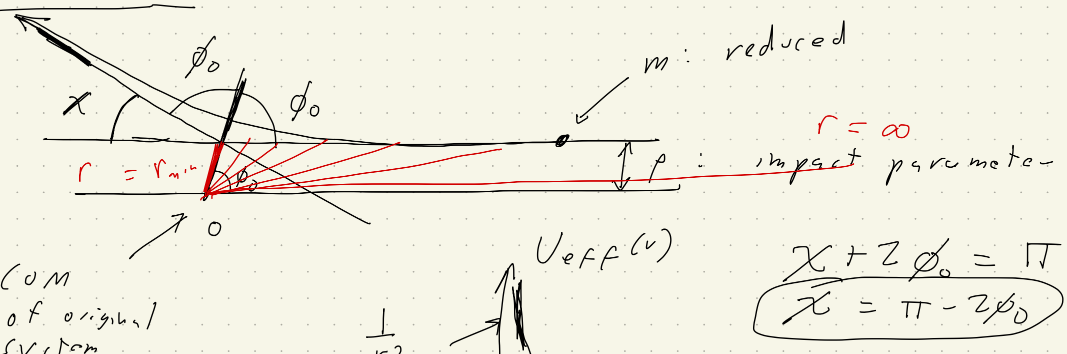
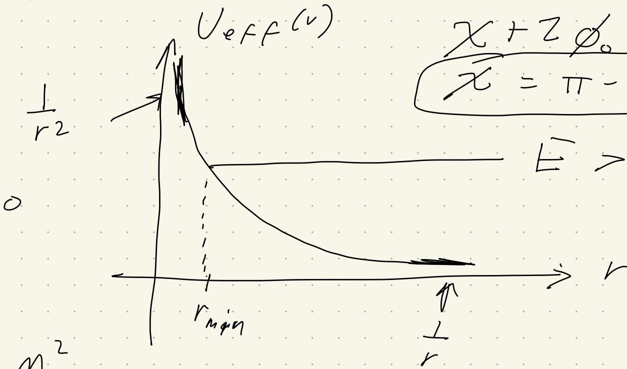
Lecture #16 Thurs 10/14



$$\chi + 2\phi_0 = \pi$$

$$\boxed{\chi = \pi - 2\phi_0}$$



$U(r) = \frac{\alpha}{r}$, $\alpha > 0$

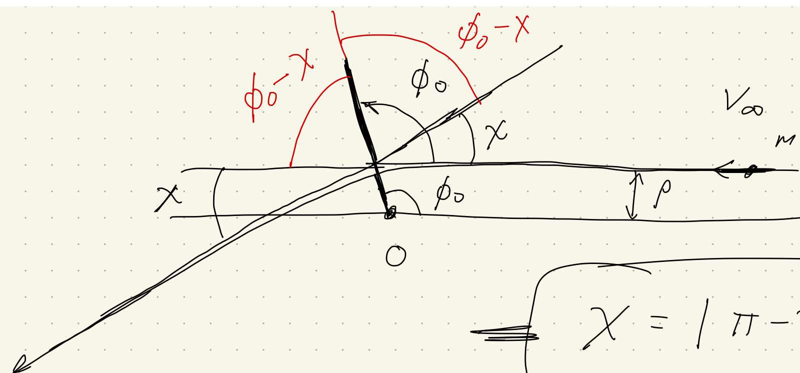
(repulsive)

