Elementary Music Theory, Musical Scales, Intervals, Cents

Piano Keyboard

For those interested in musical scales and intervals, the following is some cursory music theory. Perhaps you are interested in musical intervals such as octaves, fifths, fourths, etc., or how to translate "do, re, mi, fa, sol, la, ti, do" into a musical scale.
Also, it may be useful to know the basics behind the so-called "equal tempered scale" that has dominated Western music for about two centuries.
The musical stave system and notation are given for completeness.
Note that "concert-A" in the United States is A4 = 440 Hz.
Recognize some of the systematic features in the layout of the piano keys.

Piano Keyboard

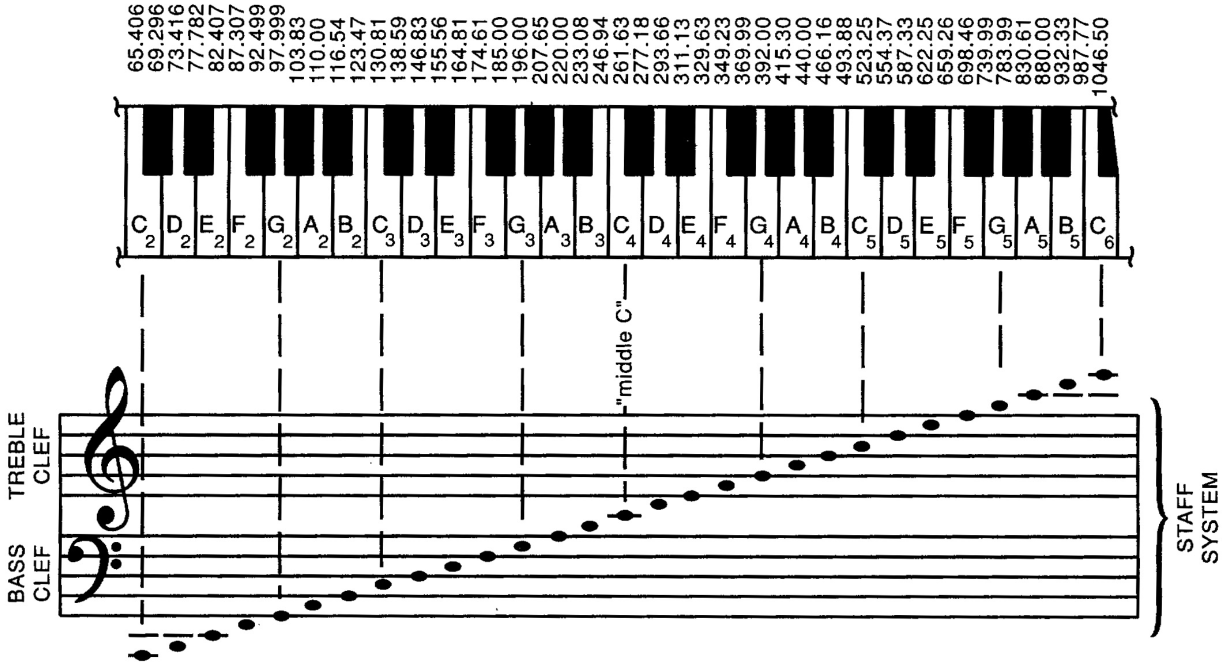

Figure. Middle section of the piano keyboard, frequencies, and notes on the musical staves. (From Berg & Stork, Fig. A-1, p. 368.)

Circle of Fifths, Pythagorean Comma, Equal Tempered Scale

Start with any musical note on the keyboard, e.g. C1, move up 12 successive fifths, i.e. C1 - G1 - D2 - A2 - E3 - B3 - F4# - C5# - G5# - D6# - A6# - F7 - C8.
If we take pure fifths, the frequency ratio C8/C1 between the ending and starting note is $(3/2)^{12}$ = 531,441/4,096 = **129.7463**.
Now start with C1 again and move up 7 octaves, i.e.
C1 - C2 - C3 - C4 - C5 - C6 - C7 - C8
The ratio C8/C1 this time is 2^7 = **128.0000**.
We see that the 12 pure fifths result in a slightly higher frequency than the 7 octaves. The ratio of the two numbers is slightly different from unity:

$$\mathbf{(3/2)^{12}/(2^7) = 129.7463/128.0000 = 1.013643}\ldots$$, called the **"**Comma of Pythagoras"

Demonstration
Choose f = 700 Hz ("7 octaves") and f = 700·1.013643 = 709.6 Hz ("12 fifths") on two signal generators. Listen to the beats. You are "hearing" the Pythagorean comma.

