

Notes: Thurs 8/27

- 1) Elliptic Functions  $\hookrightarrow$  go beyond small angle approx
- 2) Simple pendulum

Elliptic Functions:

$$\int \frac{dx}{\sqrt{1-x^2}} = ? = \sin^{-1}(x) + \text{const}$$

"   
  $\arcsin(x)$

$$\rightarrow \sin^2 \theta + \cos^2 \theta = 1$$

subst:  $x = \sin \theta \rightarrow \sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta$

$$dx = \cos \theta d\theta$$

$$\int \frac{\cos \theta d\theta}{\cos \theta} = \int d\theta = \theta = \sin^{-1}(x) \quad \begin{matrix} x = \sin \theta \\ \theta = \sin^{-1}(x) \end{matrix}$$

$$\frac{d}{d\theta} \sin \theta = \cos \theta$$

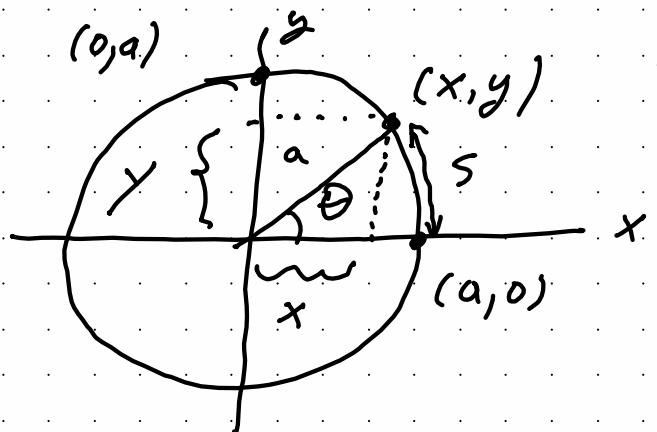
$$\lim_{h \rightarrow 0} \left( \frac{\sin(\theta+h) - \sin \theta}{h} \right)$$

$$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \dots$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

## circular functions..

$$x^2 + y^2 = a^2, \quad a = \text{radius}$$



$$\underline{\text{Def:}} \quad \sin \theta = \frac{y}{a}, \quad \cos \theta = \frac{x}{a}$$

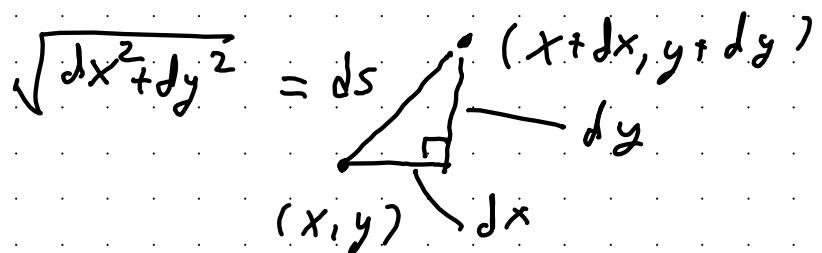
$s$ : arc length from  $(a, 0)$  to  $(x, y)$

$$s = a\theta \quad | \quad \theta = \frac{s}{a}$$

$$= \frac{1}{a} \int ds$$

$$= \frac{1}{a} \int \sqrt{dx^2 + dy^2}$$

$$= \int d\theta$$



$$\underline{\text{Given:}} \quad x^2 + y^2 = a^2$$

$$\underline{\text{Follows:}} \quad (i) \quad a^2 \cos^2 \theta + a^2 \sin^2 \theta = a^2 \rightarrow \boxed{\cos^2 \theta + \sin^2 \theta = 1}$$

$$(ii) \quad \boxed{\frac{d \sin \theta}{d \theta}} = \frac{1}{a} \frac{dy}{d \theta} = \frac{1}{a} \frac{dy}{\sqrt{dx^2 + dy^2}} = \frac{1}{\sqrt{x^2 + y^2}}$$

$$2x dx + 2y dy = 0 \rightarrow dx = -\frac{y}{x} dy \quad | \quad \Rightarrow \frac{1}{\sqrt{x^2 + y^2}} = \frac{1}{\sqrt{x^2}} = \frac{x}{a} = \boxed{\cos \theta}$$

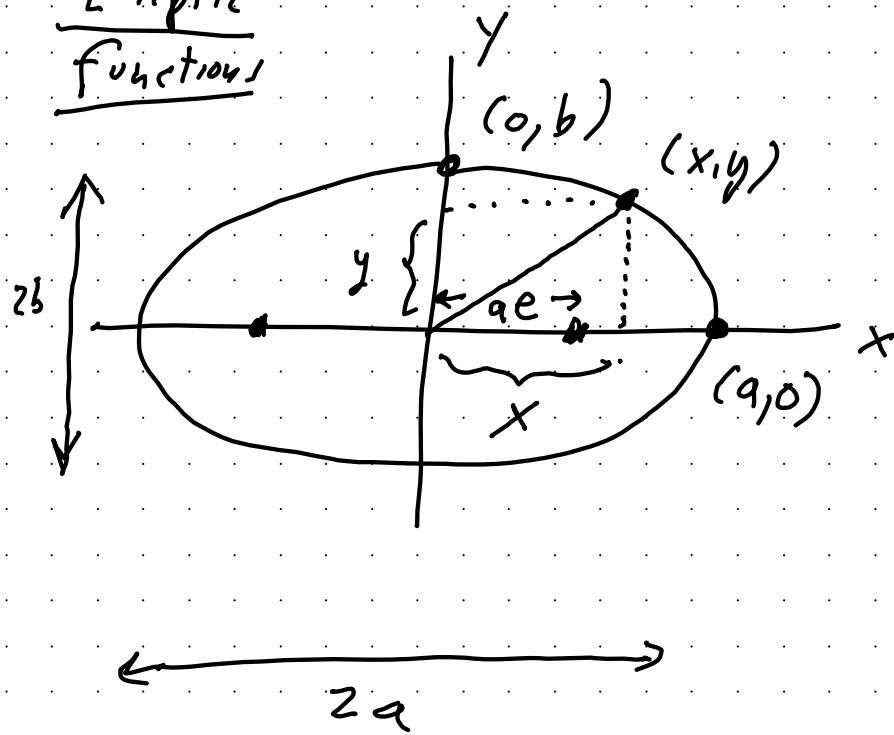
$$\frac{d(\sin \theta)}{d\theta} = \cos \theta \rightarrow \int \frac{d(\sin \theta)}{\cos \theta} = \int d\theta$$

$$x = \sin \theta$$

$$\begin{aligned}\cos \theta &= \sqrt{1 - \sin^2 \theta} \\ &= \sqrt{1 - x^2}\end{aligned}$$

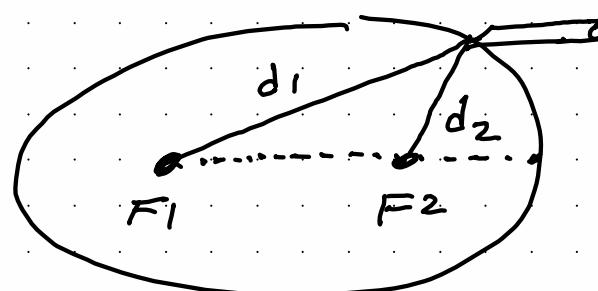
$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \theta = \sin^{-1}(x)}$$

Elliptic functions

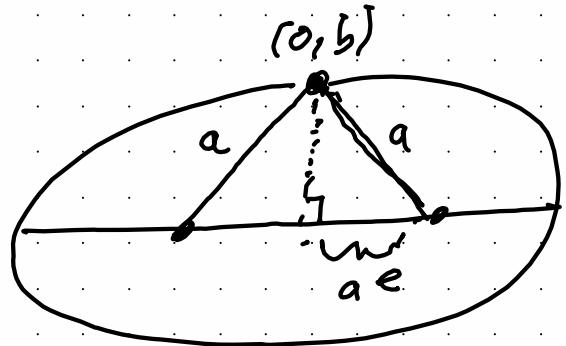


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

eccentricity :  $e = 0$  (for circle)



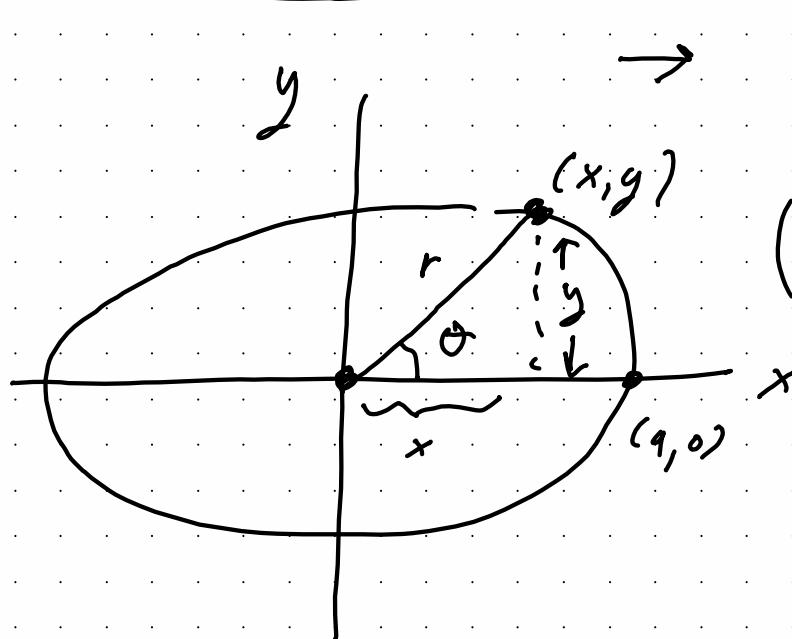
$$d_1 + d_2 = 2a$$



$$(ae)^2 + b^2 = a^2$$

$$a^2 e^2 + b^2 = a^2$$

$$e^2 = \frac{a^2 - b^2}{a^2} = 1 - \left(\frac{b}{a}\right)^2$$



$$\rightarrow e = \sqrt{1 - \left(\frac{b}{a}\right)^2} = k$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

notation used  
in elliptic  
function

$$x^2 + y^2 = r^2$$

$$x = r \cos \theta, y = r \sin \theta$$

Define:  $\operatorname{cn}(u; k) \equiv \frac{x}{a}$ ,  $\operatorname{sn}(u; k) \equiv \frac{y}{b}$

$$\operatorname{dn}(u; k) \equiv \frac{r}{a} \quad (=1 \text{ for a circle})$$

where  $u = \frac{1}{b} \int_0^\theta r d\theta$   $y = r \sin \theta$   $ds = \sqrt{dx^2 + dy^2}$   
 $(= \theta \text{ for a circle})$   $= \sqrt{dr^2 + r^2 d\theta^2}$

$$\text{Given: } \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad , \quad x^2 + y^2 = r^2 \quad dn(u; k) = \frac{r}{a}$$

$$\begin{aligned} \text{Follows: (i)} \quad & cn^2(u; k) + sn^2(u; k) = 1 \\ \text{(ii)} \quad & dn^2(u; k) + k^2 sn^2(u; k) = 1 \end{aligned} \quad u = \int_0^\theta r d\theta$$

$$(iii) \quad \frac{d}{du} sn(u; k) = cn(u; k) dn(u; k)$$

$$\frac{d}{du} cn(u; k) = -sn(u; k) dn(u; k)$$

$$\frac{d}{du} dn(u; k) = -k^2 sn(u; k) cn(u; k)$$

Integrate:  $\frac{d sn(u; k)}{dn(u; k)} = cn(u; k) dn(u; k)$

$$\int \frac{d sn(u; k)}{cn(u; k) dn(u; k)} = \int du = u$$

$$\int \frac{dx}{\sqrt{1-x^2}} = u + \text{const} = \sin^{-1}(x; k) + \cos, t$$

$$x \equiv sn(u; k)$$

Analogous to  
 $\frac{ds \sin \theta}{d\theta} = \cos \theta$

$$\frac{ds \sin \theta}{\cos \theta} = d\theta$$

$$x = \sin \theta$$

$$\int \frac{dx}{\sqrt{1-x^2}} = d\theta = \theta = \sin^{-1} x$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + \cos, t$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-k^2 x^2}} \equiv K(k) \rightarrow$$

(Complete elliptic integral of 1st kind)

related to  
Period of a pendulum  
going beyond  
small-angle  
approximation

$$\int_0^1 \frac{\sqrt{1-k^2 x^2} dx}{\sqrt{1-x^2}} \equiv E(k) \rightarrow$$

(Complete elliptic integral of 2nd kind)

circumference  
around an ellip.



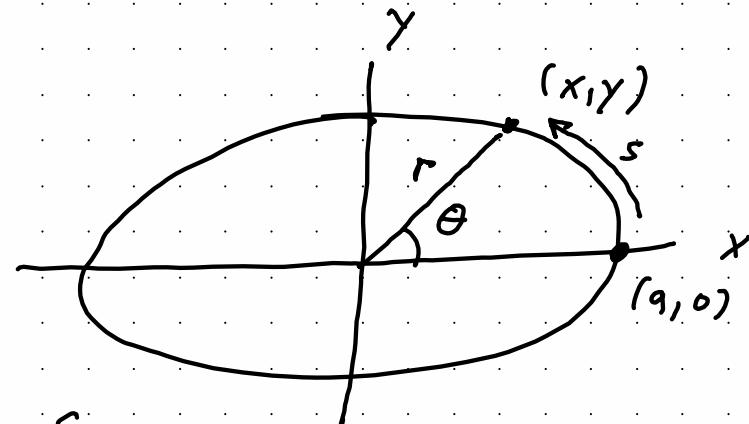
circle:  $C = 2\pi a$

Notes: Tuesday 9/1

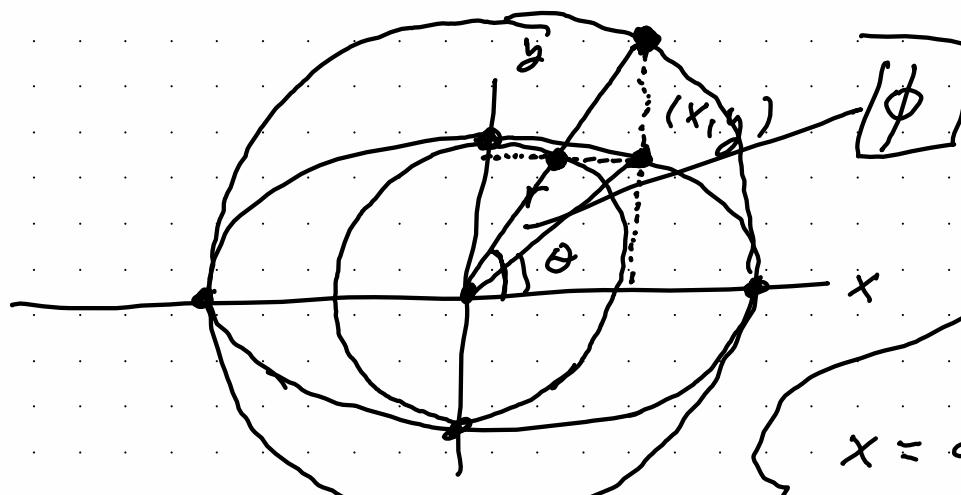
- 1) Review of elliptic functions
- 2) Simple pendulum

$$u = \frac{1}{b} \int_{(a,0)}^{(x,y)} r d\theta$$

$$bu = \int_0^\theta r d\theta \leq \int_0^\theta ds = s$$



$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\theta^2} \geq r d\theta$$



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$x = a \cos \phi$$

$$y = b \sin \phi$$

$$x = a \cos \phi, y = a \sin \phi$$

$$x = b \cos \phi, y = b \sin \phi$$

## Simple pendulum:

(i) "Freshman physics"

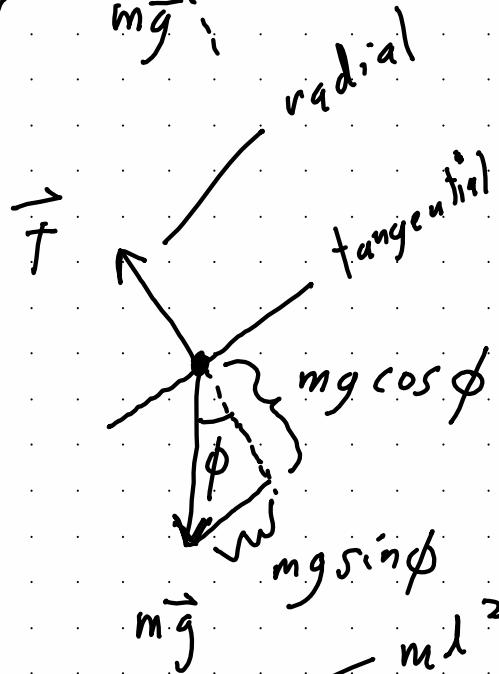
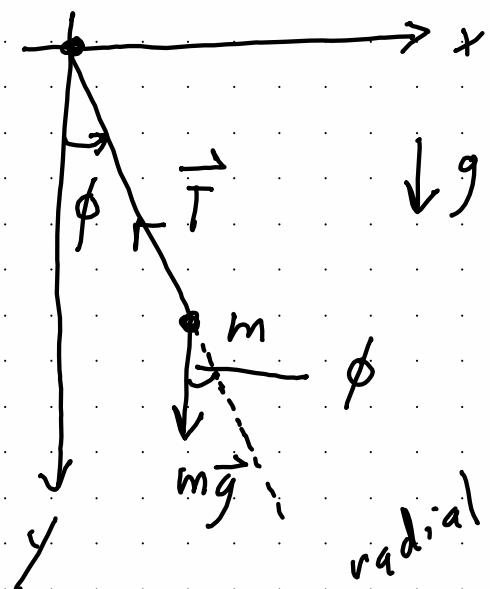
forces, free-body diagrams

→ EoM, tension

tangential:

$$-mg \sin \phi = m a_{\text{tangential}}$$

$$-g \sin \phi = m \ddot{\phi}$$



$$\text{Torque} = I \alpha - \dot{\phi}$$

$\phi$ : angular displacement [rad]

$\dot{\phi}$ : angular velocity [rad/sec]

$\ddot{\phi}$ : angular accel [rad/sec<sup>2</sup>]

$$\boxed{\ddot{\phi} = -\frac{g}{l} \sin \phi} \quad (\text{EoM})$$

radial:  $T - mg \cos \phi = m a_{\text{centripetal}}$

$$T - mg \cos \phi = m \dot{\phi}^2 l$$

$$\boxed{T = mg \cos \phi + m \dot{\phi}^2 l}$$

(ii) Small angle approximation:

$$\sin \phi \approx \phi \leftarrow \phi \ll 1 = 57 \text{ degrees}$$

$$\ddot{\phi} = -\frac{g}{l} \sin \phi \approx -\frac{g}{l} \phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l} \phi}$$

$$\phi(t) = \boxed{a} \cos(\omega t) + \boxed{b} \sin(\omega t)$$

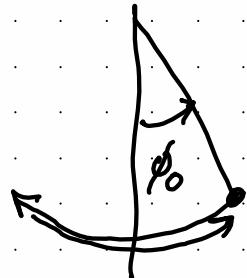
$$\text{where } \omega = \sqrt{\frac{g}{l}}$$

small angle  
approx.

determined by  
initial conditions

I.Cs: If  $\phi(0) = \phi_0$  (at rest)

then  $\boxed{\phi(t) = \phi_0 \cos(\omega t)}$



Period:  $P = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}}$

independent of  $\phi_0$  !!

(iii) Lagrangian approach       $T \equiv$  Kinetic Energy

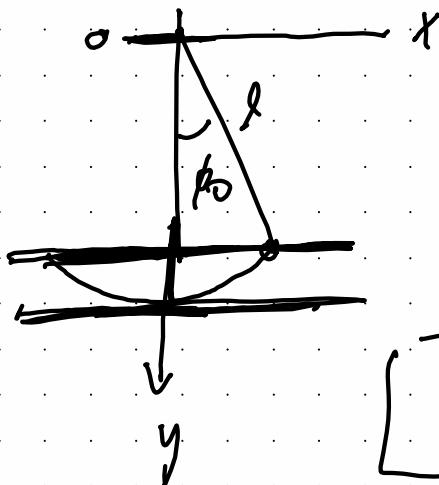
$$L = T - U$$

$U \equiv$  Potential energy

$$T = \frac{1}{2} m l^2 \dot{\phi}^2 \quad (= \frac{1}{2} m (x^2 + y^2))$$

$$y = l \cos \phi$$

$$x = l \sin \phi$$



$$U = -mg l \cos \phi + \text{const}$$

$$U = -mgy \quad \text{action}$$

$$U = mgy l (1 - \cos \phi)$$

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mg l \cos \phi$$

$$S = \int_{t_1}^{t_2} dt L(\phi, \dot{\phi}, t)$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \quad \text{Lagrange's equation}$$

$$\frac{d}{dt} (ml^2 \dot{\phi}) = -mg l \sin \phi$$

$$ml^2 \ddot{\phi} = -mg l \sin \phi \rightarrow \ddot{\phi} = -\frac{g}{l} \sin \phi \quad (\text{LHS})$$

(iv) solving  $\ddot{\phi} = -\frac{g}{l} \sin \phi$  ( $2^{\text{nd}}$  order non-linear)

$$E = \text{const}$$

$$= T + U$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 - m g l \cos \phi$$

ODE ↑

hard!!

$$E = 0 - m g l \cos \phi_0$$

release from rest

$$= -m g l \cos \phi_0$$

from  $\phi = \phi_0$

$$-m g l \cos \phi_0 = \frac{1}{2} m l^2 \dot{\phi}^2 - m g l \cos \phi$$

$$-m g l (\cos \phi_0 - \cos \phi) = \frac{1}{2} m l^2 \dot{\phi}^2$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{-2 \frac{g}{l} (\cos \phi_0 - \cos \phi)}$$

$|\phi| \leq \phi_0$

$$\int \frac{d\phi}{\sqrt{-2 \frac{g}{l} (\cos \phi_0 - \cos \phi)}} = \int dt = t + \text{const}$$

Separable  
1st order  
ODE

$$t + t_0 = \int \frac{d\phi}{\sqrt{-2 \frac{g}{l} (\cos \phi_0 - \cos \phi)}}$$

Substitution:

$$\cos \phi = 1 - 2 \sin^2 \left( \frac{\phi}{2} \right)$$

$$\cos \phi_0 = 1 - 2 \sin^2 \left( \frac{\phi_0}{2} \right)$$

$$\cos \phi = \cos \left( 2 \left( \frac{\phi}{2} \right) \right)$$

$$= \cos^2 \left( \frac{\phi}{2} \right) - \sin^2 \left( \frac{\phi}{2} \right)$$

$$= 1 - 2 \sin^2 \left( \frac{\phi}{2} \right)$$

$$\Rightarrow \cos \phi_0 - \cos \phi = -2 \left( \sin^2 \left( \frac{\phi_0}{2} \right) - \sin^2 \left( \frac{\phi}{2} \right) \right)$$

$$t + t_0 = \int \frac{d\phi}{2 \sqrt{\frac{g}{l}} \sqrt{\sin^2 \left( \frac{\phi_0}{2} \right) - \sin^2 \left( \frac{\phi}{2} \right)}}$$

$$|\phi| \leq \phi_0$$

$$= \frac{1}{2 \sqrt{\frac{g}{l}}} \int \frac{d\phi}{\sin \left( \frac{\phi_0}{2} \right) \sqrt{1 - \frac{\sin^2 \left( \frac{\phi}{2} \right)}{\sin^2 \left( \frac{\phi_0}{2} \right)}}}$$

let  $x = \sin \left( \frac{\phi}{2} \right)$

$$\sin \left( \frac{\phi_0}{2} \right)$$

$$x = \frac{\sin(\frac{\phi}{2})}{\sin(\frac{\phi_0}{2})}$$

$$\sqrt{1-x^2}$$

denominator

Find this out

①  $\phi(t) =$

② Period = ??

③ Redo the analysis using Lagrange multiplier for find tension in strings

$$t + t_0 = \int \text{---} \quad \begin{matrix} \text{integrated} \\ \text{for } \sin^{-1}(x_j/T) \end{matrix}$$

$$T = \sin\left(\frac{\phi_0}{2}\right)$$

$$dx = \frac{1}{\sin(\frac{\phi_0}{2})} \cdot \frac{1}{2} \cos\left(\frac{\phi}{2}\right) d\phi$$

$$d\phi = \frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\cos\left(\frac{\phi}{2}\right)}$$

$$= \frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1 - \sin^2\left(\frac{\phi}{2}\right)}}$$

$$= \frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1 - \sin^2\left(\frac{\phi_0}{2}\right) x^2}}$$

$$\sqrt{1 - \sin^2\left(\frac{\phi_0}{2}\right) x^2}$$

$$\hookrightarrow T^2$$

Lec #4 : Thurs 9/3

$$\phi(t) = 2 \sin^{-1} \left[ k \operatorname{sn} \left( \omega_0 \left( t + \frac{P}{4} \right); k \right) \right] \star$$

$$k \equiv \sin \left( \frac{\phi_0}{2} \right), \quad \omega_0 = \sqrt{\frac{g}{L}}$$

$$P = 4\sqrt{\frac{L}{g}} \quad K(k) = 4\sqrt{\frac{L}{g}} \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-k^2 x^2}}$$

small angle  
approx

$$P_0 = 2\pi \sqrt{\frac{L}{g}}$$

$$\frac{\sin \left( \frac{\phi}{2} \right)}{\sin \left( \frac{\phi_0}{2} \right)} = x = \operatorname{sn} \left[ \sqrt{\frac{g}{L}} (t + t_0); k \right]$$

$$\sqrt{\frac{L}{g}} \int \frac{dx}{\sqrt{1-x^2} \sqrt{1-k^2 x^2}} = \sqrt{\frac{L}{g}} \operatorname{sn}^{-1}(x; k) = t + t_0$$

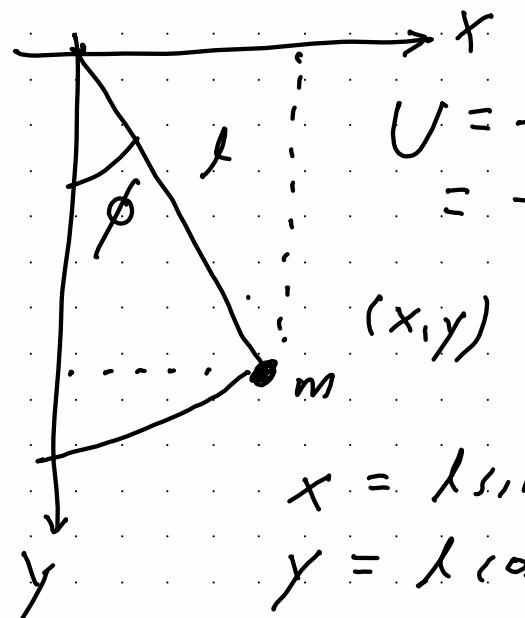
$$\operatorname{sn}^{-1}(x; k) = \sqrt{\frac{g}{L}} (t + t_0)$$

$$P = 4\sqrt{\frac{L}{g}} T(H) = 2\pi \sqrt{\frac{L}{g}} \left( 1 + \frac{1}{16} \phi_0^2 + \dots \right)$$

Problem Landau II, 1

Lagrange multiplier:

$$\varphi(x, y) = x^2 + y^2 - l^2 = 0$$



$$\begin{aligned} U &= -mg y \\ &= -mgl \cos \phi \end{aligned}$$

$$\begin{aligned} x^2 + y^2 &= l^2 \sin^2 \phi + l^2 \cos^2 \phi \\ &= l^2 \end{aligned}$$

$$\begin{aligned} x &= l \sin \phi \\ y &= l \cos \phi \end{aligned}$$

$$\begin{aligned} T &= \frac{1}{2} m (x'^2 + y'^2) \\ &= \frac{1}{2} m l^2 \dot{\phi}^2 \end{aligned}$$

$$L = T - U + \lambda \phi$$

$\uparrow$   
 $\lambda, *$

$$L(x, \dot{x}, y, \dot{y}, t)$$

$$q = (x, y) \quad \dot{q} = (\dot{x}, \dot{y})$$

$$L(r, \dot{r}, \phi, \dot{\phi}, t)$$

$$q = (r, \phi)$$

Lagrange multiplier

$\lambda(t)$   
 $r(t)$   
 $\phi(t)$

$$L(\phi, \dot{\phi}, t)$$

$$\phi(x, y) = x^2 + y^2 - l^2 = 0$$

$$\phi(r, \phi) = r - l = 0$$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + mg \underbrace{r \cos \phi}_{y} + \lambda (r - l)$$

$$r: \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} \rightarrow \cancel{m \ddot{r}} = mr \dot{\phi}^2 + mg \cos \phi + \lambda$$

$$\phi: \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \rightarrow \cancel{\frac{d}{dt} (mr^2 \dot{\phi})} = -mg r \sin \phi$$

$$\lambda: \cancel{r - l = 0} \quad \checkmark \quad \cancel{2mr \dot{r} \dot{\phi} + mr^2 \ddot{\phi}} = -mg r \sin \phi$$

$$r - l = 0 \rightarrow r = l$$

$$\dot{r} = 0$$

$$\ddot{r} = 0$$

---

$$\rightarrow \ddot{\phi} = -\frac{g}{l} \sin \phi$$

---

$$ml\dot{\phi}^2 + mg \cos \phi + \lambda = 0$$

$$\boxed{\lambda = -(mg \cos \phi + ml\dot{\phi}^2)}$$

$$\lambda = -T$$

T

$$L = T - U + \lambda \phi$$

$$U = U(x, t)$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\frac{d}{dt} (m \dot{x}) = - \frac{\partial U}{\partial x} + \lambda \frac{\partial \phi}{\partial x}$$

$$\phi(x, y, t)$$

$$U(x, y, t)$$

$$\frac{d}{dt} (m \dot{y}) = - \frac{\partial U}{\partial y} + \lambda \frac{\partial \phi}{\partial y}$$

$$\frac{d \vec{P}}{dt} = \vec{F}_{\text{net}}$$

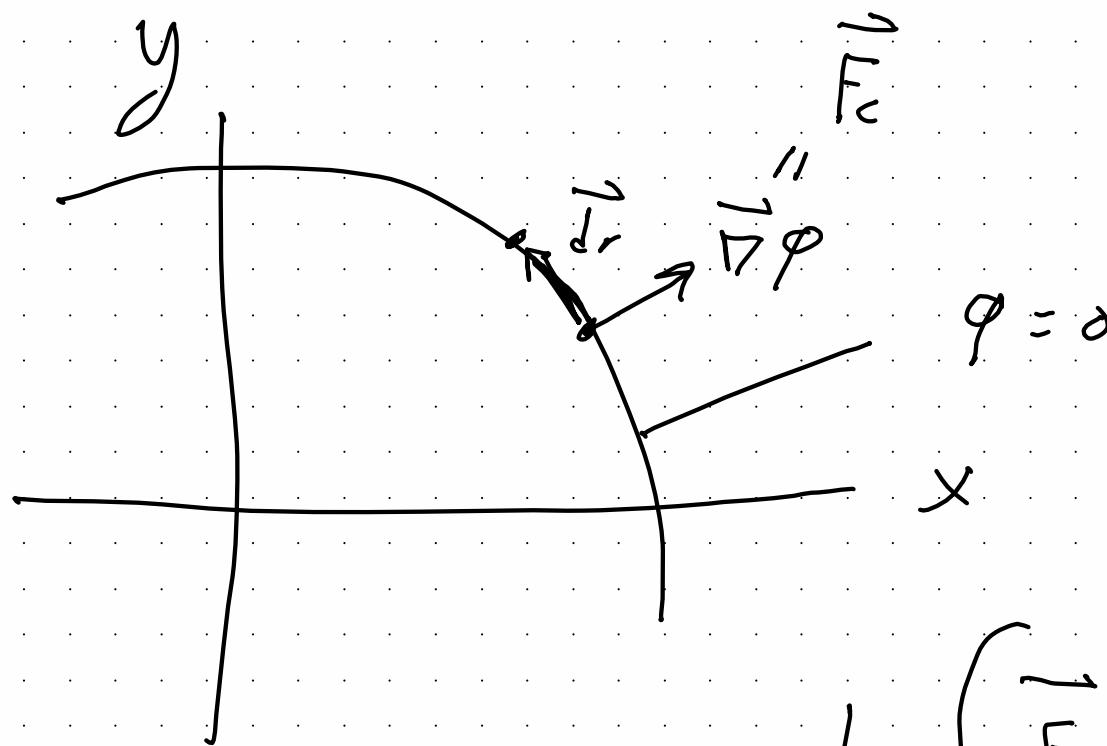
applied force      force

$$\frac{d p_x}{dt} = F_x + \lambda \frac{\partial \phi}{\partial x}$$

$$\frac{d \vec{P}}{dt} = \vec{F} + \lambda \vec{\nabla} \phi$$

$$\frac{d p_y}{dt} = F_y + \lambda \frac{\partial \phi}{\partial y}$$

$$\left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) = \vec{\nabla} \phi$$



$$\nabla \varphi \perp \varphi = \text{const}$$

$$\int \vec{F}_c \cdot d\vec{r} = \text{work}$$

$$\varphi = r - \ell$$

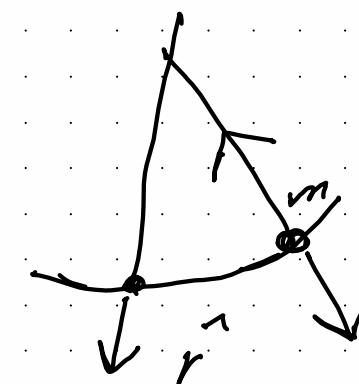
$$\nabla \varphi = \hat{r}$$

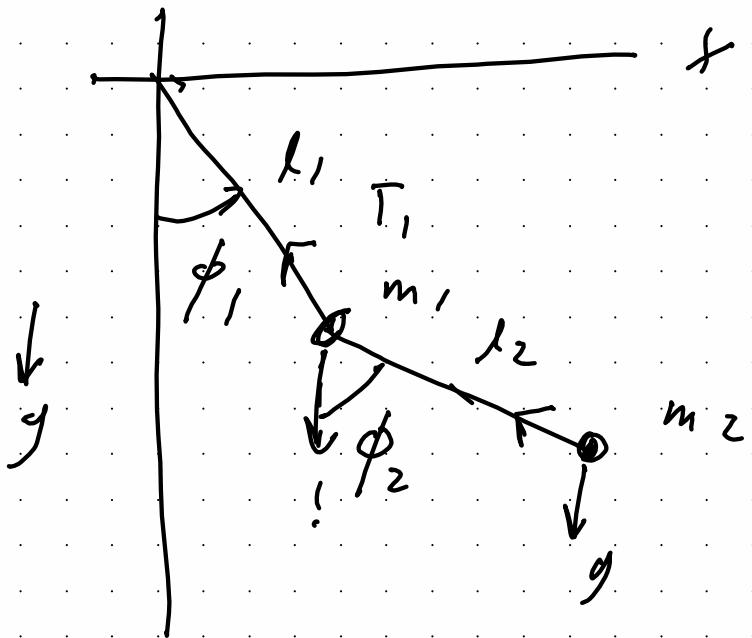
$$\nabla \varphi$$

$$(r, \varphi)$$

$$\hat{r}, \hat{\varphi}$$

$$\hat{r}$$

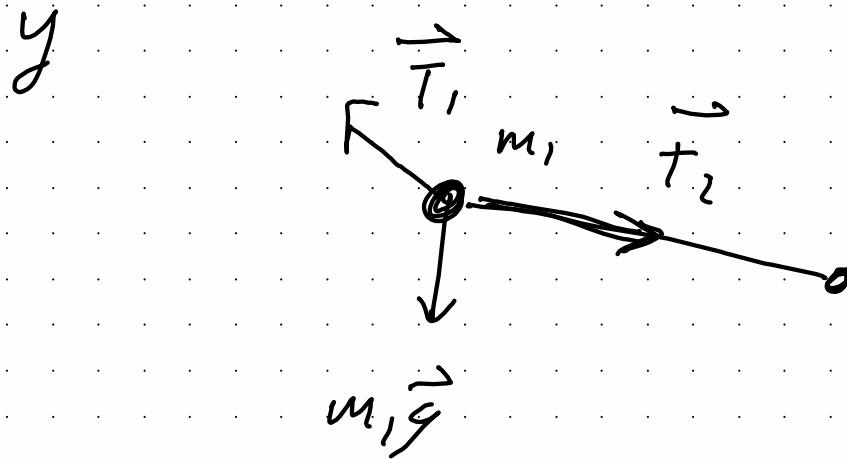




$$U = -m_1 g y_1 - m_2 g y_2$$

$\phi_1, \phi_2$

$E_{om}$



$$\begin{aligned} & T \\ & (\lambda_1 \varphi_1 + \lambda_2 \varphi_2 + \dots) \end{aligned}$$

$L$



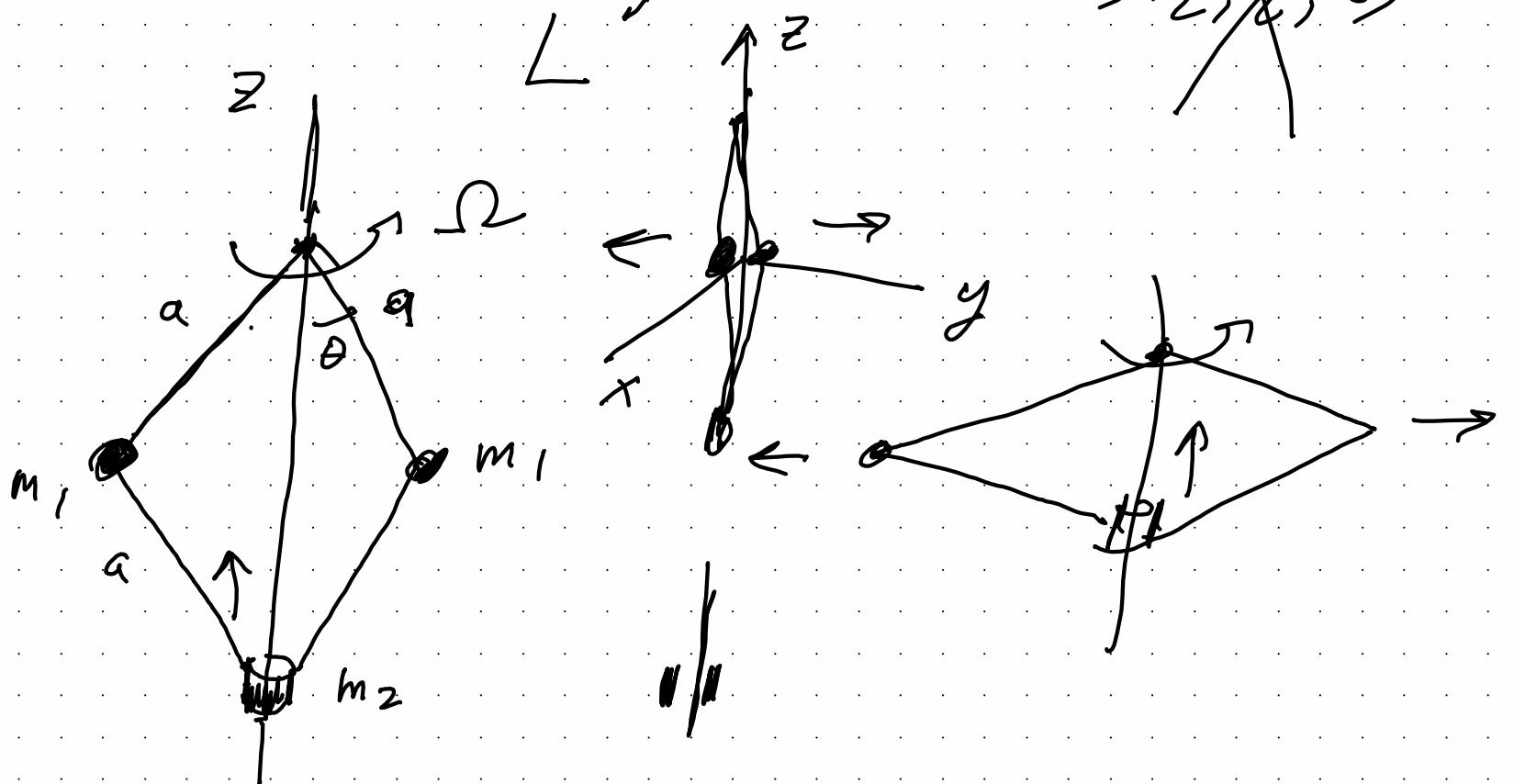
$E \circ M$



$L$

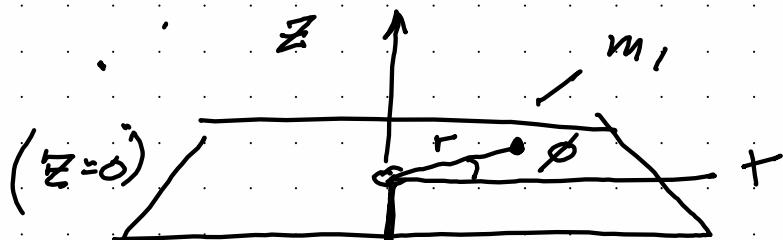
$$L \rightarrow L + \frac{d}{dt} (f(q, t))$$

