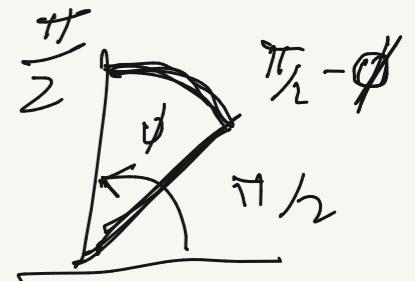


Elliptic Functions / integrals:

- i) period of a simple pendulum beyond the small-angle approximation
- ii) circumference of an ellipse

A generalization of definition of circular functions (sines, cosines) to ellipses.

Standard notation:



$$\int_0^x \frac{dt}{\sqrt{1-k^2 t^2} \sqrt{1-t^2}} = F(\phi, k) = \sin^{-1} x$$

$$\int_0^x \frac{\sqrt{1-k^2 t^2}}{\sqrt{1-t^2}} dt = E(\phi, k)$$

where $x = \sin \phi$ and $0 \leq k \leq 1$

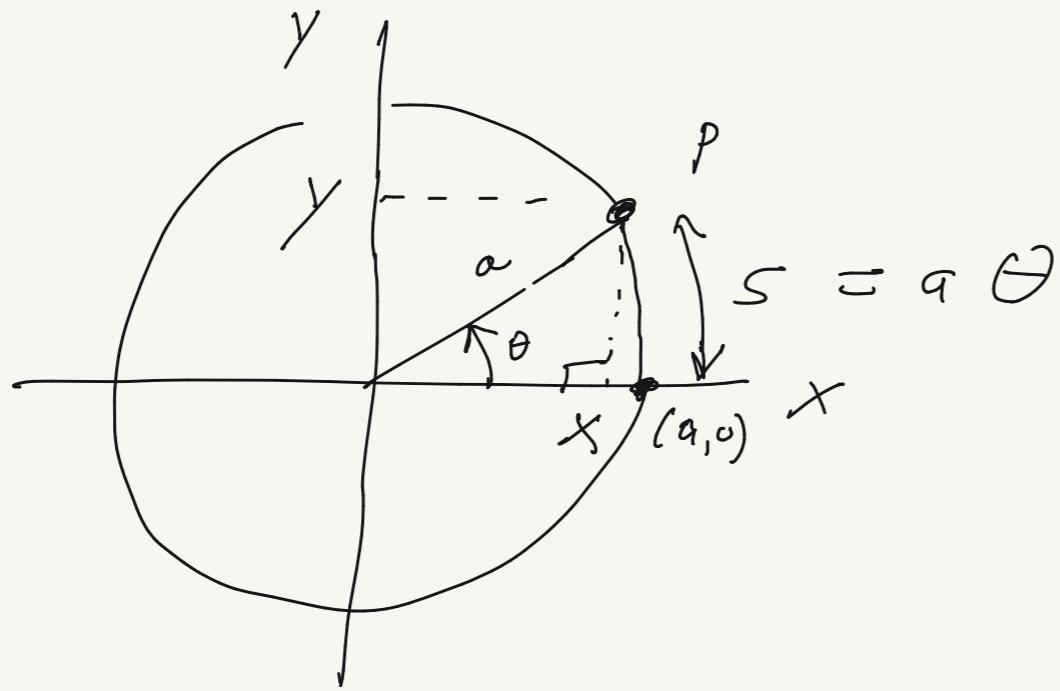
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If we change variables  $t \rightarrow \sin \theta$  in the integrals then

$$F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}$$

$$E(\phi, k) = \int_0^\phi \sqrt{1-k^2 \sin^2 \theta} d\theta$$

## Circular Functions:



$$\sin \theta = \frac{y}{a} \quad \cos \theta = \frac{x}{a}$$

where  $\theta = \frac{\text{arc length from } (a, 0) \text{ to } (x, y)}{a}$

$$= \frac{1}{a} \int_{(a, 0)}^{(x, y)} \sqrt{dx^2 + dy^2} \quad (= \int d\theta)$$

$$x^2 + y^2 = a^2 \rightarrow a^2 \cos^2 \theta + a^2 \sin^2 \theta = a^2$$

$$\rightarrow \boxed{\cos^2 \theta + \sin^2 \theta = 1}$$

## Derivatives:

$$\begin{aligned} \frac{d}{d\theta} \sin \theta &= \frac{d}{d\theta} \left( \frac{y}{a} \right) = \frac{1}{a} \frac{dy}{d\theta} = \frac{dy}{\sqrt{dx^2 + dy^2}} \\ &= \sqrt{\left( \frac{dx}{dy} \right)^2 + 1} \\ &= \frac{1}{\sqrt{\left( \frac{dy}{dx} \right)^2 + 1}} \end{aligned}$$

Now:  $x^2 + y^2 = a^2$   
 $\rightarrow 2x dx + 2y dy = 0$

$$\frac{dx}{dy} = -\frac{y}{x}$$

Then,

$$\frac{d}{d\theta} \sin \theta = \frac{1}{\sqrt{1 + \left(\frac{y}{x}\right)^2}} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} = \cos \theta$$

so  $\frac{d \sin \theta}{d\theta} = \cos \theta$

Similarly,  $\frac{d \cos \theta}{d\theta} = -\sin \theta$

Integrate :

$$\int \frac{d(\sin \theta)}{\cos \theta} = \int d\theta = \theta + \text{const}$$

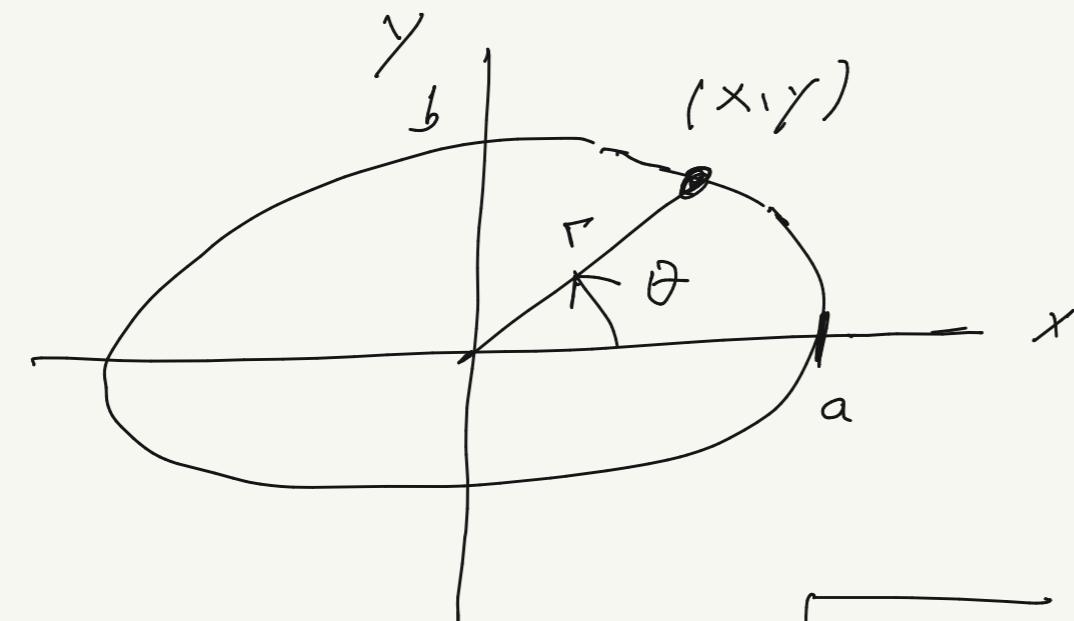
$$t = \sin \theta$$

$$\cos \theta = \sqrt{1-t^2}$$

$$dt = d(\sin \theta)$$

$$\rightarrow \boxed{\int \frac{dt}{1-t^2} = \theta + \text{const} = \sin^{-1} t + \text{const}}$$

## Different parameterization of an ellipse



$$x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Let  $a \geq b$

Eccentricity:

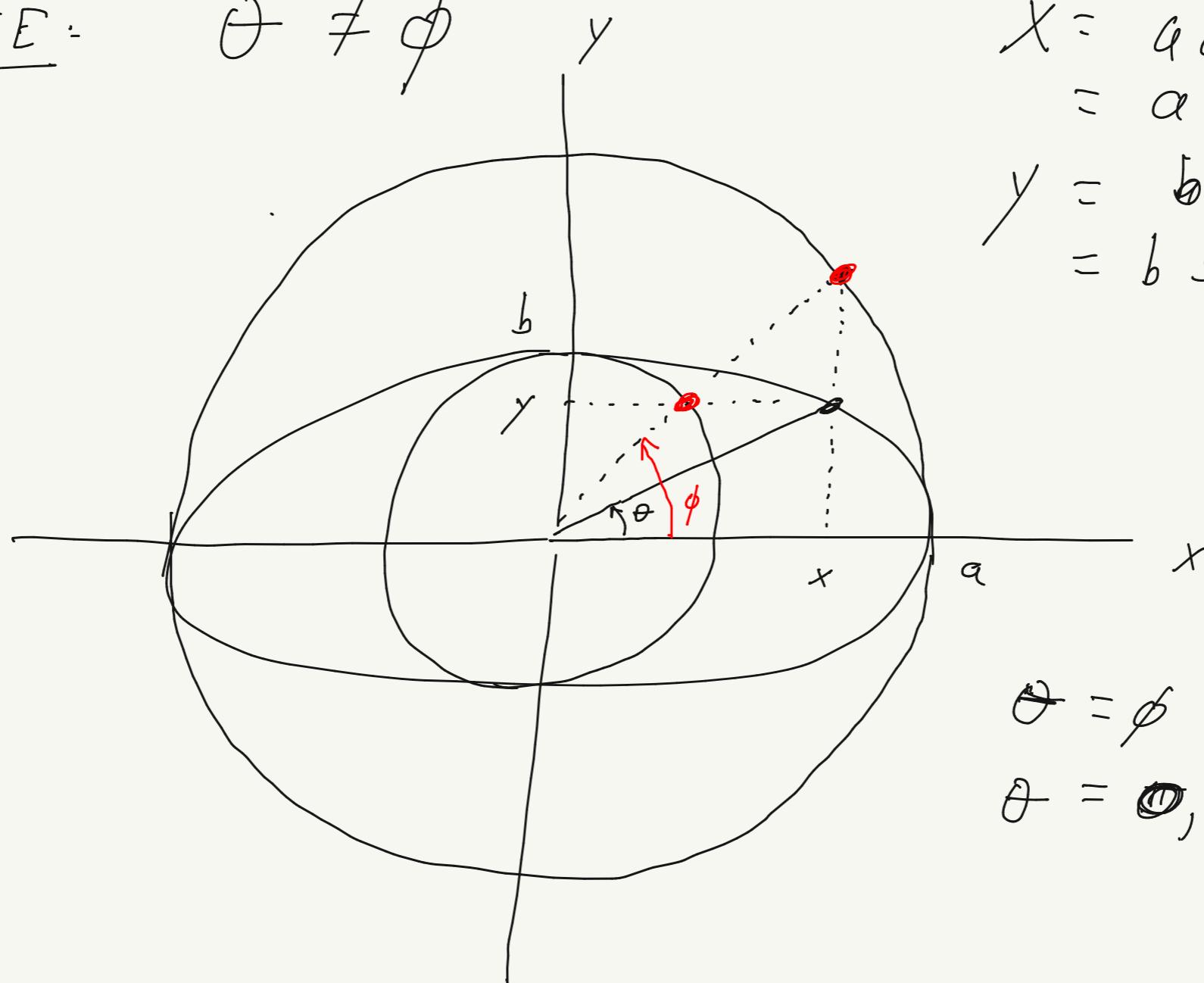
$$e^2 = 1 - \left(\frac{b}{a}\right)^2$$

( $e=0$  for a circle)

Another parameterization:

$$\begin{cases} x = a \cos \phi \\ y = b \sin \phi \end{cases} \rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Note:  $\theta \neq \phi$



$$\begin{aligned} x &= a \cos \phi \\ &= a \cos(u; \tau) \end{aligned}$$

$$\begin{aligned} y &= b \sin \phi \\ &= b \sin(u; \tau) \end{aligned}$$

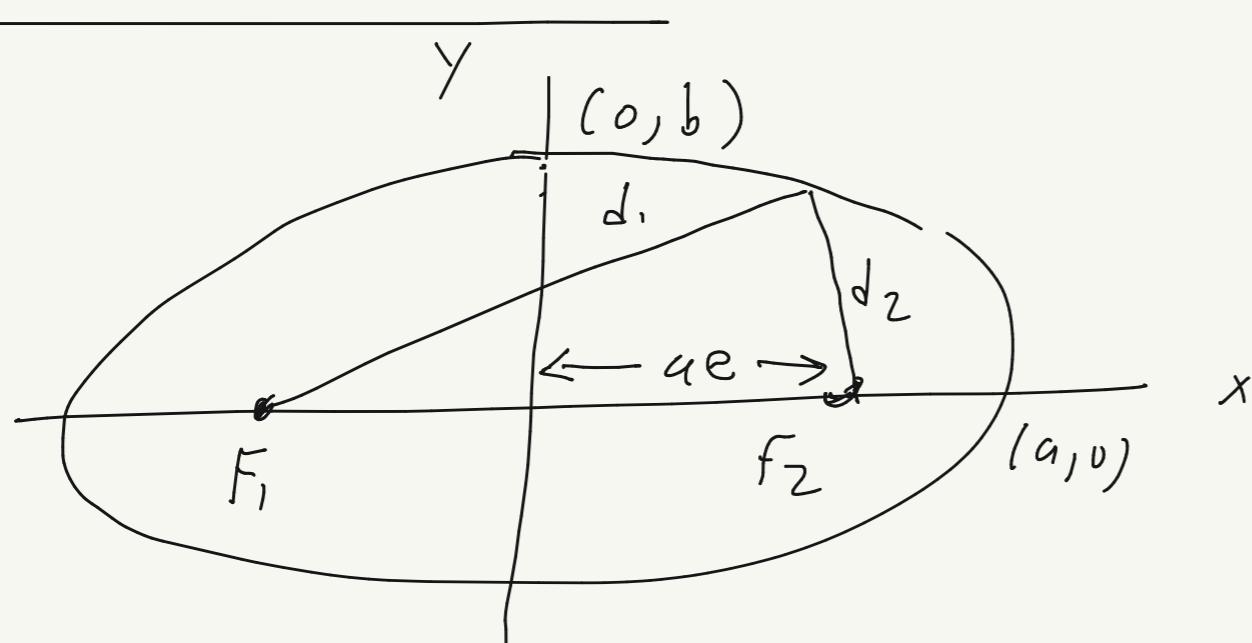
$\theta = \phi$  for

$$\theta = 0, \frac{\pi}{2}, \dots$$

$$\tan \theta = \frac{y}{x} = \left(\frac{b}{a}\right) \tan \phi \rightarrow \theta = \arctan \left[ \frac{b}{a} \tan \phi \right]$$

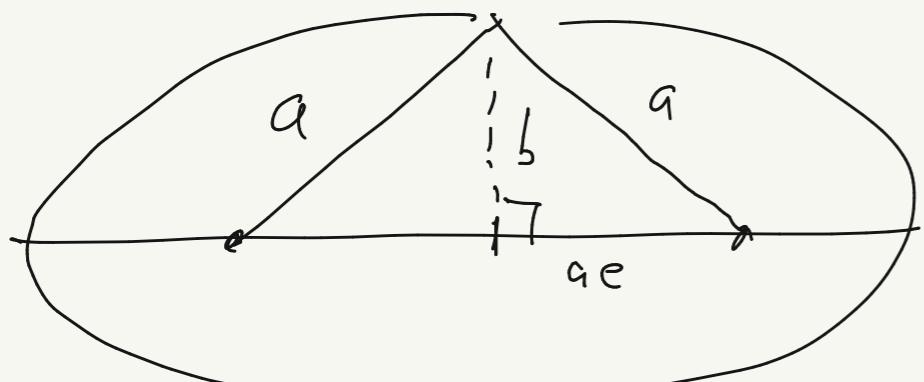
$$\phi = \arctan \left[ \frac{a}{b} \tan \theta \right]$$

Elliptic Functions :



$$d_1 + d_2 = 2a$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

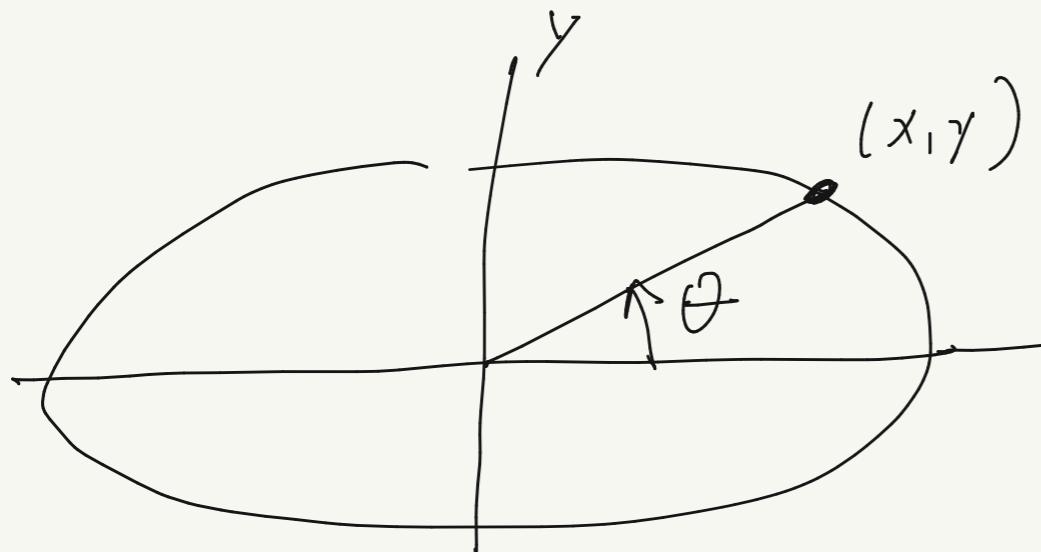


$$a^2 = b^2 + c^2$$

$$a^2(1-e^2) = b^2$$

$$b = a \sqrt{1-e^2}$$

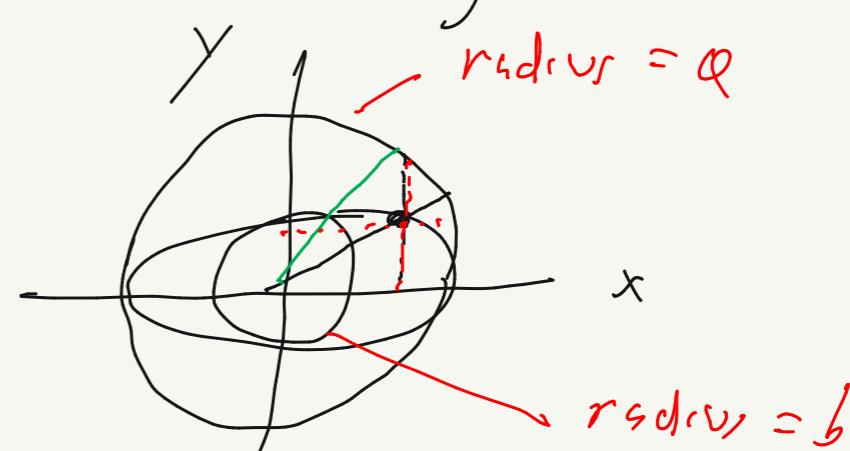
$$At (x, y), \quad \left(\frac{b}{a}\right)^2 = 1-e^2 \quad \rightarrow \quad e = \sqrt{1-\left(\frac{b}{a}\right)^2}$$



$$x = r \cos \theta$$

$$y = r \sin \theta$$

(but  $r$  changes)



$$x = a \cos \phi \quad \phi \neq \theta$$

$$y = b \sin \phi$$

Define:  $\operatorname{cn}(u; k) \equiv x/a$  where  $k = e$   
 $\operatorname{sn}(u; k) \equiv y/b$   $0 \leq k \leq 1$

$$\operatorname{dn}(u; k) \equiv r/a$$

and

where  $u = \int_b^{\theta} \sqrt{r^2 + r'^2} d\theta$

$u = \theta$  for circle

$u \neq \text{arc length}$

since  $ds = \sqrt{dr^2 + r'^2 d\theta}$

NOTE:

$$b_u = \int_0^\theta r d\theta \leq \int_0^\theta ds$$

$\leq$  arc length from  $(a, 0)$  to  $(x, y)$

Properties of  $sn, cn, dn$ :

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \rightarrow [cn^2 u + sn^2 u = 1]$$

$$x^2 + y^2 = r^2 \rightarrow a^2 cn^2 u + b^2 sn^2 u = a^2 dn^2 u$$

$$a^2(1 - sn^2 u) + b^2 sn^2 u = a^2 dn^2 u$$

$$1 - sn^2 u + \frac{b^2}{a^2} sn^2 u = dn^2 u$$

$$1 - \left(1 - \frac{b^2}{a^2}\right) sn^2 u = dn^2 u$$

$$1 - H^2 sn^2 u = dn^2 u$$

Thus,

$$[dn^2 u + H^2 sn^2 u = 1]$$

for circle  $H=0$ ,  $dn u = 1$

Derivatives:

$$\frac{d}{du} sn u = \frac{d}{du} \left( \frac{y}{b} \right) = \frac{\frac{dy}{du}}{b}$$

Now:  $b du = r d\theta$

$$\rightarrow \frac{d}{du} sn u = \frac{dy}{r d\theta}$$

$$\begin{aligned}x &= r \cos \theta & dx &= dr \cos \theta - r \sin \theta d\theta \\y &= r \sin \theta & dy &= dr \sin \theta + r \cos \theta d\theta\end{aligned}$$

$$\begin{aligned}\rightarrow -\sin \theta dx &= -\sin \theta \cos \theta dr + r \sin^2 \theta d\theta \\+\cos \theta dy &= \cos \theta \sin \theta dr + r \cos^2 \theta d\theta\end{aligned}$$


