

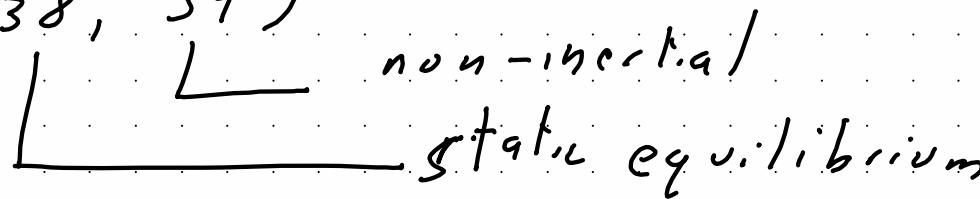
Lec #23: Tuesday Nov 10th

- Quiz #5: Thursday

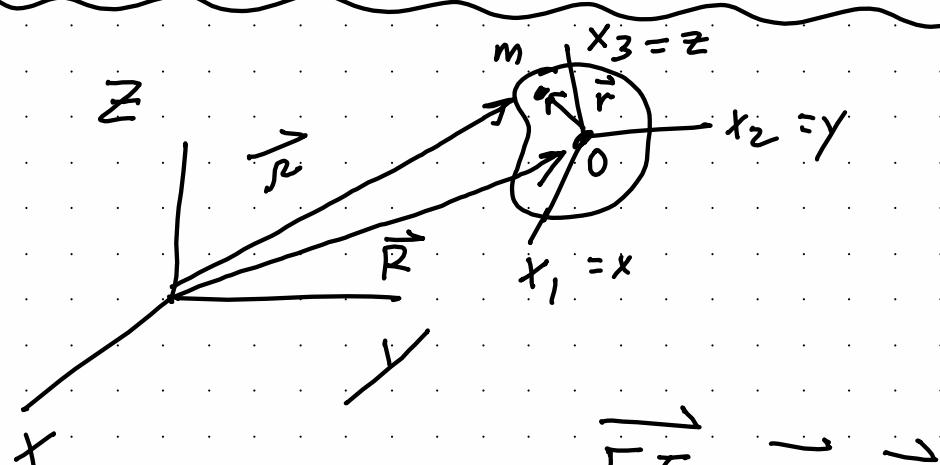
- Midterm #2: Next Thursday 11/19

- Today: Rigid body motion

(Sec 31-36, 38, 39)



O: at COM (usually)
 x_1, x_2, x_3 : fixed in RB



$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\Omega} \times \vec{r} + \dot{\vec{R}}$$

$\vec{\Omega}$: angular velocity vector

$$\vec{\Omega} = \frac{d\vec{\phi}}{dt}$$

$(\vec{R}, \vec{\phi})$: 6 DOF = q_i

$$T = \frac{1}{2} m v^2$$

$$\vec{p} = m \vec{v}$$

rotational quant. t.

$$T_{\text{rot}} = \frac{1}{2} I \omega^2$$

$$\overrightarrow{M} = I \vec{\omega} \rightarrow M_i = \sum I_{ij} \omega_j$$

I: moment of inertia

→ I_{ij} : inertia tensor

1	2	3
2		
3		

$$M_i = I_{ij} \omega_j$$

KE:

$$T = \frac{1}{2} \sum_a m_a |\vec{v}_a|^2$$

$$= \frac{1}{2} \sum_a m_a \left| \vec{V} + \vec{\omega} \times \vec{r}_a \right|^2$$

$$= \frac{1}{2} \sum_a m_a \left(|\vec{V}|^2 + |\vec{\omega} \times \vec{r}_a|^2 + 2 \vec{V} \cdot (\vec{\omega} \times \vec{r}_a) \right)$$

$$\textcircled{1} = \frac{1}{2} \sum_a m_a |\vec{V}|^2 = \boxed{\frac{1}{2} \mu V^2}$$

total mass

$$\textcircled{3} = \sum_a m_a \vec{V} \cdot (\vec{\omega} \times \vec{r}_a) = \vec{A} \cdot (\vec{B} \times \vec{C})$$

$$= \left(\sum_a m_a \vec{r}_a \right) \cdot (\vec{V} \times \vec{\omega})$$

$$= \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$= \mu \vec{R}_{com} \cdot (\vec{V} \times \vec{\omega})$$

$$= \vec{C} \cdot (\vec{A} \times \vec{B})$$

$\underbrace{\quad}_{=0}$ for

$\vec{0}$ at com

$$\vec{v}_a = \vec{V} + \vec{\omega} \times \vec{r}_a$$

$$\begin{aligned} & \left| \vec{A} + \vec{B} \right|^2 \\ &= A^2 + B^2 + 2 \vec{A} \cdot \vec{B} \end{aligned}$$

$$\begin{aligned}
 (2) &= \frac{1}{2} \sum_a m_a |\vec{\omega} \times \vec{r}_a|^2 \\
 &= \frac{1}{2} \sum_a m_a (\vec{\omega} \times \vec{r}_a) \cdot (\vec{\omega} \times \vec{r}_a) \\
 &= \frac{1}{2} \sum_a m_a \vec{\omega} \cdot (\vec{r}_a \times (\vec{\omega} \times \vec{r}_a)) \\
 &= \frac{1}{2} \sum_a m_a \vec{\omega} \cdot (\vec{\omega} r_a^2 - \vec{r}_a (\vec{r}_a \cdot \vec{\omega})) \\
 &= \frac{1}{2} \sum_a m_a (\vec{\omega}^2 r_a^2 - (\vec{\omega} \cdot \vec{r}_a)^2) \\
 &= \frac{1}{2} \sum_a m_a \left(\sum_{i,j} \vec{\omega}_i \vec{\omega}_j \delta_{ij} r_a^2 - \sum_{i,j} \vec{\omega}_i r_{ai} \vec{\omega}_j r_{aj} \right) \\
 &= \frac{1}{2} \sum_{i,j} \left(\sum_a m_a (\vec{\omega}_i^2 \delta_{ij} - r_{ai} \vec{\omega}_i \cdot r_{aj}) \right) \vec{\omega}_i \vec{\omega}_j \\
 &= \boxed{\frac{1}{2} \sum_{i,j} I_{ij} \vec{\omega}_i \vec{\omega}_j} \quad (= \frac{1}{2} \boxed{\text{I}} \boxed{\text{II}} \boxed{\text{III}} \boxed{\text{IV}})
 \end{aligned}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

