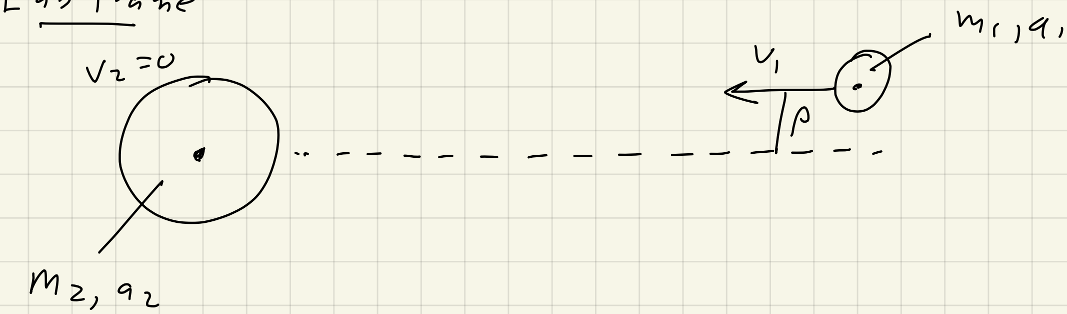
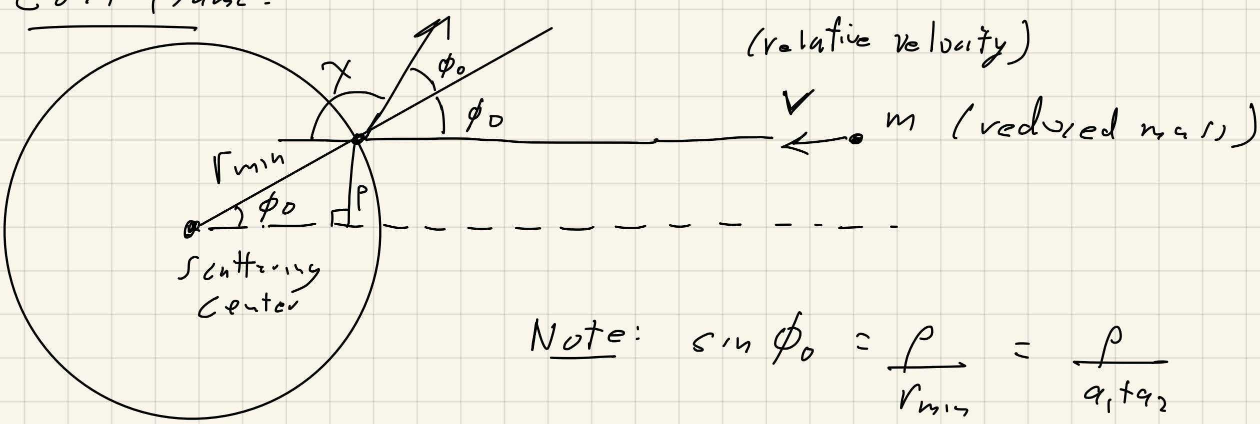
11 oct 2022

Scattering problem: 2 hard spheres, different masses and radii

Lab frame



COM frame:



Note: $\sin\phi_0 = \dfrac{\rho}{r_{min}} = \dfrac{\rho}{a_1+a_2}$

$r_{min} = a_1 + a_2$

$\chi + 2\phi_0 = \pi$

$\phi_0 = \dfrac{\pi}{2} - \dfrac{\chi}{2}$

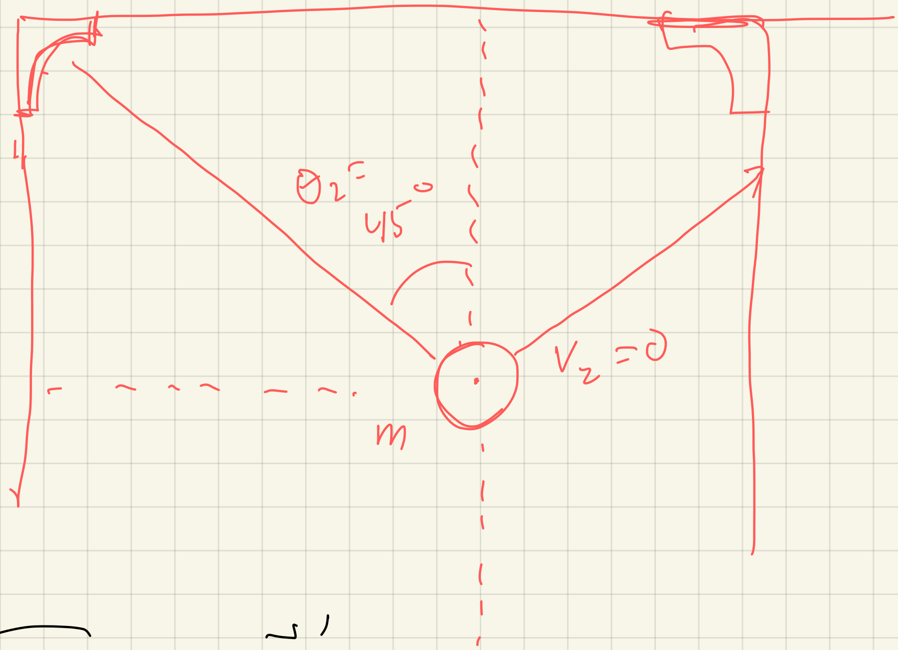
$\sin\left(\dfrac{\pi}{2} - \dfrac{\chi}{2}\right) = \dfrac{\rho}{a_1+a_2}$

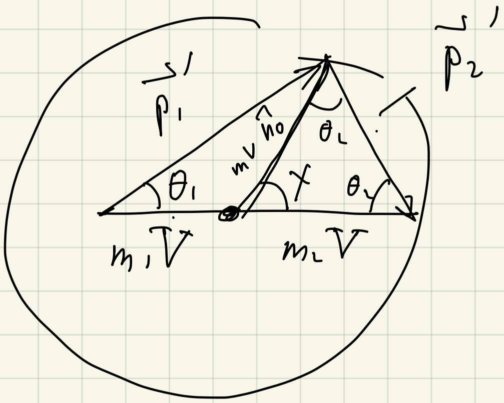
$\cos\left(\dfrac{\chi}{2}\right) = \dfrac{\rho}{a_1+a_2}$

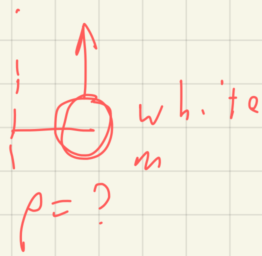
$\rightarrow \boxed{\rho = (a_1+a_2)\cos\left(\dfrac{\chi}{2}\right)}$

$\chi + 2\theta_2 = \pi$ (always, independent of the masses)

thus, $\boxed{\theta_2 = \dfrac{\pi}{2} - \dfrac{\chi}{2}}$

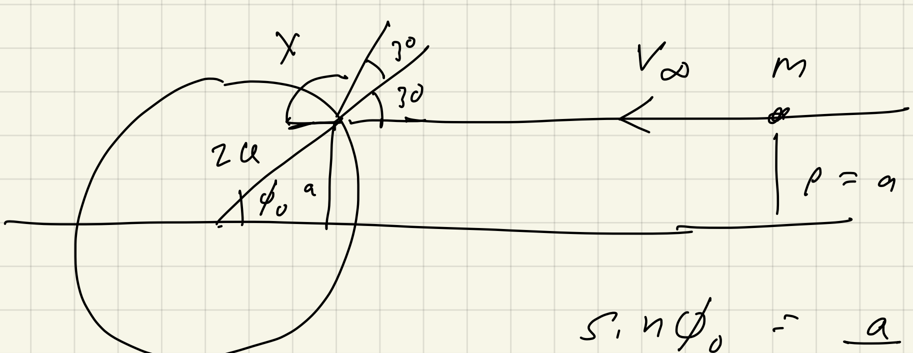






$$\boxed{\tan\theta_1 = \frac{m_2 \sin\chi}{m_2 + m_1 \cos\chi}}$$

$$2\theta_2 + \chi = \pi \longrightarrow \theta_2 = \frac{\pi}{2} - \frac{\chi}{2}$$
$$= 90 - 60$$
$$= \boxed{30^\circ}$$