The *Pythagorean comma* is the difference of 1.3643% between 12 pure fifths and 7 octaves. The comma is clearly audible. It corresponds to a difference of approximately 1/4 of a semitone, which is something to be avoided in music.
The so-called *"Circle of Fifths"* of 12 pure fifths does not close exactly into 7 octaves. But we ended with the same note (C8) on the keyboard! This forced agreement is a compromise resulting from the so-called "tempering" of the fifths.

The problem of the Pythagorean comma existed for several hundred years. Finally and ingeniously, the discrepancy was spread equally over all 12 fifths by lowering or "tempering" them all equally. This led to the ***Equal-Tempered Scale*** **(ET)** as follows. Let us designate the tempered fifth by the symbol *y*. It is no longer equal to 3/2. On the other hand, we do wish to preserve the exact ratio of 2/1 for the octave. We thus require 12 "tempered" fifths to equal exactly 7 octaves. If we call the tempered fifth "y", then

$$\mathbf{y^{12} = 2^7 \quad or \quad y = 2^{7/12} = 1.498307\ldots.}$$

The tempered fifth in equal temperament (ET) is smaller than 3/2 by 0.113%.

The 12-tone chromatic scale has 12 half-steps (semitones) in an octave. Therefore the frequency ratio between two notes a semitone apart is given by

$$\mathbf{2^{(7/12)/7} = 2^{1/12} = 1.059464\ldots}$$ = semitone spacing in the *equal-tempered (ET) scale*.

If f_0 is the frequency of a note, then the frequencies of the 12 notes n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 in the chromatic scale are

$$\mathbf{f_n = 2^{n/12} f_0}\,, \qquad \text{with } f_0 = \text{tonic and } f_{12} = 2^{12/12} f_0 = 2f_0 \text{ (octave).}$$

Revisiting Musical Fifths, Octaves, Pythagorean Scale, and Wolf Fifth

We rephrase our preceding discussion and ask:
Can one build a 12-tone scale from musical fifths and *octaves* alone? The answer is "yes", but we must use modified, not Pythagorean, fifths.
Let m and n be two positive integers. We then ask for a solution to the equation

$$(3/2)^m = 2^n \quad \text{or} \quad \frac{m}{n} = \frac{\log 2}{\log 3/2} = 1.709511.....$$

But no exact solution exists for any pair of integers (m, n). Hence one cannot pack exactly an integer number of Pythagorean fifths into an integer number of octaves, and so the circle of fifths does not quite close. We ask instead:
For which set of *small* integers (m, n) do we come close to the ratio of 1.70951…?
The following table shows the ratios m/n of the exponents for the fifths and octaves.

Number m of Fifths and Number of n of Octaves and Corresponding Ratios m/n
(Table computed by Professor Igor Volobouev, Texas Tech University)