$f(q, \dot{q}, t)$

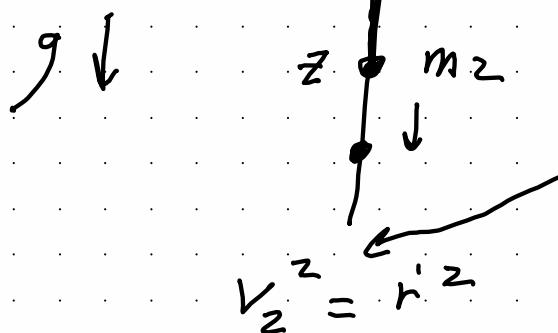


Lec #5: Tuesday 9/18

$$r - z = \ell = \text{length of string}$$



$$L = T - U$$



$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$v_2^2 = \dot{z}^2, \quad v_1^2 = \dot{r}^2 + r^2 \dot{\phi}^2 \\ (= \dot{x}^2 + \dot{y}^2, \quad x = r \cos \phi, \quad y = r \sin \phi)$$

$$\boxed{T = \frac{1}{2} m_1 (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2 \\ = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2}$$

$$U = m_2 g z = m_2 g (r - \ell) = m_2 g r - m_2 g \ell$$

$$\boxed{U = m_2 g r}$$

$$L = T - U = \boxed{\frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r}$$

constant

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r}, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \Rightarrow \text{2nd order EOMs}$$

No explicit  $t$  dependence:

$$E = \dot{r} \frac{\partial L}{\partial \dot{r}} + \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - L = \boxed{T + U}$$

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

$\downarrow$

$$= p_i$$

No explicit  $\phi$  dependence:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \text{const} = M_z$$

$$M_z = m_1 r^2 \dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{m_1 r^2}$$

$M$ : angular momentum  
(L&L notation)

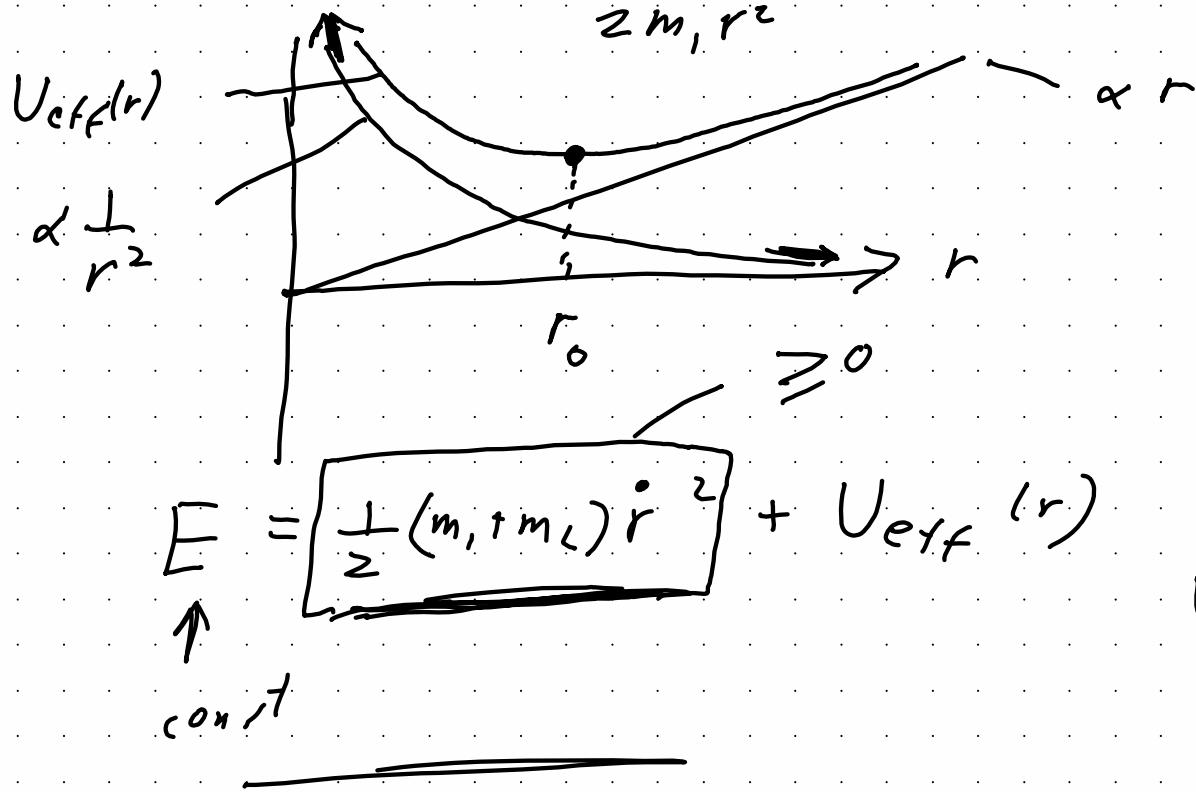
$$E = \frac{1}{2} m \dot{r}^2 + U(r)$$

$$E = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 + m_2 g r$$

$$E = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \left( \frac{M_z^2}{2 m_1 r^2} + m_2 g r \right)$$

$$U_{\text{eff}}(r) = U(r) + \frac{M_z^2}{2 m_1 r^2}$$

$$U_{eff}(r) = \frac{M_2^2}{2m_1 r^2} + m_2 g r$$



i)  $E = U_{eff, min} = U_{eff}(r_0)$

unif circular motion:  $r = r_0$ ,  $\dot{\phi} = \frac{M_2}{m_1 r_0^2}$

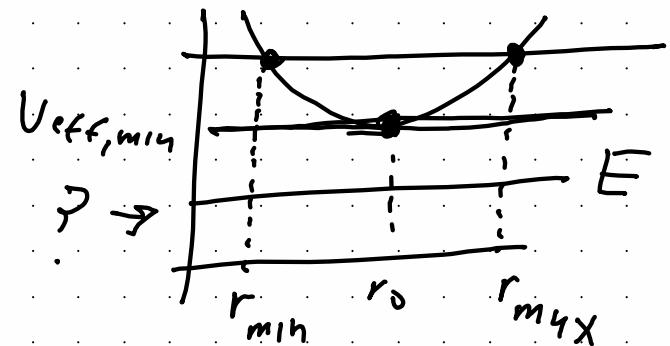
ii)  $E > U_{eff, min}$

$$E = U_{eff}(r_{min}) = U_{eff}(r_{max})$$

$$U(r) = m_2 g r$$

$r_{min}, r_{max}$ :

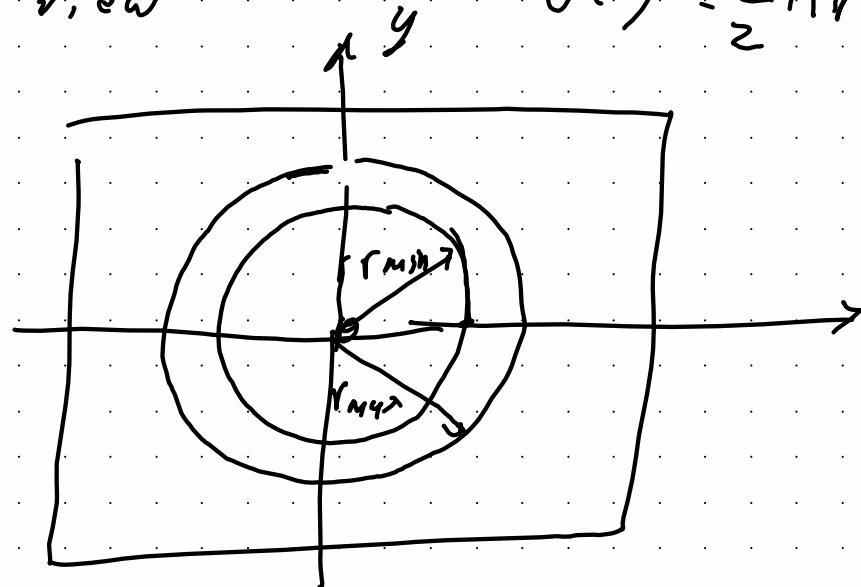
"turning points":  
( $r=0$ )



$E < U_{eff, min}$   
(not allowed)

$$E \geq U_{eff, min}$$

top view



$$U(r) = \frac{1}{2} k r^2$$

closed bound

$$U(r) = -\frac{G m_1 m_2}{r}$$

r

Newtonian gravity  
bound orbits = ellipse  
are closed

1 degree

11

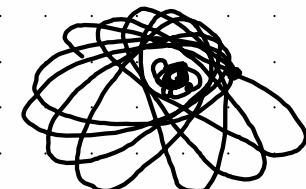
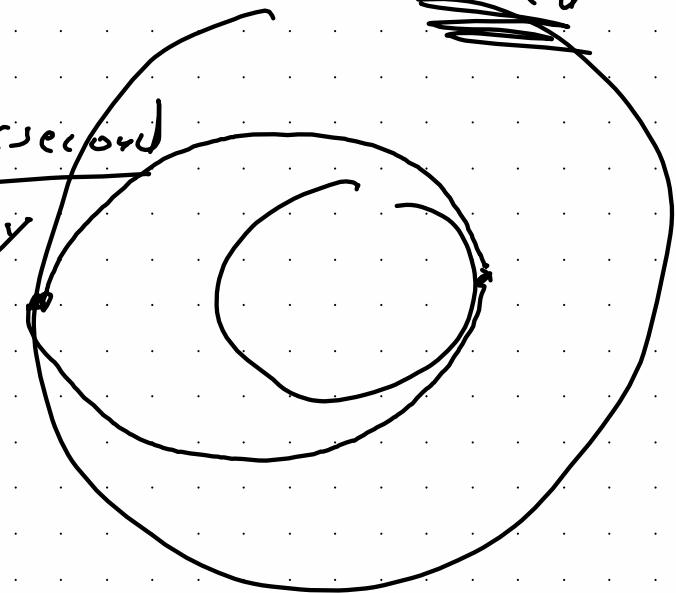
60 mins  
of arc

1 min of  
arc

11  
60 sec of arc

perihelion precession of Mercury

closest approach to sun



$$\underline{r_0} : \left. \frac{dU_{\text{eff}}}{dr} \right|_{r=r_0} = 0 \quad (\text{minimum})$$

$$0 = \left. \frac{d}{dr} \left( \frac{M_z^2}{2m_1 r^2} + m_2 gr \right) \right|_{r=r_0}$$

$$= -\frac{M_z^2}{m_1 r_0^3} + m_2 g$$

$$M_z^2 = m_1 m_2 g r_0^3$$

tells you the value of  $M_z$  needed to have a specific  $r_0$  value.

For a given  $M_z$ , this tells you what  $r_0$  equals.

## Energy equation:

$$E = \frac{1}{2} (m_1 + m_2) r^2 + \frac{M_z^2}{2m_1 r^2} + m_2 gr$$

$$\boxed{\dot{\phi} = \frac{M_z}{m_1 r^2}} \quad \leftarrow \quad \phi \text{ equation}$$

$$\frac{1}{2} (m_1 + m_2) r^2 = E - \frac{M_z^2}{2m_1 r^2} - m_2 gr$$

$$\dot{r} = \frac{dr}{dt} = \pm \sqrt{\left( \frac{2}{m_1 + m_2} \right) \left( E - \frac{M_z^2}{2m_1 r^2} - m_2 gr \right)}$$

$$\int \frac{dr}{\sqrt{\left( \frac{2}{m_1 + m_2} \right) \left( E - \frac{M_z^2}{2m_1 r^2} - m_2 gr \right)}} = \int dt = t + \text{const}$$

$r(t) \Leftrightarrow r(t)$

orbital equations:

$$r = r(\phi) \Leftrightarrow \phi = \phi(r)$$

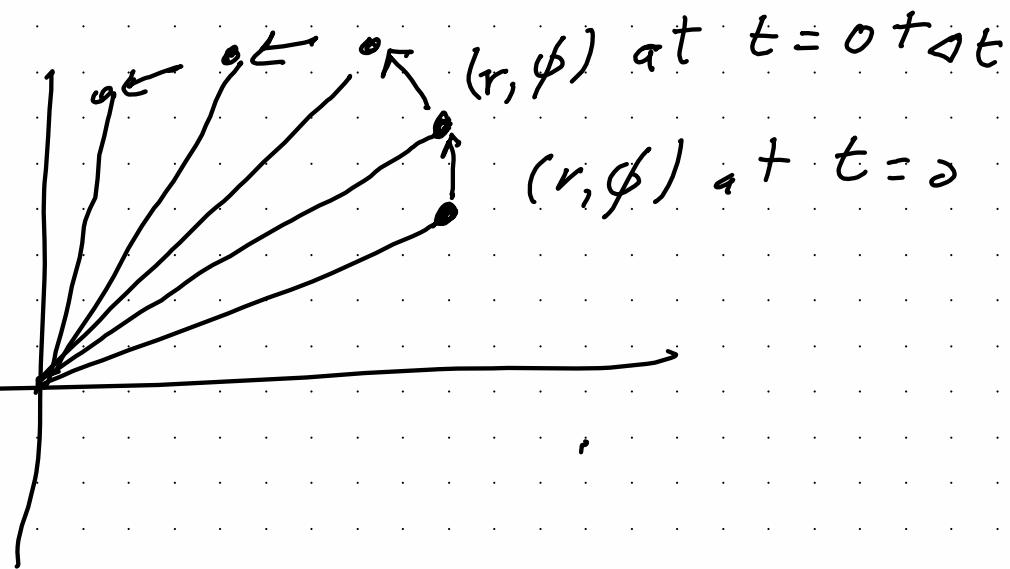
$$\frac{dr}{dt} = \dot{r} = \sqrt{\frac{2}{m_1 + m_2}} \left[ E - \frac{M_Z^2}{2m_1 r^2} - m_2 g r \right]$$

$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{M_Z}{m_1 t^2}$$

with

$$\rightarrow \frac{dr}{d\phi} = \frac{m_1 r^2}{M_Z} \sqrt{\frac{2}{m_1 + m_2}} \left[ \quad \right]$$

$$\int \frac{dr}{\frac{m_1 r^2}{M_Z} \sqrt{\left( \right) \left[ \quad \right]}} = \int d\phi = \phi + \cos t$$



Show,  $r, \phi$  at some time  $t$

Given:  $\Delta t$  need to know  $\Delta r$  and  $\Delta \phi$

$$r(t+\Delta t) = r(t) + \Delta r(t) + \dots$$

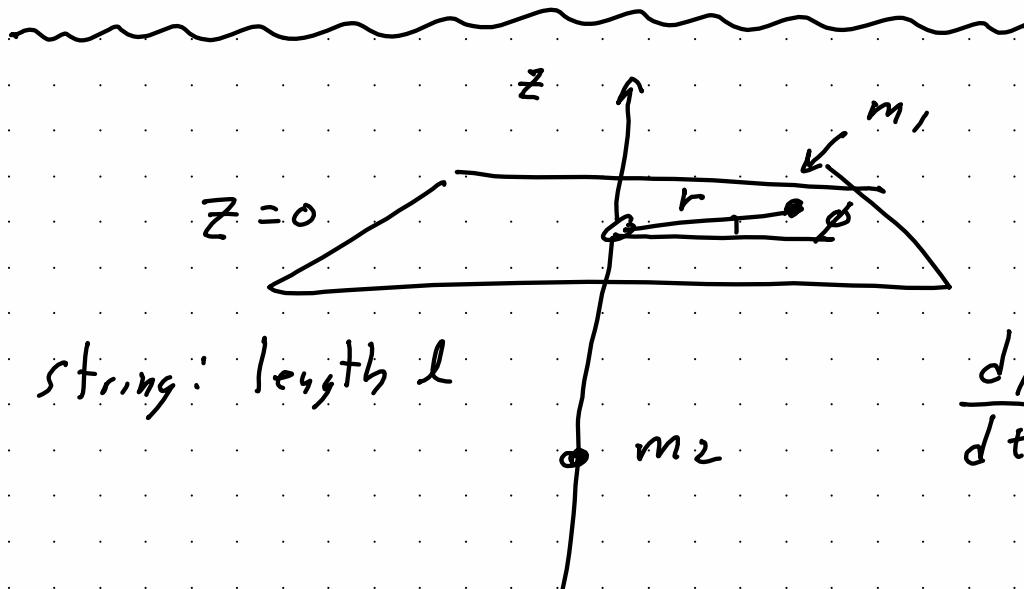
$$\phi(t+\Delta t) = \phi(t) + \Delta \phi(t) + \dots$$

(W)

ignore it if  $\Delta t$   
is suff. small

## Lecture #6 : Thursday 10 Sep

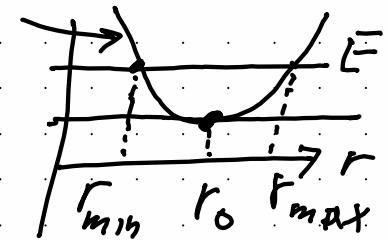
- 1) Secs 6-10 (today), sec 40 (next Tuesday)
- 2) Finish up example from last time
- 3) Conservation of  $E$ ,  $\vec{P}$ ,  $\vec{m}$
- 4) Mechanical similarity
- 5) Quiz: last 20 minutes (1:30 pm)



$$E, M_Z = \text{const}$$

$v$        $v$

$v_{\text{eff}}$



$$\frac{d\phi}{dt} = \frac{M_Z}{m_1 r^2}$$

$$\frac{dr}{dt} = \sqrt{\left(\frac{2}{m_1 + m_2}\right) \left( E - m_2 gr - \frac{M_Z^2}{2m_1 r^2}\right)} = \sqrt{\Theta}$$



$$\frac{d\phi}{dt} = \frac{M_2}{m_1 r^2} \rightarrow \Delta\phi = \Delta t \frac{M_2}{m_1 r^2}$$

$$\frac{dr}{dt} = \sqrt{\textcircled{2}} \rightarrow \Delta r = \Delta t \sqrt{\textcircled{3}}$$

$$r(0) = r_{min}$$

$$\phi(0) = 0$$

$$\phi(\Delta t) = \phi(0) + \Delta\phi$$

$$r(\Delta t) = r(0) + \Delta r$$

$$\phi(2\Delta t) = \phi(\Delta t) + \Delta\phi$$

$$r(2\Delta t) = r(\Delta t) + \Delta r$$

⋮

$$\phi(t_i) = \phi(t_{i-1}) + \Delta\phi$$

$$r(t_i) = r(t_{i-1}) + \Delta r$$

Cons. of  $E$ ,  $\vec{P}$ ,  $\vec{M}$ :

All of  $E$ ,  $\vec{P}$ ,  $\vec{M}$  conserved for a closed system

no external forces

$$U = U(\vec{r}_1 - \vec{r}_2, \vec{r}_2 - \vec{r}_3, \dots)$$

relative position vectors

Even in the presence of external forces, you can still have cons. of  $E$  and some components of  $\vec{P}$  and  $\vec{M}$ .

(i)  $U = mgx$        $\vec{F} = -m\vec{g}$

If  $U$  does not depend explicitly on time  $t$ , then  $E$  is conserved.

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

constant  
external  
field

(ii) e.g.  $\downarrow \vec{F}_g = m\vec{g}$

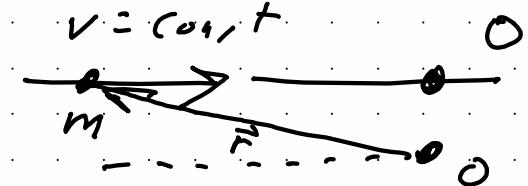
$$\xleftarrow{\hspace{1cm}} \quad x, y \quad \xrightarrow{\hspace{1cm}}$$

$$P_x = \text{const}$$

$$P_y = \text{const}$$

If  $U$  is unchanged by a translation in some direction  $\hat{E}$  then  $\vec{P} \cdot \hat{E} = \text{const}$

$$\vec{v} = \text{const}$$



(iii)  $\vec{M}$  depends on choice of origin

(a) uniform gravitational field



If  $U$  is unchanged by a rotation about a particular axis  $\hat{n}$

then  $\boxed{\vec{M} \cdot \hat{n} = \text{const}}$  (e.g.,  $\hat{n} = \hat{z}$ ,  $M_z = \text{const}$ )

$$\vec{F} = m\vec{g} \quad \uparrow z$$

$$U = m_1 g z_1 + m_2 g z_2 + \dots$$

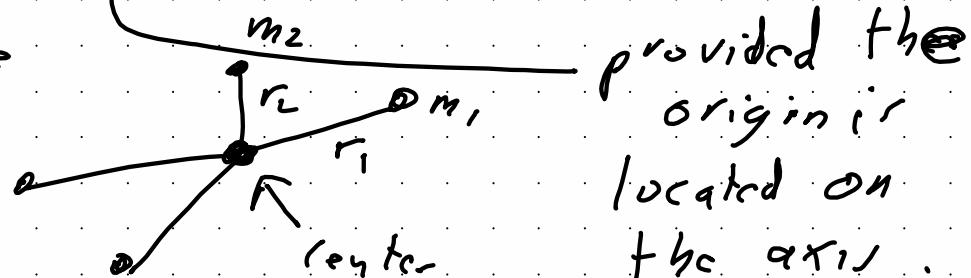
(b) central force

$$\boxed{M = \text{const}}$$

$$U = U(r)$$

$$\vec{F} = -\nabla U$$

$$= -\frac{dU}{dr} \hat{r}$$



## Mechanical similarity :

$$L \rightarrow L' = c \cdot L$$

same equations of motion

suppose we rescale position vectors  $\vec{r}_a \rightarrow \alpha \vec{r}_a$

$$U'(\vec{r}_1, \dots) = U(\alpha \vec{r}_1, \alpha \vec{r}_2, \dots) = \alpha^K U(\vec{r}_1, \vec{r}_2, \dots)$$

potential is homogeneous of degree  $K$   
w.r.t position vector

Example: i)  $U = mgx$ ,  $K = 1$

ii)  $U = \frac{1}{2}kx^2$ ,  $K = 2$

iii)  $U = -\frac{Gm_1 m_2}{r}$ ,  $K = -1$

$$\begin{aligned} U' &= mg \alpha x \\ &= \alpha mgx \\ &= \alpha U \end{aligned}$$

$$\begin{aligned} L' &= T' - U' = T' - \alpha^K U = \text{const.} \cdot L \\ &= \alpha^{K+1} T - \alpha^K U \\ &= \alpha^{K+1} (T - U) = \alpha^{K+1} L \end{aligned}$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \frac{l^2}{t^2} \quad \begin{array}{l} \text{length} \\ \text{time} \end{array}$$

$$\begin{aligned} l &\rightarrow l' = \alpha l \\ t &\rightarrow t' = \beta t \end{aligned}$$

$$T' = \frac{1}{2} m v'^2 = \frac{1}{2} m \frac{l'^2}{t'^2}$$

$$= \frac{1}{2} m \frac{\alpha^2 l^2}{\beta^2 t^2}$$

$$= \frac{\alpha^2}{\beta^2} T$$

$$\boxed{\frac{\alpha^2}{\beta^2} = \alpha^{4t}}$$

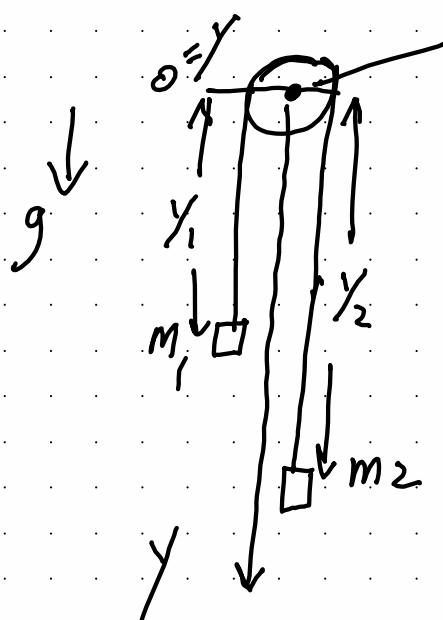
$$\beta^2 = \alpha^{2-4t}$$

$$\begin{cases} \frac{l'}{l} = \alpha \\ \frac{t'}{t} = \beta \end{cases} \quad \boxed{\begin{array}{l} U = mgy, k=1 \\ \frac{t'}{t} = \left(\frac{l'}{l}\right)^{\frac{1}{2}} \\ U = \frac{1}{2} kx^2, k=2 \\ \frac{t'}{t} = \text{const} \end{array}}$$

$$\boxed{\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-t/2}}$$

$$\begin{array}{c} \uparrow \\ \boxed{\beta = \alpha^{1-t/2}} \\ \uparrow \\ \boxed{\frac{P^2}{GJ} = \text{const}} \end{array} \quad \boxed{\begin{array}{l} U = -\frac{GM_1 M_2}{r} \\ kT = -1 \\ \left(\frac{t'}{t}\right) = \left(\frac{l'}{l}\right)^{3/2} \\ P^2 = \text{Dist}^3 \end{array}}$$

# QUIZ #1 : Atwood's machine



$$(m_1 > m_2)$$

string: length  $l$

(mass less  
inextensible  
...)

$$i) \quad L \quad ?$$

$$ii) \quad EoM$$

$$iii) \quad \text{solve EoM}$$

$$y_1 + y_2 = l \rightarrow y_2 = l - y_1 \rightarrow \dot{y}_2 = -\dot{y}_1$$

$$\begin{aligned} U &= -m_1 g y_1 - m_2 g y_2 \\ &= -m_1 g y_1 - m_2 g (l - y_1) \\ &= -m_1 g y_1 - \underbrace{m_2 g l}_{\text{ignore}} + m_2 g y_1 \end{aligned}$$

$$= \boxed{-(m_1 - m_2) g y_1} \quad \parallel \quad \dot{y}_1^2$$

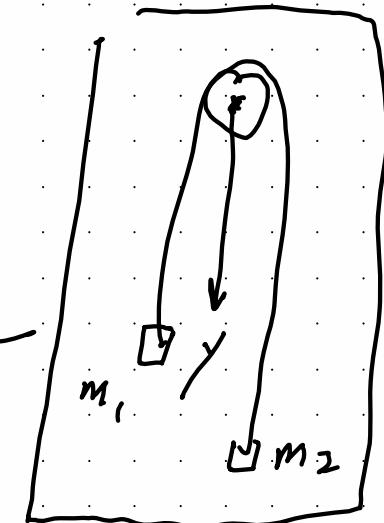
$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2$$

$$= \boxed{\frac{1}{2} (m_1 + m_2) \dot{y}_1^2}$$

# Lecture #7: Tuesday 9/15

- 1) Go over quiz #1
- 2) Modified Atwood problem
- 3) Finish mechanical similarity (sec 10)
- 4) Hamilton's equations (sec 40)

$$\downarrow a \quad (a=g)$$



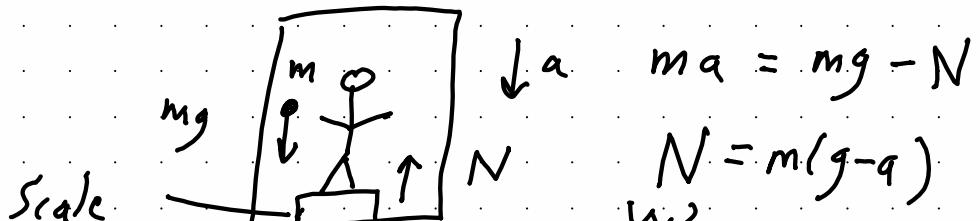
$$L = T - U$$

$$= \frac{1}{2} (m_1 + m_2) \dot{y}_1^2 + (m_1 - m_2) g y_1$$

EOMs:  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}_1} \right) = \frac{\partial L}{\partial y_1} \rightarrow \frac{(m_1 + m_2) \ddot{y}_1}{m_1 - m_2} = g$

$$\boxed{\ddot{y}_1 = \frac{(m_1 - m_2) g}{(m_1 + m_2)}}$$

$$y_1(t) = y_0 + v_0 t + \frac{1}{2} \frac{(m_1 - m_2) g t^2}{(m_1 + m_2)}$$



$$\boxed{\ddot{y}_1 = \frac{(m_1 - m_2) (g-a)}{m_1 + m_2}}$$

apparent effective weight

$$\vec{F} = \vec{m}\vec{a} \quad (\text{valid in an inertial ref frame})$$

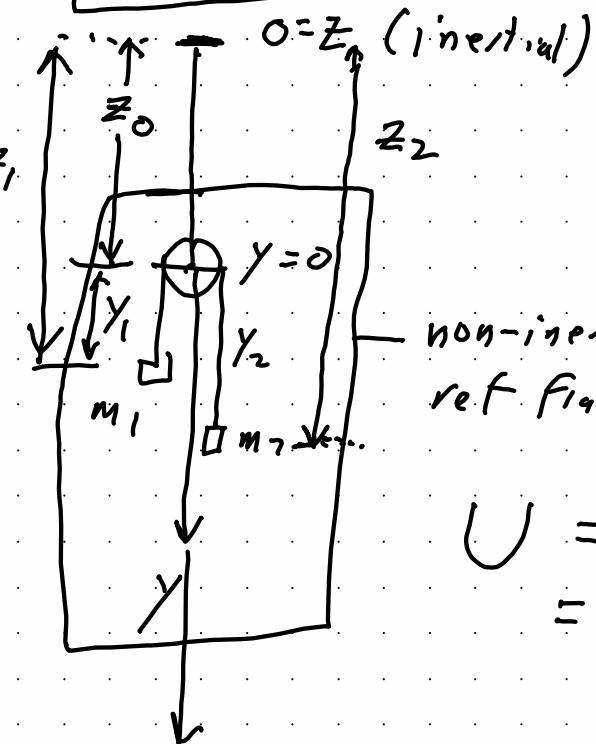
$$\vec{F} + \vec{F}_{\text{fictional}} = \vec{m}\vec{a} \quad \text{w.r.t. to a non-inertial ref. frame}$$

sec 3q (L&L)

$$L' = L + F(t)$$

$$L = T - U$$

(valid in an inertial ref frame)



non-inertial  
ref frame

$$T = \frac{1}{2}m_1 \dot{z}_1^2 + \frac{1}{2}m_2 \dot{z}_2^2$$

$$U = -m_1 g z_1 - m_2 g z_2$$

$$z_1 = z_0 + y_1 \quad | \quad y_2 = l - y_1$$

$$z_2 = z_0 + y_2$$

$$l - y_1$$

$$U = -m_1 g (z_0 + y_1) - m_2 g (z_0 + y_2)$$

$$= \boxed{-m_1 g z_0} - m_1 g y_1 \boxed{-m_2 g z_0} \boxed{-m_2 g l + m_2 g y_1}$$

"const"

prescribed function of time = ignore

$$= \boxed{-(m_1 + m_2)g y_1}$$

Do this at home:

$$T = \frac{1}{2} (m_1 + m_2) \dot{y}_1^2 + (m_1 - m_2) z_0 \dot{y}_1$$

$$L' = L + \frac{d}{dt} (f(q, t)) \rightarrow \text{same EoM}$$

$$(m_1 - m_2) z_0 \dot{y}_1 = \frac{d}{dt} [(m_1 - m_2) z_0 \dot{y}_1] - (m_1 - m_2) z_0'' y_1$$

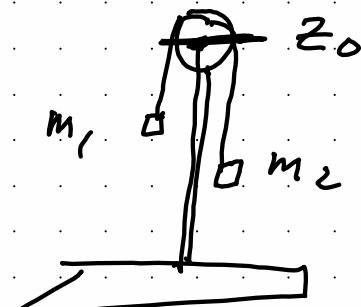
$f(y_1, t)$

$\equiv \alpha$

ignore

$$T = \frac{1}{2} (m_1 + m_2) \dot{y}_1^2 - (m_1 - m_2) \alpha y_1$$

$z_0(t)$ : given  $\underline{\underline{}}$ , not to be solved for



Hamilton's equations:

$$L(q_i, \dot{q}_i, t)$$

Hamiltonian:  $E = \sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$

$$H = H(q, p)$$

$$E(q, \dot{q}, t)$$

not here

if  $L = L(q, \dot{q})$

$$H = \left( \sum p_i \dot{q}_i - L \right) \Big|_{\dot{q} = \dot{q}(q, p)}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Example:

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

(single particle, 1-d,  
const external field)

$$p = \frac{\partial L}{\partial \dot{x}} = m \dot{x}$$



$$\dot{x} = \frac{p}{m}$$

$$H = (p \dot{x} - L) \Big|_{\dot{x} = p/m}$$

~~t~~

$$= \left( p \dot{x} - \frac{1}{2} m \dot{x}^2 + U(x) \right) \Big|_{\dot{x} = p/m} = \frac{p^2}{m} - \frac{1}{2} m \left( \frac{p}{m} \right)^2 + U(x)$$

$$H = \frac{1}{2} \frac{p^2}{m} + U(x)$$

EOMs: (Hamilton's equations) (39,6) Prob 2  
Sec 40

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad i=1, \dots, s \quad \# \text{ of DOF}$$

Lagrange's equation:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} \quad i=1, \dots, s$$

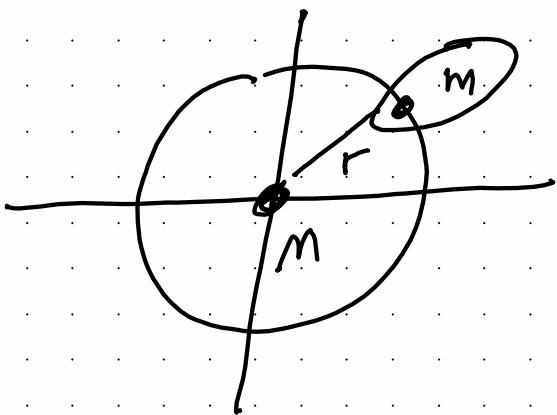
$s$ -equations, 2<sup>nd</sup> order ODE for  $q_i$

$$\begin{aligned} x &= p_m \\ p &= m\dot{x} \\ \dot{p} &= -\frac{\partial U}{\partial x} \end{aligned}$$

$\rightarrow$  2s-equations, 1<sup>st</sup> order in  $q_i, p_i$

$$H = \frac{p^2}{2m} + U(x) \rightarrow \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$$

$$L = \frac{1}{2}m\dot{x}^2 - U(x) \rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \rightarrow m\ddot{x} = -\frac{\partial U}{\partial x}$$



$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$v = \sqrt{\frac{GM}{r}}$$

$$U = -\frac{GMm}{r}$$

Problem:  $U' = cU$

~~masses = constant~~

$$r' = r, m' = m$$

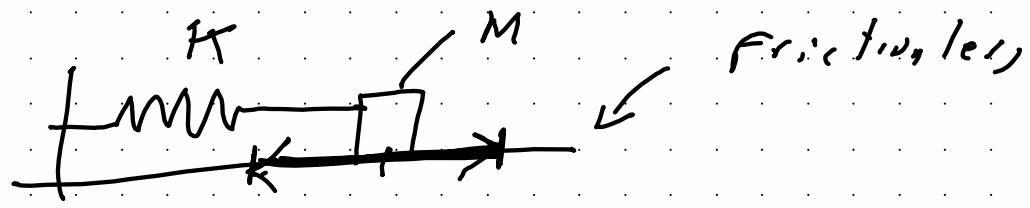
Suppose  $M \rightarrow cM = M'$

$$U' = cU$$

$$\frac{2\pi r}{T'} = v' = \sqrt{\frac{cGM}{r}}$$

$$\frac{1}{T'} \propto \sqrt{c}$$

$$\sqrt{\frac{E}{E'}} = \sqrt{\frac{U'}{U}}$$



$$U = \frac{1}{2} Kx^2$$

$$T = \frac{1}{2} m \dot{x}^2$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} Kx^2$$

$$\omega = \sqrt{\frac{K}{m}}$$

$$m \ddot{x} = -Kx$$

$$\ddot{x} = -\frac{K}{m}x$$

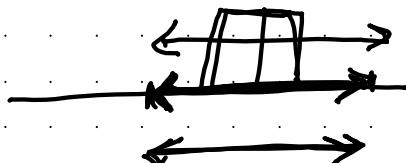
$$\rightarrow x(t) = a \cos \omega t + b \sin \omega t$$

$$U' = cU \quad (\text{for example } T' = ct)$$

$$T' = cT$$

$$\ddot{x} = -\frac{K'}{m}x$$

$$\omega' = \sqrt{\frac{K'}{m}}$$



$$T = \frac{1}{2} m \dot{x}^2$$

$$T' = cT$$

$$\frac{2\pi}{P'} = \sqrt{\frac{K'}{m}}$$

$$\frac{P}{P'} = \sqrt{\frac{U'}{U}}$$

$$L = T - U$$

$L, L'$

same  $E_{0m}$

$$L' = cL \quad = cT - cU$$

$L, L''$

$$L'' = T - cU \neq cL$$

different  $E_{0m}$

---

$$T = \frac{1}{2}mx^2, \quad U = \frac{1}{2}\kappa x^2$$

$$\frac{1}{\kappa} = c$$

$$L = T - U \rightarrow mx'' = -\kappa x$$

$$L' = cL \rightarrow \cancel{mx''} = -\cancel{c}\kappa x$$

$$mx'' = -\kappa x$$

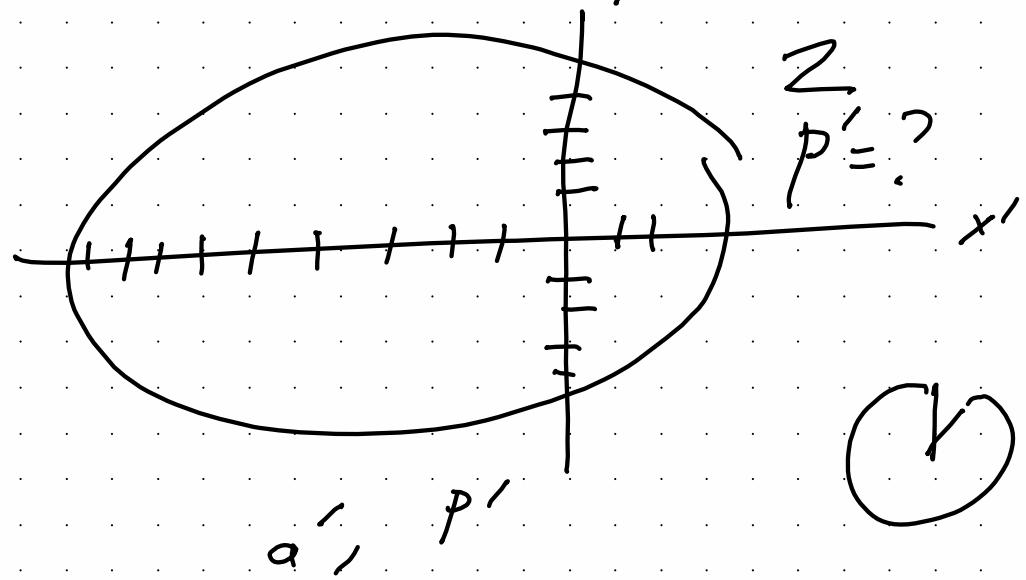
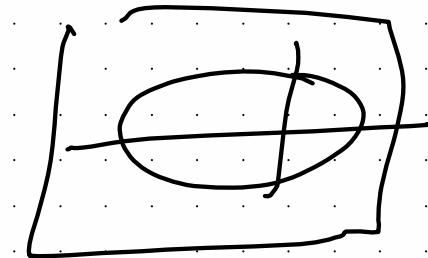
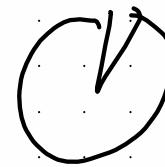
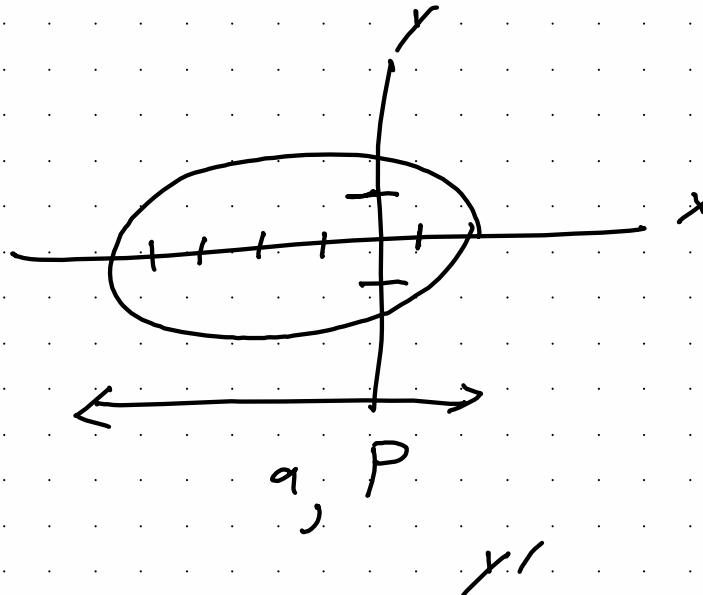
$$L'' \rightarrow mx'' = -c\kappa x$$

$$cx'' = \cancel{x''} = -c\left(\frac{\kappa}{m}\right)x$$

different  
 $E_{0m's}$

---

$$x = a \cos(\omega't) + b \sin(\omega't)$$



$$\stackrel{?}{=} p'?$$

$$\frac{p^2}{a^2} = \cos T$$



$$U' = cU$$

$$\cancel{P'} = ? \cancel{P}$$

$$m' = m$$

$$l' = l$$

$$L' = \cancel{(c\omega_0 + L)}$$

$$T' - U' = (c\omega_0 + L) - U'$$

$$T' - cU = c \cancel{(T-U)} \cancel{+ cU} = U'$$

$\cancel{+}$   
 $\cancel{-}$

$$\frac{P'}{P} = \sqrt{\frac{U}{U'}}$$

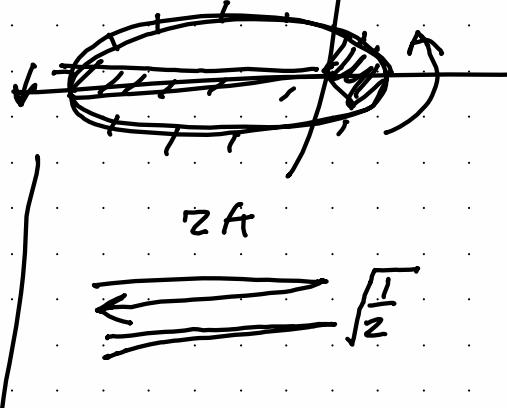
"mechanical similarity"

$$H \rightarrow H' = ck$$

Periods

$$g \rightarrow 2g$$

$$1m \left[ P' ? P \right] \frac{1}{\sqrt{2}}$$



$$T = \frac{1}{2} m \dot{x}^2$$

$$T' = \cancel{T}$$

$$U' = cU$$

$$\cancel{T' \neq cT}$$

$$L' = cL$$



Lec #8: Thu Sep 17th

Today — 1-d motion (Sec 11)

