Notes: Thurs 8/27

1) Elliptic functions
2) Simple pendulum

go beyond small angle approx

Elliptic Functions:

$$\int \frac{dx}{\sqrt{1-x^2}} = ? = \sin^{-1}(x) + \text{const}$$

$\sin^{-1}(x) = \arcsin(x)$

$\rightarrow \sin^2\theta + \cos^2\theta = 1$

subst: $x = \sin\theta \rightarrow \sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$dx = \cos\theta\, d\theta$

$$\int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \int d\theta = \theta = \sin^{-1}(x)$$

$x = \sin\theta$
$\theta = \sin^{-1}(x)$

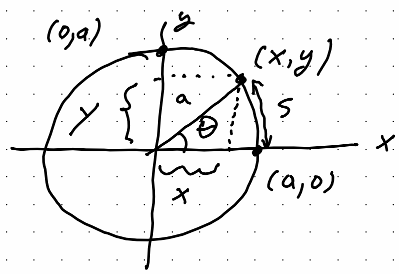
$$\frac{d}{d\theta}\sin\theta = \cos\theta \qquad \lim_{h\to 0}\left(\frac{\sin(\theta+h) - \sin\theta}{h}\right)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\sin(A+B) = \sin A\cos B + \cos A \sin B$$

circular functions:

$x^2+y^2 = a^2$, a = radius

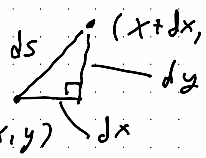


Def: $\sin\theta = \frac{y}{a}$, $\cos\theta = \frac{x}{a}$

s: arclength from $(a,0)$ to (x,y)

$s = a\theta$

$\sqrt{dx^2+dy^2} = ds$



$$\theta = \frac{s}{a} = \frac{1}{a}\int ds = \frac{1}{a}\int\sqrt{dx^2+dy^2} = \int d\theta$$

Given: $x^2+y^2 = a^2$

Follows: (i) $\cancel{a^2}\cos^2\theta + \cancel{a^2}\sin^2\theta = \cancel{a^2} \rightarrow \boxed{\cos^2\theta + \sin^2\theta = 1}$

(ii) $$\boxed{\frac{d}{d\theta}\sin\theta} = \frac{1}{a}\frac{dy}{d\theta} = \frac{1}{\cancel{a}}\frac{dy}{\frac{1}{\cancel{a}}\sqrt{dx^2+dy^2}} = \frac{1}{\sqrt{\left(\frac{y}{x}\right)^2+1}}$$

$$\cancel{2}x\,dx + \cancel{2}y\,dy = 0 \rightarrow dx = -\frac{y}{x}dy$$

$$= \frac{1}{\sqrt{\frac{x^2+y^2}{x^2}}} = \frac{x}{a} = \boxed{\cos\theta}$$

$$\frac{d(\sin\theta)}{d\theta} = \cos\theta \quad \rightarrow \quad \int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta$$

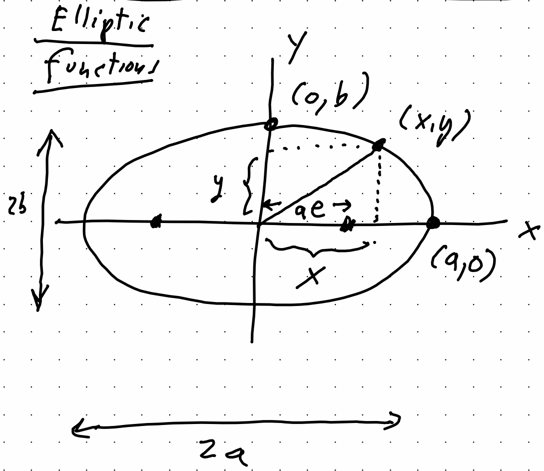
$x = \sin\theta$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$= \sqrt{1-x^2}$

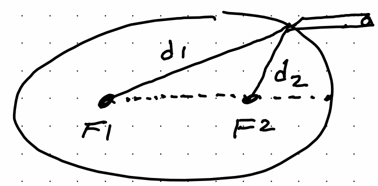
$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \theta = \sin^{-1}(x)}$$

Elliptic functions

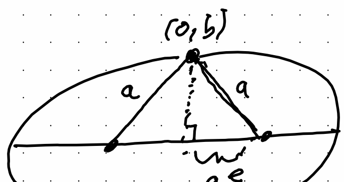


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

eccentricity : $e = 0$ (for circle)



$d_1 + d_2 = 2a$



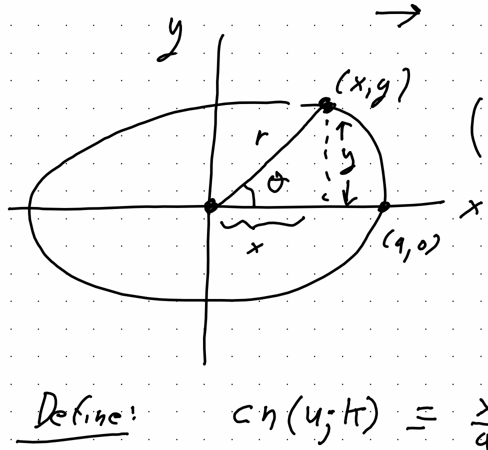
$(ae)^2 + b^2 = a^2$

$a^2e^2 + b^2 = a^2$

$e^2 = \frac{a^2-b^2}{a^2} = 1 - \left(\frac{b}{a}\right)^2$

$\rightarrow \quad e = \sqrt{1-\left(\frac{b}{a}\right)^2} \equiv k$

notation used in elliptic function



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$x^2 + y^2 = r^2$

$x = r\cos\theta, \; y = r\sin\theta$

Define: $\quad cn(u;k) \equiv \frac{x}{a} \quad , \quad sn(u;k) \equiv \frac{y}{b}$

$dn(u;k) \equiv \frac{r}{a}$ — (=1 for a circle)

where $u = \frac{1}{b}\int_0^\theta r\,d\theta$ — $y = r\sin\theta$

(= θ for a circle)

$ds = \sqrt{dx^2 + dy^2}$

$= \sqrt{dr^2 + r^2 d\theta^2}$

Given: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, $x^2 + y^2 = r^2$ $\quad dn(u;k) = \frac{r}{a}$

Follow: (i) $cn^2(u;k) + sn^2(u;k) = 1$

(ii) $dn^2(u;k) + k^2 sn^2(u;k) = 1$

$$u = \frac{1}{b}\int_0^\theta r\,d\theta$$

(iii) $\frac{d}{du} sn(u;k) = cn(u;k)\, dn(u;k)$

$\frac{d}{du} cn(u;k) = -sn(u;k)\, dn(u;k)$

$\frac{d}{du} dn(u;k) = -k^2 sn(u;k)\, cn(u;k)$

Analogous to

$\frac{d\sin\theta}{d\theta} = \cos\theta$

$\frac{d\sin\theta}{\cos\theta} = d\theta$

$x = \sin\theta$

$\int \frac{dx}{\sqrt{1-x^2}} = \int d\theta = \theta = \sin^{-1} x$

Integrate: $\frac{d\, sn(u;k)}{du} = cn(u;k)\, dn(u;k)$

$$\int \frac{d\, sn(u;k)}{cn(u;k)\, dn(u;k)} = \int du = u$$

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}} = u + \text{const} = sn^{-1}(x;k) + \text{const} \quad x \equiv sn(u;k)}$$

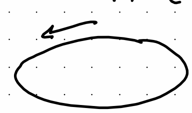
$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + \text{const}}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}} \equiv K(k) \longrightarrow$$ related to Period of a pendulum going beyond small-angle approximation

(complete elliptic integral of 1st kind)

$$\int_0^1 \frac{\sqrt{1-k^2x^2}\; dx}{\sqrt{1-x^2}} \equiv E(k) \longrightarrow$$ circumference around an ellipse

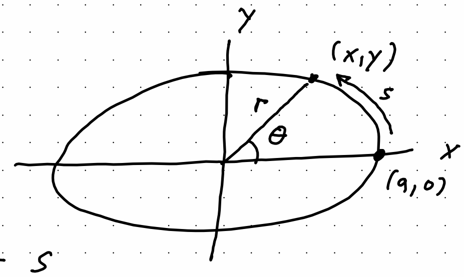
(complete elliptic integral of 2nd kind)



circle: $C = 2\pi a$

Notes: Tuesday 9/1

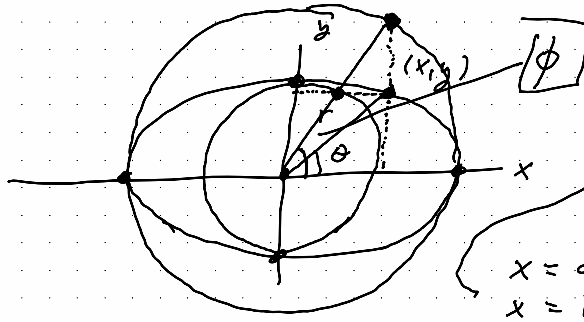
1) Review of elliptic functions
2) Simple pendulum



$$u = \frac{1}{b}\int_{(a,0)}^{(x,y)} r\,d\theta$$

$$bu = \int_0^\theta r\,d\theta \le \int_0^\theta ds = s$$

$$ds = \sqrt{dx^2+dy^2} = \sqrt{dr^2+r^2d\theta^2} \ge r\,d\theta$$



$x = r\cos\theta$

$y = r\sin\theta$

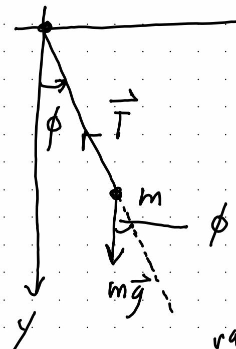
$x = a\cos\phi$

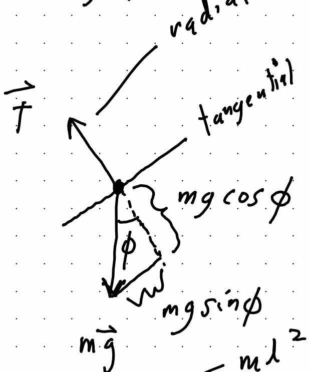
$y = b\sin\phi$

$x = a\cos\phi$, $y = \underline{\underline{a}}\sin\phi$

$x = b\cos\phi$ $y = b\sin\phi$

<u>simple pendulum:</u>





$$\text{Torque} = I\alpha \quad (I = ml^2,\ \alpha = \ddot\phi)$$

(i) "Freshman physics"

forces, free-body diagram

→ EOM, tension

<u>tangential:</u>

$$-mg\sin\phi = m a_{tangential}$$

$$-\cancel{m}g\sin\phi = \cancel{m}l\ddot\phi$$

ϕ : angular displacement [rad]

$\dot\phi$: angular velocity [rad/sec]

$\ddot\phi$: angular accel [rad/sec^2]

$$\boxed{\ddot\phi = -\frac{g}{l}\sin\phi} \quad \text{(EOM)}$$

<u>radial:</u> $T - mg\cos\phi = m a_{centripetal}$

$$T - mg\cos\phi = m\dot\phi^2 l$$

$$T = mg\cos\phi + m\dot\phi^2 l$$

(ii) Small angle approximation:

$\sin\phi \approx \phi \leftarrow \phi << \underset{\text{radian}}{1} = 57$ degrees

$$\ddot{\phi} = -\frac{g}{l}\sin\phi \approx -\frac{g}{l}\phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\phi}$$

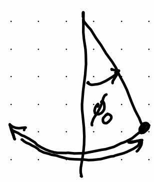
small angle approx.

$$\phi(t) = \boxed{a}\cos(\omega t) + \boxed{b}\sin(\omega t)$$

where $\omega \equiv \sqrt{\frac{g}{l}}$

determined by initial conditions

<u>ICs:</u> If $\phi(0) = \phi_0$ (at rest)



then $\boxed{\phi(t) = \phi_0\cos(\omega t)}$

<u>Period:</u> $P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$

independent of ϕ_0 !!

(iii) Lagrangian approach

$T \equiv$ Kinetic energy

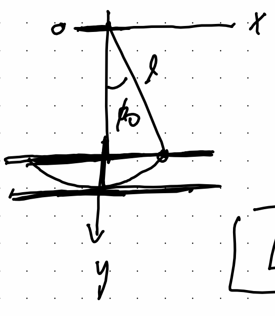
$U \equiv$ Potential energy

$$L = T - U$$

$$T = \frac{1}{2}ml^2\dot{\phi}^2 \quad \left(= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\right)$$

$y = l\cos\phi$

$x = l\sin\phi$



$$U = -mgl\cos\phi + \text{const} \leftarrow U = -mgy$$

$$U = mgl(1-\cos\phi)$$

$$\boxed{L = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi}$$

action

$$S = \int_{t_1}^{t_2} dt\, L(\phi, \dot{\phi}, t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \quad \text{Lagrange's equation}$$

$$\frac{d}{dt}\left(ml^2\dot{\phi}\right) = -mgl\sin\phi$$

$$ml^2\ddot{\phi} = -mgl\sin\phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi} \quad \text{(EOM)}$$

(iv) solving $\ddot{\phi} = -\frac{g}{l}\sin\phi$ (2nd order non-linear ODE) hard!!

$$E = \text{const}$$
$$= T + U$$
$$= \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$$

$$E = 0 - mgl\cos\phi_0$$
$$= -mgl\cos\phi_0$$

release from rest from $\phi = \phi_0$

$$-mgl\cos\phi_0 = \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$$
$$-\cancel{m}gl(\cos\phi_0 - \cos\phi) = \frac{1}{2}\cancel{m}l^2\dot{\phi}^2$$
$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{-2\frac{g}{l}(\cos\phi_0 - \cos\phi)}$$

$|\phi| \leq \phi_0$

Separable 1st order ODE

$$\int \frac{d\phi}{\sqrt{-\frac{2g}{l}(\cos\phi_0 - \cos\phi)}} = \int dt = t + \text{const}$$

$$t + t_0 = \int \frac{d\phi}{\sqrt{-2\frac{g}{l}(\cos\phi_0 - \cos\phi)}}$$

$$\frac{1}{\sqrt{a + bx^2}}$$

substitution:

$$\cos\phi = 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$
$$\cos\phi_0 = 1 - 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$\cos\phi = \cos\left(2\left(\frac{\phi}{2}\right)\right)$$
$$= \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)$$
$$= 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\rightarrow \cos\phi_0 - \cos\phi = -2\left(\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)\right)$$

$$t + t_0 = \int \frac{d\phi}{2\sqrt{\frac{g}{l}}\sqrt{\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)}}$$

$$= \frac{1}{2\sqrt{\frac{g}{l}}}\int \frac{d\phi}{\sin\left(\frac{\phi_0}{2}\right)\sqrt{1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)}}}$$

$|\phi| \leq \phi_0$

let $x = \frac{\sin(\phi/2)}{\sin\left(\frac{\phi_0}{2}\right)}$

$$\boxed{x = \frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}} \rightarrow dx = \frac{1}{\sin\left(\frac{\phi_0}{2}\right)} \frac{1}{2} \cos\left(\frac{\phi}{2}\right) d\phi$$

$\sqrt{1-x^2}$

denominator

$$d\phi = \frac{2\sin\left(\frac{\phi_0}{2}\right) dx}{\cos\left(\frac{\phi}{2}\right)} = \frac{2\sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1-\sin^2\left(\frac{\phi}{2}\right)}} = \boxed{\frac{2\sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1-\underbrace{\sin^2\left(\frac{\phi_0}{2}\right)}_{k^2} x^2}}}$$

Find this out

① $\phi(t) = \boxed{\times}$

② Period = ??

③ Redo the analysis using Lagrange multiplier for find tension in string

$t + t_0 = \int \bigcirc$ — integrand for $sn^{-1}(x; k)$ $\quad k = \sin\left(\frac{\phi_0}{2}\right)$

Lec #4 : Thurs 9/3

$$\phi(t) = 2\sin^{-1}\left[k\, sn\left(\omega_0\left(t + \frac{P}{4}\right); k\right)\right] \quad ☆$$

$$k = \sin\left(\frac{\phi_0}{2}\right), \quad \omega_0 = \sqrt{\frac{g}{l}}$$

$$P = 4\sqrt{\frac{l}{g}}\, K(k) = 4\sqrt{\frac{l}{g}} \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

small angle approx

$$P_0 = 2\pi\sqrt{\frac{l}{g}}$$

$$\frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)} = x = sn\left[\sqrt{\frac{g}{l}}(t+t_0); k\right]$$

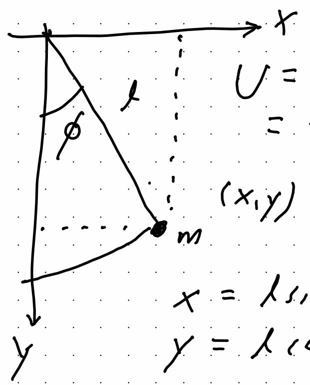
$$\sqrt{\frac{l}{g}} \int \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} = \sqrt{\frac{l}{g}}\, sn^{-1}(x; k) = t + t_0$$

$$sn^{-1}(x; k) = \sqrt{\frac{g}{l}}(t+t_0)$$

$$P = 4\sqrt{\frac{l}{g}}\,K(k) = \underbrace{2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\phi_0^2 + \dots\right)}_{\text{problem Landau 11,1}}$$

Lagrange multiplier:

$$\underbrace{\varphi(x,y)}_{\text{constraint function}} = x^2+y^2-l^2 = 0$$



$$U = -mgy = -mgl\cos\phi$$

$$x^2+y^2 = l^2\sin^2\phi + l^2\cos^2\phi = l^2$$

$$x = l\sin\phi \qquad y = l\cos\phi$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}ml^2\dot{\phi}^2$$

$$L = T - U + \lambda\varphi$$

Lagrange multiplier

$$L(x,\dot{x},y,\dot{y},t) \quad q=(x,y) \quad \dot{q}=(\dot{x},\dot{y})$$

$$L(r,\dot{r},\phi,\dot{\phi},t) \quad q=(r,\phi) \quad \dot{q}=(\dot{r},\dot{\phi})$$

$\lambda(t)$, $r(t)$, $\phi(t)$

$$L(\phi,\dot{\phi},t) \quad \varphi(x,y)=x^2+y^2-l^2=0 \quad \boxed{\varphi(r,\phi) = r-l=0}$$

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + mg\underbrace{r\cos\phi}_{y} + \lambda(r-l)}$$

r: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \rightarrow \boxed{\cancel{m\ddot{r}} = mr\dot{\phi}^2 + mg\cos\phi + \lambda}$

ϕ: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \rightarrow \frac{d}{dt}(mr^2\dot{\phi}) = -mgr\sin\phi$

λ: $r-l=0$ ✓ $\qquad \cancel{2mr\dot{r}\dot{\phi}} + mr^2\ddot{\phi} = -mgr\sin\phi$

$r - l = 0 \quad \longrightarrow \quad r = l$

$\dot{r} = 0$

$\ddot{r} = 0$

$\rightarrow \quad \ddot{\phi} = -\frac{g}{l} \sin\phi$

$m l \dot{\phi}^2 + m g \cos\phi + \lambda = 0$

$$\boxed{\lambda = -\underbrace{\left(m g \cos\phi + m l \dot{\phi}^2\right)}_{T}} \qquad \lambda = -T$$

$$L = T - U + \lambda \varphi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}(m\dot{x}) = -\frac{\partial U}{\partial x} + \lambda \frac{\partial \varphi}{\partial x}$$

$U = U(x,t)$

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

$\varphi(x, y, t)$

$U(x, y, t)$

$$\frac{d}{dt}(m\dot{y}) = -\frac{\partial U}{\partial y} + \lambda \frac{\partial \varphi}{\partial y}$$

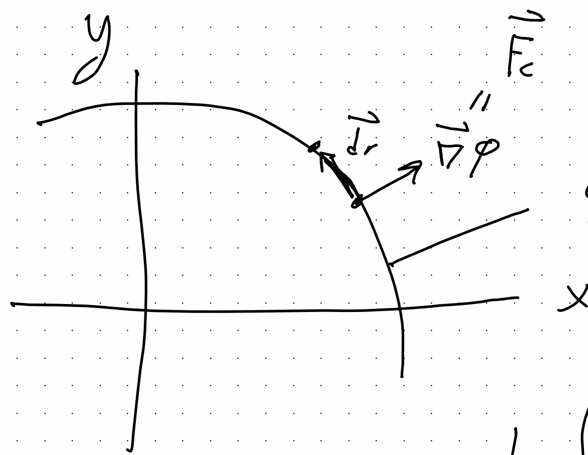
$$\boxed{\frac{d\vec{p}}{dt} = \vec{F}_{net}}$$

applied force force constraint

$$\left.\begin{aligned} \frac{dp_x}{dt} &= F_x + \lambda \frac{\partial \varphi}{\partial x} \\ \frac{dp_y}{dt} &= F_y + \lambda \frac{\partial \varphi}{\partial y} \end{aligned}\right\}$$

$$\boxed{\frac{d\vec{p}}{dt} = \vec{F} + \lambda \vec{\nabla}\varphi}$$

$$\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) = \vec{\nabla}\varphi$$



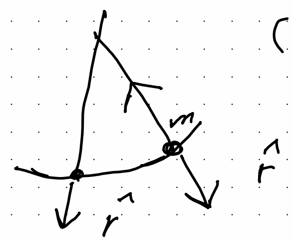
$$\int \underbrace{\vec{F}_c \cdot d\vec{r}}_{0} = \text{work}$$

$\vec{\nabla}\varphi \perp \varphi = \text{const}$

$\varphi = r - l$

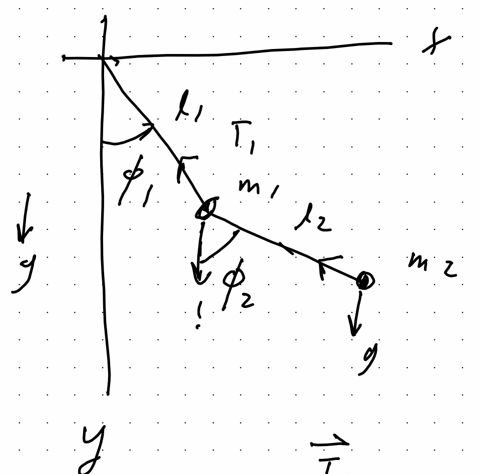
$\vec{\nabla}\varphi = \hat{r}$

$\lambda \vec{\nabla}\varphi$



(r, ϕ)

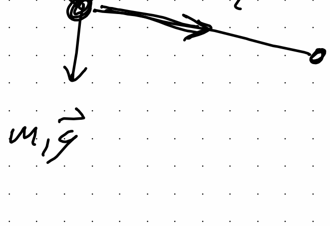
$\hat{r}, \hat{\phi}$



$U = -m_1 g y_1$
$\quad - m_2 g y_2$

$\ddot{\phi}_1 , \ddot{\phi}_2$

EOM



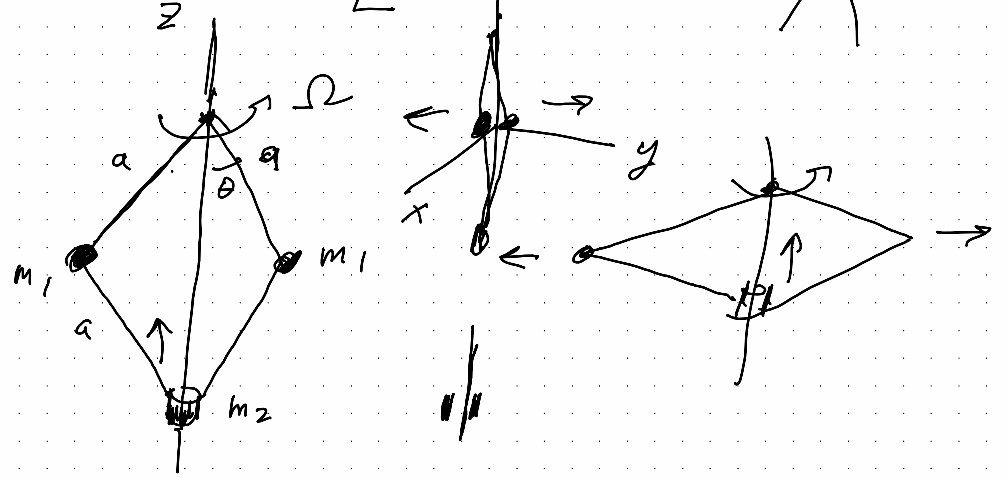
$\lambda \varphi$

$(\lambda_1 \varphi_1 + \lambda_2 \varphi_2$
$+ \cdots)$

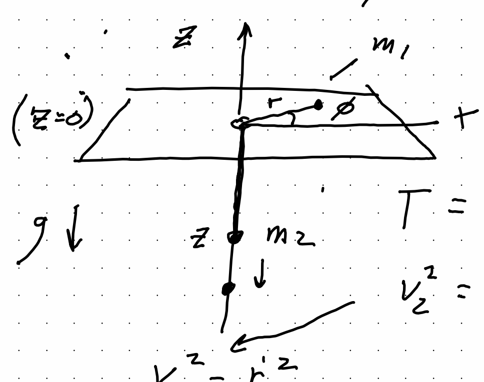
$$L \longrightarrow \text{EoM}$$

$$\underbrace{L \rightarrow L + \frac{d}{dt}\left(f(q,t)\right)}_{L'}$$

$$f(q, \cancel{\dot{q}}, t)$$



Lec #5: Tuesday 9/8



$\boxed{r - z = l}$ = length of string

$$L = T - U$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$v_2^2 = \dot{z}^2, \quad v_1^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

$$\left(= \dot{x}^2 + \dot{y}^2, \quad \begin{matrix} x = r\cos\phi \\ y = r\sin\phi \end{matrix}\right)$$

$$v_2^2 = \dot{r}^2$$

$$\boxed{\begin{aligned} T &= \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2 \\ &= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 \end{aligned}}$$

$$U = m_2 g z = m_2 g (r - l) = m_2 g r - \underbrace{m_2 g l}_{\text{constant}}$$

$$\boxed{U = m_2 g r}$$

$$L = T - U = \boxed{\frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = \frac{\partial L}{\partial r}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi} \Rightarrow \text{2nd order EOMs}$$

No explicit t dependence:

$$E = \dot r \frac{\partial L}{\partial \dot r} + \dot\phi \frac{\partial L}{\partial \dot \phi} - L = \boxed{T+U}$$

$$E = \sum_i \dot q_i \underbrace{\frac{\partial L}{\partial \dot q_i}}_{=p_i} - L$$

No explicit ϕ dependence:

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = \text{const} = M_z$$

$\vec{M}$: angular momentum (L&L notation)

$$\boxed{M_z = m_1 r^2 \dot\phi} \rightarrow \boxed{\dot\phi = \frac{M_z}{m_1 r^2}}$$

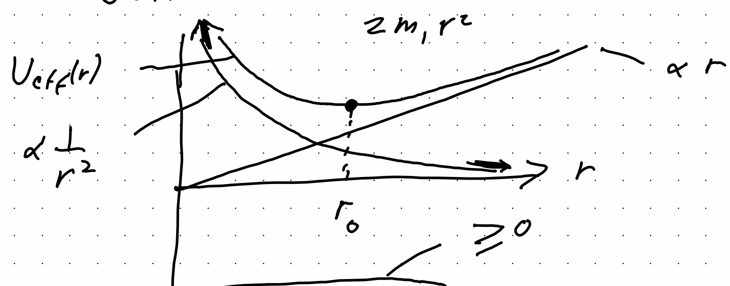
$$E = \tfrac{1}{2} m\dot x^2 + U(x)$$

$$\boxed{E = \tfrac{1}{2}(m_1+m_2)\dot r^2 + \tfrac{1}{2} m_1 r^2 \dot\phi^2 + m_2 g r}$$

$$\boxed{E = \tfrac{1}{2}(m_1+m_2)\dot r^2 + \left(\frac{M_z^2}{2m_1 r^2} + m_2 g r\right)} \leftarrow \boxed{U_{eff}(r) = U(r) + \frac{M_z^2}{2m_1 r^2}}$$

$$U_{eff}(r) = \frac{M_z^2}{2m_1 r^2} + m_2 g r$$

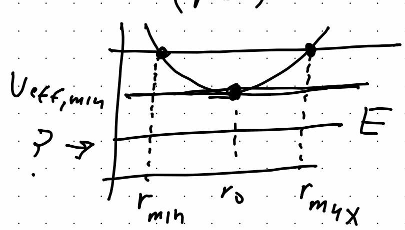
$$U(r) = m_2 g r$$



$$E = \underbrace{\boxed{\tfrac{1}{2}(m_1+m_2)\dot r^2}}_{\geq 0} + U_{eff}(r)$$

E: const

r_{min}, r_{max}: "turning points" ($\dot r = 0$)



i) $E = U_{eff,min} = U_{eff}(r_0)$

unif circular motion: $r = r_0$, $\dot\phi = \dfrac{M_z}{m_1 r_0^2}$

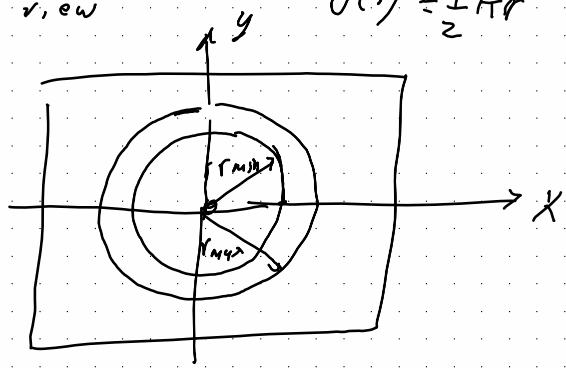
$E < U_{eff,min}$ (not allowed)

ii) $E > U_{eff,min}$

$E = U_{eff}(r_{min}) = U_{eff}(r_{max})$

$E \geq U_{eff,min}$

top view

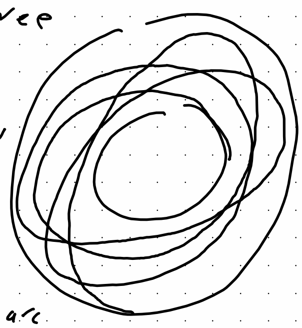


$$\underbrace{\begin{array}{l} U(r) = \frac{1}{2} k r^2 \\ U(r) = -\frac{G m_1 m_2}{r} \end{array}}_{\text{Newtonian gravity}}$$ — closed bound orbits

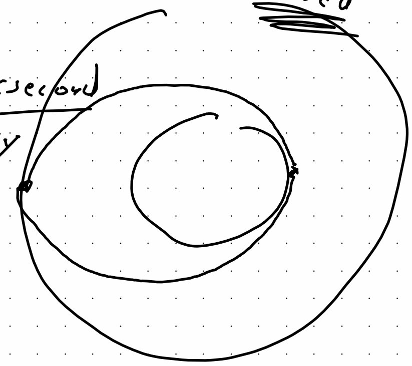
bound orbits $\equiv$ ellipse
are closed

1 degree
=
60 mins of arc

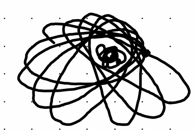
1 min of arc
=
60 sec of arc



$\frac{43 \text{ arcsecond}}{\text{century}}$



perihelion precession of Mercury
└ closest approach to sun



$\underline{r_0}$: $\left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = 0$ (minimum)

$$0 = \frac{d}{dr}\left(\frac{M_z^2}{2 m_1 r^2} + m_2 g r\right)\Big|_{r=r_0}$$

$$= -\frac{M_z^2}{m_1 r_0^3} + m_2 g$$

$$\boxed{M_z^2 = m_1 m_2 g r_0^3}$$

tells you the value of M_z needed to have a specific r_0 value.