m\n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	0.5	0.333333	0.25	0.2	0.166667	0.142857	0.125	0.111111	0.1	0.0909091	0.0833333	0.0769231	0.0714286
2	2	1	0.666667	0.5	0.4	0.333333	0.285714	0.25	0.222222	0.2	0.181818	0.166667	0.153846	0.142857
3	3	1.5	1	0.75	0.6	0.5	0.428571	0.375	0.333333	0.3	0.272727	0.25	0.230769	0.214286
4	4	2	1.33333	1	0.8	0.666667	0.571429	0.5	0.444444	0.4	0.363636	0.333333	0.307692	0.285714
5	5	2.5	1.66667	1.25	1	0.833333	0.714286	0.625	0.555556	0.5	0.454545	0.416667	0.384615	0.357143
6	6	3	2	1.5	1.2	1	0.857143	0.75	0.666667	0.6	0.545455	0.5	0.461538	0.428571
7	7	3.5	2.33333	1.75	1.4	1.16667	1	0.875	0.777778	0.7	0.636364	0.583333	0.538462	0.5
8	8	4	2.66667	2	1.6	1.33333	1.14286	1	0.888889	0.8	0.727273	0.666667	0.615385	0.571429
9	9	4.5	3	2.25	1.8	1.5	1.28571	1.125	1	0.9	0.818182	0.75	0.692308	0.642857
10	10	5	3.33333	2.5	2	1.66667	1.42857	1.25	1.11111	1	0.909091	0.833333	0.769231	0.714286
11	11	5.5	3.66667	2.75	2.2	1.83333	1.57143	1.375	1.22222	1.1	1	0.916667	0.846154	0.785714
12	12	6	4	3	2.4	2	1.71429	1.5	1.33333	1.2	1.09091	1	0.923077	0.857143
13	13	6.5	4.33333	3.25	2.6	2.16667	1.85714	1.625	1.44444	1.3	1.18182	1.08333	1	0.928571
14	14	7	4.66667	3.5	2.8	2.33333	2	1.75	1.55556	1.4	1.27273	1.16667	1.07692	1
15	15	7.5	5	3.75	3	2.5	2.14286	1.875	1.66667	1.5	1.36364	1.25	1.15385	1.07143
16	16	8	5.33333	4	3.2	2.66667	2.28571	2	1.77778	1.6	1.45455	1.33333	1.23077	1.14286
17	17	8.5	5.66667	4.25	3.4	2.83333	2.42857	2.125	1.88889	1.7	1.54545	1.41667	1.30769	1.21429
18	18	9	6	4.5	3.6	3	2.57143	2.25	2	1.8	1.63636	1.5	1.38462	1.28571
19	19	9.5	6.33333	4.75	3.8	3.16667	2.71429	2.375	2.11111	1.9	1.72727	1.58333	1.46154	1.35714
20	20	10	6.66667	5	4	3.33333	2.85714	2.5	2.22222	2	1.81818	1.66667	1.53846	1.42857
21	21	10.5	7	5.25	4.2	3.5	3	2.625	2.33333	2.1	1.90909	1.75	1.61538	1.5
22	22	11	7.33333	5.5	4.4	3.66667	3.14286	2.75	2.44444	2.2	2	1.83333	1.69231	1.57143
23	23	11.5	7.66667	5.75	4.6	3.83333	3.28571	2.875	2.55556	2.3	2.09091	1.91667	1.76923	1.64286
24	24	12	8	6	4.8	4	3.42857	3	2.66667	2.4	2.18182	2	1.84615	1.71429

In this table the m-numbers are shown in the left-most column and the n-numbers in the top row. The other columns and rows show the corresponding ratios m/n.

We see that for **m = 12 and n = 7** we have

$$\frac{m}{n} = \frac{12}{7} = 1.714286..... \quad \text{This is closest to the number } 1.709511…$$

The 10 ratios closest to the ideal value m/n = 1.709511 are:

1. m = 12, n = 7, ratio = 1.71429
2. m = 24, n = 14, ratio = 1.71429
3. m = 17, n = 10, ratio = 1.70000
4. m = 22, n = 13, ratio = 1.69231
5. m = 19, n = 11, ratio = 1.72727
6. m = 7, n = 4, ratio = 1.75000
7. m = 14, n = 8, ratio = 1.75000
8. m = 21, n = 12, ratio = 1.75000
9. m = 5, n = 3, ratio = 1.66667
10. m = 10, n = 6, ratio = 1.66667

For all other sets (m, n) for m from 1 to 24 and n from 1 to 14, the ratio m/n is larger than for 12/7 = 1.714286, although other ratios, especially 17/10 and 19/11, are also close the the ideal value of 1.709511.

The number m = 12 yields the 12 degrees in the octave of the *Chromatic Scale* in Western music, with all notes constructed solely from fifths and octaves.

In order to construct the chromatic scale, we move 12 fifths up (or down) and scale them down (or up) by octaves, so that we always land within the original octave. This yields the 12 notes (degrees) in that octave. (See next page for the detailed construction.)

How can we close the "circle of fifths" so that 12 Pythagorean fifths exactly equal 7 octaves?
We show two possibilities:
1. In the Pythagorean 12-tone scale, all fifths have values of 3/2 except the last one. This bad fifth or **"wolf fifth"** is obtained by moving up 7 octaves and 11 Pythagorean fifths of **3/2** from the same starting note and taking the ratio $2^7/(3/2)^{11}$ = **1.479810**… **≠ 3/2**.
2. In the equal tempered scale, all fifths "*y*" are modified or tempered in the same way according to $y^{12} = 2^7$, or **y = 1.498307…**, as already mentioned.

Exercise
Show that the bad "wolf fifth" deviates from a pure fifth by the comma of Pythagoras.
Answer: Take the ratio 1.500000/1.479810 = 1.01364…, which is the "Comma of Pythagoras".

