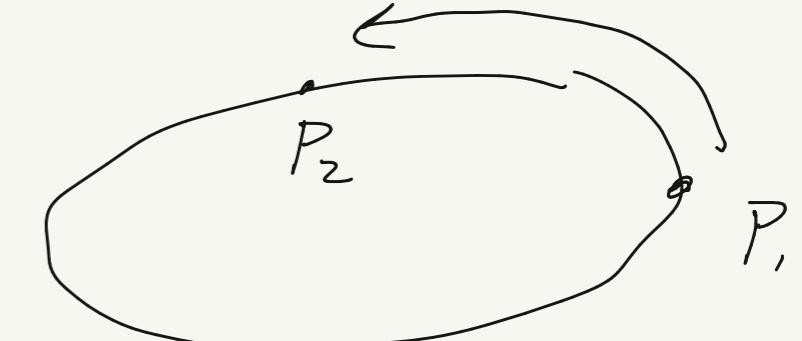


Lecture #1: Aug 24th

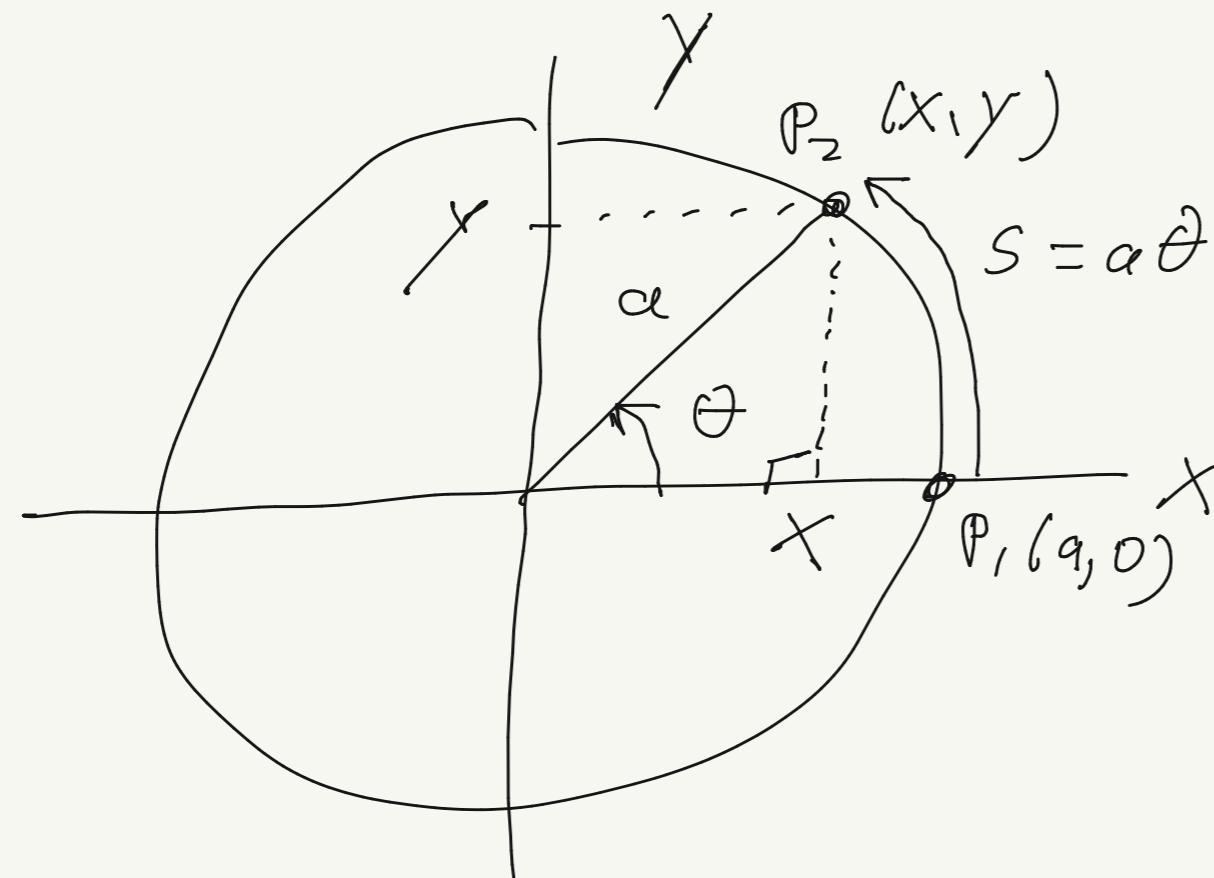
Elliptic functions / integrals:

i) circumference of an ellipse

ii) period of a simple pendulum beyond the small-angle approximation



Circular functions: sines, cosines,



$a = \text{radius}$

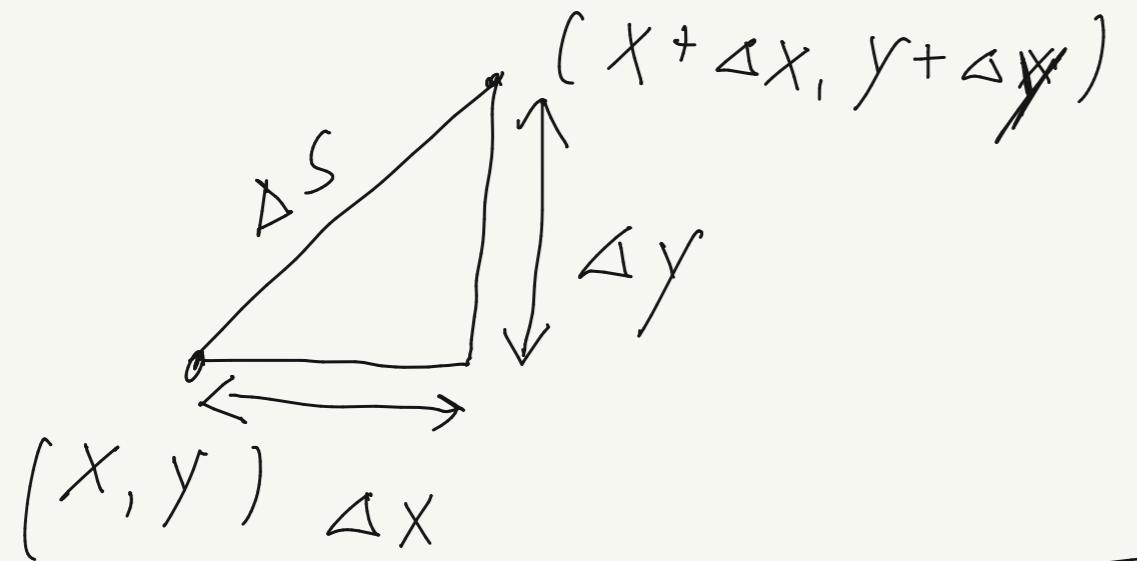
$$x^2 + y^2 = a^2$$

$$\theta = \frac{s}{a}$$

$$= \frac{1}{a} \int \sqrt{dx^2 + dy^2}$$

$$ds^2 = dx^2 + dy^2$$

$$= \int d\theta \Big|_{P_1}^{P_2}$$



$$x = a \cos \theta$$

$$y = a \sin \theta$$

$$\begin{aligned} \sin \theta &= y/a \\ \cos \theta &= x/a \end{aligned} \quad \left. \begin{array}{l} \text{def. of } \sin \theta \\ \text{def. of } \cos \theta \end{array} \right\}$$

