Standing Waves and Overtone Series

Transverse Standing Waves

Take two sine waves of the same frequency and amplitude traveling in *opposite* directions and interfering with each other. By the principle of superposition you add the two waves to obtain the resultant wave. What is the result? It is a **standing wave** with its crests and troughs not moving in the direction of the original waves. Instead, the medium oscillates up and down about an equilibrium position. The displacement does not keep moving in the direction of the original traveling waves. The two traveling waves have become a standing wave or stationary wave. It has twice the amplitude of the component waves and oscillates with the same frequency. The points of maximum and minimum displacement in the standing wave are called **antinodes** and **nodes**, respectively.

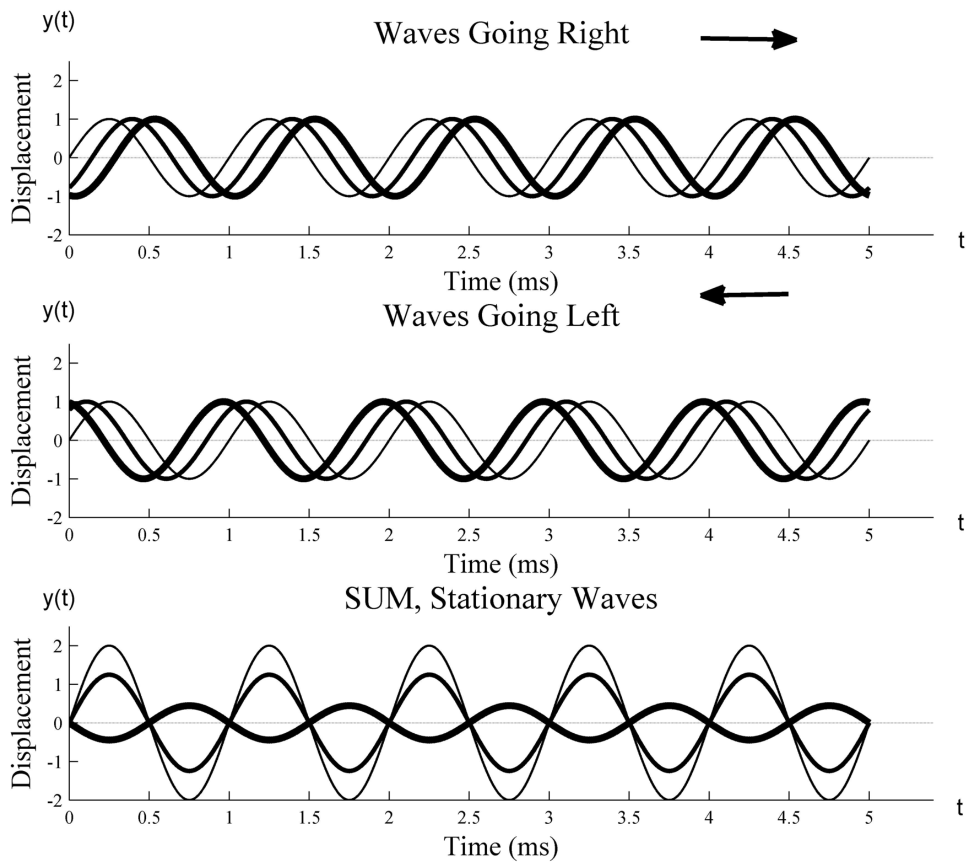


Figure. Two sinusoidal waves on a string have equal amplitude and travel in opposite directions. Dashed, light solid, and dark solid lines are for three progressing times. The standing wave at the bottom is the addition (superposition) of the two traveling waves at the three different times.

Demonstrations of Transverse Traveling and Standing Waves
1. Send a transverse traveling wave down a slinky, which at the other end is held by a person or fastened to the wall. Observe the reflected wave coming back. Keep sending waves down the slinky. Soon you will see a standing wave, provided that you hit the right frequency. (Otherwise you will get a scramble and no discernible standing wave.) By starting at low frequencies, you first will produce the fundamental mode of the wave, designated as N = 1. At integer multiples of this frequency, you will see the higher vibrational modes or harmonics. It should be fairly easy to make the slinky vibrate to about the 5th vibrational mode N = 5.
2. Demonstrate waves on the overhead projector with the little mechanical "wave machine" from the early days of TTU. Turn the shafts and see traveling waves, standing waves, and different amplitudes and phases.

Demonstration of Standing Waves on a Vibrating Rope or String
Use a stretched rope or string and show the standing wave modes, starting with the fundamental mode. Fasten one end of the wire to a vibrator, guide the other end over a pulley and add a weight. Vary the vibrator frequency to show the first few modes.