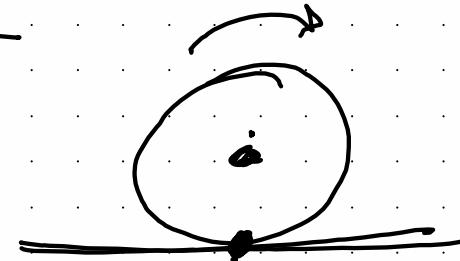
$$T = \underbrace{\frac{1}{2} m V^2}_{\text{trans}} + \underbrace{\sum_{i,j} I_{ij} \cdot \Omega_i \cdot \Omega_j}_{\text{rotational}} + \textcircled{Q}$$

$$\left(\frac{1}{2} I \dot{\theta}^2 \right)$$

Freshman physics

for COM at origin
OF RB Frame

\vec{M} : wrt COM of body

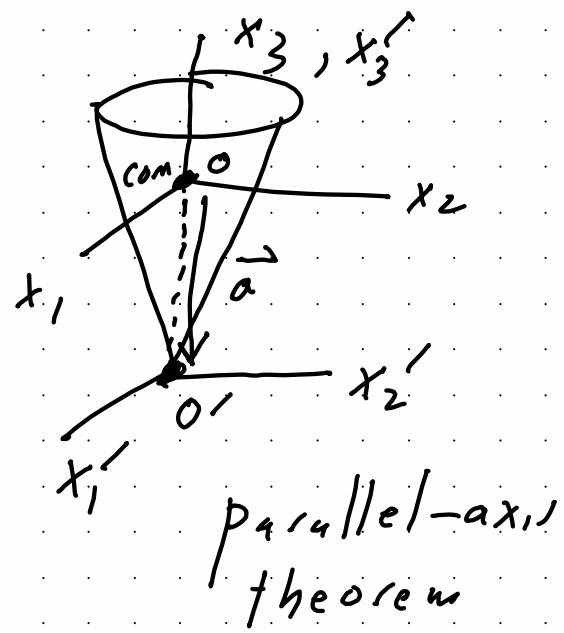


$$\begin{aligned} \vec{M} &= \sum_a \vec{r}_a \times \vec{p}_a \\ &= \sum_a m_a \vec{r}_a \times \vec{v}_a \\ &= \sum_a m_a \vec{r}_a \times (\vec{V} + \vec{\omega} \times \vec{r}_a) \\ &= \sum_a m_a \vec{r}_a \times (\vec{\omega} \times \vec{r}_a) \end{aligned}$$

$$= \sum_a m_a (\vec{\omega} \cdot \vec{r}_a)^2 - \vec{r}_a (\vec{r}_a \cdot \vec{\omega})$$

$$M_i = \sum_j I_{ij} \cdot \Omega_j$$

$$\vec{M} = I \vec{\omega} \quad (\text{Fresh. phys.})$$



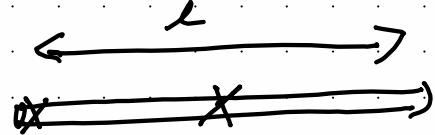
$$I_{ij}' = I_{ij} + m(a^2 \delta_{ij} - a_i a_j)$$

wrt O' wrt O (com)

$$\vec{a} : \text{Vector from } O \text{ to } O'$$

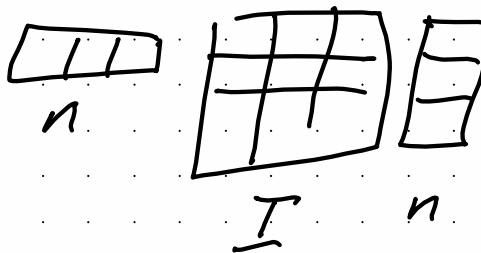
$$I(\hat{n}) = \sum_{i,j} I_{ij} n_i n_j$$

~~moment~~
of inertia



M

\hat{n} : axis of rotation



$$I = n$$

$$\boxed{\begin{aligned} I_{\text{com}} &= \frac{1}{2} M l^2 \\ I_{\text{end}} &= \frac{1}{3} M l^2 \end{aligned}}$$

$$I_{ij} = \sum_a m_a (r_a^2 \delta_{ij} - r_{ai} r_{aj})$$

3x3 real
symmetric

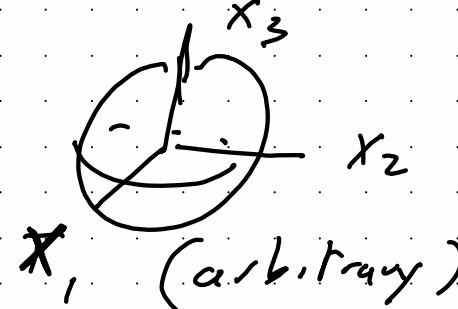
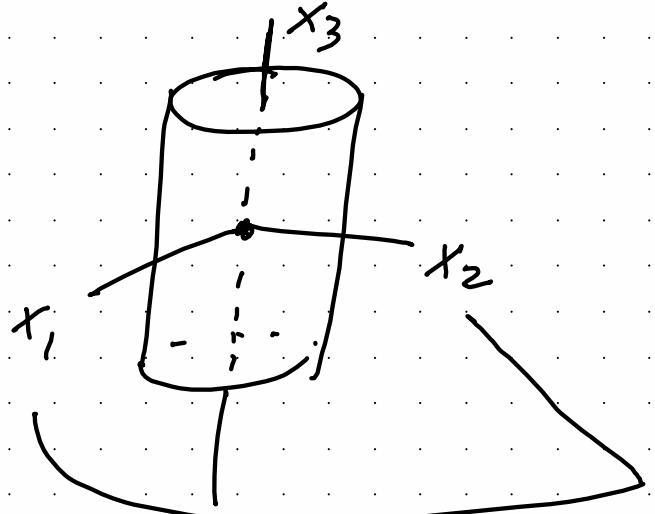
$$= \int p dV (r^2 \delta_{ij} - r_i r_j)$$

(can always be diagonalized)

$$\begin{array}{|c|c|c|} \hline I_1 & 0 & 0 \\ \hline 0 & I_2 & 0 \\ \hline 0 & 0 & I_3 \\ \hline \end{array}$$

$$I_{ij} = I_i \delta_{ij} \quad T_{\text{rot}} = \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j$$

Principle Axes: $(x_1, x_2, x_3) = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$