$$U_{eff}(r) = U(r) + \frac{M_z^2}{2mr^2}$$

$$= \frac{\alpha}{r} + \frac{M_z^2}{2mr^2}$$

$E = U_{eff}(r_{min})$

turning point



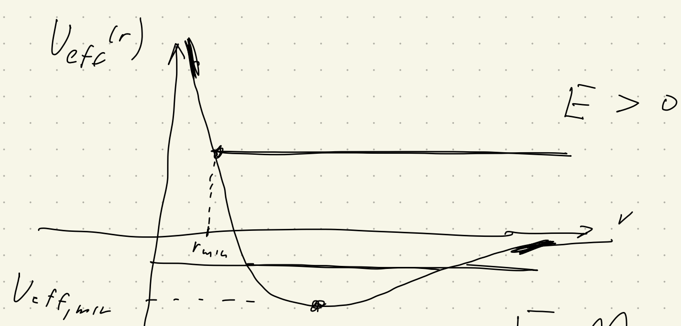
$$2(\phi_0 - \chi) + \chi = \pi$$

$$2\phi_0 - \chi = \pi$$

$$\boxed{\chi = 2\phi_0 - \pi}$$

$$\boxed{\chi = |\pi - 2\phi_0|}$$

for both repulsive & attractive



$$U(r) = -\frac{\alpha}{r}$$

$$U_{eff}(r) = -\frac{\alpha}{r} + \frac{M_z^2}{2mr^2}$$

$$\boxed{E = \frac{1}{2} m v_\infty^2, \quad M = m\rho v_\infty}$$

E, M

ρ, v_∞

Sec 14:

$$t = \pm \int \frac{dr}{\sqrt{}} + \text{const}$$

$$\Rightarrow \quad \phi = \pm \int \frac{M\,dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M^2}{r^2}}} + \text{const}$$

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M\,dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M^2}{r^2}}}$$

$E = U_{eff}(r_{min})$

$M = m\rho v_\infty$

$E = \frac{1}{2} m v_\infty^2$

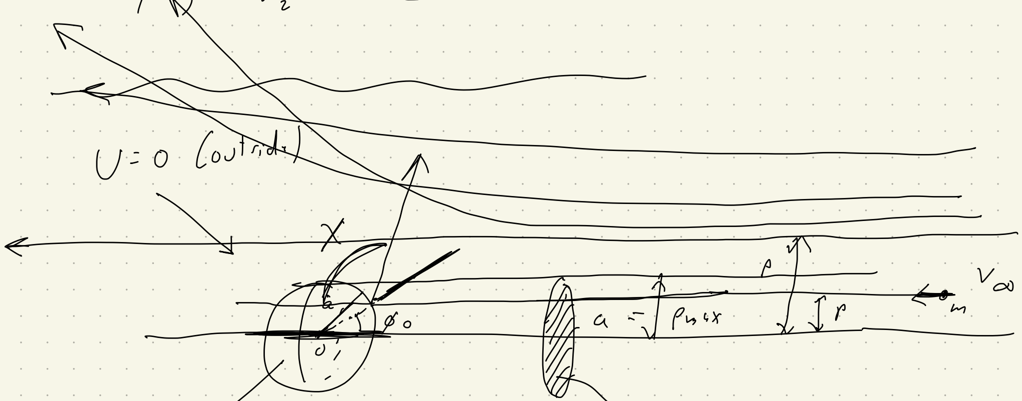
$$= \int_{r_{min}}^{\infty} \frac{m\rho v_\infty\,dr/r^2}{\sqrt{2m\left(\frac{1}{2}mv_\infty^2 - U(r)\right) - \frac{m^2\rho^2 v_\infty^2}{r^2}}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\,dr/r^2}{\sqrt{1 - \frac{U(r)}{\frac{1}{2}mv_\infty^2} - \rho^2/r^2}}$$

$$\theta_2 = \frac{1}{2}(\pi - \chi) \qquad \chi = |\pi - 2\phi_0|$$

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}$$

v_1', v_2' = . . .



