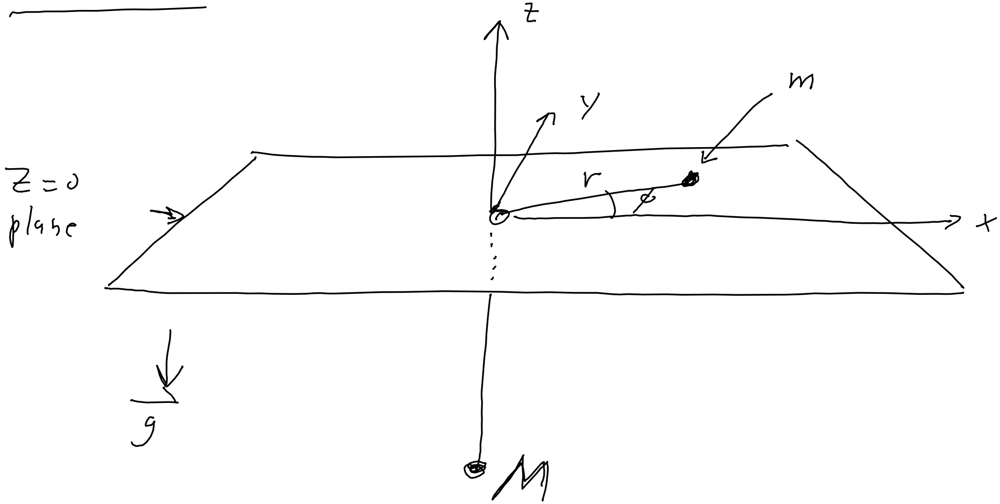
Example:



ℓ = length of string

$z = r - \ell$

$\ell = r - z$

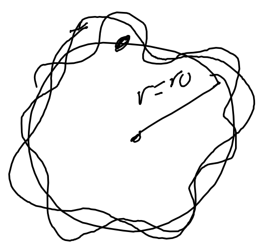
a) write down Lagrangian: $L(r, \phi, \dot{r}, \dot{\phi})$

b) determined conserved quantities

c) determine the effective potential $U_{eff}(r)$

d) qualitatively determine the allowed motions

Top view:

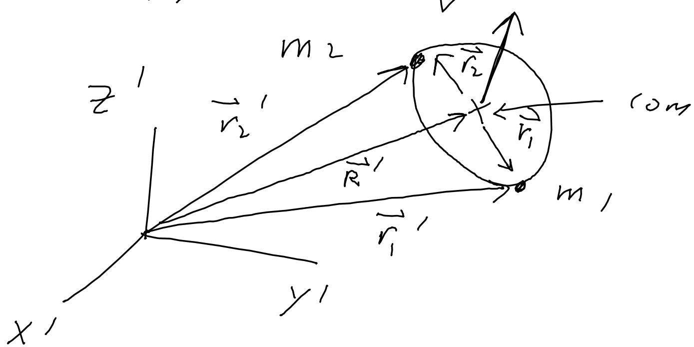


Two body problem:

m_1, m_2 $\quad \vec{V}', \quad U(|\vec{r}_1' - \vec{r}_2'|)$

$E, \vec{P}', \vec{M}'$

$E = \text{const}, \quad \vec{P}' = \text{const}, \quad \vec{M}' = \text{const}$



$m_1 + m_2$

$$\vec{P}' = m_1 \vec{v}_1' + m_2 \vec{v}_2' = \mu \vec{V}'$$

COM frame (x, y, z)

$\vec{V} = 0, \quad \vec{P} = 0,$

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = \mu \vec{R} = 0$$

$\vec{r}_1' = \vec{R}' + \vec{r}_1$

$\vec{r}_2' = \vec{R}' + \vec{r}_2$

$\vec{r}_1, \vec{r}_2$: position vector w-t COM frame

$\vec{r} = \vec{r_1} - \vec{r_2}$ relative separation vector

$0 = m_1 \vec{r_1} + m_2 \vec{r_2}$

$$\boxed{\vec{r_1} = \frac{m_2 \vec{r}}{(m_1 + m_2)}, \quad \vec{r_2} = -\frac{m_1 \vec{r}}{(m_1 + m_2)}}$$

$$T = \frac{1}{2}\left(m_1 |\dot{\vec{r_1}}|^2 + m_2 |\dot{\vec{r_2}}|^2\right)$$

$$= \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right) |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} m |\dot{\vec{r}}|^2$$

$$U = U(|\vec{r_1}' - \vec{r_2}'|) = U(|\vec{r_1} - \vec{r_2}|) = U(|\vec{r}|) = U(r)$$

$$\boxed{L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(r)}$$

two body problem

↓

one body

choose z-axis of COM frame to point along $\vec{M}$

$$\vec{M} = \vec{r} \times \vec{p}$$

$\hat{z}$, $\vec{M} = M\hat{z}$

$\vec{r}$ lies in the x-y plane

$x = r\cos\phi$

$y = r\sin\phi$

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)}$$

$$M \equiv p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \text{const} \qquad \frac{\partial L}{\partial \phi} = 0$$

