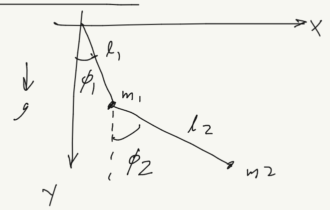
Sec 5, Prob 1:



$$x_1 = l_1 \sin\phi_1$$
$$y_1 = l_1 \cos\phi_1$$

$$x_2 = x_1 + l_2 \sin\phi_2$$
$$y_2 = y_1 + l_2 \cos\phi_2$$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_1 g l_1 \cos\phi_1 - m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$
$$= -(m_1+m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

$$T = \tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\dot{x}_1 = l_1 \dot{\phi}_1 \cos\phi_1 \quad \rightarrow \quad \dot{x}_1^2 = l_1^2 \dot{\phi}_1^2 \cos^2\phi_1$$
$$\dot{y}_1 = -l_1 \dot{\phi}_1 \sin\phi_1 \quad \rightarrow \quad \dot{y}_1^2 = l_1^2 \dot{\phi}_1^2 \sin^2\phi_1$$
$$\dot{x}_1^2 + \dot{y}_1^2 = l_1^2 \dot{\phi}_1^2$$

$$\dot{x}_2 = l_1 \dot{\phi}_1 \cos\phi_1 + l_2 \dot{\phi}_2 \cos\phi_2$$
$$\dot{y}_2 = -l_1 \dot{\phi}_1 \sin\phi_1 - l_2 \dot{\phi}_2 \sin\phi_2$$

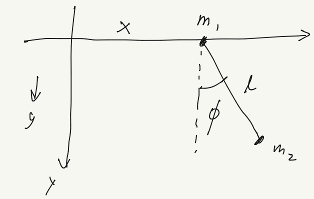
$$\rightarrow \dot{x}_2^2 = l_1^2 \dot{\phi}_1^2 \cos^2\phi_1 + l_2^2 \dot{\phi}_2^2 \cos^2\phi_2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos\phi_1 \cos\phi_2$$
$$\dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 \sin^2\phi_1 + l_2^2 \dot{\phi}_2^2 \sin^2\phi_2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin\phi_1 \sin\phi_2$$

so $\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2)$
$$= l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

Thus,
$$T = \tfrac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \tfrac{1}{2} m_2 l_1^2 \dot{\phi}_1^2 + \tfrac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$
$$= \tfrac{1}{2} (m_1+m_2) l_1^2 \dot{\phi}_1^2 + \tfrac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

$$\rightarrow L = T - U$$
$$= \tfrac{1}{2} (m_1+m_2) l_1^2 \dot{\phi}_1^2 + \tfrac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$
$$+ (m_1+m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2$$

Sec 5, Prob 2:



$$(x_1, y_1) = (x, 0)$$
$$(x_2, y_2) = (x + l\sin\phi,\ l\cos\phi)$$

$$(\dot{x}_1, \dot{y}_1) = (\dot{x}, 0)$$
$$(\dot{x}_2, \dot{y}_2) = (\dot{x} + l\dot{\phi}\cos\phi,\ -l\dot{\phi}\sin\phi)$$

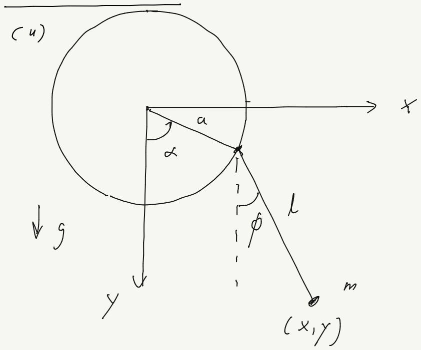
$$T = \tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$
$$= \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 (\dot{x}^2 + l^2 \dot{\phi}^2 \cos^2\phi + 2 l \dot{x} \dot{\phi} \cos\phi$$
$$+ l^2 \dot{\phi}^2 \sin^2\phi)$$
$$= \tfrac{1}{2} (m_1+m_2) \dot{x}^2 + \tfrac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \dot{x} \dot{\phi} \cos\phi$$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_2 g l \cos\phi$$

$$L = T - U$$
$$= \tfrac{1}{2} (m_1+m_2) \dot{x}^2 + \tfrac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \dot{x} \dot{\phi} \cos\phi$$
$$+ m_2 g l \cos\phi$$
$$= \tfrac{1}{2} (m_1+m_2) \dot{x}^2 + \tfrac{1}{2} m_2 (l^2 \dot{\phi}^2 + l \dot{x} \dot{\phi} \cos\phi) + m_2 g l \cos\phi$$

Sec 5, Prob 3:

(a)



$$\alpha = \gamma t$$

$$x = a\sin\alpha + l\sin\phi$$
$$y = a\cos\alpha + l\cos\phi$$

$$U = -mgy$$
$$= \underbrace{-mga\cos\alpha}_{\text{prescribed function of time (ignore)}} - mgl\cos\phi$$
$$= -mgl\cos\phi$$

$$\dot{x} = a\gamma\cos\alpha + l\dot{\phi}\cos\phi$$
$$\dot{y} = -a\gamma\sin\alpha - l\dot{\phi}\sin\phi$$

$$\dot{x}^2 = a^2\gamma^2\cos^2\alpha + l^2\dot{\phi}^2\cos^2\phi + 2al\gamma\dot{\phi}\cos\alpha\cos\phi$$
$$\dot{y}^2 = a^2\gamma^2\sin^2\alpha + l^2\dot{\phi}^2\sin^2\phi + 2al\gamma\dot{\phi}\sin\alpha\sin\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2}m\left[a^2\gamma^2 + l^2\dot{\phi}^2 + 2al\gamma\dot{\phi}\cos(\alpha-\phi)\right]$$
$$= \underbrace{\frac{1}{2}ma^2\gamma^2}_{\text{prescribed function of time (ignore)}} + \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\cos(\gamma t-\phi)$$
$$= \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\cos(\gamma t-\phi)$$

$$L = T - U$$
$$= \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\cos(\gamma t-\phi) + mgl\cos\phi$$

Now:

$$\gamma\dot{\phi}\cos(\gamma t-\phi) = \underbrace{\frac{d}{dt}\left[-\gamma\sin(\gamma t-\phi)\right]}_{\text{can ignore since total time derivative.}} + \gamma^2\cos(\gamma t-\phi)$$

Thus,

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos(\gamma t-\phi) + mgl\cos\phi$$

NOTE:

EOM should be the same for both Lagrangians:

(1st): $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi}$

$$\frac{d}{dt}\left(ml^2\dot{\phi} + mal\gamma\cos(\gamma t-\phi)\right) = mal\gamma\dot{\phi}\sin(\gamma t-\phi) - mgl\sin\phi$$

$$ml^2\ddot{\phi} - mal\gamma^2\sin(\gamma t-\phi) + \cancel{mal\gamma\dot{\phi}\sin(\gamma t-\phi)} = \cancel{mal\gamma\dot{\phi}\sin(\gamma t-\phi)} - mgl\sin\phi$$

$$\rightarrow \ddot{\phi} = \frac{a}{l}\gamma^2\sin(\gamma t-\phi) - \frac{g}{l}\sin\phi$$

agree

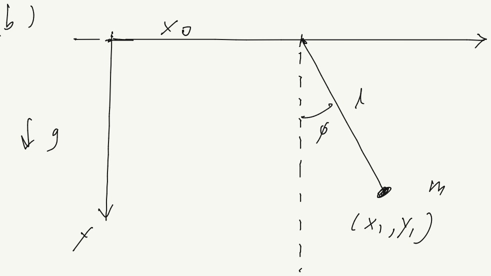
(2nd) $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi}$

$$\frac{d}{dt}\left(ml^2\dot{\phi}\right) = +mal\gamma^2\sin(\gamma t-\phi) - mgl\sin\phi$$

$$ml^2\ddot{\phi} = mal\gamma^2\sin(\gamma t-\phi) - mgl\sin\phi$$

$$\rightarrow \ddot{\phi} = \frac{a}{l}\gamma^2\sin(\gamma t-\phi) - \frac{g}{l}\sin\phi$$