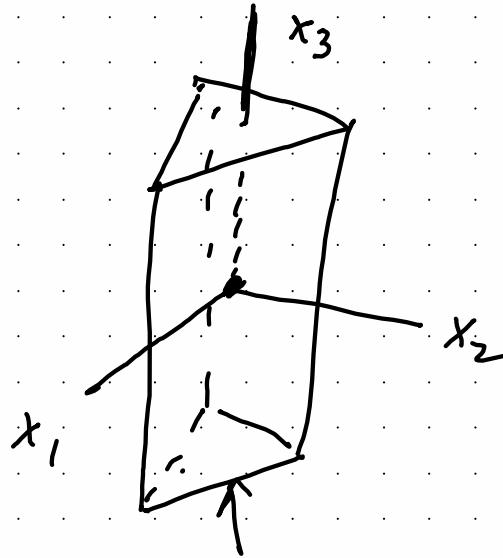


arb. tray

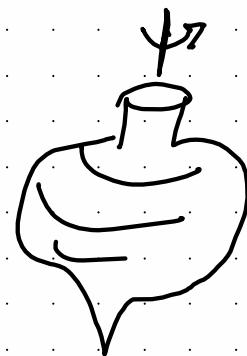
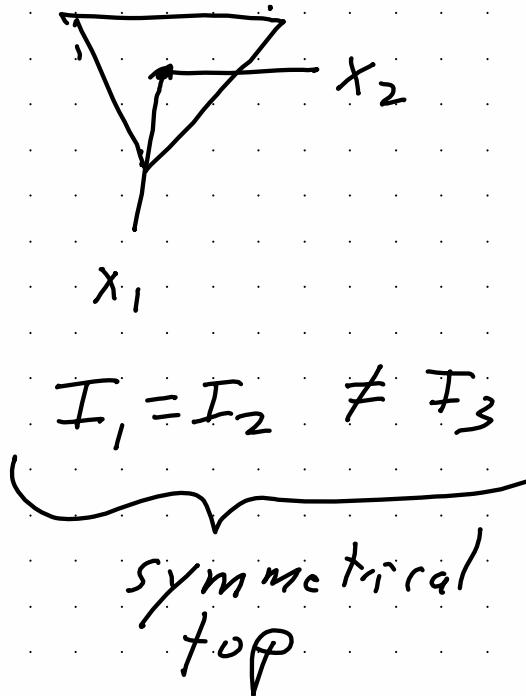
$$M_i = \sum_j I_{ij} \Omega_j$$

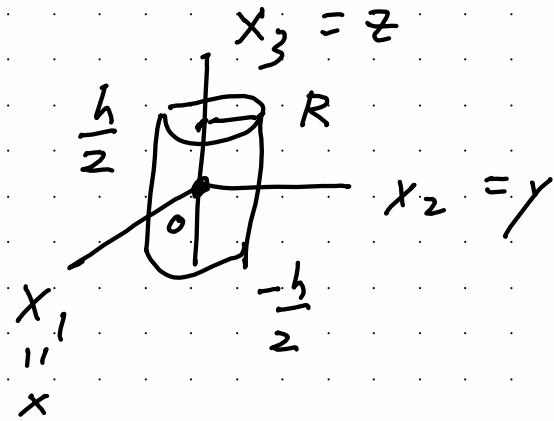
$$M_1 = I_1 \Omega_1, M_2 = I_2 \Omega_2$$

$$M_3 = I_3 \Omega_3$$



equilateral
triangle





total mass μ

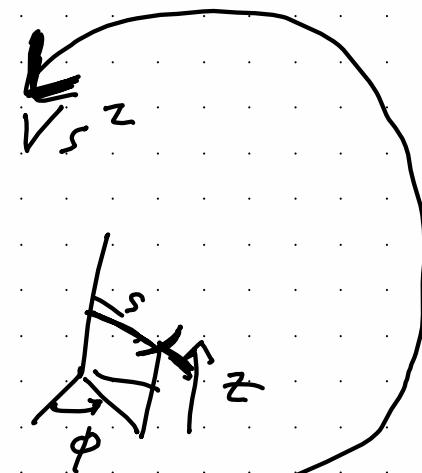
$$\rho = \frac{M}{\text{volume}} = \frac{M}{\pi R^2 h}$$

$$I_3 = I_{33} = \int \rho dV \left(r^2 \delta_{33} - \frac{r_3 r_3}{z^2} \right)$$

$$= \int \rho dV (r^2 - z^2)$$

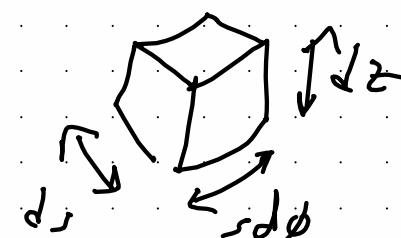
$$= \int \rho dV (x^2 + y^2) = \int \rho dV s^2$$

cylindrical: s, ϕ, z $s^2 = x^2 + y^2$



$$dV = ds s d\phi dz$$

$$= s ds d\phi dz$$

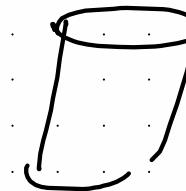


$$I_3 = \int \rho dV s^2$$

$$= \frac{M}{\pi R^2 h} \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \int_{-\frac{h}{2}}^{\frac{h}{2}} d\phi \int_{0}^{2\pi} ds s^3$$

$$= \frac{M}{\pi R^2 h} \cdot 2\pi K \frac{R^4}{4} h^2$$

$$= \boxed{\frac{1}{2} M R^2}$$



$$I_1 = I_2 \equiv I$$

$$I_1 = \int \rho dV (r^2 - x^2)$$

$$+ I_2 = \int \rho dV (r^2 - y^2)$$

$$2I = \int \rho dV (2r^2 - x^2 - y^2)$$

$$\begin{aligned} r^2 &= s^2 + z^2 \\ x^2 + y^2 &= s^2 \end{aligned}$$

$$2I = \int \rho dV (s^2 + 2z^2)$$

$$\boxed{I} = \frac{1}{2} \underbrace{\int \rho dV s^2}_{I_3} + \int \rho dV z^2$$
$$= \frac{1}{2} I_3 + \int \rho dV z^2$$

easier to
evaluate

$$\int \rho dV z^2 = \frac{M}{\pi R^2 h} \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz \cdot z^2 \int_0^R s ds$$

\curvearrowleft \curvearrowleft \curvearrowleft

$$= \frac{z^3}{3} \Big|_{-h/2}^{h/2} = \frac{h^3}{12}$$

$$= \frac{\frac{2}{3} \left(\frac{h}{2}\right)^3}{3} = \frac{h^3}{12}$$

$$= \frac{M}{\cancel{\pi R^2 h}} \cdot \cancel{\frac{2}{3} \pi} \cdot \frac{h^3}{12} \cdot \cancel{\frac{R^2}{2}}$$

$$= \boxed{\frac{M h^2}{12}}$$

$$I = \frac{1}{2} \left(\frac{1}{2} MR^2 \right) + \frac{1}{12} mh^2$$

$$= \frac{1}{4} MR^2 + \frac{1}{12} mh^2$$

$$= \frac{1}{4} M \left(R^2 + \frac{1}{3} h^2 \right) = I_1, I_2$$

$$I_3 = \frac{1}{2} MR^2$$

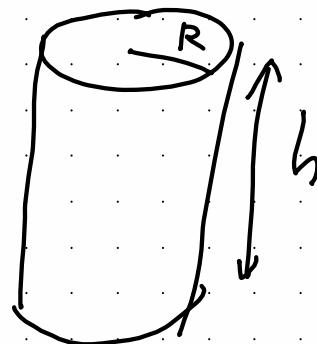


Limiting cases

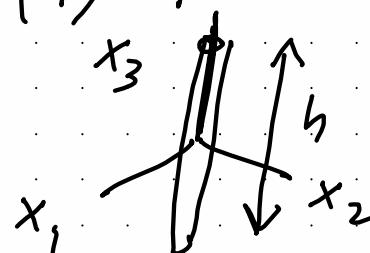
(i) Disc ($h \rightarrow 0$)

$$I_3 = \frac{1}{2} MR^2$$

$$I_1, I_2 = \frac{1}{4} MR^2$$



(i) thin rod ($R \rightarrow 0$)



$$I_3 = 0$$

$$I_1 = I_2 = \frac{1}{12} mh^2$$

$$L = T - U$$

$$= \frac{1}{2} M \dot{V}^2 + \frac{1}{2} \sum_{i,j} I_{ij} \dot{\theta}_j \dot{\theta}_i - U$$