$$\boxed{x^2 + y^2 = a^2} \rightarrow \frac{a^2(\cos^2 \theta + \sin^2 \theta)}{\cos^2 \theta + \sin^2 \theta} = a^2$$

$$\boxed{\frac{d}{d\theta} \sin \theta = \cos \theta}$$

$$\begin{aligned} \text{Proof: } \frac{d}{d\theta} \sin \theta &= \frac{d}{d\theta} \left(\frac{y}{a} \right) \\ &= \frac{1}{a} \frac{dy}{d\theta} \\ &= \frac{dy}{\sqrt{dx^2 + dy^2}} \\ &= \frac{1}{\sqrt{dx^2 + dy^2}} \quad \left. \begin{array}{l} \frac{1}{\sqrt{\left(\frac{dx}{dy} \right)^2 + 1}} \\ \text{def. of } \sec \theta \end{array} \right\} \end{aligned}$$

$$\begin{aligned} d\theta &= ds \\ d\theta &= \sqrt{dx^2 + dy^2} \end{aligned}$$

$$x^2 + y^2 = a^2 \rightarrow 2x dx + 2y dy = 0$$

$$\frac{dx}{dy} = -\frac{y}{x}$$

$$\frac{d}{d\theta} \sin \theta = \frac{1}{\sqrt{\left(\frac{y}{x}\right)^2 + 1}} = \frac{1}{\sqrt{\frac{x^2}{x^2} + 1}} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{a} = \cos \theta$$

$$\boxed{\frac{d}{d\theta} \sin \theta = \cos \theta}$$

Similarly

$$\boxed{\frac{d \cos \theta}{d \theta} = -\sin \theta}$$

$$\int \frac{d(\sin \theta)}{\cos \theta} = \int d\theta = \theta + \text{const}$$