(b)



$x_0 = a\cos\gamma t$

$\dot{x}_0 = -a\gamma\sin\gamma t$

$$x = x_0 + l\sin\phi$$

$$y = l\cos\phi$$

$$U = -mgy = -mgl\cos\phi$$

$$\dot{x} = \dot{x}_0 + l\dot{\phi}\cos\phi = -a\gamma\sin\gamma t + l\dot{\phi}\cos\phi$$

$$\dot{y} = -l\dot{\phi}\sin\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left(a^2\gamma^2\sin^2\gamma t + l^2\dot{\phi}^2\cos^2\phi - 2al\gamma\dot{\phi}\sin\gamma t\cos\phi + l^2\dot{\phi}^2\sin^2\phi\right)$$

$$= \underbrace{\frac{1}{2}ma^2\gamma^2\sin^2\gamma t}_{\text{prescribed function of time (ignore)}} + \frac{1}{2}ml^2\dot{\phi}^2 - mal\underbrace{\gamma\dot{\phi}\sin\gamma t\cos\phi}$$

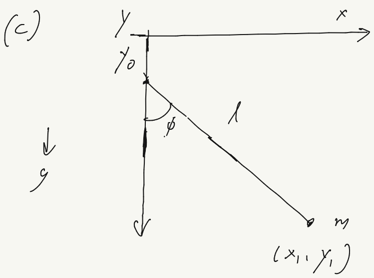
$$= \frac{d}{dt}\left(\gamma\sin\gamma t\sin\phi\right) - \gamma^2\cos\gamma t\sin\phi$$

ignore

$$= \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\sin\phi$$

$$\rightarrow L = T - U = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\sin\phi + mgl\cos\phi$$

(c)



$$y = a\cos\gamma t$$
$$x = l\sin\phi\,, \quad y = y_0 + l\cos\phi$$

$$\dot{x} = l\dot{\phi}\cos\phi$$
$$\dot{y} = \dot{y}_0 + l\dot{\phi}\sin\phi$$
$$= -a\gamma\sin\gamma t - l\dot{\phi}\sin\phi$$

$$\dot{x}^2 = l^2\dot{\phi}^2\cos^2\phi$$
$$\dot{y}^2 = a^2\gamma^2\sin^2\gamma t + l^2\dot{\phi}^2\sin^2\phi + 2al\gamma\dot{\phi}\sin\gamma t\sin\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2}m\left(a^2\gamma^2\sin^2\gamma t + l^2\dot{\phi}^2 + 2al\gamma\dot{\phi}\sin\gamma t\sin\phi\right)$$
$$= \underbrace{\frac{1}{2}ma^2\gamma^2\sin^2\gamma t}_{\text{prescribed fcn of t (ignore)}} + \frac{1}{2}ml^2\dot{\phi}^2 + \underbrace{mal\gamma\dot{\phi}\sin\gamma t\sin\phi}$$

$$= \frac{d}{dt}\left(-\gamma\sin\gamma t\cos\phi\right) + \gamma^2\cos\gamma t\cos\phi$$

total time derivative (ignore)

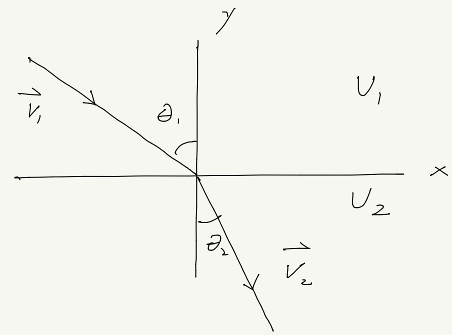
$$\simeq \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\cos\phi$$

$$U = -mgy = -mg(y + l\cos\phi)$$
$$= \underbrace{-mga\cos\gamma t}_{\text{prescribed fcn of t (ignore)}} - mgl\cos\phi$$

thus,

$$L = T - U = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\cos\phi + mgl\cos\phi$$

Sec 7, Prob 1:



- Energy is conserved
- Component of linear momentum in x-direction is also conserved

i) $$E = \frac{1}{2}mv_1^2 + U_1 = \frac{1}{2}mv_2^2 + U_2$$
$$\frac{1}{2}mv_2^2 = \frac{1}{2}mv_1^2 + (U_1 - U_2)$$
$$v_2 = \sqrt{v_1^2 + \frac{2}{m}(U_1 - U_2)}$$
$$\frac{v_2}{v_1} = \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2}mv_1^2}}$$

ii) $$p_x = \cancel{m}v_1\sin\theta_1 = \cancel{m}v_2\sin\theta_2$$
$$\rightarrow \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1}$$
$$= \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2}mv_1^2}}$$

Sec 8, Prob 1:

$$S[q] = \int_{t_1}^{t_2} dt\, L(q, \dot q, t)$$

Let inertial frame K' move with velocity $\vec V$ wrt inertial frame K.

Then:
$$\vec v_a = \vec v_a{}' + \vec V$$
$$\vec r_a = \vec r_a{}' + \vec V \cdot t$$

$$\sum_a m_a \vec r_a{}' = \mu \vec R'$$

Thus,

$$L = \frac{1}{2}\sum_a m_a |\vec v_a|^2 - U(\vec r_1, \vec r_2, \dots, t)$$
$$= \frac{1}{2}\sum_a m_a |\vec v_a{}' + \vec V|^2 - U$$
$$= \frac{1}{2}\sum_a m_a |\vec v_a{}'|^2 + \frac{1}{2}\sum_a m_a V^2 + \left(\sum_a m_a \vec v_a{}'\right)\cdot \vec V - U$$
$$= L' + \vec p\,'\cdot \vec V + \frac{1}{2}\mu V^2$$

$$\rightarrow S = \int_{t_1}^{t_2} dt\, \left(L' + \vec p\,'\cdot \vec V + \frac{1}{2}\mu V^2\right)$$
$$= S' + \vec V \cdot \sum_a m_a \vec r_a{}'\Big|_{t_1}^{t_2} + \frac{1}{2}\mu V^2\Big|_{t_1}^{t_2}$$
$$= S' + \mu \vec V \cdot \left(\vec R'(t_2) - \vec R'(t_1)\right) + \frac{1}{2}\mu V^2 (t_2 - t_1)$$

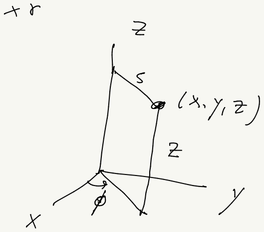
where $\vec R'$ is com of system wrt frame K'

Sec 9, Prob 1:

cylindrical coords (s, ϕ, z):

$$x = s\cos\phi,\quad y = s\sin\phi,\quad z = z$$

$$\vec M = \vec r \times \vec p = m \vec r \times \vec v = m \vec r \times \dot{\vec r}$$