The "wolf fifth" caused serious problems in musical modulation and transposition.
The problem was solved after many years by the introduction of the equal tempered scale around the time of Johann Sebastian Bach. The harmony in the pure Pythagorean intervals was slightly damaged as a consequence, but this was outweighed by getting rid of the "wolf fifth" and the ability of composing in many different musical keys.

Construction of Musical Scales from Musical Fifths and Octaves

The 12-degree (12-tone) Pythagorean chromatic scale can be constructed entirely from pure fifth (ratio 3/2 or 2/3) and octaves (2/1 or 1/2). This is done by moving up in frequency by ratios of 3/2 and dividing the result by octave ratios of 1/2 to bring the final result to within one octave from the starting frequency. A number of up to 12 steps of fifths is needed for this procedure in order to obtain the 12 notes in the chromatic scale. The final result is obtained by ordering the 12 results in ascending numerical values. The last note will then be an octave above the starting note.

Example
Construct the note $C^{\#}$ a half step above C.
Answer
Go up 7 fifths, divide by 4 octaves, and get
$C^{\#}/C = (3/2)^7/2^4 = 218700/204800 = 1.06787$.
Reminder: In equal temperament the ratio is $C^{\#}/C = 2^{1/12} = 1.05946$.

The construction of the other 11 notes in the 12-tone scale is left as an exercise for those who are more deeply interested in music theory. However, see later in this chapter for the explicit construction of the 7-note Pythagorean scale, which is a subset of the 12-note scale.

Wolf Fifth
The construction of the 12-note Pythagorean scale and using it in different musical keys is hampered by the Wolf fifth. Hence again, equal temperament (ET) is the preferred tuning in most of Western music.

Demonstrations
1. Compare a “wolf fifth” of 1.479810 with a pure fifth of 3/2 = 1.500000:
Set two signal generators to $f_1 = 400$ Hz and $f_2 = 400\text{x}1.500000 = 600.0$ Hz.
Listen to the pure fifth $f_2/f_1 = 600/400 = 1.500000$.
Set a third frequency generator to $f_3 = 400\text{x}1.479810 = 591.9$ Hz.
Listen to the “wolf fifth” $f_3/f_1 = 591.9/400 = 1.479810$.
This probably sounds bad to most people.
The “wolf” beats with a frequency of $\Delta f = f_2 – f_4 = 600 - 591.9 = 8.1$ Hz.

2. Compare an equally tempered fifth of 1.498307 with a pure fifth of 1.500000:
Set a fourth frequency generator to the fifth in equal temperament,
i.e. $f_4 = 400\text{x}1.498307… = 599.32$ Hz.
Play f_2 and f_4 together and hear very slow beats at $\Delta f = f_2 – f_4 = 0.68$ Hz.
Listen to the fifth in equal temperament, $f_2/f_1 = 599.32/400 = 1.498307$, and alternate with the pure fifth.
Many people will not be able to hear the difference.
Equal temperament has now become the standard in Western music.

Killing the Wolf, Equal Temperament and Ruining Harmony

Killing the "wolf" and "sweeping the Pythagorean comma under the rug" became ever more important as music progressed and became more complex. Changes were necessary for musical transposition and modulation, and the tuning of different instruments.

In equal temperament (ET), the discrepancy of the Pythagorean comma is spread *equally* over the 12 notes of the chromatic scale.
All pure intervals are forsaken, except the octave, whose interval remains a ratio of 2:1. We already have seen that we require 12 modified fifths to equal exactly 7 octaves, yielding a frequency ratio of $2^{(7/12)} = 1.498307\ldots$ for the fifth in equal temperament.

The modified fifth is smaller than the ratio 3/2 for a pure fifth, but only by very little. Summarizing, this fifth then finally led to the *Equal-Tempered Scale* (ET).

In the 12-tone chromatic scale, a musical fifth contains 7 half steps or semitones, for instance from C4 to G4. "Tempering" all the 7 semitone intervals to the same size means

$$2^{(7/12)/7} = \mathbf{2^{1/12} = 1.059464\ldots}$$

The semitone interval therefore is given by a frequency ratio of 1.059464, or a frequency difference of **5.9464 %.**
This interval can be divided further (useful for the tuning of instruments) into so-called **musical cents**, where $\mathbf{2^{1/12} = 100}$ **cents**.

