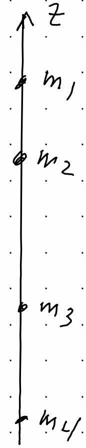
Sec 32, Prob 1:

(a)



$I_3 = 0$ since $x_a = y_a = 0$ for all masses

$$I_1 = \sum_a m_a (r_a^2 - \cancel{x_a^2}^{\,0}) = \sum_a m_a z_a^2$$

$$I_2 = \sum_a m_a (r_a^2 - \cancel{y_a^2}^{\,0}) = \sum_a m_a z_a^2$$

$$\rightarrow I_1 = I_2 \equiv I$$

$$= \sum_a m_a z_a^2$$

...

(assuming COM at $z=0$)

If COM is not at $z=0$, but at z_{com}, then:

$$I = \sum_a m_a (z_a - z_{com})^2 \,, \qquad z_{com} = \frac{1}{\mu} \sum_b m_b z_b$$

$$= \sum_a m_a (z_a^2 + z_{com}^2 - 2 z_{com} z_a)$$

$$= \sum_a m_a z_a^2 + \mu z_{com}^2 - 2 z_{com} \underbrace{\sum_a m_a z_a}_{\mu z_{com}}$$

$$= \sum_a m_a z_a^2 - \mu z_{com}^2$$

This last expression can be rewritten in terms of $l_{ab} \equiv |z_a - z_b|$ as follows:

$$I = \frac{1}{2} \sum_a m_a z_a^2 + \frac{1}{2} \sum_b m_b z_b^2 - \frac{1}{\mu} \Big(\sum_a m_a z_a\Big)\Big(\sum_b m_b z_b\Big)$$

$$= \frac{1}{2\mu} \sum_a \sum_b m_b m_a z_a^2 + \frac{1}{2\mu} \sum_a \sum_b m_a m_b z_b^2 - \frac{1}{\mu} \sum_a \sum_b m_a m_b z_a z_b$$

Thus

$$I = \frac{1}{2M} \sum_a \sum_b m_a m_b \left(z_a^2 + z_b^2 - 2 z_a z_b \right)$$

$$= \frac{1}{2M} \sum_a \sum_b m_a m_b \left(z_a - z_b \right)^2$$

$$= \frac{1}{2M} \sum_a \sum_b m_a m_b \, l_{ab}^2$$

<u>NOTE:</u> For just two masses:

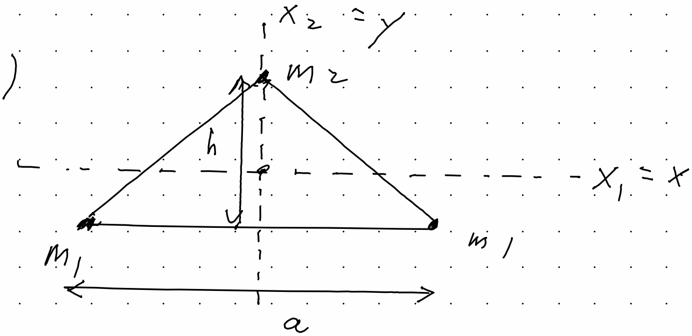
$$I = \frac{1}{2M} \left(m_1 m_2 l^2 + m_2 m_1 l^2 \right)$$

$$= \frac{m_1 m_2}{M} l^2$$

$$= m l^2 \quad \text{where} \quad m \equiv \frac{m_1 m_2}{m_1 + m_2}$$

$$l = |z_1 - z_2|$$

(b)



Assume COM at $(x_1, x_2) = (x, y) = (0, 0)$

Then $2m_1 y_1 + m_2 y_2 = 0$

where $y_2 - y_1 = h$

Thus, $2m_1 y_1 + m_2(h + y_1) = 0$

$$(2m_1 + m_2) y_1 + m_2 h = 0$$

$$y_1 = \frac{-m_2 h}{\mu}, \quad \mu = 2m_1 + m_2 = \text{total mass}$$

and
$$y_2 = y_1 + h = \frac{-m_2 h}{\mu} + h = \frac{(\mu - m_2) h}{\mu} = \frac{2m_1 h}{\mu}$$

All masses have $z_a = 0$

Thus, $I_3 = \sum_a m_a (r_a^2 - z_a^2) = \sum_a m_a (x_a^2 + y_a^2)$

$$I_1 = \sum_a m_a (r_a^2 - x_a^2) = \sum_a m_a y_a^2$$

$$I_2 = \sum_u m_u (r_u^2 - y_u^2) = \sum_u m_u x_u^2$$

Thus, $I_3 = I_1 + I_2$

so need to calculate I_1, I_2.

$$I_1 = \sum_u m_u y_u^2$$

$$= 2 m_1 y_1^2 + m_2 y_2^2$$

$$= 2 m_1 \frac{m_2^2 h^2}{\mu^2} + m_2 \frac{4 m_1^2 h^2}{\mu^2}$$

$$= \frac{2 m_1 m_2 h^2}{\mu^2} (m_2 + 2 m_1)$$

$$= \boxed{\frac{2 m_1 m_2}{\mu} h^2}$$

$$I_2 = \sum_u m_u x_u^2$$

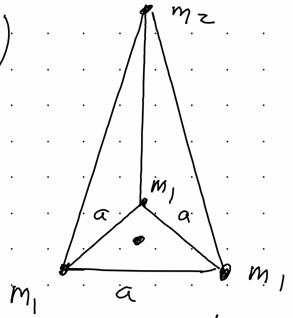
$$= m_1 \left(\frac{a}{2}\right)^2 + m_1 \left(\frac{-a}{2}\right)^2$$

$$= \boxed{\frac{m_1 a^2}{2}}$$

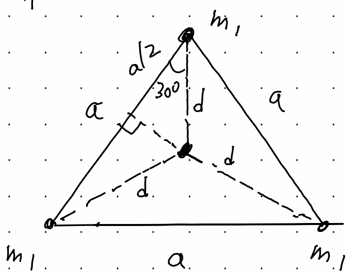
$$I_3 = I_1 + I_2$$

$$= \boxed{\frac{2 m_1 m_2}{\mu} h^2 + \frac{m_1 a^2}{2}}$$

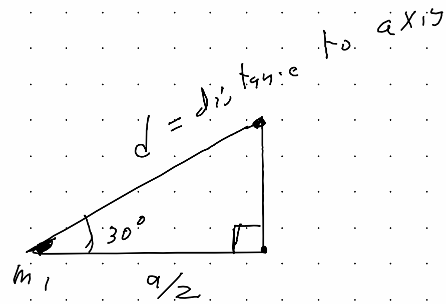
(c)



tetrahedron, height h
base: equilateral △ with side length a



base



$$\cos 30° = \frac{a}{2d} = \frac{\sqrt{3}}{2}$$

$$\to \quad d = \frac{a}{\sqrt{3}}$$

COM lies on axis of symmetry (Z-axis)
Assume COM has $Z = 0$

Then $0 = m_2 Z_2 + 3m_1 Z_1$, $\quad Z_2 - Z_1 = h$

$$= m_2 (z_1 + h) + 3m_1 z_1$$

$$= (3m_1 + m_2) z_1 + m_2 h$$

$$\to \quad Z_1 = \frac{-m_2 h}{3m_1 + m_2} = \frac{-m_2 h}{\mu}$$

$$Z_2 = h + z_1$$

$$= h - \frac{m_2 h}{\mu}$$

$$= \frac{3m_1 h}{\mu}$$

Since a tetrahedron has 3-fold rotational symmetry, the x_1 principal axes can be chosen arbitrarily in the plane $\perp$ to the symmetry axis ($x_3 \equiv z$). [x_2 is $\perp$ to x_1, x_3]

Thus, $I_1 = I_2 \equiv I$

$$I_3 = \sum_a m_a (r_a^2 - z_a^2)$$

$$= \sum_a m_a s_a^2 \quad \text{where } s^2 = r^2 - z^2$$

$$\left.\begin{aligned} I_1 &= \sum_a m_a (r_a^2 - x_a^2) \\ I_2 &= \sum_a m_a (r_a^2 - y_a^2) \end{aligned}\right\} \text{equal } (I_1 = I_2 \equiv I)$$

$$2I = I_1 + I_2$$

$$= \sum_a m_a (2r_a^2 - x_a^2 - y_a^2)$$

$$= \sum_a m_a \left(2(s_a^2 + z_a^2) - s_a^2\right)$$

$$= \sum_a m_a s_a^2 + 2\sum_a m_a z_a^2$$

$$= I_3 + 2\sum_a m_a z_a^2$$

Thus,

$$I = \frac{1}{2} I_3 + \sum_a m_a z_a^2$$

Now

$$I_3 = \sum_a m_a s_a^2$$

$$= 3 m_1 d^2$$

$$= \cancel{3} m_1 \frac{a^2}{\cancel{3}} = \boxed{m_1 a^2}$$

Also,

$$\sum_q m_q z_q^2 = 3m_1 z_1^2 + m_2 z_2^2$$

$$= 3m_1\left(-\frac{m_2}{\mu}h\right)^2 + m_2\left(\frac{3m_1}{\mu}h\right)^2$$

$$= \frac{3m_1 m_2^2 h^2}{\mu^2} + \frac{9m_1^2 m_2 h^2}{\mu}$$

$$= \frac{3m_1 m_2 h^2}{\mu^2}\underbrace{\left(m_2 + 3m_1\right)}_{\mu}$$

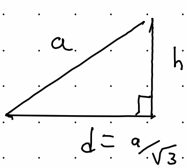
$$= \frac{3m_1 m_2 h^2}{\mu}$$

Thus,

$$I = \frac{1}{2}I_3 + \sum_q m_q z_q^2$$

$$= \boxed{\frac{1}{2}m_1 a^2 + \frac{3m_1 m_2 h^2}{\mu}}$$

Regular tetrahedron: $m_1 = m_2$



$$h^2 + \frac{a^2}{3} = a^2 \rightarrow h = \sqrt{\frac{2}{3}}\,a$$

$$\mu = 4m_1$$

$$I_3 = m_1 a^2$$

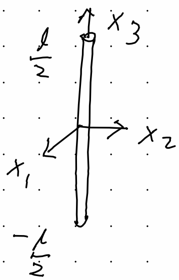
$$I = \frac{1}{2}m_1 a^2 + \frac{\cancel{3}\cancel{m_1}m_1}{4\cancel{m_1}}\left(\frac{2}{\cancel{3}}\right)a^2$$

$$= m_1 a^2 \quad (= I_1 = I_2)$$

so $I_1 = I_2 = I_3 = m_1 a^2$

Sec 32, Prob 2:

(a) Thin rod of length ℓ:



$$I_3 = \boxed{0}$$

and $I_1 = I_2 \equiv I$

$$I = \int \rho\, dV\, (r^2 - x^2)$$

$$= \int \rho\, dV\, z^2$$

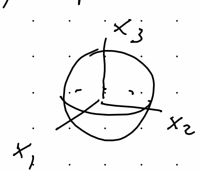
$$= \int_{-\ell/2}^{\ell/2} dz \left(\frac{\mu}{\ell}\right) z^2$$

$$= \frac{\mu}{\ell} \frac{z^3}{3} \Big|_{-\ell/2}^{\ell/2}$$

$$= \frac{\mu}{\ell} \frac{2}{3} \frac{\ell^3}{8}$$

$$= \boxed{\frac{1}{12} \mu \ell^2}$$

(b) Sphere of radius R:



$$I_1 = I_2 = I_3 = I$$

$$I = \frac{1}{3}(I_1 + I_2 + I_3)$$

$$= \frac{1}{3}\left[\int \rho\, dV\, (r^2 - x^2) + \int \rho\, dV\, (r^2 - y^2) + \int \rho\, dV\, (r^2 - z^2)\right]$$

$$= \frac{1}{3} \int \rho\, dV\, [3r^2 - x^2 - y^2 - y^2]$$

$$= \frac{2}{3} \int \rho\, dV\, r^2$$

$$I = \frac{2}{3}\int \rho\, dV\, r^2 \qquad \rho = \frac{M}{\frac{4}{3}\pi R^3}$$

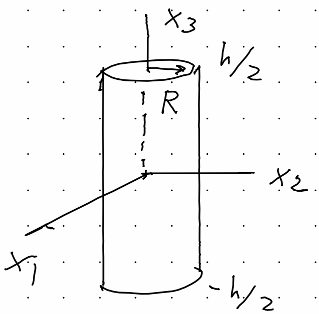
$$= \frac{2}{3}\frac{M}{\frac{4}{3}\pi R^3}\int_0^R r^4 dr \underbrace{\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi}_{4\pi}$$

$$= \frac{M}{2\pi R^3}\cdot 4\pi \int_0^R r^4 dr$$

$$= \frac{2M}{R^3}\frac{R^5}{5}$$

$$= \boxed{\frac{2}{5}MR^2}$$

(c)



$$\rho = \frac{M}{\pi R^2 \cdot h}$$

$$dV = ds\, s\, d\phi\, dz$$

where $s^2 = x^2 + y^2$

$$I_1 = I_2 \equiv I$$

$$2I = I_1 + I_2$$

$$= \int \rho\, dV (r^2 - x^2) + \int \rho\, dV (r^2 - y^2)$$

$$= \int \rho\, dV (2r^2 - s^2)$$

$$= \int \rho\, dV (2(s^2 + z^2) - s^2)$$

$$= \int \rho\, dV\, s^2 + 2\int \rho\, dV\, z^2$$

$$\rightarrow I = \frac{1}{2} I_3 + \int \rho\, dV \cdot z^2$$

$$I_3 = \int \rho \, dV \, s^2$$

$$= \frac{M}{\pi R^2 h} \int_0^R s^3 \, ds \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz$$

$$= \frac{M}{\cancel{\pi} R^2 \cancel{h}} \frac{R^4}{4} \cdot 2\cancel{\pi} \cdot \cancel{h}$$

$$= \boxed{\frac{1}{2} M R^2}$$

$$I = \frac{1}{2} I_3 + \int \rho \, dV \, z^2$$

$$\int \rho \, dV \, z^2 = \frac{M}{\pi R^2 h} \int_0^R s \, ds \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} z^2 \, dz$$

$$= \frac{M}{\cancel{\pi} \cancel{R^2} h} \frac{\cancel{R^2}}{\cancel{2}} \cdot \cancel{2}\cancel{\pi} \cdot \frac{z^3}{3} \Big|_{-h/2}^{h/2}$$

$$= \frac{M}{h} \frac{2}{3} \frac{h^3}{8}$$

$$= \frac{1}{12} M h^2$$

Thus,

$$I = \frac{1}{2} \left(\frac{1}{2} M R^2 \right) + \frac{1}{12} M h^2$$

$$= \frac{1}{4} M R^2 + \frac{1}{12} M h^2$$

$$= \boxed{\frac{1}{4} M \left(R^2 + \frac{1}{3} h^2 \right)}$$

NOTE: special limiting case:

(i) Thin rod $(R \to 0)$

$$I_3 = 0$$

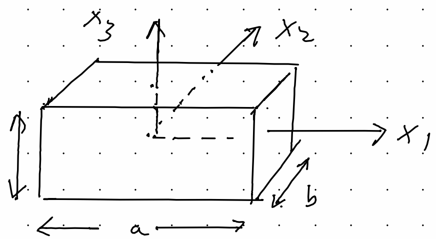
$$I_1 = I_2 = \frac{1}{12} M h^2$$

(ii) Thin disk $(h \to 0)$

$$I_3 = \frac{1}{2} M R^2$$

$$I_1 = I_2 = \frac{1}{4} M R^2$$

(d)



$$\rho = \frac{M}{abc}$$

$$dV = dx\,dy\,dz$$

$$I_1 = \int \rho\, dV\,(r^2 - x^2)$$

$$= \int \rho\, dV\,(y^2 + z^2)$$

$$= \frac{M}{abc} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz\,(y^2 + z^2)$$

$$= \frac{M}{\cancel{a}bc}\, \cancel{a} \int_{-b/2}^{b/2} dy \left(y^2 z + \frac{z^3}{3}\right)\Big|_{-c/2}^{c/2}$$

$$= \frac{M}{bc} \int_{-b/2}^{b/2} dy \left(c y^2 + \frac{2}{3}\frac{c^3}{8}\right)$$

$$= \frac{M}{bc}\left[c\frac{y^3}{3}\Big|_{-b/2}^{b/2} + \frac{1}{12} c^3 y \Big|_{-b/2}^{b/2} \right]$$

$$I_1 = \frac{M}{bc}\left[c\,\frac{2}{3}\,\frac{b^3}{8} + \frac{1}{12}\,b c^3\right]$$

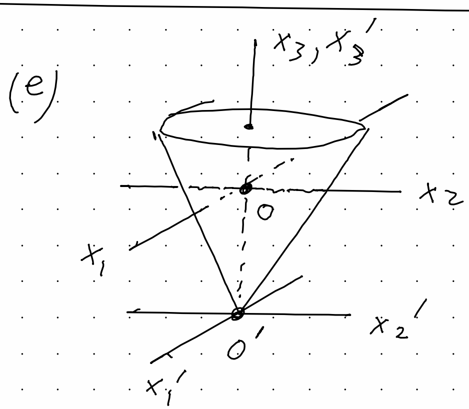
$$= \frac{M}{12\,bc}\left[c b^3 + b c^3\right]$$

$$= \frac{M}{12}\left(b^2 + c^2\right)$$

$$I_2 = \frac{M}{12}\left(c^2 + a^2\right)$$

$$I_3 = \frac{M}{12}\left(a^2 + b^2\right)$$

(e)



First calculate I'_{ij} (w.r.t. x_1', x_2', x_3').
Then calculate I_{ij} via

$$I_{ij} = I'_{ij} - M\left(a^2 \delta_{ij} - a_i a_j\right)$$

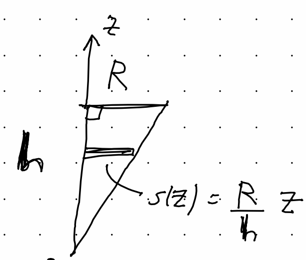
where $\vec{a} = (0, 0, -d)$

d: height of COM above O'