Thus,
$$M_x = m(y\dot z - z\dot y)$$
$$M_y = m(z\dot x - x\dot z)$$
$$M_z = m(x\dot y - y\dot x)$$

$$\dot x = \dot s\cos\phi - s\dot\phi\sin\phi$$
$$\dot y = \dot s\sin\phi + s\dot\phi\cos\phi$$
$$\dot z = \dot z$$

$$\rightarrow M_x = m\left[s\sin\phi\,\dot z - z(\dot s\sin\phi + s\dot\phi\cos\phi)\right]$$
$$= m\left[\sin\phi(s\dot z - z\dot s) - zs\dot\phi\cos\phi\right]$$

$$M_y = m\left[z(\dot s\cos\phi - s\dot\phi\sin\phi) - s\cos\phi\,\dot z\right]$$
$$= m\left[-\cos\phi(s\dot z - z\dot s) - zs\dot\phi\sin\phi\right]$$

$$M_z = m\left[s\cos\phi(\cancel{\dot s\sin\phi} + s\dot\phi\cos\phi) - s\sin\phi(\cancel{\dot s\cos\phi} - s\dot\phi\sin\phi)\right]$$
$$= m s^2 \dot\phi$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$= m^2 \Big[\sin^2\phi\,(s\dot z - z\dot s)^2 + z^2 s^2 \dot\phi^2 \cos^2\phi$$

$$- \cancel{2 z s \dot\phi \cos\phi \sin\phi\,(s\dot z - z\dot s)}$$

$$+ \cos^2\phi\,(s\dot z - z\dot s)^2 + z^2 s^2 \dot\phi^2 \sin^2\phi$$

$$+ \cancel{2 z s \dot\phi \cos\phi \sin\phi\,(s\dot z - z\dot s)}$$

$$+ s^4 \dot\phi^2 \Big]$$

$$= m^2 \left[(s\dot z - z\dot s)^2 + z^2 s^2 \dot\phi^2 + s^4 \dot\phi^2 \right]$$

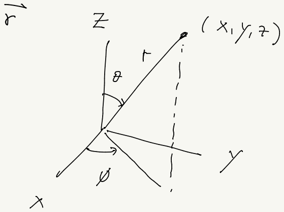
$$= m^2 \left[(s\dot z - z\dot s)^2 + s^2 (z^2 + s^2) \dot\phi^2 \right]$$

Sec 9, Prob 2 :

spherical polar coords (r, θ, ϕ) :

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$\vec M = \vec r \times \vec p = m\vec r \times \vec v = m\vec r \times \dot{\vec r}$$



Thus,
$$M_x = m(y\dot z - z\dot y)$$
$$M_y = m(z\dot x - x\dot z)$$
$$M_z = m(x\dot y - y\dot x)$$

$$\dot x = \dot r \sin\theta\cos\phi + r\dot\theta\cos\theta\cos\phi - r\dot\phi\sin\theta\sin\phi$$

$$\dot y = \dot r \sin\theta\sin\phi + r\dot\theta\cos\theta\sin\phi + r\dot\phi\sin\theta\cos\phi$$

$$\dot z = \dot r\cos\theta - r\dot\theta\sin\theta$$

$$\rightarrow M_x = m\Big[r\sin\theta\sin\phi\,(\cancel{\dot r\cos\theta} - r\dot\theta\sin\theta)$$

$$- r\cos\theta\,(\cancel{\dot r\sin\theta\sin\phi} + r\dot\theta\cos\theta\sin\phi + r\dot\phi\sin\theta\cos\phi)\Big]$$

$$= m\left[-r^2\dot\theta\sin\phi\,(\sin^2\theta + \cos^2\theta) - r^2\dot\phi\sin\theta\cos\theta\cos\phi \right]$$

$$= m\left[-r^2\dot\theta\sin\phi - r^2\dot\phi\sin\theta\cos\theta\cos\phi \right]$$

$$M_y = m\Big[r\cos\theta\,(\cancel{\dot r\sin\theta\cos\phi} + r\dot\theta\cos\theta\cos\phi - r\dot\phi\sin\theta\sin\phi)$$

$$- r\sin\theta\cos\phi\,(\cancel{\dot r\cos\theta} - r\dot\theta\sin\theta)\Big]$$

$$= m\left[r^2\dot\theta\cos\phi - r^2\dot\phi\sin\theta\cos\theta\sin\phi \right]$$

$$M_z = m\left[r\sin\theta\cos\phi\left(\cancel{\dot r\sin\theta\sin\phi} + r\dot\theta\cancel{\cos\theta\sin\phi} + r\dot\phi\sin\theta\cos\phi\right) - r\sin\theta\sin\phi\left(\cancel{\dot r\sin\theta\cos\phi} + r\dot\theta\cancel{\cos\theta\cos\phi} - r\dot\phi\sin\theta\sin\phi\right)\right]$$

$$= m\left[r^2\dot\phi\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right)\right]$$

$$= m r^2\dot\phi\sin^2\theta$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

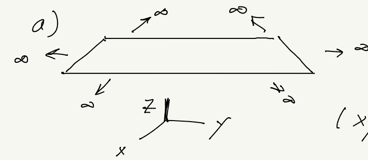
$$= m^2\big[r^4\dot\theta^2\sin^2\phi + r^4\dot\phi^2\sin^2\theta\cos^2\theta\cos^2\phi$$
$$\cancel{+2r^4\dot\theta\dot\phi\sin\theta\cos\theta\sin\phi\cos\phi}$$
$$+ r^4\dot\theta^2\cos^2\phi + r^4\dot\phi^2\sin^2\theta\cos^2\theta\sin^2\phi$$
$$\cancel{-2r^4\dot\theta\dot\phi\sin\theta\cos\theta\sin\phi\cos\phi}$$
$$+ r^4\dot\phi^2\sin^4\theta\big]$$

$$= m^2r^4\left[\dot\theta^2 + \dot\phi^2\left(\sin^2\theta\cos^2\theta + \sin^4\theta\right)\right]$$

$$= m r^4\left[\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right]$$

Sec 9, Prob 3:

a)

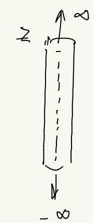


P_x, P_y = const

M_z = const

(xy-plane // to homog. plane)

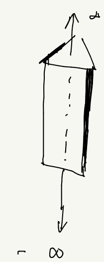
b)



P_z, M_z = const

(z-axis is axis of cylinder)

c)



infinite homog. cylinder

P_z = const

(z-axis is // to edges of prism)

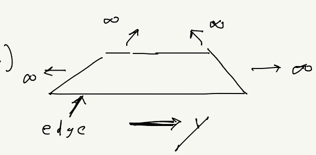
d)

two points

M_z = const

(z-axis passes through the two points)

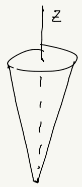
e)



∞ homog. $\frac{1}{2}$ plane

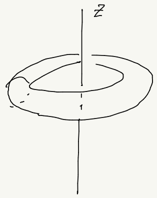
P_y = const

(y axis is // to edge of $\frac{1}{2}$ plane)

f) 

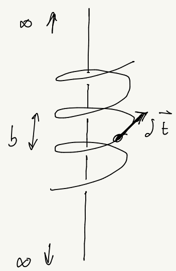
$M_z = \text{const}$

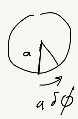
(z-axis is axis of cone)

g) 

$M_z = \text{const}$

(z-axis is axis of torus)

h) 



a = radius of helix

b = height between neighboring coils of helix

$h \equiv b/a$ (pitch of helix)

Lagrangian invariant w.r.t. translation around the helix:

$$\delta\vec{t} = a\,\delta\phi\,\hat{\phi} + \frac{b\,\delta\phi}{2\pi}\,\hat{z}$$

$$= a\,\delta\phi\left[\hat{\phi} + \frac{b/a}{2\pi}\,\hat{z}\right]$$

$$= a\,\delta\phi\left[\hat{\phi} + \frac{h}{2\pi}\,\hat{z}\right]$$

$$\delta L = \frac{\partial L}{\partial\phi}\,\delta\phi + \frac{\partial L}{\partial z}\,\delta z$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right)\delta\phi + \frac{d}{dt}\left(\frac{\partial L}{\partial\dot z}\right)\frac{h}{2\pi}\,\delta\phi$$