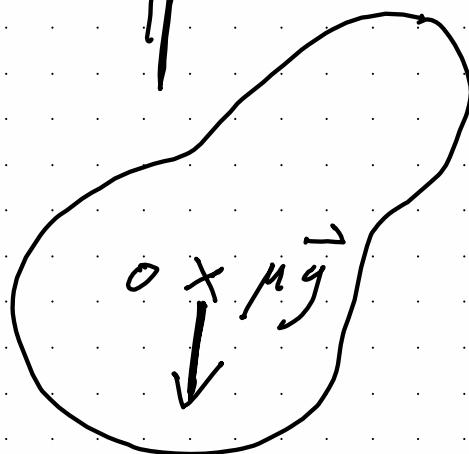
$$\vec{V} = \dot{\vec{R}}$$

$$\vec{R} = \vec{\phi}$$

$$L(\vec{R}, \dot{\vec{\phi}}, \vec{R}, \ddot{\vec{\phi}})$$

$$\vec{g}$$

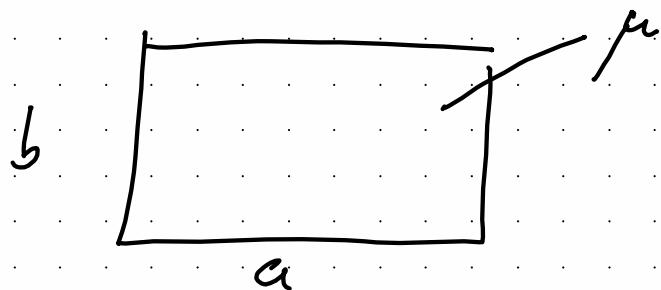
uniform
Field



Lecture #24 : Thursday 11/12

- Quiz #5 (today)
- Midterm #2 (next Thursday) (scattering, small oscillations, RB motion)
- Today's topics:
 - (1) RB EOMs
 - (2) Euler's equations
 - (3) Euler angles

Q5: Calculate the principal moments of inertia for a 2-d rectangle with side lengths a, b . uniform

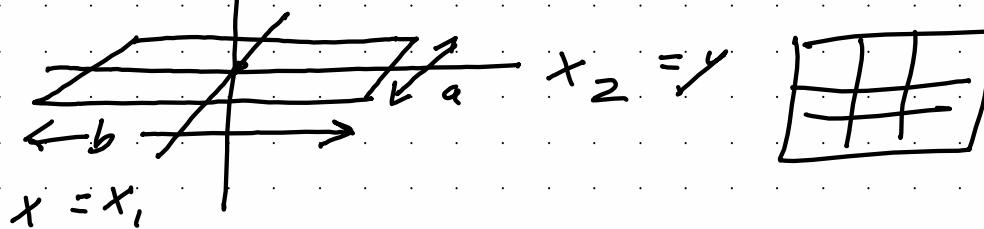


name-q5.pdf

$$z = x_3$$

$$I_{ij} = \int \rho dV (r^2 s_{ij} - r_i r_j)$$

$$r^2 = x^2 + y^2 + z^2$$



$$I_{11} = \int \underline{\rho dV} (r^2 - x^2)$$

$$I_{11} = \int \rho dV y^2$$

$$= \frac{M}{ab} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy y^2$$

$$= \frac{M}{ab} \alpha \frac{y^3}{3}$$

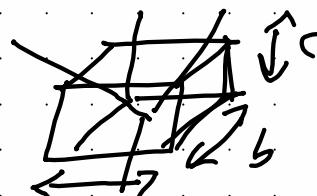
$$= \frac{M}{b} \frac{2}{3} \left(\frac{b}{2}\right)^3$$

$$= \frac{M}{b} \frac{2}{3} \frac{b^3}{8}$$

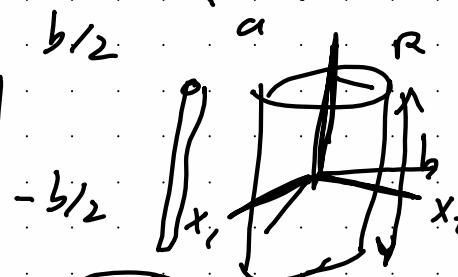
$$dm = \rho dV$$

$$= \sigma dx dy$$

$$= \frac{M}{ab} dx dy$$



$$\boxed{I_2 = \frac{1}{12} M a^2}$$

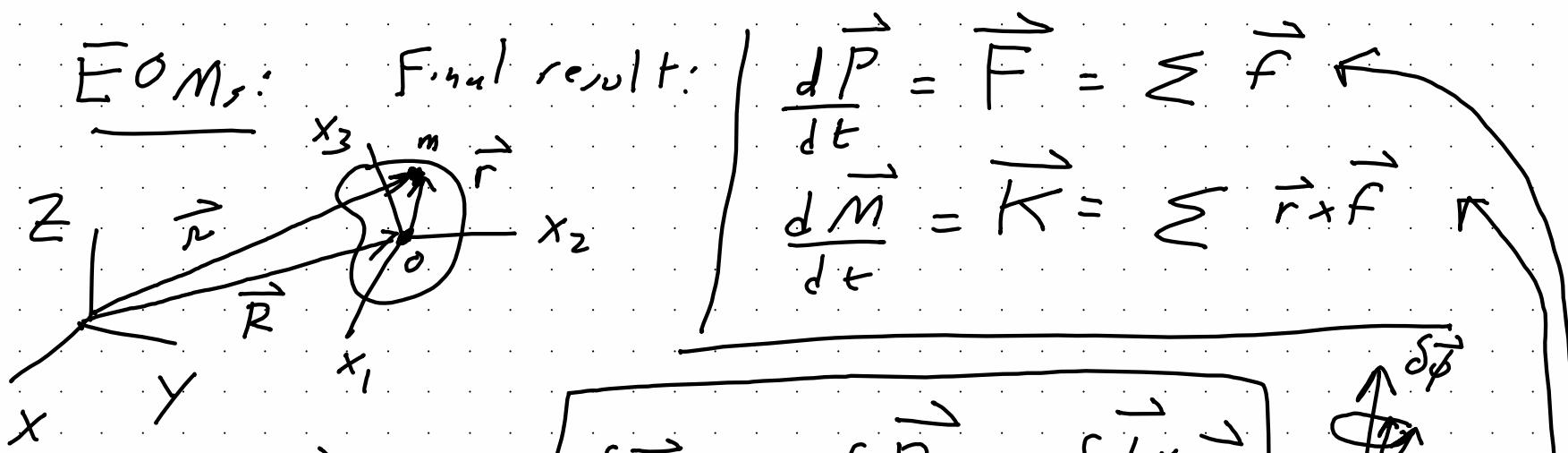


$$I_3 = \int \sigma dx dy \cdot (x^2 + y^2)$$

$$= I_1 + I_2$$

$$\boxed{I_3 = \frac{1}{12} M (a^2 + b^2)}$$

$$\boxed{= \frac{1}{12} M b^2}$$



$$\vec{r} = \vec{R} + \vec{r}, \quad \delta\vec{r} = \delta\vec{R} + \delta\phi \times \vec{r}$$

