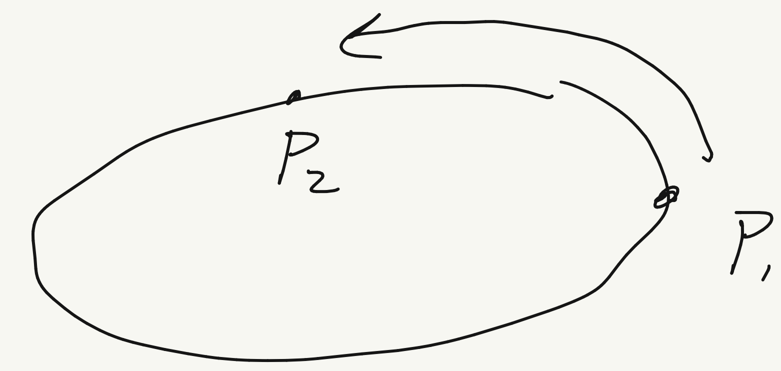
Lecture #1: Aug 24th

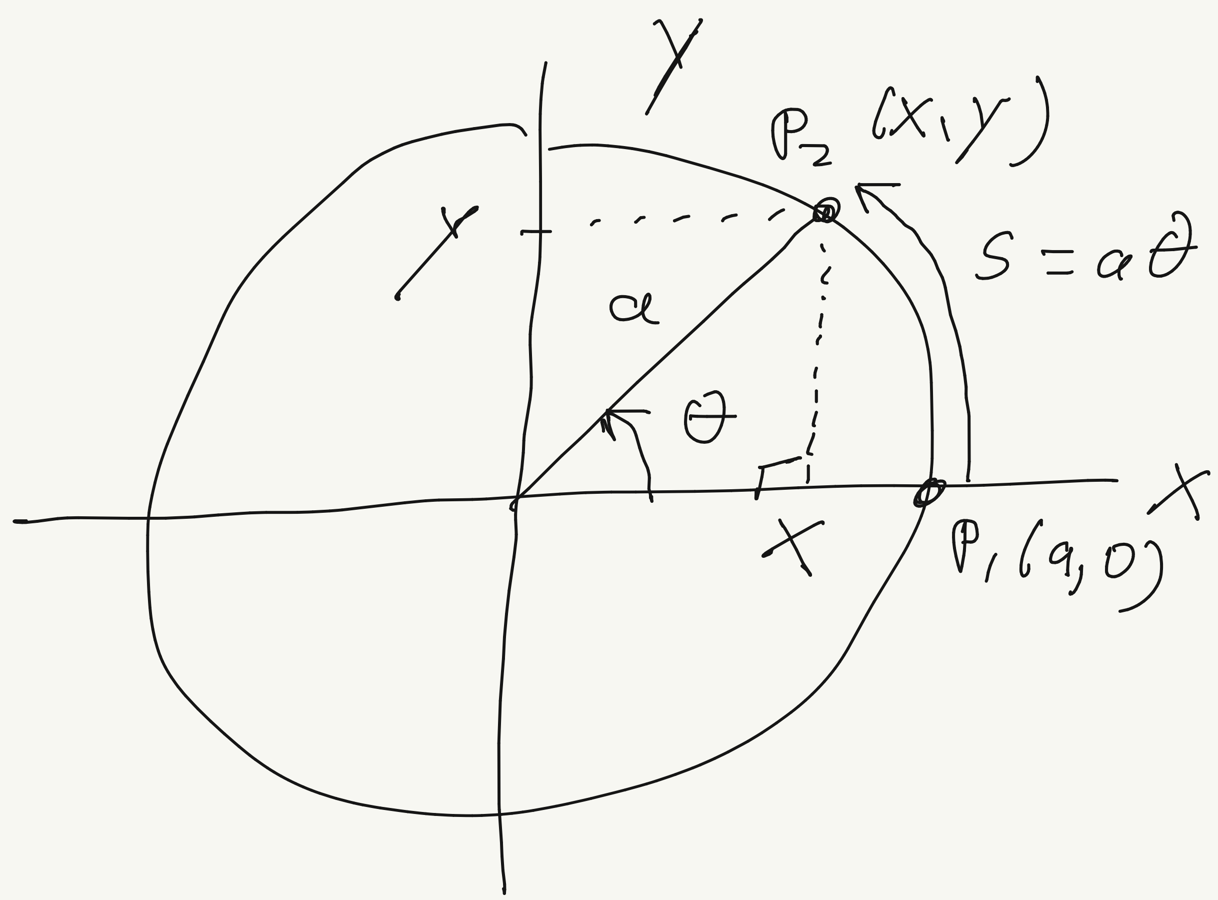
Elliptic functions / integrals:

i) circumference of an ellipse

ii) period of a simple pendulum beyond the small-angle approximation



Circular functions: sines, cosines,



a = radius

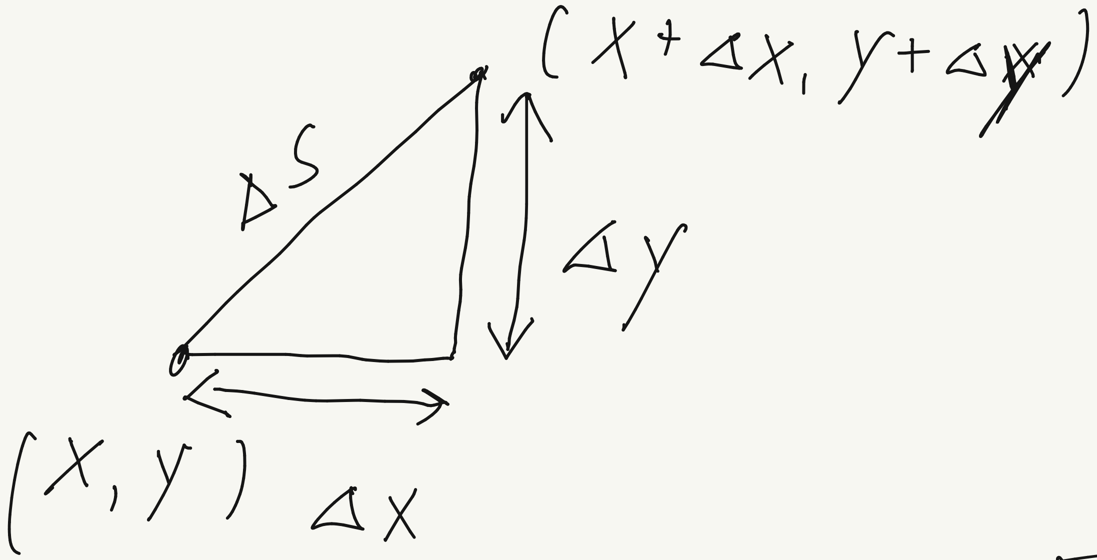
$$x^2 + y^2 = a^2$$

$$\theta = \frac{s}{a}$$

$$= \frac{1}{a} \int^{P_2} \sqrt{dx^2 + dy^2}$$

$$= \int_{P_1} d\theta$$

$$ds^2 = dx^2 + dy^2$$



$$\left.\begin{aligned}\sin\theta &\equiv y/a\\ \cos\theta &\equiv x/a\end{aligned}\right\}\ \text{definitions}$$

$$x = a\cos\theta$$
$$y = a\sin\theta$$

$$\boxed{x^2+y^2=a^2} \rightarrow \cancel{a^2}\cos^2\theta + \cancel{a^2}\sin^2\theta = \cancel{a^2}$$

$$\boxed{\cos^2\theta+\sin^2\theta = 1}$$

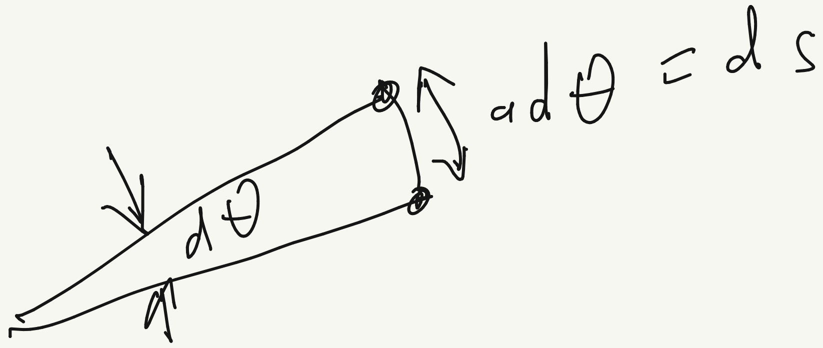
$$\boxed{\frac{d}{d\theta}\sin\theta = \cos\theta}$$