$U = \infty$

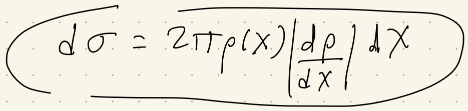
Hard sphere

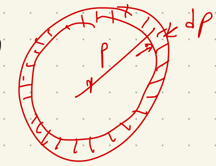
$$\sigma = \pi a^2$$

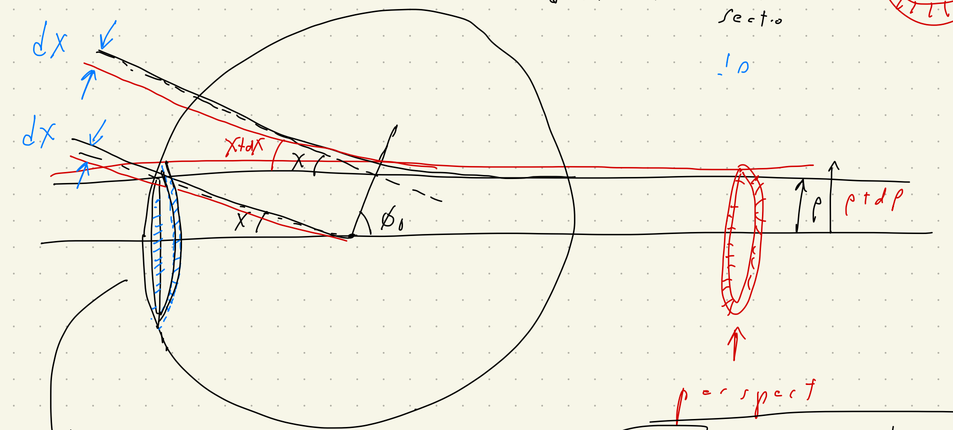
cross-section

$$d\sigma = 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi \leftarrow d\sigma = 2\pi\rho\, d\rho$$

differential cross sectio

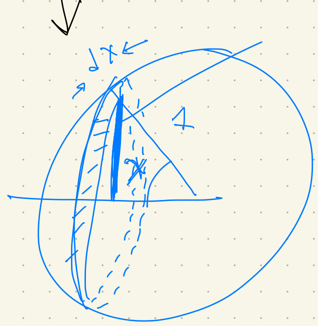






perspect

$$d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$



$$\underbrace{d\Omega}_{\text{solid angle}} = 2\pi \sin\chi\, d\chi$$

$$\frac{d\sigma}{d\Omega} = \frac{\cancel{2\pi}\rho\, d\rho}{\cancel{2\pi}\sin\chi\, d\chi} = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right|$$

$$\frac{d\sigma}{d\Omega} = \frac{\rho}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| \qquad \text{COM frame}$$

$$\frac{d\sigma_1}{d\Omega_1} = \frac{\rho}{\sin\theta_1}\left|\frac{d\rho}{d\theta_1}\right| \qquad \text{Lab frame } \theta_1$$

$$\frac{d\sigma_2}{d\Omega_2} = \frac{\rho}{\sin\theta_2}\left|\frac{d\rho}{d\theta_2}\right| \qquad \text{Lab frame } \theta_2$$

$$\frac{d\sigma_1}{d\Omega_1} = \frac{\sin\chi}{\sin\theta_1}\left|\frac{d\chi}{d\theta_1}\right| \frac{d\sigma}{d\Omega}$$

$$= \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| \frac{d\sigma}{d\Omega}$$

calculated for the COM

Lecture #17: Tues Oct 19th

Next week: start small oscillations

Today / Thurs: Q & A (collisions and scattering)

Lecture #18: Thur Oct 21st

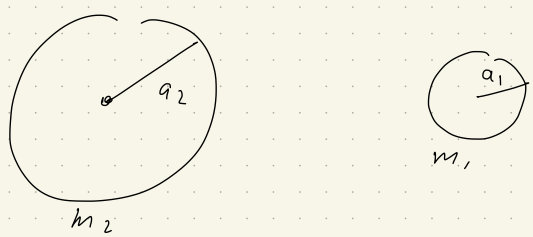
Next week: small oscillations

Today: Q&A, Quiz #3 (last 20 minutes)

$\boxed{\text{Q3\_name.pdf}}$

Quiz #3:

Two hard spheres with masses m_1, m_2 and radii a_1, a_2:



$\rho(\chi) = (a_1 + a_2)\cos\left(\frac{\chi}{2}\right)$

a) Find $\rho = \rho(\chi)$ where χ = scattering angle wrt COM frame

b) What value of ρ will give $\theta_2 = 60°$?