The frequencies of the 12 successive notes of the equal-tempered chromatic scale, starting with any note (e.g. f = C4 = 261.63 Hz), are given by

$f,\ 2^{1/12}f,\ 2^{2/12}f,\ 2^{3/12}f,\ 2^{4/12}f,\ 2^{5/12}f,\ 2^{6/12}f,\ 2^{7/12}f,\ 2^{8/12}f,\ 2^{9/12}f,\ 2^{10/12}f,\ 2^{11/12}f,\ 2^{12/12}f = 2f$

Example: The C-major *Equal-Tempered* Diatonic Scale
It is based on A4= 440.00 Hz, which yields C4 = 261.63 Hz, and is given by:

C4	D4	E4	F4	G4	A4	B4	C5
$2^{0/12}$	$2^{2/12}$	$2^{4/12}$	$2^{5/12}$	$2^{7/12}$	$2^{9/12}$	$2^{11/12}$	$2^{12/12}$
1	1.1225	1.2599	1.3348	1.4983	1.6818	1.8877	2.0000
261.63	293.66	329.63	349.23	392.00	440.00	493.88	523.25 Hz

More than a millennium had passed before the *Equal Tempered Scale* was finalized at the beginning of the 19th century. In retrospect, it does not look that complicated, neither musically nor mathematically, and it is simpler than using earlier scales. The latter do give purer harmonies that can be played by string instruments, trombone etc., but not by the piano with its fixed notes.

Construction of the *Pythagorean Diatonic Scale*

For constructing this historical scale, only musical fifths **3:2** and octaves **2:1** are needed. Start with C4 and move either up or down. We obtain

F3		C4		G4		D5		A5		E6		B6
	2/3		3/2		3/2		3/2		3/2		3/2	

Now rearrange all notes so that they fall within one single octave. For example, raise F3 by an octave to obtain F4, i.e. multiply the frequency by a factor of 2. Similarly, lower D5 to D4 by an octave and A5 to A4 by an octave, i.e. divide by a factor of 2. Next, lower E6 to E4 and B6 to B4, i.e. lower the frequency by 2 octaves or divide by $2^2 = 4$. Finally, add to this the note C5 one octave above C4. We now have arrived at all notes in the Pythagorean diatonic scale, summarized as follows:

C4 from C4	1	=	**1**
D4 from D5	3/2 x 3/2 x 1/2	=	**9/8**
E4 from E6	3/2 x 3/2 x 3/2 x 3/2 x 1/2 x 1/2	=	**81/64**
F4 from F3	2/3 x 2	=	**4/3**
G4 from G4	**3/2 given**	=	**3/2**
A4 from A5	3/2 x 3/2 x 3/2 x 1/2	=	**27/16**
B4 from B6	3/2 x 3/2 x 3/2 x 3/2 x 3/2 x 1/2 x 1/2	=	**243/128**
C5 from C4	**2/1 given**	=	**2**

The resulting frequencies, based on C4 = 261.63 Hz (not on A4 = 440 Hz) are:

C4	D4	E4	F4	G4	A4	B4	C5
do	re	mi	fa	sol	la	ti	do
1	9/8	81/64	4/3	3/2	27/16	243/128	2
261.63	294.33	331.11	348.83	392.44	441.49	496.67	523.25 Hz

Exercise
Musical fifths such as G_4/C_4 can also be obtained from harmonics (in this case from C_4), without resorting a priori to the ratio 3/2. Derive this ratio instead.
Answer: The first 3 harmonics (N = 1, 2, 3) in the harmonic series of C_4 are C_4, C_5, G_5. We see that $G_5 = 3C_4$. We also know that G_5 is an octave above G_4.
Hence $G_5 = 2G_4$ and $G_4 = G_5/2 = 3C_4/2$ and finally $G_4/C_4 = 3/2$.

Demonstration
Play the major triad C - E - D in the
Pythagorean scale 261.63 - 331.11 - 392.44 Hz, and compare with
equal temperament 261.63 - 329.63 - 392.00 Hz.
First, play the triads separately in the two tunings. Do you hear much difference?
Then play them together. Listen to the beats.

Construction of the *Pentatonic* and *Syntonic Diatonic Scales*

Pentatonic Scale
The Pythagorean Diatonic Scale contains the five notes of the *pentatonic scale* as a subset, resulting in one of the simplest and oldest musical scales.