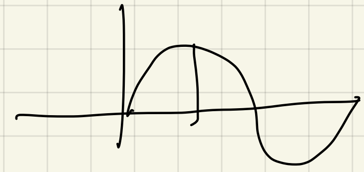


$$\sin\phi_0 = \frac{a}{2a} = \frac{1}{2}$$

$$2\phi_0 + \chi = \pi$$
$$\chi = |\pi - 2\phi_0|$$

$$\boxed{\phi_0 = 30^\circ} \rightarrow \boxed{\chi = 120^\circ}$$

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}$$



$\underline{m_1 = m_2:}$

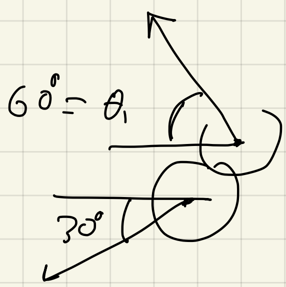
$$\tan\theta_1 = \frac{\cancel{m}\sin\chi}{\cancel{m} + \cancel{m}\cos\chi}$$

$$= \frac{\sin(120^\circ)}{1 + \cos(120^\circ)}$$

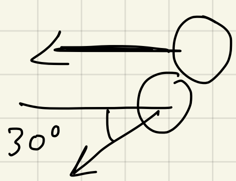
$$= \frac{\sqrt{3}/2}{1 - \frac{1}{2}}$$

$$= \sqrt{3}$$

$$\boxed{\theta_1 = 60^\circ}$$

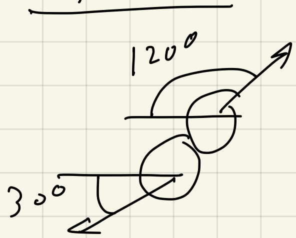


$\underline{m_1 >> m_2:}$ $\tan\theta_1 \approx 0 \rightarrow \theta_1 \approx 0$

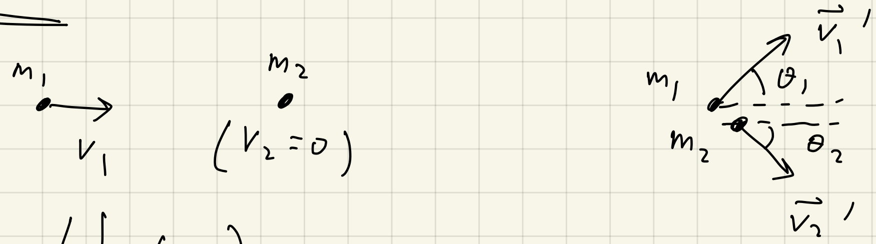


$\underline{m_1 << m_2}$ $\tan\theta_1 = \tan\chi$

$(\theta_1 \cancel{= \chi} = 120^\circ)$



Problem: Elastic collision



(before) (after)

[Lab Frame]

Suppose $m_1 = 1\,kg$, $m_2 = 2\,kg$, $v_1 = 1\,m/s$, $v_2 = 0$

$\theta_2 = 60°$

Determine:

v_1' $\left(= \frac{\sqrt{7}}{3}\ m/s \right)$

v_2' $\left(= \frac{1}{3}\ m/s \right)$

θ_1 $(\approx 41°)$

Solution: Use conservation of linear momentum and KE

1) $\cancel{\tfrac{1}{2}} m_1 v_1^2 = \cancel{\tfrac{1}{2}} m_1 (v_1')^2 + \cancel{\tfrac{1}{2}} m_2 (v_2')^2$

$$m_1 v_1^2 = m_1 (v_1')^2 + m_2 (v_2')^2$$

2) $m_1 v_1 = m_1 v_1' \cos\theta_1 + m_2 v_2' \cos\theta_2$

3) $0 = m_1 v_1' \sin\theta_1 - m_2 v_2' \sin\theta_2$

3 equations, 3 unknowns (v_1', v_2', θ_1)

(not simple to solve!!)