---

add:  $\cos \theta dy - \sin \theta dx = r d\theta$

$$\rightarrow \frac{x}{r} dy - \frac{y}{r} dx = r d\theta$$

thus,

$$\begin{aligned}\frac{d \operatorname{cnu}}{dy} &= \frac{dy}{r d\theta} = \frac{\cancel{dy}}{\cancel{x} dy - \cancel{y} dx} \\&\stackrel{!}{=} \frac{r}{x - y \frac{dx}{dy}}\end{aligned}$$

Allg:  $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \rightarrow \frac{2x dx}{a^2} + \frac{2y dy}{b^2} = 0$

$$\frac{dx}{dy} = -\frac{y}{x} \left(\frac{a}{b}\right)^2$$

$$\rightarrow \frac{d \operatorname{cnu}}{dy} = \frac{r}{x + \frac{y^2}{x} \left(\frac{a}{b}\right)^2} = \frac{r x}{x^2 + y^2 \left(\frac{a^2}{b^2}\right)}$$

$$= \frac{r}{a} \frac{x}{a} \underbrace{\frac{1}{\left(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2\right)}}_1 = \boxed{\operatorname{cnu} \cdot \operatorname{dn} u}$$

Then using

$$\sin^2 u + \cos^2 u = 1$$

we have  $\cancel{\frac{d}{du} \sin u} \frac{d}{du} \sin u + \cancel{\frac{d}{du} \cos u} \frac{d}{du} \cos u = 0$

$$\begin{aligned}\rightarrow \frac{d \cos u}{du} &= -\frac{\sin u}{\cos u} \frac{d}{du} \sin u \\ &= -\frac{\sin u}{\cos u} \cos u \cdot du \\ &= \boxed{-\sin u \cdot du}\end{aligned}$$

And using  $\sin^2 u + \pi^2 \sin^2 u = 1$ :

$$\cancel{\frac{d}{du} \sin u} \frac{d}{du} (\sin u) + \cancel{\pi^2 \sin u} \frac{d}{du} (\sin u) = 0$$

$$\begin{aligned}\rightarrow \frac{d}{du} (\sin u) &= -\pi^2 \frac{\sin u}{\cos u} \frac{d}{du} (\cos u) \\ &= -\pi^2 \frac{\sin u}{\cos u} \cos u \cdot du \\ &= \boxed{-\pi^2 \sin u \cos u}\end{aligned}$$

Summary:

$$\frac{d}{du} \sin u = \cos u \cdot du$$

$$\frac{d}{du} \cos u = -\sin u \cdot du$$

$$\frac{d}{du} \sin u = -\pi^2 \sin u \cos u$$

## Integration:

$$\frac{d \sin u}{du} = \csc u \cdot du$$

$$\int \frac{d(\sin u)}{\csc u \cdot du} = \int du = u + \text{const}$$

$$\text{Let } t = \sin u \Rightarrow \csc u = \sqrt{1-t^2}, \quad du = \sqrt{1-H^2 \sin^2 u}$$

$$\rightarrow \int \frac{dt}{\sqrt{1-t^2} \sqrt{1-H^2 t^2}} = u + \text{const}$$

But since  $t = \sin u$ :

$$\boxed{\int \frac{dt}{\sqrt{1-t^2} \sqrt{1-H^2 t^2}} = \sin^{-1}(t; H) + \text{const}}$$

In hand book:

$$\boxed{\int_0^{\sin \phi} \frac{dt}{\sqrt{1-t^2} \sqrt{1-H^2 t^2}} = F(\phi, H)}$$

Incomplete  
R.H.p.t.c  
integral of  
the 1st  
kind

$$\boxed{\int_0^{\phi} \frac{d\theta}{\sqrt{1-H^2 \sin^2 \theta}} = F(\phi, H)}$$

where  $t = \sin \theta$ ,

NOTE:  $\boxed{\sin \phi = \sin u} \star$

Complete elliptic integral of 1<sup>st</sup> kind:

$$K(\tau) = \int_0^1 \frac{dt}{\sqrt{1-t^2} \sqrt{1-\tau^2 t^2}}$$

~~~~~

Connection to simple pendulum:

$$\varphi = \frac{\omega_0}{\tau} K\left(\tau \equiv \sin\left(\frac{\phi_0}{2}\right)\right)$$

$$\omega_0 t = \operatorname{sn}^{-1} \left(x \equiv \frac{\sin\left(\frac{\phi_1}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}, \tau \equiv \sin\left(\frac{\phi_0}{2}\right) \right) + \text{const}$$

↑
 $\tau = E/\tau$

\uparrow
 $\sqrt{\frac{g}{l}}$

for $\phi = \phi_0$
when $t = 0$

$$\phi(t) = 2 \arcsin \left[\tau \operatorname{sn} \left(\omega_0 \left(t + \frac{P}{q} \right); \tau \right) \right]$$

Elliptic integral of 2nd kind (circumference of ellipse w.r.t y-axis)

$$E(\phi, \kappa) = \int_0^{\phi} dt \frac{\sqrt{1 - \kappa^2 t^2}}{\sqrt{1 - t^2}}$$

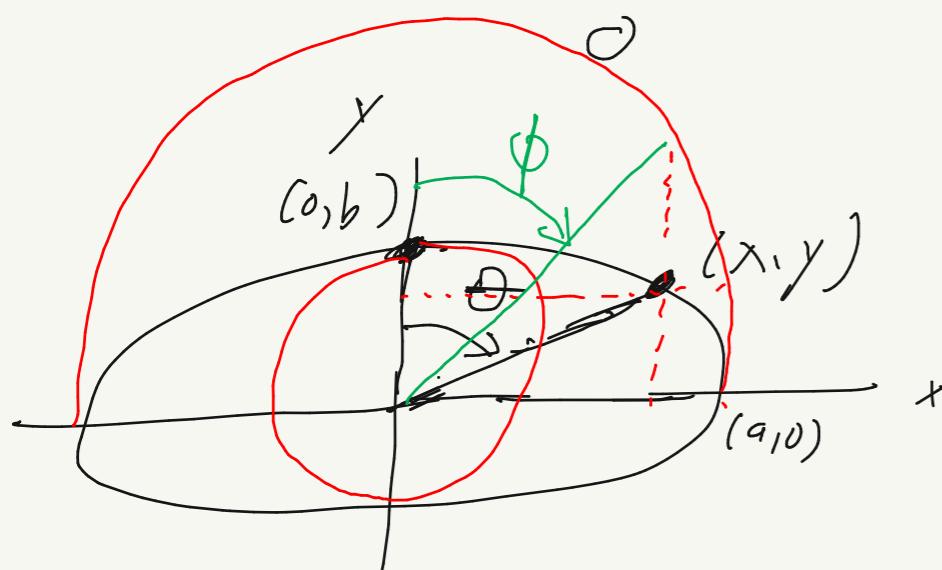
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cw

Rew., to:

$$t = \sin \bar{\phi}$$

$$dt = \cos \bar{\phi} d\bar{\phi} = \sqrt{1 - t^2} d\bar{\phi}$$

$$E(\phi, \kappa) = \int_0^{\phi} d\bar{\phi} \sqrt{1 - \kappa^2 \sin^2 \bar{\phi}} \quad (\text{scipy definition})$$



$$\kappa = e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

$$\begin{aligned} x &= r \sin \theta &= a \sin \psi &= a \sin \phi \\ y &= r \cos \theta &= b \cos \psi &= b \cos \phi \end{aligned} \quad \left. \begin{array}{l} \text{sin, cos} \\ \text{interchanged} \\ \text{from} \\ \text{previous} \\ \text{page} \end{array} \right\}$$

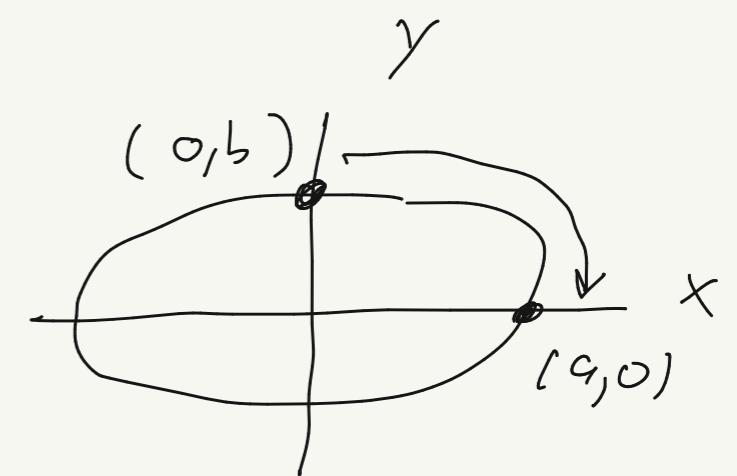
$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} d\phi \\ &= \sqrt{a^2 (1 - \sin^2 \phi) + b^2 \sin^2 \phi} d\phi \\ &= a \sqrt{1 - \left(1 - \left(\frac{b}{a}\right)^2\right) \sin^2 \phi} d\phi \\ &= a \sqrt{1 - \kappa^2 \sin^2 \phi} d\phi \end{aligned}$$

$$\begin{cases} (\phi \neq \theta) \\ \tan \theta = \frac{a}{b} \tan \phi \\ \theta = \arctan \left(\frac{a}{b} \tan \phi \right) \\ \phi = \arctan \left(\frac{b}{a} \tan \theta \right) \end{cases}$$

$$S_{(0,b) \rightarrow (x,y)} = a \int_0^{\phi} \sqrt{1 - \kappa^2 \sin^2 \bar{\phi}} d\bar{\phi} \equiv E(\phi, \kappa)$$

Complete elliptic integral of 2nd kind :

$$\begin{aligned} E(k) &= E\left(\frac{\pi}{2}, k\right) \\ &= \int_0^{\frac{\pi}{2}} d\phi \sqrt{1 - k^2 \sin^2 \phi} \\ &= \int_0^1 dt \frac{\sqrt{1 - k^2 t^2}}{\sqrt{1 - t^2}} \end{aligned}$$



Circumference:

$$C = 4a \int_0^1 dt \frac{\sqrt{1 - k^2 t^2}}{\sqrt{1 - t^2}}$$

1st order correction (for $k \ll 1$; nearly circular)

$$C \approx 4a \int_0^1 \frac{dt}{\sqrt{1 - t^2}} \left(1 - \frac{1}{2} k^2 t^2 \right)$$

$$\approx 4a \left[\underbrace{\int_0^1 \frac{dt}{\sqrt{1 - t^2}}}_{\text{circumference of unit circle}} - \frac{k^2}{2} \int_0^1 \frac{dt}{\sqrt{1 - t^2}} t^2 \right]$$

$t = \sin x$

$$\sin^{-1}(1) = \frac{\pi}{2}$$

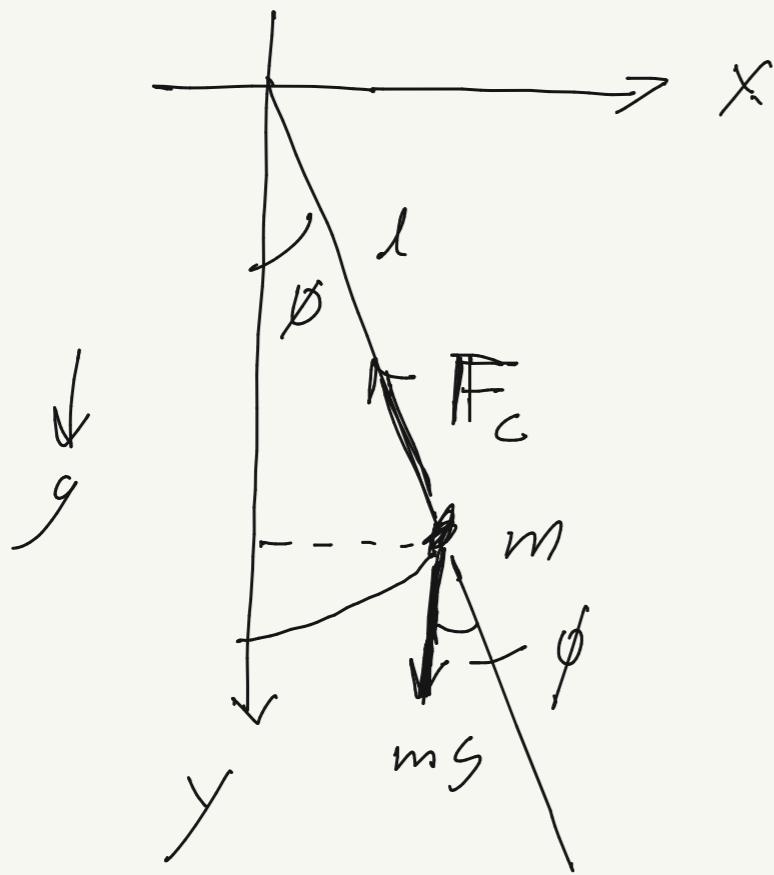
$$\approx 4a \left[\frac{\pi}{2} - \frac{k^2}{2} \int_0^{\pi/2} \frac{\cos dx \sin^2 x}{\sqrt{1 - \sin^2 x}} \right]$$

$$\approx 4a \left[\frac{\pi}{2} - \frac{k^2}{2} \int_0^{\pi/2} (1 - \cos^2 x)^{-1/2} dx \right]$$

$$= 4a \left[\frac{\pi}{2} - \frac{k^2}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right]$$

$$= 2\pi a \left(1 - \frac{k^2}{4} \right)$$

Simple pendulum (freshman physics analysis) :



$$mg \sin \phi = -m\alpha_T \\ = -m l \ddot{\phi}$$

$$\rightarrow \ddot{\phi} = -\frac{g}{l} \sin \phi$$

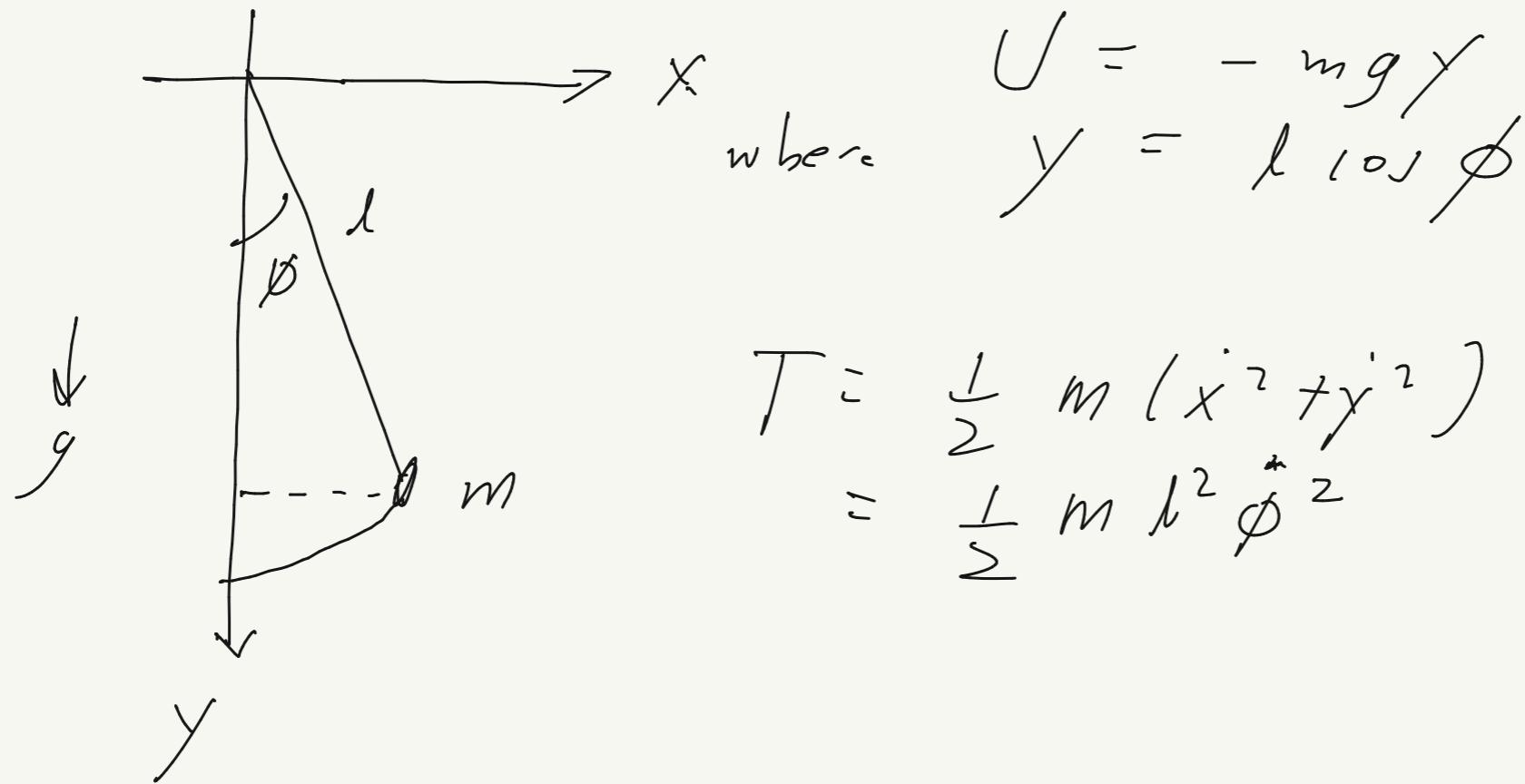
constraint force :

$$F_c - mg \cos \phi = m a_c \\ = m \omega^2 l \\ = m l \dot{\phi}^2$$

$$\text{Thus, } F_c = mg \cos \phi + ml \dot{\phi}^2$$

(non-zero at turning points as well as in vertical position - i.e., $\theta = 0$)

Period of a simple pendulum:



$$T = \frac{1}{2} m (x^2 + y^2) \\ = \frac{1}{2} m l^2 \dot{\phi}^2$$

$$L = T - U \\ = \frac{1}{2} m l^2 \dot{\phi}^2 + mg l \cos \phi$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi}$$

$$\frac{d}{dt} (m l^2 \dot{\phi}) = -m g l \sin \phi$$

$$m l^2 \ddot{\phi} = -m g l \sin \phi$$

$$\rightarrow \boxed{\ddot{\phi} = -\frac{g}{l} \sin \phi} \quad (\text{same as before})$$

Small-angle approx: $\sin \phi \approx \phi$

$$\Rightarrow \ddot{\phi} \approx -\frac{g}{l} \phi = -\omega^2 \phi$$

Sol: $\phi(t) = Ae^{i\omega t}, \quad \omega = \sqrt{\frac{g}{l}}, \quad P = \frac{2\pi}{\omega}$

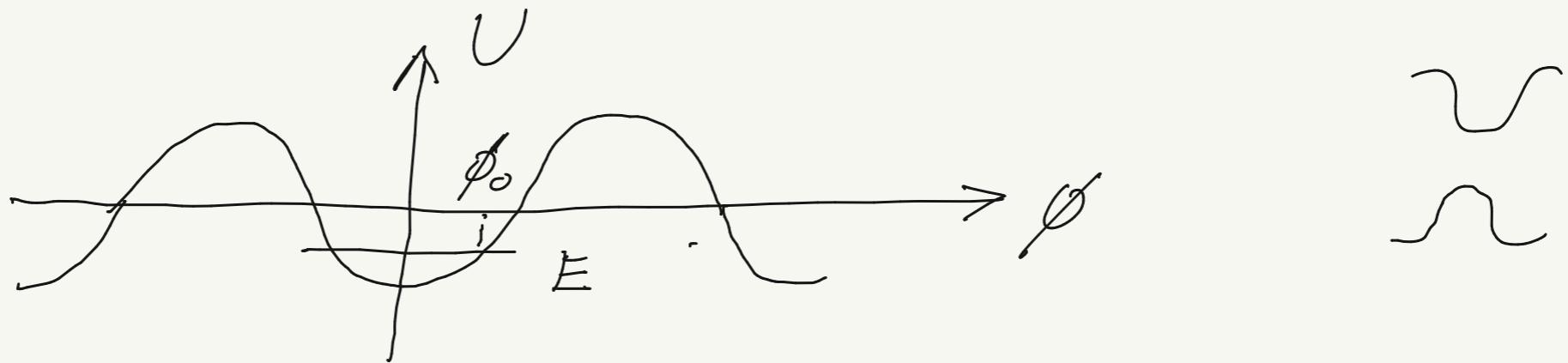
Complex

Beyond small-angle approx:

Cons. of Energy:

$$\begin{aligned} E &= T + U \\ &= \frac{1}{2} m l^2 \dot{\phi}^2 - m g l \cos \phi \end{aligned}$$

$$E = \text{const} = -m g l \cos \phi_0 \quad \text{at turning points}$$



Thus,

$$-m g l \cos \phi_0 = \frac{1}{2} m l^2 \dot{\phi}^2 - m g l \cos \phi$$

$$\begin{aligned} \frac{1}{2} m l^2 \dot{\phi}^2 &= +m g l (\cos \phi - \cos \phi_0) \\ &\geq 0 \quad \text{since } \phi \leq \phi_0 \end{aligned}$$

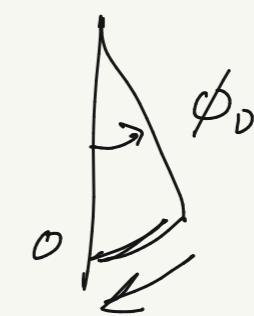
$$\rightarrow \dot{\phi} = \pm \sqrt{2 \frac{g}{l} (\cos \phi - \cos \phi_0)}$$

$$\int dt = \pm \int \frac{d\phi}{\sqrt{2 \frac{g}{l} \sqrt{\cos \phi - \cos \phi_0}}}$$

$$\rightarrow \left[t = \frac{\pm 1}{\sqrt{2} \omega_0} \int \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_0}} + \text{const} \right] \quad \omega_0 = \sqrt{\frac{g}{l}}$$

Period:

$$P = \frac{4}{\sqrt{2} \omega_0} \int_0^{\phi_0} \frac{d\phi}{\sqrt{(\omega_0\phi - \omega_0\phi_0)}}$$



$\frac{1}{4}$ th of a complete cycle

$$\text{Now: } \cos\phi = \cos\left(2\frac{\phi}{\omega_0}\right)$$

$$= \cos^2\left(\frac{\phi}{\omega_0}\right) - \sin^2\left(\frac{\phi}{\omega_0}\right)$$

$$= 1 - 2\sin^2\left(\frac{\phi}{\omega_0}\right)$$

$$\rightarrow (\omega_0\phi - \omega_0\phi_0) = -2\sin^2\left(\frac{\phi}{\omega_0}\right) + 2\sin^2\left(\frac{\phi_0}{\omega_0}\right)$$

$$= 2\sin^2\left(\frac{\phi_0}{\omega_0}\right) \left[1 - \frac{\sin^2\left(\frac{\phi}{\omega_0}\right)}{\sin^2\left(\frac{\phi_0}{\omega_0}\right)} \right]$$

$$\text{Let: } X \equiv \frac{\sin\left(\frac{\phi}{\omega_0}\right)}{\sin\left(\frac{\phi_0}{\omega_0}\right)}$$

\nearrow

$$\text{defn, } H \equiv \sin\left(\frac{\phi_0}{\omega_0}\right)$$

$$\rightarrow dx = \frac{1}{\sin\left(\frac{\phi_0}{\omega_0}\right)} \frac{1}{2} \cos\left(\frac{\phi}{\omega_0}\right) d\phi$$

$$= \frac{1}{2\sin\left(\frac{\phi_0}{\omega_0}\right)} \sqrt{1 - \sin^2\left(\frac{\phi}{\omega_0}\right)} d\phi$$

$$= \frac{1}{2H} \sqrt{1 - H^2 X^2} d\phi$$

Thus,

$$P = \frac{4}{\sqrt{2} \omega_0} \int_0^1 \frac{dx}{\sqrt{1 - H^2 X^2}} \cancel{R.H.S.}$$

$$\cancel{\sqrt{2} H} \sqrt{1 - x^2}$$

$$= \boxed{\frac{4}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1 - H^2 X^2} \sqrt{1 - x^2}}} = \frac{4}{\omega_0} \overline{E}(H)$$

complete elliptic integral
of 1st kind

Leading-order correction to period:

$$P = \frac{4}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1-H^2x^2}} \sqrt{1-x^2}$$

$$\text{Suppose } \phi_0 \ll 1 \rightarrow H \approx \sin\left(\frac{\phi_0}{2}\right) \approx \frac{\phi_0}{2} \ll 1$$

$$\begin{aligned} \text{Then, } \frac{1}{\sqrt{1-H^2x^2}} &\approx 1 + \frac{1}{2} H^2 x^2 \\ &= 1 + \frac{1}{2} \left(\frac{\phi_0}{2}\right)^2 x^2 \\ &= 1 + \frac{1}{8} \phi_0^2 x^2 \end{aligned}$$

$$\begin{aligned} \rightarrow P &\approx \frac{4}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1-x^2}} \left[1 + \frac{1}{8} \phi_0^2 x^2 \right] \\ &= \frac{4}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \frac{\phi_0^2}{2\omega_0} \int_0^1 \frac{dx}{\sqrt{1-x^2}} x^2 \\ &= \frac{4}{\omega_0} \sin^{-1}(1) + \frac{\phi_0^2}{2\omega_0} \int_0^{\pi/2} \frac{\cos \theta d\theta \sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \\ &= \frac{4}{\omega_0} \cdot \frac{\pi}{2} + \frac{\phi_0^2}{2\omega_0} \int_0^{\pi/2} d\theta \frac{1}{2} (1 - \sin 2\theta) \\ &= \frac{2\pi}{\omega_0} + \frac{\phi_0^2}{4\omega_0} \left(\frac{\pi}{2} - \frac{1}{2} \sin 2\theta \Big|_0^{\pi/2} \right) \\ &= \boxed{\frac{2\pi}{\omega_0} \left(1 + \frac{\phi_0^2}{16} \right)} \end{aligned}$$

General time dependence of oscillations:

$$\int_0^t dt = \frac{1}{2\omega_0} \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\cos(\phi) - \cos(\phi_0)}}$$

where $\phi(\omega) = \phi_0$
and $\frac{d\phi}{dt} < 0$

$$\begin{aligned} \rightarrow t &= -\frac{1}{\omega_0} \int_{\tau}^x \frac{dx}{\sqrt{1-\tau^2 x^2} \sqrt{1-x^2}} \\ &= -\frac{1}{\omega_0} \left[\int_1^0 + \int_0^x \right] \frac{dx}{\sqrt{1-\tau^2 x^2} \sqrt{1-x^2}} \\ &= +\frac{1}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1-\tau^2 x^2} \sqrt{1-x^2}} - \frac{1}{\omega_0} \int_0^x \frac{dx}{\sqrt{1-\tau^2 x^2} \sqrt{1-x^2}} \\ &= \frac{1}{\omega_0} \left[\operatorname{Erf}(x) - \operatorname{erf}^{-1}(x; \tau) \right] \end{aligned}$$

$$\text{Thus, } \operatorname{erf}^{-1}(x; \tau) = \operatorname{Erf}(x) - \omega_0 t$$

$$\rightarrow x = \operatorname{erf} \left(\frac{\operatorname{Erf}(x) - \omega_0 t; \tau}{\frac{\omega_0}{4}} \right) = \operatorname{cn}(\omega_0 t; \tau)$$

$$\text{But } x = \sin(\phi/2) / \sin(\phi_0/2) = \frac{1}{\tau} \sin(\phi/2)$$

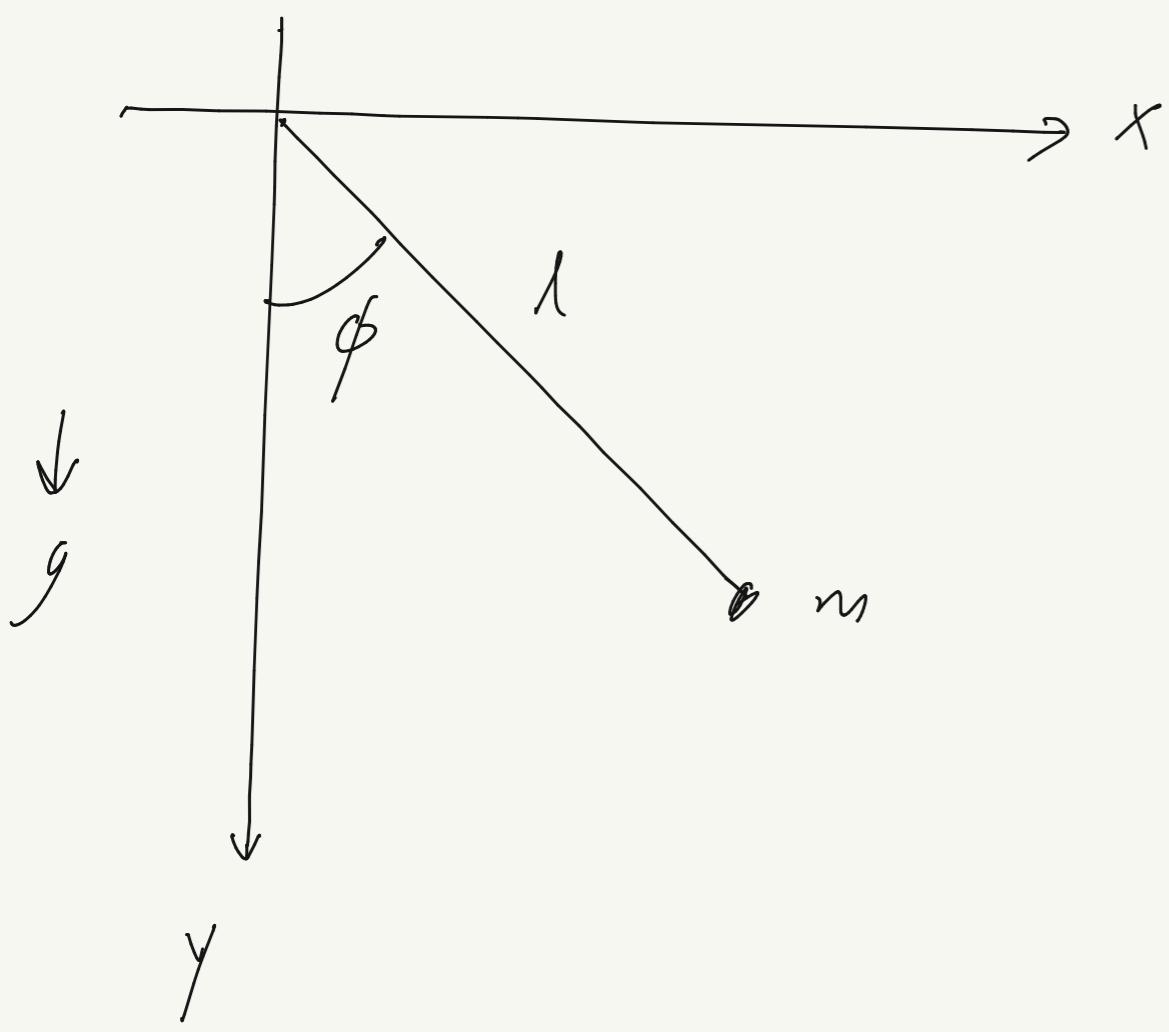
$\operatorname{erf}^{-1}(x; \tau) =$
 $\cos \theta =$
 $\sin(\frac{\pi}{2} - \theta)$

$$\rightarrow \sin\left(\frac{\phi}{2}\right) = \tau \operatorname{cn}(\omega_0 t; \tau)$$

$\phi(t) = 2 \arcsin \left[\tau \operatorname{cn}(\omega_0 t; \tau) \right]$

where $\tau = \operatorname{erf}\left(\frac{\phi_0}{2}\right)$

Lagrange multipliers:



constraint:

$$\ell = r = \sqrt{x^2 + y^2}$$

$$U = -mgy$$

$$= -mg r \cos\phi$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m (r^2 \dot{\phi}^2 + \dot{r}^2)$$

$$\varphi(r, \phi) = \ell - r = 0$$

$$L = T - U + \lambda \varphi$$

$$= \frac{1}{2} m (r^2 \dot{\phi}^2 + \dot{r}^2) + mg r \cos\phi + \lambda (\ell - r)$$

Euler-Lagrange:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} \quad (1)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \quad (2)$$

$$\varphi = 0 \quad (3)$$

$$\frac{d}{dt} (mr) = m\dot{r} + r\dot{\phi}^2 + mg \cos\phi - \lambda$$