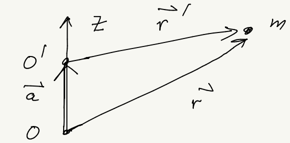
$$= \delta\phi\,\frac{d}{dt}\Big[\underbrace{p_\phi}_{M_z} + p_z\frac{h}{2\pi}\Big] \rightarrow \boxed{M_z + \frac{p_z h}{2\pi} = \text{const}}$$

<u>NOTE:</u> M_z is independent of location of origin on z-axis

$$\vec{M} = \vec{r}\times\vec{p} = m\vec{r}\times\dot{\vec{r}}$$

Change origin by shifting along $\vec{a} = a\hat{z}$

$$\vec{r} = \vec{r}\,' + \vec{a}$$



$$\vec{M} = m\vec{r}\times\dot{\vec{r}}$$

$$= m(\vec{r}\,' + \vec{a})\times\frac{d}{dt}(\vec{r}\,' + \vec{a})$$

$$= m\vec{r}\,'\times\dot{\vec{r}}\,' + m\vec{a}\times\dot{\vec{r}}\,'$$

$$= \vec{M}\,' + \vec{a}\times\vec{p}\,' \qquad (\text{for arbitrary } \vec{a})$$

Thus,

$$M_z = \vec{M}\cdot\hat{z}$$

$$= \left(\vec{M}\,' + \vec{a}\times\vec{p}\,'\right)\cdot\hat{z}$$

$$= M'_z + a\left(\hat{z}\times\vec{p}\,'\right)\cdot\hat{z}$$

$$= M'_z + a\,\cancelto{0}{(\hat{z}\times\hat{z})}\cdot\vec{p}\,'$$

$$= M'_z$$

Sec 10, Prob 1:

same path, different masses, same potential energies

$\rightarrow$ $x' = x$, $m' \neq m$, $U' = U$, $t' \neq t$

$$L = T - U = \frac{1}{2} m \dot{x}^2 - U$$

$$L' = \frac{1}{2} m' \left(\frac{dx}{dt'}\right)^2 - U$$

$$= \frac{1}{2} m' \left(\frac{t}{t'}\right)^2 \dot{x}^2 - U$$

Thus, $L' = L$ $\rightarrow$ $m' \left(\frac{t}{t'}\right)^2 = m$

$$\left(\frac{t'}{t}\right)^2 = \frac{m'}{m}$$

$$\rightarrow \quad \frac{t'}{t} = \sqrt{\frac{m'}{m}}$$

Sec 10, Prob 2:

Same path, Same mass, potential energies differing by a constant factor ($U' = c\,U$)

$\rightarrow$ $x = x'$, $m = m'$, $t' \neq t$

$$L = T - U$$

$$= \frac{1}{2} m \dot{x}^2 - U$$

$$L' = \frac{1}{2} m \left(\frac{dx}{dt'}\right)^2 - U'$$

$$= \frac{1}{2} m \left(\frac{t}{t'}\right)^2 \dot{x}^2 - c\,U$$

Thus, need $\left(\frac{t}{t'}\right)^2 = c$ to get same EOM,

$$\rightarrow \quad \frac{t'}{t} = \sqrt{\frac{1}{c}}$$

$$= \sqrt{\frac{U}{U'}}$$

Sec 40, Prob 1

Hamiltonian for a single particle

$$L = T - U$$
$$= \frac{1}{2} m |\dot{\vec{r}}|^2 - U(\vec{r}, t)$$

Cartesian:

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = p_x/m$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \rightarrow \dot{y} = p_y/m$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$\rightarrow \quad H = \left(\sum_i p_i \dot{q}_i - L \right)_{\dot{q}_i = \dot{q}_i(q, p)}$$

$$= \left(p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(x, y, z, t) \right) \Big|_{\dot{x} = p_x/m, \text{ etc}}$$

$$= \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + U(x, y, z, t)$$

cylindrical: (s, ϕ, z)

$$L = \frac{1}{2} m (\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2) - U(s, \phi, z, t)$$

$$p_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s} \rightarrow \dot{s} = p_s/m$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m s^2 \dot{\phi} \rightarrow \dot{\phi} = p_\phi / m s^2$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$\rightarrow \quad H = \left(p_s \dot{s} + p_\phi \dot{\phi} + p_z \dot{z} - L \right)_{\dot{s} = p_s/m, \text{ etc}}$$

$$= \frac{p_s^2}{m} + \frac{p_\phi^2}{m s^2} + \frac{p_z^2}{m}$$
$$- \frac{1}{2} m \left[\left(\frac{p_s}{m}\right)^2 + s^2 \left(\frac{p_\phi}{m s^2}\right)^2 + \left(\frac{p_z}{m}\right)^2 \right] + U(s, \phi, z, t)$$

$$= \frac{1}{2m} \left(p_s^2 + \frac{p_\phi^2}{s^2} + p_z^2 \right) + U(s, \phi, z, t)$$

Spherical polar: (r, θ, ϕ)

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2) - U(r, \theta, \phi, t)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \longrightarrow \dot{r} = p_r/m$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \rightarrow \dot{\theta} = p_\theta / m r^2$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\theta \, \dot{\phi} \rightarrow \dot{\phi} = p_\phi / m r^2 \sin^2\theta$$

$$\rightarrow H = \left(p_r \dot r + p_\theta \dot\theta + p_\phi \dot\phi - L\right)\Big|_{\dot r = p_r/m,\ \text{etc.}}$$

$$= p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{mr^2}\right) + p_\phi\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)$$

$$- \frac{1}{2}m\left(\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{mr^2}\right)^2 + r^2\sin^2\theta\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)^2\right)$$

$$+ U(r,\theta,\phi,t)$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r,\theta,\phi,t)$$

Sec 40, Prob 2:

For a uniformly rotating ref. frame:

$$L = \frac{1}{2}mv^2 + m\vec v\cdot(\vec\Omega\times\vec r) + \frac{1}{2}m|\vec\Omega\times\vec r|^2 - U$$

Hamiltonian

$$H = \left(\sum_i p_i \dot q_i - L\right)\Big|_{\dot q = \dot q(q,p)}$$

where

$$\vec p = \frac{\partial L}{\partial \vec v}$$

$$= m\vec v + m(\vec\Omega\times\vec r)$$

$$= m[\vec v + \vec\Omega\times\vec r] \text{ —— velocity wrt inertial frame}$$

$$\rightarrow \vec v = \frac{\vec p}{m} - \vec\Omega\times\vec r$$

thus,

$$H = \left(\vec p\cdot\vec v - L\right)\Big|_{\vec v = \vec p/m - \vec\Omega\times\vec r}$$

$$= \vec p\cdot\left(\frac{\vec p}{m} - \vec\Omega\times\vec r\right) - \frac{1}{2}m\left|\frac{\vec p}{m} - \vec\Omega\times\vec r\right|^2$$

$$- m\left(\frac{\vec p}{m} - \vec\Omega\times\vec r\right)\cdot(\vec\Omega\times\vec r) - \frac{1}{2}m|\vec\Omega\times\vec r|^2 + U$$

$$= \frac{p^2}{m} - \cancel{\vec p\cdot(\vec\Omega\times\vec r)} - \frac{1}{2}m\left(\frac{p^2}{m^2} + \cancel{|\vec\Omega\times\vec r|^2} - \cancel{\frac{2}{m}\vec p\cdot(\vec\Omega\times\vec r)}\right)$$

$$- \vec p\cdot(\vec\Omega\times\vec r) + \cancel{m|\vec\Omega\times\vec r|^2} - \cancel{\frac{1}{2}m|\vec\Omega\times\vec r|^2} + U$$