Go to COM Frame:

Relative velocity vector $\quad v = v' \quad$ where $\quad \vec{v} = \vec{v}_1 - \cancel{\vec{v}_2}^{\,0} = v_1$

so $\quad v = v' = v_1$

COM velocity $\quad \vec{V} = \dfrac{m_1 \vec{v}_1}{m_1 + m_2}$

Velocities wrt COM Frame

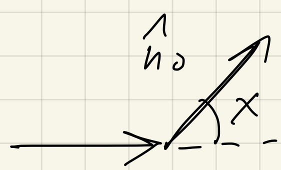
$$\vec{v}_{10} = \left(\frac{m_2}{m_1+m_2}\right)\vec{v}, \qquad \vec{v}_{20} = \left(\frac{-m_1}{m_1+m_2}\right)\vec{v}$$

$$\vec{v}_{10}' = \left(\frac{m_2}{m_1+m_2}\right)\hat{n}_0\, v, \qquad \vec{v}_{20}' = \left(\frac{-m_1}{m_1+m_2}\right)\hat{n}_0\, v$$

Velocities wrt lab frame

$\vec{v}_1$ known

$\vec{v}_2 = 0$ known



$$\vec{v}_1' = \vec{v}_{10}' + \vec{V}$$

$$= \left(\frac{m_2}{m_1+m_2}\right)\hat{n}_0\, v + \frac{m_1 \vec{v}}{m_1+m_2}$$

$$\vec{v}_2' = \vec{v}_{20}' + \vec{V}$$

$$= -\left(\frac{m_1}{m_1+m_2}\right)\hat{n}_0\, v + \frac{m_1 \vec{v}}{m_1+m_2}$$

Thus,

$$\vec{v}_1' = \left(\left(\frac{m_2}{m_1+m_2}\right)\cos\chi + \frac{m_1}{m_1+m_2}\right)v_1\,\hat{x} + \left(\frac{m_2}{m_1+m_2}\right)\sin\chi\, v_1\,\hat{y}$$

$$\vec{v}_2' = \left(-\left(\frac{m_1}{m_1+m_2}\right)\cos\chi + \frac{m_1}{m_1+m_2}\right)v_1\,\hat{x} - \left(\frac{m_1}{m_1+m_2}\right)\sin\chi\, v_1\,\hat{y}$$

so

$$\boxed{v_1' = \frac{v_1}{m_1+m_2}\sqrt{(m_2\cos\chi + m_1)^2 + m_2^2\sin^2\chi} = \frac{v_1}{m_1+m_2}\sqrt{m_2^2+m_1^2 + 2m_1m_2\underbrace{\cos\chi}_{-\cos 2\theta_2}}}$$

$$v_2' = \frac{v_1}{m_1+m_2}\sqrt{(-m_1\cos\chi + m_1)^2 + m_1^2\sin^2\chi}$$

$$= \frac{v_1}{m_1+m_2}\sqrt{m_1^2+m_1^2 - 2m_1^2\cos\chi}$$

$$= \frac{v_1}{m_1+m_2}\, m_1\sqrt{2(1-\cos\chi)}$$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$2\sin^2\theta = 1 - \cos(2\theta)$$

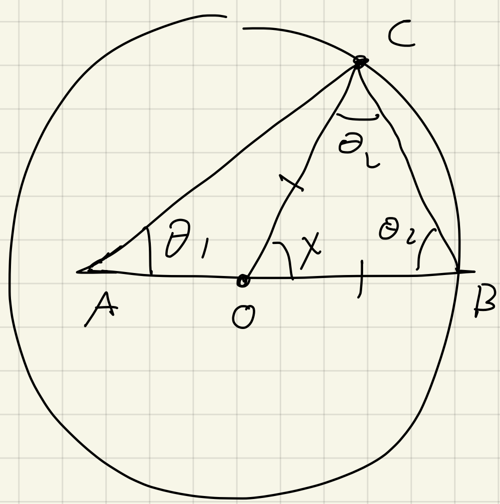
$$= \boxed{\frac{2m_1v_1}{m_1+m_2}\underbrace{\sin\left(\frac{\chi}{2}\right)}_{=\cos\theta_2}}$$

