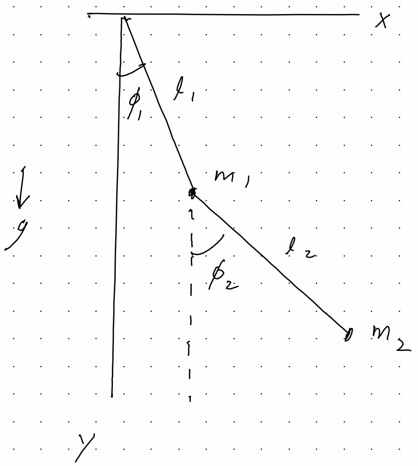
sec5, prob1



$$x_1 = l_1 \sin\phi_1$$
$$y_1 = l_1 \cos\phi_1$$
$$x_2 = x_1 + l_2 \sin\phi_2 = l_1 \sin\phi_1 + l_2 \sin\phi_2$$
$$y_2 = y_1 + l_2 \cos\phi_2 = l_1 \cos\phi_1 + l_2 \cos\phi_2$$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_1 g l_1 \cos\phi_1 - m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$
$$= -(m_1+m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\dot{x}_1 = l_1 \cos\phi_1 \dot{\phi}_1 \qquad \dot{y}_1 = -l_1 \sin\phi_1 \dot{\phi}_1$$
$$\dot{x}_1^2 = l_1^2 \cos^2\phi_1 \dot{\phi}_1^2 \qquad \dot{y}_1^2 = l_1^2 \sin^2\phi_1 \dot{\phi}_1^2$$

Thus, $\dot{x}_1^2 + \dot{y}_1^2 = l_1^2 (\sin^2\phi_1 + \cos^2\phi_1) \dot{\phi}_1^2$
$$= l_1^2 \dot{\phi}_1^2$$

$$\dot{x}_2 = l_1 \cos\phi_1 \dot{\phi}_1 + l_2 \cos\phi_2 \dot{\phi}_2$$
$$\rightarrow \dot{x}_2^2 = l_1^2 \cos^2\phi_1 \dot{\phi}_1^2 + l_2^2 \cos^2\phi_2 \dot{\phi}_2^2 + 2 l_1 l_2 \cos\phi_1 \cos\phi_2 \dot{\phi}_1 \dot{\phi}_2$$
$$\dot{y}_2 = -l_1 \sin\phi_1 \dot{\phi}_1 - l_2 \sin\phi_2 \dot{\phi}_2$$
$$\rightarrow \dot{y}_2^2 = l_1^2 \sin^2\phi_1 \dot{\phi}_1^2 + l_2^2 \sin^2\phi_2 \dot{\phi}_2^2 + 2 l_1 l_2 \sin\phi_1 \sin\phi_2 \dot{\phi}_1 \dot{\phi}_2$$

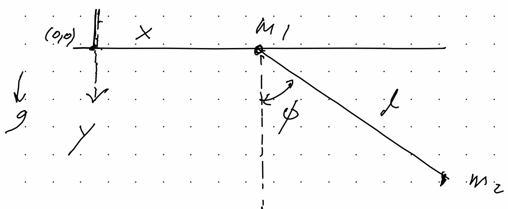
Thus,
$$\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2) \dot{\phi}_1 \dot{\phi}_2$$
$$= l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

so $T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$
$$= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2$$
$$+ m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$
$$= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

$$U = -(m_1+m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

$$L = T - U$$
$$= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$
$$+ (m_1+m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2$$

Sec 5 Prob. 2



Generalised coords: x, ϕ

$(x_1, y_1) = (x, 0)$

$(x_2, y_2) = (x + l\sin\phi,\ l\cos\phi)$

$U = -m_1 g y_1 - m_2 g y_2$

$\quad = -m_2 g l \cos\phi$

$T = \frac{1}{2} m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2(\dot{x}_2^2 + \dot{y}_2^2)$

Now: $\dot{x}_1^2 + \dot{y}_1^2 = \dot{x}^2$

$\dot{x}_2^2 + \dot{y}_2^2 = (\dot{x} + l\cos\phi\dot{\phi})^2 + (-l\sin\phi\dot{\phi})^2$

$= \dot{x}^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi} + l^2\sin^2\phi\,\dot{\phi}^2$

$= \dot{x}^2 + l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi}$

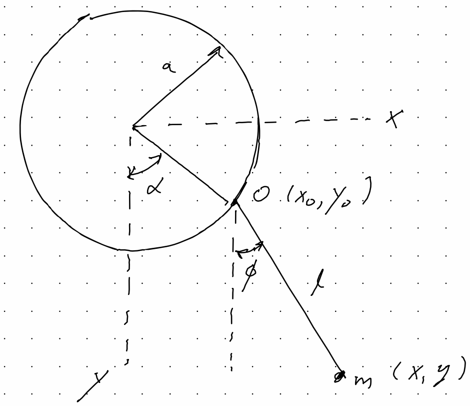
$\rightarrow T = \frac{1}{2} m_1\dot{x}^2 + \frac{1}{2} m_2(\dot{x}^2 + l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi})$

$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2\dot{\phi}^2 + m_2 l\cos\phi\,\dot{x}\dot{\phi}$

$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2\dot{\phi}^2 + m_2 l\cos\phi\,\dot{x}\dot{\phi} + m_2 g l\cos\phi$

Sec 5, Prob 3

(a)



point of support O moves along circle:

$x_0 = a\sin\alpha$, $y_0 = a\cos\alpha$

where $\alpha = \gamma t$, $\gamma = \text{const}$

pendulum bob:

(x, y): $x = x_0 + l\sin\phi$

$y = y_0 + l\cos\phi$

$U = -mgy = \underbrace{-mgy_0}_{\text{specified function of time}} - mgl\cos\phi$

[can ignore in L]

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

$\dot{x} = \dot{x}_0 + l\cos\phi\,\dot{\phi}$

$\dot{x}^2 = \dot{x}_0^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}_0\dot{\phi}$

$\dot{y} = \dot{y}_0 - l\sin\phi\,\dot{\phi}$

$\dot{y}^2 = \dot{y}_0^{\,2} + l^2\sin^2\phi\,\dot{\phi}^2 - 2l\sin\phi\,\dot{y}_0\dot{\phi}$

Thus,

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$
$$= \frac{1}{2}m\left(\dot{x}_0^{\,2} + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}_0\dot{\phi} + \dot{y}_0^{\,2} + l^2\sin^2\phi\,\dot{\phi}^2 - 2l\sin\phi\,\dot{y}_0\dot{\phi}\right)$$
$$= \frac{1}{2}m(\dot{x}_0^{\,2}+\dot{y}_0^{\,2}) + \frac{1}{2}ml^2\dot{\phi}^2 + ml\dot{\phi}(\dot{x}_0\cos\phi - \dot{y}_0\sin\phi)$$

NOTE: $\dot{x}_0^{\,2}+\dot{y}_0^{\,2} = a^2\dot{\alpha}^2 = a^2\gamma^2$

since this is a specified function of time, we can ignore it in the Lagrangian.

Thus, $$L = \frac{1}{2}ml^2\dot{\phi}^2 + ml\dot{\phi}(\dot{x}_0\cos\phi - \dot{y}_0\sin\phi) + mgl\cos\phi$$

We can rewrite the second term:

$x_0 = a\sin\alpha \rightarrow \dot{x}_0 = a\cos\alpha\,\dot{\alpha}$ $\quad(\dot{\alpha}\equiv\gamma)$

$y_0 = a\cos\alpha \rightarrow \dot{y}_0 = -a\sin\alpha\,\dot{\alpha}$

Thus,

$$ml\dot{\phi}(\dot{x}_0\cos\phi - \dot{y}_0\sin\phi) = ml\dot{\phi}a\gamma(\cos\alpha\cos\phi + \sin\alpha\sin\phi)$$
$$= ml\dot{\phi}a\gamma\cos(\phi-\alpha)$$
$$= ml\dot{\phi}a\gamma\cos(\phi-\gamma t)$$

Now: $\frac{d}{dt}\left[mla\gamma\sin(\phi-\gamma t)\right]$

$$= mla\gamma\cos(\phi-\gamma t)(\dot{\phi}-\gamma)$$
$$= mla\dot{\phi}\gamma\cos(\phi-\gamma t) - mla\gamma^2\cos(\phi-\gamma t)$$