Volume of cone:

$$V = \int dV$$

$$= \int_0^h dz \int_0^{2\pi} d\phi \int_0^{\frac{R}{h}z} s\,ds$$



$$= \cancel{2}\pi \int_0^h dz\, \frac{s^2}{\cancel{2}}\Big|_0^{Rz/h}$$

$$V = \pi \int_0^h dz \, \frac{R^2 z^2}{h^2}$$

$$= \frac{\pi R^2}{h^2} \left. \frac{z^3}{3} \right|_0^h$$

$$= \frac{\pi R^2}{h^2} \frac{h^3}{3}$$

$$= \boxed{\frac{1}{3} \pi R^2 h}$$

Thus, $\rho = \dfrac{M}{V} = \boxed{\dfrac{M}{\frac{1}{3}\pi R^2 h}}$ (mass density)

$$I_3' = \int \rho \, dV \, (r^2 - z^2)$$

$$= \int \rho \, dV \, s^2$$

$$= \rho \int_0^h dz \int_0^{2\pi} d\phi \int_0^{Rz/h} s^3 \, ds$$

$$= \frac{\rho}{4} \cdot 2\pi \int_0^h dz \left. s^4 \right|_0^{\frac{Rz}{h}}$$

$$= \frac{\rho \pi}{2} \int_0^h dz \left(\frac{R^4}{h^4} \right) z^4$$

$$= \frac{\rho \pi R^4}{2 h^4} \frac{h^5}{5}$$

$$= \frac{\rho \pi}{10} R^4 h$$

Thus,

$$I_3' = \frac{M}{\frac{1}{3}\cancel{\pi} R^2 \cancel{h}} \frac{\cancel{\pi}}{10} R^4 \cancel{h}$$

$$= \boxed{\frac{3}{10} M R^2}$$

Similar to the cylinder, we have $I_1' = I_2' \equiv I'$ where

$$I' = \frac{1}{2} I_3' + \int \rho \, dV \, z^2$$

Now:

$$\int \rho \, dV z^2 = \rho \int_0^h dz \, z^2 \int_0^{2\pi} d\phi \int_0^{Rz/h} s \, ds$$

$$= \cancel{2}\pi\rho \int_0^h dz \, z^2 \left. \frac{s^2}{\cancel{2}} \right|_0^{Rz/h}$$

$$= \pi\rho \frac{R^2}{h^2} \int_0^h dz \, z^4$$

$$= \cancel{\pi}\rho \frac{R^2}{h^2} \frac{h^5}{5}$$

$$= \frac{\pi}{5} \rho R^2 h^3$$

$$= \frac{\pi}{5} \left(\frac{M}{\frac{1}{3}\pi \cancel{R^2} h} \right) \cancel{R^2} h^3$$

$$= \frac{3}{5} M h^2$$

So

$$I' = \frac{1}{2}\left(\frac{3}{10}\mu R^2\right) + \frac{3}{5}\mu h^2$$

$$= \boxed{\frac{3}{5}\mu\left(\frac{R^2}{4} + h^2\right)} = I_1' = I_2'$$

Need to find location of COM:

$$d = \frac{1}{\mu}\int \rho \, dV \, z$$

$$= \frac{1}{\mu}\rho \int_0^h z\,dz \int_0^{2\pi} d\phi \int_0^{Rz/h} s\,ds$$

$$= \frac{1}{\mu}\rho \cdot \cancel{2}\pi \int_0^h z\,dz \, \frac{1}{\cancel{2}}\left(\frac{R}{h}\right)^2 z^2$$

$$= \frac{\pi\rho}{\mu}\frac{R^2}{h^2}\left.\frac{z^4}{4}\right|_0^h$$

$$= \frac{\pi\rho}{4\mu} R^2 h^2$$

$$= \frac{\cancel{\pi}}{4\cancel{\mu}} \cdot \frac{\cancel{\mu}}{\frac{1}{3}\cancel{\pi}\cancel{R^2}h} \cancel{R^2} h^2$$

$$= \boxed{\frac{3}{4}h}$$

Thus,

$$I_{ij} = I_{ij}' - \mu (a^2 \delta_{ij} - a_i a_j)$$

where $\vec{a} = (0, 0, -\frac{3}{4}h) \rightarrow a^2 = \frac{9}{16} h^2$

$$\rightarrow \quad I_1 = I_1' - \mu a^2$$

$$= \frac{3}{5}\mu\left(\frac{R^2}{4} + h^2\right) - \mu \frac{9}{16} h^2$$

$$= \frac{3}{20}\mu R^2 + \mu h^2 \underbrace{\left(\frac{3}{5} - \frac{9}{16}\right)}_{\frac{48-45}{80} = \frac{3}{80}}$$

$$= \boxed{\frac{3}{20}\mu\left(R^2 + \frac{h^2}{4}\right)}$$

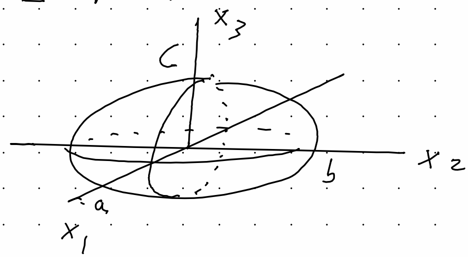
Also, $I_2 = I_1$

Finally, $I_3 = I_3' - \mu(\cancel{a^2 - a^2})^{\to 0}$

$$= I_3'$$

$$= \boxed{\frac{3}{10}\mu R^2}$$

(f) Ellipsoid with semi-axes a, b, c



$(a, b, c) \leftrightarrow (x_1, x_2, x_3)$

Define rescaled coordinates:

$$(u, v, w) \equiv \left(\frac{x_1}{a}, \frac{x_2}{b}, \frac{x_3}{c}\right)$$

so that boundary of ellipsoid

$$1 = \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 + \left(\frac{x_3}{c}\right)^2 = \underbrace{u^2 + v^2 + w^2}_{\text{unit 2-sphere}}$$

Volume:

$$V = \int dx_1 \int dx_2 \int dx_3$$

$$= abc \int du \int dv \int dw$$

$$= abc \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^1 r^2 dr$$

$$= abc \; 2\pi \cdot 2 \cdot \frac{r^3}{3}\Big|_0^1$$

$$= \boxed{\frac{4}{3}\pi abc}$$

$$\longrightarrow \rho = \frac{M}{\frac{4}{3}\pi abc}$$

$$I_3 = \int \rho\, dV\,(r^2 - z^2)$$

$$= \int \rho\, dV\,(x^2 + y^2)$$

$$= \frac{M}{\frac{4}{3}\pi abc} \iiint dx\, dy\, dz\,(x^2 + y^2)$$

$$= \frac{M}{\frac{4}{3}\pi \cancel{abc}}\, \cancel{abc} \iiint du\, dv\, dw\,(a^2 u^2 + b^2 v^2)$$

$$= \frac{M}{\frac{4}{3}\pi} \int_0^1 r^2 dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi\, \left(a^2 r^2 \sin^2\theta \cos^2\phi + b^2 r^2 \sin^2\theta \sin^2\phi\right)$$

Now:

$$\int_0^{\pi} \sin\theta\, d\theta\, \sin^2\theta = \int_{-1}^{1} d(\cos\theta)\,(1 - \cos^2\theta)$$

$$= \int_{-1}^{1} dx\,(1 - x^2)$$

$$= \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$= 2 \cdot \frac{2}{3} = \boxed{\frac{4}{3}}$$

$$\int_0^1 r^4 dr = \frac{r^5}{5}\Big|_0^1 = \boxed{\frac{1}{5}}$$

$$\int_0^{2\pi} d\phi \begin{Bmatrix} \sin^2\phi \\ \cos^2\phi \end{Bmatrix} = 2\pi \cdot \frac{1}{2} = \boxed{\pi}$$

thus,

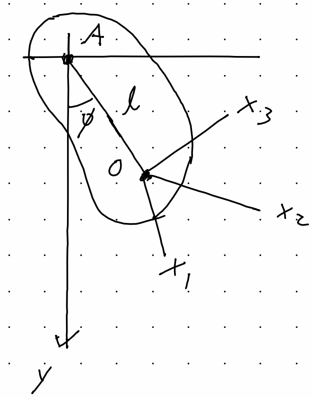
$$I_3 = \frac{M}{\cancel{\frac{4}{3}}\cancel{\pi}}\left(a^2 \cancel{\frac{4}{3}} \cdot \frac{1}{5} \cdot \cancel{\pi} + b^2 \cancel{\frac{4}{3}} \cdot \frac{1}{5} \cdot \cancel{\pi}\right)$$

$$= \boxed{\frac{M}{5}\left(a^2+b^2\right)}$$

cyclically permuting a, b, c $\rightarrow$

$$\boxed{\begin{aligned} I_1 &= \frac{M}{5}\left(b^2+c^2\right) \\ I_2 &= \frac{M}{5}\left(c^2+a^2\right) \end{aligned}}$$

Sec 32, Prob 3:



com at O

rotation axis at A, out of page

$$\vec{\Omega} = \dot{\phi}\,\hat{n}$$

$$U = \mu g l(1-\cos\phi)$$

$$\approx \frac{1}{2}\mu g l \phi^2 \quad \text{for } \phi \ll 1$$

$$L = T - U$$

$$T = \frac{1}{2}\mu V^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$\vec{\Omega} = \dot{\phi}\,\hat{n} \rightarrow \quad \Omega_1 = \dot{\phi}\,\hat{n}\cdot\hat{x}_1 = \dot{\phi}\cos\alpha$$

$$\Omega_2 = \dot{\phi}\,\hat{n}\cdot\hat{x}_2 = \dot{\phi}\cos\beta$$

$$\Omega_3 = \dot{\phi}\,\hat{n}\cdot x_3 = \dot{\phi}\cos\gamma$$

$$V = l\dot{\phi}$$

Thus,

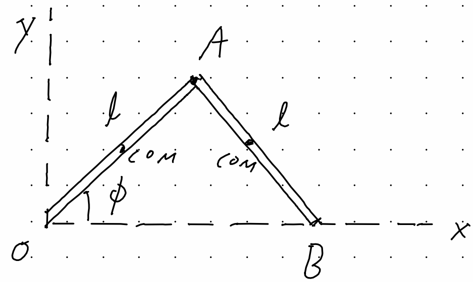
$$T = \frac{1}{2}\mu l^2\dot{\phi}^2 + \frac{1}{2}\dot{\phi}^2\left(I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)$$

so

$$L = \frac{1}{2}\dot{\phi}^2\left(\mu l^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right) - \frac{1}{2}\mu g l \phi^2$$

$$\rightarrow \omega = \sqrt{\frac{\mu g l}{\mu l^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma}}$$

