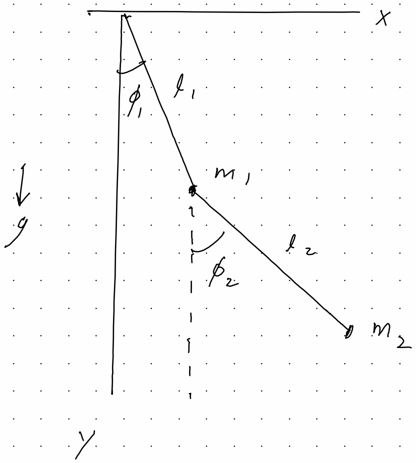
sec 5, prob 1



$$x_1 = l_1 \sin\phi_1$$
$$y_1 = l_1 \cos\phi_1$$
$$x_2 = x_1 + l_2 \sin\phi_2 = l_1 \sin\phi_1 + l_2 \sin\phi_2$$
$$y_2 = y_1 + l_2 \cos\phi_2 = l_1 \cos\phi_1 + l_2 \cos\phi_2$$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_1 g l_1 \cos\phi_1 - m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$
$$= -(m_1+m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\dot{x}_1 = l_1 \cos\phi_1 \dot{\phi}_1 \qquad \dot{y}_1 = -l_1 \sin\phi_1 \dot{\phi}_1$$
$$\dot{x}_1^2 = l_1^2 \cos^2\phi_1 \dot{\phi}_1^2 \qquad \dot{y}_1^2 = l_1^2 \sin^2\phi_1 \dot{\phi}_1^2$$

Thus, $\dot{x}_1^2 + \dot{y}_1^2 = l_1^2 (\sin^2\phi_1 + \cos^2\phi_1) \dot{\phi}_1^2$
$$= l_1^2 \dot{\phi}_1^2$$

$$\dot{x}_2 = l_1 \cos\phi_1 \dot{\phi}_1 + l_2 \cos\phi_2 \dot{\phi}_2$$
$$\rightarrow \dot{x}_2^2 = l_1^2 \cos^2\phi_1 \dot{\phi}_1^2 + l_2^2 \cos^2\phi_2 \dot{\phi}_2^2 + 2 l_1 l_2 \cos\phi_1 \cos\phi_2 \dot{\phi}_1 \dot{\phi}_2$$
$$\dot{y}_2 = -l_1 \sin\phi_1 \dot{\phi}_1 - l_2 \sin\phi_2 \dot{\phi}_2$$
$$\rightarrow \dot{y}_2^2 = l_1^2 \sin^2\phi_1 \dot{\phi}_1^2 + l_2^2 \sin^2\phi_2 \dot{\phi}_2^2 + 2 l_1 l_2 \sin\phi_1 \sin\phi_2 \dot{\phi}_1 \dot{\phi}_2$$

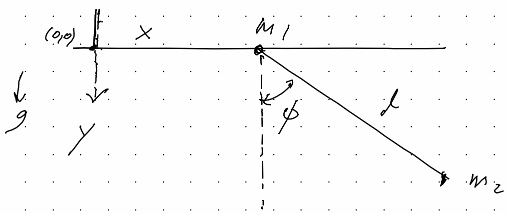
Thus,
$$\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2) \dot{\phi}_1 \dot{\phi}_2$$
$$= l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

so $T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$
$$= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2$$
$$+ m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$
$$= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

$$U = -(m_1 + m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

$$L = T - U$$
$$= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$
$$+ (m_1 + m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2$$

Sec 5 Prob. 2



Generalised coords: x, ϕ

$(x_1, y_1) = (x, 0)$

$(x_2, y_2) = (x + l\sin\phi \;,\; l\cos\phi)$

$U = -m_1 g y_1 - m_2 g y_2$

$= -m_2 g l \cos\phi$

$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$

Now: $\dot{x}_1^2 + \dot{y}_1^2 = \dot{x}^2$

$\dot{x}_2^2 + \dot{y}_2^2 = (\dot{x} + l\cos\phi\,\dot{\phi})^2 + (-l\sin\phi\,\dot{\phi})^2$

$= \dot{x}^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi}$

$+ l^2\sin^2\phi\,\dot{\phi}$

$= \dot{x}^2 + l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi}$

$\rightarrow T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}^2 + l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi})$

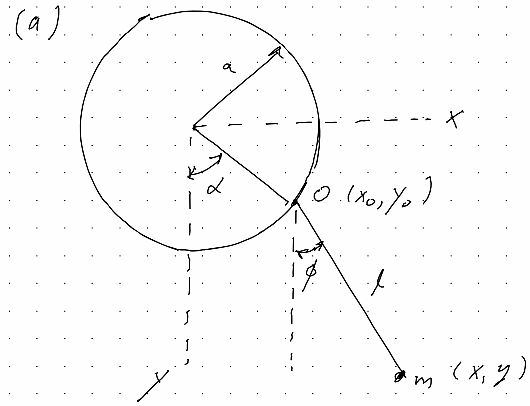
$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \cos\phi\,\dot{x}\dot{\phi}$

$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l\cos\phi\,\dot{x}\dot{\phi}$

$+ m_2 g l \cos\phi$

Sec 5, Prob 3

(a)



point of support O moves along circle:

$x_0 = a\sin\alpha \quad , \quad y_0 = a\cos\alpha$

where $\alpha = \gamma t \quad , \quad \gamma = \text{const}$

pendulum bob:

(x, y): $\quad x = x_0 + l\sin\phi$

$y = y_0 + l\cos\phi$

$U = -mgy = \underbrace{-mgy_0}_{\text{specified function of time}} - mgl\cos\phi$

[can ignore in L]

$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

$\dot{x} = \dot{x}_0 + l\cos\phi\,\dot{\phi}$

$\dot{x}^2 = \dot{x}_0^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}_0\dot{\phi}$

$\dot{y} = \dot{y}_0 - l\sin\phi\,\dot{\phi}$

$\dot{y}^2 = \dot{y}_0^2 + l^2\sin^2\phi\,\dot{\phi}^2 - 2l\sin\phi\,\dot{y}_0\dot{\phi}$

Thus,

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$
$$= \frac{1}{2}m\left(\dot{x}_0^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}_0\dot{\phi} + \dot{y}_0^2 + l^2\sin^2\phi\,\dot{\phi}^2 - 2l\sin\phi\,\dot{y}_0\dot{\phi}\right)$$
$$= \frac{1}{2}m(\dot{x}_0^2+\dot{y}_0^2) + \frac{1}{2}ml^2\dot{\phi}^2 + ml\dot{\phi}(\dot{x}_0\cos\phi - \dot{y}_0\sin\phi)$$

NOTE: $\dot{x}_0^2+\dot{y}_0^2 = a^2\dot{\alpha}^2 = a^2\gamma^2$

since this is a specified function of time, we can ignore it in the Lagrangian.

Thus, $$L = \frac{1}{2}ml^2\dot{\phi}^2 + ml\dot{\phi}(\dot{x}_0\cos\phi - \dot{y}_0\sin\phi) + mgl\cos\phi$$

We can rewrite the second term:

$x_0 = a\sin\alpha \rightarrow \dot{x}_0 = a\cos\alpha\,\dot{\alpha}$ $\quad(\dot{\alpha}\equiv\gamma)$

$y_0 = a\cos\alpha \rightarrow \dot{y}_0 = -a\sin\alpha\,\dot{\alpha}$

Thus,

$$ml\dot{\phi}(\dot{x}_0\cos\phi - \dot{y}_0\sin\phi) = ml\dot{\phi}a\gamma(\cos\alpha\cos\phi + \sin\alpha\sin\phi)$$
$$= ml\dot{\phi}a\gamma\cos(\phi-\alpha)$$
$$= ml\dot{\phi}a\gamma\cos(\phi-\gamma t)$$

Now: $\frac{d}{dt}[mla\gamma\sin(\phi-\gamma t)]$

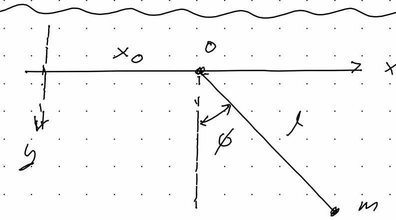
$$= mla\gamma\cos(\phi-\gamma t)(\dot{\phi}-\gamma)$$
$$= mla\dot{\phi}\gamma\cos(\phi-\gamma t) - mla\gamma^2\cos(\phi-\gamma t)$$

Thus,

$$mla\dot{\phi}\gamma\cos(\phi-\gamma t) = \frac{d}{dt}[mla\gamma\sin(\phi-\gamma t)] + mla\gamma^2\cos(\phi-\gamma t)$$

(and we can ignore the total time derivative in the Lagrangian)

$$\rightarrow L = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi + mla\gamma^2\cos(\phi-\gamma t)$$

(b) 

point O moving according to $x_0 = a\cos\gamma t$

$x = x_0 + l\sin\phi$

$y = l\cos\phi$

$U = -mgy = -mgl\cos\phi$

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{x}_0 + l\cos\phi\,\dot{\phi}\,, \qquad x_0 = a\cos\gamma t$$

$$= -a\sin(\gamma t)\,\gamma + l\cos\phi\,\dot{\phi}$$

$$\rightarrow \dot{x}^2 = a^2\gamma^2\sin^2\gamma t + l^2\cos^2\phi\,\dot{\phi}^2 - 2al\gamma\dot{\phi}\sin(\gamma t)\cos\phi$$

$$\dot{y} = -l\sin\phi\,\dot{\phi}$$

$$\rightarrow \dot{y}^2 = l^2\sin^2\phi\,\dot{\phi}^2$$

Thus, $T = \frac{1}{2}m\left(a^2\gamma^2\sin^2\gamma t + l^2\cos^2\phi\,\dot{\phi}^2 - 2al\gamma\dot{\phi}\sin(\gamma t)\cos\phi + l^2\sin^2\phi\,\dot{\phi}^2\right)$

$$= \frac{1}{2} m l^2\dot{\phi}^2 + \frac{1}{2} m a^2\gamma^2\sin^2\gamma t - mal\gamma\dot{\phi}\sin(\gamma t)\cos\phi$$

specified function of time (ignore)

$$L = \frac{1}{2}ml^2\dot{\phi}^2 - mal\gamma\dot{\phi}\sin(\gamma t)\cos\phi + mgl\cos\phi$$

2nd term:

$$-\frac{d}{dt}\left[mal\gamma\sin(\gamma t)\sin\phi\right]$$

$$= -mal\gamma^2\cos(\gamma t)\sin\phi - mal\gamma\dot{\phi}\sin(\gamma t)\cos\phi$$

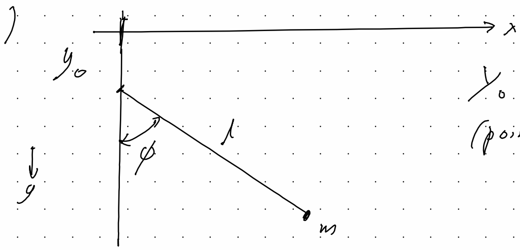
so

$$-mal\gamma\dot{\phi}\sin(\gamma t)\cos\phi = \underbrace{-\frac{d}{dt}[\;]}_{\text{ignore}} + mal\gamma^2\cos(\gamma t)\sin\phi$$

Thus, ignoring total time derivative,

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi + mal\gamma^2\cos(\gamma t)\sin\phi$$

(c)



$Y_0 = a\cos\gamma t$ (point of support)

$$x = l\sin\phi$$

$$y = y_0 + l\cos\phi$$

$$= a\cos\gamma t + l\cos\phi$$

$$U = -mgy$$

$$= \underbrace{-mga\cos\gamma t}_{} - mgl\cos\phi$$

specified function of time [can ignore]

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = l\cos\phi\,\dot{\phi}$$

$$\dot{x}^2 = l^2\cos^2\phi\,\dot{\phi}^2$$

$$\dot{y} = -a\gamma\sin(\gamma t) - l\sin\phi\,\dot{\phi}$$

$$\dot{y}^2 = \underbrace{a^2\gamma^2\sin^2(\gamma t)}_{} + l^2\sin^2\phi\,\dot{\phi}^2 + 2al\gamma\sin(\gamma t)\sin\phi\,\dot{\phi}$$

specified function of time [can ignore]

Thus, ignoring this function of time

$$T = \frac{1}{2} m l^2 \dot\phi^2 + m a l \dot\phi \sin(\gamma t) \sin\phi$$

$$\rightarrow L = T - U$$

$$= \frac{1}{2} m l^2 \dot\phi^2 + m a l \gamma \dot\phi \sin(\gamma t) \sin\phi + m g l \cos\phi$$

Rewrite 2nd term:

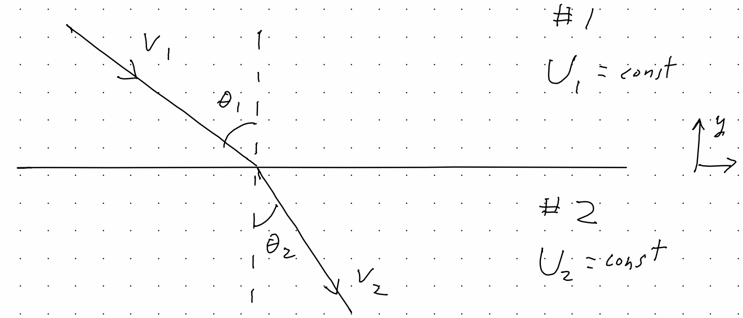
$$-\frac{d}{dt}[m a l \gamma \sin(\gamma t) \cos\phi] = -m a l \gamma^2 \cos(\gamma t) \cos\phi + m a l \gamma \sin(\gamma t) \sin\phi \, \dot\phi$$

so $m a l \gamma \dot\phi \sin(\gamma t) \sin\phi = -\frac{d}{dt}[\ ] + m a l \gamma^2 \cos(\gamma t) \cos\phi$