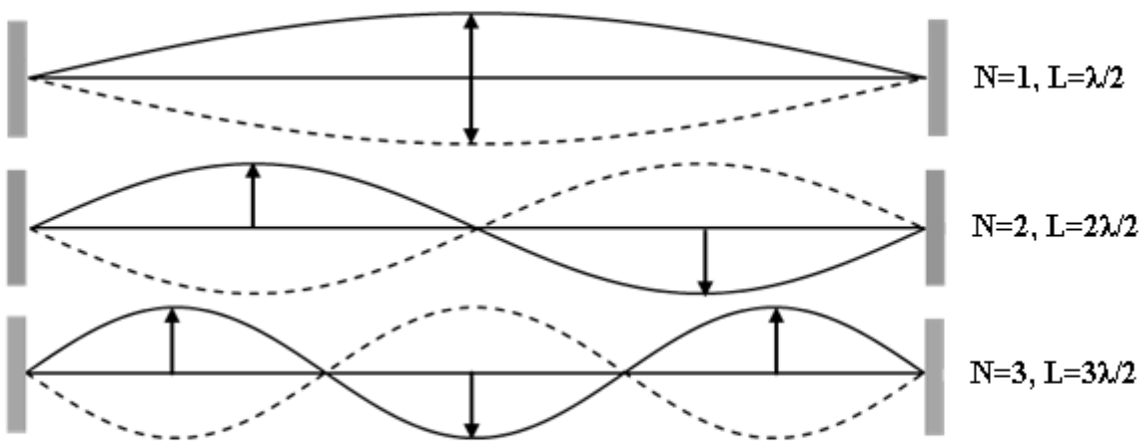


Figure. The first three standing wave modes on a stretched rope or wire.

The fundamental mode has harmonic number N = 1.
The 2nd harmonic or 1st overtone has N = 2. The harmonic overtones are N = 2, 3, 4, 5,…
For the fundamental mode we have $L = \lambda_1/2$, where λ_1 is the wavelength.
For the Nth mode we have $\mathbf{L = N\lambda_N/2}$ or $\mathbf{\lambda_N = 2L/N}$
The frequencies are:
$\mathbf{f_N = v/\lambda_N = (v/2L)N}$ or $\mathbf{f_N = Nf_1}$ or $\mathbf{f_N = f_1, 2f_1, 3f_1, 4f_1, 5f_1, \ldots}$
where $\mathbf{f_1 = v/2L}$ is the fundamental frequency and **v** the wave speed on the string.

Exercise
Continue with the above drawing and sketch the next two vibrational modes N = 4 and 5.

Overtone Series, Stretched Strings, Open and Closed Tubes

A sustained tone can be represented by its *amplitude-frequency spectrum* or simply *spectrum*. A spectrum shows the amplitude A on the y-axis versus the frequency f on the x-axis, beginning with the fundamental frequency ("pitch"), followed by the overtones.

Demonstrations

1. Play the overtone series with a PASCO frequency generator for the harmonics N = 1, 2, 3, 4, 5, 6, 7, 8. Acquire a spectrum of the harmonics on the computer.
2. Play a note with a violin or wind instrument. Acquire the *amplitude-frequency spectrum* in real time with a computer. Note the fundamental frequency and overtones.
3. Pluck the string of an Indian string instrument or violin in the middle and see the odd harmonics N =1, 3, 5... Now pluck the string very close to its fastened end and see the odd harmonics as well as even harmonics in the sound spectrum. (To understand this, draw the first 6 or so vibrational modes and imagine plucking the string in the middle or near its end.)

Harmonic Numbers, Wavelengths, and Frequencies of a Vibrating String

From the preceding drawing of the vibrational modes of a string we can see the relationship between the harmonic number and the wavelength. From the wavelength follows the frequency with the formula f = v/λ, where v is the wave speed on the string (not the speed of sound in air!).

Table. Harmonic number, wavelength, and frequency for the first six standing wave modes on a string of length L, with *v* the wave speed on the string.

