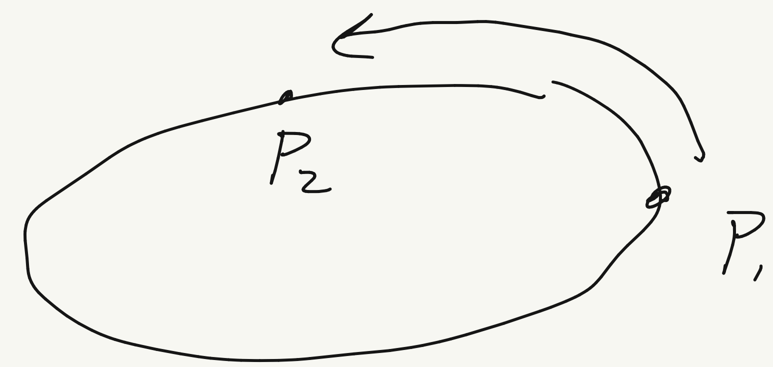
Lecture #1: Aug 24th

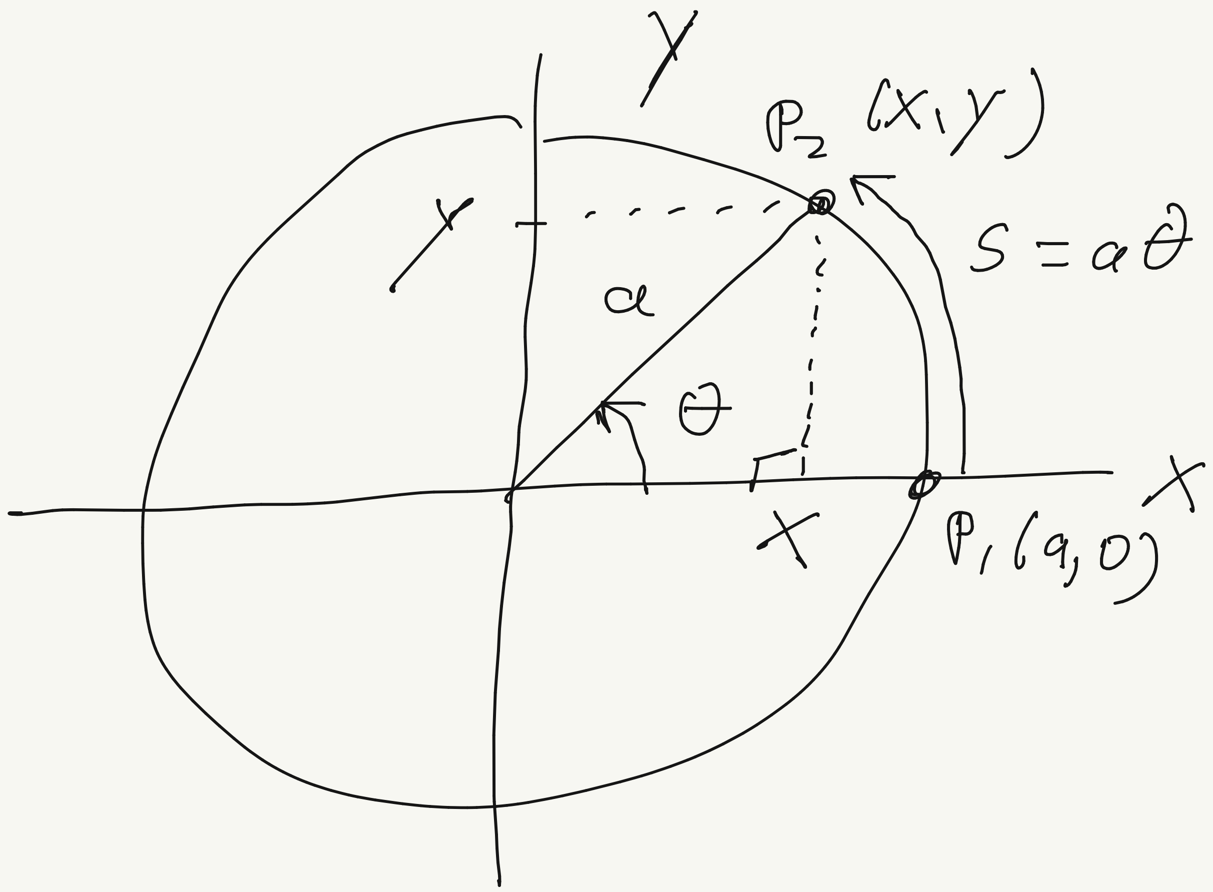
Elliptic functions / integrals:

i) circumference of an ellipse



ii) period of a simple pendulum beyond the small-angle approximation

Circular functions: sines, cosines,



a = radius

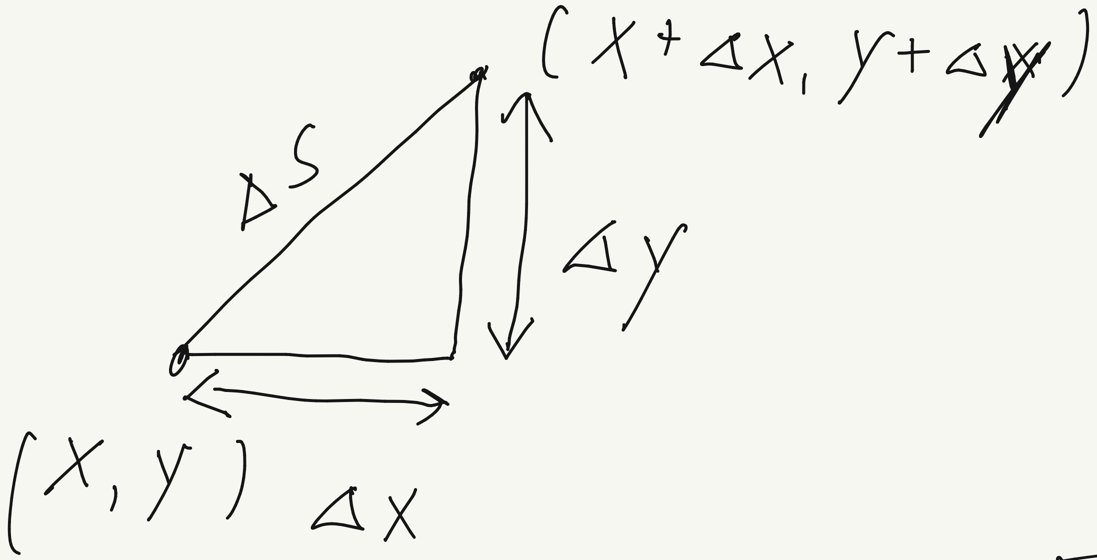
$$x^2 + y^2 = a^2$$

$$\theta = \frac{s}{a}$$

$$= \frac{1}{a}\int_{P_1}^{P_2} \sqrt{dx^2 + dy^2}$$

$$= \int d\theta$$

$$ds^2 = dx^2 + dy^2$$



$$\left.\begin{aligned} \sin\theta &\equiv y/a \\ \cos\theta &\equiv x/a \end{aligned}\right\} \text{definitions}$$

$$x = a\cos\theta$$
$$y = a\sin\theta$$

$$\boxed{x^2 + y^2 = a^2} \rightarrow \cancel{a^2}\cos^2\theta + \cancel{a^2}\sin^2\theta = \cancel{a^2}$$

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

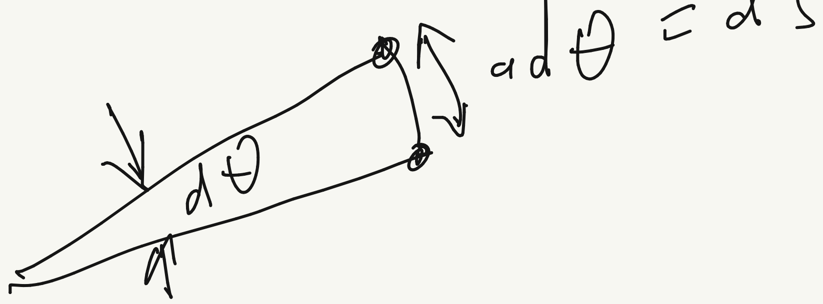
$$\boxed{\frac{d}{d\theta}\sin\theta = \cos\theta}$$

Proof: $\displaystyle \frac{d}{d\theta}\sin\theta = \frac{d}{d\theta}\left(\frac{y}{a}\right)$

$$= \frac{1}{a}\,\frac{dy}{d\theta}$$

$$= \frac{dy}{\sqrt{dx^2 + dy^2}}$$

$$= \frac{1}{\frac{\sqrt{dx^2+dy^2}}{dy}} = \frac{1}{\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}}$$



$$a\,d\theta = \sqrt{dx^2 + dy^2}$$

$$x^2+y^2=a^2 \rightarrow 2x\,dx + 2y\,dy = 0$$

$$\frac{dx}{dy} = -\frac{y}{x}$$

$$\frac{d}{d\theta}\sin\theta = \frac{1}{\sqrt{\left(\frac{-y}{x}\right)^2+1}} = \frac{1}{\sqrt{\frac{y^2}{x^2}+1}} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{a} = \cos\theta$$

$$\boxed{\frac{d}{d\theta}\sin\theta = \cos\theta}$$

Similarly $\boxed{\frac{d\cos\theta}{d\theta} = -\sin\theta}$

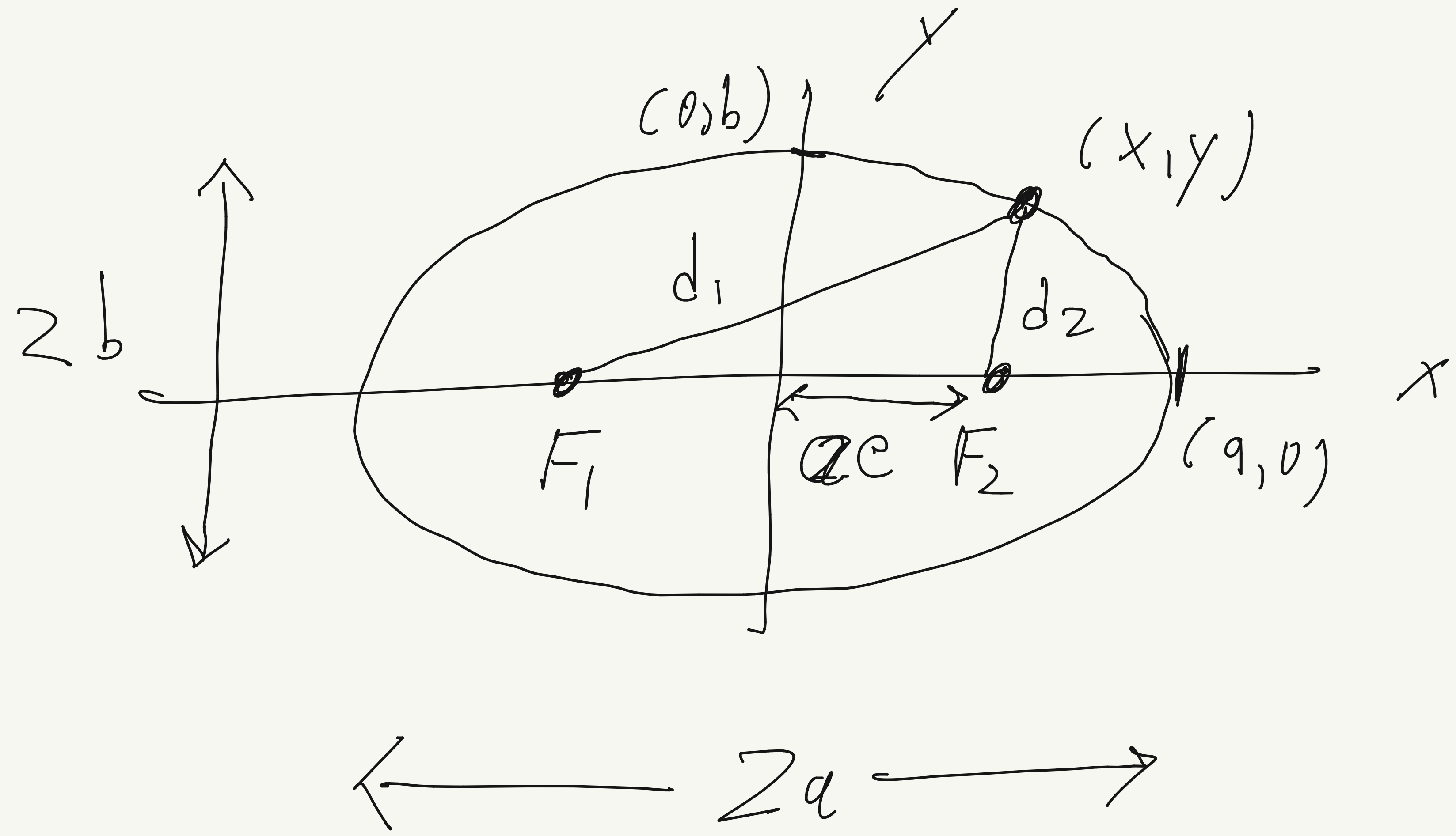
$$\int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta = \theta + \text{const}$$

$$t = \sin\theta$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-t^2}$$

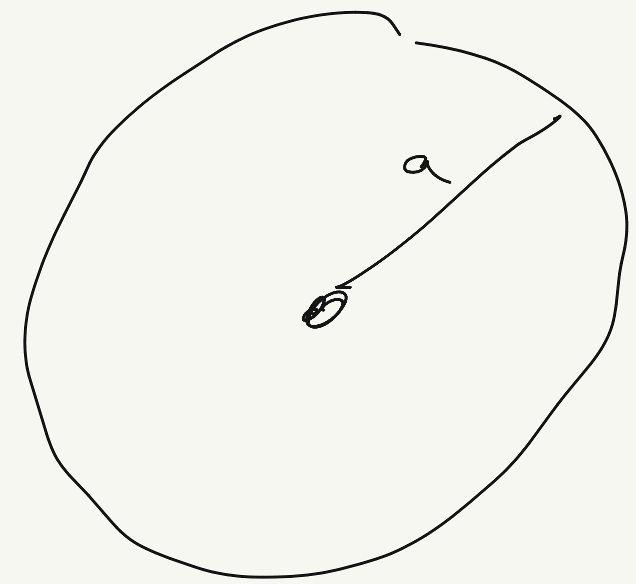
$$\boxed{\int \frac{dt}{\sqrt{1-t^2}} = \theta + \text{const} = \sin^{-1} t + \text{const}}$$

Lec #2: Aug 26th



$d_1 + d_2 = 2a$

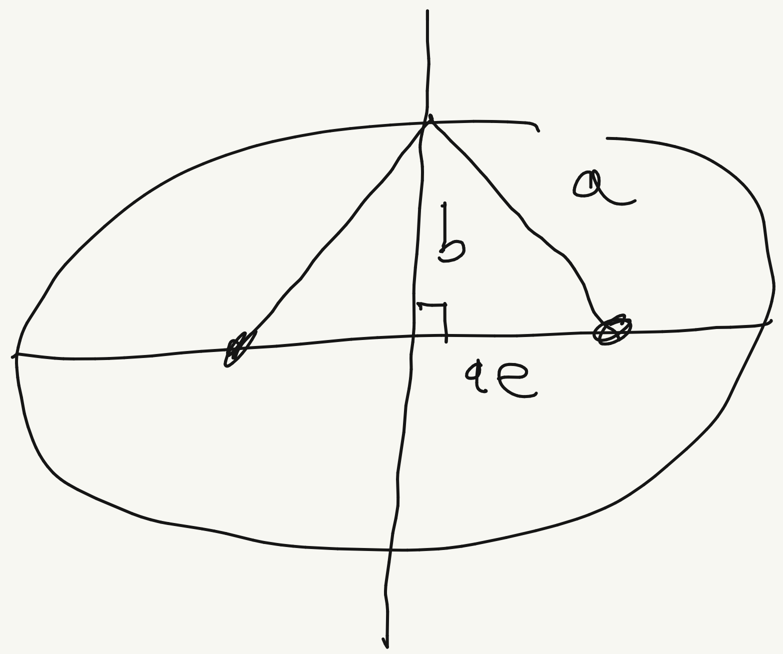
$x^2 + y^2 = a^2$



$e \overset{?}{\cancel{=}} \frac{b}{a}$

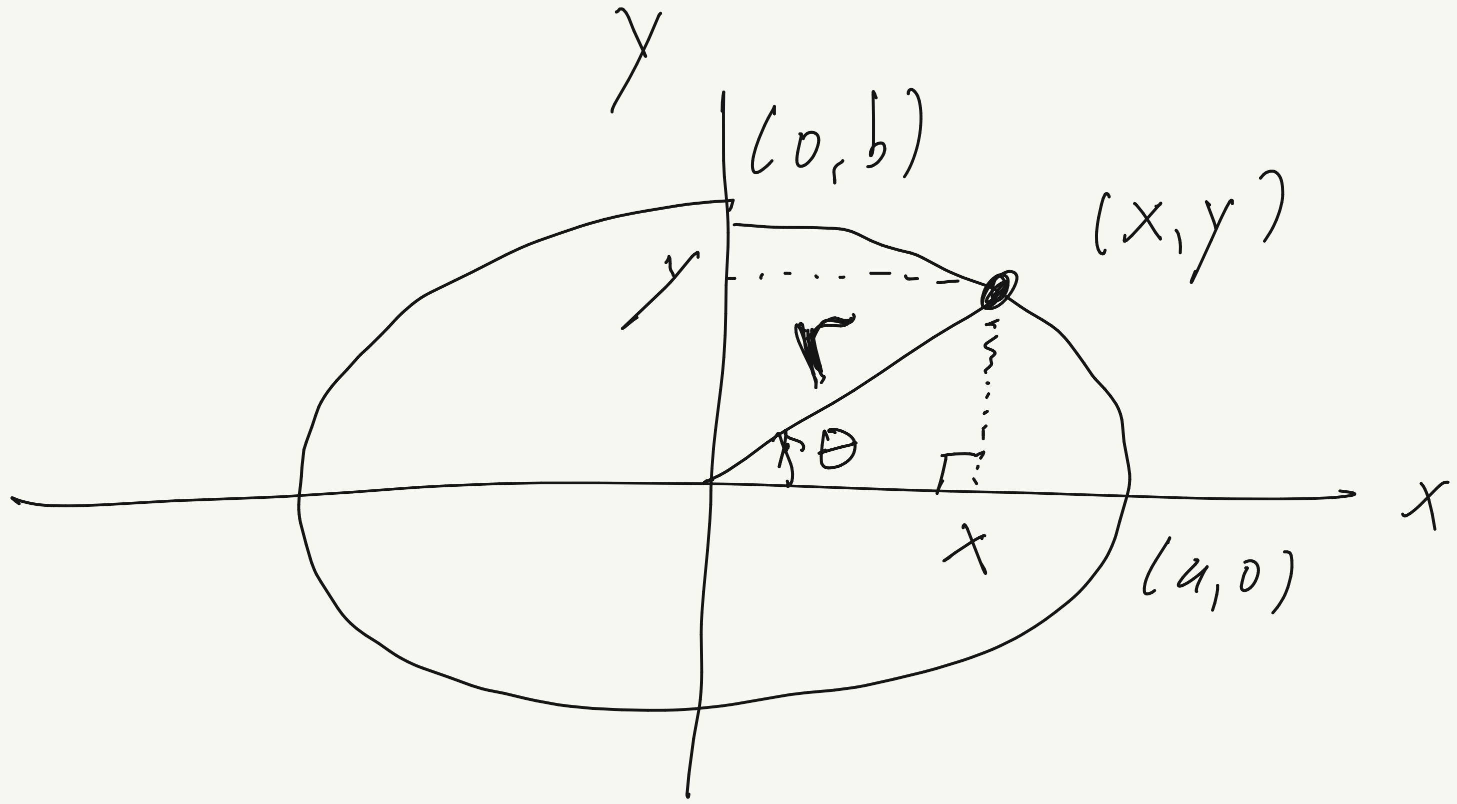
$e \cancel{=} 1 - \frac{b}{a}$

$e^2 = \frac{b^2}{1-a^2}$



$$\boxed{e = \sqrt{1 - \left(\frac{b}{a}\right)^2}}$$

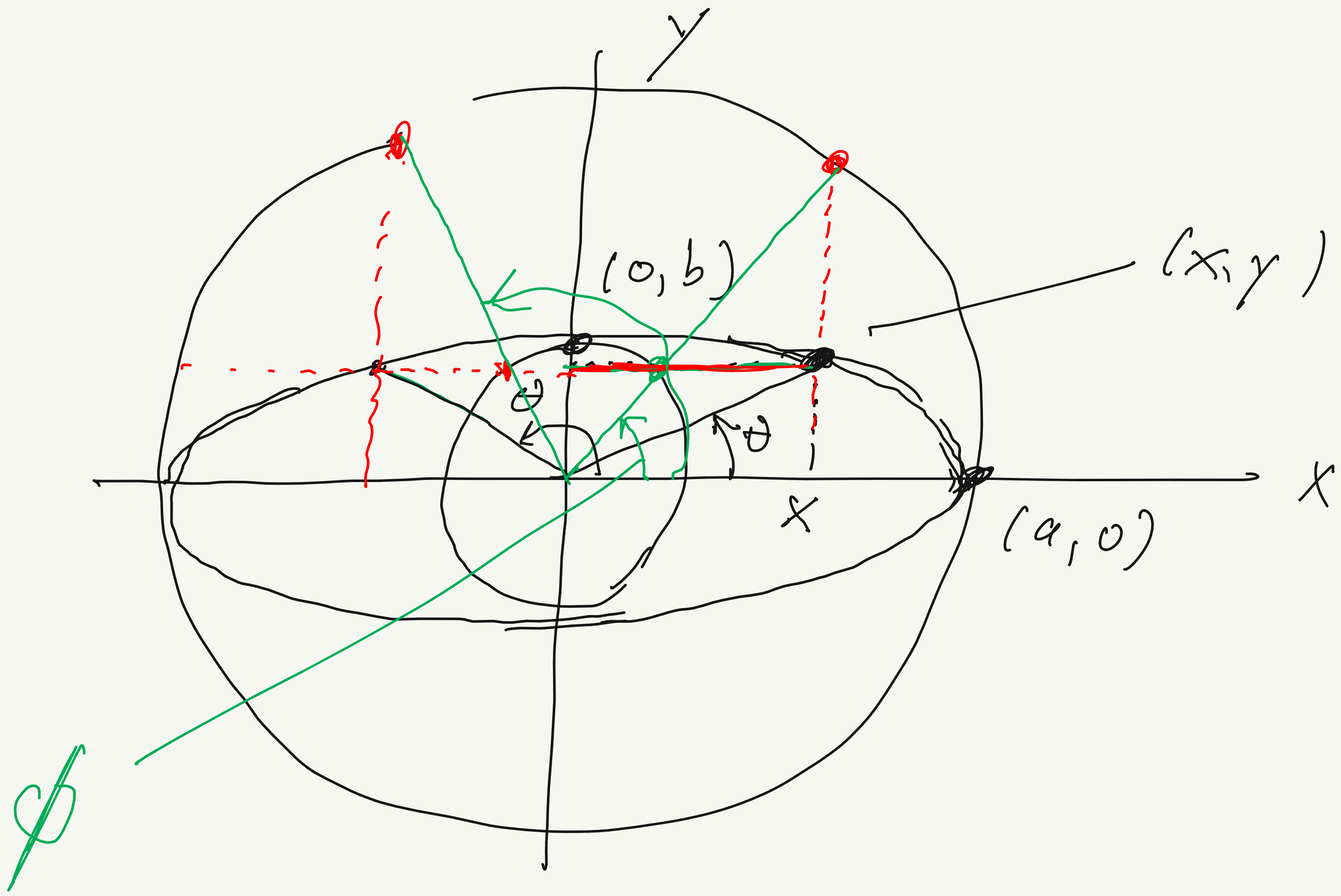
Pf:
$a^2 = b^2 + (ae)^2$
$a^2(1-e^2) = b^2$
$1 - e^2 = \left(\frac{b}{a}\right)^2$
$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$



$$x = r\cos\theta$$
$$y = r\sin\theta$$

r changes

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \longleftarrow \begin{cases} x = a\cos\phi \\ y = b\sin\phi \end{cases}$$