$$\boxed{\ddot{r} = r\dot{\phi}^2 + g \cos\phi - \frac{\lambda}{m}} \quad (1)$$

$$\frac{d}{dt} (mr^2\dot{\phi}) = -mgv \sin \phi$$

$$2mr\ddot{r}\dot{\phi} + mr^2\ddot{\phi} = -mg r \sin \phi$$

$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = -\frac{g}{r} \sin \phi$

(2)

$\ell - r = 0$

(3)

Differentiate constraint twice w.r.t time:

$$\rightarrow \dot{r} = 0, \quad \ddot{r} = 0$$

Substitute for \ddot{r} using (1) :

$$0 = \ddot{\phi} = r\dot{\phi}^2 + g \cos \phi - \frac{\lambda}{m}$$

$$\begin{aligned} \rightarrow \lambda &= mr\dot{\phi}^2 + mg \cos \phi \\ &= ml\dot{\phi}^2 + mg \cos \phi \end{aligned}$$

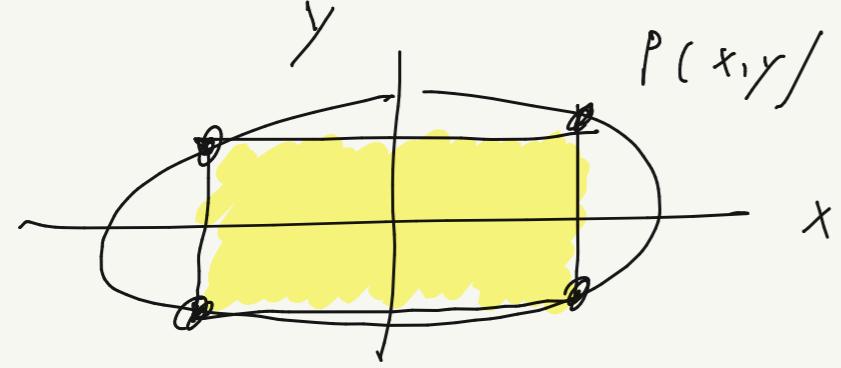
Compare with freshman physics calculation

$$F_c = mg \cos \phi + ml\dot{\phi}^2 \quad (\text{in } -\vec{r} \text{ direction})$$

Thus,

$$\overrightarrow{F}_c = \lambda \vec{\nabla} \phi \quad (\text{since } \phi = \ell - r \rightarrow \vec{\nabla} \phi = -\vec{r})$$

Another example:



Suppose you want to maximize the area of a rectangle subject to the constraint that its corners lie on an ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ (with a, b given).

Two methods:

$$\text{Area} = 4xy$$

$$\text{Constraint: } \varphi(x, y) = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$$

① Define:

$$\begin{aligned} F(x, y, \lambda) &= 4xy - \lambda \varphi(x, y) \\ &= 4xy - \lambda \left(1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 \right) \end{aligned}$$

require: $\frac{\partial F}{\partial x} = 0 \quad (1)$

$$\frac{\partial F}{\partial y} = 0 \quad (2)$$

$$\frac{\partial F}{\partial \lambda} = 0 \quad (3)$$

$$(1) \quad 0 = 4y + \frac{2\lambda x}{a^2} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad \begin{array}{l} 3 \text{ equations,} \\ 3 \text{ unknowns,} \\ (x, y, \lambda) \end{array}$$

$$(2) \quad 0 = 4x + \frac{2\lambda y}{b^2}$$

$$(3) \quad 0 = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$$

$$O = \frac{4y^2}{b^2} + \frac{2\lambda xy}{a^2 b^2}$$

$$O = \frac{4x^2}{a^2} + \frac{2\lambda yx}{b^2 a^2}$$

Subtract: $O = 4 \left[\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 \right]$

$$\rightarrow \frac{x}{a} = \pm \frac{y}{b}$$

Substitute into (3) :- $O = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$

$$2 \left(\frac{x}{a}\right)^2 = 1$$

$$\frac{x}{a} = \pm \frac{1}{\sqrt{2}}$$

$$\boxed{x = \frac{a}{\sqrt{2}}} \rightarrow \boxed{y = \frac{b}{\sqrt{2}}}$$

(2.) Reduced space method

$$F(x) \equiv 4xy \quad |$$

$$y = b \sqrt{1 - \left(\frac{x}{a}\right)^2}$$

$$= 4x b \sqrt{1 - \left(\frac{x}{a}\right)^2}$$

Solve $F'(x) = 0$

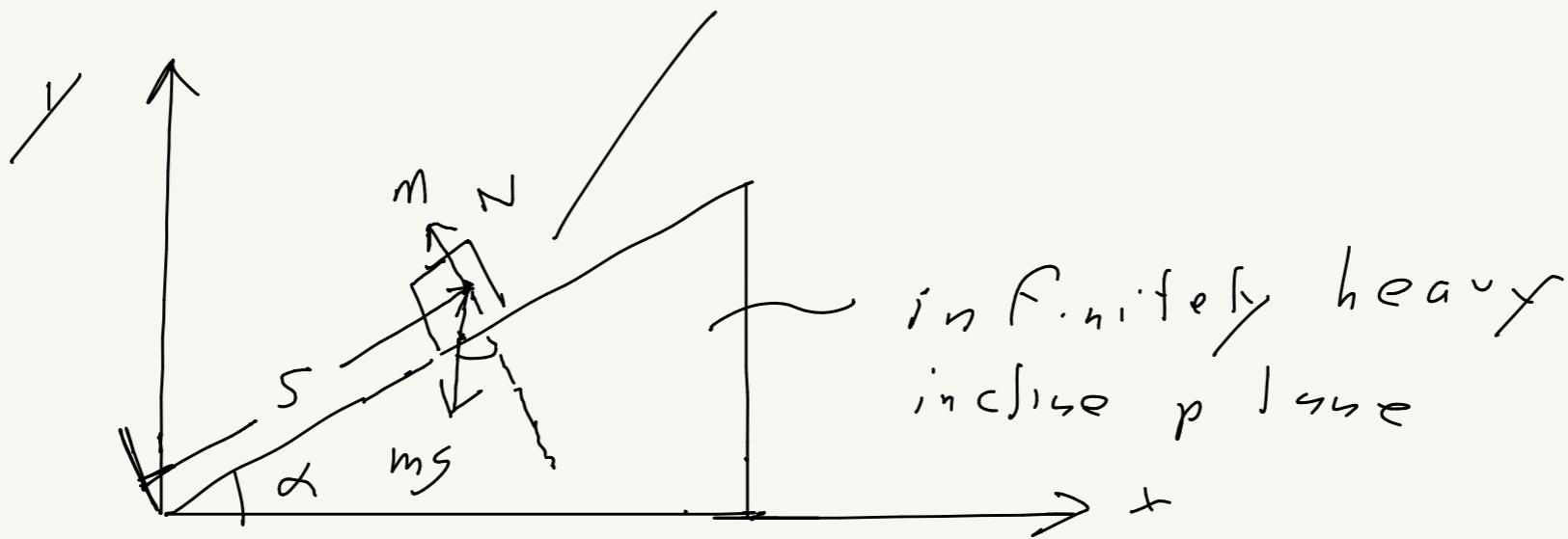
$$\begin{aligned}
 O &= F'(x) \\
 &= 4b \sqrt{1 - \left(\frac{x}{a}\right)^2} + 4x b \frac{1}{a^2} \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \left(-\frac{2x}{a^2}\right) \\
 &= 4b \left(\sqrt{1 - \left(\frac{x}{a}\right)^2} - \left(\frac{x}{a}\right)^2 \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \right) \\
 &= \frac{4b}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \left[1 - \left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^2 \right] \\
 &= \frac{4b}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \left[1 - 2\left(\frac{x}{a}\right)^2 \right]
 \end{aligned}$$

$$\begin{aligned}
 \rightarrow \left(\frac{x}{a}\right)^2 &= \frac{1}{2} \rightarrow \boxed{x = \frac{a}{\sqrt{2}}} \\
 \rightarrow \boxed{y = b \sqrt{1 - \left(\frac{x}{a}\right)^2}} &\quad \Big| \quad \frac{x}{a} = \frac{1}{\sqrt{2}} \\
 &= \frac{b}{\sqrt{2}}
 \end{aligned}$$

(same result as Lagrange multiplier method)

Another example

no Friction



$$x = s \cos \alpha$$

$$y = s \sin \alpha$$

$$\begin{aligned} m \ddot{s} &= -mg \sin \alpha \\ \ddot{s} &= -g \sin \alpha \end{aligned} \quad \text{Eqn}$$

$$N - mg \cos \alpha = 0$$

$$\rightarrow [N = mg \cos \alpha] \quad \text{constraint force}$$

(normal force)

Using Lagrangian multipliers:

(x, y) degrees of freedom

Constraint: $\frac{y}{x} = \tan \alpha \Rightarrow \varphi = y - x \tan \alpha = 0$

Lagrangian:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mg y \end{aligned}$$

Enforce constraint

$$\begin{aligned} L' &= L + \lambda \varphi(x, y) \\ &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mg y + \lambda (y - x \tan \alpha) \end{aligned}$$

EOMs:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} \quad (1)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) = \frac{\partial L}{\partial y} - \lambda \frac{\partial \varphi}{\partial y} \quad (2)$$

$$\varphi(x, y) = 0 \quad (3)$$

$$(1) \quad m \ddot{x} = -\lambda \tan \alpha$$

$$\rightarrow \ddot{x} = -\frac{\lambda}{m} \tan \alpha$$

$$(2) \quad m \ddot{y} = -mg + \lambda$$

$$\rightarrow \ddot{y} = -g + \frac{\lambda}{m}$$

$$(3) \quad y - x \tan \alpha = 0$$

$$y = x \tan \alpha$$

Differentiate constraint twice wrt t:

$$\ddot{y} = \ddot{x} + \tan \alpha$$

Substitute from (1) and (2):

$$-g + \frac{\lambda}{m} = -\frac{\lambda}{m} \tan \alpha + \tan \alpha$$

$$\frac{\lambda}{m} (1 + \tan^2 \alpha) = g$$

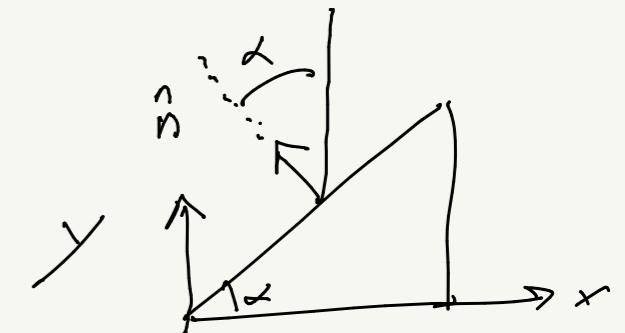
$$\begin{aligned} s^2 + c^2 &= 1 \\ 1 + \tan^2 \alpha &= \sec^2 \alpha \end{aligned}$$

$$\lambda = \frac{mg}{\sec^2 \alpha} = \boxed{mg \cos^2 \alpha}$$

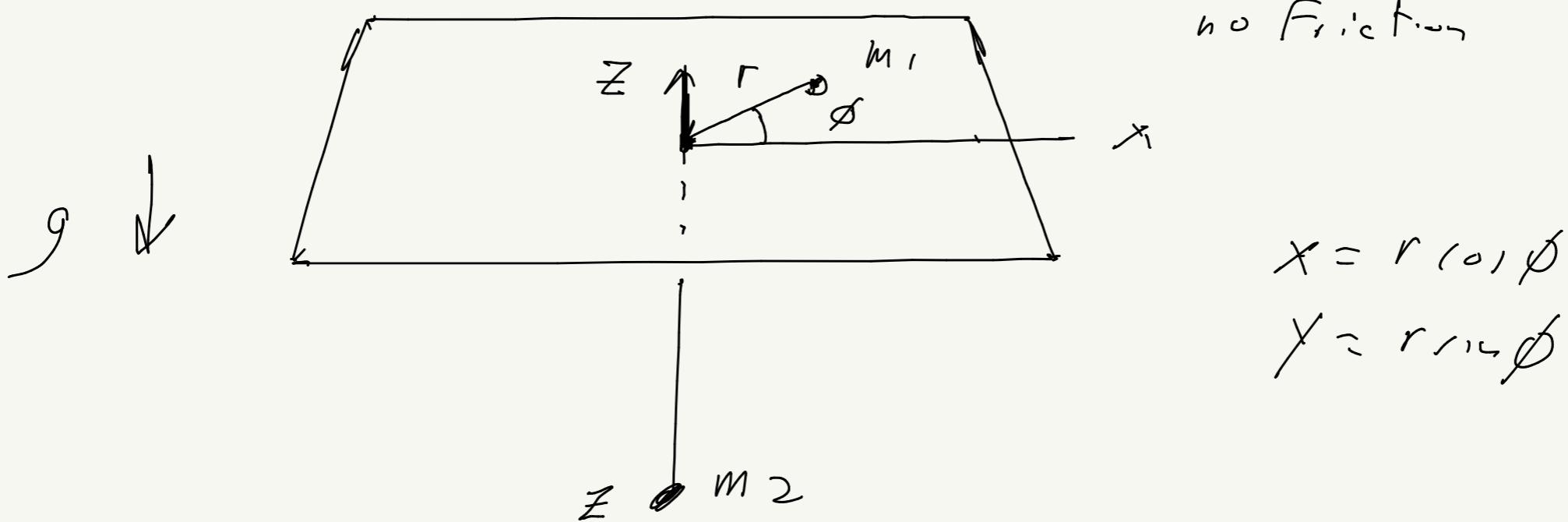
(normal force):

$$\begin{aligned}\overrightarrow{F}_c &= \lambda \nabla \phi \\ &= mg \cos^2 \alpha \left(\hat{y} - \hat{x} \tan \alpha \right) \\ &= mg \cos \alpha \left(\cos \alpha \hat{y} - \sin \alpha \hat{x} \right) \\ &= mg \cos \alpha \hat{n}\end{aligned}$$

where $\hat{n} = -\sin \alpha \hat{x} + \cos \alpha \hat{y}$
is unit vector \perp to surface of
inclined plane



Conservation of momentum example



$$\underline{\text{Constraint:}} \quad \mathbf{r} - \mathbf{z} = \ell$$

$$\begin{aligned} T &= \sum m_i (\dot{x}^2 + \dot{y}^2) + \sum m_i \dot{z}^2 \\ &= \sum m_i (r^2 + r^2 \dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2 \\ &= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 \end{aligned}$$

$$U = m_2 g z = m_2 g (r - \ell) = m_2 g r \underbrace{+ \text{const}}_{\text{is no } \omega}$$

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r \end{aligned}$$

No t -dependence:

$$\begin{aligned} E &= T + U = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 + m_2 g r \\ &= \text{const} \end{aligned}$$

No ϕ -dependence

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} = \text{const} = M z$$

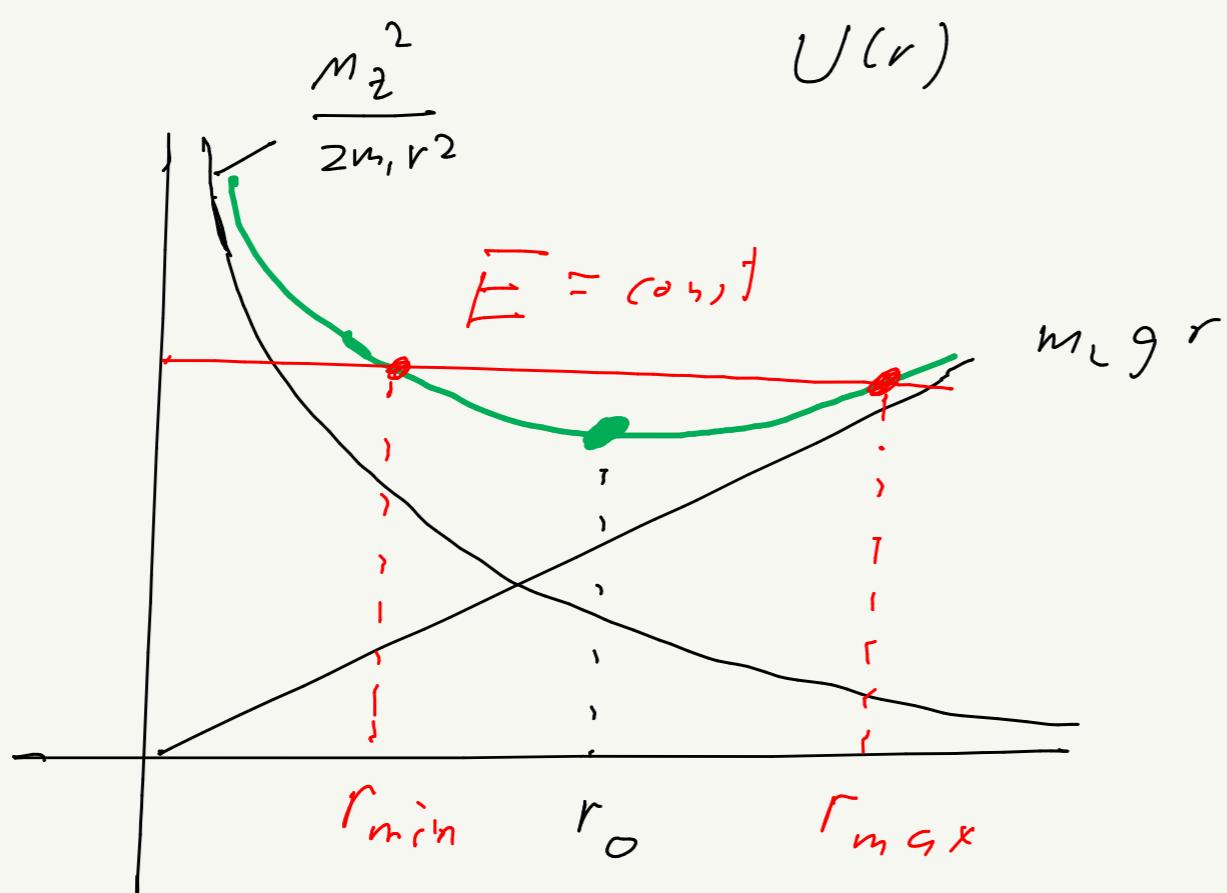
$$\text{Thrust} \quad \phi = \frac{M_2}{m_1 r^2}$$

$$E = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \left(\frac{M_2^2}{m_1^2 r^4} \right) + m_2 g r$$

$$= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{M_2^2}{2 m_1 r^2} + m_2 g r$$

$$= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + U_{\text{eff}}(r)$$

$$U_{\text{eff}}(r) = \frac{M_2^2}{2 m_1 r^2} + m_2 g r$$



r_{\min}, r_{\max} : turning points

$E = U_{\text{eff}}(r_0) \rightarrow$ stable circular orbit ($v = v_0$)

$$E \geq U_{\text{eff}}(r_0)$$

$$\begin{aligned}
 O &= \frac{d U_{\text{eff}}}{dr} \Big|_{r=r_0} \\
 &= \left(-\frac{M_z^2}{m_1 r^3} + m_2 g \right) \Big|_{r=r_0} \\
 &= \frac{-M_z^2}{m_1 r_0^3} + m_2 g
 \end{aligned}$$

$$M_z^2 = m_1 m_2 g r_0^3$$

$$\text{so } \boxed{M_z = \sqrt{m_1 m_2 g r_0^3}}$$

angular momentum
necessary to
give x_0 a
circular orbit at $r=r_0$

$$\begin{aligned}
 U_{\text{eff}, m_1} &= U_{\text{eff}}(r_0) \\
 &= \left(\frac{M_z^2}{2 m_1 r_0^2} + m_2 g r_0 \right) \Big|_{r=r_0} \\
 &= \cancel{\frac{m_1 m_2 g r_0^3}{2 m_1 r_0^2}} + m_2 g r_0 \\
 &= \frac{3}{2} m_2 g r_0
 \end{aligned}$$

Form:

$$E = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + U_{\text{eff}}(r)$$

$$= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{M_2^2}{2m_1 r^2} + m_2 g r$$

$$\rightarrow \frac{2}{(m_1 + m_2)} \left(E - \frac{M_2^2}{2m_1 r^2} - m_2 g r \right) = \dot{r}^2$$

$$\frac{dr}{dt} = \pm \sqrt{\left(\frac{2}{m_1 + m_2} \right) \left(E - \frac{M_2^2}{2m_1 r^2} - m_2 g r \right)}$$

$$\int \pm dt = \int \frac{dr}{\sqrt{\frac{2}{m_1 + m_2} \left(E - \frac{M_2^2}{2m_1 r^2} - m_2 g r \right)}}$$

Integrate to get $t = t(r)$

Orbit equation:

$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{M_2}{m_1 r^2}$$

$$\text{Thus, } \frac{dr}{d\phi} \frac{M_2}{m_1 r^2} = \pm \sqrt{\text{ell}}$$

$$\frac{dr}{r^2} = \pm \frac{m_1}{M_2} \sqrt{\text{ell}} d\phi$$

$$\int \pm d\phi = \int \frac{dr/r^2}{\frac{m_1}{M_2} \sqrt{\frac{2}{m_1 + m_2}} \left(E - \frac{m_2^2}{2m_1 r^2} - m_2 g r \right)}$$

Can try to evaluate these integrals analytically making substitutions of the type

$$u = \frac{1}{r} \rightarrow du = -\frac{1}{r^2} dr, \text{ etc.}$$



Numerically:

$$dr = \pm \sqrt{\Theta} dt, \quad \Theta = \frac{2}{m_1 + m_2} \left(E - \frac{m_2^2}{2m_1 r^2} - m_2 g r \right)$$

$$d\phi = \frac{M_2}{m_1 r^2} dt$$

So start the system at time $t = t_0$ with

some values of r and ϕ for given E, M_2

$$\underline{t = t_0}: \quad r(t_0), \phi(t_0)$$

$$\underline{t = t_0 + \Delta t}: \quad r(t_0) + \Delta r, \phi(t_0) + \Delta \phi$$

$$\text{where } \Delta r = \pm \sqrt{\Theta} \Delta t, \quad \Delta \phi = \frac{M_2}{m_1 r(t_0)} \Delta t$$

!

(repeat)

Vivial theorem:

Assume potential energy is a homogeneous function of degree κ :

$$U(\alpha \vec{r}_1, \alpha \vec{r}_2, \dots) = \alpha^\kappa U(\vec{r}_1, \vec{r}_2, \dots)$$

Theorem:

$$\sum_a \frac{\partial U}{\partial \vec{r}_a} \cdot \vec{v}_a = \kappa U$$

Note:

$$\sum_a \frac{\partial T}{\partial v_a} \cdot \vec{v}_a = 2T = \sum_a \frac{\partial L}{\partial \vec{v}_a} \cdot \vec{v}_a = \sum_a \vec{p}_a \cdot \vec{v}_a$$

$$U = mgy \rightarrow \kappa = 1$$

$$U = -\frac{G m_1 m_2}{r} \rightarrow \kappa = -1$$

$$U = \frac{1}{2} \kappa x^2 \rightarrow \kappa = 2$$

