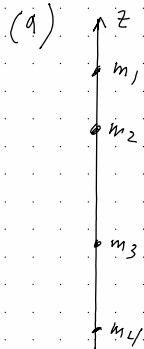


Sec 32, Prob 1:



$$I_3 = 0 \text{ since } x_a = y_a = 0 \\ \text{for all masses}$$

$$I_1 = \sum_a m_a (r_a^2 - \ell_a^2) = \sum_a m_a z_a^2$$

$$I_2 = \sum_a m_a (r_a^2 - \ell_a^2) = \sum_a m_a z_a^2$$

$$\rightarrow I_1 = I_2 = I$$

$$= \sum_a m_a z_a^2$$

(assuming COM at  $z=0$ )

If COM is not at  $z=0$ , but at  $z_{\text{COM}}$ ,  
then:

$$I = \sum_a m_a (z_a - z_{\text{COM}})^2, \quad z_{\text{COM}} = \frac{1}{m} \sum_b m_b z_b \\ = \sum_a m_a (z_a^2 + z_{\text{COM}}^2 - 2z_{\text{COM}} z_a) \\ = \sum_a m_a z_a^2 + m z_{\text{COM}}^2 - 2z_{\text{COM}} \sum_a m_a z_a \\ = \sum_a m_a z_a^2 - m z_{\text{COM}}^2$$

This last expression can be written  
in terms of  $\ell_{ab} = |z_a - z_b|$  as follows:

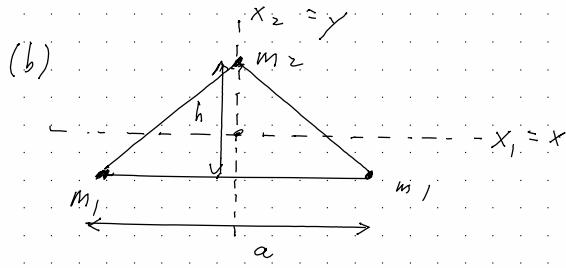
$$I = \frac{1}{2} \sum_a m_a z_a^2 + \frac{1}{2} \sum_b m_b z_b^2 - \frac{1}{m} \left( \sum_a m_a z_a \right) \left( \sum_b m_b z_b \right) \\ = \frac{1}{2m} \sum_{a,b} m_a m_b z_a^2 + \frac{1}{2m} \sum_{a,b} m_a m_b z_b^2 - \frac{1}{m} \sum_a m_a z_a \sum_b m_b z_b$$

Thus

$$I = \frac{1}{2m} \sum_{a,b} m_a m_b (z_a^2 + z_b^2 - 2z_a z_b) \\ = \frac{1}{2m} \sum_{a,b} m_a m_b (z_a - z_b)^2 \\ = \frac{1}{2m} \sum_{a,b} m_a m_b \ell_{ab}^2$$

NOTE: For just two masses:

$$I = \frac{1}{2m} (m_1 m_2 \ell^2 + m_2 m_1 \ell^2) \\ = \frac{m_1 m_2}{m} \ell^2 \\ = m \ell^2 \quad \text{where } m = \frac{m_1 m_2}{m_1 + m_2} \\ \ell = |z_1 - z_2|$$



Assume COM at  $(x_1, x_2) = (x, y) = (0, 0)$

$$\text{Then } 2m_1y_1 + m_2y_2 = 0$$

$$\text{where } y_2 - y_1 = h$$

$$\text{thus, } 2m_1y_1 + m_2(h + y_1) = 0$$

$$(2m_1 + m_2)y_1 + m_2h = 0$$

$$y_1 = \frac{-m_2h}{\mu}, \quad \mu = 2m_1 + m_2 \\ = \text{total mass}$$

$$\text{and } y_2 = y_1 + h \\ = \frac{-m_2h}{\mu} + h \\ = \frac{(\mu - m_2)h}{\mu} \\ = \frac{2m_1h}{\mu}$$

