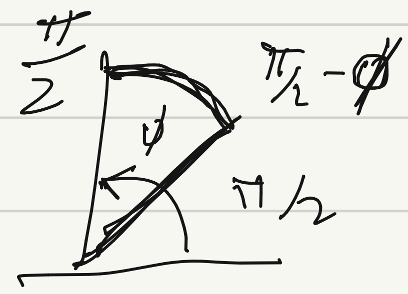
Elliptic functions / integrals:

i) period of a simple pendulum beyond the small-angle approximation

ii) circumference of an ellipse

A generalization of definition of circular functions (sines, cosines) to ellipses.



standard notation:

$$\int_0^x \frac{dt}{\sqrt{1-k^2t^2}\sqrt{1-t^2}} = F(\phi, k) = sn^{-1}x$$

$$\int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt = E(\phi, k)$$

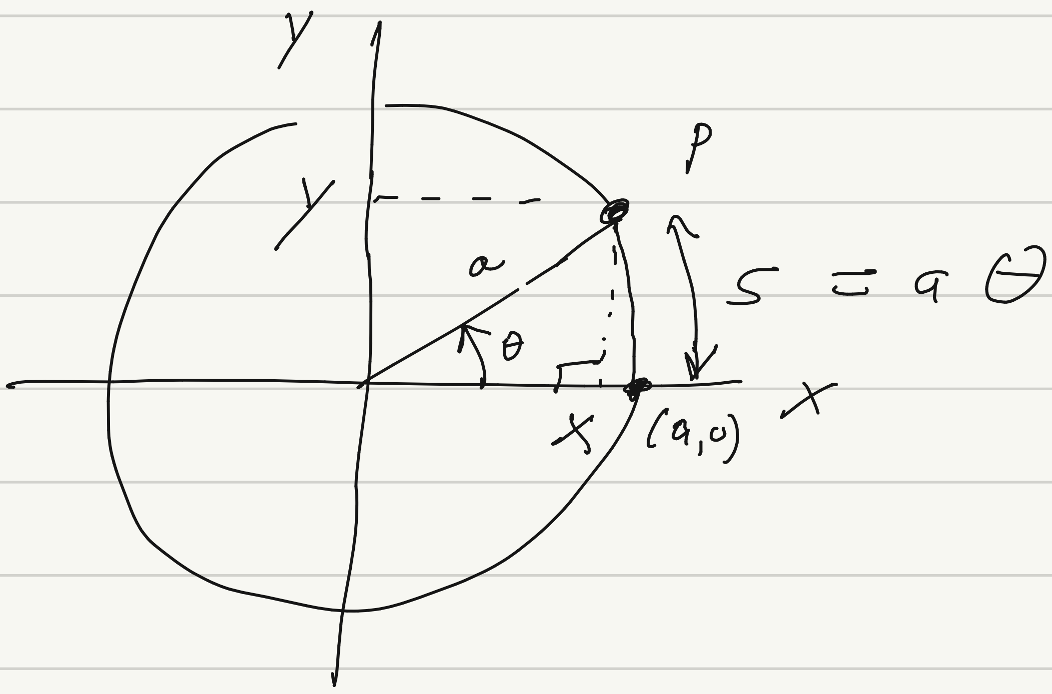
where $x = \sin\phi$ and $0 \leq k \leq 1$

If we change variables $t \to \sin\theta$ in the integrals then

$$F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$E(\phi, k) = \int_0^\phi \sqrt{1-k^2\sin^2\theta}\, d\theta$$

Circular functions:



$$\sin\theta \equiv \frac{y}{a}, \qquad \cos\theta \equiv \frac{x}{a}$$

where $\theta \equiv \dfrac{\text{arc length from } (a,0) \text{ to } (x,y)}{a}$

$$= \frac{1}{a}\int_{(a,0)}^{(x,y)} \sqrt{dx^2+dy^2} \qquad \left(= \int d\theta \right)$$

$$x^2+y^2 = a^2 \quad \rightarrow \quad a^2\cos^2\theta + a^2\sin^2\theta = a^2$$

$$\rightarrow \quad \boxed{\cos^2\theta + \sin^2\theta = 1}$$

Derivatives:

$$\frac{d}{d\theta}\sin\theta = \frac{d}{d\theta}\left(\frac{y}{a}\right) = \frac{1}{a}\frac{dy}{d\theta} = \frac{dy}{\sqrt{dx^2+dy^2}}$$

$$= \frac{1}{\sqrt{\left(\frac{dx}{dy}\right)^2+1}}$$

$$= \frac{1}{\sqrt{\left(\frac{-y}{x}\right)^2+1}}$$

Now: $x^2+y^2=a^2$

$$\rightarrow \quad 2x\,dx + 2y\,dy = 1$$

$$\frac{dx}{dy} = \frac{-y}{x}$$

Thus,

$$\frac{d}{d\theta}\sin\theta = \frac{1}{\sqrt{1+\left(\frac{-y}{x}\right)^2}} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta$$

so $\frac{d\sin\theta}{d\theta} = \cos\theta$

Similarly, $\frac{d\cos\theta}{d\theta} = -\sin\theta$

<u>Integrate</u>:

$$\int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta = \theta + \text{const}$$

