - U(\theta)) - \frac{M_z^2}{m^2 l^4 \sin^2 \theta}}$$

$$\frac{d\theta}{dt} = \dot{\theta} = \sqrt{\frac{2}{ml^2} (E + mgl_{cos}\theta)} - \frac{M_z^2}{m^2 l^4 \sin^2 \theta}$$

$$\rightarrow dt = \frac{d\theta}{\sqrt{\dots}}$$

$$t = \int \frac{d\theta}{\sqrt{\frac{2}{ml^2} (E + mgl_{cos}\theta)} - \frac{M_z^2}{m^2 l^4 \sin^2 \theta}} + \text{const}$$

P4th: use  $M_z = ml^2 \sin^2 \theta \phi$

$$\text{Thus, } \frac{d\theta}{dt} = \frac{d\theta}{d\phi} \frac{d\phi}{dt} \\ = \frac{d\theta}{d\phi} \frac{M_z}{ml^2 \sin^2 \theta}$$

$$\text{Thus, } \frac{d\theta}{d\phi} = \frac{d\theta}{dt} \frac{ml^2 \sin^2 \theta}{M_z} = \sqrt{\dots} \frac{ml^2 \sin^2 \theta}{M_z}$$

$$d\phi = \frac{d\theta M_z}{\sqrt{ml^2 \sin^2 \theta}}$$

$$\phi = \int \frac{M_z d\theta / ml^2 \sin^2 \theta}{\sqrt{\frac{2}{ml^2} (E + mgl_{cos}\theta)} - \frac{M_z^2}{m^2 l^4 \sin^2 \theta}} + \text{const}$$

Turning points: (where  $\dot{\theta} = 0$ )

$$E = V_{\text{eff}}(r)$$

$$= \frac{M_z^2}{2m_1^2 \sin^2 \theta} - m_g l \cos \theta$$

$$\rightarrow 2E_m l^2 \sin^2 \theta = M_z^2 - 2m_g^2 l^3 \sin^2 \theta \cos \theta$$

$$2E_m l^2 (1 - \cos^2 \theta) = M_z^2 - 2m_g^2 l^3 (1 - \cos^2 \theta) \cos \theta$$

$$2E_m l^2 - 2E_m l^2 \cos^2 \theta$$

$$= M_z^2 - 2m_g^2 l^3 \cos \theta + 2m_g^2 l^3 \cos^3 \theta$$

$$\text{Thus, } 2m_g^2 l^3 \cos^3 \theta + 2E_m l^2 \cos^2 \theta - 2m_g^2 l^3 \cos \theta$$

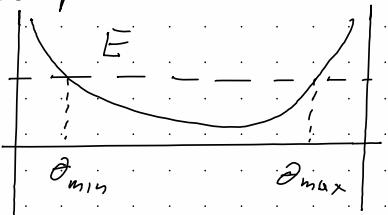
$$+ (M_z^2 - 2E_m l^2) = 0$$

Divide by  $2m_g^2 l^3$ :

$$\rightarrow \boxed{\cos^3 \theta + \frac{E}{m_g l} \cos^2 \theta - \cos \theta + \left( \frac{M_z^2}{2m_g^2 l^3} - \frac{E}{m_g l} \right) = 0}$$

Cubic equation for  $X \equiv \cos \theta$

Effective potential looks like:

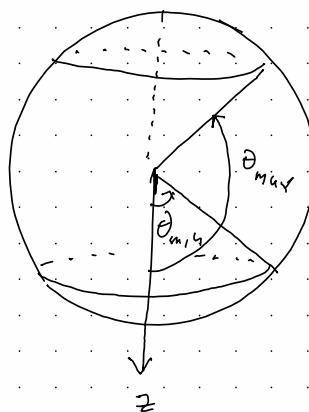


$$\theta = 0$$

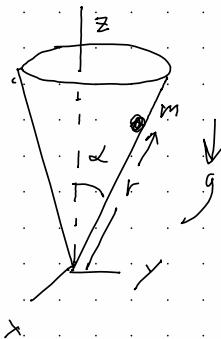
$$X = +1$$

$$\theta = \pi$$

$$X = -1$$



Sec 14, Prob. 2



spherical polar coords.  $(r, \theta, \phi)$

Constraint  $\theta = \alpha$

Generalized coords:  $(r, \phi)$

$$T = \frac{1}{2}m(r^2 + r^2\dot{\phi}^2 + r^2 \sin^2 \alpha \dot{\phi}^2)$$
$$= \frac{1}{2}m(r^2 + r^2 \sin^2 \alpha \dot{\phi}^2)$$

$$U = mgz$$

$$= mg r \cos \alpha$$

$$L = T - U$$

$$= \frac{1}{2}m(r^2 + r^2 \sin^2 \alpha \dot{\phi}^2) - mg r \cos \alpha$$

$E = \text{const}$  (since no explicit t dependence)

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2 \alpha \dot{\phi} = M_Z = \text{const}$$

(since no explicit  $\phi$  dependence)

$$E = T + U$$

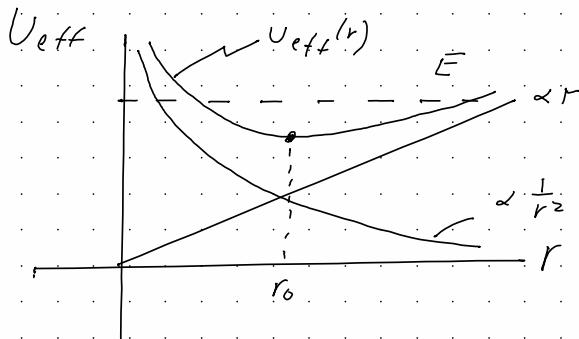
$$= \frac{1}{2}m(r^2 + r^2 \sin^2 \alpha \dot{\phi}^2) + mgr \cos \alpha$$

$$= \frac{1}{2}mr^2 + \frac{1}{2}mr^2 \sin^2 \alpha \left( \frac{M_Z^2}{m^2 r^4 \sin^4 \alpha} \right) + mgr \cos \alpha$$

$$= \frac{1}{2}mr^2 + \frac{M_Z^2}{2mr^2 \sin^2 \alpha} + mgr \cos \alpha$$

$$= \frac{1}{2}mr^2 + U_{\text{eff}}(r)$$

$$U_{\text{eff}}(r) = \frac{M_2^2}{2mr^2 \sin^2 \alpha} + mgyr \cos \alpha$$



Bound orbits for  $E > U_{\text{eff}, \min} = U_{\text{eff}}(r_0)$

$r_0$ : stable circular orbit



$$r/r_0 = \cos \alpha$$

$t$ -equation:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}(E - mgyr \cos \alpha) - \frac{M_2^2}{mr^2 \sin^2 \alpha}}} + C_1 t$$

$$\text{Using } M_2 = mr^2 \sin^2 \alpha \phi$$

$$\rightarrow \frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{M_2}{mr^2 \sin^2 \alpha}$$

$\phi$ -equation:

$$\phi = \int \frac{\left(\frac{M_2}{\sin^2 \alpha}\right) dr}{r^2} + C_2 \phi$$

$$\int \frac{\sqrt{2m(E - mgyr \cos \alpha)} - \frac{M_2^2}{r^2 \sin^2 \alpha}}{r^2} dr$$

Turning points:  $r = r_{\min}, r_{\max}$

Determined by effective potential

$$E = V_{\text{eff}}(r)$$

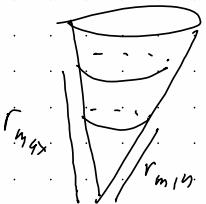
$$= \frac{M_2^2}{2mV^2 \sin^2 \alpha} + mg r \cos \alpha$$

$$\rightarrow Z_m E r^2 \sin^2 \alpha = M_2^2 + Z_m^2 g r^3 \sin^2 \alpha \cos \alpha$$