Harmonic number N	Wavelength λ	Frequency $\left(f = \frac{v}{\lambda}\right)$
1	$\lambda_1 = 2L$	$f_1 = \frac{v}{2L}$
2	$\lambda_2 = L = \frac{1}{2}\lambda_1$	$f_2 = \frac{v}{L} = 2f_1$
3	$\lambda_3 = \frac{2}{3}L = \frac{1}{3}\lambda_1$	$f_3 = \frac{v}{\frac{2}{3}L} = 3f_1$
4	$\lambda_4 = \frac{1}{2}L = \frac{1}{4}\lambda_1$	$f_4 = \frac{v}{\frac{1}{2}L} = 4f_1$
5	$\lambda_5 = \frac{2}{5}L = \frac{1}{5}\lambda_1$	$f_5 = \frac{v}{\frac{2}{5}L} = 5f_1$
6	$\lambda_6 = \frac{1}{3}L = \frac{1}{6}\lambda_1$	$f_6 = \frac{v}{\frac{1}{3}L} = 6f_1$

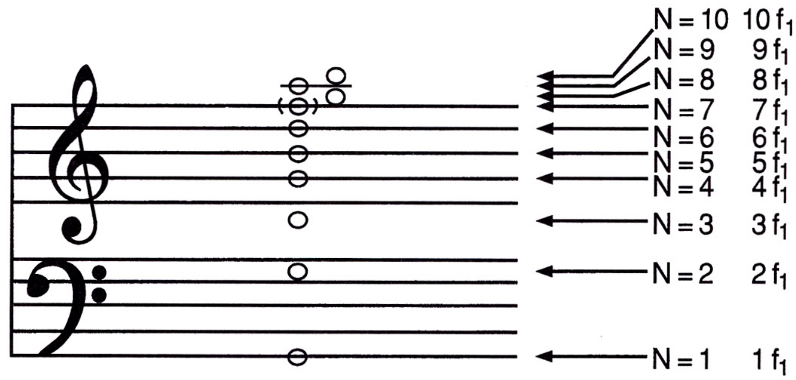


Figure. The first 10 harmonics in the overtone series, starting with the note G2 on the musical stave system. The first 6 notes correspond to the lowest 6 modes of a string vibrating at the fundamental G2 (i.e. N = 1). (From Berg & Stork, 3rd edition, Fig. 3-12, p. 76.) The symbols on the staves are the "treble clef" (top) and "bass clef" (bottom).

Harmonic Series, Frequencies, Musical Intervals and Notes

N	f	Interval	Note	Frequency Hz
1	f_1	Unison	G2	98.0
2	$2f_1$	One octave	G3	196.0
3	$3f_1$	One octave + perfect fifth	D4	294.0
4	$4f_1$	Two octaves	G4	392.0
5	$5f_1$	Two octaves + major third	B4	490.0
6	$6f_1$	Two octaves + perfect fifth	D5	588.0
7	$7f_1$	Two octaves + minor seventh	F5	686.0
8	$8f_1$	Three octaves	G5	784.0

Challenge Question for the Musically Interested
How can you get a musical third (ratio 5:4) from the column labeled "Interval" in the above table without making explicit use of the first two columns?
Answer: ____

Demonstrations
1. Play the overtone series on the keyboard, using the note G2 as the fundamental. Call out the notes (see above Table).
2. Start with C2, play the overtone series, and call out the notes.
3. Use a function generator to play the harmonic series 100, 200, 300, 400, 500, 600, 700, 800 Hz.
4. Use additional function generators to play the musical intervals 2:1, 3:2, 4:3, 5:4, 6:5.

Exercises

1. Construct a table similar to the earlier one for G2, but this time with C2 as the fundamental frequency, for which f_{C2} = 65.406 Hz. Consult the piano keyboard.
2. Draw two musical staffs, treble clef above bass clef. Insert the first 8 notes of the C2 overtone series.
3. Take the frequency ratio for the N = 3 and N = 2 entries in the above table. What is the resulting value and musical interval? Can you also obtain the name of this interval from the column labeled "Interval"?

Construction of Musical Intervals

Example: Start with the tonic C4 ("middle C")

N =	1		2		3		4		5		6
	f_1		$2f_1$		$3f_1$		$4f_1$		$5f_1$		$6f_1$
	C4		C5		G5		C6		E6		G6
		C5/C4		G5/C5		C6/G5		E6/C6		G6/E6	
		2/1		3/2		4/3		5/4		6/5	
		octave		fifth		fourth		major third		minor third	

Demonstration

Visualize the intervals on a keyboard:
Start with C4, count 8 white keys, including C4, and arrive at C5 an octave above C4.
From there, count 5 white keys, including C5, and arrive at G5, a fifth above C5.
From there, count 4 white keys, including G5, and arrive at C6, a fourth above G5.
From there, count 3 white keys, including C6, and arrive at E6, a third above C6.

Question

Name the musical intervals C4/C3, G4/C4, C5/G4, E5/C5, G5/E5.