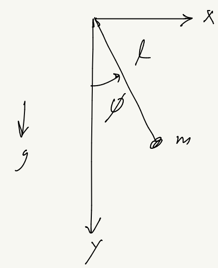
$$H = \frac{p^2}{2m} - \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + U$$

$$= \frac{p^2}{2m} - \vec{\Omega}\cdot(\vec{r}\times\vec{p}) + U$$

$$= \frac{p^2}{2m} - \vec{\Omega}\cdot\vec{M} + U$$

Sec 11, Prob 1:

Simple pendulum:



Cons. of energy:

$$E = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$$
$$= \text{const.}$$

Assume pendulum bob released from rest from $\phi = \phi_0$.

$$\rightarrow E = -mgl\cos\phi_0$$

Thus, $-mgl\cos\phi_0 = \frac{1}{2} ml^2\dot{\phi}^2 - mgl\cos\phi$

$$\frac{d\phi}{dt} \equiv \dot{\phi} = \pm\sqrt{\frac{2g}{l}(\cos\phi - \cos\phi_0)}$$
$$= \pm\sqrt{2}\,\omega_0\sqrt{\cos\phi - \cos\phi_0}$$

where $\omega_0 \equiv \sqrt{\frac{g}{l}}$ (angular freq in small-angle approximation)

Separable differential equation:

$$\sqrt{2}\,\omega_0\int dt = \pm\int\frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$

$\frac{1}{4}$ period for $\phi = \phi_0 \rightarrow \phi = 0$

$$\sqrt{2}\,\omega_0\frac{P}{4} = \int_0^{\phi_0}\frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$

$$\rightarrow \quad P = \frac{1}{\omega_0}\frac{4}{\sqrt{2}}\int_0^{\phi_0}\frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$

substitute:

$$\cos\phi = \cos^2(\phi/2) - \sin^2(\phi/2)$$
$$= 1 - 2\sin^2(\phi/2)$$
$$\cos\phi_0 = 1 - 2\sin^2(\phi_0/2)$$

$$\rightarrow \sqrt{\ } = \sqrt{2}\sqrt{\sin^2(\phi_0/2) - \sin^2(\phi/2)}$$
$$= \sqrt{2}\,\sin(\phi_0/2)\sqrt{1 - \frac{\sin^2(\phi/2)}{\sin^2(\phi_0/2)}}$$
$$= \sqrt{2}\,\sin\left(\frac{\phi_0}{2}\right)\sqrt{1-x^2}$$

where $x \equiv \dfrac{\sin(\phi/2)}{\sin(\phi_0/2)}$

NOTE: $\phi = 0,\ \phi_0 \rightarrow x = 0, 1$

$$dx = \frac{1}{k}\frac{1}{2}\cos\left(\frac{\phi}{2}\right)d\phi$$
$$= \frac{1}{2k}\sqrt{1-\sin^2(\phi/2)}\,d\phi$$
$$= \frac{1}{2k}\sqrt{1-k^2x^2}\,d\phi, \quad k \equiv \sin(\phi_0/2)$$

Thus, $$d\phi = \frac{2k\,dx}{\sqrt{1-k^2x^2}}$$

$$\rightarrow \quad P = \frac{1}{\omega_0}\frac{4}{\cancel{\sqrt{2}}}\int_0^1 \frac{\cancel{2k}\,dx}{\sqrt{1-k^2x^2}}\ \frac{1}{\cancel{\sqrt{2}}\ \cancel{k}\sqrt{1-x^2}}$$
$$= \frac{4}{\omega_0}\int_0^1\frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$
$$= \frac{4}{\omega_0}K(k)$$

complete elliptic integral of the 1st kind.

Expand $K(k)$ keeping 1st non-zero correction:

$$k = \sin\left(\frac{\phi_0}{2}\right) \approx \frac{\phi_0}{2}$$

$$K(k) = \int_0^1\frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$
$$\cong \int_0^1\frac{dx}{\sqrt{1-x^2}}\left(1 + \frac{1}{2}k^2x^2\right)$$
$$= \int_0^1\frac{dx}{\sqrt{1-x^2}} + \frac{1}{2}k^2\int_0^1\frac{dx\ x^2}{\sqrt{1-x^2}}$$
$$= \sin^{-1}(1) + \frac{1}{2}k^2\int_0^1\frac{dx\ x^2}{\sqrt{1-x^2}}$$

Now. $\sin^{-1}(1) = \pi/2$

$$\int_0^1 dx \, \frac{x^2}{\sqrt{1-x^2}} = \int_0^{\pi/2} \frac{\cancel{\cos\theta}\, d\theta \; \sin^2\theta}{\cancel{\sqrt{1-\sin^2\theta}}}$$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$$= \int_0^{\pi/2} \sin^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}(1-\cos 2\theta)$$

(Using $\cos 2\theta = 1 - 2\sin^2\theta \rightarrow \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$)

$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{2}\cancel{\sin 2\theta}\Big|_0^{\pi/2}\right)$$

$$= \frac{\pi}{4}$$

Thus,

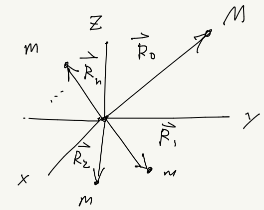
$$K(k) \approx \frac{\pi}{2} + \frac{1}{2}k^2\frac{\pi}{4}$$

$$= \frac{\pi}{2}\left(1 + \frac{1}{4}k^2\right) \qquad k \approx \frac{\phi_0}{2}$$

$$= \frac{\pi}{2}\left(1 + \frac{1}{16}\phi_0^2\right)$$

$$\rightarrow P = \frac{4}{\omega_0}K(k) \cong \frac{2\pi}{\omega_0}\left(1 + \frac{1}{16}\phi_0^2\right)$$

Sec 13, Prob 1:



$\vec{R}_0$: position vector for mass M

$\vec{R}_i$, $i = 1, 2, \dots, n$: position vectors for n masses all with mass m

COM frame: $\vec{0} = M\vec{R}_0 + m\sum_i \vec{R}_i$

Relative position vectors:

$$\vec{r}_i \equiv \vec{R}_i - \vec{R}_0$$

Thus,

$$\vec{0} = M\vec{R}_0 + m\sum_i\left(\vec{R}_0 + \vec{r}_i\right)$$

$$= \underbrace{(M + nm)}_{\text{total mass } \mu}\vec{R}_0 + m\sum_i \vec{r}_i$$

$$\rightarrow \vec{R}_0 = -\frac{m}{\mu}\sum_i \vec{r}_i$$

Potential energy:

$$U = U\left(|\vec{R}_1 - \vec{R}_2|, \dots, |\vec{R}_i - \vec{R}_0|, \dots, |\vec{R}_n - \vec{R}_0|\right)$$

$$= \underbrace{U\left(|\vec{r}_1 - \vec{r}_2|, \dots, |\vec{r}_i|, \dots, |\vec{r}_n|\right)}_{\text{depends only on the relative position vectors } \vec{r}_1, \vec{r}_2, \dots, \vec{r}_n}$$

Kinetic energy:

$$T = \frac{1}{2} M |\dot{\vec{R}}_0|^2 + \frac{1}{2} m \sum_i |\dot{\vec{R}}_i|^2$$

Now: $\dot{\vec{R}}_i = \dot{\vec{r}}_i + \dot{\vec{R}}_0$

$$\rightarrow |\dot{\vec{R}}_i|^2 = |\dot{\vec{r}}_i|^2 + |\dot{\vec{R}}_0|^2 + 2 \dot{\vec{r}}_i \cdot \dot{\vec{R}}_0$$

and $\dot{\vec{R}}_0 = -\frac{m}{\mu} \sum_i \dot{\vec{r}}_i$

$$\rightarrow |\dot{\vec{R}}_0|^2 = \frac{m^2}{\mu^2} \left|\sum_i \dot{\vec{r}}_i\right|^2$$

thus,

$$T = \frac{1}{2} \frac{M m^2}{\mu^2} \left|\sum_i \dot{\vec{r}}_i\right|^2 + \frac{1}{2} m \sum_i |\dot{\vec{r}}_i|^2 + \frac{1}{2} m n |\dot{\vec{R}}_0|^2 + m \sum_i \dot{\vec{r}}_i \cdot \dot{\vec{R}}_0$$