Thus, ignoring total time derivative

$$L = \frac{1}{2} m l^2 \dot\phi^2 + m g l \cos\phi + m a l \gamma^2 \cos(\gamma t) \cos\phi$$

Sec 7, Prob 1



#1

U_1 = const

#2

U_2 = const

- Energy conserved, since no time dependence.
- Also momentum in x-direction (// to interface) is conserved, since no x-dependence of the potential

$$U(x,y) = \begin{cases} U_1 & y \ge 0 \\ U_2 & y < 0 \end{cases}$$

v_1: given

$$E = \frac{1}{2} m v_1^2 + U_1 = \frac{1}{2} m v_2^2 + U_2$$

$$\rightarrow \frac{1}{2} m v_2^2 = \frac{1}{2} m v_1^2 + (U_1 - U_2)$$

$$v_2^2 = v_1^2 + \frac{2(U_1 - U_2)}{m}$$

so, $$v_2 = v_1 \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

The angles θ_1, θ_2 are related by

$$p_{1x} = p_{2x}$$

$$\cancel{m} v_1 \sin\theta_1 = \cancel{m} v_2 \sin\theta_2$$

thus, $\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{v_2}{v_1} = \sqrt{1 + \dfrac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$

Sec 8, Prob 1

Transformation of action $S = \int L\,dt$

K, K': two inertial frames

K' moves with velocity $\vec{V}$ w.r.t K

Assume that K, K' coincide at $t=0$ so $\vec{r}_a = \vec{r}_a{}'$ wrt these two frames

Now: $\vec{v}_a = \vec{V} + \vec{v}_a{}'$

$$L = T - U$$
$$= \sum_a \frac{1}{2} m_a |\vec{v}_a|^2 - U(\vec{r}_1, \vec{r}_2, \ldots, t)$$

$$|\vec{v}_a|^2 = |\vec{V} + \vec{v}_a{}'|^2$$
$$= |\vec{V}|^2 + |\vec{v}_a{}'|^2 + 2\vec{V} \cdot \vec{v}_a{}'$$

so

$$L = \sum_a \frac{1}{2} m_a \left(|\vec{V}|^2 + |\vec{v}_a{}'|^2 + 2\vec{V} \cdot \vec{v}_a{}'\right) - U$$
$$= \frac{1}{2}\mu V^2 + T' + \vec{V} \cdot \sum_a m_a \vec{v}_a{}' - U$$

$$= T' - U + \frac{1}{2}\mu V^2 + \vec{P}' \cdot \vec{V}$$
$$= L' + \frac{1}{2}\mu V^2 + \vec{P}' \cdot \vec{V}$$

where $\vec{P}'$ = total momentum wrt K'

$\mu = \sum_a m_a$ = total mass

$$S = \int_{t_1}^{t_2} L\,dt$$

$$= \int_{t_1}^{t_2} \left(L' + \tfrac{1}{2}\mu V^2 + \vec{P}'\cdot\vec{V}\right) dt$$

$$= S' + \underbrace{\tfrac{1}{2}\mu V^2 (t_2 - t_1)}_{\text{doesn't change EOMs}} + \vec{V}\cdot\int_{t_1}^{t_2}\vec{P}'\,dt$$

Now:

$$\vec{V}\cdot\int_{t_1}^{t_2}\vec{P}'\,dt = \vec{V}\cdot\int_{t_1}^{t_2}\sum_a m_a \vec{v}_a'\,dt$$

$$= \vec{V}\cdot\sum_a m_a \int_{t_1}^{t_2}\left(\frac{d\vec{r}_a}{dt}\right)dt$$

$$= \vec{V}\cdot\sum_a m_a \vec{r}_a\Big|_{t_1}^{t_2}$$

$$= \vec{V}\cdot\left(\mu\vec{R}(t_2) - \mu\vec{R}(t_1)\right)$$

$$= \mu\vec{V}\cdot\left(\underbrace{\vec{R}(t_2) - \vec{R}(t_1)}_{\text{difference in COM position}}\right)$$

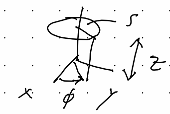
So:

$$S = S' + \tfrac{1}{2}\mu V^2(t_2 - t_1) + \mu\vec{V}\cdot\left(\vec{R}(t_2) - \vec{R}(t_1)\right)$$

Sec 9, Prob. 1

cylindrical coordinates (s, ϕ, z)

$$s^2 = x^2 + y^2$$



$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = z$$

$$\vec{M} = \vec{r}\times\vec{p} = m\vec{r}\times\dot{\vec{r}}$$

Now: $M_x = m(y\dot{z} - z\dot{y})$

$$M_y = m(z\dot{x} - x\dot{z})$$
$$M_z = m(x\dot{y} - y\dot{x})$$
$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}$$

$$\dot{z} = \dot{z}$$
$$\dot{y} = \dot{s}\sin\phi + s\cos\phi\,\dot{\phi}$$
$$\dot{x} = \dot{s}\cos\phi - s\sin\phi\,\dot{\phi}$$

Thus, $M_x = m\left(s\sin\phi\,\dot{z} - z(\dot{s}\sin\phi + s\cos\phi\,\dot{\phi})\right)$

$$= m(s\sin\phi\,\dot{z} - z\sin\phi\,\dot{s} - zs\cos\phi\,\dot{\phi})$$

$$M_y = m\left(z(\dot{s}\cos\phi - s\sin\phi\,\dot{\phi}) - s\cos\phi\,\dot{z}\right)$$

$$= m(z\cos\phi\,\dot{s} - zs\sin\phi\,\dot{\phi} - s\cos\phi\,\dot{z})$$

$$M_z = m\left[s\cos\phi\,(\cancel{\dot{s}\sin\phi} + s\cos\phi\,\dot{\phi}) - s\sin\phi\,(\cancel{\dot{s}\cos\phi} - s\sin\phi\,\dot{\phi})\right]$$

$$= ms^2\dot{\phi}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$= m^2 \Big\{ \left(\sin\phi\,(s\dot z - z\dot s) - zs\cos\phi\,\dot\phi\right)^2$$

$$+ \left(-\cos\phi\,(s\dot z - z\dot s) - zs\sin\phi\,\dot\phi\right)^2$$

$$+ \left(s^2\dot\phi\right)^2 \Big\}$$

$$= m^2 \Big\{ \sin^2\phi\,(s\dot z - z\dot s)^2 + z^2 s^2 \cos^2\phi\,\dot\phi^2$$

$$\cancel{- 2zs\sin\phi\cos\phi\,\dot\phi\,(s\dot z - z\dot s)}$$

$$+ \cos^2\phi\,(s\dot z - z\dot s)^2 + z^2 s^2 \sin^2\phi\,\dot\phi^2$$

$$\cancel{+ 2zs\sin\phi\cos\phi\,\dot\phi\,(s\dot z - z\dot s)}$$

$$+ s^4\dot\phi^2 \Big\}$$

$$= m^2 \left\{ (s\dot z - z\dot s)^2 + z^2 s^2 \dot\phi^2 + s^4\dot\phi^2 \right\}$$

$$= m^2 \left[(s\dot z - z\dot s)^2 + s^2\dot\phi^2 (z^2 + s^2) \right]$$

Sec 9, Prob 2

repeat for spherical polar coords.

$M_x = m(y\dot z - z\dot y)$, cyclic

$M^2 = M_x^2 + M_y^2 + M_z^2$

Now: $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$\rightarrow \dot x = \dot r\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot\theta - r\sin\theta\sin\phi\,\dot\phi$

$\dot y = \dot r\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot\theta + r\sin\theta\cos\phi\,\dot\phi$

$\dot z = \dot r\cos\theta - r\sin\theta\,\dot\theta$

Thus,

$$M_x = m(y\dot z - z\dot y)$$

$$= m\Big\{ r\sin\theta\sin\phi\,(\cancel{\dot r\cos\theta} - r\sin\theta\,\dot\theta)$$

$$- r\cos\theta\,(\cancel{\dot r\sin\theta\sin\phi} + r\cos\theta\sin\phi\,\dot\theta + r\sin\theta\cos\phi\,\dot\phi)\Big\}$$

$$= m\left\{ -r^2\sin^2\theta\sin\phi\,\dot\theta - r^2\cos^2\theta\sin\phi\,\dot\theta - r^2\sin\theta\cos\theta\cos\phi\,\dot\phi \right\}$$

$$= m\left\{ -r^2\sin\phi\,\dot\theta - r^2\sin\theta\cos\theta\cos\phi\,\dot\phi \right\}$$

$$= -mr^2\left[\sin\phi\,\dot\theta + \sin\theta\cos\theta\cos\phi\,\dot\phi \right]$$

$$M_y = m(z\dot{x} - x\dot{z})$$

$$= m\{ r\cos\theta\,(\cancel{\dot{r}\sin\theta\cos\phi} + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi})$$

$$- r\sin\theta\cos\phi\,(\cancel{\dot{r}\cos\theta} - r\sin\theta\,\dot{\theta}) \}$$

$$= m\{ r^2\cos^2\theta\cos\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\sin\phi\,\dot{\phi} + r^2\sin^2\theta\cos\phi\,\dot{\theta} \}$$

$$= m[r^2\cos\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\sin\phi\,\dot{\phi}]$$

$$= mr^2 [\cos\phi\,\dot{\theta} - \sin\theta\cos\theta\sin\phi\,\dot{\phi}]$$

$$M_z = m(x\dot{y} - y\dot{x})$$

$$= m\{ r\sin\theta\cos\phi\,(\cancel{\dot{r}\sin\theta\sin\phi} + \cancel{r\cos\theta\sin\phi\,\dot{\theta}} + r\sin\theta\cos\phi\,\dot{\phi})$$

$$- r\sin\theta\sin\phi\,(\cancel{\dot{r}\sin\theta\cos\phi} + \cancel{r\cos\theta\cos\phi\,\dot{\theta}} - r\sin\theta\sin\phi\,\dot{\phi}) \}$$

$$= m[r^2\sin^2\theta\cos^2\phi\,\dot{\phi} + r^2\sin^2\theta\sin^2\phi\,\dot{\phi}]$$

$$= mr^2\sin^2\theta\,\dot{\phi}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$= m^2r^4 [\sin\phi\,\dot{\theta} + \sin\theta\cos\theta\cos\phi\,\dot{\phi}]^2$$

$$+ m^2r^4 [\cos\phi\,\dot{\theta} - \sin\theta\cos\theta\sin\phi\,\dot{\phi}]^2$$

$$+ m^2r^4\sin^4\theta\,\dot{\phi}^2$$

cross terms will cancel

$$M^2 = m^2r^4 \{ \sin^2\phi\,\dot{\theta}^2 + \sin^2\theta\cos^2\theta\cos^2\phi\,\dot{\phi}^2$$

$$+ \cos^2\phi\,\dot{\theta}^2 + \sin^2\theta\cos^2\theta\sin^2\phi\,\dot{\phi}^2$$

$$+ \sin^4\theta\,\dot{\phi}^2 \}$$

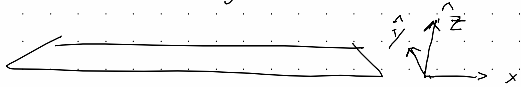
$$= m^2r^4 [\dot{\theta}^2 + \sin^2\theta\cos^2\theta\,\dot{\phi}^2 + \sin^4\theta\,\dot{\phi}^2]$$

$$= m^2r^4 [\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2 (\cos^2\theta + \sin^2\theta)]$$

$$= m^2r^4 [\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2]$$

Sec 9, Prob 3

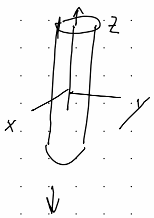
a) Infinite homogeneous plane



P_x, P_y conserved

M_z conserved, where origin is <u>anywhere</u> in (x,y) plane

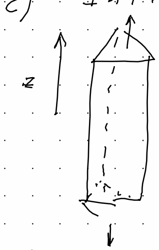
b) Infinite homogeneous cylinder



P_z conserved

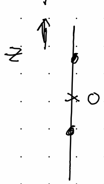
M_z conserved, origin <u>anywhere</u> on z-axis

c) Infinite homog prism

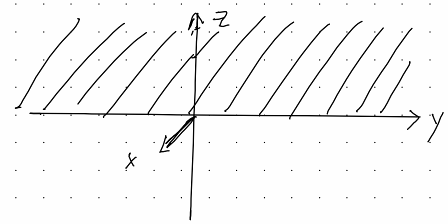


P_z conserved

d) two points : M_z conserved, origin at midpoint of line connecting the two points



e)



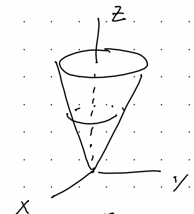
infinite, homog. half plane

$x = 0,\ z > 0$

$-\infty < y < \infty$

P_y conserved

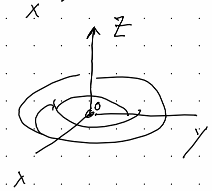
f)



homogeneous cone

M_z conserved, origin <u>anywhere</u> on z-axis.