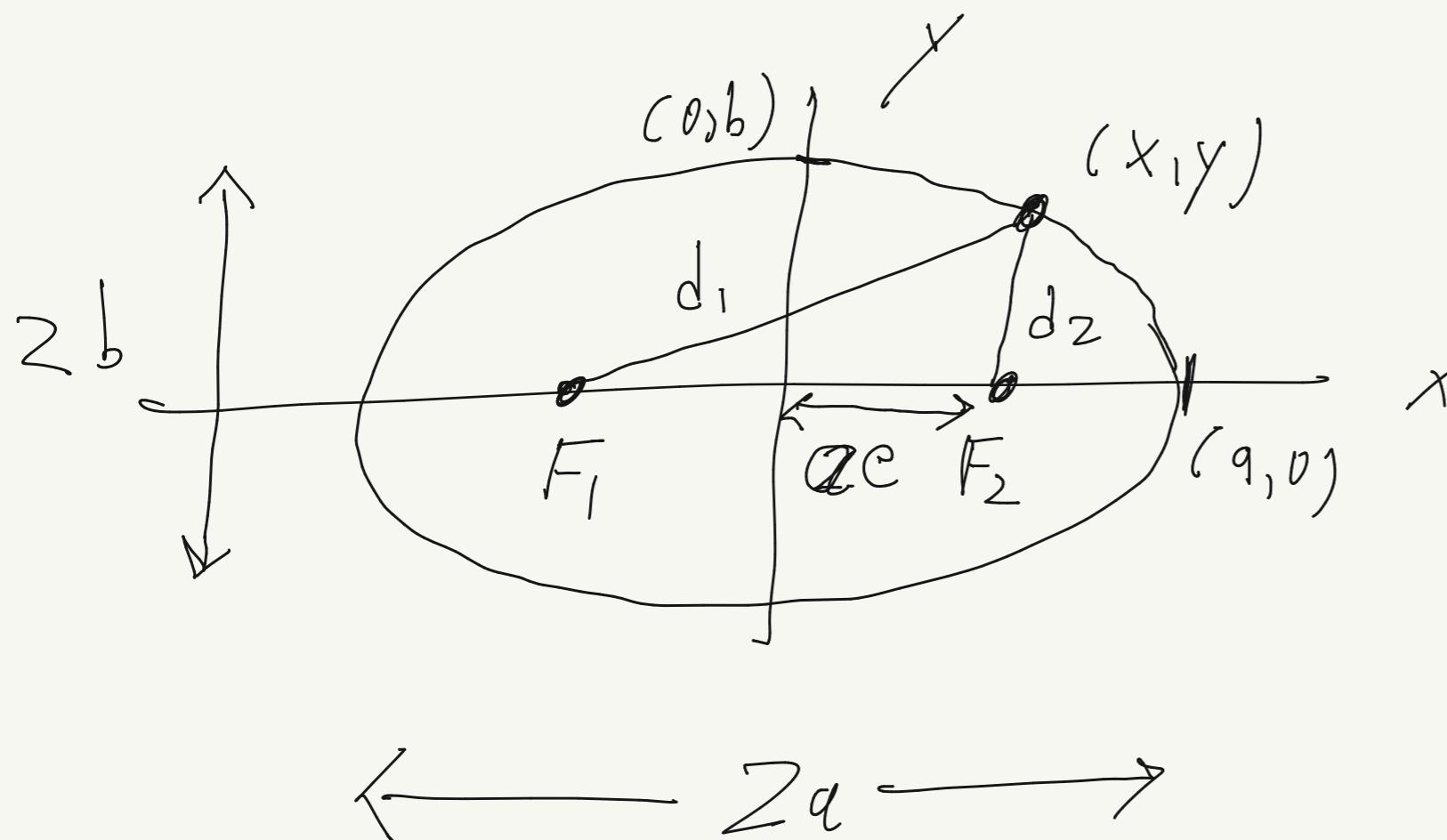
$$t = \sin \theta$$

$$\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - t^2}$$

$$\int \frac{dt}{\sqrt{1-t^2}} = \theta + \text{const}$$

$$\boxed{\sin^{-1} t + \text{const}}$$

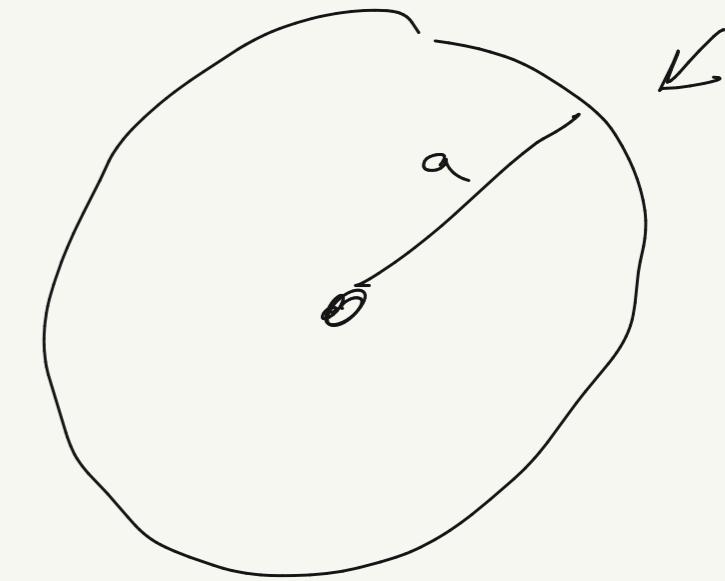
Lec #2 : Aug 26th



$$d_1 + d_2 = 2a$$

$$x^2 + y^2 = a^2$$

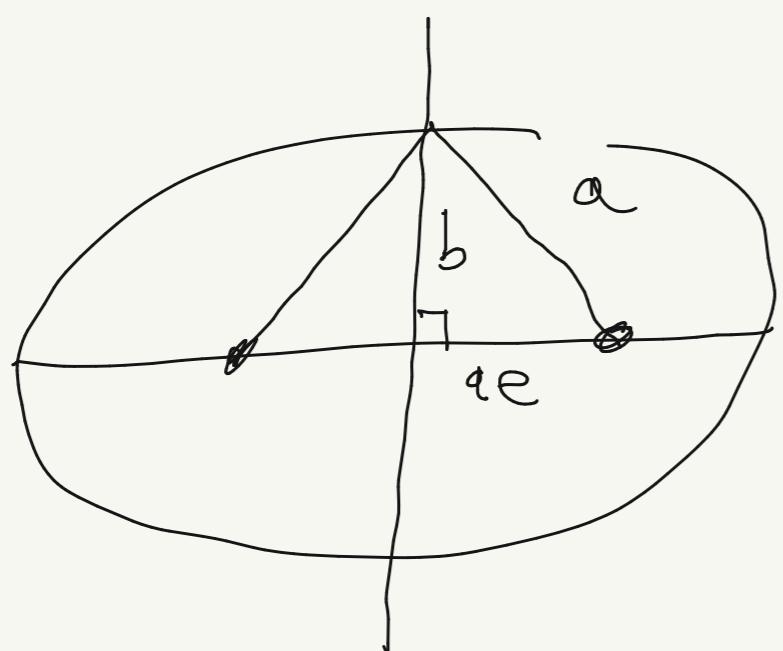
$$e \neq \frac{b}{a}$$



$$e \neq 1 - \frac{b}{a}$$

$$e^2 = \frac{b^2}{1-a^2}$$

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