Sec. 32, Prob 4



two uniform rods

$$I_{com} = \frac{1}{12} M l^2$$

$$I_{end} = \frac{1}{3} M l^2$$

$$T = T_1 + T_2$$

$$T_1 = \frac{1}{2} M \left(\frac{l}{2}\right)^2 \dot{\phi}^2 + \frac{1}{2} I_{com} \dot{\phi}^2$$

$$= \frac{1}{8} M l^2 \dot{\phi}^2 + \frac{1}{24} M l^2 \dot{\phi}^2$$

$$= \left(\frac{1}{8} + \frac{1}{24}\right) M l^2 \dot{\phi}^2$$

$$= \frac{1}{6} M l^2 \dot{\phi}^2$$

$$T_2 = \frac{1}{2} M V^2 + \frac{1}{2} I_{com} \dot{\phi}^2$$

Now: $V^2 = \dot{x}^2 + \dot{y}^2$

$$x = \frac{3}{2} l \cos\phi \quad , \quad y = \frac{l}{2} \sin\phi$$

$$\dot{x} = -\frac{3}{2} l \sin\phi \, \dot{\phi} \quad , \quad \dot{y} = \frac{l}{2} \cos\phi \, \dot{\phi}$$

$$\rightarrow V^2 = \frac{9}{4} l^2 \sin^2\phi \, \dot{\phi}^2 + \frac{l^2}{4} \cos^2\phi \, \dot{\phi}^2$$

$$= 2 l^2 \sin^2\phi \, \dot{\phi}^2 + \frac{l^2}{4} \dot{\phi}^2$$

$$= 2 l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{8}\right)$$

so

$$T_2 = \frac{1}{\cancel{2}} \mu \cancel{2} l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{8} \right) + \frac{1}{24} \mu l^2 \dot{\phi}^2$$

$$= \mu l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{8} + \frac{1}{24} \right)$$

$$= \mu l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{6} \right)$$

Thus,

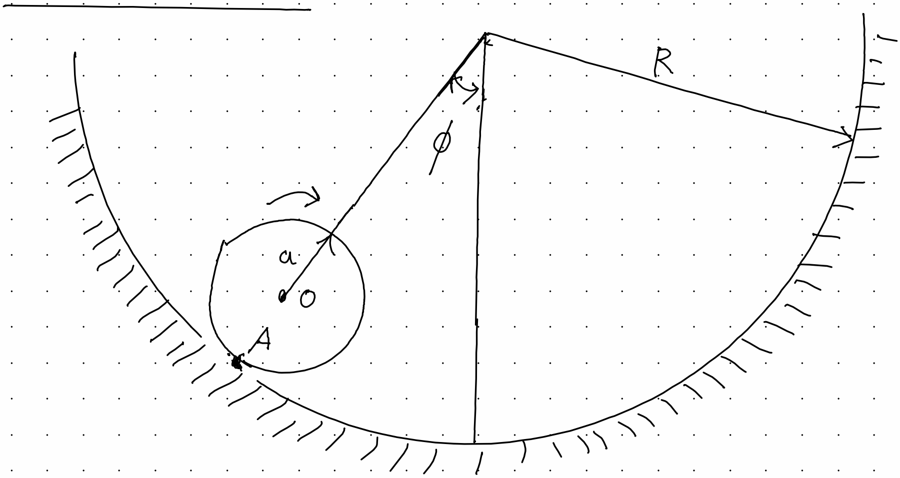
$$T = T_1 + T_2$$

$$= \frac{1}{6} \mu l^2 \dot{\phi}^2 + \mu l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{6} \right)$$

$$= \mu l^2 \dot{\phi}^2 \left(\frac{1}{3} + \sin^2\phi \right)$$

$$= \frac{1}{3} \mu l^2 \dot{\phi}^2 \left(1 + 3 \sin^2\phi \right)$$

Sec 32, Prob 6:



Homogeneous cylinder of radius a, mass μ:

$$I_3 = \frac{1}{2}\mu a^2 \quad \text{(about com)}$$

$$V = -(R-a)\dot{\phi} \quad \text{(velocity of com)}$$

Instantaneous axis of rotation at A:

$$0 = \vec{V} + \vec{\Omega} \times (-a\hat{n}), \quad \vec{\Omega} \text{ into page}$$

so $V = \Omega a$

Thus, $\Omega a = (R-a)|\dot{\phi}|$

$$\Omega = \left(\frac{R-a}{a}\right)|\dot{\phi}|$$

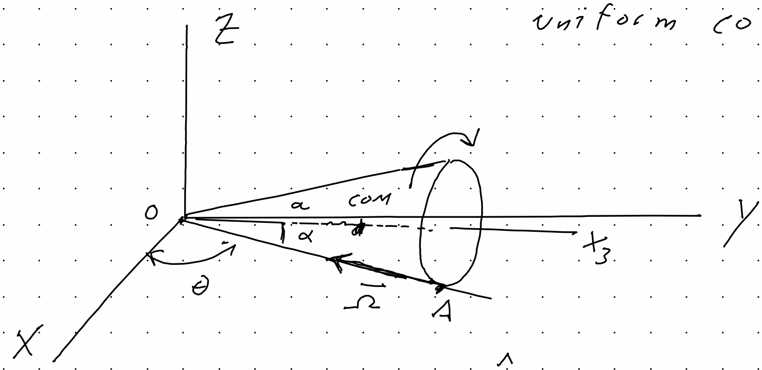
$$T = \frac{1}{2}\mu V^2 + \frac{1}{2} I_3 \Omega^2$$

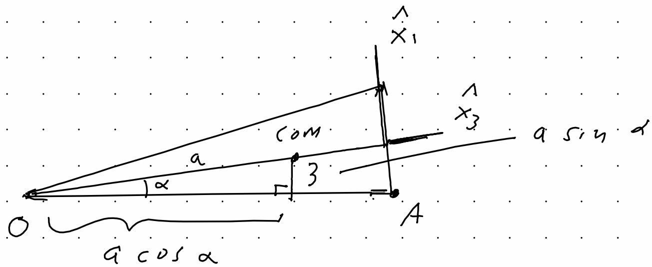
$$= \frac{1}{2}\mu\left(\frac{R-a}{\cancel{a}}\right)^2 \cancel{a^2}\dot{\phi}^2 + \frac{1}{4}\mu \cancel{a^2}\left(\frac{R-a}{\cancel{a}}\right)^2\dot{\phi}^2$$

$$= \mu(R-a)^2\dot{\phi}^2\left(\frac{1}{2}+\frac{1}{4}\right) = \boxed{\frac{3}{4}\mu(R-a)^2\dot{\phi}^2}$$

sec 32, Prob 7.

radius R, height h
uniform cone





V = velocity of COM
$= a\cos\alpha\,\dot{\theta}$

OA : instantaneous axis of rotation

$0 = \vec{V} - a\sin\alpha\,\vec{\Omega}\times\hat{n}$ $\quad (\hat{n} = \hat{z})$

$0 = V - a\sin\alpha\,\Omega$

thus, $a\sin\alpha\,\Omega = V = a\cos\alpha\,\dot{\theta}$

$$\boxed{\Omega = \cot\alpha\,\dot{\theta}}$$

$\vec{\Omega}$: directed from A to O

$\Omega_3 = -\Omega\cos\alpha = -\dfrac{\cos^2\alpha}{\sin\alpha}\dot{\theta}$

$\Omega_1 = \Omega\sin\alpha = \cos\alpha\,\dot{\theta}$

Thus,

$$(I_2 = I_1)$$

$$T = \frac{1}{2}\mu V^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_1 \Omega_2^2 + I_3 \Omega_3^2\right)$$

$$= \frac{1}{2}\mu a^2 \cos^2\alpha\, \dot{\theta}^2 + \frac{1}{2}\left(I_1 \cos^2\alpha\, \dot{\theta}^2 + I_3 \frac{\cos^4\alpha}{\sin^2\alpha}\dot{\theta}^2\right)$$

Now:

$$I_1 = I_2 = \frac{3}{20}\mu\left(R^2 + \frac{1}{4}h^2\right)$$

$$I_3 = \frac{3}{10}\mu R^2$$

Also:

$$a = \frac{3}{4}h$$

$$\tan\alpha = \frac{R}{h} \rightarrow R = h\tan\alpha$$

Thus,

$$T = \frac{1}{2}\mu\left(\frac{9}{16}\right)h^2\cos^2\alpha\,\dot{\theta}^2$$

$$+ \frac{1}{2}\left[\frac{3}{20}\mu\left(h^2\tan^2\alpha + \frac{1}{4}h^2\right)\cos^2\alpha\,\dot{\theta}^2\right.$$

$$\left. + \frac{3}{10}\mu h^2 \tan^2\alpha \frac{\cos^4\alpha}{\sin^2\alpha}\dot{\theta}^2\right]$$

$$= \mu h^2 \dot{\theta}^2\left[\frac{9}{32}\cos^2\alpha + \frac{3}{40}\underbrace{\sin^2\alpha}_{(1-\cos^2\alpha)} + \frac{3}{160}\cos^2\alpha\right.$$

$$\left. + \frac{3}{20}\cos^2\alpha\right]$$

$$= \mu h^2\dot{\theta}^2\left[\frac{3}{40} + \cos^2\alpha\left(\frac{9}{32} - \frac{3}{40} + \frac{3}{160} + \frac{3}{20}\right)\right]$$