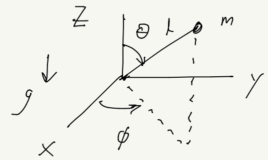
$$= \frac{1}{2} m \sum_i |\dot{\vec{r}}_i|^2 + \frac{1}{2} \frac{M m^2}{\mu^2} \left|\sum_i \dot{\vec{r}}_i\right|^2 + \frac{1}{2} m n \frac{m^2}{\mu^2} \left|\sum_i \dot{\vec{r}}_i\right|^2 - \frac{m^2}{\mu} \left|\sum_i \dot{\vec{r}}_i\right|^2$$

$$= \frac{1}{2} m \sum_i |\dot{\vec{r}}_i|^2 + \frac{1}{2} \frac{m^2}{\mu^2} \left|\sum_i \dot{\vec{r}}_i\right|^2 (\underbrace{M + mn}_{\mu} - 2\mu)$$

$$= \frac{1}{2} m \sum_i |\dot{\vec{r}}_i|^2 - \frac{1}{2} \frac{m^2}{\mu} \left|\sum_i \dot{\vec{r}}_i\right|^2$$

$$\rightarrow \boxed{L = \frac{1}{2} m \sum_i |\dot{\vec{r}}_i|^2 - \frac{1}{2} \frac{m^2}{\mu} \left|\sum_i \dot{\vec{r}}_i\right|^2 - U(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n)}$$

Sec 14, Prob 1



$$x = l \sin\theta \cos\phi$$
$$y = l \sin\theta \sin\phi$$
$$z = l \cos\theta$$

$$U = mgz = mgl\cos\theta$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m (l^2 \dot{\theta}^2 + l^2 \sin^2\theta \, \dot{\phi}^2)$$

$$L = T - U = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2) - mgl\cos\theta$$

No explicit t dependence or ϕ dependence

$$\rightarrow E \equiv T + U = \text{const}, \quad M_z \equiv p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \text{const}$$

$$E = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2) + mgl\cos\theta$$

$$M_z = m l^2 \sin^2\theta \, \dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{m l^2 \sin^2\theta}$$

Thus,

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m l^2 \sin^2\theta \left(\frac{M_z^2}{m^2 l^4 \sin^4\theta}\right) + mgl\cos\theta$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 + \underbrace{\frac{M_z^2}{2 m l^2 \sin^2\theta} + mgl\cos\theta}_{\text{effective potential} \equiv U_{eff}(\theta)}$$

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + U_{eff}(\theta)$$

$$\rightarrow \dot{\theta} = \pm \sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}$$

so $$\frac{d\theta}{\pm\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}} = dt$$

$$\rightarrow \boxed{t = \int \frac{d\theta}{\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}} + \text{const}}$$

$$\longrightarrow t = t(\theta)$$

Trajectory:

To find $\theta = \theta(\phi)$ write

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d\theta}{d\phi}\dot{\phi} = \frac{d\theta}{d\phi}\frac{M_z}{ml^2 \sin^2\theta}$$

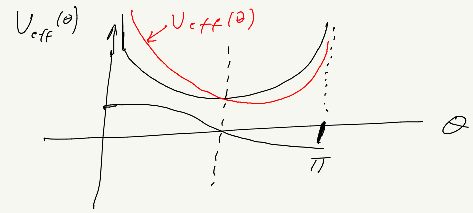
so $$\frac{d\theta}{d\phi}\frac{M_z}{ml^2 \sin^2\theta} = \pm\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}$$

$$\frac{d\theta / \sin^2\theta}{\pm\frac{ml^2}{M_z}\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}} = d\phi$$

$$\rightarrow \boxed{\phi = \int \frac{d\theta/\sin^2\theta}{\sqrt{\frac{2ml^2}{M_z^2}\left(E - U_{eff}(\theta)\right)}} + \text{const}}$$

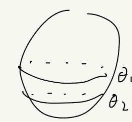
$$U_{eff}(\theta) = \frac{M_z^2}{2ml^2\sin^2\theta} + mgl\cos\theta$$

$$= \frac{M_z^2}{2ml^2(1-\cos^2\theta)} + mgl\cos\theta$$



Turning points: $\theta = \theta_1, \theta_2$ for which

$$E - U_{eff}(\theta) = 0$$



$$E = U_{eff}(\theta)$$

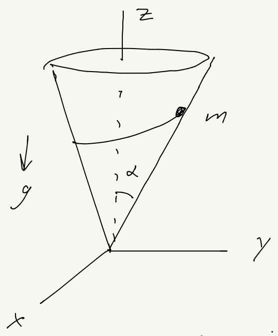
$$= \frac{M_z^2}{2ml^2(1-\cos^2\theta)} + mgl\cos\theta$$

$$2ml^2E(1-\cos^2\theta) = M_z^2 + 2m^2gl^3\cos\theta(1-\cos^2\theta)$$

$$2ml^2E - 2ml^2E\cos^2\theta = M_z^2 + 2m^2gl^3\cos\theta - 2m^2gl^3\cos^3\theta$$

$$\rightarrow \underbrace{\left(2ml^2E - M_z^2\right) - 2m^2gl^3\cos\theta - 2ml^2E\cos^2\theta + 2m^2gl^3\cos^3\theta = 0}_{\text{Cubic equation for } \cos\theta}$$

Sec 14, Prob 2



spherical coords: (r, θ, ϕ)

$$\theta = \alpha$$

$$\rightarrow \quad x = r \sin\alpha \cos\phi, \quad y = r \sin\alpha \sin\phi, \quad z = r\cos\alpha$$

$$T = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$= \tfrac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\, \dot{\phi}^2)$$
$$= \tfrac{1}{2} m (\dot{r}^2 + r^2 \sin^2\alpha\, \dot{\phi}^2)$$

$$U = mgz = mgr\cos\alpha$$

$$L = T - U$$

No explicit t, ϕ dependence $\rightarrow$

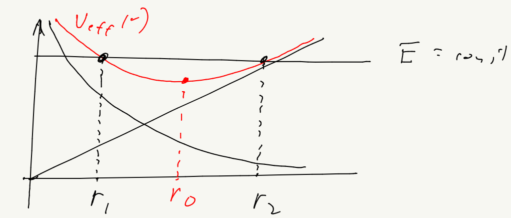
$$E = T + U = \text{const}$$

$$M_z \equiv p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\alpha\, \dot{\phi} = \text{const}$$

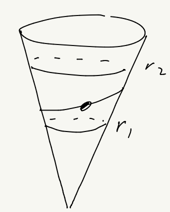
$$\rightarrow \quad \dot{\phi} = \frac{M_z}{m r^2 \sin^2\alpha}$$

$$E = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m r^2 \sin^2\alpha \left(\frac{M_z}{m r^2 \sin^2\alpha}\right)^2 + mgr\cos\alpha$$
$$= \tfrac{1}{2} m \dot{r}^2 + \frac{M_z^2}{2 m r^2 \sin^2\alpha} + mgr\cos\alpha$$
$$= \tfrac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = \frac{M_z^2}{2 m r^2 \sin^2\alpha} + mgr\cos\alpha$$