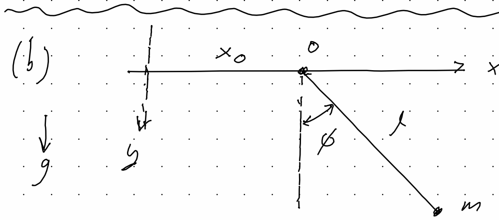
Thus,

$$mla\dot{\phi}\gamma\cos(\phi-\gamma t) = \frac{d}{dt}\left[mla\gamma\sin(\phi-\gamma t)\right] + mla\gamma^2\cos(\phi-\gamma t)$$

(and we can ignore the total time derivative in the Lagrangian)

$$\rightarrow L = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi + mla\gamma^2\cos(\phi-\gamma t)$$

(b)



point O moving according to $x_0 = a\cos\gamma t$

$x = x_0 + l\sin\phi$

$y = l\cos\phi$

$U = -mgy = -mgl\cos\phi$

$$T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{x}_0 + l\cos\phi\,\dot{\phi}\,, \qquad x_0 = a\cos\gamma t$$

$$= -a\sin(\gamma t)\,\gamma + l\cos\phi\,\dot{\phi}$$

$$\rightarrow \dot{x}^2 = a^2\gamma^2\sin^2\gamma t + l^2\cos^2\phi\,\dot{\phi}^2 - 2al\gamma\dot{\phi}\sin(\gamma t)\cos\phi$$

$$\dot{y} = -l\sin\phi\,\dot{\phi}$$

$$\rightarrow \dot{y}^2 = l^2\sin^2\phi\,\dot{\phi}^2$$

Thus, $$T = \tfrac{1}{2} m\big(a^2\gamma^2\sin^2\gamma t + l^2\cos^2\phi\,\dot{\phi}^2 - 2al\gamma\dot{\phi}\sin(\gamma t)\cos\phi + l^2\sin^2\phi\,\dot{\phi}^2\big)$$

$$= \tfrac{1}{2} m l^2\dot{\phi}^2 + \tfrac{1}{2} m a^2\gamma^2\sin^2\gamma t - m a l\gamma\dot{\phi}\sin(\gamma t)\cos\phi$$

specified function of time (ignore)

$$L = \tfrac{1}{2} m l^2\dot{\phi}^2 - m a l\gamma\dot{\phi}\sin(\gamma t)\cos\phi + mgl\cos\phi$$

2nd term:

$$-\frac{d}{dt}\big[m a l\gamma\sin(\gamma t)\sin\phi\big] = -m a l\gamma^2\cos(\gamma t)\sin\phi - m a l\gamma\dot{\phi}\sin(\gamma t)\cos\phi$$

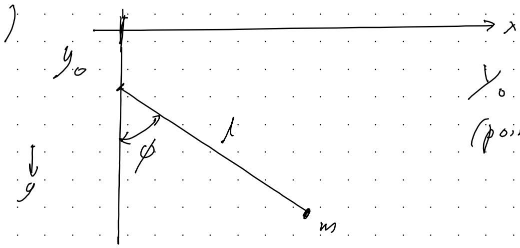
so

$$-m a l\gamma\dot{\phi}\sin(\gamma t)\cos\phi = \underbrace{-\frac{d}{dt}[\;\;]}_{\text{ignore}} + m a l\gamma^2\cos(\gamma t)\sin\phi$$

Thus, ignoring total time derivatives,

$$L = \tfrac{1}{2} m l^2\dot{\phi}^2 + mgl\cos\phi + m a l\gamma^2\cos(\gamma t)\sin\phi$$

(c)



$y_0 = a\cos\gamma t$ (point of support)

$$x = l\sin\phi$$

$$y = y_0 + l\cos\phi$$

$$= a\cos\gamma t + l\cos\phi$$

$$U = -mgy$$

$$= \underbrace{-mga\cos\gamma t}_{} - mg\,l\cos\phi$$

specified function of time [can ignore]

$$T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = l\cos\phi\,\dot{\phi}$$

$$\dot{x}^2 = l^2\cos^2\phi\,\dot{\phi}^2$$

$$\dot{y} = -a\gamma\sin(\gamma t) - l\sin\phi\,\dot{\phi}$$

$$\dot{y}^2 = \underbrace{a^2\gamma^2\sin^2(\gamma t)}_{} + l^2\sin^2\phi\,\dot{\phi}^2 + 2al\gamma\sin(\gamma t)\sin\phi\,\dot{\phi}$$

specified function of time [can ignore]

Thus, ignoring this function of time

$$T = \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \dot{\phi} \sin(\gamma t) \sin\phi$$

$$\rightarrow L = T - U$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \gamma \dot{\phi} \sin(\gamma t) \sin\phi + m g l \cos\phi$$

Rewrite 2nd term:

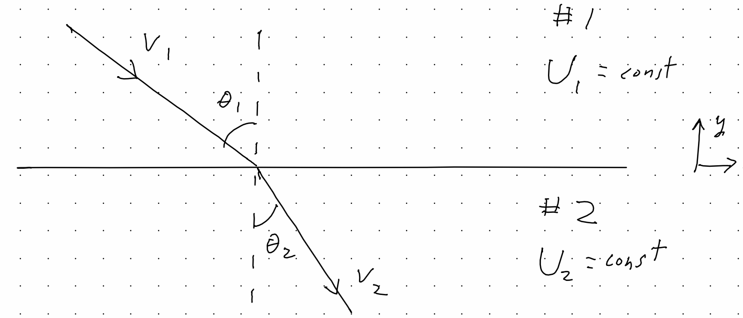
$$-\frac{d}{dt}\left[m a l \gamma \sin(\gamma t) \cos\phi \right] = -m a l \gamma^2 \cos(\gamma t) \cos\phi + m a l \gamma \sin(\gamma t) \sin\phi \, \dot{\phi}$$

so $m a l \gamma \dot{\phi} \sin(\gamma t) \sin\phi = -\frac{d}{dt}[\;\;] + m a l \gamma^2 \cos(\gamma t) \cos\phi$

Thus, ignoring total time derivative

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \cos\phi + m a l \gamma^2 \cos(\gamma t) \cos\phi$$

Sec 7, Prob 1



- Energy conserved, since no time dependence.
- Also momentum in x-direction (// to interface) is conserved, since no x-dependence of the potential

$$U(x,y) = \begin{cases} U_1 & y \geq 0 \\ U_2 & y < 0 \end{cases}$$

v_1 : given

$$E = \frac{1}{2} m v_1^2 + U_1 = \frac{1}{2} m v_2^2 + U_2$$

$$\rightarrow \frac{1}{2} m v_2^2 = \frac{1}{2} m v_1^2 + (U_1 - U_2)$$

$$v_2^2 = v_1^2 + \frac{2(U_1 - U_2)}{m}$$

so, $$v_2 = v_1 \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

The angles θ_1, θ_2 are related by

$$p_{1x} = p_{2x}$$

$$\cancel{m} V_1 \sin\theta_1 = \cancel{m} V_2 \sin\theta_2$$

thus, $$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

<u>Sec 8, Prob 1</u>

Transformation of action $S = \int L\,dt$

K, K': two inertial frames

K' moves with velocity $\vec{V}$ wrt K

Assume that K, K' coincide at $t=0$ so

$\vec{r}_a = \vec{r}_a{}'$ wrt these two frames

<u>Now:</u> $$\vec{v}_a = \vec{V} + \vec{v}_a{}'$$

$$L = T - U$$

$$= \sum_a \frac{1}{2} m_a |\vec{v}_a|^2 - U(\vec{r}_1, \vec{r}_2, \dots, t)$$

$$|\vec{v}_a|^2 = |\vec{V} + \vec{v}_a{}'|^2$$

$$= |\vec{V}|^2 + |\vec{v}_a{}'|^2 + 2\vec{V}\cdot\vec{v}_a{}'$$

so

$$L = \sum_a \frac{1}{2} m_a \left(|\vec{V}|^2 + |\vec{v}_a{}'|^2 + 2\vec{V}\cdot\vec{v}_a{}'\right) - U$$

$$= \frac{1}{2}\mu V^2 + T' + \vec{V}\cdot\sum_a m_a \vec{v}_a{}' - U$$

$$= T' - U + \frac{1}{2}\mu V^2 + \vec{P}'\cdot\vec{V}$$

$$= L' + \frac{1}{2}\mu V^2 + \vec{P}'\cdot\vec{V}$$

where $\vec{P}'$ = total momentum wrt K'