<u>Proof</u>: $\frac{d}{d\theta}\sin\theta = \frac{d}{d\theta}\left(\frac{y}{a}\right)$

$$= \frac{1}{a}\,\frac{dy}{d\theta}$$

$$= \frac{dy}{\sqrt{dx^2+dy^2}}$$

$$= \frac{1}{\frac{\sqrt{dx^2+dy^2}}{dy}} = \frac{1}{\sqrt{\left(\frac{dx}{dy}\right)^2+1}}$$



$$a\,d\theta = \sqrt{dx^2+dy^2}$$

$$x^2+y^2=a^2 \rightarrow 2x\,dx+2y\,dy=0$$

$$\frac{dx}{dy}=-\frac{y}{x}$$

$$\frac{d}{d\theta}\sin\theta=\frac{1}{\sqrt{\left(\frac{-y}{x}\right)^2+1}}=\frac{1}{\sqrt{\frac{y^2}{x^2}+1}}=\frac{x}{\sqrt{x^2+y^2}}=\frac{x}{a}=\cos\theta$$

$$\boxed{\frac{d}{d\theta}\sin\theta=\cos\theta}$$

Similarly $\boxed{\frac{d\cos\theta}{d\theta}=-\sin\theta}$

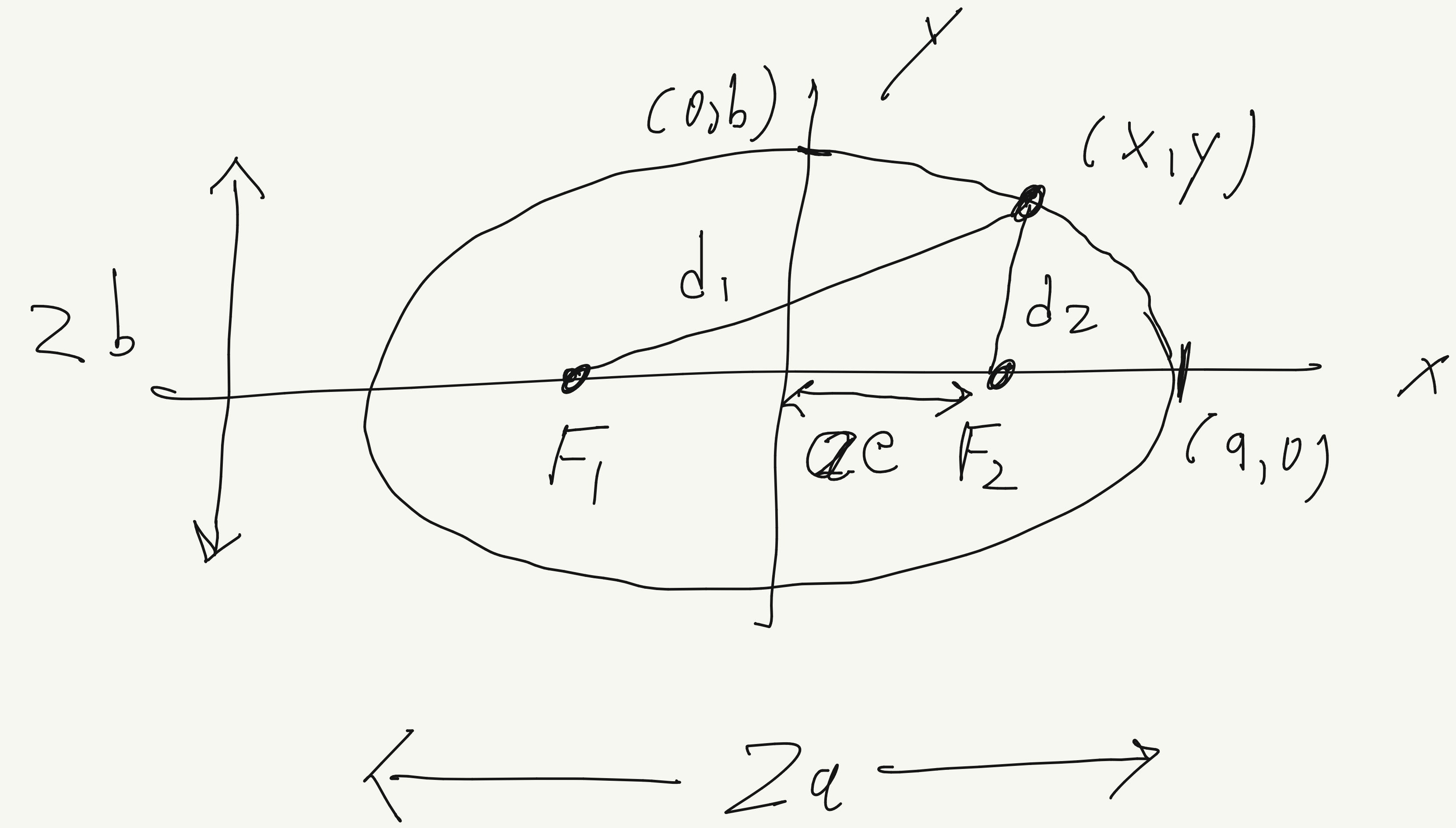
$$\int\frac{d(\sin\theta)}{\cos\theta}=\int d\theta=\theta+\text{const}$$

$$\boxed{\int\frac{dt}{\sqrt{1-t^2}}=\theta+\text{const}=\sin^{-1}t+\text{const}}$$