Turning points: $r = r_1, r_2$ when $E = U_{eff}(r)$



$$E = \frac{M_z^2}{2 m r^2 \sin^2\alpha} + mgr\cos\alpha$$

$$2mE r^2 \sin^2\alpha = M_z^2 + 2 m^2 g r^3 \sin^2\alpha \cos\alpha$$

$$\underbrace{0 = M_z^2 - 2mEr^2\sin^2\alpha - 2m^2 g r^3 \sin^2\alpha\cos\alpha}_{\text{cubic equation for } r}$$

$$r_1 \le r \le r_2$$

Integrals for $t = t(r)$, $\phi = \phi(r)$:

$$E = \tfrac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

$$\pm\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)} = \frac{dr}{dt}$$

$$\rightarrow \quad \boxed{t = \pm\int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}} + \text{const}}$$

Orbit equation:

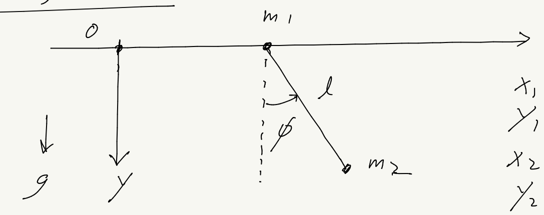
$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{d}{d\phi}\,\frac{M_z}{mr^2\sin^2\alpha}$$

so
$$\frac{dr}{d\phi}\,\frac{M_z}{mr^2\sin^2\alpha} = \pm\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}$$

$$\frac{dr/r^2}{\frac{m\sin^2\alpha}{M_z}\sqrt{\quad}} = \pm\, d\phi$$

$$\rightarrow \boxed{\phi = \frac{\pm M_z}{\sin^2\alpha}\int \frac{dr/r^2}{\sqrt{2m\left(E - U_{eff}(r)\right)}} + \text{const}}$$

Sec 14, Prob 3:



$x_1 = x$
$y_1 = 0$
$x_2 = x + l\sin\phi$
$y_2 = l\cos\phi$

$$U = -mgy_2$$
$$= -mgl\cos\phi$$

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$
$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left[\left(\dot{x} + l\dot{\phi}\cos\phi\right)^2 + l^2\dot{\phi}^2\sin^2\phi\right]$$
$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left[\dot{x}^2 + 2l\dot{x}\dot{\phi}\cos\phi + l^2\dot{\phi}^2\cos^2\phi + l^2\dot{\phi}^2\sin^2\phi\right]$$
$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\phi}^2 + m_2l\dot{x}\dot{\phi}\cos\phi$$

$$L = T - U$$
$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\phi}^2 + m_2l\dot{x}\dot{\phi}\cos\phi + mgl\cos\phi$$

No explicit t, x dependence $\rightarrow$

$$E = T + U = \text{const}$$
$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2l\dot{\phi}\cos\phi = \text{const}$$

Since $p_x = \text{const}$, we can work in the x-com frame where $x_{com} = 0$ and $p_x = 0$

In this frame:

$$0 = x_{com} = \frac{m_1 x + m_2 (x + l\sin\phi)}{m_1 + m_2}$$

$$= \frac{(m_1+m_2)x + m_2 l \sin\phi}{m_1+m_2}$$

$$= x + \left(\frac{m_2}{m_1+m_2}\right) l \sin\phi$$

$$\rightarrow \boxed{x = -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi}$$

Thus, $\dot{x} = -\left(\frac{m_2}{m_1+m_2}\right) l \dot{\phi} \cos\phi$

$$\rightarrow E = \frac{1}{2}(m_1+m_2)\frac{m_2^2}{(m_1+m_2)^2} l^2 \dot{\phi}^2 \cos^2\phi + \frac{1}{2} m_2 l^2 \dot{\phi}^2 - \frac{m_2^2 l^2}{m_1+m_2}\dot{\phi}^2 \cos^2\phi - m_2 g l \cos\phi$$

$$= \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left(1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right) - m_2 g l \cos\phi$$

$$= \frac{1}{2}\left(\frac{m_2}{m_1+m_2}\right) l^2 \dot{\phi}^2 \left(m_1 + m_2 \sin^2\phi\right) - m_2 g l \cos\phi$$

Solve for $\dot{\phi}$:

$$E + m_2 g l \cos\phi = \frac{1}{2}\left(\frac{m_2}{m_1+m_2}\right) l^2 \dot{\phi}^2 \left(m_1 + m_2 \sin^2\phi\right)$$

$$\pm\sqrt{\frac{\frac{2}{l^2}\left(\frac{m_1+m_2}{m_2}\right)\left(E + m_2 g l \cos\phi\right)}{m_1 + m_2 \sin^2\phi}} = \frac{d\phi}{dt}$$

$$\rightarrow \boxed{t = \pm\sqrt{\frac{l^2}{2}\left(\frac{m_2}{m_1+m_2}\right)} \int d\phi \sqrt{\frac{m_1 + m_2 \sin^2\phi}{E + m_2 g l \cos\phi}} + \text{const}}$$

NOTE: In the x-com frame $x = -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi$

$$x_2 = x + l\sin\phi$$
$$= -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi + l\sin\phi$$
$$= \left(\frac{m_1}{m_1+m_2}\right) l \sin\phi$$
$$\equiv b \sin\phi$$

$$y_2 = l\cos\phi \equiv a\cos\phi$$

Thus,

$$\left(\frac{x_2}{b}\right)^2 + \left(\frac{y_2}{a}\right)^2 = \sin^2\phi + \cos^2\phi = 1$$

So m_2 traces out an ellipse in the x-com frame.

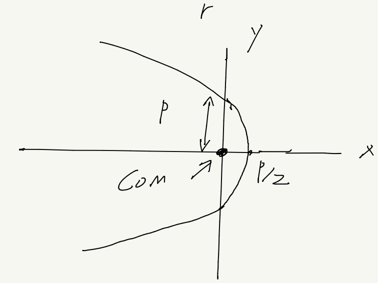
Sec 15, Prob 1:

$E = 0$ (parabola), $e = 1$

$U = -\alpha/r$

$$\left.\begin{aligned} \frac{p}{r} &= 1 + e\cos\phi \\ &= 1 + \cos\phi \end{aligned}\right\}$$

$\phi = 0 \rightarrow \frac{p}{r} = 2 \rightarrow r_{min} = \frac{p}{2}$



Time dependence:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{2}{m}\left(\cancel{E}^{\,0} + \frac{\alpha}{r}\right) - \frac{M^2}{m^2r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{M^2}{m^2r^2}}} + \text{const}$$

Recall: $p = \frac{M^2}{m\alpha} \rightarrow M^2 = m\alpha p$

$$\rightarrow t = \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{m\alpha p}{m^2 r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{\alpha}{m}}\sqrt{\frac{2}{r} - \frac{p}{r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha p}} \int \frac{r\,dr}{\sqrt{\frac{2r}{p} - 1}} + \text{const}$$

Let: $\frac{2r}{p} - 1 = \xi^2$ $\left(\xi = 0 \rightarrow \frac{2r}{p} - 1 = 0 \rightarrow r = p/2 \right)$

Thus, $\frac{\cancel{2}dr}{p} = \cancel{2}\xi\,d\xi$

$dr = p\xi\,d\xi$

Also: $\frac{2r}{p} = 1 + \xi^2 \rightarrow r = \frac{p}{2}(1+\xi^2)$

$$\rightarrow t = \sqrt{\frac{m}{\alpha p}} \int \frac{\frac{p}{2}(1+\xi^2)\,p\cancel{\xi}\,d\xi}{\cancel{\sqrt{\xi^2}}} + \text{const}$$

$$= \frac{p^2}{2}\sqrt{\frac{m}{\alpha p}} \int (1+\xi^2)\,d\xi + \text{const}$$

so $t = \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}}\left(\xi + \frac{\xi^3}{3}\right) + \text{const}$

choose const $= 0$ so that $t = 0 \leftrightarrow \xi = 0$

Thus,
$$\boxed{\begin{aligned} r &= \frac{p}{2}(1+\xi^2) \\ t &= \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}}\left(\xi + \frac{1}{3}\xi^3\right) \end{aligned}}$$