$\mu = \sum_a m_a$ = total mass

$$S = \int_{t_1}^{t_2} L\,dt$$

$$= \int_{t_1}^{t_2} \left(L' + \tfrac{1}{2}\mu V^2 + \vec{P}'\cdot\vec{V}\right) dt$$

$$= S' + \underbrace{\tfrac{1}{2}\mu V^2 (t_2 - t_1)}_{\text{doesn't change EOMs}} + \vec{V}\cdot\int_{t_1}^{t_2} \vec{P}'\,dt$$

Now:

$$\vec{V}\cdot\int_{t_1}^{t_2} \vec{P}'\,dt = \vec{V}\cdot\int_{t_1}^{t_2} \sum_a m_a \vec{v}_a{}'\,dt$$

$$= \vec{V}\cdot\sum_a m_a \int_{t_1}^{t_2} \left(\frac{d\vec{r}_a}{dt}\right) dt$$

$$= \vec{V}\cdot\sum_a m_a \vec{r}_a \Big|_{t_1}^{t_2}$$

$$= \vec{V}\cdot\left(\mu\vec{R}(t_2) - \mu\vec{R}(t_1)\right)$$

$$= \mu\vec{V}\cdot\Big(\underbrace{\vec{R}(t_2) - \vec{R}(t_1)}_{\text{difference in COM position}}\Big)$$

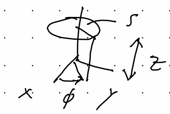
So:

$$S = S' + \tfrac{1}{2}\mu V^2 (t_2 - t_1) + \mu\vec{V}\cdot\left(\vec{R}(t_2) - \vec{R}(t_1)\right)$$

Sec 9, Prob. 1

cylindrical coordinates (s, ϕ, z)

$s^2 = x^2 + y^2$



$x = s\cos\phi$

$y = s\sin\phi$

$z = z$

$$\vec{M} = \vec{r}\times\vec{p} = m\vec{r}\times\dot{\vec{r}}$$

Now: $M_x = m(y\dot{z} - z\dot{y})$

$M_y = m(z\dot{x} - x\dot{z})$

$M_z = m(x\dot{y} - y\dot{x})$

$M = \sqrt{M_x^2 + M_y^2 + M_z^2}$

$\dot{z} = \dot{z}$

$\dot{y} = \dot{s}\sin\phi + s\cos\phi\,\dot{\phi}$

$\dot{x} = \dot{s}\cos\phi - s\sin\phi\,\dot{\phi}$

Thus, $M_x = m\left(s\sin\phi\,\dot{z} - z(\dot{s}\sin\phi + s\cos\phi\,\dot{\phi})\right)$

$= m(s\sin\phi\,\dot{z} - z\sin\phi\,\dot{s} - zs\cos\phi\,\dot{\phi})$

$M_y = m\left(z(\dot{s}\cos\phi - s\sin\phi\,\dot{\phi}) - s\cos\phi\,\dot{z}\right)$

$= m(z\cos\phi\,\dot{s} - zs\sin\phi\,\dot{\phi} - s\cos\phi\,\dot{z})$

$M_z = m\big[s\cos\phi(\cancel{\dot{s}\sin\phi} + s\cos\phi\,\dot{\phi})$

$- s\sin\phi(\cancel{\dot{s}\cos\phi} - s\sin\phi\,\dot{\phi})\big]$

$= ms^2\dot{\phi}$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$= m^2 \Big\{ \left(\sin\phi\,(s\dot z - z\dot s) - zs\cos\phi\,\dot\phi\right)^2$$

$$+ \left(-\cos\phi\,(s\dot z - z\dot s) - zs\sin\phi\,\dot\phi\right)^2$$

$$+ \left(s^2\dot\phi\right)^2 \Big\}$$

$$= m^2 \Big\{ \sin^2\phi\,(s\dot z - z\dot s)^2 + z^2 s^2 \cos^2\phi\,\dot\phi^2$$

$$\cancel{- 2zs\sin\phi\cos\phi\,\dot\phi\,(s\dot z - z\dot s)}$$

$$+ \cos^2\phi\,(s\dot z - z\dot s)^2 + z^2 s^2 \sin^2\phi\,\dot\phi^2$$

$$\cancel{+ 2zs\sin\phi\cos\phi\,\dot\phi\,(s\dot z - z\dot s)}$$

$$+ s^4\dot\phi^2 \Big\}$$

$$= m^2 \left\{ (s\dot z - z\dot s)^2 + z^2 s^2 \dot\phi^2 + s^4\dot\phi^2 \right\}$$

$$= m^2 \left[(s\dot z - z\dot s)^2 + s^2\dot\phi^2 (z^2 + s^2) \right]$$

<u>Sec 9, Prob 2</u>

repeat for spherical polar coords.

$$M_x = m(y\dot z - z\dot y) \quad , \quad \text{cyclic}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

<u>Now:</u>

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$\rightarrow \quad \dot x = \dot r\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot\theta - r\sin\theta\sin\phi\,\dot\phi$$

$$\dot y = \dot r\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot\theta + r\sin\theta\cos\phi\,\dot\phi$$

$$\dot z = \dot r\cos\theta - r\sin\theta\,\dot\theta$$

Thus,

$$M_x = m(y\dot z - z\dot y)$$

$$= m\Big\{ r\sin\theta\sin\phi\,(\cancel{\dot r\cos\theta} - r\sin\theta\,\dot\theta)$$

$$- r\cos\theta\,(\cancel{\dot r\sin\theta\sin\phi} + r\cos\theta\sin\phi\,\dot\theta + r\sin\theta\cos\phi\,\dot\phi) \Big\}$$

$$= m\left\{ -r^2\sin^2\theta\sin\phi\,\dot\theta - r^2\cos^2\theta\sin\phi\,\dot\theta - r^2\sin\theta\cos\theta\cos\phi\,\dot\phi \right\}$$

$$= m\left\{ -r^2\sin\phi\,\dot\theta - r^2\sin\theta\cos\theta\cos\phi\,\dot\phi \right\}$$

$$= -mr^2\left[\sin\phi\,\dot\theta + \sin\theta\cos\theta\cos\phi\,\dot\phi \right]$$

$$M_y = m(z\dot{x} - x\dot{z})$$

$$= m\{ r\cos\theta\,(\cancel{\dot{r}\sin\theta\cos\phi} + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi})$$

$$- r\sin\theta\cos\phi\,(\cancel{\dot{r}\cos\theta} - r\sin\theta\,\dot{\theta})$$

$$= m\{ r^2\cos^2\theta\cos\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\sin\phi\,\dot{\phi}$$

$$+ r^2\sin^2\theta\cos\phi\,\dot{\theta} \}$$

$$= m[r^2\cos\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\sin\phi\,\dot{\phi}]$$

$$= mr^2[\cos\phi\,\dot{\theta} - \sin\theta\cos\theta\sin\phi\,\dot{\phi}]$$

$$M_z = m(x\dot{y} - y\dot{x})$$

$$= m\{ r\sin\theta\cos\phi\,(\cancel{\dot{r}\sin\theta\sin\phi} + \cancel{r\cos\theta\sin\phi\,\dot{\theta}}$$

$$+ r\sin\theta\cos\phi\,\dot{\phi})$$

$$- r\sin\theta\sin\phi\,(\cancel{\dot{r}\sin\theta\cos\phi} + \cancel{r\cos\theta\cos\phi\,\dot{\theta}}$$

$$- r\sin\theta\sin\phi\,\dot{\phi}) \}$$

$$= m[r^2\sin^2\theta\cos^2\phi\,\dot{\phi} + r^2\sin^2\theta\sin^2\phi\,\dot{\phi}]$$

$$= mr^2\sin^2\theta\,\dot{\phi}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$= m^2r^4[\sin\phi\,\dot{\theta} + \sin\theta\cos\theta\cos\phi\,\dot{\phi}]^2$$

$$+ m^2r^4[\cos\phi\,\dot{\theta} - \sin\theta\cos\theta\sin\phi\,\dot{\phi}]^2$$