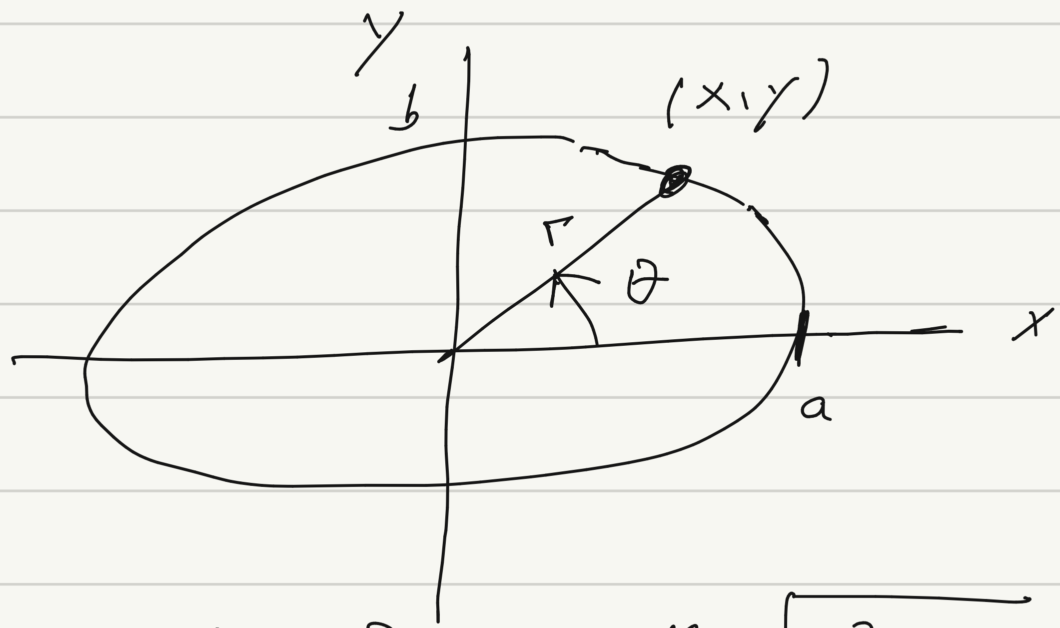
$t = \sin\theta$

$\cos\theta = \sqrt{1-t^2}$

$dt = d(\sin\theta)$

$$\rightarrow \boxed{\int \frac{dt}{1-t^2} = \cancel{\theta} + \text{const} = \sin^{-1} t + \text{const}}$$

Different parametrization of an ellipse:



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Let $a \geq b$

Eccentricity:

$$e^2 = 1 - \left(\frac{b}{a}\right)^2$$

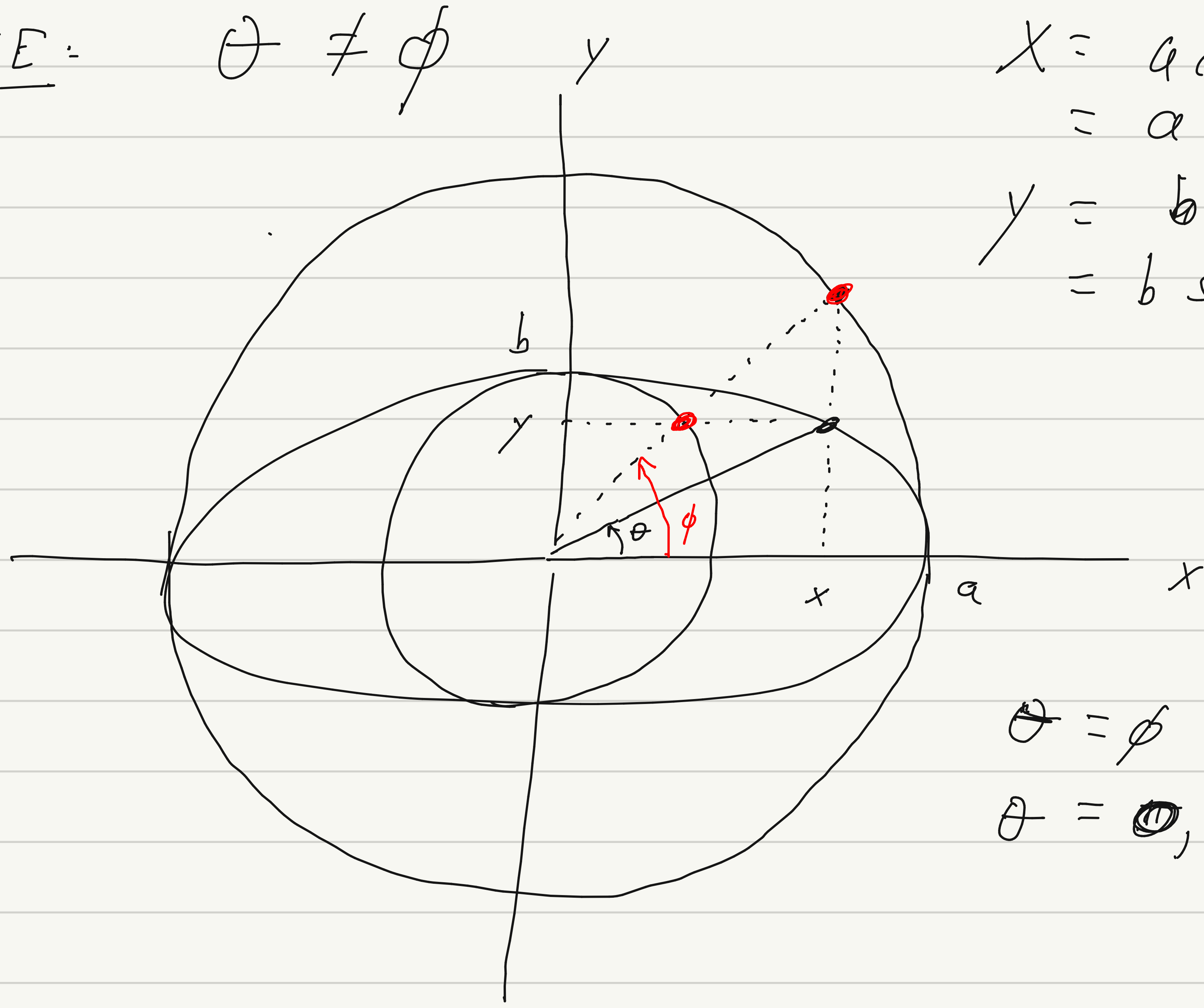
($e = 0$ for a circle)

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\theta$$

Another parameterization:

$$\left.\begin{aligned} x &= a\cos\phi \\ y &= b\sin\phi \end{aligned}\right\} \rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