$$t=\sin\theta$$

$$\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-t^2}$$

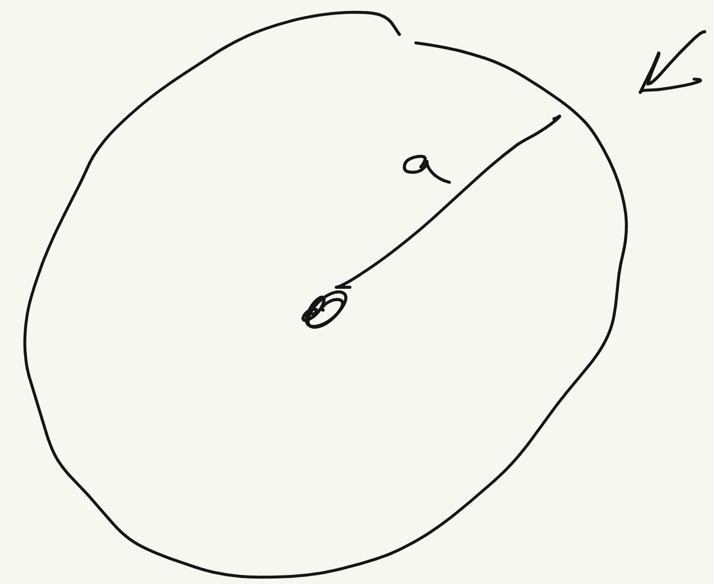
Lec #2: Aug 26th



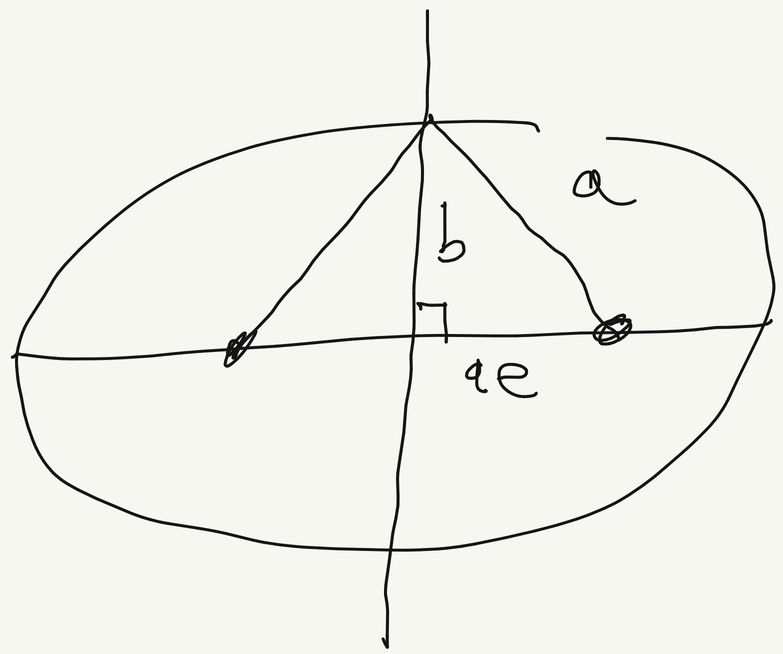
$$d_1 + d_2 = 2a$$

$$e \neq \frac{b}{a}$$

$$x^2 + y^2 = a^2$$



$$e \neq 1 - \frac{b}{a}$$



$$e^2 = \frac{b^2}{1-a^2}$$

$$\boxed{e = \sqrt{1 - \left(\frac{b}{a}\right)^2}}$$

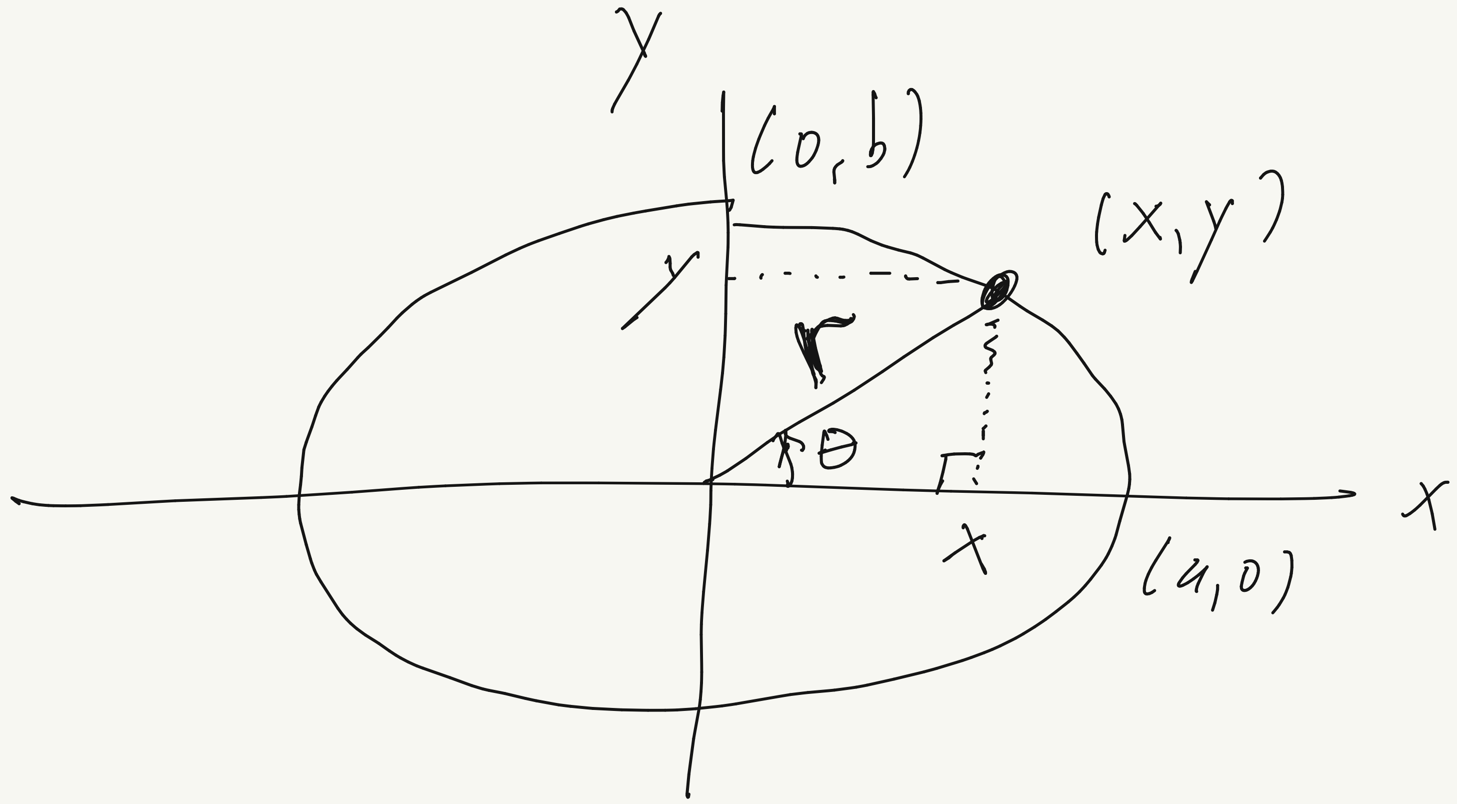
Pf:

$$a^2 = b^2 + (ae)^2$$

$$a^2(1-e^2) = b^2$$

$$1 - e^2 = \left(\frac{b}{a}\right)^2$$

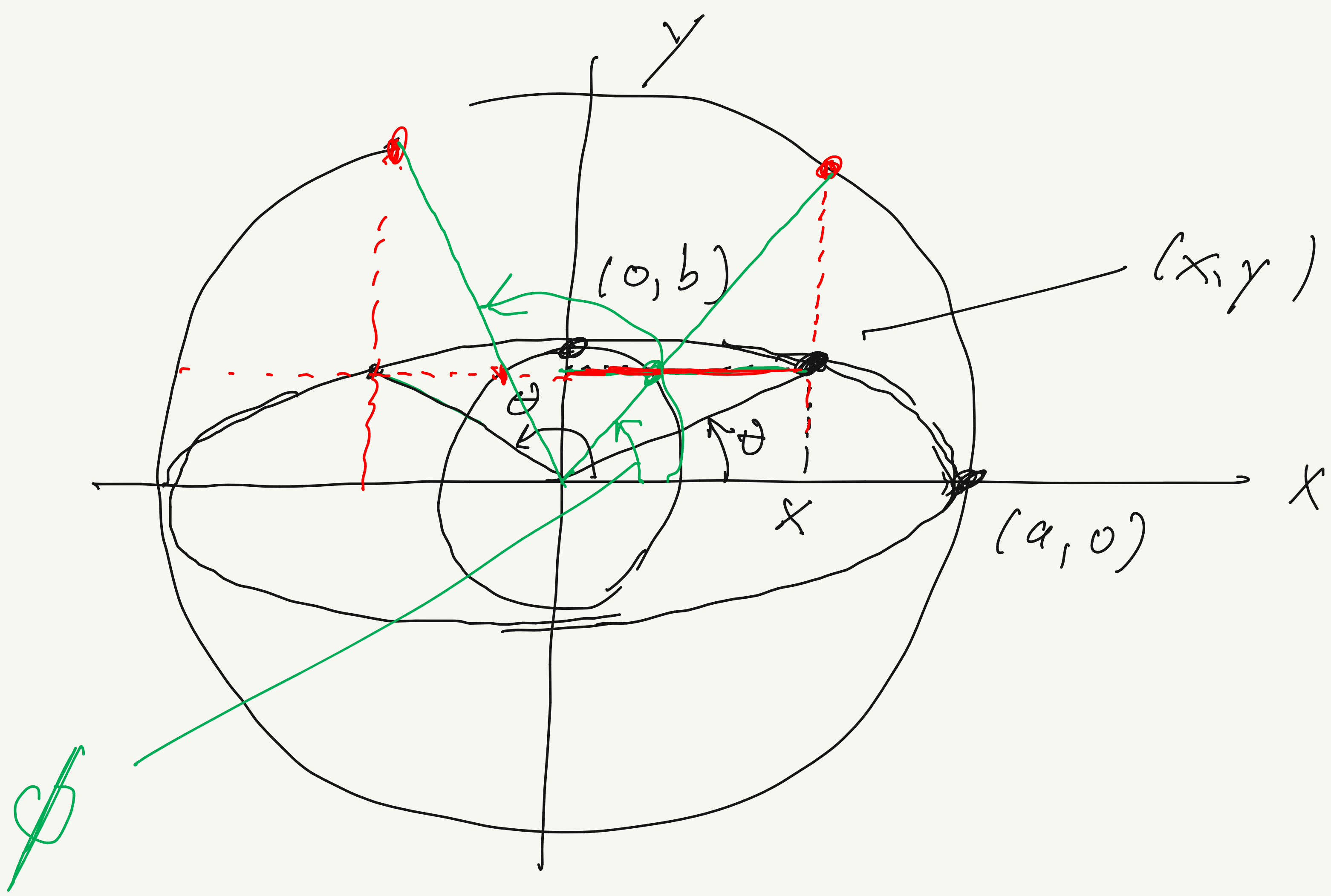
$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$