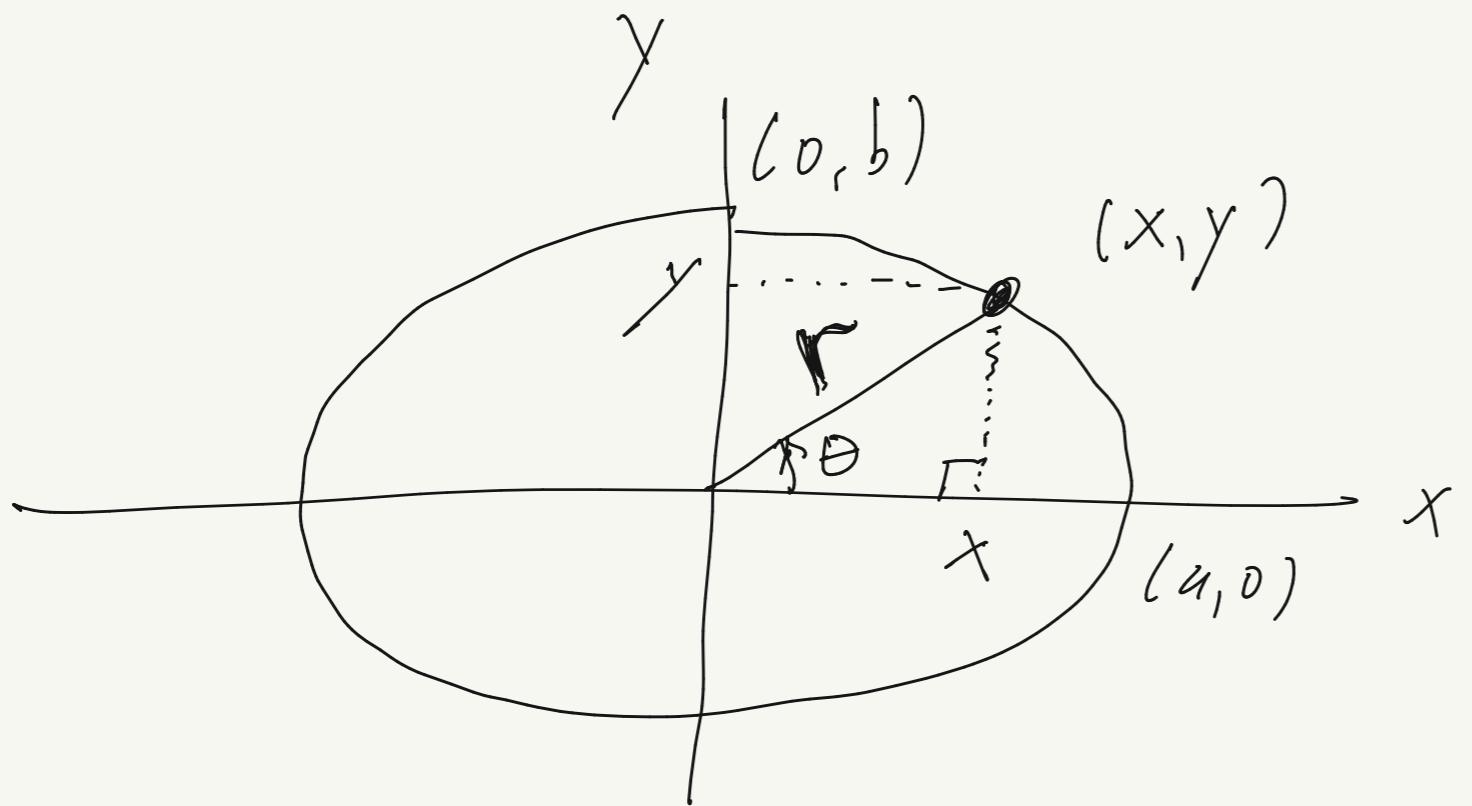


PF: $a^2 = b^2 + (ae)^2$

$$a^2(1-e^2) = b^2$$

$$1-e^2 = \left(\frac{b}{a}\right)^2$$

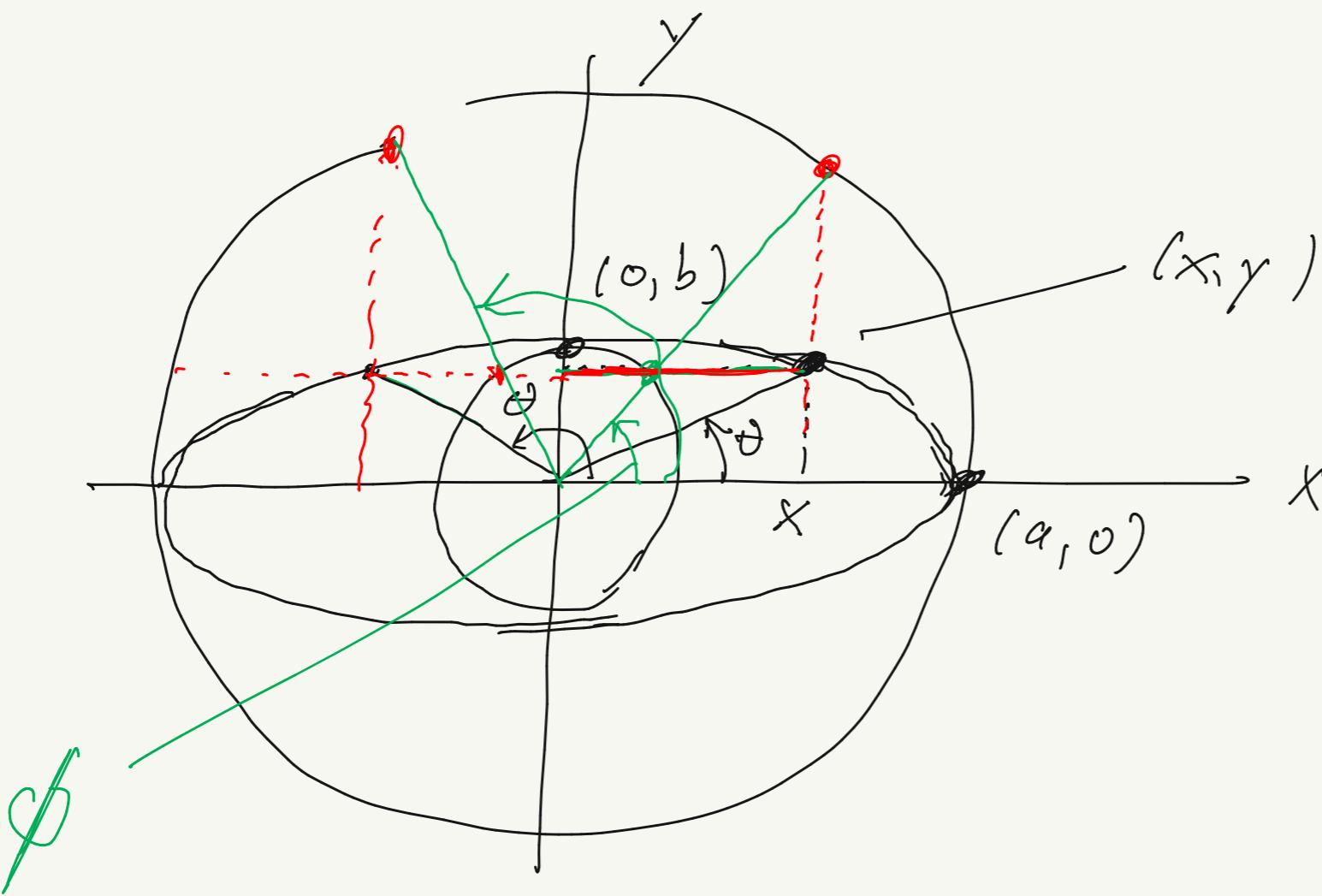
$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

r changes



$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

$$0 < e < 1$$

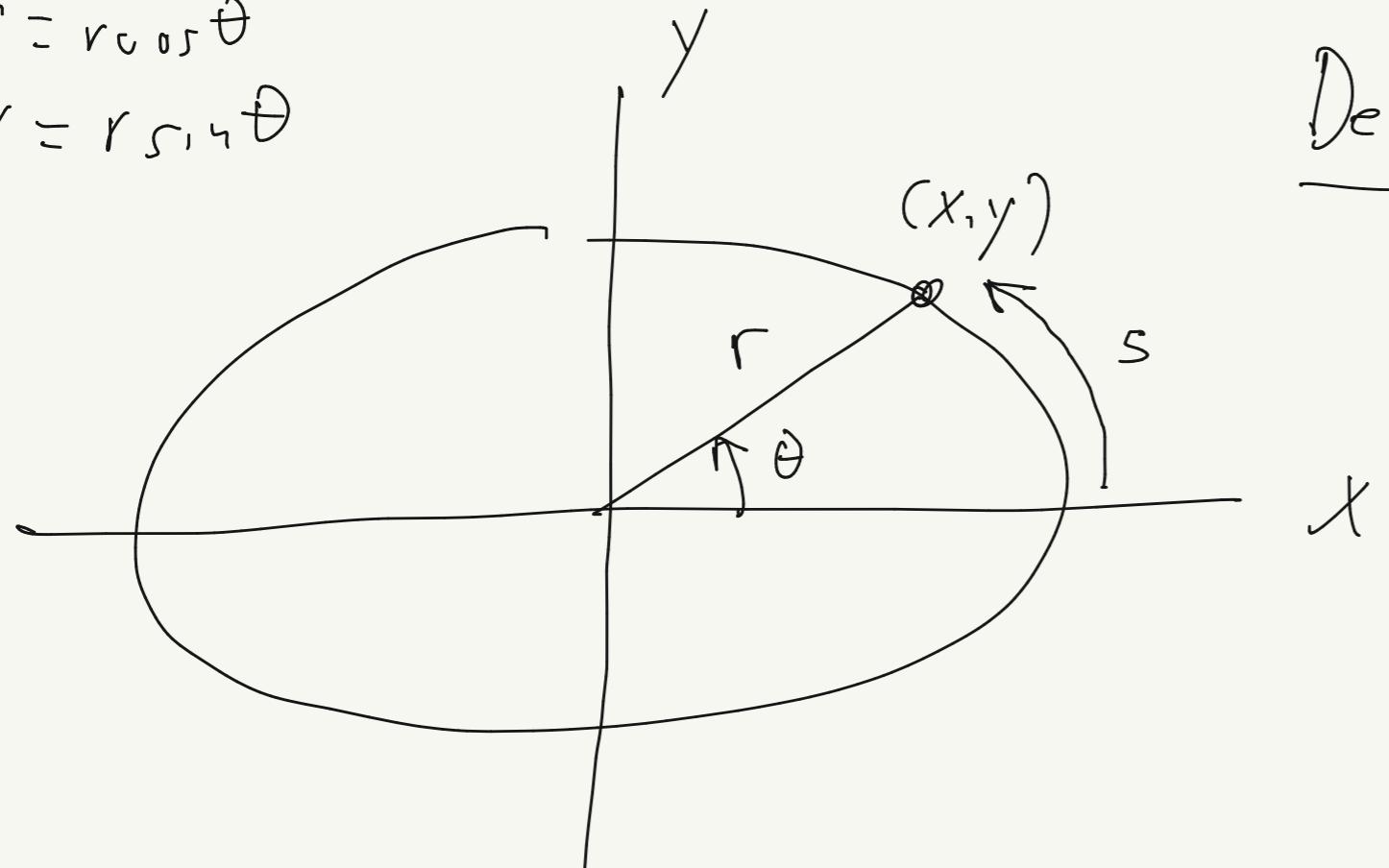
$e = 0$ circle

$e \approx 1$ parabola

$e > 1$ hyperbola

$$x = r \cos \theta$$

$$y = r \sin \theta$$



Defn:

$$\begin{cases} \operatorname{cn}(u; k) = x/a \\ \operatorname{sn}(u; k) = y/b \\ \operatorname{dn}(u; k) = r/a \end{cases}$$

$$k = e, \quad 0 < k < 1$$

\curvearrowright
Modulus

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{dr^2 + r^2 d\theta^2} \end{aligned}$$

$$\oint u = \int_0^\theta r d\theta < \# S$$

$$u \equiv \frac{1}{b} \int_0^\theta r d\theta$$

\uparrow
not θ , not arc length

$$\begin{aligned} x &= r \cos \theta, \quad y = r \sin \theta \\ dx &= dr \cos \theta - r \sin \theta d\theta \\ dy &= dr \sin \theta + r \cos \theta d\theta \end{aligned}$$

$\operatorname{cn}(u)$ $\operatorname{sn}(u)$	pendulum $k = \sin\left(\frac{\phi_0}{2}\right)$
--	---

Property:

$$\boxed{Cn^2 u + Sn^2 u = 1} \quad \leftarrow \quad \left(\frac{x}{a} \right)^2 + \left(\frac{y}{b} \right)^2 = 1$$
$$\leftarrow \quad x^2 + y^2 = r^2$$

$$\begin{aligned} d n^2 u &= Cn^2 u + \left(\frac{b}{a} \right)^2 Sn^2 u \\ &= 1 - Sn^2 u + \left(\frac{b}{a} \right)^2 Sn^2 u \\ &= 1 - Sn^2 u \left(1 - \left(\frac{b}{a} \right)^2 \right) \\ &= 1 - H^2 Sn^2 u \end{aligned}$$

$$\boxed{d n^2 u + H^2 Sn^2 u = 1}$$

$$\frac{d}{du} \sin u = \frac{1}{b} \frac{dy}{du}$$

$$= \frac{dy}{r d\theta}$$

$$u = \int_0^\theta (r d\theta)$$

$$du = \frac{r d\theta}{b} \rightarrow b du = r d\theta$$



$$x = r \cos \theta, y = r \sin \theta$$

$$\begin{aligned} dx &= dr \cos \theta - r \sin \theta d\theta & \rightarrow \sin \theta dx = \sin \theta \cos \theta dr - r \sin^2 \theta d\theta \\ dy &= dr \sin \theta + r \cos \theta d\theta & \rightarrow -\cos \theta dy = -\cos \theta \sin \theta dr - r \cos^2 \theta d\theta \end{aligned}$$

add

$$\sin \theta dx - \cos \theta dy = -r d\theta$$

$$\frac{y}{r} dx - \frac{x}{r} dy = -r d\theta$$

$$\rightarrow \boxed{rd\theta = \frac{-y}{r} dx + \frac{x}{r} dy}$$

$$\frac{d}{du} \sin u = \frac{dy}{-\frac{y}{r} dx + \frac{x}{r} dy}$$

$$= \frac{r}{-\frac{y}{r} \frac{dx}{dy} + x}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \cancel{\int \frac{x dx}{a^2} + \cancel{\int y dy}}_{\text{= 0}} = 0$$

$$\frac{dx}{dy} = -\frac{y}{x} \frac{a^2}{b^2}$$

$$\frac{d \sin u}{du} = \frac{r}{-y \left(\frac{-y}{x}\right) \frac{a^2}{b^2} + x} = \frac{r}{y^2 \left(\frac{a}{b}\right)^2 + x^2}$$

$$= \frac{r}{a} \frac{x}{a} \left(\frac{1}{\left(\frac{y}{b}\right)^2 + \left(\frac{x}{a}\right)^2} \right)$$

$$= \frac{\sin u \cdot \cos u}{1}$$

$$\boxed{\frac{d}{du} \sin u = \cos u \cdot \frac{d}{du} u}$$

$$\frac{d}{du} \boxed{\operatorname{cn} u} = -\operatorname{sn} u \cdot \operatorname{dn} u$$

$$\frac{d}{du} \operatorname{dn} u = -\pi^2 \operatorname{sn} u \cdot \operatorname{cn} u$$

$$\frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \cdot \operatorname{dn} u$$

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1$$

$$\operatorname{dn}^2 u + \pi^2 \operatorname{sn}^2 u = 1$$

$$\int \frac{d(\operatorname{cn} \theta)}{\operatorname{cos} \theta} = \int d\theta = \theta$$

$$\int \frac{dt}{\sqrt{1-t^2}} = \operatorname{sn}^{-1} t \quad \theta = \operatorname{cn}^{-1} t$$

Integrate!

$$\int \frac{d(\operatorname{sn} u)}{\operatorname{cn} u \cdot \operatorname{dn} u} = \int du = u + \operatorname{const}$$

$$t = \operatorname{sn} u$$

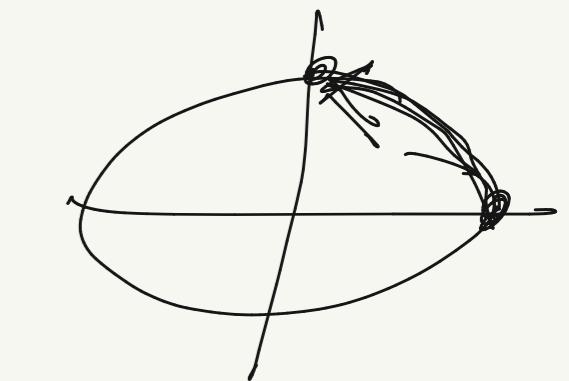
$$\int \frac{dt}{\sqrt{1-t^2} \sqrt{1-\pi^2 t^2}} = \operatorname{sn}^{-1}(t; \pi) + \operatorname{const}$$

$$F(\phi, k) = \int_0^{\sin \phi} \frac{dt}{\sqrt{1-t^2} \sqrt{1-k^2 t^2}}$$

$$\left. \begin{aligned} \sin u &= \sin \phi \\ \frac{y}{b} &= \sin \phi \\ \cnu u &= \cos \phi \end{aligned} \right\}$$

incomplete elliptic integral of 1st kind (angular dependent, period of a simple pendulum)

$$E(\phi, k) = \int_0^{\sin \phi} \frac{dt}{\sqrt{1-t^2} \sqrt{1-k^2 t^2}}$$



incomplete elliptic integral of 2nd kind (arc length along ellipse)