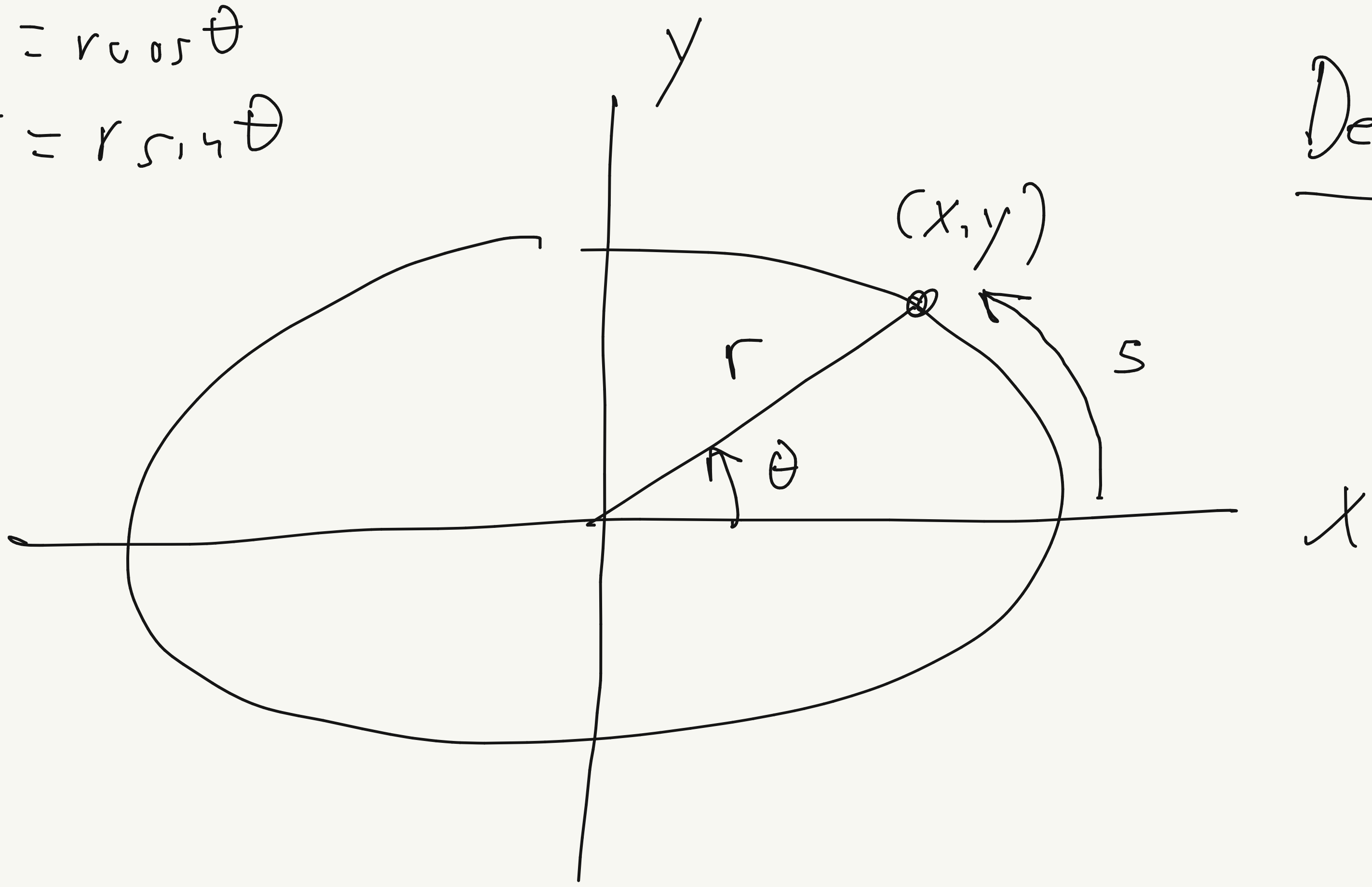
$0 < e < 1$

$e = 0$ circle

$e = 1$ parabola

$e > 1$ hyperbola

$x = r\cos\theta$
$y = r\sin\theta$



$$k = e \quad , \quad \underbrace{0 < k < 1}_{\text{modulus}}$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\theta^2}$$

$$bu = \int_0^\theta r\,d\theta < \cancel{\theta}\ s$$

Define:

$$\boxed{\begin{aligned} cn(u;k) &\equiv x/a \\ sn(u;k) &\equiv y/b \\ dn(u;k) &\equiv \frac{r}{a} \end{aligned}}$$

$$u \equiv \frac{1}{b}\int_0^\theta r\,d\theta$$

not θ, not arc length

$$x = r\cos\theta \ , \ y = r\sin\theta$$
$$dx = dr\cos\theta - r\sin\theta\,d\theta$$
$$dy = dr\sin\theta + r\cos\theta\,d\theta$$

$cn(u)$ $sn(u)$	pendulum $k = \sin\left(\frac{\phi_0}{2}\right)$

Property:

$$\boxed{cn^2 u + sn^2 u = 1} \leftarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\leftarrow x^2 + y^2 = r^2$$

$$dn^2 u = cn^2 u + \left(\frac{b}{a}\right)^2 sn^2 u$$

$$= 1 - sn^2 u + \left(\frac{b}{a}\right)^2 sn^2 u$$

$$= 1 - sn^2 u \left(1 - \left(\frac{b}{a}\right)^2\right)$$

$$= 1 - k^2 sn^2 u$$

$$\boxed{dn^2 u + k^2 sn^2 u = 1}$$

$$\frac{d}{du}\,\mathrm{sn}\,u = \frac{1}{b}\frac{dy}{du}$$

$$= \frac{dy}{r\,d\theta}$$

$$u = \frac{1}{b}\int_0^\theta r\,d\theta$$

$$du = \frac{r\,d\theta}{b} \rightarrow b\,du = r\,d\theta$$

$x = r\cos\theta \;,\; y = r\sin\theta$

$dx = dr\cos\theta - r\sin\theta\,d\theta \longrightarrow \sin\theta\,dx = \sin\theta\cos\theta\,dr - r\sin^2\theta\,d\theta$

$dy = dr\sin\theta + r\cos\theta\,d\theta \longrightarrow -\cos\theta\,dy = -\cos\theta\sin\theta\,dr - r\cos^2\theta\,d\theta$

add $\quad \sin\theta\,dx - \cos\theta\,dy = -r\,d\theta$

$\frac{y}{r}dx - \frac{x}{r}dy = -r\,d\theta$

$$\rightarrow \boxed{r\,d\theta = \frac{-y}{r}dx + \frac{x}{r}dy}$$

$$\frac{d}{du}\,\mathrm{sn}\,u = \frac{dy}{\frac{-y}{r}dx + \frac{x}{r}dy} = \frac{r}{-y\frac{dx}{dy} + x}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad \cancel{2}\frac{x\,dx}{a^2} + \cancel{2}\frac{y\,dy}{b^2} = 0$$

$$\frac{dx}{dy} = -\frac{y}{x}\frac{a^2}{b^2}$$

$$\frac{d\,\mathrm{sn}\,u}{du} = \frac{r}{-y\left(-\frac{y}{x}\right)\frac{a^2}{b^2} + x} = \frac{r\,x}{y^2\left(\frac{a}{b}\right)^2 + x^2}$$

$$= \frac{r}{a}\,\frac{x}{a}\left(\frac{1}{\left(\frac{y}{b}\right)^2 + \left(\frac{x}{a}\right)^2}\right)$$

$$= \frac{\mathrm{dn}\,u \cdot \mathrm{cn}\,u}{1}$$

$$\boxed{\frac{d}{du}\,\mathrm{sn}\,u = \mathrm{cn}\,u \cdot \mathrm{dn}\,u}$$

$$\frac{d}{du}\boxed{cn}\,u = -\,sn\,u \cdot dn\,u$$

$$\frac{d}{du}\,dn\,u = -k^2\,sn\,u \cdot cn\,u$$

$$\frac{d}{du}\,sn\,u = cn\,u \cdot dn\,u$$

$$sn^2u + cn^2u = 1$$

$$dn^2u + k^2 sn^2u = 1$$

$$\int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta = \theta$$

$t = \sin\theta$

$\theta = \sin^{-1}t$

$$\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}t$$

Integrate!

$$\int \frac{d(sn\,u)}{cn\,u \cdot dn\,u} = \int du = u + const$$

$t = sn\,u$

$$\boxed{\int \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}} = \underline{sn^{-1}}(t; k) + const}$$

$$\underbrace{F(\phi, k)}_{\text{incomplete elliptic integral of 1st kind}} \equiv \int_0^{\sin\phi} \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}}$$

$\mathrm{sn}\, u = \sin\phi$

$\frac{y}{b} = \sin\phi$

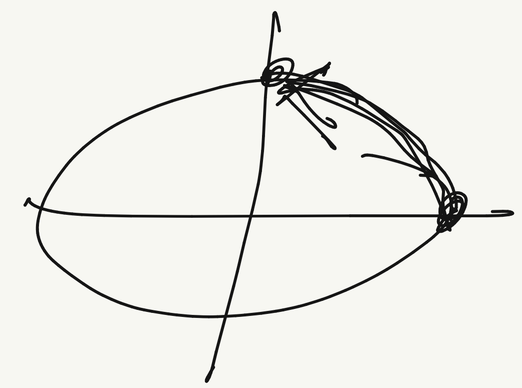
$\mathrm{cn}\, u = \cos\phi$

(angular dependence, period of a simple pendulum)

$$\underbrace{E(\phi, k)}_{\text{incomplete elliptic integral of 2nd kind}} \equiv \int_0^{\sin\phi} \frac{dt\,\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}$$

$\frac{x}{a} = \cos\phi$

(arclength along ellipse)



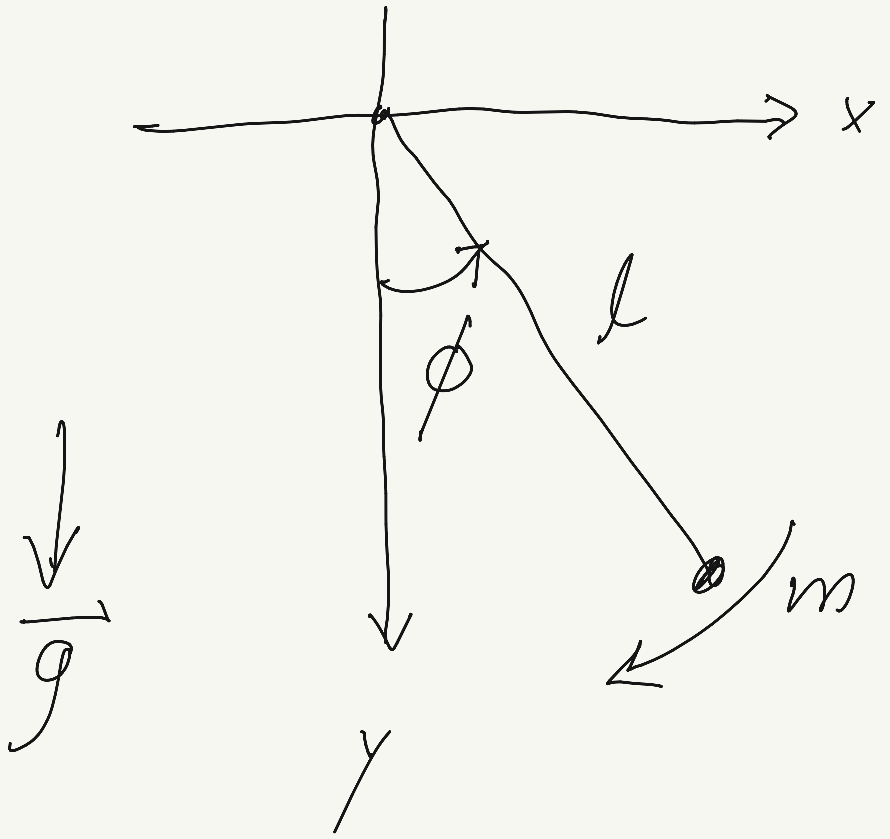
$\phi = \pi/2$

$$\left.\begin{aligned} \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} &= K(k) \\ \int_0^1 \frac{dt}{\sqrt{1-t^2}}\sqrt{1-k^2t^2} &= E(k) \end{aligned}\right\}$$

complete elliptic integrals of 1st and 2nd kind

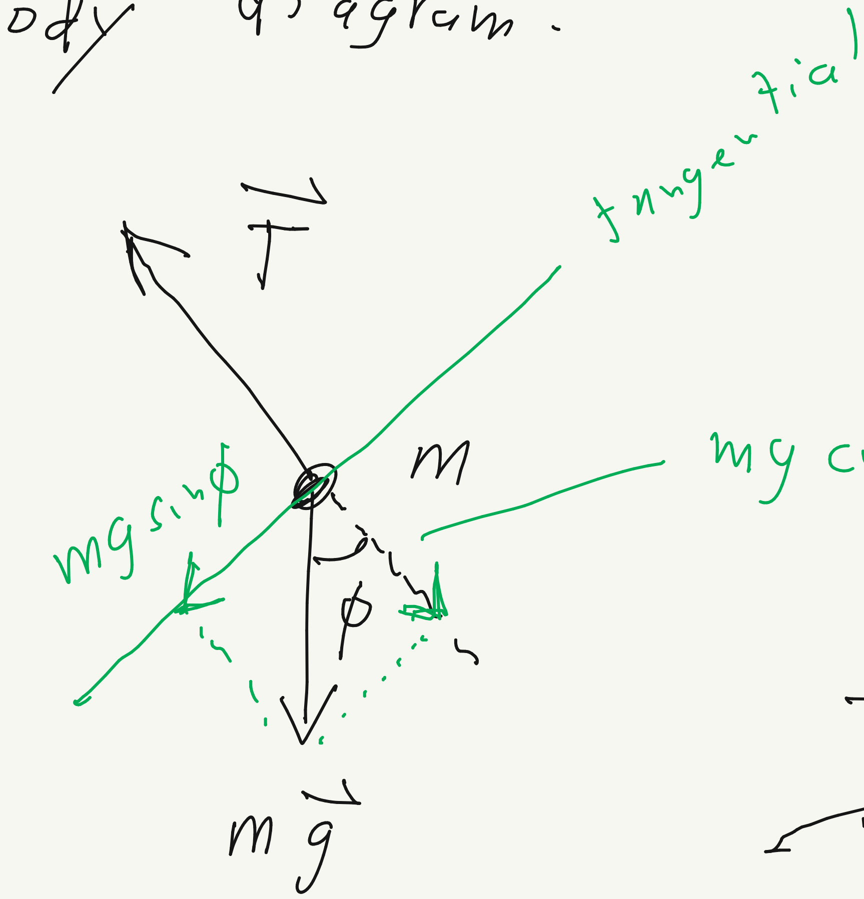
Lec #3: Aug 31st

Simple pendulum: ("Freshman physics")



$$\boxed{N = mg\cos\alpha}$$

Free-body diagram:



$$a_c = \frac{v^2}{r} = \omega^2 r$$
$$a_t = \alpha r$$
$$\alpha = \ddot{\phi}, \quad \omega = \dot{\phi}$$

$$\cancel{T = mg\cos\phi}$$

1) $T - mg\cos\phi = m\underbrace{\dot{\phi}^2 \ell}_{\text{centripetal acceleration}}$

$$\boxed{T = m\dot{\phi}^2 \ell + mg\cos\phi}$$

2) $mg\sin\phi = -m\, a_{\text{tangential}}$

$$= -m\ell\ddot{\phi}$$

$$\boxed{\ddot{\phi} = -\frac{g}{\ell}\sin\phi} \quad \text{EOM}$$

$$\ddot{\phi} = -\frac{g}{l} \sin\phi$$

(hard to solve ; 2^{nd} order non-linear ODE)

Small-angle approx: $\phi \ll 1$ rad $\cong 57^\circ$ (π radians $= 180^\circ$)

$$\rightarrow \quad \ddot{\phi} = -\frac{g}{l}\phi \qquad \omega_0 \equiv \sqrt{\frac{g}{l}}$$

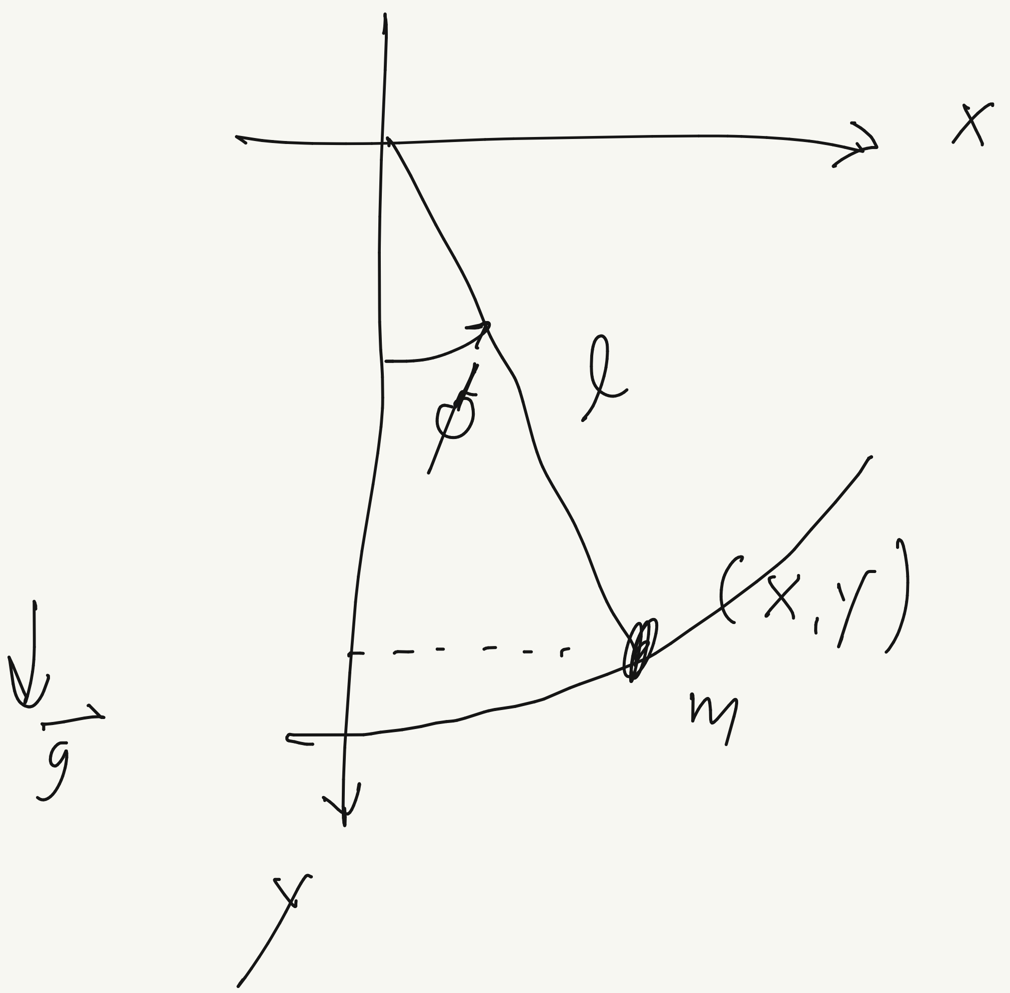
complex

$$\phi(t) = A e^{-i\sqrt{\frac{g}{l}}t}$$

$$= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

$$= a \cos(\omega_0 t + \alpha)$$

Lagrangian formalism:



$$L = T - U$$

$$U = -mgy$$
$$= -mgl\cos\phi$$

$$y = l\cos\phi$$
$$x = l\sin\phi$$
$$\Downarrow$$
$$\dot{x} = l\dot{\phi}\cos\phi$$
$$\dot{y} = -l\dot{\phi}\sin\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2}ml^2\dot{\phi}^2$$

$$\boxed{L = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi} \quad \cancel{+U(t)}$$

$$\Downarrow$$

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$$\frac{d}{dt}\left(ml^2\dot{\phi}\right) = -mgl\sin\phi$$

$$ml^2\ddot{\phi} = -mgl\sin\phi$$

$$\ddot{\phi} = \frac{-g}{l}\sin\phi$$

(x,y) $\quad x = r\cos\phi$

(r,ϕ) $\quad y = r\sin\phi$

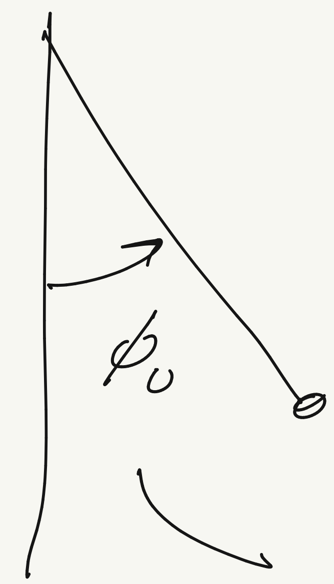
$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

L does not depend explicitly on time t $\rightarrow$ E is conserved.