NOTE: $\theta \neq \phi$



$$x = a\cos\phi = a\,\mathrm{cn}(u;k)$$
$$y = b\sin\phi = b\,\mathrm{sn}(u;k)$$

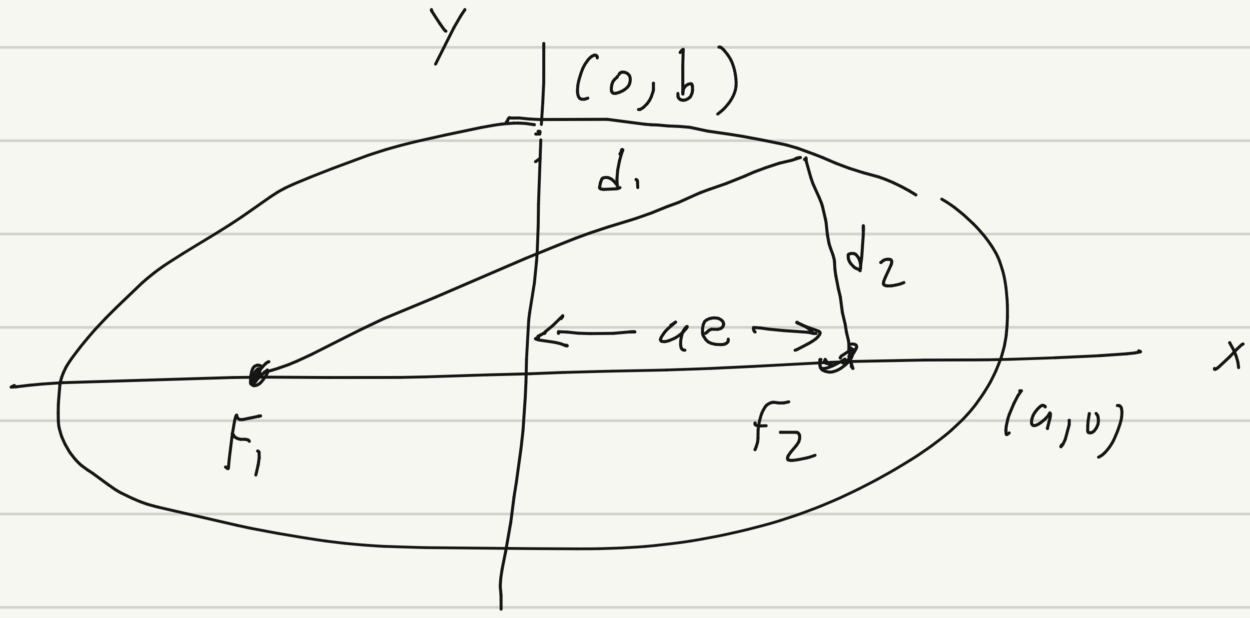
$\theta = \phi$ for

$\theta = 0, \frac{\pi}{2}, \ldots$

$$\tan\theta = \frac{y}{x} = \left(\frac{b}{a}\right)\tan\phi \rightarrow \theta = \arctan\left[\frac{b}{a}\tan\phi\right]$$

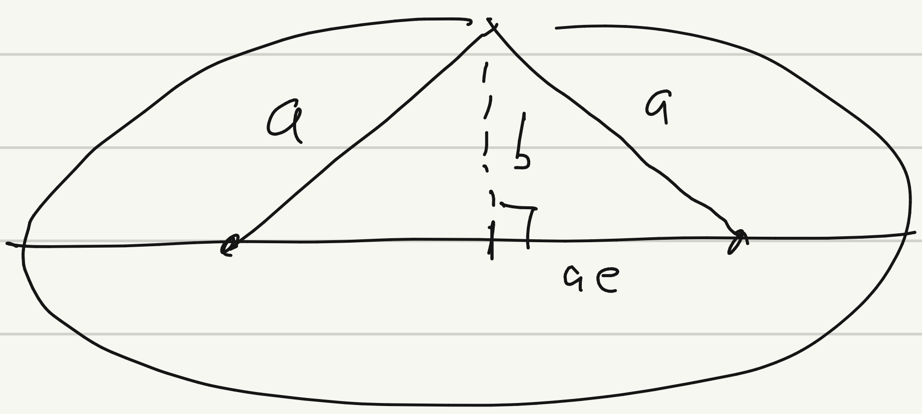
$$\phi = \arctan\left[\frac{a}{b}\tan\theta\right]$$

Elliptic functions:



$d_1 + d_2 = 2a$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

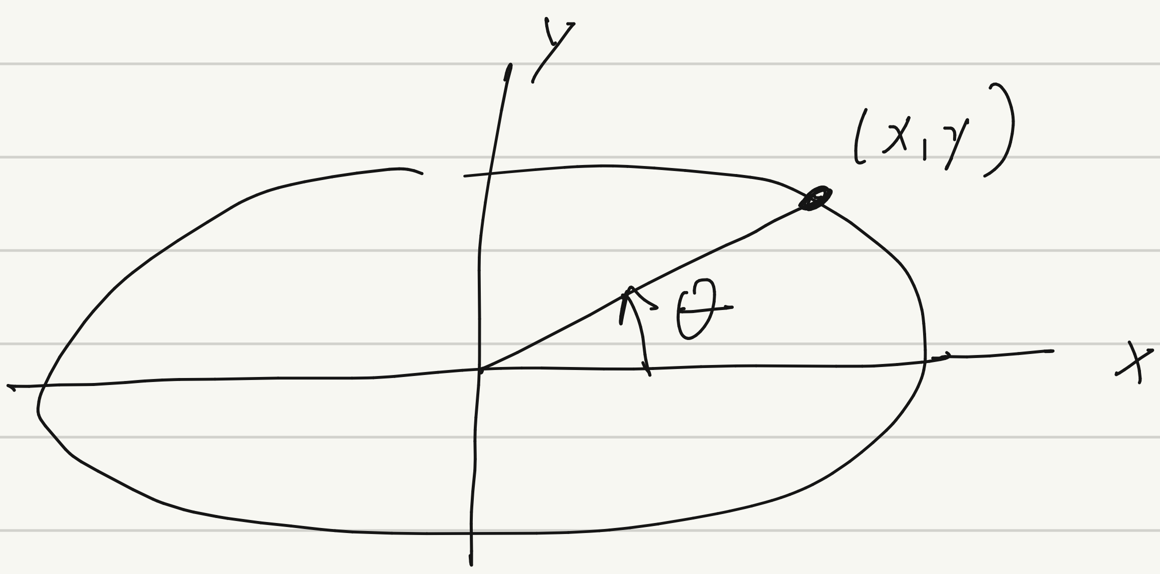


$a^2 = b^2 + a^2 e^2$

$a^2(1-e^2) = b^2$

$b = a\sqrt{1-e^2}$

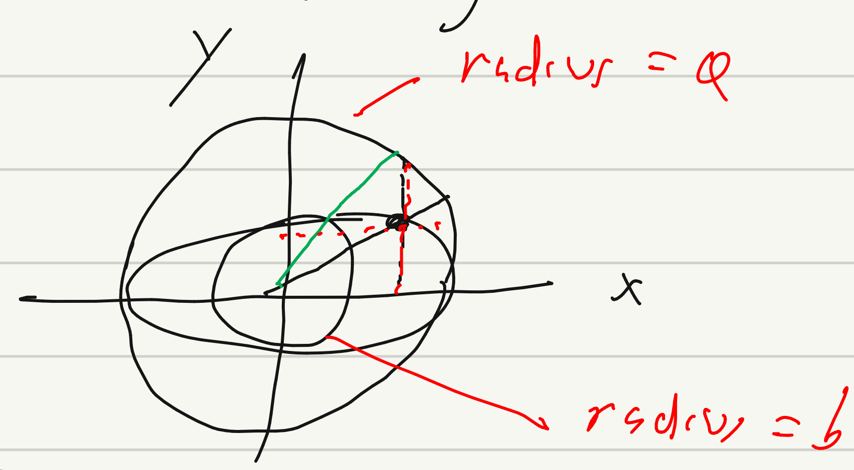
Also, $\left(\frac{b}{a}\right)^2 = 1 - e^2 \rightarrow \boxed{e = \sqrt{1 - \left(\frac{b}{a}\right)^2}}$



$x = r\cos\theta$

$y = r\sin\theta$

(but r changes)



$x = a\cos\phi$

$y = b\sin\phi$

$\phi \neq \theta$

Define: $cn(u;k) \equiv x/a$

$sn(u;k) \equiv y/b$

$dn(u;k) \equiv r/a$

where $k = e$

$0 \leq k \leq 1$

~~and~~ where $u = \frac{1}{b}\int_0^\theta r\,d\theta$ — $u = \theta$ for circle

$u \neq$ arc length

since $ds = \sqrt{dr^2 + r^2 d\theta}$

NOTE: $\quad bu = \int_0^\theta r\,d\theta \leq \int_0^\theta ds$

$\leq$ arc length from $(a,0)$ to (x,y)

Properties of sn, cn, dn:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \rightarrow \boxed{cn^2 u + sn^2 u = 1}$$

$$x^2 + y^2 = r^2 \rightarrow a^2 cn^2 u + b^2 sn^2 u = a^2 dn^2 u$$

$$a^2(1 - sn^2 u) + b^2 sn^2 u = a^2 dn^2 u$$

$$1 - sn^2 u + \frac{b^2}{a^2} sn^2 u = dn^2 u$$

$$1 - \left(1 - \frac{b^2}{a^2}\right) sn^2 u = dn^2 u$$

$$1 - k^2 sn^2 u = dn^2 u$$

Thus, $\boxed{dn^2 u + k^2 sn^2 u = 1}$

for circle $k = 0$, $dn\, u = 1$

Derivatives:

$$\frac{d}{du} sn\, u = \frac{d}{du}\left(\frac{y}{b}\right) = \frac{dy}{b\,du}$$

Now: $b\,du = r\,d\theta$

$$\rightarrow \frac{d}{du} sn\, u = \frac{dy}{r\,d\theta}$$

$$x = r\cos\theta \quad \rightarrow \quad dx = dr\cos\theta - r\sin\theta\, d\theta$$
$$y = r\sin\theta \qquad\qquad dy = dr\sin\theta + r\cos\theta\, d\theta$$

$$\rightarrow \quad -\sin\theta\, dx = -\sin\theta\cos\theta\, dr + r\sin^2\theta\, d\theta$$
$$+\cos\theta\, dy = \cos\theta\sin\theta\, dr + r\cos^2\theta\, d\theta$$

add: $\cos\theta\, dy - \sin\theta\, dx = r\, d\theta$

$$\rightarrow \quad \frac{x}{r}dy - \frac{y}{r}dx = r\, d\theta$$

thus,

$$\frac{d\,\mathrm{sn}\,u}{du} = \frac{dy}{r\,d\theta} = \frac{d\cancel{y}}{\frac{x}{r}dy - \frac{y}{r}d\cancel{x}}$$

$$= \frac{r}{x - y\frac{dx}{dy}}$$

Also: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \rightarrow \frac{2x\,dx}{a^2} + \frac{2y\,dy}{b^2} = 0$

$$\frac{dx}{dy} = -\frac{y}{x}\left(\frac{a}{b}\right)^2$$

$$\rightarrow \quad \frac{d\,\mathrm{sn}\,u}{du} = \frac{r}{x + \frac{y^2}{x}\left(\frac{a}{b}\right)^2} = \frac{r x}{x^2 + y^2\left(\frac{a^2}{b^2}\right)}$$

$$= \frac{r}{a}\frac{x}{a}\frac{1}{\underbrace{\left(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2\right)}_{1}} = \boxed{\mathrm{cn}\,u \cdot \mathrm{dn}\,u}$$