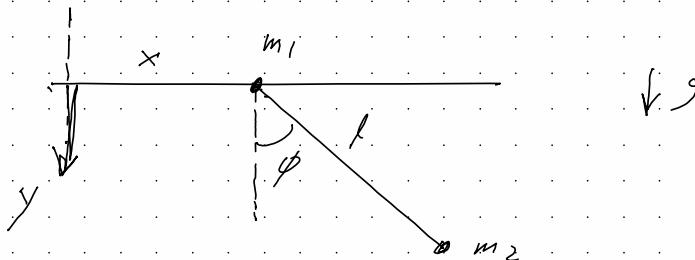
$$0 = Z_m^2 g r^3 \sin^2 \alpha \cos \alpha - Z_m E r^2 \sin^2 \alpha + M_2^2$$

$$= r^3 - \left( \frac{E}{mg \cos \alpha} \right) r^2 + \frac{M_2^2}{Z_m^2 g \sin^2 \alpha \cos \alpha}$$

cubic equation again



Sect 14, Prob 3



From Sect 5, Prob 2 we have

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) + m_2gl\cos\phi$$

No dependence on  $x$ :

$$\rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi = \text{const}$$

( $x$ -component of total momentum)

No explicit  $t$ -dependence

$$\begin{aligned} \rightarrow E &= T + U = \text{const} \\ &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) - m_2gl\cos\phi \end{aligned}$$

Work in frame where  $\text{com}_x = 0$ :

$$\begin{aligned} \text{com}_x &= m_1x + m_2(x + l\sin\phi) \\ &= (m_1 + m_2)x + m_2l\sin\phi \end{aligned}$$

$$\text{com}_x = 0 \rightarrow x = -\left(\frac{m_2}{m_1 + m_2}\right)l\sin\phi$$

$$\dot{x} = - \left( \frac{m_2}{m_1 + m_2} \right) l \cos \phi \dot{\phi}$$

for,

$$E = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 (l^2 \dot{\phi}^2 + 2l \dot{x} \dot{\phi} \cos \phi) - m_2 g l \cos \phi$$

$$= \frac{1}{2} (m_1 + m_2) \frac{m_2^2}{(m_1 + m_2)^2} l^2 \cos^2 \phi \dot{\phi}^2$$

$$+ \frac{1}{2} m_2 l^2 \dot{\phi}^2 - m_2 \left( \frac{m_2}{m_1 + m_2} \right) l^2 \cos^2 \phi \dot{\phi}^2 - m_2 g l \cos \phi$$

$$= \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left[ 1 - \left( \frac{m_2}{m_1 + m_2} \right) \cos^2 \phi \right] - m_2 g l \cos \phi$$

1-d problem:

$$\underline{E + m_2 g l \cos \phi} = \frac{1}{2} m_2 l^2 \dot{\phi}^2$$

$$\left[ 1 - \left( \frac{m_2}{m_1 + m_2} \right) \cos^2 \phi \right]$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{\frac{2}{m_2 l^2} \left( E + m_2 g l \cos \phi \right)} \frac{1}{1 - \left( \frac{m_2}{m_1 + m_2} \right) \cos^2 \phi}$$

$$\rightarrow dt = \frac{d\phi}{\sqrt{\dots}}$$

$$= d\phi \sqrt{\frac{1 - \left(\frac{m_2}{m_1+m_2}\right) \cos^2 \phi}{\frac{2}{m_2 l^2} (E + m_2 g l \cos \phi)}}$$

$$= d\phi \sqrt{\frac{m_2 l^2}{2} \frac{1}{m_1+m_2}} \sqrt{\frac{(m_1+m_2) - m_2 \cos^2 \phi}{E + m_2 g l \cos \phi}}$$

$$= d\phi \sqrt{\frac{m_2}{m_1+m_2}} \sqrt{\frac{l^2}{2}} \sqrt{\frac{m_1+m_2 \sin^2 \phi}{E + m_2 g l \cos \phi}}$$

so

$$t = \sqrt{\left(\frac{m_2}{m_1+m_2}\right) \frac{l^2}{2}} \int d\phi \sqrt{\frac{m_1+m_2 \sin^2 \phi}{E + m_2 g l \cos \phi}} + \text{const}$$