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$$
$$= T + U$$

<u>Example:</u>

$$E = \frac{\partial L}{\partial \dot{\phi}}\phi - L$$
$$= ml^2\dot{\phi}\,\dot{\phi} - \left(\frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi\right)$$
$$= \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$$
$$= T + U$$



initial displacement

$$E = -mgl\cos\phi_0$$

$$-\cancel{m}gl\cos\phi_0 = \frac{1}{2}\cancel{m}l^2\dot{\phi}^2 - \cancel{m}gl\cos\phi$$
$$\dot{\phi}^2 = \frac{2}{l^2}\left(gl\cos\phi - gl\cos\phi_0\right)$$

L&L 11.1

$$\dot{\phi}^2 = 2\frac{g}{\ell}\left(\cos\phi - \cos\phi_0\right)$$

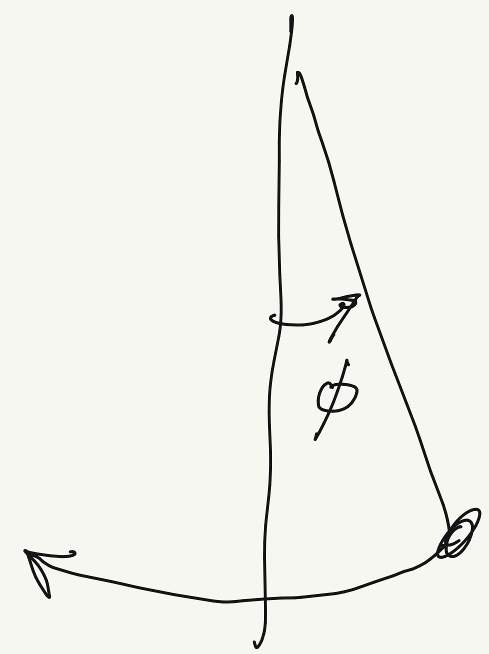
$$\frac{d\phi}{dt} = \dot{\phi} = \pm\sqrt{2}\underbrace{\sqrt{\frac{g}{\ell}}}_{\omega_0}\sqrt{\cos\phi - \cos\phi_0}$$

$$\int_0^t dt = \int_{\phi_0}^{\phi} \frac{-d\phi}{\sqrt{2}\,\omega_0\sqrt{\cos\phi - \cos\phi_0}}$$

$$\omega_0 t + \text{const} = \pm\int \frac{d\phi}{\sqrt{2}\sqrt{\cos\phi - \cos\phi_0}}$$

$$\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}t + \text{const}$$

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} =$$



$$\cos\phi = \cos\left(2\cdot\frac{\phi}{2}\right) = \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right) = 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\cos\phi_0 = 1 - 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$\rightarrow \sqrt{\cos\phi - \cos\phi_0} = \sqrt{2}\sqrt{\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)} = \sqrt{2}\left|\sin\left(\frac{\phi_0}{2}\right)\right|\sqrt{1 - \frac{\sin^2(\phi/2)}{\sin^2(\phi_0/2)}}$$

$$k = \left|\sin\left(\frac{\phi_0}{2}\right)\right| \qquad (0 < k < 1)$$

$$x = \frac{\sin\left(\frac{\phi}{2}\right)}{\left|\sin\left(\frac{\phi_0}{2}\right)\right|} = \frac{\sin\left(\frac{\phi}{2}\right)}{k} \longrightarrow dx = \frac{1}{k}\frac{1}{2}\cos\left(\frac{\phi}{2}\right)d\phi$$

$$= \frac{1}{2k}\sqrt{1-\sin^2\left(\frac{\phi}{2}\right)}\,d\phi$$

$$= \frac{1}{2k}\sqrt{1-k^2x^2}\,d\phi$$

$$\boxed{\sqrt{2}\sqrt{\cos\phi - \cos\phi_0} = 2k\sqrt{1-x^2}}$$

$$\boxed{d\phi = \frac{2k\,dx}{\sqrt{1-k^2x^2}}}$$

$$\omega_0 t + \text{const} = \pm\int \frac{\cancel{2k}\,dx}{\sqrt{1-k^2x^2}\;\cancel{2k}\sqrt{1-x^2}}$$

$$= \int_0^x \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

$$= \mathrm{sn}^{-1}(x; k)$$

$$\left\{\begin{array}{l} x = \dfrac{\sin\left(\frac{\phi}{2}\right)}{k} \\ k = \left|\sin\left(\frac{\phi_0}{2}\right)\right| \end{array}\right.$$

$$t = 0 \longleftrightarrow \phi = \phi_0$$

$$\text{const} = \int_0^1 \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}} \equiv \underbrace{K(k)}_{\text{complete elliptic integral of the 1st kind}}$$

$$\omega_0 t + K(k) = \mathrm{sn}^{-1}\left(\frac{\sin\left(\frac{\phi}{2}\right)}{k};\ k\right) \qquad \Big|\ \underbrace{k \equiv \sin\left(\frac{\phi_0}{2}\right)}$$

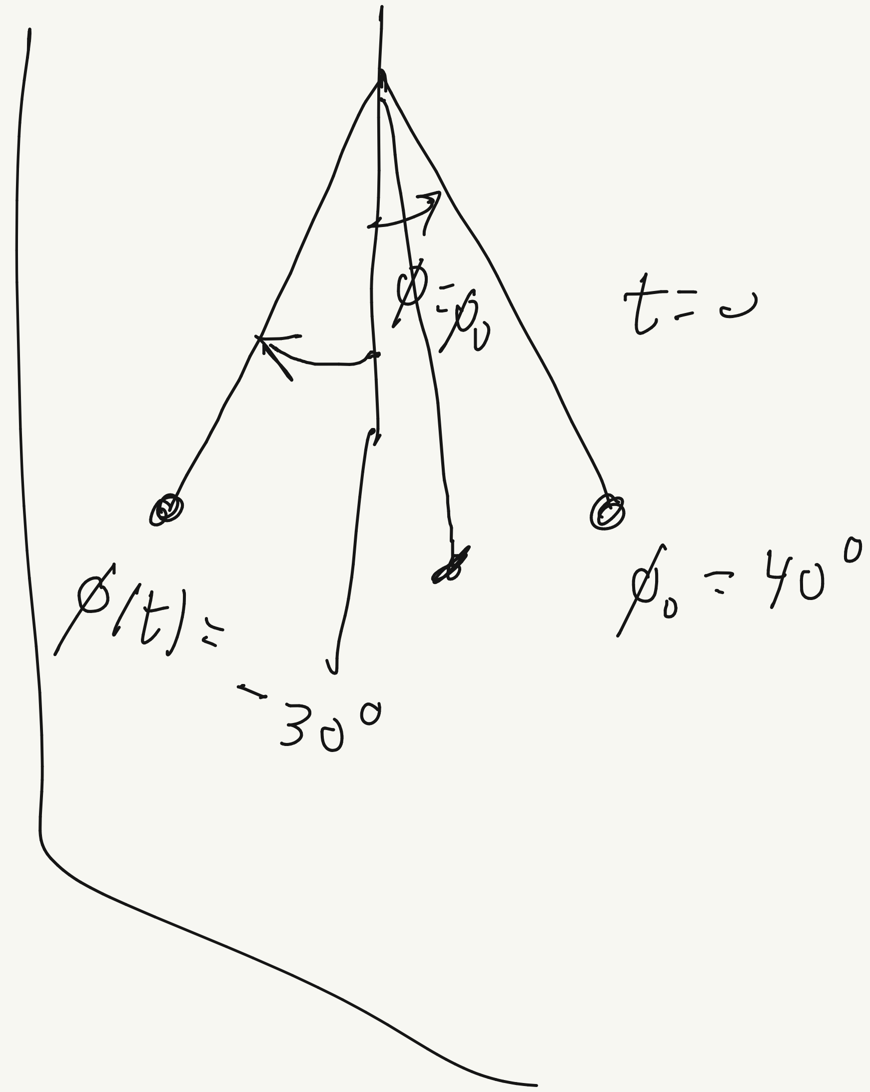
$$\mathrm{sn}\left(\omega_0 t + K(k); k\right) = \frac{1}{k}\sin\left(\frac{\phi}{2}\right)$$

$$\longrightarrow \boxed{\phi(t) = 2\arcsin\left(k\ \mathrm{sn}(\omega_0 t + K(k); k)\right)}$$

$$P = \frac{4}{\omega_0} K(k) \longrightarrow \frac{\omega_0 P}{4} = K(k)$$

Lec #4: 2 Sep 2021

$$\omega_0 \int_0^t dt = -\int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\Theta}}$$



$$\omega_0 t = -\left[\int_{\phi_0}^{0} + \int_0^{\phi} \right] \frac{d\phi}{\sqrt{\Theta}}$$

$$= + \int_0^{\phi_0} \frac{d\phi}{\sqrt{\Theta}} - \int_0^{\phi} \frac{d\phi}{\sqrt{\Theta}}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} - \int_0^x \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

$$x = \frac{\sin\left(\frac{\phi}{2}\right)}{\left|\sin\left(\frac{\phi_0}{2}\right)\right|}$$

$$k = \left|\sin\left(\frac{\phi_0}{2}\right)\right|$$

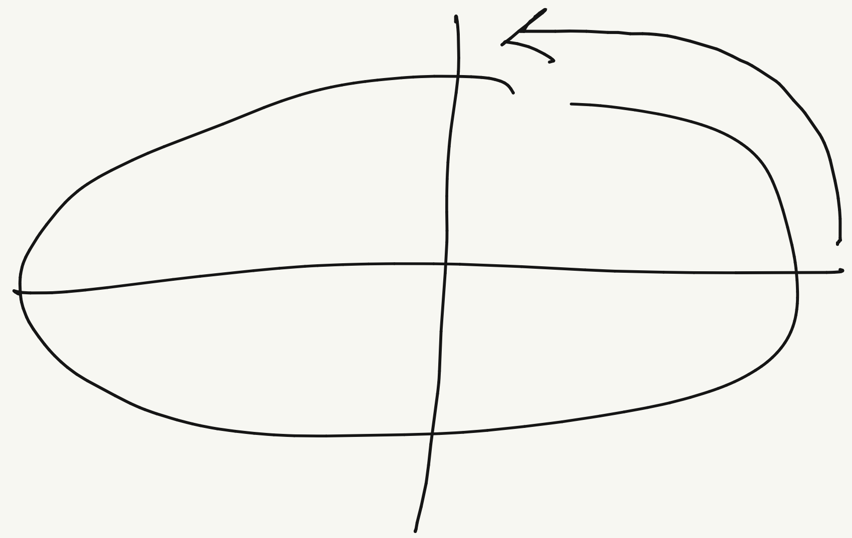
$$= K(k) - sn^{-1}(x; k)$$

$$sn^{-1}(x; k) = K(k) - \omega_0 t$$

$$\omega_0 = \sqrt{\frac{g}{l}}$$

$$\frac{\sin\left(\frac{\phi}{2}\right)}{k} = x = sn\left[K(k) - \omega_0 t ; k \right) = cn(\omega_0 t; k)$$

$$\rightarrow \boxed{\phi(t) = 2 \arcsin\left(k \, sn(K(k) - \omega_0 t; k) \right)}$$



$$\frac{1}{\sqrt{1-k^2x^2}} \approx 1 + \frac{1}{2}k^2x^2$$

$$(1+\epsilon)^p \approx 1 + p\epsilon$$

$$\boxed{\omega_0 \frac{P}{4} = K(k)} \rightarrow \boxed{P = \frac{4}{\omega_0} K(k)}$$

$$K(k) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}}$$

$\underline{k=0:}$

$$K(0) = \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} 1 = \pi/2$$

$$P = \frac{4}{\omega_0}\frac{\pi}{2} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}$$

$0 < k << 1$

$$K(k) = \int_0^1 \frac{dx}{\sqrt{1-x^2}}\left(1 + \frac{1}{2}k^2x^2\right) = \frac{\pi}{2} + \int_0^1 \frac{dx\ \frac{1}{2}k^2x^2}{\sqrt{1-x^2}}$$

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$$

<u>pf:</u> $$\sin\left(\frac{\pi}{2}-\theta\right) = \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}\cos\theta - \cancel{\cos\left(\frac{\pi}{2}\right)}\sin\theta$$

$$= \cos\theta$$

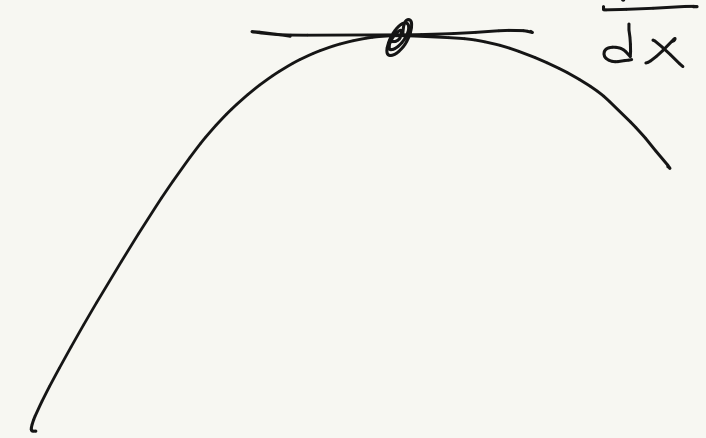
$$\sin(-\theta) = \sin\theta \quad , \quad \cos(-\theta) = \cos\theta$$

$$\mathrm{sn}(K(k) - x;\ k) = \mathrm{cn}(x)$$

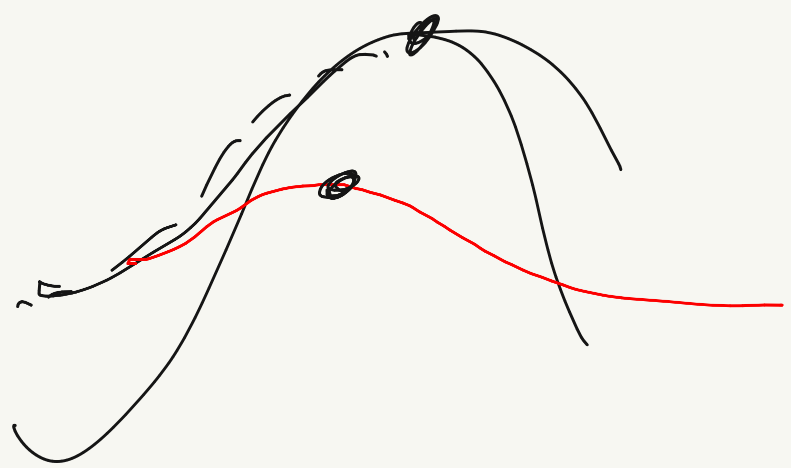
Lagrange multipliers

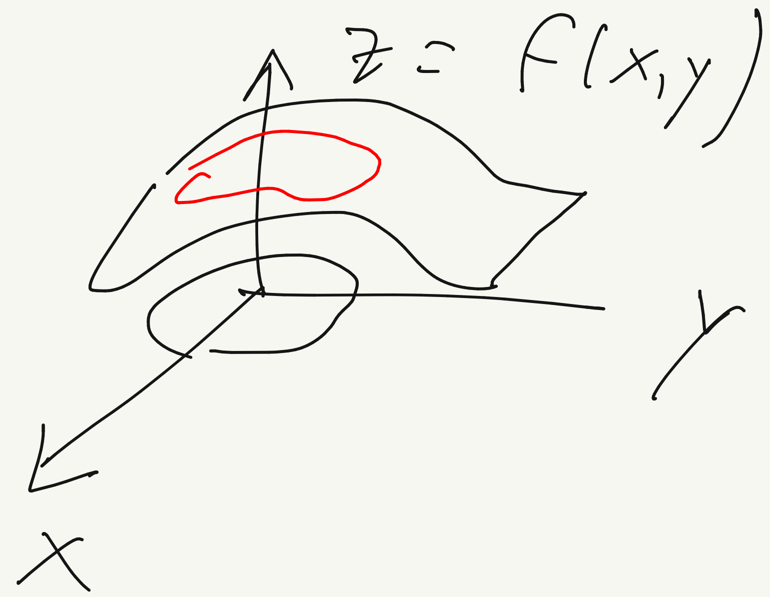
$f(x)$: max or min ? $\frac{df}{dx} = 0$

$f(x,y)$: " $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$



$\left.\frac{df}{dx}\right|_{x_0} = 0$





$f(x,y)$: Find extreme value subject to constraint $\underbrace{\phi(x,y) = 0}$?