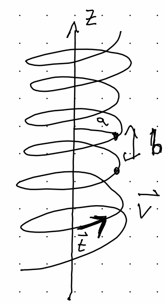
g)



homogeneous circular torus

M_z conserved with $\hat z$ directed as shown and origin at O.

h)



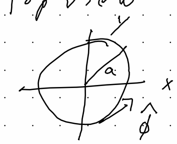
Field constant along helix:

$\Delta z = b$ when $\Delta\phi = 2\pi$ at $s^2 = x^2 + y^2 = a^2$

$$\frac{\Delta\phi}{2\pi} = \frac{\Delta z}{b}$$

$$\rightarrow \Delta z = \left(\frac{b}{2\pi}\right)\Delta\phi$$

top view



$$\vec t = a\Delta\phi\,\hat\phi + \Delta z\,\hat z$$

$$= a\Delta\phi\,\frac{(x\hat y - y\hat x)}{s} + \Delta z\,\hat z$$

$$\vec{t} = \Delta\phi\,(x\hat{y} - y\hat{x}) + \left(\frac{b}{2\pi}\right)\Delta\phi\,\hat{z}$$

$$= \Delta\phi\left[x\hat{y} - y\hat{x} + \left(\frac{b}{2\pi}\right)\hat{z}\right]$$

Field unchanged if you move along $\vec{t}$.

Thus, $\vec{P}\cdot\vec{t} = \text{const}$

$$\vec{P}\cdot\vec{t} = \Delta\phi\left[xP_y - yP_x + \left(\frac{b}{2\pi}\right)P_z\right]$$

$$= \Delta\phi\left[M_z + \frac{b}{2\pi}P_z\right]$$

so $M_z + \frac{b}{2\pi}P_z = \text{const}$

where z = axis of helix

$b = \Delta z$ for $\Delta\phi = 2\pi$ at $s = a$

Sec 10, prob 1

Different masses, same path, same potential energy

$$L_1 = \frac{1}{2}m_1 v_1^2 - U$$

$$L_2 = \frac{1}{2}m_2 v_2^2 - U$$

Thus,

$$m_1 v_1^2 = m_2 v_2^2$$

$$\frac{m_1}{t_1^2} = \frac{m_2}{t_2^2}$$

$$\rightarrow \left(\frac{t_2}{t_1}\right)^2 = \frac{m_2}{m_1}$$

or $$\boxed{\frac{t_2}{t_1} = \sqrt{\frac{m_2}{m_1}}}$$

Sec 10, Prob 2:

Same path, mass but potential energies differing by a constant

$$L_1 = \frac{1}{2} m v_1^2 - U_1$$
$$L_2 = \frac{1}{2} m v_2^2 - U_2$$

Thus, $\dfrac{v_1^2}{v_2^2} = \dfrac{U_1}{U_2}$

$$\rightarrow \frac{(l/t_1)^2}{(l/t_2)^2} = \frac{U_1}{U_2}$$

so $\dfrac{t_2}{t_1} = \sqrt{\dfrac{U_1}{U_2}}$

sec 40 - Prob 1

single particle in a constant external field

$$L = \frac{1}{2} m v^2 - U(\vec{r})$$

a) Cartesian coords (x, y, z)

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x,y,z)$$

$$\rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = p_x/m$$
$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \rightarrow \dot{y} = p_y/m$$
$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$H = \left(\sum_i p_i \dot{q}_i - L\right)\Big|_{\dot{q} = \dot{q}(q,p)}$$

$$= \left(p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(x,y,z)\right)\Big|_{\dot{x} = p_x/m \text{ etc}}$$

$$= p_x\left(\frac{p_x}{m}\right) + p_y\left(\frac{p_y}{m}\right) + p_z\left(\frac{p_z}{m}\right) - \frac{1}{2} m\left(\left(\frac{p_x}{m}\right)^2 + \left(\frac{p_y}{m}\right)^2 + \left(\frac{p_z}{m}\right)^2\right) + U(x,y,z)$$

$$= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + U(x,y,z)$$

b) cylindrical coords (s, ϕ, z), $s^2 = x^2 + y^2$

$$L = \frac{1}{2} m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2) - U(s,\phi,z)$$

$$\rightarrow p_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s} \rightarrow \dot{s} = p_s/m$$
$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ms^2\dot{\phi} \rightarrow \dot{\phi} = p_\phi/ms^2$$
$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$H = \left(p_s \dot{s} + p_\phi \dot{\phi} + p_z \dot{z} - \frac{1}{2}m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2) + U(s,\phi,z)\right)\Big|$$

$$= p_s\left(\frac{p_s}{m}\right) + p_\phi\left(\frac{p_\phi}{ms^2}\right) + p_z\left(\frac{p_z}{m}\right)$$

$\dot{s} = p_s/m$ etc

$$-\frac{1}{2}m\left(\left(\frac{p_s}{m}\right)^2 + s^2\left(\frac{p_\phi}{ms^2}\right)^2 + \left(\frac{p_z}{m}\right)^2\right) + U(s,\phi,z)$$

$$= \frac{1}{2m}\left(p_s^2 + \frac{p_\phi^2}{s^2} + p_z^2\right) + U(s,\phi,z)$$

---

c) spherical polar coords (r,θ,ϕ)

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - U(r,\theta,\phi)$$

$$\rightarrow p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \rightarrow \dot{r} = p_r/m$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \rightarrow \dot{\theta} = p_\theta/mr^2$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi} \rightarrow \dot{\phi} = p_\phi/mr^2\sin^2\theta$$

$$H = \left(p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) + U(r,\theta,\phi)\right)\Big|$$

$$= p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{mr^2}\right) + p_\phi\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)$$

$\dot{r} = p_r/m$ etc.

$$-\frac{1}{2}m\left(\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{mr^2}\right)^2 + r^2\sin^2\theta\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)^2\right) + U(r,\theta,\phi)$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r,\theta,\phi)$$

<u>sec 40 - prob 2</u>

$$L = \frac{1}{2}mv^2 + m\vec{v}\cdot(\vec{\Omega}\times\vec{r}) + \frac{1}{2}m|\vec{\Omega}\times\vec{r}|^2 - m\vec{W}\cdot\vec{r} - U$$

restrict to uniformly rotating frame of reference $\vec{W} = 0$, $\dot{\vec{\Omega}} = 0$

$$\rightarrow L = \frac{1}{2}mv^2 + m\vec{v}\cdot(\vec{\Omega}\times\vec{r}) + \frac{1}{2}m|\vec{\Omega}\times\vec{r}|^2 - U(\vec{r})$$

Now: $H = \vec{p}\cdot\vec{v} - L$

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + m\vec{\Omega}\times\vec{r} = m(\vec{v} + \vec{\Omega}\times\vec{r})$$

$$\longrightarrow \vec{v} = \frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}$$

Thus,

$$H = \vec{p}\cdot\left(\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right) - \frac{1}{2}m\left|\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right|^2$$

$$- m\left(\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right)\cdot(\vec{\Omega}\times\vec{r})$$

$$- \frac{1}{2}m|\vec{\Omega}\times\vec{r}|^2 + U(\vec{r})$$

$$= \frac{|\vec{p}|^2}{m} - \cancel{\vec{p}\cdot(\vec{\Omega}\times\vec{r})} - \frac{1}{2}m\left(\frac{|\vec{p}|^2}{m^2} + \cancel{|\vec{\Omega}\times\vec{r}|^2} - \cancel{2\frac{\vec{p}\cdot(\vec{\Omega}\times\vec{r})}{m}}\right)$$

$$- \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + m\cancel{|\vec{\Omega}\times\vec{r}|^2} - \frac{1}{2}m\cancel{|\vec{\Omega}\times\vec{r}|^2} + U(\vec{r})$$

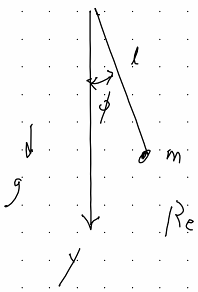
$$= \frac{|\vec{p}|^2}{2m} - \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + U(\vec{r})$$

$$= \frac{|\vec{p}|^2}{2m} - \vec{\Omega}\cdot\underbrace{(\vec{r}\times\vec{p})}_{\vec{M}\text{: angular momentum}} + U(\vec{r})$$

Sec. 11, Prob 1

Simple pendulum: $U = -mgy$

$$= -mgl\cos\phi$$



Also, $T = \frac{1}{2} m l^2 \dot\phi^2$

Thus, $E = T + U$

$$= \frac{1}{2} m l^2 \dot\phi^2 - mgl\cos\phi$$

Release from rest at $\phi = \phi_0$

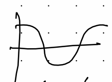
$$E = -mgl\cos\phi_0$$

$$\rightarrow \frac{1}{2} m l^2 \dot\phi^2 - mgl\cos\phi = -mgl\cos\phi_0$$

$$T(E) = 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{E - U(\phi)}}$$

$$= 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{-mgl\cos\phi_0 + mgl\cos\phi}}$$

$$= 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$



$\phi < \phi_0$

$\cos\phi > \cos\phi_0$

Now: $\cos\phi = \cos\left(2\frac{\phi}{2}\right)$

$$= \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

Also, $\cos\phi_0 = 1 - 2\sin^2\left(\frac{\phi_0}{2}\right)$

$$T(E) = 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{2\left(\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)\right)}}$$

$$= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2\left(\frac{\phi_0}{2}\right)\left(1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)}\right)}}$$

Let: $x = \dfrac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)} \rightarrow dx = \dfrac{d\phi}{2} \dfrac{\cos\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}$

$$= \frac{d\phi}{2} \frac{\sqrt{1 - \sin^2\left(\frac{\phi}{2}\right)}}{\sin\left(\frac{\phi_0}{2}\right)}$$

$$= \frac{d\phi}{2} \frac{\sqrt{1 - \sin^2\left(\frac{\phi_0}{2}\right)x^2}}{\sin\left(\frac{\phi_0}{2}\right)}$$

Thus,

$$T(E) = 2\sqrt{\frac{l}{g}} \int_0^1 \frac{2\,dx\,\cancel{\sin\left(\frac{\phi_0}{2}\right)}}{\sqrt{1 - \sin^2\left(\frac{\phi_0}{2}\right)x^2}\;\cancel{\sin\left(\frac{\phi_0}{2}\right)}\sqrt{1 - x^2}}$$

$$= 4\sqrt{\frac{l}{g}} \int_0^1 \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}}, \quad k = \sin\left(\frac{\phi_0}{2}\right)$$

$$= 4\sqrt{\frac{l}{g}}\, K(k)$$

where $K(k)$ = complete elliptic integral of the 1st kind.

Approximation:

$$T(E) = 4\sqrt{\frac{l}{g}} \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

For $k \ll 1$: $\frac{1}{\sqrt{1-k^2x^2}} \approx 1 + \frac{1}{2}k^2x^2$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \approx \int_0^1 \frac{dx}{\sqrt{1-x^2}}\left(1 + \frac{1}{2}k^2x^2\right)$$

Now: $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin(1) = \boxed{\frac{\pi}{2}}$

$$\frac{1}{2}k^2 \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^2}} = \frac{1}{2}k^2 \int_0^{\pi/2} \frac{\sin^2\theta\, \cancel{\cos\theta}\, d\theta}{\cancel{\sqrt{1-\sin^2\theta}}}$$

Let: $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$x^2 = \sin^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\rightarrow \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$\rightarrow \frac{1}{2}k^2 \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^2}} = \frac{1}{2}k^2\,\frac{1}{2}\int_0^{\pi/2} (1-\cos 2\theta)\, d\theta$$

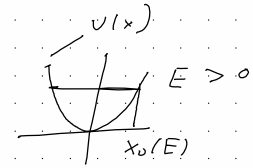
$$= \frac{1}{4}k^2\left[\frac{\pi}{2} - \cancel{\frac{\sin 2\theta}{2}}\Big|_0^{\pi/2}\right]$$

$$= \boxed{\frac{1}{8}k^2\pi}$$

$k = \sin\left(\frac{\phi_0}{2}\right) \approx \frac{\phi_0}{2}$

Thus, $T(E) \approx 4\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \frac{1}{8}k^2\pi + \cdots\right) = \boxed{2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\phi_0^2 + \cdots\right)}$

Sec 11, Prob 2:



(a) $U = A|x|^n$

$$T(E) = 2\sqrt{2m}\int_0^{x_0(E)} \frac{dx}{\sqrt{E - Ax^n}}$$

where $U(x_0) = E$

$A x_0^n = E$

$\rightarrow x_0 = \left(E/A\right)^{1/n}$

$$\sqrt{E - Ax^n} = \sqrt{E}\sqrt{1 - \left(\frac{A}{E}\right)x^n}$$

$$= \sqrt{E}\sqrt{1-t^n} \quad \text{where } t = \left(\frac{A}{E}\right)^{\frac{1}{n}} x$$

$$dt = \left(\frac{A}{E}\right)^{\frac{1}{n}} dx$$

$$t_0 = \left(\frac{A}{E}\right)^{\frac{1}{n}} x_0 = \left(\frac{A}{E}\right)^{\frac{1}{n}}\left(\frac{E}{A}\right)^{\frac{1}{n}} = 1$$

Thus,

$$T(E) = 2\frac{\sqrt{2m}}{\sqrt{E}}\int_0^1 \left(\frac{E}{A}\right)^{\frac{1}{n}} \frac{dt}{\sqrt{1-t^n}}$$

Now let $u = t^n$

$du = n t^{n-1}\,dt = n\, u^{\frac{n-1}{n}}\, dt$

$= n\, u^{1-\frac{1}{n}}\, dt$

$dt = \frac{1}{n} u^{\frac{1}{n}-1}\, du$, $t = 0, 1 \rightarrow u = 0, 1$

Thus

$$T(E) = 2\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 \frac{1}{n}\,\frac{u^{\frac{1}{n}-1}\,du}{\sqrt{1-u}}$$

$$= \frac{2}{n}\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 u^{\frac{1}{n}-1}(1-u)^{-\frac{1}{2}}\,du$$