Now:  $x_2 = x + l \sin \phi$

$$y_2 = l \cos \phi$$

using  $x = -\left(\frac{m_2}{m_1+m_2}\right) l \sin \phi$

$$\rightarrow x_2 = \left[ -\left(\frac{m_2}{m_1+m_2}\right) l \sin \phi + l \cos \phi \right] = \left(\frac{m_1}{m_1+m_2}\right) l \sin \phi$$

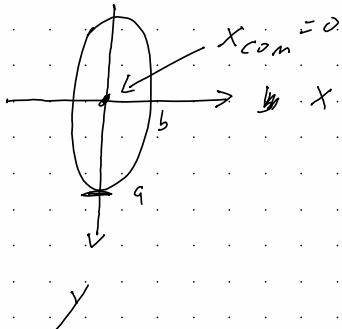
$$x_2 = \left( \frac{m_1}{m_1 + m_2} \right) l \sin \phi = b \sin \phi$$

$$y_2 = l \cos \phi = a \cos \phi$$

$$\left(\frac{y_2}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = \cos^2 \phi + \sin^2 \phi = 1$$

which is an ellipse with semi-major and semi-minor axes:

$$a = l, \quad b = l \left( \frac{m_1}{m_1 + m_2} \right)$$



If  $m_1 \gg m_2$   
then  $a = b \approx l$   
so that  $m_2$  moves  
along a circular  
arc of radius  $l$ .

Sec 15, Prob, 1

$$U = -\frac{\alpha}{r}, \quad E = 0 \quad \rightarrow \quad e = 1$$

$$\frac{p}{r} = 1 + \cos\phi$$

when  $\phi = 0$ ,  $p = 2 \cdot r$   
so  $r_{min} = \frac{p}{2}$

NOTE:  $p = r + r\cos\phi$   
 $= \sqrt{x^2 + y^2} + x$

$$\rightarrow (p-x)^2 = x^2 + y^2$$

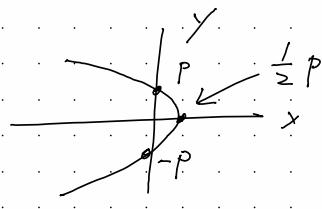
$$p^2 - 2px = x^2 + y^2$$

$$p^2 - y^2 = 2px$$

$$\rightarrow \boxed{x = \frac{p^2 - y^2}{2p}} \quad \text{parabola}$$

when  $y=0$ ,  $x = \frac{1}{2}p$

$$x = 0, \quad y = \pm p$$



a

Time equation:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}[E - U(r)] - \frac{p^2}{r^2}}} \quad t \text{ const}$$

$$U(r) = -\frac{\alpha}{r}, \quad p = \frac{M^2}{m\alpha}, \quad e=1, \quad E=0$$

$$\rightarrow t = \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{M^2}{m^2 r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{m\alpha p}{m^2 r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{dr}{\sqrt{\frac{2}{r} - \frac{p}{r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{r dr}{\sqrt{2r-p}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha p}} \int \frac{r dr}{\sqrt{\frac{2r-1}{p}}} + \text{const}$$

Definic:  $\frac{2r-1}{p} = \xi^2 > 0 \quad (-\infty < \xi < \infty)$

$$2r = p(1+\xi^2)$$

$$\boxed{r = \frac{p}{2}(1+\xi^2)} \rightarrow dr = p\xi d\xi$$

$$r dr = \frac{p^2}{2}(\xi + \xi^3) d\xi$$

Thus,

$$t = \sqrt{\frac{m}{\alpha p}} \int \frac{p^2 / (\xi + \xi^3) d\xi}{\sqrt{\xi^2}} + \text{const}$$

$$= \sqrt{\frac{mp^3}{\alpha}} \cdot \frac{1}{2} \int (1 + \xi^2) d\xi + \text{const}$$

$$= \frac{1}{2} \sqrt{\frac{mp^3}{\alpha}} \left( \xi + \frac{1}{3}\xi^3 \right) + \text{const}$$

choose const, so that at  $t=0 \Leftrightarrow \xi = 0$  (const=0)

$$\boxed{t = \frac{1}{2} \sqrt{\frac{mp^3}{\alpha}} \left( \xi + \frac{1}{3}\xi^3 \right)}$$