For a given M_z, this tells you what r_0 equals.

Energy equation:

$$E = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{M_z^2}{2m_1 r^2} + m_2 g r$$

$$\boxed{\dot{\phi} = \frac{M_z}{m_1 r^2}} \leftarrow \phi \text{ equation}$$

$$\frac{1}{2}(m_1+m_2)\dot{r}^2 = E - \frac{M_z^2}{2m_1 r^2} - m_2 g r$$

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}$$

$$\int \frac{dr}{\sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}} = \int dt = t + \text{const}$$

$$t(r) \leftrightarrow r(t)$$

orbital equation:

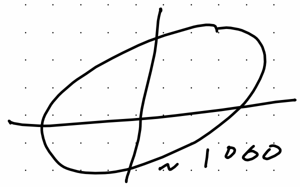
$$r = r(\phi) \leftrightarrow \phi = \phi(r)$$

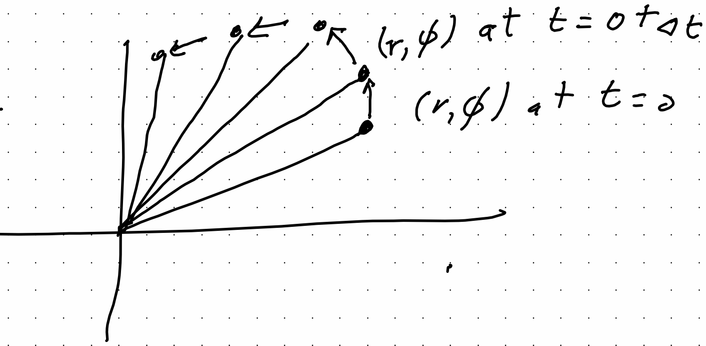
$$\frac{dr}{dt} = \dot{r} = \sqrt{\left(\frac{2}{m_1+m_2}\right)\left[E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right]}$$

$$\frac{dr}{dt} = \underbrace{\frac{dr}{d\phi}}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M_z}{m_1 r^2}$$

$$\rightarrow \frac{dr}{d\phi} = \frac{m_1 r^2}{M_z}\sqrt{\left(\frac{2}{m_1+m_2}\right)\left[\quad\right]}$$

$$\int \frac{dr}{\frac{m_1 r^2}{M_z}\sqrt{(\;)[\quad]}} = \int d\phi = \phi + \text{const}$$





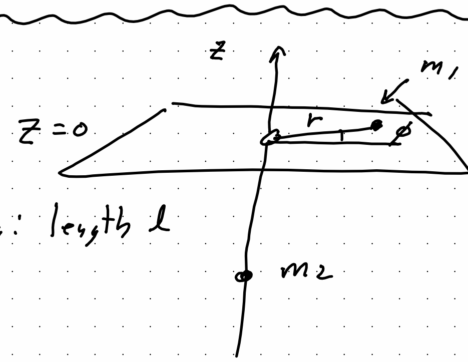
Know: r, ϕ at some time t

given: Δt need to know Δr and $\Delta\phi$

$$r(t+\Delta t) = r(t) + \Delta r(t) + \cdots$$
$$\phi(t+\Delta t) = \phi(t) + \Delta\phi(t) + \underbrace{\cdots}_{\text{ignore if } \Delta t \text{ is suff. small}}$$

Lecture #6 : Thursday 10 Sep

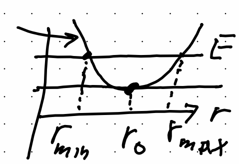
1) Secs 6–10 (today), Sec 40 (next Tuesday)
2) Finish up example from last time
3) Conservation of E, $\vec{P}$, $\vec{M}$
4) Mechanical similarity
5) Quiz: last 20 minutes (1:30pm)



string: length l

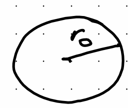
E, M_z = const

U_{eff}



$$\frac{d\phi}{dt} = \frac{M_z}{m_1 r^2}$$

$$\frac{dr}{dt} = \sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E - \underbrace{m_2 g r - \frac{M_z^2}{2m_1 r^2}}\right)} = \sqrt{0}$$



$$\frac{d\phi}{dt} = \frac{M_z}{m_1 r^2} \rightarrow \Delta\phi = \Delta t \, \frac{M_z}{m_1 r^2}$$

$$\frac{dr}{dt} = \sqrt{\quad} \rightarrow \Delta r = \Delta t \sqrt{\quad}$$

$$r(0) = r_{min}$$
$$\phi(0) = 0$$

$$\phi(\Delta t) = \phi(0) + \Delta\phi$$
$$r(\Delta t) = r(0) + \Delta r$$

$$\phi(2\cdot\Delta t) = \phi(\Delta t) + \Delta\phi$$
$$r(2\cdot\Delta t) = r(\Delta t) + \Delta r$$

$$\vdots$$

$$\phi(t_i) = \phi(t_{i-1}) + \Delta\phi$$
$$r(t_i) = r(t_{i-1}) + \Delta r$$

Cons. of $E, \vec{P}, \vec{M}$:

All of E, $\vec{P}$, $\vec{M}$ conserved for a closed system

no external Forces

$U = U(\vec{r}_1 - \vec{r}_2, \vec{r}_2 - \vec{r}_3, \cdots)$

relative position vectors.

Even in the presence of external Forces, you can still have cons. of E and some components of $\vec{P}$ and $\vec{M}$.

(i) $U = mgy$ $\quad \vec{F} = -m\vec{g}$

If U does not depend explicitly on time t, then E is conserved.

$$E \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

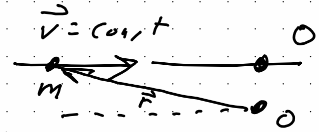
constant external Field

(ii) e.g., $\downarrow \vec{F}_g = m\vec{g}$

$\longleftrightarrow$ x, y

$P_x = \text{const}$
$P_y = \text{const}$

If U is unchanged by a translation in some direction $\hat{t}$ then $\vec{P}\cdot\hat{t} = \text{const}$



(iii) $\vec{M}$ depends on choice of origin

(a) uniform gravitational field $\vec{F} = m\vec{g}$

$U = m_1 g z_1 + m_2 g z_2 + \dots$

If U is unchanged by a rotation about a particular axis $\hat{n}$ then $\boxed{\vec{M}\cdot\hat{n} = \text{const}}$ (e.g., $\hat{n} = \hat{z}$, $M_z = \text{const}$)

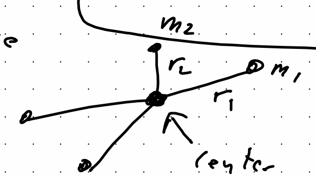
provided the origin is located on the axis.

(b) central force

$U = U(r)$

$\vec{F} = -\vec{\nabla} U = -\frac{dU}{dr}\hat{r}$

$\boxed{\vec{M} = \text{const}}$



Mechanical similarity:

$L \to L' = c\cdot L$ same equations of motion

suppose we rescale position vectors $\vec{r}_a \to \alpha\vec{r}_a$

$U(\vec{r}_i') \equiv U(\alpha\vec{r}_1, \alpha\vec{r}_2, \dots) = \alpha^k U(\vec{r}_1, \vec{r}_2, \dots)$

potential is homogenous of degree k w.r.t. position vector

Example: i) $U = mgy$, $k = 1$
ii) $U = \frac{1}{2}kx^2$, $k = 2$
iii) $U = -\frac{Gm_1m_2}{r}$, $k = -1$

$U' = mg\alpha y = \alpha mgy = \alpha U$

$L' = T' - U' = T' - \alpha^k U = \text{const}\cdot L$
$= \alpha^k T - \alpha^k U$
$= \alpha^k (T - U) = \alpha^k L$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \frac{l^2}{t^2}$$

length

time

$$l \to l' = \alpha l$$
$$t \to t' = \beta t$$

$$\frac{l'}{l} = \alpha \qquad \frac{t'}{t} = \beta$$

$U = mgy$, $k = 1$

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{\frac{1}{2}}$$

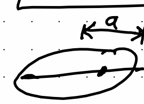
$U = \frac{1}{2} k x^2$, $k = 2$

$$t'/t = \text{const}$$

$$T' = \frac{1}{2} m v'^2 = \frac{1}{2} m \frac{l'^2}{t'^2}$$
$$= \frac{1}{2} m \frac{\alpha^2 l^2}{\beta^2 t^2}$$
$$= \frac{\alpha^2}{\beta^2} T$$

$$\boxed{\frac{\alpha^2}{\beta^2} = \alpha^k} \rightarrow \boxed{\beta = \alpha^{1-k/2}} \rightarrow \boxed{\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-k/2}}$$

$$\beta^2 = \alpha^{2-k}$$



$$\boxed{\frac{P^2}{a^3} = \text{const}}$$

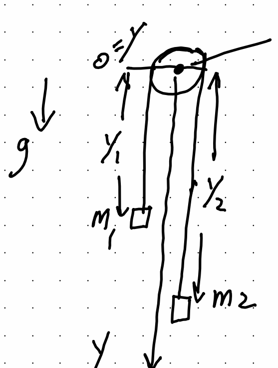
$U = -\frac{G m_1 m_2}{r}$

$k = -1$

$$\left(\frac{t'}{t}\right) = \left(\frac{l'}{l}\right)^{3/2}$$

$$P^2 = \text{Dist}^3$$

QUIZ #1: Atwood's machine



massless
frictionless
pulley

i) L ?

ii) EOM

iii) solve EOM

$(m_1 > m_2)$

string: length l
(massless
inextensible
...)

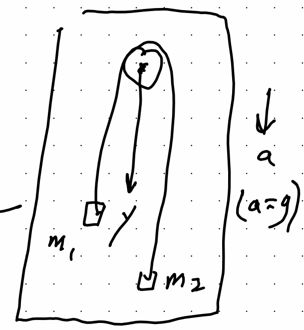
$$y_1 + y_2 = l \rightarrow y_2 = l - y_1 \rightarrow \dot{y}_2 = -\dot{y}_1$$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_1 g y_1 - m_2 g (l - y_1)$$
$$= -m_1 g y_1 \underbrace{- m_2 g l}_{\text{ignore}} + m_2 g y_1$$
$$= \boxed{-(m_1 - m_2) g y_1}$$

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \underbrace{\dot{y}_2^2}_{= \dot{y}_1^2}$$
$$= \boxed{\frac{1}{2}(m_1 + m_2) \dot{y}_1^2}$$

Lecture #7: Tuesday 9/15

1) Go over quiz #1
2) Modified Atwood problem
3) Finish mechanical similarity (sec 10)
4) Hamilton's equations (sec 40)

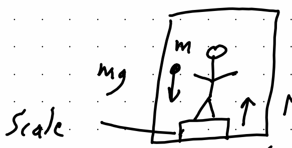


$$L = T - U$$
$$= \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 + (m_1 - m_2) g y_1$$

EOMs: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) = \frac{\partial L}{\partial y_1} \rightarrow (m_1 + m_2)\ddot{y}_1 = (m_1 - m_2) g$

$$\boxed{\ddot{y}_1 = \frac{(m_1 - m_2) g}{(m_1 + m_2)}}$$

$$y_1(t) = y_0 + v_0 t + \frac{1}{2}\frac{(m_1 - m_2)}{(m_1 + m_2)} g t^2$$



$ma = mg - N$

$\underbrace{N}_{w} = m(g - a)$

apparent or effective weight

$$\boxed{\ddot{y}_1 = \frac{(m_1 - m_2)}{m_1 + m_2}(g - a)}$$

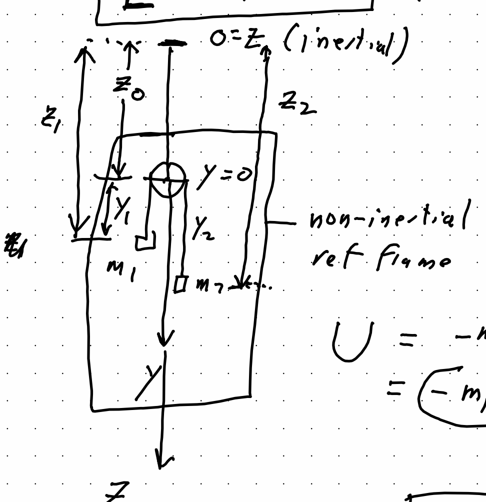
$\vec{F} = m\vec{a}$ (valid in an inertial ref frame)

$\vec{F} + \vec{F}_{\text{fictitious}} = m\vec{a}$ ——— w.r.t. to a non-inertial ref. frame

sec 39 (L&L)

$L' = L + f(t)$

$\boxed{L = T - U}$ (valid in an inertial ref frame)



$$T = \frac{1}{2} m_1 \dot{z}_1^2 + \frac{1}{2} m_2 \dot{z}_2^2$$

$$U = -m_1 g z_1 - m_2 g z_2$$

$z_1 = z_0 + y_1$ | $y_2 = l - y_1$

$z_2 = z_0 + y_2$

$$U = -m_1 g (z_0 + y_1) - m_2 g (z_0 + \underbrace{y_2}_{l - y_1})$$

$$= -m_1 g z_0 - m_1 g y_1 - m_2 g z_0 \underbrace{- m_2 g l}_{\text{const}} + m_2 g y_1$$

prescribed function of time = ignore

$$= \boxed{-(m_1 - m_2) g y_1}$$

Do this at home:

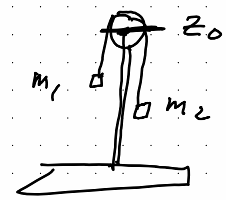
$$T = \frac{1}{2}(m_1+m_2)\dot{y}_1^2 + (m_1-m_2)\dot{z}_0\dot{y}_1$$

$$L' = L + \frac{d}{dt}\left(f(q,t)\right) \longrightarrow \text{same EOM}$$

$$(m_1-m_2)\dot{z}_0\dot{y}_1 = \underbrace{\frac{d}{dt}\Big[\underbrace{(m_1-m_2)\dot{z}_0 y_1}_{f(y_1,t)}\Big]}_{\text{ignore}} \underbrace{- (m_1-m_2)\ddot{z}_0 y_1}_{\ddot{z}_0\,\equiv\, a}$$

$$\boxed{T = \frac{1}{2}(m_1+m_2)\dot{y}_1^2 - (m_1-m_2)a y_1}$$

$z_0(t)$: given, not to be solved for



Hamilton's equations:

$L(q,\dot{q},t)$

Hamiltonian: $E = \underbrace{\sum \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L}_{E(q,\dot{q},t)}$

$H = H(q,p)$

not here if $L = L(q,\dot{q})$

$$H = \left(\sum_i p_i \dot{q}_i - L\right)\Bigg|_{\dot{q} = \dot{q}(q,p)}$$

$$\boxed{p_i \equiv \frac{\partial L}{\partial \dot{q}_i}}$$

Example:

$L = \frac{1}{2}m\dot{x}^2 - U(x)$ (single particle, 1-d, const external field)

$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \longrightarrow \dot{x} = \frac{p}{m}$

$$H = (p\dot{x} - L)\Big|_{\dot{x}=p/m}, \quad \cancel{E}$$

$$= \left(p\dot{x} - \frac{1}{2}m\dot{x}^2 + U(x)\right)\Big|_{\dot{x}=p/m} = \frac{p^2}{m} - \frac{1}{2}m\left(\frac{p}{m}\right)^2 + U(x)$$

$$H = \frac{1}{2}\frac{p^2}{m} + U(x)$$

EOMs: (Hamilton's equations)   ((39,6) Prob 2 Sec 40)

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, \dots, s$$

w # of DOF

Lagrange's equation:

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}}_{\text{s-equations, 2nd order ODE for } q_i} \qquad i = 1, \dots, s$$

2s-equations, 1st order in $\dot{q}_i, \dot{p}_i$

$\dot{x} = p/m$

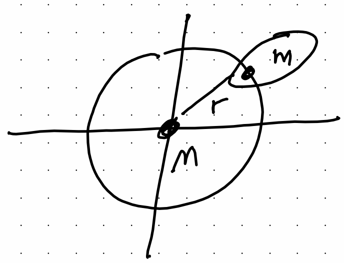
$p = m\dot{x}$

$\dot{p} = -\frac{\partial U}{\partial x}$

$$\boxed{m\ddot{x} = -\frac{\partial U}{\partial x}}$$

$$H = \frac{p^2}{2m} + U(x) \to \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$$

$$L = \frac{1}{2}m\dot{x}^2 - U(x) \to \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \to m\ddot{x} = -\frac{\partial U}{\partial x}$$



$$U = -\frac{GMm}{r}$$

problem: $U' = cU$

~~masses = equal~~

$r' = r$, $m' = m$

$$\frac{GM\cancel{m}}{r^2} = \frac{\cancel{m}v^2}{r}$$

$$\boxed{v = \sqrt{\frac{GM}{r}}}$$

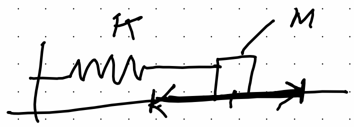
Suppose $M \to cM = M'$

$$U' = cU$$

$$\frac{2\pi r}{T'} = v' = \sqrt{\frac{cGM}{r}}$$

$$\frac{1}{T'} \propto \sqrt{c}$$

$$\sqrt{\frac{t}{t'}} = \sqrt{\frac{U'}{U}}$$



frictionless

$U = \frac{1}{2} K x^2$

$T = \frac{1}{2} m \dot{x}^2$ $\qquad L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2$ $\qquad w = \sqrt{\frac{K}{m}}$

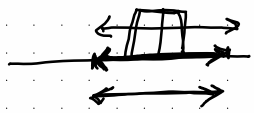
$m \ddot{x} = -Kx$

$\ddot{x} = -\frac{K}{m} x \rightarrow x(t) = a \cos wt + b \sin wt$

$U' = cU$ $\quad$ (for example $K' = cK$)

$T' = cU$

$\ddot{x} = -\frac{K'}{m} x$ $\quad , \quad w' = \sqrt{\frac{K'}{m}}$



$T = \frac{1}{2} m \dot{x}^2$

$T' = cT$

$\frac{2\pi}{P'} = \sqrt{\frac{K'}{m}}$

$\frac{P}{P'} = \sqrt{\frac{U'}{U}}$

$L = T - U$ $\qquad$ L, L' same EOMs

$L' = cL \quad = cT - cU$

L, L'' different EOMs

$L'' = T - cU \neq cL$

$\frac{1}{t'^2} = c$

$T = \frac{1}{2} m \dot{x}^2 , \quad U = \frac{1}{2} K x^2$

$L = T - U \rightarrow m \ddot{x} = -Kx$

$L' = cL \rightarrow \cancel{c} m \ddot{x} = -\cancel{c} K x$

$m \ddot{x} = -Kx$

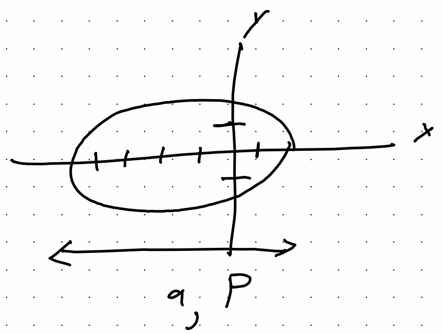
$\ddot{x} = -\frac{K}{m} x$

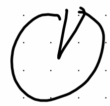
$L'' \longrightarrow m \ddot{x} = -cKx$

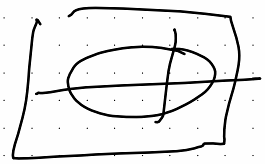
$c x'' = \ddot{x} = -c\left(\frac{K}{m}\right) x$

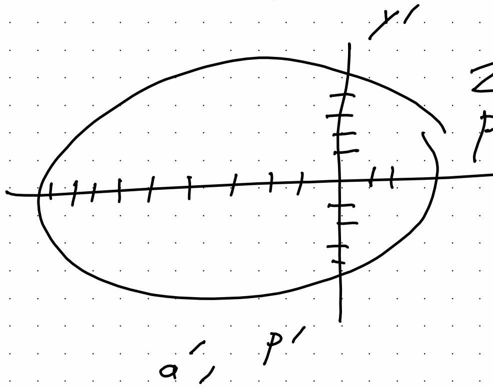
differ EOMs

$x = a \cos(w't) + b \sin(w't)$





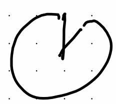




2

$P' = ?$

$\frac{P^2}{a^3} = \text{const}$



t'

$U' = cU$

$P' = ?\, P$

$m' = m$

$l' = l$

"mechanical similarity"

$L' = \text{const}\ L$

$T' - U' = (\text{const})(T - U)$

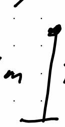
$T' - cU = c(T - U) \quad cU = U'$

$\frac{P'}{P} = \sqrt{\frac{U}{U'}}$

$k \to k' = ck$

Periods

$g \to 2g$



1m $P' ?\, P$

$\frac{1}{\sqrt{2}}$

2A

$\sqrt{\frac{1}{2}}$

k

$T = \frac{1}{2} m \dot{x}^2$

$\cancel{T' = T}$

$U' = cU$

$\cancel{L' \neq cL}$

$L' = cL$

Lec #8: Thur Sep 17th

Today — 1-d motion (Sec 11)

Next two weeks — central force (Sec 13-15)

☆ Midterm 1 — Tues Oct 6th

$$\underline{T = \frac{1}{2} a(q)\,\dot q^2}$$

$$T = \frac{1}{2} \sum_{j,k} a_{jk}(q)\, \dot q_j\, \dot q_k \quad ??$$

[single particle] $j = 1,2,3$

$$U(q_1, \dots, q_s, t)$$

$$T = \frac{1}{2} m |\vec v|^2$$

$$= \frac{1}{2} m (\dot x^2 + \dot y^2 + \dot z^2) \qquad \text{cartesian } (x,y,z)$$

$$= \frac{1}{2} m (\dot r^2 + r^2 \dot\theta^2 + r^2 \sin^2\theta\, \dot\phi^2) \qquad \text{sph. polar } (r,\theta,\phi)$$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

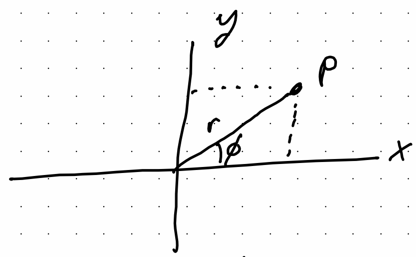
$q_1 = r$

$q_2 = \theta$

$q_3 = \phi$

$$a_{ij} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & ? \end{bmatrix} \qquad ? = r^2 \sin^2\theta$$

$\underline{a_{11} = m}$, $a_{22} = mr^2$, $a_{33} = mr^2\sin^2\theta$, $a_{12} = 0$, $a_{13} = 0$

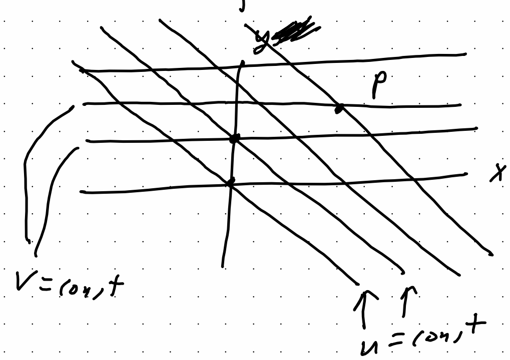


(x,y)

(r,ϕ)

$\dot x = \dot u - \dot v$

$\dot y = \dot v$



$u = x + y$

~~$v = x - y$~~

$v = y$

$x = u - v$

$y = v$

$u = \text{const}$?

$x + y = \text{const}$

$$\boxed{y = \text{const} - x}$$

b m

$$T = \frac{1}{2} m (\dot u^2 + \dot v^2) \quad ??$$

$$T = \frac{1}{2} m (\dot x^2 + \dot y^2) \quad ✓$$

$$= \frac{1}{2} m (\dot u^2 + \dot v^2 - 2\dot u \dot v + \dot v^2)$$

$$= \frac{1}{2} m (\dot u^2 + 2\dot v^2 - 2\dot u \dot v)$$

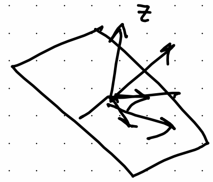
$$a_{ij} = m \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$T = \frac{1}{2} m |\vec{v}|^2$

$\vec{v} = \sum_{i=1}^{3} v_i \hat{e}_i$

$\hat{e}_1 \cdot \hat{e}_1 = 1$

$\hat{e}_i \cdot \hat{e}_j \neq 0$

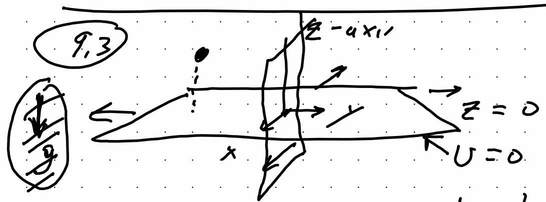


$|\vec{v}|^2 = \vec{v} \cdot \vec{v}$

$= \left(\sum_i v_i \hat{e}_i\right) \cdot \left(\sum_j v_j \hat{e}_j\right)$

$= \sum_{i,j} v_i v_j \underbrace{\hat{e}_i \cdot \hat{e}_j}_{\text{might not}} = \delta_{ij}$ (orthogonal)

9.3

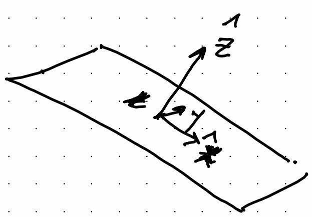


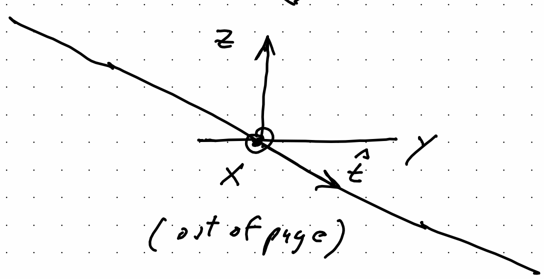
(a) $U(\cancel{x}, \cancel{y}, z)$

$mgz = U(z)$, $\vec{F} = -m\vec{g}$

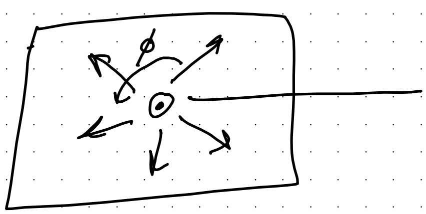
$\delta L = 0$ if you displace in $\hat{x}, \hat{y}$ → P_x, P_y = const, M_z = const

$\vec{F} = -\vec{\nabla} U = -\frac{dU}{dz} \hat{z}$



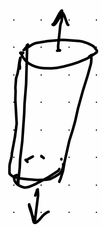


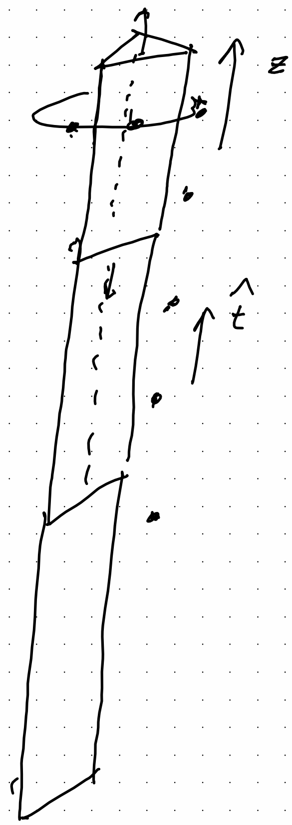
$\vec{P} \cdot \hat{t}$ = constant



$\hat{z}$ out

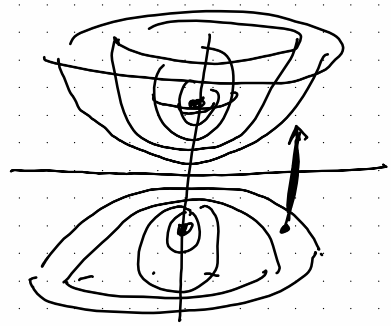
$\vec{M} \cdot \hat{z}$ = const





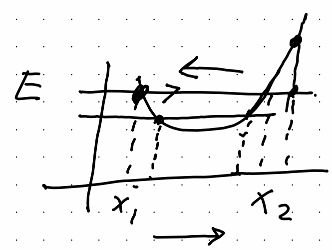
$U(x, y, \cancel{z})$

$\vec{P} \cdot \hat{t} = \text{const} = P_z$



$L = \frac{1}{2} m \dot{x}^2 - U(x)$

$E = \underbrace{\frac{1}{2} m \dot{x}^2}_{\text{positive}} + U(x) = \text{const}$