Recall: $\frac{p}{r} = 1 + \cos\phi$

so $1 + \cos\phi = \frac{2}{1+\xi^2}$

$$\boxed{\cos\phi = \frac{2}{1+\xi^2} - 1 = \frac{1-\xi^2}{1+\xi^2}}$$

Cartesian: $x = r\cos\phi$

$$= \frac{p}{2}\cancel{(1+\xi^2)}\left(\frac{1-\xi^2}{\cancel{1+\xi^2}}\right)$$

$$= \frac{p}{2}(1-\xi^2)$$

$$y = r\sin\phi = \frac{p}{2}(1+\xi^2)\sqrt{1 - \left(\frac{1-\xi^2}{1+\xi^2}\right)^2}$$

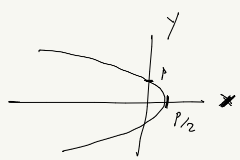
$$y = \frac{p}{2}\cancel{(1+\xi^2)}\,\frac{1}{\cancel{(1+\xi^2)}}\sqrt{(1+\xi^2)^2 - (1-\xi^2)^2}$$

$$= \frac{p}{2}\sqrt{\cancel{1}+\cancel{\xi^4}+2\xi^2 - (\cancel{1}+\cancel{\xi^4}-2\xi^2)}$$

$$= \frac{p}{2}\sqrt{4\xi^2}$$

$$= p\xi$$

Thus, $$\boxed{\begin{aligned} x &= \frac{p}{2}(1-\xi^2) \\ y &= p\xi \end{aligned}}$$



NOTE:

$$x = \frac{p}{2} - \frac{p}{2}\xi^2$$

$$= \frac{p}{2} - \frac{y^2}{2p}$$

parabola

$x = y^2$

$\underline{y=0}$: $x = p/2$

$\underline{x=0}$: $y = \pm p$

<u>Sec 15, Prob 3:</u>

$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}} \qquad (14.10)$$

Consider a small perturbation $\delta U(r)$ to the potential energy:

$$\rightarrow \Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U-\delta U) - M^2/r^2}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - M^2/r^2 - 2m\delta U}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}\sqrt{1 - \frac{2m\delta U}{2m(E-U) - M^2/r^2}}}$$

$$\approx 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}}\left(1 + \frac{m\delta U}{2m(E-U) - M^2/r^2}\right)$$

$$= \underbrace{2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}}}_{\Delta\phi_0} + \underbrace{2\int_{r_{min}}^{r_{max}} \frac{Mm\,\delta U\,dr/r^2}{\left(2m(E-U) - M^2/r^2\right)^{3/2}}}_{\delta\phi}$$

Now: $\dfrac{\partial}{\partial M}\left(\dfrac{1}{\sqrt{2m(E-U)-M^2/r^2}}\right) = -\dfrac{1}{2}\dfrac{(-2M/r^2)}{(2m(E-U)-M^2/r^2)^{3/2}}$

$$= \frac{M/r^2}{(2m(E-U)-M^2/r^2)^{3/2}}$$

Thus,

$$\delta\phi = \frac{\partial}{\partial M}\left(\int_{r_{min}}^{r_{max}} \frac{2m\,\delta U\,dr}{\sqrt{2m(E-U)-M^2/r^2}}\right)$$

Consider the case $U(r) = -\alpha/r$

Then: $\Delta\phi_0 = 2\pi$ (since a bound orbit is an ellipse, which is closed)

Also: $\delta\phi = \dfrac{\partial}{\partial M}\left(\displaystyle\int_{r_{min}}^{r_{max}} \frac{2m\,\delta U\,dr}{\sqrt{2m(E+\alpha/r)-M^2/r^2}}\right)$

In the integral, we can use the solution for the unperturbed motion:

$$\frac{p}{r} = 1+e\cos\phi \rightarrow -\frac{p}{r^2}dr = -e\sin\phi\,d\phi$$

$$\text{so} \quad dr = \frac{e}{p}r^2\sin\phi\,d\phi$$

$$r = r_{min}, r_{max} \leftrightarrow \phi = 0, \pi$$

$$\sqrt{} = \sqrt{2m\left(E+\frac{\alpha}{r}\right) - \frac{M^2}{r^2}}$$

$$= \sqrt{2mE + \frac{2m\alpha(1+e\cos\phi)}{p} - M^2\frac{(1+e\cos\phi)^2}{p^2}}$$

$$= \frac{1}{p}\sqrt{2mEp^2 + 2m\alpha p(1+e\cos\phi) - M^2(1+e\cos\phi)^2}$$

Recall: $p = \dfrac{M^2}{m\alpha}$

$$e = \sqrt{1+\frac{2EM^2}{m\alpha^2}}$$

Thus, $2mEp^2 = 2mE\dfrac{M^4}{m^2\alpha^2} = \left(\dfrac{2EM^2}{m\alpha^2}\right)M^2$

$$2m\alpha p = 2M^2$$

$$\rightarrow \sqrt{} = \frac{m\alpha}{M^2}\sqrt{\left(\frac{2EM^2}{m\alpha^2}\right)M^2 + 2M^2(\cancel{1}+\cancel{e\cos\phi}) - M^2(\cancel{1}+e^2\cos^2\phi + \cancel{2e\cos\phi})}$$

$$= \frac{m\alpha}{M}\sqrt{\underbrace{\frac{2EM^2}{m\alpha^2}+1}_{e^2} - e^2\cos^2\phi}$$

$$= \frac{m\alpha e}{M}\sin\phi$$

Thus,

$$\delta\phi = \frac{\partial}{\partial M}\left(\int_0^\pi \frac{2\cancel{m}\,\delta U\, r^2\, \frac{\cancel{e}}{p}\,\cancel{\sin\phi}\; d\phi}{\frac{\cancel{m}\alpha\cancel{e}}{M}\,\cancel{\sin\phi}}\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2M}{\alpha p}\int_0^\pi r^2\,\delta U\, d\phi\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m}{M}\int_0^\pi r^2\,\delta U\, d\phi\right) \qquad \left(\text{using } p = \frac{M^2}{m\alpha}\right)$$

where $\frac{p}{r} = 1 + e\cos\phi$

(a) $\delta U = \frac{\beta}{r^2}$

$$\rightarrow \quad \delta\phi = \frac{\partial}{\partial M}\left(\frac{2m}{M}\int_0^\pi \cancel{r^2}\,\frac{\beta}{\cancel{r^2}}\, d\phi\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m}{M}\,\beta\pi\right)$$

$$= -\frac{2\pi m\beta}{M^2} \quad \boxed{= \frac{-2\pi\beta}{\alpha p}}$$

(b) $\delta U = \frac{\gamma}{r^3}$

$$\rightarrow \quad \delta\phi = \frac{\partial}{\partial M}\left(\frac{2m}{M}\int_0^\pi r^2\,\frac{\gamma}{r^3}\, d\phi\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m\gamma}{M}\int_0^\pi \frac{d\phi}{r}\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m\gamma}{M}\,\frac{1}{p}\int_0^\pi (1 + e\cos\phi)\, d\phi\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m\gamma}{Mp}\left[\pi + \underbrace{e\sin\phi\Big|_0^\pi}_{0}\right]\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2\pi m\gamma}{Mp}\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2\pi m^2\gamma\alpha}{M^3}\right) \longleftarrow \text{using } p = \frac{M^2}{m\alpha}$$

$$= -\frac{6\pi m^2\gamma\alpha}{M^4}$$

$$= \frac{-6\pi\cancel{m^2}\gamma\alpha}{p^2\cancel{m^2}\alpha^2}$$

$$= \boxed{\frac{-6\pi\gamma}{p^2\alpha}}$$