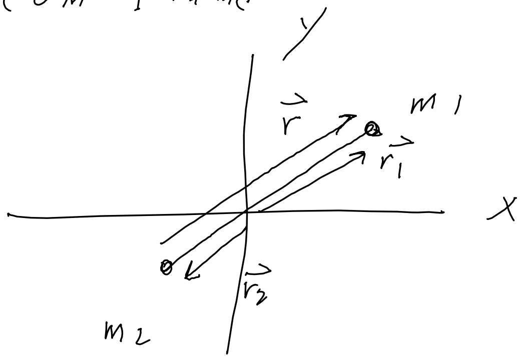
$$M = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi}$$

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + U(r) = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L$$

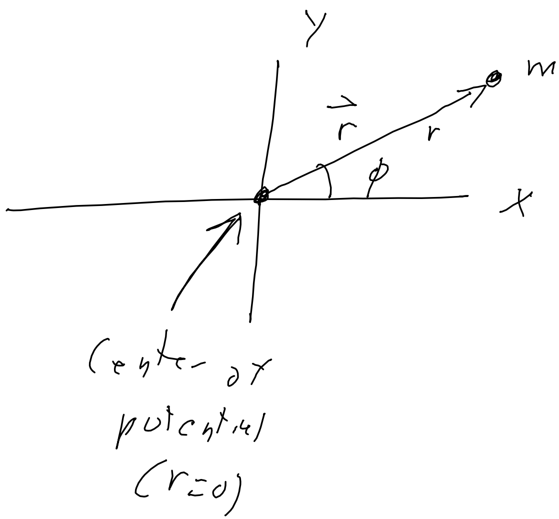
$$= \frac{1}{2}m\dot{r}^2 + \frac{M^2}{2mr^2} + U(r)$$

$$= \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$$

COM frame



y
x
$\vec{r}$
m_1
$\vec{r}_1$
$\vec{r}_2$
m_2



y
x
m
$\vec{r}$
r
ϕ
Center of potential (r=0)

$$E = \frac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

$$\dot{\phi} = \frac{M}{mr^2}$$

$$\pm\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)} = \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M}{mr^2}$$

$$= \pm\sqrt{}$$

$$dt = \frac{\pm\, dr}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}}$$

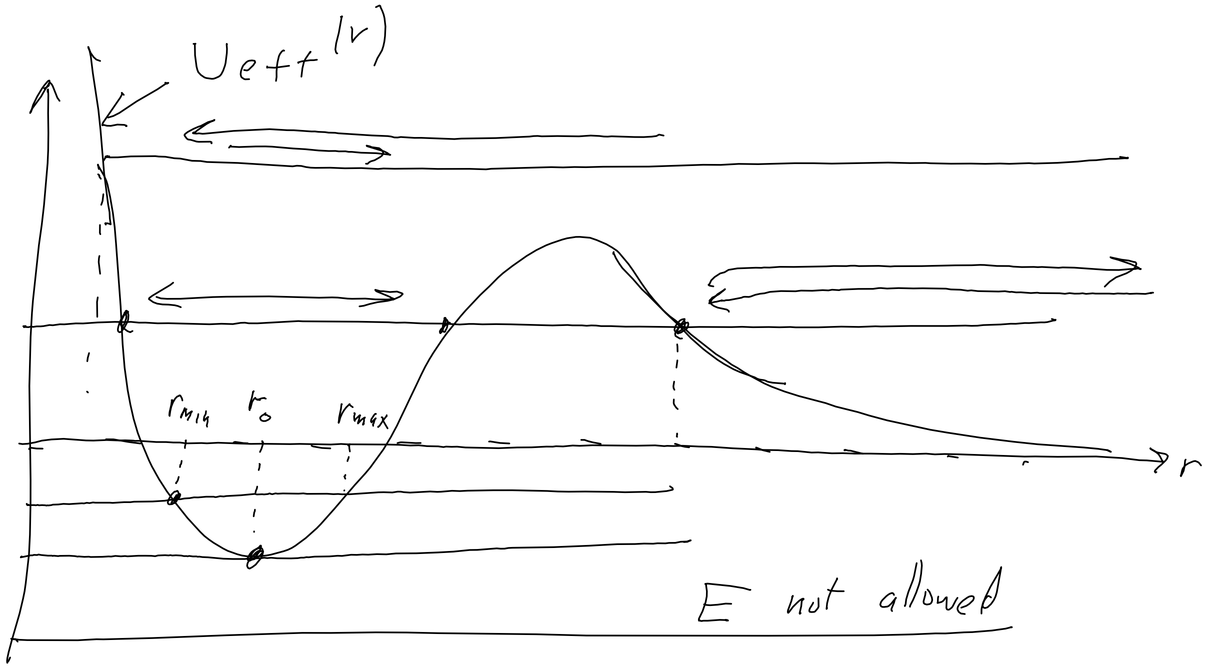
$$\boxed{t = \pm\int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}} + \text{const}}$$