Next two weeks — central force (Sec 13-15)

\* Midterm 1 — Tues Oct 6<sup>th</sup>

$$\underline{T = \frac{1}{2} m(\dot{q})^2}$$

$$T = \frac{1}{2} \sum_{j=1}^N q_j \dot{q}_j \quad [ \text{single particle} ]$$

$$\bar{T} = \frac{1}{2} m \vec{v}^2$$

$$U(q_1, \dots, q_N, t)$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad \text{cartesian } (x, y, z)$$

$$= \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \quad \begin{matrix} \text{sp. polar} \\ (r, \theta, \phi) \end{matrix}$$

$$x = r \sin \theta \cos \phi$$

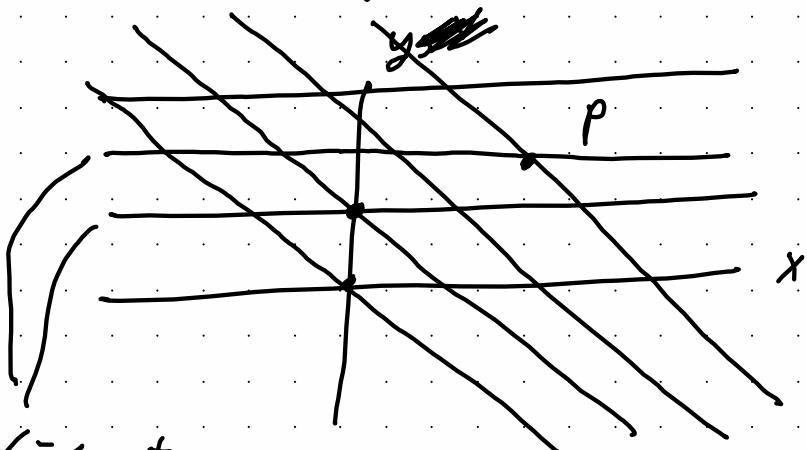
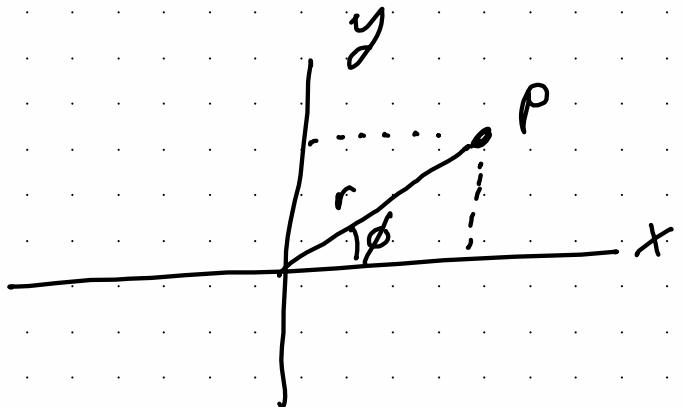
$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\left[ \begin{array}{l} q_1 = r \\ q_2 = \theta \\ q_3 = \phi \end{array} \right] \quad a_{ij} = m \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$r^2 \sin^2 \theta$$

$$\underline{\underline{a_{11} = m}}, \quad \underline{\underline{q_{22} = mr^2}}, \quad \underline{\underline{q_{33} = mr^2 \sin^2 \theta}}, \quad \underline{\underline{q_{12} = 0}}, \quad \underline{\underline{q_{13} = 0}}$$



$$V = \text{const}, t$$

$$\begin{matrix} \uparrow & \uparrow \\ u & = \text{const} & + \end{matrix}$$

$$T = \frac{1}{2} m (\dot{u}^2 + \dot{v}^2) ??$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \checkmark$$

$$= \frac{1}{2} m (\dot{u}^2 + \dot{v}^2 - 2\dot{u}\dot{v} + \dot{v}^2)$$

$$= \frac{1}{2} m (\dot{u}^2 + 2\dot{v}^2 - 2\dot{u}\dot{v})$$

$$(x, y)$$

$$(r, \phi)$$

$$\begin{matrix} \dot{x} = \ddot{u} - \ddot{v} \\ \dot{y} = \ddot{v} \end{matrix}$$

$$u = x + y$$

$$\cancel{u = x - y}$$

$$v = y$$

$$x = u - v$$

$$y = v$$

$$u = \text{const} ?$$

$$x + y = \text{const}$$

$$y = \text{const} - x$$

b      m

$$a_{ij} = m \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$T = \frac{1}{2} m |\vec{v}|^2$$

$$\vec{v} = \sum_{i=1}^3 v_i \hat{e}_i$$

$$|\vec{v}|^2 = \vec{v} \cdot \vec{v}$$

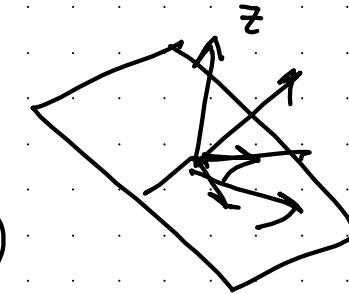
$$= (\sum_i v_i \hat{e}_i) \cdot (\sum_j v_j \hat{e}_j)$$

$$= \sum_{i,j} v_i v_j \hat{e}_i \cdot \hat{e}_j$$

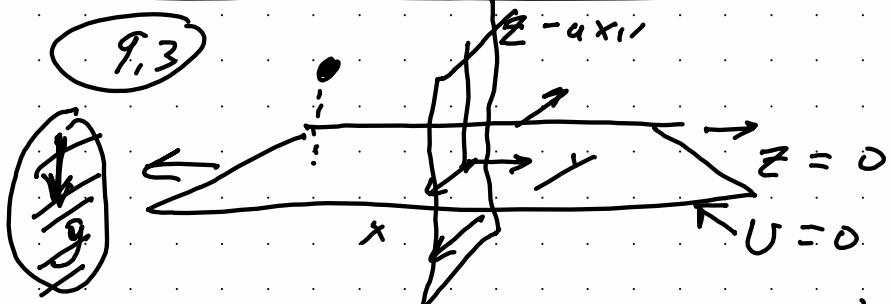
$\underbrace{m_i g + n_i t}_{\text{might not}} = \delta_{ij} \quad (\text{orthogonal})$

$$\hat{e}_i \cdot \hat{e}_i = 1$$

$$\hat{e}_i \cdot \hat{e}_j \neq 0$$



9.3



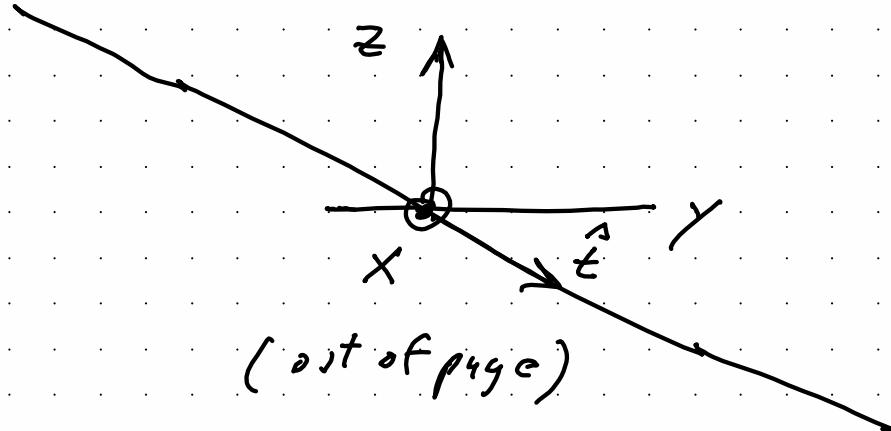
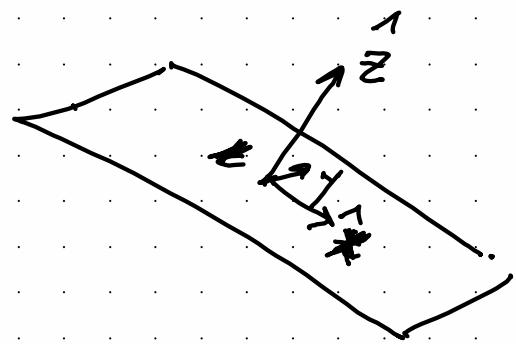
$$(a) U(x, y, z)$$

$$m g z = U(z), \vec{F} = -m \vec{g}$$

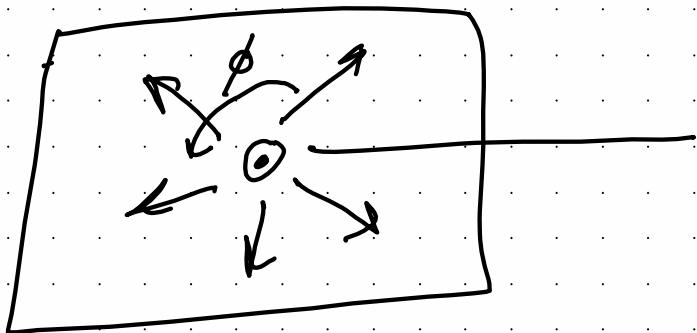
$\delta L = 0$ ; if you displace in  $\hat{x}, \hat{y}$   $\rightarrow [P_x, P_y] = \text{const}$

$$\vec{F} = -\vec{\nabla} U = -\frac{dU}{dz} \hat{z}$$

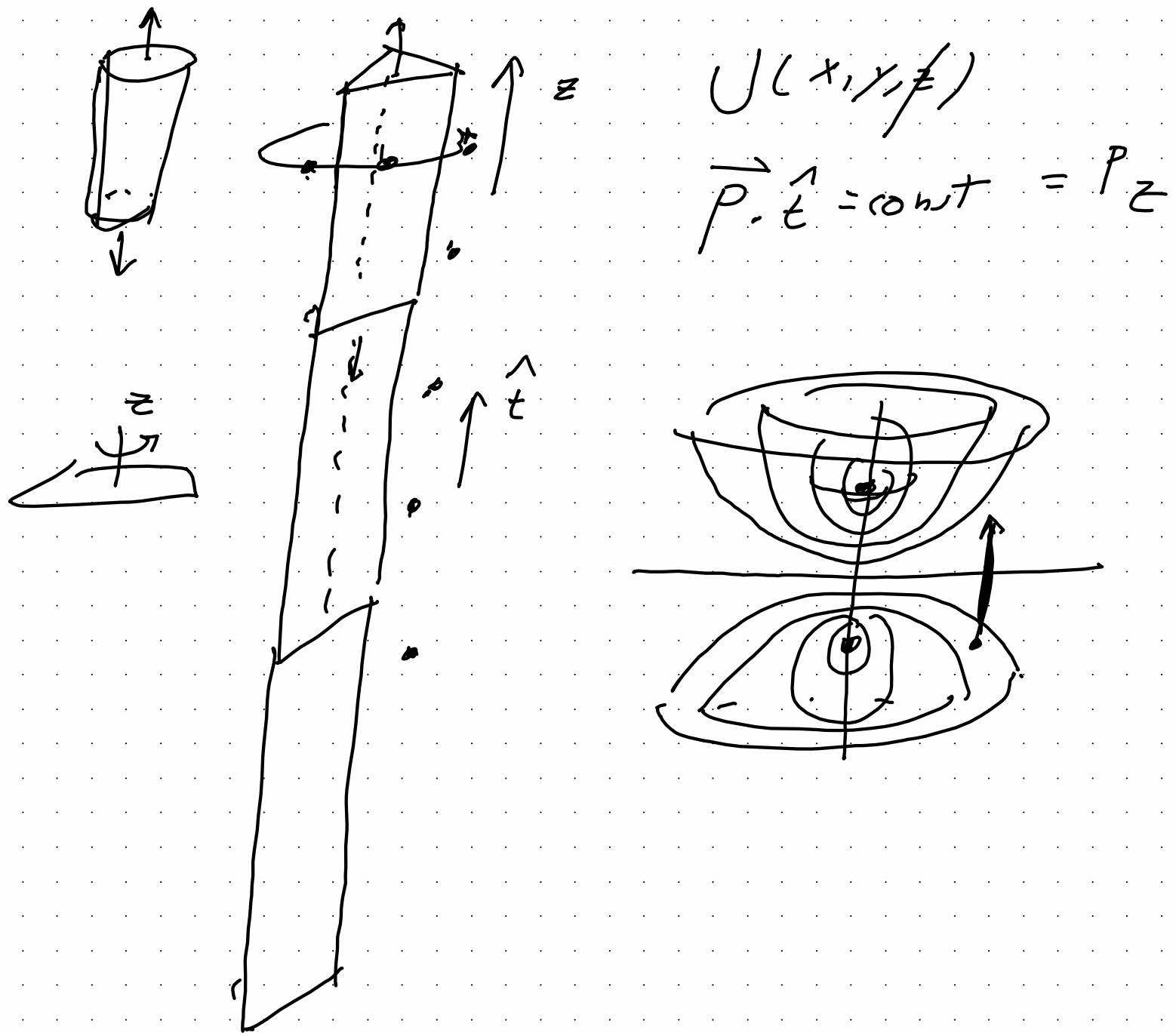
$M_z = \text{const}$



$$\vec{P} \cdot \hat{t} = \text{constant}$$

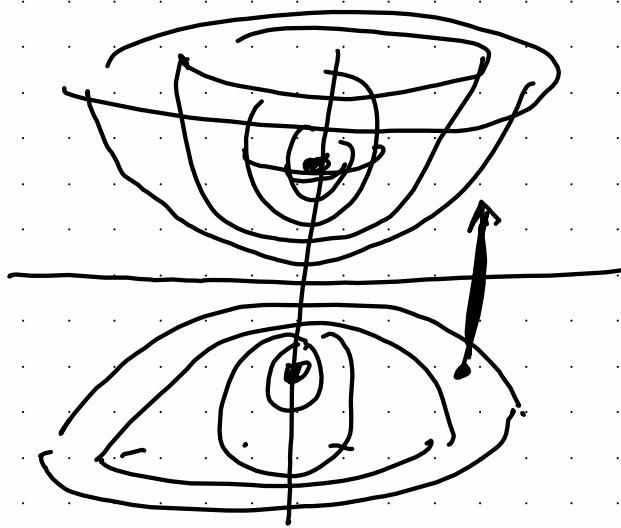


$$\vec{\Sigma} \text{ out} \\ \vec{M} \cdot \hat{z} = \text{constant}$$



$$U(x, y, f)$$

$$\vec{P} \cdot \hat{e} = \text{const} = P_z$$



$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$(E) = \frac{1}{2} m \dot{x}^2 + U(x) = \text{const}$$

positive

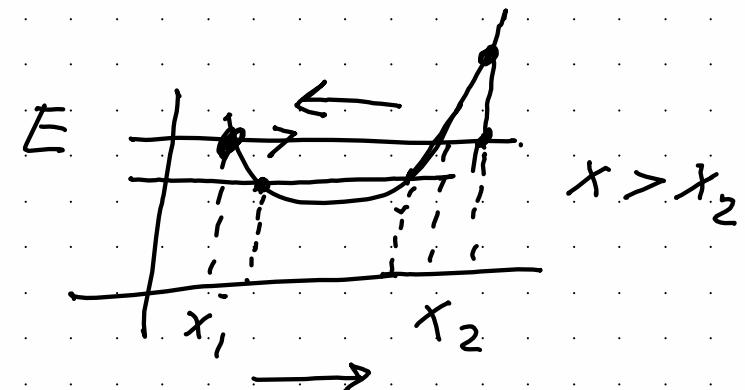
$$\frac{1}{2} m \dot{x}^2 = E - U(x)$$

$$\frac{dx}{dt} = \dot{x} = \pm \sqrt{\frac{2}{m}(E - U(x))} \quad \text{— separable, 1st order}$$

$$\int \pm \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}} = \int dt$$

$t = \int \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}} + \text{const}$

$$\Delta x = \pm \Delta t \sqrt{\frac{2}{m}(E - U(x))}$$



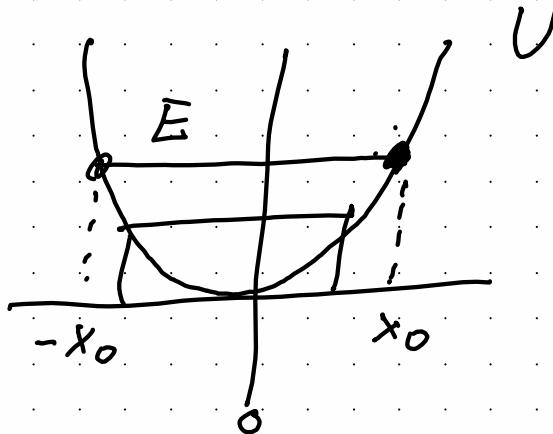
ODE  $\frac{dx}{dt} = \sqrt{\frac{2}{m}(E - U(x))}$

$$T(E) = 2 \int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}}$$

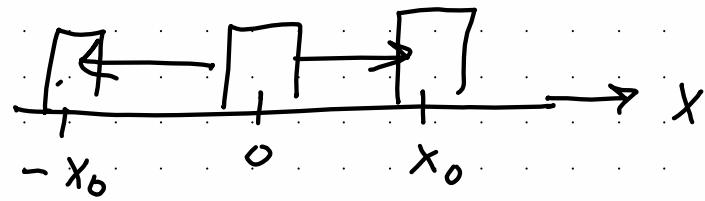
$$U(x) = \frac{1}{2} k x^2$$

$$U(\phi) = mg \cos \phi$$

$$\phi = \frac{g}{l} \sin \phi$$



$$U = \frac{1}{2} kx^2$$



→ Beta Functions

$$E = \frac{1}{2} kx_0^2$$

$$x_0 = \sqrt{\frac{2E}{k}}$$

$$E = U(x_1) = U(x_2)$$

Prob 2

$$(a) U = A|x|^n$$

$$(b) U = -\frac{U_0}{\cosh^2(\alpha x)}$$

$$(c) U = U_0 + \tan^2(\alpha x)$$

$$x_2(E)$$

$$\frac{dx}{\sqrt{E - U(x)}}$$

$$x_1(E) \uparrow \frac{1}{2} kx_0^2 - \frac{1}{2} kx^2$$

$$\frac{1}{2} k(x_0^2 - x^2)$$

$$\frac{dx}{\sqrt{x_0^2 - x^2}}$$

Lecture #9: Tues 9/22

40%

60%

- 1) Midterm #1: Tues Oct 6<sup>th</sup> (short answer; long problems)
- 2) Next 4 lecture (central force problem)  
Sec 13 - 15

General Formalism:

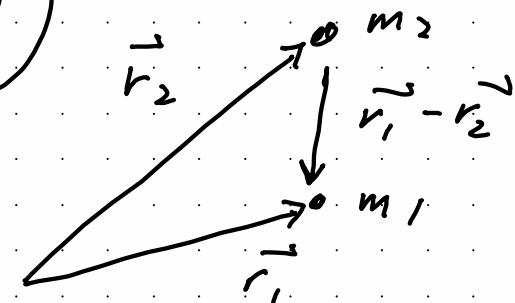
Two interacting particles (no external forces)

$m_1, m_2$

$$U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$$

$$L = T - U$$

$$\begin{aligned} &= \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 \\ &\quad - U(|\vec{r}_1 - \vec{r}_2|) \end{aligned}$$



i)  $L$  unchanged by a translation

$$\begin{aligned} \vec{r}_1 &\rightarrow \vec{r}_1 + \delta \vec{x} \\ \vec{r}_2 &\rightarrow \vec{r}_2 + \delta \vec{x} \end{aligned}$$

$$\vec{p} = m_1 \vec{v}_1 + m_2 \vec{v}_2 = \text{const}$$

com moves with const velocity

choose ref frame such that COM at orig'.

(  
inertial)

III  
COM Frame

$$\frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \vec{R}_{\text{COM}} = 0$$

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

Define:  $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$

$$U(|\vec{r}_1 - \vec{r}_2|)$$

$$U = U(r), \quad r = |\vec{r}|$$

~~\*\*\*~~

$$\vec{r}_2 = \vec{r}_1 - \vec{r}$$

$$m_1 \vec{r}_1 + m_2 (\vec{r}_1 - \vec{r}) = 0$$

$$(m_1 + m_2) \vec{r}_1 = m_2 \vec{r} = 0$$

$$\vec{r}_1 = \left( \frac{m_2}{m_1 + m_2} \right) \vec{r}$$

$$T = \frac{1}{2} m_1 |\vec{r}_1|^2 + \frac{1}{2} m_2 |\vec{r}_2|^2$$

$$\vec{r}_2 = \left( \frac{m_2}{m_1 + m_2} \right) \vec{r} - \vec{r}$$

$$\vec{r}_2 = - \left( \frac{m_1}{m_1 + m_2} \right) \vec{r}$$

$$T = \frac{1}{2} m_1 \left( \frac{m_2}{m_1 + m_2} \right)^2 |\dot{\vec{r}}|^2 + \frac{1}{2} m_2 \left( \frac{-m_1}{m_1 + m_2} \right)^2 |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)^2} |\dot{\vec{r}}|^2 \cancel{(m_2 + m_1)}$$

$$= \frac{1}{2} \left( \frac{m_1 m_2}{m_1 + m_2} \right) |\dot{\vec{r}}|^2$$

reduced mass :  $m$

$$= \frac{1}{2} m |\dot{\vec{r}}|^2$$

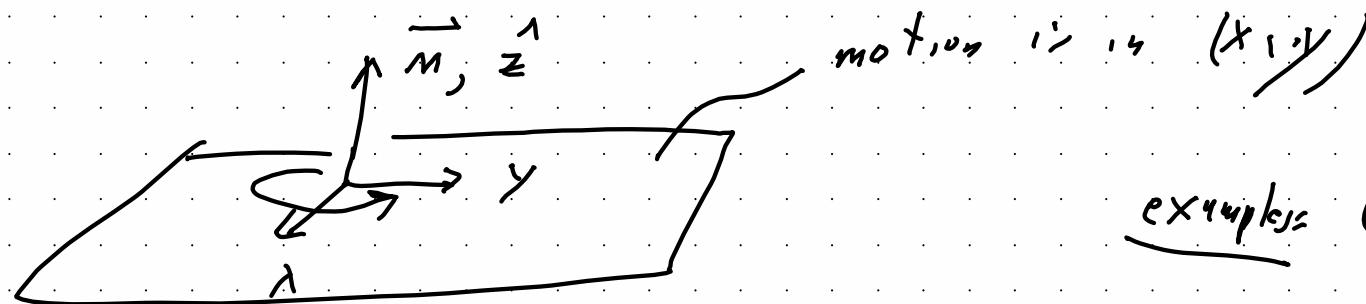
central potential

$$\boxed{L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(r)}$$

"effective"  
one body Lagrangian

ii)  $L$  unchanged under  $\tau$  rotations

$$\rightarrow \overrightarrow{M} = \text{const} \quad (\overrightarrow{M} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v})$$



motion is in  $(x, y)$

Newtonian gravity  
example:  $U(r) = -\frac{Gm_1m_2}{r}$

$$L = \perp m /dot{r})^2 - U(r)$$

$$U(r) = \frac{1}{2}kr^2$$

$$= \frac{1}{2}m(r^2 + r^2\dot{\phi}^2) - U(r)$$

Space oscillator

i) No explicit t-dependence

$$\boxed{E = T + U = \text{const}}$$

$E, M$  constants  
of the motion

ii) No  $\phi$  dependence

$$M_z = p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \text{const} = mr^2\ddot{\phi} = M$$

$$\boxed{mr^2\ddot{\phi} = M}$$

$$\left. \begin{array}{l} E = \frac{1}{2}m(r^2 + r^2\dot{\phi}^2) + U(r), \\ M = mr^2\dot{\phi} \end{array} \right\} \rightarrow \boxed{\dot{\phi} = \frac{M}{mr^2}}$$

$$\left. \begin{array}{l} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} \\ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L_{\phi}}{\partial \phi} \end{array} \right\} \rightarrow E = \frac{1}{2}mr^2 + \left( \frac{M^2}{2mr^2} + U(r) \right)$$

$\underbrace{\hspace{10em}}_{U_{\text{eff}}(r)}$

$$\begin{aligned} \frac{dr}{dt} &= \dot{r} = \sqrt{\frac{2}{m} \left( E - U(r) - \frac{M^2}{2mr^2} \right)} \\ &= \sqrt{\frac{2}{m} (E - U(r)) - \frac{M^2}{m^2 r^2}} \end{aligned}$$

~~dt~~

$$dt = \sqrt{\frac{dr}{\frac{2}{m} (E - U(r)) - \frac{M^2}{m^2 r^2}}} + \text{const}$$

$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{M}{mr^2} = \sqrt{\quad}$$

$\int \frac{\frac{dr}{d\phi} M}{mr^2} dt + \text{const}$  =  $\int d\phi = \phi$

$$\phi = \int \frac{M dr/r^2}{\sqrt{2m(E - U(r)) - M^2/r^2}} + \text{const}$$

$$\phi = \phi(r) \iff r = r(\phi)$$

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

$$m_1 + m_2$$

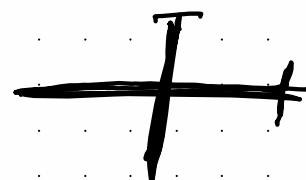
$$\begin{aligned} r &= r(\phi) \\ r &= r(t) \\ \phi &= \phi(t) \end{aligned} \quad \left. \begin{array}{l} r(t) \\ \phi(t) \end{array} \right\} \vec{r}(t)$$

Example:  $U(r) = \frac{1}{2}kr^2$  (space oscillator)  $(r, \phi)$

$$\begin{aligned}\text{Egy: } L &= \frac{1}{2}m(r'^2 + r^2\dot{\phi}^2) - \frac{1}{2}kr^2 \quad (x, y) \\ &= \frac{1}{2}m(x'^2 + y'^2) - \frac{1}{2}k(x^2 + y^2) \\ &= \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) + \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2\right)\end{aligned}$$

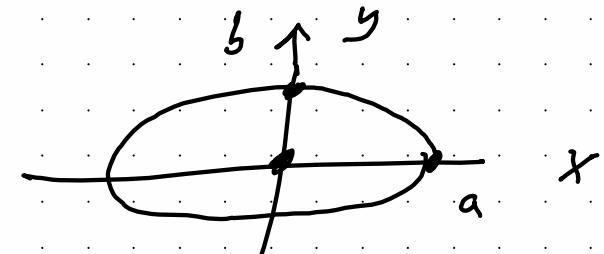
$$x(t) = a \cos(\omega t + \alpha), \quad \omega = \sqrt{\frac{k}{m}}$$

$$y(t) = b \sin(\omega t + \beta), \quad \underline{\alpha}$$



→ closed orbit

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



ellipse with center at origin

$U = \frac{-\alpha}{r}$ ,  $U = \frac{1}{2}kr^2$  are only two potentials that have closed bound orbits.

Harter:  $U = \frac{1}{2} k r^2$

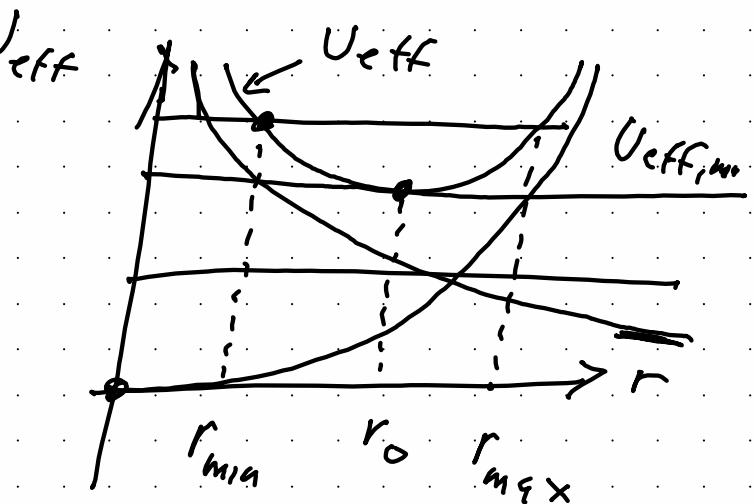
$$U_{\text{eff}} = U(r) + \frac{M^2}{2mr^2}$$

$$= \frac{1}{2} k r^2 + \frac{M^2}{2mr^2}$$

$$\phi = \int \frac{M dr/r^2}{\sqrt{2m(E - \frac{1}{2}kr^2) - \frac{M^2}{r^2}}} + \text{const}$$

substitutes

$$u = \frac{1}{r}, du = -\frac{dr}{r^2}$$



bound orbits

$$v = u^2$$

$$dv = 2u du$$

$$\phi = - \int \frac{M du}{\sqrt{2m(E - \frac{1}{2}\frac{k}{u^2}) - M^2 u^2}} = - \int \frac{Mu du}{\sqrt{2m(Eu^2 - \frac{k}{2}) - M^2 u^2}} + \text{const}$$

+ const

$$\phi = -\frac{1}{2} \int \frac{dv M}{\sqrt{2m(Ev - \frac{k}{2}) - M^2 v^2}} + \text{const}$$

Complete the square:  $-M^2 v^2 + 2mEv - mk$

$$= -M^2 \left( v^2 - \frac{2mEv}{M^2} + \frac{mk}{M^2} \right)$$

$$= -M^2 \left( \left( v - \frac{mE}{M^2} \right)^2 - \frac{m^2 E^2}{M^4} + \frac{mk}{M^2} \right)$$

$$= -M^2 \left( (v-A)^2 - B^2 \right)$$

$$= M^2 (B^2 - (v-A)^2)$$

$$A = \frac{mE}{M^2}, \quad B^2 = A^2 - \frac{mk}{M^2}$$

$$\phi = -\frac{1}{2} \int \frac{dv}{\sqrt{B^2 - (v-A)^2}} + C_{\text{const}}$$

3rd substitution:

$$v-A = B \sin \theta$$

$$dv = B \cos \theta d\theta$$

$$v-A = \pm B$$

$$\frac{1}{r^2} - A = \pm B$$

$$r_{\max}$$

$$\frac{1}{r^2} = A \pm B$$

$$\int \frac{dx}{\sqrt{1-x^2}}$$

$$x = \sin \theta$$

$$dx = \cos \theta d\theta$$

$$\int \frac{\cos \theta d\theta}{\sqrt{1-\sin^2 \theta}}$$

$$= \theta$$

$$= \sin^{-1}(x)$$

$$\phi = -\frac{1}{2} \int \frac{B \cos \theta d\theta}{\sqrt{B^2 - B^2 \sin^2 \theta}} + C_{\text{const}}$$

$$= -\frac{1}{2} \theta + C_{\text{const}}$$

$$= -\frac{1}{2} \sin^{-1} \left( \frac{v-A}{B} \right) + C_{\text{const}}$$

$$= -\frac{1}{2} \sin^{-1} \left( \frac{\frac{1}{r^2} - A}{B} \right) + C_{\text{const}}$$

$$\frac{1}{r_{\max}^2} = A - B$$

$$\frac{1}{r_{\min}^2} = A + B \quad v = 4^2$$

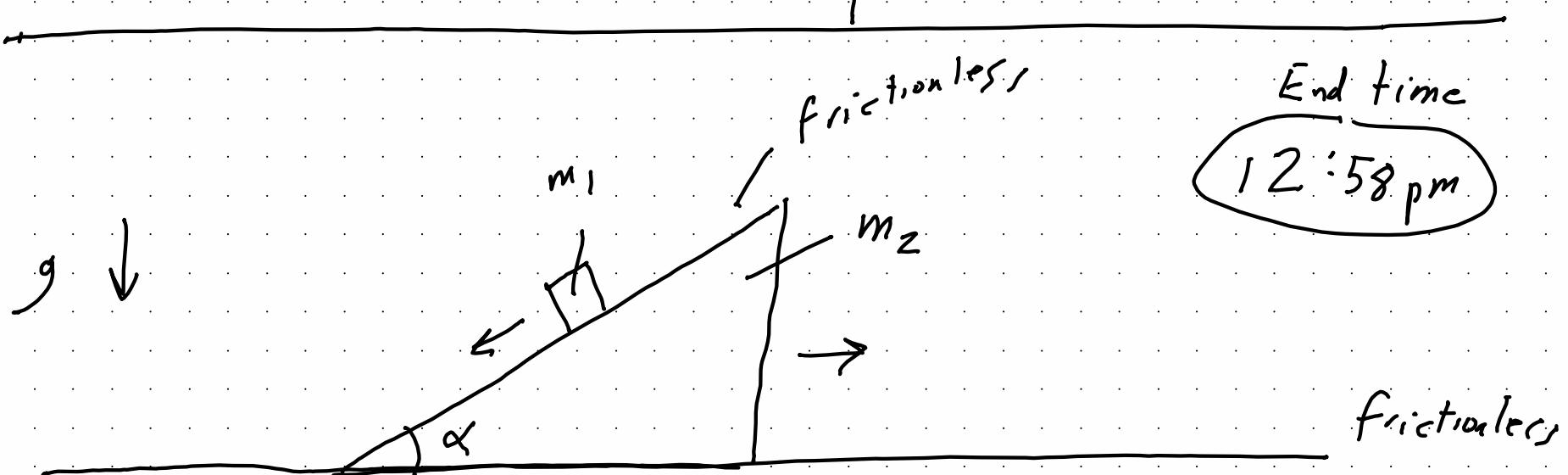
$$4 = \frac{1}{r}$$

$$v = \frac{1}{r^2}$$

Lecture #10: 9/24

- Finsler spring oscillator
- Quiz #2
- Next week: Q&A

lastname - 9.2.pdf



i) generalized coord)

ii)  $L = T - V$

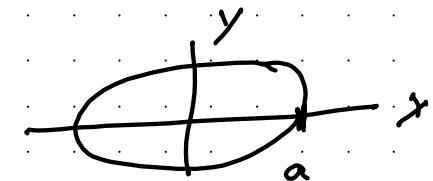
iii) EOMs

iv) special limiting cases:  $\alpha = 0$ ,  $\alpha = \pi/2$   
 $m_1 \gg m_2$ ,  $m_1 \ll m_2$

$$\phi = -\frac{1}{2} \sin^{-1} \left( \frac{\frac{1}{r^2} - A}{B} \right) + \underline{\text{const}} \quad E, M$$

choose const so that  $\phi=0 \leftrightarrow r=r_{\max}$

$$\frac{1}{r_{\max}^2} = A - B$$



$$\frac{\frac{1}{r_{\max}^2} - A}{+B} = \frac{-B}{B} = -1$$

$$\begin{aligned} 0 &= -\frac{1}{2} \sin^{-1}(-1) + \text{const} \\ &= -\frac{\pi}{2} + \text{const} \\ &= \frac{\pi}{4} + \text{const} \end{aligned}$$

$$\boxed{\text{const} = -\frac{\pi}{4}}$$

$$\phi = -\frac{1}{2} \sin^{-1} \left( \frac{\frac{1}{r^2} - A}{B} \right) - \frac{\pi}{4}$$

$$\begin{aligned} -2(\phi + \frac{\pi}{4}) &= \sin^{-1} \left( \frac{\frac{1}{r^2} - A}{B} \right) \\ -\sin \left( 2\phi + \frac{\pi}{2} \right) &= \left( \frac{\frac{1}{r^2} - A}{B} \right) \end{aligned}$$

$$-\sin\left(2\phi + \frac{\pi}{2}\right) = \left(\frac{1}{r^2} - A\right)/B$$

$$-B \cos(2\phi) = \frac{1}{r^2} - A$$

$$\boxed{\frac{1}{r^2} = A - B \cos(2\phi)}$$

$$\cos(2\phi) = \cos^2\phi - \sin^2\phi$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(2\phi) = \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2$$

$$\boxed{\frac{1}{r^2} = A - B \left(\left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2\right)}$$

$$1 = Ar^2 - B(x^2 - y^2)$$

$$= A(x^2 + y^2) - B(x^2 - y^2) =$$

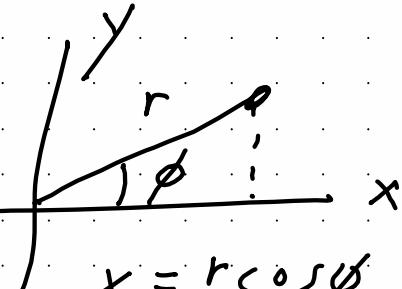
$$A = mE/m^2, B^2 = A^2 - mk/m^2$$

$$\sin\left(2\phi + \frac{\pi}{2}\right)$$

$$= \sin(2\phi) \cos\left(\frac{\pi}{2}\right)$$

$$+ \cos(2\phi) \sin\left(\frac{\pi}{2}\right)$$

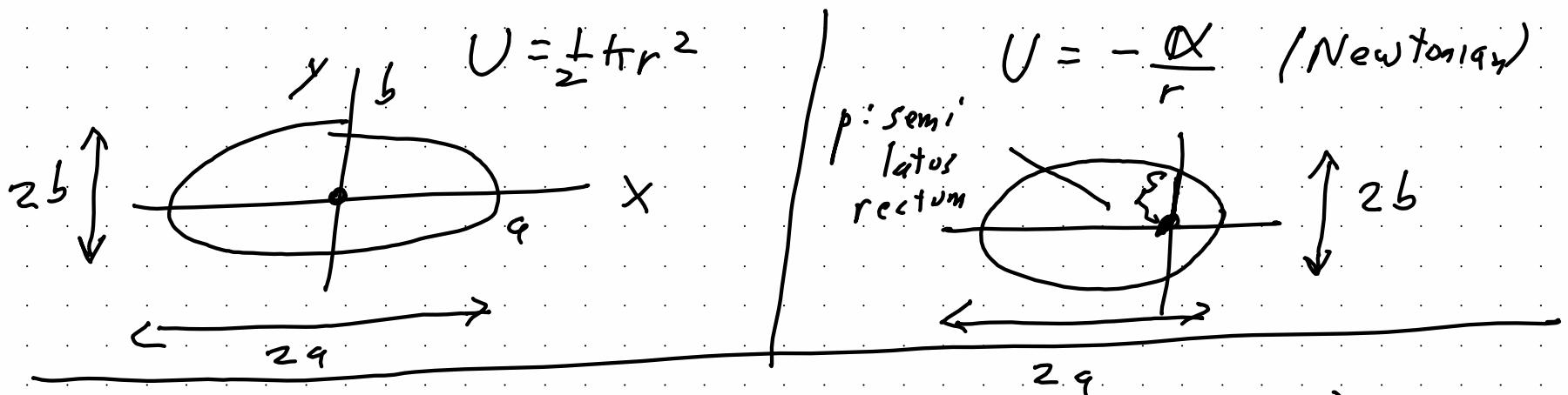
$$= \cos(2\phi) = 0$$



$$\boxed{x^2 + y^2 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1}$$

$$\boxed{\frac{1}{a^2} = A - B, \frac{1}{b^2} = A + B}$$

$$(A-B)x^2 + (A+B)y^2 = 1$$



$$E = \underbrace{\frac{1}{2} m r^2}_{=0, +} + \frac{M^2}{2mr^2} + \frac{1}{2} k r^2 \quad \begin{pmatrix} r=a \\ r=b \end{pmatrix}$$

$$\boxed{E = \frac{M^2}{2ma^2} + \frac{1}{2} k a^2}$$

$$\boxed{E = \frac{M^2}{2mb^2} + \frac{1}{2} k b^2}$$

$$U = -\frac{k}{r} \quad (\text{Newtonian})$$

$$\rightarrow E_a^2 = \frac{M^2}{2m} + \frac{1}{2} k a^2$$

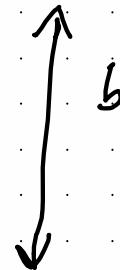
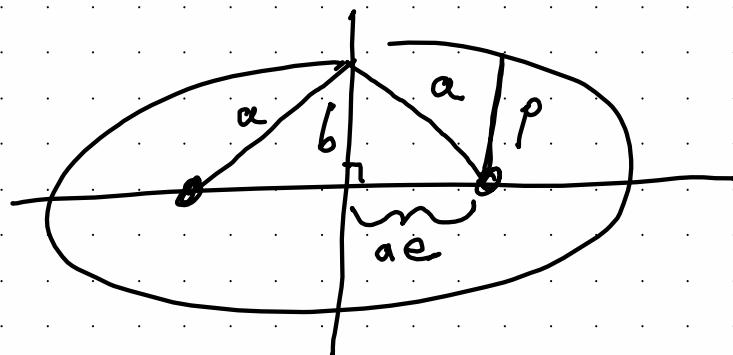
$$\rightarrow E_b^2 = \frac{M^2}{2m} + \frac{1}{2} k b^2$$

$$\rightarrow E = \frac{1}{2} k (a^2 + b^2)$$

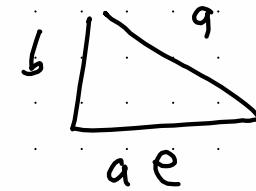
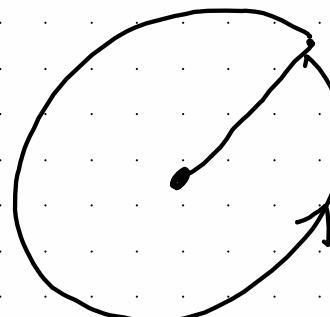
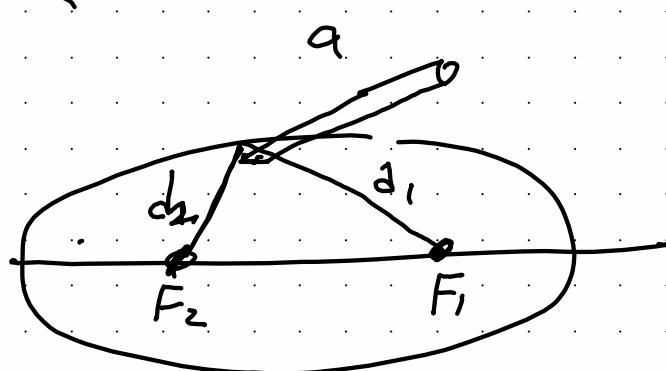
$$\rightarrow M^2 = m k a^2 b^2$$

$$\text{subtract } E(a^2 + b^2) = \frac{1}{2} k (a^4 - b^4)$$

$$(a^2 + b^2)(a^2 - b^2) \rightarrow$$



$e = 0$  circle  
 $e = 1$  line  
 dimensionless



$$a^2 = b^2 + a^2 e^2$$

$$e = \sqrt{\frac{a^2 - b^2}{a^2}}$$

$$d_1 + d_2 = 2a$$

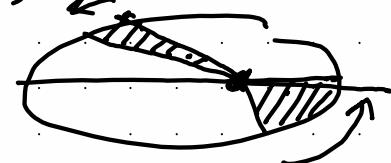
Kepler's Law: (For  $V = -\alpha/r$ )

$$b = a \sqrt{1-e^2}$$

i) elliptical orbits with sun at one focus

ii) equal areas in equal times

$$\frac{P^2}{a^3} = \text{const}$$



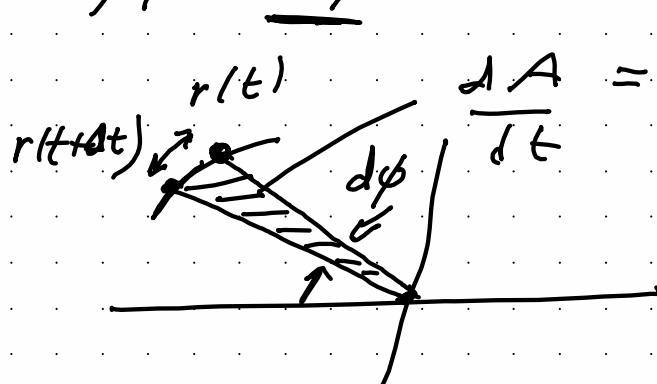
$$U = \frac{1}{2} k r^2$$

"Kepler's laws": (i) elliptical orbits with "Sun" at center  
of the ellipse  
(ii) equal areas in equal times