Another Way to Memorize the Names of Musical Intervals

Example: Start with the note C (e.g. C4). Count the first 5 white keys on the piano keyboard, including C4. You land at G4 and you are a musical fifth up from C4!
Similarly:

Third:	do, re, **mi** — 1, 2, **3**
Fourth:	do, re, mi, **fa** — 1, 2, 3, **4**
Fifth:	do, re, mi, fa, **sol** — 1, 2, 3, 4, **5**
Octave:	do, re, mi, fa, sol, la, ti, **do** — 1, 2, 3, 4, 5, 6, 7, **8**

Vibrating Strings - Mersenne's Law

The frequency of the fundamental mode of a vibrating stretched string of length L is given by Mersenne's law (Marin Mersenne 1588-1648)

$$f = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

where F is the tension in Newton (N) and μ the linear mass density of the string. We have $\mu = m/L$ (kg/meter), where m is the mass of the string and L its length.

Demonstrations

1. Fasten a string at one end. Guide the other end over a pulley and hang a weight on the string. Pluck the string and listen to the tone produced.
2. Change the tension on the string with different weights. Note the change in pitch. Doubling the weight raises the pitch by $\sqrt{2}$ in accordance with Mersenne's law.
3. Alternatively, use a sonometer with several strings under tension. Change the tension or the effective length. Listen to the change in pitch.

Sample Calculation of the Fundamental Frequency of a Stretched String

Calculate the fundamental frequency f_1 (N = 1) for the G2 string of a cello.
The effective length of the cello string between the nut and bridge is L = 0.70 m.
The string is made of steel wire with a mass density $\rho = 7900$ kg/m^3.
The wire diameter of the G2 string is D = 0.97 mm = 0.97 x 10^{-3} m.
A weight of mass of 11.2 kg provides the tension in the string.
First calculate the linear mass density of the string: Use $\mu = m/L = \rho V/L$, where the cylindrical volume V of the string is $V = \pi r^2 L$, or $V = \pi D^2 L/4$.
Answer: $\mu = \rho\pi D^2/4 = 7900$ (kg/m^3) x $\pi(0.97 \times 10^{-3}$ m$)^2/4$ = **5.84 x 10^{-3} kg/m**.
Tension F = 9.8 m/s^2 x 11.2 kg = **110 N**.
Put the values for L, μ, and F into Mersenne's formula. Verify that **f = 98.0 Hz** (be sure to do that!). This is the pitch or fundamental frequency of the musical note G2.

The tension in the strings of a cello, violin, guitar, and other string instruments can be quite high, i.e. on the order of 100 N. The above is a typical example for a cello.

Exercise

The lowest note on the cello is C2 with f = 65.41 Hz. A C2-string has a linear mass density μ = 18 g/m. Calculate the tension in the string. For the length of the string between nut and bridge use L = 0.70 m. (P.S.: The diameter of the string is D = 1.8 mm, not needed here.)
Answer: Verify that F = 151 N.

Exercise for Guitar Players (optional)

The strings of a guitar are E3, A3, D4, G4, B4, E5. Find their thicknesses, lengths, and material of which they are made. Calculate their linear mass densities. Calculate the string tensions.

Wave Speed on a String

Consider the fundamental frequency of a vibrating string: $f = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$

From this we can obtain the speed of the wave on a stretched string.
Remember that for the fundamental mode we have $\lambda = 2L$. For the wave speed we have seen earlier that $v = \lambda f$. Therefore the wave speed is

$$v = \sqrt{\frac{F}{\mu}}$$

Example

Consider the G2 string of a cello. Assume a tension of F = 110 N and a linear mass density $\mu = 5.84\text{x}10^{-3}$ kg/m. Substituting these values in the above formula yields v = 137 m/s = 307 mph. This is the speed with which waves travel down the wire, are reflected back and forth at the ends, and form standing waves on the string.

How do the String Parameters Affect the Fundamental Frequency or Pitch?
Exercise

Rewrite Mersenne's law and show that $f = \frac{1}{\sqrt{\pi}LD}\sqrt{\frac{F}{\rho}}$,

where L is the length of the string, D the diameter, and ρ the usual mass density in kg/m^3 (*not* linear mass density μ) of the string. For instance ρ = 7900 kg/m^3 for steel wire.

We see that the frequency increases with increasing string tension F and decreases with increasing length L, diameter D, and density ρ. For a given string instrument such as a violin, the length is given and the mass density depends on the chosen string material. The adjustable parameters then are the tension F in the wire and its diameter D. Properly choosing the values for F and D is the task of s*tring scaling* so that one arrives at the needed pitch and quality of sound.