1) "reduced space method":

solve constraint $\phi(x,y) = 0 \rightarrow y = g(x)$

$F(x) = f(x,y)\Big|_{y=g(x)}$

e.g., $\underbrace{x^2 + y^2 = 1}$

$2x\,dx + 2y\,dy = 0$

$dx = -\frac{y}{x}\,dy$

2) "method of Lagrange multipliers"

$F(x,y,\lambda) = f(x,y) + \lambda\,\phi(x,y)$ $\begin{cases} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0 , \quad \frac{\partial F}{\partial \lambda} = 0 \end{cases}$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0 \qquad F(x, y, \lambda) = f(x,y) + \lambda \varphi(x,y)$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0$$

$$\frac{\partial F}{\partial \lambda} = \varphi(x,y) = 0$$

$$\varphi(r, \phi) = 0 = r - l$$

$$\underbrace{L(r, \phi, \dot r, \dot \phi, t) + \lambda \overbrace{(r - l)}^{\varphi(r,\phi)}} = L'$$

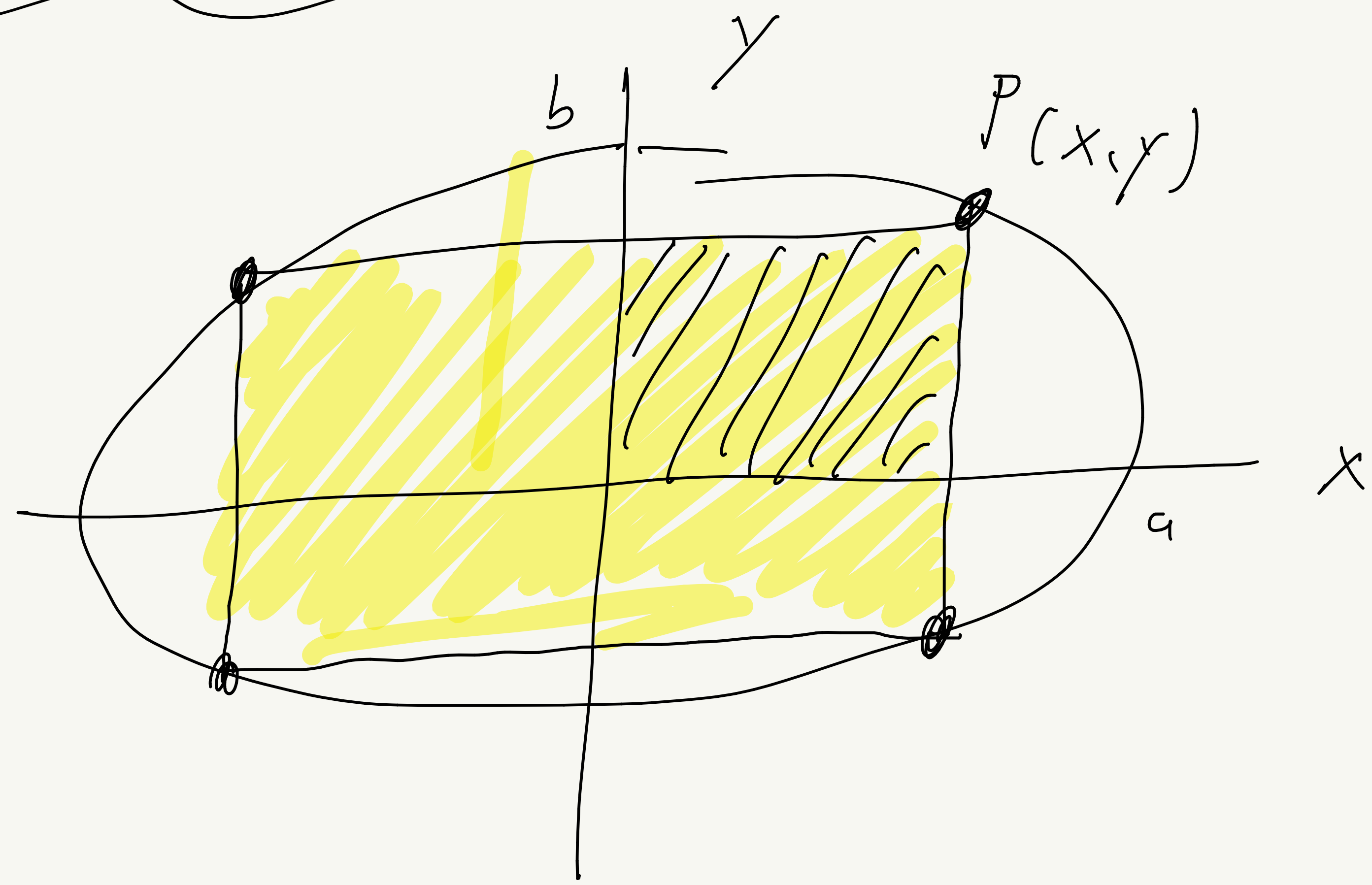
$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot q}\right) = \frac{\partial L'}{\partial q}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q} + \lambda \frac{\partial \varphi}{\partial q}$$

$$\frac{dp}{dt} = -\frac{\partial U}{\partial q} + \lambda \frac{\partial \varphi}{\partial q} = F_{app} + F_{constraint}$$

Lec #5: 9/7

1) Lagrange multipliers
2) Example : 2-d oscillating orbit



Maximize the area of a rectangle whose corners lie on the ellipse

$$\rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$A(x,y) = 4xy$$

$$\varphi(x,y) = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 0$$

$$F(x,y,\lambda) = 4xy + \lambda\,\varphi(x,y)$$

$$\boxed{\begin{aligned} \frac{\partial F}{\partial x} &= 4y + \lambda\left(-\frac{2x}{a^2}\right) = 0 \\ \frac{\partial F}{\partial y} &= 4x + \lambda\left(-\frac{2y}{b^2}\right) = 0 \\ \frac{\partial F}{\partial \lambda} &= 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 0 \end{aligned}}$$

$$\frac{\partial F}{\partial x} = 4y + \lambda\left(-\frac{2x}{a^2}\right) = 0 \longrightarrow 4y^2a^2 - 2\lambda xy = 0$$

$$\frac{\partial F}{\partial y} = 4x + \lambda\left(-\frac{2y}{b^2}\right) = 0 \longrightarrow 4x^2b^2 - 2\lambda yx = 0$$

subtract

$$\frac{\partial F}{\partial \lambda} = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 0$$

$$4y^2a^2 - 4x^2b^2 = 0$$

$$y^2a^2 = x^2b^2$$

$$\rightarrow \boxed{\frac{y}{b} = \pm\frac{x}{a}}$$

$$0 = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^2$$

$$= 1 - 2\left(\frac{x}{a}\right)^2$$

$$\frac{x}{a} = \frac{1}{\sqrt{2}} \longrightarrow \boxed{x = \frac{a}{\sqrt{2}}}, \quad \boxed{y = \frac{b}{\sqrt{2}}} \rightarrow A_{max} = 4xy$$

$$= 4\frac{a}{\sqrt{2}}\frac{b}{\sqrt{2}}$$

$$= \boxed{2ab}$$

Reduced space method

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$y = b\sqrt{1-\left(\frac{x}{a}\right)^2}$$

$$F(x) = 4xy\Big|_{y=b\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

$$= 4b\,x\sqrt{1-\left(\frac{x}{a}\right)^2}$$

maximize:

$$0 = F'(x) = 4b\sqrt{} + 4bx\left(\frac{1}{\cancel{2}}\right)\frac{1}{\sqrt{}}\left(-\frac{\cancel{2}x}{a^2}\right)$$

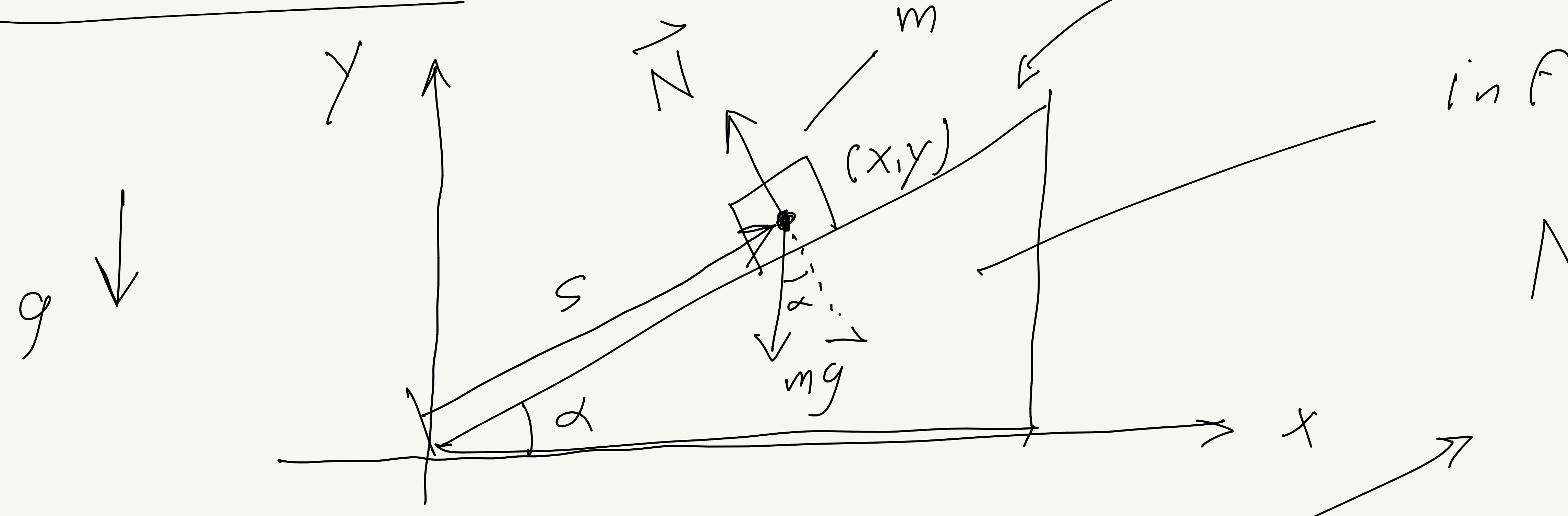
$$= 4b\left(\sqrt{} - \left(\frac{x}{a}\right)^2\frac{1}{\sqrt{}}\right)$$

$$= \frac{4b}{\sqrt{}}\left(1-\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^2\right)$$

$$= \frac{4b}{\sqrt{}}\left(1 - 2\left(\frac{x}{a}\right)^2\right) \longrightarrow \left(\frac{x}{a}\right)^2 = \frac{1}{2}$$

$$x = \pm\frac{a}{\sqrt{2}}$$

Mechanics problem:



Freshman physics analysis:

$$N - mg\cos\alpha = 0$$

$$\boxed{N = mg\cos\alpha}$$

$$m\ddot{s} = -mg\sin\alpha$$

$$\boxed{\ddot{s} = -g\sin\alpha}$$

Lagrangian analysis:

$$x = s\cos\alpha, \quad y = s\sin\alpha$$

$$\frac{y}{x} = \tan\alpha$$

$$y = x\tan\alpha$$

$$\boxed{\phi(x,y) = y - x\tan\alpha = 0}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$U = mgy$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$L' = L + \lambda \varphi$$

$$F = A + \lambda \varphi$$

$$= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(y - x\tan\alpha)$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}}\right) = \frac{\partial L'}{\partial x} \longrightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} + \lambda \frac{\partial \varphi}{\partial x}}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} + \lambda \frac{\partial \varphi}{\partial y}}$$

$$\boxed{\begin{aligned} \frac{dp_x}{dt} &= F_x + \lambda \frac{\partial \varphi}{\partial x} \\ \frac{dp_y}{dt} &= F_y + \lambda \frac{\partial \varphi}{\partial y} \end{aligned}}$$

$$\boxed{\frac{d\vec{p}}{dt} = \vec{F} + \lambda \vec{\nabla} \varphi}$$

$$\boxed{\varphi(x,y) = y - x\tan\alpha = 0}$$

$$\frac{d}{dt}(m\dot{x}) = 0 + \lambda(-\tan\alpha) \rightarrow m\ddot{x} = -\lambda \tan\alpha$$

$$\ddot{x} = -\frac{\lambda}{m}\tan\alpha$$

$$\frac{d}{dt}(m\dot{y}) = -mg + \lambda \longrightarrow \ddot{y} = -g + \frac{\lambda}{m}$$

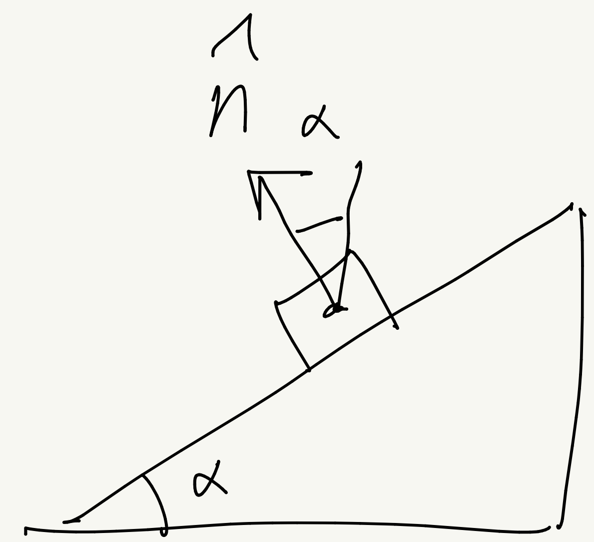
$$y - x\tan\alpha = 0$$

$$\ddot{y} - \ddot{x}\tan\alpha = 0 \rightarrow \ddot{y} = \ddot{x}\tan\alpha$$

$$\left(-g + \frac{\lambda}{m}\right) = -\frac{\lambda}{m}\tan\alpha \cdot \tan\alpha$$

$$\varphi = y - x\tan\alpha$$

$$\frac{\lambda}{m}\underbrace{\left(1 + \tan^2\alpha\right)}_{\sec^2\alpha = \frac{1}{\cos^2\alpha}} = g$$



$$\frac{\lambda}{m} = g\cos^2\alpha$$

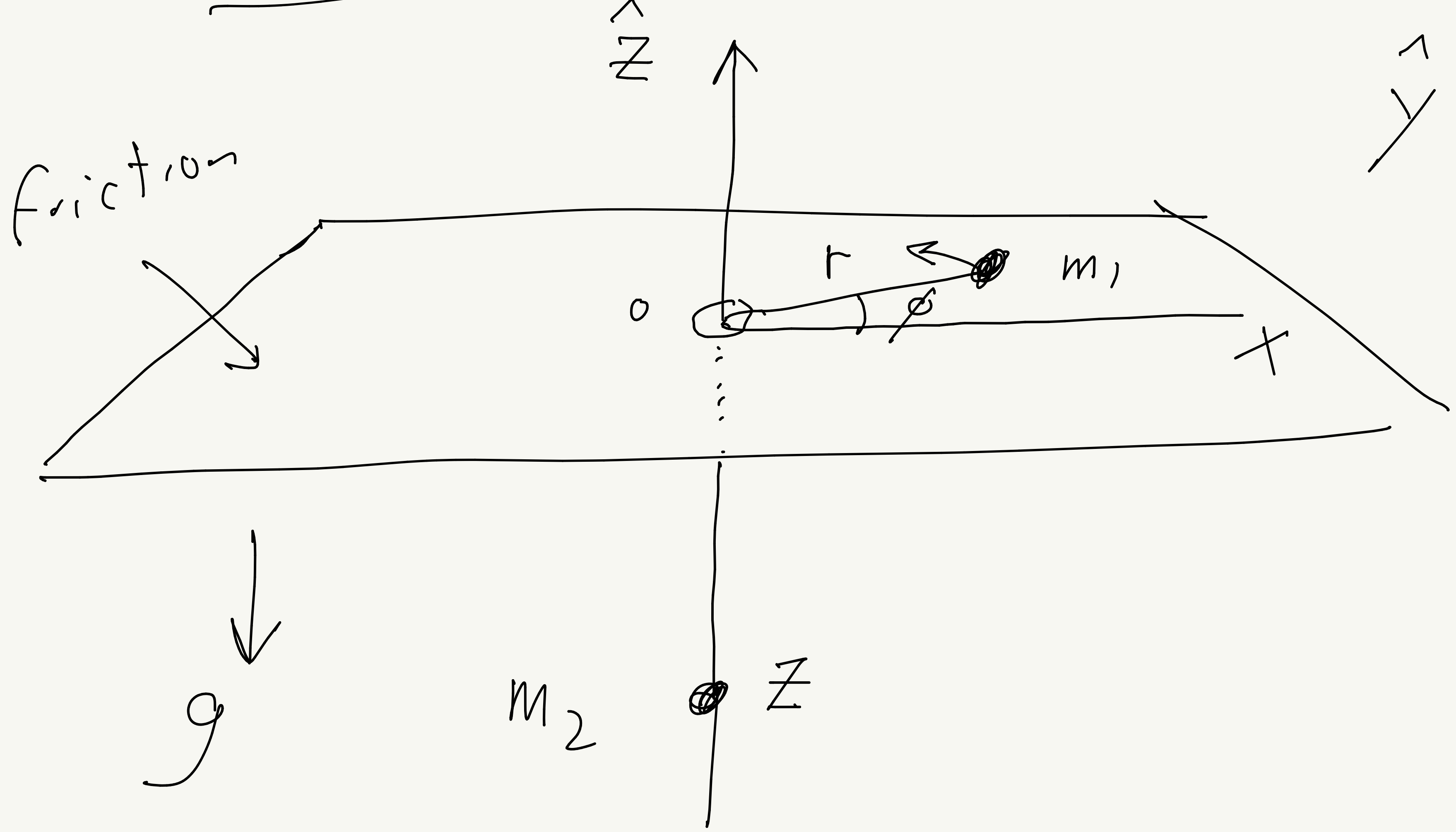
$$\lambda = mg\cos^2\alpha$$

$$\vec{F_c} = \lambda\vec{\nabla}\varphi$$
$$= \lambda\left(\underbrace{\hat{y} - \hat{x}\tan\alpha}_{\text{unit}}\right)$$

$$\tilde{F_c} = \lambda\left(\hat{y} - \hat{x}\frac{\sin\alpha}{\cos\alpha}\right)$$
$$= \frac{\lambda}{\cos\alpha}\left(\cos\alpha\,\hat{y} - \hat{x}\sin\alpha\right)$$
$$= \frac{\lambda}{\cos\alpha}\hat{n} = \boxed{mg\cos\alpha\,\hat{n}}$$

② 2-d example:

string length $= \ell$



$\hat{y}$: into page

$$x = r\cos\phi$$
$$y = r\sin\phi$$

g

$$r - Z = \ell$$

$$Z = r - \ell$$

$$T = \frac{1}{2} m_1 (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_2 \dot{Z}^2$$
$$= \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2$$
$$= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2$$

$$U = m_2 g Z = m_2 g (r - \ell) = m_2 g r \cancel{- m_2 g \ell}$$

ignore (const)

$$\boxed{\begin{aligned} L &= T - U \\ &= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r \end{aligned}}$$

(r, ϕ)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \to 0 \qquad = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = \text{const} = \boxed{m_1 r^2 \dot{\phi} = M_z}$$

No explicit time-dependence:

$$E = \text{const} = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \underbrace{T + U}$$

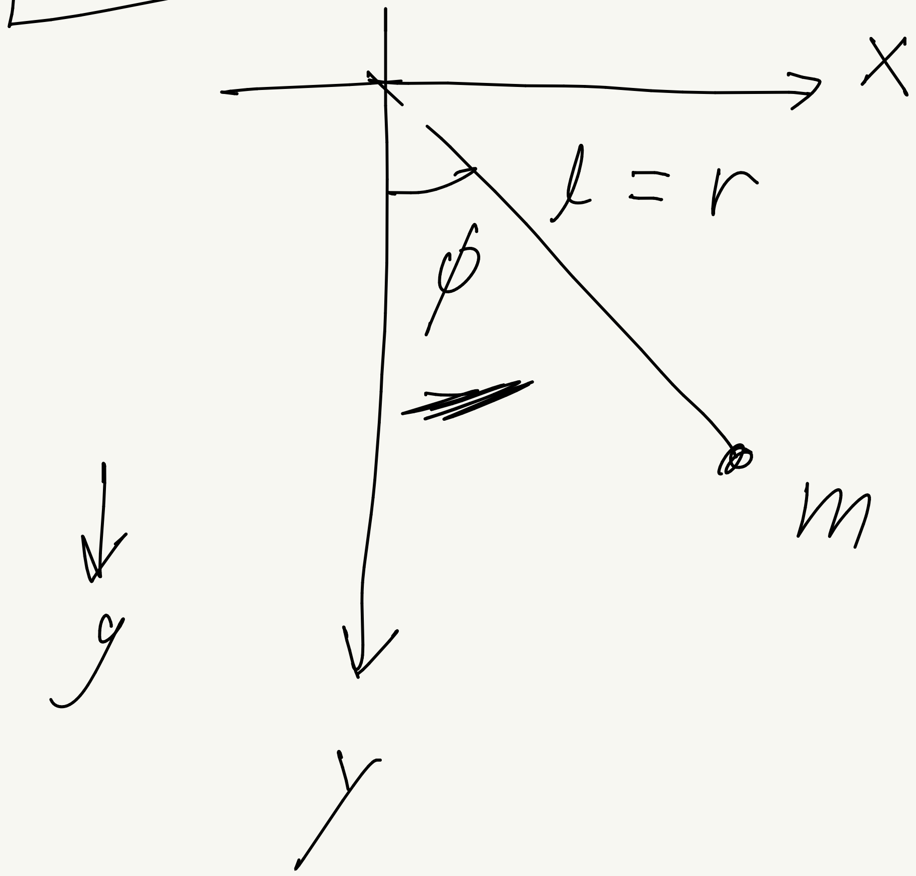
$$\left.\begin{aligned} E &= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\phi}^2 + m_2 g r \\ M_z &= m_1 r^2 \dot{\phi} \longrightarrow \dot{\phi} = \frac{M_z}{m_1 r^2} \end{aligned}\right\} \underline{\underline{\text{const.}}}$$

$$\rightarrow E = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\left(\frac{M_z}{m_1 r^2}\right)^2 + m_2 g r$$

$$= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \underbrace{\frac{M_z^2}{2m_1 r^2} + m_2 g r}_{\underline{\underline{U_{eff}(r)}}}$$

Lec #6: Sept. 9th

$\boxed{\text{QUIZ \#1}}$



Calculate tension in the string using the method of Lagrange multipliers.