$x > x_2$

$\frac{1}{2} m \dot{x}^2 = E - U(x)$

$\frac{dx}{dt} = \dot{x} = \pm \sqrt{\frac{2}{m}(E - U(x))}$ — separable 1st order ODE

$\int \frac{dx}{\pm\sqrt{\frac{2}{m}(E-U(x))}} = \int dt$

$$\boxed{t = \int \frac{dx}{\sqrt{\frac{2}{m}(E-U(x))}} + \underline{\text{const}}}$$

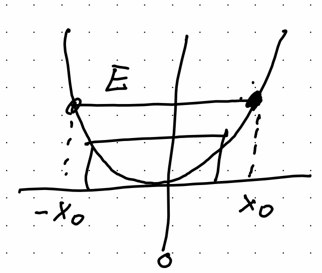
$$\boxed{T(E) = 2 \int_{x_1 = x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{\frac{2}{m}(E-U(x))}}}$$

$\Delta x = \pm \Delta t \underline{\sqrt{\frac{2}{m}(E - U(x))}}$

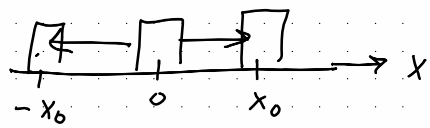
$U(x) = \frac{1}{2} k x^2$

$U(\phi) = mg\cos\phi$

$\ddot{\phi} = \frac{-g}{\ell} \sin\phi$



$U = \frac{1}{2} kx^2$



Prob 2

(a) $U = A|x|^n$

(b) $U = \dfrac{-U_0}{\cosh^2(\alpha x)}$

(c) $U = U_0 \tan^2(\alpha x)$

$$\int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E - U(x)}}$$

$E \leftarrow \frac{1}{2} k x_0^2 - \frac{1}{2} kx^2$

$\frac{1}{2} k (x_0^2 - x^2)$

$$\frac{dx}{\sqrt{x_0^2 - x^2}}$$

→ Beta Function

$E = \frac{1}{2} k x_0^2$

$x_0 = \sqrt{\dfrac{2E}{k}}$

$E = U(x_1) = U(x_2)$

Lecture #9: Tues 9/22

1) Midterm #1 : Tues at 6^{th} (short answer; long problems) 40% 60%

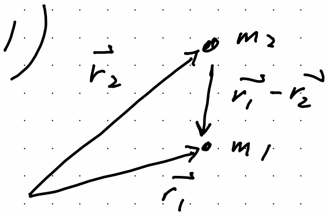
2) Next 4 lecture (central force problem) Sec 13 - 15

General Formalism:

Two interacting particles (no external forces)

m_1, m_2

$U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$



$L = T - U$

$= \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 - U(|\vec{r}_1 - \vec{r}_2|)$

i) L unchanged by a translation $\vec{r}_1 \to \vec{r}_1 + \delta\vec{x}$, $\vec{r}_2 \to \vec{r}_2 + \delta\vec{x}$

$\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2 = \text{const}$

COM moves with const velocity

choose inertial ref frame suchthat COM at origin.
''' COM Frame

$$\frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1+m_2} = \vec{R}_{com} = 0$$

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

$U(|\vec{r}_1 - \vec{r}_2|)$

<u>Define:</u> $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$

$U = U(r)\,,\quad r = |\vec{r}|$

$$\vec{r}_2 = \vec{r}_1 - \vec{r}$$

$$m_1 \vec{r}_1 + m_2(\vec{r}_1 - \vec{r}) = 0$$

$$(m_1+m_2)\vec{r}_1 - m_2 \vec{r} = 0$$

$$\boxed{\vec{r}_1 = \left(\frac{m_2}{m_1+m_2}\right)\vec{r}}$$

$$\vec{r}_2 = \left(\frac{m_2}{m_1+m_2}\right)\vec{r} - \vec{r}$$

$$\boxed{\vec{r}_2 = -\left(\frac{m_1}{m_1+m_2}\right)\vec{r}}$$

$$T = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2$$

$$T = \frac{1}{2} m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 |\dot{\vec{r}}|^2 + \frac{1}{2} m_2 \left(\frac{-m_1}{m_1+m_2}\right)^2 |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} \frac{m_1 m_2}{(m_1+m_2)^{\cancel{2}}} |\dot{\vec{r}}|^2 \cancel{(m_2 + m_1)}$$

$$= \frac{1}{2} \underbrace{\left(\frac{m_1 m_2}{m_1+m_2}\right)}_{\text{reduced mass}: \ m} |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} m |\dot{\vec{r}}|^2$$

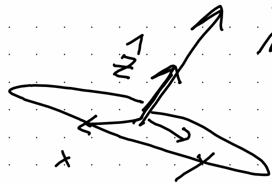
central potential

$$\boxed{L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(r)}$$

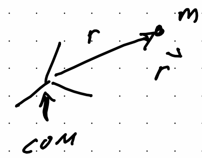
"effective" one body Lagrangian

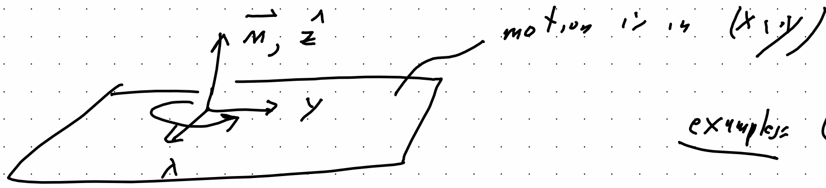
ii) L unchanged under a rotation

$\rightarrow$ <u>$\vec{M} = \text{const}$</u> $\quad (\vec{M} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v})$



$$\underbrace{\vec{M} = \vec{r}\times\vec{p}}_{\perp}$$





examples: $U(r) = -\frac{Gm_1m_2}{r}$ Newtonian gravity

$U(r) = \frac{1}{2}kr^2$ space oscillator

$$\boxed{L = \frac{1}{2}m|\dot{\vec{r}}|^2 - U(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)}$$

i) No explicit t-dependence

$E = T + U = \text{const}$

E, M constants of the motion

ii) No ϕ dependence

$$M_z = p_\phi = \frac{\partial L}{\partial\dot{\phi}} = \text{const} = \boxed{mr^2\dot{\phi} \equiv M}$$

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + U(r),$$

$$M = mr^2\dot{\phi} \longrightarrow \boxed{\dot{\phi} = \frac{M}{mr^2}}$$

$$E = \frac{1}{2}m\dot{r}^2 + \underbrace{\left(\frac{M^2}{2mr^2} + U(r)\right)}_{U_{eff}(r)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right) = \frac{\partial L}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi} = 0$$

$$\frac{dr}{dt} = \dot{r} = \sqrt{\frac{2}{m}\left(E - U(r) - \frac{M^2}{2mr^2}\right)}$$

$$= \sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2r^2}}$$

$$\int dt = \boxed{t = \int \frac{dr}{\sqrt{\frac{2}{m}(E-U(r)) - \frac{M^2}{m^2r^2}}} + \text{const}}$$

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M}{mr^2} = \sqrt{}$$

$$\text{const} + \int \frac{\frac{dr\,M}{mr^2}}{\sqrt{}} = \int d\phi = \phi$$

$$\boxed{\phi = \int \frac{M\,dr/r^2}{\sqrt{2m(E-U(r)) - M^2/r^2}} + \text{const}}$$

$$\phi = \phi(r) \longleftrightarrow r = r(\phi)$$

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

$$\left.\begin{matrix} r = r(\phi) \\ r = r(t) \\ \phi = \phi(t) \end{matrix}\right\} \vec{r}(t)$$

Example: $U(r) = \frac{1}{2}kr^2$ (space oscillator) (r, ϕ)

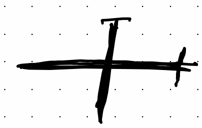
Easy: $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}kr^2$ (x, y)

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2)$$

$$= \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) + \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2\right)$$

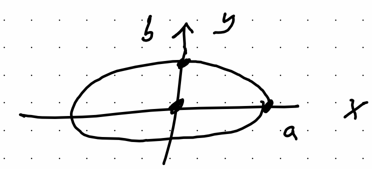
$$x(t) = a\cos(\omega t + \alpha \xrightarrow{} 0), \quad \omega = \sqrt{\frac{k}{m}}$$

$$y(t) = b\sin(\omega t + \beta \xrightarrow{} 0),$$



$\rightarrow$ closed orbit

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



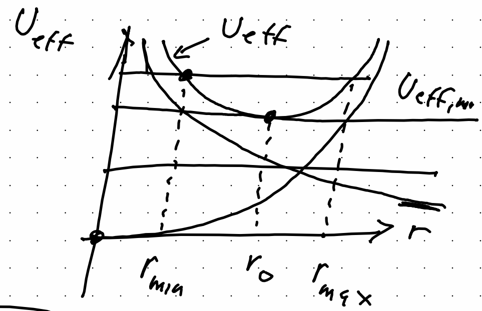
ellipse with center at origin

$U = -\frac{\alpha}{r}$, $U = \frac{1}{2}kr^2$ are only two potentials that have closed bound orbits

Harder: $U = \frac{1}{2} k r^2$

$$U_{eff} = U(r) + \frac{M^2}{2mr^2}$$

$$= \frac{1}{2} k r^2 + \frac{M^2}{2mr^2}$$



$$\phi = \int \frac{M\,dr/r^2}{\sqrt{2m\left(E - \frac{1}{2} k r^2\right) - \frac{M^2}{r^2}}} + const$$ bound orbits

substitutes

$u = \frac{1}{r}$, $du = \frac{-dr}{r^2}$ $v = u^2$, $dv = 2u\,du$

$$\phi = -\int \frac{M\,du}{\sqrt{2m\left(E - \frac{1}{2}\frac{k}{u^2}\right) - M^2 u^2}} + const = -\int \frac{M\,u\,du}{\sqrt{2m\left(Eu^2 - \frac{k}{2}\right) - M^2 u^4}} + const$$

$$\phi = -\frac{1}{2}\int \frac{dv\,M}{\sqrt{2m\left(Ev - \frac{k}{2}\right) - M^2 v^2}} + const$$

Complete the square:

$$-M^2 v^2 + 2mEv - mk$$

$$= -M^2\left(v^2 - \frac{2mEv}{M^2} + \frac{mk}{M^2}\right)$$

$$= -M^2\left(\left(v - \frac{mE}{M^2}\right)^2 - \frac{m^2E^2}{M^4} + \frac{mk}{M^2}\right)$$

$$= -M^2\left((v-A)^2 - B^2\right)$$

$$= M^2\left(B^2 - (v-A)^2\right)$$

$$A = \frac{mE}{M^2}, \quad B^2 = A^2 - \frac{mk}{M^2}$$

$$\phi = -\frac{1}{2}\int \frac{dv}{\sqrt{B^2-(v-A)^2}} + \text{const}$$

$$\int \frac{dx}{\sqrt{1-x^2}}$$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$$\int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\sqrt{1-\sin^2\theta}}} = \theta = \sin^{-1}(x)$$

3rd substitution:

$$v - A = B\sin\theta$$
$$dv = B\cos\theta\, d\theta$$

$v - A = \pm B$

$\frac{1}{r^2} - A = \pm B$

r_{max}

$\frac{1}{r^2} = A \pm B$

$$\phi = -\frac{1}{2}\int \frac{\cancel{B\cos\theta}\, d\theta}{\cancel{\sqrt{B^2 - B^2\sin^2\theta}}} + \text{const}$$
$$= -\frac{1}{2}\theta + \text{const}$$
$$= -\frac{1}{2}\sin^{-1}\left(\frac{v-A}{B}\right) + \text{const}$$
$$= -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right) + \text{const}$$

$\frac{1}{r_{max}^2} = A - B$

$\frac{1}{r_{min}^2} = A + B$

$v = u^2$

$u = \frac{1}{r}$

$v = \frac{1}{r^2}$

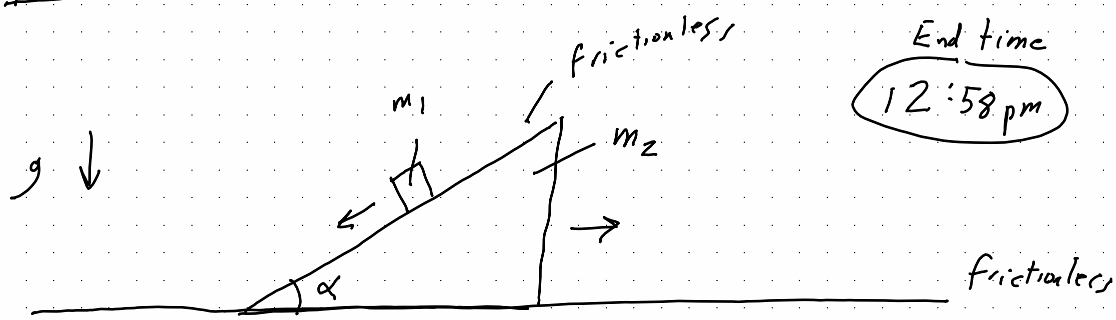
Lecture #10: 9/24

— Finish space oscillator
— Quiz #2
— Next week: Q&A

lastname_q2.pdf

End time

12:58 pm



i) generalized coords

ii) $L = T - U$

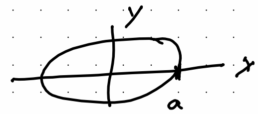
iii) EOMs

iv) special limiting cases: $\alpha = 0$, $\alpha = \pi/2$

$m_1 \gg m_2$, $m_1 \ll m_2$

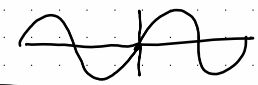
$$\phi = -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right) + \underline{\underline{\text{const}}} \qquad E, M$$

choose const so that $\phi = 0 \longleftrightarrow r = r_{max}$



$$\frac{1}{r_{max}^2} = A - B$$

$$\frac{\frac{1}{r_{max}^2} - A}{+B} = \frac{-B}{B} = -1$$



$$0 = \underbrace{-\frac{1}{2}\sin^{-1}(-1)}_{-\frac{\pi}{2}} + \text{const}$$

$$= \frac{\pi}{4} + \text{const}$$

$$\boxed{\text{const} = -\frac{\pi}{4}}$$

$$\phi = -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right) - \frac{\pi}{4}$$

$$-2\left(\phi + \frac{\pi}{4}\right) = \sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right)$$

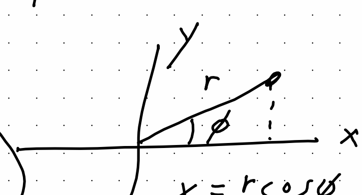
$$-\sin\left(2\phi + \frac{\pi}{2}\right) = \left(\frac{1}{r^2} - A\right)/B$$

$$-\sin\left(2\phi + \frac{\pi}{2}\right) = \left(\frac{1}{r^2} - A\right)/B$$

$$-B\cos(2\phi) = \frac{1}{r^2} - A$$

$$\boxed{\frac{1}{r^2} = A - B\cos(2\phi)}$$

$$\sin\left(2\phi + \frac{\pi}{2}\right)$$
$$= \sin(2\phi)\cancel{\cos\left(\frac{\pi}{2}\right)}^{\,0}$$
$$+ \cos(2\phi)\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}$$
$$= \cos(2\phi)$$



$x = r\cos\phi$

$y = r\sin\phi$

$$\cos(2\phi) = \cos^2\phi - \sin^2\phi$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(2\phi) = \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2$$

$$\left[\frac{1}{r^2} = A - B\left(\left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2\right)\right] \times r^2$$

$$1 = Ar^2 - B(x^2 - y^2)$$

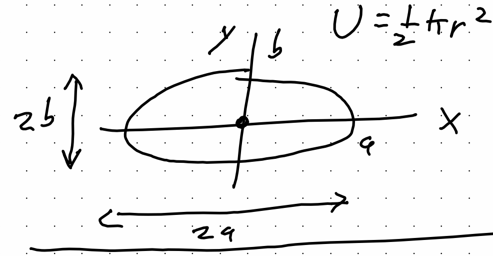
$$= A(x^2+y^2) - B(x^2-y^2) = (A-B)x^2 + (A+B)y^2 = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

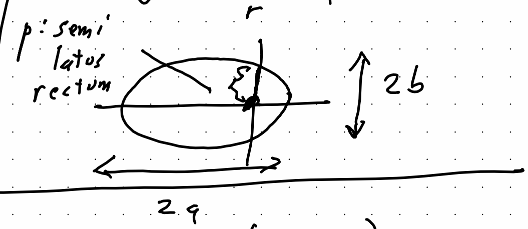
$$\frac{1}{a^2} = A - B, \quad \frac{1}{b^2} = A + B$$

$$A = mE/M^2, \quad B^2 = A^2 - mk/M^2$$

$U = \frac{1}{2}kr^2$



$U = -\frac{\alpha}{r}$ (Newtonian)



$$E = \underbrace{\frac{1}{2}m\dot{r}^2}_{\substack{=0 \text{ at} \\ r=a,\ r=b}} + \frac{M^2}{2mr^2} + \frac{1}{2}kr^2 \qquad \begin{pmatrix} r=a \\ r=b \end{pmatrix}$$

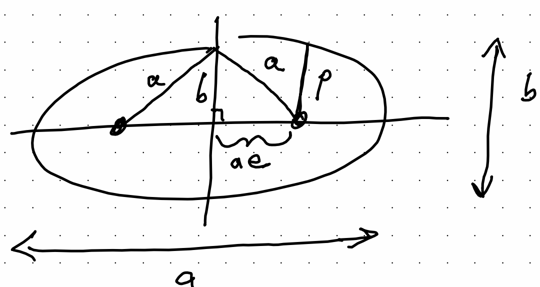
$$\boxed{\begin{aligned} E &= \frac{M^2}{2ma^2} + \frac{1}{2}ka^2 \\ E &= \frac{M^2}{2mb^2} + \frac{1}{2}kb^2 \end{aligned}}$$

$$\rightarrow Ea^2 = \frac{M^2}{2m} + \frac{1}{2}ka^4$$

$$\rightarrow Eb^2 = \frac{M^2}{2m} + \frac{1}{2}kb^2$$

$$\boxed{E = \frac{1}{2}k(a^2+b^2)}$$

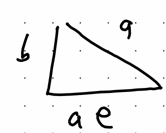
subtract $E(\cancel{a^2-b^2}) = \frac{1}{2}k\underbrace{(a^4-b^4)}_{(\cancel{a^2-b^2})(a^2+b^2)}$ $\rightarrow$ $\boxed{M^2 = mka^2b^2}$

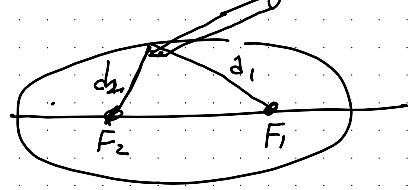


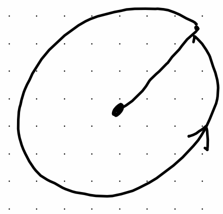
$e = 0$ circle

$e = 1$ line

dimensionless







$a^2 = b^2 + a^2e^2$

$e = \sqrt{\frac{a^2-b^2}{a^2}}$

$d_1 + d_2 = 2a$

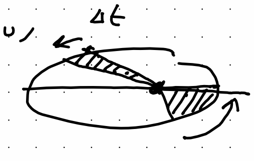
$\boxed{b = a\sqrt{1-e^2}}$

Kepler's laws (for $U = -\alpha/r$)

i) elliptical orbits with sun at one focus

ii) equal areas in equal times

iii) $\frac{P^2}{a^3} = \text{const}$



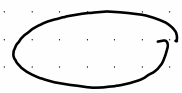
$U = \frac{1}{2} k r^2$

"Kepler's laws": (i) elliptical orbits with "Sun" at center of the ellipse

(ii) equal areas in equal times

(iii)

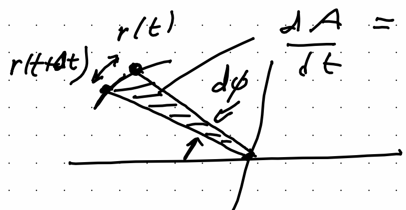
true for any central potential



$$\frac{dA}{dt} = \frac{M}{2m}$$

$$\int dA = \int \frac{M}{2m}\, dt$$

$$M = m r^2 \dot{\phi} \equiv \text{const}$$



$$\frac{dA}{dt} = \frac{1}{2}\, r\, r\frac{d\phi}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt}$$

height

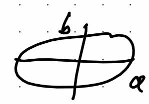
$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{M}{2m} = \text{const}$$

$$\frac{dA}{dt} = \text{const}$$

$$A = \frac{M}{2m} P$$

$A = \pi r^2$ circle

$= \pi a b$



$$\pi a b = \frac{M}{2m} P$$

$$P = \frac{2m}{M} \pi a b \,, \qquad M^2 = m k a^2 b^2$$

$$M = \sqrt{mk}\, a b$$

$$= \frac{2m\, \pi \cancel{ab}}{\sqrt{mk}\, \cancel{ab}}$$

$$= \frac{2\pi}{\sqrt{k/m}}$$

$$= \frac{2\pi}{\omega} = \text{const} \quad (\text{for any } a, b)$$

$P = \text{const}$ indep of a, b !!

Lec #11: Tuesday 9/29

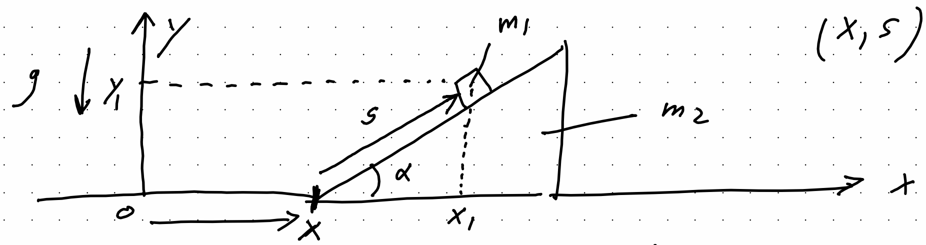
— Midterm #1: Next Tuesday Oct 6^{th}

— Trial midterm on Blackboard (do it before Thursday)

— Today: a) Quiz #2

b) Finish $U = \frac{1}{2} k r^2$

c) Q&A



$L = T - U \qquad T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$

$x_1 = X + s\cos\alpha \rightarrow \dot{x}_1 = \dot{X} + \dot{s}\cos\alpha$

$y_1 = s\sin\alpha \rightarrow \dot{y}_1 = \dot{s}\sin\alpha$

$x_2 = X \longrightarrow \dot{x}_2 = \dot{X}$

$y_2 = 0 \longrightarrow \dot{y}_2 = 0$

$\dot{x}_1^2 = (\dot{X} + \dot{s}\cos\alpha)^2 = \dot{X}^2 + \underline{\dot{s}^2\cos^2\alpha} + 2\dot{X}\dot{s}\cos\alpha$

$\dot{y}_1^2 = \underline{\dot{s}^2 \sin^2\alpha}$

$T = \frac{1}{2} m_1 (\dot{X}^2 + \dot{s}^2 + 2\dot{X}\dot{s}\cos\alpha) + \frac{1}{2} m_2 \dot{X}^2$

$= \frac{1}{2}(m_1 + m_2)\dot{X}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1 \dot{X}\dot{s}\cos\alpha$

$U = m_1 g y_1 = m_1 g s \sin\alpha$

$$\boxed{L = \frac{1}{2}(m_1+m_2)\dot{X}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1 \dot{X}\dot{s}\cos\alpha - m_1 g s \sin\alpha}$$

EOMs: X: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = \cancel{\frac{\partial L}{\partial X}} \rightarrow 0$ $\quad | \quad E = T + U$

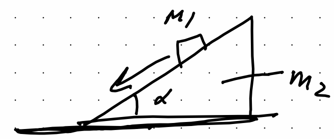
$$\boxed{P_X = (m_1+m_2)\dot{X} + m_1 \dot{s}\cos\alpha = \text{const}}$$

$$\rightarrow \boxed{(m_1+m_2)\ddot{X} + m_1 \ddot{s}\cos\alpha = 0}$$

s:

$\frac{d}{dt}(\cancel{m_1}\dot{s} + \cancel{m_1}\dot{X}\cos\alpha) = -\cancel{m_1} g \sin\alpha \qquad \boxed{\ddot{s} + \ddot{X}\cos\alpha = -g\sin\alpha}$

$$\ddot{x} = -\left(\frac{m_1}{m_1+m_2}\right)\ddot{s}\cos\alpha$$



$$\ddot{s} = -g\sin\alpha\left(\frac{m_1+m_2}{m_1\sin^2\alpha+m_2}\right)$$

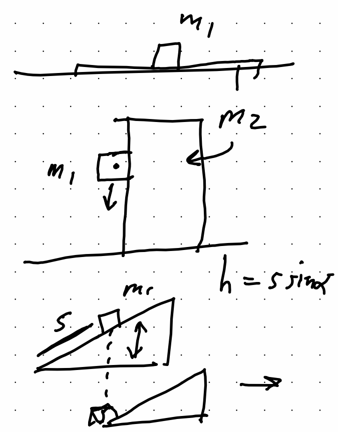
---

i) $m_2 \gg m_1$: $\ddot{s} \approx -g\sin\alpha$

ii) $\alpha = 0$ $\quad \ddot{s} = 0, \ \ddot{x} = 0$

iii) $\alpha = \pi/2$ $\quad \ddot{s} = -g, \ \ddot{x} = 0$

iv) $m_1 \gg m$: $\begin{cases} \ddot{s} \approx -\dfrac{g}{\sin\alpha} \\ \ddot{x} \approx +\dfrac{g}{\tan\alpha} \end{cases}$

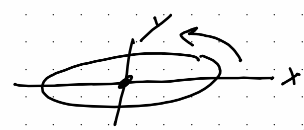


$$\ddot{h} = -g = \ddot{s}\sin\alpha \rightarrow \ddot{s} = \frac{-g}{\sin\alpha}$$

Space Oscillator: $U = \frac{1}{2}kr^2$ $\quad$ $U = \frac{-\alpha}{r}$

"Kepler's" law:

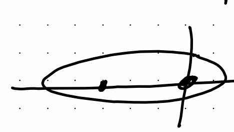
(i) ellipses with origin at center of ellipse



(ii) equal areas in equal times — applies to all central potentials

(iii) $P = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}}$ indep. of a, b

$\frac{P^2}{a^3}$ = const $\quad$ ordinary Kepler



$$t = -\int \frac{dr}{\sqrt{\frac{2}{m}(E-U(r)) - \frac{M^2}{m^2r^2}}} + \text{const}$$

angular moment

reduced mass $\quad$ $\frac{1}{2}kr^2$ $\quad$ $E = \frac{1}{2}k(a^2+b^2)$

want t to increase as r decreases. $\quad$ $M^2 = mka^2b^2$

$$t = -\int \frac{dr}{\sqrt{\frac{\cancel{2}}{m}\left(\frac{1}{\cancel{2}}k(a^2+b^2) - \frac{1}{\cancel{2}}kr^2\right) - \frac{mka^2b^2}{m^2 r^2}}}$$

(r,ϕ)
(x,y)

$$= -\frac{1}{\sqrt{\frac{k}{m}}}\int \frac{dr}{\sqrt{(a^2+b^2) - r^2 - \frac{a^2b^2}{r^2}}}$$

$r^2 = x^2+y^2$

$$= -\frac{1}{\sqrt{\frac{k}{m}}}\int \frac{r\,dr}{\sqrt{r^2(a^2+b^2) - r^4 - a^2b^2}}$$

$$= - \frac{1}{\omega}\int \frac{r\,dr}{\sqrt{-\underbrace{(r^4 - r^2(a^2+b^2) + a^2b^2)}_{=(r^2-a^2)(r^2-b^2)}}}$$

$b<r<a$

$\sqrt{a^2(1-\sin^2\theta)} = a\cos\theta$

$$\int \frac{dx}{\sqrt{a^2-x^2}}$$

$x = a\sin\theta$

substitution

$$\boxed{r^2 = a^2\cos^2\xi + b^2\sin^2\xi}$$

$\boxed{x = a\cos\xi}$ $\boxed{y = b\sin\xi}$

$\rightarrow r^2 = a^2\cos^2\xi + b^2\sin^2\xi$

Differentiates

$$2r\,dr = -2a^2\cos\xi\sin\xi + 2b^2\sin\xi\cos\xi$$

$$= -2\sin\xi\cos\xi(a^2-b^2)$$

$$r\,dr = -\sin\xi\cos\xi(a^2-b^2)$$

$$\sqrt{-(r^2-a^2)(r^2-b^2)} = \sqrt{-(\underbrace{a^2\cos^2\xi + b^2\sin^2\xi - a^2}_{-a^2\sin^2\xi})(\underbrace{a^2\cos^2\xi + b^2\sin^2\xi - b^2}_{-b^2\cos^2\xi})}$$

$$= \sqrt{\underbrace{-(-)}_{+}(a^2-b^2)\sin^2\xi\,(a^2-b^2)\cos^2\xi}$$

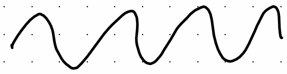
$$= \boxed{(a^2-b^2)\sin\xi\cos\xi}$$

$t = \frac{1}{\omega}\int d\xi + \text{const} \rightarrow \boxed{\omega t = \xi} + \underbrace{\text{const}}_{=0}$

$$\boxed{\begin{aligned} x &= a\cos(\omega t) \\ y &= b\sin(\omega t)\end{aligned}} \rightarrow r^2 = x^2+y^2$$

$$= a^2\cos^2(\omega t) + b^2\sin^2(\omega t)$$

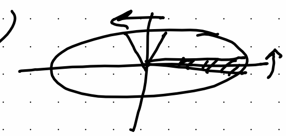
$$\underbrace{\boxed{r = \sqrt{a^2\cos^2(\omega t) + b^2\sin^2(\omega t)}}}_{r(t)}$$