All moments about the base are zero

$$\text{Thus, } I_3 = \sum_a m_a(r_a^2 - z_a^2) = \sum_a m_a(x_a^2 + y_a^2)$$

$$I_1 = \sum_a m_a(r_a^2 - x_a^2) = \sum_a m_a y_a^2$$

$$I_2 = \sum_a m_a(r_a^2 - x_a^2) \approx \sum_a m_a x_a^2$$

$$\text{Thus, } I_3 = I_1 + I_2$$

so need to calculate  $I_1, I_2$

$$I_1 = \sum_a m_a y_a^2$$

$$= 2m_1 y_1^2 + m_2 y_2^2$$

$$= 2m_1 \frac{m_2^2 h^2}{\mu^2} + m_2 \frac{4m_1^2 h^2}{\mu^2}$$

$$= \frac{2m_1 m_2 h^2 (m_2 + 2m_1)}{\mu^2}$$

$$= \boxed{\frac{2m_1 m_2 h^2}{\mu}}$$

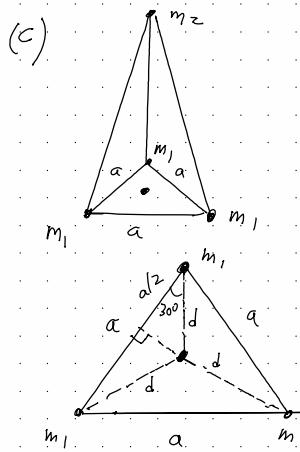
$$I_2 = \sum_a m_a x_a^2$$

$$= m_1 \left(\frac{a}{2}\right)^2 + m_2 \left(-\frac{a}{2}\right)^2$$

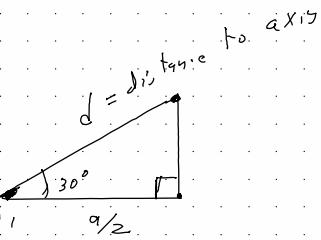
$$= \boxed{\frac{m_1 a^2}{2}}$$

$$I_3 = I_1 + I_2$$

$$= \boxed{\frac{2m_1 m_2 h^2 + m_1 a^2}{\mu}}$$



tetrahedron, height  $h$   
base: equilateral  $\triangle$  with  
side length  $a$



$$\cos 30^\circ = \frac{a}{2d} = \frac{\sqrt{3}}{2}$$

$$\rightarrow d = \frac{a}{\sqrt{3}}$$

COM lies on axis of symmetry ( $Z$ -axis)

Assume COM has  $Z=0$

$$\text{Then } O = m_2 Z_2 + 3m_1 Z_1, \quad Z_2 - Z_1 = h$$

$$= m_2 (Z_1 + h) + 3m_1 Z_1$$

$$= (3m_1 + m_2) Z_1 + m_2 h$$

$$\rightarrow Z_1 = \frac{-m_2 h}{3m_1 + m_2} = -\frac{m_2 h}{\mu}$$

$$Z_2 = h + Z_1$$

$$= h - \frac{m_2 h}{\mu}$$

$$= \frac{3m_1 h}{\mu}$$

Since a tetrahedron has 3-fold rotational symmetry,  
the  $x_1$  principal axes can be chosen  
arbitrarily in the plane  $\perp$  to the  
symmetry axis ( $x_3 \equiv Z$ ). [ $x_2$  is  $\perp$  to  $x_1, x_3$ ]

thus,  $I_1 = I_2 \equiv I$

$$I_3 = \sum_a m_a (r_a^2 - z_a^2)$$

$$= \sum_a m_a s_a^2 \quad \text{where } s^2 = r^2 - z^2$$

$$I_1 = \sum_a m_a (r_a^2 - x_a^2)$$

$$I_2 = \sum_a m_a (r_a^2 - y_a^2) \quad \Rightarrow \text{equal} (I_1 = I_2 \equiv I)$$

$$2I = I_1 + I_2$$

$$= \sum_a m_a (2r_a^2 - x_a^2 - y_a^2)$$

$$= \sum_a m_a (2(s_a^2 + z_a^2) - s_a^2)$$

$$= \sum_a m_a s_a^2 + 2 \sum_a m_a z_a^2$$

$$= I_3 + 2 \sum_a m_a z_a^2$$

thus,

$$I = \frac{1}{2} I_3 + \sum_a m_a z_a^2$$

Now

$$I_3 = \sum_a m_a s_a^2$$

$$= 3m_1 d^2$$

$$= 3m_1 \frac{a^2}{4} = \boxed{m_1 a^2}$$

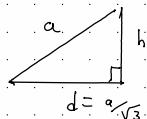
A 1/20,

$$\begin{aligned}\sum_q m_q z_q^2 &= 3m_1 z_1^2 + m_2 z_2^2 \\&= 3m_1 \left(-\frac{m_2 h}{M}\right)^2 + m_2 \left(\frac{3m_1 h}{M}\right)^2 \\&= \frac{3m_1 m_2^2 h^2}{M^2} + \frac{9m_1^2 m_2 h^2}{M} \\&= \frac{3m_1 m_2 h^2}{M^2} \underbrace{(m_2 + 3m_1)}_{M} \\&= \frac{3m_1 m_2 h^2}{M}\end{aligned}$$