Now:

$$\frac{9}{32} - \frac{3}{40} + \frac{3}{160} + \frac{3}{20}$$

$$= \frac{1}{160}\left[45 - 12 + 3 + 24\right]$$

$$= \frac{60}{160}$$

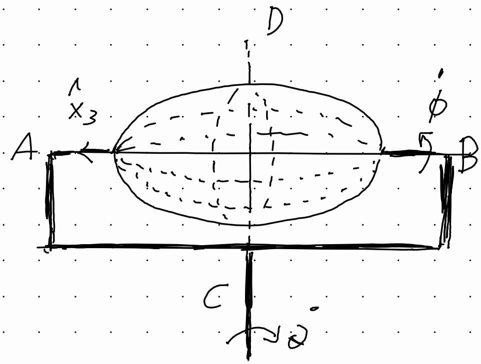
$$= \frac{15}{40}$$

Thus,

$$T = \mu h^2 \dot{\theta}^2 \left[\frac{3}{40} + \frac{15}{40}\cos^2\alpha\right]$$

$$= \boxed{\frac{3}{40}\mu h^2 \dot{\theta}^2 \left[1 + 5\cos^2\alpha\right]}$$

Sec 32, Prob 9.



homogeneous ellipsoid with principal moments of inertia I_1, I_2, I_3

$$\vec{\Omega} = \vec{\dot{\phi}} + \vec{\dot{\theta}}$$

Now: $\vec{\dot{\phi}} = \dot{\phi}\,\hat{x}_3$

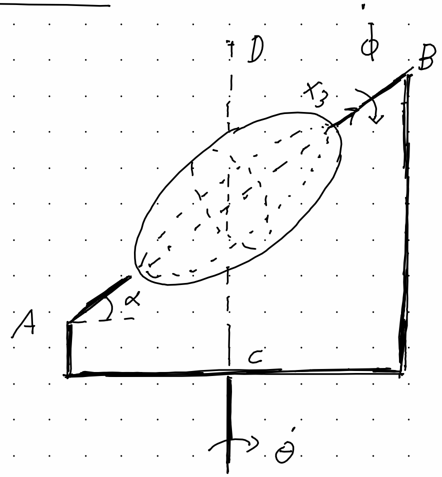
$$\vec{\dot{\theta}} = \dot{\theta}\left[\cos\phi\,\hat{x}_1 + \sin\phi\,\hat{x}_2\right]$$

so $\vec{\Omega} = \dot{\theta}\cos\phi\,\hat{x}_1 + \dot{\theta}\sin\phi\,\hat{x}_2 + \dot{\phi}\,\hat{x}_3$

$$T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$= \frac{1}{2}\left[\left(I_1\cos^2\phi + I_2\sin^2\phi\right)\dot{\theta}^2 + I_3\dot{\phi}^2\right]$$

Sec 32, Prob 10.



Uniform ellipsoid with $I_1 = I_2$ (circular cross section)

$$\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta}$$

$$\vec{\dot\phi} = \dot\phi\,\hat{x}_3$$

$$\vec{\dot\theta} = \dot\theta\left[\cos\left(\tfrac{\pi}{2}-\alpha\right)\hat{x}_3 + \sin\left(\tfrac{\pi}{2}-\alpha\right)\left(\cos\phi\,\hat{x}_1 + \sin\phi\,\hat{x}_2\right)\right]$$

Now! $\cos\left(\frac{\pi}{2}-\alpha\right) = \cancel{\cos\left(\frac{\pi}{2}\right)}^{\,0}\cos\alpha + \sin\left(\frac{\pi}{2}\right)\sin\alpha$

$= \sin\alpha$

$\sin\left(\frac{\pi}{2}-\alpha\right) = \sin\frac{\pi}{2}\cos\alpha - \cancel{\cos\left(\frac{\pi}{2}\right)}^{\,0}\sin\alpha$

$= \cos\alpha$

So $\vec{\dot\theta} = \dot\theta\left[\sin\alpha\,\hat{x}_3 + \cos\alpha\cos\phi\,\hat{x}_1 + \cos\alpha\sin\phi\,\hat{x}_2\right]$

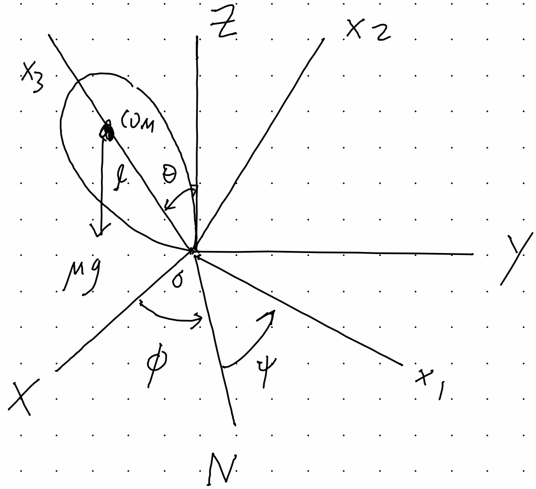
$$\rightarrow \vec{\Omega} = \dot\theta\cos\alpha\left(\cos\phi\,\hat{x}_1 + \sin\phi\,\hat{x}_2\right) + \left(\dot\phi + \dot\theta\sin\alpha\right)\hat{x}_3$$

$$T = \frac{1}{2}\left(I_1\Omega_1^2 + I_1\Omega_2^2 + I_3\Omega_3^2\right)$$

$$= \frac{1}{2}\left[I_1\,\dot\theta^2\cos^2\alpha\left(\cos^2\phi + \sin^2\phi\right) + I_3\left(\dot\phi + \dot\theta\sin\alpha\right)^2\right]$$

$$= \frac{1}{2}\left[I_1\cos^2\alpha\,\dot\theta^2 + I_3\left(\dot\phi + \dot\theta\sin\alpha\right)^2\right]$$

Sec. 35, prob 1:



symmetrical top:

$I_1 = I_2$, I_3 (wrt principal axes passing through COM)

$I_1' = I_1 + Ml^2$
$I_2' = I_1'$
$I_3' = I_3$

(wrt axes passing through O, which is displaced from the COM by l in the $-x_3$ direction)

$L = T - U$

$U = mgZ = mgl\cos\theta$

$T = \frac{1}{2}\left(I_1'\Omega_1^2 + I_2'\Omega_2^2 + I_3'\Omega_3^2\right)$

$= \frac{1}{2}\left[I_1'(\Omega_1^2+\Omega_2^2) + I_3\Omega_3^2\right]$

Now: (From (35.1))

$$\Omega_1 = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$
$$\Omega_2 = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$$
$$\Omega_3 = \dot\phi \cos\theta + \dot\psi$$

so $\Omega_3^2 = \left(\dot\phi \cos\theta + \dot\psi\right)^2$

$$\Omega_1^2 = \dot\phi^2 \sin^2\theta \sin^2\psi + \dot\theta^2 \cos^2\psi + 2\dot\theta\dot\psi \sin\theta \sin\psi \cos\psi$$
$$\Omega_2^2 = \dot\phi^2 \sin^2\theta \cos^2\psi + \dot\theta^2 \sin^2\psi - 2\dot\theta\dot\psi \sin\theta \cos\psi \sin\psi$$

$$\longrightarrow \quad \Omega_1^2 + \Omega_2^2 = \dot\phi^2 \sin^2\theta + \dot\theta^2$$

Thus,

$$L = \frac{1}{2}\left[I_1'\left(\dot\phi^2 \sin^2\theta + \dot\theta^2\right) + I_3\left(\dot\phi \cos\theta + \dot\psi\right)^2\right] - \mu g l \cos\theta$$

---

1) No explicit t-dependence

$$E = T + U = \text{const}$$

2) No explicit ϕ dependence:

$$p_\phi \equiv \frac{\partial L}{\partial \dot\phi} = I_1' \sin^2\theta\, \dot\phi + I_3\left(\dot\phi \cos\theta + \dot\psi\right)\cos\theta$$
$$= \text{const}$$

(see next page) $= M_z$

3) No explicit ψ dependence:

$$p_\psi \equiv \frac{\partial L}{\partial \dot\psi} = I_3\left(\dot\phi \cos\theta + \dot\psi\right) = \text{const} \equiv M_3$$

To show that $p_\phi = M_Z$:

$$M_Z = \vec{M} \cdot \hat{z}$$
$$= (M_1 \hat{x}_1 + M_2 \hat{x}_2 + M_3 \hat{x}_3) \cdot \hat{z}$$
$$= M_1 \hat{x}_1 \cdot \hat{z} + M_2 \hat{x}_2 \cdot \hat{z} + M_3 \hat{x}_3 \cdot \hat{z}$$

Now: $\hat{x}_3 \cdot \hat{z} = \cos\theta$

$\hat{x}_1 \cdot \hat{z} = \sin\theta \sin\psi$

$\hat{x}_2 \cdot \hat{z} = \sin\theta \cos\psi$

Thus,

$$M_z = M_1 \sin\theta \sin\psi + M_2 \sin\theta \cos\psi + M_3 \cos\theta$$
$$= I_1' \Omega_1 \sin\theta \sin\psi + I_1' \Omega_2 \sin\theta \cos\psi + I_3 \Omega_3 \cos\theta$$
$$= I_1' (\sin^2\theta \sin^2\psi \dot{\phi} + \cancel{\sin\theta \sin\psi} \cos\psi \dot{\theta}) + I_1' (\sin^2\theta \cos^2\psi \dot{\phi} - \cancel{\sin\theta \cos\psi} \sin\psi \dot{\theta}) + I_3 (\dot{\phi} \cos\theta + \dot{\psi}) \cos\theta$$
$$= I_1' \sin^2\theta \dot{\phi} + I_3 \cos\theta (\dot{\phi} \cos\theta + \dot{\psi})$$