(iii)

true for any  
central potential

$M = mr^2\dot{\phi} = \text{const}$



$$\frac{dA}{dt} = \text{const}$$



$$\frac{dA}{dt} = \frac{M}{zm}$$

$$\int dA = \int \frac{M}{zm} dt$$

$$\frac{dA}{dt} = \frac{1}{2} r \frac{dr}{dt} \frac{d\phi}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt}$$

height

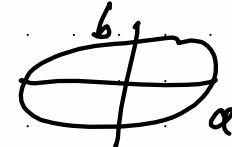
$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{M}{zm} = \text{const}$$

$$A = \frac{M}{2m} P$$

$$A = \pi r^2 \quad \text{circle}$$

$$= \pi ab$$

$$\pi ab = \frac{M}{2m} P$$



$$P = \frac{2m}{M} \pi ab$$

$$M^2 = m \pi a^2 b^2$$

$$M = \sqrt{m \pi a b}$$

$$= \frac{2m \pi ab}{\sqrt{m \pi a b}}$$

$$= \frac{2\pi}{\sqrt{K/m}}$$

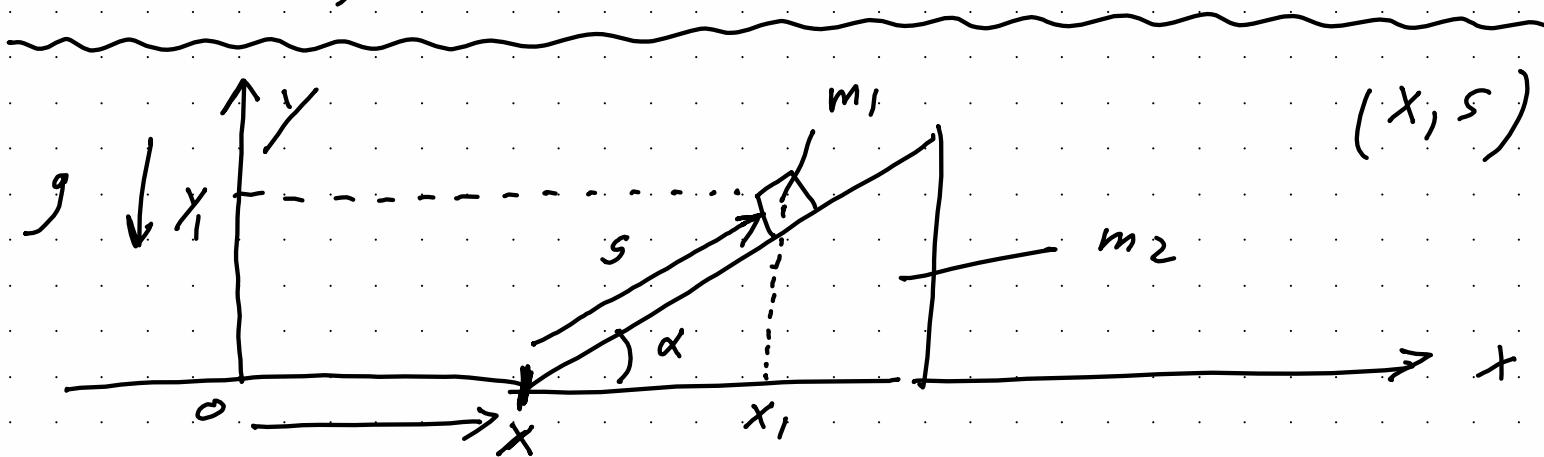
$$= \frac{2\pi}{\omega} = \text{const} \quad (\text{for any } a, b)$$

$$P = \text{const}$$

indep of  $a, b$  !!

Lec #11: Tuesday 9/29

- Midterm #1: Next Tuesday Oct 6<sup>th</sup>
- Trial midterm on Blackboard (do it before Thursday)
- Today:
  - a) Quiz #2
  - b) F. nsh  $U = \frac{1}{2} k r^2$
  - c) Q & A



$$L = T - U \quad T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$x_1 = x + s \cos \alpha \rightarrow \dot{x}_1 = \dot{x} + s \cos \alpha$$

$$y_1 = s \sin \alpha \rightarrow \dot{y}_1 = s \sin \alpha$$

$$x_2 = x \rightarrow \dot{x}_2 = \dot{x}$$

$$y_2 = 0 \rightarrow \dot{y}_2 = 0$$

$$\dot{x}_1^2 = (\dot{x} + s \cos \alpha)^2 = \dot{x}^2 + \underline{s^2 \cos^2 \alpha} + 2\dot{x}s \cos \alpha$$

$$\dot{x}_1^2 = \underline{s^2 \sin^2 \alpha}$$

$$T = \frac{1}{2} m_1 (\dot{x}^2 + \dot{s}^2 + 2\dot{x}s \cos \alpha) + \frac{1}{2} m_2 \dot{x}^2$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1 \dot{x} s \cos \alpha$$

$$U = m_1 g y_1 = m_1 g s \sin \alpha$$

$$L = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1 \dot{x} s \cos \alpha - m_1 g s \sin \alpha$$

Euler's:  $x$ :  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \cancel{\frac{\partial L}{\partial x}} \rightarrow 0 \quad | \quad E = T + U$

$$P_x = (m_1 + m_2) \dot{x} + m_1 s \cos \alpha = \text{const}$$

$s$ :

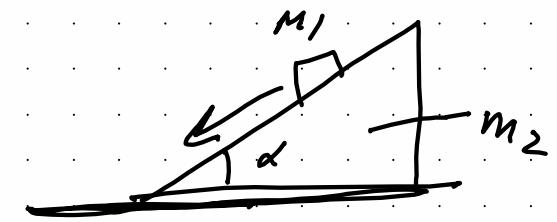
$$\rightarrow \boxed{(m_1 + m_2) \ddot{x} + m_1 \ddot{s} \cos \alpha = 0}$$

$$\frac{d}{dt} (m_1 \dot{s} + m_1 \dot{x} \cos \alpha) = -m_1 g \sin \alpha$$

$$\dot{s} + \dot{x} \cos \alpha = -g \sin \alpha$$

$$\ddot{x} = - \left( \frac{m_1}{m_1 + m_2} \right) \ddot{s} \cos \alpha$$

$$\ddot{s} = -g \sin \alpha \left( \frac{m_1 + m_2}{m_1 \sin^2 \alpha + m_2} \right)$$



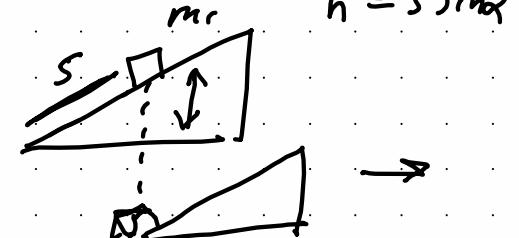
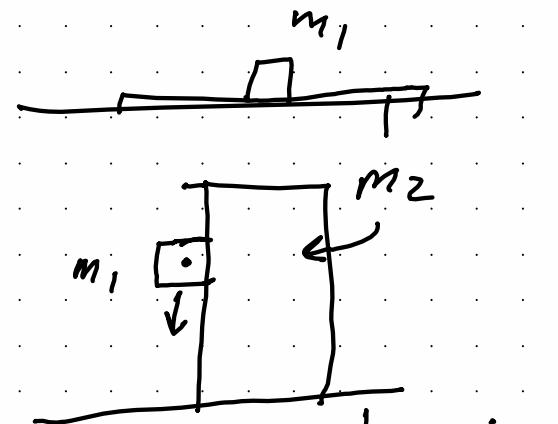
i)  $m_2 \gg m_1$ :  $\ddot{s} \approx -g \sin \alpha$

ii)  $\alpha = 0$   $\ddot{s} = 0, \ddot{x} = 0$

iii)  $\alpha = \frac{\pi}{2}$   $\ddot{s} = -g, \ddot{x} = 0$

iv)  $m_1 \gg m$ :  $\left\{ \begin{array}{l} \ddot{s} \approx -\frac{g}{\sin \alpha} \\ \ddot{x} \approx +\frac{g}{\tan \alpha} \end{array} \right.$

$h = -g = \ddot{s} \sin \alpha \rightarrow \ddot{s} = -\frac{g}{\sin \alpha}$



Space oscillator:  $U = \frac{1}{2}kr^2$

$$U = -\frac{\alpha}{r}$$

"Kepler's" law:

(i) ellipses with origin at center of ellipse

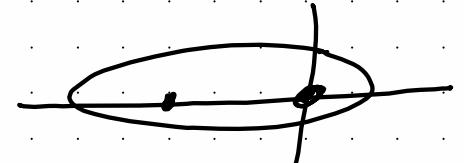


(ii) equal areas in equal times — approach to all central potentials

(iii)  $P = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}}$  indep. of  $a, b$

Central potential

$$\frac{P^2}{a^2} = \text{const} + \text{ordinary Kepler}$$



$$t = \int \frac{dr}{\sqrt{\frac{2}{m}(E-U(r)) - \frac{M^2}{m^2r^2}}} + \text{const}$$

reduced mass,

want  $t$  to increase as  $r$  decrease,

$$\frac{1}{2}kr^2 \quad E = \frac{1}{2}k(a^2+b^2) \\ M^2 = m^2a^2b^2$$

$$t = - \int \frac{dr}{\sqrt{\frac{1}{m} \left( \frac{1}{2} k(a^2 + b^2) - \frac{1}{2} kr^2 \right) - \frac{m + a^2 b^2}{m^2 r^2}}}$$

$(t, \phi)$   
 $(x, y)$

$$= - \frac{1}{\sqrt{\frac{k}{m}}} \int \frac{dr}{\sqrt{(a^2 + b^2) - r^2 - \frac{a^2 b^2}{r^2}}}$$

$$\begin{aligned} r^2 &= \\ x^2 + y^2 &\end{aligned}$$

$$= - \frac{1}{\sqrt{\frac{k}{m}}} \int \frac{r dr}{\sqrt{r^2(a^2 + b^2) - r^4 - a^2 b^2}}$$

$$= - \frac{1}{\sqrt{\cancel{a^2(1 - \sin^2 \theta)}}} \int \frac{r dr}{\sqrt{- (r^4 - r^2(a^2 + b^2) + a^2 b^2)}}$$

$$\begin{aligned} \cancel{a^2(1 - \sin^2 \theta)} &= a_{10, \theta} \\ \frac{dx}{\sqrt{a^2 - x^2}} &\\ x = a \sin \theta &\end{aligned}$$

$$\text{subst. } t = \frac{x}{a} \quad = (r^2 - a^2)(r^2 - b^2)$$

$$b < r < a$$

$$r^2 = a^2 \cos^2 \xi + b^2 \sin^2 \xi$$

$$x = a \cos \xi$$

$$y = b \sin \xi$$

$$\rightarrow r^2 = a^2 \cos^2 \xi + b^2 \sin^2 \xi$$

Differentiates

$$\begin{aligned} 2rdr &= -2a^2 \cos \xi \sin \xi + 2b^2 \sin \xi \cos \xi \\ &= -2 \sin \xi \cos \xi (a^2 - b^2) \end{aligned}$$

$$r dr = -\sin \xi \cos \xi (a^2 - b^2)$$

$$\sqrt{-(r^2 - a^2)(r^2 - b^2)} = \sqrt{- (a^2 \cos^2 \xi + b^2 \sin^2 \xi - a^2)(a^2 \cos^2 \xi + b^2 \sin^2 \xi - b^2)}$$
$$-\underline{a^2 \sin^2 \xi} \quad \quad \quad -\underline{b^2 \cos^2 \xi}$$

$$= \sqrt{- (-1/(a^2 - b^2)) \sin^2 \xi (a^2 - b^2) \cos^2 \xi}$$

$$= \boxed{(a^2 - b^2) \sin \xi \cos \xi}$$

$$t = \frac{1}{\omega} \int d\xi + \text{const} \rightarrow \boxed{\omega t = \xi} + \frac{\text{const}}{\omega} = 0$$

$$\begin{cases} x = a \cos(\omega t) \\ y = b \sin(\omega t) \end{cases}$$

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= a^2 \cos^2(\omega t) + b^2 \sin^2(\omega t) \end{aligned}$$

$$r = \sqrt{a^2 \cos^2(\omega t) + b^2 \sin^2(\omega t)}$$

$$\phi = \phi(t)$$

$$(x, y) \leftrightarrow (r, \phi)$$

$$\begin{cases} x = r \cos \phi \\ y = r \sin \phi \end{cases}$$

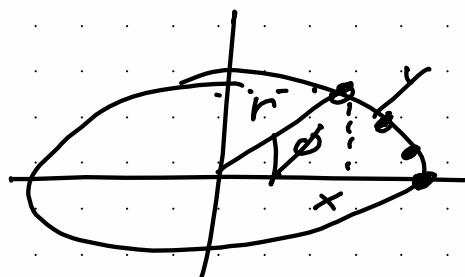
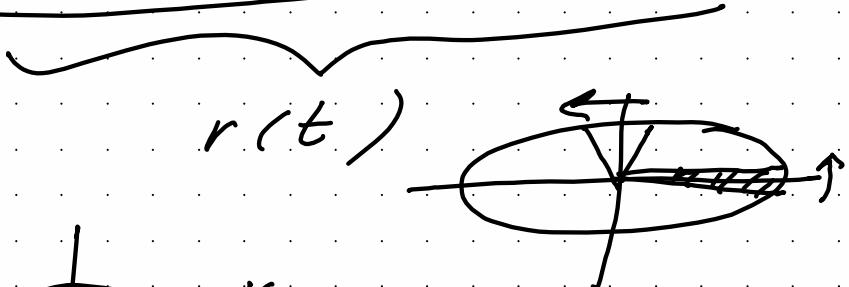
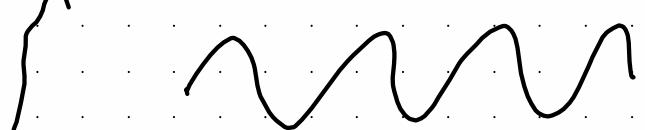
$$r \cos \phi = a \cos(\omega t)$$

$$r \sin \phi = b \sin(\omega t)$$

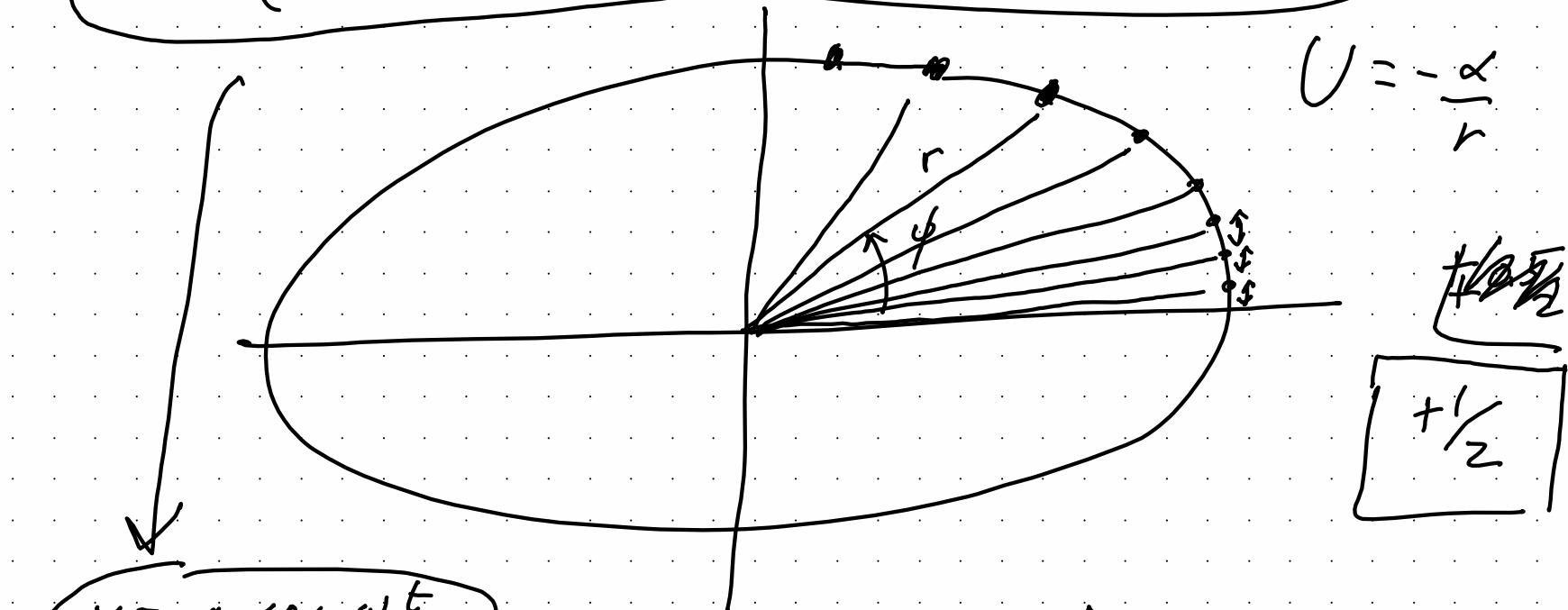
Divide 2<sup>nd</sup> by 1<sup>st</sup>:  $\tan \phi = \frac{b}{a} \tan(\omega t)$

$$\phi = \tan^{-1} \left( \frac{b}{a} \tan(\omega t) \right)$$

$$\rightarrow \phi = \omega t \text{ if } b = a$$



$$L = \left( \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2 \right) + \left( \frac{1}{2} m \dot{y}^2 - \frac{1}{2} K y^2 \right)$$



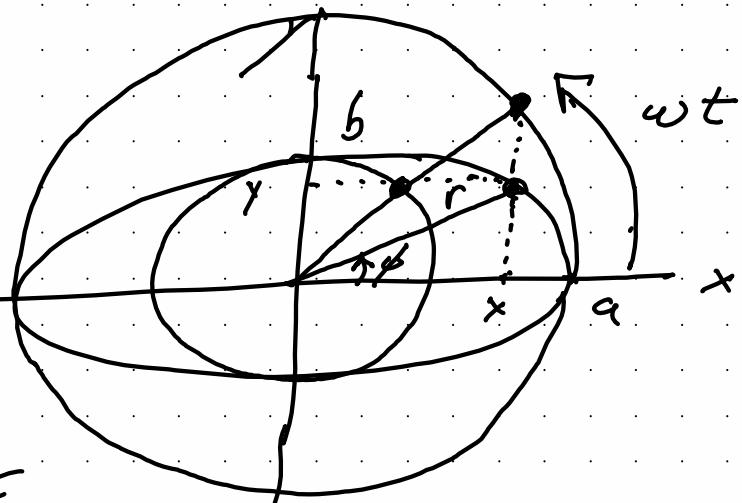
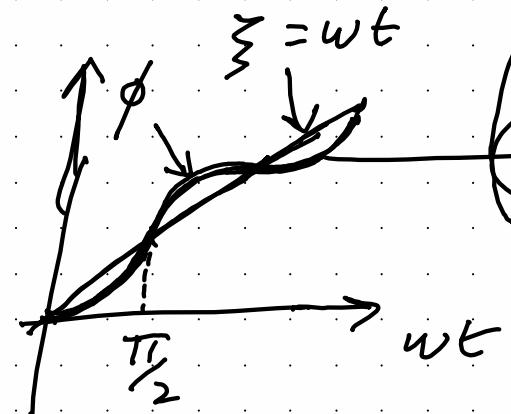
$$U = -\frac{\alpha}{r}$$

~~+1/2~~

$$+\frac{1}{2}$$

$$\begin{aligned} x &= a \cos \omega t \\ y &= b \sin \omega t \end{aligned}$$

$$\begin{aligned} x &= r \cos \phi \\ y &= r \sin \phi \end{aligned}$$



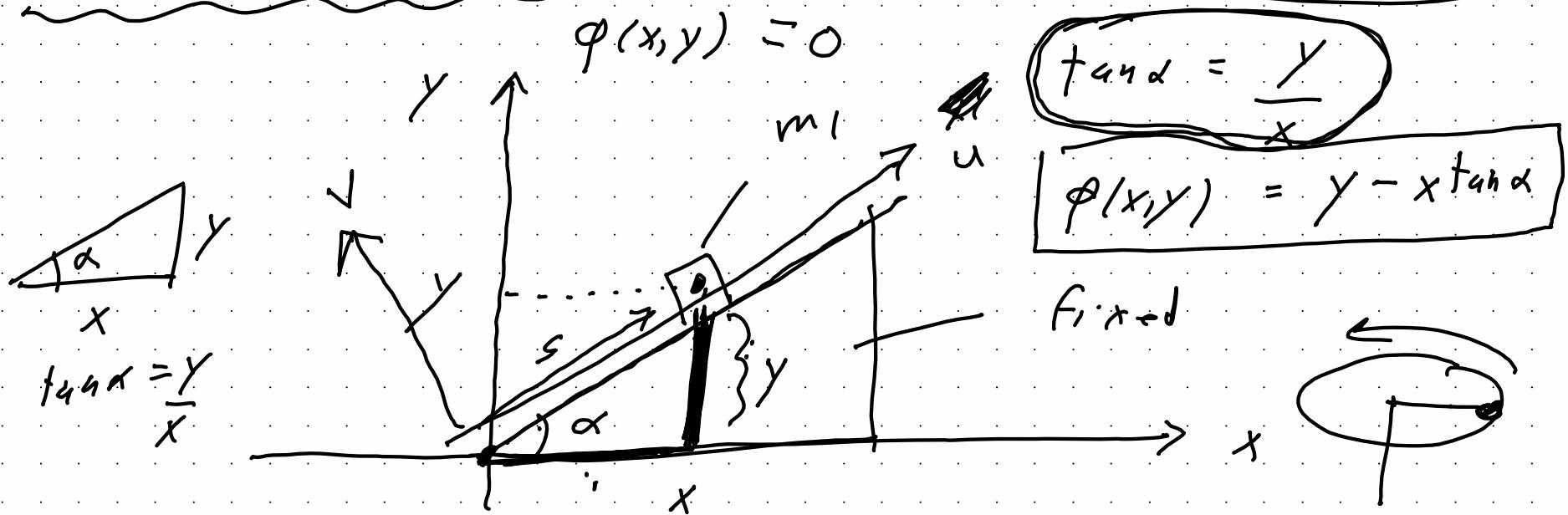
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## Lec #12:

- Midterm Exam 1 : Tuesday Oct 6 + 5
- Do trial exam by end of the day today
- Today : Q&A

$$x^2 + y^2 - r_0^2 = 0$$

$$r - r_0 = 0$$



$$T = \frac{1}{2} m_1 (x^2 + y^2), \quad U = m_1 g y \quad (x, y)$$

$$L = T - U + \underline{\underline{(y - x \tan \alpha)}} \quad \checkmark \quad (r, \phi)$$

L-OMs:

$$x : \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \rightarrow m_1 \ddot{x} = -\lambda \tan \alpha$$

$$y : \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) = \frac{\partial L}{\partial y} \rightarrow m_1 \ddot{y} = -mg + \lambda$$

constraint:

$$\dot{y} - \dot{x} \tan \alpha = 0$$

$$\ddot{y} - \ddot{x} \tan \alpha = 0$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2$$

$$(m_1 \ddot{x} = -\lambda \tan \alpha)_{\text{true}}$$

$$+ m_1 \dot{x} \tan \alpha = -mg + \lambda$$

$$0 = -\lambda \tan^2 \alpha - \lambda$$

$$+ m_1 g$$

$$= -\lambda (1 + \tan^2 \alpha) + mg$$

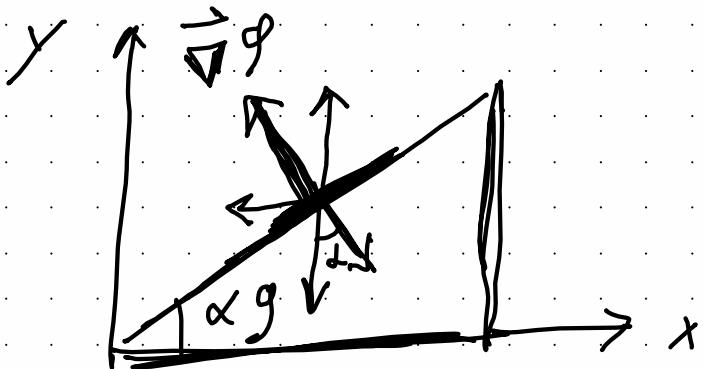
$$L = \frac{1}{2} m_1 (\dot{x}^2 + \dot{y}^2) - mg y + \lambda (\dot{y} - \dot{x} \tan \alpha)$$

$$\lambda = \frac{mg}{\sec^2 \alpha} \propto mg \cos \alpha$$

$$\varphi(x, y) = y - x \tan \alpha$$

$$\vec{\nabla} \varphi = \frac{\partial \varphi}{\partial x} \hat{x} + \frac{\partial \varphi}{\partial y} \hat{y}$$

$$= -\tan \alpha \hat{x} + \hat{y} = \frac{-\sin \alpha \hat{x} + \cos \alpha \hat{y}}{\cos \alpha}$$



$$= \frac{-\sin \alpha \hat{x} + \cos \alpha \hat{y}}{\cos \alpha}$$

$\vec{N} = mg \cos \alpha$   
 ↓ to incline

$$\vec{\nabla} \varphi \propto -\sin \alpha \hat{x} + \cos \alpha \hat{y}$$

$$y = x \tan \alpha$$

$$dy = \tan \alpha dx \rightarrow \boxed{\cos \alpha dy = \sin \alpha dx}$$

$$\vec{\nabla} \varphi \cdot d\vec{r} = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy = -\tan \alpha dx + \frac{1}{\tan \alpha} dx = 0$$

$$(u, v)$$

$$\varphi(u, v) = v = 0$$

$$L = T - U + \lambda v$$

$$v=0$$

$$L = T - U + \lambda \varphi$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt} (m \dot{x}) = - \frac{\partial U}{\partial x}$$

$$\boxed{\frac{d \vec{p}}{dt} = - \frac{\partial U}{\partial x} = F}$$

$$\frac{d \vec{p}}{dt} = F$$

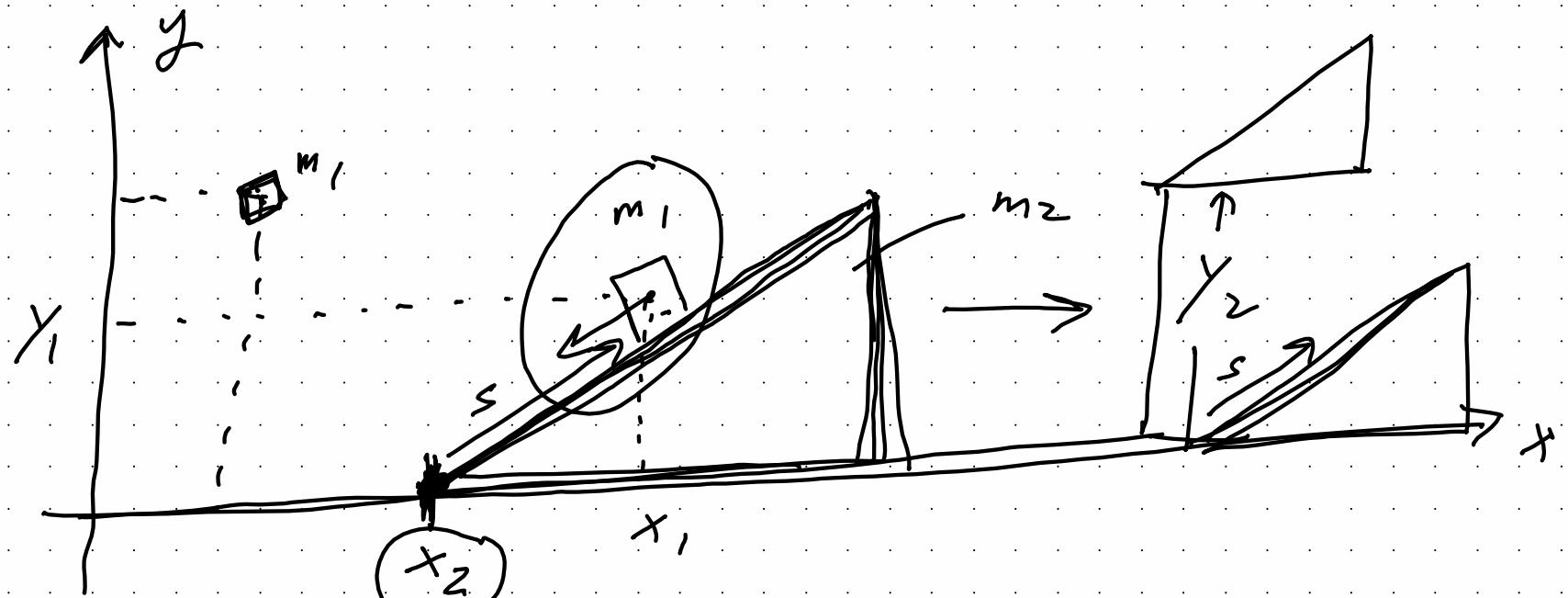
$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = - \frac{\partial L}{\partial r}$$

$$\frac{d \vec{p}}{dt} = - \frac{\partial U}{\partial r} + \lambda \frac{\partial \varphi}{\partial r}$$

$$\frac{d \vec{p}}{dt} = - \vec{\nabla} U + \vec{\nabla} \varphi$$

$\vec{F}_{\text{constant}}$   
 $\text{applied}$



$(x_1, y_1, x_2, y_2)$

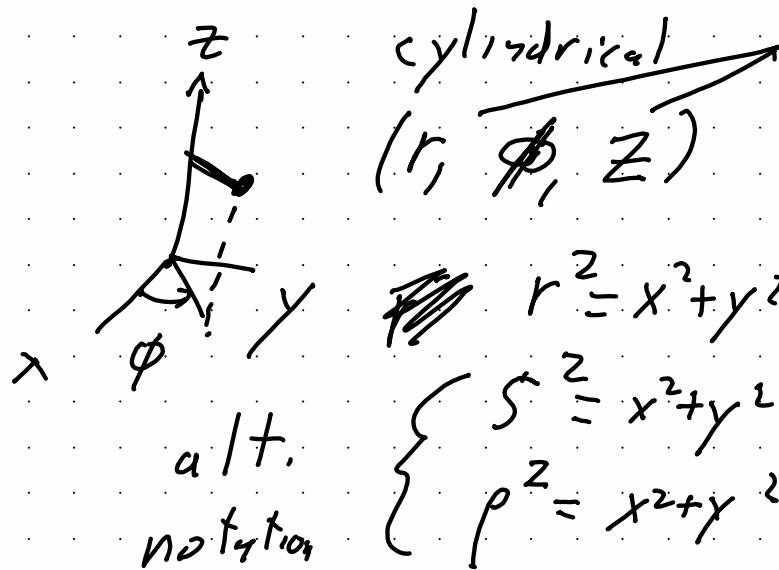
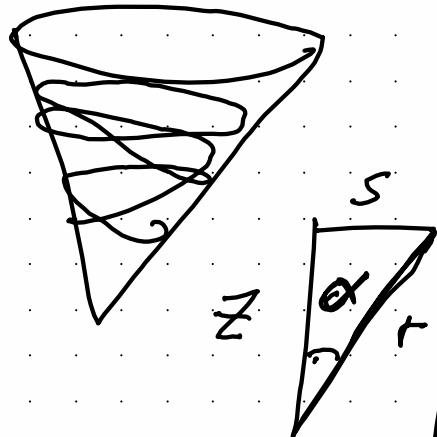
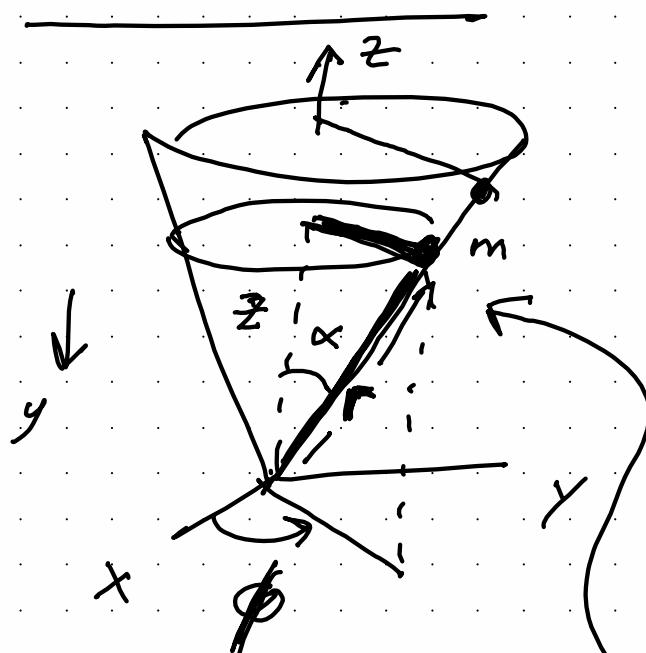
$\approx$

$m_1$

$y_2 = 0$

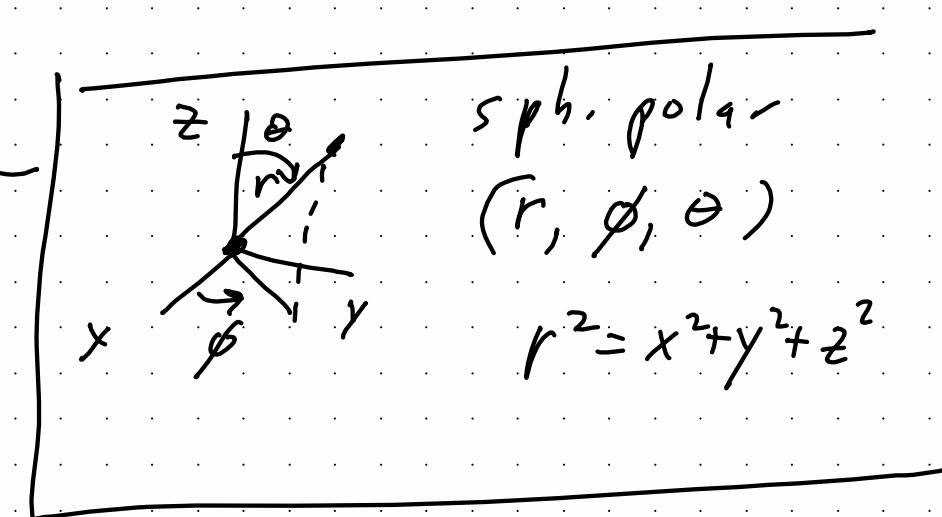
$$\underline{T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)}$$

Sec 14, Prob 2:



$$\cancel{r^2 = x^2 + y^2}$$

$$\left\{ \begin{array}{l} s^2 = x^2 + y^2 \\ \rho^2 = x^2 + y^2 \end{array} \right.$$



$$r^2 = x^2 + y^2 + z^2$$

$$\theta = \alpha$$

$$\phi, s$$

$$\phi, z$$

$$\tan \alpha = \frac{s}{z}$$

$$T = \frac{1}{2} m \left( r^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right)$$

$\downarrow$

$\theta$        $\sin^2 \alpha$

sp<sup>b</sup>. pol.,  
 $\theta = \alpha$

$$T = \frac{1}{2} m \left( s^2 + s^2 \dot{\phi}^2 + \dot{z}^2 \right)$$

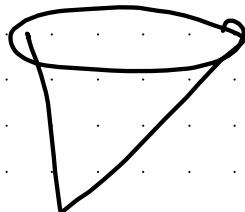
$\curvearrowleft$

cyl. coord,  
 $s, \phi, z$

~~theta~~

$\tan \alpha = \frac{s}{z}$

plane polar  
coord



$$L = T - V + \lambda \varphi$$

$$\varphi = s - z \tan \alpha$$

$$z = \frac{s}{\tan \alpha}$$

$$\dot{z} = \frac{s}{\tan \alpha}$$

$s = z \tan \alpha$ 
 $s = z \tan \alpha$

$$f(x, y) = e^{-\frac{1}{2}(x^2+y^2)}$$

No constraint:  $\left\{ \begin{array}{l} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{array} \right.$

$$\begin{aligned} g(x_1, y_1, \lambda) &= f(x_1, y_1) + \lambda \varphi(x_1, y_1) \\ \frac{\partial g}{\partial x} &= 0 \\ \frac{\partial g}{\partial y} &= 0, \quad \frac{\partial g}{\partial \lambda} = 0 \end{aligned}$$