$$L = T - U = \frac{1}{2} \mu \vec{V}^2 + \underbrace{\frac{1}{2} I_{ij} \Omega_i \Omega_j}_{\delta\Omega_K} - U(\vec{r})$$

$$\delta L = \mu \vec{V} \cdot \delta\vec{V} + \underbrace{\frac{1}{2} I_{ij} \delta\Omega_i \delta\Omega_j}_{\delta\Omega_K} - \underbrace{\frac{\partial U}{\partial \vec{r}} \cdot \delta\vec{r}}$$

$$- \underbrace{\sum \frac{\partial U}{\partial \vec{r}} \cdot (\delta\vec{R} + \delta\phi \times \vec{r})}$$

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \vec{V}} \right) = \frac{\partial L}{\partial \vec{R}}}$$

$$, \quad \boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \vec{r}} \right) = \frac{\partial L}{\partial \vec{\phi}}}$$

$$\frac{1}{2} \sum_{i,j} I_{ij} \Delta \Omega_j \Omega_j + \frac{1}{2} \sum_{i,j} I_{ij} \Omega_j \Delta \Omega_j$$

$$= " + \frac{1}{2} \sum_{j,i} I_{ji} \Omega_j \Delta \Omega_i$$

$$= " + \frac{1}{2} \sum_{j,i} I_{ji} \Omega_j \Delta \Omega_i$$

swap

$$= \frac{1}{2} \sum_{i,j} I_{ij} \Omega_j \Delta \Omega_i + \frac{1}{2} \sum_{i,j} I_{ij} \Omega_j \Delta \Omega_i$$

$$= \boxed{\sum_{i,j} I_{ij} \Omega_j \Delta \Omega_i}$$

$$\delta L = \underbrace{M \cdot \vec{\delta V} + \sum_{i,j} I_i \cdot R_j \cdot \delta R_i}_{\leq M; \delta R_i} - \left(\frac{\partial U}{\partial R} \right) \cdot \vec{\delta R} - \underbrace{\sum \frac{\partial U}{\partial \vec{r}} \cdot (\vec{\delta \phi} \times \vec{r})}_{\begin{aligned} &= - \sum \delta \vec{\phi} \cdot \left(\vec{r} \times \frac{\partial U}{\partial \vec{r}} \right) \\ &= - \vec{\delta \phi} \cdot \left(\sum \vec{r} \times \frac{\partial U}{\partial \vec{r}} \right) \end{aligned}}$$

$$\boxed{\delta L = \vec{P} \cdot \vec{\delta V} + \vec{M} \cdot \vec{\delta R} + (\sum \vec{F}) \cdot \vec{\delta R} + \vec{\delta \phi} \cdot (\sum \vec{r} \times \vec{F})}$$

$$\vec{P} = \frac{\partial L}{\partial \vec{V}}, \quad \vec{M} = \frac{\partial L}{\partial \vec{R}}, \quad \sum \vec{F} = \frac{\partial L}{\partial \vec{R}}, \quad \sum \vec{r} \times \vec{F} = \frac{\partial L}{\partial \vec{\phi}}$$

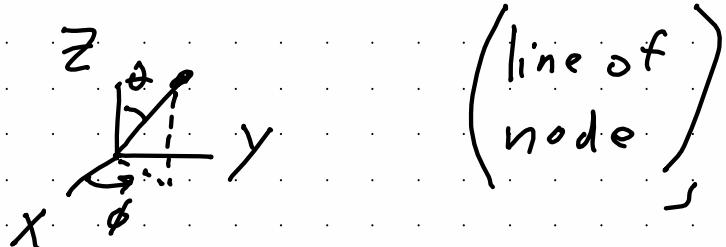
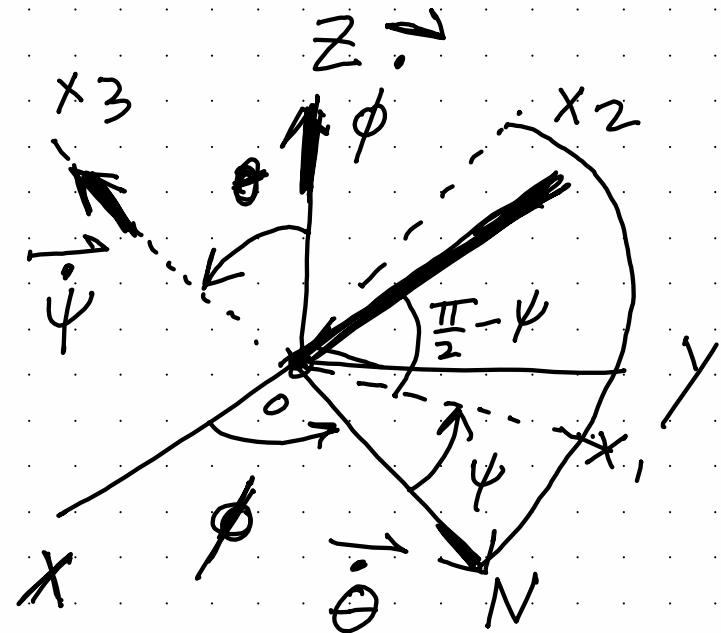
$$\frac{d}{dt} \left(\frac{\partial L}{\partial \vec{V}} \right) = \frac{\partial L}{\partial \vec{R}} \rightarrow \boxed{\frac{d \vec{P}}{dt} = \sum \vec{F} = \vec{F}} \quad \left| \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \vec{R}} \right) = \frac{\partial L}{\partial \vec{\phi}} \rightarrow \boxed{\frac{d \vec{M}}{dt} = \sum \vec{r} \times \vec{F} = \vec{F}} \right.$$

Euler's equations / Euler angles

$$\vec{R}, \vec{\phi}$$

Ω_i : i th component of $\vec{\Omega}$

wrt \hat{x}_i



$$\vec{\Omega} = \dot{\phi} \hat{x}_3 + \dot{\theta} \hat{x}_1 + \dot{\psi} \hat{x}_2$$

$$\dot{\psi} = \dot{\psi} \hat{x}_3$$

$$\dot{\theta} = \dot{\theta} \cos \psi \hat{x}_1 - \dot{\theta} \sin \psi \hat{x}_2$$

$$\dot{\phi} = \dot{\phi} \cos \theta \hat{x}_3$$

$$+ \dot{\phi} \sin \theta (\underline{\sin \psi \hat{x}_1} + \underline{\cos \psi \hat{x}_2})$$

$$\cos(\frac{\pi}{2} - \psi) \quad \sin(\frac{\pi}{2} - \psi)$$

Announcements

- Midterm II is this Thursday
- Today:
 - i) Euler angles
 - ii) Euler's equation for RB motion
 - iii) Free rotation with $\vec{\Omega} = \text{const}$
 - iv) II of a symmetric top ($I_1 = I_2$)
 - v) Heavy symmetrical top with one point fixed [prob. 35.1]