Piano Strings

The piano poses special challenges for "string scaling" because of its wide frequency range of more than 7 octaves. Ideally, the tension in the strings should be rather constant across the entire piano to avoid warping of the frame. For the high notes, short strings with low linear mass density are used. For the low notes, the strings are made of thicker and longer wire. But that is not sufficient. Therefore one wraps these strings with additional wire to make them more massive without greatly increasing their rigidity.

Demonstrations

1. Show strings for string instruments and piano (wrapped and unwrapped strings).
2. Show a violin or guitar.
3. Suspend a lead block from a string. Guide the string horizontally over a pulley and fasten the string at its end. Pluck the string, listen to the sound and demonstrate the dependence of the frequency f_1 on the quantities F, L, and D in Mersenne's. Do so by varying the weight F, the length L, and using another string with a different diameter D.

Overtone Series, Standing Waves in Air, Open and Closed Tubes

Below are shown standing sound waves in the air column of a tube. They are the longitudinal analog to standing transverse waves on a string. The standing waves are a superposition of waves traveling in opposite directions in the tube as a result of reflections at the ends.

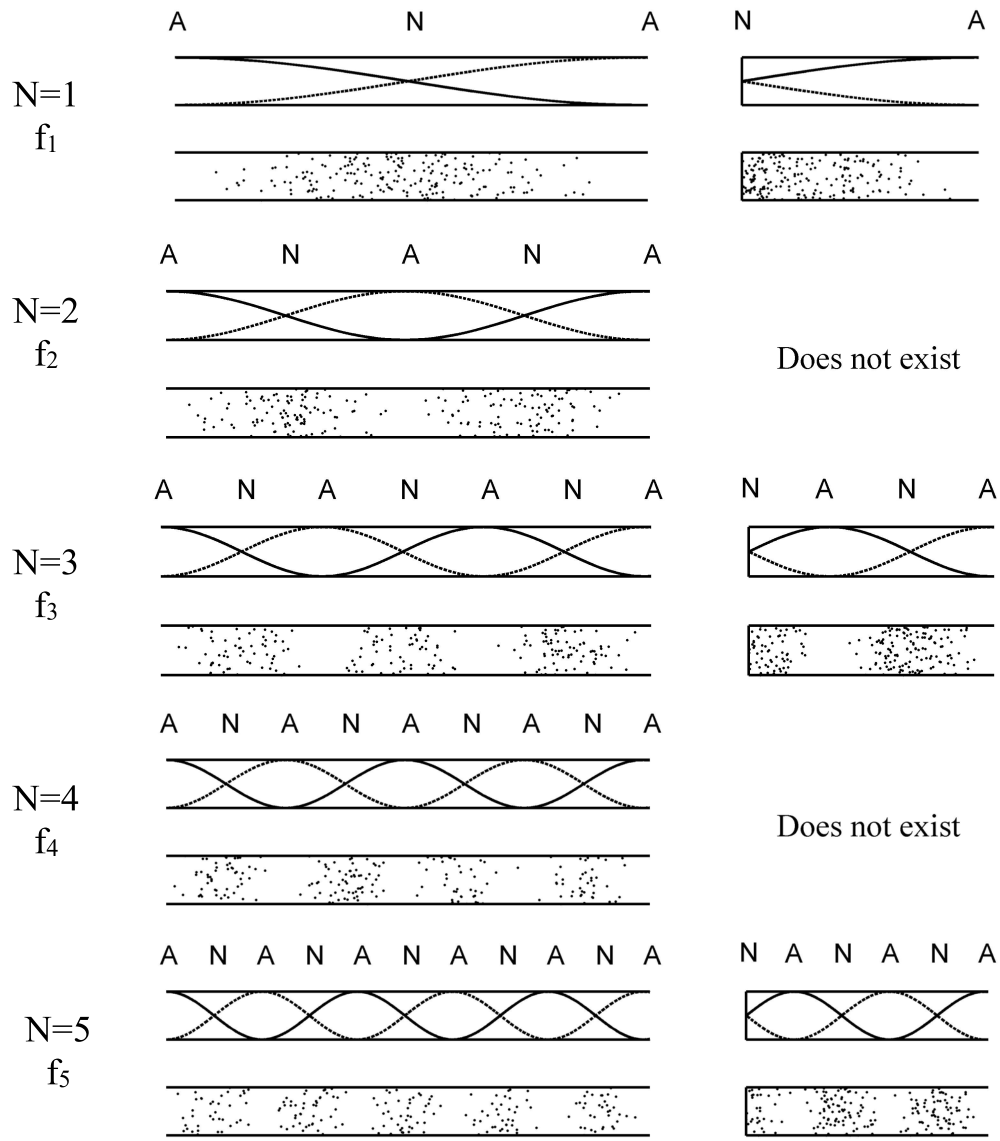


Figure. Standing longitudinal waves inside a cylindrical tube. “N” are the nodes and “A” the antinodes of the motion of air molecules. Left: First 5 resonance modes (harmonics) in a tube open at both ends. Right: The first 3 existing resonance modes in a tube with one end open and the other end closed (stopped). Note that only the odd harmonics N = 1, 3, 5 … occur in this case. The dotted scatter plots show where sand particles or small styrofoam pellets would accumulate.