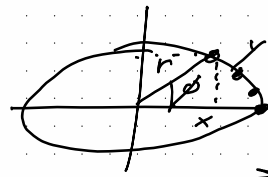


$\phi = \phi(t)$

$(x,y) \longleftrightarrow (r,\phi)$

$$\boxed{\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi\end{aligned}}$$





$r\cos\phi = a\cos(\omega t)$

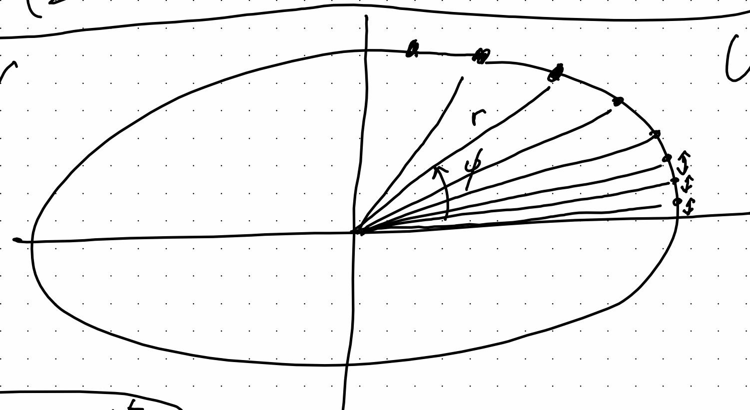
$r\sin\phi = b\sin(\omega t)$

Divide 2nd by 1st: $\tan\phi = \frac{b}{a}\tan(\omega t)$ $\rightarrow \boxed{\phi = \tan^{-1}\left(\frac{b}{a}\tan(\omega t)\right)}$

$\rightarrow \phi = \omega t$ if $b = a$

$$L = \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) + \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2\right)$$

$U = -\frac{\alpha}{r}$



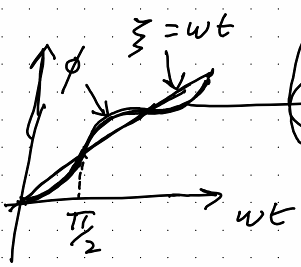
$\boxed{+\frac{1}{2}}$

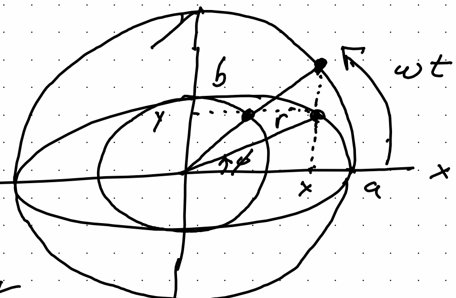
$x = a\cos\omega t$

$y = b\sin\omega t$

$x = r\cos\phi$

$y = r\sin\phi$





Lec #12:

- Midterm Exam 1 : Tuesday Oct 6th
- Do trial exam by end of the day today
- Today : Q&A

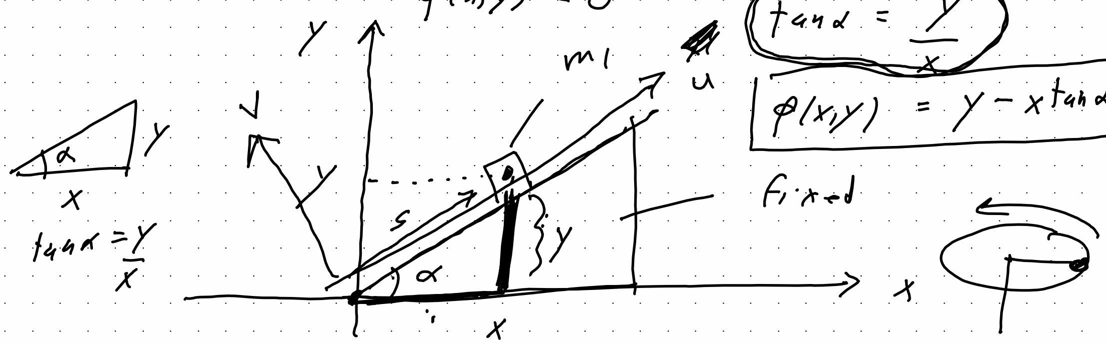
$$x^2+y^2-r_0^2=0$$

$$r-r_0=0$$

$$\varphi(x,y)=0$$

$$\tan\alpha=\frac{y}{x}$$

$$\boxed{\varphi(x,y)=y-x\tan\alpha}$$



$$T=\frac{1}{2}m_1(\dot{x}^2+\dot{y}^2)\quad,\quad U=m_1gy$$

$$L=T-U+\lambda(y-x\tan\alpha)$$

(x,y) ✓

(r,ϕ)

EOMs:

x : $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right)=\frac{\partial L}{\partial x}\rightarrow$ $m_1\ddot{x}=-\lambda\tan\alpha$ ✓

y : $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{y}}\right)=\frac{\partial L}{\partial y}\rightarrow$ $m_1\ddot{y}=-m_1g+\lambda$ ✓

constraint : ✓ $\boxed{y-x\tan\alpha=0}$

$$\boxed{\ddot{y}-\ddot{x}\tan\alpha=0}$$

$$(m_1\ddot{x}=-\lambda\tan\alpha)\tan\alpha$$

$$+\,m_1\ddot{x}\tan\alpha=-m_1g+\lambda$$

$$0=-\lambda\tan^2\alpha-\lambda+m_1g$$

$$=-\lambda(1+\tan^2\alpha)+mg$$

$\sin^2+\cos^2=1$

$\tan^2+1=\sec^2$

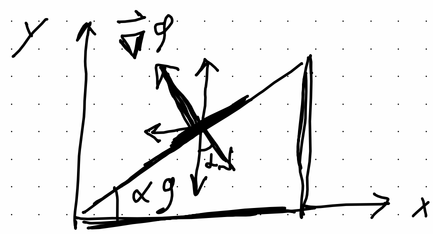
$$\boxed{L=\frac{1}{2}m_1(\dot{x}^2+\dot{y}^2)-mgy+\lambda(y-x\tan\alpha)}$$

$$\lambda=\frac{mg}{\sec^2\alpha}\propto mg\cos\alpha$$

$$\varphi(x,y) = y - x\tan\alpha$$

$$\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{x} + \frac{\partial\varphi}{\partial y}\hat{y}$$

$$= -\tan\alpha\,\hat{x} + \hat{y} = -\frac{\sin\alpha}{\cos\alpha}\hat{x} + \hat{y}$$



$$= \frac{-\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}}{\cos\alpha}$$

$$\boxed{\begin{array}{l}\vec{N} = mg\cos\alpha \\ \perp \text{ to incline}\end{array}}$$

$$\vec{\nabla}\varphi \propto -\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}$$

$$y = x\tan\alpha$$

$$dy = \tan\alpha\,dx \rightarrow \boxed{\cos\alpha\,dy = \sin\alpha\,dx}$$

$$\vec{\nabla}\varphi\cdot d\vec{r} = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = -\cancel{\tan\alpha\,dx} + 1\,\cancel{\tan\alpha}\,dx$$

$$= 0$$

(u,v) $\qquad \varphi(u,v) = v = 0$

$$L = T - U + \lambda v$$

$$v = 0$$

$$\boxed{L = T - U + \lambda\varphi}$$

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - U(x)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}(m\dot{x}) = -\frac{\partial U}{\partial x}$$

$$\boxed{\frac{dp}{dt} = -\frac{\partial U}{\partial x} = F}$$

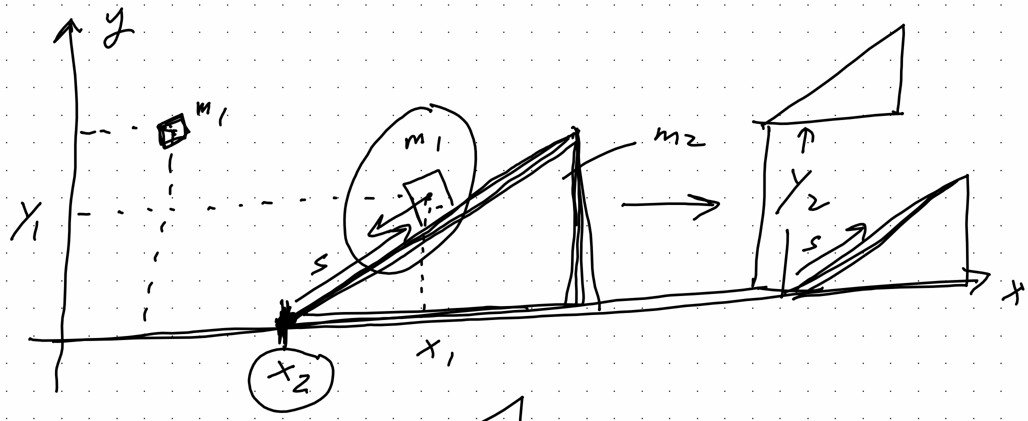
$$\frac{dp}{dt} = F$$

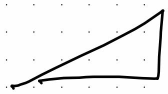
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\vec{r}}}\right) = -\frac{\partial L}{\partial\vec{r}}$$

$$\frac{d\vec{p}}{dt} = -\frac{\partial U}{\partial\vec{r}} + \lambda\frac{\partial\varphi}{\partial\vec{r}}$$

$$\frac{d\vec{p}}{dt} = -\vec{\nabla}U + \vec{\nabla}\varphi$$

$$= \underbrace{\vec{F}_{\text{applied}} + \vec{F}_{\text{constraint}}}$$





$y_2 = 0$

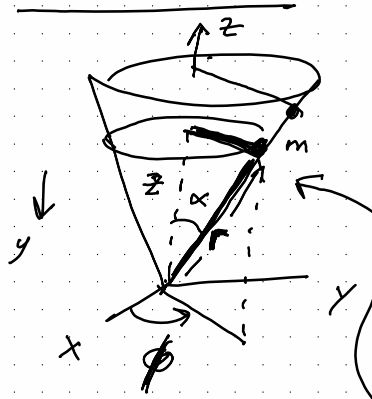
(x_1, y_1, x_2, y_2)

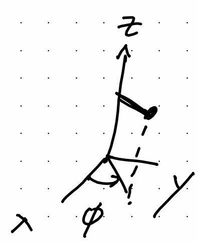
m_1

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \cancel{\dot{y}_2^2})$$

$\dot{y}_2 \to 0$

Sec 14, Prob 2:





~~cylindrical~~

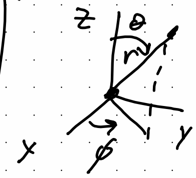
(r, ϕ, z)

$r^2 = x^2 + y^2$

alt. notation

$s^2 = x^2 + y^2$

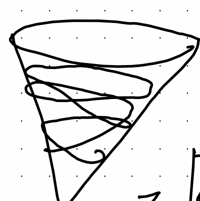
$\rho^2 = x^2 + y^2$

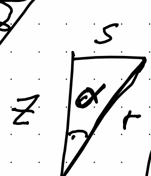


sph. polar

(r, ϕ, θ)

$r^2 = x^2 + y^2 + z^2$





$\boxed{\tan\alpha = \frac{s}{z}}$

$\theta = \alpha$

(ϕ, s) (ϕ, z)

$$T = \frac{1}{2}m\left(\dot r^2 + \underbrace{\cancel{r^2\dot\theta^2}}_{0} + r^2\underbrace{\sin^2\theta}_{\sin^2\alpha}\,\dot\phi^2\right)$$

sph. polar

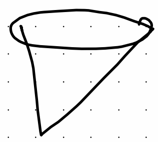
$\theta = \alpha$

$$T = \frac{1}{2}m\left(\underbrace{\dot s^2 + s^2\dot\phi^2}_{\text{plane polar coord}} + \dot z^2\right)$$

cyl. coord,

s, ϕ, z

$\tan\alpha = \dfrac{s}{z}$



$z = \dfrac{s}{\tan\alpha}$

$\dot z = \dfrac{\dot s}{\tan\alpha}$

$s = z\tan\alpha$

$\dot s = \dot z\tan\alpha$

$L = T - U + \lambda\varphi$

$\varphi = s - z\tan\alpha$

$$f(x,y) = e^{-\frac{1}{2}(x^2+y^2)}$$

No constraint: $\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases}$

$g(x,y,\lambda)$

$\equiv f(x,y) + \lambda\varphi(x,y)$

$\dfrac{\partial g}{\partial x} = 0$

$\dfrac{\partial g}{\partial y} = 0, \quad \dfrac{\partial g}{\partial \lambda} = 0$

suppose constraint $\quad y = \frac{1}{2}x^2 + 1$

$$F(x) = f(x,y)\Big|_{y=\frac{1}{2}x^2+1}$$

$\boxed{\dfrac{dF}{dx} = 0} \qquad \underbrace{\boxed{\varphi = y - \frac{1}{2}x^2 - 1}}_{=0}$

$L = T - U + \lambda\varphi$

x, y, λ

COM:

$m_1, \; m_2, \; m_3, \; \vec r_1, \; \vec r_2, \; \vec r_3$

$$\vec R_{com} = \frac{m_1 \vec r_1 + m_2 \vec r_2 + m_3 \vec r_3}{m_1 + m_2 + m_3}$$

$$0 = \frac{\sum_a m_a \vec r_a}{\sum_b m_b}$$

$\vec P = M_{tot} \dot{\vec R}_{com}$

$= \text{const}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec r}_a}\right) = 0$$

$$\vec P = \sum_a \frac{\partial L}{\partial \dot{\vec r}_a} = \text{const}$$

~~Rcom~~ closed system: $\vec P = \text{const}$

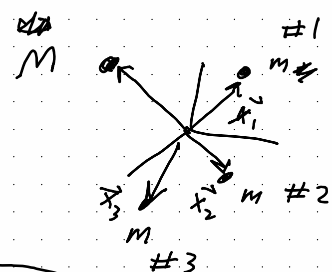
$$L = \frac{1}{2} \sum_a m_a |\dot{\vec r}_a|^2 - U(|\vec r_1 - \vec r_2|, |\vec r_1 - \vec r_3|, \text{etc.})$$

$\vec r_a \to \vec r_a + \delta \vec x$

$L \to L$

$$\frac{\partial L}{\partial \vec r} = 0$$

$M, \; \underbrace{m, \dots, m}_{n}$



$\vec X \quad \vec x_1, \dots, \vec x_n$

$$M \vec X + m \sum_{a=1}^{n} \vec x_a = 0$$

$$\boxed{\begin{aligned} \vec r_1 &= \vec x_1 - \vec X \\ \vec r_2 &= \vec x_2 - \vec X \\ &\vdots \\ \vec r_n &= \vec x_n - \vec X \end{aligned}}$$

$\vec r_1, \vec r_2$

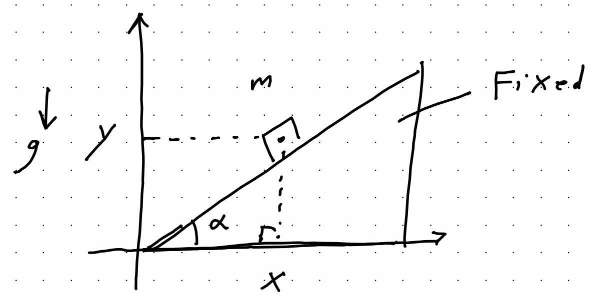
$\vec r = \vec r_1 - \vec r_2$

1 hr + 20 mm

80 mm

120 mins

2:35 pm



$\tan\alpha = \dfrac{y}{x}$

so $y = x\tan\alpha$

$\underbrace{\varphi(x,y) = y - x\tan\alpha = 0}_{\text{constraint}}$

$$L = T - U + \lambda\varphi$$
$$= \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy + \lambda(y - x\tan\alpha)$$

EOMs:

x: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) = \dfrac{\partial L}{\partial x} \rightarrow m\ddot{x} = -\lambda\tan\alpha \quad (1)$

y: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{y}}\right) = \dfrac{\partial L}{\partial y} \rightarrow m\ddot{y} = -mg + \lambda \quad (2)$

λ: $y - x\tan\alpha = 0 \rightarrow y = x\tan\alpha \quad (3)$

Differentiate constraint twice:

$$\ddot{y} = \ddot{x}\tan\alpha$$

substitute for $\ddot{y}$, $\ddot{x}$ using (1) and (2)

$$-g + \frac{\lambda}{m} = -\frac{\lambda}{m}\tan\alpha \cdot \tan\alpha$$

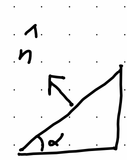
$$\frac{\lambda}{m}\big(\underbrace{1+\tan^2\alpha}_{\sec^2\alpha}\big) = g$$

$$\boxed{\lambda = \frac{mg}{\sec^2\alpha}}$$

constraint Force:

$\vec{F}_c = \lambda\vec{\nabla}\varphi$ where $\varphi = y - x\tan\alpha$

$$= \frac{mg}{\sec^2\alpha}\left(-\tan\alpha\,\hat{x} + \hat{y}\right)$$
$$= \frac{mg}{\sec^2\alpha}\left(-\frac{\sin\alpha}{\cos\alpha}\,\hat{x} + \hat{y}\right)$$
$$= mg\cos\alpha\left(-\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}\right)$$
$$= \boxed{mg\cos\alpha\;\hat{n}}$$



(where $\hat{n} = -\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}$ is $\perp$ to incline)

Return to EOMs:

$$\ddot{x} = -\frac{\lambda}{m} \tan\alpha$$

$$\ddot{y} = -g + \frac{\lambda}{m}$$

where $\lambda = \dfrac{mg}{\sec^2\alpha} = mg\cos^2\alpha$

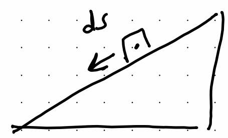
$$\rightarrow \quad \ddot{x} = -g\cos^2\alpha\tan\alpha$$

$$= -g\sin\alpha\cos\alpha$$

$$\ddot{y} = -g + g\cos^2\alpha$$

$$= -g(1-\cos^2\alpha)$$

$$= -g\sin^2\alpha$$



$ds = \sqrt{dx^2+dy^2}$

Acceleration down the incline:

$$\ddot{s} = -\sqrt{\ddot{x}^2+\ddot{y}^2} = -g\sin\alpha\sqrt{\cos^2\alpha+\sin^2\alpha} = \boxed{-g\sin\alpha}$$

standard result

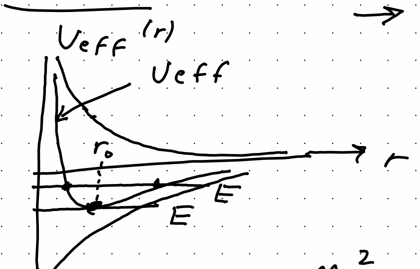
Lecture #14: Thurs Oct 8^{th}

- midterm 1: Avg $\approx 14/20$
- midterm 2: Nov 19^{th} (before Thanksgiving)
- oral Final

Collisions (2) & Scattering (3) (Sec 16 - 20)

Scattering: closed system, two bodies

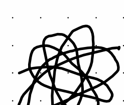
$\rightarrow$ central force problem

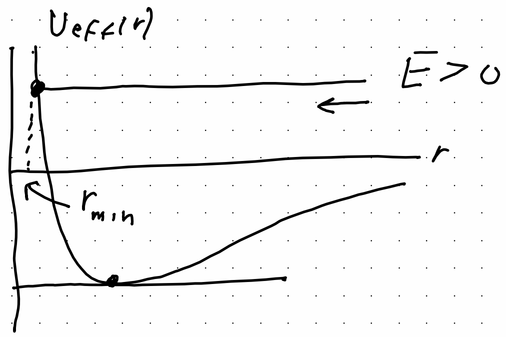


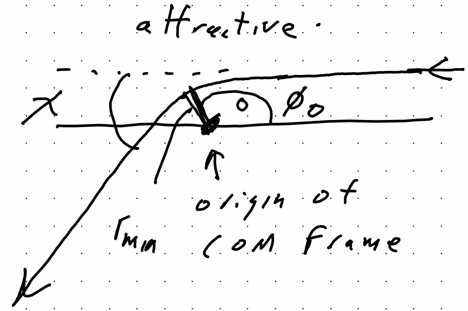
$$U_{eff} = -\alpha/r + \frac{M_z^2}{2mr^2}$$

$E = U_{eff,min} \rightarrow r = r_0$ (circular)

$U_{eff,min} < E < 0 \rightarrow$ $\underbrace{\text{bound orbit}}_{\text{ellipse for } -\alpha/r}$

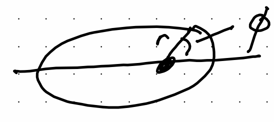


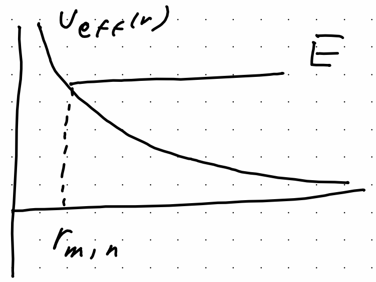


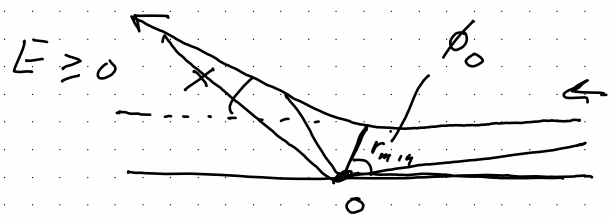


$$U(r) = +\alpha/r$$

$$U_{eff}(r) = \alpha/r + M_z^2/2mr^2$$







$$\int_0^{\phi_0} d\phi = \int_{r_{min}}^{\infty} \frac{M_z\, dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M_z^2}{r^2}}} \qquad (14.7)\ L\&L$$

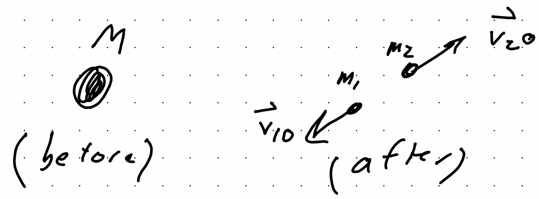
$$\boxed{\phi_0 = \int_{r_{min}}^{\infty} \frac{M_z\, dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M_z^2}{r^2}}}}$$

$E, M_z, U(r)$

Fig 18 in L & L

Collisions: (Sec 16, 17)

- Elastic collisions of two particles
- Spontaneous disintegration of a single particle



Analyze in COM Frame (to begin with)

$$\underbrace{\vec{p}_{tot,0}}_{\text{COM}} = \vec{0}$$

cons of momentum

$\vec{p}_0$: momentum of mass m_1 after disintegration

$-\vec{p}_0$: " m_2

cons. of energy

$$\underbrace{E_i}_{\substack{\text{internal} \\ \text{energy of} \\ \text{mass } M = m_1 + m_2}} = \underbrace{E_{1i}}_{\substack{\text{internal} \\ \text{energy} \\ \text{of } m_1}} + \underbrace{T_{10}}_{\substack{\text{KE} \\ \text{in} \\ \text{COM} \\ \text{frame}}} + E_{2i} + T_{20}$$

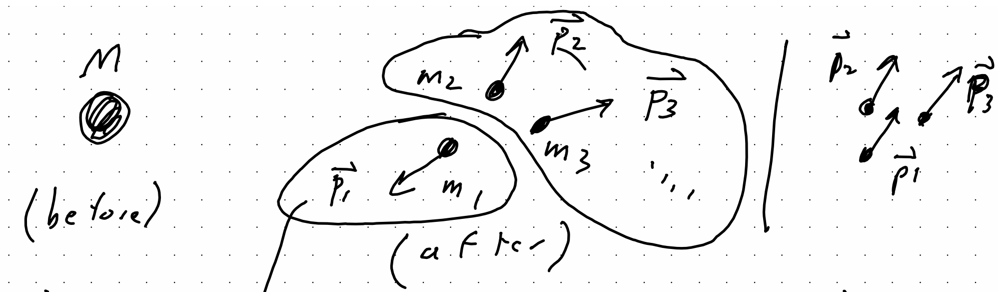
$$T_{10} = \frac{1}{2} m_1 |\vec{v}_{10}|^2 = \frac{|\vec{p}_0|^2}{2m_1}$$

$$E_i = E_{1i} + \frac{|\vec{p}_0|^2}{2m_1} + E_{2i} + \frac{|\vec{p}_0|^2}{2m_2}$$

$$\underbrace{E_i - E_{1i} - E_{2i}}_{\substack{\epsilon = \text{disintegration energy} \\ \geq 0}} = \frac{p_0^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

$$= \frac{p_0^2}{2}\left(\frac{m_1 + m_2}{m_1 m_2}\right) = \frac{p_0^2}{2m}$$

$$p_0 = \sqrt{2m\epsilon} \rightarrow v_{10} = p_0/m_1, \quad v_{20} = p_0/m_2$$



$\vec{p}_{tot} = 0$

$0 = \vec{p}_{tot} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots$

$\vec{p}_1 = \vec{p}_0$

$T_{10} = \frac{p_0^2}{2m}$

$\underline{Q:}$ what condition is there on the velocities of $m_2, m_3, \ldots$ such that KE of m_1 is largest?

Consider COM of $m_2, m_3, \ldots$

E_i' : internal energy of $m_2, m_3, \ldots$

$\vec{p}_2 + \vec{p}_3 + \cdots = -\vec{p}_0$

$KE = \frac{p_0^2}{2\underbrace{(M-m_1)}_{(m_2+m_3+\cdots)}}$

---

cons. of energy:

$$\underbrace{E_i}_{\text{int. energy of } M} = \underbrace{E_{1i} + \frac{p_0^2}{2m_1}}_{\text{int. + KE of } m_1} + \underbrace{E_i' + \frac{p_0^2}{2(M-m_1)}}_{\text{int + KE of } m_2, m_3, \ldots}$$

$$E_i - E_{1i} - E_i' = \frac{p_0^2}{2}\left(\frac{1}{m_1} + \frac{1}{M-m_1}\right)$$

$$E_i - E_{1i} - E_i' = \frac{p_0^2}{2}\,\frac{M}{m_1(M-m_1)}$$

$E_i' \neq E_{2i} + E_{3i} + \ldots$

$$\boxed{T_{10} = \frac{p_0^2}{2m_1} = \left(\frac{M-m_1}{M}\right)\left(E_i - E_{1i} - E_i'\right)}$$

T_{10} ~~is~~ maximum when E_i' is minimum.

$$E_i' + \frac{p_0^2}{2(M-m_1)} = \underbrace{E_{2i} + \frac{p_2^2}{2m_2} + E_{3i} + \frac{p_3^2}{2m_3} + \cdots}_{\text{int energy + KE of masses } m_2,\ m_3,\ \ldots}$$

$\vec{p}_2 = m_2\vec{v}_2$, etc.

$$= E_{2i} + \tfrac{1}{2}m_2|\vec{v}_2|^2 + E_{3i} + \tfrac{1}{2}m_2|\vec{v}_3|^2 + \cdots$$

$$\underset{\text{minimized}}{E_i'} = (E_{2i} + E_{3i} + \cdots) + \left(\tfrac{1}{2}m_2|\vec{v}_2|^2 + \tfrac{1}{2}m_3|\vec{v}_3|^2 + \cdots\right) - \frac{|\vec{p}_2 + \vec{p}_3 + \cdots|^2}{2(m_2+m_3+\cdots)}$$

$= |\vec{v}_0|^2$

$$= (E_{2i} + E_{3i} + \cdots) + \left(\tfrac{1}{2}m_2|\vec{v}_2|^2 + \tfrac{1}{2}m_3|\vec{v}_3|^2 + \cdots\right) - \frac{1}{2}\frac{|m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots|^2}{m_2+m_3+\cdots}$$

$$|\vec{v}_2 + \vec{v}_3|^2 = |\vec{v}_2|^2 + |\vec{v}_3|^2 + 2\vec{v}_2\cdot\vec{v}_3$$

IF $\vec{v}_2 = \vec{v}_3 = \cdots \equiv \vec{v}_0$

then:

$$\tfrac{1}{2}m_2|\vec{v}_2|^2 + \tfrac{1}{2}m_3|\vec{v}_3|^2 + \cdots = \tfrac{1}{2}(m_2+m_3+\cdots)|\vec{v}_0|^2$$

and

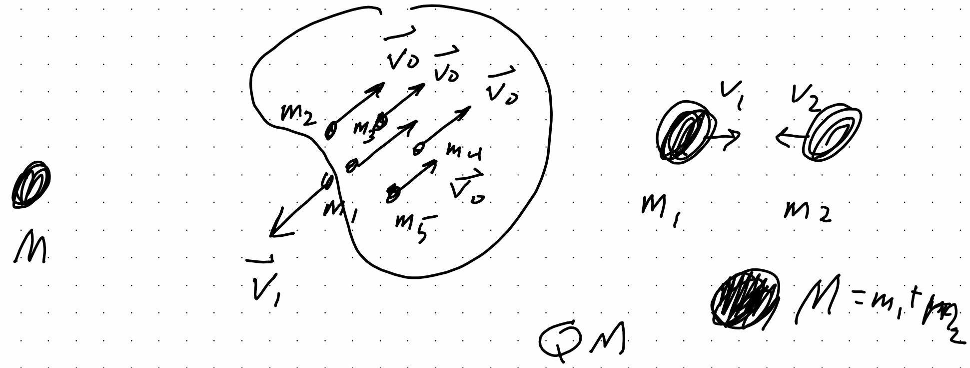
$$\frac{1}{2}\frac{(m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots)^2}{m_2+m_3+\cdots}$$

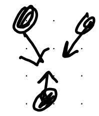
$$= \frac{1}{2}|\vec{v}_0|^2\frac{(m_2+m_3+\cdots)^{\cancel{2}}}{\cancel{m_2+m_3+\cdots}}$$

$$= \tfrac{1}{2}|\vec{v}_0|^2(m_2+m_3+\cdots)$$

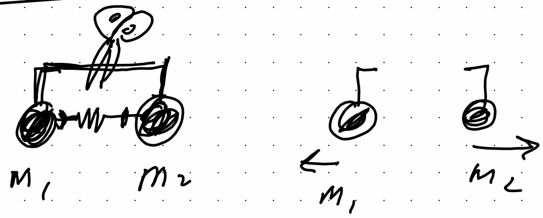
$=$

$$T_{10,\max} = \left(\frac{M-m_1}{M}\right)\epsilon$$





Figs 14	Figs 15, 16



Lec #15: Tues 10/13

- solutions to midterm #1 posted
- next two weeks : Sec 16 – 20
  Collisions and scattering