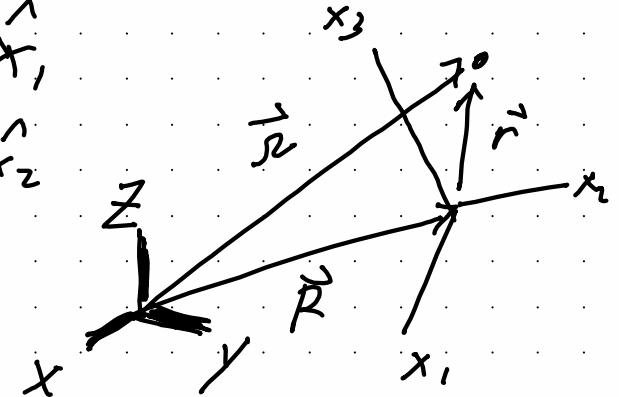


$$\vec{\Omega} = (\dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi) \hat{x}_1 \\ + (\dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi) \hat{x}_2 \\ + (\dot{\psi} + \dot{\phi} \cos \theta) \hat{x}_3$$

$$\Omega_1 = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi$$

$$\Omega_2 = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi$$

$$\Omega_3 = \dot{\psi} + \dot{\phi} \cos \theta$$



$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}$$

~~+ Coriolis~~

$$\frac{d' \vec{r}}{dt}$$

wrt
RB

Euler's equations: (wrt RB axes) \vec{A} : any vector

$$\frac{d\vec{P}}{dt} = \sum \vec{F} = \vec{F}$$

$$\frac{d\vec{M}}{dt} = \sum \vec{r} \times \vec{F} = \vec{K}$$

$$\frac{d\vec{A}}{dt} = \frac{d' \vec{A}}{dt} + \vec{\Omega} \times \vec{A}$$

wrt inertial frame wrt rotating frame angular velocity of the rotating frame

$$\left(\frac{d' \vec{A}}{dt} \right)_i = \frac{d A_i}{dt} = \dot{A}_i$$

cartesian components wrt rotating frame

$$\vec{a} = \frac{d^2 \vec{r}}{dt^2}$$

$$= \ddot{x} \hat{x} + \ddot{y} \hat{y} + \ddot{z} \hat{z}$$
~~$$= \ddot{r} \hat{r}$$~~

$$\vec{A} = \sum_i A_i \hat{x}_i$$

$$\frac{d \vec{A}}{dt} = \sum_i \left(\frac{d A_i}{dt} \right) \hat{x}_i + \sum_i A_i \frac{d \hat{x}_i}{dt}$$

$$\frac{d' \vec{A}}{dt}$$

$$\vec{r} \times \vec{A}$$

$$\frac{d\vec{A}}{dt} = \frac{d' \vec{A}}{dt} + \vec{\omega} \times \vec{A} \quad , \quad \left(\frac{d' \vec{A}}{dt} \right)_i = \dot{A}_i$$

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d' \vec{P}}{dt} + \vec{\omega} \times \vec{P}$$

$$\begin{aligned} \rightarrow \boxed{F_1} &= \dot{P}_1 + (\vec{\omega} \times \vec{P})_1 \\ &= \dot{P}_1 + \Omega_2 P_3 - \Omega_3 P_2 \\ &= \mu(V_1 + \Omega_2 V_3 - \Omega_3 V_2) \end{aligned}$$

$$\vec{P} = \mu \vec{V}$$

similar equations
for F_2, F_3

$$\vec{K} = \frac{d\vec{M}}{dt} = \frac{d' \vec{M}}{dt} + \vec{\omega} \times \vec{M}$$

$$M_i = I_i \cdot \Omega_i$$

$$\begin{aligned} \boxed{K_1} &= \dot{M}_1 + \Omega_2 M_3 - \Omega_3 M_2 \\ &= I_1 \dot{\Omega}_1 + \Omega_2 I_3 \Omega_3 - \Omega_3 F_2 \Omega_2 \\ &= I_1 \dot{\Omega}_1 + \Omega_2 \Omega_3 (I_3 - I_2) \end{aligned}$$

* (similar
equations)
for K_2, K_3)

Free rotation: $\dot{H}_i = 0, \dot{F}_i = 0$

$$\ddot{\Omega} = I_1 \dot{\Omega}_1 + \Omega_2 \Omega_3 (I_3 - I_2), \quad \cancel{\text{not}}$$

$$\ddot{\Omega} = I_2 \dot{\Omega}_2 + \Omega_3 \Omega_1 (I_1 - I_3)$$

$$\ddot{\Omega} = I_3 \dot{\Omega}_3 + \Omega_1 \Omega_2 (I_2 - I_1)$$

Free rotation with $\vec{\Omega} = \text{const}$:

$$\ddot{\Omega} = \Omega_2 \Omega_3 (I_3 - I_2)$$

$$\ddot{\Omega} = \Omega_3 \Omega_1 (I_1 - I_3)$$

$$\ddot{\Omega} = \Omega_1 \Omega_2 (I_2 - I_1)$$

$\Omega_1 = \text{const}$, $I_1 < I_2 < I_3$
 $\Omega_2 = 0$ \leftarrow stable
 $\Omega_3 = 0$ \leftarrow unstable

$\Omega_2 = \text{const}, \Omega_1 = 0, \Omega_3 = 0$

$$\begin{cases} \frac{d \vec{\Omega}}{dt} = 0 \\ \frac{d' \vec{\Omega}}{dt} = 0 \\ \frac{d \vec{\Omega}}{dt} = \frac{d' \vec{\Omega}}{dt} + \vec{\omega} \times \vec{\Omega} \end{cases}$$

$$\frac{d \vec{m}}{dt} \neq \frac{d' \vec{m}}{dt}$$

$\Omega_3 = \text{const}, \Omega_1 = 0, \Omega_2 = 0$

$\Omega_1 = \text{const}$, $\Omega_2 = 0$, $\Omega_3 = 0$: exact

$$\boxed{\begin{aligned}\Omega_1 &= \text{const} + \epsilon_1 \\ \Omega_2 &= \epsilon_2 \\ \Omega_3 &= \epsilon_3\end{aligned}}$$

$\epsilon_{1,2,3}$: small time dependent perturbations

Keep 0th and 1st order terms.