We can solve M_z, M_3 for $\dot\phi, \dot\psi$:

$$M_z = I_1' \sin^2\theta\, \dot\phi + I_3(\dot\phi \cos\theta + \dot\psi)\cos\theta$$

$$M_3 = I_3(\dot\phi\cos\theta + \dot\psi)$$

$$\rightarrow \quad M_z = I_1' \sin^2\theta\, \dot\phi + M_3 \cos\theta$$

so $I_1' \sin^2\theta\, \dot\phi = M_z - M_3\cos\theta$

$$\boxed{\dot\phi = \frac{M_z - M_3\cos\theta}{I_1' \sin^2\theta}}$$

And:

$$\frac{M_3}{I_3} = \dot\phi\cos\theta + \dot\psi$$

$$\rightarrow \quad \boxed{\dot\psi = \frac{M_3}{I_3} - \dot\phi\cos\theta = \frac{M_3}{I_3} - \left(\frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}\right)\cos\theta}$$

Also:

$$E = T + U$$

$$= \frac{1}{2}\left[I_1'(\sin^2\theta\,\dot\phi^2 + \dot\theta^2) + I_3(\dot\phi\cos\theta + \dot\psi)^2\right]$$

$$+ \mu g l \cos\theta$$

can be rewritten as

$$E = \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2} I_1' \sin^2\theta \, \dot{\phi}^2 + \frac{1}{2} I_3 \left(\frac{M_3}{I_3}\right)^2 + \mu g l \cos\phi$$

$$= \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2} I_1' \sin^2\theta \frac{(M_z - M_3\cos\theta)^2}{I_1'^2 \sin^4\theta} + \frac{1}{2}\frac{M_3^2}{I_3} + \mu g l \cos\phi$$

$$= \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2}\frac{M_3^2}{I_3} + \frac{1}{2}\frac{(M_z - M_3\cos\theta)^2}{I_1' \sin^2\theta} + \mu g l \cos\theta$$

Now: $\frac{1}{2}\frac{M_3^2}{I_3} = \text{const}$

Also: $\mu g l \cos\theta = -\mu g l (1-\cos\theta) + \mu g l$

so

$$\underbrace{E - \frac{1}{2}\frac{M_3^2}{I_3} - \mu g l}_{E'} = \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2}\frac{(M_z - M_3\cos\theta)^2}{I_1' \sin^2\theta} - \mu g l (1-\cos\theta)$$

$$\longrightarrow \quad E' = \frac{1}{2} I_1' \dot{\theta}^2 + U_{eff}(\theta)$$

where $\boxed{U_{eff}(\theta) = \frac{1}{2}\frac{(M_z - M_3\cos\theta)^2}{I_1' \sin^2\theta} - \mu g l (1-\cos\theta)}$

EOM:

$$E' = \frac{1}{2} I_1' \dot{\theta}^2 + U_{eff}(\theta)$$

$$\dot{\theta}^2 = \frac{2}{I_1'} \left(E' - U_{eff}(\theta) \right)$$

$$\frac{d\theta}{dt} = \sqrt{\frac{2}{I_1'} \left(E' - U_{eff}(\theta) \right)}$$

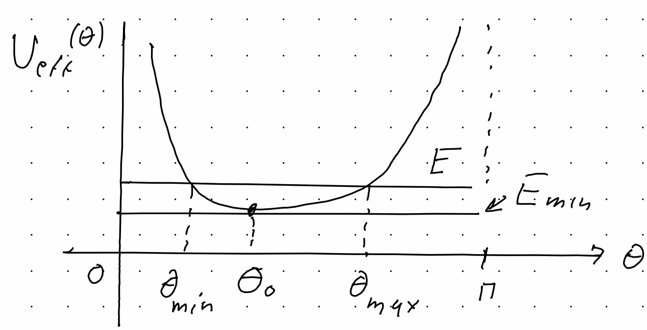
$$\longrightarrow \int dt = \int \frac{d\theta}{\sqrt{\frac{2}{I_1'} \left(E' - U_{eff}(\theta) \right)}}$$

$$\boxed{t = \int \frac{d\theta}{\sqrt{\frac{2}{I_1'} \left(E' - U_{eff}(\theta) \right)}} + \text{const}}$$

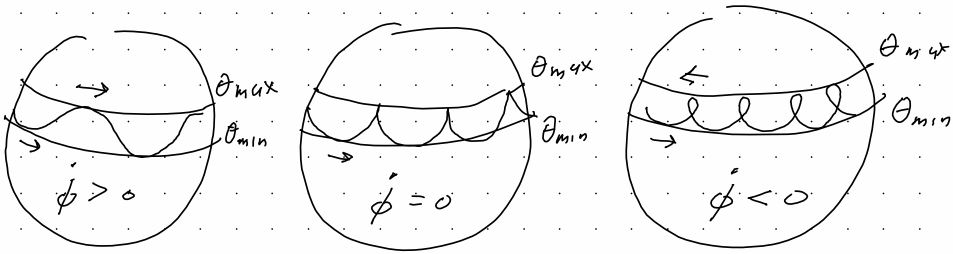
Effective potential:

$$U_{eff}(\theta) = \frac{1}{2} \frac{(M_z - M_3 \cos\theta)^2}{I_1' \sin^2\theta} - \mu g l (1 - \cos\theta)$$

For $M_z \neq M_3$, $U_{eff}(\theta) \rightarrow \infty$ as $\theta \rightarrow 0, \pi$



For $E > E_{min}$, θ varies between θ_{min} and θ_{max}. The motion of the x_3-axis of the top can have the following three forms depending on the sign of $\dot{\phi}$ when $\theta = \theta_{max}$



This motion is called nutation.

Sec 3.5, Prob 2

For rotation of a top around a vertical axis, to be stable, we need $\left.\frac{d^2 U_{eff}}{d\theta^2}\right|_{\theta=0} > 0$

Now:

$$U_{eff}(\theta) = \frac{(M_z - M_3 \cos\theta)^2}{2 I_1' \sin^2\theta} - \mu g l (1-\cos\theta)$$

Also:

$$M_z = I_1' \sin^2\theta\, \dot{\phi} + I_3 (\dot{\phi} \cos\theta + \dot{\psi}) \cos\theta$$

$$M_3 = I_3 (\dot{\phi} \cos\theta + \dot{\psi})$$

In the limit $\theta \to 0$

$$\left.\begin{aligned} M_z &\simeq I_3 (\dot{\phi} + \dot{\psi}) \\ M_3 &\simeq I_3 (\dot{\phi} + \dot{\psi}) \end{aligned}\right\} \text{ so they are equal in this limit}$$

Thus,

$$U_{eff}(\theta) \simeq \frac{M_3^2 (1-\cos\theta)^2}{2 I_1' \sin^2\theta} - \mu g l (1-\cos\theta)$$

$$\simeq \frac{M_3^2 \left(\frac{\theta^2}{2}\right)^2}{2 I_1' \theta^2} - \mu g l \frac{\theta^2}{2}$$

$$= \left(\frac{1}{8} \frac{M_3^2}{I_1'} - \frac{1}{2} \mu g l\right) \theta^2$$

Thus

$$U_{eff}(\theta) \simeq \left(\frac{1}{8} \frac{M_3}{I_1'} - \frac{1}{2} \mu g l \right) \theta^2$$

$$\mapsto \quad U_{eff}(0) = 0$$

$$\left. \frac{dU_{eff}}{d\theta} \right|_{\theta=0} = 2 \left(\frac{1}{8} \frac{M_3}{I_1'} - \frac{1}{2} \mu g l \right) \theta \Big|_{\theta=0} = 0$$

$$\left. \frac{d^2 U_{eff}}{d\theta} \right|_{\theta=0} = 2 \left(\frac{1}{8} \frac{M_3}{I_1'} - \frac{1}{2} \mu g l \right)$$

Need $\left. \frac{d^2 U_{eff}}{d\theta} \right|_{\theta=0} > 0$ for stable rotation!

$$\frac{1}{8} \frac{M_3^2}{I_1'} - \frac{1}{2} \mu g l > 0$$

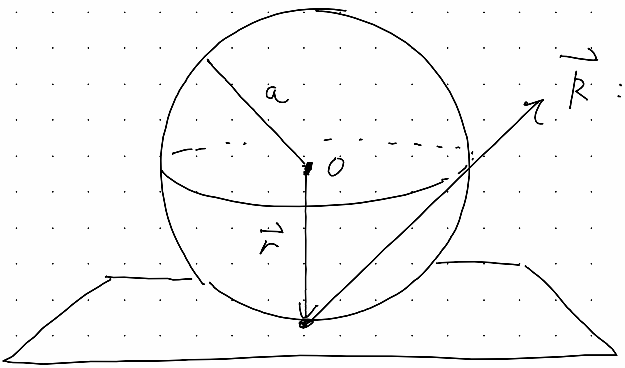
$$\boxed{M_3^2 > 4 \mu g l I_1'}$$

Equivalently, since $M_3 = I_3 \Omega_3$:

$$\boxed{\Omega_3^2 > \frac{4 \mu g l I_1'}{I_3^2}}$$

Sec 3.8, Prob 1:

Homogeneous sphere (radius a) rolling without slipping on a horizontal surface, subject to applied force $\vec{F}$ and torque $\vec{K}$:



$\vec{R}$: reaction force

$I = \frac{2}{5} \mu a^2$

(homog sphere about com)

$$\frac{d\vec{P}}{dt} = \vec{F} + \vec{R} \longrightarrow \mu \frac{d\vec{V}}{dt} = \vec{F} + \vec{R}$$

$$\frac{d\vec{M}}{dt} = \vec{K} + \vec{r} \times \vec{R} \rightarrow I \frac{d\vec{\Omega}}{dt} = \vec{K} - a\hat{z} \times \vec{R}$$

Rolling without slipping:

$$0 = \vec{V} + \vec{\Omega} \times \vec{r}$$
$$= \vec{V} - a\vec{\Omega} \times \hat{z}$$

so $\vec{V} = a\vec{\Omega} \times \hat{z}$

Using $(\vec{A} \times \vec{B})_1 = A_2 B_3 - A_3 B_2$, etc

$$\boxed{\begin{aligned} V_x &= a\Omega_y \\ V_y &= -a\Omega_x \\ V_z &= 0 \end{aligned}}$$

no motion off surface

Combine:

$$\mu \frac{d\vec{V}}{dt} = \vec{F} + \vec{R} \qquad (1)$$

$$I \frac{d\vec{\Omega}}{dt} = \vec{K} - a\hat{z} \times \vec{R} \qquad (2)$$

$$V_x = a\Omega_y, \; V_y = -a\Omega_x, \; V_Z = 0 \qquad \text{(constraint)}$$

Take timederivative of constraint equation:

$$\frac{dV_x}{dt} = a\frac{d\Omega_y}{dt}, \quad \frac{dV_y}{dt} = -a\frac{d\Omega_x}{dt}$$

substitute from (1), (2) into these two equations

$$\frac{1}{\mu}(F_x + R_x) = \frac{a}{I}(K_y - aR_x)$$

$$\frac{1}{\mu}(F_y + R_y) = \frac{-a}{I}(K_x + aR_y)$$

Thus,

$$F_x + R_x = \frac{\mu a}{I}K_y - \frac{\mu a^2}{I}R_x$$

$$R_x\left(\frac{I + \mu a^2}{I}\right) = \frac{\mu a}{I}K_y - F_x$$

$$R_x \frac{\frac{7}{5}\cancel{\mu a^2}}{\frac{2}{5}\cancel{\mu a^2}} = \frac{\cancel{\mu a}}{\frac{2}{5}\mu a^2}K_y - F_x$$

$$\rightarrow \boxed{R_x = \frac{5}{7}\frac{K_y}{a} - \frac{2}{7}F_x}$$

Similarly,

$$F_y + R_y = -\frac{\mu a}{I} K_x - \frac{\mu a^2}{I} R_y$$

$$R_y\left(\frac{I + \mu a^2}{I}\right) = -\frac{\mu a}{I} K_x - F_y$$

$$R_y \frac{\frac{7}{5}\cancel{\mu a^2}}{\frac{2}{5}\cancel{\mu a^2}} = -\frac{\cancel{\mu a}}{\frac{2}{5}\cancel{\mu} a^2} K_x - F_y$$

$$\rightarrow \boxed{R_y = -\frac{5}{7}\frac{K_x}{a} - \frac{2}{7} F_y}$$

Also,

$$\mu \frac{d\cancel{V_z}^{\,0}}{dt} = F_z + R_z$$

$$\rightarrow \boxed{R_z = -F_z}$$

Using these expression for R_x, R_y we can write down EOMs for V_x, V_y

$$\mu \frac{dV_x}{dt} = F_x + R_x$$

$$= F_x + \frac{5}{7}\frac{K_y}{a} - \frac{2}{7} F_x$$

$$= \frac{5}{7}\left(F_x + \frac{K_y}{a}\right)$$

Similarly

$$\mu \frac{dV_y}{dt} = F_y + R_y$$

$$= F_y - \frac{5}{7}\frac{K_x}{a} - \frac{2}{7}F_y$$

$$= \frac{5}{7}\left(F_y - \frac{K_x}{a}\right)$$

Summary:

$$R_x = \frac{5}{7}\frac{K_y}{a} - \frac{2}{7}F_x$$

$$R_y = -\frac{5}{7}\frac{K_x}{a} - \frac{2}{7}F_y$$

$$R_z = -F_z$$

$$\mu \frac{dV_x}{dt} = \frac{5}{7}\left(F_x + \frac{K_y}{a}\right)$$

$$\mu \frac{dV_y}{dt} = \frac{5}{7}\left(F_y - \frac{K_x}{a}\right)$$

$$V_z = 0$$

$$\Omega_x = -\frac{V_y}{a}$$

$$\Omega_y = \frac{V_x}{a}$$

$$I \frac{d\Omega_z}{dt} = K_z$$

Example:

Suppose: $\vec{F} = -\mu g \hat{z} + F_0 \hat{x}$

$\vec{K} = 0$

Then: $F_x = F_0$, $F_y = 0$, $F_z = -\mu g$

Solution from previous page gives:

$$R_x = -\frac{2}{7} F_0$$

$$R_y = 0$$

$$R_z = \mu g \quad \text{(normal force upward)}$$

$$\mu \frac{dV_x}{dt} = \frac{5}{7} F_0 \rightarrow V_x = \frac{5}{7} \frac{F_0}{\mu} t + V_{x0}$$

$$\mu \frac{dV_y}{dt} = 0 \rightarrow V_y = \text{const} = V_{y0}$$

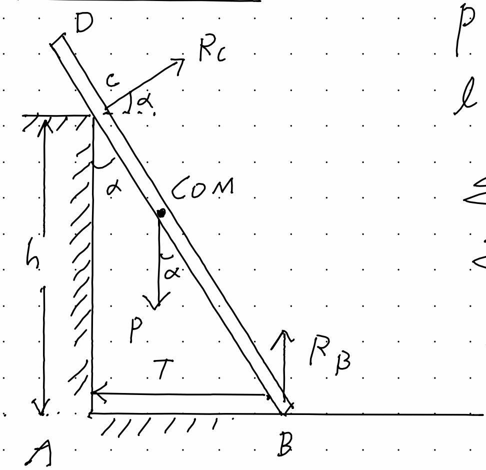
$$V_z = 0$$

$$\Omega_x = -\frac{V_{y0}}{a}$$

$$\Omega_y = \frac{V_x}{a} = \frac{5}{7} \frac{F_0}{\mu a} t + \frac{V_{x0}}{a}$$

$$I \frac{d\Omega_z}{dt} = 0 \rightarrow \Omega_z = \text{const} = \Omega_{z0}$$

Sec 38, Prob 2



P = weight = mg

l : length of uniform rod

$$\sum \vec{F} = 0$$

$$\sum \vec{r} \times \vec{F} = 0$$

including reaction force T, R_C, R_B

horizontal direction

$$-T + R_C \cos\alpha = 0 \qquad (1)$$

vertical direction

$$R_C \sin\alpha - P + R_B = 0 \qquad (2)$$

torques around B :

$$\frac{l}{2} \sin\alpha \, P - \frac{h}{\cos\alpha} R_C = 0 \qquad (3)$$

$\cos\alpha = \frac{h}{d}$

$\rightarrow d = \frac{h}{\cos\alpha}$

Thus,

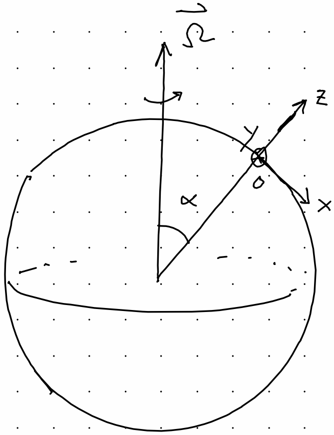
$$\frac{h}{\cos\alpha} R_C = \frac{l}{2} \sin\alpha \, P$$

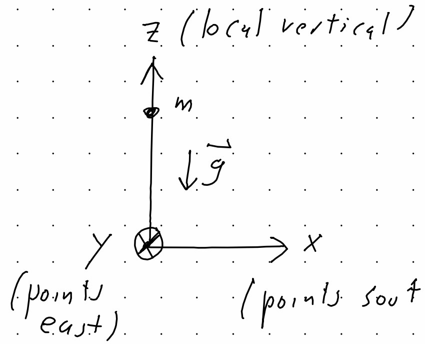
$$\boxed{R_C = \frac{lP}{2h} \sin\alpha \cos\alpha}$$

$$\boxed{T = R_C \cos\alpha}$$

$$\boxed{R_B = P - R_C \sin\alpha}$$

Sec 39, prob 1:





(α: location on Earth relative to North pole)

Newton's 2nd law in rotating frame

$$m\vec{a} = m\vec{g} - m\vec{W} - 2m\vec{\Omega}\times\vec{v} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

where $\vec{W} = \vec{\Omega}\times(\vec{\Omega}\times\vec{R})$

= acceleration of origin of rotating frame

Thus,

$$m\vec{a} = m\vec{g} \underbrace{- m\vec{\Omega}\times(\vec{\Omega}\times(\vec{r}+\vec{R}))}_{\substack{2^{nd}\text{ order in }\vec{\Omega} \\ [\text{ignore}]}} - 2m\vec{\Omega}\times\vec{v}$$

$$\approx m\vec{g} - 2m\vec{\Omega}\times\vec{v}$$

Now: $\vec{g} = -g\hat{z}$

$$\vec{\Omega} = \Omega\cos\alpha\,\hat{z} - \Omega\sin\alpha\,\hat{x}$$

$$(\vec{\Omega}\times\vec{v})_x = \cancel{\Omega_y}^{\to 0} v_z - \Omega_z \dot{y}$$
$$= -\Omega_z \dot{y}$$
$$(\vec{\Omega}\times\vec{v})_y = \Omega_z v_x - \Omega_x v_z$$
$$= \Omega_z \dot{x} - \Omega_x \dot{z}$$
$$(\vec{\Omega}\times\vec{v})_z = \Omega_x v_y - \cancel{\Omega_y}^{\to 0} v_x$$
$$= \Omega_x \dot{y}$$

Thus,

$$\ddot{x} = +2\Omega_z \dot{y}$$
$$\ddot{y} = -2(\Omega_z \dot{x} - \Omega_x \dot{z})$$
$$\ddot{z} = -g + \Omega_x \dot{y}$$

Want to solve these equations to 1^{st} order in Ω

The 0^{th} order solution is

$$\left.\begin{aligned} x_0(t) &= 0 \\ y_0(t) &= 0 \\ z_0(t) &= h - \tfrac{1}{2}gt^2 \end{aligned}\right\} \text{drop from vertical height } h$$

Write $x(t) = x_0(t) + x_1(t)$, etc.