$$+ m^2r^4\sin^4\theta\,\dot{\phi}^2$$

cross terms will cancel

$$M^2 = m^2r^4\{ \sin^2\phi\,\dot{\theta}^2 + \sin^2\theta\cos^2\theta\cos^2\phi\,\dot{\phi}^2$$

$$+ \cos^2\phi\,\dot{\theta}^2 + \sin^2\theta\cos^2\theta\sin^2\phi\,\dot{\phi}^2$$

$$+ \sin^4\theta\,\dot{\phi}^2 \}$$

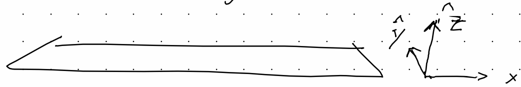
$$= m^2r^4[\dot{\theta}^2 + \sin^2\theta\cos^2\theta\,\dot{\phi}^2 + \sin^4\theta\,\dot{\phi}^2]$$

$$= m^2r^4[\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2(\cos^2\theta + \sin^2\theta)]$$

$$= m^2r^4[\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2]$$

Sec 9, Prob 3

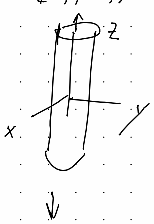
a) Infinite homogeneous plane



P_x, P_y conserved

M_z conserved, where origin is anywhere in (x,y) plane

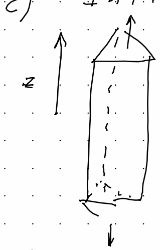
b) Infinite homogeneous cylinder



P_z conserved

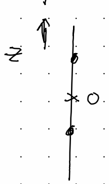
M_z conserved, origin anywhere on z-axis

c) Infinite homog prism

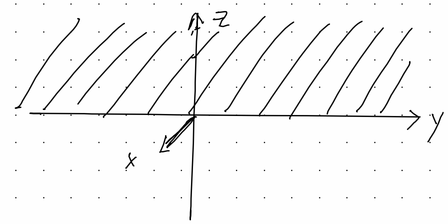


P_z conserved

d) two points: M_z conserved, origin at midpoint of line connecting the two points



e)



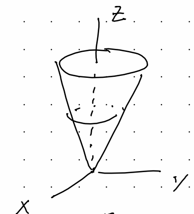
infinite, homog. half plane

$x=0,\ z>0$

$-\infty < y < \infty$

P_y conserved

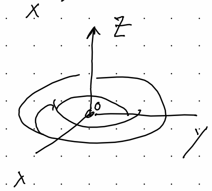
f)



homogeneous cone

M_z conserved, origin anywhere on z-axis.

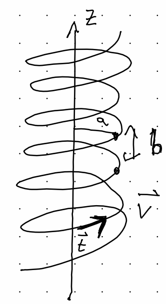
g)



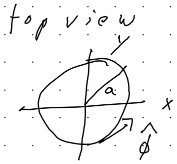
homogeneous circular torus

M_z conserved with $\hat z$ directed as shown and origin at O.

h)



top view



Field constant along helix:

$\Delta z = b$ when $\Delta\phi = 2\pi$ at $s^2 = x^2 + y^2 = a^2$

$$\frac{\Delta\phi}{2\pi} = \frac{\Delta z}{b}$$

$$\rightarrow \quad \Delta z = \left(\frac{b}{2\pi}\right)\Delta\phi$$

$$\vec t = a\Delta\phi\,\hat\phi + \Delta z\,\hat z$$

$$= \phi\Delta\phi\,\underbrace{(x\hat y - y\hat x)}_{s} + \Delta z\,\hat z$$

$$\vec{t} = \Delta\phi\,(x\hat{y} - y\hat{x}) + \left(\frac{b}{2\pi}\right)\Delta\phi\,\hat{z}$$

$$= \Delta\phi\left[x\hat{y} - y\hat{x} + \left(\frac{b}{2\pi}\right)\hat{z}\right]$$

Field unchanged if you move along $\vec{t}$.

Thus, $\vec{P}\cdot\vec{t} = \text{const}$

$$\vec{P}\cdot\vec{t} = \Delta\phi\left[xP_y - yP_x + \left(\frac{b}{2\pi}\right)P_z\right]$$

$$= \Delta\phi\left[M_z + \frac{b}{2\pi}P_z\right]$$

so $M_z + \frac{b}{2\pi}P_z = \text{const}$

where z = axis of helix

$b = \Delta z$ for $\Delta\phi = 2\pi$ at $s = a$

Sec 10, prob 1

Different masses, same path, same potential energy

$$L_1 = \frac{1}{2}m_1 v_1^2 - U$$

$$L_2 = \frac{1}{2}m_2 v_2^2 - U$$

Thus, $m_1 v_1^2 = m_2 v_2^2$

$$\frac{m_1}{t_1^2} = \frac{m_2}{t_2^2}$$

$$\rightarrow \left(\frac{t_2}{t_1}\right)^2 = \frac{m_2}{m_1}$$

or $\boxed{\frac{t_2}{t_1} = \sqrt{\frac{m_2}{m_1}}}$

Sec 10, Prob 2:

Same path, mass but potential energies differing by a constant

$$L_1 = \frac{1}{2} m v_1^2 - U_1$$
$$L_2 = \frac{1}{2} m v_2^2 - U_2$$

Thus, $\dfrac{v_1^2}{v_2^2} = \dfrac{U_1}{U_2}$

$$\rightarrow \frac{(l/t_1)^2}{(l/t_2)^2} = \frac{U_1}{U_2}$$

so $\dfrac{t_2}{t_1} = \sqrt{\dfrac{U_1}{U_2}}$

Sec 40 - Prob 1

single particle in a constant external field

$$L = \frac{1}{2} m v^2 - U(\vec{r})$$

a) Cartesian coords (x, y, z)

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x,y,z)$$

$$\rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = p_x/m$$
$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \rightarrow \dot{y} = p_y/m$$
$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$H = \left(\sum_i p_i \dot{q}_i - L\right)\Big|_{\dot{q} = \dot{q}(q,p)}$$
$$= \left(p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(x,y,z)\right)\Big|_{\dot{x} = p_x/m \text{ etc}}$$
$$= p_x\left(\frac{p_x}{m}\right) + p_y\left(\frac{p_y}{m}\right) + p_z\left(\frac{p_z}{m}\right) - \frac{1}{2}m\left(\left(\frac{p_x}{m}\right)^2 + \left(\frac{p_y}{m}\right)^2 + \left(\frac{p_z}{m}\right)^2\right) + U(x,y,z)$$
$$= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + U(x,y,z)$$

b) cylindrical coords (s, ϕ, z), $s^2 = x^2 + y^2$

$$L = \frac{1}{2} m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2) - U(s, \phi, z)$$

$$\rightarrow p_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s} \rightarrow \dot{s} = p_s/m$$
$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ms^2\dot{\phi} \rightarrow \dot{\phi} = p_\phi/ms^2$$
$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$H = \left(p_s \dot{s} + p_\phi \dot{\phi} + p_z \dot{z} - \frac{1}{2} m (\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2) + U(s,\phi,z) \right)\Big|$$

$\dot{s} = p_s/m$ etc

$$= p_s \left(\frac{p_s}{m}\right) + p_\phi \left(\frac{p_\phi}{m s^2}\right) + p_z \left(\frac{p_z}{m}\right)$$

$$- \frac{1}{2} m \left(\left(\frac{p_s}{m}\right)^2 + s^2 \left(\frac{p_\phi}{m s^2}\right)^2 + \left(\frac{p_z}{m}\right)^2 \right) + U(s,\phi,z)$$

$$= \frac{1}{2m} \left(p_s^2 + \frac{p_\phi^2}{s^2} + p_z^2 \right) + U(s,\phi,z)$$

---

c) spherical polar coords (r, θ, ϕ)

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2) - U(r,\theta,\phi)$$

$$\rightarrow p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \rightarrow \dot{r} = p_r/m$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \rightarrow \dot{\theta} = p_\theta / m r^2$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\theta \, \dot{\phi} \rightarrow \dot{\phi} = p_\phi / m r^2 \sin^2\theta$$

$$H = \left(p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2) + U(r,\theta,\phi) \right)\Big|$$

$\dot{r} = p_r/m$ etc.