For each tube drawn in the figure, the air displacement (or velocity) is drawn on the vertical axis as a function of position along the tube, which is the horizontal axis. Note the close analogy and differences with a vibrating string discussed earlier!
Understand what displacement "nodes" and "antinodes" mean and that the air *pressure fluctuations* in the tube are largest at the displacement nodes N and smallest at the antinodes A.

At the **nodes N**: The displacement and the velocity of the air molecules is a minimum. The pressure fluctuation Δp is a maximum, i.e. there is a pressure antinode. There is no phase change in the pressure Δp upon reflection.
At the **antinodes A**: The displacement or velocity is a maximum.
The pressure fluctuation Δp (not shown above) is a minimum, i.e. there is a pressure node. There also is a phase change of 180^0 in Δp upon reflection.

Open and Closed (Stopped) Tubes
At the open end of a tube there will be a displacement antinode. In contrast, at a closed end, there will be a displacement node for the air molecules (no motion).

Wavelength and Frequencies of the Longitudinal Modes in Tubes
For a cylindrical tube of effective length L and the preceding figure, we have the following for the wavelengths λ_N, frequencies f_N, and harmonic number N of the N-th vibrational mode (using $\lambda f = v$ for the speed of sound again):

Tube open at both ends **("open tube")**. All harmonics are allowed (same as for strings):
$\boldsymbol{\lambda_N = 2L/N}$, N = 1, 2, 3, 4… (all integers) and $\boldsymbol{f_N = v/\lambda_N = Nv/2L}$
For the first vibrational mode N = 1 in the open tube, we have
$\lambda_1 = 2L$ (from the Figure we can see that $L = \lambda/2$) and $\boldsymbol{f_1 = v/2L}$ (fundamental frequency).
Examples: Harmonica, flute, Indian flute, recorder, open organ pipes.

Tube closed at one end **("stopped tube")**. Strictly, only the odd harmonics are allowed:
$\boldsymbol{\lambda_N = 4L/N}$, N = 1, 3, 5, 7… (odd integers) and $\boldsymbol{f_N = v/\lambda_N = Nv/4L}$
For the first vibrational mode N = 1 in the closed tube, we have (different from strings)
$\lambda_1 = 4L$ (from the Figure we can see that $L = \lambda/4$) and $\boldsymbol{f_1 = v/4L}$ (fundamental frequency).
Examples: Clarinet, didgeridoo, closed organ pipes, other closed pipes.

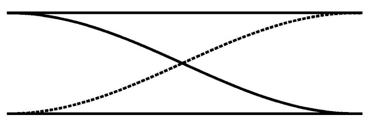
Exercise
1. Make a table of the harmonic frequencies of a pipe, open at both ends, for the first five harmonics N = 1, 2, 3, 4, 5 (see also the table for a vibrating string on p. 3-3). Add an additional column of the allowed harmonic frequencies for N = 1, 3, 5 of a pipe with one end closed and one end open.
2. Show that closed and open cylindrical tubes of the same length L have no common harmonics. (Neglect the end corrections for the tubes.)
Answer: Use the two expressions for f_N above and substitute the values for N. We obtain **1**, 3, 5, 7…•(v/4L) and f_N = **2**, 4, 6, 8,…•(v/4L), respectively.

We see that the two sets have no common frequencies between them.

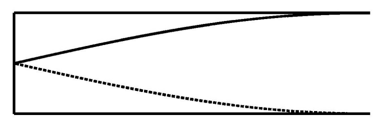
Exercise on Standing Waves in Open and Closed Tubes

Practice by drawing plots of the displacement of air molecules and the pressure variation in the tube. Note that a plot for velocity of the air molecules would look the same as for displacement and no separate drawing is required. Make drawings of the fundamental vibrational mode of air molecules in an open and closed tube**.** Also make corresponding drawings of the pressure variations in the tubes.