$2\theta_2 + \chi = \pi \rightarrow \chi = \pi - 2\theta_2$
$= 60°$

$\rho = (a_1 + a_2)\cos(30°)$
$= \frac{\sqrt{3}}{2}(a_1 + a_2)$

Lec #19: 10/26

Small oscillation: $\boxed{\text{Sec } 21}$, 22, 23

free oscillation in 1-d

Forced oscillation in 1d

more than 1d

Damping: Sec 25 (not covered)

SHM: (simple harmonic motion) is sinusoidal

$$\omega = \frac{2\pi}{T}$$

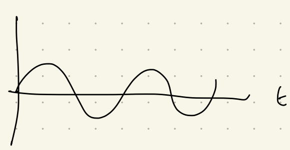
$$x(t) = a \cos(\omega t + \alpha)$$

amplitude

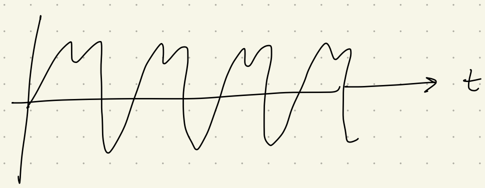
phase

angular freq

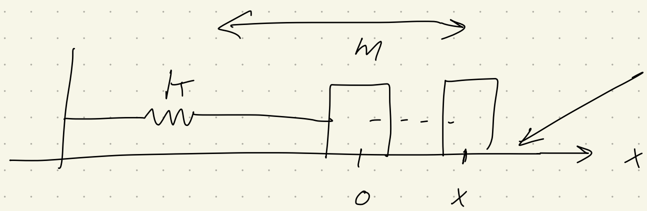
SHM



periodic



Example:



no friction

$$U = \frac{1}{2} k x^2$$

$$F = -\frac{dU}{dx} = -kx$$

$$F = -kx$$

linear

restoring

$$F = m\ddot{x} = -kx \quad \rightarrow \quad \ddot{x} = -\frac{k}{m} x$$

soln: $x = c_1 \cos \omega t + c_2 \sin \omega t$

$$\omega = \sqrt{\frac{k}{m}}$$

Alternative: $x = a \cos(\omega t + \alpha)$

$$x = \mathrm{Re}[A e^{i\omega t}], \quad A = a e^{i\alpha}$$

$$T = \frac{1}{2} m \dot{x}^2$$
$$U = \frac{1}{2} k x^2$$
$$\Big\} \quad L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

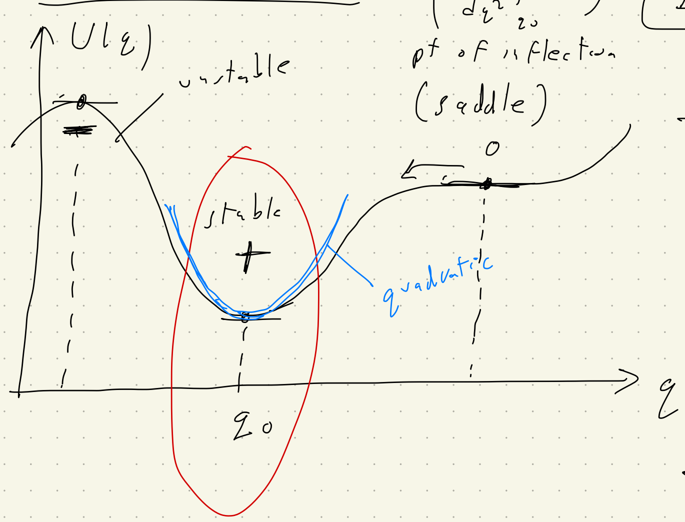
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow m\ddot{x} = -kx$$

$$U(q) = q^3$$
$$q_0 = 0$$

More generally:

$$\boxed{L = \frac{1}{2} a(q)\, \dot{q}^2 - U(q)}$$

$\left(\left.\frac{d^2U}{dq^2}\right|_{q_0} = 0\right)$ pt of inflection (saddle)