C4	D4	F4	G4	A4	C5
1	9/8	4/3	3/2	27/16	2
261.63	294.33	348.83	392.44	441.49	523.25 Hz

Exercise
Construct the pentatonic scale C D F G A C yourself from fifths and octaves without copying the preceding information.

An example for the pentatonic scale in Western music is the beautiful melodic main theme in Dvorak's "New World Symphony", which uses only these 5 notes.

Syntonic Diatonic Scale
We now construct the so-called C-major syntonic diatonic scale in analogy to the Pythagorean scale, but this time with pure major thirds **5/4 (**according to Claudius Ptolemy) in addition to Pythagorean fifths **3/2** and octaves **2/1.** Since we are given these 3 intervals to start with and thus already have 3 notes in the scale, we construct the remaining 5 notes with the simplest possible rational numbers. The most complicated rational number of these will be 15/8 for the note B4. All other ratios will only contain single-digit numbers. Starting again with C4, the procedure is this:

C4 from C4	1	=	**1**
D4 from D5	3/2 x 3/2 x 1/2	=	**9/8**
E4 from E6	**5/4 given**	**=**	**5/4**
F4 from F3	2/3 x 2	=	**4/3**
G4 from G4	**3/2 given**	=	**3/2**
A4 from F4	4/3 x 5/4	=	**5/3**
B4 from G4	3/2 x 5/4	=	**15/8**
C5 from C4	**2/1 given**	=	**2**

Summary (with frequencies based on C4 = 261.63 Hz)

C4	D4	E4	F4	G4	A4	B4	C5
do	re	mi	fa	sol	la	ti	do
1	9/8	5/4	4/3	3/2	5/3	15/8	2
261.63	294.33	327.03	348.83	392.44	436.05	490.55	523.25 Hz

The *syntonic diatonic scale* is also called a *just major scale*.
It goes back to Claudius Ptolemy (Roman citizen in Egypt, mathematician, astronomer and astrologer, 90-168 A.D.)

Frequencies of Three Diatonic Musical Scales. Musical Cents

All frequencies are based on C4 = 261.63 Hz.

do	re	mi	fa	sol	la	ti	do
Equal Temperament							
C4	D4	E4	F4	G4	A4	B4	C5
$2^{0/12}=1$	$2^{2/12}$	$2^{4/12}$	$2^{5/12}$	$2^{7/12}$	$2^{9/12}$	$2^{11/12}$	$2^{12/12}=2$
1	1.1225	1.2599	1.3348	1.4983	1.6818	1.8877	2.0000
261.63	293.66	329.63	349.23	392.00	440.00	493.88	523.25 Hz
Pythagorean Scale							
C4	D4	E4	F4	G4	A4	B4	C5
1	9/8	81/64	4/3	3/2	27/16	243/128	2
261.63	294.33	331.11	348.83	392.44	441.49	496.67	523.25 Hz
Just Major (Syntonic Diatonic) Scale							
C4	D4	E4	F4	G4	A4	B4	C5
1	9/8	5/4	4/3	3/2	5/3	15/8	2
261.63	294.33	327.03	348.83	392.44	436.05	490.55	523.25 Hz

We see that the frequency discrepancies between the three scales generally are within ±1% or less. The greatest discrepancies are in the notes E4, A4, B4, the smallest in D4, F4, G4. These differences are audible when comparing the same notes from different scales. One easily can hear beats between the same notes (demonstration). Equal temperament contains compromises by equalizing all semitone intervals. We have become quite used to this in spite of the slight destruction of harmony and the surrender of pure intervals.

Musical Cents

The smallest musical interval in Western music is the spacing between discrete semitones or half steps in equal temperament, see for instance the keys on the piano. However, deviations from this occur in mistuned instruments and in different temperaments. In order to quantify such small variations, the *musical cent scale* is used. It is defined by

$$\mathbf{c = 1200 \frac{\log f_2/f_1}{\log 2}},$$

where "*c*" is the number of musical "cents" for the interval f_2/f_1 between two frequencies f_2 and f_1. Obviously, one octave with $f_2/f_1 = 2$ is 1200 cents.

Intervals of 10 cents or even less at mid-frequencies are audible to many listeners.

Exercise: Show that a half step $2^{1/12}$ in equal temperament is 100 cents.

Proof: $f_2/f_1 = 2^{1/12} \rightarrow c = 1200\cdot\log(2^{1/12})/\log 2 = 1200\cdot(1/12)\cdot\log 2/\log 2 = 100$ cents.