$x = r\cos\theta$
$y = r\sin\theta$

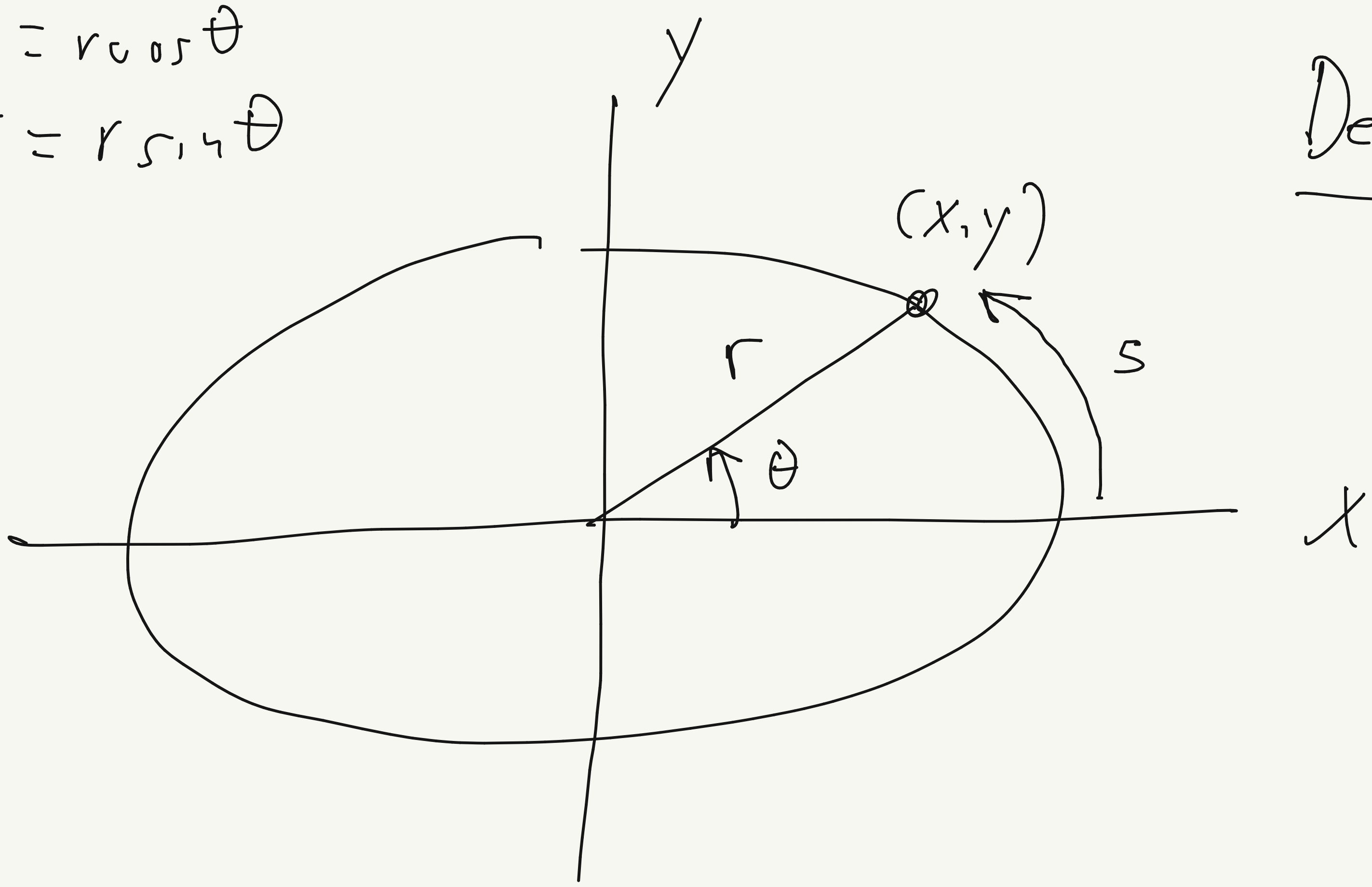
r changes

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \longleftarrow \begin{cases} x = a\cos\phi \\ y = b\sin\phi \end{cases}$$



$0 < e < 1$
$e = 0$ circle
$e = 1$ parabola
$e > 1$ hyperbola

$x = r\cos\theta$
$y = r\sin\theta$



Define:

$$\boxed{\begin{aligned} \mathrm{cn}(u;k) &\equiv x/a \\ \mathrm{sn}(u;k) &\equiv y/b \\ \mathrm{dn}(u;k) &\equiv \frac{r}{a} \end{aligned}}$$

$$u \equiv \frac{1}{b}\int_0^\theta r\,d\theta$$

not θ, not arc length

$$k = e \quad , \quad \underbrace{0 < k < 1}_{\text{modulus}}$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\theta^2}$$

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$
$$dx = dr\cos\theta - r\sin\theta\,d\theta$$
$$dy = dr\sin\theta + r\cos\theta\,d\theta$$

$$bu = \int_0^\theta r\,d\theta < \cancel{b}\, s$$

	pendulum
$\mathrm{cn}(u)$ $\mathrm{sn}(u)$	$k = \sin\left(\frac{\phi_0}{2}\right)$

Property:

$$\boxed{cn^2 u + sn^2 u = 1} \quad \longleftarrow \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\longleftarrow \quad x^2 + y^2 = r^2$$

$$dn^2 u = cn^2 u + \left(\frac{b}{a}\right)^2 sn^2 u$$

$$= 1 - sn^2 u + \left(\frac{b}{a}\right)^2 sn^2 u$$

$$= 1 - sn^2 u \left(1 - \left(\frac{b}{a}\right)^2\right)$$

$$= 1 - k^2 sn^2 u$$

$$\boxed{dn^2 u + k^2 sn^2 u = 1}$$

$$\frac{d}{du}\,\mathrm{sn}\,u = \frac{1}{b}\frac{dy}{du}$$

$$= \frac{dy}{r\,d\theta}$$

$$u = \frac{1}{b}\int_0^{\theta} r\,d\theta$$

$$du = \frac{r\,d\theta}{b} \rightarrow b\,du = r\,d\theta$$

$$x = r\cos\theta\ ,\quad y = r\sin\theta$$

$$dx = dr\cos\theta - r\sin\theta\,d\theta \rightarrow \sin\theta\,dx = \sin\theta\cos\theta\,dr - r\sin^2\theta\,d\theta$$

$$dy = dr\sin\theta + r\cos\theta\,d\theta \rightarrow -\cos\theta\,dy = -\cos\theta\sin\theta\,dr - r\cos^2\theta\,d\theta$$

add

$$\sin\theta\,dx - \cos\theta\,dy = -r\,d\theta$$

$$\frac{y}{r}dx - \frac{x}{r}dy = -r\,d\theta$$

$$\rightarrow \boxed{r\,d\theta = \frac{-y}{r}dx + \frac{x}{r}dy}$$

$$\frac{d}{du}\,\mathrm{sn}\,u = \frac{dy}{\frac{-y}{r}dx + \frac{x}{r}dy} = \frac{r}{-y\frac{dx}{dy} + x}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad \cancel{2}\,\frac{x\,dx}{a^2} + \cancel{2}\,\frac{y\,dy}{b^2} = 0$$

$$\frac{dx}{dy} = -\frac{y}{x}\,\frac{a^2}{b^2}$$

$$\frac{d\,\mathrm{sn}\,u}{du} = \frac{r}{-y\left(-\frac{y}{x}\right)\frac{a^2}{b^2} + x} = \frac{r\,x}{y^2\left(\frac{a}{b}\right)^2 + x^2}$$

$$= \frac{r}{a}\,\frac{x}{a}\left(\frac{1}{\left(\frac{y}{b}\right)^2 + \left(\frac{x}{a}\right)^2}\right)$$

$$= \frac{\mathrm{dn}\,u \cdot \mathrm{cn}\,u}{1}$$

$$\boxed{\frac{d}{du}\,\mathrm{sn}\,u = \mathrm{cn}\,u \cdot \mathrm{dn}\,u}$$

$$\frac{d}{du}\boxed{cn}\,u = -sn\,u \cdot dn\,u$$

$$\frac{d}{du}\,dn\,u = -k^2\,sn\,u \cdot cn\,u$$

$$\frac{d}{du}\,sn\,u = cn\,u \cdot dn\,u$$

$$sn^2 u + cn^2 u = 1$$

$$dn^2 u + k^2 sn^2 u = 1$$

$$\int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta = \theta$$

$t = \sin\theta$

$\theta = \sin^{-1} t$

$$\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t$$

Integrate!

$$\int \frac{d(sn\,u)}{cn\,u \cdot dn\,u} = \int du = u + \text{const}$$

$t = sn\,u$

$$\boxed{\int \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}} = \underline{sn^{-1}}(t;k) + \text{const}}$$

$$F(\phi, k) \equiv \int_0^{\sin\phi} \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2 t^2}}$$