Figures 14, 15, 16

Elastic collision of two particles

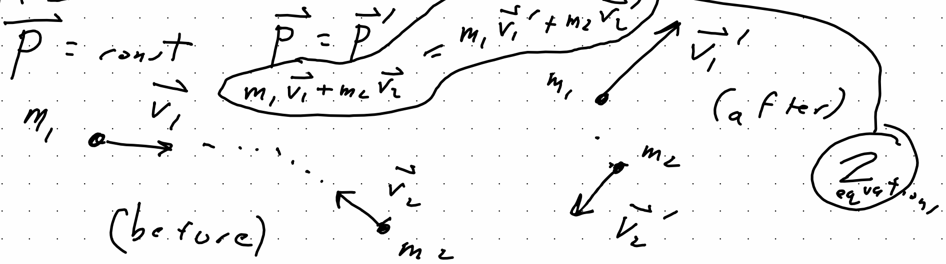
closed system: E, $\vec{P}$, $\vec{M}$ are conserved

✓ i) KE = const [ignore internal energies]

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 (v_1')^2 + \frac{1}{2}m_2 (v_2')^2$$ 1

✓ ii) $\vec{P}$ = const $\quad \vec{P} = \vec{P}'$

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1{}' + m_2\vec{v}_2{}'$$ 2 equations



After: $\vec{v}_1{}'$, $\vec{v}_2{}'$: $\boxed{\begin{array}{l}4 \text{ DOF} \\ 3 \text{ equations}\end{array}}$

Elastic collision:

$\vec{v} \equiv \vec{v}_1 - \vec{v}_2 = \vec{v}_{10} - \vec{v}_{20}$ relative velocity vector

$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 = \vec{r}_{10} - \vec{r}_{20}$ relative position vector

$\vec{v} = \dot{\vec{r}}$

$\vec{v}_1 = \vec{V} + \vec{v}_{10}$

$\vec{v}_2 = \vec{V} + \vec{v}_{20}$

$\vec{r}_1 = \vec{R} + \vec{r}_{10}$

$\vec{r}_2 = \vec{R} + \vec{r}_{20}$

$\vec{v}_1$: velocity of particle 1 wrt lab frame before collision

$\vec{v}_{10}$: " wrt COM frame before collision

$\vec{v}_1{}'$, $\vec{v}_{10}{}'$, $\vec{v}_2{}'$, $\vec{v}_{20}{}'$; velocities after collision

$$T_0 = \frac{1}{2} m_1 |\dot{\vec{r}}_{10}|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_{20}|^2$$

$$\vec{v}_{10} = \dot{\vec{r}}_{10} = \left(\frac{m_2}{m_1+m_2}\right)\dot{\vec{r}} \qquad \dot{\vec{r}} = \vec{v} \qquad \vec{r} \equiv \vec{r}_{10} - \vec{r}_{20}$$

$$\vec{v}_{20} = \dot{\vec{r}}_{20} = -\left(\frac{m_1}{m_1+m_2}\right)\dot{\vec{r}}$$

$$T_0 = \frac{1}{2} m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 |\dot{\vec{r}}|^2 + \frac{1}{2} m_2 \left(\frac{-m_1}{m_1+m_2}\right)^2 |\dot{\vec{r}}|^2$$

$$= \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)|\dot{\vec{r}}|^2$$

$$= \frac{1}{2} m |\dot{\vec{r}}|^2 = \frac{1}{2} m |\vec{v}|^2$$

$T_0 = T_0'$

$\cancel{\tfrac{1}{2}}\cancel{m}|\vec{v}|^2 = \cancel{\tfrac{1}{2}}\cancel{m}|\vec{v}'|^2$

$\boxed{v = v'}$

~~relative~~ magnitude of relative velocity vector = const

Before collision

$$\vec{V}_{10} = \left(\frac{m_2}{m_1+m_2}\right)\underbrace{\vec{V}}_{\vec{v}_1}, \quad \vec{V}_{20} = \left(\frac{-m_1}{m_1+m_2}\right)\underbrace{\vec{V}}_{\vec{v}_1}$$

$$\vec{V} = \vec{V}_1 - \cancel{\vec{V}_2}$$

$$\underbrace{\left|\begin{array}{l}\vec{V}_1 = \vec{V} \\ \vec{V}_2 = 0\end{array}\right.}_{\text{lab Frame}}$$

After collision:

$$\vec{V}_{10}' = \left(\frac{m_2}{m_1+m_2}\right) V \hat{n}_0$$

V — magnitude of $\vec{V}$

$\hat{n}_0$ — unit vector pointing in some direction (angle $= \chi$ wrt COM Frame)

$$\vec{V}_{20}' = \left(\frac{-m_1}{m_1+m_2}\right) V \hat{n}_0$$

$$\boxed{\vec{V}_1' = \vec{V}_{10}' + \vec{V} = \left(\frac{m_2}{m_1+m_2}\right) V\hat{n}_0 + \frac{m_1\vec{V}_1 + m_2\vec{V}_2}{m_1+m_2}}$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \rightarrow \vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}$$

$$\vec{V}_1' = \vec{V}_{10}' + \vec{V} = \left(\frac{m_2}{m_1+m_2}\right) V\hat{n}_0 + \frac{m_1\vec{V}_1 + m_2\vec{V}_2}{m_1+m_2} = \vec{V}_1'$$

$$\vec{V}_2' = \vec{V}_{20}' + \vec{V} = \left(\frac{-m_1}{m_1+m_2}\right) V\hat{n}_0 + \frac{m_1\vec{V}_1 + m_2\vec{V}_2}{m_1+m_2} = \vec{V}_2'$$

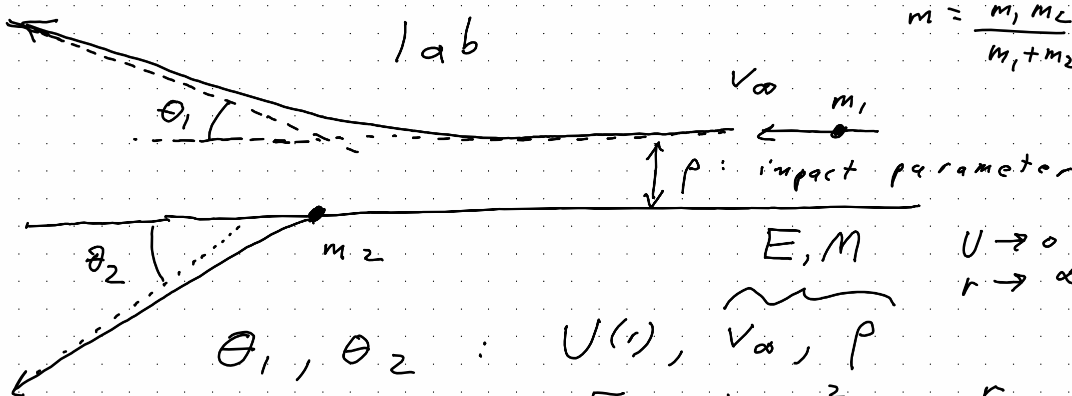
$$\boxed{\vec{p}_1' = m_1\vec{v}_1' = m V\hat{n}_0 + m_1\underbrace{\left(\frac{\vec{p}_1+\vec{p}_2}{m_1+m_2}\right)}_{\vec{V}}}$$

$$\boxed{\vec{p}_2' = m_2\vec{v}_2' = -m V\hat{n}_0 + m_2\left(\frac{\vec{p}_1+\vec{p}_2}{m_1+m_2}\right)}$$

Lec #16: Thurs 10/15

- Quiz #3 next week
- General remarks about scattering
- Example: Prob 18.1 (Hard-sphere)

$$m = \frac{m_1 m_2}{m_1 + m_2}$$



E, M $U \to 0$, $r \to \infty$

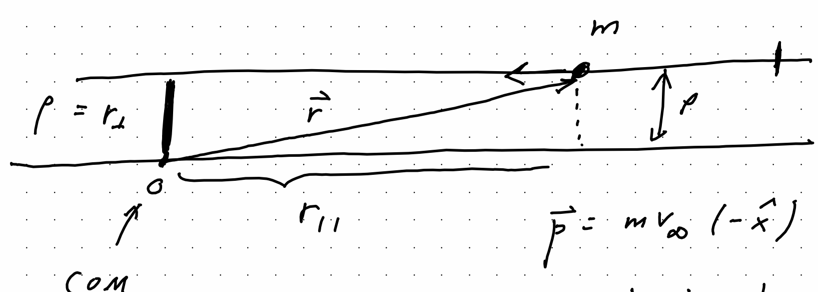
θ_1, θ_2 : $U(r)$, v_∞, ρ

χ : scattering angle w/t COM frame

$$E = \frac{1}{2} m v_\infty^2$$

$$M = \rho m v_\infty \qquad \Big|\; |\vec{r} \times \vec{p}| = M$$

$r_\perp$

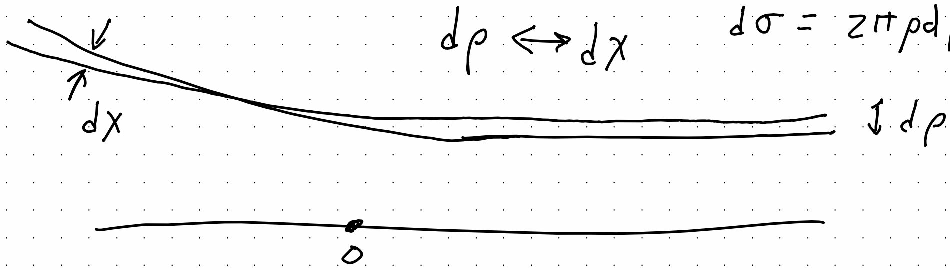


$$\vec{p} = m v_\infty (-\hat{x})$$

$$\boxed{\begin{aligned} E &= \tfrac{1}{2} m v_\infty \\ M &= \rho m v_\infty \end{aligned}}$$

$$\begin{aligned} M &= |\vec{r} \times \vec{p}| \\ &= r_\perp m v_\infty \\ &= \rho m v_\infty \end{aligned}$$

$d\rho \leftrightarrow d\chi$ $d\sigma = 2\pi \rho \, d\rho$



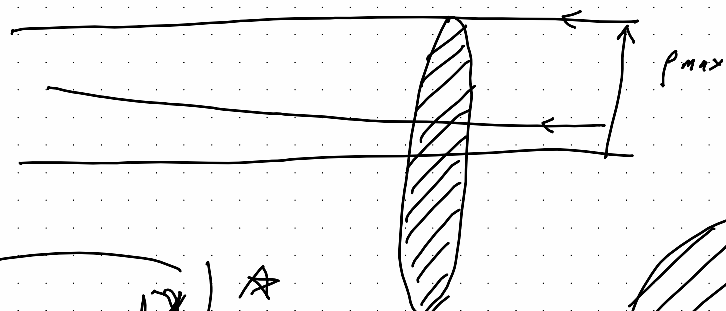
$$d\sigma = 2\pi\, \rho(\chi) \left|\frac{d\rho}{d\chi}\right| d\chi$$

$$d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega = \rho\left|\frac{d\rho}{d(\cos\chi)}\right| d\Omega$$

$$\boxed{\frac{d\sigma}{d\Omega} = \rho\left|\frac{d\rho}{d(\cos\chi)}\right|}$$ in COM frame

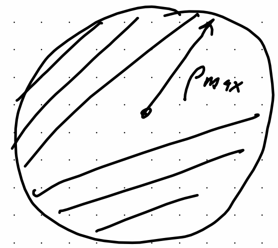
$$\boxed{\frac{d\sigma_1}{d\Omega_1}} = \rho\left|\frac{d\rho}{d(\cos\theta_1)}\right|$$ in lab frame θ_1

$$\left.\begin{aligned} &= \frac{d\sigma}{d\Omega}\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| \\ \frac{d\sigma_2}{d\Omega_2} &= \frac{d\sigma}{d\Omega}\left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right| \end{aligned}\right\}$$ wrt lab frame (θ_1, θ_2)

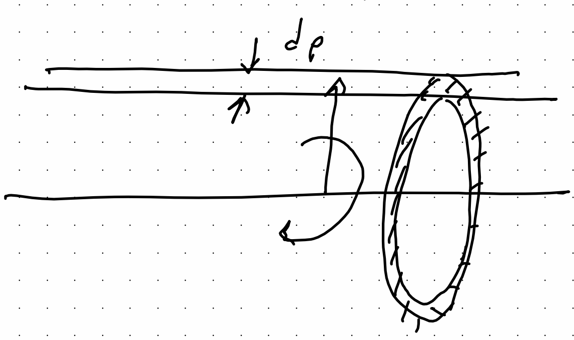


$$\boxed{d\rho \Leftrightarrow d\chi}$$ ☆ ☆

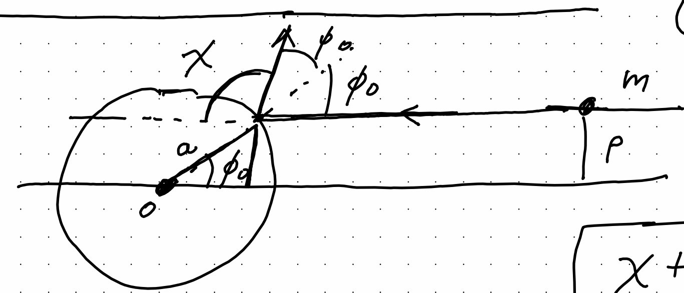
perspective



$$\sigma = \pi \rho_{max}^2$$



Hard sphere: Prob 18.1 $\quad U(r) = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases}$



↑ radius a

$$\boxed{\chi \longleftrightarrow \rho}$$

$$\boxed{\chi + 2\phi_0 = \pi}$$

$$\phi_0 = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\rho = a \sin\phi_0 = a\sin\left(\frac{\pi}{2} - \frac{\chi}{2}\right) = a\cos\left(\frac{\chi}{2}\right)$$

$$\frac{d\rho}{d\chi} = -\frac{a}{2}\sin\left(\frac{\chi}{2}\right)$$

$$d\sigma = 2\pi\rho\, d\rho = 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi$$

$$d\sigma = \cancel{2}\pi\; a\cos\left(\frac{\chi}{2}\right)\frac{a}{\cancel{2}}\sin\left(\frac{\chi}{2}\right) d\chi$$

$$= \pi a^2 \sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right) d\chi$$

$$= \boxed{\frac{\pi a^2}{2}\sin\chi\, d\chi}$$

$$d\Omega = 2\pi\sin\chi\, d\chi \quad \rightarrow \quad \sin\chi\, d\chi = \frac{1}{2\pi}\, d\Omega$$

$$d\sigma = \frac{\cancel{\pi} a^2}{2}\cdot\frac{1}{2\cancel{\pi}}\, d\Omega$$

$$= \boxed{\frac{1}{4}a^2\, d\Omega}$$ — uniform distribution

Total cross-section:

$$\sigma_{tot} \equiv \sigma = \int_{\text{unit sphere}} d\sigma = \frac{1}{4}a^2 \underbrace{\int_{\text{sphere}} d\Omega}_{=4\pi} = \boxed{\pi a^2}$$

$$\frac{d\sigma_1}{d\Omega_1} = \frac{d\sigma}{d\Omega}\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|$$

$\frac{1}{4}a^2$

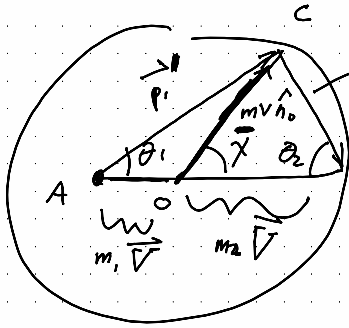
$$\frac{d\sigma_2}{d\Omega_2} = \frac{d\sigma}{d\Omega}\left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right|$$

$$\boxed{\begin{aligned}&\chi = \pi - 2\theta_2\\ &\cos\chi = -\frac{m_1}{m_2}\sin^2\theta_1 \pm \cos\theta_1\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}\end{aligned}}$$

$$\sin\theta_1\, d\theta_1 = -d(\cos\theta_1)$$

Lecture #17 Tue, 10/20

- Quiz #3 – this week
- Q&A: $\vec{p}_1{}' + \vec{p}_2{}' = \vec{P}_{tot}{}'$
  $= \vec{P}_{tot}$



$\vec{p}_1 + \cancel{\vec{p}_2} \to 0$
$= \vec{p}_1$

$m_1 < m_2$, $v_2 = 0$

$m_1 \to$ $\quad\quad$ m_2

$$m_1\vec{V} + m_2\vec{V} = (m_1+m_2)\vec{V} = \vec{P}_{tot}$$

$$\vec{V} = \frac{\vec{p}_1 + \cancel{\vec{p}_2}^{\to 0}}{m_1+m_2} = \frac{\vec{p}_1}{m_1+m_2} = \frac{m_1\vec{v}_1}{m_1+m_2}$$

COM velocity

$$m_1\vec{V} = m_1\left(\frac{m_1\vec{v}_1}{m_1+m_2}\right) = \left(\frac{m_1^2}{m_1+m_2}\right)\vec{v}_1$$

$\vec{v}_2 = 0 \to \boxed{\vec{p}_2 = 0}$

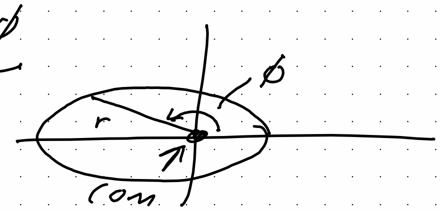
$$m_1 \vec{r_1} + m_2 \vec{r_2} = 0$$

$$\vec{r} = \vec{r_1} - \vec{r_2}$$

$$T = \frac{1}{2} m_1 |\dot{\vec{r_1}}|^2 + \frac{1}{2} m_2 |\dot{\vec{r_2}}|^2$$

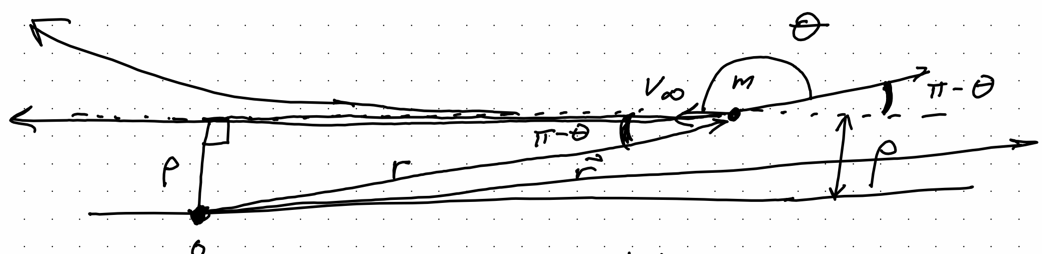
$$= \frac{1}{2} m |\dot{\vec{r}}|^2 \qquad m = \frac{m_1 m_2}{m_1 + m_2}$$

$\underbrace{r \text{ vs } \phi}$



$$\vec{r_1} = \frac{m_2}{m_1+m_2} \vec{r}$$

$$\vec{r_2} = \frac{-m_1}{m_1+m_2} \vec{r}$$



$U = 0$

$$M = \rho m v_\infty$$

$$\vec{M} = \vec{r} \times \vec{p} \qquad \vec{p} = m\vec{v_\infty}$$

$$= m \vec{r} \times \vec{v_\infty}$$

$$M = |\vec{M}| = m \underbrace{r v_\infty \sin\theta}$$

$$= m v_\infty \underbrace{r \sin(\pi - \theta)}$$

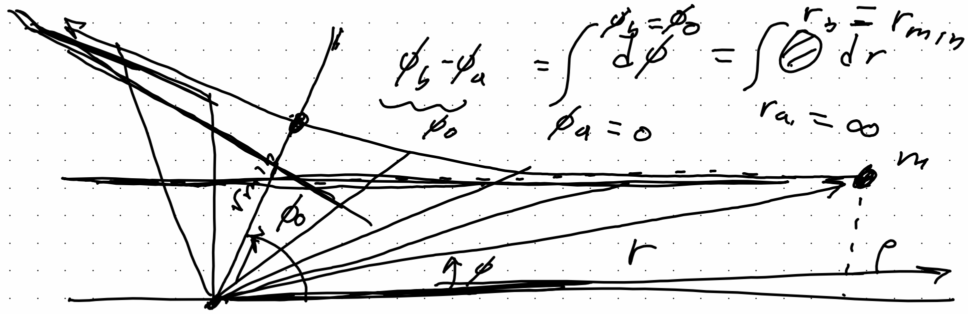
$$= \rho m v_\infty \quad \rho$$

$\sin(\pi - \theta)$

$= \sin\pi \cos\theta$

$- \cos\pi \sin\theta$

$= \sin\theta$



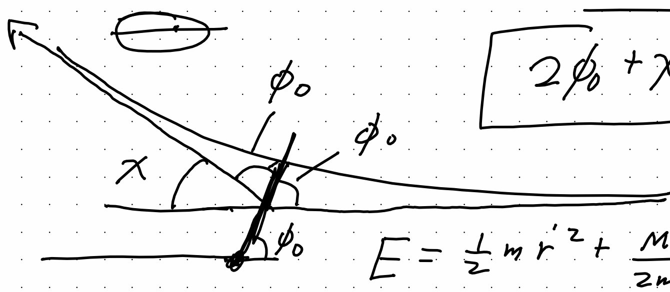
$$\underbrace{\phi_b-\phi_a}_{\phi_0} = \int_{\phi_a=0}^{\phi_b=\phi_0} d\phi = \int_{r_a=\infty}^{r_b=r_{min}} (-)\, dr$$

$$U(r) = \frac{\alpha}{r}$$

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M\,dr/r^2}{\pm\sqrt{\underbrace{2m(E-U) - M^2/r^2}_{>0}}}$$

$$M = \rho m v_\infty$$

$$E = \frac{1}{2} m v_\infty^2$$



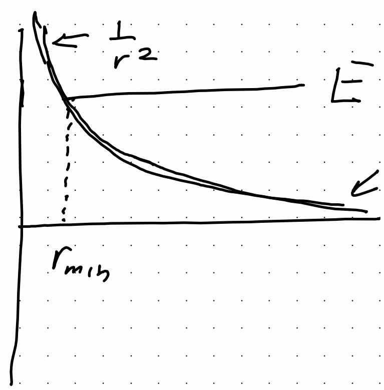
$$\boxed{2\phi_0 + \chi = \pi}$$

$$\boxed{\begin{array}{c} v_\infty \\ \hline \rho \end{array}}$$

$$\chi \leftrightarrow \rho$$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} + U(r)$$

$$\dot{r} = \pm\sqrt{\ }$$



$$U = \frac{\alpha}{r}$$

$\alpha > 0$

coulomb scattering

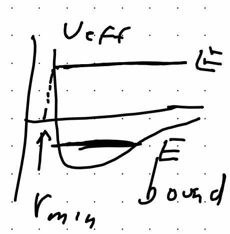
$+\ +$

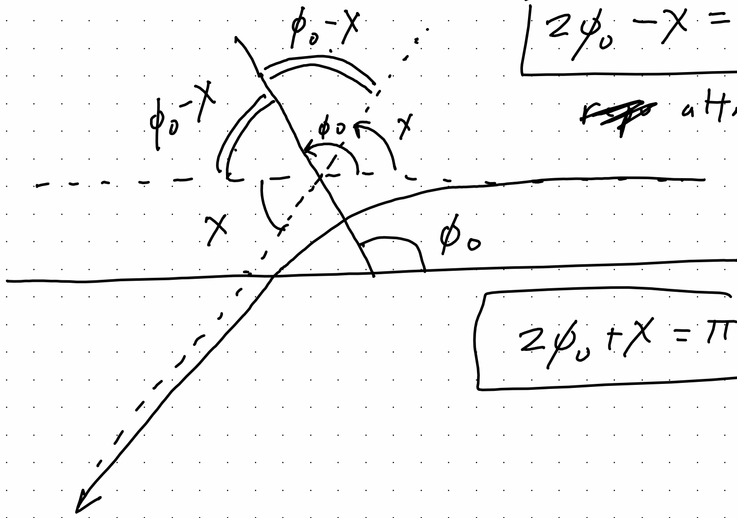
$-\ -$

$$U_{eff} = U(r) + \frac{M^2}{2mr^2}$$

$$E = U_{eff}(r_{min})$$

$$U = -\frac{\alpha}{r}$$





$2(\phi_0-\chi)+\chi=\pi$

$$\boxed{2\phi_0-\chi=\pi}$$

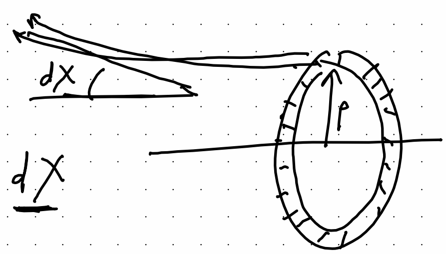
~~rep~~ attractive

$$\boxed{2\phi_0+\chi=\pi}$$

$d\sigma = 2\pi\rho\, d\rho$

$= 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi$

$= \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$

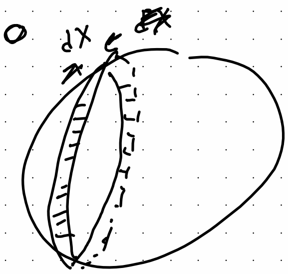


$d\rho > 0$

$d\chi < 0$

$d\Omega = 2\pi \sin\chi\, d\chi$

do



wrt COM frame

to go to lab frame

$\chi \rightarrow \theta_1$

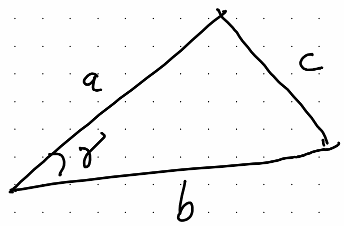
or θ_2

$d\sigma_1$, $d\sigma_2$

$d\theta_1$, $d\theta_2$, $d\Omega_1$, $d\Omega_2$

Lecture #18: Thurs 10/22

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$



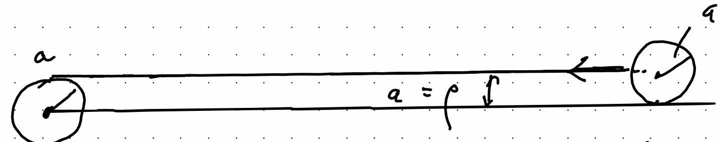
$\cos^2 x + \sin^2 x = 1$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

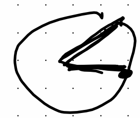
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

QUIZ #3: name_q3.pdf

lab Frame



$(m_1 = m_2)$



(hard sphere scattering)

1) calculate χ (scattering angle of the reduced mass w.r.t COM Frame)

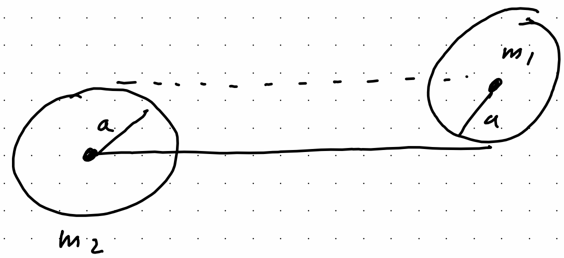
2) calculate θ_1, θ_2 (scattering angles w.r.t lab Frame)

$$\boxed{\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}}, \quad \boxed{\theta_2 = \frac{1}{2}(\pi - \chi)}$$

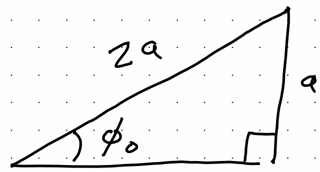
elastic scattering m_2 initially at rest

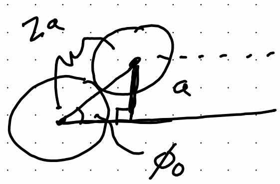
3) How do θ_1, θ_2 change if masses m_1 and m_2 change? $(m_1 >> m_2\ ;\ m_1 << m_2)$

$$U(r) = \begin{cases} \infty & r < 2a \\ 0 & r > 2a \end{cases}$$



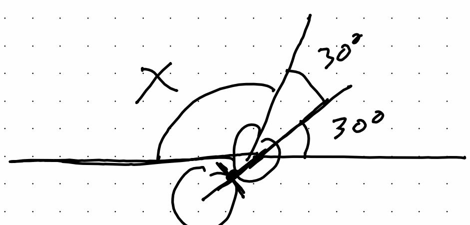
$\vec{r} = \vec{r}_1 - \vec{r}_2$





$\sin\phi_0 = \frac{a}{2a} = \frac{1}{2}$

$\boxed{\phi_0 = \pi/6 = 30^\circ}$



$\boxed{\chi = 120^\circ} = \frac{2\pi}{3}$

independent of m_1 and m_2

$\frac{2\pi}{3} = 120^\circ$

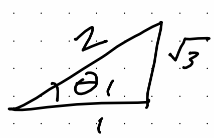
2) $\theta_2 = \frac{1}{2}(\pi - \chi) = \frac{1}{2}(180^\circ - 120^\circ) = \boxed{30^\circ}$ $\boxed{\theta_2 \parallel}$

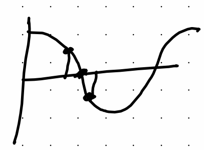
$$\tan\theta_1 = \frac{\cancel{m_2}\sin(120^\circ)}{\cancel{m_1} + \cancel{m_2}\cos(120^\circ)} = \frac{\sin(120^\circ)}{1 + \cos(120^\circ)}$$