Demonstration: Play *pure* thirds or fifths C4-E4 or C4-G4 and compare with *equal-tempered* thirds and fifths. Note the resulting beats!

Demonstration of Musical Fifths and Thirds with a 4-Chime Set

A set of 4 chimes was custom-built by *Music of the Spheres* in Austin, Texas, to demonstrate musical intervals. The chimes were made of aluminum tubing with an outside diameter of 6.34 cm and inside diameter 5.37 cm.

The available intervals with these chimes are a pure Ptolemaic major third A3-C$^{\#}$4 with an interval ratio of 5/4, as well as this third in equal temperament with the ratio $2^{4/12}$, and lasstly a pure Pythagorean fifth A3-E4 with the ratio 3/2.

Table. Details of the 4-chime set: Musical notes available, corresponding tuning and frequencies, and lengths of the four chimes

A3		220.00 Hz	chime length 129.65 cm
C$^{\#}$4 pure, just Ptolemaic	220.00 x 5/4	= 275.00 Hz	115.65 cm
C$^{\#}$4 ET, equal temperament	220.00 x $2^{4/12}$	= 277.18 Hz	115.20 cm
E4 pure, Pythagorean	220.00 x 3/2	= 330.00 Hz	105.45 cm

Demonstrations

1. Strike the two chimes C$^{\#}$4 “pure” and C$^{\#}$4 “ET” with a soft beater at their center to excite the fundamental vibrational mode. Listen to the beat frequency of about 2.2 Hz (14 cents or about 1/7 of a semitone).
2. Strike the above two chimes near their ends to excite the 2nd vibrational mode. (Higher modes may be excited at the same time.) Listen to the beats again. You should predominantly hear beats with a 2.756 higher frequency than before, i.e. about 6.0 Hz. This is due to the fact that the 2nd mode is inharmonic with a 2.756-times higher frequency than the fundamental mode. In addition, you may still hear the beats at 2.2 Hz from the fundamental mode as before.
3. Strike the three chimes A3, C$^{\#}$4 “*pure*”, and E4 “pure” together and listen to the major triad.
4. Strike the three chimes A3, C$^{\#}$4 “*ET*”, and E4 “pure” together and listen to the major triad. Do you hear a difference?
5. Strike all 4 chimes together and listen to the major triad, but now containing the beats between C$^{\#}$4 “*pure*” and C$^{\#}$4 “*ET*”.

The Comma of Didymus

How much do the musical major thirds of Pythagoras (81/16) and Claudius Ptolemy (5/4) differ?
The answer is that they differ by the so-called **Comma of Didymus**, which we can obtain numerically in the following way.

We start with the diatonic scale

do	re	mi	fa	sol	la	ti	do
C4	D4	E4	F4	G4	A4	B4	C5

We have for the intervals of the diatonic scale of Pythagoras and its major third E4/C4:

1/1	9/8	**81/64**	4/3	**3/2**	27/16	243/128	**2/1**

For the syntonic diatonic scale of Claudius Ptolemy we have:

1/1	9/8	**5/4**	4/3	**3/2**	5/3	15/8	**2/1**

From this follow for the ratio of the major thirds of Pythagoras and Ptolemy:

$$\frac{81/64}{5/4} = \frac{81}{80} = 1.0125.$$

This is the *"Comma of Didymus"*, also called the *"syntonic comma"*.

We see that the Pythagorean major third is "sharp" with respect to the Ptolemaic major third by

$$c = 1200\frac{\log f_2/f_1}{\log 2} = 1200\frac{\log 81/80}{\log 2} = 21.5 \text{ cents}.$$

The difference of 21.5 cents is about 1/5 of a semitone, with easily audible beats between the two intervals.

A Musical Comma at Texas Tech University

There is a work of art, a big shiny metal sphere, between the Student Union Building and Library at Texas Tech University, called “Comma”. Push in, at the same time, the two pistons on opposite sides of the sphere. Two slightly different high-pitched tones are generated. You should hear a beat frequency between them, called the “Comma” here.

The plaque “Comma” next to the sculpture does not say which musical comma is meant. It probably does not mean a ratio between musical intervals as we have discussed in this chapter. More likely, it is the smallness of the beat frequency compared to the frequencies of the two generating tones that is the “Comma” here. (The high-pitched tones represent the ringing of the sun, moved up many octaves.)
It is rather amazing how this sculpture at TTU by Po Shu Wang combines elements from art, music, and science!