$$= p_r \left(\frac{p_r}{m}\right) + p_\theta \left(\frac{p_\theta}{m r^2}\right) + p_\phi \left(\frac{p_\phi}{m r^2 \sin^2\theta}\right)$$

$$- \frac{1}{2} m \left(\left(\frac{p_r}{m}\right)^2 + r^2 \left(\frac{p_\theta}{m r^2}\right)^2 + r^2 \sin^2\theta \left(\frac{p_\phi}{m r^2 \sin^2\theta}\right)^2 \right) + U(r,\theta,\phi)$$

$$= \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta} \right) + U(r,\theta,\phi)$$

sec 40 - prob 2

$$L = \frac{1}{2} m v^2 + m \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + \frac{1}{2} m |\vec{\Omega} \times \vec{r}|^2 - m \vec{W} \cdot \vec{r} - U$$

restrict to uniformly rotating frame of reference $\vec{W} = 0$, $\dot{\vec{\Omega}} = 0$

$$\rightarrow L = \frac{1}{2} m v^2 + m \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + \frac{1}{2} m |\vec{\Omega} \times \vec{r}|^2 - U(\vec{r})$$

Now: $H = \vec{p} \cdot \vec{v} - L$

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + m \vec{\Omega} \times \vec{r} = m(\vec{v} + \vec{\Omega} \times \vec{r})$$

$$\longrightarrow \vec{v} = \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r}$$

Thus,

$$H = \vec{p} \cdot \left(\frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r} \right) - \frac{1}{2} m \left| \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r} \right|^2$$

$$- m \left(\frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r} \right) \cdot (\vec{\Omega} \times \vec{r})$$

$$- \frac{1}{2} m |\vec{\Omega} \times \vec{r}|^2 + U(\vec{r})$$

$$= \frac{|\vec{p}|^2}{m} - \cancel{\vec{p} \cdot (\vec{\Omega} \times \vec{r})} - \frac{1}{2} m \left(\frac{|\vec{p}|^2}{m^2} + \cancel{|\vec{\Omega} \times \vec{r}|^2} - \cancel{2 \frac{\vec{p} \cdot (\vec{\Omega} \times \vec{r})}{m}} \right)$$

$$- \vec{p} \cdot (\vec{\Omega} \times \vec{r}) + m \cancel{|\vec{\Omega} \times \vec{r}|^2} - \frac{1}{2} m \cancel{|\vec{\Omega} \times \vec{r}|^2} + U(\vec{r})$$

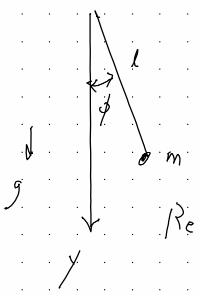
$$= \frac{|\vec{p}|^2}{2m} - \vec{p} \cdot (\vec{\Omega} \times \vec{r}) + U(\vec{r})$$

$$= \frac{|\vec{p}|^2}{2m} - \vec{\Omega} \cdot \underbrace{(\vec{r} \times \vec{p})}_{\vec{M}\text{: angular momentum}} + U(\vec{r})$$

Sec. 11, Prob 1

Simple pendulum: $U = -mgy$

$= -mgl\cos\phi$



Also, $T = \frac{1}{2} m l^2 \dot{\phi}^2$

Thus, $E = T + U$

$= \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$

Release from rest at $\phi = \phi_0$

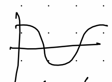
$E = -mgl\cos\phi_0$

$\rightarrow \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi = -mgl\cos\phi_0$

$$T(E) = 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{E - U(\phi)}}$$

$$= 4\sqrt{\frac{\cancel{m} l^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{-mgl\cos\phi_0 + mgl\cos\phi}}$$

$$= 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$



$\phi < \phi_0$

$\cos\phi > \cos\phi_0$

Now: $\cos\phi = \cos(2\frac{\phi}{2})$

$= \cos^2(\frac{\phi}{2}) - \sin^2(\frac{\phi}{2})$

$= 1 - 2\sin^2(\frac{\phi}{2})$

Also, $\cos\phi_0 = 1 - 2\sin^2(\frac{\phi_0}{2})$

$$T(E) = 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{2\left(\sin^2(\frac{\phi_0}{2}) - \sin^2(\frac{\phi}{2})\right)}}$$

$$= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2(\frac{\phi_0}{2})\left(1 - \frac{\sin^2(\frac{\phi}{2})}{\sin^2(\frac{\phi_0}{2})}\right)}}$$

Let: $x = \frac{\sin(\frac{\phi}{2})}{\sin(\frac{\phi_0}{2})} \rightarrow dx = \frac{d\phi}{2} \frac{\cos(\frac{\phi}{2})}{\sin(\frac{\phi_0}{2})}$

$= \frac{d\phi}{2} \frac{\sqrt{1 - \sin^2(\frac{\phi}{2})}}{\sin(\frac{\phi_0}{2})}$

$= \frac{d\phi}{2} \frac{\sqrt{1 - \sin^2(\frac{\phi_0}{2})x^2}}{\sin(\frac{\phi_0}{2})}$

Thus,

$$T(E) = 2\sqrt{\frac{l}{g}} \int_0^1 \frac{2\,dx\,\cancel{\sin(\frac{\phi_0}{2})}}{\sqrt{1 - \sin^2(\frac{\phi_0}{2})x^2}\;\cancel{\sin(\frac{\phi_0}{2})}\sqrt{1 - x^2}}$$

$$= 4\sqrt{\frac{l}{g}} \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}, \quad k \equiv \sin(\frac{\phi_0}{2})$$

$$= 4\sqrt{\frac{l}{g}}\, K(k)$$

where $K(k)$ = complete elliptic integral of the 1st kind.

Approximation:

$$T(E) = 4\sqrt{\frac{l}{g}} \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

For $k \ll 1$: $\quad \frac{1}{\sqrt{1-k^2x^2}} \approx 1 + \frac{1}{2}k^2x^2$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \approx \int_0^1 \frac{dx}{\sqrt{1-x^2}}\left(1 + \frac{1}{2}k^2x^2\right)$$

Now: $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin(1) = \boxed{\frac{\pi}{2}}$

$$\frac{1}{2}k^2 \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^2}} = \frac{1}{2}k^2 \int_0^{\pi/2} \frac{\sin^2\theta\,\cancel{\cos\theta}\,d\theta}{\cancel{\sqrt{1-\sin^2\theta}}}$$

Let: $x = \sin\theta$ $\qquad \cos 2\theta = 1 - 2\sin^2\theta$

$dx = \cos\theta\, d\theta$ $\qquad \rightarrow \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$x^2 = \sin^2\theta$

$$\rightarrow \frac{1}{2}k^2\int_0^1 \frac{x^2\,dx}{\sqrt{1-y^2}} = \frac{1}{2}k^2\,\frac{1}{2}\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{4}k^2\left[\frac{\pi}{2} - \frac{\sin 2\theta}{2}\Big|_0^{\pi/2}\right]$$

$$= \boxed{\frac{1}{8}k^2\pi}$$

$k = \sin\left(\frac{\phi_0}{2}\right) \approx \frac{\phi_0}{2}$

Thus, $T(E) = 4\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \frac{1}{8}k^2\pi + \cdots\right) = \boxed{2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\phi_0^2 + \cdots\right)}$

Sec 13, Prob 1

$\vec{X}$: position vector of M

$\vec{x}_a$: $a = 1, 2, \cdots, n$ position vector of n masses all with mass m.

Closed system $\rightarrow$ linear momentum conserved

$\rightarrow$ CoM Frame

$$M\vec{X} + m(\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_n) = 0$$

$$\text{or} \quad M\vec{X} + m\sum_{a=1}^{n}\vec{x}_a = 0 \qquad (1)$$

Taking time derivative $\rightarrow$ $M\dot{\vec{X}} + \sum_a \dot{\vec{x}}_a = 0. \quad (2)$

Define relative position vectors:

$\vec{r}_1 \equiv \vec{x}_1 - \vec{X}$

$\vec{r}_2 = \vec{x}_2 - \vec{X}$

etc.

or $\vec{r}_a \equiv \vec{x}_a - \vec{X}, \quad a = 1, 2, \cdots, n. \qquad (3)$

Summing up (3):

$$\sum \vec{r}_a = \sum(\vec{x}_a - \vec{X})$$

$$= \sum \vec{x}_a - n\vec{X}$$

$$= -\frac{M}{m}\vec{X} - n\vec{X}$$

$$= -\frac{(M + nm)}{m}\vec{X} = -\frac{\mu}{m}\vec{X}$$

total mass

Thus, $\vec{X} = -\frac{m}{\mu}\sum_a \vec{r}_a$

and $\vec{x}_a = \vec{r}_a + \vec{X}$

give you $\vec{x}_a, \vec{X}$ in terms of $\vec{r}_a$.