Equilibrium: $(q = q_0)$

$$0 = F = -\left.\frac{dU}{dq}\right|_{q_0}$$

stable:

$$\left.\frac{d^2U}{dq^2}\right|_{q_0} > 0$$

$U = $ const

Expansion: (about q_0)

$$U(q) = \underbrace{U(q_0)}_{\substack{\text{const} \\ \text{(ignore in } L)}} + \underbrace{\cancel{\left.\frac{dU}{dq}\right|_{q_0}}}_{\text{equilibrium}}^{0} \cdot (q - q_0) + \frac{1}{2} \underbrace{\left.\frac{d^2U}{dq^2}\right|_{q_0}}_{k > 0} (q - q_0)^2$$
$$\underbrace{+ \frac{1}{3!} \left.\frac{d^3U}{dq^3}\right|_{q_0} (q - q_0)^3 + \cdots}_{\text{ignore for } q - q_0 \text{ small}}$$

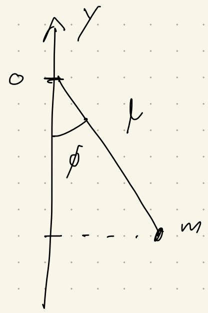
$$\boxed{U(q) \approx \frac{1}{2} k x^2}$$

$$x \equiv q - q_0$$
$$k = \left.\frac{d^2U}{dq^2}\right|_{q_0}$$

$$T = \frac{1}{2} a(q)\, \dot{q}^2$$
$$= \frac{1}{2} a(q_0)\, \dot{x}^2$$
$$\approx \frac{1}{2} m \dot{x}^2$$

$$\boxed{L \approx \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2}$$

Simple pendulum:



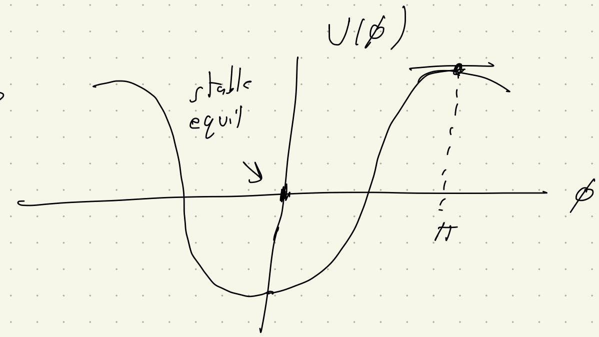
$U = mgy$

$= -mgl\cos\phi$

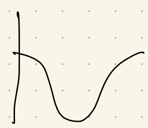
$T = \frac{1}{2} m l^2 \dot{\phi}^2$

$U = -mgl\cos\phi$

$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi$



$\phi_0 = 0$ (equilibrium)



$\cos\phi = 1 - \frac{1}{2}\phi^2 + \dots$

$$\cos\phi = \underbrace{\cos 0}_{1} + \cancel{\underbrace{\frac{d(\cos\phi)}{d\phi}}_{\sin\phi \to 0}}\bigg|_{\phi=0} \cdot \phi + \frac{1}{2}\frac{d^2(\cos\phi)}{d\phi^2}\bigg|_{\phi=0} \phi^2 + \dots$$

$= 1 - \frac{1}{2}\phi^2 + \dots$

$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\left(1 - \frac{1}{2}\phi^2\right)$

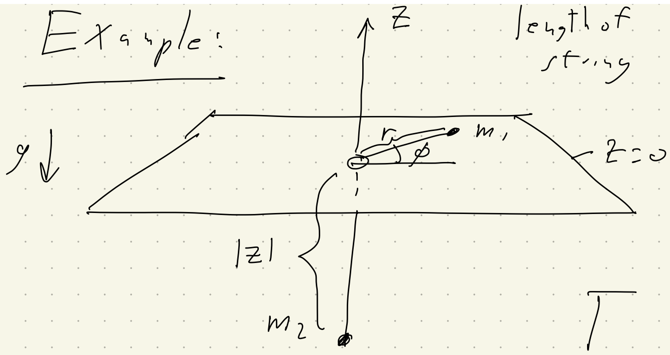
$= \frac{1}{2} \underset{\substack{\| \\ M}}{(ml^2)} \dot{\phi}^2 - \frac{1}{2} \underset{\substack{\| \\ K}}{(mgl)} \phi^2 + \underbrace{mgl}_{\text{ignore}}$