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Constraint: $\varphi = l - r = 0$

$U = -mgy$
$= -mgr\cos\phi$

$$\vec{F}_c = \lambda \vec{\nabla} \varphi$$

$$L = T - U$$
$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + mgr\cos\phi$$

Eric
Niramay
Muhammad

$$(1) \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} + \lambda \frac{\partial \varphi}{\partial r}$$

$$(2) \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} + \lambda \frac{\partial \varphi}{\partial \phi}$$

$\varphi = 0$
$\rightarrow r = l \quad (3)$

$$\frac{d}{dt}(m\dot{r}) = mr\dot{\phi}^2 + mg\cos\phi - \lambda$$

$$\boxed{m\ddot{r} = mr\dot{\phi}^2 + mg\cos\phi - \lambda}$$

(1) $ml^2\ddot{\phi} = -mgl\sin\phi$

$$\frac{d}{dt}(mr^2\dot{\phi}) = -mgr\sin\phi$$

$$\boxed{2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} = -mgr\sin\phi}$$

(2) $\ddot{\phi} = \frac{-g}{l}\sin\phi$

(3)

$$\boxed{r = l} \rightarrow \boxed{\dot{r} = 0}, \boxed{\ddot{r} = 0}$$

$\varphi = l - r$

$$0 = ml\dot{\phi}^2 + mg\cos\phi - \lambda$$

$$\Longrightarrow \boxed{\lambda = ml\dot{\phi}^2 + mg\cos\phi}, \quad \boxed{\vec{\nabla}\varphi = -\hat{r}}$$

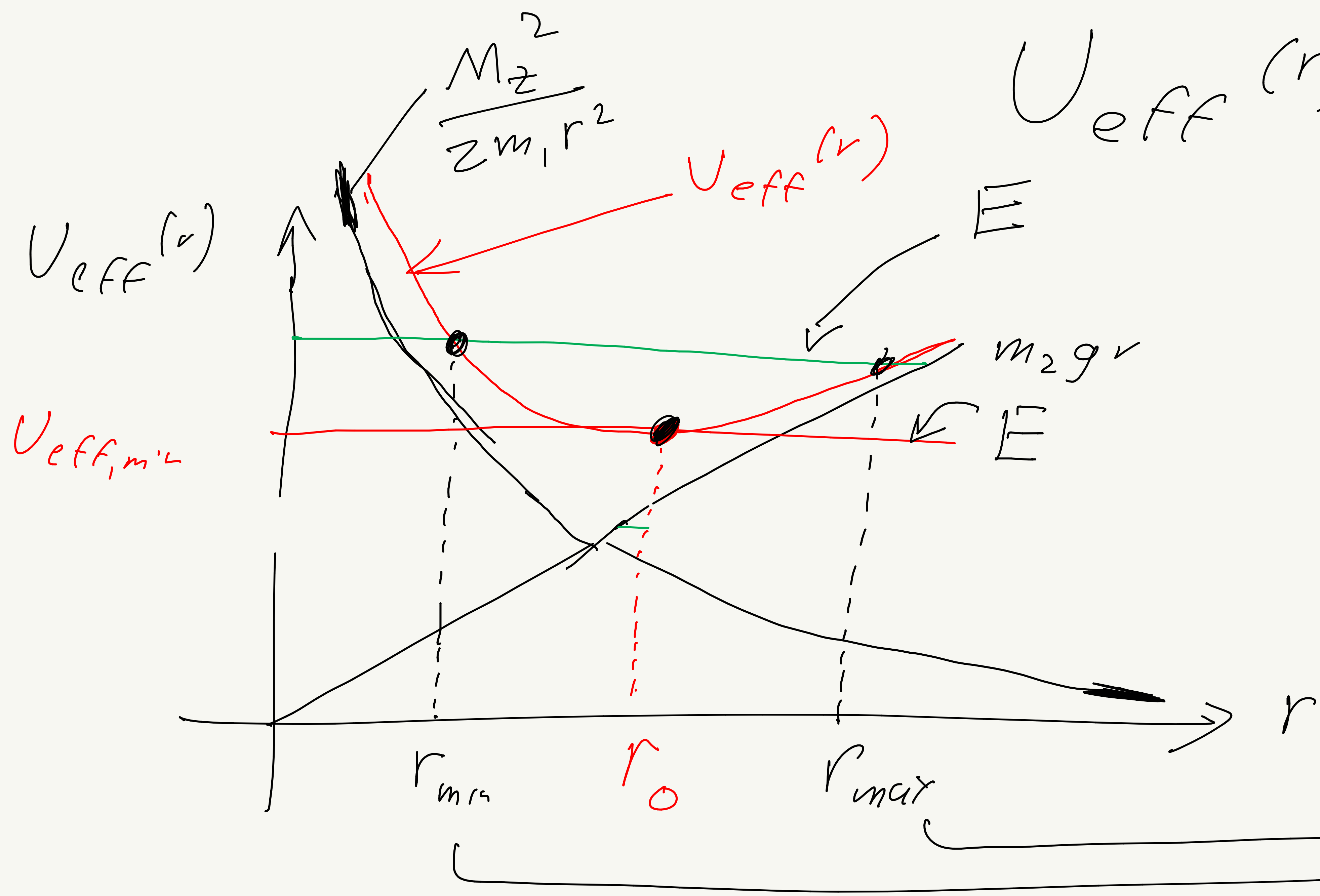
$$\boxed{\begin{aligned}&\lambda - mg\cos\phi \\ &= ml\dot{\phi}^2\end{aligned}}$$

$$\boxed{\vec{F_c} = -(ml\dot{\phi}^2 + mg\cos\phi)\hat{r}}$$

Revisit: oscillating orbit example

$$E = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \left(\frac{M_z}{m_1 r^2}\right)^2 + m_2 g r$$

$$= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \underbrace{\frac{M_z^2}{2 m_1 r^2} + m_2 g r}_{U_{eff}(r)}$$



$E \geq U_{eff,min}$

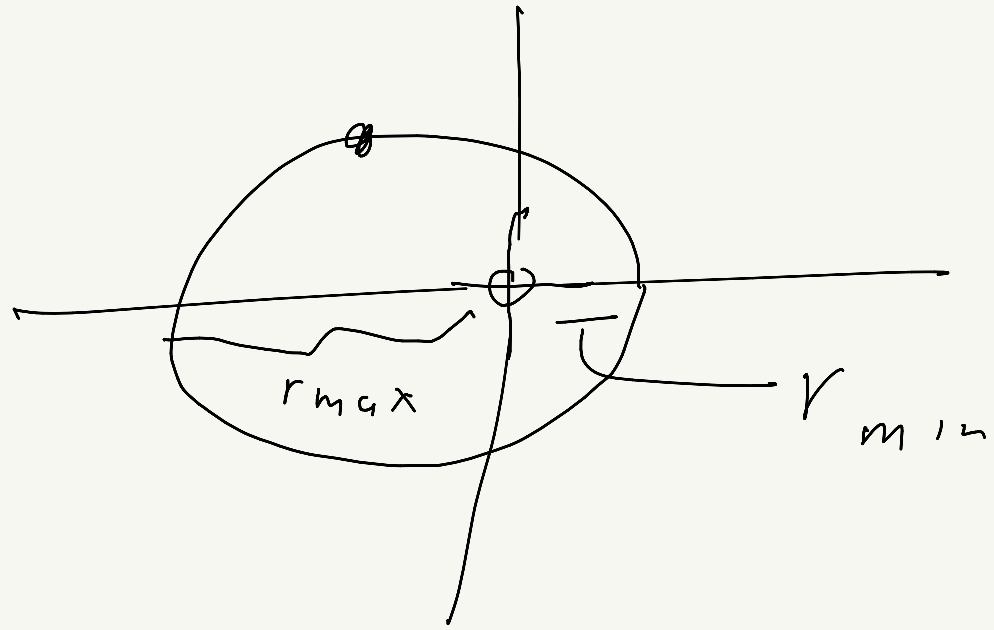
i) $E = U_{eff,min}$

$\rightarrow$ stable circular orbit $r = r_0$

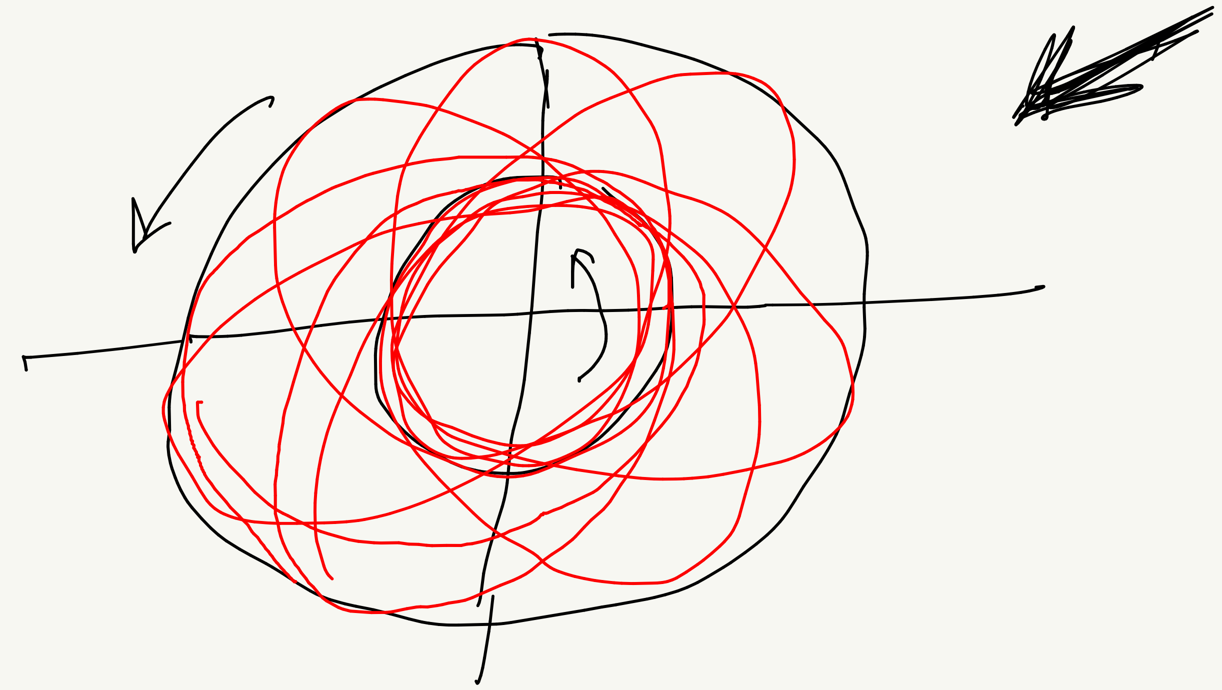
ii) $E > U_{eff,min}$

$\rightarrow$ bounded motion between r_{min} and r_{max}

turning points



closed bound orbit



bound orbit that's not closed

r_0 : minimum of the effective potential

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = \left.\left(m_2 g - \frac{M_z^2}{m_1 r^3}\right)\right|_{r=r_0}$$

$$0 = m_2 g - \frac{M_z^2}{m_1 r_0^3}$$

$$\boxed{M_z^2 = m_1 m_2 g r_0^3}$$

$$E = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + U_{eff}(r),$$

$$U_{eff}(r) = m_2 g r + \frac{M_z^2}{2 m_1 r^2}$$

$$E - U_{eff}(r) = \frac{1}{2}(m_1 + m_2)\dot{r}^2$$

$$\frac{dr}{dt} = \dot{r} = \pm\sqrt{\left(\frac{2}{m_1 + m_2}\right)\left(E - U_{eff}(r)\right)}$$

$$\boxed{\int dt = \pm\int \frac{dr}{\sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E - U_{eff}(r)\right)}}} \rightarrow t = t(r)$$

$$M_z = m_1 r^2 \dot{\phi} \longrightarrow \boxed{\dot{\phi} = \frac{M_z}{m_1 r^2}}$$

$$\frac{dr}{dt} = \pm\sqrt{\quad}, \qquad \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{M_z}{m_1 r^2}\frac{dr}{d\phi}$$

$$\frac{M_z}{m_1 r^2}\frac{dr}{d\phi} = \pm\sqrt{\oslash}$$

orbit equation

$$\int \frac{M_z\, dr}{m_1 r^2 \sqrt{\oslash}} = \int \pm\, d\phi \longrightarrow \begin{array}{l}\phi = \phi(r)\\ r = r(\phi)\end{array}$$

$$\oslash = \frac{2}{m_1 + m_2}\left(E - U_{eff}(r)\right)$$

$$\frac{dr}{dt} = \pm\sqrt{\oslash}$$

$$\frac{d\phi}{dt} = \frac{M_z}{m_1 r^2}$$

~~start~~ Choose $E, M_z, m_1, m_2,$ etc.

start system off at some values of r and ϕ at $t=0$

$$\Delta r = \pm\sqrt{\oslash}\,\Delta t$$

$$\Delta\phi = \frac{M_z}{m_1 r^2}\Delta t$$

step 1: r_1, ϕ_1

step 2: $r_2 = r_1 + \Delta r, \quad \phi_2 = \phi_1 + \Delta\phi$

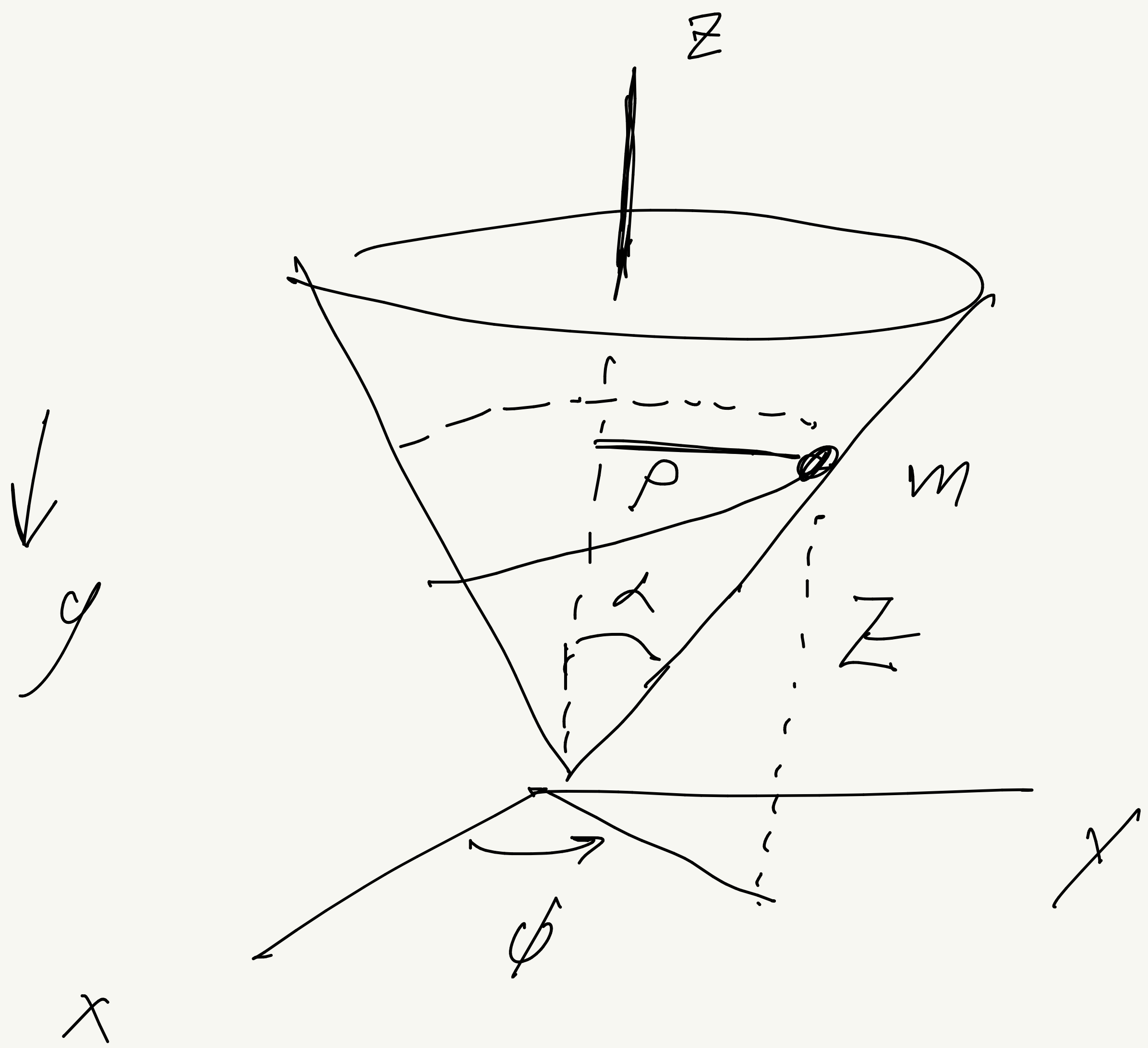
(repeat)

Lec #7: Sep 14^{th}

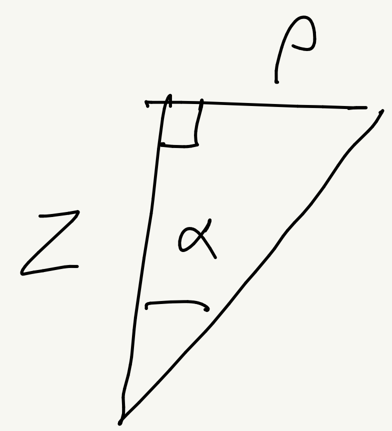
Lec #8: Sep 16^{th}

Q & A: Determine constraint force using method of Lagrange multiplier

$$\phi(\rho, \phi, z) = \rho - z\tan\alpha = 0$$



cylindrical coord. (ρ, ϕ, z)



$$\boxed{\tan\alpha = \frac{\rho}{z}} \rightarrow \rho = z\tan\alpha$$

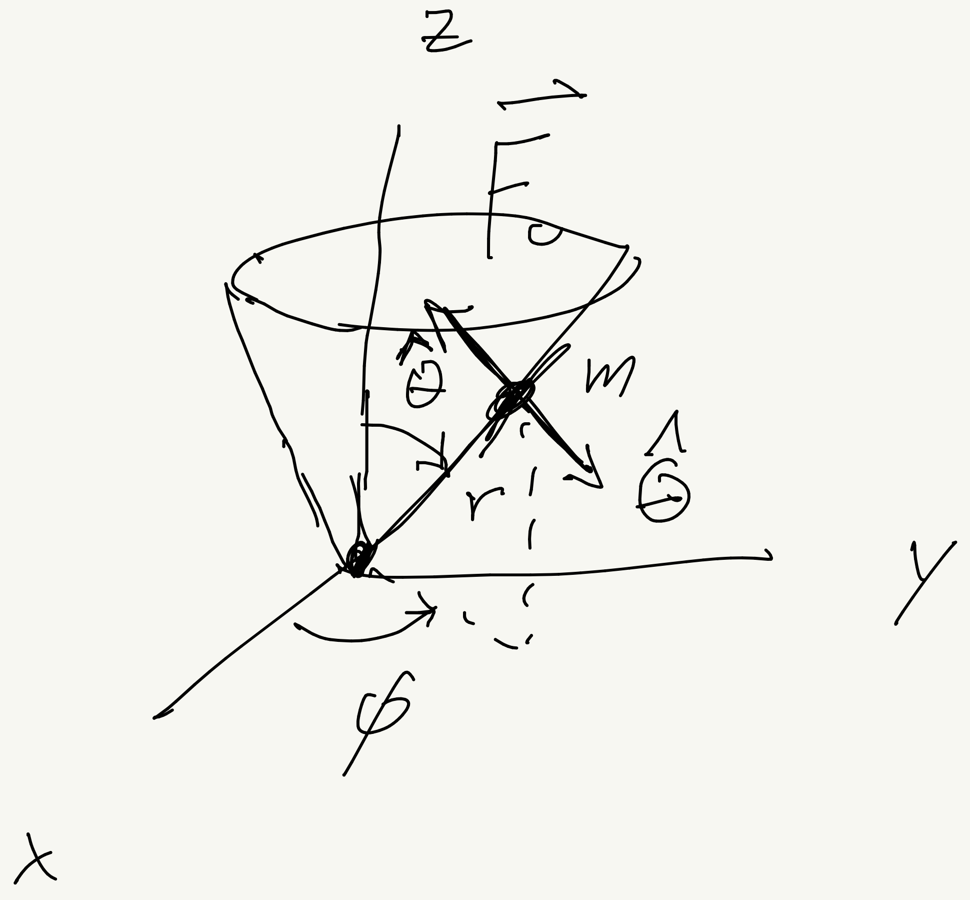
$$\dot{\rho} = \dot{z}\tan\alpha$$

$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) \leftarrow$$

$$= \frac{1}{2}m\left(\dot{z}^2\tan^2\alpha + z^2\tan^2\alpha\,\dot{\phi}^2 + \dot{z}^2\right)$$

$$= \frac{1}{2}m\left(\dot{z}^2(1+\tan^2\alpha) + z^2\tan^2\alpha\,\dot{\phi}^2\right)$$

$$U = mgz$$



$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \underbrace{\dot{\theta}^2}_{0} + r^2 \underbrace{\sin^2\theta}_{\alpha} \dot{\phi}^2 \right)$$

$$\boxed{\theta = \alpha} = \text{const}$$

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \sin^2\alpha \, \dot{\phi}^2 \right)$$

$$U = mgz = mgr\cos\alpha$$

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$\underbrace{\Phi(r,\theta,\phi) = \alpha - \theta}_{\text{constraint}} = 0$$

$$L' = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right) - mgr\cos\theta + \lambda(\alpha - \theta)$$

(r, ϕ) $\qquad \phi, \; r = l$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot r}\right) = \frac{\partial L'}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot\theta}\right) = \frac{\partial L'}{\partial \theta}$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot\phi}\right) = \frac{\partial L'}{\partial \phi}$$

$$\varphi = \alpha - \theta = 0 \rightarrow \boxed{\theta = \alpha}$$

$$L' = \underbrace{L}_{T-U} + \lambda\varphi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = \frac{\partial L}{\partial r} + \lambda \frac{\partial \varphi}{\partial r}$$

etc.

Differentiate constraint twice w/t time

$$\dot\varphi = 0 \quad , \quad \ddot\varphi = 0$$

$$\dot\theta = 0 \qquad \ddot\theta = 0$$

solve for λ (algebraic)

$$\vec{F}_c = \lambda \vec\nabla \varphi = \underline{\underline{-\lambda \hat\theta}}$$

Hamiltonian:

$$L = T - U$$

$$\underbrace{H} = \left(\sum_i p_i \dot{q}_i - L \right)$$

$$\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$$

$$\| \; E(q, \dot{q})$$

$$\left(\frac{dE}{dt} = 0 \text{ if } \frac{\partial L}{\partial t} = 0 \right)$$

$$\dot{q} = \dot{q}(q, p)$$

$$H(q, p, t)$$

$q \equiv q_i$ generalized coord, $i = 1, \dots, s$

$p \equiv p_i$ generalized momenta $i = 1, \dots, s$

\# DOF

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \quad (\text{def'n}) \quad = f(q, \dot{q})$$

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

Energy conserved ? Yes because $\frac{\partial L}{\partial t} = 0$

$$E = \frac{1}{2} m \dot{x}^2 + U(x)$$

p_x, p_y, p_z

Momentum conjugate to x ?