<u>KE:</u> $$T = \frac{1}{2}\sum_a m|\dot{\vec{x}}_a|^2 + \frac{1}{2}M|\dot{\vec{X}}|^2$$

$$= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a + \dot{\vec{X}}|^2 + \frac{1}{2}M|\dot{\vec{X}}|^2$$

$$= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a|^2 + \frac{1}{2}\sum_a m\, 2\dot{\vec{r}}_a\cdot\dot{\vec{X}} + \frac{1}{2}\sum_a m|\dot{\vec{X}}|^2 + \frac{1}{2}M|\dot{\vec{X}}|^2$$

$$= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a|^2 + m\Big(\sum_a \dot{\vec{r}}_a\Big)\cdot\dot{\vec{X}} + \frac{1}{2}nm|\dot{\vec{X}}|^2 + \frac{1}{2}M|\dot{\vec{X}}|^2$$

$$= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a|^2 - \mu|\dot{\vec{X}}|^2 + \frac{1}{2}\underbrace{(nm+M)}_{\mu}|\dot{\vec{X}}|^2$$

$$= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a|^2 - \frac{1}{2}\mu|\dot{\vec{X}}|^2$$

Now rewrite last term:

$$-\frac{1}{2}\mu|\dot{\vec{X}}|^2 = -\frac{1}{2}\mu\frac{m^2}{\mu^2}\Big|\sum_a \dot{\vec{r}}_a\Big|^2 = -\frac{1}{2}\frac{m^2}{\mu}\Big|\sum_a \dot{\vec{r}}_a\Big|^2$$

Thus, $$T = \frac{1}{2}m\sum_a|\dot{\vec{r}}_a|^2 - \frac{1}{2}\frac{m^2}{\mu}\Big|\sum_a \dot{\vec{r}}_a\Big|^2$$

Potential energy

$$U = U(|\vec{x}_1-\vec{x}_2|, |\vec{x}_1-\vec{x}_3|, \dots, |\vec{x}_1-\vec{x}_n|, \dots, |\vec{x}_1-\vec{X}|, |\vec{x}_2-\vec{X}|, \dots, |\vec{x}_n-\vec{X}|)$$

$$= U(|\vec{r}_1-\vec{r}_2|, |\vec{r}_1-\vec{r}_3|, \dots, |\vec{r}_1-\vec{r}_n|, \dots, |\vec{r}_1|, |\vec{r}_2|, \dots, |\vec{r}_n|)$$

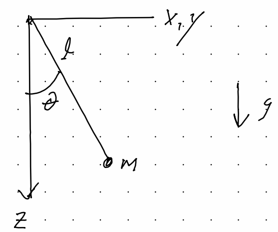
which depends only on the relative position vectors $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$

Thus,

$$L = \frac{1}{2}m\sum_a|\dot{\vec{r}}_a|^2 - \frac{1}{2}\frac{m^2}{\mu}\Big|\sum_a \dot{\vec{r}}_a\Big|^2 - U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n)$$

Sec 14, Prob. 1

Spherical pendulum:



$$T = \tfrac{1}{2} m l^2 (\dot\theta^2 + \sin^2\theta\, \dot\phi^2)$$

$$U = -mg\, z$$
$$= -mgl \cos\theta$$

Thus,

$$L = T - U$$
$$= \tfrac{1}{2} m l^2 (\dot\theta^2 + \sin^2\theta\, \dot\phi^2) + mgl\cos\theta$$

No t-dependence $\rightarrow$ $E = T + U = \text{const}$

No ϕ-dependence $\rightarrow$ $\dfrac{\partial L}{\partial \dot\phi} = m l^2 \sin^2\theta\, \dot\phi \equiv M_z = \text{const}$

$$E = \tfrac{1}{2} m l^2 (\dot\theta^2 + \sin^2\theta\, \dot\phi^2) - mgl\cos\theta$$
$$= \tfrac{1}{2} m l^2 \left(\dot\theta^2 + \sin^2\theta \frac{M_z^2}{m^2 l^4 \sin^4\theta}\right) - mgl\cos\theta$$
$$= \tfrac{1}{2} m l^2 \dot\theta^2 + \underbrace{\frac{M_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta}_{U_{eff}(\theta)}$$

$$E = \tfrac{1}{2} m l^2 \dot\theta^2 + \frac{M_z^2}{2ml^2\sin^2\theta} + U(\theta)$$

$$\rightarrow \quad \tfrac{1}{2} m l^2 \dot\theta^2 = E - U(\theta) - \frac{M_z^2}{2ml^2\sin^2\theta}$$

$$\dot\theta = \sqrt{\frac{2}{ml^2}(E - U(\theta)) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}$$

$$\frac{d\theta}{dt} = \dot\theta = \sqrt{\frac{2}{ml^2}(E + mgl\cos\theta) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}$$

$$\rightarrow \quad dt = \frac{d\theta}{\sqrt{\quad}}$$

$$\boxed{t = \int \frac{d\theta}{\sqrt{\frac{2}{ml^2}(E + mgl\cos\theta) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}} + \text{const}}$$

Path: use $M_z = m l^2 \sin^2\theta\, \dot\phi$

Thus,
$$\frac{d\theta}{dt} = \frac{d\theta}{d\phi}\frac{d\phi}{dt}$$
$$= \frac{d\theta}{d\phi}\frac{M_z}{ml^2\sin^2\theta}$$

Thus,
$$\frac{d\theta}{d\phi} = \frac{d\theta}{dt}\frac{ml^2\sin^2\theta}{M_z} = \sqrt{\quad}\;\frac{ml^2\sin^2\theta}{M_z}$$

$$d\phi = \frac{d\theta\, M_z}{\sqrt{\quad}\; ml^2\sin^2\theta}$$

$$\boxed{\phi = \int \frac{M_z\, d\theta / ml^2\sin^2\theta}{\sqrt{\frac{2}{ml^2}(E + mgl\cos\theta) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}} + \text{const}}$$

Turning points: (where $\dot{\theta} = 0$)

$$E = U_{eff}(r)$$
$$= \frac{M_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta$$

$$\rightarrow 2Eml^2\sin^2\theta = M_z^2 - 2m^2gl^3\sin^2\theta\cos\theta$$
$$2Eml^2(1-\cos^2\theta) = M_z^2 - 2m^2gl^3(1-\cos^2\theta)\cos\theta$$
$$2Eml^2 - 2Eml^2\cos^2\theta$$
$$= M_z^2 - 2m^2gl^3\cos\theta + 2m^2gl^3\cos^3\theta$$

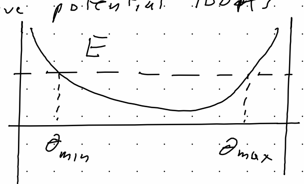
Thus, $2m^2gl^3\cos^3\theta + 2Eml^2\cos^2\theta - 2m^2gl^3\cos\theta + (M_z^2 - 2Eml^2) = 0$

Divide by $2m^2gl^3$:

$$\rightarrow \boxed{\cos^3\theta + \frac{E}{mgl}\cos^2\theta - \cos\theta + \left(\frac{M_z^2}{2m^2gl^3} - \frac{E}{mgl}\right) = 0}$$

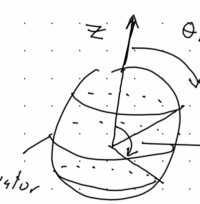
Cubic equation for $x \equiv \cos\theta$

Effective potential looks like:



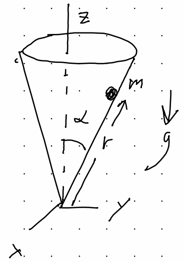
θ_{min} θ_{max}

$\theta = 0$, $x = +1$ $\theta = \pi$, $x = -1$



θ_{min} θ_{max} equator

Sec 14, Prob. 2



spherical polar coords (r, θ, ϕ)

Constraint $\theta = \alpha$

Generalized coords: (r, ϕ)