Then using $\quad sn^2 u + cn^2 u = 1$

we have $\quad \cancel{2}\, sn\,u \dfrac{d}{du} sn\,u + \cancel{2}\, cn\,u \dfrac{d}{du} cn\,u = 0$

$$\rightarrow \quad \frac{d\, cn\,u}{du} = -\frac{sn\,u}{cn\,u} \frac{d}{du} sn\,u$$

$$= -\frac{sn\,u}{\cancel{cn\,u}} \cancel{cn\,u} \cdot dn\,u$$

$$= \boxed{-sn\,u \cdot dn\,u}$$

And using $\quad dn^2 u + k^2 sn^2 u = 1$:

$$\cancel{2}\, dn\,u \frac{d(dn\,u)}{du} + \cancel{2} k^2 sn\,u \frac{d}{du}(sn\,u) = 0$$

$$\rightarrow \quad \frac{d}{du}(dn\,u) = -k^2 \frac{sn\,u}{dn\,u} \frac{d(sn\,u)}{du}$$

$$= -k^2 \frac{sn\,u}{\cancel{dn\,u}} cn\,u \cdot \cancel{dn\,u}$$

$$= \boxed{-k^2\, sn\,u\, cn\,u}$$

summary:

$$\frac{d}{du} sn\,u = cn\,u \cdot dn\,u$$

$$\frac{d}{du} cn\,u = -sn\,u \cdot dn\,u$$

$$\frac{d}{du} dn\,u = -k^2\, sn\,u\, cn\,u$$

Integrate:

$$\frac{d\,\mathrm{sn}\,u}{du} = \mathrm{cn}\,u \cdot \mathrm{dn}\,u$$

$$\int \frac{d(\mathrm{sn}\,u)}{\mathrm{cn}\,u \cdot \mathrm{dn}\,u} = \int du = u + \text{const}$$

Let $t = \mathrm{sn}\,u \rightarrow \mathrm{cn}\,u = \sqrt{1-t^2}$, $\mathrm{dn}\,u = \sqrt{1-k^2\,\mathrm{sn}^2 u}$

$$\rightarrow \int \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}} = u + \text{const}$$

But since $t = \mathrm{sn}\,u$:

$$\boxed{\int \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}} = \mathrm{sn}^{-1}(t;k) + \text{const}}$$

In handbooks:

$$\boxed{\int_0^{\sin\phi} \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}} = F(\phi,k)}$$

Incomplete Elliptic integral of the 1st kind

$$\downarrow$$

$$\boxed{\int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\phi,k)}$$

where $t = \sin\theta$.

NOTE: $\boxed{\sin\phi = \mathrm{sn}\,u}$ ☆

Complete elliptic integral of 1st kind:

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}}$$

Connection to simple pendulum:

$$P = \frac{4}{\omega_0} K\left(k \equiv \sin\left(\frac{\phi_0}{2}\right)\right)$$

$$\omega_0 t = sn^{-1}\left(x \equiv \frac{\sin(\phi/2)}{\sin\left(\frac{\phi_0}{2}\right)},\ k \equiv \sin\left(\frac{\phi_0}{2}\right)\right) + \text{const}$$

$\omega_0 = \sqrt{\frac{g}{l}}$

$\text{const} = -K(k)$ for $\phi = \phi_0$ when $t = 0$

$$\phi(t) = 2 \arcsin\left[k\, sn\left(\omega_0\left(t + \frac{P}{4}\right); k\right)\right]$$

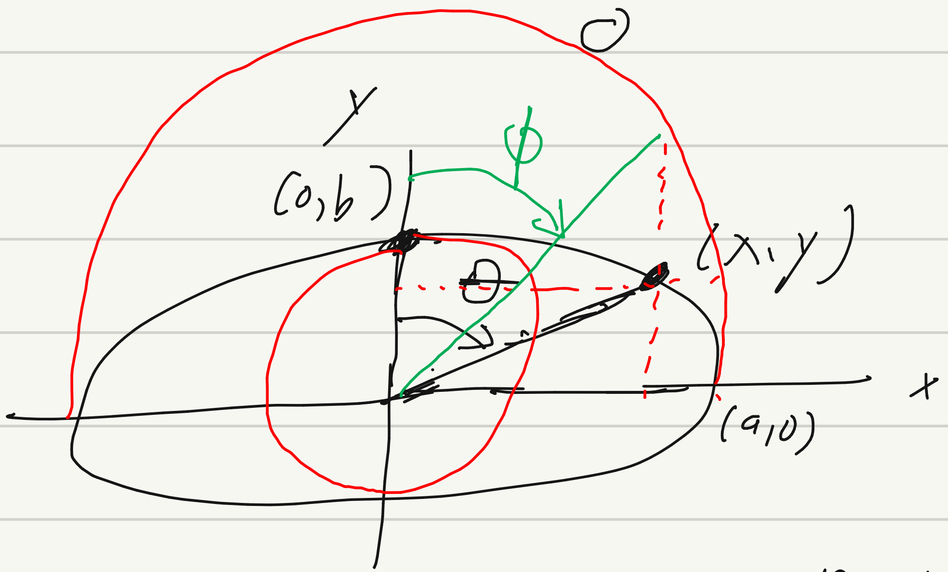
Elliptic integral of 2^{nd} kind (circumference of ellipse wrt y-axis) $\xrightarrow{}$ cw

$$E(\phi, k) \equiv \int_0^{\sin\phi} dt \, \frac{\sqrt{1-k^2 t^2}}{\sqrt{1-t^2}}$$