$$\phi = \frac{\pi}{2}$$

$$\int_0^1 \frac{dt}{\sqrt{1-t^2} \sqrt{1-k^2 t^2}} = K(k)$$

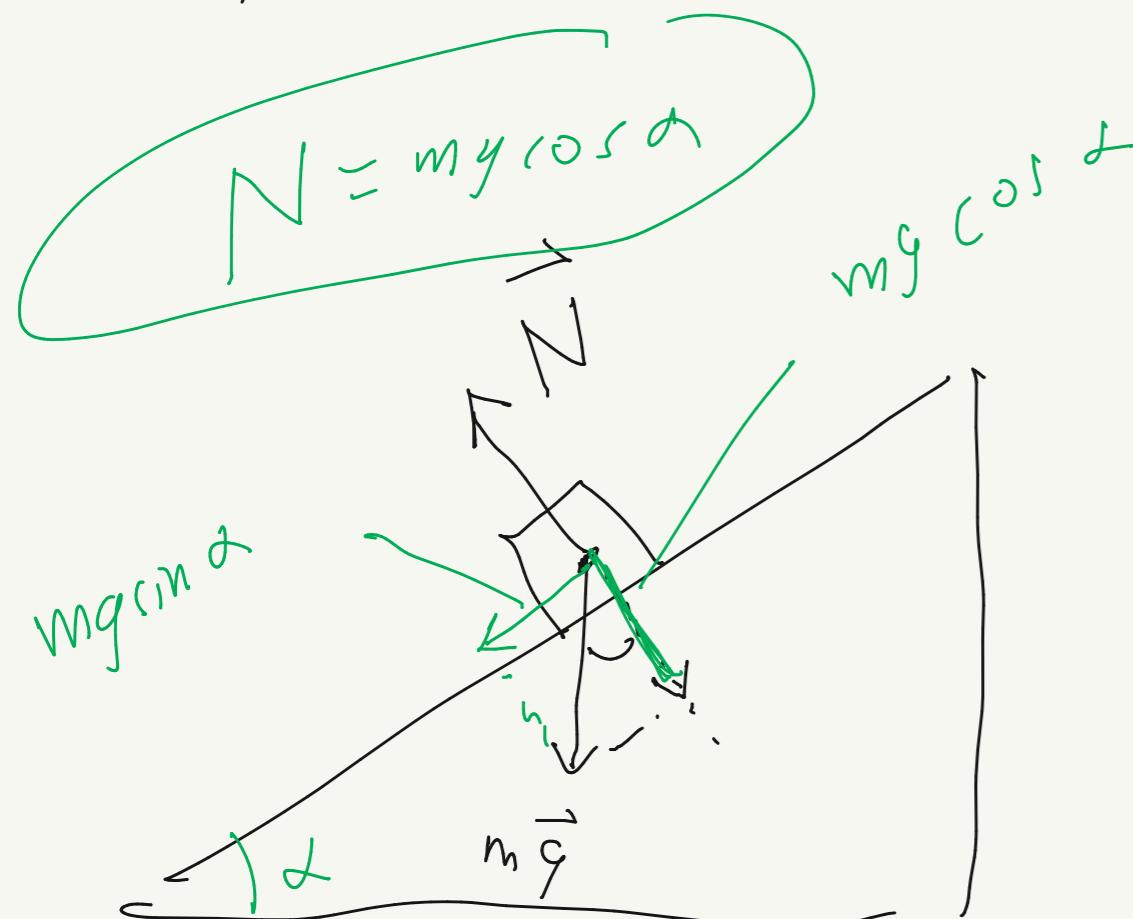
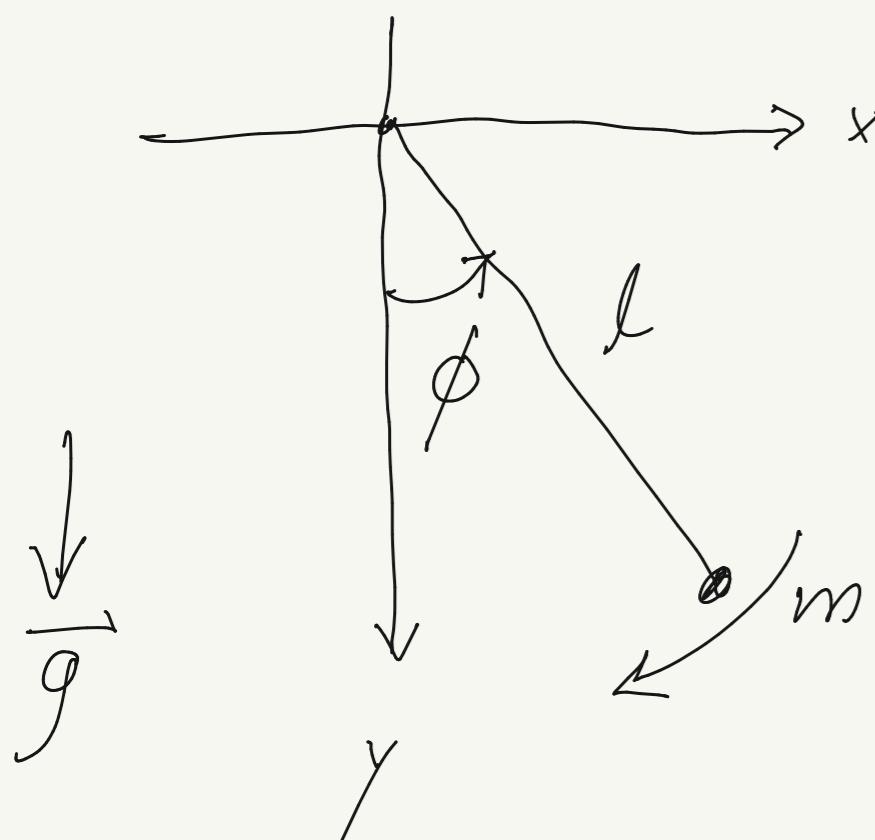
$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} \sqrt{1-k^2 t^2} = E(k)$$