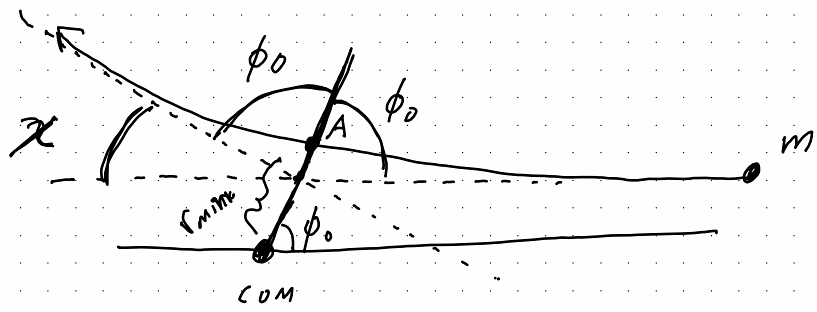
$$= \frac{\sqrt{3}/2}{1 - 1/2}$$

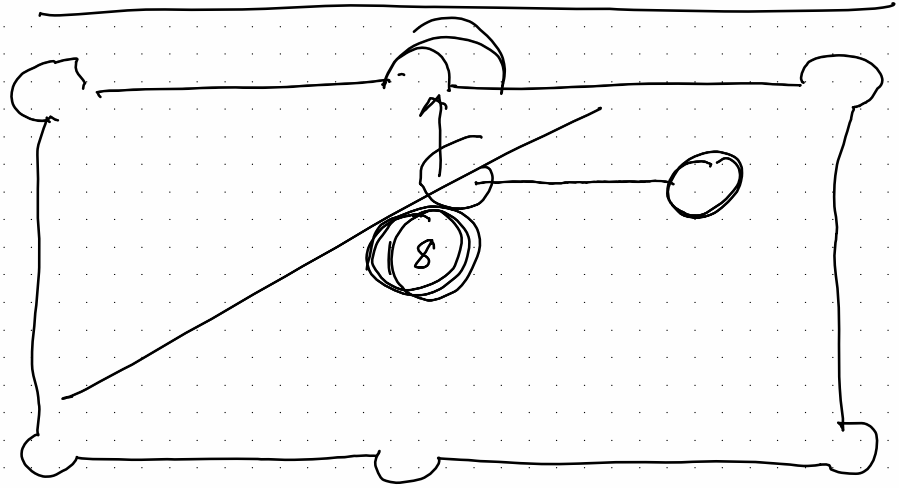
$$= \sqrt{3}$$

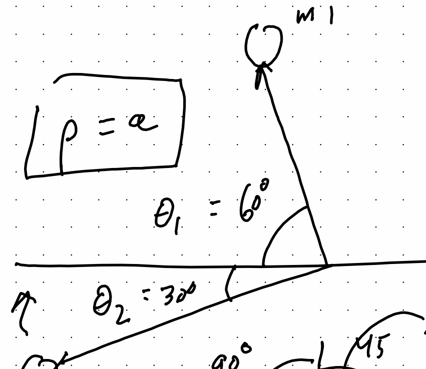
$\boxed{\theta_1 = 60^\circ}$

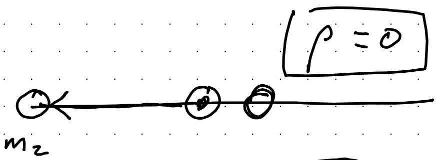




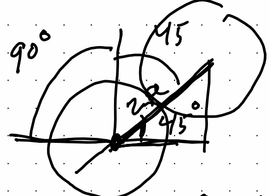






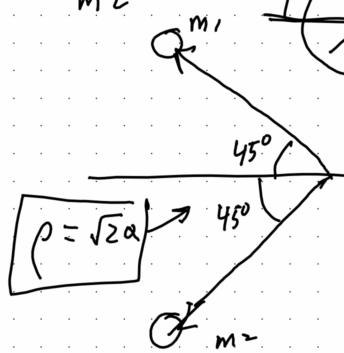


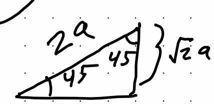
$6 \to 5$



$\uparrow$ increase : 1

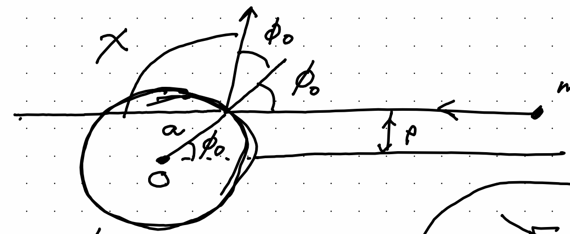
decrease : 3





$\chi = \pi/2$

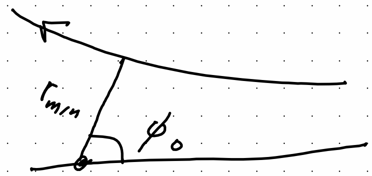
$$\theta_2 = \frac{1}{2}(\pi - \chi)$$
$$= \frac{1}{2}\left(\pi - \frac{\pi}{2}\right) = \pi/4$$
$$\theta_1 = \pi/4$$



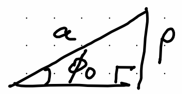
radius a

$$2\phi_0 + \chi = \pi$$

$$\boxed{\chi = \pi - 2\phi_0}$$



$$U(r) = \begin{cases} 0 & r > a \\ \infty & r < a \end{cases}$$

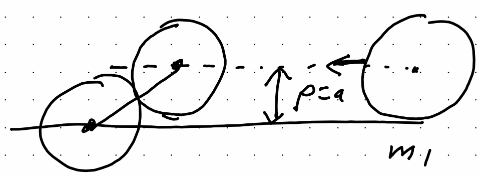


$$\sin\phi_0 = \frac{\rho}{a}$$

$$\phi_0 = \sin^{-1}(\rho/a)$$

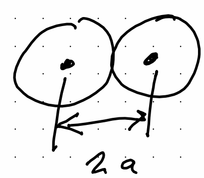
$$\boxed{\begin{aligned} \tan\theta_1 &= \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi} \\ \theta_2 &= \frac{1}{2}(\pi - \chi) \end{aligned}}$$

Prob 18.1

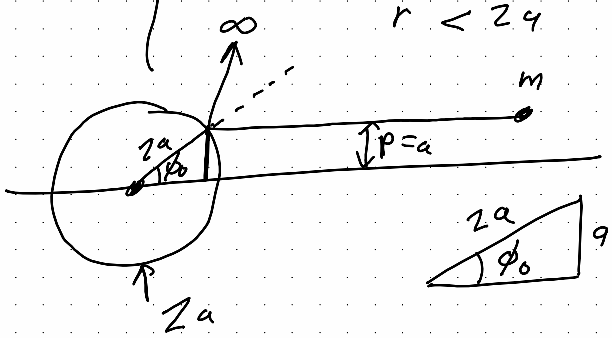


lab Frame

$m_1 = m_2$ $\qquad \rho = a$



$$U(r) = \begin{cases} 0 & r > 2a \\ \infty & r < 2a \end{cases}$$



CoM Fram

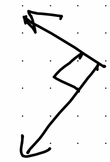
$$\sin\phi_0 = \frac{1}{2}$$

$$\phi_0 = 30^\circ$$

$$\chi = \pi - 2\phi_0 \quad = 180^\circ - 2\cdot 30^\circ = \boxed{120^\circ}$$

independent of m_1, m_2

$$\boxed{\begin{aligned} \tan\theta_1 &= \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi} \\ \theta_2 &= \frac{1}{2}(\pi - \chi) \end{aligned}}$$



$$\theta_2 = \frac{1}{2}(180^\circ - 120^\circ) = \boxed{30^\circ} \quad (\text{indep. of } m_1, m_2)$$

$$\tan\theta_1 = \frac{\sin(120^\circ)}{1+\cos(120^\circ)} = \frac{\sqrt{3}/2}{1-1/2} = \sqrt{3}$$

$$\boxed{\theta_1 = 60^\circ}$$

$$m_1 \ll m_2 \rightarrow \tan\theta_1 \approx \tan\chi \longrightarrow \boxed{\theta_1 = 120^\circ}$$

$$m_1 \gg m_2 \rightarrow \tan\theta_1 \approx \frac{m_2}{m_1}\sin\chi \longrightarrow \boxed{\theta_1 \approx 0^\circ}$$

Lec #19: Tuesday 10/27

- Small oscillation (next 3 classes)

sec 21, 22, 23

(Free oscillation in 1-d) (Forced oscillation in 1-d) several dimensions

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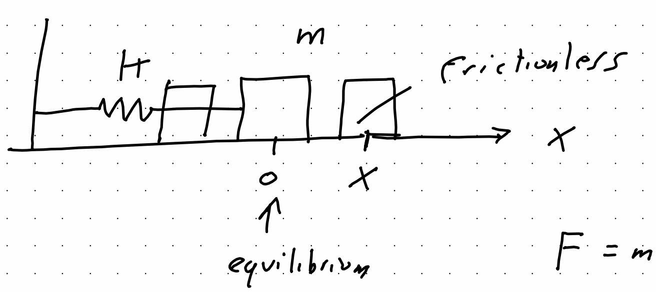


$$F = -kx$$

k: spring constant

$$F = m\ddot{x}$$

$$-kx = m\ddot{x}$$

$$\rightarrow \ddot{x} = -\frac{k}{m}x \longrightarrow x(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$$

$\omega = \sqrt{\frac{k}{m}}$ (angular freq)

$$x(t) = \underbrace{c_1 \cos \omega t + c_2 \sin \omega t}_{\text{determined by ICs.}}$$

$$x(t) = \underset{\text{amplitude}}{a} \cos(\omega t + \underset{\text{initial phase}}{\alpha})$$

$$x(t) = \mathrm{Re}\left[\underset{\text{complex}}{A} e^{i\omega t} \right], \quad A = a e^{i\alpha}$$

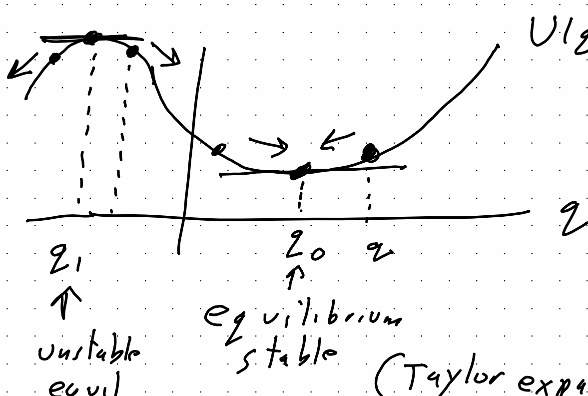
$$\mathrm{Re}\left[a e^{i\alpha} e^{i\omega t}\right] = \mathrm{Re}\left[a\, e^{i(\omega t + \alpha)}\right]$$
$$= \mathrm{Re}\left[a \left(\cos(\omega t+\alpha) + i \sin(\omega t + \alpha)\right)\right]$$
$$= a \cos(\omega t + \alpha)$$

Small oscillation:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \ldots$$

1-d system: q generalized coord

$$L = \frac{1}{2} a(q)\, \dot{q}^2 - U(q)$$

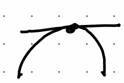


$$F = -\frac{dU}{dq}$$

$$F(q_0) = 0$$

$$\left.\frac{dU}{dq}\right|_{q_0} = 0$$

(Taylor expansion)



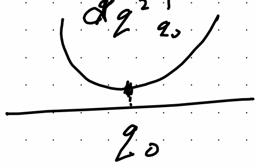
$$U(q) = U(q_0) + \underbrace{\left.\frac{dU}{dq}\right|_{q_0}}_{0}(q - q_0) + \frac{1}{2}\left.\frac{d^2U}{dq^2}\right|_{q_0}(q-q_0)^2 + \ldots$$

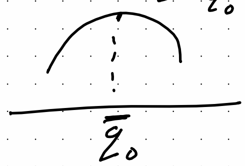
Small displacement away from equilibrium

$$\frac{|q - q_0|}{q_0} << 1 \quad (\text{ignore } O(3))$$

$$\boxed{U(q) \simeq U(q_0) + \frac{1}{2} \left.\frac{d^2U}{dq^2}\right|_{q_0} (q-q_0)^2}$$

$$\left.\frac{d^2U}{dq^2}\right|_{q_0} > 0 \qquad\qquad \left.\frac{d^2U}{dq^2}\right|_{q_0} < 0$$





$$k = \left.\frac{d^2U}{dq^2}\right|_{q_0}$$

$$U(q) \simeq \underbrace{U(q_0)}_{\text{const}} + \frac{1}{2} k \underbrace{(q-q_0)^2}_{x^2}$$

$$F = -\frac{dU}{dq}$$

$$= -k(q-q_0)$$

$$= -kx$$

$$U = \frac{1}{2}kx^2$$

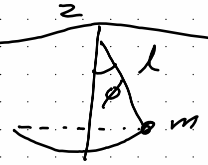
$$L = \frac{1}{2} a(q) \dot{q}^2 - U(q)$$

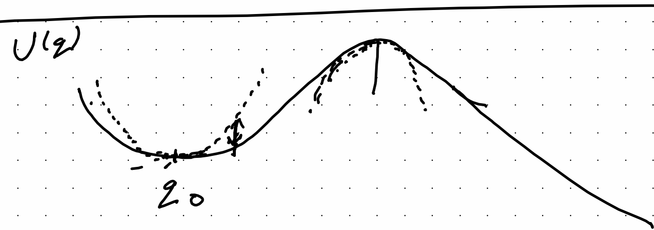
$$= \frac{1}{2} \underset{m}{\underset{\parallel}{a(q_0)}} \dot{x}^2 - \underbrace{U(q_0)}_{\text{const}} - \frac{1}{2}kx^2$$

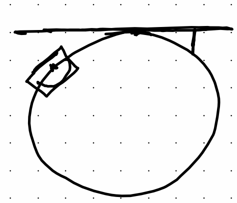
$$x = q - q_0 \rightarrow \dot{x} = \dot{q}$$

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2}kx^2 \xrightarrow[\text{EOM}]{} m\ddot{x} = -kx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

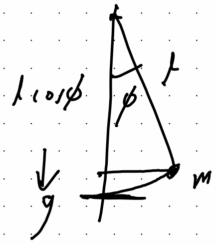






Example:

simple pendulum



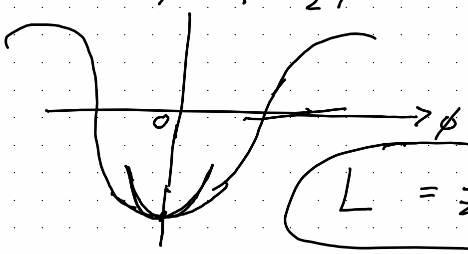
$$L = \tfrac{1}{2} m l^2 \dot\phi^2 - mgl(1-\cos\phi)$$
$$= \tfrac{1}{2} m l^2 \dot\phi^2 + mgl\cos\phi$$
$$= T - U$$

$\uparrow y$

mgy

$$y = l - l\cos\phi$$
$$= l(1-\cos\phi)$$

$$U(\phi) = -mgl\cos\phi \approx -mgl\left(1-\tfrac{1}{2}\phi^2\right)$$
$$\cos\phi = 1 - \tfrac{1}{2}\phi^2 + \dots \qquad = \underbrace{-mgl}_{\text{const}} + \tfrac{1}{2} mgl\phi^2$$

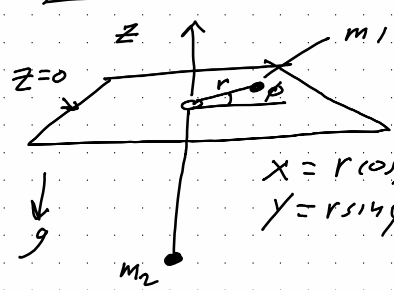


$$\omega = \sqrt{\frac{K}{M}} = \sqrt{\frac{\cancel{m}gl}{\cancel{m}l^2}} = \boxed{\sqrt{\frac{g}{l}}}$$

$\phi = 0$

$$\boxed{L = \tfrac{1}{2}\underbrace{(ml^2)}_{M}\dot\phi^2 - \tfrac{1}{2}\underbrace{(mgl)}_{K}\phi^2}$$

Example: 2-d problem



$$T = \tfrac{1}{2} m_1 (\overbrace{\dot r^2 + r^2\dot\phi^2}^{\dot x^2 + \dot y^2}) + \tfrac{1}{2} m_2 \underbrace{\dot z^2}_{\dot r^2}$$

$x = r\cos\phi$
$y = r\sin\phi$

$$\boxed{T = \tfrac{1}{2}(m_1+m_2)\dot r^2 + \tfrac{1}{2} m_1 r^2\dot\phi^2}$$

string: length l

$r + |z| = l$

$r - z = l$

$z = r - l$

$\rightarrow \dot z = \dot r$

$$\boxed{U = m_2 g z} = m_2 g(r - \underset{\text{const}}{l})$$

$$L = T - U$$
$$\boxed{L = \tfrac{1}{2}(m_1+m_2)\dot r^2 + \tfrac{1}{2} m_1 r^2\dot\phi^2 - m_2 g r}$$

$\uparrow z \quad \downarrow \vec g$

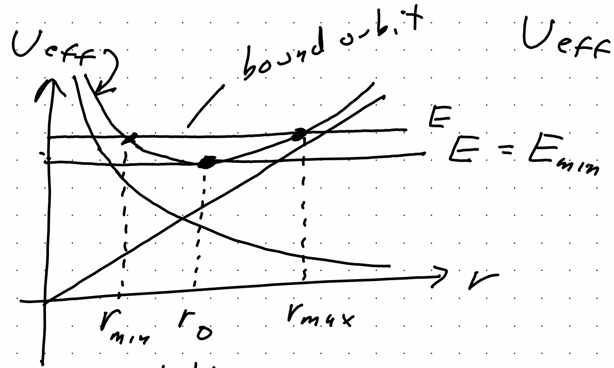
No ϕ dependence

$$\dot\phi = \frac{M_z}{m_1 r^2} \leftarrow M_z = \frac{\partial L}{\partial\dot\phi} = m_1 r^2\dot\phi = \text{const}$$

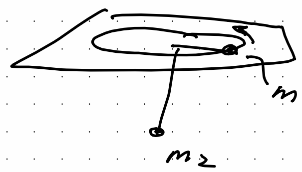
No explicit timedependence $\rightarrow$ $E = \text{const}$

$$E = T + U$$
$$= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\phi}^2 + m_2 g r$$
$$= \boxed{\frac{1}{2}(m_1+m_2)\dot{r}^2} + \underbrace{\frac{M_z^2}{2m_1 r^2} + m_2 g r}_{U_{eff}(r)} - m_2 g l$$

not effect EOMs



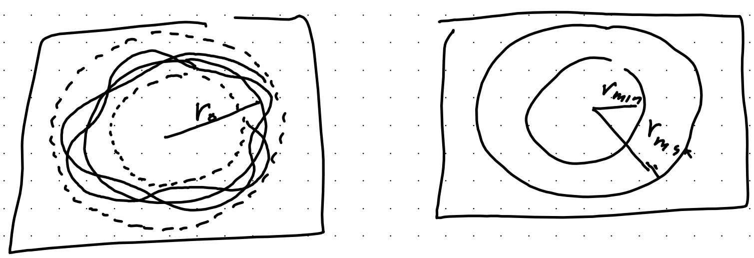
stable equilibrium



$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0}, \quad U_{eff}(r) = \frac{M_z^2}{2m_1 r^2} + m_2 g r$$
$$= \frac{-M_z^2}{m_1 r_0^3} + m_2 g$$

$$\boxed{M_z^2 = m_1 m_2 g r_0^3}$$

Small oscillations around r_0: $\quad \frac{|r-r_0|}{r_0} \ll 1$



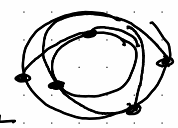
top view

$$\frac{dU_{eff}}{dr} = \frac{-M_z^2}{m_1 r^3} + m_2 g$$

$$\rightarrow \left.\frac{d^2 U_{eff}}{dr^2}\right|_{r_0} = \frac{3M_z^2}{m_1 r_0^4}, \qquad M_z^2 = m_1 m_2 g r_0^3$$

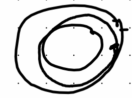
$$K = \frac{3 \cancel{m_1} m_2 g r_0^3}{\cancel{m_1} r_0^4} = \boxed{\frac{3 m_2 g}{r_0}}$$

$$E = \frac{1}{2}\underline{(m_1+m_2)}\dot{r}^2 + \underbrace{U_{eff}(r_0)}_{const} + \frac{1}{2}K(r-r_0)^2$$



$$x = r - r_0$$

$$E = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}Kx^2 + const$$



$$\omega_r = \sqrt{\frac{K}{m_1+m_2}} = \boxed{\sqrt{\frac{3 m_2 g}{(m_1+m_2) r_0}}}$$

$$\omega_\phi = \dot{\phi}\Big|_{r_0} \qquad M_z = m_1 r^2 \dot{\phi}$$

$$= \frac{M_z}{m_1 r_0^2}$$

$$= \frac{\sqrt{m_1 m_2 g r_0^3}}{m_1 r_0^2} = \frac{\sqrt{m_1 m_2 g r_0^3}}{\sqrt{m_1^2 r_0^4}}$$

$$= \boxed{\sqrt{\frac{m_2 g}{m_1 r_0}}} \qquad 0 \rightarrow 2\pi$$

not equal in general

$$\boxed{\omega_r = \sqrt{\frac{3 m_2 g}{(m_1+m_2) r_0}}}$$

For $\omega_r = \omega_\phi$

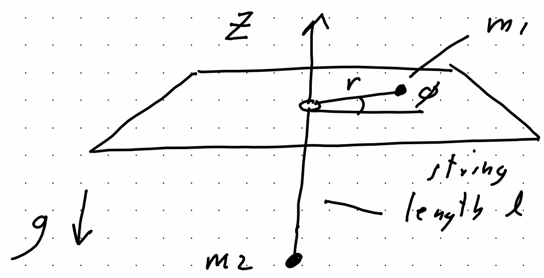
$$\frac{3}{m_1+m_2} = \frac{1}{m_1}$$

$$m_1 + m_2 = 3m_1$$

closed bound orbits

Lecture #20

- Quiz #4 – Tuesday
- Today: (i) Finish up example from last time
  (ii) Forced oscillations



$r = r_0$ (stable circular orbit)

$\omega_r = ?$

small oscillation
$|r - r_0| \ll r_0$

$$\omega_r = \sqrt{\frac{3 m_2 g}{(m_1 + m_2) r_0}}, \quad \omega_\phi = \sqrt{\frac{m_2 g}{m_1 r_0}} = \dot{\phi}\Big|_{r_0}$$

$\omega_r = \omega_\phi \rightarrow$ closed orbit $\quad \frac{3}{m_1 + m_2} = \frac{1}{m_1} \rightarrow 3m_1 = m_1 + m_2$

$m_1 = \frac{1}{2} m_2$

Forced oscillations: (1-d)

$$m\ddot{x} = -kx + F(t)$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}, \quad \omega \equiv \sqrt{\frac{k}{m}}$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + x F(t)$$

General sol'n

$$x(t) = \underbrace{x_h(t)}_{\text{general soln to } F(t)=0} + \underbrace{x_p(t)}_{\text{particular soln}}$$

general soln to $F(t)=0$: $a\cos(\omega t + \alpha)$

particular soln: any solution to the equation with RHS $= F(t)$

Example: $F(t) = f\cos(\gamma t + \beta)$

$$x_p(t) = b\cos(\gamma t + \beta) = \frac{f}{m}\left(\frac{1}{\omega^2 - \gamma^2}\right)$$

$$-b\gamma^2 \cancel{\cos(\gamma t + \beta)} + \omega^2 b \cancel{\cos(\gamma t + \beta)} = \frac{f}{m} \cancel{\cos(\gamma t + \beta)}$$

$$b = \frac{f}{m}\left(\frac{1}{\omega^2 - \gamma^2}\right)$$

Resonance: $\gamma = \omega$

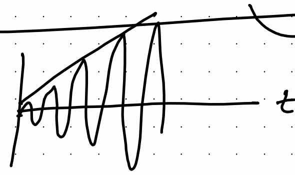
$$x(t) = a\cos(\omega t + \alpha) + \frac{f}{\underbrace{m(\omega^2-\gamma^2)}_{0}}\underbrace{\left(\cos(\gamma t+\beta) - \cos(\omega t+\beta)\right)}_{\text{particular solution}}$$

$$\frac{0}{0}$$

L'Hopital's

$$\frac{\left.\frac{d}{d\gamma}(\text{num})\right|_{\gamma=\omega}}{\left.\frac{d}{d\gamma}(\text{den})\right|_{\gamma=\omega}} = \frac{+f\,t\sin(\omega t+\beta)}{+2m\omega}$$

$$\boxed{x(t) = a\cos(\omega t+\alpha) + \underbrace{\frac{f\,t}{2m\omega}\sin(\omega t+\beta)}}$$



after some time t, this term no longer $\rightarrow$ invalidates small osc. approx.

General $F(t)$:

$$F(t) = \int_{-\infty}^{\infty} d\gamma \underbrace{\tilde{F}(\gamma)}_{\text{complex}} e^{i\gamma t}$$

Examples

$$F(t) = f_2\sin(2\omega t) + f_3\sin(3\omega t) + f_4\sin(4\omega t)$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m} \quad (2^{nd}\text{ order})$$

$$\left.\begin{aligned}\xi &= \dot{x} + i\omega x\\ \dot{\xi} &= \ddot{x} + i\omega\dot{x}\end{aligned}\right\} \rightarrow \boxed{\dot{\xi} - i\omega\xi = \frac{F(t)}{m}} \quad (1^{st}\text{ order})$$

by Boas "Math method"

$$\dot{\xi} - i\omega\xi = \frac{F(t)}{m}$$

$$y' + P(x)\,y = Q(x)$$

$$y' = \frac{dy}{dx}$$

$$\boxed{dy + \big(P(x)y - Q(x)\big)dx = 0}$$ — not exact!

"Exact" differential:

If exact $1\,dy + \big(P(x)y - Q(x)\big)dx = dU(x,y)$

$$dU(x,y) = \left(\frac{\partial U}{\partial x}\right)dx + \left(\frac{\partial U}{\partial y}\right)dy$$

$$\rightarrow \frac{\partial^2 U}{\partial y\,\partial x} = \frac{\partial^2 U}{\partial x\,\partial y}$$

$$\frac{\partial 1}{\partial x} \overset{?}{=} P(x)$$

$$0 \neq P(x)$$

Claim:

$$\mu(x)\Big[\overbrace{dy + \big(P(x)y - Q(x)\big)dx}^{=0 \text{ (original equation)}}\Big] = dU$$

integrating Factor

$$\text{RHS} = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

$$\mu(x) = \frac{\partial U}{\partial y}$$

$$\mu(x)\big(P(x)y - Q(x)\big) = \frac{\partial U}{\partial x}$$

$$\frac{\partial^2 U}{\partial x\,\partial y} = \frac{\partial^2 U}{\partial y\,\partial x} \rightarrow \mu'(x) = \mu(x)\,P(x)$$

$$\int \frac{\mu'}{\mu} = \int P(x)\,dx$$

$$\ln\mu = \int P(x)\,dx \rightarrow \mu = e^{\int P(x)dx}$$

$$I(x) = \int P(x)\,dx$$

$$\mu(x) = e^{I(x)}, \quad I(x) = \int dx\, P(x)$$

$$\frac{\partial U}{\partial y} = \mu(x)$$

$$U(x,y) = \mu(x)\, y + g(x) = C$$

const

$$\frac{\partial U}{\partial x} = \mu(x)\left(P(x)y - Q(x)\right)$$

$$\cancel{\mu'(x)y} + g'(x) = \mu(x)\left(\cancel{P(x)y} - Q(x)\right)$$

$$g'(x) = -\mu(x)\, Q(x)$$

$$\rightarrow \quad g(x) = -\int dx\, Q(x)\mu(x) = -\int dx\, Q(x) e^{I(x)}$$

$$\boxed{y = \frac{1}{\mu(x)}\left(C - g(x)\right) = e^{-I(x)}\left(C + \int dx\, Q(x) e^{I(x)}\right)}, \quad \mu(x) = e^{I(x)}, \quad I(x) = \int dx\, P(x)$$

$$y' + P(x)y = Q(x)$$

$$x(t) = \mathrm{Im}\left(\frac{\xi(t)}{\omega}\right)$$
$$\dot{x}(t) = \mathrm{Re}(\xi(t))$$

$$\dot{\xi} - i\omega\xi = \frac{F(t)}{m}$$

$$\xi = \dot{x} + i\omega x$$

$$y(x) \rightarrow \xi(t)$$
$$P(x) \rightarrow -i\omega$$
$$I(x) \rightarrow I(t) = -i\omega t$$
$$Q(x) \rightarrow F(t)/m$$

$$\boxed{\xi(t) = e^{i\omega t}\left[\xi_0 + \int_0^t d\bar{t}\, e^{-i\omega\bar{t}} F(\bar{t})/m\right]}$$

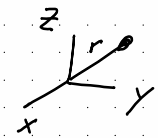
Lecture #21: Tues 11/3

Q4: on Thursday

Thursday – rigid body motion / non-inertial reference frames

Sec 23: Free oscillations in 2-d or higher

Prob 3, Sec 23: $U = \frac{1}{2} k r^2$ — space oscillator

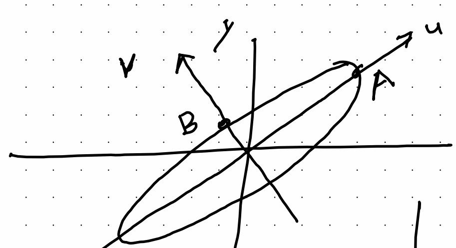


motion is in 2-d plane (x, y)

$$L = T - U$$
$$= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k (x^2 + y^2)$$
$$= \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2\right) + \left(\frac{1}{2} m\dot{y}^2 - \frac{1}{2} ky^2\right)$$

2 independent oscillation in the x, y-directions

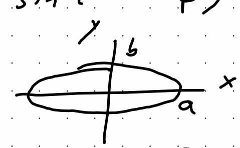
Ang freq: $\omega_x = \sqrt{\frac{k}{m}}$, $\omega_y = \sqrt{\frac{k}{m}}$ $\rightarrow \omega_x = \omega_y$



most general motion

$$x = a \cos(\omega t + \alpha)$$
$$y = b \sin(\omega t + \beta)$$

$\alpha = \beta = 0$

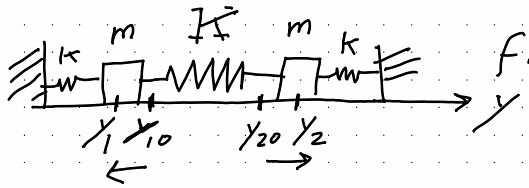


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Normal coordinates:

N independent oscillations with frequency $\omega_1, \omega_2, \ldots, \omega_N$

Example:



frictionless

$i, k = 1, 2, \cdots N$, $N = 2$

$$\left.\begin{aligned} x_1 &= y_1 - y_{10} \\ x_2 &= y_2 - y_{20} \end{aligned}\right\}$$ small deviations away from equilibrium

$$(y_2 - y_1) - (y_{20} - y_{10}) = (y_2 - y_{20}) - (y_1 - y_{10}) = x_2 - x_1$$

$$L = T - U$$

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) = \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k$$

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} K (x_2 - x_1)^2$$

$$(x_1^2 + x_2^2 - 2 x_1 x_2)$$

$$= \frac{1}{2} (k + K) x_1^2 + \frac{1}{2} (k + K) x_2^2 - K x_1 x_2 = \frac{1}{2} \sum_{i,k} k_{ik} x_i x_k$$

$$T = \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$= \frac{1}{2} [m_{11} \dot{x}_1^2 + m_{12} \dot{x}_1 \dot{x}_2 + m_{21} \dot{x}_2 \dot{x}_1 + m_{22} \dot{x}_2^2]$$

$$= \frac{1}{2} [m_{11} \dot{x}_1^2 + m_{22} \dot{x}_2^2 + 2 m_{12} \dot{x}_1 \dot{x}_2]$$

$\rightarrow m_{11} = m, \quad m_{22} = m, \quad m_{12} = 0 = m_{21}$

$$\boxed{m_{ik} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}}$$

$$U = \frac{1}{2} \sum_{i,k} k_{ik} x_i x_k = \frac{1}{2} (k + K) x_1^2 + \frac{1}{2} (k + K) x_2^2 - 2 K x_1 x_2$$

$$\boxed{K_{ik} = \begin{pmatrix} k + K & -K \\ -K & k + K \end{pmatrix}}$$

$$L = T - U$$

$$= \frac{1}{2}\sum_{j,k} m_{jk}\dot{x}_j\dot{x}_k - \frac{1}{2}\sum_{j,k} k_{jk} x_j x_k \qquad \Big|\ \delta_{ji},\ \delta_{ki}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \qquad (i = 1, 2, \cdots N)$$

$$\boxed{\sum_k m_{ik}\ddot{x}_k = -\sum_k k_{ik} x_k}$$

$$\underset{m}{\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}} \underset{\dot{x}}{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}} = \begin{bmatrix} \ \\ \ \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} \dot{x}_1 & \dot{x}_2 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$\| $$

$$\frac{1}{2}\sum_{i,k} m_{ik}\dot{x}_i\dot{x}_k$$

$$\frac{1}{2}\sum_{j,l} m_{jl}\dot{x}_j\dot{x}_l$$

$$\frac{\partial}{\partial x_1}\left(k_{11}x_1^2 + k_{22}x_2^2 + 2k_{12}x_1x_2\right)\left(-\frac{1}{2}\right)$$

$$= -\left(k_{11}x_1 + k_{12}x_2\right) = -\sum_k k_{1k}x_k$$

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) &= \frac{\partial L}{\partial x_1} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) &= \frac{\partial L}{\partial x_2} \end{aligned}\right\} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$

$$i = 1, 2$$

$$\sum_k m_{ik}\ddot{x}_k = -\sum_k k_{ik}x_k$$

$$\sum_k \left(m_{ik}\ddot{x}_k + k_{ik}x_k\right) = 0$$

<u>Trial solution:</u> $x_k = A_k e^{i\omega t}$

$$\ddot{x}_k = -\omega^2 A_k e^{i\omega t}$$

$$\sum_k \underbrace{\left(-m_{ik}\omega^2 + k_{ik}\right)}_{\text{matrix}} A_k e^{i\omega t} = 0 \quad \text{0 vector}$$

$$\begin{bmatrix} \ & \ \\ \ & \ \end{bmatrix}\begin{bmatrix} \ \\ \ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$M^{-1}\left(M\,\underline{v} = \begin{bmatrix}0\\0\end{bmatrix}\right)$$

$$\underline{v} = M^{-1}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$k_{ik} - \omega^2 m_{ik}$ must be a singular matrix

$$\boxed{\det(k_{ik} - \omega^2 m_{ik}) = 0}$$ characteristic equation for angular freq

eigenvalues $\rightarrow$ $\omega_1^2, \omega_2^2, \ldots, \omega_N^2$

(normal mode freqs)

eigenvector $\rightarrow$ $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_N}$

(normal mode oscillation)

$$\det\left(\begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} (k+K)-\omega^2 m & -K \\ -K & (k+K)-\omega^2 m \end{bmatrix} = 0$$

$$\oslash\,\omega^4 + \oslash\,\omega^2 + \oslash = 0$$

$\rightarrow$ quadratic equation for ω^2 $(\equiv \lambda)$

$$\oslash\,\lambda^2 + \oslash\,\lambda + \oslash = 0$$

solve quadratic equation: $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\boxed{\begin{aligned} \omega_+^2 &= \frac{k+2K}{m} \\ \omega_-^2 &= \frac{k}{m} \end{aligned}}$$

$\underline{\omega_+^2}$:

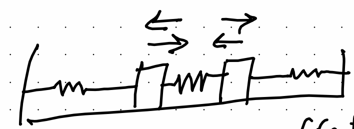
$$\begin{pmatrix} k+\bar{K}-m\omega_+^2 & -\bar{K} \\ -\bar{K} & k+\bar{K}-m\omega_+^2 \end{pmatrix} \underbrace{\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}}_{V_+} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

V_+ : eigenvector assoc. with ω_+

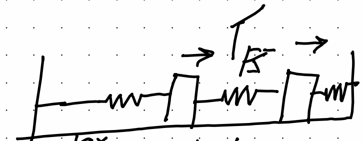
$$A_2 = -A_1$$

$$\boxed{V_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Normal mode oscillation



unaffected



$\underline{\omega_-^2}$:

$$\boxed{V_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

complex constant determined by initial conditions

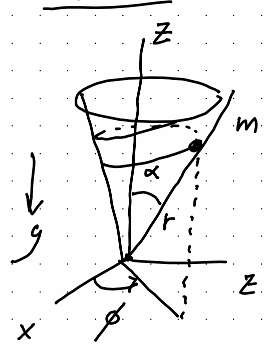
general solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathrm{Re}\left(C_+ V_+ e^{i\omega_+ t} + C_- V_- e^{i\omega_- t} \right)$$

Lec 22: Nov 5^{th}

- Quiz #4 today (Quiz #5 - next thursday, Quiz #6 - last day of class)
- Rigid body motion: Sec 31-36
  Non-inertial ref frames: Sec 38, 39

Q4: name_q4.pdf



a) Write down the Lagrangian

b) Find r_0 for a stable circular orbit for fixed angular momentum M_z.

c) Find angular freq of small radial oscillations about r_0.

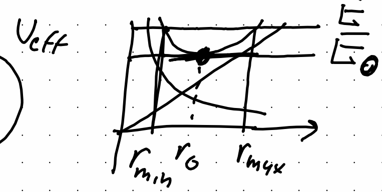
$$z = r\cos\alpha, \quad T = \frac{1}{2}m\left(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2 + \cancel{r^2\dot{\theta}^2}\right)$$

$$E = \frac{1}{2} m \dot{r}^2 + \underbrace{U_{eff}(r)}$$

$$\frac{M_z^2}{2mr^2 \sin^2\alpha} + mgr\cos\alpha$$

$M_z^2 = m^2 g r_0^3 \cos\alpha \sin^2\alpha$

$M_z = mr^2 \sin^2\alpha \, \dot{\phi}$



$\omega = \sqrt{\frac{k}{m}}$

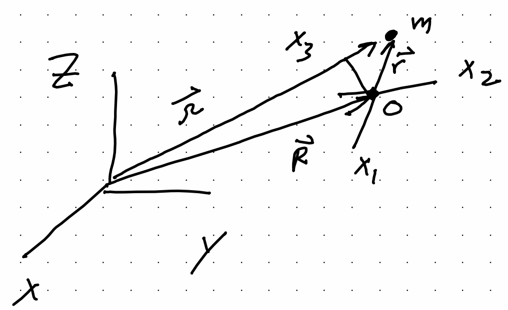
$x = q - q_0$

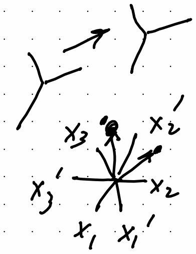
$$k = \left.\frac{d^2 U_{eff}}{dr^2}\right|_{r_0} = \frac{3mg\cos\alpha}{r_0}$$

$$\omega_r = \sqrt{\frac{k}{m}} = \boxed{\sqrt{\frac{3g\cos\alpha}{r_0}}}$$

$$\dot{\phi} = \frac{M_z}{mr^2 \sin^2\alpha} = \frac{M_z}{mr_0^2 \sin^2\alpha} = \frac{\sqrt{m^2 g r_0^3 \cos\alpha \sin^2\alpha}}{\sqrt{m^2 r_0^4 \sin^4\alpha}}$$

$$\omega_\phi = \sqrt{\frac{g}{r_0} \frac{\cos\alpha}{\sin^2\alpha}} \neq \omega_r$$





$\vec{V}$, $\vec{\Omega}$

linear velocity — depend on O

ang velocity — does not depend on O

$\vec{\rho} = \vec{R} + \vec{r}$

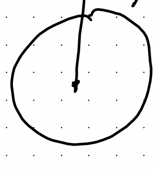
$$\boxed{\vec{v} = \frac{d\vec{\rho}}{dt} = \vec{V} + \underbrace{\vec{\Omega} \times \vec{r}}}$$

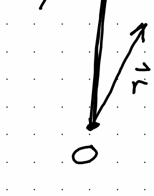
angular velocity vector

translational velocity of O

rotational velocity

$\vec{R} \to \vec{R} + d\vec{R}$, $\frac{d\vec{R}}{dt} = \vec{V} \, \delta\vec{\phi}$

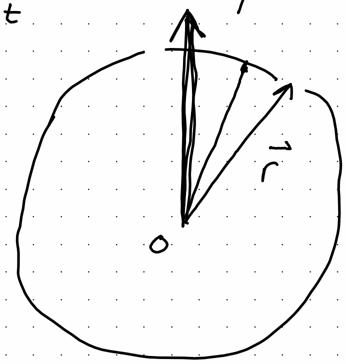




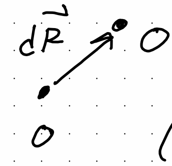
infinitesimal rotational about an axis through an infinitesimal angle $\delta\phi$

$v = \frac{dx}{dt}$

$\delta\vec{\phi}$

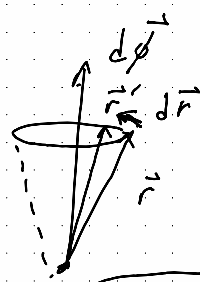


$\vec{V} = \frac{d\vec{R}}{dt}$ — 3 DOF associated with origin of body frame



$d\vec{R}$ $\quad (\phi, \theta, \psi) = \vec{\phi}$: 3 DOF assoc with orientation of body frame wrt inertial frame

$\frac{\to 0}{\to 0}$



$d\vec{r} = d\vec{\phi} \times \vec{r}$

$\frac{d\vec{r}}{dt} = \underbrace{\frac{d\vec{\phi}}{dt}}_{\vec{\Omega}} \times \vec{r}$

$$\boxed{\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}}$$

Lec #23: Tuesday Nov 10th

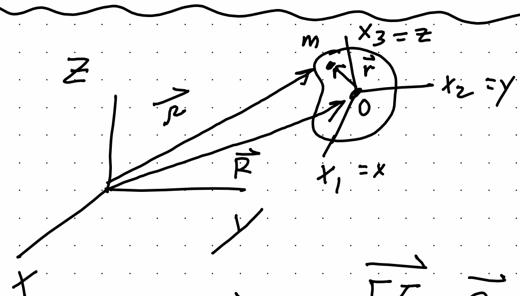
— Quiz #5: Thursday

— Midterm #2: Next Thursday 11/19 (Scattering, small oscillations, some rigid body)

— Today: Rigid body motion
(Sec 31–36, 38, 39)

39 — non-inertial

38 — static equilibrium



O: at COM (usually)

x_1, x_2, x_3: fixed in RB

$\left.\begin{aligned} \vec{V} &= \frac{d\vec{R}}{dt} \\ \vec{\Omega} &= \frac{d\vec{\phi}}{dt} \end{aligned}\right\} \dot{q}_i$

$\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}$

$\vec{\Omega}$: angular velocity vector

$(\vec{R}, \vec{\phi})$: 6 DOF $= q_i$

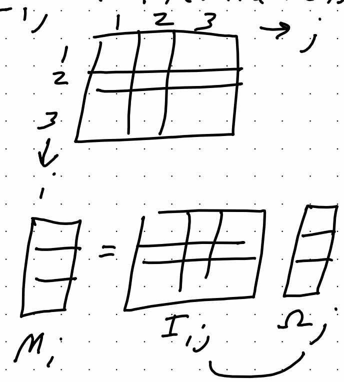
rotational quantities

$$T = \frac{1}{2} m v^2 \quad , \quad T_{rot} = \frac{1}{2} I \Omega^2$$

$$\vec{p} = m\vec{v} \qquad \vec{M} = I\vec{\Omega} \rightarrow M_i = \sum_j I_{ij}\Omega_j$$

I : moment of inertia

$\longrightarrow$ I_{ij} : inertia tensor



KE:

$$T = \frac{1}{2}\sum_a m_a |\vec{v}_a|^2 \quad , \quad \vec{v}_a = \vec{V} + \vec{\Omega}\times\vec{r}_a$$

$$= \frac{1}{2}\sum_a m_a |\vec{V} + \vec{\Omega}\times\vec{r}_a|^2$$

$$= \frac{1}{2}\sum_a m_a \left(|\vec{V}|^2 + |\vec{\Omega}\times\vec{r}_a|^2 + 2\vec{V}\cdot(\vec{\Omega}\times\vec{r}_a) \right)$$

$$|\vec{A}+\vec{B}|^2 = A^2 + B^2 + 2\vec{A}\cdot\vec{B}$$

① $= \frac{1}{2}\sum_a m_a |\vec{V}|^2 = \boxed{\frac{1}{2}\mu V^2}$

total mass

③ $= \sum_a m_a \vec{V}\cdot(\vec{\Omega}\times\vec{r}_a)$

$= \left(\sum_a m_a \vec{r}_a\right)\cdot(\vec{V}\times\vec{\Omega})$

$= \underbrace{\mu \vec{R}_{com}}_{=0 \text{ for } \boxed{O \text{ at COM}}} \cdot (\vec{V}\times\vec{\Omega})$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$② = \frac{1}{2}\sum_a m_a |\vec{\Omega}\times\vec{r_a}|^2$$

$$= \frac{1}{2}\sum_a m_a (\vec{\Omega}\times\vec{r_a})\cdot(\vec{\Omega}\times\vec{r_a})$$

$$= \frac{1}{2}\sum_a m_a \ \vec{\Omega}\cdot\left(\vec{r_a}\times(\vec{\Omega}\times\vec{r_a})\right)$$

$$= \frac{1}{2}\sum_a m_a \ \vec{\Omega}\cdot\left(\vec{\Omega}\ r_a^2 - \vec{r_a}\,(\vec{r_a}\cdot\vec{\Omega})\right)$$

$$= \frac{1}{2}\sum_a m_a \left(\Omega^2 r_a^2 - (\vec{\Omega}\cdot\vec{r_a})^2\right)$$

$$= \frac{1}{2}\sum_a m_a \left(\sum_{i,j}\Omega_i\Omega_j\delta_{ij}\, r_a^2 - \sum_{i,j}\Omega_i r_{ai}\Omega_j r_{aj}\right)$$

$$= \frac{1}{2}\sum_{i,j}\left(\sum_a m_a\left(r_a^2\delta_{ij} - r_{ai} r_{aj}\right)\right)\Omega_i\Omega_j$$

$$= \boxed{\frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\Omega_j} \quad \left(= \frac{1}{2} \ \cdots \right)$$

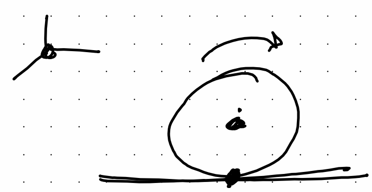
$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\boxed{T = \underbrace{\frac{1}{2}\mu V^2}_{\text{trans}} + \underbrace{\frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\Omega_j}_{\text{rotational}}} \quad + \ ② \quad \left(\frac{1}{2} I \Omega^2\right)$$

Freshman physics

for COM at origin
OF RB Frame

$\vec{M}$: wrt COM of body



$$\vec{M} = \sum_a \vec{r_a}\times\vec{p_a}$$

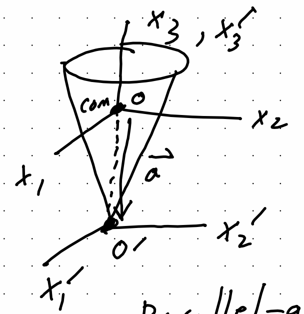
$$= \sum_a m_a \ \vec{r_a}\times\vec{v_a}$$

$$= \sum_a m_a \ \vec{r_a}\times(\vec{V} + \vec{\Omega}\times\vec{r_a})$$

$$= \sum_a m_a \ \vec{r_a}\times(\vec{\Omega}\times\vec{r_a})$$

$$= \sum_a m_a \left(\vec{\Omega}\, r_a^2 - \vec{r_a}\,(\vec{r_a}\cdot\vec{\Omega})\right)$$

$$\boxed{M_i = \sum_j I_{ij}\,\Omega_j} \qquad \vec{M} = I\vec{\Omega} \ \text{(Fresh. physics)}$$



$\underbrace{I'_{ij}}_{\text{wrt } O'}$, $\underbrace{I_{ij}}_{\text{wrt } O \text{ (com)}}$

parallel-axis theorem

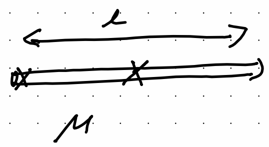
$$I'_{ij} = I_{ij} + M(a^2\delta_{ij} - a_i a_j)$$

$\vec{a}$: vector from O to O'

$$\underbrace{I(\hat{n})}_{\text{moment of inertia}} = \sum_{i,j} I_{ij} \underbrace{n_i n_j}_{}$$

$\hat{n}$: axis of rotation

n I n



$$\boxed{\begin{aligned} I_{com} &= \frac{1}{12} M l^2 \\ I_{end} &= \frac{1}{3} M l^2 \end{aligned}}$$

$$\boxed{\underbrace{I_{ij}}_{3\times 3 \text{ real symmetric}} = \sum_a m_a (r_a^2 \delta_{ij} - r_{ai} r_{aj}) = \int \underbrace{\rho\, dV}_{dm} (r^2 \delta_{ij} - r_i r_j)}$$

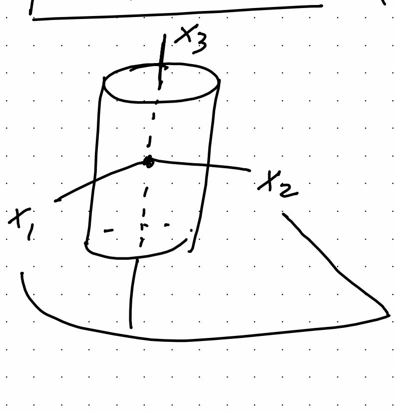
can always be diagonalized

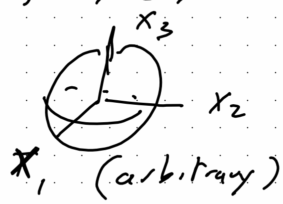
$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$I_{ij} = I_i \delta_{ij}$$

$$T_{rot} = \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j = \frac{1}{2}(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$$

principle axes: (x_1, x_2, x_3)

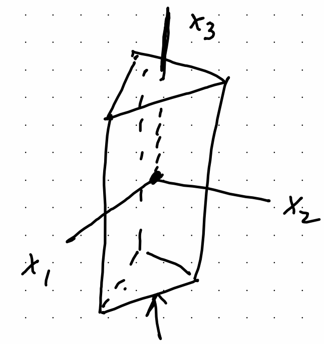




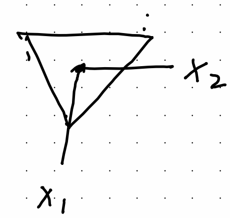
$$M_i = \sum_j I_{ij} \Omega_j$$

$M_1 = I_1 \Omega_1$, $M_2 = I_2 \Omega_2$

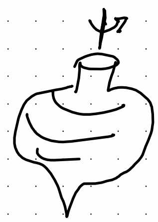
$M_3 = I_3 \Omega_3$

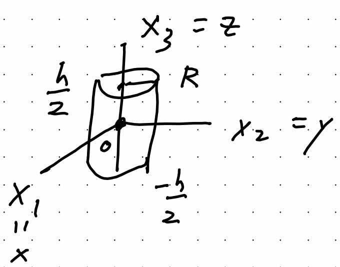


equilateral triang



$$\underbrace{I_1 = I_2 \neq I_3}_{\text{symmetrical top}}$$





total mass M

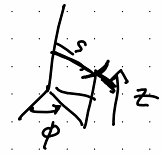
$$\rho = \frac{M}{\text{volume}} = \frac{M}{\pi R^2 h}$$

$$I_3 = I_{33} = \int \rho\, dV \left(r^2 \underbrace{\delta_{33}}_{1} - \underbrace{r_3 r_3}_{z^2} \right)$$

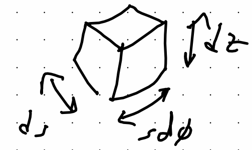
$$= \int \rho\, dV \left(r^2 - z^2 \right)$$

$$= \int \rho\, dV \left(x^2 + y^2 \right) = \int \rho\, dV s^2$$

<u>cylindrical:</u> s, ϕ, z $\quad s^2 = x^2 + y^2$



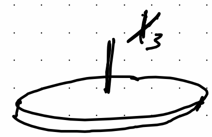
$$dV = ds\ s\,d\phi\ dz$$
$$= s\,ds\,d\phi\,dz$$



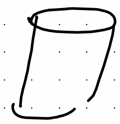
$$I_3 = \int \rho \, dV \, s^2$$

$$= \frac{M}{\pi R^2 h} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-h/2}^{h/2} dz}_{h} \underbrace{\int_0^{R} s^3 \, ds}_{\frac{R^4}{4}}$$

$$= \frac{M}{\cancel{\pi} R^{\cancel{2}} \cancel{h}} \cdot 2\cancel{\pi} \cancel{h} \, \frac{R^{\cancel{4} \to 2}}{4}$$



$$= \boxed{\frac{1}{2} M R^2}$$



$$I_1 = I_2 \equiv I$$

$$I_1 = \int \rho \, dV (r^2 - x^2)$$

$$+ \; I_2 = \int \rho \, dV (r^2 - y^2)$$

$$2I = \int \rho \, dV (2r^2 - x^2 - y^2)$$

$$\left(\begin{array}{l} r^2 = s^2 + z^2 \\ x^2 + y^2 = s^2 \end{array} \right)$$

$$2I = \int \rho \, dV (s^2 + 2z^2)$$

$$\boxed{I} = \frac{1}{2} \underbrace{\int \rho \, dV \, s^2}_{I_3} + \int \rho \, dV \, z^2$$

$$\boxed{= \frac{1}{2} I_3 + \int \rho \, dV \, z^2}$$

easier to evaluate

$$\int \rho\, dV\, z^2 = \frac{M}{\pi R^2 h} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-h/2}^{h/2} dz \cdot z^2}_{\left.\frac{z^3}{3}\right|_{-h/2}^{h/2}} \underbrace{\int_0^R s\, ds}_{\frac{R^2}{2}}$$

$$= \frac{2}{3}\left(\frac{h}{2}\right)^3$$

$$= \frac{h^3}{12}$$

$$= \frac{M}{\cancel{\pi} \cancel{R^2} h} \cdot \cancel{2}\cancel{\pi} \cdot \frac{h^3}{12} \cdot \frac{\cancel{R^2}}{\cancel{2}}$$

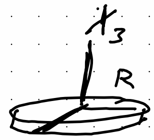
$$= \boxed{\frac{M h^2}{12}}$$

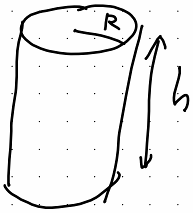
$$I = \frac{1}{2}\left(\frac{1}{2} M R^2\right) + \frac{M h^2}{12}$$

$$= \frac{1}{4} M R^2 + \frac{1}{12} M h^2$$

$$= \frac{1}{4} M \left(R^2 + \frac{1}{3} h^2 \right) = I_1, I_2$$

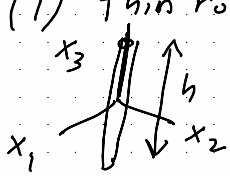
$$\boxed{I_3 = \frac{1}{2} M R^2}$$





Limiting cases: (i) Disk ($h \to 0$) $I_3 = \frac{1}{2} M R^2$, $I_1, I_2 = \frac{1}{4} M R^2$

(ii) thin rod ($R \to 0$)



$I_3 = 0$

$I_1 = I_2 = \frac{1}{12} M h^2$

$$L = T - U$$

$$= \frac{1}{2}\mu V^2 + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\Omega_j - U$$

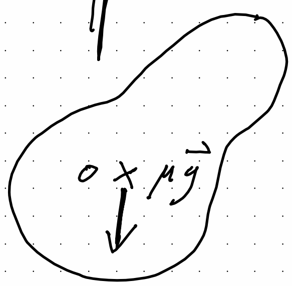
$\vec{V} = \dot{\vec{R}}$

$\vec{\Omega} = \dot{\vec{\phi}}$

$L(\vec{R}, \vec{\phi}, \dot{\vec{R}}, \dot{\vec{\phi}})$

$\vec{g} \downarrow$

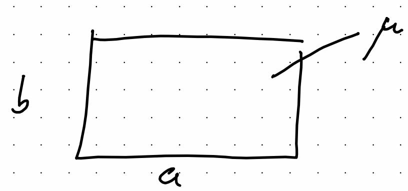
uniform
Field



Lecture #24 : Thursday 11/12

- Quiz #5 (today)
- Midterm #2 (next Thursday) (scattering, small oscillations, RB motion)
- Today's topics:
  (1) RB EOMs
  (2) Euler's equations
  (3) Euler angles

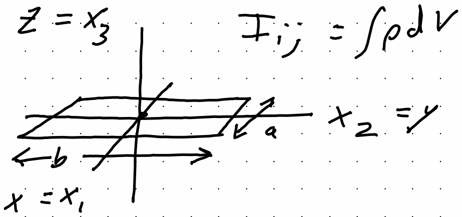
Q5: Calculate the principal moments of inertia for a 2-d rectangle with sidelengths a, b. uniform

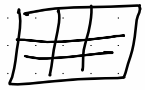


name_Q5.pdf

$z = x_3$ $\qquad I_{ij} = \int \rho\, dV\,(r^2 \delta_{ij} - r_i r_j)$

$r^2 = x^2 + y^2 + z^2$





$$I_1 = \int \rho\, dV\,(r^2 - x^2)$$

$$I_{11} = \int \rho\, dV\, y^2$$

$$= \frac{M}{ab}\int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\; y^2$$

$$= \frac{M}{\cancel{a}b}\,\cancel{a}\,\frac{y^3}{3}\Big|_{-b/2}^{b/2}$$

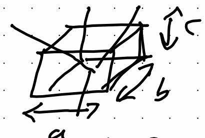
$$= \frac{M}{b}\,\frac{2}{3}\left(\frac{b}{2}\right)^3$$

$$= \frac{M}{b}\,\frac{2}{3}\,\frac{b^3}{8} = \boxed{\frac{1}{12} M b^2}$$

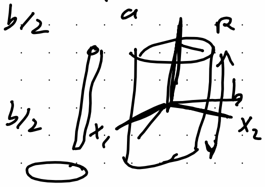
$dm = \rho\, dV$

$= \sigma\, dx\, dy$

$= \frac{M}{ab}\, dx\, dy$



$$\boxed{I_2 = \frac{1}{12} M a^2}$$



$I_3 = \int \sigma\, dx\, dy\, (x^2 + y^2)$

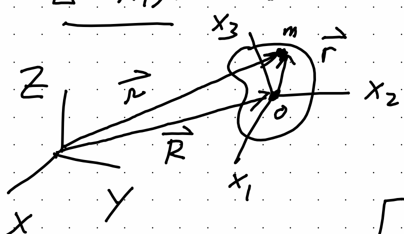
$= I_1 + I_2$

$$\boxed{I_3 = \frac{1}{12} M (a^2 + b^2)}$$

EOMs: $\quad$ Final result:

$$\frac{d\vec{P}}{dt} = \vec{F} = \sum \vec{f}$$

$$\frac{d\vec{M}}{dt} = \vec{K} = \sum \vec{r} \times \vec{f}$$



$$\vec{\imath} = \vec{R} + \vec{r}\,, \quad \boxed{\delta\vec{\imath} = \delta\vec{R} + \underbrace{\delta\vec{\phi}}_{\delta\Omega_k} \times \vec{r}}$$

$$L = T - U$$

$$= \frac{1}{2} M |\vec{V}|^2 + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i \Omega_j - U(\vec{\imath})$$

$$\delta L = M\,\vec{V}\cdot\delta\vec{V} + \sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i\; \boxed{-\sum \frac{\partial U}{\partial \vec{\imath}}\cdot\delta\vec{\imath}}$$

$$-\sum \frac{\partial U}{\partial \vec{\imath}}\cdot\left(\delta\vec{R} + \delta\vec{\phi}\times\vec{r}\right)$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{V}}\right) = \frac{\partial L}{\partial \vec{R}}}\;, \quad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{\Omega}}\right) = \frac{\partial L}{\partial \vec{\phi}}}$$

$$\frac{1}{2}\sum_{i,j} I_{ij}\,\delta\Omega_i\,\Omega_j + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\delta\Omega_j$$

$$= \quad '' \quad + \frac{1}{2}\sum_{j,i} I_{ji}\,\Omega_j\,\delta\Omega_i$$

$$= \quad '' \quad + \frac{1}{2}\sum_{\underbrace{j,i}_{\text{swap}}} I_{ij}\,\Omega_j\,\delta\Omega_i$$

$$= \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i$$

$$= \boxed{\sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i}$$

$$\delta L = m\vec{V}\cdot\delta\vec{V} + \sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i$$