Recall: Beta Function

Gamma Function

$$B(x,y) = \int_0^1 dt\, t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

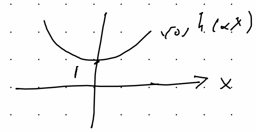
(where $Re(x)>0$, $Re(y)>0$)

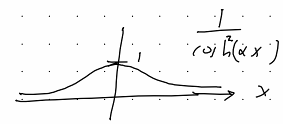
Thus,

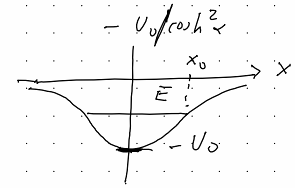
$$\boxed{T(E) = \frac{2}{n}\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} = \frac{2}{n}\sqrt{\frac{2\pi m}{E}}\left(\frac{E}{A}\right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}}$$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(b) $U = -U_0/\cosh^2 \alpha x$







$-U_0 \le E < 0$

Thus, $E = -|E|$

$|E| < U_0$

$$T(E) = 2\sqrt{2m}\int_0^{x_0(E)} \frac{dx}{\sqrt{E-U(x)}}$$

where x_0 given by

$$E = U(x_0) = \frac{-U_0}{\cosh^2(\alpha x_0)}$$

$$\rightarrow \cosh^2(\alpha x_0) = \frac{-U_0}{E}$$

$$\cosh(\alpha x_0) = \sqrt{\frac{U_0}{|E|}}$$

$$\alpha x_0 = \cosh^{-1}\left(\sqrt{\frac{U_0}{|E|}}\right)$$

$$\boxed{x_0 = \frac{1}{\alpha}\cosh^{-1}\left(\sqrt{\frac{U_0}{|E|}}\right)}$$

$$\sqrt{E-U(x)} = \sqrt{E + \frac{U_0}{\cosh^2(\alpha x)}} \qquad ch^2 - sh^2 = 1$$

$$= \frac{1}{\cosh(\alpha x)} \sqrt{E\cosh^2(\alpha x) + U_0}$$

$$= \frac{1}{\cosh(\alpha x)} \sqrt{E\left(1+\sinh^2(\alpha x)\right) + U_0}$$

$$= \frac{1}{\cosh(\alpha x)} \sqrt{(E+U_0) + E\sinh^2(\alpha x)}$$

$$= \frac{1}{\cosh(\alpha x)} \sqrt{(U_0-|E|) - |E|\sinh^2(\alpha x)}$$

thus

$$T(E) = 2\sqrt{2m}\int_0^{x_0} \frac{dx}{\sqrt{E-U(x)}}$$

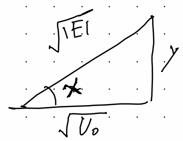
$$= 2\sqrt{2m}\int_0^{x_0} \frac{\cosh(\alpha x)\,dx}{\sqrt{(U_0-|E|) - |E|\sinh^2(\alpha x)}}$$

<u>Let</u>: $t = \sinh(\alpha x) \longrightarrow t_0 = \sinh(\alpha x_0)$

$dt = \alpha\cosh(\alpha x)\,dx$

so $t_0 = \sinh\left(\cosh^{-1}\sqrt{\frac{U_0}{|E|}}\right)$

$$= \sqrt{\frac{U_0-|E|}{|E|}}$$



$U_0 - y^2 = |E|$

$y = \sqrt{U_0-|E|}$

$$T(E) = 2\sqrt{2m}\int_0^{\sqrt{\frac{U_0-|E|}{|E|}}} \frac{dt}{\alpha\sqrt{(U_0-|E|) - |E|t^2}}$$

$$= 2\sqrt{2m}\int_0^{\sqrt{\frac{U_0-|E|}{|E|}}} \frac{dt}{\alpha\sqrt{U_0-|E|}\sqrt{1 - \frac{|E|}{U_0-|E|}t^2}}$$

<u>Define</u>: $\sin u = \sqrt{\frac{|E|}{U_0-|E|}}\, t$

$du\,\cos u = \sqrt{\frac{|E|}{U_0-|E|}}\, dt$

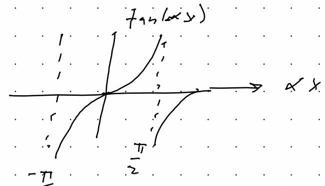
$$T(E) = \frac{2\sqrt{2m}}{\alpha\cancel{\sqrt{U_0-|E|}}}\int_0^{\pi/2} \sqrt{\frac{\cancel{U_0-|E|}}{|E|}}\,\frac{\cancel{\cos u}\,du}{\cancel{\sqrt{1-\sin^2 u}}}$$

$$= \frac{\cancel{2}}{\alpha}\sqrt{\frac{2m}{|E|}}\,\frac{\pi}{\cancel{2}}$$

$$= \boxed{\frac{\pi}{\alpha}\sqrt{\frac{2m}{|E|}}}$$

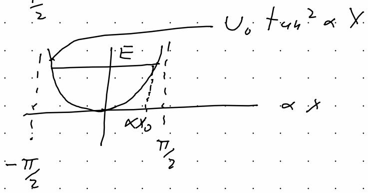
(c) $U = U_0 \tan^2 \alpha x$

$$T = 2\sqrt{2m} \int_0^{x_0} \frac{dx}{\sqrt{E - U(x)}}$$



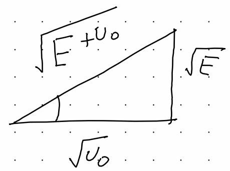
where x_0 determined by

$$E = U(x_0)$$
$$= U_0 \tan^2(\alpha x_0)$$



$$\rightarrow \quad \alpha x_0 = \tan^{-1}\left(\sqrt{\frac{E}{U_0}}\right)$$

$$\boxed{x_0 = \frac{1}{\alpha} \tan^{-1}\left(\sqrt{\frac{E}{U_0}}\right)}$$



$$\sqrt{E - U(x)} = \sqrt{E - U_0 \tan^2(\alpha x)}$$
$$= \sqrt{E - U_0 \frac{\sin^2(\alpha x)}{\cos^2(\alpha x)}}$$
$$= \frac{1}{\cos \alpha x} \sqrt{E \cos^2(\alpha x) - U_0 \sin^2(\alpha x)}$$
$$= \frac{1}{\cos \alpha x} \sqrt{E\left(1 - \sin^2(\alpha x)\right) - U_0 \sin^2(\alpha x)}$$
$$= \frac{1}{\cos \alpha x} \sqrt{E - (U_0 + E)\sin^2(\alpha x)}$$
$$= \frac{\sqrt{E}}{\cos \alpha x} \sqrt{1 - \left(\frac{U_0 + E}{E}\right) \sin^2(\alpha x)}$$

Thus

$$T(E) = 2\sqrt{2m} \int_0^{x_0} \frac{\cos \alpha x \, dx}{\sqrt{E} \sqrt{1 - \left(\frac{U_0 + E}{E}\right) \sin^2 \alpha x}}$$

Let $u = \sin \alpha x$

$$du = \alpha \cos \alpha x \, dx$$

$$x = 0, x_0 \rightarrow \quad u = 0$$
$$u_0 = \sin(\alpha x_0)$$
$$= \sin\left(\tan^{-1} \sqrt{\frac{E}{U_0}}\right)$$
$$= \sqrt{\frac{E}{E + U_0}}$$

$$\rightarrow T(E) = 2\sqrt{\frac{2m}{E}} \frac{1}{\alpha} \int_0^{u_0} \frac{du}{\sqrt{1 - \frac{U_0 + E}{E} u^2}}$$

Let $\sqrt{\frac{U_0 + E}{E}}\, u = \sin \theta \quad \rightarrow \quad \sqrt{\frac{U_0 + E}{E}}\, du = \cos \theta \, d\theta$

$$u = 0 \rightarrow \theta = 0$$
$$u = u_0 \rightarrow \quad \sin \theta_0 = \sqrt{\frac{U_0 + E}{E}} \sqrt{\frac{E}{U_0 + E}} = 1 \rightarrow \theta_0 = \frac{\pi}{2}$$

$$\rightarrow T(E) = 2\sqrt{\frac{2m}{E}} \frac{1}{\alpha} \int_0^{\pi/2} \frac{\cancel{\cos \theta}\, d\theta}{\sqrt{\frac{U_0 + E}{E}} \cancel{\sqrt{1 - \sin^2 \theta}}}$$

$$= \frac{2}{\alpha} \sqrt{\frac{2m}{E}} \sqrt{\frac{E}{U_0 + E}} \frac{\pi}{2} = \boxed{\frac{\pi}{\alpha} \sqrt{\frac{2m}{E}} \sqrt{\frac{E}{U_0 + E}}}$$

Sec 13, Prob 1

$\vec{X}$: position vector of M

$\vec{x}_a$: $a = 1, 2, \cdots, n$ position vector of n masses all with mass m.

Closed system $\rightarrow$ linear momentum conserved

$\rightarrow$ CoM frame

$$M\vec{X} + m(\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_n) = 0$$

or $$M\vec{X} + m\sum_{a=1}^{n} \vec{x}_a = 0 \qquad (1)$$

Taking time derivative $\rightarrow$ $M\dot{\vec{X}} + \sum_a \dot{\vec{x}}_a = 0$. (2)

Define relative position vectors:

$\vec{r}_1 \equiv \vec{x}_1 - \vec{X}$

$\vec{r}_2 \equiv \vec{x}_2 - \vec{X}$

etc.

or $\vec{r}_a \equiv \vec{x}_a - \vec{X}$, $a = 1, 2, \cdots, n$. (3)

Summing up (3):

$$\begin{aligned}\sum \vec{r}_a &= \sum(\vec{x}_a - \vec{X}) \\ &= \sum \vec{x}_a - n\vec{X} \\ &= -\frac{M}{m}\vec{X} - n\vec{X} \\ &= -\frac{(M + nm)}{m}\vec{X} = -\frac{\mu}{m}\vec{X}\end{aligned}$$

total mass

Thus, $\vec{X} = -\frac{m}{\mu}\sum_a \vec{r}_a$

and $\vec{x}_a = \vec{r}_a + \vec{X}$

give you $\vec{x}_a, \vec{X}$ in terms of $\vec{r}_a$.

KE: $$\begin{aligned}T &= \frac{1}{2}\sum_a m|\dot{\vec{x}}_a|^2 + \frac{1}{2}M|\dot{\vec{X}}|^2 \\ &= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a + \dot{\vec{X}}|^2 + \frac{1}{2}M|\dot{\vec{X}}|^2 \\ &= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a|^2 + \frac{1}{2}\sum_a m\, 2\dot{\vec{r}}_a \cdot \dot{\vec{X}} \\ &\quad + \frac{1}{2}\sum_a m|\dot{\vec{X}}|^2 + \frac{1}{2}M|\dot{\vec{X}}|^2 \\ &= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a|^2 + m\Big(\sum_a \dot{\vec{r}}_a\Big)\cdot\dot{\vec{X}} \\ &\quad + \frac{1}{2}nm|\dot{\vec{X}}|^2 + \frac{1}{2}M|\dot{\vec{X}}|^2 \\ &= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a|^2 - \mu|\dot{\vec{X}}|^2 + \frac{1}{2}\underbrace{(nm + M)}_{\mu}|\dot{\vec{X}}|^2 \\ &= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a|^2 - \frac{1}{2}\mu|\dot{\vec{X}}|^2\end{aligned}$$

Now rewrite last term:

$$-\frac{1}{2}\mu|\dot{\vec{X}}|^2 = -\frac{1}{2}\mu\frac{m^2}{\mu^2}\Big|\sum_a \dot{\vec{r}}_a\Big|^2 = -\frac{1}{2}\frac{m^2}{\mu}\Big|\sum_a \dot{\vec{r}}_a\Big|^2$$

Thus, $$T = \frac{1}{2}m\sum_a |\dot{\vec{r}}_a|^2 - \frac{1}{2}\frac{m^2}{\mu}\Big|\sum_a \dot{\vec{r}}_a\Big|^2$$

Potential energy

$$U = U\big(|\vec{x}_1 - \vec{x}_2|, |\vec{x}_1 - \vec{x}_3|, \cdots, |\vec{x}_1 - \vec{x}_n|, \ldots, |\vec{x}_1 - \vec{X}|, |\vec{x}_2 - \vec{X}|, \cdots, |\vec{x}_n - \vec{X}| \big)$$

$$= U\big(|\vec{r}_1 - \vec{r}_2|, |\vec{r}_1 - \vec{r}_3|, \cdots, |\vec{r}_1 - \vec{r}_n|, \ldots, |\vec{r}_1|, |\vec{r}_2|, \cdots, |\vec{r}_n| \big)$$

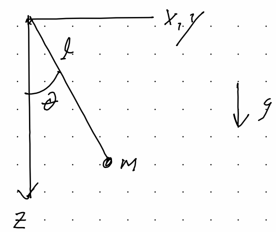
which depends only on the relative position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$

Thus,

$$L = \frac{1}{2} m \sum_a |\dot{\vec{r}}_a|^2 - \frac{1}{2}\frac{m^2}{M} \Big| \sum_a \dot{\vec{r}}_a \Big|^2 - U(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n)$$

Sec 14, Prob. 1

spherical pendulum:



$$T = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2)$$

$$U = -mgz$$
$$= -mgl\cos\theta$$

Thus,

$$L = T - U$$
$$= \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + mgl\cos\theta$$