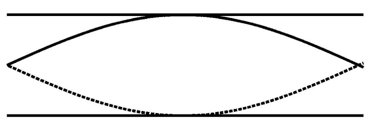
Open Tube:
Air displacement or velocity in the tube



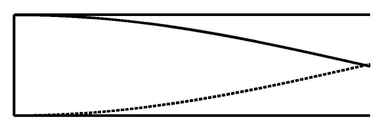
Closed Tube: (Closed at one end)
Air displacement or velocity in the tube



Open Tube:
Pressure variation in the tube



Closed Tube: (Closed at one end)
Pressure variation in the tube



Exercise
Indicate in the top two drawings the displacement nodes and antinodes and in the bottom two drawings the pressure nodes and antinodes. Be sure to understand the differences and relationships between displacement and pressure nodes and antinodes.

Questions
1. Why must the closed end of a tube be a node for the displacement of air molecules?
Answer: ____
2. Consider an "open tube" of length L. For the fundamental mode with the longest wavelength λ and lowest frequency f, what is the relationship between L and λ?
Answer: $L = \lambda/2$
3. Consider a "closed tube" of length L. For the fundamental mode with the longest wavelength λ and lowest frequency f, what is the relationship between L and λ?
Answer: $L = \lambda/4$

Exercise
Show that, for an open tube of length L and a closed tube of length L/2, the fundamental frequencies are the same.

Example of Air Resonances in a 3-Inch Iron Pipe

We investigated the resonances in a vertical pipe made of thick-walled iron. The pipe was set in concrete in the ground. It formerly had been used as the mast of a satellite dish antenna. After decommissioning the antenna the pipe was used for acoustics experiments. It could be closed off with a removable well-fitting plastic cap at the top. When the cap was pulled off rapidly, sound rushed into the pipe through the now open end. The resulting booming sound had a resonance spectrum shown in the figure below.

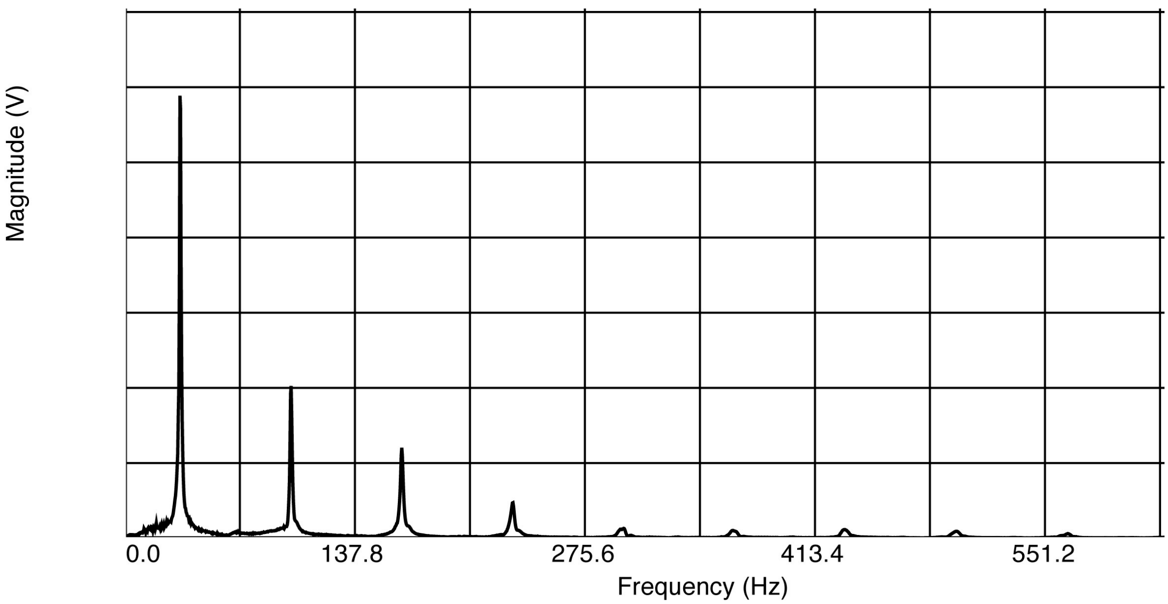


Figure. Example of the resonances in a vertical iron pipe with the top open and the bottom end closed. Only odd harmonics occurred, as expected, whose amplitudes decreased with increasing harmonic number. Nine odd harmonics can be seen. (P.S.: The decrease in the amplitudes of the first 3 odd harmonics resembles that of a square wave.)