$\vec{A}\cdot(\vec{B}\times\vec{C})$
$\vec{B}\cdot(\vec{C}\times\vec{A})$

$$-\left(\sum \frac{\partial U}{\partial \vec{r}}\right)\cdot\delta\vec{R} - \sum \frac{\partial U}{\partial \vec{r}}\cdot(\delta\vec{\phi}\times\vec{r})$$

$$= -\sum \delta\vec{\phi}\cdot\left(\vec{r}\times\frac{\partial U}{\partial\vec{r}}\right)$$

$$= -\delta\vec{\phi}\cdot\left(\sum \vec{r}\times\frac{\partial U}{\partial\vec{r}}\right)$$

$$\sum_i M_i\,\delta\Omega_i = \vec{M}\cdot\delta\vec{\Omega}$$

$$dF = \bigcirc\, dx + \bigcirc\, dy$$

$$\boxed{\delta L = \vec{P}\cdot\delta\vec{V} + \vec{M}\cdot\delta\vec{\Omega} + \left(\sum\vec{f}\right)\cdot\delta\vec{R} + \delta\vec{\phi}\cdot\left(\sum\vec{r}\times\vec{f}\right)}$$

$$\vec{P} = \frac{\partial L}{\partial\vec{V}}, \quad \vec{M} = \frac{\partial L}{\partial\vec{\Omega}}, \quad \sum\vec{f} = \frac{\partial L}{\partial\vec{R}}, \quad \sum\vec{r}\times\vec{f} = \frac{\partial L}{\partial\vec{\phi}}$$

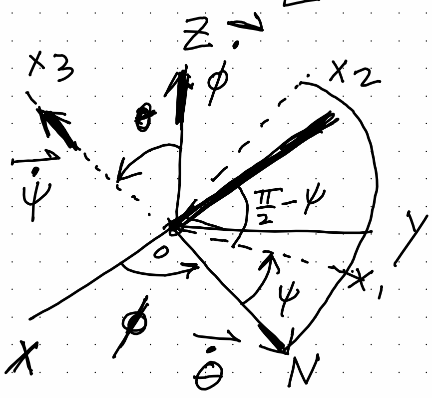
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\vec{V}}\right) = \frac{\partial L}{\partial\vec{R}} \to \boxed{\frac{d\vec{P}}{dt} = \sum\vec{f} \equiv \vec{F}} \quad \Bigg| \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\vec{\Omega}}\right) = \frac{\partial L}{\partial\vec{\phi}} \to \boxed{\frac{d\vec{M}}{dt} = \sum\vec{r}\times\vec{f} = \vec{K}}$$

Euler's equations / Euler angles ☆

$\vec{R}, \vec{\phi}$

Ω_i : ith component of $\vec{\Omega}$

wrt $\hat{x}_i$

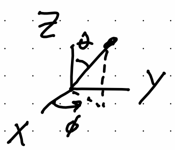


$$\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta} + \vec{\dot\psi}$$

$$\vec{\dot\psi} = \dot\psi\, \hat{x}_3$$

$$\vec{\dot\theta} = \dot\theta \cos\psi\, \hat{x}_1 - \dot\theta \sin\psi\, \hat{x}_2$$

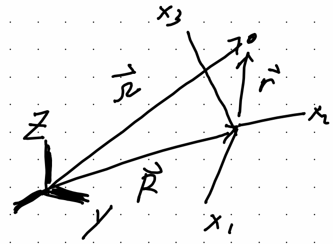
$$\vec{\dot\phi} = \dot\phi \cos\theta\, \hat{x}_3 + \dot\phi \sin\theta \left(\underbrace{\sin\psi}_{\cos(\frac{\pi}{2}-\psi)} \hat{x}_1 + \underbrace{\cos\psi}_{\sin(\frac{\pi}{2}-\psi)} \hat{x}_2 \right)$$



Announcements

- Midterm II is this thursday
- Today:
  - i) Euler angles
  - ii) Euler's equation for RB motion
  - iii) Free rotation with $\vec{\Omega}$ = const
  - iv) " of a symmetric top ($I_1 = I_2$)
  - v) Heavy symmetrical top with one point fixed [prob 35.1]

$$\vec{\Omega} = (\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)\,\hat{x}_1 + (\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)\,\hat{x}_2 + (\dot\psi + \dot\phi \cos\theta)\,\hat{x}_3$$



$$\boxed{\begin{aligned} \Omega_1 &= \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi \\ \Omega_2 &= \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi \\ \Omega_3 &= \dot\psi + \dot\phi \cos\theta \end{aligned}}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}$$

$\frac{d'\vec{r}}{dt}$ wrt RB

Euler's equations:

(wrt RB axes) $\vec{A}$: any vector

$$\frac{d\vec{P}}{dt} = \sum \vec{f} \equiv \vec{F}$$

$$\frac{d\vec{M}}{dt} = \sum \vec{r} \times \vec{f} = \vec{K}$$

$$\boxed{\underbrace{\frac{d\vec{A}}{dt}}_{\text{wrt inertial frame}} = \underbrace{\frac{d'\vec{A}}{dt}}_{\text{wrt rotating frame}} + \underbrace{\vec{\Omega}}_{\text{angular velocity of the rotating frame}} \times \vec{A}}$$

$$\underbrace{\left(\frac{d'\vec{A}}{dt}\right)_i}_{\text{cartesian components wrt rotating frame}} = \frac{dA_i}{dt} \equiv \dot{A}_i$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}\,\hat{x} + \ddot{y}\,\hat{y} + \ddot{z}\,\hat{z}$$

$$\cancel{= \ddot{r}\,\hat{r}}$$

$$\vec{A} = \sum_i \underline{\underline{A_i}}\ \underline{\underline{\hat{x}_i}}$$

$$\frac{d\vec{A}}{dt} = \underbrace{\sum_i \left(\frac{dA_i}{dt}\right)\hat{x}_i}_{\frac{d'\vec{A}}{dt}} + \underbrace{\sum_i A_i \frac{d\hat{x}_i}{dt}}_{\vec{\Omega} \times \vec{A}}$$

$$\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\Omega}\times\vec{A} \quad , \quad \left(\frac{d'\vec{A}}{dt}\right)_i = \dot{A}_i$$

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d'\vec{P}}{dt} + \vec{\Omega}\times\vec{P}$$

$$\vec{P} = \mu\vec{V}$$

$$\rightarrow \boxed{F_1 = \dot{P}_1 + (\vec{\Omega}\times\vec{P})_1 = \dot{P}_1 + \Omega_2 P_3 - \Omega_3 P_2 = \mu(\dot{V}_1 + \Omega_2 V_3 - \Omega_3 V_2)}$$

similar equations for F_2, F_3

$$\boxed{\vec{K} = \frac{d\vec{M}}{dt}} = \frac{d'\vec{M}}{dt} + \vec{\Omega}\times\vec{M}$$

$$M_i = I_i \Omega_i$$

$$K_x = \dot{M}_x$$

$$\boxed{K_1 = \dot{M}_1 + \Omega_2 M_3 - \Omega_3 M_2 = I_1\dot{\Omega}_1 + \Omega_2 I_3 \Omega_3 - \Omega_3 I_2 \Omega_2 = I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_2)}$$

*(similar equations for K_2, K_3)

Free rotation: $K_i = 0, \ F_i = 0$

$$0 = I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_2) \ \cancel{, etc.}$$
$$0 = I_2\dot{\Omega}_2 + \Omega_3\Omega_1(I_1 - I_3)$$
$$0 = I_3\dot{\Omega}_3 + \Omega_1\Omega_2(I_2 - I_1)$$

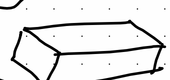
Free rotation with $\vec{\Omega} = \text{const}$:

$$\frac{d\vec{\Omega}}{dt} = 0$$
$$\frac{d'\vec{\Omega}}{dt} = 0$$
$$\boxed{\frac{d\vec{\Omega}}{dt} = \frac{d'\vec{\Omega}}{dt}} + \cancel{\vec{\Omega}\times\vec{\Omega}}$$

$$\boxed{\frac{d\vec{M}}{dt} \neq \frac{d'\vec{M}}{dt}}$$

$$0 = \Omega_2\Omega_3(I_3 - I_2)$$
$$0 = \Omega_3\Omega_1(I_1 - I_3)$$
$$0 = \Omega_1\Omega_2(I_2 - I_1)$$

$\boxed{\Omega_1 = \text{const}, \ \Omega_2 = 0, \ \Omega_3 = 0}$ ← stable , $I_1 < I_2 < I_3$



unstable ↓

$\boxed{\Omega_2 = \text{const}, \ \Omega_1 = 0, \ \Omega_3 = 0}$ $\boxed{\Omega_3 = \text{const}, \ \Omega_1 = 0, \ \Omega_2 = 0}$

$\Omega_1 = \text{const},\ \Omega_2 = 0,\ \Omega_3 = 0$: exact

$$\boxed{\begin{aligned} \Omega_1 &= \text{const} + \epsilon_1 \\ \Omega_2 &= \epsilon_2 \\ \Omega_3 &= \epsilon_3 \end{aligned}}$$

$\epsilon_{1,2,3}$: small time dependent perturbations

Keep 0^{th} and 1^{st} order terms.
Ignore 2^{nd} order, e.g. $\epsilon_2 \epsilon_3$

$$0 = I_1 \frac{d}{dt}(\underset{\to 0}{\text{const}} + \epsilon_1) + \underbrace{\epsilon_2 \epsilon_3 (I_3 - I_2)}_{2^{nd}\text{ order} \to \text{ignore}}$$

$$\approx I \dot{\epsilon}_1$$

$$\to \epsilon_1 = \text{const} \to \Omega_1 = \text{const}$$

$$\left.\begin{aligned} 0 &= I_2 \dot{\epsilon}_2 + \epsilon_3 \Omega_1 (I_1 - I_3) \\ 0 &= I_3 \dot{\epsilon}_3 + \Omega_1 \epsilon_2 (I_2 - I_1) \end{aligned}\right\} \text{coupled } 1^{st} \text{ order diff. equations}$$

Differentiate

$$0 = I_2 \ddot{\epsilon}_2 + \dot{\epsilon}_3 \Omega_1 (I_1 - I_3)$$

$$= I_2 \ddot{\epsilon}_2 - \frac{\Omega_1 \epsilon_2 (I_2 - I_1)}{I_3} \Omega_1 (I_1 - I_3)$$

$I_1 < I_2 < I_3$

$$0 = I_2 \ddot{\epsilon}_2 + \dot{\epsilon}_3 \Omega_1 (I_1 - I_3)$$

$$= I_2 \ddot{\epsilon}_2 - \frac{\Omega_1 \epsilon_2 (I_2 - I_1)}{I_3} \Omega_1 (I_1 - I_3)$$

$$= \ddot{\epsilon}_2 + \underbrace{\frac{\Omega_1^2 (I_2 - I_1)(I_3 - I_1)}{I_2 I_3}}_{\omega^2} \epsilon_2$$

$$\boxed{\begin{matrix} (I_3 - I_2) \\ (I_1 - I_2) \end{matrix}}$$

$$= \ddot{\epsilon}_2 + \omega^2 \epsilon_2$$

$$\ddot{\epsilon}_2 = -\omega^2 \epsilon_2 \longrightarrow \text{SHM}$$

$$\begin{aligned} \epsilon_2 &= A \cos \omega t + B \sin \omega t \\ &= C \cos(\omega t + \alpha) \end{aligned}$$

Similarly,

$$\ddot{\epsilon}_3 = -\omega^2 \epsilon_3 \longrightarrow \epsilon_3 = D \cos(\omega t + \beta)$$

(ϵ_2, ϵ_3 are bound by their initial deviations away from 0)

For $\Omega_2 = \text{const}$ solution

perturbations

$$\ddot{\epsilon}_1 = +\omega^2 \epsilon_1$$

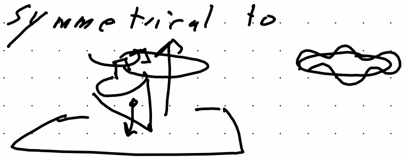
$$\ddot{\epsilon}_3 = +\omega^2 \epsilon_3$$

$$\rightarrow \quad \epsilon_1(t) = \underbrace{A e^{\omega t}}_{\text{grows exponentially}} + B \underbrace{e^{-\omega t}}_{\text{damped exponentially}}$$

$\longrightarrow$ instability

Lecture #27   Tues 11/24

- Midterm II – Avg $\approx$ 12/20 (solns posted)
- Quiz #6 – Next Tuesday (last day)
- Oral Final – google doc sign up sheet ??
  Sat 12/5 1:30pm – 4:00pm
- Today, next time:
  i) Free rotation of a symmetric top
  ii) motion in a non-inertial ref. frame
- On your own:
  (i) Sec 35, Prob 1: motion of a heavy symmetrical top
  (ii) static equil:
  (Sec 38)



Free rotation of a symmetric top:

no force, torque $\to$ $\vec{P} = \text{const}$, $\vec{M} = \text{const}$

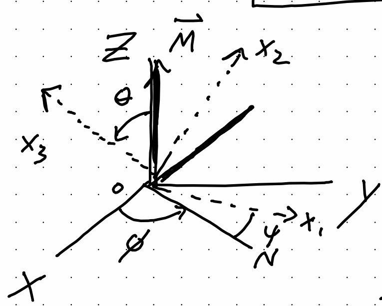
Euler's equations: $\quad I_1 = I_2$

$$0 = I_1\dot{\Omega}_1 + \Omega_2 \overset{\text{const}}{\Omega_3} (I_3 - \overset{I_1}{\overset{\|}{I_2}}) \qquad (1)$$

$$0 = I_2\dot{\Omega}_2 + \Omega_3\Omega_1(I_1 - I_3) \qquad (2)$$

$$0 = I_3\dot{\Omega}_3 + \Omega_1\Omega_2\cancel{(I_2 - I_1)} \to 0 \qquad (3)$$

$$\dot{\Omega}_3 = 0 \to \boxed{\Omega_3 = \text{const}}$$



$$\Omega_1 = \dot{\phi}\sin\theta\sin\psi + \cancel{\dot{\theta}\cos\psi} \to 0$$

$$\Omega_2 = \dot{\phi}\sin\theta\cos\psi - \cancel{\dot{\theta}\sin\psi} \to 0$$

$$\Omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$$

$$I_1\Omega_1 = M_1 = M\sin\theta\sin\psi \qquad (4)$$

$$I_1\Omega_2 = M_2 = M\sin\theta\cos\psi \qquad (5)$$

$$I_3\Omega_3 = M_3 = M\cos\theta \qquad (6)$$

$$\Omega_3 = \text{const} \to \boxed{\theta = \text{const}}, \quad \cos\theta = \frac{I_3\Omega_3}{M}$$

$$0 = I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_1)$$

$$0 = \dot{\Omega}_1 + \Omega_2\left(\frac{\Omega_3(I_3 - I_1)}{I_1}\right) \longleftarrow \boxed{\omega \equiv \frac{\Omega_3(I_3 - I_1)}{I_1}}$$

$$0 = I_1\dot{\Omega}_2 + \Omega_1\Omega_3(I_1 - I_3) \qquad M_1 = I_1\Omega_1,\ M_2 = I_1\Omega_2$$

$$0 = \dot{\Omega}_2 - \Omega_1\left(\frac{\Omega_3(I_3 - I_1)}{I_1}\right) \qquad \sqrt{\Omega_1^2 + \Omega_2^2} = a$$

$$0 = \dot{\Omega}_1 + \Omega_2\omega$$

$$0 = \dot{\Omega}_2 - \Omega_1\omega$$

$$0 = \ddot{\Omega}_1 + \dot{\Omega}_2\omega = \ddot{\Omega}_1 + \Omega_1\omega^2 \qquad \boxed{\Omega_2 = a\sin(\omega t + \alpha)}$$

$$\ddot{\Omega}_1 = -\omega^2\Omega_1 \to \boxed{\Omega_1 = a\cos(\omega t + \alpha)}$$

$$\xi = \Omega_1 + i\Omega_2$$

$$\dot{\xi} = \dot{\Omega}_1 + i\dot{\Omega}_2$$

$$\longrightarrow \dot{\xi} = i\omega\xi \to \xi(t) = \underset{\text{complex}}{A}e^{i\omega t}$$

$$I_1 \dot\phi \cancel{\sin\theta \sin\psi} = M \cancel{\sin\theta \sin\psi} \qquad \boxed{\theta = \text{const}}$$

$$\boxed{\dot\phi = \frac{M}{I_1} = \text{const}} \rightarrow \boxed{\phi(t) = \phi_0 + \frac{M}{I_1} t}$$

$$I_3 \left(\dot\psi + \cos\theta \dot\phi\right) = M \cos\theta$$

$$\boxed{-\omega} = \dot\psi = \frac{M\cos\theta}{I_3} - \cos\theta \dot\phi$$

$$= \frac{M\cos\theta}{I_3} - \cos\theta \frac{M}{I_1}$$

$$= M\cos\theta \left(\frac{1}{I_3} - \frac{1}{I_1}\right) = \text{const}$$

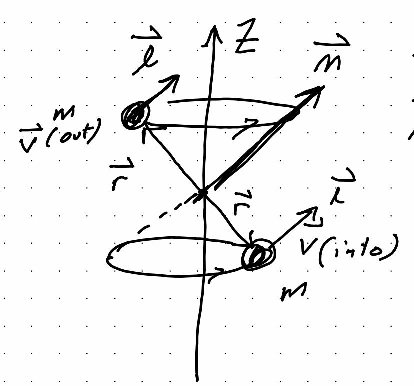
$$\rightarrow \boxed{\psi(t) = \psi_0 + M\cos\theta\left(\frac{1}{I_3} - \frac{1}{I_1}\right)}$$

$$\omega = \frac{\Omega_3 (I_3 - I_1)}{I_1}$$

$$\left.\begin{aligned} M_3 &= I_3 \Omega_3 \\ M\cos\theta &= M_3 \end{aligned}\right\}$$

$$\Omega_3 = \frac{M\cos\theta}{I_3}$$

$$\omega = \frac{M\cos\theta (I_3 - I_1)}{I_1 I_3} = M\cos\theta\left(\frac{1}{I_1} - \frac{1}{I_3}\right) \qquad \boxed{-\dot\psi}$$



$\vec\Omega = \text{const}$? Yes $\vec\Omega = \Omega \hat{z}$

$\vec{M} = \text{const}$? No $\vec M$ precesses around $\hat z$ axis

$$\vec{l} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v}$$

---

wobble vs spin freq

$\dot\phi$ — precession freq of 3-axis around $\vec M$ (or $\hat z$)

Ω_3 — Component of angular along 3-axis

$$\dot\phi = \frac{M}{I_1}$$

$$\Omega_3 = \frac{M\cos\theta}{I_3}$$

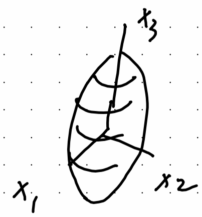
$$\boxed{\frac{\Omega_3}{\dot\phi} = \frac{M\cos\theta}{I_3} \frac{I_1}{M} = \frac{I_1 \cos\theta}{I_3}}$$

$$\frac{\Omega_3}{\dot\phi} = \frac{I_1}{I_3}\cos\theta \longrightarrow \frac{I_1}{I_3} \qquad (\theta \to 0)$$

$\frac{\text{spin}}{\text{wobble}}$

$I_1 = I_3$ sphere

$$\boxed{\frac{\Omega_3}{\dot\phi} = 1}$$

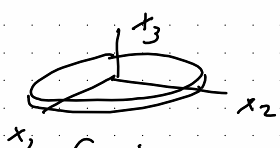


football

$I_1 > I_3$

football

$$\boxed{\frac{\Omega_3}{\dot\phi} > 1}$$



frisbee

$$\underset{\frac{1}{4}MR^2}{I_1} < \underset{\frac{1}{2}MR^2}{I_3}$$

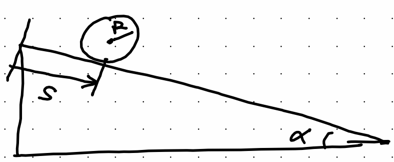
$I_1 = \frac{1}{2} I_3$ (uniform disk)

frisbee

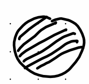
$$\boxed{\frac{\Omega_3}{\dot\phi} = \frac{1}{2} < 1}$$

Lecture #28: name = q6.pdf

- course evaluations (please submit by EOB today)
- sign-up for oral final exam date/time
- Quiz #6



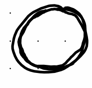
uniform disk $I_3 = \frac{1}{2}MR^2$



a) calculate KE of the uniform disk as it rolls without slipping down the incline

$$T = \text{something}(M, \dot s, R)$$

b) repeat for a hoop



$$I_3 = MR^2$$

$$T = \text{something}$$

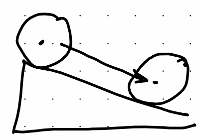
Which object will reach the bottom first? (or do both reach the bottom at the same time)

$$T = \frac{1}{2}\mu V^2 + \frac{1}{2}\sum_{i,j} I_{ij}\Omega_i\Omega_j$$
$$= \frac{1}{2}\mu V^2 + \frac{1}{2}\sum_{i} I_i\Omega_i^2$$
$$= \frac{1}{2}\mu V^2 + \frac{1}{2} I_3\Omega_3^2$$

$$V = \dot s, \quad \Omega_3 \stackrel{?}{=} \dot s \longrightarrow \Omega_3 = \frac{\dot s}{R}$$

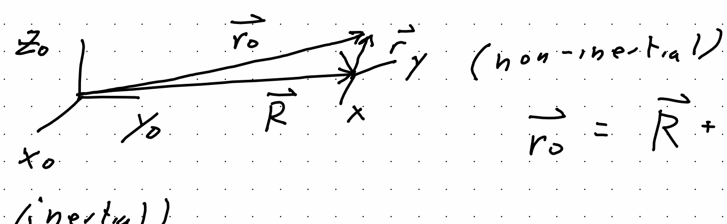
$$T = \frac{1}{2}\mu \dot s^2 + \frac{1}{2} I_3 \frac{\dot s^2}{R^2}$$
$$= \frac{1}{2}\mu \dot s^2 + \frac{1}{2}\left(\frac{1}{2}\mu \cancel{R^2}\right)\frac{\dot s^2}{\cancel{R^2}}$$
$$= \boxed{\frac{3}{4}\mu \dot s^2} \quad \text{(unif disk)} \longrightarrow \dot s = \sqrt{\frac{4}{3}\frac{I}{\mu}}$$

$$\boxed{T = \mu \dot s^2} \quad \text{(hoop)}$$



$$\longrightarrow \dot s = \sqrt{\frac{I}{\mu}}$$

Motion in a non-inertial ref frame (sec 39)



$$\vec{r_0} = \vec R + \vec r$$

$$m\vec{a_0} = \vec F \quad \text{(2}^{\text{nd}}\text{ law in inertial frame)}$$

$$m\vec a = \vec F \; ??$$

$$\boxed{\left.\frac{d\vec A}{dt}\right|_0 = \underbrace{\frac{d\vec A}{dt}}_{\text{wrt rotating frame}} + \vec\Omega \times \vec A} \longleftarrow \text{use this}$$

$$\vec{a_0} = \left.\frac{d\vec{v_0}}{dt}\right|_0, \quad \vec{v_0} = \left.\frac{d\vec{r_0}}{dt}\right|_0$$

$$\vec{v_0} = \left.\frac{d}{dt}(\vec R + \vec r)\right|_0 = \left.\frac{d\vec R}{dt}\right|_0 + \left.\frac{d\vec r}{dt}\right|_0 = \boxed{\vec V + \overset{\frac{d\vec r}{dt}}{\vec v} + \vec\Omega \times \vec r}$$

$$\vec{v_0} = \vec{V} + \vec{v} + \vec{\Omega}\times\vec{r}$$

$$\vec{a_0} = \frac{d\vec{v_0}}{dt}\Big|_0$$

$$= \frac{d\vec{V}}{dt}\Big|_0 + \frac{d\vec{v}}{dt}\Big|_0 + \frac{d\vec{\Omega}}{dt}\Big|_0 \times \vec{r} + \vec{\Omega}\times\frac{d\vec{r}}{dt}\Big|_0$$

$$= \underbrace{\vec{W}}_{\text{accel of origin of non-inertial frame}} + \Big(\underbrace{\frac{d\vec{v}}{dt}}_{=\vec{a}} + \vec{\Omega}\times\vec{v}\Big) + \Big(\underbrace{\frac{d\vec{\Omega}}{dt}}_{=\dot{\vec{\Omega}}} + \cancelto{0}{\vec{\Omega}\times\vec{\Omega}}\Big)\times\vec{r}$$

$$+ \vec{\Omega}\times\Big(\underbrace{\frac{d\vec{r}}{dt}}_{=\vec{v}} + \vec{\Omega}\times\vec{r}\Big)$$

$$\boxed{\vec{a}_0 = \vec{a} + \underbrace{\vec{W}}_{\text{trans accel.}} + \underbrace{\dot{\vec{\Omega}}\times\vec{r}}_{\text{rot. accel.}} + \underbrace{2\vec{\Omega}\times\vec{v}}_{\text{coriolis}} + \underbrace{\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}_{\text{centrifugal}}}$$

$$m\vec{a_0} = \vec{F}$$

$$m(\vec{a} + \dots) = \vec{F}$$

$$\rightarrow \boxed{m\vec{a} = \vec{F} - m\vec{W} - m\dot{\vec{\Omega}}\times\vec{r} - 2m\vec{\Omega}\times\vec{v} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$$

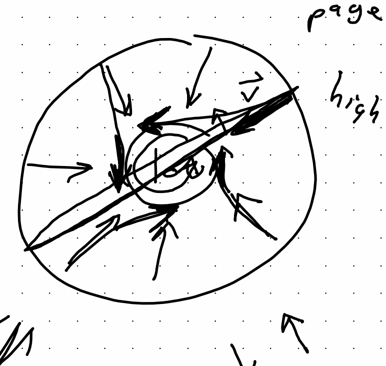
$$\frac{d\vec{p}}{dt} = \sum f = \vec{F}$$

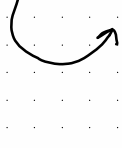
$$\boxed{\underbrace{\frac{d'\vec{p}}{dt}}_{\text{wrt rigid body frame}} + \vec{\Omega}\times\vec{p} = \vec{F}} \longrightarrow \boxed{\frac{d'\vec{M}}{dt} + \vec{\Omega}\times\vec{M} = \vec{K}}$$

$$\frac{d'\vec{p}}{dt} = \vec{F} - \vec{\Omega}\times\vec{p}$$

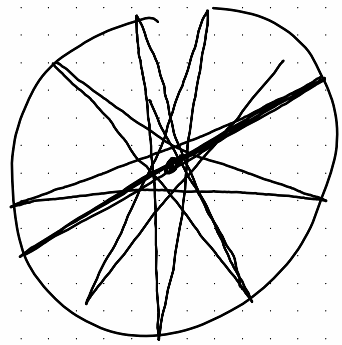
ccw

$\odot\vec{\Omega}$ out of page

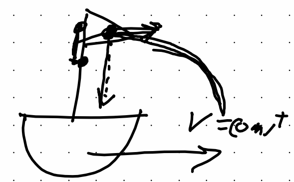




cyclonic



Example: merry-go-round

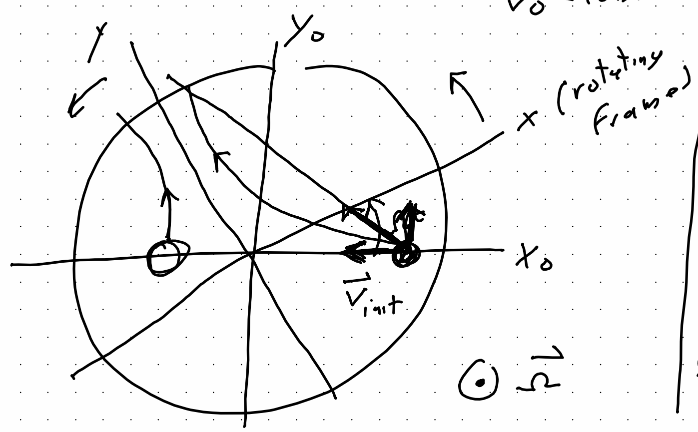


$$m\vec{a}_0 = \vec{0}$$

$$\vec{v}_0 = \text{const}$$

solve 2nd law in merry-go-round frame:

$$m\frac{d\vec{v}}{dt} = -2m\vec{\Omega}\times\vec{v} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$



$\odot\ \vec{\Omega}$

$$\vec{r} = x\hat{x} + y\hat{y}$$

$$\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y}$$

$$\frac{d\vec{v}}{dt} = \ddot{x}\hat{x} + \ddot{y}\hat{y}$$

$$\vec{\Omega} = \Omega\hat{z}$$

$$\boxed{\begin{aligned}\ddot{x} &= 2\Omega\dot{y} + \Omega^2 x \\ \ddot{y} &= -2\Omega\dot{x} + \Omega^2 y\end{aligned}}$$

$$\left.\begin{aligned}\xi &= x + iy \\ \dot{\xi} &= \dot{x} + i\dot{y} \\ \ddot{\xi} &= \ddot{x} + i\ddot{y}\end{aligned}\right\}\quad \ddot{\xi} + i2\Omega\dot{\xi} - \Omega^2\xi = 0$$

Guess: $\xi = e^{i\lambda t}$

$\lambda = -\Omega$ (double root)

$$\boxed{\xi = (A + Bt)\,e^{-i\Omega t}}$$

complex const determined by initial conditions.

$\xi(t) = x(t) + iy(t)$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$