No t-dependence $\rightarrow$ $E = T + U = \text{const}$

No ϕ-dependence $\rightarrow$ $\frac{\partial L}{\partial \dot{\phi}} = m l^2 \sin^2\theta\, \dot{\phi} \equiv M_z = \text{const}$

$$E = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) - mgl\cos\theta$$

$$= \frac{1}{2} m l^2 \left(\dot{\theta}^2 + \sin^2\theta \frac{M_z^2}{m^2 l^4 \sin^4\theta}\right) - mgl\cos\theta$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 + \underbrace{\frac{M_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta}_{U_{eff}(\theta)}$$

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{M_z^2}{2ml^2\sin^2\theta} + U(\theta)$$

$$\rightarrow \frac{1}{2} m l^2 \dot{\theta}^2 = E - U(\theta) - \frac{M_z^2}{2ml^2\sin^2\theta}$$

$$\dot{\theta} = \sqrt{\frac{2}{ml^2}(E - U(\theta)) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}$$

$$\frac{d\theta}{dt} = \dot{\theta} = \sqrt{\frac{2}{ml^2}\left(E + mgl\cos\theta\right) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}$$

$$\rightarrow dt = \frac{d\theta}{\sqrt{\quad}}$$

$$\boxed{t = \int \frac{d\theta}{\sqrt{\frac{2}{ml^2}\left(E + mgl\cos\theta\right) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}} + \text{const}}$$

Path: use $M_z = ml^2 \sin^2\theta\, \dot{\phi}$

Thus, $$\frac{d\theta}{dt} = \frac{d\theta}{d\phi}\frac{d\phi}{dt} = \frac{d\theta}{d\phi}\frac{M_z}{ml^2\sin^2\theta}$$

Thus, $$\frac{d\theta}{d\phi} = \frac{d\theta}{dt}\frac{ml^2\sin^2\theta}{M_z} = \sqrt{\quad}\;\frac{ml^2\sin^2\theta}{M_z}$$

$$d\phi = \frac{d\theta\, M_z}{\sqrt{\quad}\; ml^2\sin^2\theta}$$

$$\boxed{\phi = \int \frac{M_z\, d\theta / ml^2\sin^2\theta}{\sqrt{\frac{2}{ml^2}\left(E + mgl\cos\theta\right) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}} + \text{const}}$$

Turning points: (where $\dot{\theta} = 0$)

$$E = U_{eff}(r) = \frac{M_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta$$

$$\rightarrow 2Eml^2\sin^2\theta = M_z^2 - 2m^2gl^3\sin^2\theta\cos\theta$$

$$2Eml^2(1-\cos^2\theta) = M_z^2 - 2m^2gl^3(1-\cos^2\theta)\cos\theta$$

$$2Eml^2 - 2Eml^2\cos^2\theta = M_z^2 - 2m^2gl^3\cos\theta + 2m^2gl^3\cos^3\theta$$

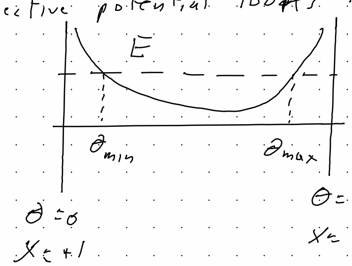
Thus, $$2m^2gl^3\cos^3\theta + 2Eml^2\cos^2\theta - 2m^2gl^3\cos\theta + \left(M_z^2 - 2Eml^2\right) = 0$$

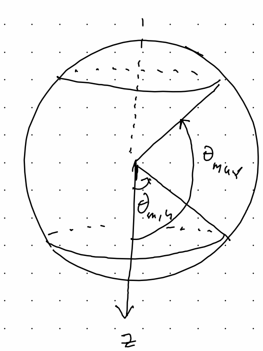
Divide by $2m^2gl^3$:

$$\rightarrow \boxed{\cos^3\theta + \frac{E}{mgl}\cos^2\theta - \cos\theta + \left(\frac{M_z^2}{2m^2gl^3} - \frac{E}{mgl}\right) = 0}$$

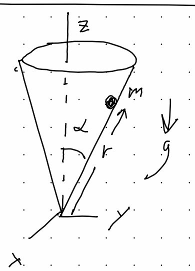
Cubic equation for $x \equiv \cos\theta$

Effective potential looks like:





Sec 14, Prob. 2



spherical polar coords (r, θ, ϕ)

Constraint $\theta = \alpha$

Generalized coords: (r, ϕ)

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \underbrace{\dot{\theta}^2}_{0} + r^2 \sin^2 \underbrace{\theta}_{\alpha} \dot{\phi}^2\right)$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \sin^2\alpha \, \dot{\phi}^2\right)$$

$$U = mgz$$
$$= mgr\cos\alpha$$

$$L = T - U$$
$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \sin^2\alpha \, \dot{\phi}^2\right) - mgr\cos\alpha$$

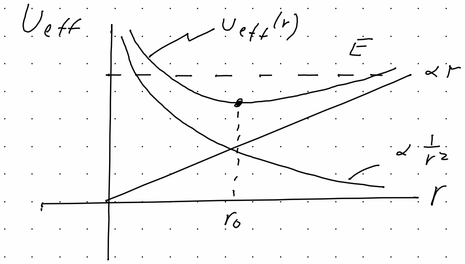
$E = \text{const}$ (since no explicit t dependence)

$$\frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\alpha \, \dot{\phi} \equiv M_z = \text{const}$$

(since no explicit ϕ depend.)

$$E = T + U$$
$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \sin^2\alpha \, \dot{\phi}^2\right) + mgr\cos\alpha$$
$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \sin^2\alpha \left(\frac{M_z^2}{m^2 r^4 \sin^4\alpha}\right) + mgr\cos\alpha$$
$$= \frac{1}{2} m \dot{r}^2 + \frac{M_z^2}{2 m r^2 \sin^2\alpha} + mgr\cos\alpha$$
$$= \frac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = \frac{M_z^2}{2 m r^2 \sin^2\alpha} + mgr\cos\alpha$$



Bound orbits for $E > U_{eff,min} = U_{eff}(r_0)$

r_0: stable circular orbit

$r = r_0 = \text{const}$

t-equation:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - mgr\cos\alpha\right) - \frac{M_z^2}{m^2 r^2 \sin^2\alpha}}} + \text{const}$$

Using $M_z = m r^2 \sin^2\alpha \, \dot{\phi}$

$$\rightarrow \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M_z}{m r^2 \sin^2\alpha}$$

ϕ-equation:

$$\phi = \int \frac{\left(\frac{M_z}{\sin^2\alpha}\right)\frac{dr}{r^2}}{\sqrt{2m\left(E - mgr\cos\alpha\right) - \frac{M_z^2}{r^2 \sin^2\alpha}}} + \text{const}$$

Turning points: $r = r_{min}, r_{max}$

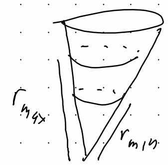
Determined by effective potential

$$E = U_{eff}(r)$$
$$= \frac{M_z^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha$$

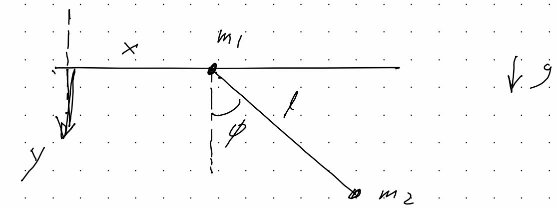
$$\to \quad 2mEr^2\sin^2\alpha = M_z^2 + 2m^2 g r^3 \sin^2\alpha \cos\alpha$$

$$0 = 2m^2 g r^3 \sin^2\alpha \cos\alpha - 2mEr^2\sin^2\alpha + M_z^2$$
$$= r^3 - \left(\frac{E}{mgr\cos\alpha}\right) r^2 + \frac{M_z^2}{2m^2 g \sin^2\alpha \cos\alpha}$$

cubic equation agin



Sec 14, Prob 3



From Sec 5, Prob 2 we have

$$L = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2} m_2 (l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) + m_2 g l \cos\phi$$

No dependence on x:

$$\to \quad P_x \equiv \frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2 l \dot{\phi}\cos\phi$$
$$= \text{const}$$

(x-component of total momentum)

No explicit t-dependence

$$\to \quad E = T + U = \text{const}$$
$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2} m_2 (l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi)$$
$$- m_2 g l \cos\phi$$

Work in frame where $COM_x = 0$:

$$COM_x = m_1 x + m_2(x + l\sin\phi)$$
$$= (m_1+m_2)x + m_2 l \sin\phi$$

$$COM_x = 0 \quad \to \quad x = -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi$$

$$\dot{x} = -\left(\frac{m_2}{m_1+m_2}\right) l \cos\phi \, \dot{\phi}$$

thus,

$$E = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2\left(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi\right) - m_2 g l \cos\phi$$

$$= \frac{1}{2}(m_1+m_2)\frac{m_2^2}{(m_1+m_2)^2} l^2 \cos^2\phi \, \dot{\phi}^2 + \frac{1}{2}m_2 l^2 \dot{\phi}^2 - m_2\left(\frac{m_2}{m_1+m_2}\right) l^2 \cos^2\phi \, \dot{\phi}^2 - m_2 g l \cos\phi$$

$$= \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left[1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right] - m_2 g l \cos\phi$$

1-d problem:

$$\frac{E + m_2 g l \cos\phi}{\left[1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right]} = \frac{1}{2} m_2 l^2 \dot{\phi}^2$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{\frac{\frac{2}{m_2 l^2}\left(E + m_2 g l \cos\phi\right)}{1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi}}$$

$$\rightarrow \quad dt = \frac{d\phi}{\sqrt{}} = d\phi \sqrt{\frac{1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi}{\frac{2}{m_2 l^2}\left(E + m_2 g l \cos\phi\right)}}$$

$$= d\phi \sqrt{\frac{m_2 l^2}{2}\frac{1}{m_1+m_2}} \sqrt{\frac{(m_1+m_2) - m_2\cos^2\phi}{E + m_2 g l \cos\phi}}$$

$$= d\phi \sqrt{\frac{m_2}{m_1+m_2}} \sqrt{\frac{l^2}{2}} \sqrt{\frac{m_1 + m_2 \sin^2\phi}{E + m_2 g l \cos\phi}}$$

so

$$\boxed{t = \sqrt{\left(\frac{m_2}{m_1+m_2}\right)\frac{l^2}{2}} \int d\phi \sqrt{\frac{m_1 + m_2 \sin^2\phi}{E + m_2 g l \cos\phi}} + \text{const}}$$

Now: $x_2 = x + l\sin\phi$

$y_2 = l\cos\phi$

Using $x = -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi$

$$\rightarrow \quad x_2 = \left[-\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi + l\sin\phi\right] = \left(\frac{m_1}{m_1+m_2}\right) l \sin\phi$$

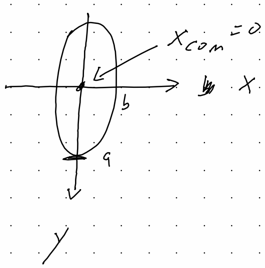
$$x_2 = \left(\frac{m_1}{m_1+m_2}\right) l \sin\phi = b \sin\phi$$

$$y_2 = l \cos\phi = a \cos\phi$$

$$\left(\frac{y_2}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = \cos^2\phi + \sin^2\phi = 1$$

which is an ellipse with semi-major and semi-minor axes:

$$a = l, \quad b = l\left(\frac{m_1}{m_1+m_2}\right)$$



If $m_1 >> m_2$ then $a = b \approx l$ so that m_2 moves along a circular arc of radius l.

Sec 15, Prob 1

$U = -\frac{\alpha}{r}$, $E = 0 \longrightarrow e = 1$

$$\frac{p}{r} = 1 + \cos\phi$$

when $\phi = 0$, $p = 2 \cdot r$

so $r_{min} = \frac{p}{2}$

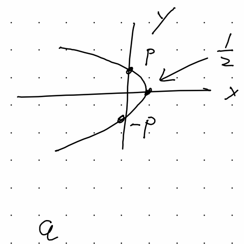
NOTE: $p = r + r\cos\phi$
$= \sqrt{x^2+y^2} + x$

$\rightarrow (p-x)^2 = x^2 + y^2$

$p^2 + \cancel{x^2} - 2px = \cancel{x^2} + y^2$

$p^2 - y^2 = 2px$

$\rightarrow \boxed{x = \frac{p^2 - y^2}{2p}}$ parabola



when $y = 0$, $x = \frac{1}{2}p$
$x = 0$, $y = \pm p$

Time equation:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}[E - U(r)] - \frac{M^2}{m^2 r^2}}}$$

t const

$U(r) = -\frac{\alpha}{r}$, $p = \frac{M^2}{m\alpha}$, $e=1$, $E=0$

$$\rightarrow \quad t = \int \frac{dr}{\sqrt{\frac{2}{m}\frac{\alpha}{r} - \frac{M^2}{m^2 r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{2}{m}\frac{\alpha}{r} - \frac{m\alpha p}{m^2 r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{dr}{\sqrt{\frac{2}{r} - \frac{p}{r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{r\,dr}{\sqrt{2r-p}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha p}} \int \frac{r\,dr}{\sqrt{\frac{2r}{p}-1}} + \text{const}$$

Define: $\frac{2r}{p} - 1 = \xi^2 > 0 \quad (-\infty < \xi < \infty)$

$2r = p(1+\xi^2)$

$\boxed{r = \frac{p}{2}(1+\xi^2)} \rightarrow dr = p\xi\, d\xi$

$r\,dr = \frac{p^2}{2}(\xi + \xi^3)\,d\xi$

Thus,

$$t = \sqrt{\frac{m}{\alpha p}} \int \frac{\frac{p^2}{2}(\xi+\xi^3)\,d\xi}{\sqrt{\xi^2}} + \text{const}$$

$$= \sqrt{\frac{mp^3}{\alpha}}\,\frac{1}{2} \int (1+\xi^2)\,d\xi + \text{const}$$

$$= \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}}\left(\xi + \frac{1}{3}\xi^3\right) + \text{const}$$

choose constant so that $t=0 \leftrightarrow \xi = 0$ (const $=0$)

so, $\boxed{t = \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}}\left(\xi + \frac{1}{3}\xi^3\right)}$

Now:

$$\frac{p}{r} = 1 + \cos\phi$$

$$p = r(1+\cos\phi)$$

$$p = \frac{p}{2}(1+\xi^2)(1+\cos\phi)$$

$$2 = 1 + \xi^2 + \cos\phi + \xi^2\cos\phi$$

$$1 - \xi^2 = (1+\xi^2)\cos\phi$$

$$\rightarrow \boxed{\cos\phi = \frac{1-\xi^2}{1+\xi^2}}$$

$$\boxed{x = r\cos\phi = \frac{p}{2}\cancel{(1+\xi^2)}\left(\frac{1-\xi^2}{\cancel{1+\xi^2}}\right) = \frac{p}{2}(1-\xi^2)}$$

Also,

$$x^2+y^2 = r^2$$

$$\rightarrow y^2 = r^2 - x^2$$

$$= \frac{p^2}{4}(1+\xi^2)^2 - \frac{p^2}{4}(1-\xi^2)^2$$

$$= \frac{p^2}{4}\left(\cancel{1}+\cancel{\xi^4}+2\xi^2-\cancel{1}-\cancel{\xi^4}+2\xi^2\right)$$

$$= p^2\xi^2$$

so $\boxed{y = p\xi}$

<u>Sec 15, Prob 3:</u>

$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}}$$

This is the change in ϕ as r goes from r_{min} to r_{max} and then back to r_{min}.