$$d\phi = \pm \frac{dr\, M/mr^2}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}}$$

$$u = \frac{1}{r}$$

$$du = -\frac{1}{r^2}\, dr$$

$$\boxed{\phi = \pm \int \frac{M\, dr/r^2}{m\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}} + \text{const}}$$

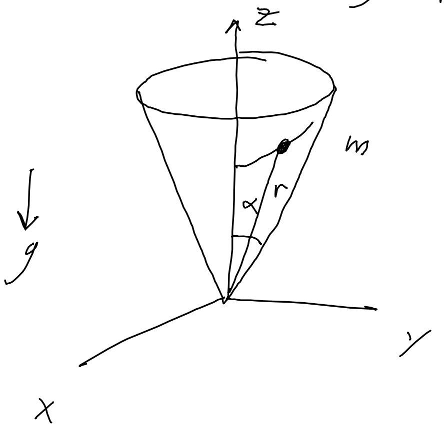


$$E = \frac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

$$= \frac{1}{2} m \dot{r}^2 + \left(\frac{M^2}{2mr^2} + U(r) \right)$$

Quiz #3:

Write down and plot the effective potential for the following problem



(use sph. polar coordinates)

$$U = mgz$$
$$= mgr\cos\alpha$$
$$T = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2)$$

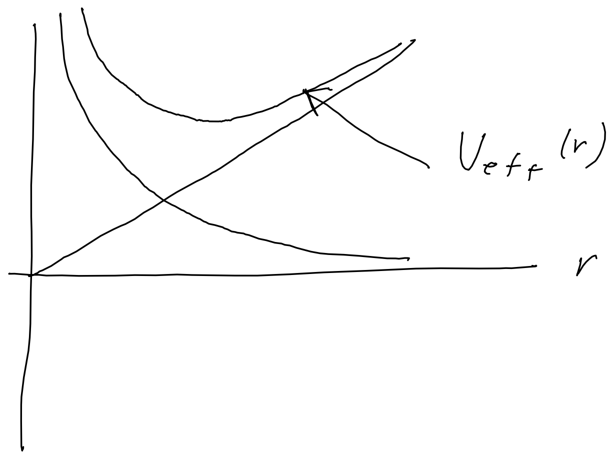
$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2) - mgr\cos\alpha$$

No ϕ dependence — $p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\alpha\,\dot{\phi} = \text{const}$

No explicit t dependence —

$$\text{const} = E = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L$$
$$= T + U$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2) + mgr\cos\alpha$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{p_\phi^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha$$

$$\boxed{U_{eff}(r) = \frac{p_\phi^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha}$$



Oct 6th Thursday – Midterm #1

$$+20 = \underbrace{8}_{\text{short answer question}} + \underbrace{12}_{\text{2 longer problems}}$$

- ellipses $(a, b, e, p, \ldots)$

  p — semi-latus rectum

- $E, M \longleftrightarrow$ a, b / a, p

-

$$\sqrt{1-\left(\frac{b}{a}\right)^2} = e$$

- $$\underbrace{\frac{p}{r} = 1 + e\cos\phi}_{\text{an ellipse ?}}$$

-

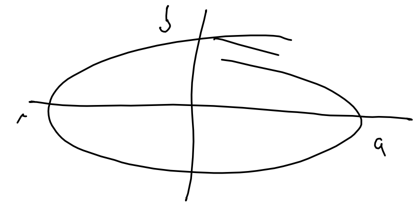
$$x = r\cos\phi$$
$$y = r\sin\phi$$

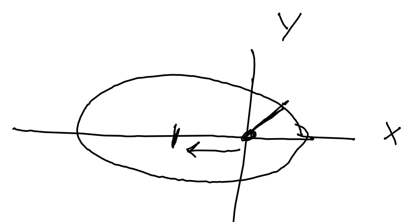
- other things ??