$\omega = \sqrt{\frac{K}{M}}$

$= \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}$

$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2$

Example:



length of string — $\ell = r + |z| = r - z$

z: location of m_2

r: for m_1

$$\boxed{z = r - \ell \rightarrow \dot{z} = \dot{r}}$$

$$T = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}m_2\dot{z}^2$$

$$U = m_2 g z = m_2 g(r-\ell)$$
$$= m_2 g r - m_2 g \ell \quad \text{(ignore) const}$$

$$\boxed{L = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\phi}^2 - m_2 g r}$$

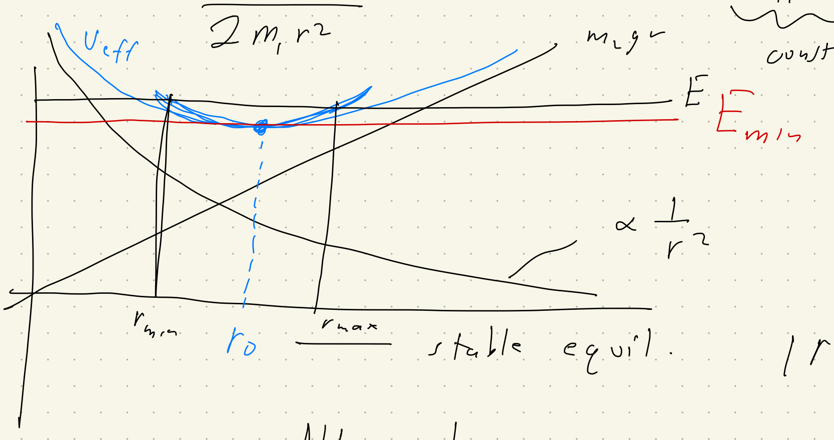
i) $M_z = \dfrac{\partial L}{\partial \dot{\phi}} = m_1 r^2\dot{\phi} \rightarrow \dot{\phi} = \dfrac{M_z}{m_1 r^2}$

ii) $E = T + U = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \underbrace{\frac{1}{2}m_1 r^2\dot{\phi}^2}_{\frac{M_z^2}{2m_1 r^2}} + m_2 g r$

$$E = \frac{1}{2}\underbrace{(m_1+m_2)}_{M}\dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = \frac{M_z^2}{2m_1 r^2} + m_2 g r = \underbrace{U_{eff}(r_0)}_{\text{const}} + \frac{1}{2}k(r-r_0)^2$$

$$k = \left.\frac{d^2 U_{eff}}{dr^2}\right|_{r=r_0}$$



$|r - r_0| \ll r_0$

$r_0 = ?$ $\quad 0 = \left.\dfrac{dU_{eff}}{dr}\right|_{r=r_0}$

$$= -\frac{M_z^2}{m_1 r_0^3} + m_2 g \rightarrow \boxed{M_z^2 = m_1 m_2 g r_0^3}$$

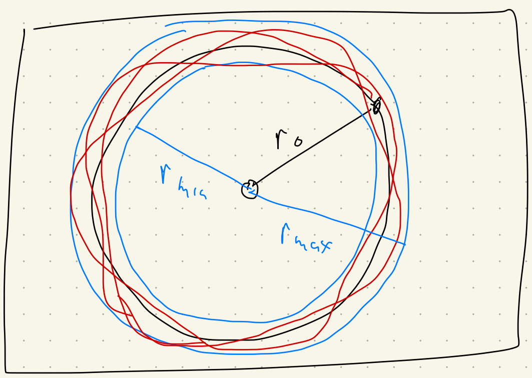
$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r=r_0} = \frac{3M_z^2}{m_1 r_0^4}$$