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \longrightarrow \dot{x} = p/m$$

$$H = (p\dot{x} - L)_{\dot{x} = p/m}$$

$$= \left(p\dot{x} - \left(\frac{1}{2} m\dot{x}^2 - U(x)\right)\right)$$

$$= p \cdot p/m - \frac{1}{2} m \left(\frac{p}{m}\right)^2 + U(x)$$

$$= p^2/2m + U(x)$$

Hamilton's equations:

$$\begin{cases} \dot{p}_i = -\dfrac{\partial H}{\partial q_i} & \underline{i = 1, \dots s} \\ \dot{q}_i = \dfrac{\partial H}{\partial p_i} & i = 1, \dots, s \end{cases}$$

3"

2s 1st-order ordinary differential equations (coupled)

Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

$i = 1, 2, \dots, s$

$L = L(q, \dot{q})$

s 2nd order ordinary diff equations (coupled)

$$L = \frac{1}{2} m\dot{x}^2 - U(x)$$

$$H = \frac{p^2}{2m} + U(x)$$

EL equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\boxed{m\ddot{x} = -\frac{\partial U}{\partial x}} \qquad (ma = F)$$

Hamilton's equations:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad (p = m\dot{x})$$

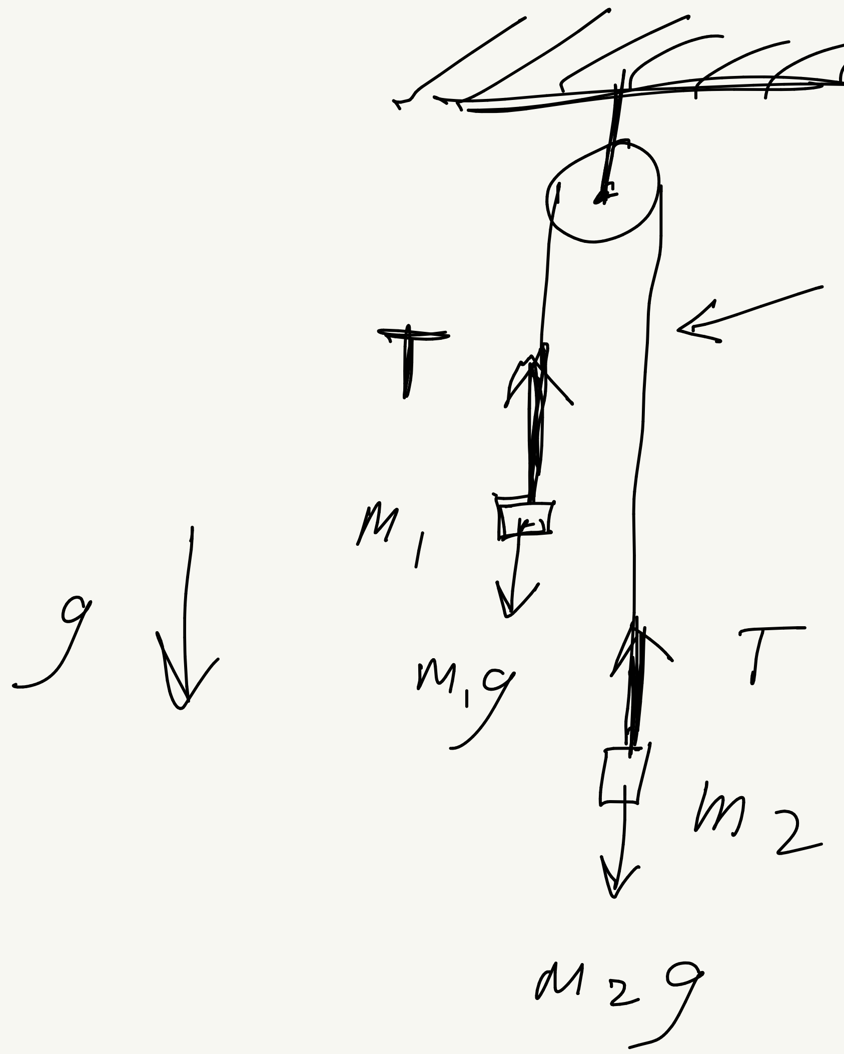
$$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$$

$$\boxed{m\ddot{x} = -\frac{\partial U}{\partial x}}$$

Atwood's machine:

$$a = \frac{g(m_1 - m_2)}{m_1 + m_2}$$

(down)



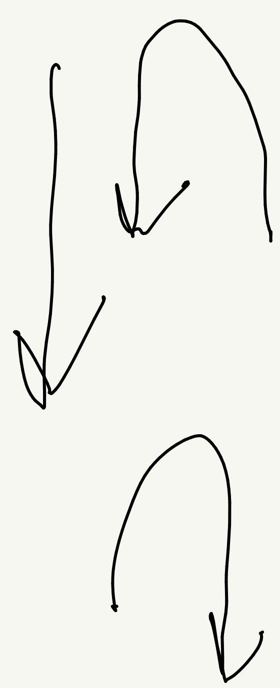
string = length l

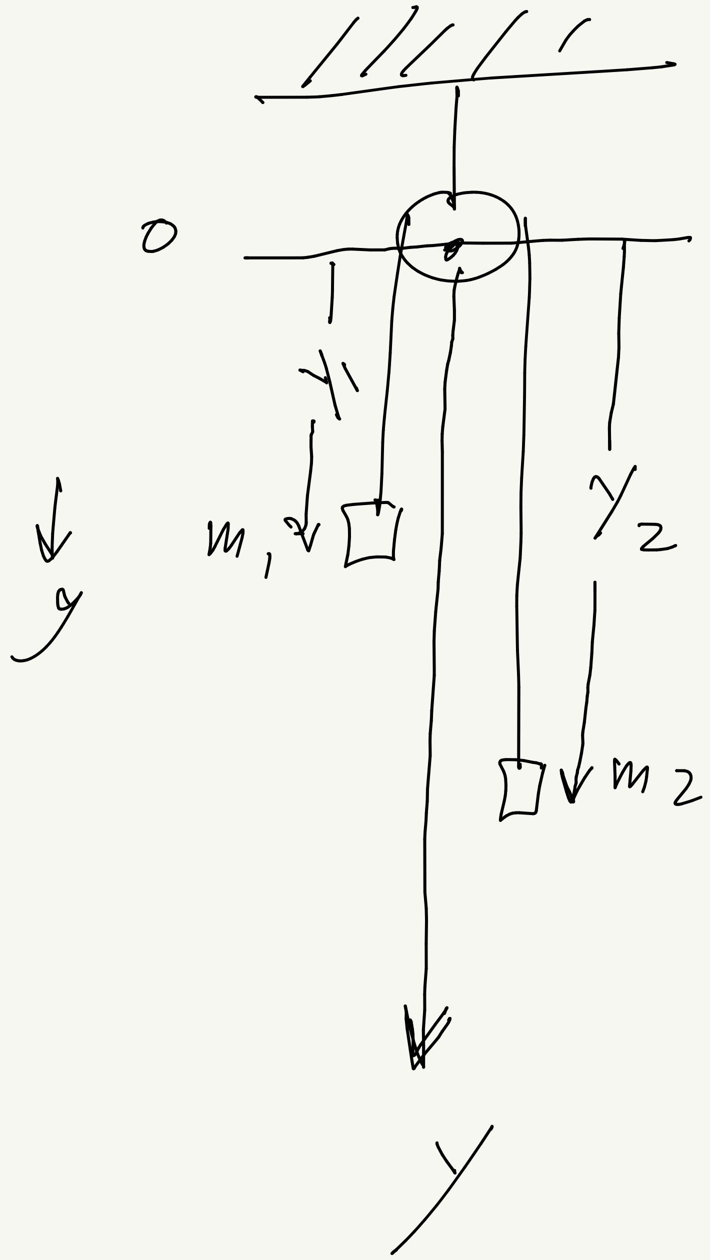
Net force down:

$$\left.\begin{aligned} m_1 g - T &= m_1 a \\ m_2 g - T &= -m_2 a \end{aligned}\right\}$$

subtract $(m_1 - m_2)g = (m_1 + m_2)a$

$$\boxed{a = \frac{(m_1 - m_2)g}{m_1 + m_2}}$$





$$L = T - U$$

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2$$

$$y_2 = l - y_1$$

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_1^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{y}_1^2$$

$$\varphi = l - (y_1 + y_2)$$

$$U = -m_1 g y_1 - m_2 g y_2$$

$$= -m_1 g y_1 - m_2 g (l - y_1)$$

$$= -(m_1 - m_2) g y_1 \underbrace{- m_2 g l}_{\text{const ignore}}$$

$$y_1 + y_2 = l$$

$$y_2 = l - y_1$$

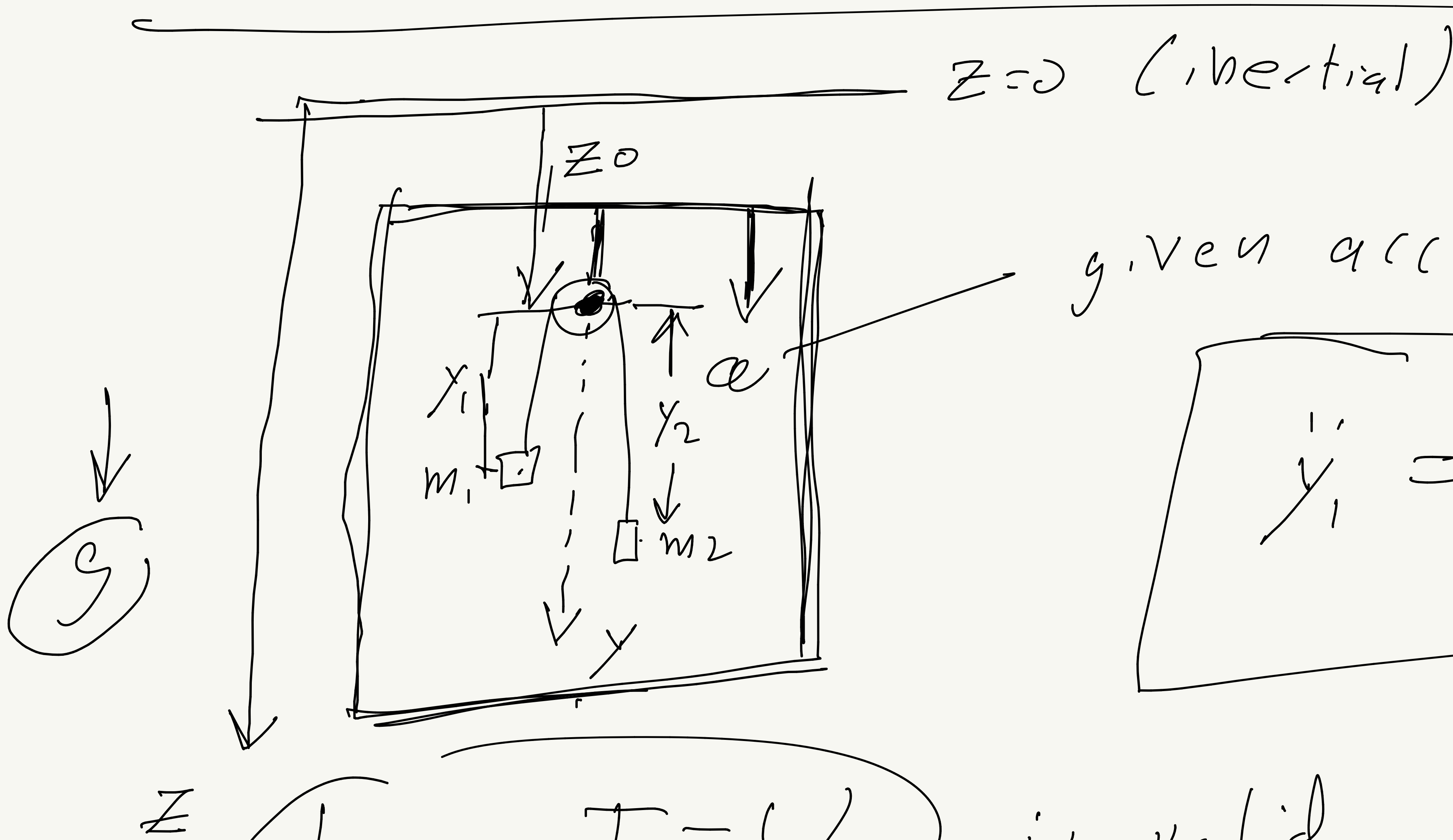
$$\dot{y}_2 = -\dot{y}_1$$

$$\boxed{L = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 + (m_1 - m_2) g y_1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) = \frac{\partial L}{\partial y_1} \rightarrow (m_1 + m_2)\ddot{y}_1 = (m_1 - m_2) g$$

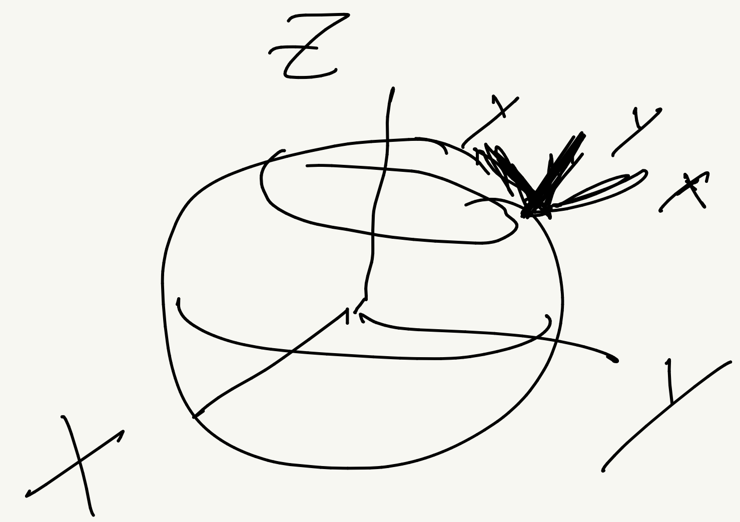
$$\boxed{\ddot{y}_1 = \frac{(m_1 - m_2) g}{m_1 + m_2}}$$

Analyze Atwood's machine in an accelerating reference frame



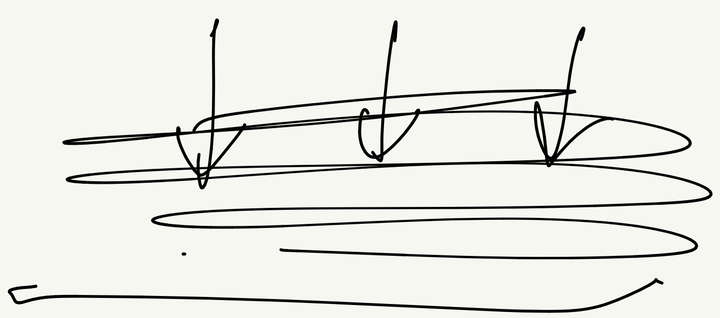
$$\boxed{Z_0 = \frac{1}{2} a t^2}$$

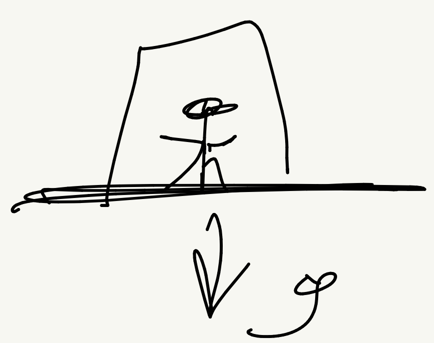
$$\boxed{\ddot{y}_1 = ??}$$

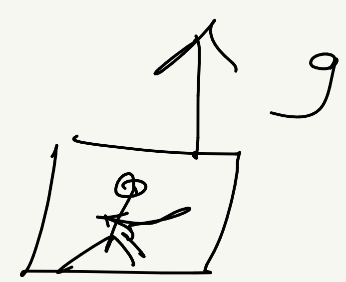


$L = T - U$ is valid only w.r.t an inertial reference frame

$\vec{F} = m\vec{a}$ (only wrt inertial frame)







Lec #9: Sep 21st

$$\vec{F} = m\vec{a}$$ (valid only in an inertial Frame)

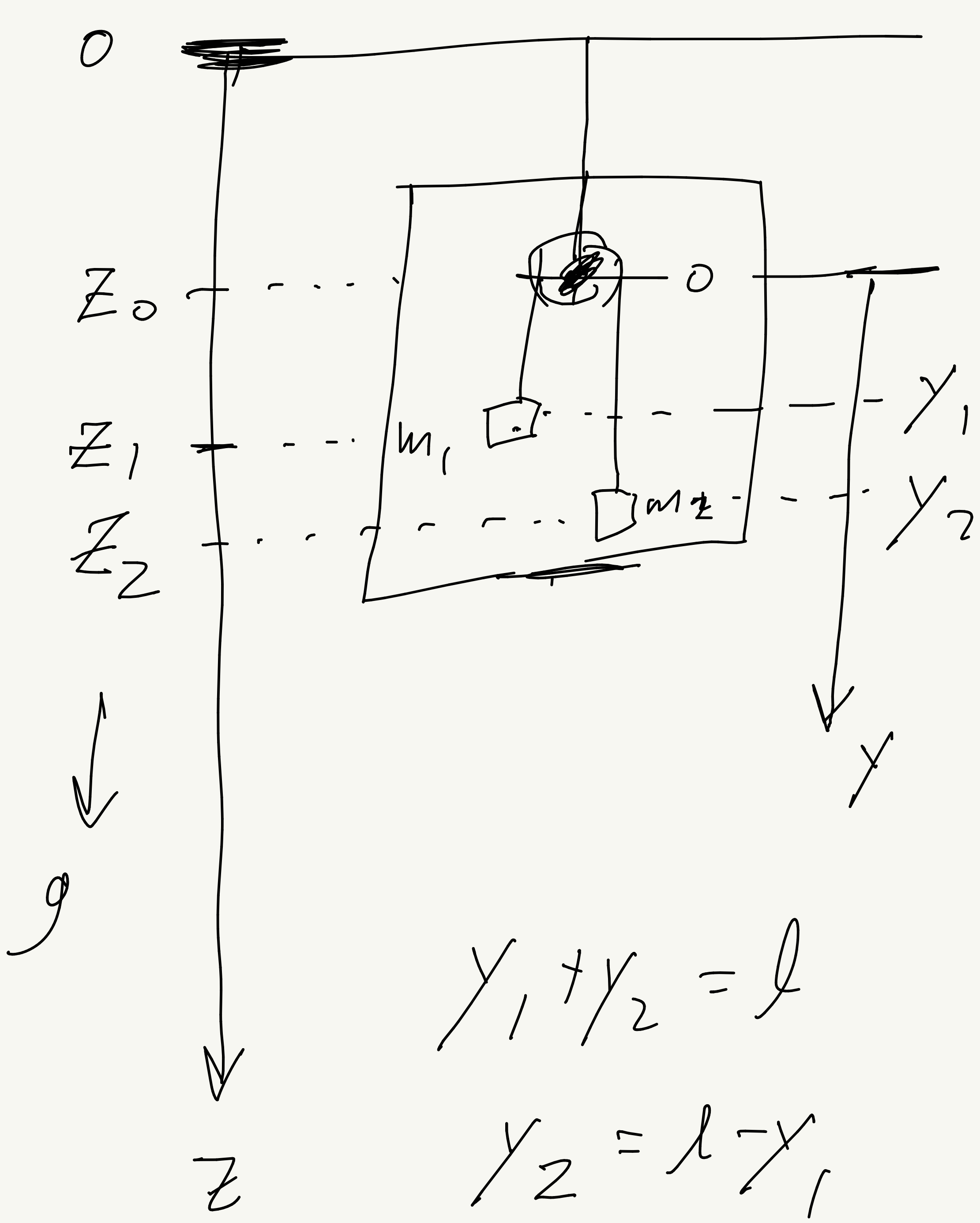
$$\vec{F} + \underbrace{\vec{F}_{fictitious}}_{} = m\vec{a}$$ wrt a non-inertial Frame

coriolis Force
+ centrifugal Force
+ linear acceleration
+ angular acceleration

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

same form of EOMs in inertial and non-inertial ref Frame

$$L = T - U$$ —— valid only in an inertial frame



$$Z_1 = Z_0 + y_1$$
$$Z_2 = Z_0 + y_2$$

$$\underbrace{Z_0 = \frac{1}{2} a t^2}_{\text{unif acceleration}}$$

$$y_1 + y_2 = l$$
$$y_2 = l - y_1$$

$$T = \frac{1}{2} m_1 \dot{z}_1^2 + \frac{1}{2} m_2 \dot{z}_2^2$$
$$= \frac{1}{2} m_1 (\dot{y}_1^2 + a^2 t^2 + 2 a t \dot{y}_1)$$
$$+ \frac{1}{2} m_2 (\dot{y}_2^2 + a^2 t^2 + 2 a t \dot{y}_2)$$

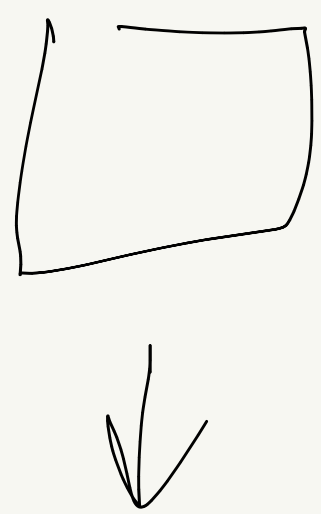
$$\dot{z}_1 = \dot{z}_0 + \dot{y}_1$$
$$= a t + \dot{y}_1$$
$$\dot{z}_2 = a t + \dot{y}_2$$

$$= \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 + \frac{1}{2}(m_1 + m_2) a^2 t^2$$
$$+ \underbrace{a t (m_1 \dot{y}_1 + m_2 \dot{y}_2)}$$

$$= \underbrace{\frac{d}{dt}\left(a t (m_1 y_1 + m_2 y_2)\right)}_{\text{ignore}} - a (m_1 y_1 + m_2 y_2)$$