A closed bound orbit would have $\Delta\phi = 2\pi m/n$ for m, n integers.

<u>Consider:</u> $U = -\frac{\alpha}{r} + \delta U$ where $|\delta U| \ll \left|\frac{\alpha}{r}\right|$

For $\delta U = 0$, $\Delta\phi = 2\pi$

$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-(U+\delta U)) - M^2/r^2}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - 2m\delta U - M^2/r^2}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{\left(2m(E-U) - \frac{M^2}{r^2}\right)\left(1 - \frac{2m\,\delta U}{2m(E-U) - M^2/r^2}\right)}}$$

$$\approx 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}}\left[1 + \frac{m\,\delta U}{2m(E-U) - M^2/r^2}\right]$$

For $U = -\alpha/r$:

$$\Delta\phi \approx 2\pi + 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2 \quad m\,\delta U}{\left[2m(E-U) - M^2/r^2\right]^{3/2}}$$

$$\approx 2\pi + \delta\phi$$

where

$$\delta\phi \equiv \int_{r_{min}}^{r_{max}} \frac{2m\,\delta U\, M\,dr/r^2}{\left[2m(E-U) - M^2/r^2\right]^{3/2}}$$

$$= \frac{\partial}{\partial M}\left[\int_{r_{min}}^{r_{max}} \frac{2m\,\delta U\,dr}{\sqrt{2m(E-U) - M^2/r^2}}\right]$$

Evaluate terms in integrand along unperturbed path since δU is already small

$$\frac{p}{r} = 1 + e\cos\phi \quad , \quad p = a(1-e^2)$$

$$\rightarrow \quad -\frac{p}{r^2}\,dr = -e\sin\phi\,d\phi \quad \rightarrow \quad \boxed{dr = \frac{r^2}{p} e \sin\phi\, d\phi}$$

$$\sqrt{} = \sqrt{2mE + \frac{2m\alpha}{r} - \frac{M^2}{r^2}}$$

$$= \sqrt{-2m|E| + \frac{4m|E|}{(1-e^2)}\frac{p}{r} - \frac{2m|E|}{(1-e^2)}\frac{p^2}{r^2}}$$

$$\sqrt{} = \sqrt{2m|E|}\sqrt{-1 + \frac{2}{(1-e^2)}(1+e\cos\phi) - \frac{1}{(1-e^2)}(1+e\cos\phi)^2}$$

$$= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}}\sqrt{-(\cancel{1}-e^2) + 2(\cancel{1}+\cancel{e\cos\phi}) - (\cancel{1}+\cancel{2e\cos\phi}+e^2\cos^2\phi)}$$

$$= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}}\sqrt{e^2(1-\cos^2\phi)}$$

$$= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}}\, e\sin\phi$$

Thus,

$$\boxed{\delta\phi = \frac{\partial}{\partial M}\left[2m\int_0^{\pi} \frac{\delta U\, r^2\, \cancel{e\sin\phi}\, d\phi}{\sqrt{\frac{2m|E|}{1-e^2}}\;\cancel{e\sin\phi}\; p}\right]
= \frac{\partial}{\partial M}\left[\frac{2m}{p\sqrt{\frac{2m|E|}{1-e^2}}}\int_0^{\pi} d\phi\, r^2\,\delta U\right]
= \frac{\partial}{\partial M}\left[\frac{2m}{M}\int_0^{\pi} d\phi\, r^2\,\delta U\right]}$$

using (15.6): $\dfrac{M}{\sqrt{2m|E|}} = \dfrac{p}{\sqrt{1-e^2}}$

Evaluate:

$$\delta\phi = \frac{\partial}{\partial M}\left[\frac{2m}{M}\int_0^\pi d\phi\, r^2 \delta U\right]$$

for (a) $\delta U = \beta/r^2$, (b) $= \delta U = \frac{\gamma}{r^3}$

(a) $$\delta\phi = \frac{\partial}{\partial M}\left[\frac{2m}{M}\int_0^\pi d\phi\, \beta\right]$$

$$= 2\pi\beta m \frac{\partial}{\partial M}\left(\frac{1}{M}\right)$$

$$= -\frac{2\pi\beta m}{M^2}$$

$$= \boxed{-\frac{2\pi\beta}{\alpha p}}$$

$\xleftarrow{\text{Recall:}}$ $p = \frac{M^2}{m\alpha}$

$\alpha p = \frac{M^2}{m}$

(b) $$\delta\phi = \frac{\partial}{\partial M}\left[\frac{2m}{M}\int_0^\pi d\phi \frac{\gamma}{r}\right]$$

$$= \frac{\partial}{\partial M}\left[\frac{2m\gamma}{M}\int_0^\pi d\phi\left(\frac{1+e\cos\phi}{p}\right)\right]$$

$$= 2m\gamma \frac{\partial}{\partial M}\left[\frac{1}{Mp}\left(\pi + \overset{0}{\cancel{e\sin\phi\Big|_0^\pi}}\right)\right]$$

Now: $\frac{1}{Mp} = \frac{m\alpha}{M^3} \rightarrow \frac{\partial}{\partial M}\left(\frac{1}{Mp}\right) = -\frac{3m\alpha}{M^4}$

So $$\delta\phi = -6\pi\gamma\frac{m^2\alpha}{M^4}$$

$$= -6\pi\gamma\alpha\left(\frac{1}{p\alpha}\right)^2$$

$$= \boxed{\frac{-6\pi\gamma}{p^2\alpha}}$$

sec 16, Prob 2

From (16.7) we have

$$p(\theta_0)d\theta_0 = \frac{1}{2}\sin\theta_0\, d\theta_0 = -\frac{1}{2}d(\cos\theta_0)$$

where θ_0 is the angle of one of the emitted particles in the com frame.

⇒ We would like to find $p(\theta)d\theta$, where θ is the angle of one of the emitted particle in the lab frame.

⇒ Since $p(\theta)d\theta = p(\theta_0)d\theta_0$

we just need to find θ_0 as a function of θ.

This is given by (16.6) which we first derive.

Proof: Given $\tan\theta = \dfrac{v_0\sin\theta_0}{v_0\cos\theta_0 + V}$

we have

$$\tan\theta(v_0\cos\theta_0 + V) = v_0\sqrt{1-\cos^2\theta_0}$$

square both sides

$$\tan^2\theta(v_0^2\cos^2\theta_0 + V^2 + 2v_0 V\cos\theta_0) = v_0^2(1-\cos^2\theta_0)$$

$$\underbrace{(1+\tan^2\theta)}_{\sec^2\theta}v_0^2\cos^2\theta_0 + 2v_0V\tan^2\theta\cos\theta_0 + (V^2\tan^2\theta - v_0^2) = 0$$

quadratic equation for $\cos\theta_0$

$$\rightarrow \cos\theta_0 = \frac{-2v_0V\tan^2\theta \pm \sqrt{4v_0^2V^2\tan^4\theta - 4v_0^2\sec^2\theta(V^2\tan^2\theta - v_0^2)}}{2v_0^2\sec^2\theta}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \frac{1}{\sec^2\theta}\sqrt{\left(\frac{V}{v_0}\right)^2\tan^4\theta - \sec^2\theta\left(\left(\frac{V}{v_0}\right)^2\tan^2\theta - 1\right)}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \frac{1}{\sec\theta}\sqrt{\left(\frac{V}{v_0}\right)^2\left(\frac{\sin^4\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}\right) + 1}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{\left(\frac{V}{v_0}\right)^2\frac{\sin^2\theta(\sin^2\theta - 1)}{\cos^2\theta} + 1}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{1 - \left(\frac{V}{v_0}\right)^2\sin^2\theta}$$

(for $v_0 > V$ take $+\sqrt{\ }$, for $v_0 < V$ take both signs.)

Now differentiate both sides:

$$d(\cos\theta_0) = -\frac{V}{v_0}2\sin\theta\cos\theta\, d\theta \mp \sin\theta\, d\theta\sqrt{\ } \pm \cos\theta\left(\frac{1}{\cancel{2}}\right)\frac{1}{\sqrt{\ }}(-\cancel{2})\left(\frac{V}{v_0}\right)^2\sin\theta\cos\theta\, d\theta$$

$$= \sin\theta\, d\theta\left[-2\frac{V}{v_0}\cos\theta \mp \sqrt{\ } \mp \left(\frac{V}{v_0}\right)^2\cos^2\theta\frac{1}{\sqrt{\ }}\right]$$

Now:

$$\sqrt{} + \left(\frac{V}{v_0}\right)^2 \cos^2\theta \, \frac{1}{\sqrt{}}$$

$$= \frac{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta + \left(\frac{V}{v_0}\right)^2 \cos^2\theta}{\sqrt{}}$$

$$= \frac{1 + \left(\frac{V}{v_0}\right)^2 (\cos^2\theta - \sin^2\theta)}{\sqrt{}}$$

$$= \frac{1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta}{\sqrt{}}$$

Thus,

$$d(\cos\theta_0) = \underbrace{-\sin\theta\, d\theta}_{d(\cos\theta)} \left[2\frac{V}{v_0}\cos\theta \pm \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

So:

$$p(\theta)\, d\theta = \frac{1}{2}\sin\theta\, d\theta \left[2\frac{V}{v_0}\cos\theta \pm \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

Now: for $v_0 > V$, take $+$ sign $(\theta \in [0, \pi])$

for $v_0 < V$, as θ_0 increases from 0 to π

θ increases from $0 \mapsto \theta_{max}$

θ decreases from $\theta_{max} \mapsto 0$

Thus, for $v_0 < V$ need to take the difference of the $+$ and $-$ expressions:

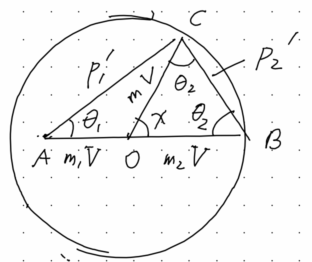
$$p(\theta) d\theta = \frac{1}{2}\sin\theta d\theta \left[2\frac{V}{v_0}\cos\theta + \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

$$- \frac{1}{2}\sin\theta d\theta \left[2\frac{V}{v_0}\cos\theta - \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

$$= \frac{\sin\theta d\theta \left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}}$$

which is valid for $0 \le \theta \le \theta_{max} = \sin^{-1}\left(\frac{v_0}{V}\right)$

Sec 17, Prob 1:



$$\chi + 2\theta_2 = \pi$$
$$\chi = \pi - 2\theta_2$$

From the above figure:

$$(m_2 v_2')^2 = 2(mv)^2 - 2(mv)^2 \cos\chi$$
$$= 2m^2v^2\left(1 - \cos(\pi - 2\theta_2)\right)$$
$$= 2m^2v^2\Big[1 - \big(\underbrace{\cos(\pi)}_{-1}\cos(2\theta_2) + \cancel{\sin(\pi)}^{0}\sin(2\theta_2)\big)\Big]$$
$$= 2m^2v^2\left(1 + \cos(2\theta_2)\right)$$

$$\rightarrow \quad v_2' = \sqrt{2}\left(\frac{m}{m_2}\right)v\sqrt{1 + (\cos^2\theta_2 - \sin^2\theta_2)}$$
$$= \sqrt{2}\left(\frac{m_1}{m_1+m_2}\right)v\sqrt{2\cos^2\theta_2}$$
$$= 2\left(\frac{m_1}{m_1+m_2}\right)V\cos\theta_2$$

Thus, $\left(\frac{v_2'}{V}\right) = \left(\frac{2m_1}{m_1+m_2}\right)\cos\theta_2$

Also,

$$(mv)^2 = (m_1 V)^2 + (m_1 v_1')^2 - 2m_1^2 v_1' V\cos\theta_1$$

$$\rightarrow \quad (m_1 v_1')^2 - 2m_1 V\cos\theta_1 (m_1 v_1') + m_1^2 V^2 - m^2 V^2 = 0$$

Now: $V = \frac{m_1 v_1 + m_2 \cancel{v_2}^{0}}{m_1+m_2} = \frac{m_1 v}{m_1+m_2}$