$$\begin{aligned} \text{Thm, } \kappa U &= \sum_a \frac{\partial U}{\partial \vec{v}_a} \cdot \vec{r}_a \\ &= - \sum_a \frac{\partial L}{\partial \vec{v}_a} \cdot \vec{r}_a \\ &= - \sum_a \frac{d}{dt} \left(\frac{\partial L}{\partial \vec{v}_a} \right) \cdot \vec{r}_a \\ &= - \frac{d}{dt} \left(\sum_a \vec{p}_a \cdot \vec{r}_a \right) + \sum_a \vec{p}_a \cdot \vec{v}_a \end{aligned}$$

Now:

$$\sum_a \vec{p}_a \cdot \vec{v}_a = \sum_a m_a |\vec{v}_a|^2 = 2T$$

Thus,

$$H \bar{U} = -\frac{d}{dt} \left(\sum_a \vec{p}_a \cdot \vec{r}_a \right) + 2T$$

Also assume that the motion is bounded, so

$$\sum_a \vec{p}_a \cdot \vec{r}_a = \text{finite for all } t.$$

Then if we take the time average:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt f(t) = \bar{f}$$

We find:

$$H \bar{U} = - \lim_{T \rightarrow \infty} \frac{1}{T} \left(\sum_a \vec{p}_a \cdot \vec{r}_a \Big|_{t=0} - \sum_a \vec{p}_a \cdot \vec{r}_a \Big|_{t=T} \right)$$

$\underbrace{}$ $\underbrace{}$
 f_{ini} f_{fin}

$$+ 2 \bar{T}$$

$$= 2 \bar{T}$$

so

$$H \bar{U} = 2 \bar{T}$$

Hamiltonian mechanics:

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \quad (= \text{const if } L \text{ does not depend explicitly on } t)$$

$$= E(q, \dot{q}, t)$$

Proof: Suppose $\frac{\partial L}{\partial t} = 0$

$$\rightarrow \frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right)$$

$$= \sum_i \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right]$$

$$= \sum_i \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right]$$

$$= \frac{d}{dt} \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right)$$

Thus, $\frac{d}{dt} \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right) = 0$

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = \text{const}$$

Hamiltonian: $H = \left(\sum_i p_i \dot{q}_i - L \right) \Big|_{\dot{q} = \dot{q}(q, p)}$

Hamiltonian is a function of (q_i, p_i, t)

EOMs:

$$\begin{aligned} dL &= \sum_i \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right) + \frac{\partial L}{\partial t} \\ &= \sum_i (p_i dq_i + \dot{p}_i d\dot{q}_i) + \frac{\partial L}{\partial t} \\ &= \sum_i (\dot{p}_i dq_i + d(p_i \dot{q}_i) - dp_i \dot{q}_i) + \frac{\partial L}{\partial t} \end{aligned}$$

$$d(L - \sum_i p_i \dot{q}_i) = \sum_i (p_i dq_i - \dot{q}_i dp_i) + \frac{\partial L}{\partial t}$$

$$d(\underbrace{\sum_i p_i \dot{q}_i - L}_{H}) = \sum_i (-p_i dq_i + \dot{q}_i dp_i) - \frac{\partial L}{\partial t}$$

$$\rightarrow \boxed{\frac{\partial H}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

Hamilton's equations,
2s 1st order equations

Conservation
of
energy

Example:

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$p = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \rightarrow \dot{x} = \frac{p}{m}$$

$$\rightarrow H = \left(p \dot{x} - L \right) |$$
$$\dot{x} = p/m$$

$$= \frac{p^2}{m} - \left(\frac{1}{2} m \left(\frac{p}{m} \right)^2 + U(x) \right)$$

$$= \frac{p^2}{2m} + U(x)$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$
$$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$$

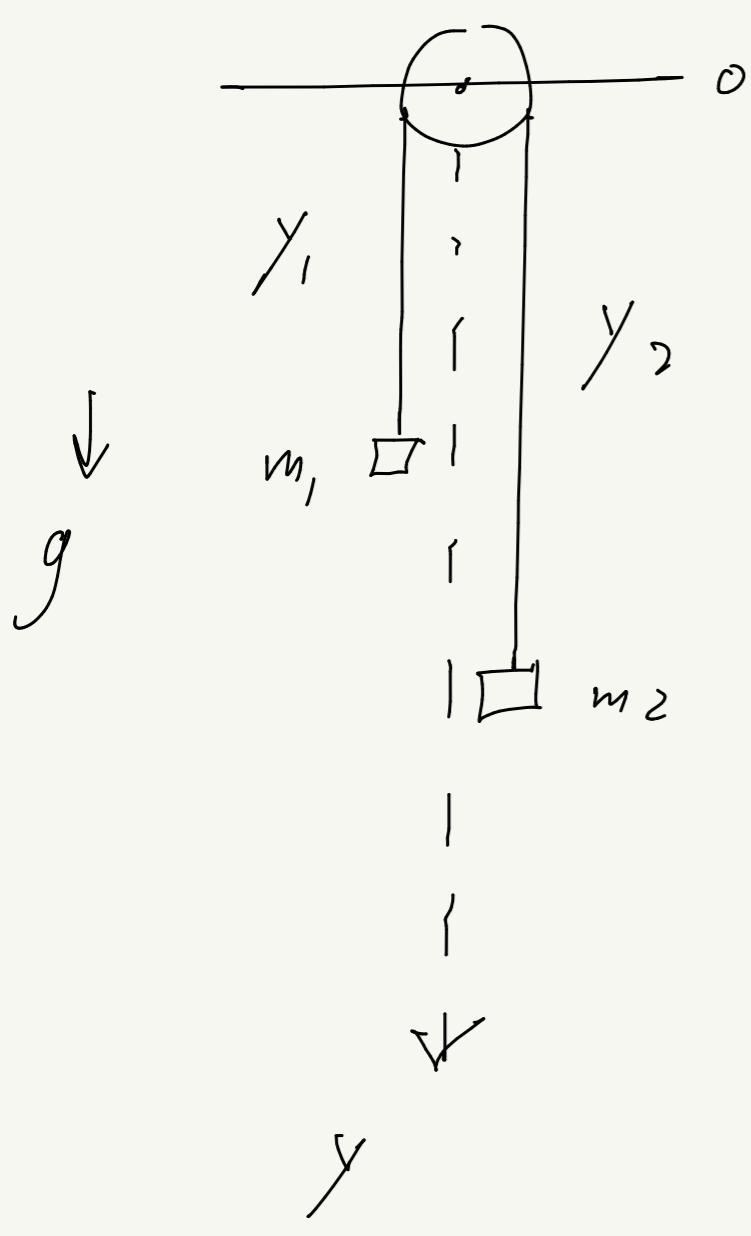
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Combine:

$$p = m \dot{x}$$
$$\rightarrow \dot{p} = m \ddot{x}$$
$$\rightarrow m \ddot{x} = -\frac{\partial U}{\partial x}$$

Atwood machine

$$y_1 + y_2 = l \rightarrow y_2 = l - y_1$$



$$\begin{aligned} U &= -m_1 gy_1 - m_2 gy_2 \\ &= -m_1 gy_1 - m_2 g(1-y_1) \\ &= -(m_1 - m_2)gy_1 - m_2 g \underbrace{l}_{\text{const}}, \\ &\quad \text{igno-} \end{aligned}$$

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2$$

$$= \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_1^2$$

$$= \frac{1}{2} (m_1 + m_2) \dot{y}_1^2$$

$$L = T - U$$

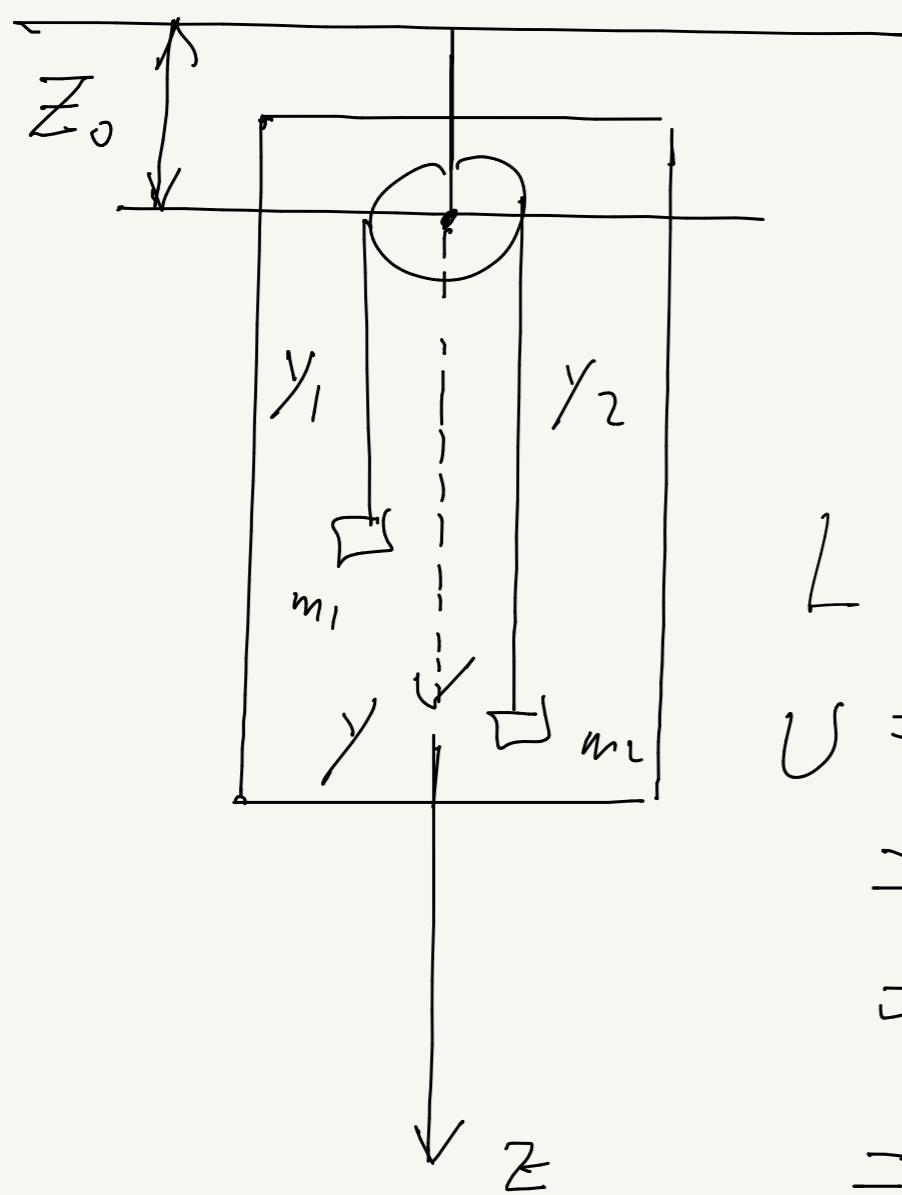
$$= \frac{1}{2} (m_1 + m_2) \dot{y}_1^2 + (m_1 - m_2) gy_1,$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}_1} \right) = \frac{\partial L}{\partial y_1}$$

$$(m_1 + m_2) \ddot{y}_1 = (m_1 - m_2) g$$

$$\ddot{y}_1 = \frac{(m_1 - m_2)}{(m_1 + m_2)} g$$

What happens if the Atwood machine is in an accelerating reference frame?



$$z_0 = \frac{1}{2} a t^2$$

$$z_1 = z_0 + y_1$$

$$z_2 = z_0 + y_2$$

$$L = T - U \quad (\text{w.r.t inertial frame})$$

$$U = -m_1 g z_1 - m_2 g z_2$$

$$= -m_1 g y_1 - m_2 g y_2 - \underbrace{(m_1 + m_2) g z_0}_{\text{prescribed function of time}}$$

$$= -(m_1 - m_2) g y_1 - \underbrace{m_2 g l}_{\text{const}} \quad \begin{matrix} \text{prescribed} \\ \text{function} \\ \text{of time} \\ \text{(ignore)} \end{matrix}$$

$$= -(m_1 - m_2) g y_1 \quad \begin{matrix} \text{const} \\ \text{(ignore)} \end{matrix}$$

$$y_1 + y_2 = l$$

$$T = \frac{1}{2} (m_1 \dot{z}_1^2 + m_2 \dot{z}_2^2)$$

$$= \frac{1}{2} m_1 (a^2 t^2 + \dot{y}_1^2 + 2 a t \dot{y}_1)$$

$$+ \frac{1}{2} m_2 (a^2 t^2 + \dot{y}_2^2 + 2 a t \dot{y}_2)$$

$$= \underbrace{\frac{1}{2} (m_1 + m_2) a^2 t^2}_{\text{prescribed function of } t} + \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 + a t (m_1 y_1 + m_2 y_2)$$

$$\dot{z}_1 = \dot{z}_0 + \dot{y}_1$$

$$= a t + \dot{y}_1$$

$$\dot{z}_2 = \dot{z}_0 + \dot{y}_2$$

$$= a t + \dot{y}_2$$

prescribed function of t
(ignore)

$$\underline{\text{Now:}} \quad a_t(m_1\dot{y}_1 + m_2\dot{y}_2) = \frac{d}{dt} [a_t(m_1y_1 + m_2y_2)] - a(m_1y_1 + m_2y_2)$$

thus,

$$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 - a(m_1y_1 + m_2y_2) + \underbrace{\text{total time derivative}}_{(\text{ignore})}$$

$$= \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - a(m_1y_1 + m_2(1-y_1))$$

$$= \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - a(m_1 - m_2)y_1 - a m_2 l$$

ignore

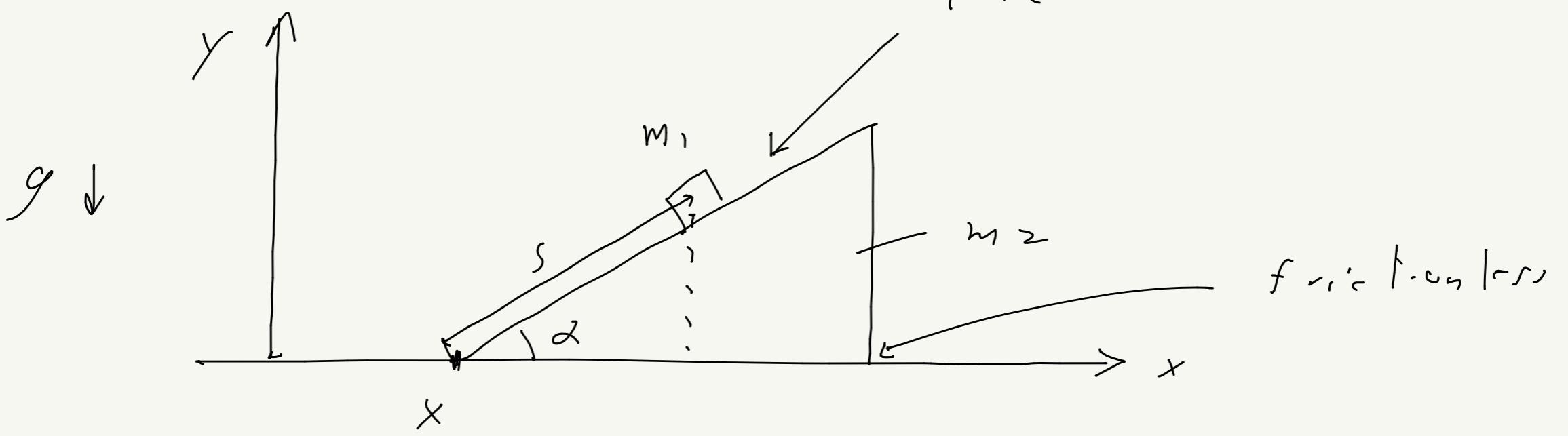
$$L = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - a(m_1 - m_2)y_1 + (m_1 - m_2)gy_1$$

$$= \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 + (m_1 - m_2)(g - a)y_1.$$

$$\rightarrow \boxed{\dot{y}_1 = \frac{(m_1 - m_2)(g - a)}{m_1 + m_2}}$$

so the acceleration of gravity g got modified by the accelerations of the non-inertial reference frame

Ques # 2:



$$x_2 = x$$

$$y_2 = 0$$

$$\dot{x}_1 = x + s \cos \alpha$$

$$\dot{y}_1 = s \sin \alpha$$

$$\ddot{x}_1 = \ddot{x} + s \cos \alpha$$

$$\ddot{y}_1 = s \sin \alpha$$

$$U = m_1 g y$$

$$= m_1 g s \sin \alpha$$

$$\dot{x}_2 = \dot{x}$$

$$\dot{y}_2 = 0$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_1 (\dot{x}^2 + \underbrace{s^2 \cos^2 \alpha}_{\text{constant}} + 2 \dot{x} s \cos \alpha + \underbrace{s^2 \sin^2 \alpha}_{\text{constant}}) + \frac{1}{2} m_2 \dot{x}^2$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_1 s^2 + m_1 \dot{x} s \cos \alpha$$

$$\boxed{\begin{aligned} L &= T - U \\ &= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_1 s^2 + m_1 \dot{x} s \cos \alpha - m_1 g s \sin \alpha \end{aligned}}$$

i) L does not depend explicitly on t
 $\rightarrow E = T + U = \text{const}$

ii) L does not depend on x
 $\rightarrow p_x \equiv \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2) \dot{x} + m_1 s \cos \alpha$
 $= \text{const}$.

Thus, can choose a reference frame such
 that $p_x = 0$ and $x_{\text{com}} = 0$

$$\begin{aligned} x_{\text{com}} &= m_1 x_1 + m_2 x_2 \\ &= m_1 (x + s \cos \alpha) + m_2 x \\ &= (m_1 + m_2) x + m_1 s \cos \alpha \end{aligned}$$

$$x_{\text{com}} = 0 \quad \rightarrow \boxed{x = -\left(\frac{m_1}{m_1 + m_2}\right) s \cos \alpha}$$

Central Force motion:

Reduction of 2-body problem

$$L = \frac{1}{2} m_1 |\vec{r}_1|^2 + \frac{1}{2} m_2 |\vec{r}_2|^2 - U(|\vec{r}_1 - \vec{r}_2|)$$

- i) No explicit time dependence $\rightarrow E = T + U$ conserved
 - ii) Translationally invariant ($\vec{r}_c \rightarrow \vec{r}_c + \vec{\delta}_x$) \rightarrow total momentum $\vec{P} = \sum_a m_a \vec{v}_a = \text{constant}$
 - iii) Rotationally invariant ($\vec{r}_c \rightarrow \vec{r}_c + \vec{\delta}\phi \times \vec{r}_c$
 $\vec{v}_c \rightarrow \vec{v}_c + \vec{\delta}\phi \times \vec{v}_c$)
- \rightarrow total angular momentum conserved $\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a$

Using $\vec{P} = \text{const}$ \rightarrow can go to com frame where
 $\sum_a m_a \vec{r}_a = 0 \iff m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$

Define: (relative position vector)
 $\vec{r} = \vec{r}_1 - \vec{r}_2$

Then $U = U(r)$, $r = |\vec{r}| = |\vec{r}_1 - \vec{r}_2|$

and $\vec{r} = m_1 \vec{r}_1 + m_2 \vec{r}_2$
 $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$\begin{aligned} \vec{r}_c &= \vec{r}_1 - \vec{r}_2 \quad \rightarrow \quad \vec{r} = m_1 \vec{r}_1 + m_2 (\vec{r}_1 - \vec{r}) \\ &= (m_1 + m_2) \vec{r}_1 - m_2 \vec{r} \end{aligned}$$

so
$$\boxed{\vec{r}_1 = \frac{m_2 \vec{r}}{m_1 + m_2}}$$

$$\begin{aligned}
 \vec{r}_2 &= \vec{r}_1 - \vec{r} \\
 &= \frac{m_2}{m_1 + m_2} \vec{r}_1 - \vec{r} \\
 &= \left(\frac{-m_1}{m_1 + m_2} \right) \vec{r}
 \end{aligned}$$

$\left. \vec{r}_2 = -\frac{m_1}{m_1 + m_2} \vec{r} \right\}$

$$\begin{aligned}
 T^{\text{ho}} &= \\
 T &= \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 \\
 &= \frac{1}{2} m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 |\dot{\vec{r}}|^2 + \frac{1}{2} m_2 \left(\frac{-m_1}{m_1 + m_2} \right)^2 |\dot{\vec{r}}|^2 \\
 &= \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)^2} [\cancel{m_1 + m_2}] |\dot{\vec{r}}|^2 \\
 &= \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) |\dot{\vec{r}}|^2 \\
 &= \frac{1}{2} m |\dot{\vec{r}}|^2
 \end{aligned}$$

where $m \equiv \left(\frac{m_1 m_2}{m_1 + m_2} \right)$ = reduced mass,

$\rightarrow \boxed{L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(r)}$

Using $\vec{M} = \text{const}$ \rightarrow take \vec{z} along \vec{M} .

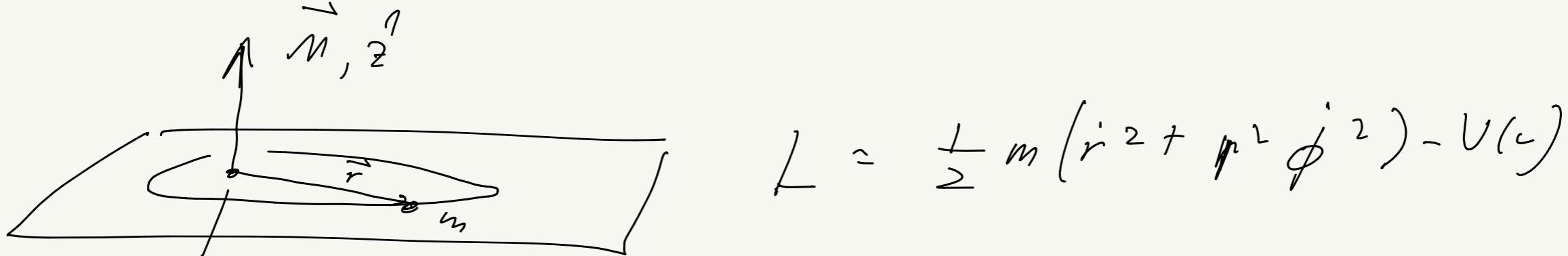
Since

$$\begin{aligned}\vec{M} &= \sum_i \vec{r}_i \times \vec{p}_i \\ &= m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2 \\ &= m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 \vec{r} \times \vec{v} + m_2 \left(\frac{-m_1}{m_1 + m_2} \right)^2 \vec{r} \times \vec{v} \\ &= \frac{m_1 m_2}{(m_1 + m_2)} \underbrace{[\vec{r}_2 + \vec{r}_1]}_{\vec{r}} \vec{r} \times \vec{v}\end{aligned}$$