$\operatorname{sn} u = \sin\phi$

$\frac{y}{b} = \sin\phi$

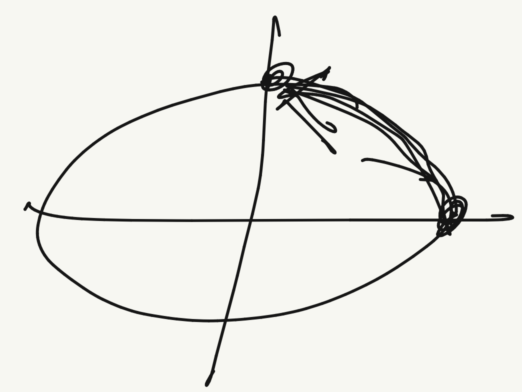
$\operatorname{cn} u = \cos\phi$

incomplete
elliptic integral of 1st kind (angular dependence, period of a simple pendulum)

$\frac{x}{a} = \cos\phi$

$$E(\phi, k) \equiv \int_0^{\sin\phi} \frac{dt\,\sqrt{1-k^2 t^2}}{\sqrt{1-t^2}}$$

incomplete
elliptic integral of 2nd kind (arclength along ellipse)



$\phi = \pi/2$

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} = K(k)$$

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}}\sqrt{1-k^2t^2} = E(k)$$

complete elliptic integrals of 1st and 2nd kind

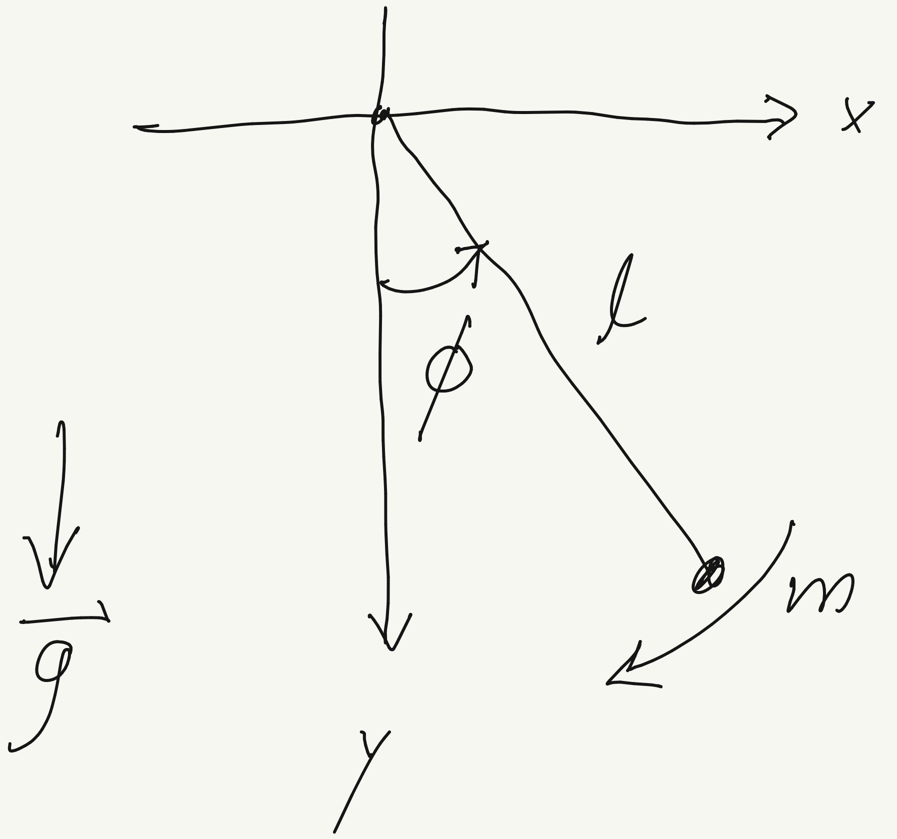
Lec #3: Aug 31^{st}

Simple pendulum: ("Freshman physics")

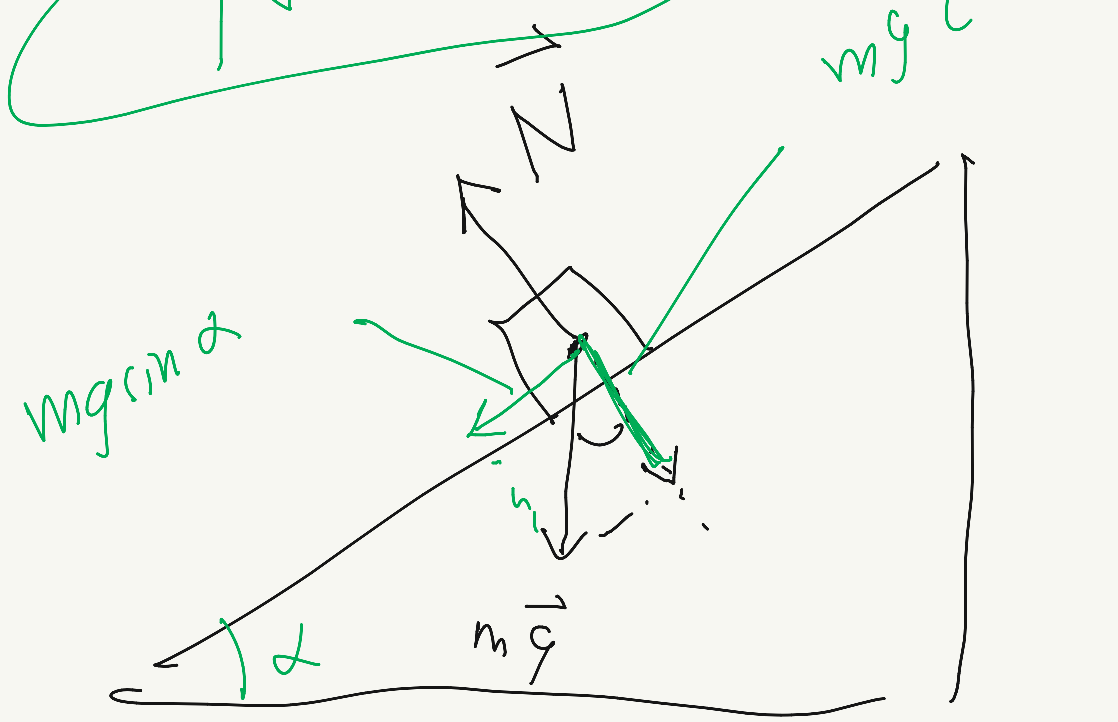
$$a_c = \frac{v^2}{r} = \omega^2 r$$

$$a_t = \alpha r$$

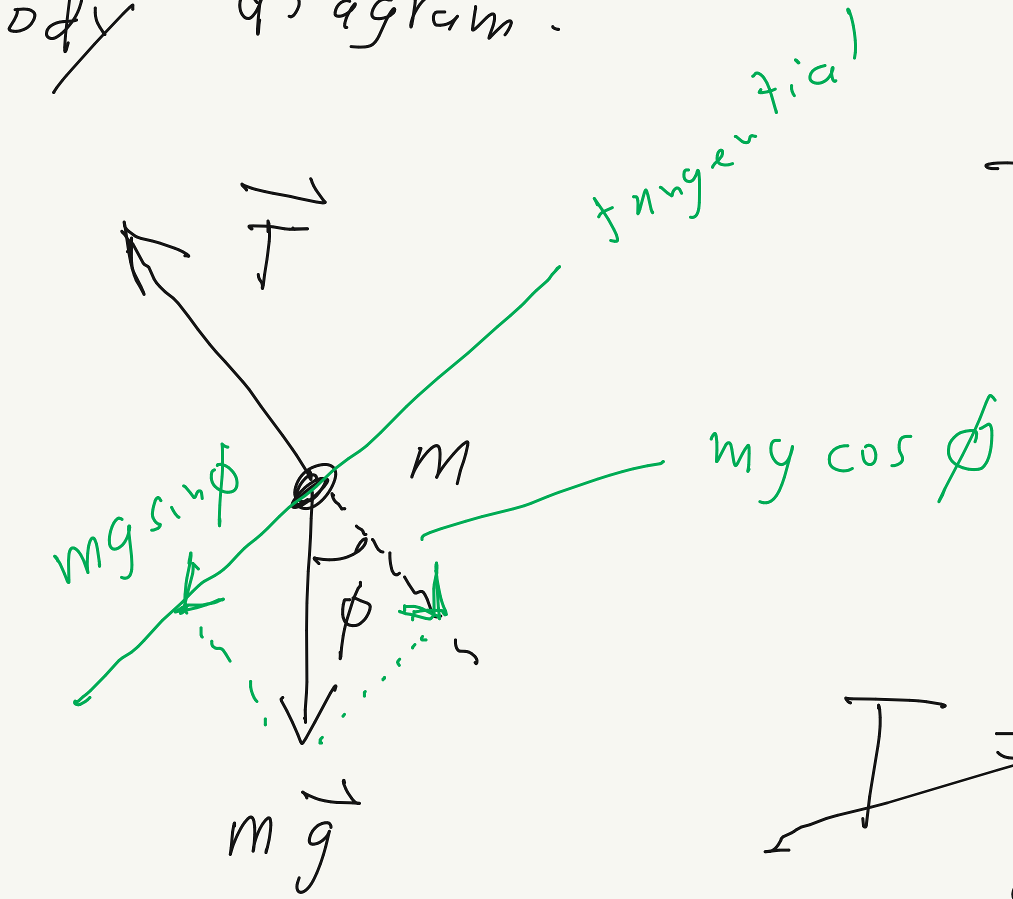
$$\alpha = \ddot{\phi}, \quad \omega = \dot{\phi}$$