(ignore) prescribed function of time

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 - a (m_1 y_1 + m_2 y_2)$$



$$L = \underbrace{T - U}_{\text{wrt inertial frame}}$$

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 - a(m_1 y_1 + m_2 y_2)$$

$$y_2 = l - y_1 \quad \rightarrow \quad \dot{y}_2 = -\dot{y}_1$$

$$\boxed{\begin{aligned} T &= \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - a(m_1 y_1 + m_2 l - m_2 y_1) \\ &= \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - a(m_1 - m_2) y_1 \end{aligned}} \underbrace{- a m_2 l}_{\text{ignore}}$$

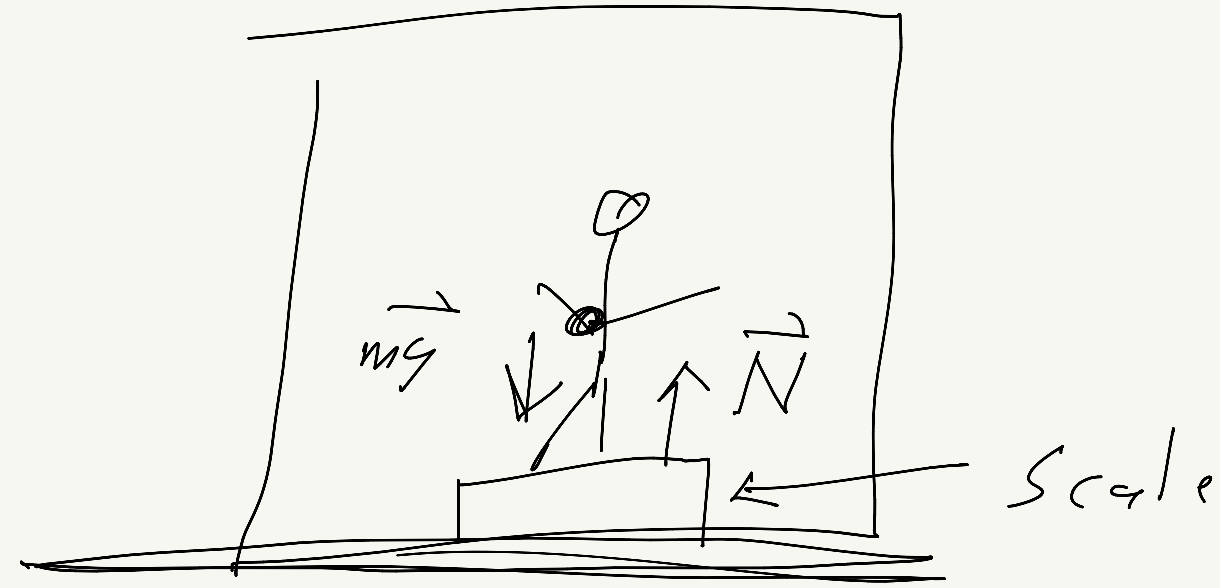
$$\boxed{\begin{aligned} U &= -m_1 g z_1 - m_2 g z_2 \\ &= -m_1 g (z_0 + y_1) - m_2 g (z_0 + y_2) \\ &= -m_1 g y_1 - m_2 g y_2 \underbrace{- (m_1 + m_2) g z_0}_{\text{ignore}} \\ &= -(m_1 - m_2) g y_1 \end{aligned}} \underbrace{- m_2 g l}_{\text{ignore}}$$

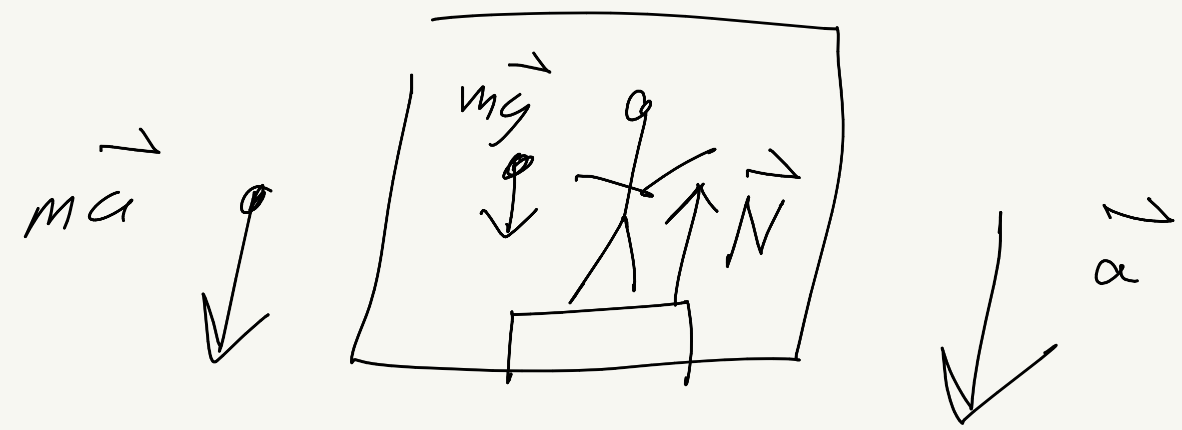
$$\boxed{\begin{aligned} L &= T - U = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - a(m_1 - m_2) y_1 + (m_1 - m_2) g y_1 \\ &= \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 + (m_1 - m_2)\underline{(g - a)} y_1 \end{aligned}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) = \frac{\partial L}{\partial y_1}$$

$$(m_1 + m_2)\ddot{y}_1 = (m_1 - m_2)(g - a)$$

$$\rightarrow \boxed{\ddot{y}_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)(g - a)}$$





$$m\vec{a} = \vec{F}$$
$$= m\vec{g} + \vec{N}$$

$$\vec{N} = -m(\vec{g} - \vec{a})$$

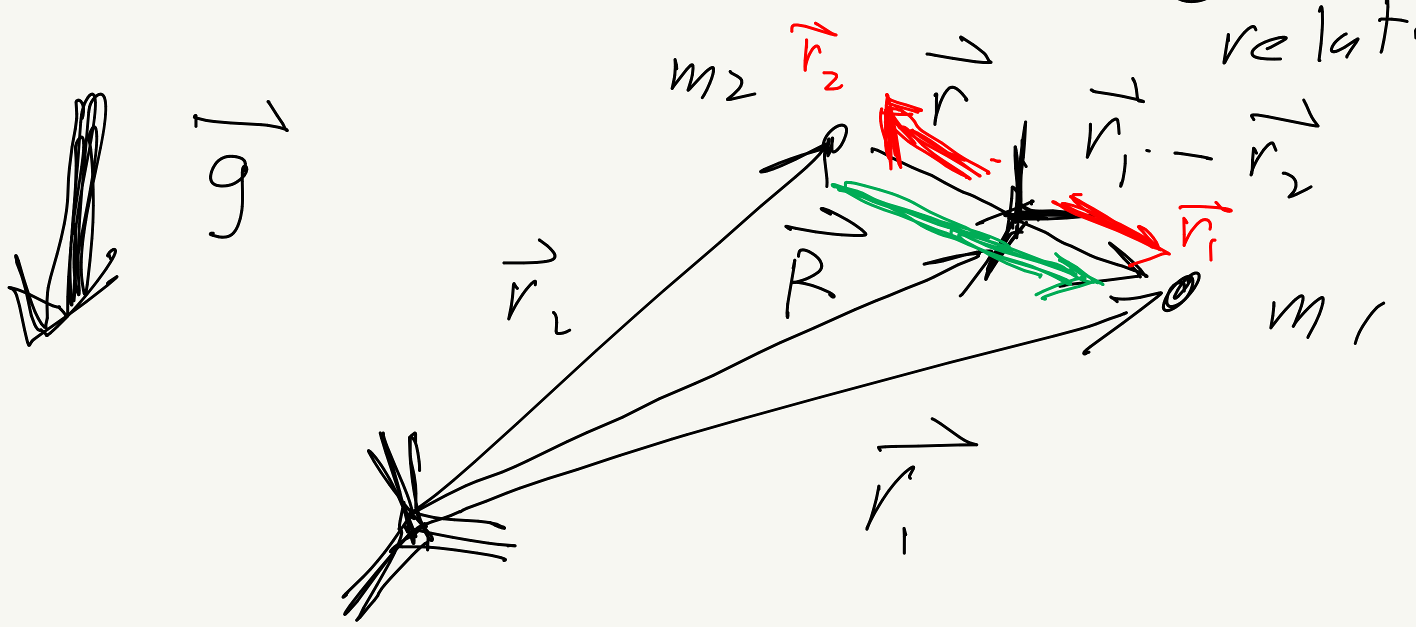
$$\vec{N} = 0 \quad \text{if } \vec{a} = \vec{g}$$

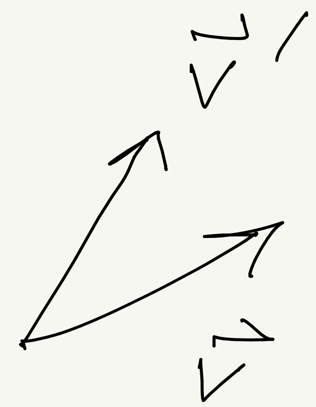
Central force motion:

Two masses, closed system,
interact via a central potential

$$U = U(\underbrace{|\vec{r}_1 - \vec{r}_2|})$$

relative separation vector $\vec{r} = \vec{r}_1 - \vec{r}_2$





e.g.-)

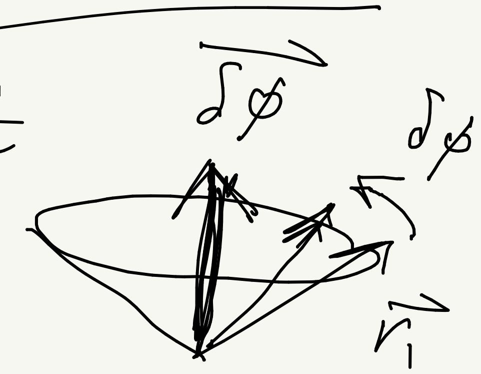
$$U = \frac{1}{2} k r^2$$

$$U = -\frac{G m_1 m_2}{r}$$

$$r = |\vec{r}_1 - \vec{r}_2|$$

spring oscillator

$t \to t + \delta t$



$$L = T - U$$

$$= \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 - U(|\vec{r}_1 - \vec{r}_2|)$$

(1) no explicit time dependence $\rightarrow$ $E = T + U = \text{const}$

(2) translational invariance $\vec{r}_a \to \vec{r}_a + \delta\vec{x}$: $\vec{P} = \sum_a m_a \vec{v}_a = \text{const}$ $\left(\begin{array}{l} \vec{r}_1 + \delta\vec{x} \\ \vec{r}_2 + \delta\vec{x} \end{array}\right)$

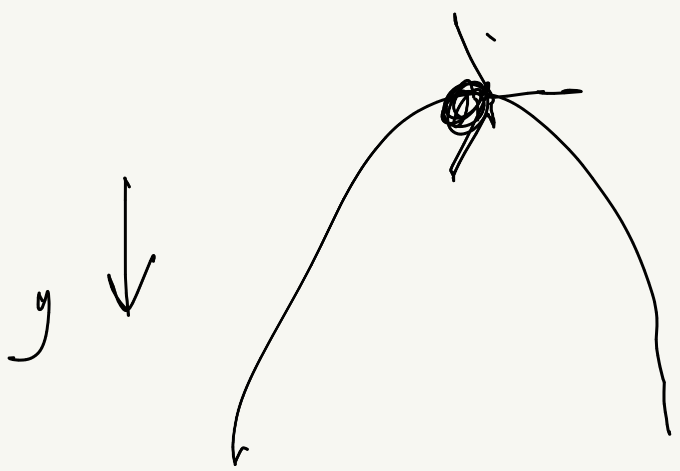
(3) rotational invariance : $\left(\begin{array}{l} \vec{r}_a \to \vec{r}_a + \vec{\delta\phi} \times \vec{r}_a \\ \vec{v}_a \to \vec{v}_a + \vec{\delta\phi} \times \vec{v}_a \end{array}\right)$ $\Big|$ $\underbrace{L + \delta L}_{L'}$

$$\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a = \text{const}$$

$\vec{P}$ = const   COM moves with const velocity

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2$$

$$= \frac{d}{dt}\left(\underbrace{\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}}_{\vec{R}}\right)\cdot(m_1+m_2)$$



$$= (m_1+m_2)\boxed{\frac{d\vec{R}}{dt}}$$

We can go to the COM Frame.

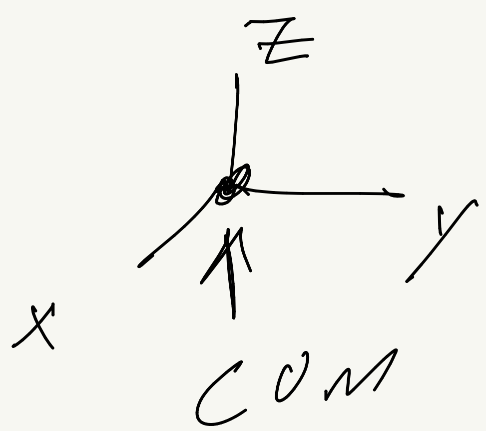
$(\vec{P} = 0$

$\frac{d\vec{R}}{dt} = 0$

$\vec{R}$ = const

$= \vec{0}$

w Origin



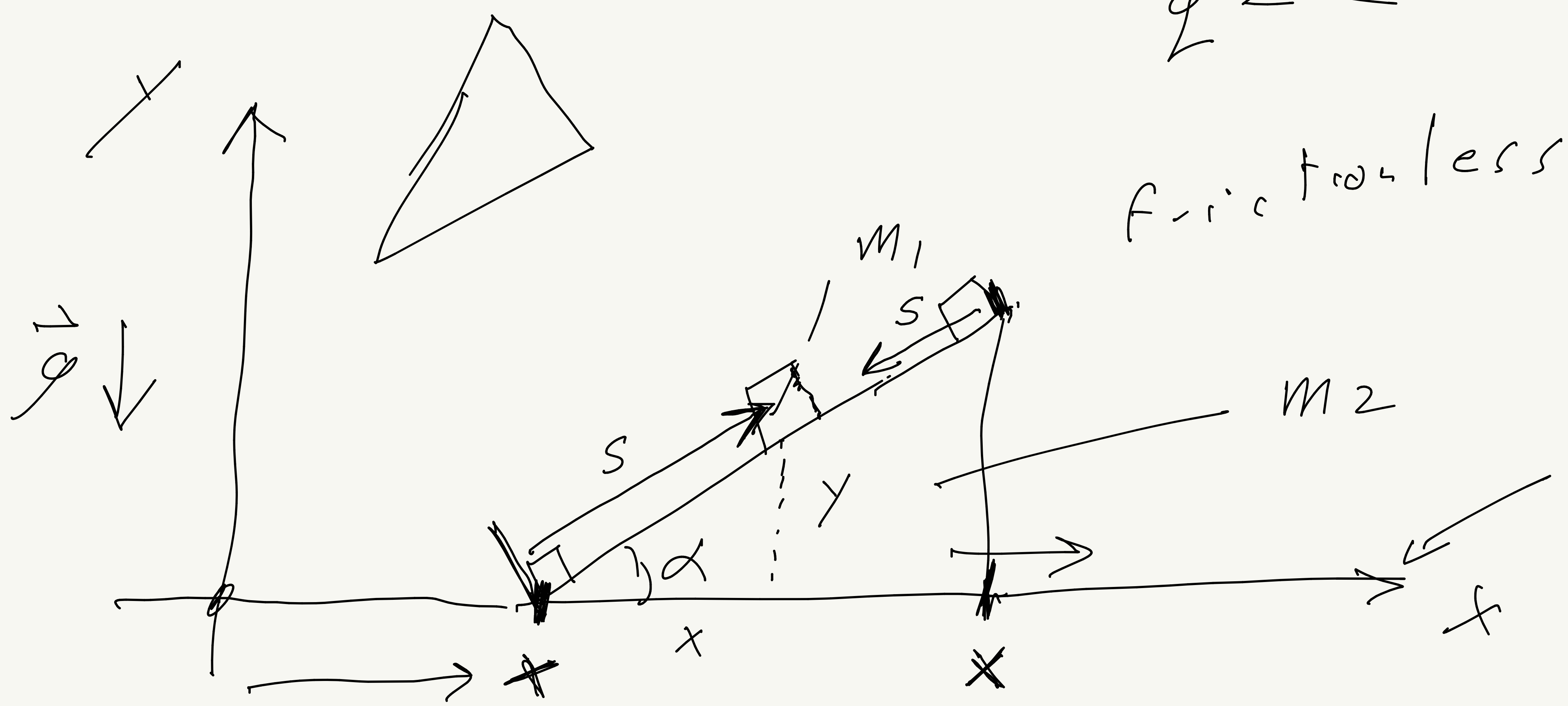
$$\boxed{m_1\vec{r}_1 + m_2\vec{r}_2 = 0}$$

$$\boxed{\vec{r} \equiv \vec{r}_1 - \vec{r}_2}$$

QUIZ #2:

joseph.d.romano@ttu.edu

q2_Firstname_lastname.pdf



$\tan\alpha = \frac{y}{x}$

1) Write down Lagrangian in terms of generalized coords

2) what quantities (if any) are conserved? why?

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\begin{cases} x_2 = X \\ y_2 = 0 \\ x_1 = X + s\cos\alpha \\ y_1 = s\sin\alpha \end{cases}$$

Lec #10: (Th 9/23)

Midterm 1: Tues Oct 5^{th}

Q2: $\dot{x}_1 = \dot{x} + \dot{s}\cos\alpha$ $\qquad\qquad$ $x_1 = x + s\cos\alpha$

$\dot{y}_1 = \dot{s}\sin\alpha$

$\dot{x}_2 = \dot{x}$

$\dot{y}_2 = 0$

$$T = \frac{1}{2}m_1\left(\left(\dot{x}^2 + \dot{s}^2\cos^2\alpha + 2\dot{x}\dot{s}\cos\alpha\right) + \dot{s}^2\sin^2\alpha\right)$$
$$+ \frac{1}{2}m_2\dot{x}^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_1\dot{s}^2 + m_1\dot{x}\dot{s}\cos\alpha$$

$$U = m_1 g y_1 = m_1 g s\sin\alpha$$

$L = T - U$, $\quad E = T + U$ conserved (no explicit t dependence)

$\frac{\partial L}{\partial x} = 0 \longrightarrow \frac{\partial L}{\partial \dot{x}} = P_x$ = conserved (no x dependence)

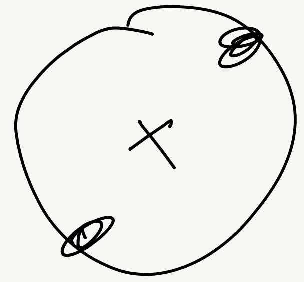
COM: $m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \qquad (1)$

$\vec{r}_1 - \vec{r}_2 = \vec{r} \qquad (2) \longrightarrow \qquad \vec{r}_2 = \vec{r}_1 - \vec{r}$