Suppose constraint  $y = \frac{1}{2}x^2 + 1$

$$F(x) = f(x, y) \Big| y = \frac{1}{2}x^2 + 1$$

$$L = T - U + \lambda \varphi$$

$$x_1, y_1, \lambda$$

$$\boxed{\frac{dF}{dx} = 0}$$

$$\boxed{\varphi = y - \frac{1}{2}x^2 - 1 = 0}$$

COM:

$$m_1, m_2, m_3, \vec{r}_1, \vec{r}_2, \vec{r}_3 \quad \left. \right\} \vec{P} = M_{\text{tot}} \vec{R}_{\text{com}}$$

$$\vec{R}_{\text{com}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3}{m_1 + m_2 + m_3} = \text{const}$$

$$0 = \sum_a m_a \vec{r}_a$$

$\sum_b m_b$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}_a} \right) = 0$$

$$\vec{P} = \sum_a \frac{\partial L}{\partial \dot{r}_a} = \text{const}$$

~~closed system~~:  $\vec{P} = \text{const}$

$$L = \frac{1}{2} \sum_a m_a |\vec{r}_a|^2 - U(\vec{r}_1 - \vec{r}_2), (\vec{r}_2 - \vec{r}_3), \dots$$

$$\vec{r}_a \rightarrow \vec{r}_a + \delta \vec{x}$$

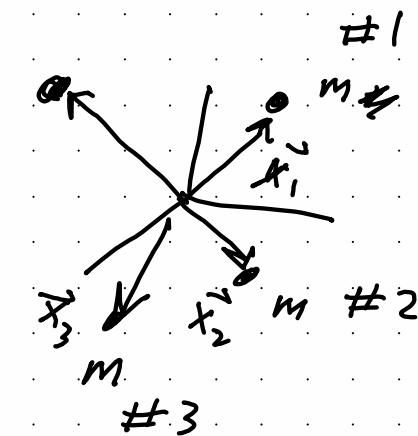
$$L \rightarrow L$$

$$\frac{\partial L}{\partial \vec{r}} = 0$$

$$M, \underbrace{m_1, \dots, m_n}_n$$

$$\overrightarrow{X} \quad \overrightarrow{x_1}, \dots, \overrightarrow{x_n}$$

$$M \overrightarrow{X} + m \sum_{a=1}^n \overrightarrow{x_a} = 0$$



$$\overrightarrow{r_1}, \overrightarrow{r_2}$$

$$\overrightarrow{r} = \overrightarrow{r_1} - \overrightarrow{r_2}$$

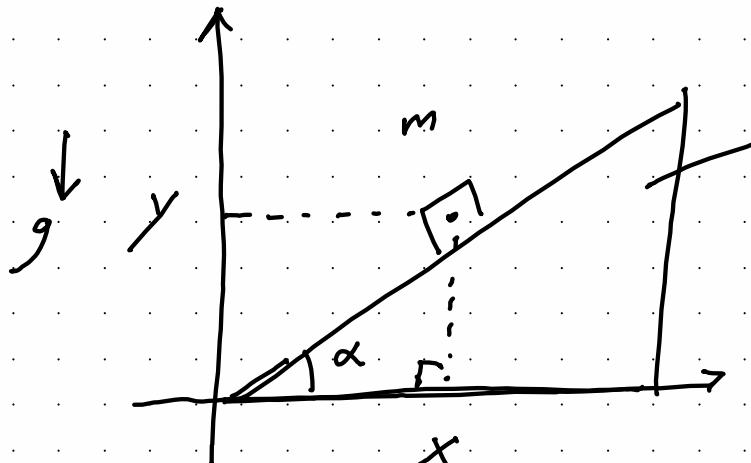
$$\begin{aligned}\overrightarrow{r_1} &= \overrightarrow{x_1} - \overrightarrow{X} \\ \overrightarrow{r_2} &= \overrightarrow{x_2} - \overrightarrow{X} \\ &\vdots \\ \overrightarrow{r_n} &= \overrightarrow{x_n} - \overrightarrow{X}\end{aligned}$$

1 hr + 20 min

80 mm

120 mm/s

2 : 35 pm



Fixed

$$\tan \alpha = \frac{y}{x}$$

$$so \quad y = x \tan \alpha$$

$$\phi(x, y) = y - x \tan \alpha = 0$$

constraint

$$L = T - U + \lambda \phi$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mg y + \lambda(y - x \tan \alpha)$$

Eoms:

$$x: \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \rightarrow m \ddot{x} = -\lambda \tan \alpha \quad (1)$$

$$y: \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) = \frac{\partial L}{\partial y} \rightarrow m \ddot{y} = -mg + \lambda \quad (2)$$

$$\lambda: y - x \tan \alpha = 0 \rightarrow y = x \tan \alpha \quad (3)$$

Differentiate constraint twice:

$$\ddot{y} = \ddot{x} \tan \alpha$$

Subst. to eq for  $\ddot{y}$ ,  $\ddot{x}$  using (1) and (2)

$$-g + \frac{1}{m} = -\frac{1}{m} \tan \alpha \cdot \tan \alpha$$

$$\frac{1}{m} (1 + \tan^2 \alpha) = g$$

$\tan^2 \alpha$

$$\lambda = \frac{mg}{\sec^2 \alpha}$$

constraint Force:

$$\overrightarrow{F}_c = \lambda \nabla \phi \quad \text{where} \quad \phi = y - x \tan \alpha$$

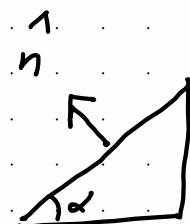
$$= \frac{mg}{\sec^2 \alpha} \left( -\tan \alpha \hat{x} + \hat{y} \right)$$

$$= \frac{mg}{\sec^2 \alpha} \left( -\frac{\sin \alpha}{\cos \alpha} \hat{x} + \hat{y} \right)$$

$$= mg \cos \alpha \left( -\sin \alpha \hat{x} + \cos \alpha \hat{y} \right)$$

$$= mg \cos \alpha \hat{n}$$

(where  $\hat{n} = -\sin \alpha \hat{x} + \cos \alpha \hat{y}$   
is  $\perp$  to incline)



Return to Eoms:

$$\ddot{x} = -\frac{\lambda}{m} \tan \alpha$$

$$\ddot{y} = -g + \frac{\lambda}{m}$$

where  $\lambda = \frac{mg}{\sec^2 \alpha} = mg \cos^2 \alpha$

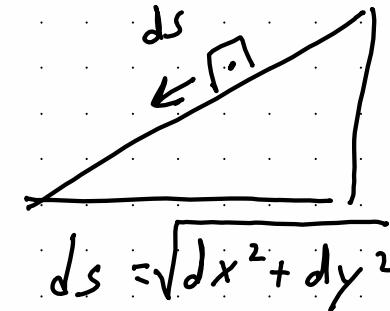
$$\rightarrow \ddot{x} = -g \cos^2 \alpha \tan \alpha$$

$$= -g \sin \alpha \cos \alpha$$

$$\ddot{y} = -g + g \cos^2 \alpha$$

$$= -g (1 - \cos^2 \alpha)$$

$$= -g \sin^2 \alpha$$



Acceleration down the incline:

$$\ddot{s} = -\sqrt{\ddot{x}^2 + \ddot{y}^2} = -g \sin \alpha \sqrt{\cos^2 \alpha + \sin^2 \alpha} = \boxed{-g \sin \alpha}$$

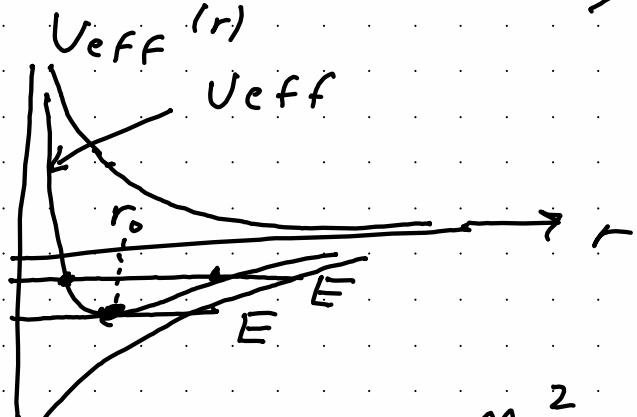
*s standard result*

Lecture #14: Thurs Oct 8<sup>th</sup>

- midterm 1: Avg  $\approx 14/20$
- midterm 2: Nov 19<sup>th</sup> (before Thanksgiving)
- org final

Collisions & Scattering (Sec 16-20)  
(2) (3)

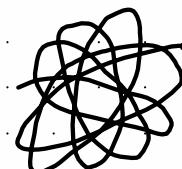
Scattering: closed system, two bodies,  
 $\rightarrow$  central force problem



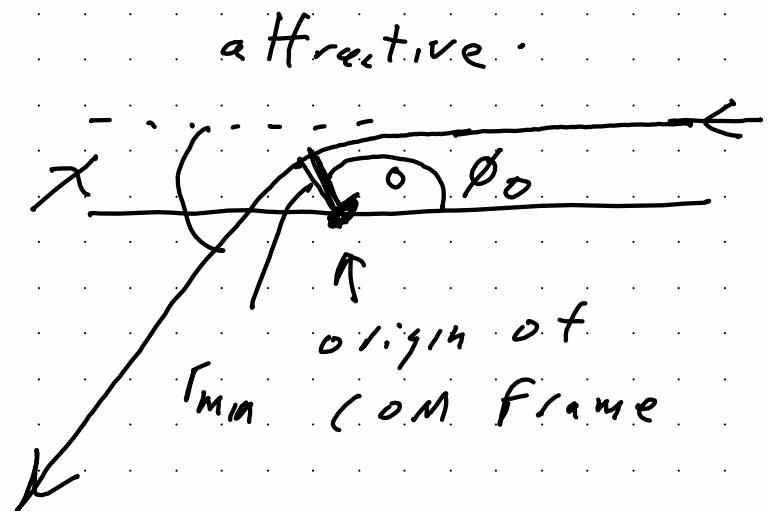
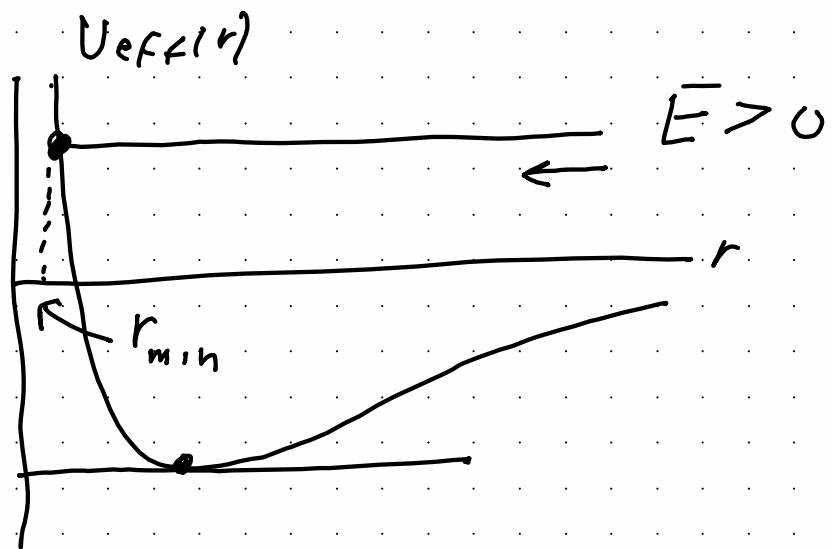
$$U_{\text{eff}} = -\frac{\alpha}{r} + \frac{M_2^2}{2mr^2}$$

$$E = U_{\text{eff}, \min} \rightarrow r = r_0 \quad (\text{circular})$$

$U_{\text{eff}, \min} < E < 0 \rightarrow$  bound orbit

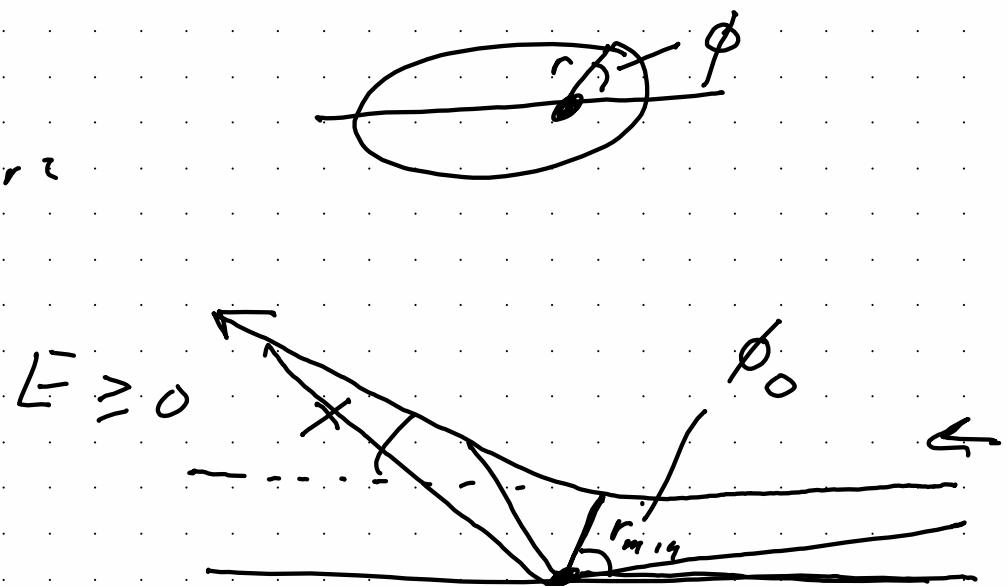
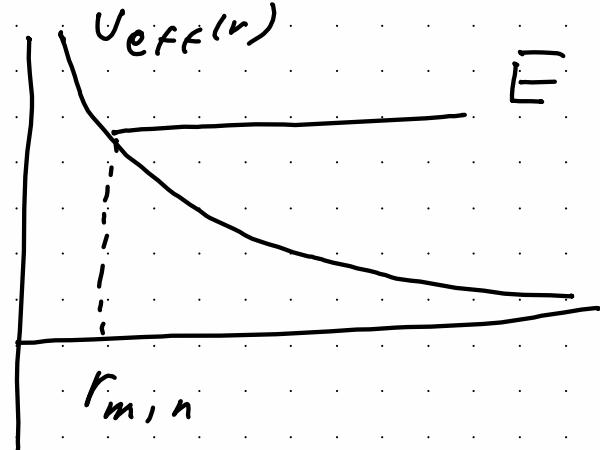


ellipse  
for  $-\alpha/r$



$$U(r) = +\alpha/r$$

$$U_{\text{eff}}(r) = \alpha/r + M_z^2/2mr^2$$



$$\int_0^{\phi_0} d\phi = \int_{r_{min}}^{\infty} \frac{M_e dr / r^2}{\sqrt{2m(E - U(r)) - \frac{M_e^2}{r^2}}} \quad (14.7) L \& L$$

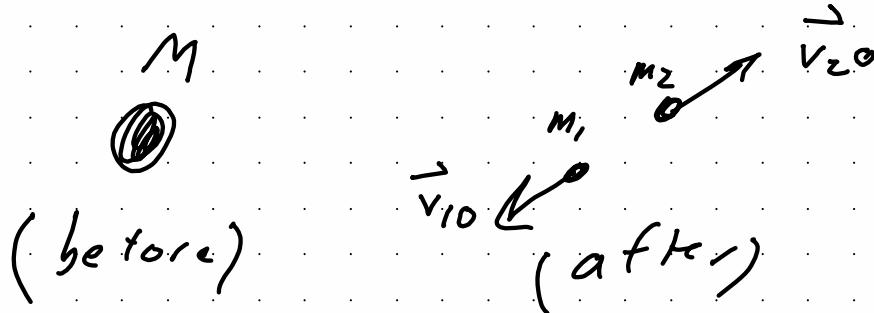
$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M_e dr / r^2}{\sqrt{2m(E - U(r)) - \frac{M_e^2}{r^2}}} \quad E, M_e, U(r)$$

Fig 18 in L & L

## Collisions: (Sec 16, 17)

- Elastic collisions of two particles

(\*) Spontaneous disintegration of a single particle



Analyze in COM Frame (to begin with)

$$\vec{p}_{\text{tot}, 0} = \vec{0}$$

$L_{\text{COM}}$

cons. of momentum

$$\begin{aligned} \vec{p}_0 \\ - \vec{p}_0 \end{aligned}$$

: momentum of mass  $m_1$ , after disintegration

11

$m_2$

cons. of energy

$$E_i = E_{1c} + T_{10} + E_{2c} + T_{20}$$

in  
internal  
energy of  
mass  $M = m_1 + m_2$

intergal KE  
of  $m_1$  in  
frame  $= \frac{1}{2} m_1 |\vec{v}_{10}|^2$   
 $= \frac{|\vec{p}_0|^2}{2m_1}$

$$E_i = E_{1c} + \frac{|\vec{p}_0|^2}{2m_1} + E_{2c} + \frac{|\vec{p}_0|^2}{2m_2}$$

$$\underbrace{E_i - E_{1c} - E_{2c}}_{\text{Energy}} = \frac{p_0^2}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right)$$

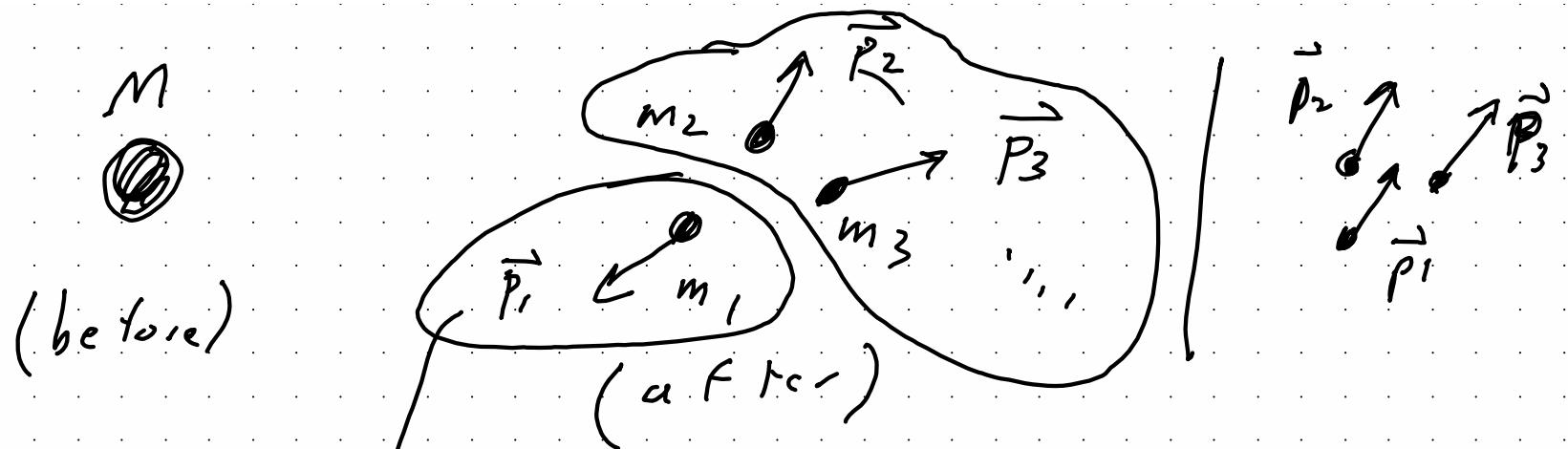
$\epsilon$  = d/integration

$\geq 0$  energy

$$= \frac{p_0^2}{2} \left( \frac{m_1 + m_2}{m_1 m_2} \right) = \frac{p_0^2}{2m}$$

$$\rho_0 = \sqrt{2m\epsilon}$$

$$\rightarrow v_{10} = p_0/m_1, v_{20} = p_0/m_2$$



$$\vec{p}_{\text{tot}} = 0$$

$$\vec{p}_1 = \vec{p}_0$$

$$T_{1,0} = \frac{p_0^2}{2m}$$

Q: what conditions are there on the velocities of  $m_2, m_3, \dots$  such that KE of  $m_1$  is largest?

Consider COM of  $m_2, m_3, \dots$

$E_i^{\text{int}}$ : internal energy of  $m_2, m_3, \dots$

$$\vec{p}_2 + \vec{p}_3 + \dots = -\vec{p}_0$$

$$KE = \frac{p_0^2}{2(M-m_1)}$$

$$(m_2 + m_3 + \dots)$$

Cons. of energy.

$$E_i = E_{i,i} + \frac{p_0^2}{2m_1}$$

ht. energy  
of  $M$

ht. + KE  
of  $m_1$

$$+ E_{i'} + \frac{p_0^2}{2(M-m_1)}$$

int + KE  
of  $m_2, m_3, \dots$

$$E_i - E_{i,i} - E_{i'} = \frac{p_0^2}{2} \left( \frac{1}{m_1} + \frac{1}{M-m_1} \right)$$

$$E_i - E_{i,i} - E_{i'} = \frac{p_0^2}{2} \frac{M}{m_1(M-m_1)}$$

$E_{2i} + E_{3i} + \dots$

X

$$T_{10} = \frac{p_0^2}{2m_1} = \left( \frac{M-m_1}{M} \right) (E_i - E_{i,i} - E_{i'})$$

$T_{10}$  ~~is~~ maximum when  $E_{i'}$  is minimum

$$E_i' + \frac{p_0^2}{2(M-m_i)} = E_{2i} + \frac{p_2^2}{2m_2} + E_{3i} + \frac{p_3^2}{2m_3} + \dots$$

$\vec{p}_2 = m_2 \vec{v}_2, \text{ etc.}$

int energy + KE of  
masses  $m_2, m_3, \dots$

$$= E_{2i} + \frac{1}{2} m_2 |\vec{v}_2|^2 + E_{3i} + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots$$

$$E_i' = (E_{2i} + E_{3i} + \dots) + \left( \frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots \right)$$

W  $\frac{|\vec{p}_2 + \vec{p}_3 + \dots|^2}{2(m_2 + m_3 + \dots)} = |\vec{v}_0|^2$

$$= (E_{2i} + E_{3i} + \dots) + \left( \frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots \right)$$

$\ominus \frac{1}{2} \frac{|\vec{m}_2 \vec{v}_2 + \vec{m}_3 \vec{v}_3 + \dots|^2}{M_2 + M_3 + \dots}$

$\begin{aligned} & |\vec{v}_2 + \vec{v}_3|^2 \\ &= |\vec{v}_2|^2 + |\vec{v}_3|^2 \\ &+ 2 \vec{v}_2 \cdot \vec{v}_3 \end{aligned}$

$$\text{If } \vec{v}_2 = \vec{v}_3 = \dots \equiv \vec{v}_0$$

then:  $\frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots$

$$= \left( \frac{1}{2} (m_2 + m_3 + \dots) \right) |\vec{v}_0|^2$$

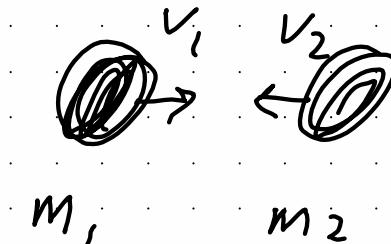
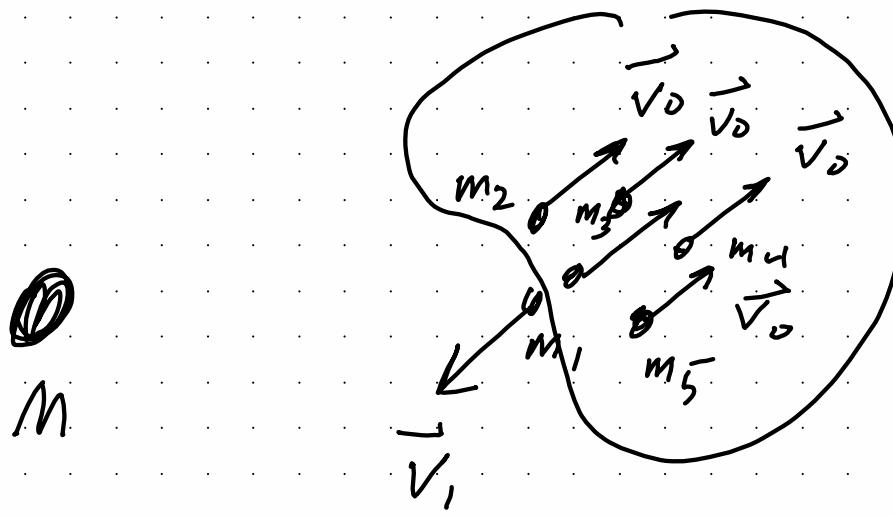
and

$$\frac{1}{2} \frac{(m_2 \vec{v}_2 + m_3 \vec{v}_3 + \dots)^2}{m_2 + m_3 + \dots}$$

$$= \frac{1}{2} |\vec{v}_0|^2 \frac{(m_2 + m_3 + \dots)^2}{m_2 + m_3 + \dots}$$

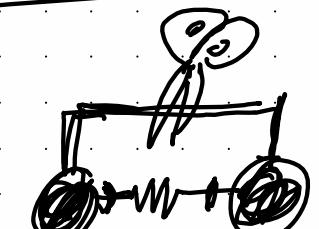
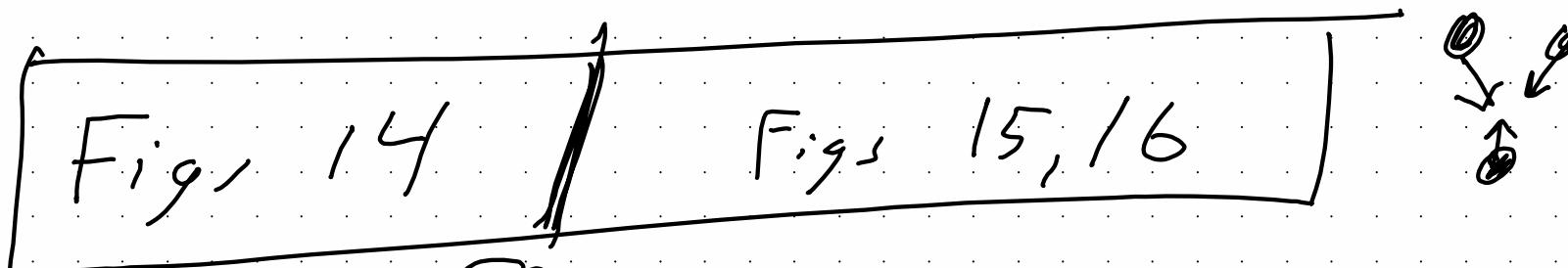
$$= \frac{1}{2} |\vec{v}_0|^2 (m_2 + m_3 + \dots)$$

$$T_{IO_1, \max} = \left( \frac{M-m_1}{M} \right) \in$$

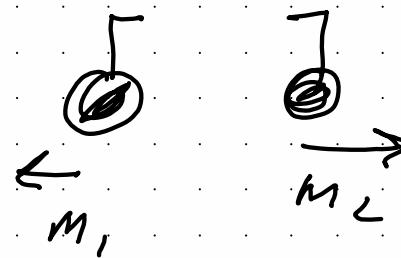


$QM$

$$M = m_1 + m_2$$



$m_1$



$m_1$

Lec #15: Tues 10/13

— solutions to midterm #1 posted

— next two weeks: Sec 16-20

## Collisions and scattering

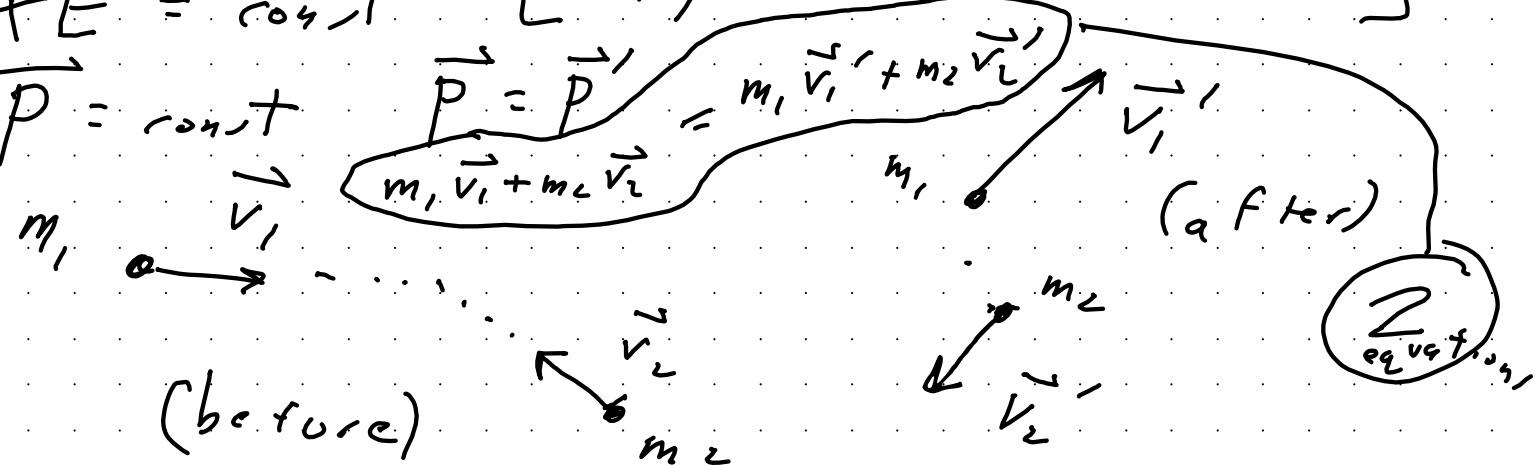
Figures 14, 15, 16

Elastic collision of two particles

Closed system:  $E$ ,  $\vec{P}$ ,  $\vec{M}$  are conserved

✓ i)  $\cancel{KE} = \text{const}$  [ignore internal energies]

✓ ii)  $\vec{P} = \text{const}$



After:  $\vec{v}_1'$ ,  $\vec{v}_2'$  : 4 DOF  
3 equations

Elastic collision:

$$\vec{V} \equiv \vec{v}_1 - \vec{v}_2 = \vec{v}_{10} - \vec{v}_{20} \quad \text{relative velocity vector}$$

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 = \vec{r}_{10} - \vec{r}_{20} \quad \text{relative position vector}$$

$$\vec{v}_1 = \dot{\vec{r}}$$

$$\vec{v}_1 = \vec{V} + \vec{v}_{10}$$

$$\vec{v}_2 = \vec{V} + \vec{v}_{20}$$

$$\vec{r}_1 = \vec{R} + \vec{r}_{10}$$

$$\vec{r}_2 = \vec{R} + \vec{r}_{20}$$

$\vec{v}_1$ : velocity of particle 1 wrt lab frame  
before collision

$\vec{V}_{10}$ : " wrt COM Frame  
before collision

$\vec{v}_1'$ ,  $\vec{v}_{10}'$ ,  $\vec{v}_2'$ ,  $\vec{v}_{20}'$ ; velocities after collision

$$T_0 = \frac{1}{2} m_1 |\dot{\vec{r}}_{10}|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_{20}|^2$$

$$\begin{aligned} \vec{v}_{10} &= \dot{\vec{r}}_{10} = \left( \frac{m_2}{m_1 + m_2} \right) \dot{\vec{r}} = \vec{V} & \vec{r} &= \vec{r}_{10} - \vec{r}_{20} \\ \vec{v}_{20} &= \dot{\vec{r}}_{20} = -\left( \frac{m_1}{m_1 + m_2} \right) \dot{\vec{r}} \end{aligned}$$

$$T_0 = \frac{1}{2} m_1 \left( \frac{m_2}{m_1 + m_2} \right)^2 |\dot{\vec{r}}|^2$$

$$+ \frac{1}{2} m_2 \left( \frac{-m_1}{m_1 + m_2} \right)^2 |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} \left( \frac{m_1 m_2}{m_1 + m_2} \right) |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} m |\dot{\vec{r}}|^2 = \frac{1}{2} m |\vec{V}|^2$$

$$T_0 = T_0'$$

$$\frac{1}{2} m |\vec{V}|^2$$

$$= \frac{1}{2} m' |\vec{V}'|^2$$

$$\boxed{V = V'}$$

~~relative~~  
magnitude  
of relative  
velocity  
vector = const

Before collision

$$\overrightarrow{V}_{1,0} = \left( \frac{m_2}{m_1 + m_2} \right) \overrightarrow{V}, \quad \overrightarrow{V}_{2,0} = \left( \frac{-m_1}{m_1 + m_2} \right) \overrightarrow{V}$$

$$\overrightarrow{V} = \overrightarrow{V}_1 - \overrightarrow{V}_2$$

$$\begin{cases} \overrightarrow{V}_1' = \overrightarrow{V} \\ \overrightarrow{V}_2' = 0 \end{cases}$$

Lab Frame

After collision:

$$\overrightarrow{V}'_{1,0} = \left( \frac{m_2}{m_1 + m_2} \right) V \hat{n}_0$$

magnitude  
of  $\overrightarrow{V}'$

unit vector  
pointing in  
some direction  
(angle =  $\chi$  w.r.t.  
COM frame)

$$\overrightarrow{V}'_{2,0} = \left( \frac{-m_1}{m_1 + m_2} \right) V \hat{n}_0$$

$$\boxed{\overrightarrow{V}'_1 = \overrightarrow{V}'_{1,0} + \overrightarrow{V} = \left( \frac{m_2}{m_1 + m_2} \right) V \hat{n}_0 + \frac{m_1 \overrightarrow{V}_1 + m_2 \overrightarrow{V}_2}{m_1 + m_2}}$$

$$\overrightarrow{R} = \frac{m_1 \overrightarrow{r}_1 + m_2 \overrightarrow{r}_2}{m_1 + m_2} \rightarrow \overrightarrow{V} = \frac{m_1 \overrightarrow{v}_1 + m_2 \overrightarrow{v}_2}{m_1 + m_2}$$

$$\vec{v}_1' = \vec{v}_{10} + \vec{V} = \left( \frac{m_2}{m_1+m_2} \right) \vec{v}_{n_0}^\wedge + \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1+m_2} = \vec{v}_1'$$

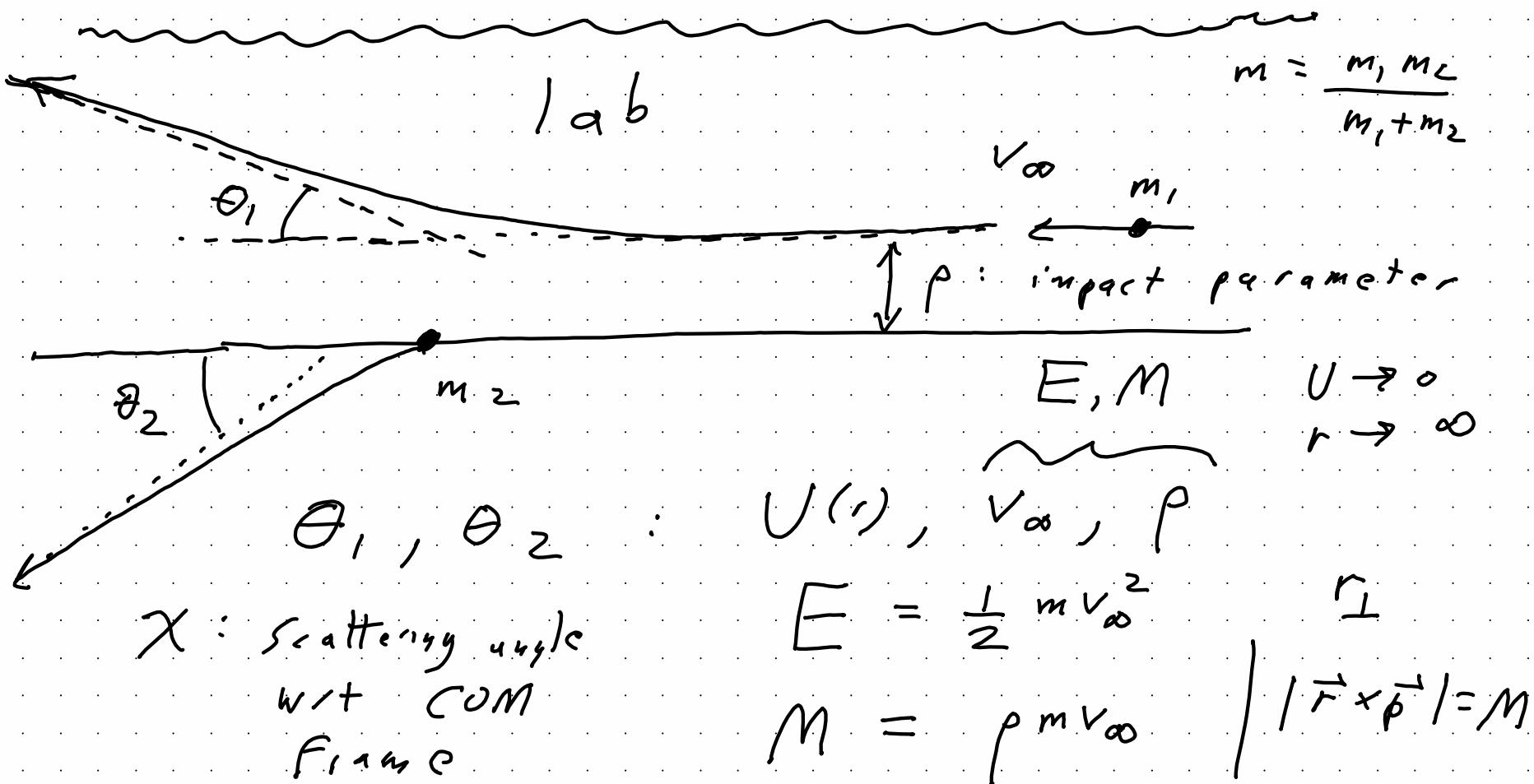
$$\vec{v}_2' = \vec{v}_{20} + \vec{V} = \left( \frac{-m_1}{m_1+m_2} \right) \vec{v}_{n_0}^\wedge + \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1+m_2} = \vec{v}_2'$$

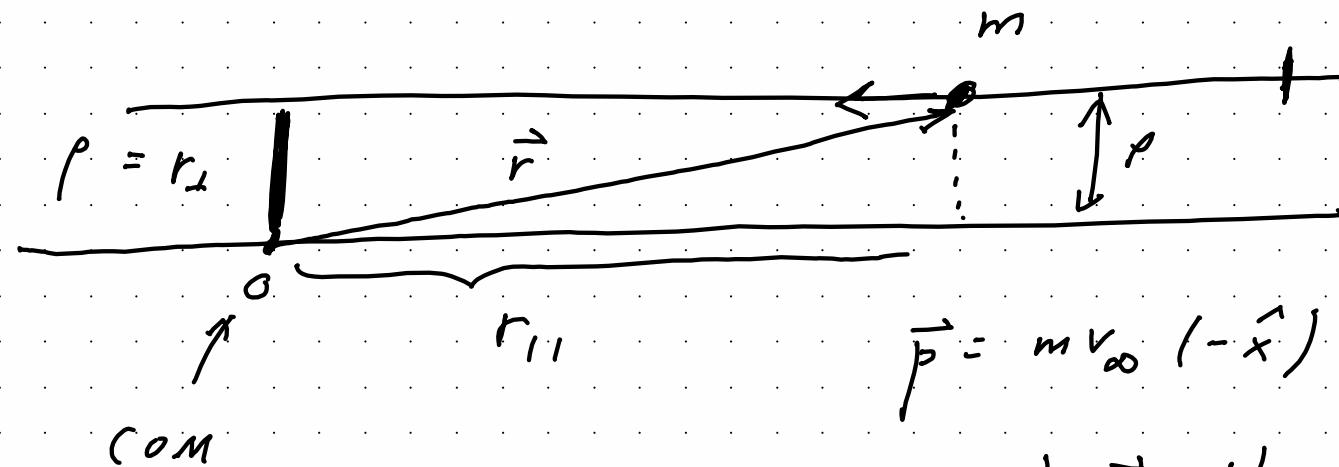
$$\vec{p}_1' = m_1 \vec{v}_1' = m \vec{v}_{n_0}^\wedge + m_1 \left( \frac{\vec{p}_1 + \vec{p}_2}{m_1+m_2} \right)$$

$$\vec{p}_2' = m_2 \vec{v}_2' = -m \vec{v}_{n_0}^\wedge + m_2 \left( \frac{\vec{p}_1 + \vec{p}_2}{m_1+m_2} \right)$$

Lec #16: Thurs 10/15

- Quiz #3 next week
- General remarks about scattering
- Example: Prob 18.1 (Hard-sphere)





COM

$$\boxed{E = \frac{1}{2} m v_\infty^2}$$

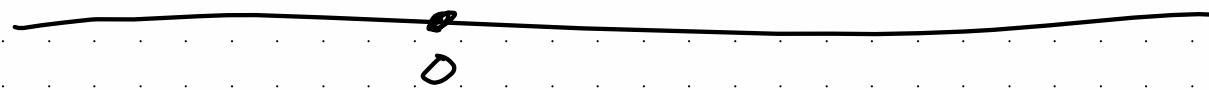
$$M = \rho m v_\infty$$

$$M = |\vec{r} \times \vec{p}|$$

$$= r_\perp m v_\infty$$

$$= \rho m v_\infty$$

$$dp \leftrightarrow dx \quad d\sigma = 2\pi p dp$$



$$d\sigma = 2\pi \rho(x) \left| \frac{d\rho}{dx} \right| dx$$

$$d\sigma = \frac{\rho(x)}{\sin x} \left| \frac{d\rho}{dx} \right| d\Omega = \rho \left| \frac{d\rho}{d(\cos x)} \right| d\Omega$$

$$\boxed{\frac{d\sigma}{d\Omega} = \rho \left| \frac{d\rho}{d(\cos x)} \right|}$$