$$= m \vec{r} \times \vec{v}$$

we have $\vec{r}, \vec{v} \perp$ to \vec{M} .

\rightarrow so motion is in xy plane



com

(cent. of potential)

But, not only is direction of \vec{M} constant, so is its magnitude: $M = M_z$

$$\begin{aligned}\rightarrow p_\phi &\equiv \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = \text{const} \quad \left(\text{since } \frac{\partial L}{\partial \phi} = 0 \right) \\ &\equiv M_z\end{aligned}$$

Integral equations :

Cons. of energy

$$E = T + U = \text{const}$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + U(r)$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \left(\frac{M_Z}{mr^2} \right)^2) + U(r)$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{M_Z^2}{2mr^2} + U(r)$$

$$= \frac{1}{2} m \dot{r}^2 + U_{\text{eff}}(r)$$

$$U_{\text{eff}}(r) = U(r) + \frac{M_Z^2}{2mr^2}$$

Motion in 1-d subject to $U = U_{\text{eff}}(r)$

$$\frac{dr}{dt} = \dot{r} = \pm \sqrt{\frac{2}{m} (E - U_{\text{eff}}(r))}$$

$$dt = \pm \frac{dr}{\sqrt{\frac{2}{m} (E - U_{\text{eff}}(r))}}$$

(14.6)

$$t = \pm \int \frac{dr}{\sqrt{\frac{2}{m} (E - U(r)) - \frac{M_Z^2}{m^2 r^2}}} + \text{const}$$

Orbit equation: $\phi = \phi(r)$

$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{M_2}{mr^2}$$

Thus, $\frac{dr}{d\phi} \frac{M_2}{mr^2} = \pm \sqrt{\frac{2}{m}(E - U_{eff}(r))}$

$$\rightarrow \downarrow \phi = \pm \frac{M_2 dr/r^2}{\sqrt{\frac{2m}{m}(E - U(r)) - \frac{M_2^2}{r^2}}}$$

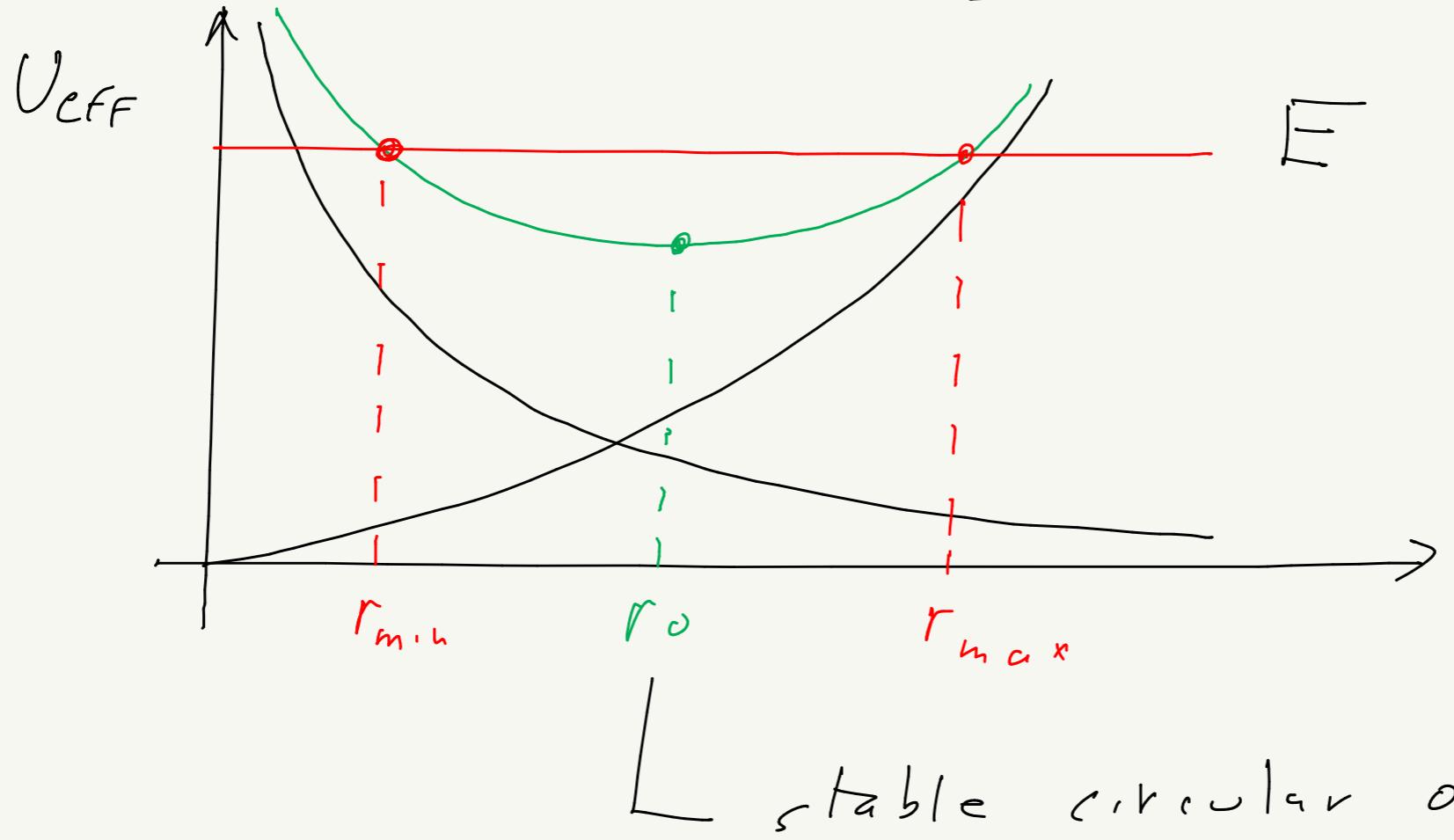
$$= \pm \frac{M_2 dr/r^2}{\sqrt{\frac{2m}{m}(E - U(r)) - M_2^2/r^2}}$$

$$\left. \begin{aligned} \phi &= \pm M_2 \int \frac{dr/r^2}{\sqrt{\frac{2m}{m}(E - U(r)) - M_2^2/r^2}} + C_1, \\ &\quad C_1 \end{aligned} \right\} (14.7)$$

Example: spring of stiffness k $U(r) = \frac{1}{2} k r^2$

$$U_{\text{eff}}(r) = U(r) + \frac{M_z^2}{2m r^2}$$

$$= \frac{1}{2} k r^2 + \frac{M_z^2}{2m r^2}$$



stable circular orbit

$$\phi = M_z \int \frac{dr/r^2}{\sqrt{2m(E - U(r)) - \frac{M_z^2}{r^2}}} + C_1 e^{i\omega t}$$

$$= M_z \int \frac{dr/r^2}{\sqrt{2mE - mkr^2 - \frac{M_z^2}{r^2}}}$$

Let: $u = 1/r \rightarrow du = -\frac{1}{r^2} dr$

$$\phi = -M_z \int \frac{du}{\sqrt{\frac{2mE}{u^2} - \frac{mk}{u} - M_z^2 u^2}}$$

$$\phi = -M_T \int \frac{u du}{\sqrt{2mE_u^2 - m\hbar\Gamma - M_Z^2 u^4}} + \text{const}$$

Let: $V = u^2$, $dv = 2u du$

$$\phi = -\frac{M_Z}{2} \int \frac{dv}{\sqrt{2mE_v - m\hbar\Gamma - M_Z^2 v^2}} + \text{const}$$

Complete the square:

$$\begin{aligned} -M_Z^2 v^2 + 2mE_v - m\hbar\Gamma &= -M_Z^2 \left(v^2 - \frac{2mE_v}{M_Z^2} + \frac{m\hbar\Gamma}{M_Z^2} \right) \\ &= -M_Z^2 \left[\left(v - \frac{mE}{M_Z^2} \right)^2 - \frac{m^2 E^2}{M_Z^4} + \frac{m\hbar\Gamma}{M_Z^2} \right] \end{aligned}$$

$$= -M_Z^2 \left[(v - A)^2 - B^2 \right]$$

$$A = \frac{mE}{M_Z^2}, \quad B^2 = A^2 - \frac{m\hbar\Gamma}{M_Z^2}$$

$$\rightarrow \phi = -\frac{1}{2} \int \frac{dv}{\sqrt{B^2 - (v - A)^2}} + \text{const}$$

$$\text{Let } V - A = B \sin \theta$$

$$\rightarrow \sqrt{V^2 - A^2} = B \cos \theta$$

$$\rightarrow \sqrt{B^2 - (V-A)^2} = B \sqrt{1 - \sin^2 \theta} = B \cos \theta$$

Sol:

$$\phi = -\frac{1}{2} \int \frac{\cancel{B \cos \theta} d\theta}{\cancel{B \cos \theta}} + \text{const}$$

$$= -\frac{1}{2} \theta + \text{const}$$

$$= -\frac{1}{2} \sin^{-1} \left(\frac{V-A}{B} \right) + \text{const}$$

$$= -\frac{1}{2} \sin^{-1} \left(\frac{\frac{1}{r^2} - A}{B} \right) + \text{const}$$

$$\begin{aligned} v &= u^2 \\ u &= \frac{1}{r} \\ \rightarrow v &= \frac{1}{r^2} \end{aligned}$$

Choose const so that $\phi = 0 \iff r = r_{\max}$.

Turning points $r = r_{\min}, r_{\max}$ correspond to zeros of the $\sqrt{-}$

$$0 = B^2 - (V-A)^2$$

$$= B^2 - \left(\frac{1}{r^2} - A \right)^2$$

$$\pm B = \frac{1}{r^2} - A$$

$$\rightarrow \frac{1}{r^2} = A \pm B, \quad \frac{1}{r_{\max}^2} = A - B$$

Thus,

$$\begin{aligned}0 &= -\frac{1}{2} \sin^{-1} \left(\frac{A-B}{B} \right) + \text{const} \\&= -\frac{1}{2} \sin^{-1}(-1) + \text{const} \\&= -\frac{1}{2} \left(-\frac{\pi}{2} \right) + \text{const}\end{aligned}$$

$$\boxed{\text{const} = -\frac{\pi}{4}}$$

Thus, $\phi = -\frac{1}{2} \sin^{-1} \left(\frac{\frac{1}{r^2} - A}{B} \right) - \frac{\pi}{4}$

$$-2 \left(\phi + \frac{\pi}{4} \right) = \sin^{-1} \left(\frac{\frac{1}{r^2} - A}{B} \right)$$

$$-\sin \left(2\phi + \frac{\pi}{2} \right) = \frac{\frac{1}{r^2} - A}{B}$$

$$-\left[\sin \left(2\phi \right) \cos \left(\frac{\pi}{2} \right) + \cos \left(2\phi \right) \underbrace{\sin \left(\frac{\pi}{2} \right)}_1 \right] = \frac{\frac{1}{r^2} - A}{B}$$

$$-\cos \left(2\phi \right) = \frac{\frac{1}{r^2} - A}{B}$$

$$\boxed{A - B \cos(2\phi) = \frac{1}{r^2}}$$

actually an ellipse
with origin at center

$$\frac{1}{r^2} = A - B \cos(2\phi) \quad , \quad B^2 \equiv A^2 - \frac{m \hbar^2}{M_2^2}$$

$$1 = Ar^2 - Br^2 (\cos^2 \phi - \sin^2 \phi)$$

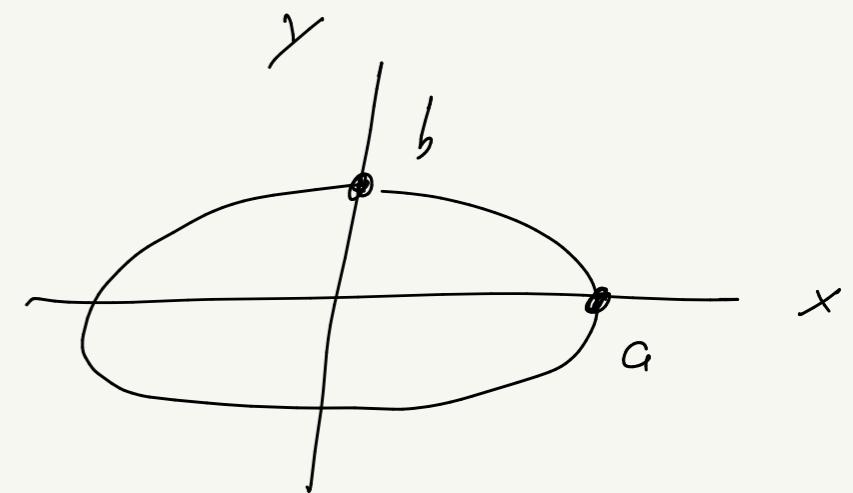
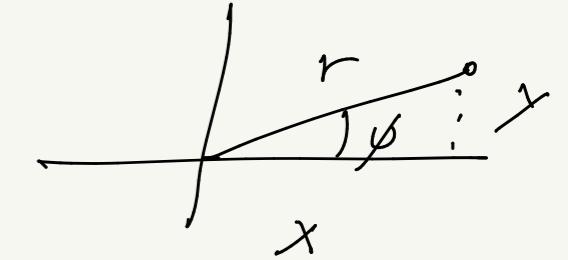
$$= A(x^2 + y^2) - B(x^2 - y^2)$$

$$= (A - B)x^2 + (A + B)y^2$$

$$= \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

where $a^2 = \frac{1}{A-B} = r_{max}^2$

$$b^2 = \frac{1}{A+B} = r_{min}^2$$



$R_{circular}$: at turning points $E = U_{eff}(r)$

$$E = \frac{1}{2} \hbar r^2 + \frac{M_2^2}{2mr^2}$$

$$\rightarrow E = \frac{1}{2} \hbar a^2 + \frac{M_2^2}{2ma^2} \quad \leftarrow \text{multiply by } a^2$$

$$\rightarrow E = \frac{1}{2} \hbar b^2 + \frac{M_2^2}{2mb^2} \quad \leftarrow \text{multiply by } b^2$$

then subtract

$$E(a^2 - b^2) = \frac{1}{2} \hbar (a^4 - b^4)$$

$$= \frac{1}{2} \hbar (a^2 - b^2)(a^2 + b^2)$$

$\rightarrow E = \frac{1}{2} \hbar (a^2 + b^2)$

$E = \frac{1}{2} \hbar a^2 + \frac{1}{2} \hbar b^2$

NOTE: $a^2 + b^2 = \frac{1}{A-B} + \frac{1}{A+B}$

$$= \frac{A+B + A-B}{A^2 - B^2}$$

$$= \frac{2A}{A^2 - B^2}$$

$$= \frac{2 \cancel{\hbar} E / \cancel{m_z^2}}{\cancel{\hbar} / \cancel{m_z^2}}$$

$$= \frac{2E}{\hbar}$$

$\rightarrow E = \frac{1}{2} \hbar (a^2 + b^2)$

subtract: $O = \frac{1}{2} \hbar (a^2 - b^2) + \frac{m_z^2}{2m} \left(\frac{1}{a^2} - \frac{1}{b^2} \right)$

$$= \frac{1}{2} \hbar (a^2 - b^2) + \frac{m_z^2}{2ma^2 b^2} (b^2 - a^2)$$

$$= \frac{1}{2} (a^2 - b^2) \left[\hbar - \frac{m_z^2}{ma^2 b^2} \right]$$

$$\rightarrow \boxed{M_z^2 = \hbar m a^2 b^2} \rightarrow \boxed{M_z = \sqrt{\hbar m} ab}$$

NOTE: $\frac{M_z^2}{\hbar m} = a^2 b^2$

$$= \frac{1}{A-B} \cdot \frac{1}{A+B}$$

$$= \frac{1}{A^2 - B^2}$$

$$= \frac{1}{\frac{m\hbar}{M_z^2}}$$

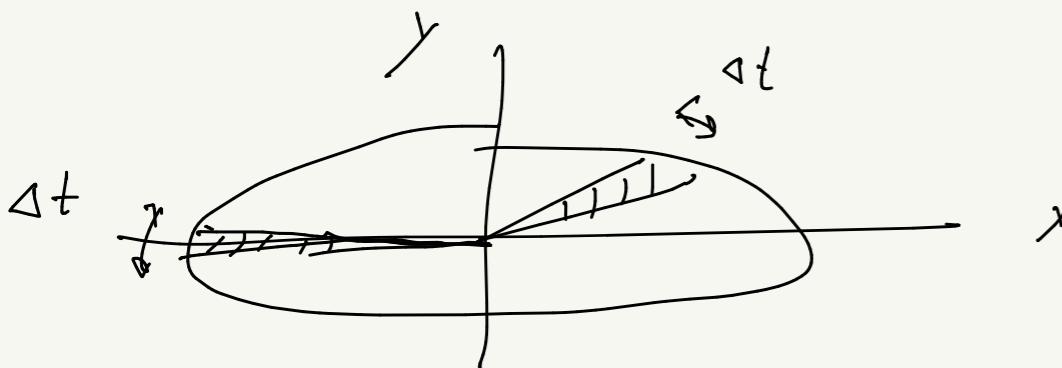
$$= \frac{M_z^2}{m\hbar} \quad \checkmark$$

All central potentials have equal areas in

equal times,

$$dA = \frac{1}{2} r(\tau) d\phi$$

$$dA = \frac{1}{2} r^2 d\phi$$



$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt} = \frac{1}{2} \frac{M_z}{m} = \text{constant}$$

$$M_z = mr^2 \dot{\phi}$$

Integrate :

$$dA = \frac{1}{2} \frac{M_2}{m} dt$$

$$A = \frac{1}{2} \frac{M_2}{m} P$$

$$\pi_{ab} = \frac{1}{2} \frac{M_2}{m} P$$

$$P = 2\pi ab \frac{m}{M_2}$$

$$= 2\pi \frac{M_2}{\sqrt{k/m}} \frac{m}{M_2}$$

$$= \frac{2\pi}{\sqrt{k/m}}$$

$$= \frac{2\pi}{\omega}$$

Thus "Kepler's 3 law" for square oscillator potential :

1) orbits are ellipses with centre of ellipse at origin of the potential

2) radius vector sweeps out equal areas in equal time

3) $P = \text{const} = \frac{2\pi}{\sqrt{k/m}}$ (where $\frac{P^2}{a^3} = \text{const}$)
independent of size of ellipse.

Time dependence of o.b.t

$$t = - \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M_z^2}{m^2 r^2}}} + \text{const}$$

$$E = \frac{1}{2} \hbar^2 (a^2 + b^2), \quad M_z^2 = m \hbar^2 a^2 b^2$$

$$U = \frac{1}{2} \hbar^2 r^2$$

$$\rightarrow \sqrt{ } = \sqrt{\frac{2}{m} \left(\frac{1}{2} \hbar^2 (a^2 + b^2) - \frac{1}{2} \hbar^2 r^2 \right) - \frac{m \hbar^2 a^2 b^2}{m^2 r^2}}$$

$$= \sqrt{\frac{\hbar^2}{m} (a^2 + b^2 - r^2)} - \frac{\hbar^2 a^2 b^2}{m r^2}$$

$$= \sqrt{\frac{\hbar^2}{m}} \sqrt{a^2 + b^2 - r^2 - \frac{a^2 b^2}{r^2}}$$

$$= \sqrt{\frac{\hbar^2}{m}} \left(\frac{1}{r} \right) \sqrt{-r^4 + r^2(a^2 + b^2) - a^2 b^2}$$

$$= \sqrt{\frac{\hbar^2}{m}} \left(\frac{1}{r} \right) \sqrt{-(r^2 - a^2)(r^2 - b^2)}$$

$$x = a \cos \xi \\ y = b \sin \xi$$

$$r^2 = x^2 + y^2 = a^2 \cos^2 \xi + b^2 \sin^2 \xi$$

$$r^2 - a^2 = a^2 (\cos^2 \xi - 1) + b^2 \sin^2 \xi = (b^2 - a^2) \sin^2 \xi$$

$$r^2 - b^2 = a^2 (\cos^2 \xi + b^2 (\sin^2 \xi - 1)) = (a^2 - b^2) \cos^2 \xi$$

Thus,

$$\sqrt{\frac{F}{m}} \left(\frac{1}{r} \right) \sqrt{-[b^2 - a^2] \sin^2 \xi + [a^2 - b^2] \cos^2 \xi}$$

$$= \sqrt{\frac{F}{m}} \left(\frac{1}{r} \right) (a^2 - b^2) \sin \xi \cos \xi$$

$$\rightarrow t = - \int \frac{dr}{\sqrt{\frac{h}{m} \left(\frac{1}{r} \right) \left(a^2 - b^2 \right)}} + \text{const}$$

$$= -\frac{1}{w} \int \frac{r dr}{(a^2 - b^2) \sin^2(\theta) +}$$

$$\text{Now: } r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

$$\begin{aligned} \rightarrow Z r dr &= 2a^2 \cos \xi (-\sin \xi) d\xi + 2b^2 \sin \xi \cos \xi d\xi \\ &= -2(a^2 - b^2) \sin \xi \cos \xi d\xi \end{aligned}$$

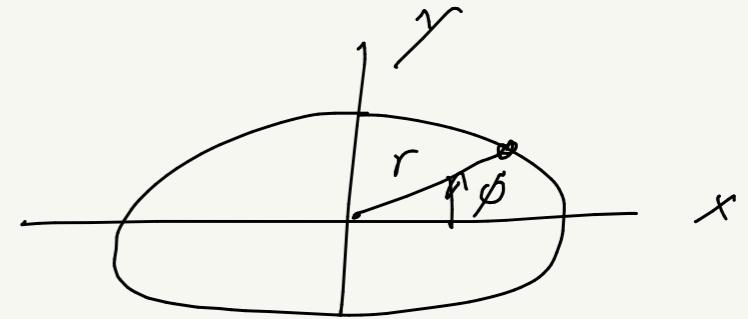
$$\rightarrow r dr = -(a^2 - b^2) \sin z \cos z dz$$

$$\rightarrow t = -\frac{1}{\omega} \int \frac{-\cancel{(a^2+b^2)} \sin \varphi \cos \varphi d\varphi}{\cancel{(a^2+b^2)} \sin \varphi \cos \varphi} + C_1, +$$

$$= -\frac{\xi}{\omega} + \text{const} + \underbrace{\dots}_{L} = 0 \quad \text{for } t=0$$

$$+ h\nu, \boxed{z = wt} \quad , \quad \begin{cases} x = a \cos(wt) \\ y = b \sin(wt) \end{cases}$$

$$\begin{aligned} X &= a \cos(\omega t) &= r \cos \phi \\ Y &= b \sin(\omega t) &= r \sin \phi \end{aligned}$$



$$\begin{aligned} r^2 &= a^2 \cos^2 \xi + b^2 \sin^2 \xi \\ &= a^2 \cos^2(\omega t) + b^2 \sin^2(\omega t) \end{aligned}$$

$$\tan \phi = \frac{y}{x} = \frac{b}{a} \frac{\sin(\omega t)}{\cos(\omega t)} = \frac{b}{a} \tan(\omega t)$$

$$\phi = \tan^{-1} \left[\frac{b}{a} \tan(\omega t) \right]$$

Relation between ϕ and ω :

$$M_z = m r^2 \dot{\phi}$$