$$x^2 + y^2 = R^2$$

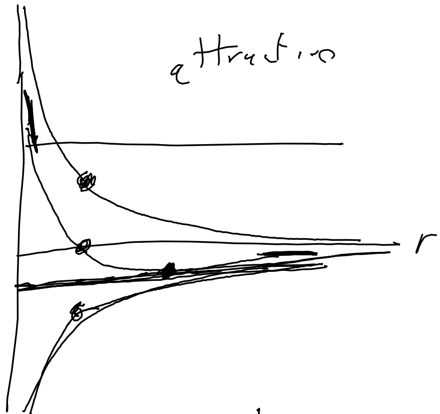
$$\left(\frac{x+ae}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



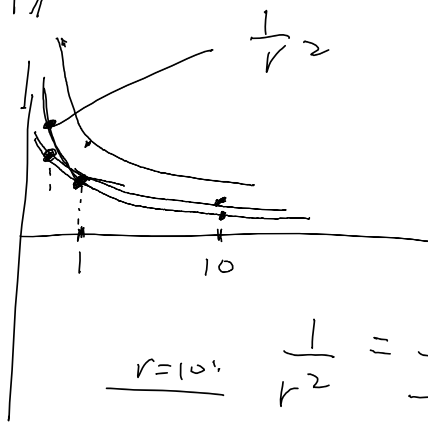


$$\left.\begin{aligned} x &= -ae \\ y &= 0 \end{aligned}\right\}$$ center of the ellipse



$$U_{eff} = -\frac{\alpha}{r} + \frac{M}{2m r^2}$$

$$= -\frac{1}{r} + \frac{1}{r^2}$$

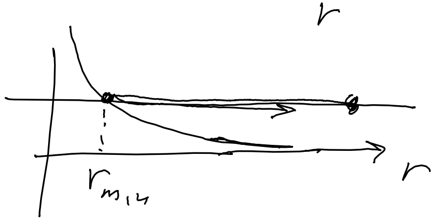


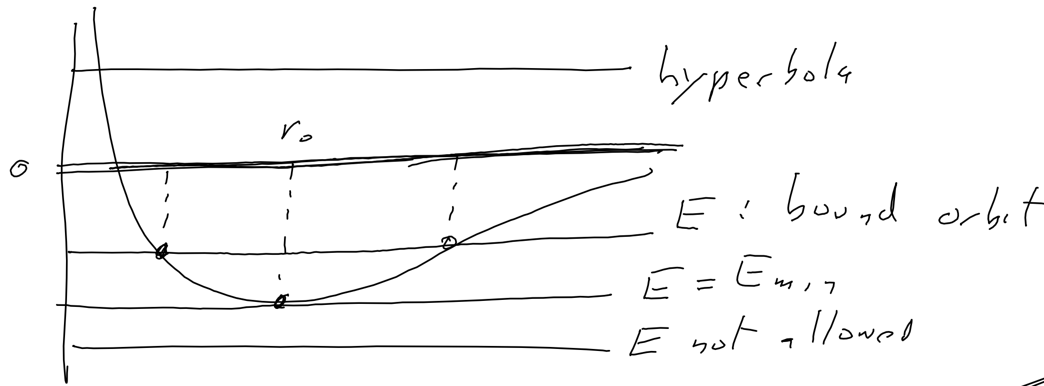
$$U_{eff} = \frac{1}{r} + \frac{1}{r^2}$$

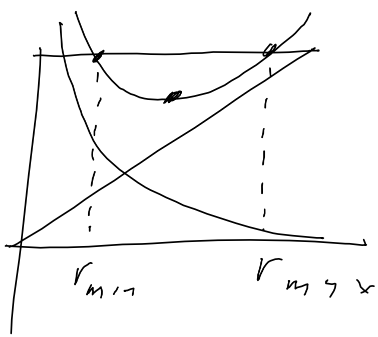
$r = 10$: $\frac{1}{r^2} = \frac{1}{100}$

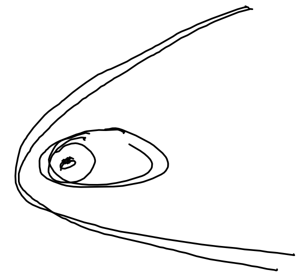
$\frac{1}{r} = \frac{1}{10}$

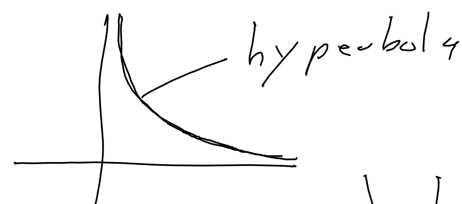
$$r = \frac{1}{2} \begin{cases} \frac{1}{r^2} = \frac{1}{\left(\frac{1}{2}\right)^2} = 4 \\ \frac{1}{r} = \frac{1}{\left(\frac{1}{2}\right)} = 2 \end{cases}$$