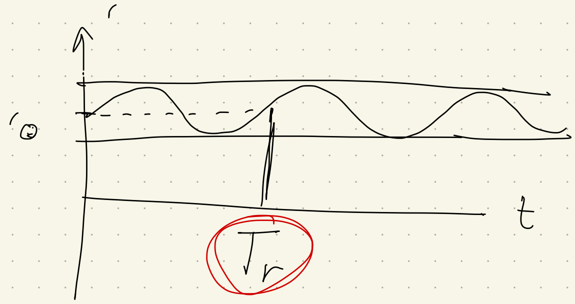
$$= \frac{3\cancel{m_1} m_2 g r_0^3}{\cancel{m_1} r_0^4}$$

$$= \boxed{\frac{3m_2 g}{r_0} = K}$$

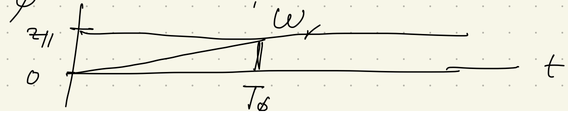
$$\omega_r = \sqrt{\frac{K}{\underbrace{m_1+m_2}_{=M}}}$$

$$= \sqrt{\frac{3m_2 g}{(m_1+m_2)r_0}}$$





$$T_r = \frac{2\pi}{\omega_r}$$



$$T_\phi = \frac{2\pi}{\omega_\phi} = \frac{2\pi}{\left.\dot\phi\right|_{r_0}}$$

$$\dot\phi = \frac{M_z}{m_1 r^2}$$

$$\omega_\phi = \left.\dot\phi\right|_{r=r_0} = \frac{M_z}{m_1 r_0^2} = \frac{\sqrt{m_1 m_2 g r_0^3}}{m_1 r_0^2}$$

$$\boxed{\omega_\phi = \sqrt{\frac{m_2 g}{m_1 r_0}}}$$

compare to

$$\boxed{\omega_r = \sqrt{\frac{3m_2 g}{(m_1+m_2) r_0}}}$$

If $\frac{3}{m_1+m_2} = \frac{1}{m_1}$ then $\omega_r = \omega_\phi$

$$3m_1 = m_1 + m_2$$

$$\boxed{2m_1 = m_2}$$

Lec #20: 10/28

Forced oscillation:

$$m\ddot{x} = -kx + F(t)$$

$$\ddot{x} + \frac{k}{m}x = \frac{F(t)}{m}$$

$$\boxed{\ddot{x} + \omega^2 x = \frac{F(t)}{m}}, \quad \omega \equiv \sqrt{\frac{k}{m}}$$

$U(x) = \frac{1}{2}kx^2 - x\,F(t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$L = T - U$$

$$\frac{\partial L}{\partial x} = -kx + F(t)$$

general sol'n:

$$x(t) = x_h(t) + x_p(t)$$

x_h: homogeneous $(F(t)=0)$

x_p: particular (any solution for $F(t)$)

$$\boxed{x_h(t) = a\cos(\omega t + \alpha)}$$

a, α: two constants (Initial condition)

Suppose: $F(t) = f\cos(\gamma t + \beta)$

$$\ddot{x}_p + \omega^2 x_p = \frac{f}{m}\cos(\gamma t + \beta)$$

<u>Guess!!</u> $x_p(t) = b\cos(\gamma t + \beta)$

$$-b\gamma^2\cancel{\cos(\gamma t + \beta)} + \omega^2 b\,\cancel{\cos(\gamma t + \beta)} = \frac{f}{m}\cancel{\cos(\gamma t + \beta)}$$

$$b(\omega^2 - \gamma^2) = \frac{f}{m}$$

$$\to \boxed{b = \frac{f}{m(\omega^2 - \gamma^2)}}$$

$+ \; 10\cos(\omega t)$

$$\boxed{x_p(t) = \frac{f}{m(\omega^2 - \gamma^2)}\cos(\gamma t + \beta)}$$

$$\boxed{x(t) = a\cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)}\cos(\gamma t + \beta)}$$

$$x(t) = a\cos(\omega t + \alpha) + \underbrace{\frac{f}{m(\omega^2 - \gamma^2)}\Big[\cos(\gamma t + \beta) - \cos(\omega t + \beta)\Big]}_{\frac{0}{0}}$$