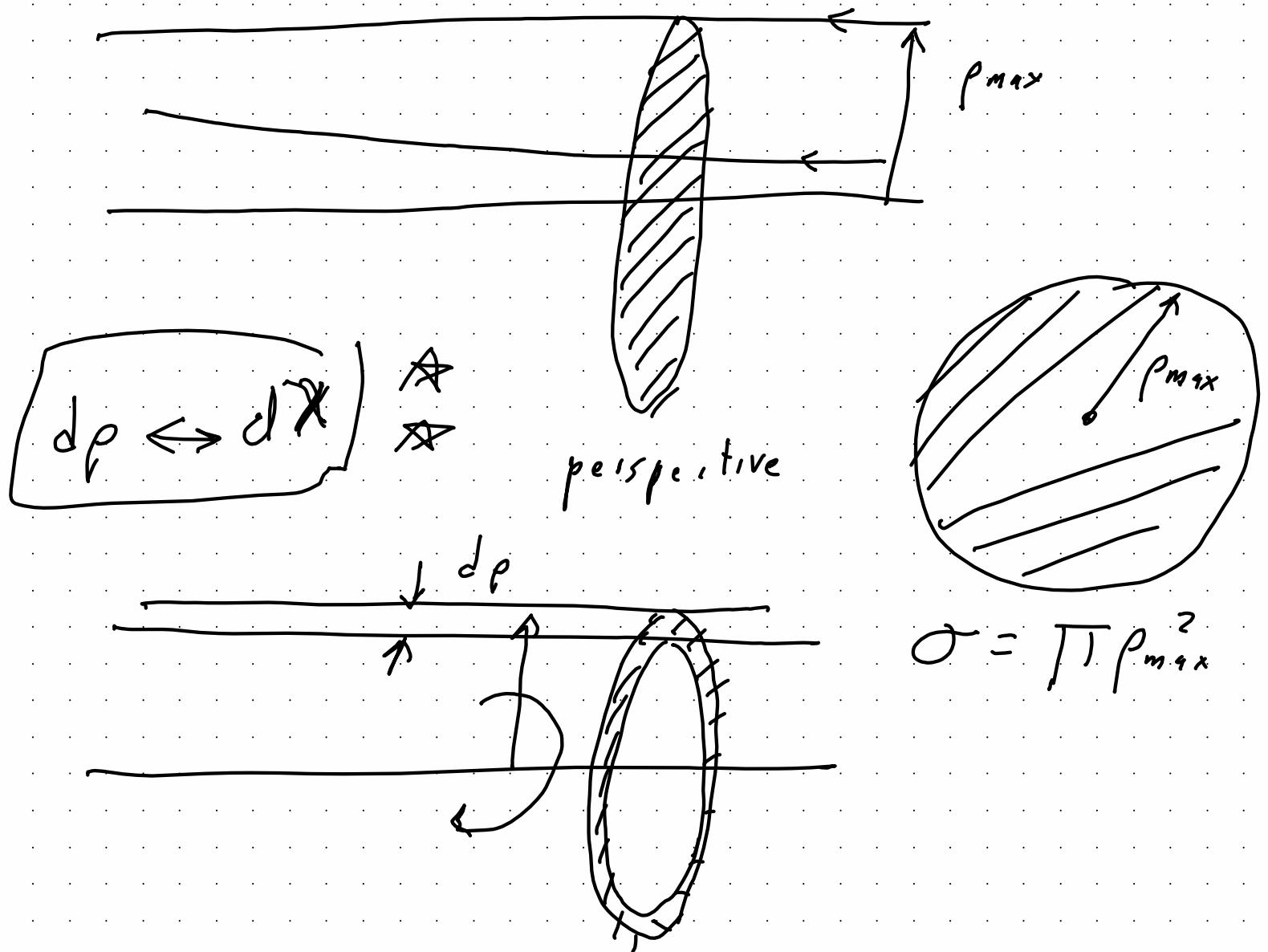
in COM frame

$$\boxed{\frac{d\sigma_1}{d\Omega_1} = \rho \left| \frac{d\rho}{d(\cos \theta_1)} \right|} \quad \text{in lab frame } \Theta_1$$

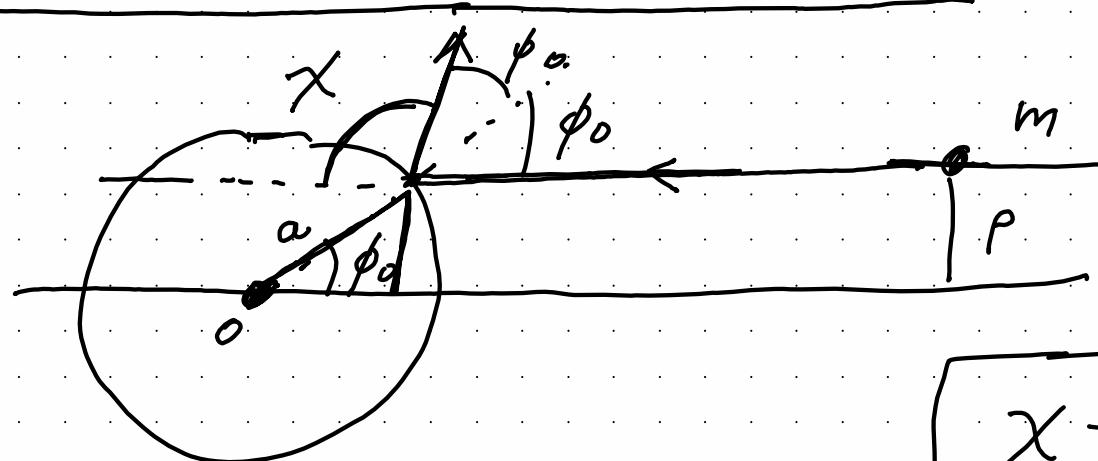
$$= \frac{d\sigma}{d\Omega} \left| \frac{d(\cos x)}{d(\cos \theta_1)} \right|$$

$$\boxed{\frac{d\sigma_2}{d\Omega_2} = \frac{d\sigma}{d\Omega} \left| \frac{d(\cos x)}{d(\cos \theta_2)} \right|}$$

wrt  
lab frame  
( $\theta_1, \theta_2$ )



Hard sphere: Prob 18.1  $U(r) = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases}$



$$x + 2\phi_0 = \pi$$

$$\boxed{x \leftrightarrow \rho}$$

↑  
radius  $a$

$$\boxed{\begin{aligned} \rho &= a \sin \phi_0 \\ &= a \sin\left(\frac{\pi}{2} - \frac{x}{2}\right) \\ &= a \cos\left(\frac{x}{2}\right) \end{aligned}}$$

$$\boxed{\begin{aligned} \phi_0 &= \frac{\pi}{2} - \frac{x}{2} \\ \frac{d\rho}{dx} &= -\frac{a}{2} \sin\left(\frac{x}{2}\right) \end{aligned}}$$

$$d\sigma = 2\pi \rho d\rho = 2\pi \rho(x) \left| \frac{d\rho}{dx} \right| dx$$

$$d\sigma = \cancel{\pi} a \cos\left(\frac{x}{2}\right) \cancel{\frac{a}{2}} \sin\left(\frac{x}{2}\right) dx$$

$$= \pi a^2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) dx$$

$$= \boxed{\frac{\pi a^2}{2} \sin x dx}$$

$$d\Omega = 2\pi \sin x dx \rightarrow \sin x dx = \frac{1}{2\pi} d\Omega$$

$$d\sigma = \frac{\pi a^2}{2} \cdot \frac{1}{2\pi} d\Omega$$

$$= \boxed{\frac{1}{4} a^2 d\Omega}$$

uniform distribution

Total cross-section:

$$\sigma_{tot} = \sigma = \int_{unit\ sphere} d\sigma$$

$$= \frac{1}{4} a^2 \int_{sphere} d\Omega = \boxed{\pi a^2} \quad \text{|| } 4\pi$$

$$\frac{d\sigma_1}{d\Omega_1} = \frac{d\sigma}{d\Omega} \left| \frac{d(\cos X)}{d(\cos\theta_1)} \right|$$

$$\frac{d\sigma_2}{d\Omega_2} = \frac{\frac{1}{4}a^2}{d\Omega} \left| \frac{d(\cos X)}{d(\cos\theta_2)} \right|$$

$$X = \pi - 2\theta_2$$

$$\cos X = -\frac{m_1}{m_2} \sin^2 \theta_1 \pm \cos \theta_1 \sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2 \theta_1}$$

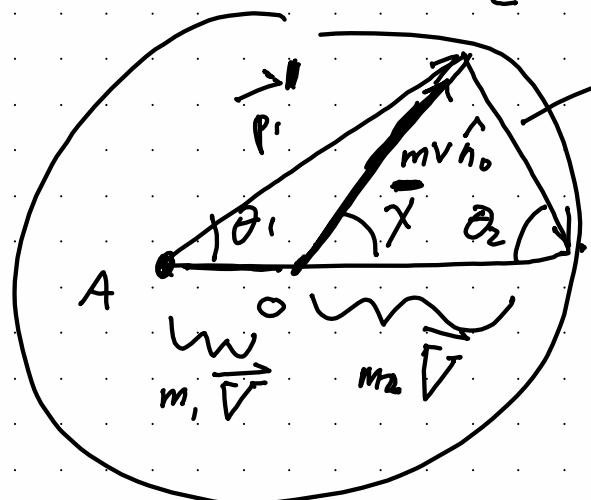
$$\sin \theta_1 d\theta_1 = -d(\cos\theta_1)$$

## Lecture #17

Tue, 10/20

- Q & A #3 - this week

- Q & A :  $\vec{p}_1 + \vec{p}_2 = \vec{P}_{\text{tot}}$



$$m_1 < m_2, v_2 = 0$$



$m_1$

$$\vec{v}_2 = 0$$



$m_2$

$$\begin{aligned} & m_1 \vec{V} + m_2 \vec{V} \\ & = (m_1 + m_2) \vec{V} = \vec{P}_{\text{tot}} \end{aligned}$$

$$\vec{V} = \frac{\vec{p}_1 + \vec{p}_2}{m_1 + m_2}$$

$$\left( \text{com velocity} \right) = \frac{\vec{p}_1}{m_1 + m_2}$$

$$= \frac{m_1 \vec{v}_1}{m_1 + m_2}$$

$$m_1 \vec{V} = m_1 \left( \frac{m_1 \vec{v}_1}{m_1 + m_2} \right)$$

$$\boxed{\vec{p}_2 = 0}$$

$$= \left( \frac{m_1^2}{m_1 + m_2} \right) \vec{v}_1$$

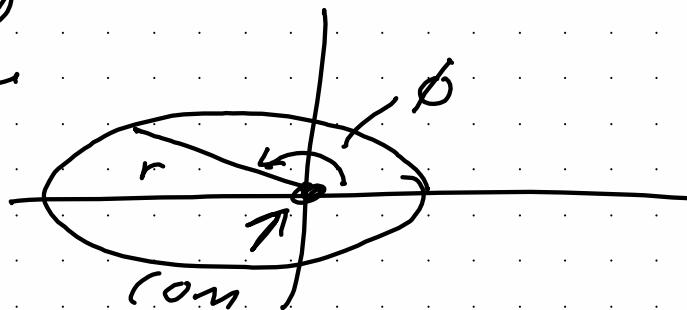
$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$T = \frac{1}{2} m_1 |\vec{r}_1|^2 + \frac{1}{2} m_2 |\vec{r}_2|^2$$

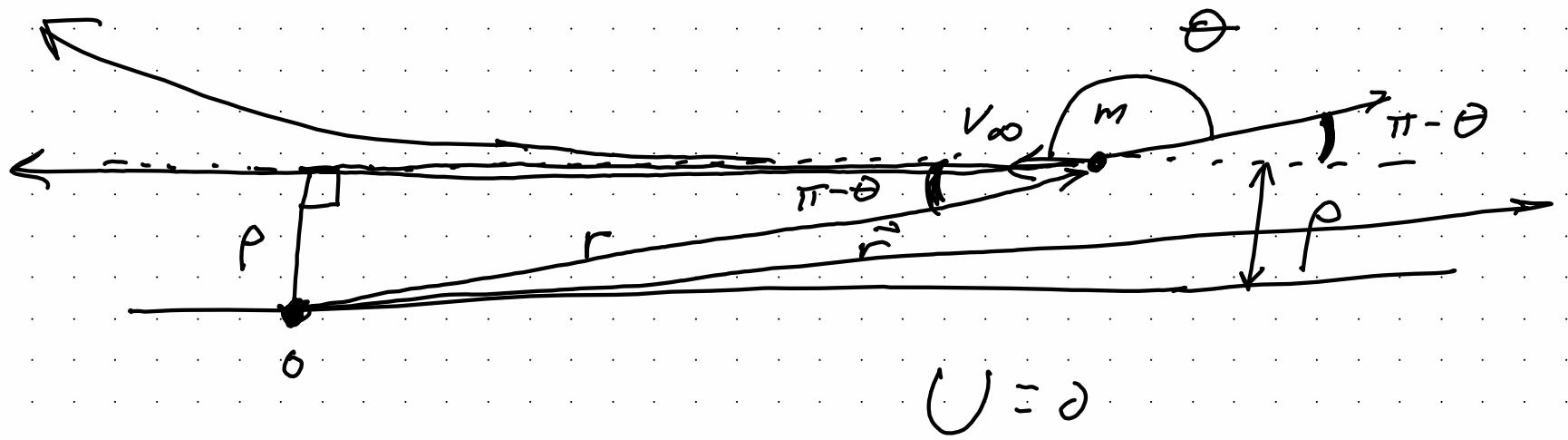
$$= \frac{1}{2} m |\vec{r}|^2$$

$$m = \frac{m_1 m_2}{m_1 + m_2}$$



$$\vec{r}_1 = \frac{m_2}{m_1 + m_2} \vec{r}$$

$$\vec{r}_2 = \frac{-m_1}{m_1 + m_2} \vec{r}$$



$$M = \rho m v_\infty$$

$$\vec{M} = \vec{r} \times \vec{P}$$

$$= m \vec{r} \times \vec{v}_\infty$$

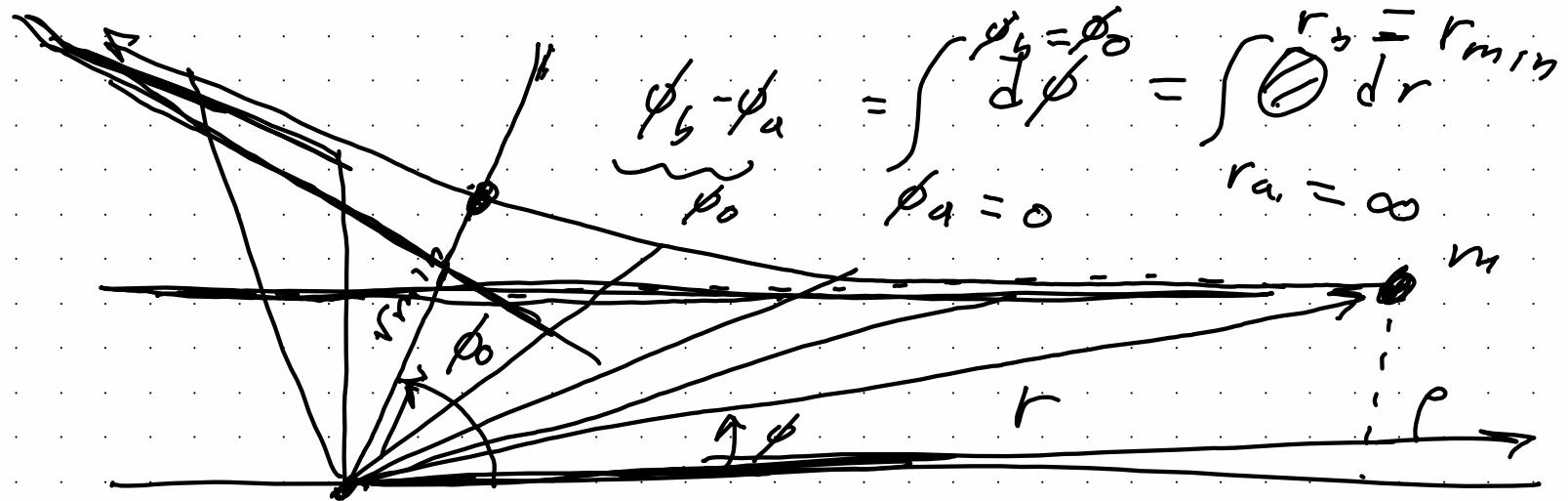
$$M = |\vec{M}| = m \underbrace{r v_\infty}_{1} \sin \theta$$

$$= m v_\infty \underbrace{r \sin(\pi - \theta)}_{1}$$

$$= \rho m v_\infty \rho$$

$$\vec{P} = m \vec{v}_\infty$$

$$\begin{aligned} \sin(\pi - \theta) &= \sin \theta \\ &= \sin \theta \\ &= \sin \theta \end{aligned}$$

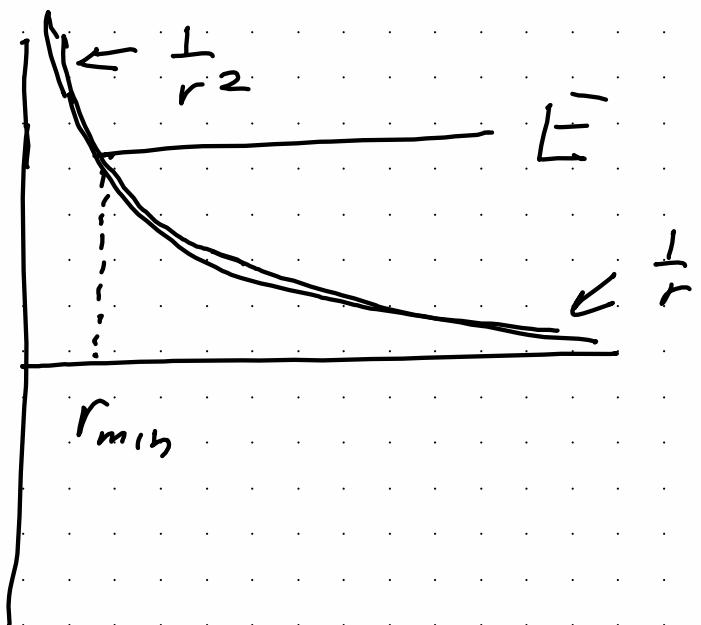


$$\phi_0 = \int_{r_{min}}^{\infty} \frac{m dr / r^2}{\pm \sqrt{2m(E-U) - m^2/r^2}}$$

$U(r) = \frac{\alpha}{r}$   
 $M = \rho m v_\infty$   
 $E = \frac{1}{2} m v_\infty^2$

$2\phi_0 + \chi = \pi$   
 $V_\infty$   
 $\rho$

$\chi \leftrightarrow \rho$   
 $\phi_0$   
 $\phi_0$   
 $\chi$   
 $\phi_0$   
 $E = \frac{1}{2} m r'^2 + \frac{m^2}{2mr^2} + U(r)$   
 $r' = \pm \sqrt{}$



$$U = \frac{\alpha}{r}$$

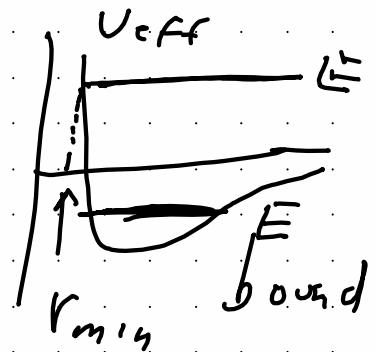
$$\alpha > 0$$

Coulomb scattering  
++  
--

$$U_{\text{eff}} = U(r) + \frac{p^2}{2mr^2}$$

$$E = U_{\text{eff}}(r_{\min})$$

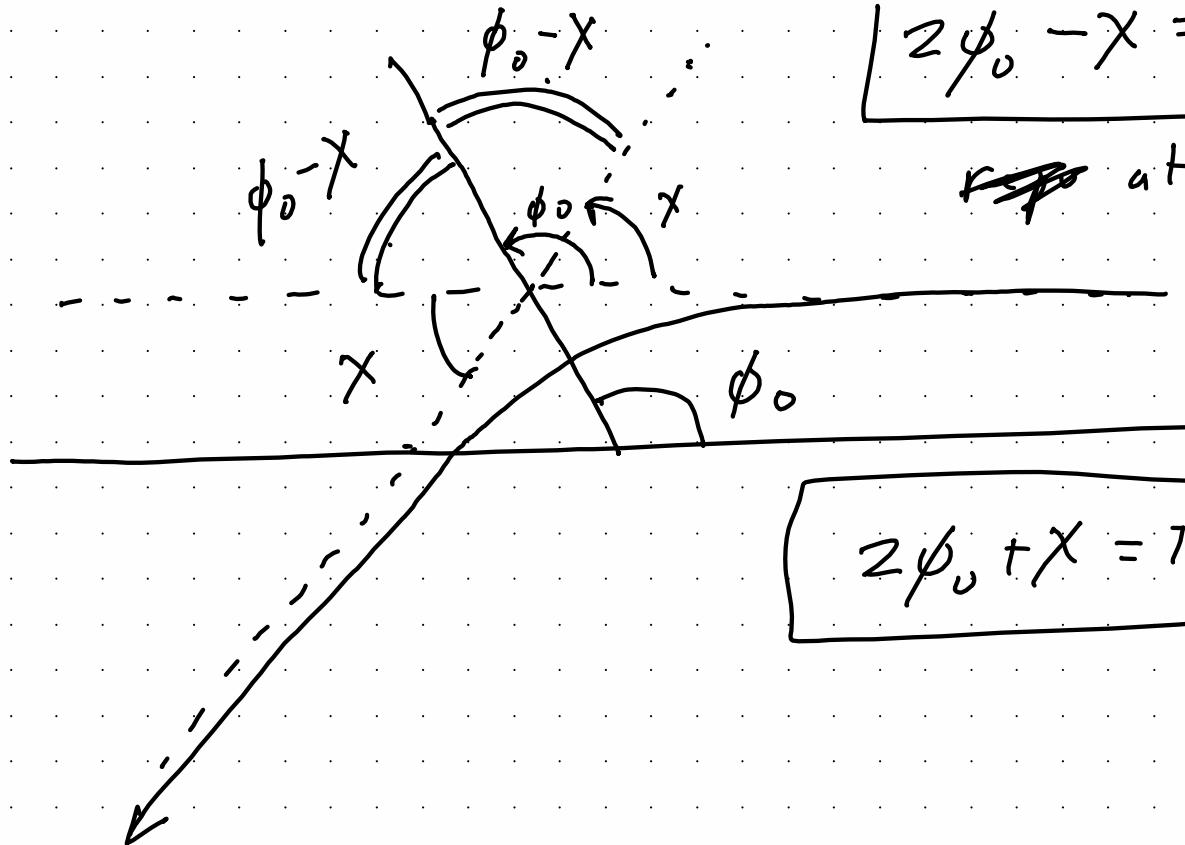
$$U = -\frac{\alpha}{r}$$



$$2(\phi_0 - x) + x = \pi$$

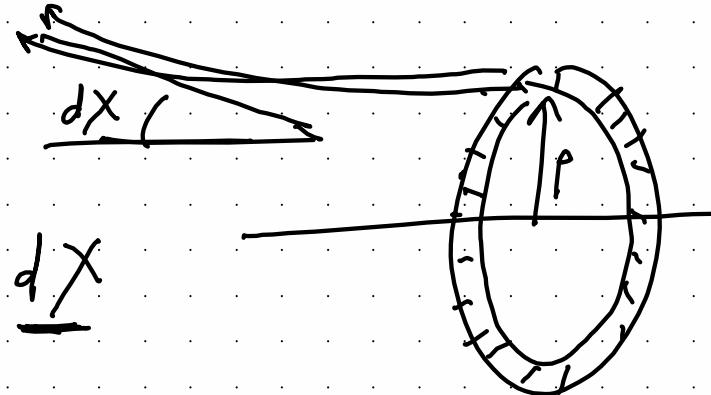
$$2\phi_0 - x = \pi$$

~~rep~~ attractive



$$2\phi_0 + x = \pi$$

$$\text{d}\sigma = 2\pi \rho d\rho$$



$$= 2\pi \rho(x) \left| \frac{d\rho}{dx} \right| dx$$

$$= \frac{\rho(x)}{\sin x} \left| \frac{d\rho}{dx} \right| dx$$

~~d\rho > 0~~

~~dx < 0~~

w.r.t Com Frame

To go to lab frame

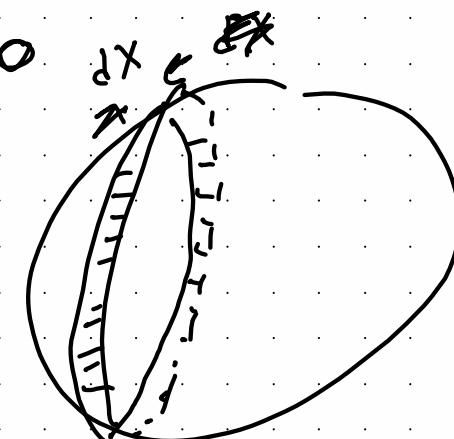
$$x \rightarrow \theta_1$$

$$\text{or } \theta_2$$

$$d\sigma_1, d\sigma_2$$

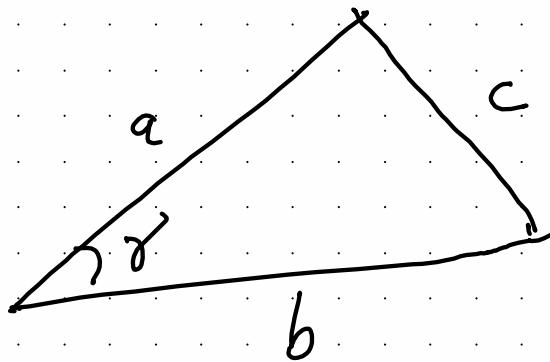
$$d\theta_1, d\theta_2, d\Omega_1, d\Omega_2$$

$$d\Omega = 2\pi \sin x dx$$



Lecture #18: Thurs 10/22

$$c^2 = a^2 + b^2 - 2ab \cos \gamma \quad \underline{\hspace{10mm}}$$



$$\cos^2 x + \sin^2 x = 1$$

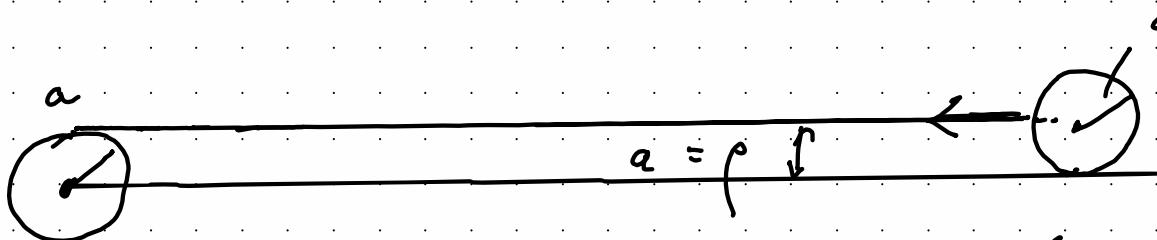
$$\sin(A \pm B) = \sin A \cos B \\ \mp \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \\ \mp \sin A \sin B$$

QVIZ #3:

Lab Frame

name-q3.pdf



$m_2$

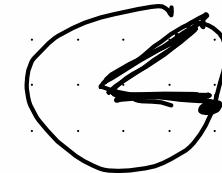
$v_2 = 0$

$(m_1 = m_2)$

$m_1$

$v_1 = v_0$

(hard sphere scattering)



1) calculate  $\chi$  (scattering angle of the reduced mass w.r.t COM Frame)

2) calculate  $\theta_1, \theta_2$  (scattering angles w.r.t Lab Frame)

$$\tan \theta_1 = \frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi}$$

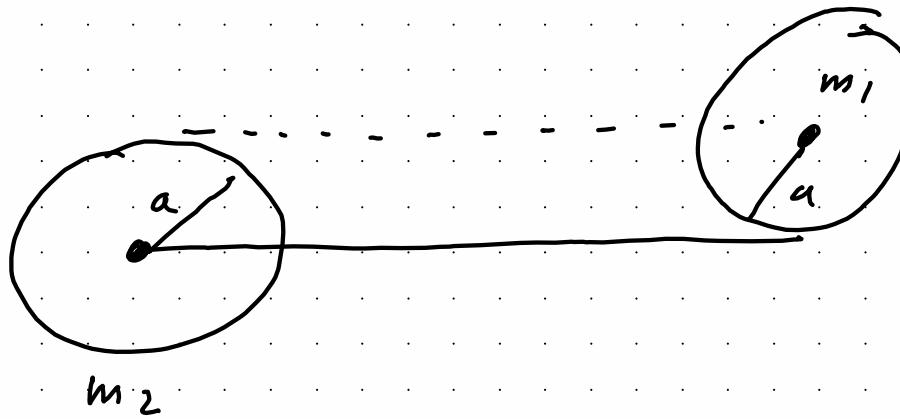
$$\theta_2 = \frac{1}{2}(\pi - \chi)$$

elast scattering

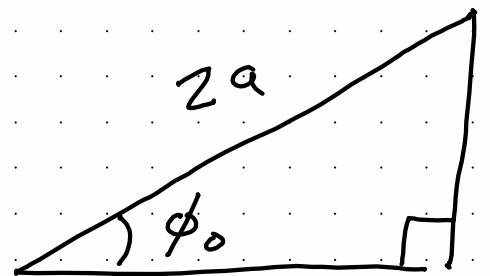
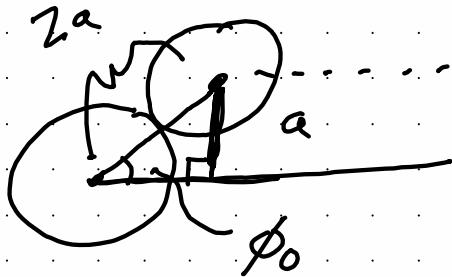
3) How do  $\theta_1, \theta_2$  change if masses  $m_1$  and  $m_2$  change? ( $m_1 \gg m_2$ ;  $m_1 \ll m_2$ )

$m_2$  at rest

$$U(r) = \begin{cases} \infty & r < 2a \\ 0 & r > 2a \end{cases}$$

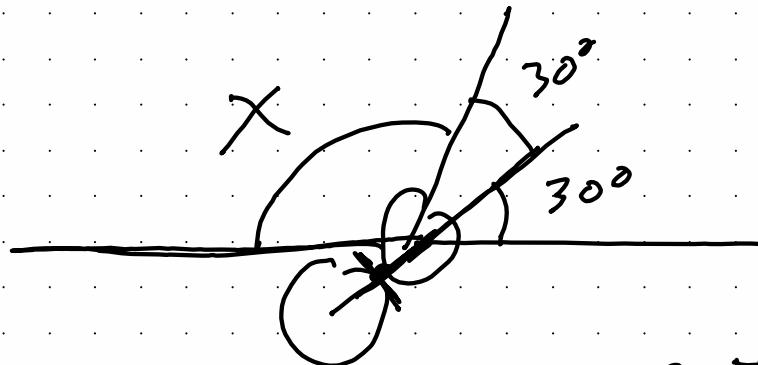


$$\vec{r} = \vec{r}_1 - \vec{r}_2$$



$$\sin \phi_0 = \frac{a}{2a} = \frac{1}{2}$$

$$\boxed{\phi_0 = \pi/6 = 30^\circ}$$



$$x = 120^\circ = \frac{2\pi}{3}$$

independent of  $m_1$  and  $m_2$

$$\frac{2\pi}{3} = 120^\circ$$

$$\begin{array}{|c|} \hline \theta_2 \\ \hline 11 \\ \hline \end{array}$$

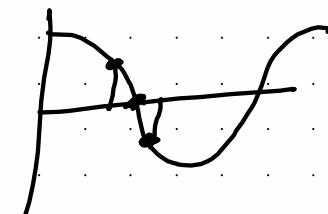
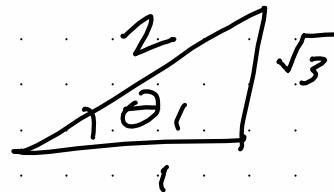
$$2) \quad \theta_2 - \frac{1}{2}(\pi - x) = \frac{1}{2}(180^\circ - 120^\circ) = 30^\circ$$

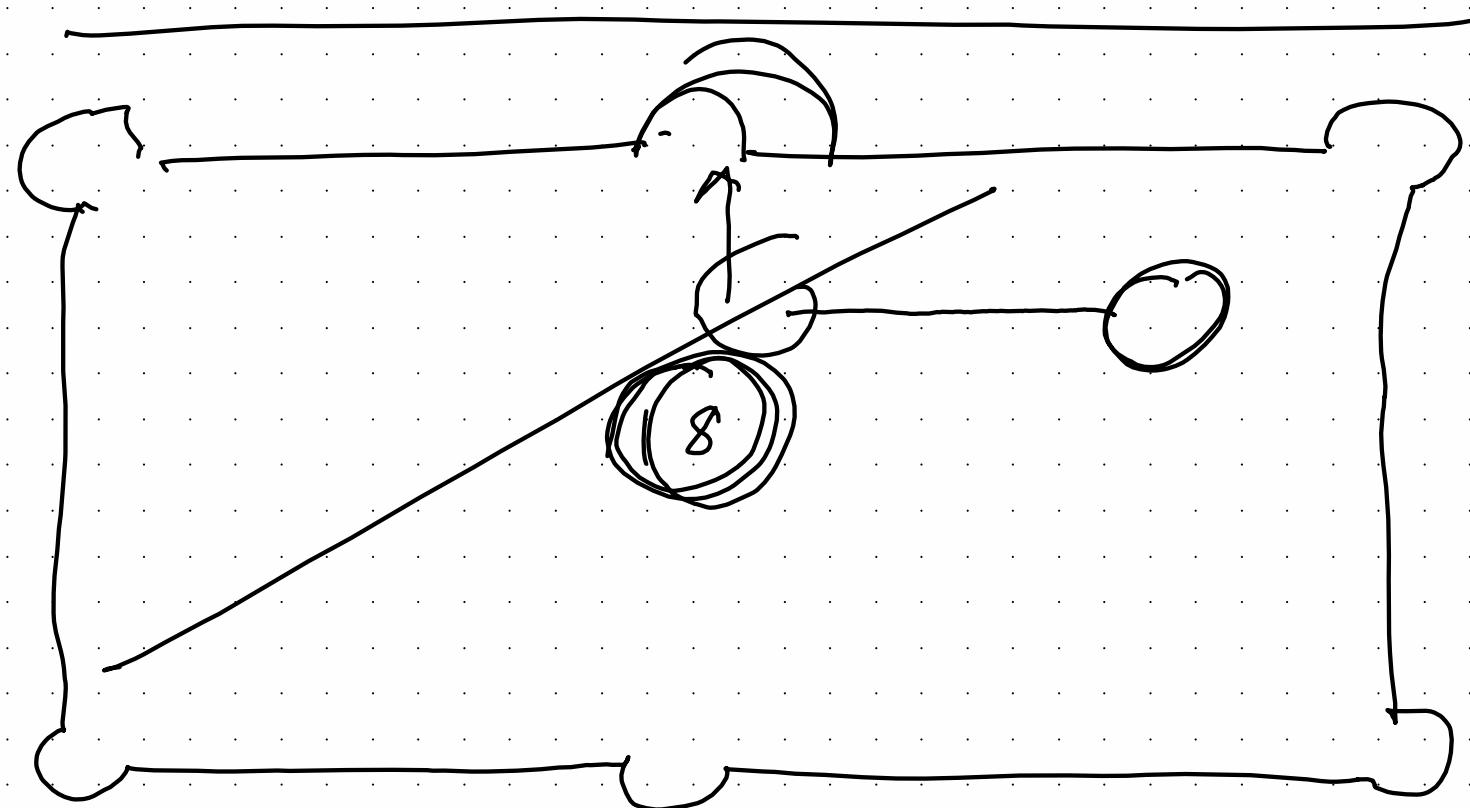
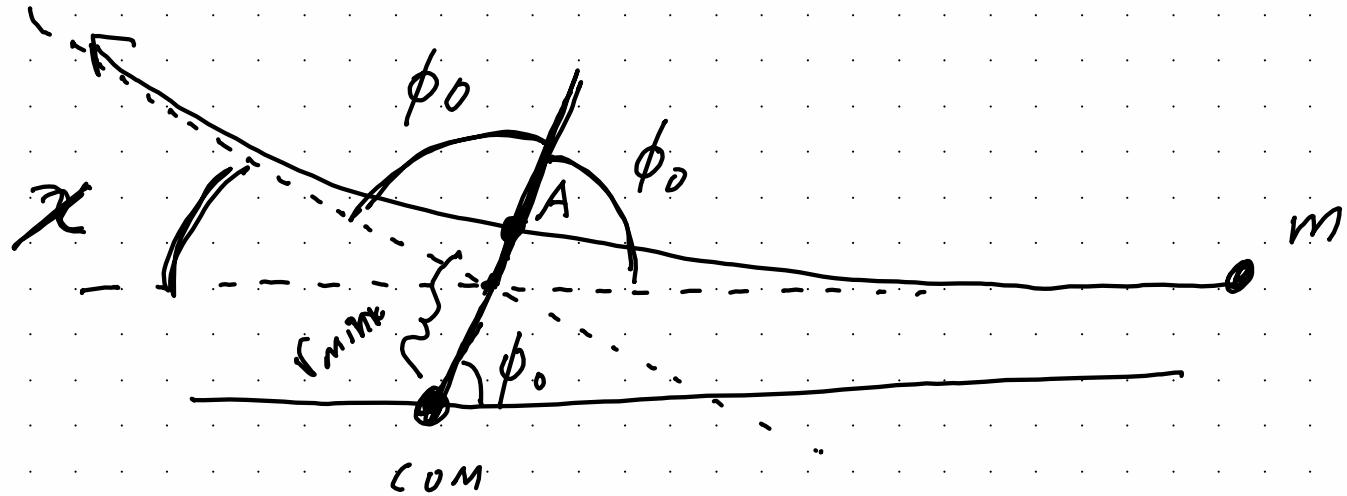
$$\tan \theta_1 = \frac{m_2 \sin(120^\circ)}{m_1 + m_2 \cos(120^\circ)} = \frac{\sin(120^\circ)}{1 + \cos(120^\circ)}$$