$$M_z = \sqrt{m \Gamma} ab = \sqrt{\frac{\Gamma}{m}} m ab = \omega m ab$$

$$\text{Thus, } mr^2 \dot{\phi} = \omega m ab$$

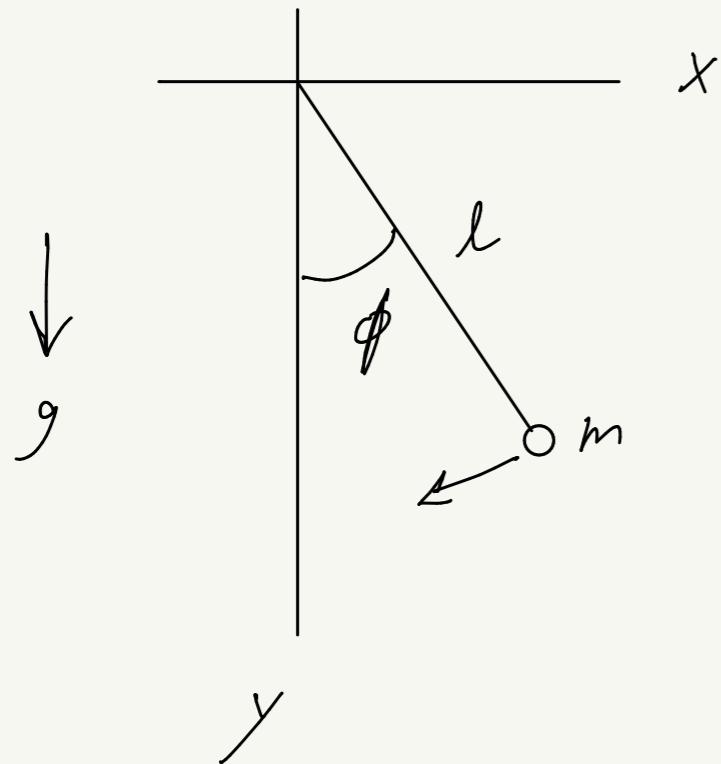
$$\dot{\phi} = \frac{\omega ab}{r^2}$$

Constraint Forces: (simple pendulum)

$$E = T + U$$

$\Delta E = W_{nc}$ = work done by non-conservative forces, i.e. friction

Constraint force does no (virtual) work



$$\phi = r - l = 0$$

$$T = \frac{1}{2} m(r^2 + r^2\dot{\phi}^2)$$

$$U = -mgx$$

$$= -mgy \cos \phi$$

$$L = T - U$$

$$1) \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} + \lambda \frac{\partial \phi}{\partial r}$$

$$2) \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} + \lambda \frac{\partial \phi}{\partial \phi}$$

$$3) \phi = r - l = 0 \rightarrow r = l \Rightarrow \dot{r} = 0, \ddot{r} = 0$$

$$\text{Then, } mr\ddot{\phi} = mr\dot{\phi}^2 + mg \cos \phi + \lambda \quad (1)$$

$$\frac{d}{dt}(mr^2\dot{\phi}) = -mgyr \sin \phi \quad (2)$$

$$\rightarrow 2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} = -mgyr \sin \phi$$

Impose constraint: $r = l, \dot{r} = 0, \ddot{r} = 0$

$$\rightarrow \lambda = -ml\dot{\phi}^2 - mg \cos \phi, \dot{\phi} = -\frac{g}{l} \sin \phi$$

Constraint Force

$$\vec{F}_c = \lambda \vec{\nabla} \varphi \\ = (-m l \dot{\varphi}^2 - mg \cos \varphi) \hat{r}$$

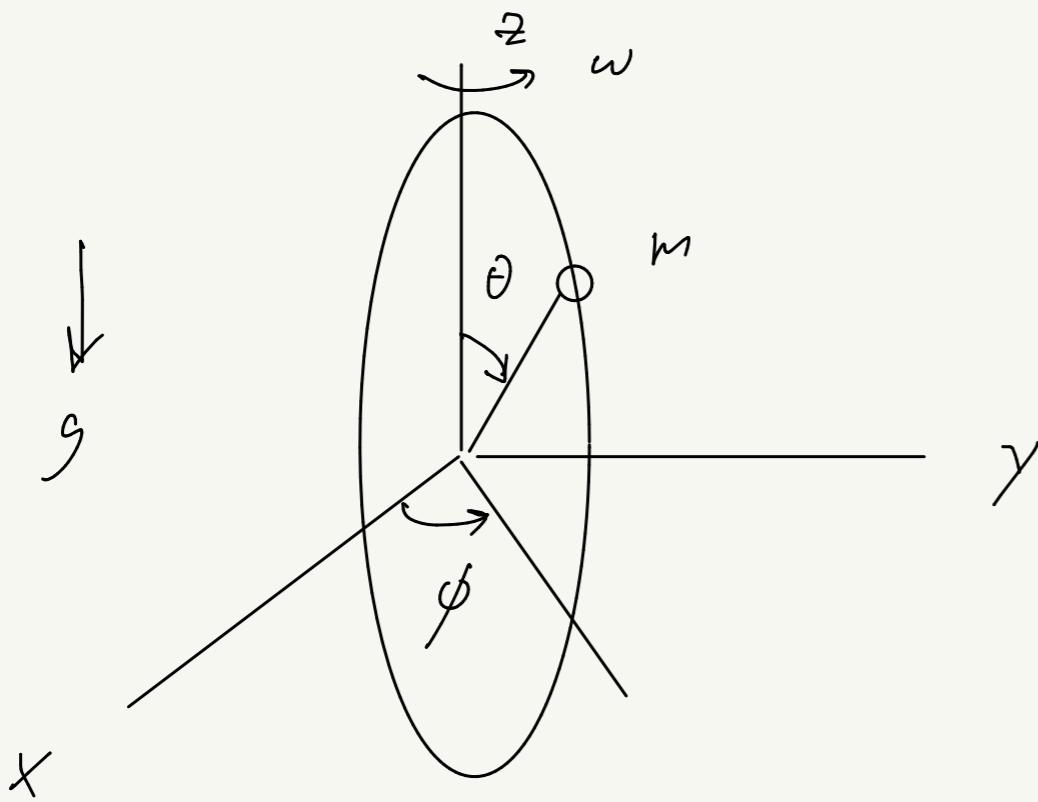
Virtual (and actual) displacement

$$\delta \vec{r} = \lambda \delta \varphi \hat{\phi}$$

$$\rightarrow \vec{F}_c \cdot \vec{\delta r} = (-m l \dot{\varphi}^2 - mg \cos \varphi) \hat{r} \cdot \lambda \delta \varphi \hat{\phi} \\ = 0$$

Since $\hat{r} \perp \hat{\phi}$.

Rotating hoop:



$$\phi = \omega t \quad (\text{specific}) \rightarrow \dot{\phi} = \omega$$

$$r = R \rightarrow \dot{r} = 0$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2)$$

$$= \frac{1}{2} m (R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2 \theta)$$

$$U = mgz$$

$$= mgR \cos \theta$$

$$L = \frac{1}{2} m (R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2 \theta) - mgR \cos \theta$$

no explicit t dependence $\Rightarrow h = \frac{\partial L}{\partial \dot{\theta}} \dot{\theta} - L$

$$= \text{const}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m R^2 \dot{\theta}$$

$$\rightarrow h = m R^2 \dot{\theta}^2 - \frac{1}{2} m R^2 \dot{\theta}^2 - \frac{1}{2} m R^2 \omega^2 \sin^2 \theta + mgR \cos \theta$$

$$= \frac{1}{2} m R^2 \dot{\theta}^2 - \frac{1}{2} m R^2 \omega^2 \sin^2 \theta + mgR \cos \theta$$

Note: $h = \text{const}$ but $h \neq T + U \equiv E$

Determining constraint force:

$$\varphi_1 \equiv r - R = 0$$

$$\varphi_2 \equiv \phi - \omega t = 0$$

$$\begin{aligned}\vec{F}_c &= \lambda_1 \vec{\nabla} \varphi_1 + \lambda_2 \vec{\nabla} \varphi_2 \\ &= \lambda_1 \vec{r} + \lambda_2 \frac{1}{r_{\sin \theta}} \vec{\phi}\end{aligned}$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2)$$

$$U = mg r \cos \theta$$

$$L = T - U$$

$$1) \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} + \lambda_1 \frac{\partial \varphi_1}{\partial r} + \lambda_2 \frac{\partial \varphi_2}{\partial r}$$

$$2) \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta} + \lambda_1 \frac{\partial \varphi_1}{\partial \theta} + \lambda_2 \frac{\partial \varphi_2}{\partial \theta}$$

$$3) \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} + \lambda_1 \frac{\partial \varphi_1}{\partial \phi} + \lambda_2 \frac{\partial \varphi_2}{\partial \phi}$$

$$4) r - R = 0 \rightarrow r = R \rightarrow \dot{r} = 0, \ddot{r} = 0$$

$$5) \phi - \omega t = 0 \rightarrow \dot{\phi} = \omega t \rightarrow \dot{\phi} = \omega, \ddot{\phi} = 0$$

$$\frac{d}{dt} (mr) = mr \dot{\theta}^2 + mr \sin^2 \theta \dot{\phi}^2 - mg r \cos \theta + \lambda_1$$

$\rightarrow \boxed{r \ddot{r} = r \dot{\theta}^2 + r \sin^2 \theta \dot{\phi}^2 - g \cos \theta + \frac{\lambda_1}{m}}$

$$\frac{d}{dt} (mr^2 \dot{\theta}) = mr^2 \sin \theta \cos \theta \dot{\phi}^2 + mg r \sin \theta$$

$2mr \dot{r} \dot{\theta} + mr^2 \ddot{\theta} = mr^2 \sin \theta \cos \theta \dot{\phi}^2 + mg \sin \theta$

$\boxed{2r \dot{r} \dot{\theta} + r^2 \ddot{\theta} = r^2 \sin \theta \cos \theta \dot{\phi}^2 + g \sin \theta}$

$$\frac{d}{dt} (mr^2 \sin^2 \theta \dot{\phi}) = + \lambda_2$$

$2mr \dot{r} \sin^2 \theta \dot{\phi} + 2mr^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi}$

$+ mr^2 \sin^2 \theta \ddot{\phi} = \lambda_2$

$$U_{1c} \quad \dot{r} = 0, \quad \ddot{r} = 0, \quad r = R, \quad \dot{\phi} = \omega t, \quad \ddot{\phi} = \omega, \quad \dddot{\phi} = 0$$

$$\rightarrow \dot{\theta} = R \dot{\theta}^2 + R \sin^2 \theta \omega^2 - g \cos \theta + \frac{\lambda_1}{R}$$

$$\dot{\theta} + R^2 \ddot{\theta} = R^2 \omega^2 \sin \theta \cos \theta + g \sin \theta$$

$$\dot{\theta} + 2m R^2 \sin \theta \cos \theta \dot{\theta} \omega + 0 = \lambda_2$$

$$\rightarrow \boxed{\lambda_2 = 2m R^2 \sin \theta \cos \theta \dot{\theta}}$$

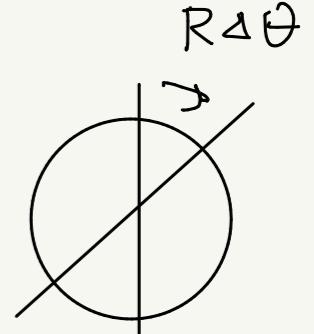
$$\boxed{\lambda_1 = -m R^2 \dot{\theta}^2 - m R^2 \sin^2 \theta \omega^2 + mg \cos \theta}$$

$$\vec{F}_c = \lambda_1 \vec{r} + \lambda_2 \frac{1}{R_{\sin \theta}} \hat{\phi}$$

$$= (-m R^2 \dot{\theta}^2 - m R^2 \sin^2 \theta \omega^2 + mg \cos \theta) \vec{r}$$

$$+ 2m R \omega \dot{\theta} \cos \theta \hat{\phi}$$

Virtual displacement: (constant time)



$$\delta \vec{r} = R \delta \theta \hat{\theta}$$

$$\rightarrow \vec{F}_c \cdot \delta \vec{r} = 0$$

Actual displacement:

$$\vec{r} = \underbrace{\delta \theta}_{\vec{r}} \vec{r} + R \sin \theta \hat{\theta} + R \cos \theta \sin \theta \hat{\phi}$$

$$= R \dot{\theta} \delta t \vec{\theta} + R \sin \theta \omega \delta t \vec{\phi}$$

$$= \delta t (\dot{\theta} \vec{\theta} + R \omega \sin \theta \vec{\phi})$$

$$\vec{F}_c \cdot \delta \vec{r} = \delta t Z m R \omega \dot{\theta} \cos \theta \omega \sin \theta$$

$$= \delta t Z m R^2 \omega^2 \sin \theta \cos \theta$$

$$= \delta [m R^2 \omega^2 \sin^2 \theta]$$

Thus,

$$\vec{F}_c = - \frac{\partial U_c}{\partial r}, \quad U_c = - m R^2 \omega^2 \sin^2 \theta$$

$$W_c = \Delta T + \Delta U = \Delta E$$

$$E = \frac{1}{2} m (R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2 \theta) + mg R \cos \theta$$

$$\Delta E = \frac{1}{2} m (R^2 (\dot{\theta}_2^2 - \dot{\theta}_1^2) + R^2 \omega^2 (\sin^2 \theta_2 - \sin^2 \theta_1)) \\ + mg R (\cos \theta_2 - \cos \theta_1)$$

$$W_c = m R^2 \omega^2 (\sin^2 \theta_2 - \sin^2 \theta_1) \\ = -\Delta U_c$$

Thus, $\boxed{\Delta = \Delta T + \Delta U + \Delta U_c = \Delta h}$

where $h = T + U + U_c$

$$= \frac{1}{2} m (R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2 \theta) \\ + mg R \cos \theta - m R^2 \omega^2 \sin^2 \theta$$

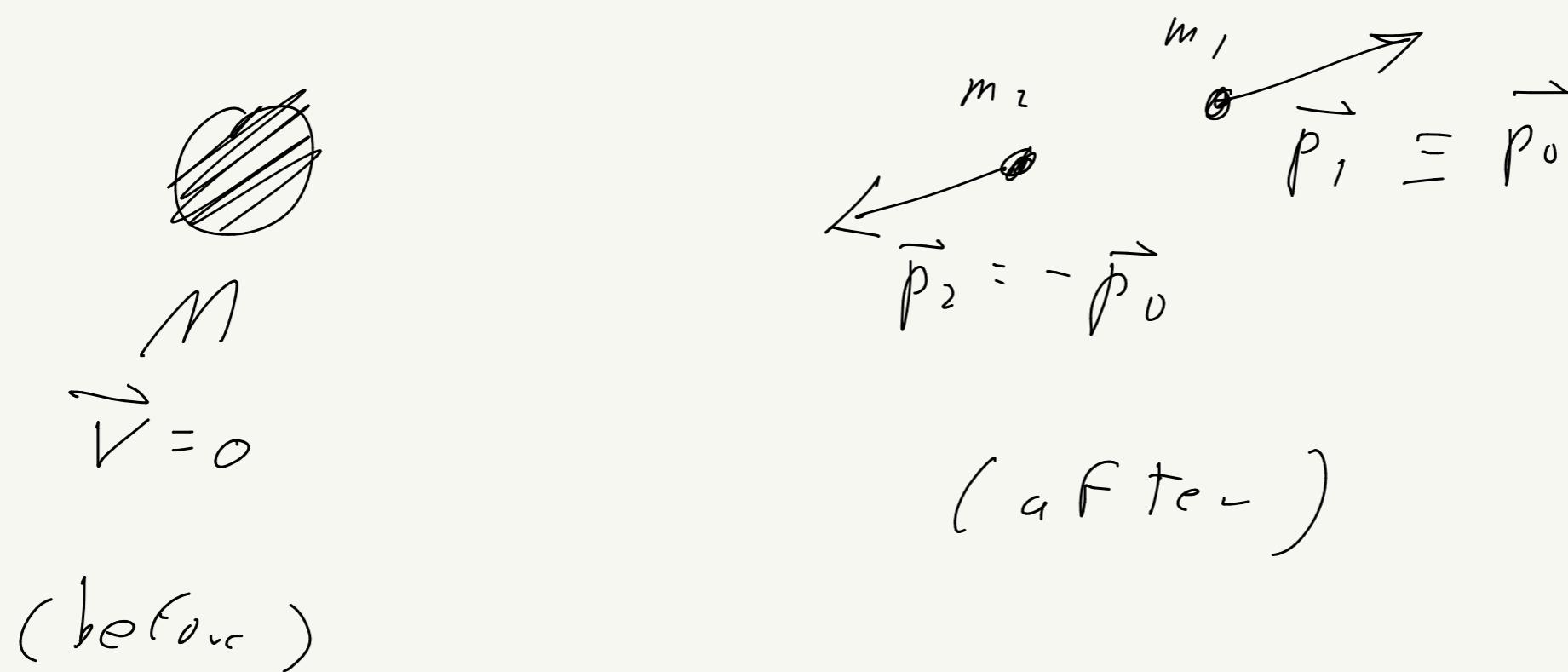
$$= \frac{1}{2} m (R^2 \dot{\theta}^2 - R^2 \omega^2 \sin^2 \theta) + mg R \cos \theta$$

Collisions / Scattering

Start with spontaneous disintegration of a single particle of mass M into two particles with mass m_1, m_2 .

Closed system: \rightarrow energy, momentum, angular momentum conserved

\rightarrow COM Frame



Cons. of energy.

$$\begin{aligned} E_i &= E_{1i} + T_{10} + E_{2i} + T_{20} \\ &= E_{1i} + E_{2i} + \frac{p_0^2}{2m_1} + \frac{p_0^2}{2m_2} \end{aligned}$$

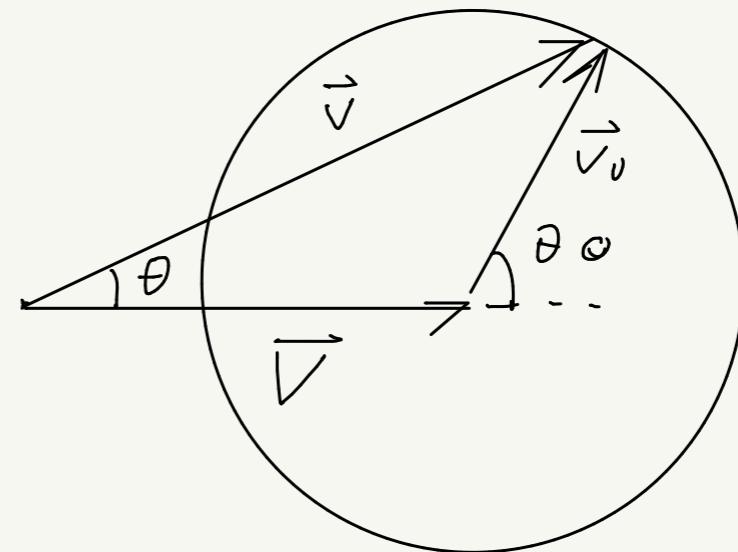
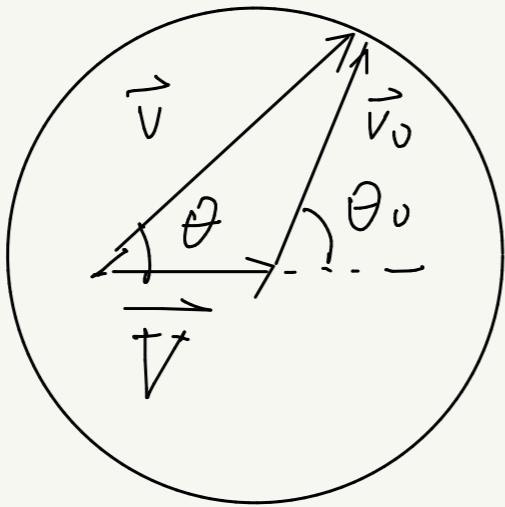
$$\begin{aligned} \underbrace{E_i - E_{1i} - E_{2i}}_{\text{dissipation energy}} &= \frac{p_0^2}{2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \quad \text{reduced mass} \\ &\rightarrow E = \frac{p_0^2}{2m}, \quad m = \frac{m_1 m_2}{m_1 + m_2} \end{aligned}$$

$$S_0, \quad p_0 = \sqrt{2mE}$$

$$\rightarrow v_1 = \frac{p_0}{m_1}, \quad v_2 = \frac{p_0}{m_2}$$

wavy line

Lab Frame:



\vec{v} : w.r.t Lab Frame (for m_1 or m_2)

\vec{V}_0 : w.r.t Com Frame (for m_1 or m_2)

\vec{V} : Velocity of Com (velocity of mass, M
w.r.t Lab)

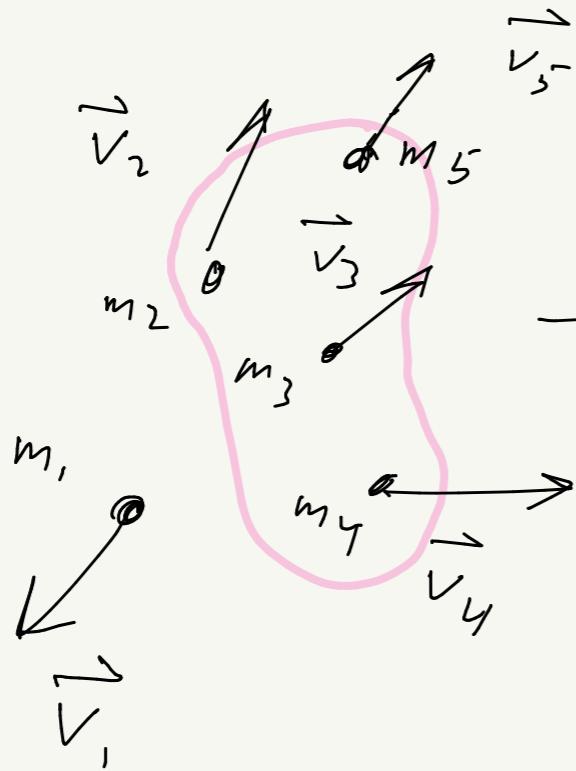
$$\rightarrow \boxed{V_0^2 = v^2 + V^2 - 2vV_{10} \cos \theta}$$

$$v \sin \theta = V_0 \sin \theta_0$$

$$v \cos \theta = V_0 \cos \theta_0 + V$$

$$\rightarrow \boxed{\tan \theta = \frac{V_0 \sin \theta_0}{V_0 \cos \theta_0 + V}}$$

Disintegration into massless two particles



Center of COM
of the
system of particles
 m_2, m_3, \dots, m_N

COM Frame:

$$\vec{p}_0 = m_1 \vec{v}_1 \quad , \quad -\vec{p}_0 = m_2 \vec{v}_2 + m_3 \vec{v}_3 + \dots + m_N \vec{v}_N$$

$$E_i' = E_{i,i} + \frac{\vec{p}_0^2}{2m_1} + E_{i,i}' + \frac{\vec{p}_0^2}{2(M-m_1)}$$

$$E_i - E_{i,i} - E_{i,i}' = \frac{\vec{p}_0^2}{2} \left(\frac{1}{m_1} + \frac{1}{M-m_1} \right)$$

$$= \frac{\vec{p}_0^2}{2} \frac{M}{m_1(M-m_1)}$$

$$\rightarrow \frac{\vec{p}_0^2}{2m_1} = \left(\frac{M-m_1}{M} \right) (E_i - E_{i,i} - E_{i,i}')$$

This is maximized when $E_{i,i}'$ is minimized

Now:

$$E_i' + \frac{p_0^2}{2(M-m_1)} = E_{2i} + \frac{1}{2} m_2 |\vec{v}_2|^2 + E_{3i} + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots$$

T^{ho} , $E_i' = E_{2i} + E_{3i} + \dots + \frac{1}{2} m_1 |\vec{v}_1|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots - \frac{1}{2} \frac{|\vec{m}_2 \vec{v}_2 + \vec{m}_3 \vec{v}_3 + \dots|^2}{(m_2 + m_3 + \dots)}$

Now: $\frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots = T'$
 $\approx \text{KE of } m_2, m_3, \dots$
 $wrt \text{ original frame}$

$$\frac{1}{2} \frac{|\vec{m}_2 \vec{v}_2 + \vec{m}_3 \vec{v}_3 + \dots|^2}{(m_2 + m_3 + \dots)} = \frac{1}{2} \frac{|\vec{p}_0|^2}{(m_2 + m_3 + \dots)} = T_{com}'$$

$= \text{KE at COM of } m_2, m_3, \dots$
 $wrt \text{ original frame}$

T^{ho} , $T' - T_{com}' = T_0'$
 $= \text{KE of } m_2, m_3, \dots wrt \text{ COM of } m_2, m_3, \dots$

\geq_0

$$\boxed{E_i' = E_{2i} + E_{3i} + \dots + T_0'}$$

$E_i' = \min \text{ when } T_0' = 0 \rightarrow m_2, m_3, \dots \text{ move with the same velocity}$

Elastic collision of two particles:

i) Elastic: ignore internal energies,

$$\rightarrow E = \text{const} \Leftrightarrow T = \text{const}$$

ii) Momentum and angular momentum also conserved