complete elliptic integrals of 1st and 2nd kind

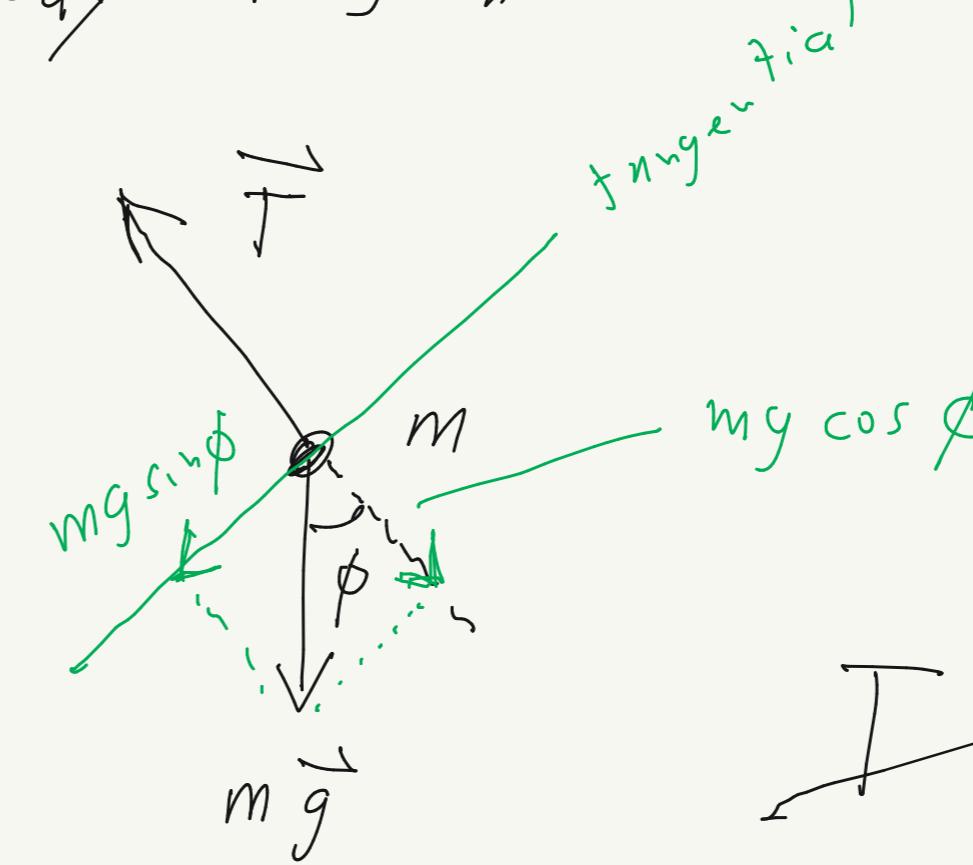
Lec #3: Aug 31st

Simpl. pendulum:

("Freshman physics")



Free-body diagram:



$$1) T - mg \cos \phi = m \dot{\phi}^2 l$$

$$[T = m \dot{\phi}^2 l + mg \cos \phi]$$

$$2) mg \sin \phi = -m \alpha_{\text{tangential}}$$

$$\ddot{\phi} = -\frac{g}{l} \sin \phi$$

EOM

$$a_c = \frac{v^2}{r} = \omega^2 r$$

$$a_t = \alpha r$$

$$\alpha = \ddot{\phi}, \omega = \dot{\phi}$$

$$T = mg \cos \phi$$

cent. petal acceleration

$$\ddot{\phi} = -\frac{g}{l} \sin \phi \quad \left(\text{hard to solve; } 2^{\text{nd}} \text{ order non-linear ODE} \right)$$

Small-angle approx: $\phi \ll 1$ rad $\approx 57^\circ$ (π radians = 180°)

$$\rightarrow \ddot{\phi} = -\frac{g}{l} \phi \quad \omega_0 = \sqrt{\frac{g}{l}}$$

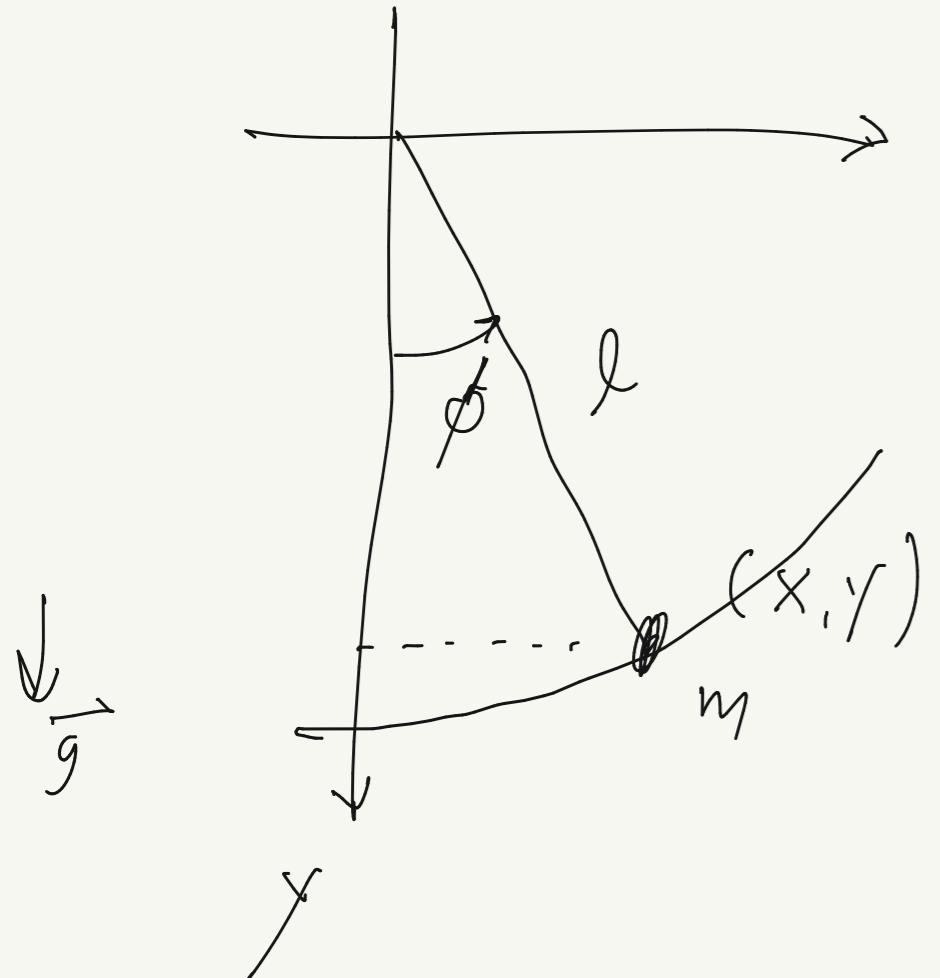
$$\phi(t) = A e^{-i \sqrt{\frac{g}{l}} t}$$

Complex

$$= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

$$= a \cos(\omega_0 t + \alpha)$$

Lagrangian formulation:



$$L = T - U$$

$$U = -mgx \\ = -mgl \cos\phi$$

$$y = l \cos\phi \\ x = l \sin\phi$$

$$T = \frac{1}{2}m(x^2 + y^2) \\ = \frac{1}{2}m l^2 \dot{\phi}^2$$

$$\dot{x} = l\dot{\phi} \cos\phi \\ \dot{y} = -l\dot{\phi} \sin\phi$$

$$L = \frac{1}{2}m l^2 \dot{\phi}^2 + mgl \cos\phi$$

~~Final~~

$$(x, y) \quad x = r \cos\phi \\ (r, \phi) \quad y = r \sin\phi$$

$$T = \frac{1}{2}m(r^2 + r^2\dot{\phi}^2)$$

Lagrange's equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi}$$

$$\frac{d}{dt}(ml^2\dot{\phi}) = -mgl \sin\phi$$

$$ml^2\ddot{\phi} = -mgl \sin\phi$$

$$\ddot{\phi} = -\frac{g}{l} \sin\phi$$

L does not depend explicitly on time $t \rightarrow E$ is conserved.