Ignore 2nd order, e.g. $\epsilon_1 \epsilon_3$

$$0 = I_1 \frac{d}{dt} (\text{const} + \epsilon_1) + \underbrace{\epsilon_2 \epsilon_3 (F_3 - I_2)}_{\text{2nd order} \rightarrow \text{ignore}}$$

$$\approx I_1 \dot{\epsilon}_1$$

$$\rightarrow \epsilon_1 = \text{const} \rightarrow \boxed{\Omega_1 = \text{const}}$$

$$\left. \begin{aligned} 0 &= I_2 \dot{\epsilon}_2 + \epsilon_3 \Omega_1 (I_1 - I_3) \\ 0 &= I_3 \dot{\epsilon}_3 + \Omega_1 \epsilon_2 (I_2 - I_1) \end{aligned} \right\} \begin{array}{l} \text{coupled} \\ \text{1st order} \\ \text{diff. equations} \end{array}$$

Differentiate ..

$$0 = I_2 \ddot{\epsilon}_2 + \epsilon_3 \Omega_1 (I_1 - I_3)$$

$$= I_2 \ddot{\epsilon}_2 - \frac{\Omega_1 \epsilon_2 (I_2 - I_1) \Omega_1 (I_1 - I_3)}{I_3}$$

$$I_1 < I_2 < I_3$$

$$\begin{aligned}
 O &= I_2 \ddot{\epsilon}_2 + \epsilon_3 \omega_1 (I_1 - I_3) \\
 &= I_2 \ddot{\epsilon}_2 - \frac{\omega_1 \epsilon_2 (I_2 - I_1) \omega_1 (I_1 - I_3)}{I_3} \\
 &= \ddot{\epsilon}_2 + \frac{\omega^2 (I_2 - I_1)(I_3 - I_1)}{I_2 I_3} \epsilon_2 \\
 &= \ddot{\epsilon}_2 + \omega^2 \epsilon_2
 \end{aligned}$$

$(I_3 - I_2)$
 $(I_1 - I_2)$

$$\ddot{\epsilon}_2 = -\omega^2 \epsilon_2 \rightarrow \sin \omega t \quad \epsilon_2 = A \cos \omega t + B \sin \omega t$$

Similarly,

$$\ddot{\epsilon}_3 = -\omega^2 \epsilon_3 \rightarrow \epsilon_3 = D \cos(\omega t + \beta)$$

$(\epsilon_2, \epsilon_3 \text{ are bound by their initial deviations away from } O)$

For $\Omega_2 = \text{const}$ solution

perturbations

$$\epsilon_1'' = +\omega^2 \epsilon_1$$

$$\epsilon_3'' = +\omega^2 \epsilon_3$$

$$\rightarrow \epsilon_1(t) = A e^{wt} + B e^{-i\omega t}$$

w damped

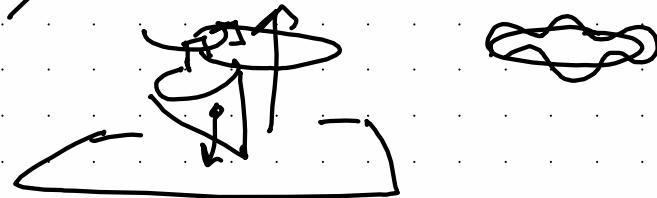
grows exponentially exponentially

~~→ instability~~

Lecture # 27

Tue, 11/24

- Midterm II - Avg $\approx 12/20$ (solutions posted)
- Quiz #6 - Next Tuesday (1st day)
- Oral Final - google doc sign up sheet ??
Sat 12/15 1:30pm - 4:00pm
- Today, next time:
 - i) Free rotation of a symmetric top
 - ii) motion in a non-inertial ref. frame
- On your own:
 - (i) Sec 35, Prob 1: motion of a heavy symmetric top
 - (ii) static equil:
(Sec 38)



Free rotation of a symmetric top:

↓
no force, torque $\rightarrow \vec{P} = \text{const}, \vec{M} = \text{const}$

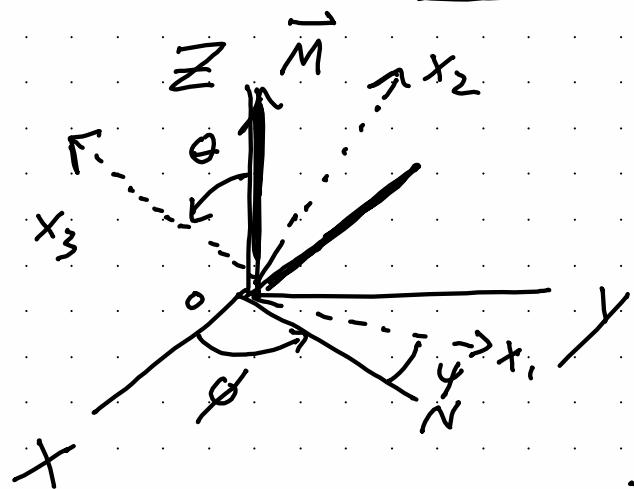
Euler's equations:

$$\dot{\Omega}_1 = I_1 \dot{\Omega}_1 + I_2 \dot{\Omega}_3 (I_3 - I_2) \quad (1)$$

$$\dot{\Omega}_2 = I_2 \dot{\Omega}_2 + I_3 \dot{\Omega}_1 (I_1 - I_3) \quad (2)$$

$$\dot{\Omega}_3 = I_3 \dot{\Omega}_3 + I_1 \dot{\Omega}_2 (I_2 - I_1) \quad (3)$$

$$\dot{\Omega}_3 = 0 \rightarrow \boxed{\Omega_3 = \text{const}}$$



$$\Omega_1 = \dot{\psi} \sin \theta \sin \psi + \dot{\theta} \cos \psi$$

$$\Omega_2 = \dot{\psi} \sin \theta \cos \psi - \dot{\theta} \sin \psi$$

$$\underline{\Omega_3 = \dot{\psi} + \dot{\theta} \cos \theta}$$

$$I_1 \Omega_1 = M_1 = M \sin \theta \sin \psi \quad (4)$$

$$I_1 \Omega_2 = M_2 = M \sin \theta \cos \psi \quad (5)$$

$$I_3 \Omega_3 = M_3 = M \cos \theta \quad (6)$$

$$\Omega_3 = \text{const} \rightarrow \boxed{\theta = \text{const}}, \cos\theta = \frac{I_3 \Omega_3}{m}$$

$$\dot{\theta} = I_1 \dot{\Omega}_1 + \Omega_2 \Omega_3 (I_3 - I_1)$$

$$\dot{\theta} = \dot{\Omega}_1 + \Omega_2 \left(\frac{\Omega_3 (I_3 - I_1)}{I_1} \right) \leftarrow \boxed{\omega = \frac{\Omega_3 (I_3 - I_1)}{I_1}}$$

$$\dot{\theta} = I_1 \dot{\Omega}_2 + \Omega_1 \Omega_3 (I_1 - I_3)$$

$$M_1 = I_1 \Omega_1, M_2 = I_1 \Omega_2$$

$$\dot{\theta} = \dot{\Omega}_2 - \Omega_1 \left(\frac{\Omega_3 (I_3 - I_1)}{I_1} \right) \sqrt{\Omega_1^2 + \Omega_2^2} = a$$