Recall that

$\chi + 2\theta_2 = \pi$

so $\chi = \pi - 2\theta_2$

$\frac{\chi}{2} = \frac{\pi}{2} - \theta_2$



$$\sin\left(\frac{\chi}{2}\right) = \sin\left(\frac{\pi}{2} - \theta_2\right) = \cos\theta_2$$

$$\cos\chi = \cos(\pi - 2\theta) = \cos\pi\cos(2\theta_2) = -\cos(2\theta_2)$$

Substitute:

$m_1 = 1\,kg$, $m_2 = 2\,kg$, $v_1 = 1\,m/s$, $v_2 = 0$, $\theta_2 = 60°$

$$\rightarrow \quad v_1' = \left(\frac{1}{1+2}\right)\sqrt{1^2 + 2^2 - 2\cdot 1\cdot 2\,\underbrace{\cos(120°)}_{-0.5}}$$

$$= \frac{1}{3}\sqrt{1+4+\frac{4}{2}}$$

$$= \boxed{\frac{\sqrt{7}}{3}}$$

$$\rightarrow \quad v_2' = \frac{2\cdot 1\cdot 1}{1+2}\,\underbrace{\cos 60°}_{\frac{1}{2}}$$

$$= \boxed{\frac{1}{3}}$$

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi} \quad , \quad \chi = \pi - 2\theta_2 = 180° - 2\cdot 60° = 60°$$

$$= \frac{2\cdot\sin 60°}{1+2\cos 60}$$

$$= \frac{2\left(\sqrt{3}/2\right)}{1+2\left(\frac{1}{2}\right)}$$

$$= \frac{\sqrt{3}}{2}$$

$$\rightarrow \quad \theta' = \arctan\left(\frac{\sqrt{3}}{2}\right) = \boxed{41°}$$