Now:

$$\frac{p}{r} = 1 + \cos\phi$$

$$p = r(1 + \cos\phi)$$

$$\phi = \frac{p}{r}(1 + \xi^2)(1 + \cos\phi)$$

$$z = 1 + \xi^2 + \cos\phi + \xi^2 \cos\phi$$

$$1 - \xi^2 = (1 + \xi^2) \cos\phi$$

$$\rightarrow \boxed{\cos\phi = \frac{1 - \xi^2}{1 + \xi^2}}$$

$$\boxed{x = r \cos \varphi}$$

$$= \frac{p}{2} \left( 1 + \xi^2 \right) \left( \frac{1 - \xi^2}{1 + \xi^2} \right)$$

$$= \frac{p}{2} (1 - \xi^2)$$

Also,

$$x^2 + y^2 = r^2$$

$$\rightarrow y^2 = r^2 - x^2$$

$$= \frac{p^2}{4} (1 + \xi^2)^2 - \frac{p^2}{4} (1 - \xi^2)^2$$

$$= \frac{p^2}{4} (x + \xi^2 + 2\xi^2 - x - \xi^2 + 2\xi^2)$$

$$= p^2 \xi^2$$

so  $\boxed{y = p\xi}$

Sec 15, Prob 3:

$$\Delta\phi = 2 \int_{r_{min}}^{r_{max}} \frac{M dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}}$$

This is the change in  $\phi$  as  $r$  goes from  $r_{min}$  to  $r_{max}$  and then back to  $r_{min}$ .

A closed bound orbit would have  $\Delta\phi = 2\pi n/m$  for  $m, n$  integers.

Consider:  $U = -\frac{\alpha}{r} + \delta U$  where  $|\delta U| \ll |\frac{\alpha}{r}|$

For  $\delta U = 0$ ,  $\Delta\phi = 2\pi$

$$\Delta\phi = 2 \int_{r_{min}}^{r_{max}} \frac{M dr/r^2}{\sqrt{2m(E - (U + \delta U)) - M^2/r^2}}$$

$$= 2 \int_{r_{min}}^{r_{max}} \frac{M dr/r^2}{\sqrt{2m(E-U) - 2m\delta U - M^2/r^2}}$$

$$= 2 \int_{r_{min}}^{r_{max}} \frac{M dr/r^2}{\sqrt{(2m(E-U) - M^2/r^2)} \left( 1 - \frac{2m\delta U}{2m(E-U) - M^2/r^2} \right)}$$

$$\approx 2 \int_{r_{min}}^{r_{max}} \frac{M dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}} \left[ 1 + \frac{m\delta U}{2m(E-U) - M^2/r^2} \right]$$

For  $U = -\alpha/r$

$$\Delta\phi \approx 2\pi + 2 \int_{r_{min}}^{r_{max}} \frac{M dr/r^2}{\left[2m(E-U) - M^2/r^2\right]^{3/2}} d\delta U$$

$$\approx 2\pi + \delta\phi$$

where

$$\delta\phi \equiv \int_{r_{min}}^{r_{max}} \frac{2m\delta U M dr/r^2}{\left[2m(E-U) - M^2/r^2\right]^{3/2}}$$

$$= \frac{2}{2M} \left[ \int_{r_{min}}^{r_{max}} \frac{2m\delta U dr}{\sqrt{2m(E-U) - M^2/r^2}} \right]$$