$$N = mg\cos\alpha$$



Free-body diagram:



$$\cancel{T = mg\cos\phi}$$

1) $$T - mg\cos\phi = m\underbrace{\dot{\phi}^2 \ell}_{\text{centripetal acceleration}}$$

$$\boxed{T = m\dot{\phi}^2 \ell + mg\cos\phi}$$

2) $$mg\sin\phi = -m\,a_{\text{tangential}}$$

$$= -m\ell\ddot{\phi}$$

$$\boxed{\ddot{\phi} = -\frac{g}{\ell}\sin\phi}$$ EOM

$$\ddot{\phi} = -\frac{g}{l} \sin\phi$$ (hard to solve; 2^{nd} order non-linear ODE)

Small-angle approx: $\phi \ll 1$ rad $\cong 57^\circ$ (π radians $= 180^\circ$)

$$\rightarrow \quad \ddot{\phi} = -\frac{g}{l}\phi \qquad \omega_0 \equiv \sqrt{\frac{g}{l}}$$

complex

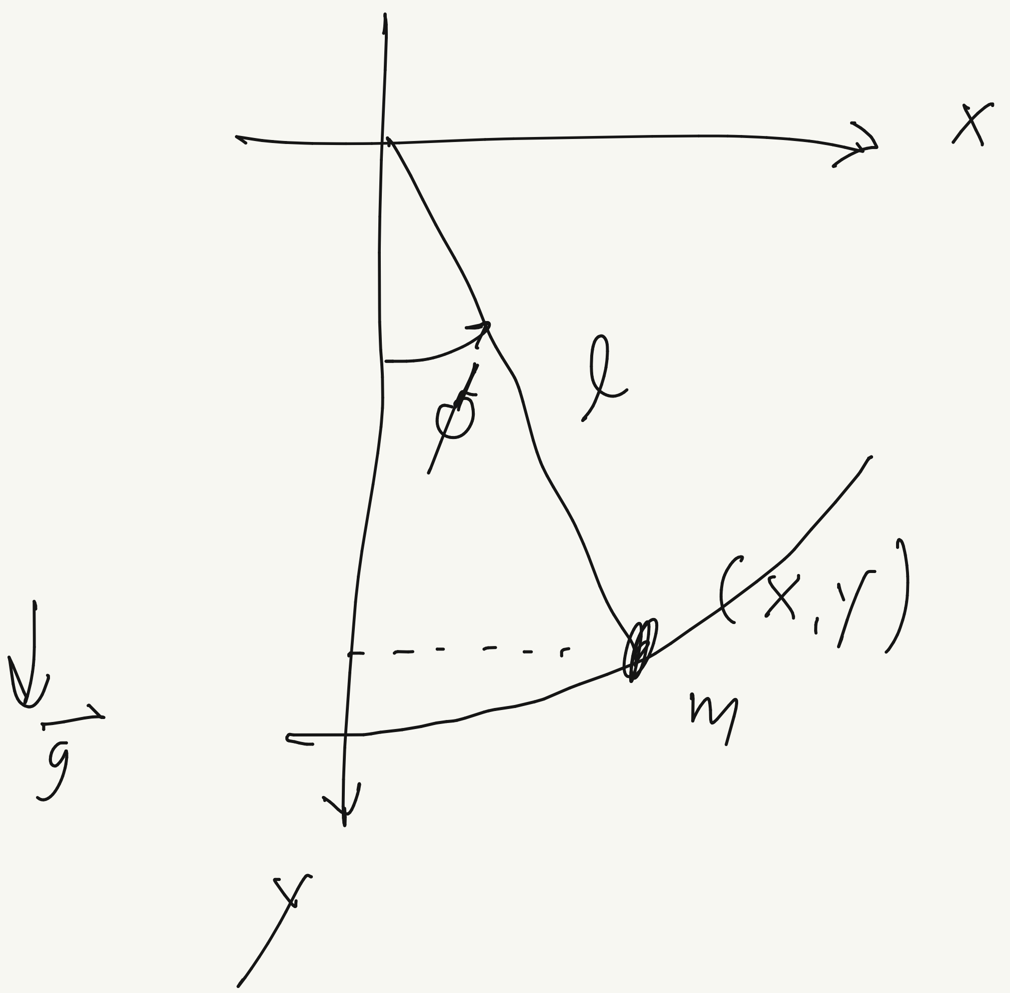
$$\phi(t) = A e^{-i\sqrt{\frac{g}{l}}t}$$

$$= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

$$= a \cos(\omega_0 t + \alpha)$$

Lagrangian formalism:

$$L = T - U$$



$$U = -mgy = -mgl\cos\phi$$

$$y = l\cos\phi$$
$$x = l\sin\phi$$
$$\downarrow$$
$$\dot{x} = l\dot{\phi}\cos\phi$$
$$\dot{y} = -l\dot{\phi}\sin\phi$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}ml^2\dot{\phi}^2$$

$$\boxed{L = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi} \quad \cancel{+U(t)}$$

(x,y) $\quad x = r\cos\phi$

(r,ϕ) $\quad y = r\sin\phi$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi}$$

$$\frac{d}{dt}(ml^2\dot{\phi}) = -mgl\sin\phi$$

$$ml^2\ddot{\phi} = -mgl\sin\phi$$

$$\ddot{\phi} = \frac{-g}{l}\sin\phi$$

L does not depend explicitly on time t $\rightarrow$ E is conserved.

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$$

$$= T + U$$

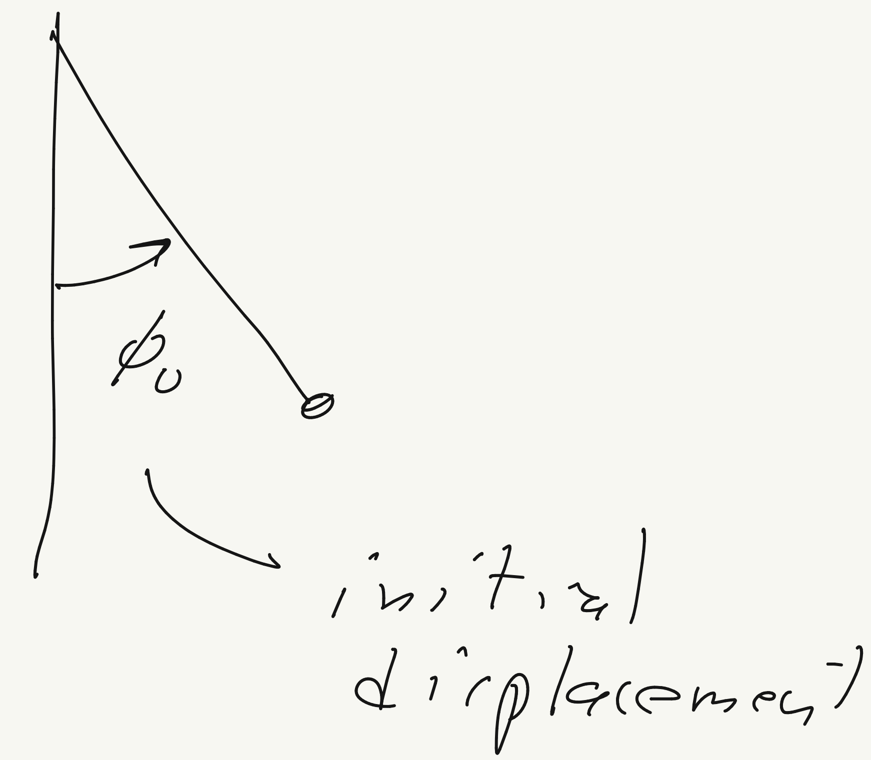
<u>Example:</u>

$$\boxed{E} = \frac{\partial L}{\partial \dot\phi}\phi - L$$

$$= ml^2\dot\phi\,\dot\phi - \left(\frac{1}{2}ml^2\dot\phi^2 + mgl\cos\phi\right)$$