$$\vec{p}_1{}' = m_1 \vec{v}_1{}' = m_1(\vec{v}_{10} + \vec{V})$$

$$\vec{p}_{10}{}' = m \vec{v}_{10}{}'$$

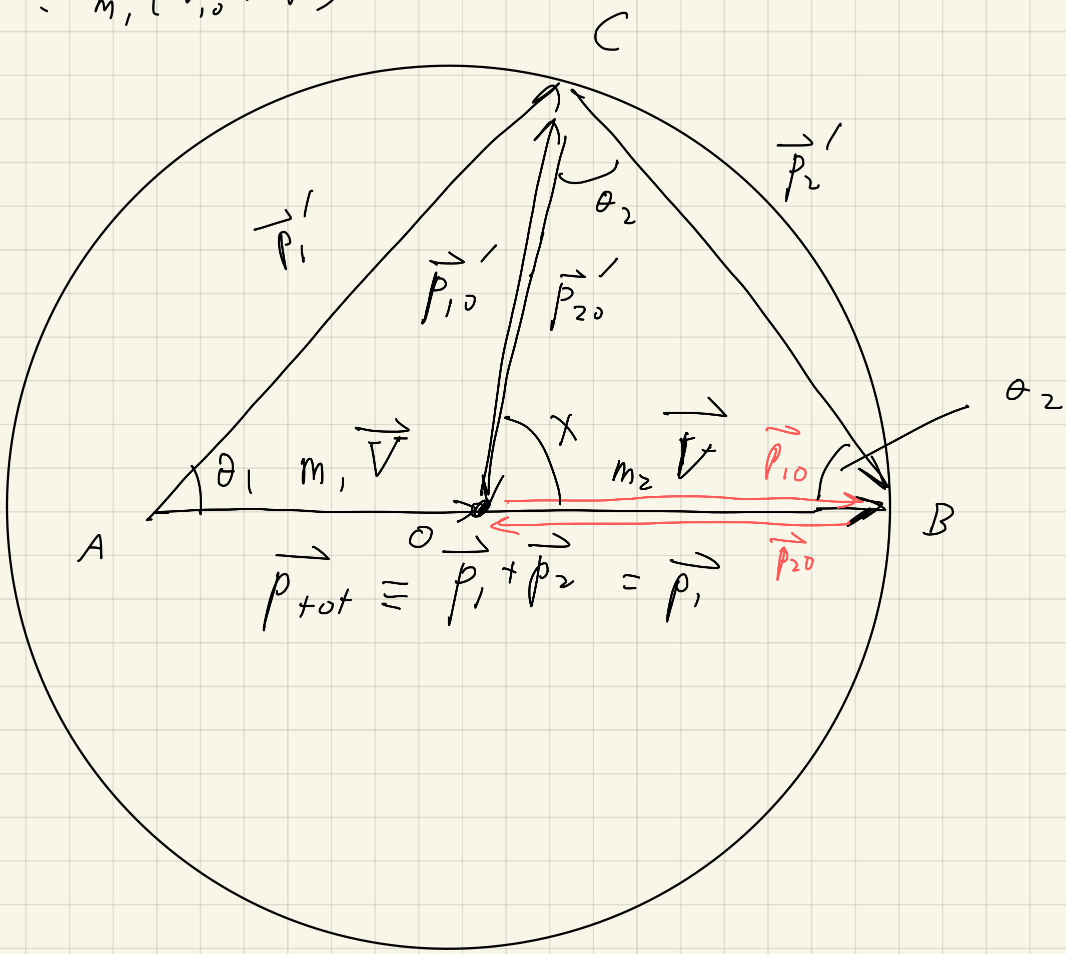
$$\vec{p}_1 = m_1 \vec{v}_1$$

$$\vec{p}_{10} = m_1 \vec{v}_{10}$$

$$\vec{p}_2 = m_2 \vec{v}_2$$

$$\vec{p}_1 = m_1 \vec{v}_1$$

(Lab frame has $\vec{v}_2 = 0$)



$$\vec{p}_{tot} \equiv \vec{p}_1 + \vec{p}_2 = \vec{p}_1$$

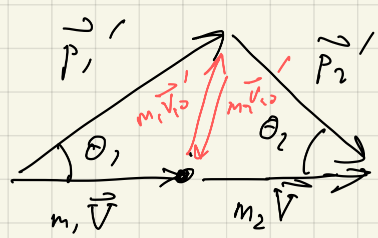
$$\vec{p}_{10} = m_1 \vec{v}_{10} = \frac{m_1 m_2 \vec{v}}{m_1 + m_2} = m_2 \vec{V}$$

$$\vec{p}_{20} = -\vec{p}_{10} \quad (\text{in order that } \vec{p}_{tot,0} = \vec{0})$$

Given; v_1, $v_2 = 0$, θ_2

Find : v_1', v_2', θ_1

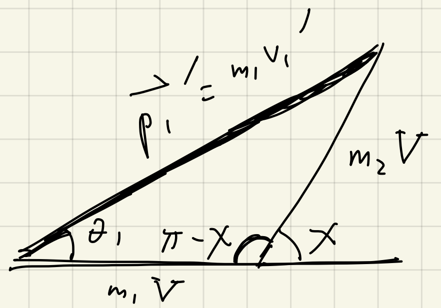
Know: $\vec{V} = \dfrac{m_1 \vec{v}_1}{m_1 + m_2}$



$$m_1 v_{10}' = m_1 \left(\frac{m_2 V}{m_1 + m_2}\right) = \frac{m_1 m_2 v_1}{m_1 + m_2}$$