$$\dot{\theta} = \dot{\Omega}_1 + \Omega_2 \omega$$

$$\ddot{\theta} = \ddot{\Omega}_1 - \Omega_1 \omega$$

$$\begin{aligned} \dot{\theta} &= \dot{\Omega}_1 + \Omega_2 \omega \\ &= \ddot{\Omega}_1 + \Omega_1 \omega^2 \quad \boxed{\Omega_2 = a \sin(\omega t + \phi)} \\ \ddot{\Omega}_1 &= -\omega^2 \Omega_1 \rightarrow \boxed{\Omega_1 = a \cos(\omega t + \alpha)} \end{aligned}$$

$$\ddot{\xi} = \Omega_1 + i \Omega_2$$

$$\dot{\xi} = \dot{\Omega}_1 + i \dot{\Omega}_2$$

$$\dot{\xi} = i \omega \xi \rightarrow \xi(t) = A e^{i \omega t}$$

↑
complex

$$I_1 \dot{\phi} = M \cos \theta \sin \psi \quad \theta = \text{const}$$

$$\left[\dot{\phi} = \frac{M}{I_1} = \text{const} \right] \rightarrow \boxed{\phi(t) = \phi_0 + \frac{M}{I_1} t}$$

$$I_3 (\dot{\psi} + \cos \theta \dot{\phi}) = M_{103} \theta$$

$$\begin{aligned} -\omega &= \dot{\psi} = \frac{M_{103} \theta}{I_3} - \cos \theta \dot{\phi} \\ &= \frac{M_{103} \theta}{I_3} - \cos \theta \frac{M}{I_1} \\ &= M_{103} \theta \left(\frac{1}{I_3} - \frac{1}{I_1} \right) \end{aligned}$$

$\omega = \frac{I_3 (I_3 - I_1)}{I_1}$

$M_3 = I_3 \Omega_3 \}$

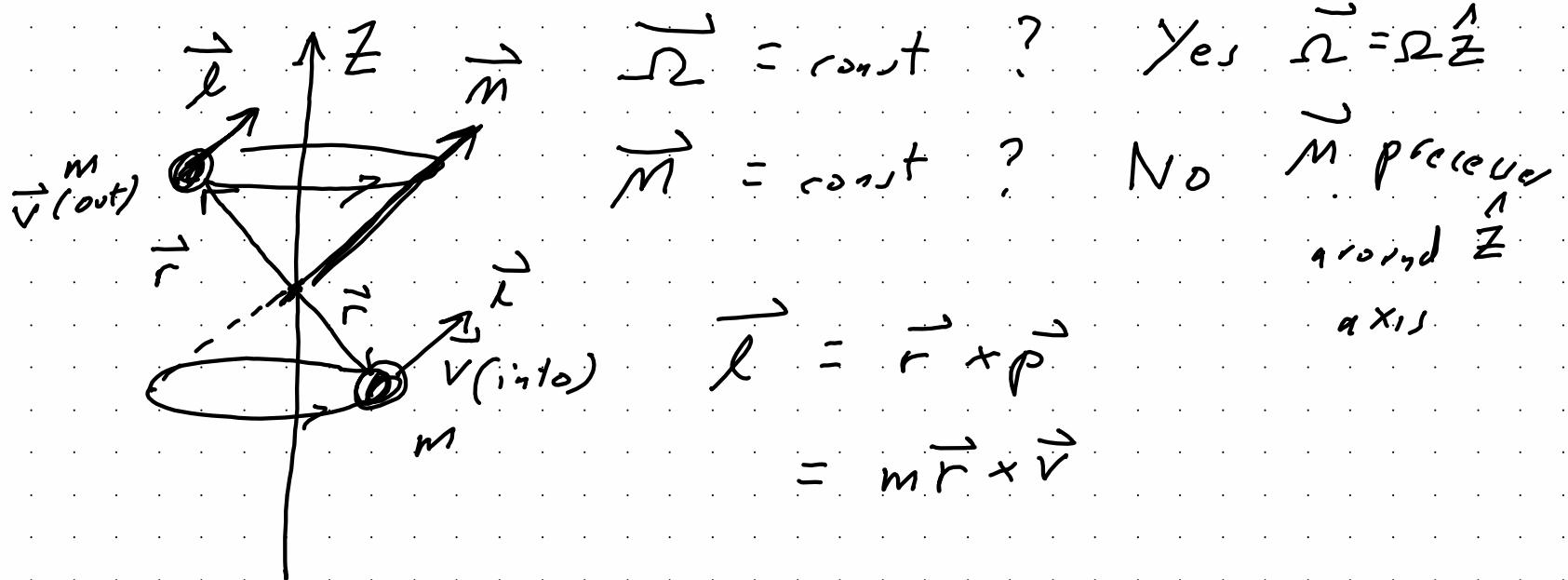
$M_{103} \theta = M_3 \}$

$\Omega_3 = \frac{M_{103} \theta}{I_3}$

$\omega = M_{103} \theta (I_3 - I_1)$ -ψ

$$\rightarrow \boxed{\psi(t) = \psi_0 + M_{103} \theta \left(\frac{1}{I_3} - \frac{1}{I_1} \right) t}$$

$= m \cos \theta \left(\frac{1}{I_1} - \frac{1}{I_3} \right) t$



wobble \leftrightarrow spin freq $\dot{\phi} = \frac{M}{I_1}$

$\dot{\phi}$
 \curvearrowright
 precession freq of
 3-ax.
 around \vec{M}
 $(\text{or } \hat{Z})$

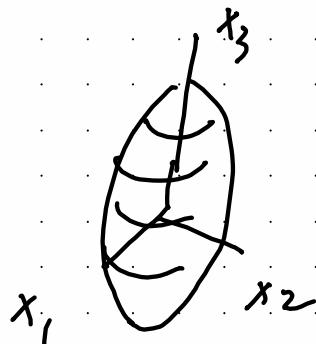
ω_3
 \curvearrowright
 Component of angular along 3-ax.

$$\frac{\omega_3}{\dot{\phi}} = \frac{M \cos \theta}{I_3} \frac{I_1}{\sin \theta} = \frac{I_1 \cos \theta}{I_3}$$

$$\frac{I_3}{\dot{\phi}} = \frac{I_1}{I_3} \cos \theta \rightarrow \frac{I_1}{I_3} \quad (\theta \gg 0)$$

w

spin
wobble



football

$$I_1 > I_3$$

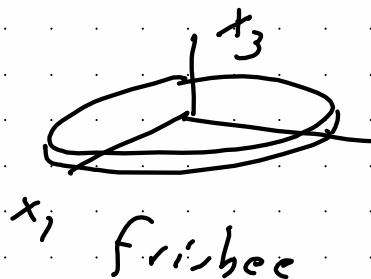
football

$$\frac{I_3}{\dot{\phi}} > 1$$

$$I_1 = I_3$$

sphere

$$\frac{I_3}{\dot{\phi}} = 1$$



frisbee

$$I_1 < I_3$$

$$\uparrow \quad \uparrow$$

$$\frac{1}{4}\mu R^2 \quad \frac{1}{2}\mu R^2 \quad (\text{uniform dist})$$

$$I_1 = \frac{1}{2} I_3$$

frisbee

$$\frac{I_3}{\dot{\phi}} = \frac{1}{2} < 1$$