$$= \frac{1}{2}ml^2\dot\phi^2 - mgl\cos\phi$$

$$= T + U$$



$$E = -mgl\cos\phi_0$$

$$-\cancel{m}gl\cos\phi_0 = \frac{1}{2}\cancel{m}l^2\dot\phi^2 - \cancel{m}gl\cos\phi$$

$$\dot\phi^2 = \frac{2}{l^2}\left(gl\cos\phi - gl\cos\phi_0\right)$$

L&L 11.1

$$\dot{\phi}^2 = 2\,\frac{g}{l}\left(\cos\phi - \cos\phi_0\right)$$

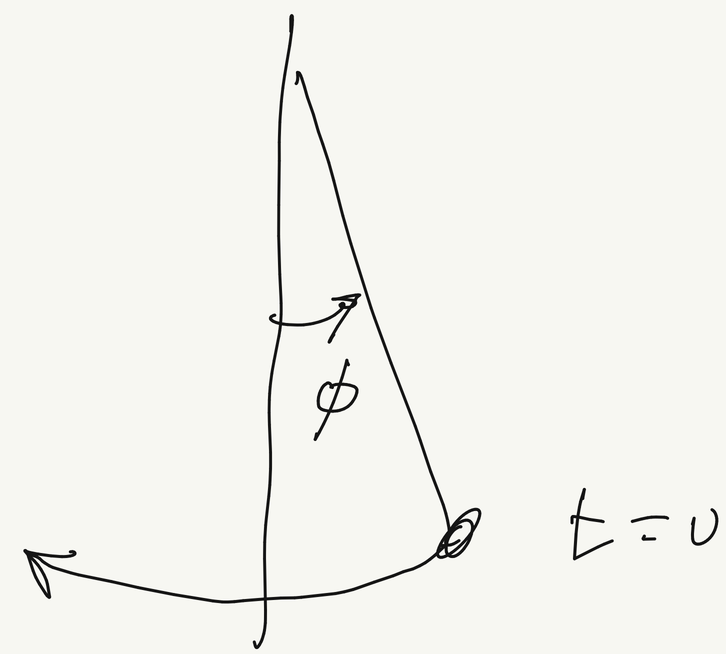
$$\frac{d\phi}{dt} = \dot{\phi} = \pm\sqrt{2}\,\underbrace{\sqrt{\frac{g}{l}}}_{\omega_0}\,\sqrt{\cos\phi - \cos\phi_0}$$

$$\int_0^t dt = \int_{\phi_0}^{\phi} \frac{-\,d\phi}{\sqrt{2}\,\omega_0\sqrt{\cos\phi - \cos\phi_0}}$$

$$\omega_0 t + \text{const} = \pm\int \frac{d\phi}{\sqrt{2}\sqrt{\cos\phi - \cos\phi_0}}$$

$$\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t + \text{const}$$

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} =$$



$$\cos\phi = \cos\left(2\cdot\frac{\phi}{2}\right) = \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right) = 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\cos\phi_0 = 1 - 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$\rightarrow \sqrt{\cos\phi - \cos\phi_0} = \sqrt{2}\sqrt{\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)} = \sqrt{2}\left|\sin\left(\frac{\phi_0}{2}\right)\right|\sqrt{1 - \frac{\sin^2(\phi/2)}{\sin^2(\phi_0/2)}}$$

$$k = \left| \sin\left(\frac{\phi_0}{2}\right) \right| \qquad (0 < k < 1)$$

$$x = \frac{\sin\left(\frac{\phi}{2}\right)}{\left|\sin\left(\frac{\phi_0}{2}\right)\right|} = \frac{\sin\left(\frac{\phi}{2}\right)}{k} \rightarrow dx = \frac{1}{k}\frac{1}{2}\cos\left(\frac{\phi}{2}\right) d\phi$$

$$= \frac{1}{2k}\sqrt{1-\sin^2\left(\frac{\phi}{2}\right)}\, d\phi$$

$$= \frac{1}{2k}\sqrt{1-k^2x^2}\, d\phi$$

$$\boxed{d\phi = \frac{2k\,dx}{\sqrt{1-k^2x^2}}}$$

$$\boxed{\sqrt{2}\sqrt{\cos\phi - \cos\phi_0} = 2k\sqrt{1-x^2}}$$

$$\omega_0 t + \text{const} = \pm \int \frac{\cancel{2k}\,dx}{\sqrt{1-k^2x^2}\;\cancel{2k}\sqrt{1-x^2}}$$

$$= \int_0^x \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

$$= \mathrm{sn}^{-1}(x; k)$$

$$\begin{cases} x = \dfrac{\sin\left(\frac{\phi}{2}\right)}{k} \\ k = \left|\sin\left(\frac{\phi_0}{2}\right)\right| \end{cases}$$

$t = 0 \longleftrightarrow \phi = \phi_0$

$$\text{const} = \int_0^1 \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}} \equiv \underbrace{K(k)}_{\substack{\text{complete elliptic}\\ \text{integral of the}\\ \text{1st kind}}}$$

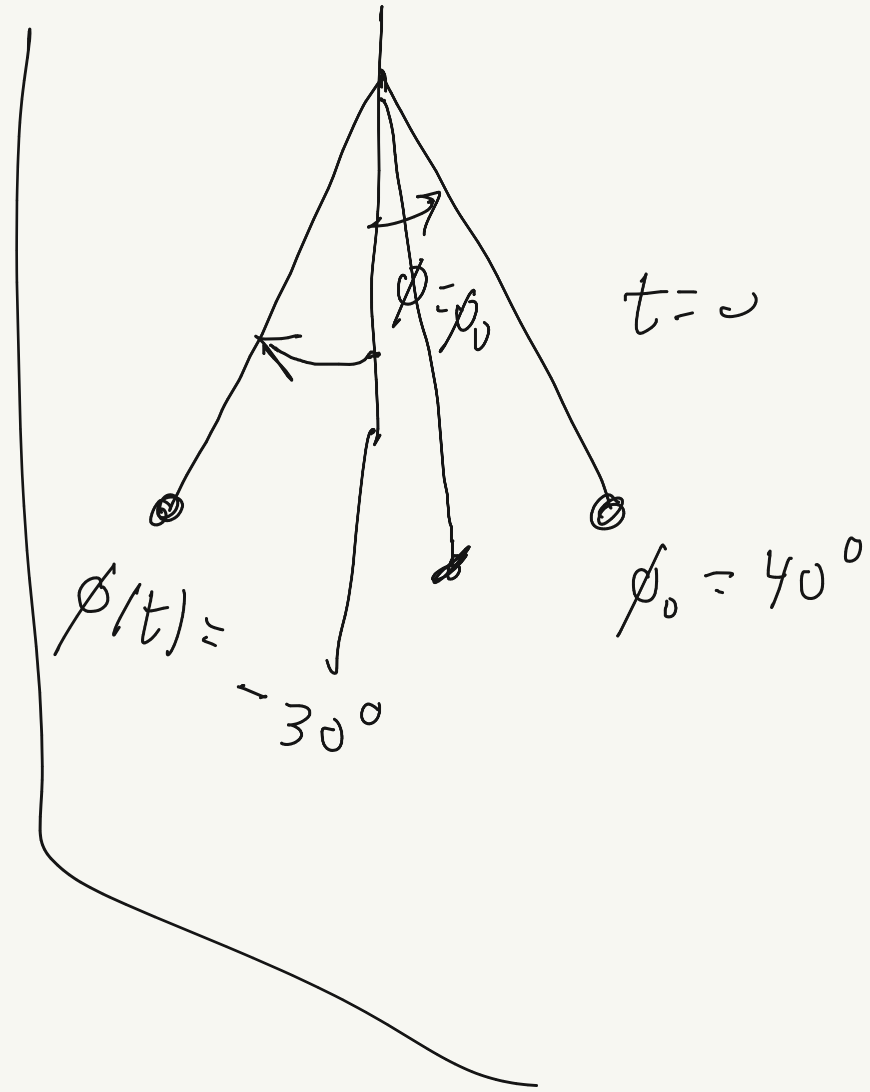
$$\omega_0 t + K(k) = sn^{-1}\left(\frac{\sin\left(\frac{\phi}{2}\right)}{k};\ k\right) \qquad \Big|\ \underbrace{k \equiv \sin\left(\frac{\phi_0}{2}\right)}$$

$$sn\left(\omega_0 t + K(k); k\right) = \frac{1}{k}\sin\left(\frac{\phi}{2}\right)$$

$$\longrightarrow \boxed{\phi(t) = 2\arcsin\left(k\, sn(\omega_0 t + K(k); k)\right)}$$

$$P = \frac{4}{\omega_0} K(k) \longrightarrow \frac{\omega_0 P}{4} = K(k)$$

Lec #4: 2 Sep 2021



$$\omega_0 \int_0^t dt = -\int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\ }}$$

$$\omega_0 t = -\left[\int_{\phi_0}^{0} + \int_0^{\phi}\right]\frac{d\phi}{\sqrt{\ }}$$

$$= +\int_0^{\phi_0}\frac{d\phi}{\sqrt{\ }} - \int_0^{\phi}\frac{d\phi}{\sqrt{\ }}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} - \int_0^x \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}}$$

$$= K(k) - sn^{-1}(x;k)$$

$$x = \frac{\sin\left(\frac{\phi}{2}\right)}{\left|\sin\left(\frac{\phi_0}{2}\right)\right|}$$