Evaluate terms in integrand along unperturbed path since  $\delta U$  is already small

$$\frac{p}{r} = 1 + e \cos\phi \quad , \quad p = a(1-e^2)$$

$$\rightarrow -\frac{p}{r^2} dr = -e \sin\phi d\phi \rightarrow \boxed{dr = \frac{r^2 e \sin\phi}{p} d\phi}$$

$$\sqrt{\dots} = \sqrt{\frac{2mE + 2m\alpha}{r} - \frac{M^2}{r^2}}$$

$$= \sqrt{-2m|E| + \frac{4m|E|}{(1-e^2)} \frac{p}{r} - \frac{2m|E|\frac{p^2}{r^2}}{(1-e^2) \frac{p^2}{r^2}}}$$

$$\begin{aligned}
 \sqrt{\cdot} &= \sqrt{2m|E|} \sqrt{-1 + \frac{2}{(1-e^2)} (1+e^{i\cos\phi}) - \frac{1}{(1-e^2)} (1+e^{i\cos\phi})^2} \\
 &= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}} \sqrt{-(1-e^2) + 2(1+e^{i\cos\phi}) - (1+2e^{i\cos\phi} + e^{2i\cos\phi})} \\
 &= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}} \sqrt{e^2(1-\cos^2\phi)} \\
 &= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}} e \sin\phi
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \boxed{\delta\phi} &= \frac{d}{dm} \left[ 2m \int_0^\pi \frac{\delta U r^2 e^{\sin\phi} d\phi}{\sqrt{\frac{2m|E|}{1-e^2} e^{\sin\phi} - p}} \right] \\
 &= \frac{d}{dm} \left[ \frac{2m}{p \sqrt{\frac{2m|E|}{1-e^2}}} \int_0^\pi d\phi r^2 \delta U \right] \\
 &= \frac{d}{dm} \left[ \frac{2m}{M} \int_0^\pi d\phi r^2 \delta U \right]
 \end{aligned}$$

using (15.6):  $\frac{M}{\sqrt{2m|E|}} = \frac{p}{\sqrt{1-e^2}}$

Evaluate:

$$\delta\phi = \frac{\partial}{\partial M} \left[ \frac{2m}{M} \int_0^\pi d\phi r^2 \delta U \right]$$

For (a)  $\delta U = \beta/r^2$ , (b)  $\delta U = \gamma/r^3$

$$(a) \delta\phi = \frac{\partial}{\partial M} \left[ \frac{2m}{M} \int_0^\pi d\phi \beta \right]$$

$$= 2\pi \beta m \frac{\partial}{\partial M} \left( \frac{1}{M} \right)$$

$$= -\frac{2\pi \beta m}{M^2}$$

Recall:  $P = \frac{M^2}{m\alpha}$

$$= \boxed{-\frac{2\pi \beta}{\alpha P}}$$

$$\leftarrow \alpha P = \frac{M^2}{m}$$

$$(b) \delta\phi = \frac{\partial}{\partial M} \left[ \frac{2m}{M} \int_0^\pi d\phi \frac{\gamma}{r} \right]$$

$$= \frac{\partial}{\partial M} \left[ \frac{2m\gamma}{M} \int_0^\pi d\phi \left( \frac{1 + e^{i\phi}}{P} \right) \right]$$

$$= 2m\gamma \frac{\partial}{\partial M} \left[ \frac{1}{MP} \left( \pi + e^{i\phi} \Big|_0^\pi \right) \right]$$

Now:  $\frac{1}{MP} = \frac{m\alpha}{M^3} \rightarrow \frac{\partial}{\partial M} \left( \frac{1}{MP} \right) = -\frac{3m\alpha}{M^4}$

$$\oint \phi = -6\pi \gamma \frac{m^2 \alpha}{M^4}$$

$$= -6\pi \gamma \alpha \left( \frac{L}{p \alpha} \right)^2$$

$$= \boxed{-6\pi \gamma \frac{L^2}{p^2 \alpha}}$$