Rewrite: $t = \sin\bar\phi$

$dt = \cos\bar\phi \, d\bar\phi = \sqrt{1-t^2} \, d\bar\phi$

$$E(\phi, k) \equiv \int_0^{\phi} d\bar\phi \sqrt{1-k^2\sin^2\bar\phi} \quad \text{(scipy definition)}$$



$$k = e = \sqrt{1-\left(\frac{b}{a}\right)^2}$$

$x = r\sin\theta = a\sin u = a\sin\phi$

$y = r\cos\theta = b\cos u = b\cos\phi$

} sin, cos interchanged from previous page

$ds = \sqrt{dx^2+dy^2}$

$= \sqrt{a^2\cos^2\phi + b^2\sin^2\phi} \; d\phi$

$= \sqrt{a^2(1-\sin^2\phi) + b^2\sin^2\phi} \; d\phi$

$= a\sqrt{1-\left(1-\left(\frac{b}{a}\right)^2\right)\sin^2\phi} \; d\phi$

$= a\sqrt{1-k^2\sin^2\phi} \; d\phi$

$(\phi \neq \theta)$

$\tan\theta = \frac{a}{b}\tan\phi$

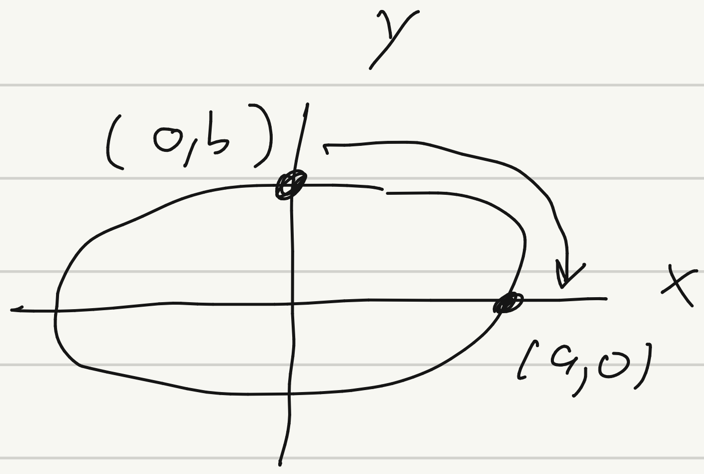
$\theta = \arctan\left(\frac{a}{b}\tan\phi\right)$

$\phi = \arctan\left(\frac{b}{a}\tan\theta\right)$

$$\boxed{S_{(0,b)\to(x,y)} = a\int_0^{\phi}\sqrt{1-k^2\sin^2\bar\phi} \; d\bar\phi} \equiv E(\phi, k)$$

Complete elliptic integral of 2^{nd} kind :

$$E(k) \equiv E\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} d\phi \sqrt{1 - k^2 \sin^2\phi} = \int_0^1 dt \, \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}$$



Circumference :

$$C = 4a \int_0^1 dt \, \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}$$

1^{st} order correction (for $k \ll 1$; nearly circular)

$$C \cong 4a \int_0^1 \frac{dt}{\sqrt{1-t^2}} \left(1 - \frac{1}{2}k^2t^2\right)$$

$$= 4a\left[\underbrace{\int_0^1 \frac{dt}{\sqrt{1-t^2}}}_{\sin^{-1}(1) = \pi/2} - \frac{k^2}{2}\int_0^1 \frac{dt \;\, t^2}{\sqrt{1-t^2}}\right] \qquad t = \sin x$$

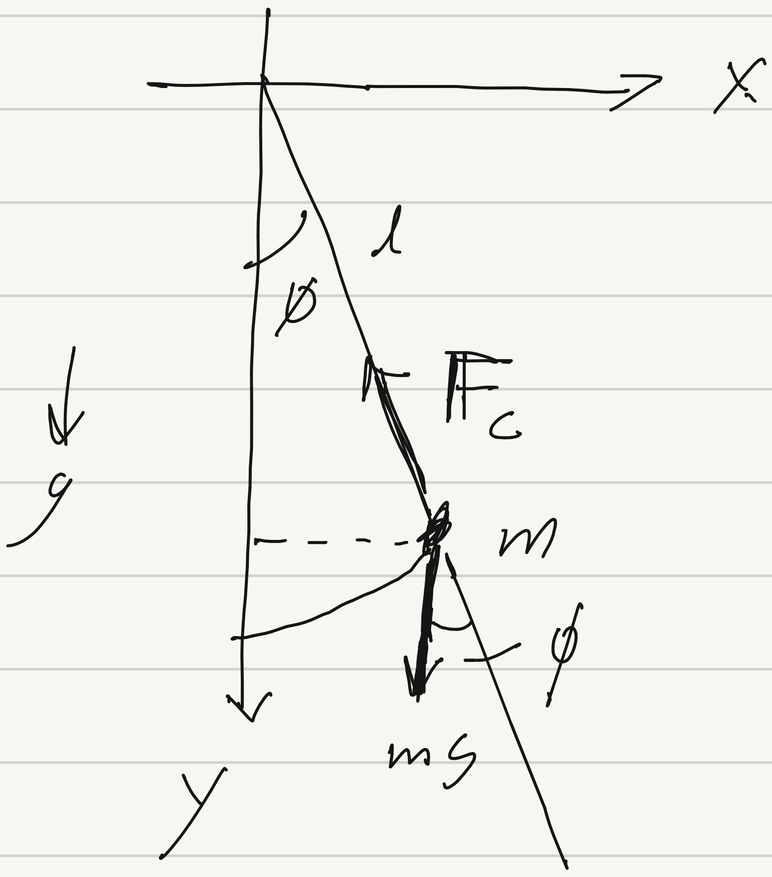
$$= 4a\left[\frac{\pi}{2} - \frac{k^2}{2}\int_0^{\pi/2} \frac{\cancel{\cos x}\, dx \, \sin^2 x}{\cancel{\sqrt{1-\sin^2 x}}}\right]$$

$$= 4a\left[\frac{\pi}{2} - \frac{k^2}{2}\,\frac{1}{2}\int_0^{\pi/2}\left(1 - \cancel{\cos 2x}^{\,0}\right) dx\right]$$

$$= 4a\left[\frac{\pi}{2} - \frac{k^2}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= 2\pi a\left(1 - k^2/4\right)$$