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\underbrace{\dot{\theta}^2}_{0} + r^2\sin^2\underbrace{\theta}_{\alpha}\,\dot{\phi}^2)$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2)$$

$$U = mgz$$
$$= mgr\cos\alpha$$

$$L = T - U$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2) - mgr\cos\alpha$$

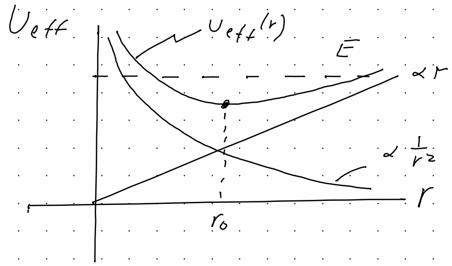
$E = $ const (since no explicit t dependence)

$$\frac{\partial L}{\partial\dot{\phi}} = mr^2\sin^2\alpha\,\dot{\phi} \equiv M_z = \text{const}$$

(since no explicit ϕ dependence)

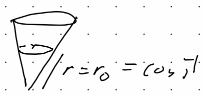
$$E = T + U$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2) + mgr\cos\alpha$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\sin^2\alpha\left(\frac{M_z^2}{m^2r^4\sin^4\alpha}\right) + mgr\cos\alpha$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{M_z^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha$$
$$= \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = \frac{M_z^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha$$



Bound orbits for $E > U_{eff,min} = U_{eff}(r_0)$

r_0 : stable circular orbit



t-equation:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}(E - mgr\cos\alpha) - \frac{M_z^2}{m^2r^2\sin^2\alpha}}} + const$$

Using $M_z = mr^2\sin^2\alpha\,\dot{\phi}$

$$\rightarrow \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M_z}{mr^2\sin^2\alpha}$$

ϕ-equation:

$$\phi = \int \frac{\left(\frac{M_z}{\sin^2\alpha}\right)\frac{dr}{r^2}}{\sqrt{2m(E - mgr\cos\alpha) - \frac{M_z^2}{r^2\sin^2\alpha}}} + const$$

Turning points: $r = r_{min}, r_{max}$

Determined by effective potential

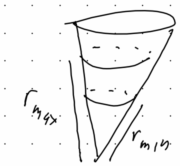
$$E = U_{eff}(r) = \frac{M_z^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha$$

$$\rightarrow 2mEr^2\sin^2\alpha = M_z^2 + 2m^2gr^3\sin^2\alpha\cos\alpha$$

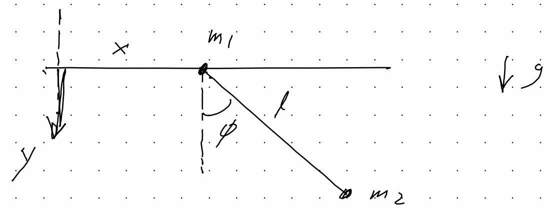
$$0 = 2m^2gr^3\sin^2\alpha\cos\alpha - 2mEr^2\sin^2\alpha + M_z^2$$

$$= r^3 - \left(\frac{E}{mgr\cos\alpha}\right)r^2 + \frac{M_z^2}{2m^2g\sin^2\alpha\cos\alpha}$$

cubic equation agin



Sec 14, Prob 3



From Sec 5, Prob 2 we have

$$L = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) + m_2 g l\cos\phi$$

No dependence on x:

$$\rightarrow \quad P_x \equiv \frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2 l\dot{\phi}\cos\phi = \text{const}$$

(x-component of total momentum)

No explicit t-dependence

$$\rightarrow \quad E = T + U = \text{const}$$
$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) - m_2 g l\cos\phi$$

Work in frame where $COM_x = 0$:

$$COM_x = m_1 x + m_2(x + l\sin\phi) = (m_1+m_2)x + m_2 l\sin\phi$$

$$COM_x = 0 \rightarrow x = -\left(\frac{m_2}{m_1+m_2}\right) l\sin\phi$$

$$\dot{x} = -\left(\frac{m_2}{m_1+m_2}\right) l\cos\phi\,\dot{\phi}$$

thus,

$$E = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) - m_2 g l\cos\phi$$

$$= \frac{1}{2}(m_1+m_2)\frac{m_2^2}{(m_1+m_2)^2} l^2\cos^2\phi\,\dot{\phi}^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 - m_2\left(\frac{m_2}{m_1+m_2}\right) l^2\cos^2\phi\,\dot{\phi}^2 - m_2 g l\cos\phi$$

$$= \frac{1}{2}m_2 l^2\dot{\phi}^2\left[1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right] - m_2 g l\cos\phi$$

1-d problem:

$$\frac{E + m_2 g l\cos\phi}{\left[1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right]} = \frac{1}{2}m_2 l^2\dot{\phi}^2$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{\frac{\frac{2}{m_2 l^2}\left(E + m_2 g l\cos\phi\right)}{1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi}}$$

$$\rightarrow \quad dt = \frac{d\phi}{\sqrt{}}$$

$$= d\phi \sqrt{\frac{1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi}{\frac{2}{m_2 l^2}\left(E + m_2 g l \cos\phi\right)}}$$

$$= d\phi \sqrt{\frac{m_2 l^2}{2}\,\frac{1}{m_1+m_2}}\;\sqrt{\frac{(m_1+m_2) - m_2\cos^2\phi}{E + m_2 g l \cos\phi}}$$

$$= d\phi \sqrt{\frac{m_2}{m_1+m_2}}\;\sqrt{\frac{l^2}{2}}\;\sqrt{\frac{m_1 + m_2\sin^2\phi}{E + m_2 g l \cos\phi}}$$

so

$$\boxed{t = \sqrt{\left(\frac{m_2}{m_1+m_2}\right)\frac{l^2}{2}} \int d\phi \sqrt{\frac{m_1 + m_2\sin^2\phi}{E + m_2 g l \cos\phi}} + \text{const}}$$

Now: $x_2 = x + l\sin\phi$

$y_2 = l\cos\phi$

Using $x = -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi$

$$\rightarrow \quad x_2 = \left[-\left(\frac{m_2}{m_1+m_2}\right) l\sin\phi + l\sin\phi\right] = \left(\frac{m_1}{m_1+m_2}\right) l \sin\phi$$

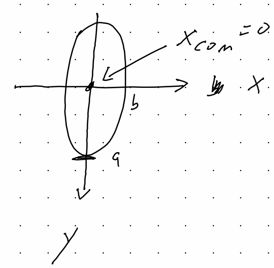
$$x_2 = \left(\frac{m_1}{m_1+m_2}\right) l \sin\phi \equiv b\sin\phi$$

$$y_2 = l\cos\phi \equiv a\cos\phi$$

$$\left(\frac{y_2}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = \cos^2\phi + \sin^2\phi = 1$$

which is an ellipse with semi-major and semi-minor axes:

$$a = l\,, \quad b = l\left(\frac{m_1}{m_1+m_2}\right)$$



If $m_1 >> m_2$ then $a = b \approx l$ so that m_2 moves along a circular arc of radius l.

Sec 15, Prob. 1

$U = -\frac{\alpha}{r}$, $E = 0$ $\longrightarrow$ $e = 1$

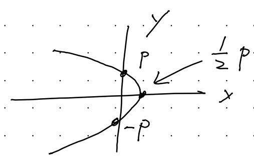
$$\frac{p}{r} = 1 + \cos\phi$$

when $\phi = 0$, $p = 2 \cdot r$

so $r_{min} = \frac{p}{2}$

NOTE: $p = r + r\cos\phi$

$= \sqrt{x^2+y^2} + x$



$\rightarrow$ $(p-x)^2 = x^2+y^2$

$p^2 + \cancel{x^2} - 2px = \cancel{x^2} + y^2$

$p^2 - y^2 = 2px$

$\rightarrow$ $\boxed{x = \frac{p^2-y^2}{2p}}$ parabola

a

when $y = 0$, $x = \frac{1}{2}p$

$x = 0$, $y = \pm p$

Time equation:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}[E - U(r)] - \frac{M^2}{m^2r^2}}} + \text{const}$$