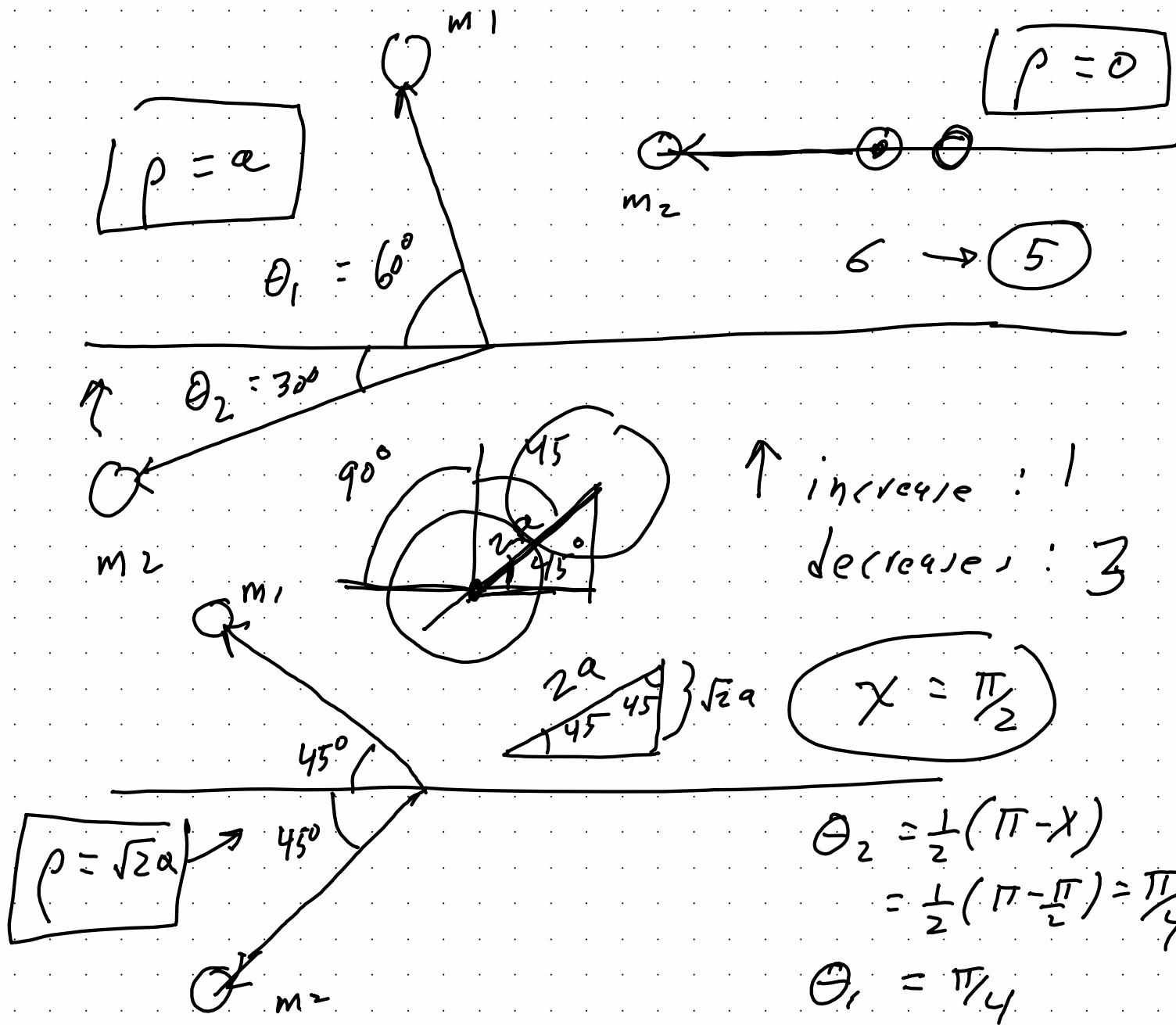
$$= \frac{\sqrt{3}/2}{1 - 1/2}$$

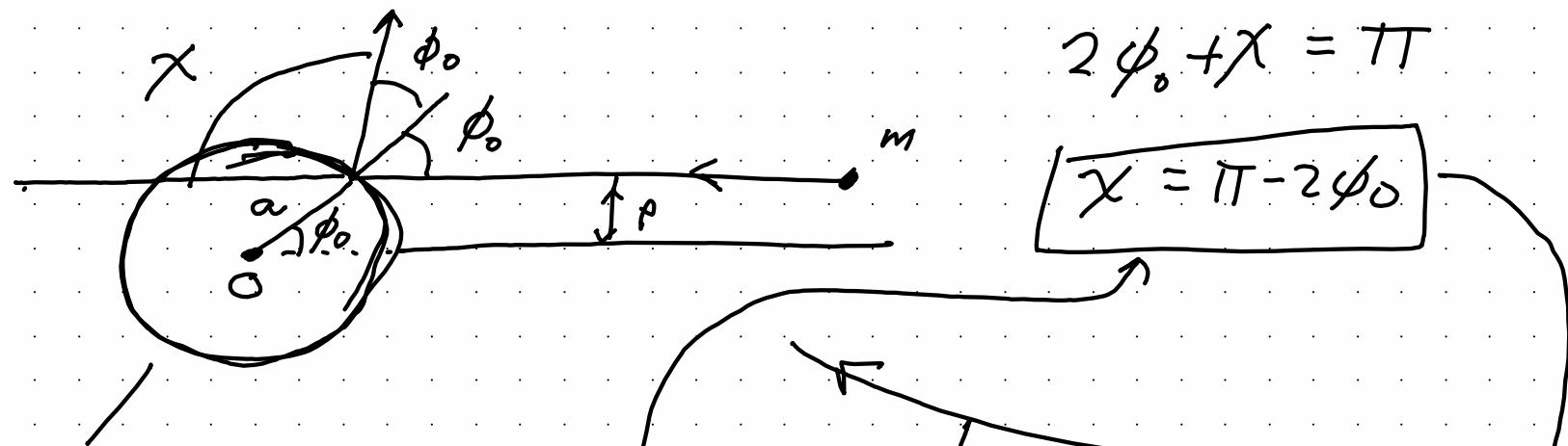
$$= \sqrt{3}$$

$$\boxed{\theta_1 = 60^\circ}$$



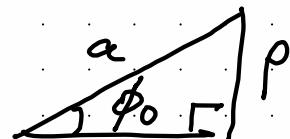






radius  $a$

$$U(r) = \begin{cases} 0 & r > a \\ \infty & r < a \end{cases}$$



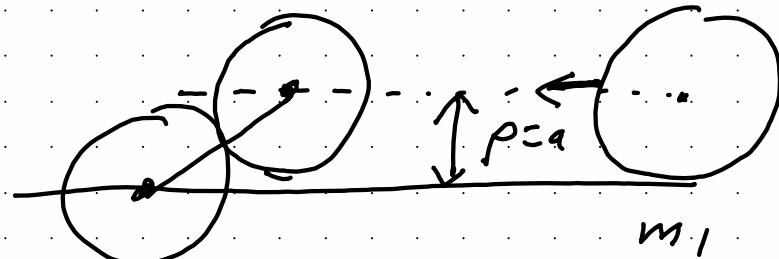
$$\sin \phi_0 = \frac{\rho}{a}$$

$$\phi_0 = \sin^{-1}\left(\frac{\rho}{a}\right)$$

$$\tan \theta_1 = \frac{m_2 \sin x}{m_1 + m_2 \cos x}$$

$$\theta_2 = \frac{1}{2}(\pi - x)$$

Prob 18.1

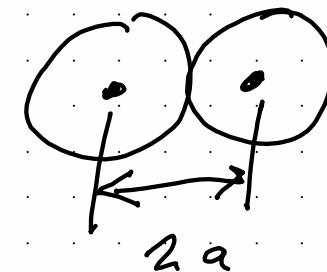


1<sub>a,b</sub>  
Frame

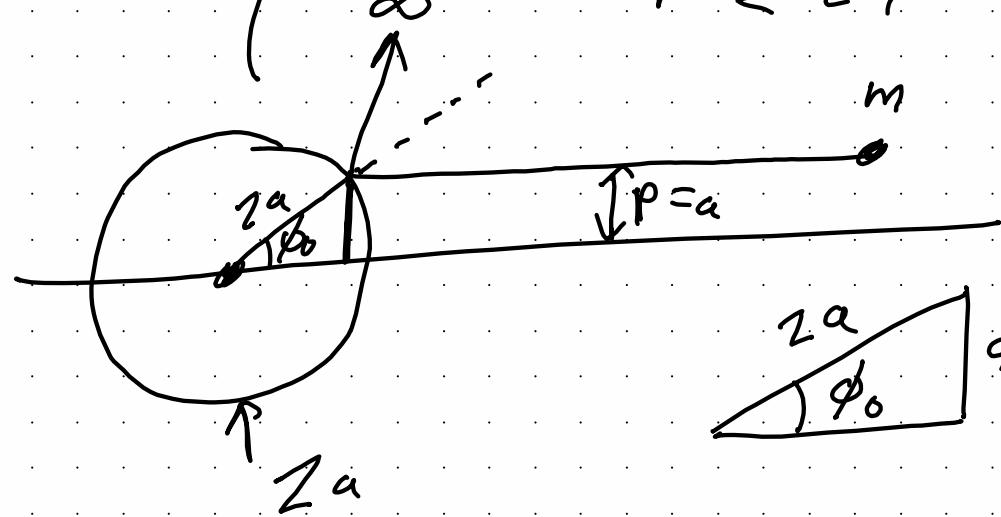
$m_2$   
( $v_2 = 0$ )

$$m_1 = m_2$$

$$p = a$$



$$U(r) = \begin{cases} 0 & r > 2a \\ \infty & r < 2a \end{cases}$$



COM

frame

$$\sin \phi_0 = \frac{1}{2}$$

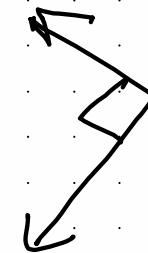
$$\phi_0 = 30^\circ$$

$$X = \pi - 2\phi_0 = 180^\circ - 2 \cdot 30^\circ = \boxed{120^\circ}$$

independent  
of  $m_1, m_2$

$$\tan \theta_1 = \frac{m_2 \sin X}{m_1 + m_2 \cos X}$$

$$\theta_2 = \frac{1}{2}(\pi - X)$$



$$\theta_2 = \frac{1}{2}(180^\circ - 120^\circ) = \boxed{30^\circ} \quad (\text{ind.p. of } m_1, m_2)$$

$$\tan \theta_1 = \frac{\sin(120^\circ)}{1 + \cos(120^\circ)} = \frac{\sqrt{3}/2}{1 - 1/2} = \sqrt{3}$$

$\theta_1 = 60^\circ$

$$m_1 \ll m_2 \rightarrow \tan \theta_1 \approx \tan X$$

$$m_1 \gg m_2 \rightarrow \tan \theta_1 \approx \frac{m_2}{m_1} \sin X \rightarrow$$

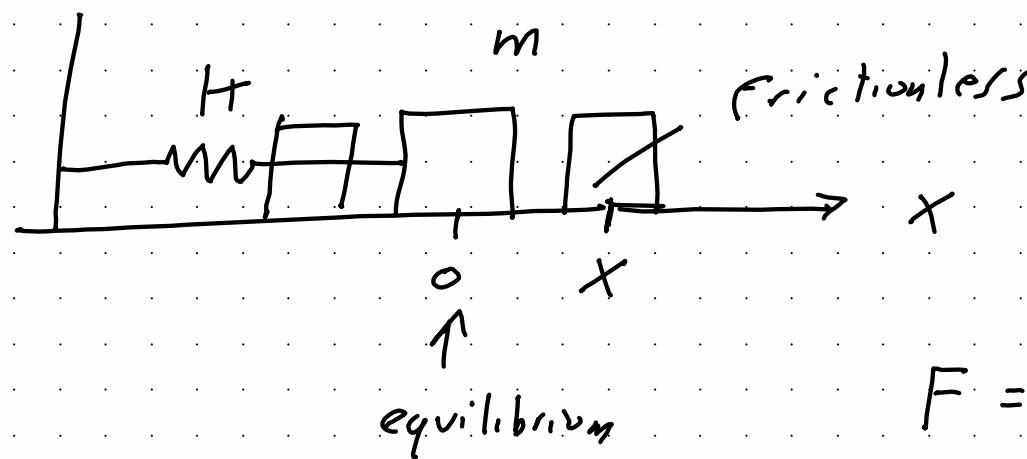
$\theta_1 = 120^\circ$

$\theta_1 \approx 0^\circ$

Lec #19: Tuesday 10/27

- Small oscillations (next 3 classes)

Sec 21, 22, 23  
(Free oscillation)  
in 1-d      (Forced oscillation)  
in 1-d      Several dimensions



$$F = -kx$$

|  
Spring constant

$$F = m \ddot{x}$$

$$-kx = m \ddot{x}$$

$$\rightarrow \ddot{x} = -\frac{k}{m}x \rightarrow x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$
$$\omega = \sqrt{\frac{k}{m}} \text{ (Angular freq)}$$

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

[ ] determined by ICs.

$$x(t) = a \cos(\omega t + \alpha)$$

| initial phase  
amplitude

$$x(t) = \operatorname{Re} [A e^{i\omega t}], A = a e^{i\alpha}$$

[ ] complex

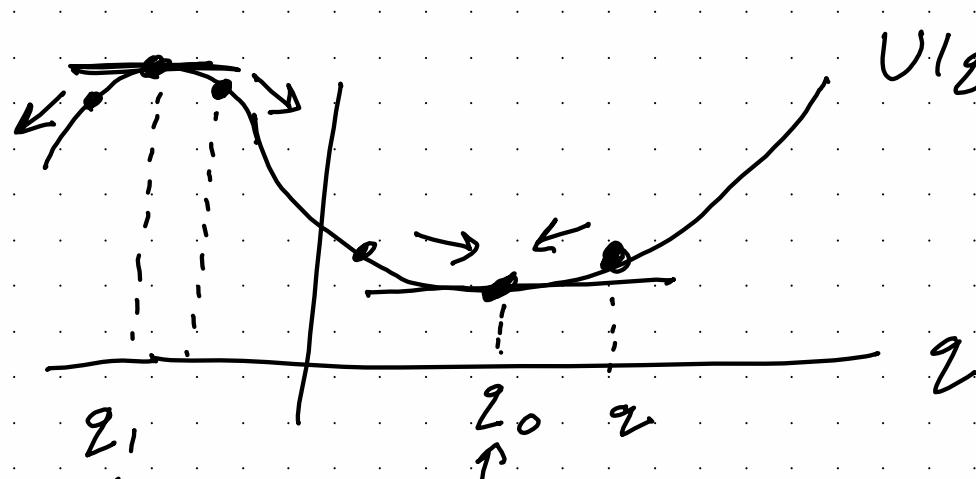
$$\begin{aligned}\operatorname{Re} [a e^{i\alpha} e^{i\omega t}] &= \operatorname{Re} [a e^{i(\omega t + \alpha)}] \\ &= \operatorname{Re} [a (\cos(\omega t + \alpha) \\ &\quad + i \sin(\omega t + \alpha))] \\ &= a \cos(\omega t + \alpha)\end{aligned}$$

Small oscillation:  $f(x) = f(x_0) + f'(x_0)(x - x_0)$

$$+ \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots$$

1-d system: 2 generalized coord

$$L = \frac{1}{2} \alpha(q) \dot{q}^2 - U(q)$$



$$U(q)$$

$$F = -\frac{dU}{dq}$$

$$F(q_0) = 0$$

$$\frac{dU}{dq} \Big|_{q_0} = 0$$

unstable  
equil       $\overset{\text{equilibrium}}{\underset{\text{stable}}{\text{stable}}}$

(Taylor expansion)

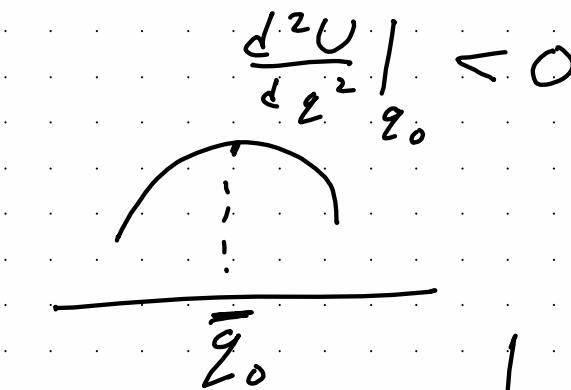
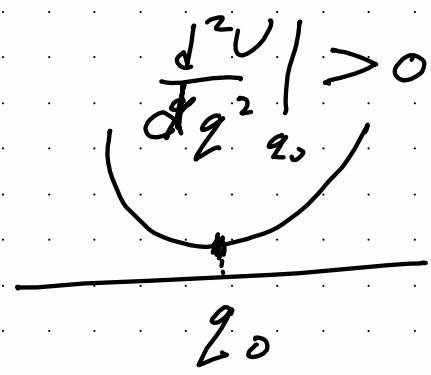


$$U(q) = \underline{U(q_0)} + \cancel{\frac{dU}{dq} \Big|_{q_0}} (q - q_0) + \frac{1}{2} \cancel{\frac{d^2U}{dq^2} \Big|_{q_0}} (q - q_0)^2 + \dots$$

Small displacement away from equilibrium

$$\frac{|q-q_0| \ll 1}{q_0} \quad (\text{ignore } O(3))$$

$$U(q) \approx U(q_0) + \frac{1}{2} \left. \frac{d^2 U}{dq^2} \right|_{q_0} (q-q_0)^2$$



$$T = \left. \frac{d^2 U}{dq^2} \right|_{q_0}$$

$$U(q) \approx U(q_0) + \underbrace{\frac{1}{2}}_{\text{const}} T (q-q_0)^2$$

$$\begin{aligned} F &= -\frac{dU}{dq} \\ &= -T (q-q_0) \\ &= -Tx \\ U &= \frac{1}{2} Tx^2 \end{aligned}$$

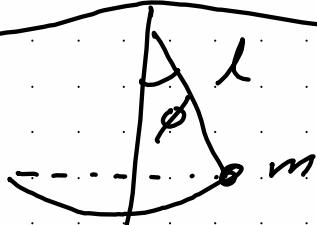
$$L = \frac{1}{2} q(\varphi) \dot{\varphi}^2 - U(\varphi)$$

$$= \frac{1}{2} q(g_0) \dot{x}^2 - U(g_0) - \frac{1}{2} kx^2$$

m  
const

$$x = \varphi - g_0 \rightarrow \dot{x} = \dot{\varphi}$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2$$

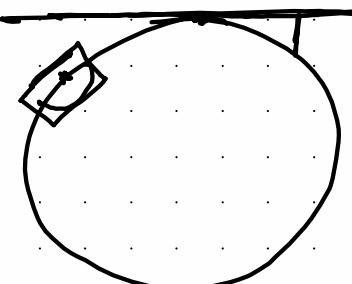
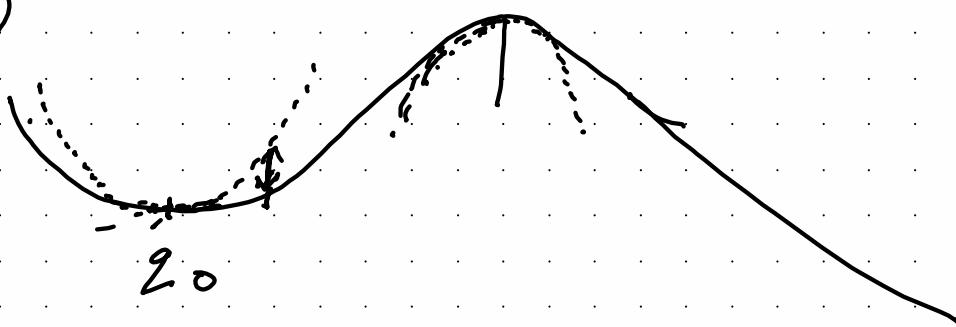


$$m \ddot{x} = -kx$$

EOM

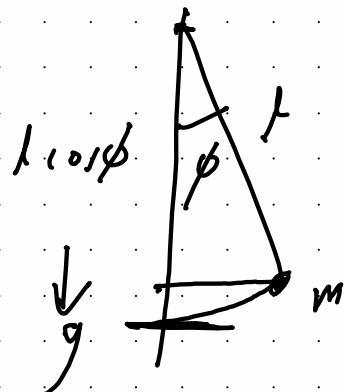
$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

$$U(x)$$



Example:

Simple pendulum



$$L = \frac{1}{2} m l^2 \dot{\phi}^2 - mg l (1 - \cos \phi)$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + mg l \cos \phi$$

$$= T - U$$

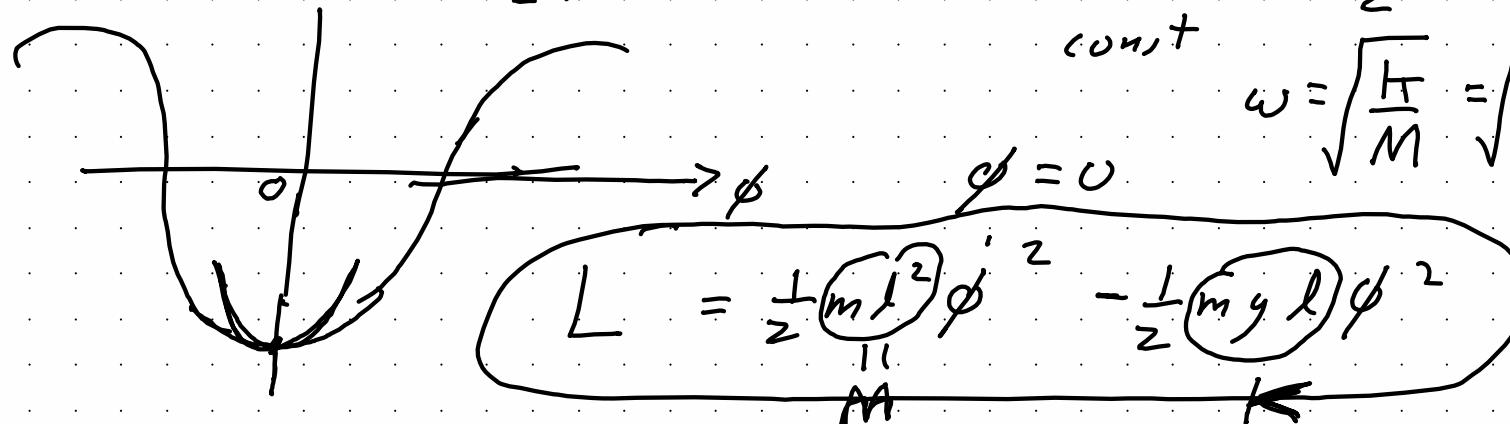
$$y = l = l \cos \phi$$

$$= l (1 - \cos \phi)$$

$$-mg l \left( 1 - \frac{1}{2} \dot{\phi}^2 \right)$$

$$U(\phi) = -mg l \cos \phi \Leftarrow$$

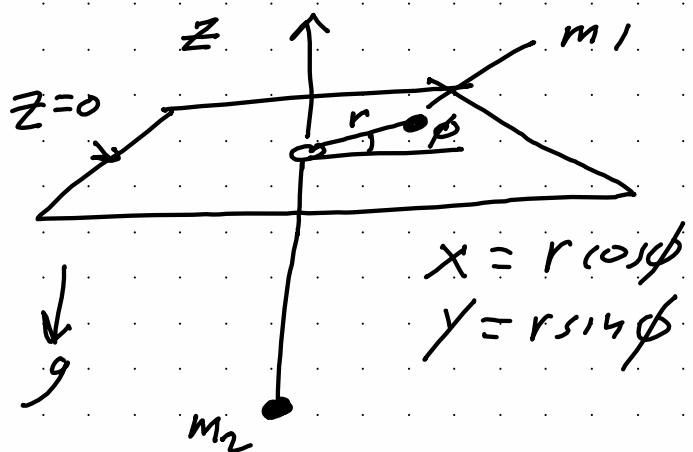
$$\cos \phi = 1 - \frac{1}{2} \dot{\phi}^2 + \dots = -\underbrace{mg l}_{\text{const}} + \frac{1}{2} mg l \dot{\phi}^2$$



$$L = \frac{1}{2} (m l^2) \dot{\phi}^2 - \frac{1}{2} (m g l) \dot{\phi}^2$$

$$\omega = \sqrt{\frac{U}{M}} = \sqrt{\frac{mg l}{ml^2}} = \sqrt{\frac{g}{l}}$$

Example: 2-d problem



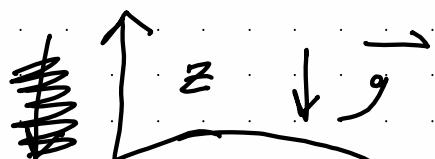
string: length  $l$

$$r + |z| = l$$

$$r - z = l$$

$$\boxed{z = r - l}$$

$$\rightarrow \dot{z} = \dot{r}$$



$$\dot{\phi} = \frac{M_z}{m_1 r^2}$$

$$T = \frac{1}{2} m_1 (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{1}{2} m_2 \dot{z}^2$$

$$\boxed{T = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2}$$

$$\boxed{U = m_2 g z} = m_2 g (r - l)$$

$\uparrow$   
const

$$\boxed{L = T - U}$$

$$\boxed{L = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r}$$

No  $\phi$  dependence

$$\leftarrow M_z = \frac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} = \text{const}$$

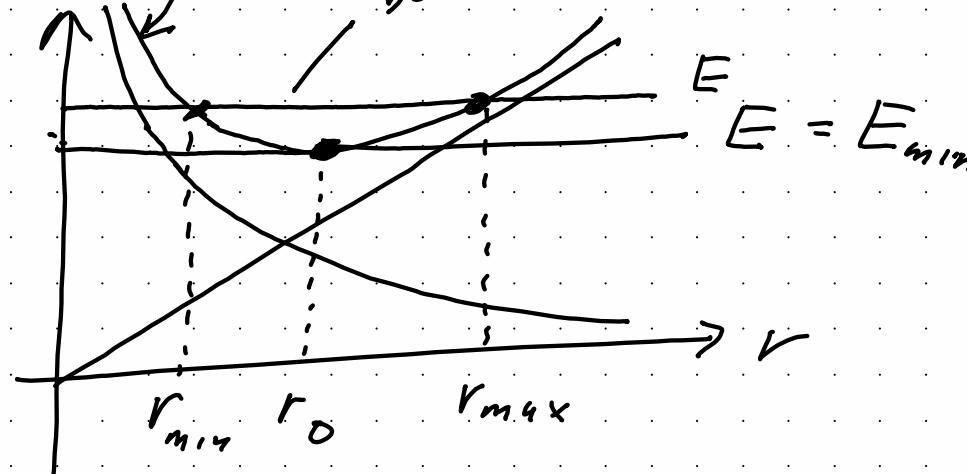
No explicit time dependence  $\rightarrow E = \text{const}$

$$E = T + U$$

$$= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 + m_2 g r$$

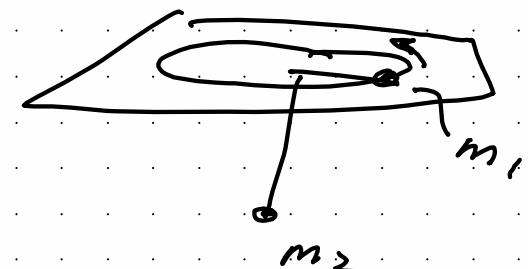
$$= \boxed{\frac{1}{2} (m_1 + m_2) \dot{r}^2} + \frac{M_z^2}{2 m_1 r^2} + m_2 g r - m_2 g \theta$$

$V_{\text{eff}}$  bound orbit  $V_{\text{eff}}(r)$



stable equilibrium

not effect  
 $E_{\text{oms}}$

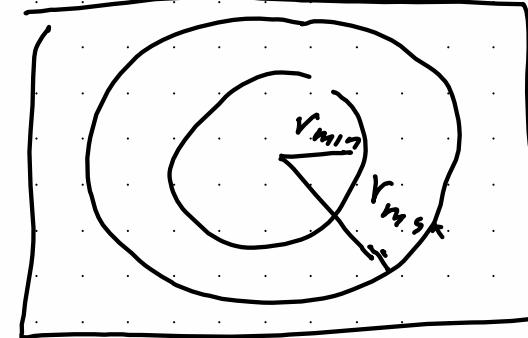
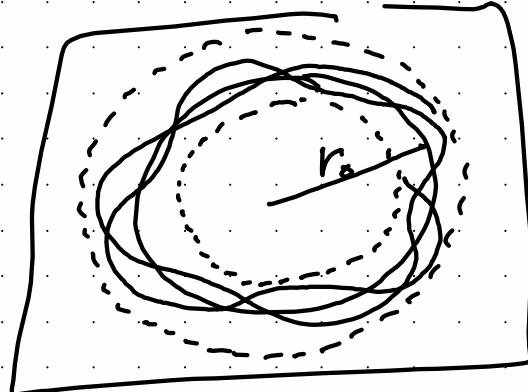


$$O = \left. \frac{dU_{eff}}{dr} \right|_{r=r_0}, \quad U_{eff}(r) = \frac{M_Z^2}{2m_1 r^2} + m_2 g r$$

$$= \frac{-M_Z^2}{m_1 r_0^3} + m_2 g$$

$$M_Z^2 = m_1 m_2 g r_0^3$$

Small oscillations around  $r_0$ :  $\frac{|r-r_0|}{r_0} \ll 1$



top view

$$\frac{dU_{\text{eff}}}{dr} = -\frac{M_2^2}{m_1 r^3} + m_2 g$$

$$\rightarrow \left. \frac{d^2 U_{\text{eff}}}{dr^2} \right|_{r_0} = \frac{3 M_2^2}{m_1 r_0^4}, \quad M_2^2 = m_1 m_2 g r_0^3$$

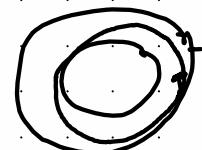
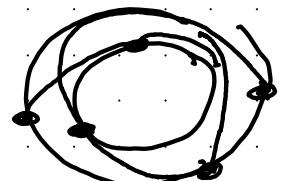
$$K = \frac{3 m_1 m_2 g r_0^3}{m_1 r_0^4} = \boxed{\frac{3 m_2 g}{r_0}}$$

$$E = \frac{1}{2} \cancel{(m_1 + m_2)} r^{-2} + \underbrace{U_{\text{eff}}(r_0) + \frac{1}{2} K (r - r_0)^2}_{\text{const}}$$

$$x = r - r_0$$

$$E = \frac{1}{2} (m_1 + m_2) \dot{x}^{-2} + \frac{1}{2} K x^2 + \text{const}$$

$$\omega_r = \sqrt{\frac{K}{m_1 + m_2}} = \boxed{\sqrt{\frac{3 m_2 g}{(m_1 + m_2) r_0}}}$$



$$\omega_\phi = \dot{\phi} |_{r_0}$$

$$M_z = m_1 r^2 \dot{\phi}$$

$$= \frac{M_z}{m_1 r_0^2}$$

$$= \frac{\sqrt{m_1 m_2 g r_0^3}}{m_1 r_0^2}$$

$$= \sqrt{\frac{m_2 g}{m_1 r_0}}$$

$$\boxed{\omega_r = \sqrt{\frac{3 m_2 g}{(m_1 + m_2) r_0}}}$$

$$= \frac{\sqrt{m_1 m_2 g r_0^3}}{\sqrt{m_1^2 r_0^4}}$$

$$0 \rightarrow 2\pi$$

not equal is seen!

For  $\omega_r = \omega_\phi$

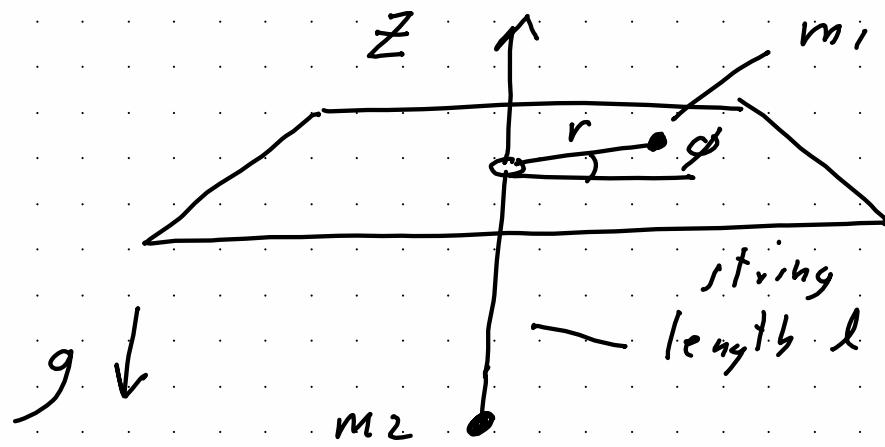
$$\frac{3}{m_1 + m_2} = \frac{1}{m_1}$$

$$\boxed{m_1 + m_2 = 3m_1}$$

closed  
bound  
Orbit

## Lecture #20

- Quiz #4 - Tuesday
- Today : (i) Finish up example from last time  
 (ii) Forced oscillations



$r = r_0$  (stable circular orbit)

$\omega_r = ?$   
 small oscillation  
 $|r - r_0| \ll r_0$

$$\omega_r = \sqrt{\frac{3m_2 g}{(m_1 + m_2)r_0}}, \quad \omega_\phi = \sqrt{\frac{m_2 g}{m_1 r_0}} = \dot{\phi} / r_0$$

$$\omega_r = \omega_\phi \rightarrow \text{closed orbit} \quad \frac{3}{m_1 + m_2} = \frac{1}{m_1} \rightarrow \frac{3m_1}{m_1 + m_2} = m_1 + m_2$$

$m_1 = \frac{1}{2}m_2$

Forced oscillations: (1-d)

$$m\ddot{x} = -kx + F(t)$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}, \quad \omega = \sqrt{\frac{k}{m}}$$

$$\begin{aligned} L &= \frac{1}{2} m \dot{x}^2 \\ &\quad - \frac{1}{2} k x^2 + x F(t) \end{aligned}$$

General sol'n:

$$x(t) = \underbrace{x_h(t)}_{\text{general soln to } F(t)=0} + \underbrace{x_p(t)}_{\text{particular soln}}$$

general soln to  $F(t)=0$

$$a \cos(\omega t + \alpha)$$

any solution to the  
equation with RHS =  $F(t)$

Example:  $F(t) = f \cos(\gamma t + \beta)$

$$x_p(t) = b \cos(\gamma t + \beta) = \frac{f}{m} \left( \frac{1}{\omega^2 - \gamma^2} \right)$$

$$-b\gamma^2 \cos(\gamma t + \beta) + \omega^2 b \cos(\gamma t + \beta) = \frac{f}{m} \cos(\gamma t + \beta)$$

$$b = \frac{f}{m} \left( \frac{1}{\omega^2 - \gamma^2} \right)$$

Resonance:  $\gamma = \omega$

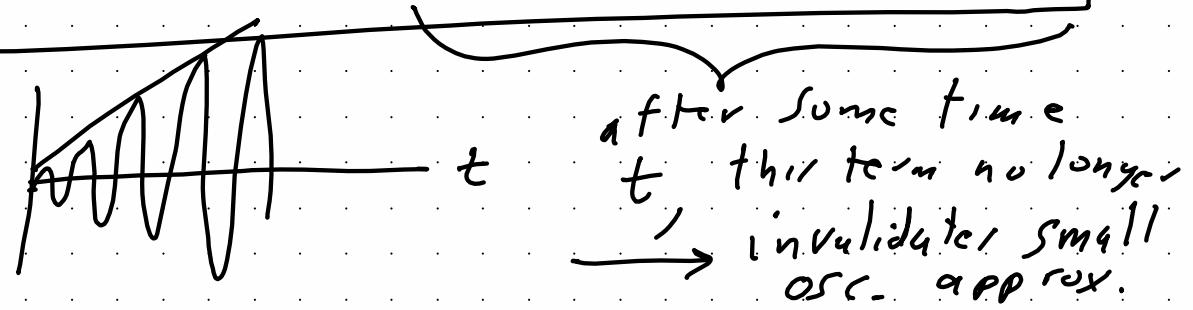
$$x(t) = a \cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)} (\cos(\gamma t + \beta) - \cos(\omega t + \beta))$$

particular solution

L'Hopital's

$$\frac{\frac{d}{d\gamma} (\text{num})}{\frac{d}{d\gamma} (\text{den})} \Big|_{\gamma=\omega} = \frac{+ft \sin(\omega t + \beta)}{+2m\omega}$$

$$x(t) = a \cos(\omega t + \alpha) + \frac{ft}{2m\omega} \sin(\omega t + \beta)$$



General  $F(t)$ :

$$F(t) = \int_{-\infty}^{\infty} dx \underset{\text{complex}}{\sim} F(x) e^{ixt}$$

Examples

$$F(t) = f_1 \sin(2\omega t) + f_2 \sin(3\omega t) + f_3 \sin(4\omega t)$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m} \quad (2^{\text{nd}} \text{o.-der})$$

$$\begin{aligned}\xi &= \dot{x} + i\omega x \\ \ddot{\xi} &= \ddot{x} + i\omega \dot{x}\end{aligned}\quad \left.\right\}$$

$$\ddot{\xi} - i\omega \dot{\xi} = \frac{F(t)}{m} \quad \boxed{\begin{array}{l} \text{(1st} \\ \text{order)} \end{array}}$$

$$y' + P(x)y = Q(x)$$

$$y' = \frac{dy}{dx}$$

Boas  
"Math method"

$$\ddot{x} - \omega^2 = \frac{F(t)}{m}$$

$$dy + (P(x)y - Q(x))dx = 0$$

not exact.

"Exact" differential:

If exact ①  $dy + (P(x)y - Q(x))dx = dU(x, y)$

$$dU(x, y) = \left( \frac{\partial U}{\partial x} \right) dx + \left( \frac{\partial U}{\partial y} \right) dy$$

$$\rightarrow \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y} \quad \left| \begin{array}{l} \frac{\partial}{\partial x} = P(x) \\ 0 \neq P(x) \end{array} \right.$$

Claim:  $\overbrace{= 0 \text{ (original equation)}}$

$$\mu(x) \left[ dy + (P(x)y - Q(x))dx \right] = dU$$

w

integrating  
factor

$$RHS = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

$$\mu(x) = \frac{\partial U}{\partial y}$$

$$\mu(x)(P(x)y - Q(x)) = \frac{\partial U}{\partial x}$$

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x} \rightarrow \mu'(x) = \mu(x) P(x)$$

$I(x)$   
" "

$$\int \frac{\mu'}{\mu} = \int P(x) dx$$

$\int P(x) dx$

~~A~~

$$\ln \mu = \int P(x) dx \rightarrow \mu = e^{\int P(x) dx}$$

$$\mu(x) = e^{\int I(x) dx}, \quad I(x) = \int dx P(x)$$

$$\frac{\partial U}{\partial y} = \mu(x)$$

$$U(x, y) = \mu(x)y + g(x) = C$$

$$\frac{\partial U}{\partial x} = \mu(x)(P(x)y - Q(x))$$

$$\cancel{\mu'(x)}y + g'(x) = \mu(x)(\cancel{P(x)}y - Q(x))$$

$$g'(x) = -\mu(x)Q(x)$$

$$\rightarrow g(x) = - \int dx Q(x) \mu(x) = - \int dx Q(x) e^{\int I(x) dx}$$

$$y = \frac{1}{\mu(x)} (C - g(x)) , \quad \mu(x) = e^{I(x)}$$

$$= e^{-I(x)} \left( C + \int dx Q(x) e^{I(x)} \right) , \quad I(x) = \int dx P(x)$$

$$y' + P(x)y = Q(x)$$

$$\begin{cases} x(t) = \operatorname{Im}\left(\frac{\xi(t)}{\omega}\right) \\ \dot{x}(t) = \operatorname{Re}(\xi(t)) \end{cases}$$

$$\ddot{\xi} - i\omega \dot{\xi} = F(t)$$

$$\ddot{\xi} = \ddot{x} + i\omega \dot{x}$$

$$y_{ik} \rightarrow \xi(t)$$

$$P(x) \rightarrow -i\omega$$

$$I(x) \rightarrow I(t) = \underline{-i\omega t}$$

$$Q(x) \rightarrow F(t)/m$$

$$\xi(t) = e^{i\omega t} \left[ \xi_0 + \int_0^t d\bar{t} e^{-i\omega \bar{t}} F(\bar{t})/m \right]$$

Lecture #21: Tues 11/3

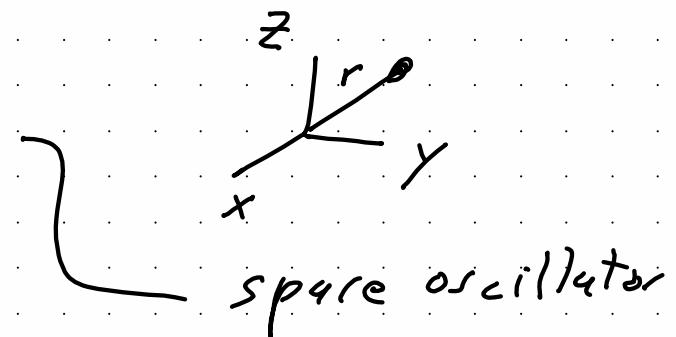
Q4: on Thursday

Thursday — rigid body motion / non-inertial reference

Sec 23: Free oscillations frames  
in 2-d or higher

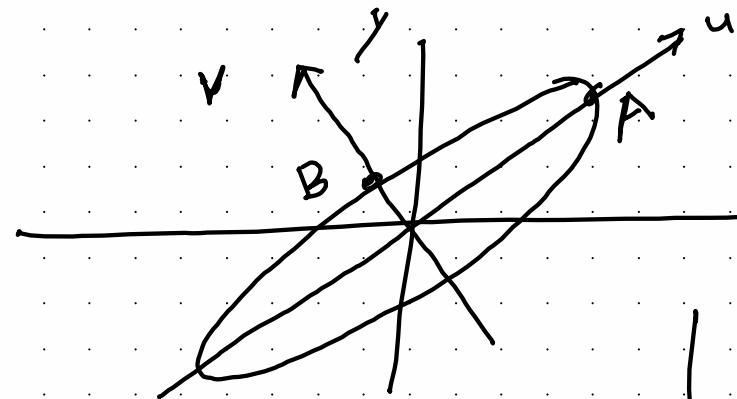
Prob 3, Sec 23:  $U = \frac{1}{2} \pi r^2$

motion is in 2-d plane  
( $x, y$ )



$$\begin{aligned} L &= T - U \\ &= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} \pi (x^2 + y^2) \\ &= \left( \frac{1}{2} m \dot{x}^2 - \frac{1}{2} \pi x^2 \right) + \left( \frac{1}{2} m \dot{y}^2 - \frac{1}{2} \pi y^2 \right) \end{aligned}$$

2 independent oscillations in the  $x, y$ -directions,  
Any freq:  $\omega_x = \sqrt{\frac{\pi}{m}}, \omega_y = \sqrt{\frac{\pi}{m}} \rightarrow \omega_x = \omega_y$

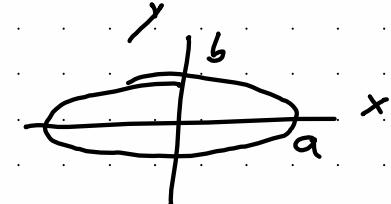


most general motion

$$x = a \cos(\omega t + \alpha)$$

$$y = b \sin(\omega t + \beta)$$

$$\alpha = \beta = 0$$

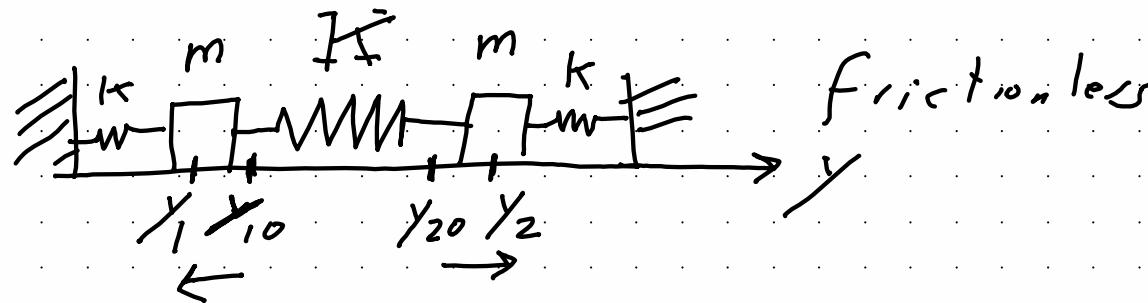


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Normal coordinates:

N independent oscillations with frequency  $\omega_1, \omega_2, \dots, \omega_N$

Example:



$$i, \tau = 1, 2, \dots, N$$

$$\begin{aligned} x_1 &= y_1 - y_{10} \\ x_2 &= y_2 - y_{20} \end{aligned} \quad \left. \begin{array}{l} 3 \text{ small deviations away from} \\ \text{equil. br.} \end{array} \right.$$

$$L = T - U$$

$$\begin{aligned} T &= \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 \\ &= \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) \end{aligned}$$

$$\begin{aligned} & (y_2 - y_1) - (y_{20} - y_{10}) \\ &= (x_2 - x_{20}) - (x_1 - x_{10}) \\ &= x_2 - x_1 \end{aligned}$$

$$\begin{aligned} U &= \frac{1}{2} K x_1^2 + \frac{1}{2} K x_2^2 + \frac{1}{2} K (x_2 - x_1)^2 \\ &= \frac{1}{2} (K + K) x_1^2 + \frac{1}{2} (K + K) x_2^2 \\ &\quad - K x_1 x_2 \end{aligned}$$

$$\frac{1}{2} \sum_{i,\tau} m_i K_i \dot{x}_i \dot{x}_{\tau}$$

$$\begin{aligned} & (x_1^2 + x_2^2 - 2 x_1 x_2) \\ &= \frac{1}{2} \sum_{i,\tau} K_i x_i x_{\tau} \end{aligned}$$

$$\begin{aligned}
 T &= \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 \\
 &= \frac{1}{2} [m_{11} \dot{x}_1^2 + m_{12} \dot{x}_1 \dot{x}_2 + m_{21} \dot{x}_2 \dot{x}_1 + m_{22} \dot{x}_2^2] \\
 &= \frac{1}{2} [m_{11} \dot{x}_1^2 + m_{22} \dot{x}_2^2 + 2m_{12} \dot{x}_1 \dot{x}_2]
 \end{aligned}$$

$$\rightarrow m_{11} = m, \quad m_{22} = m, \quad m_{12} = \theta = m_{21}$$

$$m_{ik} = \begin{array}{|c|c|} \hline m & 0 \\ \hline 0 & m \\ \hline \end{array}$$

$$\begin{aligned}
 U &= \frac{1}{2} \sum_{i,k} K_{ik} x_i x_k = \frac{1}{2} (K + EI) x_1^2 + \frac{1}{2} (K + EI) x_2^2 \\
 &\quad - 2EI x_1 x_2
 \end{aligned}$$

$$K_{ik} = \begin{array}{|c|c|} \hline K + EI & -EI \\ \hline -EI & K + EI \\ \hline \end{array}$$

$$L = T - U$$

$$= \frac{1}{2} \sum_{j,k} m_{jk} \dot{x}_j \dot{x}_k - \frac{1}{2} \sum_{j,k} K_{jk} x_j x_k \quad \left| \begin{array}{l} \delta_{jk} \\ \delta_{kj} \end{array} \right.$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) = \frac{\partial L}{\partial x_i}$$

$$\boxed{\sum_k m_{ik} \ddot{x}_k = - \sum_k K_{ik} x_k}$$

wave

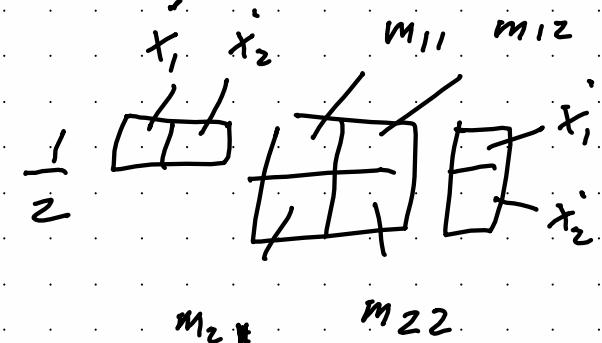
$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$m$        $x$

$$\frac{\partial}{\partial x_1} \left( K_{11} x_1^2 + K_{22} x_2^2 + 2K_{12} x_1 x_2 \right) \left( \frac{1}{2} \right)$$

$$= - (K_{11} x_1 + K_{22} x_2) = - \sum_k K_{ik} x_k$$

$$(i=1, 2, \dots, N)$$



$$\frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k$$

$$\frac{1}{2} \sum_{j,k} m_{jk} \dot{x}_j \dot{x}_k$$

$$\left. \begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) &= \frac{\partial L}{\partial x_i} \\ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_2} \right) &= \frac{\partial L}{\partial x_2} \end{aligned} \right\} \quad \begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) &= \frac{\partial L}{\partial x_i} \\ i = 1, 2 \end{aligned}$$

$$\sum_K m_{iK} \ddot{x}_K = - \sum_K k_{iK} x_K$$

$$\sum_K (m_{iK} \ddot{x}_K + k_{iK} x_K) = 0$$

Trial solution:

$$x_K = A_K e^{i\omega t}$$

$$\ddot{x}_K = -\omega^2 A_K e^{i\omega t}$$

0 vector

$$\sum_K (-m_{iK} \omega^2 + k_{iK}) A_K e^{i\omega t} = 0'$$

matrix

$$\boxed{\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}} \quad \boxed{0} = \boxed{\begin{array}{|c|} \hline 0 \\ \hline \end{array}}$$

$$M^{-1}(M \underline{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix})$$

$$\underline{v} = M^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$


---

$K_{ii} - \omega^2 m_{ii}$  must be a singular matrix

$$\det(K_{ii} - \omega^2 m_{ii}) = 0$$

characteristic equation  
for angular freq

Eigenvalues  $\rightarrow \omega_1^2, \omega_2^2, \dots, \omega_N^2$

(normal mode freqs)

Eigenvector  $\rightarrow \underline{v}_1, \underline{v}_2, \dots, \underline{v}_N$

(normal mode oscillation)

$$\det \begin{pmatrix} \begin{array}{|c|c|} \hline h+EI & -EI \\ \hline -EI & h+EI \\ \hline \end{array} & -\omega^2 \begin{array}{|c|c|} \hline m & 0 \\ \hline 0 & m \\ \hline \end{array} \end{pmatrix} = 0$$

$$\det \begin{pmatrix} \begin{array}{|c|c|} \hline (h+EI) - \omega^2 m & -EI \\ \hline -EI & (h+EI) - \omega^2 m \\ \hline \end{array} \end{pmatrix} = 0$$

$$\textcircled{1} \quad \omega^4 + \textcircled{2} \quad \omega^2 + \textcircled{3} = 0$$

$\rightarrow$  quadratic equation for  $\omega^2 (\equiv \lambda)$

$$\textcircled{1} \quad \lambda^2 + \textcircled{2} \quad \lambda + \textcircled{3} = 0$$

solve quadratic equation:

$$\boxed{\omega_+^2 = \frac{h+2EI}{m}}$$

$$\omega_-^2 = \frac{k}{m}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

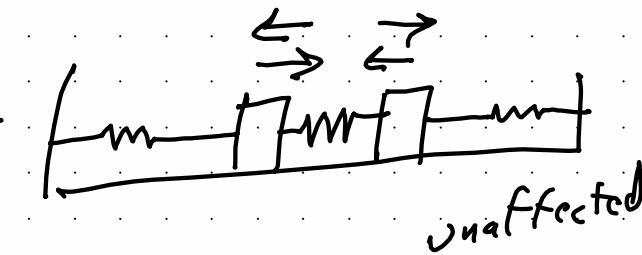
$$\underline{\omega_f^2} : \begin{vmatrix} k+I - m\omega_f^2 & -II \\ -II & k+I - m\omega_f^2 \end{vmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A_2 = -A_1$$

$$V_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$V_+$  : eigenvector  
assoc. with  $\omega_+$

Normal mode oscillations



$$\underline{\omega_-^2} : V_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

General solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \text{Re} \left( C_+ V_+ e^{i\omega_f t} + C_- V_- e^{-i\omega_f t} \right)$$

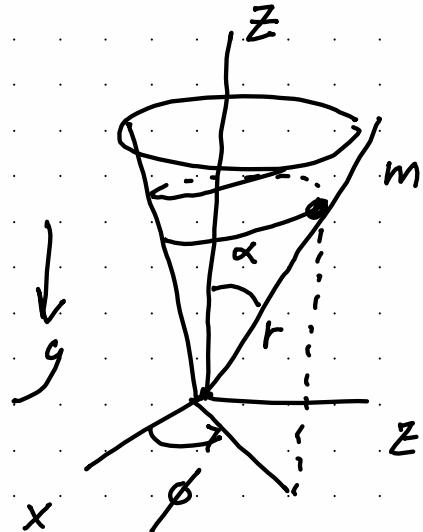
complex constant determined by  
initial conditions

Lec 22: Nov 5<sup>th</sup>

- Quiz #4 today (Quiz #5 - next Thursday  
Quiz #6 - last day of class)
- Rigid body motion: Sec 31-36  
Non-inertial ref frames: Sec 38, 39

Q4:

name - q4.pdf



- a) Lagrangian *Write down the*
- b) Find  $r_0$  for a stable circular orbit for fixed angular momentum  $M_z$ .
- c) Find freq of small radial oscillations about  $r_0$ .

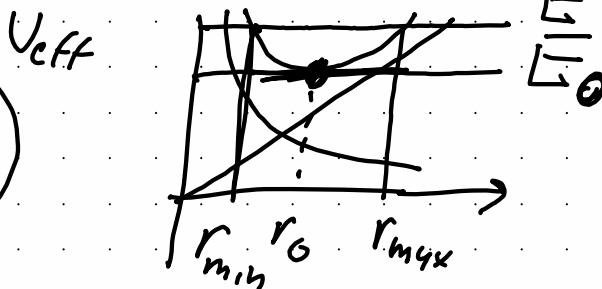
$$z = r \cos \alpha, T = \frac{1}{2} m (r^2 + r^2 \sin^2 \alpha \dot{\phi}^2 + \cancel{r^2 \dot{\theta}^2})$$

$$E = \frac{1}{2}mr^2 + U_{\text{eff}}(r)$$

$$M_z^2 = m^2 g r_0^3 \cos \alpha \sin^2 \alpha$$

$$\frac{M_z^2}{2mr^2 \sin^2 \alpha} + mg r \cos \alpha$$

$$U_{\text{eff}} = M_z^2 = mr^2 \sin^2 \alpha \dot{\phi}$$



$$\omega = \sqrt{\frac{k}{m}}$$

$$x = q - z_0$$

~~At~~

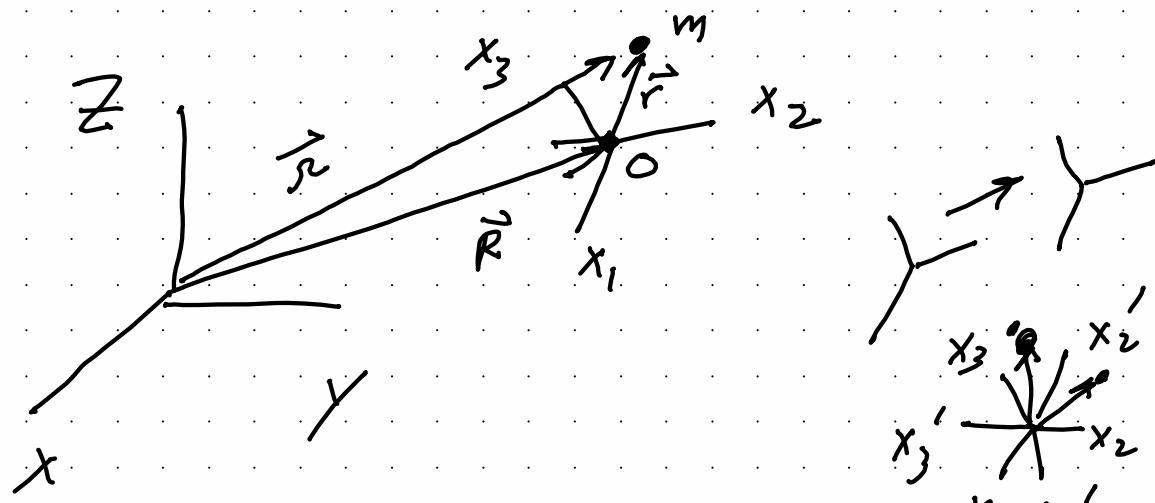
$$k = \left. \frac{d^2 U_{\text{eff}}}{dr^2} \right|_{r_0} = \frac{3mg \cos \alpha}{r_0}$$



$$\omega_r = \sqrt{\frac{k}{m}} = \sqrt{\frac{3g \cos \alpha}{r_0}}$$

$$\omega_\phi = \sqrt{\frac{g \cos \alpha}{r_0 \sin^2 \alpha}} \neq \omega_r$$

$$\begin{aligned} \dot{\phi} &= \frac{M_z}{mr^2 \sin^2 \alpha} \\ &= \frac{M_z}{mr_0^2 \sin^2 \alpha} \\ &= \frac{\sqrt{m^2 g r_0^3 \cos \alpha \sin^2 \alpha}}{\sqrt{m^2 r_0^4 \sin^4 \alpha}} \end{aligned}$$



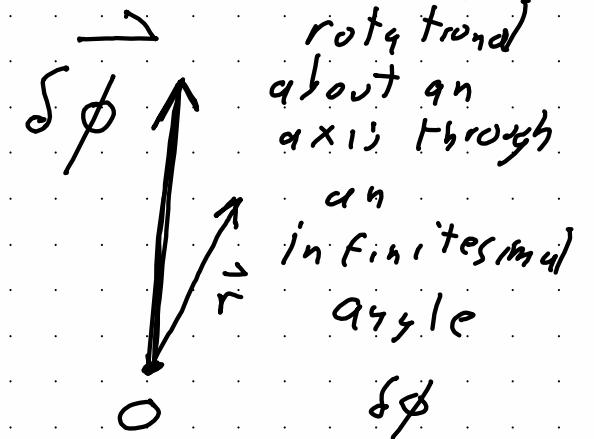
$$\vec{r} = \vec{R} + \vec{r}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\omega} \times \vec{r}$$

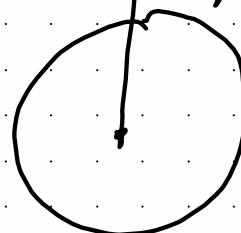
translational  
velocity of  
O

rotational  
velocity

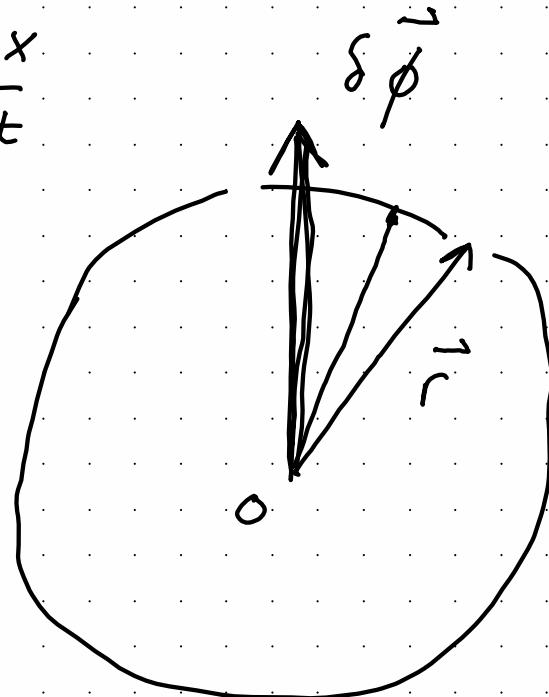
infinitesimal  
rotational  
about an  
axis through  
an



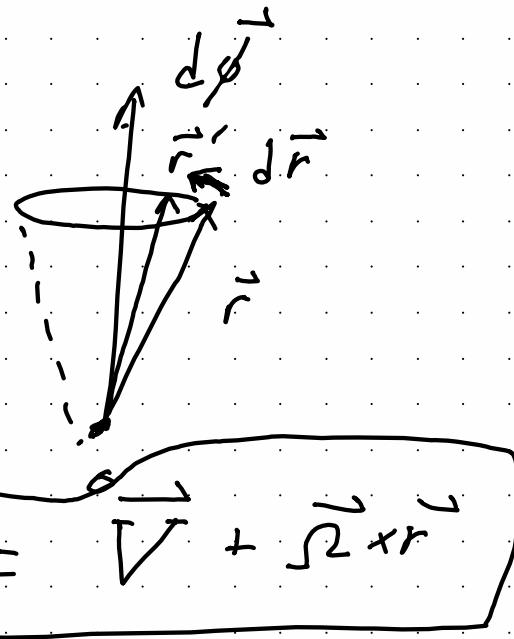
$$\vec{R} \rightarrow \vec{R} + d\vec{R}, \quad \frac{d\vec{R}}{dt} = \vec{V} \quad \delta\phi$$



$$\textcircled{1} \quad \vec{V} = \frac{dx}{dt}$$



$$\frac{d\vec{r}}{dt}$$



$$\boxed{\vec{V} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\omega} \times \vec{r}}$$

$$\vec{V} = \frac{d\vec{R}}{dt} \quad 3 \text{ DOF}$$

associated  
with Oris,

$$\vec{dR} \rightarrow O$$

$$O \quad (\phi, \theta, \psi)$$

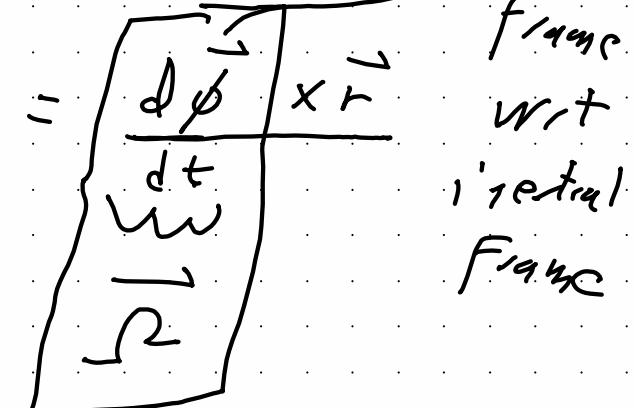
$$\frac{d\vec{R}}{dt}$$

$$\vec{\phi}: 3 \text{ DOF}$$

assoc  
w.tu  
orientat.

$$\vec{dr} = \vec{d\phi} \times \vec{r}$$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{\phi}}{dt} \times \vec{r}$$



w.r.t  
inertial  
Frame

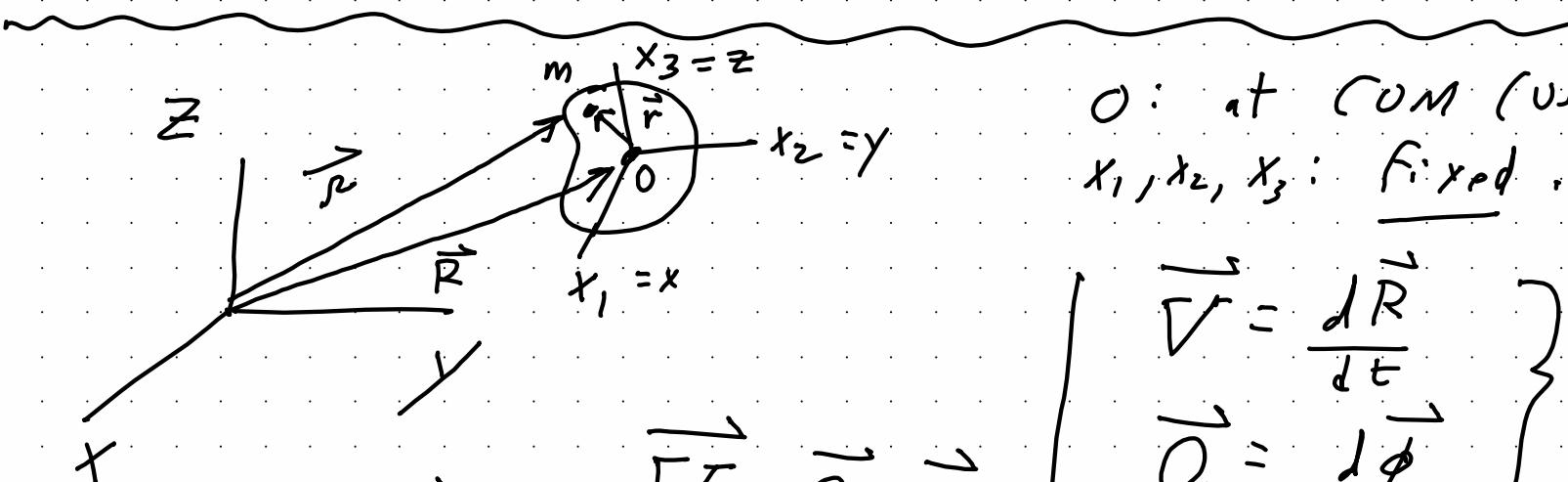
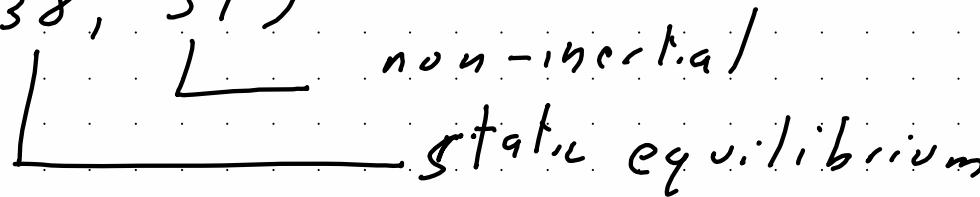
Lec #23: Tuesday Nov 10<sup>th</sup>

- Quiz #5: Thursday

- Midterm #2: Next Thursday 11/19

- Today: Rigid body motion

(Sec 31-36, 38, 39)



$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}$$

$\vec{\Omega}$ : angular velocity vector

O: at COM (usually)  
 $x_1, x_2, x_3$ : fixed in RB

$$\begin{aligned}\vec{V} &= \frac{d\vec{R}}{dt} \\ \vec{\Omega} &= \frac{d\vec{\phi}}{dt}\end{aligned}\quad \left. \begin{array}{l} \\ \end{array} \right\} q_i$$

$$(\vec{R}, \vec{\phi}): 6 \text{ DOF} = q_i$$

$$T = \frac{1}{2} m v^2$$

$$\vec{p} = m \vec{v}$$

rotational quant. t.

$$T_{\text{rot}} = \frac{1}{2} I \omega^2$$

$$\overrightarrow{M} = I \vec{\omega} \rightarrow M_i = \sum I_{ij} \omega_j$$

I: moment of inertia

→  $I_{ij}$ : inertia tensor

1	2	3
2		
3		

$$M_i = I_{ij} \omega_j$$

KE:

$$T = \frac{1}{2} \sum_a m_a |\vec{v}_a|^2$$

$$= \frac{1}{2} \sum_a m_a \left| \vec{V} + \vec{\omega} \times \vec{r}_a \right|^2$$

$$= \frac{1}{2} \sum_a m_a \left( |\vec{V}|^2 + |\vec{\omega} \times \vec{r}_a|^2 + 2 \vec{V} \cdot (\vec{\omega} \times \vec{r}_a) \right)$$

$$\textcircled{1} = \frac{1}{2} \sum_a m_a |\vec{V}|^2 = \boxed{\frac{1}{2} \mu V^2}$$

total mass

$$\textcircled{3} = \sum_a m_a \vec{V} \cdot (\vec{\omega} \times \vec{r}_a) = \vec{A} \cdot (\vec{B} \times \vec{C})$$

$$= \left( \sum_a m_a \vec{r}_a \right) \cdot (\vec{V} \times \vec{\omega})$$

$$= \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$= \mu \vec{R}_{com} \cdot (\vec{V} \times \vec{\omega})$$

$$= \vec{C} \cdot (\vec{A} \times \vec{B})$$

$\underbrace{\quad}_{=0}$  for

$\vec{0}$  at com

$$\vec{v}_a = \vec{V} + \vec{\omega} \times \vec{r}_a$$

$$\begin{aligned} & \left| \vec{A} + \vec{B} \right|^2 \\ &= A^2 + B^2 + 2 \vec{A} \cdot \vec{B} \end{aligned}$$

$$\begin{aligned}
 (2) &= \frac{1}{2} \sum_a m_a |\vec{\omega} \times \vec{r}_a|^2 \\
 &= \frac{1}{2} \sum_a m_a (\vec{\omega} \times \vec{r}_a) \cdot (\vec{\omega} \times \vec{r}_a) \\
 &= \frac{1}{2} \sum_a m_a \vec{\omega} \cdot (\vec{r}_a \times (\vec{\omega} \times \vec{r}_a)) \\
 &= \frac{1}{2} \sum_a m_a \vec{\omega} \cdot (\vec{\omega} r_a^2 - \vec{r}_a (\vec{r}_a \cdot \vec{\omega})) \\
 &= \frac{1}{2} \sum_a m_a (\vec{\omega}^2 r_a^2 - (\vec{\omega} \cdot \vec{r}_a)^2) \\
 &= \frac{1}{2} \sum_a m_a \left( \sum_{i,j} \vec{\omega}_i \vec{\omega}_j \delta_{ij} r_a^2 - \sum_{i,j} \vec{\omega}_i r_{ai} \vec{\omega}_j r_{aj} \right) \\
 &= \frac{1}{2} \sum_{i,j} \left( \sum_a m_a (\vec{\omega}_i^2 \delta_{ij} - r_{ai} \vec{\omega}_i \cdot r_{aj}) \right) \vec{\omega}_i \vec{\omega}_j \\
 &= \boxed{\frac{1}{2} \sum_{i,j} I_{ij} \vec{\omega}_i \vec{\omega}_j} \quad (= \frac{1}{2} \boxed{\text{I}} \boxed{\text{II}} \boxed{\text{III}} \boxed{\text{IV}})
 \end{aligned}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

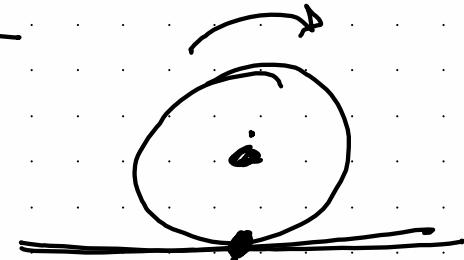
$$T = \underbrace{\frac{1}{2} m V^2}_{\text{trans}} + \underbrace{\sum_{i,j} I_{ij} \cdot \Omega_i \cdot \Omega_j}_{\text{rotational}} + \textcircled{Q}$$

$$\left( \frac{1}{2} I \dot{\theta}^2 \right)$$

Freshman physics

for COM at origin  
OF RB Frame

$\vec{M}$ : wrt COM of body

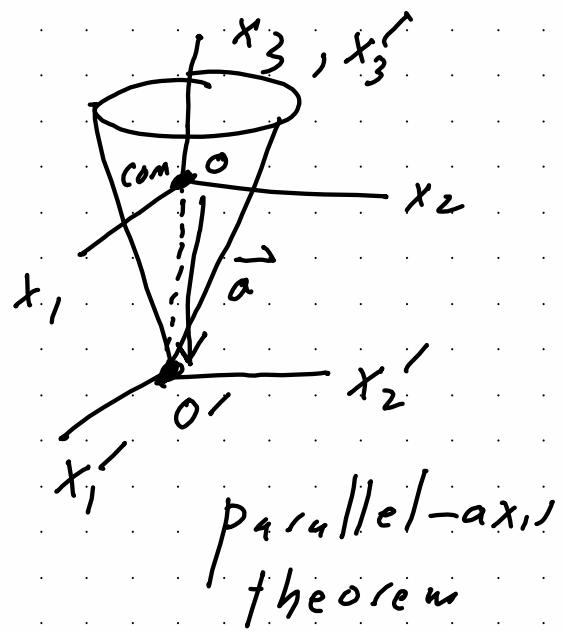


$$\begin{aligned} \vec{M} &= \sum_a \vec{r}_a \times \vec{p}_a \\ &= \sum_a m_a \vec{r}_a \times \vec{v}_a \\ &= \sum_a m_a \vec{r}_a \times (\vec{V} + \vec{\omega} \times \vec{r}_a) \\ &= \sum_a m_a \vec{r}_a \times (\vec{\omega} \times \vec{r}_a) \end{aligned}$$

$$= \sum_a m_a (\vec{\omega} \cdot \vec{r}_a)^2 - \vec{r}_a (\vec{r}_a \cdot \vec{\omega})$$

$$M_i = \sum_j I_{ij} \cdot \Omega_j$$

$$\vec{M} = I \vec{\omega} \quad (\text{Fresh. phys.})$$



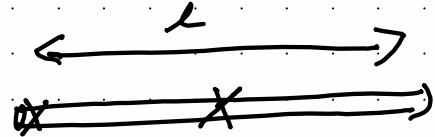
$$I_{ij}' = I_{ij} + m(a^2 \delta_{ij} - a_i a_j)$$

wrt  $O'$       wrt  $O$  (com)

$$\vec{a} : \text{Vector from } O \text{ to } O'$$

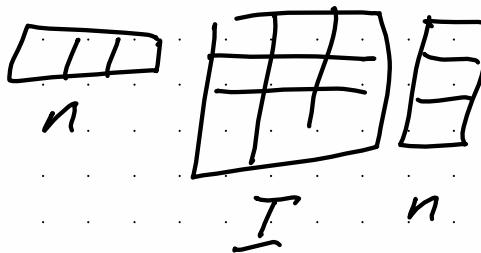
$$I(\hat{n}) = \sum_{i,j} I_{ij} n_i n_j$$

~~moment~~  
of inertia



$M$

$\hat{n}$  : axis of rotation



$$I \cdot n$$

$$I_{\text{com}} = \frac{1}{2} M l^2$$

$$I_{\text{end}} = \frac{1}{3} M l^2$$

$$I_{ij} = \sum_a m_a (r_a^2 \delta_{ij} - r_{ai} r_{aj})$$

3x3 real  
symmetric

$$= \int p dV (r^2 \delta_{ij} - r_i r_j)$$

(can always be diagonalized)

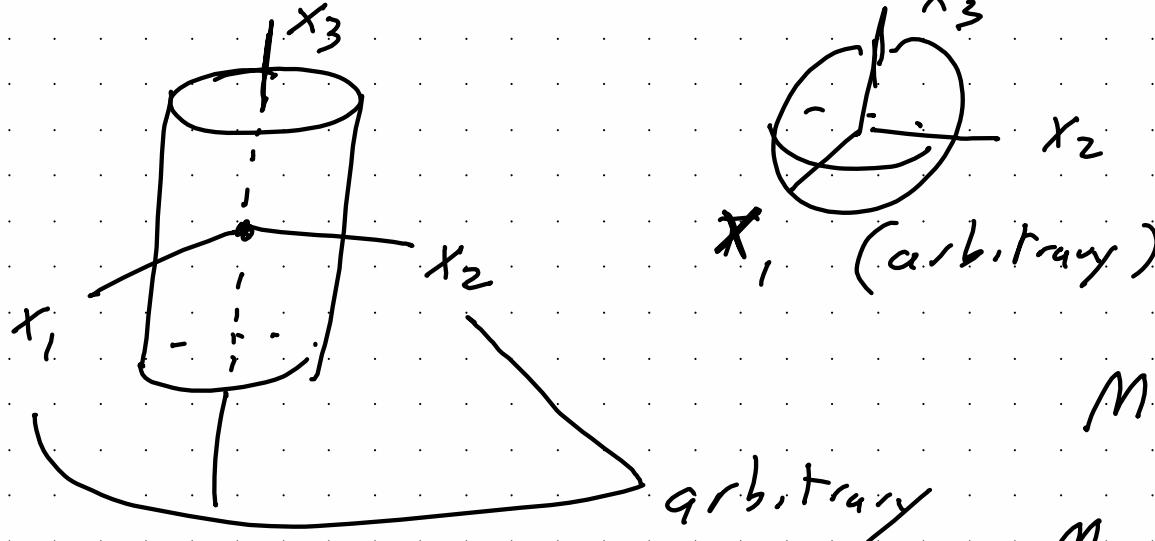
$$\begin{array}{|c|c|c|} \hline I_1 & 0 & 0 \\ \hline 0 & I_2 & 0 \\ \hline 0 & 0 & I_3 \\ \hline \end{array}$$

$$I_{ij} = I_i \delta_{ij}$$

$$T_{\text{rot}} = \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j$$

Principle Axes:  $(x_1, x_2, x_3)$

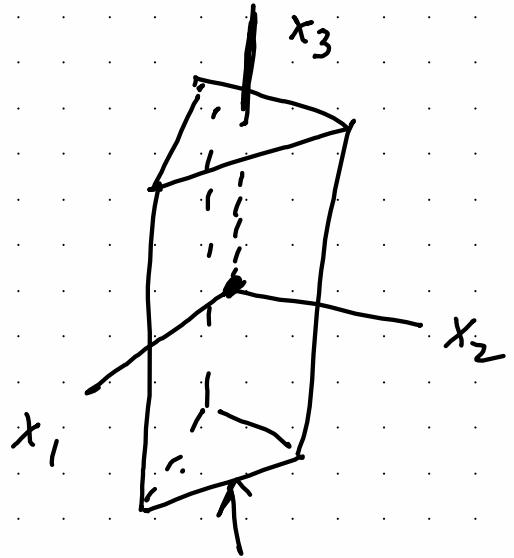
$$= \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$$



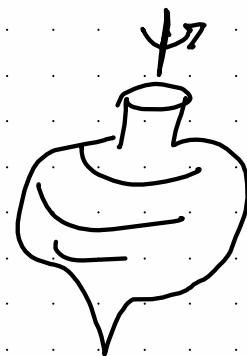
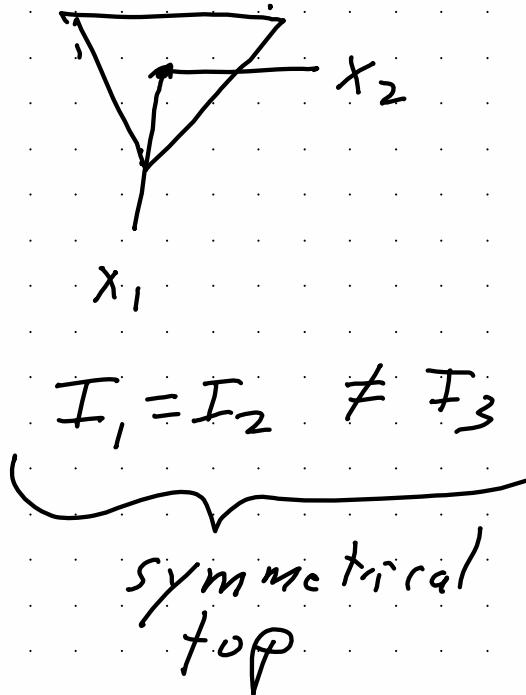
$$M_i = \sum_j I_{ij} \Omega_j$$

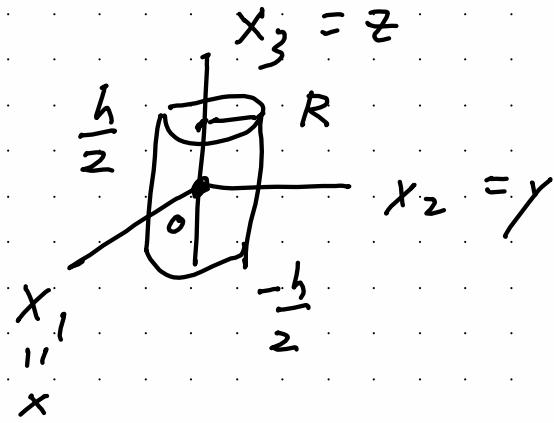
$$M_1 = I_1 \Omega_1, M_2 = I_2 \Omega_2$$

$$M_3 = I_3 \Omega_3$$



equilateral  
triangle





total mass  $\mu$

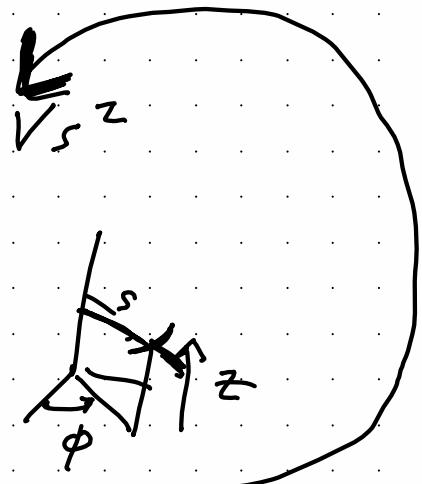
$$\rho = \frac{M}{\text{volume}} = \frac{M}{\pi R^2 h}$$

$$I_3 = I_{33} = \int \rho dV \left( r^2 \delta_{33} - \frac{r_3 r_3}{z^2} \right)$$

$$= \int \rho dV (r^2 - z^2)$$

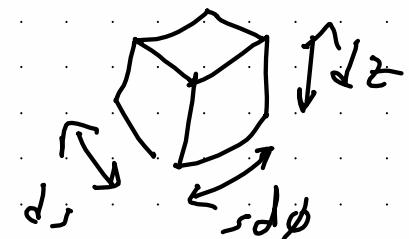
$$= \int \rho dV (x^2 + y^2) = \int \rho dV s^2$$

cylindrical:  $s, \phi, z$        $s^2 = x^2 + y^2$



$$dV = ds s d\phi dz$$

$$= s ds d\phi dz$$

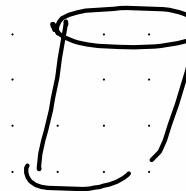


$$I_3 = \int \rho dV s^2$$

$$= \frac{M}{\pi R^2 h} \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \int_{-\frac{h}{2}}^{\frac{h}{2}} d\phi \int_{0}^{2\pi} ds s^3$$

$$= \frac{M}{\pi R^2 h} \cdot 2\pi K \frac{R^4}{4} h^2$$

$$= \boxed{\frac{1}{2} M R^2}$$



$$I_1 = I_2 \equiv I$$

$$I_1 = \int \rho dV (r^2 - x^2)$$

$$+ I_2 = \int \rho dV (r^2 - y^2)$$


---

$$2I = \int \rho dV (2r^2 - x^2 - y^2)$$

$$\begin{aligned} r^2 &= s^2 + z^2 \\ x^2 + y^2 &= s^2 \end{aligned}$$

$$2I = \int \rho dV (s^2 + 2z^2)$$

$$\boxed{I} = \frac{1}{2} \underbrace{\int \rho dV s^2}_{I_3} + \int \rho dV z^2$$
$$= \frac{1}{2} I_3 + \int \rho dV z^2$$

easier to  
evaluate

$$\int \rho dV z^2 = \frac{M}{\pi R^2 h} \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz \cdot z^2 \int_0^R s ds$$

$\curvearrowleft$        $\curvearrowleft$        $\curvearrowleft$

$$= \frac{z^3}{3} \Big|_{-h/2}^{h/2} = \frac{h^3}{12}$$

$$= \frac{\frac{2}{3} \left(\frac{h}{2}\right)^3}{3} = \frac{h^3}{12}$$

$$= \frac{M}{\cancel{\pi R^2 h}} \cdot \cancel{\frac{2}{3} \pi} \cdot \frac{h^3}{12} \cdot \cancel{\frac{R^2}{2}}$$

$$= \boxed{\frac{M h^2}{12}}$$

$$I = \frac{1}{2} \left( \frac{1}{2} MR^2 \right) + \frac{1}{12} mh^2$$

$$= \frac{1}{4} MR^2 + \frac{1}{12} mh^2$$

$$= \frac{1}{4} M \left( R^2 + \frac{1}{3} h^2 \right) = I_1, I_2$$

$$I_3 = \frac{1}{2} MR^2$$

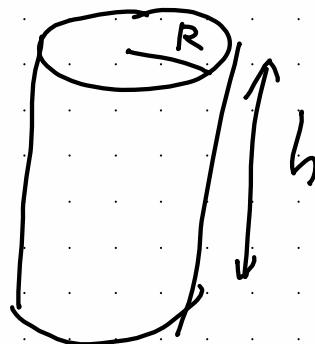


Limiting cases

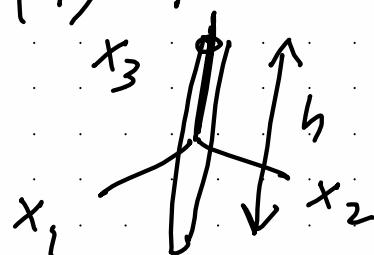
(i) Disc ( $h \rightarrow 0$ )

$$I_3 = \frac{1}{2} MR^2$$

$$I_1, I_2 = \frac{1}{4} MR^2$$



(i) thin rod ( $R \rightarrow 0$ )



$$I_3 = 0$$

$$I_1 = I_2 = \frac{1}{12} mh^2$$

$$L = T - U$$

$$= \frac{1}{2} M \dot{V}^2 + \frac{1}{2} \sum_{i,j} I_{ij} \dot{\theta}_j \dot{\theta}_i - U$$

$$\vec{V} = \dot{\vec{R}}$$

$$\vec{R} = \vec{\phi}$$

$$L(\vec{R}, \dot{\vec{\phi}}, \vec{R}, \ddot{\vec{\phi}})$$

$$\vec{g}$$

uniform  
Field