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$$

$$= T + U$$

Example:

$$\boxed{E} = \frac{\partial L}{\partial \dot{\phi}} \phi - L$$

$$= ml^2 \dot{\phi} \ddot{\phi} - \left(\frac{1}{2} ml^2 \dot{\phi}^2 + mgl \cos \phi \right)$$

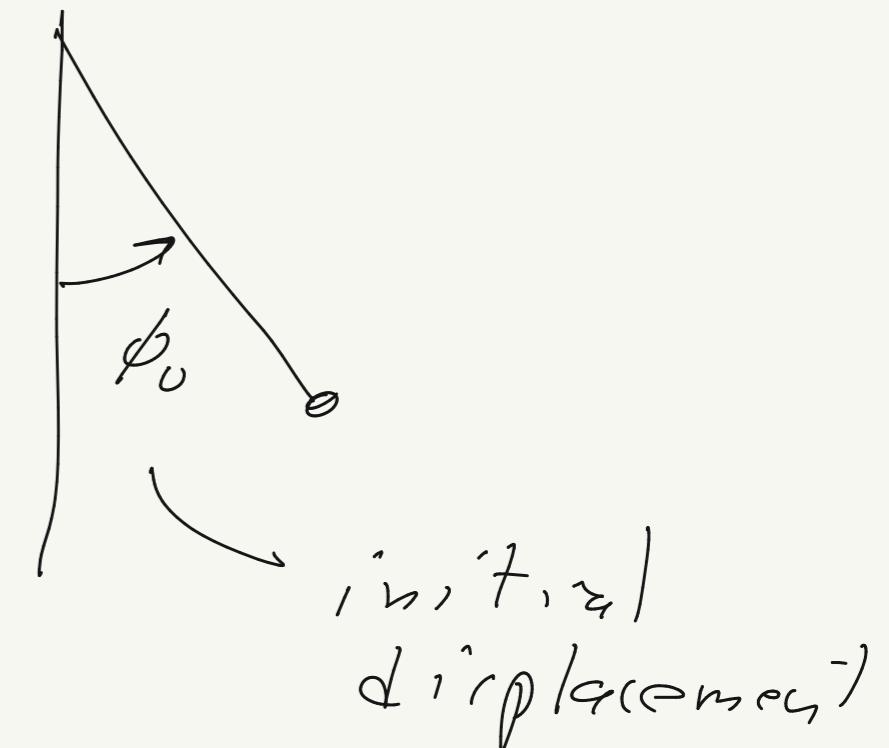
$$= \frac{1}{2} ml^2 \dot{\phi}^2 - mgl \cos \phi$$

$$= T + U$$

$$E = -mgl \cos \phi_0$$

$$-mgl \cos \phi_0 = \frac{1}{2} ml^2 \dot{\phi}^2 - mgl \cos \phi$$

$$\dot{\phi}^2 = \frac{2}{l^2} (gl \cos \phi - gl \cos \phi_0)$$



L & L 11.1

$$\dot{\phi}^2 = 2 \frac{g}{l} (\cos \phi - \cos \phi_0)$$

$$\frac{d\phi}{dt} = \dot{\phi} = \pm \sqrt{2 \sqrt{\frac{g}{l}} \sqrt{\cos \phi - \cos \phi_0}}$$

$\sqrt{\frac{g}{l}}$
 w_0

$$\int dt = \int \frac{\pm d\phi}{\sqrt{2 w_0 \sqrt{\cos \phi - \cos \phi_0}}}$$

$$w_0 t + \text{const} = \pm \int \frac{d\phi}{\sqrt{2 \sqrt{\cos \phi - \cos \phi_0}}}$$

$$\cos \phi = \cos \left(2 \cdot \frac{\phi}{2} \right) = \cos^2 \left(\frac{\phi}{2} \right) - \sin^2 \left(\frac{\phi}{2} \right) = 1 - 2 \sin^2 \left(\frac{\phi}{2} \right)$$

$$\cos \phi_0$$

$$= 1 - 2 \sin^2 \left(\frac{\phi_0}{2} \right)$$

$$\Rightarrow \sqrt{\cos \phi - \cos \phi_0} = \sqrt{2} \sqrt{\sin^2 \left(\frac{\phi_0}{2} \right) - \sin^2 \left(\frac{\phi}{2} \right)} = \sqrt{2} \left| \sin \left(\frac{\phi_0}{2} \right) \right| \sqrt{1 - \frac{\sin^2 \left(\frac{\phi_0}{2} \right)}{\sin^2 \left(\frac{\phi_0}{2} \right)}}$$

L & L 11.1

$$\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t + \text{const}$$

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} =$$

$$k = \left| \sin\left(\frac{\phi_0}{2}\right) \right| \quad (0 < \tau < 1)$$

$$x = \frac{\sin\left(\frac{\phi}{2}\right)}{\left| \sin\left(\frac{\phi_0}{2}\right) \right|} = \frac{\sin\left(\frac{\phi}{2}\right)}{\tau} \rightarrow dx = \frac{1}{\tau} \frac{1}{2} \cos\left(\frac{\phi}{2}\right) d\phi$$

$$\sqrt{2} \sqrt{\cos\phi - \cos\phi_0} = 2\tau \sqrt{1-x^2}$$

$$dx = \frac{1}{2\tau} \sqrt{1-\sin^2\left(\frac{\phi}{2}\right)} d\phi$$

$$d\phi = \frac{2\tau dx}{\sqrt{1-\tau^2 x^2}}$$

$$w_0 t + \text{const} = \pm \int \frac{2\tau dx}{\sqrt{1-\tau^2 x^2}} \rightarrow 2\tau \int \frac{dx}{\sqrt{1-x^2}}$$

$$= \int_0^x \frac{dx}{\sqrt{1-x^2} \sqrt{1-\tau^2 x^2}}$$

$$= \sin^{-1}(x; \tau)$$

$$\left\{ \begin{array}{l} x = \sin\left(\frac{\phi}{2}\right) \\ \tau = \left| \sin\left(\frac{\phi_0}{2}\right) \right| \end{array} \right.$$

$$t = 0 \iff \phi = \phi_0$$

$$\text{const} = \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-H^2 x^2}} = K(H)$$

complete elliptic
integral of the
1st kind

$$\omega_0 t + E(K) = \sin^{-1} \left(\frac{\sin(\frac{\phi}{2})}{K} \right); \quad | \quad K \equiv \sin(\frac{\phi_0}{2})$$

$$\sin(\omega_0 t + E(K); K) = \frac{1}{K} \sin\left(\frac{\phi}{2}\right)$$

→ $\phi(t) = 2 \arcsin \left(K \sin(\omega_0 t + E(K); K) \right)$

$$P = \frac{4}{\omega_0} E(K) \quad \rightarrow \quad \frac{\omega_0 P}{4} = E(K)$$