$$\vec{p}_{tot} = (m_1 + m_2)\vec{V} = m_1 \vec{V} + m_2 \vec{V}$$

$$m_2 \vec{V} = m_2 \left(\frac{m_1 V}{m_1 + m_2}\right)$$



$$(m_2 V)^2 = (m_1 V)^2 + (m_1 v_1')^2 - 2\, m_1 V\, m_1 v_1' \cos\theta_1$$

$$\left(\frac{m_2 m_1}{m_1+m_2}\right)^2 v_1^2 = \left(\frac{m_1 m_1}{m_1+m_2}\right)^2 v_1^2 + m_1^2 v_1'^2 - 2 m_1^2 \left(\frac{m_1}{m_1+m_2}\right) v_1 v_1' \cos\theta_1$$

$$0 = v_1'^2 + \frac{(m_1^2 - m_2^2)}{(m_1+m_2)^2} v_1^2 - 2\left(\frac{m_1}{m_1+m_2}\right) v_1 v_1' \cos\theta_1$$

$$\rightarrow \quad v_1' = \frac{\cancel{2}\left(\frac{m_1}{m_1+m_2}\right) v_1 \cos\theta_1 \pm \sqrt{\cancel{4}\left(\frac{m_1}{m_1+m_2}\right)^2 v_1^2 \cos^2\theta_1 - \cancel{4}\frac{(m_1^2-m_2^2)\, v_1^2}{(m_1+m_2)^2}}}{\cancel{2}}$$

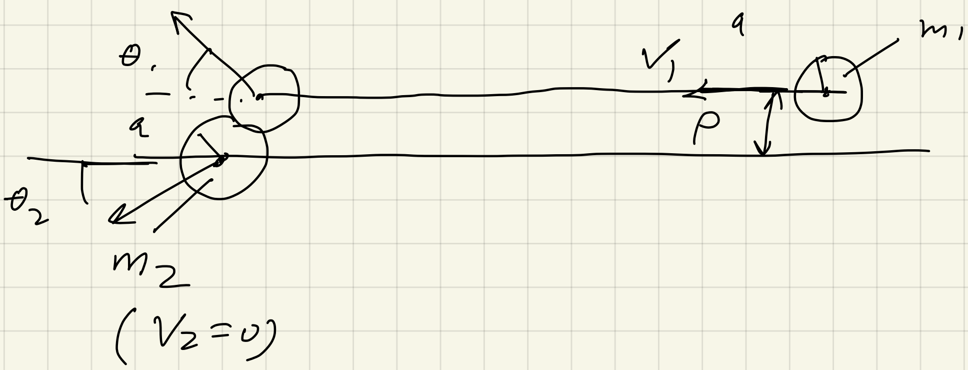
$$= \frac{v_1}{m_1+m_2}\left[m_1 \cos\theta_1 \pm \underbrace{\sqrt{m_1^2 \cos^2\theta_1 - (m_1^2 - m_2^2)}}_{\sqrt{-m_1^2 \sin^2\theta_1 + m_2^2}} \right]$$

$$= \left(\frac{v_1}{m_1+m_2}\right)\left[m_1 \cos\theta_1 \pm \sqrt{m_2^2 - m_1^2 \sin^2\theta_1} \right]$$

Problem, L&L Section 17

Quiz #5

(lab frame)

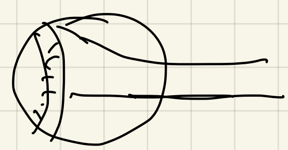


Analyze in COM

1) determine relationship between ρ and χ (scattering angle in COM)