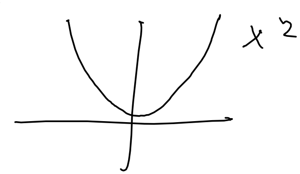


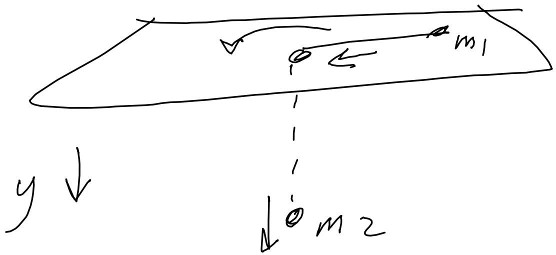

hyperbola
r_0
0
E : bound orbit
E = E min
E not allowed


r_{min}
r_{max}




hyperbola

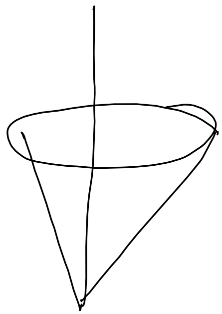

x^2



$$U = m_2 g r$$

$$U_{eff}(r) = \frac{p_\phi^2}{2m_1 r^2} + m_2 g r$$

$$\neq \frac{1}{r}, \; r^2$$



$$\frac{1}{\sqrt{1+\epsilon}} = (1+\epsilon)^{-1/2} \approx 1 - \frac{1}{2}\epsilon$$

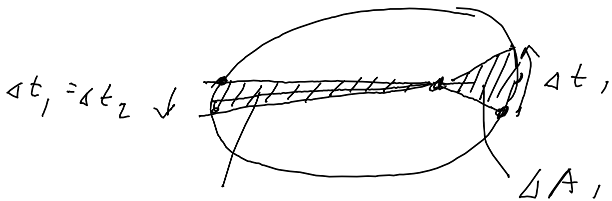
$$\boxed{(1+\epsilon)^p \approx 1 + p\epsilon \quad \text{for } |\epsilon| \ll 1}$$

Hidden

QUIZ #4:

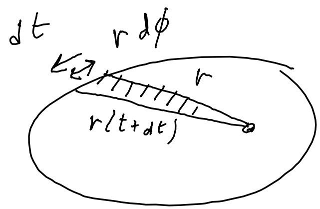
State and prove Kepler's 2^{nd} law

"Equal areas in equal times"



$\Delta A_2 \simeq \Delta A_1$

$\boxed{+1}$

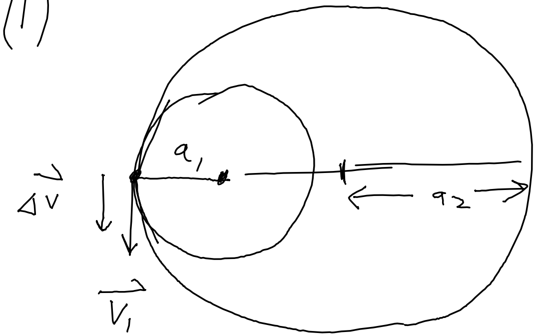


$$dA \simeq \frac{1}{2} \text{ base} \times \text{height}$$
$$= \frac{1}{2} r\,d\phi \; r$$
$$= \frac{1}{2} r^2 d\phi$$
$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi}$$
$$= \frac{1}{2} \frac{M}{m}$$
$$= \text{const}$$

$M = m r^2 \dot{\phi}$

$\boxed{+1}$

(1)



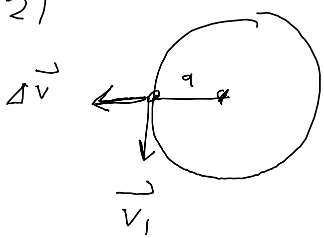
$\vec{V}_2 = \vec{V}_1 + \Delta\vec{v}$

$|\Delta\vec{v}| = \frac{V_1}{10}$

a) Find a_2 in terms of a_1

b) Find e_2

(2)



$|\Delta\vec{v}| = \frac{V_1}{\sqrt{2}}$

a) Find a_2 in terms of a_1

b) Find e_2

c) sketch the final ellipse

(3) calculate the minimum value of $|\vec{V}_2|$ needed to unbound the orbit.