Simple pendulum (Freshman physics analysis):



$$mg\sin\phi = -ma_T$$
$$= -ml\ddot{\phi}$$

$$\rightarrow \ddot{\phi} = -\frac{g}{l}\sin\phi$$

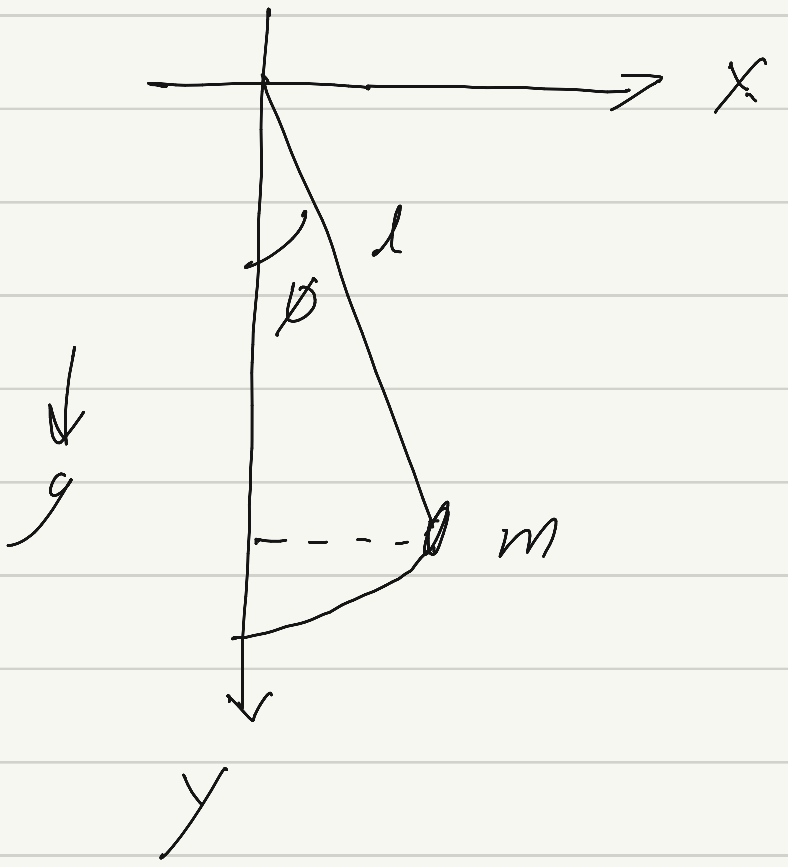
constraint force:

$$F_c - mg\cos\phi = ma_c$$
$$= m\omega^2 l$$
$$= ml\dot{\phi}^2$$

Thus, $F_c = mg\cos\phi + ml\dot{\phi}^2$

(non-zero at turning points as well as in vertical position — i.e., $\theta = 0$)

Period of a simple pendulum:



$U = -mgy$

where $y = l\cos\phi$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$= \frac{1}{2}ml^2\dot{\phi}^2$

$L = T - U$

$= \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$

$\frac{d}{dt}\left(ml^2\dot{\phi}\right) = -mgl\sin\phi$

$ml^2\ddot{\phi} = -mgl\sin\phi$

$\rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi}$ (same as before)

Small-angle approx: $\sin\phi \simeq \phi$

$\Rightarrow \dot{\phi} \simeq -\frac{g}{l}\phi = -\omega^2\phi$

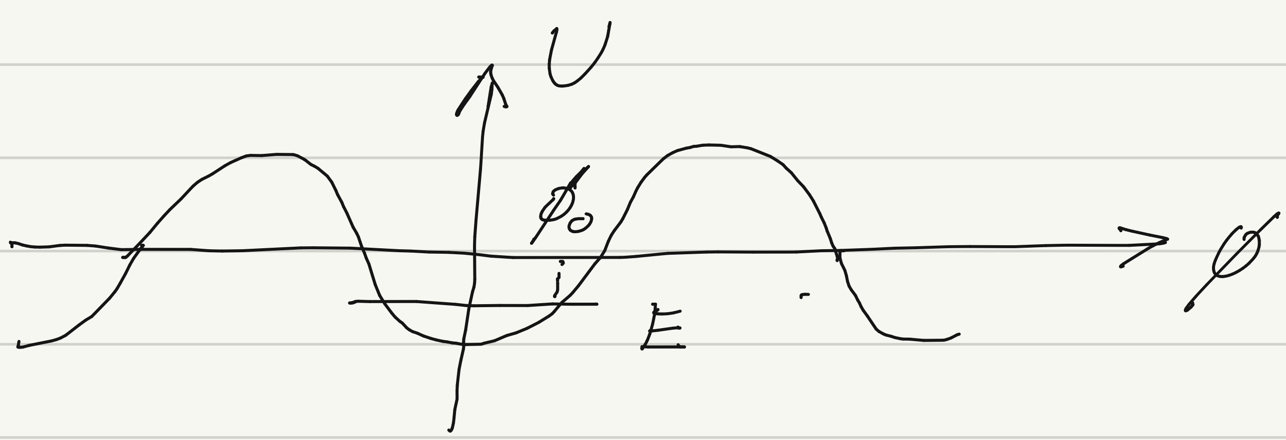
sol'n: $\phi(t) = \underset{\text{complex}}{A}e^{i\omega t}$, $\omega \equiv \sqrt{\frac{g}{l}}$, $P = \frac{2\pi}{\omega}$

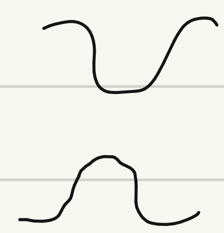
Beyond small-angle approx:

Cons. of energy:

$$E = T + U$$
$$= \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$$

$E = \text{const} = -mgl\cos\phi_0$ at turning points





Thus,

$$-mgl\cos\phi_0 = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$$

$$\frac{1}{2} m l^2 \dot{\phi}^2 = +mgl\,(\underbrace{\cos\phi - \cos\phi_0}_{\geq 0 \text{ since } \phi \leq \phi_0})$$

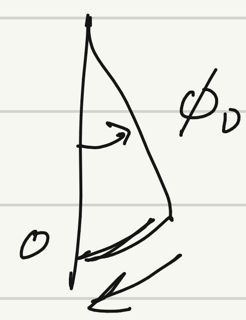
$$\longrightarrow \quad \dot{\phi} = \pm\sqrt{2\,\frac{g}{l}\left(\cos\phi - \cos\phi_0\right)}$$

$$\int dt = \pm\int \frac{d\phi}{\sqrt{2\,\frac{g}{l}}\sqrt{\cos\phi - \cos\phi_0}}$$

$$\rightarrow \quad \boxed{t = \frac{1}{\sqrt{2}\,\omega_0}\int \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}} + \text{const}}, \quad \omega_0 \equiv \sqrt{\frac{g}{l}}$$