\rightarrow motions in a plane (2-d)

4 D.O.F.: final velocities of m_1, m_2
(magnitude, direction in 2-d)

Constraints: $T = \text{const} \rightarrow 1$

$\vec{P} = \text{const} \rightarrow 2$ (in 2-d)

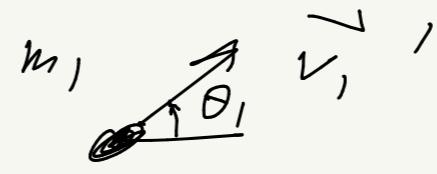
so 3 equations but 4 unknowns.

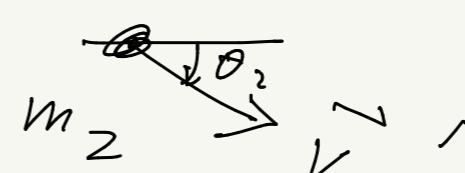
Example:

m_1, m_2 (initially at rest)

$$(\vec{v}_2 = 0)$$

m_1  m_2 (before)

 (θ_1)

 (θ_2)

$$\vec{P} = m_1 \vec{v}_1 \hat{x}$$

$$= m_1 \vec{v}_1' + m_2 \vec{v}_2'$$

$$= (m_1 v_1' \cos \theta_1 + m_2 v_2' \cos \theta_2) \hat{x} + (m_1 v_1' \sin \theta_1 - m_2 v_2' \sin \theta_2) \hat{y}$$

$$\left. \begin{array}{l} \text{So} \\ m_1 v_1 = m_1 v_1' \cos \theta_1 + m_2 v_2' \cos \theta_2 \\ 0 = m_1 v_1' \sin \theta_1 - m_2 v_2' \sin \theta_2 \end{array} \right\}$$

$$A \mid_{J_0} T = \frac{1}{2} m_1 |\vec{v}_1|^2$$

$$= \frac{1}{2} m_1 |\vec{v}_1'|^2 + \frac{1}{2} m_2 |\vec{v}_2'|^2$$

$$\rightarrow \left. m_1 v_1^2 = m_1 v_1'^2 + m_2 v_2'^2 \right\}$$

Numerical example:

$$m_1 = 1 \text{ kg}$$

$$v_1 = 1 \text{ m/s}$$

$$m_2 = 2 \text{ kg}$$

$$v_2 = 0$$

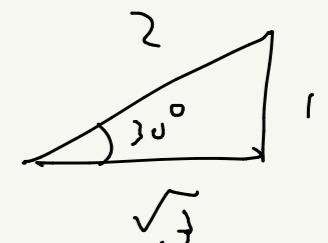
$$v_1' = ?? \quad \theta_1 = ??$$

$$v_2' = ?? \quad \theta_2 = 60^\circ$$

Use above \Rightarrow equations:

$$1 = v_1' \cos \theta_1 + 2 v_2' \underbrace{\cos 60^\circ}_{\frac{1}{2}}$$

$$\rightarrow \left. 1 = v_1' \cos \theta_1 + v_2' \right\}$$



$$0 = v_1' \sin \theta_1 - 2 v_2' \underbrace{\sin 60^\circ}_{\frac{\sqrt{3}}{2}}$$

$$\left. 0 = v_1' \sin \theta_1 - \sqrt{3} v_2' \right\} \frac{\sqrt{3}}{2}$$

$$\left. 1 - v_1'^2 = 2 v_2'^2 \right\}$$

$$\text{Now: } v_1' \sin \theta_1 = \sqrt{3} v_2'$$

$$v_1' \cos \theta_1 = 1 - v_2'$$

$$\rightarrow v_1'^2 \sin^2 \theta_1 + v_1'^2 \cos^2 \theta_1 = 3 v_2'^2 + 1 - 2v_2' + v_2'^2$$

$$v_1'^2 = 4v_2'^2 - 2v_2' + 1$$

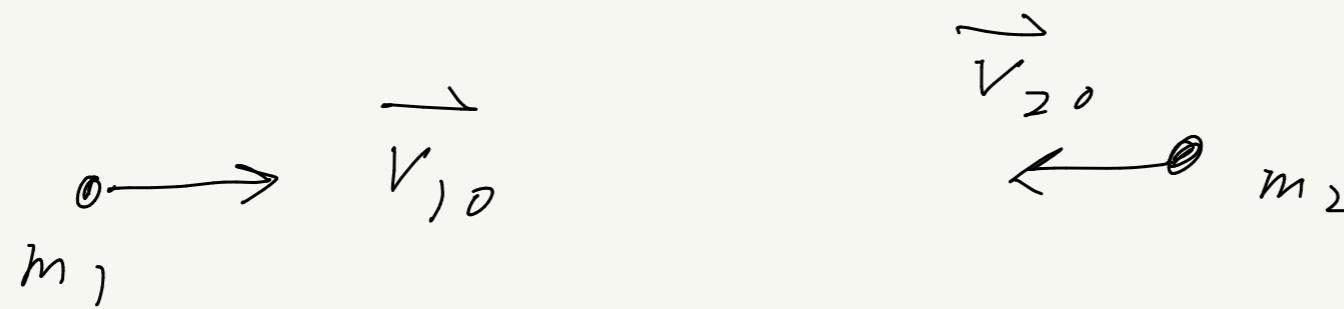
$$\text{Also: } v_1'^2 = 1 - 2v_2'^2$$

$$\begin{aligned} \text{subtract: } 0 &= 6v_2'^2 - 2v_2' \\ &= 2v_2' (3v_2' - 1) \end{aligned} \rightarrow \boxed{v_2' = \frac{1}{3} \text{ m/s}}$$

$$\rightarrow v_1'^2 = 1 - \frac{2}{9} = \frac{7}{9} \rightarrow \boxed{v_1' = \frac{\sqrt{7}}{3} \text{ m/s}}$$

$$\text{Finally: } \sin \theta_1 = \frac{\sqrt{3} v_2'}{v_1'} = \frac{\sqrt{3} \frac{1}{3}}{\frac{\sqrt{7}}{3}} = \frac{\sqrt{3}}{\sqrt{7}} \rightarrow \boxed{\theta_1 = 41^\circ}$$

Simpler to analyze in COM Frame:



$$\vec{P}_0 = 0 = m_1 \vec{v}_{10} + m_2 \vec{v}_{20}$$

$$\vec{V} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} \quad (\text{velocity of COM wrt lab frame})$$

$$\begin{aligned} \text{Relative position vs. time: } \vec{r} &= \vec{r}_{10} - \vec{r}_{20} \quad (= \vec{r}_1 - \vec{r}_2) \\ \text{velocity vs. time: } \vec{V} &= \vec{v}_{10} - \vec{v}_{20} \quad (= \vec{V}_1 - \vec{V}_2) \end{aligned}$$

Recall:

$$\vec{r}_{10} = \frac{m_2}{m_1 + m_2} \vec{r}$$

$$\vec{r}_{20} = \frac{-m_1}{m_1 + m_2} \vec{r}$$

$$\rightarrow \vec{v}_{10} = \frac{m_2}{m_1 + m_2} \vec{v}, \quad \vec{v}_{20} = \frac{-m_1}{m_1 + m_2} \vec{v}$$

$$T = \frac{1}{2} m_1 |\vec{v}_{10}|^2 + \frac{1}{2} m_2 |\vec{v}_{20}|^2$$

$$= \frac{1}{2} m |\vec{v}|^2, \quad m = \frac{m_1 m_2}{m_1 + m_2}$$

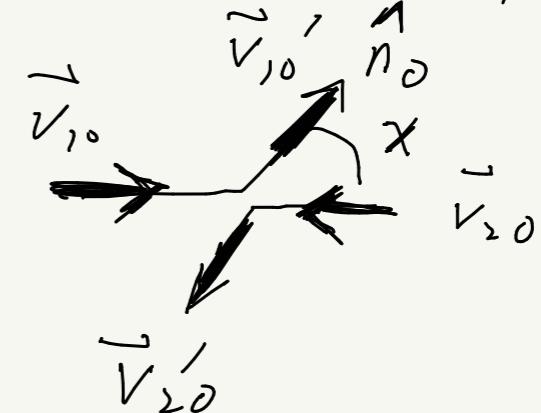
$$T' = \frac{1}{2} m |\vec{v}'|^2$$

Exact: $T = T' \rightarrow v = v'$

$$\rightarrow \vec{v}_{10} = \vec{v}'_{10}$$

$$\rightarrow \vec{v}_{20} = \vec{v}'_{20}$$

The relative velocity vector is unchanged in magnitude \Rightarrow magnitudes of individual velocities are unchanged.



$$\vec{v}'_{10} = \frac{m_2}{m_1 + m_2} \vec{v}$$

$$\vec{v}'_{20} = \frac{-m_1}{m_1 + m_2} \vec{v}$$

$$\text{In } \text{I}_{AB} \text{ frame}$$

$$\vec{v}_1' = \vec{V} + \vec{v}_{10}'$$

$$\vec{v}_2' = \vec{V} + \vec{v}_{20}'$$

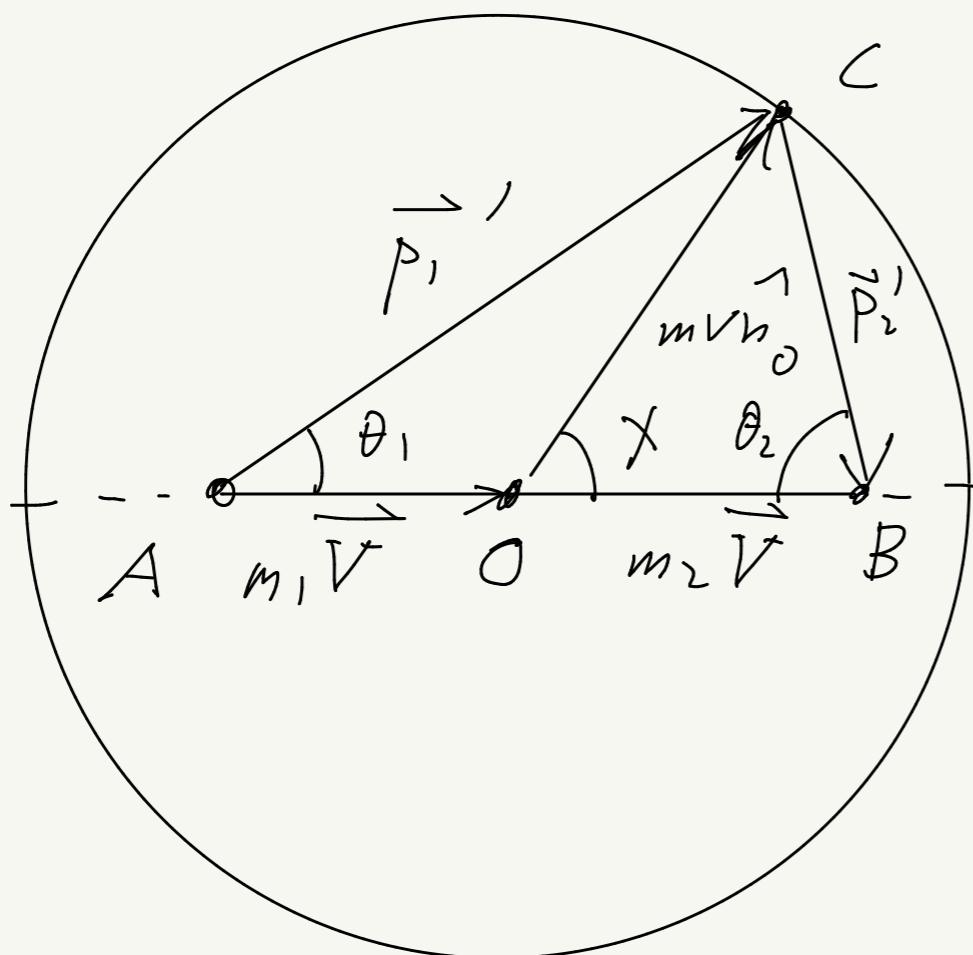
$$T_{h01}, \quad \vec{v}_1' = \left(\frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} \right) + \left(\frac{m_2}{m_1 + m_2} \right) \hat{n}_0' V$$

$$\vec{v}_2' = \left(\frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} \right) - \left(\frac{m_1}{m_1 + m_2} \right) \hat{n}_0' V$$

Convert to momentum:

$$\vec{p}_1' = m_1 \vec{v}_1' = m_1 \vec{V} + m \hat{n}_0' V$$

$$\vec{p}_2' = m_2 \vec{v}_2' = m_2 \vec{V} - m \hat{n}_0' V$$



$$AB: \vec{P} = (m_1 + m_2) \vec{V}$$

$$AO: m_1 \vec{V}$$

$$BO: m_2 \vec{V}$$

$$\text{Suppose } V_2 = 0$$

$$\text{Then } \vec{OB} = m_2 \left(\frac{m_1 \vec{V}_1 + \cancel{m_2 \vec{V}_2}}{m_1 + m_2} \right)$$

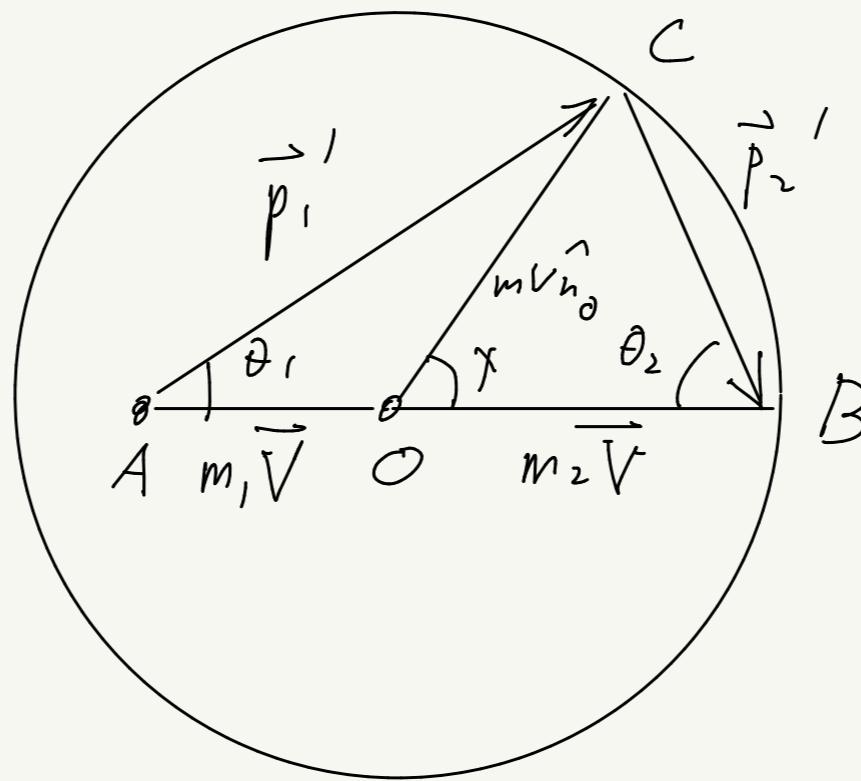
$$= m \vec{V}_1$$

$$= m \vec{V}$$

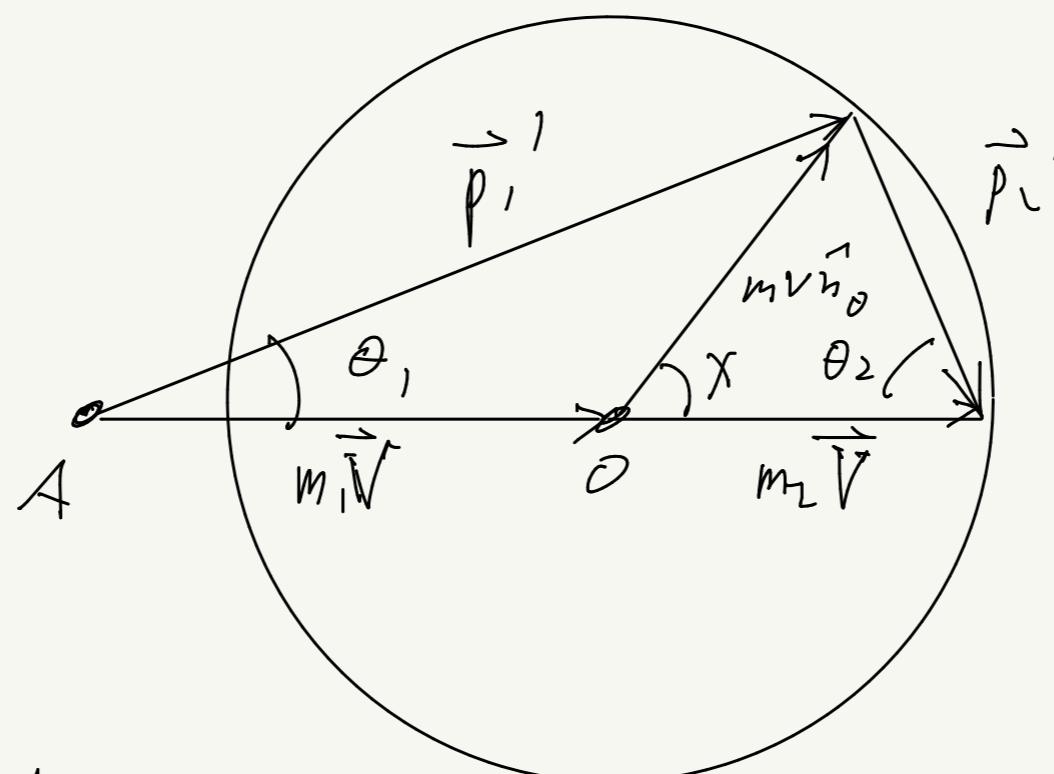
$$|\vec{OB}| = |\vec{OC}|$$

$\int F_{\theta} d\vec{V}_2 = 0$:

$$m_1 < m_2$$



$$m_1 > m_2$$



Relate angles:

$$\chi + 2\theta_2 = \pi \rightarrow \boxed{\theta_2 = \frac{\pi - \chi}{2}}$$

$$\tan \theta_1 = \frac{m_1 V \sin \chi}{m_1 V + m_2 V \cos \chi}$$

$$= \frac{\left(\frac{m_1 m_2}{m_1 + m_2} \right) \times \sin \chi}{m_1 \left(\frac{m_1 \cancel{\chi}}{m_1 + m_2} \right) + \frac{m_1 m_2 \cancel{\chi}}{m_1 + m_2} \cos \chi}$$

$$= \boxed{\frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi}} = \boxed{\tan \theta_1}$$

(17.4)

Suppose $m_1 = m_2 : (b_1 \parallel_{\text{red}} b_2 \parallel_{\text{r}})$

$$\tan \theta_1 = \frac{\sin x}{1 + \cos x}$$

$$= \frac{\cancel{\sin}(\frac{x}{2}) \cancel{\cos}(\frac{x}{2})}{\cancel{\sin^2}(\frac{x}{2})}$$

$$= \tan(\frac{x}{2})$$

$$\tan \theta_1 = \frac{x}{2}$$

$$\text{Also } \theta_2 = \frac{\pi}{2} - \frac{x}{2}$$

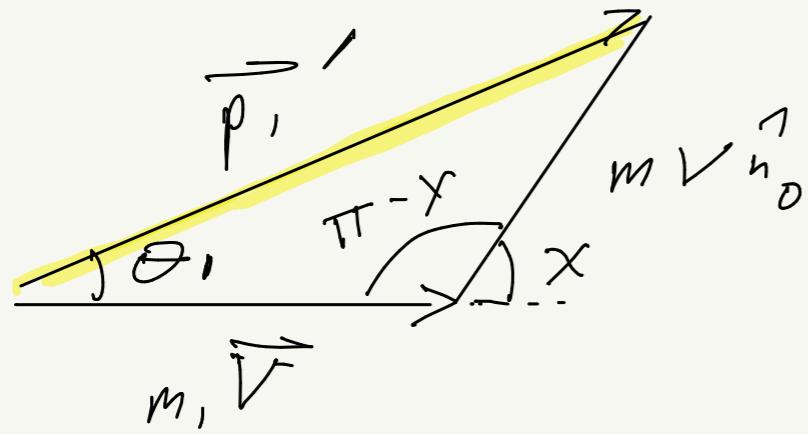
$$\rightarrow \theta_1 + \theta_2 = \frac{\pi}{2}$$

Now:

$$\begin{aligned} \cos(x) &= \cos(2 \frac{x}{2}) \\ &= \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2}) \\ &= 2\cos^2(\frac{x}{2}) - 1 \\ 1 + \cos x &= 2\cos^2(\frac{x}{2}) \\ 1 - \cos x &= 2\sin^2(\frac{x}{2}) \end{aligned}$$



Relate velocities:



$$\vec{v} = \vec{v}_1 \quad (\text{since } \vec{v}_2 = 0)$$

$$\text{Thru, } m_1^2 v_1'^2 = m_1^2 V^2 + m_2^2 v^2 - 2 m_1 V m_2 v \cos(\pi - x)$$

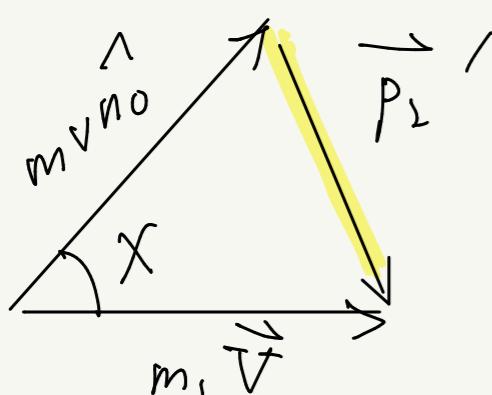
$$\rightarrow m_1^2 v_1'^2 = m_1^2 \left(\frac{m_1 V}{m_1 + m_2} \right)^2 + \left(\frac{m_1 m_2}{m_1 + m_2} \right)^2 v^2$$

$$+ 2 m_1 \left(\frac{m_1 V}{m_1 + m_2} \right) \left(\frac{m_1 m_2}{m_1 + m_2} \right) v \cos x$$

$$v_1' = \sqrt{\left(\frac{V}{m_1 + m_2} \right)^2 m_1^2 + m_2^2 + 2 m_1 m_2 \cos x}$$

Also:

$$m_2^2 v_2'^2 = 2 m_1^2 V^2 - 2 m_1^2 v^2 \cos x$$



$$= \frac{2 m_1^2 m_2^2}{(m_1 + m_2)^2} V^2 (1 - \cos x)$$

$$v_2' = \left(\frac{2 m_1 V}{m_1 + m_2} \right) \sin\left(\frac{x}{2}\right) \quad (17.5)$$

Resolve example problem:

$$m_1 = 1 \text{ kg}, \quad m_2 = 2 \text{ kg}, \quad v_1 = 1 \text{ m/s}, \quad v_2 = 0, \quad \theta_2 = 60^\circ$$

Find v_1' , v_2' , θ_1

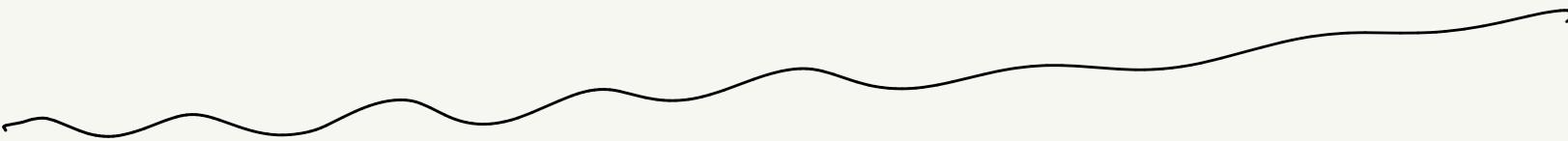
$$\chi + 2\theta_2 = \pi \rightarrow \chi = \pi - 2\left(\frac{\pi}{3}\right) = \frac{\pi}{3}$$

so $\boxed{\chi = 60^\circ}$

$$v = v_1 = 1 \text{ m/s}$$

$$v_2' = \left(\frac{2m_1 v}{m_1 + m_2} \right) \cos\left(\frac{\chi}{2}\right) = \frac{2}{3} \cos 30^\circ = \boxed{\frac{1}{3}} \text{ m/s}$$

$$v_1' = \left(\frac{v}{m_1 + m_2} \right) \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos \chi}$$
$$= \frac{1}{3} \sqrt{1 + 4 + 2 \cdot \cancel{2} \underbrace{\cos 60^\circ}_{\frac{1}{2}}}$$
$$= \boxed{\frac{\sqrt{7}}{3}} \text{ m/s}$$



Revisit original equations:

$$m_1 v_1 = m_1 v_1' \cos \theta_1 + m_2 v_2' \cos \theta_2 \quad (1)$$

$$0 = m_1 v_1' \sin \theta_1 - m_2 v_2' \sin \theta_2 \quad (2)$$

$$m_1 v_1^2 = m_1 v_1'^2 + m_2 v_2'^2 \quad (3)$$

$$\rightarrow m_1 v_1' \sin \theta_1 = m_2 v_2' \sin \theta_2$$

$$m_1 v_1' \cos \theta_1 = m_1 v_1 - m_2 v_2' \cos \theta_2$$

Square and add:

$$\begin{aligned} m_1^2 v_1'^2 &= m_2^2 v_2'^2 \sin^2 \theta_2 + m_1^2 v_1^2 + m_2^2 v_2'^2 \cos^2 \theta_2 \\ &\quad - 2 m_1 m_2 v_1 v_2' \cos \theta_2 \\ &= m_1^2 v_1^2 + m_2^2 v_2'^2 - 2 m_1 m_2 v_1 v_2' \cos \theta_2 \end{aligned}$$

Also:

$$m_1^2 v_1'^2 = m_1^2 v_1^2 - m_1 m_2 v_2'^2$$

$$\begin{aligned} \rightarrow 0 &= (m_2^2 + m_1 m_2) v_2'^2 - 2 m_1 m_2 v_1 v_2' \cos \theta_2 \\ &= m_2 (m_1 + m_2) v_2'^2 - 2 m_1 m_2 v_1 v_2' \cos \theta_2 \\ &= v_2'^2 - 2 \left(\frac{m_1}{m_1 + m_2} \right) v_2' v_1 \cos \theta_2 \\ &= v_2' \left[v_2' - 2 \left(\frac{m_1}{m_1 + m_2} \right) v_1 \cos \theta_2 \right] \end{aligned}$$

$$Th_{v_1}, \quad \boxed{v_2' = \frac{2m_1 v_1}{m_1 + m_2} \cos \theta_2}$$

$$v_{1,y} \quad m_1 v_1'^2 = m_1 v_1'^2 + m_2 v_2'^2$$

$$\rightarrow v_1'^2 = v_1^2 - \frac{m_2}{m_1} v_2'^2$$

$$= v_1^2 - \frac{m_2}{m_1} \frac{4m_1^2 v_1^2}{(m_1 + m_2)^2} \cos^2 \theta_2$$

$$= \frac{v_1^2}{(m_1 + m_2)} \left[(m_1 + m_2)^2 - 4m_1 m_2 \cos^2 \theta_2 \right]$$

$$Th_{v_1}, \quad \boxed{v_1' = \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \left(1 - 2\cos^2 \theta_2 \right)}}$$

Note:

$$\text{If we write } \theta_2 = \frac{\pi}{2} - \frac{\chi}{2} \quad , \quad \text{then}$$

$$\cos \theta_2 = \cos \left(\frac{\pi}{2} - \frac{\chi}{2} \right) = \cancel{\cos \frac{\pi}{2}} \cos \frac{\chi}{2} + \sin \frac{\pi}{2} \sin \frac{\chi}{2} = \sin \frac{\chi}{2}$$

$$1 - 2\cos^2 \theta_2 = 1 - 2 \sin^2 \left(\frac{\chi}{2} \right) = \cos \chi$$

$$\cos 2\theta = 1 - 2\sin^2 \theta$$

$$2\sin^2 \theta = 1 - \cos 2\theta$$

Expression for θ_1 :

$$m_1 v_1' \sin \theta_1 = m_2 v_2' \sin \theta_2$$

$$m_1 v_1' \cos \theta_1 = m_1 v_1 - m_2 v_2' \cos \theta_2$$

Divide:

$$\tan \theta_1 = \frac{m_2 v_2' \sin \theta_2}{m_1 v_1 - m_2 v_2' \cos \theta_2}$$

$$= \frac{m_2 \left(\frac{2m_1 v_1}{m_1 + m_2} \right) \cos \theta_2 \sin \theta_2}{\cancel{m_1 v_1} - m_2 \left(\frac{\cancel{2m_1 v_1}}{m_1 + m_2} \right) \cos^2 \theta_2}$$

$$= \boxed{\frac{m_2 \sin(2\theta_2)}{m_1 + m_2 (1 - \cos^2 \theta_2)}}$$

$$\text{For } \theta_2 = \frac{\pi}{2} - \frac{x}{2} :$$

$$\sin(2\theta_2) = \sin(\pi - x) = \sin \pi \cos x - \cos \pi \sin x = \sin x$$

$$1 - \cos^2 \theta_2 = \cos x$$