L'hopital's

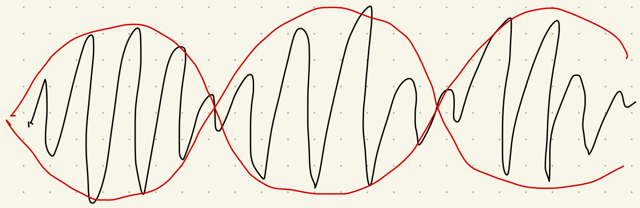
$$= \frac{\frac{d}{d\gamma}(\text{num})\Big|_{\gamma \to \omega}}{\frac{d}{d\gamma}(\text{den})\Big|_{\gamma \to \omega}}$$

$$= \frac{+\,f t \sin(\omega t + \beta)}{+\,2m\omega}$$

at resonance $(\gamma = \omega)$

$$= \frac{ft}{2m\omega}\sin(\omega t + \beta)$$

$$\boxed{x(t) = a\cos(\omega t + \alpha) + \frac{ft}{2m\omega}\sin(\omega t + \beta)}$$



$$\frac{\omega + \gamma}{2}$$

$$|\omega - \gamma| = \omega_{beat}$$

General: for arbitrary $F(t)$

$$F(t) = \mathrm{Re}\underbrace{\int_{-\infty}^{\infty} d\gamma\, \tilde{F}(\gamma)\, e^{i\gamma t}}_{\text{Fourier transform}}$$

$$\boxed{\ddot{x} + \omega^2 x = \frac{F(t)}{m}}$$

$$\boxed{y' + P(x)y = Q(x)}$$

Math methods

$y = y(x)$

Let: $\xi = \dot{x} + i\omega x$

complex

$$\dot{\xi} = \ddot{x} + i\omega\dot{x}$$
$$= \ddot{x} + i\omega(\xi - i\omega x)$$
$$= \ddot{x} + i\omega\xi + \omega^2 x$$

$$\rightarrow \dot{\xi} - i\omega\xi = \ddot{x} + \omega^2 x = \frac{F(t)}{m}$$

Homog: $\dot{\xi}_h - i\omega\xi_h = 0 \rightarrow \xi_h(t) = A e^{i\omega t}$

complex constant

replace

Guess: $\boxed{\xi_p(t) = A(t)e^{i\omega t}}$

$$\boxed{\dot{\xi}_p - i\omega \xi_p = \frac{F(t)}{m}}, \quad \xi_p = A(t)e^{i\omega t}$$

$$\rightarrow \dot{A}e^{i\omega t} + \cancel{iA\omega e^{i\omega t}} \cancel{-i\omega A e^{i\omega t}} = \frac{F(t)}{m}$$

$$\dot{A} = \frac{e^{-i\omega t}F(t)}{m}$$

$$A(t) = \int dt \, \frac{F(t)}{m} e^{-i\omega t} + \text{const}$$

$$\boxed{\xi(t) = e^{i\omega t}\left[\int_0^t d\bar{t} \, \frac{F(\bar{t})}{m} e^{-i\omega \bar{t}} + \xi_0\right]}$$

$$\xi = \dot{x} + i\omega x \rightarrow \boxed{x(t) = \frac{1}{\omega} \operatorname{Im}(\xi(t))}$$

Complex constant (I.C.'s)

$$y' + P(x)y = Q(x) \qquad f(x)\,dx = g(y)\,dy$$

$$\frac{dy}{dx} + P(x)y - Q(x) = 0$$

$$\boxed{1\,dy + (P(x)y - Q(x))\,dx = 0} \neq dU$$

$$dU(x,y) = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

$$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y}$$

M. Boas ~~Mathemat~~

$$\boxed{\mu(x)\left[dy + (P(x)y - Q(x))dx\right] = dU}$$

$$\frac{d\mu}{dx} = \frac{\partial}{\partial y}\left(P(x)\,y - Q(x) \right)\mu(x)$$

$$\frac{d\mu}{dx} = P(x)\,\mu(x)$$

$$\int \frac{d\mu}{\mu} = \int P(x)\,dx$$

$$\ln \mu = \int P(x)\,dx$$

$$\mu(x) = e^{\int P(x)\,dx}$$