Period:

$$P = \frac{4}{\sqrt{2}\,\omega_0}\int_0^{\phi_0}\frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$



Now: $\cos\phi = \cos\left(2\,\frac{\phi}{2}\right)$

$$= \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\rightarrow \quad \cos\phi - \cos\phi_0 = -2\sin^2\left(\frac{\phi}{2}\right) + 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$= 2\sin^2\left(\frac{\phi_0}{2}\right)\left[1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)}\right]$$

Let: $x \equiv \dfrac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}$ $\rightarrow$ $dx = \dfrac{1}{\sin\left(\frac{\phi_0}{2}\right)}\,\dfrac{1}{2}\cos\left(\frac{\phi}{2}\right)d\phi$

define $k \equiv \sin\left(\frac{\phi_0}{2}\right)$

$$= \frac{1}{2\sin\left(\frac{\phi_0}{2}\right)}\sqrt{1 - \sin^2\left(\frac{\phi}{2}\right)}\,d\phi$$

$$= \frac{1}{2k}\sqrt{1 - k^2x^2}\,d\phi$$

Thus,

$$P = \frac{4}{\cancel{\sqrt{2}}\,\omega_0}\int_0^1 \frac{dx\;\cancel{2}\cancel{k}}{\sqrt{1-k^2x^2}}\;\frac{1}{\cancel{\sqrt{2}}\cancel{k}\sqrt{1-x^2}}$$

$$= \boxed{\frac{4}{\omega_0}\int_0^1 \frac{dx}{\sqrt{1-k^2x^2}\sqrt{1-x^2}}} = \frac{4}{\omega_0}K(k)$$

complete elliptic integral of 1st kind

Leading-order correction to period:

$$P = \frac{4}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1-k^2x^2}\,\sqrt{1-x^2}}$$

suppose $\phi_0 << 1 \quad \rightarrow \quad k \equiv \sin\left(\frac{\phi_0}{2}\right) \approx \frac{\phi_0}{2} << 1$

Thus,

$$\frac{1}{\sqrt{1-k^2x^2}} \approx 1 + \frac{1}{2} k^2 x^2$$

$$= 1 + \frac{1}{2}\left(\frac{\phi_0^2}{4}\right) x^2$$

$$= 1 + \frac{1}{8} \phi_0^2 x^2$$

$$\rightarrow \quad P \approx \frac{4}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1-x^2}} \left[1 + \frac{1}{8}\phi_0^2 x^2 \right]$$

$$= \frac{4}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \frac{\phi_0^2}{2\omega_0} \int_0^1 \frac{dx\, x^2}{\sqrt{1-x^2}}$$

$$= \frac{4}{\omega_0} \sin^{-1}(1) + \frac{\phi_0^2}{2\omega_0} \int_0^{\pi/2} \frac{\cancel{\cos\theta}\, d\theta \, \sin^2\theta}{\cancel{\sqrt{1-\sin^2\theta}}}$$

$$= \frac{4}{\omega_0} \cdot \frac{\pi}{2} + \frac{\phi_0^2}{2\omega_0} \int_0^{\pi/2} d\theta \, \frac{1}{2}\left(1 - \cos 2\theta\right)$$

$$= \frac{2\pi}{\omega_0} + \frac{\phi_0^2}{4\omega_0}\left(\frac{\pi}{2} - \frac{1}{2} \cancelto{0}{\sin 2\theta \Big|_0^{\pi/2}} \right)$$

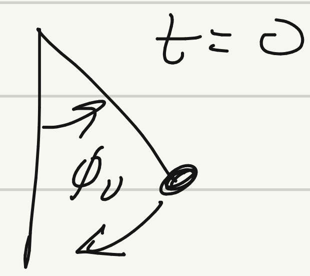
$$= \boxed{\frac{2\pi}{\omega_0}\left(1 + \frac{\phi_0^2}{16}\right)}$$

General time dependence of oscillation:

$$t = \frac{1}{\sqrt{2}\,w_0} \int \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}} + \text{const} \quad , \quad w_0 \equiv \sqrt{\frac{g}{l}}$$

$$= \frac{1}{w_0} \int \frac{dx}{\sqrt{1-k^2x^2}\,\sqrt{1-x^2}} + \text{const}$$

$$= \frac{1}{w_0}\left[\mathrm{sn}^{-1}\left(x = \frac{\sin(\phi/2)}{\sin(\phi_0/2)} ; k = \sin\left(\frac{\phi_0}{2}\right)\right) + \text{const}\right]$$



choose const so that $t=0 \Leftrightarrow \phi = \phi_0$

$$0 = \mathrm{sn}^{-1}(1, k) + \text{const}$$

$$\rightarrow \quad \text{const} = -\mathrm{sn}^{-1}(1,k) = -F\left(\frac{\pi}{2}, k\right) \equiv -K(k)$$

$$= -\frac{w_0}{4} P$$

Thus,

$$w_0 t = \mathrm{sn}^{-1}(x; k) - \frac{w_0}{4} P$$

$$w_0\left(t + \frac{P}{4}\right) = \mathrm{sn}^{-1}(x; k)$$

$$\mathrm{sn}\left[w_0\left(t + \frac{P}{4}\right); k = \sin\left(\frac{\phi_0}{2}\right)\right] = x = \frac{\sin(\phi/2)}{\sin(\phi_0/2)}$$

$$\rightarrow \boxed{\phi(t) = 2 \arcsin\left[\underbrace{\sin\left(\frac{\phi_0}{2}\right)}_{k} \mathrm{sn}\left(w_0\left(t + \frac{P}{4}\right); k = \sin\left(\frac{\phi_0}{2}\right)\right)\right]}$$