Thus,

$$(m_1 v_1')^2 - 2\left(\frac{m_1^2 v}{m_1+m_2}\right)\cos\theta_1\, m_1 v_1' + \frac{m_1^2 m_1^2 v^2}{(m_1+m_2)^2} - \frac{m_1^2 m_2^2}{(m_1+m_2)^2}v^2 = 0$$

$$\rightarrow \quad (v_1')^2 - 2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1\, v_1' + \frac{m_1^2 - m_2^2}{(m_1+m_2)^2}v^2 = 0$$

$$(v_1')^2 - 2\left(\frac{m_1 V}{m_1+m_2}\right)\cos\theta_1\, v_1' + \left(\frac{m_1 - m_2}{m_1+m_2}\right)v^2 = 0$$

$$\left(\frac{v_1'}{v}\right)^2 - 2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1\left(\frac{v_1'}{v}\right) + \left(\frac{m_1-m_2}{m_1+m_2}\right) = 0$$

Quadratic equation

$$\frac{v_1'}{v} = \frac{2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \sqrt{4\left(\frac{m_1}{m_1+m_2}\right)^2\cos^2\theta_1 - 4\left(\frac{m_1-m_2}{m_1+m_2}\right)}}{2}$$

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_1^2\cos^2\theta_1 - (m_1-m_2)(m_1+m_2)}$$

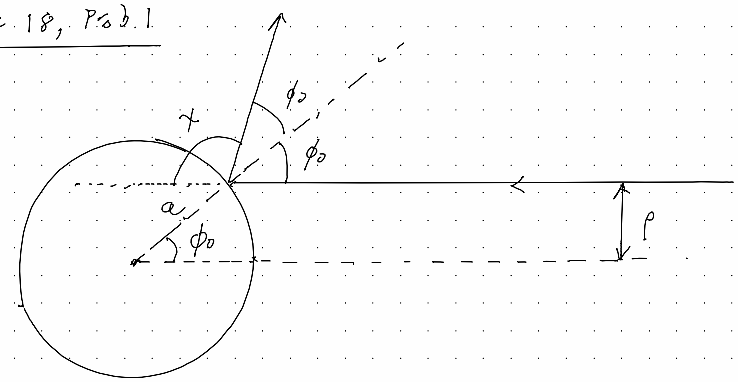
$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_1^2(\cos^2\theta_1 - 1) + m_2^2}$$

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_2^2 - m_1^2\sin^2\theta_1}$$

The $+$ sign holds for $m_1 < m_2$

$+/-$ signs hold for $m_1 > m_2$

Sec. 18, Prob. 1



$$\chi + 2\phi_0 = \pi \quad \rightarrow \quad \phi_0 = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\rho = a\sin\phi_0$$

$$= a\sin\left(\frac{\pi}{2} - \frac{\chi}{2}\right)$$

$$= a\left(\underset{\overset{\shortparallel}{1}}{\sin\left(\frac{\pi}{2}\right)}\cos\left(\frac{\chi}{2}\right) - \cancelto{0}{\cos\left(\frac{\pi}{2}\right)}\sin\left(\frac{\chi}{2}\right)\right)$$

$$= a\cos\left(\frac{\chi}{2}\right)$$

$$d\sigma = 2\pi\rho\, d\rho$$

$$= 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi$$

$$= \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$

where $d\Omega$ = solid angle

$$= \underset{\substack{\uparrow \\ \text{integral over} \\ d\phi}}{2\pi}\sin\chi\, d\chi$$

$$\rho = a\cos\left(\frac{\chi}{2}\right)$$

$$d\rho = -a \cdot \frac{1}{2} \sin\left(\frac{\chi}{2}\right) d\chi$$

$$= -\frac{a}{2} \sin\left(\frac{\chi}{2}\right) d\chi$$

thus,

$$d\sigma = \frac{\rho(\chi)}{\sin\chi} \left| \frac{d\rho}{d\chi} \right| d\Omega$$

$$= \frac{a\cos(\chi/2)}{\sin\chi} \cdot \frac{a}{2} \sin\left(\frac{\chi}{2}\right) d\Omega$$

$$= \frac{a^2}{2} \frac{\sin(\chi/2)\cos(\chi/2)}{\sin\chi} d\Omega$$

$$= \boxed{\frac{a^2}{4} d\Omega} \qquad \left(\text{since } \sin\chi = 2\sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)\right)$$

Total cross section

$$\sigma = \int d\sigma = \frac{a^2}{4}\int d\Omega = \frac{a^2}{4} \cdot 4\pi = \boxed{\pi a^2}$$

Now calculate differential cross section in lab frame for both m_1 and m_2

Use the result that

$$d\sigma_1 = \frac{\rho(\theta_1)}{\sin\theta_1} \left| \frac{d\rho}{d\theta_1} \right| d\Omega_1 = \rho \left| \frac{d\rho}{d(\cos\theta_1)} \right| d\Omega_1$$

Compare to:

$$d\sigma = \rho \left| \frac{d\rho}{d(\cos\chi)} \right| d\Omega$$

$$\rightarrow \frac{d\sigma_1}{d\Omega_1} = \rho \left| \frac{d\rho}{d(\cos\theta_1)} \right|$$

$$= \left| \frac{d(\cos\chi)}{d(\cos\theta_1)} \right| \frac{d\sigma}{d\Omega}$$

So we need to evaluate:

$$\frac{d(\cos\chi)}{d(\cos\theta_1)} \quad \text{and} \quad \frac{d(\cos\chi)}{d(\cos\theta_2)}$$

Start with θ_2: (17.4)

$$\theta_2 = \frac{1}{2}(\pi - \chi) \rightarrow \boxed{\chi = \pi - 2\theta_2}$$

$$\rightarrow \cos\chi = \cos(\pi - 2\theta_2)$$

$$= \cos\pi\cos(2\theta_2) + \cancel{\sin\pi}^{\,0}\sin(2\theta_2)$$

$$= -\cos(2\theta_2)$$

$$= -(\cos^2\theta_2 - \sin^2\theta)$$

$$= -(2\cos^2\theta_2 - 1)$$

$$= -2\cos^2\theta_2 + 1$$

Thus, $\boxed{d(\cos\chi) = -4\cos\theta_2 \, d(\cos\theta_2)}$

Thus,

$$\frac{d\sigma_2}{d\Omega_2} = \frac{d\sigma}{d\Omega}\left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right|$$

$$= \frac{1}{4}a^2 \cdot |4\cos\theta_2|$$

$$= a^2|\cos\theta_2|$$

so $\boxed{d\sigma_2 = a^2|\cos\theta_2|\,d\Omega_2}$

Now consider θ_1:

From (17.4):

$$\tan\theta_1 = \frac{m_2\sin\chi}{m_1 + m_2\cos\chi}$$

Compare with

$$\tan\theta = \frac{v_0\sin\theta_0}{V + v_0\cos\theta_0} \qquad (16.5)$$

make identifications: $\theta\to\theta_1$, $v_0\to m_2$, $V\to m_1$

then we can write down from (16.6).

$$\cos\chi = -\frac{m_1}{m_2}\sin^2\theta_1 \pm \cos\theta_1\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}$$

[See also sec 16, Prob. 2 where we derived this for θ and θ_0]

We also worked out the derivative:

$$d(\cos\theta_0) = d(\cos\theta)\left[2\frac{V}{v_0}\cos\theta \pm \frac{1+\left(\frac{V}{v_0}\right)^2\cos 2\theta}{\sqrt{1-\left(\frac{V}{v_0}\right)^2\sin^2\theta}}\right]$$

So we can similarly write dow

$$d(\cos\chi) = d(\cos\theta_1)\left[2\frac{m_1}{m_2}\cos\theta_1 \pm \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\right]$$

For $m_1 < m_2$: take + sign

For $m_1 > m_2$: As χ increases from 0 to π, θ_1 increases from 0 to θ_{max}, then θ_1 decreases from θ_{max} to 0. In that case

$$d(\cos\chi) = d(\cos\theta_1)[\cancel{\oplus} + \ominus] - d(\cos\theta_1)[\cancel{\oplus} - \ominus]$$

$$= 2\,d(\cos\theta_1)\frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

Use: (for both)

$$d\sigma_1 = \left(\frac{d\sigma}{d\Omega}\right)\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|d\Omega_1$$

$$= \frac{1}{4}a^2\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|d\Omega_1$$

Thus,

For $m_1 < m_2$:

$$d\sigma_1 = \frac{1}{4} a^2 \left[2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1 + \left(\frac{m_1}{m_2}\right)^2 \cos(2\theta_1)}{\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1}} \right] d\Omega_1$$

For $m_1 > m_2$:

$$d\sigma_1 = \frac{1}{4} a^2 \cdot 2 \; \frac{1 + \left(\frac{m_1}{m_2}\right)^2 \cos(2\theta_1)}{\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1}} \; d\Omega_1$$

$$= \frac{a^2}{2} \; \frac{1 + \left(\frac{m_1}{m_2}\right)^2 \cos(2\theta_1)}{\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1}} \; d\Omega_1$$

Sec. 18, Prob 2:

Hard sphere scattering again.
Calculate $d\sigma$ in terms of $d\epsilon$ where
ϵ = energy lost by scattered particle.

Now: ϵ = energy lost by scattered particle
= energy gained by m_2
$= \frac{1}{2} m_2 (v_2')^2$

From Fig. 16., we have (law of cosines):

$$(m_2 v_2')^2 = (mv)^2 + (mv)^2 - 2(mv)^2 \cos\chi$$
$$= 2(mv)^2 [1 - \cos\chi]$$
$$= 2(mv)^2 \, 2\sin^2\left(\frac{\chi}{2}\right)$$

So $m_2 v_2' = 2 mv \sin\left(\frac{\chi}{2}\right)$

$$\rightarrow \quad v_2' = \left(\frac{m}{m_2}\right) v \sin\left(\frac{\chi}{2}\right)$$
$$= \left(\frac{m_1}{m_1 + m_2}\right) v \sin\left(\frac{\chi}{2}\right) \qquad (17.5)$$

NOTE: $d\sigma = \frac{1}{4} a^2 \, d\Omega$
$= \frac{1}{4} a^2 \, 2\pi \sin\chi \, d\chi$
$= \frac{\pi a^2}{2} \left| d(\cos\chi) \right|$

So we would like to relate $d\epsilon$ and $d(\cos\chi)$.

Now: $\epsilon = \frac{1}{2} m_2 (v_2')^2$

$$= \frac{1}{2} m_2 \frac{4 m_1^2 v^2}{(m_1+m_2)^2} \sin^2\left(\frac{\chi}{2}\right)$$

$$= \frac{2 m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 \sin^2\left(\frac{\chi}{2}\right) \quad (\text{since } v = v_\infty)$$

$$\equiv \epsilon_{max} \sin^2\left(\frac{\chi}{2}\right)$$

where $\epsilon_{max} \equiv \frac{2 m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2$

$$= 4\left(\frac{m_1}{m_1+m_2}\right) \frac{m v_\infty^2}{2}$$

$$= 4\left(\frac{m_1}{m_1+m_2}\right) E$$

Thus, $d\epsilon = \epsilon_{max}\, 2 \sin\left(\frac{\chi}{2}\right) \cos\left(\frac{\chi}{2}\right) \frac{d\chi}{2}$

$$= \frac{1}{2} \epsilon_{max} \sin\chi \, d\chi$$

$$= \frac{1}{2} \epsilon_{max} |d(\cos\chi)|$$

So $d\sigma = \frac{\pi a^2}{2} |d(\cos\chi)|$

$$= \frac{\pi a^2}{2} \frac{2}{\epsilon_{max}} d\epsilon = \boxed{\frac{\pi a^2}{\epsilon_{max}} d\epsilon}$$

which is a uniform distribution wrt ϵ.