Exercise

The inside diameter of the pipe is D = 7.6 cm (Radius R= 3.8 cm).
The measured frequencies of the first 4 odd harmonics are:
$f_1 = 33.2$ Hz
$f_3 = 99.6$ Hz
$f_5 = 166$ Hz
$f_7 = 233$ Hz
For this pipe with one end closed and the other end open, we have for the effective length $L_{eff} = L + 0.6 \cdot R$, and furthermore $f_1 = v/\lambda = v/(4 \cdot L_{eff})$ or $L_{eff} = v/4f_1$ (use v = 346 m/s).
Calculate the effective length L_{eff} of the pipe from the measured frequencies f_1, f_2, or f_3.
From this calculated effective length L_{eff} calculate the actual length L and compare with the measured length of 2.56 m.
Answer: Calculated: L = 2.58 m, measured: L = 2.56.

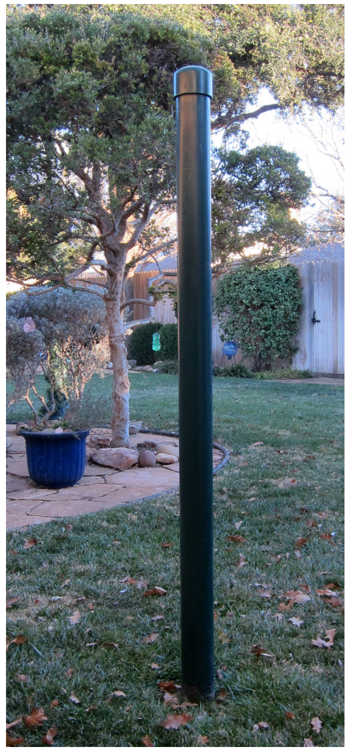


Figure. Top: Satellite mast of 3-inch diameter iron pipe. Bottom: Plug and cap in the top of the mast. Pulling the plug out rapidly produced deep-bass air resonances in the pipe. The lower end of the pipe was deep in the ground and acted acoustically as a closed end.

Demonstrations with Longitudinal Waves

1. With a "longitudinal wave spring", show simulated standing wave modes in a tube. Mount the spring vertically and drive it with a vibrator from the lower end. At certain discrete frequencies, longitudinal/compressional standing waves can be seen.

2. Use a plastic tube ("plosive aerophone") and slap one end with your palm ("closed tube"). Listen to the pitch. Acquire the sound spectrum and note the odd harmonics. Next, slap the tube and immediately remove your hand. The tube acts as a "closed" tube first and "open tube" next in quick succession. Now see odd and even harmonics in the sound spectrum. Also, the pitch from the open tube is about an octave higher. Explain!

3. Eight students play a set of 8 slap tubes tuned to C3, D3, E3, F3, G3, A3, B3, C4.

4. Hold a corrugated plastic tube ("boom whacker") at one end and whirl the other end around in a circle. Listen to the harmonics as you change the speed of rotation. Only discrete harmonics are heard. Note that the fundamental N=1 (approximately A3) is missing. The mode N=2 (A4) is weak. The harmonics N = 3, 4, 5 (E5, A5, C6#) are stronger. By changing the speed of rotation (or swirling around two of these tubes) we can hear musical fifths 3:2, fourths 4:3, and major thirds 5:4.
Question: The harmonic frequencies do not change with the speed of rotation? Why?
Answer: ____

Question: Calculate the fundamental frequency of the plastic tubes for a length of 76 cm and radius of the two openings 1.9 cm and 1.2 cm, respectively. (Use v = 346 m/s.)
Answer: The effective length is $L_{eff} = 76 + 0.6(1.9 + 1.2) = 77.9$ cm = 0.779 m.
The fundamental frequency is $f_1 = v/(2L_{eff}) = 346/(2 \times 0.779) = 222$ Hz ≈ A3.

5. Demonstrate the harmonics from a Native Indian flute, train whistle, organ pipes, etc.

6. Play a didgeridoo. Acquire the sound spectrum and note the odd harmonics. Play a different didgeridoo of the same length and compare the spectrum.
Question: Why do the two didgeridoos not sound quite the same?
Answer: ____

7. Show a custom-built "string-tube device" with a string stretched over a wooden bridge that rests on a plug closing one end of a 4-inch plastic pipe. The bridge divides the string into two segments in the ratio 3:1. The pipe length is tuned to F3 (N = 1). The first existing overtone is C5 (N = 3). When the string is tuned to F3, the sound becomes louder. The odd string harmonics are amplified by the tube resonances.

8. Cut a balloon in half and stretch it over a large plastic pipe. Pinch the mouthpiece and blow into it. Does the sound have a pitch? Is it a simple sound?

9. Show longitudinal waves by cranking the shaft of a mechanical wave maker.