Then

$$\ddot{x}_1 = 2\Omega_z \dot{y}_1 \approx 0$$
$$\ddot{y}_1 = -2(\Omega_z \dot{x}_1 - \Omega_x(-gt + \dot{z}_1)) \approx -2\Omega_x g t$$
$$\cancel{-g} + \ddot{z}_1 = \cancel{-g} + \Omega_x \dot{y}_1 \approx 0$$

Thus, only need to solve

$$\ddot{y}_1 = -2\Omega_x g t$$

$$\rightarrow \dot{y}_1 = -\Omega_x g t^2$$

$$y_1 = -\frac{1}{3}\Omega_x g t^3$$

Now: $\Omega_x = -\Omega \sin\alpha$

$T = \sqrt{\frac{2h}{g}}$ (0^{th} order solution to $0 = h - \frac{1}{2} g T^2$

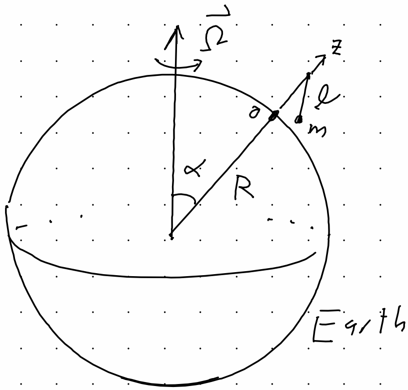
so object hits the ground at

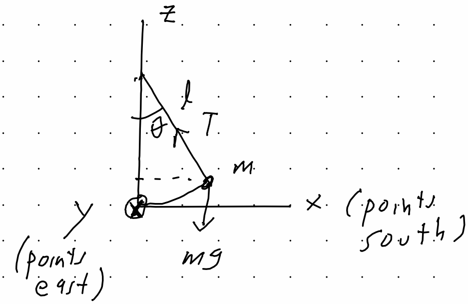
$$y = y_1 = +\frac{1}{3}\Omega \sin\alpha \left(\frac{2h}{g}\right)^{3/2}$$

indicates that the object hits the ground to the East of the dropping location.

Sec 39, Prob 3.

Foucault's pendulum, small oscillations





Newton's 2nd law in rotating frame:

$$m\vec{a} = \vec{T} + m\vec{g} - m\vec{W} - 2m\vec{\Omega}\times\vec{v} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

Now $\vec{W}$ = acceleration of origin O

$$= \vec{\Omega}\times(\vec{\Omega}\times\vec{R})$$

Thus,

$$m\vec{a} = \vec{T} + m\vec{g} \underbrace{- m\vec{\Omega}\times(\vec{\Omega}\times(\vec{r}+\vec{R}))}_{\substack{2^{nd}\text{ order in }\Omega \\ [\text{ignore}]}} - 2m\vec{\Omega}\times\vec{V}$$

$$\approx \vec{T} + m\vec{g} - 2m\vec{\Omega}\times\vec{V}$$

Now: $T = T\cos\theta\,\hat{z} - T\sin\theta\cos\phi\,\hat{x} - T\sin\theta\sin\phi\,\hat{y}$

$$\approx T\hat{z} - T\left(\frac{x}{l}\right)\hat{x} - T\left(\frac{y}{l}\right)\hat{y}$$

assuming $\theta \ll 1$ (small oscillation)

$\vec{g} = -g\hat{z}$

$$(\vec{\Omega}\times\vec{v})_x = \Omega_y v_z - \Omega_z v_y$$
$$= \cancel{\Omega_y}^{\to 0}\dot{z} - \Omega_z\dot{y}$$
$$\approx -\Omega_z\dot{y}$$

$$(\vec{\Omega}\times\vec{v})_y = \Omega_z v_x - \Omega_x v_z$$
$$= \Omega_z\dot{x} - \Omega_x\dot{z}$$
$$\approx \Omega_z\dot{x}$$

$$(\vec{\Omega}\times\vec{v})_z = \Omega_x v_y - \cancel{\Omega_y}^{\to 0} v_x$$
$$= \Omega_x\dot{y}$$

Thus,

$$m\vec{a} = \vec{T} + m\vec{g} - 2m\vec{\Omega}\times\vec{v}$$

becomes

$$m\ddot{x} = -T\left(\frac{x}{l}\right) + 2m\Omega_z\dot{y}$$
$$m\ddot{y} = -T\left(\frac{y}{l}\right) - 2m\Omega_z\dot{x}$$
$$0 = m\ddot{z} = T - mg - 2m\Omega_x\dot{y}$$

Now: $\Omega_x\dot{y} \sim \Omega\dfrac{D}{P}$

where D = max displacement in xy-plane
P = period $= 2\pi\sqrt{\dfrac{l}{g}} = \dfrac{2\pi}{\omega}$

Now $\frac{\Omega D}{P} \simeq \frac{\Omega D}{2\pi}\,\omega$

$$\ll \frac{l\,\omega^2}{2\pi}$$

since $\Omega \ll \omega$
$D \ll l$ (small oscillations)

$$= \frac{\cancel{l}}{2\pi}\,\frac{g}{\cancel{l}}$$

$$= \frac{g}{2\pi}$$

Thus, $m\Omega_x \dot{y} \ll mg$

so we can ignore these terms relative to mg

$$\rightarrow \quad 0 \approx T - mg \quad \rightarrow \quad \boxed{T \approx mg}$$

$$\cancel{m}\ddot{x} \approx -\cancel{m}g\,\frac{x}{l} + 2\cancel{m}\Omega_z \dot{y}$$

$$\boxed{\ddot{x} \approx -\omega^2 x + 2\Omega_z \dot{y}}$$

$$\cancel{m}\ddot{y} \approx -\cancel{m}g\,\frac{y}{l} - 2\cancel{m}\Omega_z \dot{x}$$

$$\boxed{\ddot{y} \approx -\omega^2 x - 2\Omega_z \dot{x}}$$

just need to solve these two equation

Standard "trick":

Define $\zeta = x + iy$ (complex valued)

then $\dot{\zeta} = \dot{x} + i\dot{y}$

$$\ddot{\zeta} = \ddot{x} + i\ddot{y}$$

so $\ddot{\zeta} = -\omega^2 \zeta + 2\Omega_z(\dot{y} - i\dot{x})$

$$= -\omega^2 \zeta - 2i\Omega_z(\dot{x} + i\dot{y})$$

$$= -\omega^2 \zeta - 2i\Omega_z \dot{\zeta}$$

thus,

$$\ddot{\zeta} + 2i\Omega_z \dot{\zeta} + \omega^2 \zeta = 0$$

Trial solution:

$$\zeta(t) = e^{i\lambda t}$$

$$\rightarrow \quad -\lambda^2 - 2\lambda\Omega_z + \omega^2 = 0$$

$$\lambda^2 + 2\lambda\Omega_z - \omega^2 = 0$$

solve quadratic:

$$\lambda_\pm = \frac{-2\Omega_z \pm \sqrt{4\Omega_z^2 + 4\omega^2}}{2}$$

$$\approx -\Omega_z \pm \omega \quad (\text{since } \Omega \ll \omega)$$

General solution:

$$\zeta(t) = A e^{-i(\Omega_z - \omega)t} + B e^{-i(\Omega_z + \omega)t}$$

$$= e^{-i\Omega_z t}\left(A e^{i\omega t} + B e^{-i\omega t}\right)$$

$$= e^{-i\Omega_z t}\, \zeta_0(t)$$

where $\zeta_0(t)$ = general solution of small oscillation problem in an inertial frame.

The factor $e^{-i\Omega_z t}$ causes the plane of oscillation of the pendulum to precess with angular frequency

$$\Omega_z = \Omega \cos\alpha$$

angle of location on Earth

$$\begin{pmatrix} \alpha = 0 \leftrightarrow \text{N Pole} \\ \alpha = \pi/2 \leftrightarrow \text{Equator} \end{pmatrix}$$

Precession period:

$$P_{\text{precession}} = \frac{2\pi}{\Omega_z}$$

$$= \frac{2\pi}{\Omega\cos\alpha}$$

$$\Omega = 1\,\text{revolution}/24\,\text{hrs}$$

$$= \frac{2\pi\,\text{rad}}{24\,\text{hrs}}$$

so $P_{\text{precession}} = \dfrac{24\,\text{hrs}}{\cos\alpha}$

$$= \begin{cases} 24\,\text{hrs} & \text{at N Pole} \\ \infty & \text{at equator (so no precession)} \end{cases}$$