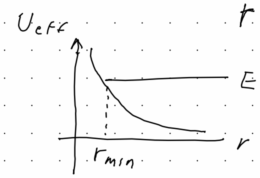
Sec 18, Prob. 4

Effective cross section to "fall" to center of $U(r) = -\alpha/r^2$ $(\alpha > 0)$

$$U_{eff}(r) = U(r) + \frac{M^2}{2mr^2}$$

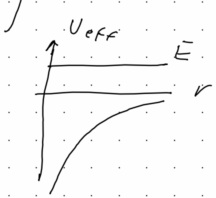
$$= -\frac{\alpha}{r^2} + \frac{M^2}{2mr^2}$$

$$= \frac{1}{r^2}\left(\frac{M^2}{2m} - \alpha\right)$$



$\frac{M^2}{2m} - \alpha > 0$

(don't fall to center since $r_{min} > 0$)



$\frac{M^2}{2m} - \alpha < 0$

Fall to center ($r=0$)

For a given $E = \frac{1}{2} m v_\infty^2 > 0$ need

$$\frac{M^2}{2m} - \alpha < 0$$

$$\frac{M^2}{2m} < \alpha$$

$$\rightarrow \quad M_{max} = \sqrt{2m\alpha}$$

Cross section: $\sigma = \pi \rho_{max}^2$, $M = \rho m v_\infty$

Thus,

$$\sigma = \pi \rho_{max}^2 \quad , \qquad \rho_{max} = \frac{M_{max}}{m v_\infty}$$

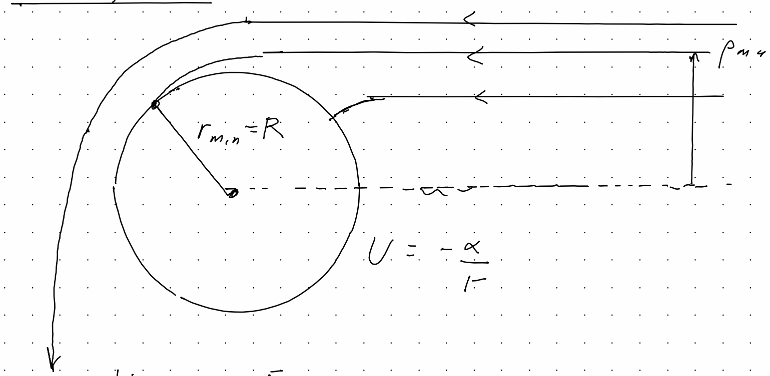
$$= \pi \frac{M_{max}^2}{m^2 v_\infty^2}$$

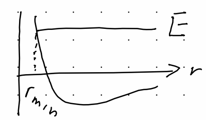
$$= \pi \frac{2 m \alpha}{m^2 v_\infty^2}$$

$$= \frac{\pi \alpha}{\frac{1}{2} m v_\infty^2}$$

$$= \boxed{\frac{\pi \alpha}{E}}$$

<u>Sec 18, Prob 6</u>





Turning point at $r = R$

$$0 = E - U_{eff}(R)$$

$$= E - U(R) - \frac{M_{max}^2}{2 m R^2}$$

$$= E + \frac{\alpha}{R} - \frac{M_{max}^2}{2 m R^2}$$

$$\rightarrow \frac{M_{max}^2}{2 m R^2} = E + \frac{\alpha}{R}$$

$$M_{max} = \rho_{max} m v_\infty \quad , \qquad E = \frac{1}{2} m v_\infty^2$$

Thus,

$$\sigma = \pi \rho_{max}^2$$

$$= \pi \frac{M_{max}^2}{m^2 v_\infty^2}$$

$$= \pi \frac{1}{m^2 v_\infty^2} \quad 2mR^2\left(E + \frac{\alpha}{R}\right)$$

$$= \pi R^2 \underbrace{\left(\frac{2}{m v_\infty^2}\right)}_{\frac{1}{E}}\left(E + \frac{\alpha}{R}\right)$$

$$= \boxed{\pi R^2\left(1 + \frac{\alpha}{ER}\right)}$$

where $E = \frac{1}{2} m v_\infty^2 = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right) v_\infty^2$

and $\alpha \equiv G m_1 m_2$

Sec 19, Prob. 1:

$$U = \frac{\alpha}{r^2}, \quad \alpha > 0$$

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M\, dr/r^2}{\sqrt{2m[E - U(r)] - M^2/r^2}}$$

Substitute: $E = \frac{1}{2} m v_\infty^2$

$M = \rho m v_\infty$

$$\rightarrow \phi_0 = \int_{r_{min}}^{\infty} \frac{\rho m v_\infty\, dr/r^2}{\sqrt{2m\left[\frac{1}{2} m v_\infty^2 - U(r)\right] - \rho^2 m^2 v_\infty^2/r^2}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - 2U(r)/m v_\infty^2 - \rho^2/r^2}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \rho^2/r^2 - 2U(r)/m v_\infty^2}}$$

Substitute: $U(r) = \frac{\alpha}{r^2}$

$$\sqrt{\quad} = \sqrt{1 - \rho^2/r^2 - \left(\frac{2\alpha}{m v_\infty^2}\right)\frac{1}{r^2}}$$

$$= \sqrt{1 - \left(\rho^2 + \frac{2\alpha}{m v_\infty^2}\right)\frac{1}{r^2}} = \sqrt{1 - \frac{A^2}{r^2}}$$

$$A^2 = \rho^2 + \frac{2\alpha}{m v_\infty^2}$$

Thus,

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1-\frac{A^2}{r^2}}}$$

$$= \int_{A}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1-\frac{A^2}{r^2}}}$$

Let: $u = \frac{1}{r} \rightarrow du = -\frac{1}{r^2}\,dr$

$$\frac{A^2}{r^2} = A^2 u^2$$

$$\phi_0 = -\int_{1/A}^{0} \frac{\rho\, du}{\sqrt{1-A^2u^2}}$$

$$= \int_{0}^{1/A} \frac{\rho\, du}{\sqrt{1-A^2u^2}}$$

Let: $\sin\theta = Au$

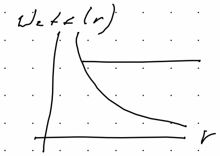
$\cos\theta\, d\theta = A\, du$

$u = 0, \frac{1}{A} \rightarrow \theta = 0, \pi/2$

$\sqrt{1-A^2u^2} = \sqrt{1-\sin^2\theta} = \cos\theta$

$$\rightarrow \phi_0 = \int_0^{\pi/2} \frac{\rho\, \cancel{\cos\theta}\, d\theta/A}{\cancel{\cos\theta}} = \frac{\rho}{A}\frac{\pi}{2}$$

$$\boxed{\phi_0 = \frac{\pi}{2}\frac{\rho}{\sqrt{\rho^2 + \frac{2\alpha}{m v_\infty^2}}}}$$



Repulsive scattering: $\chi + 2\phi_0 = \pi$

$\chi = \pi - 2\phi_0$

$$\chi = \pi - \pi\frac{\rho}{\sqrt{\rho^2 + \frac{2\alpha}{m v_\infty^2}}} = \pi\left[1 - \frac{1}{\sqrt{1+\frac{2\alpha}{\rho^2 m v_\infty^2}}}\right]$$

$$\left(\frac{\pi\rho}{\sqrt{\quad}}\right)^2 = (\pi-\chi)^2$$

$$\frac{\pi^2\rho^2}{\rho^2 + \frac{2\alpha}{m v_\infty^2}} = (\pi-\chi)^2$$

$$\pi^2\rho^2 = (\pi-\chi)^2\rho^2 + (\pi-\chi)^2\frac{2\alpha}{m v_\infty^2}$$

$$\left(\pi^2 - (\pi-\chi)^2\right)\rho^2 = (\pi-\chi)^2\frac{2\alpha}{m v_\infty^2}$$

$$\left(\cancel{\pi^2} - \cancel{\pi^2} - \chi^2 + 2\pi\chi\right)\rho^2 = (\pi-\chi)^2\frac{2\alpha}{m v_\infty^2}$$

$$\rho^2 = \frac{(\pi-\chi)^2}{2\pi\chi-\chi^2}\frac{2\alpha}{m v_\infty^2}$$

$$\boxed{\rho = \frac{(\pi-\chi)}{\sqrt{2\pi\chi-\chi^2}}\sqrt{\frac{2\alpha}{m v_\infty^2}}}$$

Differential cross-section:

$$d\sigma = 2\pi\rho\, d\rho$$

$$= 2\pi\, \rho(\chi) \left|\frac{d\rho}{d\chi}\right| d\chi$$

$$= \frac{\rho(\chi)}{\sin\chi} \left|\frac{d\rho}{d\chi}\right| d\Omega \quad , \quad d\Omega = 2\pi \sin\chi\, d\chi$$

Thus,

$$\frac{d\rho}{d\chi} = \sqrt{\frac{2\alpha}{m v_\infty^2}}\ \frac{-\sqrt{2\pi\chi-\chi^2} - \frac{1}{2\sqrt{\ }}(2\pi-2\chi)(\pi-\chi)}{2\pi\chi-\chi^2}$$

$$= -\sqrt{\frac{2\alpha}{m v_\infty^2}}\ \frac{\sqrt{2\pi\chi-\chi^2} + \frac{(\pi-\chi)^2}{\sqrt{\ }}}{2\pi\chi-\chi^2}$$

$$= -\sqrt{\frac{2\alpha}{m v_\infty^2}}\ \frac{2\pi\chi-\chi^2 + (\pi-\chi)^2}{(2\pi\chi-\chi^2)^{3/2}}$$

$$= -\sqrt{\frac{2\alpha}{m v_\infty^2}}\ \frac{\cancel{2\pi\chi} - \cancel{\chi^2} + \pi^2 + \cancel{\chi^2} - \cancel{2\pi\chi}}{(2\pi\chi-\chi^2)^{3/2}}$$

$$= -\sqrt{\frac{2\alpha}{m v_\infty^2}}\ \frac{\pi^2}{(2\pi\chi-\chi^2)^{3/2}}$$

So,

$$d\sigma = \frac{(\pi-\chi)}{\sqrt{2\pi\chi-\chi^2}} \sqrt{\frac{2\alpha}{m v_\infty^2}}\ \frac{1}{\sin\chi} \sqrt{\frac{2\alpha}{m\alpha}}\ \frac{\pi^2}{(2\pi\chi-\chi^2)^{3/2}}\, d\Omega$$

$$= \boxed{\left(\frac{2\alpha}{m v_\infty^2}\right) \frac{d\Omega}{\sin\chi}\ \frac{\pi^2(\pi-\chi)}{(2\pi\chi-\chi^2)^{3/2}}}$$

Sec 20, Prob 1   Small-angle scattering

Start with (18.4):

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \rho^2/r^2 - 2U/m v_\infty^2}}$$

Assume U is weak so that $2U/m v_\infty^2 \ll 1$

$$\frac{1}{\sqrt{\ }} = \frac{1}{\sqrt{(1-\rho^2/r^2)\left(1 - \frac{2U/m v_\infty^2}{(1-\rho^2/r^2)}\right)}}$$

$$\approx \frac{1}{\sqrt{1-\rho^2/r^2}} \left(1 + \frac{1}{\cancel{2}} \frac{\cancel{2}U/m v_\infty^2}{1-\rho^2/r^2}\right)$$

$$= \frac{1}{\sqrt{1-\rho^2/r^2}} + \frac{U/m v_\infty^2}{(1-\rho^2/r^2)^{3/2}}$$

Can replace r_{min} limit by ρ:

$$\int_{\rho}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1-\rho^2/r^2}} = -\int_{1/\rho}^{0} \frac{\rho\, du}{\sqrt{1-\rho^2 u^2}}$$

let: $u = \frac{1}{r}$

$du = -\frac{1}{r^2}\, dr$

$$= \int_0^{1/\rho} \frac{\rho\, du}{\sqrt{1-\rho^2 u^2}}$$

let $\rho u = \sin\theta$

$\rho\, du = \cos\theta\, d\theta$

$u = \frac{1}{\rho} \rightarrow \theta = \pi/2$

$$= \boxed{\frac{\pi}{2}}$$

Then,

$$\phi_0 \simeq \frac{\pi}{2} + \frac{1}{m v_\infty^2} \int_\rho^\infty \frac{\rho\, dr/r^2\, U(r)}{(1-\rho^2/r^2)^{3/2}}$$

$$= \frac{\pi}{2} + \frac{1}{m v_\infty^2} \frac{\partial}{\partial \rho} \left[\int_\rho^\infty \frac{U(r)\, dr}{\sqrt{1-\rho^2/r^2}} \right]$$

Now:

$$\int_\rho^\infty \frac{U(r)\, dr}{\sqrt{1-\rho^2/r^2}} = u v \Big|_\rho^\infty - \int_\rho^\infty v\, du$$

where $u = U(r)$

$$dv = \frac{dr}{\sqrt{1-\rho^2/r^2}} = \frac{r\, dr}{\sqrt{r^2-\rho^2}} \qquad x = r^2-\rho^2, \quad dx = 2r\,dr$$

$$= \frac{dx/2}{\sqrt{x}}$$

$$\rightarrow v = \frac{1}{2}\int \frac{dx}{\sqrt{x}} = \sqrt{x} + \text{const}$$

$$= \sqrt{r^2-\rho^2} + \text{const}$$

So:

$$\int_\rho^\infty \frac{U(r)\, dr}{\sqrt{1-\rho^2/r^2}} = \cancel{U(r)\sqrt{r^2-\rho^2}\Big|_\rho^\infty} - \int_\rho^\infty \left(\frac{dU}{dr}\right) dr \sqrt{r^2-\rho^2}$$

assuming
$U(r) \rightarrow 0$ faster
than $\frac{1}{r}$ as $r \rightarrow \infty$

So

$$\phi_0 = \frac{\pi}{2} + \frac{1}{m v_\infty^2} \frac{\partial}{\partial \rho}\left[-\int_\rho^\infty \frac{dU}{dr}\, dr \sqrt{r^2-\rho^2} \right]$$

$$= \frac{\pi}{2} + \frac{1}{m v_\infty^2} (-) \int_\rho^\infty \frac{dU}{dr}\, dr \frac{1}{\cancel{2}\sqrt{r^2-\rho^2}} (-\cancel{2}\rho)$$

$$= \frac{\pi}{2} + \frac{\rho}{m v_\infty^2} \int_\rho^\infty dr \frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

Scattering angle χ:

$$2\phi_0 + \chi = \pi$$

$$\chi = \pi - 2\phi_0$$

$$\rightarrow \chi = \cancel{\pi} - 2\left(\cancel{\frac{\pi}{2}} + \frac{\rho}{m v_\infty^2} \int_\rho^\infty dr \frac{dU/dr}{\sqrt{r^2-\rho^2}}\right)$$

$$= -\frac{2\rho}{m v_\infty^2} \int_\rho^\infty dr \frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

In terms of θ_1:

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi} \rightarrow \theta_1 \simeq \frac{m_2\chi}{m_1+m_2}$$

Thus,

$$\theta_1 \simeq \left(\frac{m_2}{m_1+m_2}\right)\chi$$

$$\simeq \left(\frac{m_2}{m_1+m_2}\right)\left(\frac{-2\rho}{m v_\infty^2}\right)\int_\rho^\infty dr\,\frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

$$= \frac{-2\rho}{m_1 v_\infty^2}\int_\rho^\infty dr\,\frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

which is Eq. (20.3)