$U(r) = -\frac{\alpha}{r}$, $p = \frac{M^2}{m\alpha}$, $e = 1$, $E = 0$

$$\rightarrow \quad t = \int \frac{dr}{\sqrt{\frac{2}{m}\frac{\alpha}{r} - \frac{M^2}{m^2r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{2}{m}\frac{\alpha}{r} - \frac{m\alpha p}{m^2r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{dr}{\sqrt{\frac{2}{r} - \frac{p}{r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{r\,dr}{\sqrt{2r - p}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha p}} \int \frac{r\,dr}{\sqrt{\frac{2r}{p} - 1}} + \text{const}$$

Define: $\frac{2r}{p} - 1 = \xi^2 > 0$ $\quad (-\infty < \xi < \infty)$

$2r = p(1+\xi^2)$

$\boxed{r = \frac{p}{2}(1+\xi^2)}$ $\rightarrow$ $dr = p\xi\, d\xi$

$r\,dr = \frac{p^2}{2}(\xi + \xi^3)\, d\xi$

Thus,

$$t = \sqrt{\frac{m}{\alpha p}} \int \frac{\frac{p^2}{2}(\xi + \xi^3)\,d\xi}{\sqrt{\xi^2}} + \text{const}$$

$$= \sqrt{\frac{mp^3}{\alpha}} \; \frac{1}{2} \int (1+\xi^2)\,d\xi + \text{const}$$

$$= \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}} \left(\xi + \frac{1}{3}\xi^3\right) + \text{const}$$

choose constant so that $t=0 \iff \xi = 0$ (const $=0$)

so, $$\boxed{t = \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}}\left(\xi + \frac{1}{3}\xi^3\right)}$$

Now:

$$\frac{p}{r} = 1 + \cos\phi$$

$$p = r(1+\cos\phi)$$

$$\cancel{p} = \frac{\cancel{p}}{2}(1+\xi^2)(1+\cos\phi)$$

$$2 = 1 + \xi^2 + \cos\phi + \xi^2\cos\phi$$

$$1 - \xi^2 = (1+\xi^2)\cos\phi$$

$$\rightarrow \boxed{\cos\phi = \frac{1-\xi^2}{1+\xi^2}}$$

$$\boxed{\begin{aligned} x &= r\cos\phi \\ &= \frac{p}{2}\cancel{(1+\xi^2)}\left(\frac{1-\xi^2}{\cancel{1+\xi^2}}\right) \\ &= \frac{p}{2}(1-\xi^2) \end{aligned}}$$

Also,

$$x^2 + y^2 = r^2$$

$$\rightarrow y^2 = r^2 - x^2$$

$$= \frac{p^2}{4}(1+\xi^2)^2 - \frac{p^2}{4}(1-\xi^2)^2$$

$$= \frac{p^2}{4}\left(\cancel{1} + \cancel{\xi^4} + 2\xi^2 - \cancel{1} - \cancel{\xi^4} + 2\xi^2\right)$$

$$= p^2\xi^2$$

so $$\boxed{y = p\xi}$$

Sec 15, Prob 3:

$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U)-M^2/r^2}}$$

This is the change in ϕ as r goes from r_{min} to r_{max} and then back to r_{min}.

A closed bound orbit would have $\Delta\phi = 2\pi m/n$ for m, n integers.

Consider: $U = -\frac{\alpha}{r} + \delta U$ where $|\delta U| \ll |\frac{\alpha}{r}|$

For $\delta U = 0$, $\Delta\phi = 2\pi$

$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-(U+\delta U))-M^2/r^2}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U)-2m\delta U-M^2/r^2}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{\left(2m(E-U)-\frac{M^2}{r^2}\right)\left(1-\frac{2m\delta U}{2m(E-U)-M^2/r^2}\right)}}$$

$$\approx 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U)-M^2/r^2}}\left[1+\frac{m\delta U}{2m(E-U)-M^2/r^2}\right]$$

For $U = -\alpha/r$:

$$\Delta\phi \approx 2\pi + 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2\;\; m\delta U}{\left[2m(E-U)-M^2/r^2\right]^{3/2}}$$

$$\approx 2\pi + \delta\phi$$

where

$$\delta\phi \equiv \int_{r_{min}}^{r_{max}} \frac{2m\delta U\, M\,dr/r^2}{\left[2m(E-U)-M^2/r^2\right]^{3/2}}$$

$$= \frac{\partial}{\partial M}\left[\int_{r_{min}}^{r_{max}} \frac{2m\,\delta U\,dr}{\sqrt{2m(E-U)-M^2/r^2}}\right]$$

Evaluate terms in integrand along unperturbed path since δU is already small

$$\frac{p}{r} = 1+e\cos\phi \quad , \quad p = a(1-e^2)$$

$$\rightarrow -\frac{p}{r^2}dr = -e\sin\phi\, d\phi \rightarrow \boxed{dr = \frac{r^2}{p}e\sin\phi\, d\phi}$$

$$\sqrt{} = \sqrt{2mE + \frac{2m\alpha}{r} - \frac{M^2}{r^2}}$$

$$= \sqrt{-2m|E| + \frac{4m|E|}{(1-e^2)}\frac{p}{r} - \frac{2m|E|}{(1-e^2)}\frac{p^2}{r^2}}$$

$$\sqrt{\ } = \sqrt{2m|E|}\sqrt{-1 + \frac{2}{(1-e^2)}(1+e\cos\phi) - \frac{1}{(1-e^2)}(1+e\cos\phi)^2}$$

$$= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}}\sqrt{-(\cancel{1}-e^2) + 2(\cancel{1}+\cancel{e\cos\phi}) - (\cancel{1}+\cancel{2e\cos\phi}+e^2\cos^2\phi)}$$

$$= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}}\sqrt{e^2(1-\cos^2\phi)}$$

$$= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}}\, e\sin\phi$$

Thus,

$$\delta\phi = \frac{\partial}{\partial m}\left[2m\int_0^{\pi}\frac{\delta U\, r^2\,\cancel{e\sin\phi}\, d\phi}{\sqrt{\frac{2m|E|}{1-e^2}}\,\cancel{e\sin\phi}\, p}\right]$$

$$= \frac{\partial}{\partial m}\left[\frac{2m}{p\sqrt{\frac{2m|E|}{1-e^2}}}\int_0^{\pi} d\phi\, r^2\delta U\right]$$

$$= \boxed{\frac{\partial}{\partial m}\left[\frac{2m}{M}\int_0^{\pi} d\phi\, r^2\delta U\right]}$$

using (15.6): $\frac{M}{\sqrt{2m|E|}} = \frac{p}{\sqrt{1-e^2}}$

Evaluate:

$$\delta\phi = \frac{\partial}{\partial m}\left[\frac{2m}{M}\int_0^{\pi} d\phi\, r^2\delta U\right]$$

for (a) $\delta U = \beta/r^2$, (b) $= \delta U = \frac{\gamma}{r^3}$

(a)
$$\delta\phi = \frac{\partial}{\partial m}\left[\frac{2m}{M}\int_0^{\pi} d\phi\,\beta\right]$$

$$= 2\pi\beta m\frac{\partial}{\partial m}\left(\frac{1}{M}\right)$$

$$= -\frac{2\pi\beta m}{M^2}$$

$$= \boxed{-\frac{2\pi\beta}{\alpha p}}$$

Recall: $p = \frac{M^2}{m\alpha}$

$\leftarrow$ $\alpha p = \frac{M^2}{m}$

(b)
$$\delta\phi = \frac{\partial}{\partial m}\left[\frac{2m}{M}\int_0^{\pi} d\phi\,\frac{\gamma}{r}\right]$$

$$= \frac{\partial}{\partial m}\left[\frac{2m\gamma}{M}\int_0^{\pi} d\phi\left(\frac{1+e\cos\phi}{p}\right)\right]$$

$$= 2m\gamma\frac{\partial}{\partial m}\left[\frac{1}{Mp}\left(\pi + \cancelto{0}{e\sin\phi\Big|_0^{\pi}}\right)\right]$$

Now: $\frac{1}{Mp} = \frac{m\alpha}{M^3} \rightarrow \frac{\partial}{\partial m}\left(\frac{1}{Mp}\right) = -\frac{3m\alpha}{M^4}$

So $\delta\phi = -6\pi\gamma \dfrac{m^2\alpha}{M^4}$

$$= -6\pi\gamma\alpha\left(\frac{1}{p\alpha}\right)^2$$

$$= \boxed{\frac{-6\pi\gamma}{p^2\alpha}}$$