2) determine differential cross section

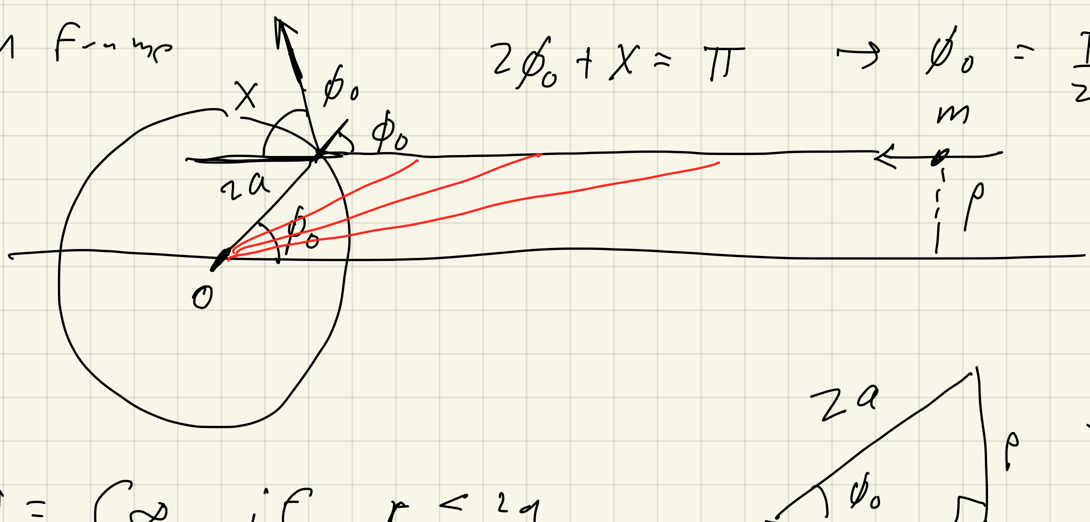
$$\frac{d\sigma}{d\Omega}$$



3) $$\sigma_{tot} = \int_{4\pi} \frac{d\sigma}{d\Omega}\, d\Omega$$

COM frame

$2\phi_0 + \chi = \pi \rightarrow \phi_0 = \frac{\pi}{2} - \frac{\chi}{2}$



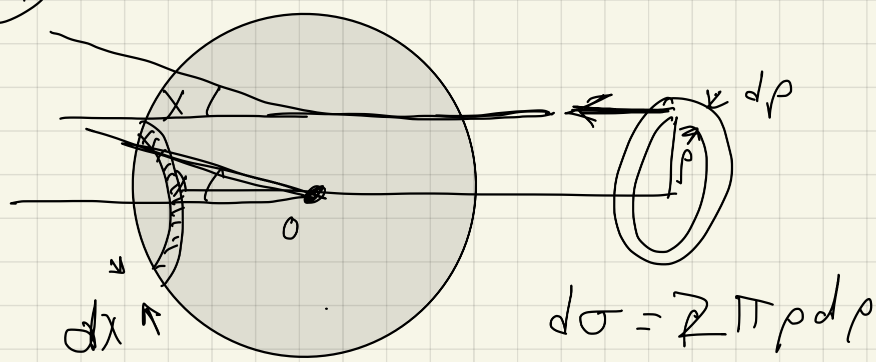
$$\sin\phi_0 = \frac{\rho}{2a}$$

$$U = \begin{cases} \infty & \text{if } r \le 2a \\ 0 & \text{if } r > 2a \end{cases}$$

$$\boxed{\rho = 2a\sin\phi_0 = 2a\sin\left(\frac{\pi}{2}-\frac{\chi}{2}\right) = 2a\cos\left(\frac{\chi}{2}\right)}$$

(+1)

2) $\dfrac{d\sigma}{d\Omega}$



$d\sigma = 2\pi\rho\, d\rho$

$$2\pi\sin\chi\, d\chi = d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{\cancel{2\pi}\rho\, d\rho}{\cancel{2\pi}\sin\chi\, d\chi} = \frac{\rho}{\sin\chi}\left|\frac{d\rho}{d\chi}\right|$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{2a\cos\left(\frac{\chi}{2}\right)}{\sin\chi}\ \cancel{2}a\frac{1}{\cancel{2}}\sin\left(\frac{\chi}{2}\right) = \frac{a^2\ \cancel{2\sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)}}{\cancel{\sin\chi}} = a^2}$$

(+0.5)

3) $$\boxed{\sigma_{tot} = \int_{\text{sphere}} \frac{d\sigma}{d\Omega}\, d\Omega = a^2 \underbrace{\int d\Omega}_{4\pi} = 4\pi a^2} = \pi(2a)^2$$

$\sin\theta\, d\theta\, d\phi$

(0.5)