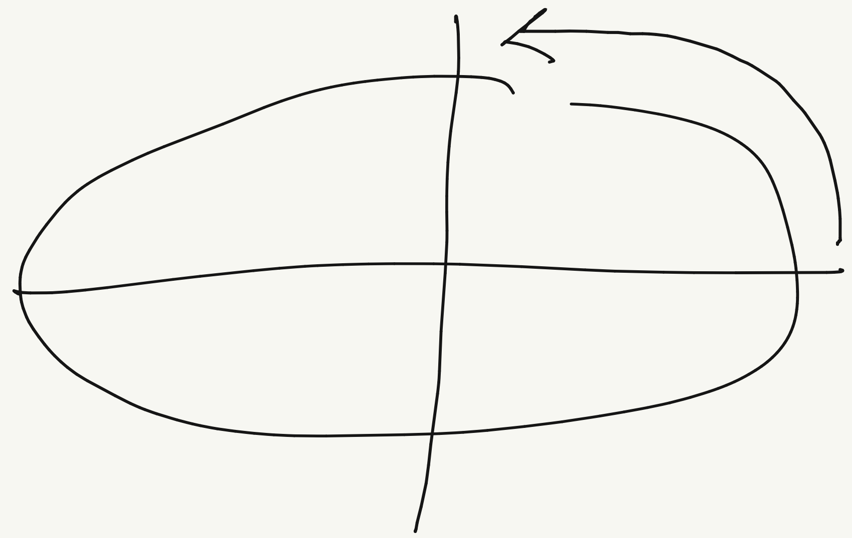
$$k = \left|\sin\left(\frac{\phi_0}{2}\right)\right|$$

$$sn^{-1}(x;k) = K(k) - \omega_0 t$$

$$\frac{\sin\left(\frac{\phi}{2}\right)}{k} = x = sn\left(K(k) - \omega_0 t;\, k\right) = cn(\omega_0 t; k)$$

$$\omega_0 = \sqrt{\frac{g}{l}}$$

$$\rightarrow \boxed{\phi(t) = 2 \arcsin\left(k\, sn\left(K(k) - \omega_0 t; k\right)\right)}$$



$$\frac{1}{\sqrt{1-k^2x^2}} \approx 1 + \frac{1}{2}k^2x^2$$

$$(1+\epsilon)^p \approx 1 + p\epsilon$$

$$\boxed{\omega_0 \frac{P}{4} = K(k)} \rightarrow \boxed{P = \frac{4}{\omega_0} K(k)}$$

$$K(k) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

$\underline{k=0:}$

$$K(0) = \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} 1 = \pi/2$$

$$P = \frac{4}{\omega_0}\frac{\pi}{2} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}$$

$0<k<<1$

$$K(k) = \int_0^1 \frac{dx}{\sqrt{1-x^2}}\left(1 + \frac{1}{2}k^2x^2\right) = \frac{\pi}{2} + \int_0^1 \frac{dx\ \frac{1}{2}k^2x^2}{\sqrt{1-x^2}}$$

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$$

pf: $$\sin\left(\frac{\pi}{2}-\theta\right) = \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}\cos\theta - \cancel{\cos\left(\frac{\pi}{2}\right)}\sin\theta$$

$$= \cos\theta$$

$$\sin(-\theta) = -\sin\theta \quad , \quad \cos(-\theta) = \cos\theta$$

$$\mathrm{sn}(K(k) - x;\, k) = \mathrm{cn}(x)$$

Lagrange multipliers

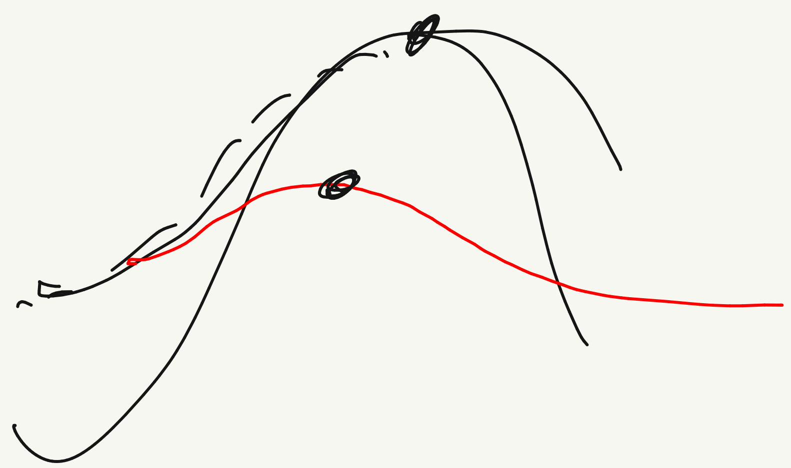
$f(x)$: max or min ?

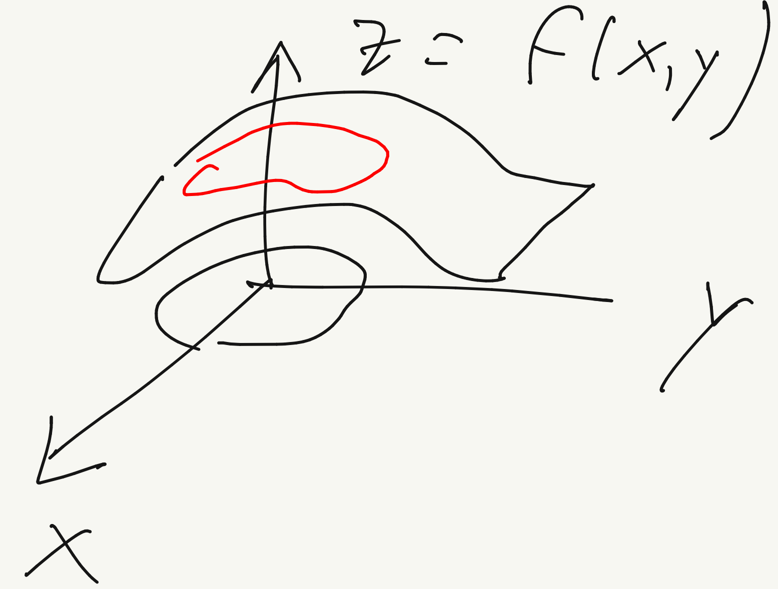
$f(x,y)$: "

$$\frac{df}{dx} = 0$$

$$\frac{\partial f}{\partial x} = 0 , \quad \frac{\partial f}{\partial y} = 0$$

$$\left.\frac{df}{dx}\right|_{x_0} = 0$$





$f(x,y)$: Find extreme value subject to constraint $\underbrace{\varphi(x,y)=0}$?

1) "reduced space method":

solve constraint $\varphi(x,y)=0 \rightarrow y = g(x)$

$$F(x) = f(x,y)\Big|_{y=g(x)}$$

e.g., $\underbrace{x^2+y^2=1}$

$$2x\,dx + 2y\,dy = 0$$

$$dx = \frac{-y}{x}\,dy$$

2) "method of Lagrange multipliers"

$$F(x,y,\lambda) = f(x,y) + \lambda\,\varphi(x,y) \quad \begin{cases} \dfrac{\partial F}{\partial x} = 0 \\[2ex] \dfrac{\partial F}{\partial y} = 0 , \quad \dfrac{\partial F}{\partial \lambda} = 0 \end{cases}$$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0$$

$$F(x, y, \lambda) = f(x, y) + \lambda \varphi(x, y)$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0$$

$$\frac{\partial F}{\partial \lambda} = \varphi(x, y) = 0$$

$$\varphi(r, \phi) = 0 = r - l$$

$$\underbrace{L(r, \phi, \dot{r}, \dot{\phi}, t) + \lambda \overbrace{(r - l)}^{\varphi(r,\phi)}} = L'$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) = \frac{\partial L'}{\partial q}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} + \lambda \frac{\partial \varphi}{\partial q}$$

$$\frac{dp}{dt} = -\frac{\partial U}{\partial q} + \lambda \frac{\partial \varphi}{\partial q} = F_{app} + F_{constraint}$$