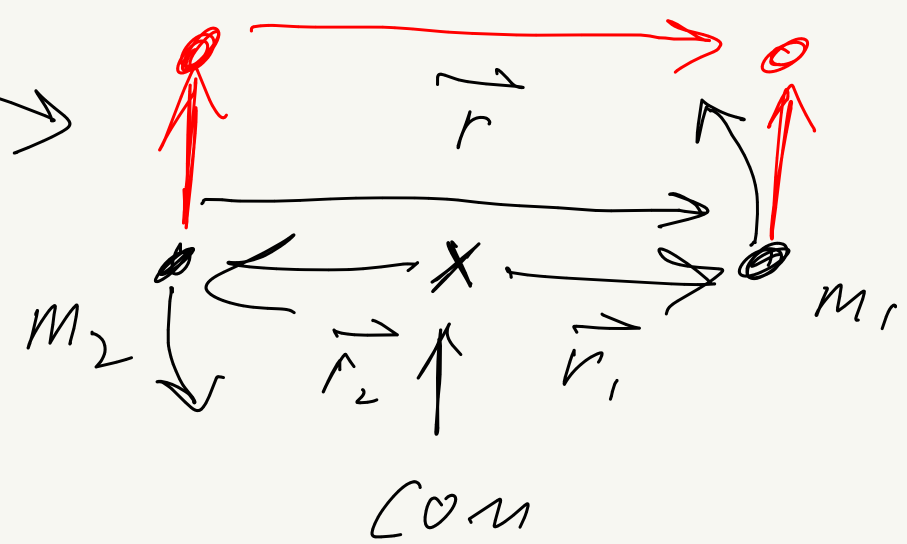
$$m_1\vec{r}_1 + m_2(\vec{r}_1 - \vec{r}) = 0$$

$$(m_1+m_2)\vec{r}_1 - m_2\vec{r} = 0$$

$\vec{r} = \vec{r}_1 - \vec{r}_2$



$\dot{\vec{r}} \neq 0$ for rotational motion $\longrightarrow$



$$\boxed{\begin{aligned}\vec{r}_1 &= \left(\frac{m_2}{m_1+m_2}\right)\vec{r} \\ \vec{r}_2 &= -\left(\frac{m_1}{m_1+m_2}\right)\vec{r}\end{aligned}}$$

in COM frame

$$\dot{\vec{r}}_1 = \left(\frac{m_2}{m_1+m_2}\right)\dot{\vec{r}}, \qquad \dot{\vec{r}}_2 = -\left(\frac{m_1}{m_1+m_2}\right)\dot{\vec{r}}$$

$$T = \frac{1}{2}m_1|\dot{\vec{r}}_1|^2 + \frac{1}{2}m_2|\dot{\vec{r}}_2|^2 + \ldots$$

$$= \frac{1}{2}m|\dot{\vec{r}}|^2 + \cancel{\frac{1}{2}(m_1+m_2)|\vec{V}|^2}$$

$= 0$ in com frame

$$\boxed{m \equiv \frac{m_1 m_2}{m_1+m_2}}$$

define

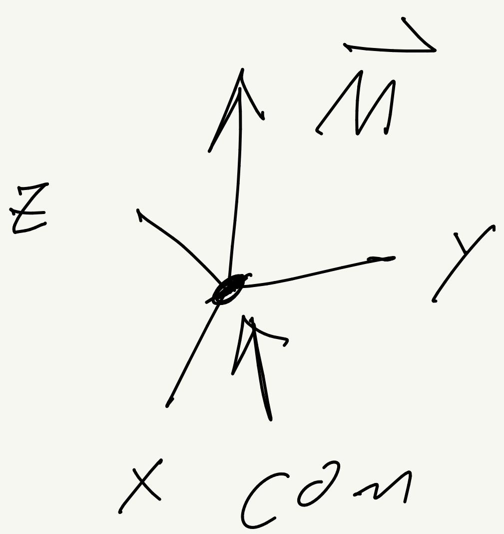
reduced mass of system

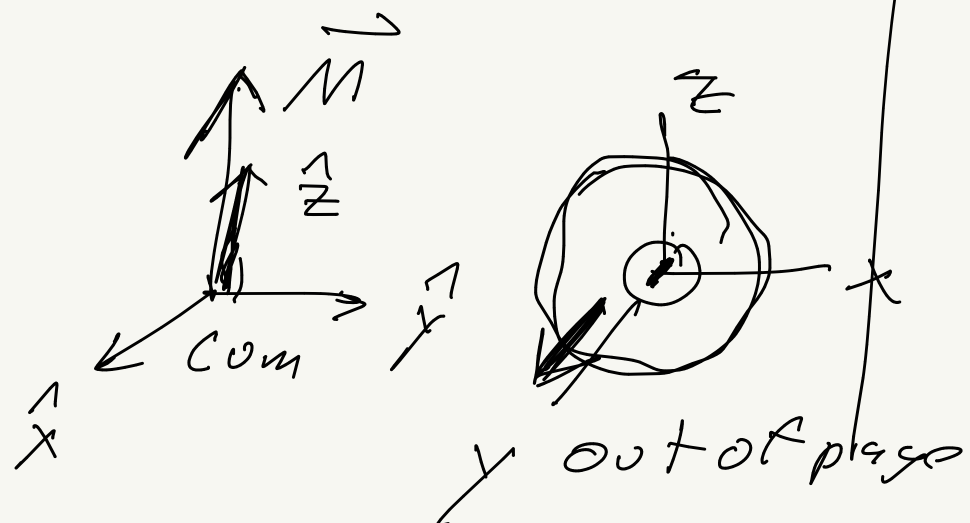
$U = U(|\vec r_1 - \vec r_2|) = U(|\vec r|) = U(r)$

$$\boxed{L = T - U = \frac{1}{2} m |\dot{\vec r}|^2 - U(r)}$$

(effective one-body problem)

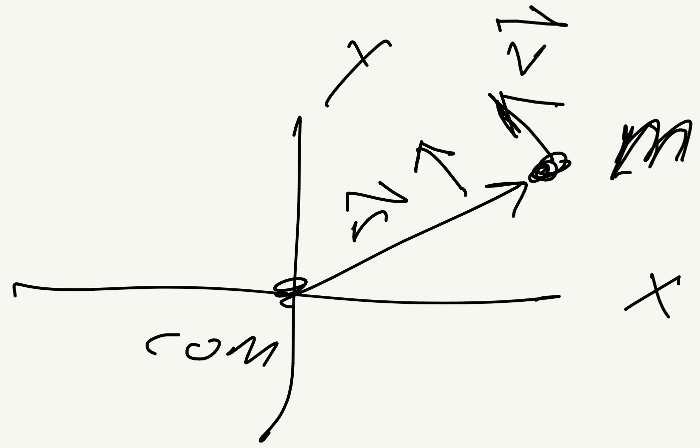
ii) cons. of angular momentum





$$\begin{aligned}\vec M &= \text{const} \\ &= m_1 \vec r_1 \times \vec v_1 + m_2 \vec r_2 \times \vec v_2 \\ &\;\;\vdots \\ &= m \vec r \times \vec v \\ &= m \vec r \times \dot{\vec r}\end{aligned}$$

$\vec r, \vec v$ must lie in the xy plane



$\vec r = \vec r_1 - \vec r_2$

$$L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(r)$$

$$= \frac{1}{2} m (\underbrace{\dot{r}^2 + r^2\dot{\phi}^2}_{\dot{x}^2+\dot{y}^2}) - U(r)$$

using plane polar coordinates (r, ϕ)

$$\frac{\partial L}{\partial \phi} = 0 \rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{const} = m r^2 \dot{\phi}$$

$$\frac{\partial L}{\partial \dot{\phi}} \equiv p_\phi = M = M_z$$

$$\boxed{M_z = m r^2 \dot{\phi}} = \text{const}$$

not total mass

magnitude of angular momentum vector $\vec{M}$.

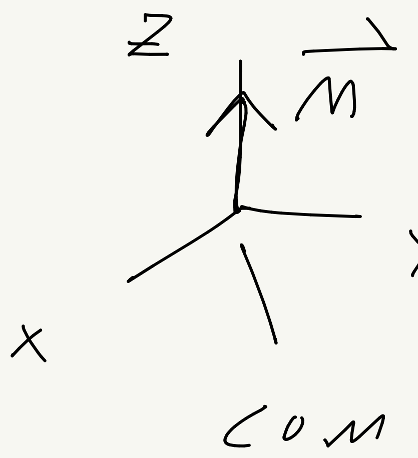
$$\frac{d\vec{M}}{dt} = 0$$

i) cons. of linear momentum $\rightarrow$ COM Frame

$$L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(r) \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

ii) cons. of angular momentum $\rightarrow$ chose $\hat{z}$-axis to point along $\vec{M}$



$$\vec{M} = \vec{r} \times \vec{p}$$
$$= m \vec{r} \times \dot{\vec{r}}$$

$\rightarrow$ motion $(\vec{r}, \dot{\vec{r}})$ lies in the x-y plane

Using plane-polar coord (r, ϕ):

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r)$$

$$\frac{\partial L}{\partial \phi} = 0 \rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{const} = \boxed{m r^2 \dot{\phi} = M_z}$$

$$\boxed{\dot{\phi} = \frac{M_z}{m r^2}}$$

$$\frac{\partial L}{\partial \dot{\phi}} = p_\phi = M_z = M$$

iii) cons of energy $\quad E = T + U = \text{const}$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 + U(r) = \frac{1}{2} m \dot{r}^2 + \underbrace{\left(\frac{M_z^2}{2 m r^2} + U(r) \right)}_{U_{eff}(r)}$$

$$\boxed{E = \frac{1}{2} m \dot{r}^2 + U_{eff}(r)}$$

where $\underbrace{U_{eff}(r) = U(r) + \frac{M_z^2}{2mr^2}}$

$$E - U_{eff}(r) = \frac{1}{2} m \dot{r}^2$$

$$\pm\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)} = \dot{r} = \frac{dr}{dt} \longleftarrow \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt}$$

$$\pm\sqrt{} = \frac{dr}{d\phi}\frac{M_z}{mr^2} \rightarrow \text{next page}$$

$$\int dt = \pm\int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}}$$

$$\boxed{t = \pm\int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U(r)\right) - \frac{M_z^2}{m^2 r^2}}} + \text{const}}$$

(14.6) L&L

$\rightarrow t = t(r)$

$r = r(t)$

$$\frac{d\phi}{dt} = \frac{M_z}{mr^2} \rightarrow \int d\phi = \int \frac{M_z}{mr^2(t)}\, dt$$

$$\rightarrow \phi = \int \frac{M_z}{mr^2(t)}\, dt + \text{const}$$

$$\int d\phi = \int \frac{\pm M_z \, dr/r^2}{m\sqrt{\frac{2}{m}(E-U(r)) - \frac{M_z^2}{m^2 r^2}}}$$

(14.7) L&L

$$\boxed{\phi = \pm M_z \int \frac{dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M_z^2}{r^2}}} + \text{const}}$$

$$\phi = \phi(r) \longleftrightarrow r = r(\phi)$$

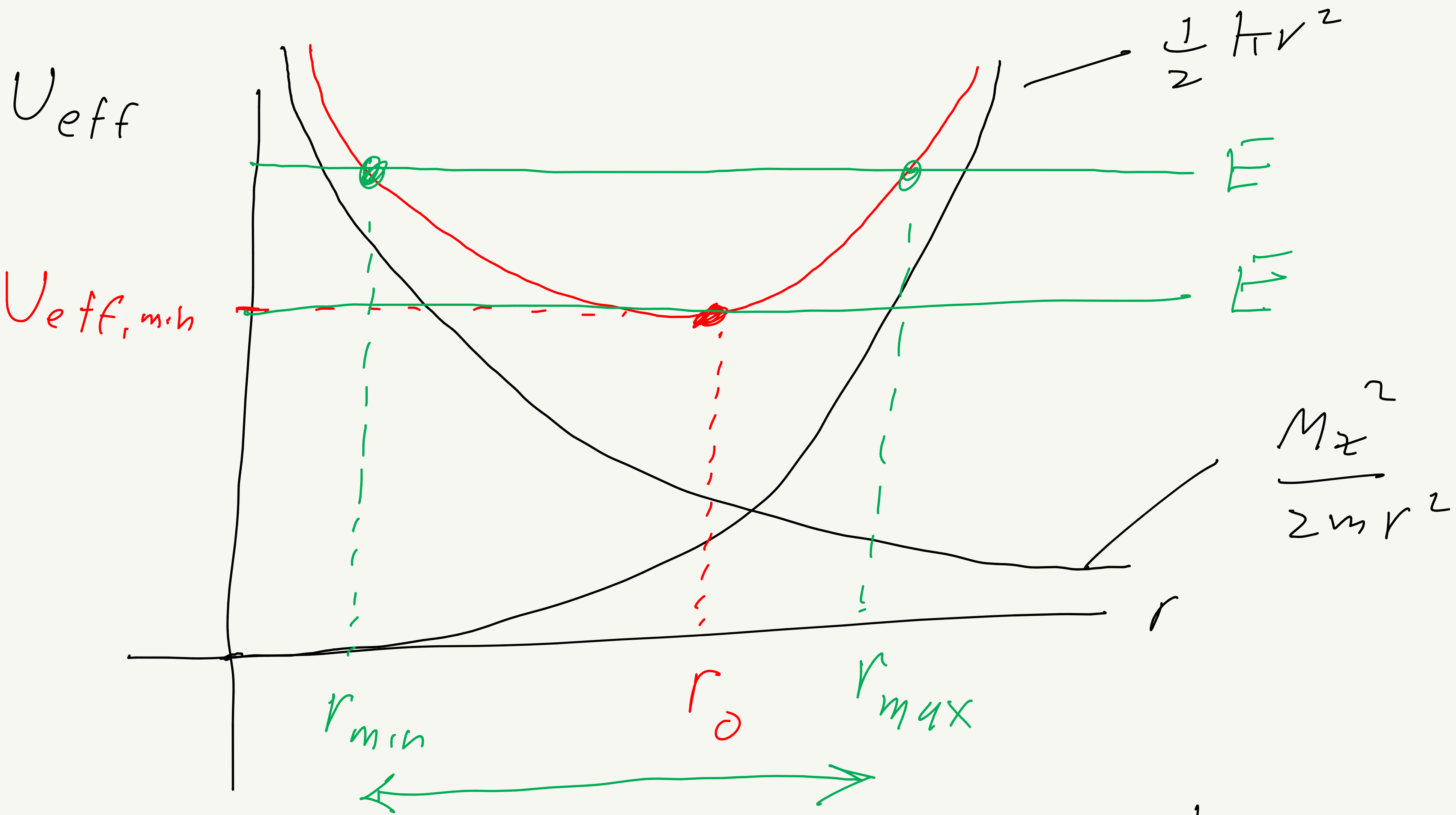
☆☆ $U(r) = \frac{1}{2} k r^2$ (space oscillator)

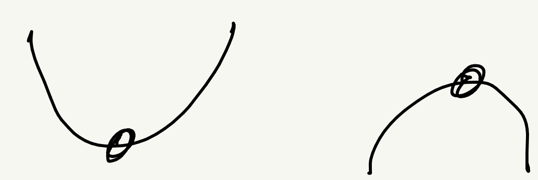
$U(r) = -\frac{\alpha}{r} = \frac{-G m_1 m_2}{r}$ (Newtonian gravity)

$$U_{eff}(r) = U(r) + \frac{M_z^2}{2mr^2}$$

$$= \frac{1}{2}kr^2 + \frac{M_z^2}{2mr^2}$$

$$\left.\frac{dU_{eff}}{dr}\right|_{r_0} = 0$$





$E = U_{eff,min} \longrightarrow$ stable circular orbit with $r = r_0$

$E > U_{eff,min} \longrightarrow$ bounded motion between r_{min} and r_{max} ($r_{min} \leq r \leq r_{max}$)

$$\phi = M_z \int \frac{dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M_z^2}{r^2}}} + \text{const}$$

$$= M_z \int \frac{dr/r^2}{\sqrt{2m(E - \frac{1}{2}kr^2) - \frac{M_z^2}{r^2}}} + \text{const}$$

$$= M_z \int \frac{dr/r^2}{\sqrt{2mE - mkr^2 - M_z^2/r^2}} + \text{const}$$

$$\sqrt{} = \sqrt{a + bx + cx^2}$$

e.g. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + \text{const}$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$\sqrt{} = \sqrt{1 - \sin^2\theta}$

$= \cos\theta$

$\sin^{-1} x = \theta = \int d\theta$

$$\boxed{u = \frac{1}{r}} \rightarrow du = -\frac{1}{r^2}\,dr \rightarrow dr/r^2 = -du$$

$$\sqrt{} = \sqrt{2mE - \frac{mk}{u^2} - M_z^2 u^2}$$

$$= \frac{1}{u}\sqrt{2mEu^2 - mk - M_z^2 u^4}$$

$$\phi = -M_z \int \frac{u\,du}{\sqrt{2mEu^2 - mk - M_z^2 u^4}} + \text{const}$$

$$\boxed{v = u^2} \rightarrow dv = 2u\,du \rightarrow u\,du = \frac{dv}{2}$$

$$\sqrt{} = \sqrt{2mEv - mk - M_z^2 v^2}$$

$$\boxed{\phi = -\frac{M_z}{2}\int \frac{dv}{\sqrt{2mEv - mk - M_z^2 v^2}}} \boxed{+\text{const}}$$

Complete the square:

$$-M_z^2 v^2 + 2mEv - mK = -M_z^2\left(v^2 - \frac{2mE}{M_z^2}v + \frac{mK}{M_z^2}\right)$$

$$= -M_z^2\left(\left(v - \frac{mE}{M_z^2}\right)^2 - \frac{m^2E^2}{M_z^4} + \frac{mK}{M_z^2}\right)$$

$$= -M_z^2\left((v-A)^2 - B^2\right)$$

$$= M_z^2\left(B^2 - (v-A)^2\right)$$

$$A = \frac{mE}{M_z^2}$$

$$B^2 \equiv A^2 - \frac{mK}{M_z^2}$$

$E \geq U_{eff,min}$

$$\phi = -\frac{\cancel{M_z}}{2}\int \frac{dv}{\sqrt{\cancel{M_z}^2\left(B^2-(v-A)^2\right)}} + \text{const}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x$$

Let: $v - A = B\sin\theta$

$dv = B\cos\theta\, d\theta$

$$B^2 - (v-A)^2 = B^2 - B^2\sin^2\theta$$

$$= B^2\cos^2\theta$$

$$\sqrt{} = B\cos\theta$$

$$\phi = -\frac{1}{2}\int d\theta + \text{const}$$

$$= -\frac{1}{2}\theta + \text{const}$$

$$= -\frac{1}{2}\sin^{-1}\left(\frac{v-A}{B}\right) + \text{const}$$

$$= -\frac{1}{2}\sin^{-1}\left(\frac{1/r^2 - A}{B}\right) + \text{const}$$

$$\boxed{v - A = \pm B \rightarrow \sin\theta = \pm 1}$$

$-\pi/2$

$v - A = B\sin\theta$

$$\sin\theta = \frac{v-A}{B}$$

$$\theta = \sin^{-1}\left(\frac{v-A}{B}\right)$$

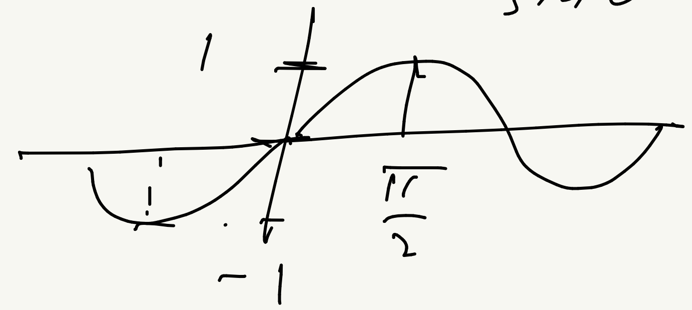
$v = u^2 = 1/r^2$

$u = \frac{1}{r}$

choose const so that $\phi = 0 \longleftrightarrow r = r_{max}$

$$0 = -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r_{max}^2} - A}{B}\right) + \text{const}$$

$\sin\theta$



$$\frac{1}{r_{max}^2} - A = B$$

$$v - A = \pm B$$

$$\frac{1}{r_{max}^2} - A = \pm B$$

$$\frac{1}{r_{max}^2} = \underline{A \pm B} = A - B$$

$$\frac{1}{r_{max}^2} - A = -B$$

$$0 = -\frac{1}{2}\sin^{-1}(-1) + \text{const}$$

$$= -\frac{1}{2}\left(-\frac{\pi}{2}\right) + \text{const}$$

$$= \frac{\pi}{4} + \text{const}$$

$$\text{const} = -\pi/4$$

$$\phi = -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2} - A}{B}\right) - \frac{\pi}{4}$$

$$\phi + \frac{\pi}{4} = -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2} - A}{B}\right)$$

$$-2\left(\phi + \frac{\pi}{4}\right) = \sin^{-1}\left(\frac{\frac{1}{r^2} - A}{B}\right)$$

$$-\sin\left(2\left(\phi + \frac{\pi}{4}\right)\right) = \frac{\frac{1}{r^2} - A}{B}$$

$\sin(-x)$
$= -\sin x$

$$LHS = -\sin\left(2\phi + \pi/2\right)$$

$$= -\left(\sin(2\phi)\underbrace{\cancel{\cos(\pi/2)}}_{\to 0} + \cos(2\phi)\underbrace{\sin(\pi/2)}_{=1}\right)$$

$$= -\cos 2\phi$$

$$-\cos 2\phi = \frac{\frac{1}{r^2} - A}{B}$$

$$-B\cos 2\phi = \frac{1}{r^2} - A$$

$$\boxed{\frac{1}{r^2} = A - B\cos 2\phi}$$

ellipse with center of ellipse at origin

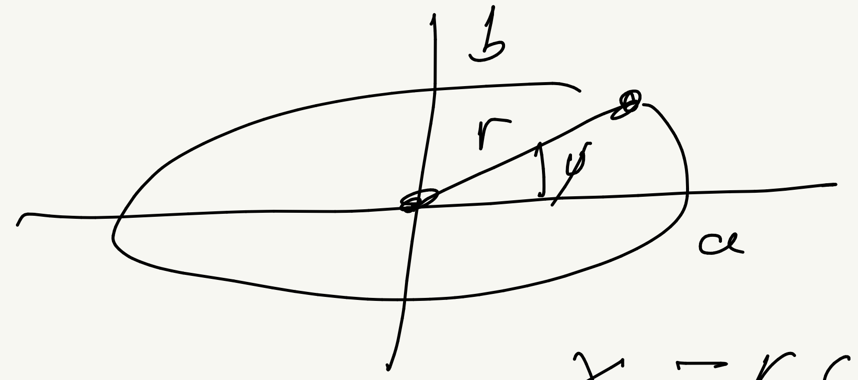
$$1 = Ar^2 - Br^2\underbrace{\cos 2\phi}_{\cos^2\phi - \sin^2\phi}$$

$$= Ar^2 - Br^2(\cos^2\phi - \sin^2\phi)$$

$$= A(x^2+y^2) - B(x^2-y^2)$$

$$= (A-B)x^2 + (A+B)y^2$$

$$= \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$x = r\cos\phi$$
$$y = r\sin\phi$$

$$x^2 + y^2 = r^2$$

$$\frac{1}{a^2} = A - B$$

$$\frac{1}{b^2} = A + B$$

$$a = \frac{1}{\sqrt{A-B}}, \quad b = \frac{1}{\sqrt{A+B}}$$