Thus,

$$\begin{aligned}I &= \frac{l}{2} I_3 + \sum_q m_q z_q^2 \\&= \boxed{\frac{1}{2} m_1 q^2 + \frac{3m_1 m_2 h^2}{M}}\end{aligned}$$

Regular tetrahedron:  $m_1 = m_2$

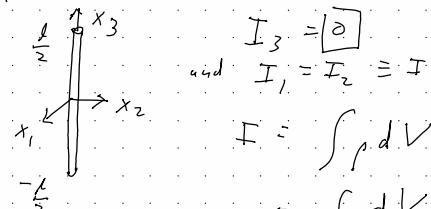


$$\begin{aligned}h^2 + \frac{a^2}{3} &= a^2 \rightarrow h = \sqrt{\frac{2}{3}} a \\M &= 4m_1\end{aligned}$$

$$\begin{aligned}I_3 &= m_1 q^2 \\I &= \frac{1}{2} m_1 q^2 + \frac{3m_1 m_1}{4m_1} \left(\frac{\sqrt{2}}{3} a\right)^2 \\&= m_1 q^2 (= I_1 = I_2)\end{aligned}\quad \left.\begin{array}{l} \text{so } I_1 = I_2 \\ = I_3 = m_1 q^2 \end{array}\right\}$$

Sec 32, Prob 2:

(a) Thin rod of length  $l$ :



$$I = \int \rho dV (r^2 - x^2)$$

$$= \int \rho dV z^2$$

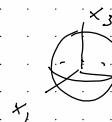
$$= \int dz \left(\frac{M}{l}\right) z^2$$

$$= \frac{M}{l} \frac{z^3}{3} \Big|_{-l/2}^{l/2}$$

$$= \frac{M}{l} \frac{2}{3} \frac{l^3}{8}$$

$$= \boxed{\frac{1}{12} M l^2}$$

(b) Sphere of radius  $R$ :



$$I_1 = I_2 = I_3 = I$$

$$I = \frac{1}{3}(I_1 + I_2 + I_3)$$

$$= \frac{1}{3} \left[ \int \rho dV (r^2 - x^2) + \int \rho dV (r^2 - y^2) \right]$$

$$+ \int \rho dV (r^2 - z^2) \right]$$

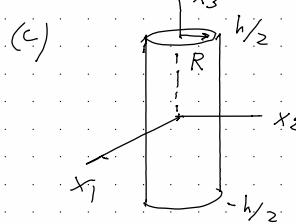
$$= \frac{1}{3} \int \rho dV [3r^2 - x^2 - y^2 - z^2]$$

$$= \frac{2}{3} \int \rho dV r^2$$

$$\begin{aligned} I &= \frac{2}{3} \int \rho dV \cdot r^2 \\ &= \frac{2}{3} \frac{M}{4\pi R^3} \int_0^R r^4 dr \int_0^\pi \int_0^{2\pi} \rho ds d\theta d\phi \\ &= \frac{M}{2\pi R^3} \cdot \frac{4\pi}{3} \int_0^R r^4 dr \end{aligned}$$

$$= \frac{2M}{R^3} \cdot \frac{R^5}{5}$$

$$= \boxed{\frac{2}{5} M R^2}$$



$$\rho = \frac{M}{\pi R^2 \cdot h}$$

$$dV = ds \, d\phi \, dz$$

where  $s^2 = x^2 + y^2$

$$I_1 = I_2 = I$$

$$2I = I_1 + I_2$$

$$= \int \rho dV (r^2 - x^2) + \int \rho dV (r^2 - y^2)$$

$$= \int \rho dV (2r^2 - s^2)$$

$$= \int \rho dV (2(s^2 + z^2) - s^2)$$

$$= \int \rho dV s^2 + 2 \int \rho dV z^2$$

$$\rightarrow I = \frac{1}{2} I_3 + \int \rho dV \cdot z^2$$

$$\rho = \frac{M}{\frac{4}{3} \pi R^3}$$

$$\begin{aligned} I_3 &= \int \rho dV \cdot z^2 \\ &= \frac{M}{\pi R^2 h} \int_0^R s^3 ds \int_0^\pi \int_{-h/2}^{h/2} dz \\ &= \frac{M}{\pi R^2 h} \cdot \frac{R^4}{4} \cdot 2\pi \cdot h \\ &= \boxed{\frac{1}{2} M R^2} \end{aligned}$$

$$\begin{aligned} I &= \frac{1}{2} I_3 + \int \rho dV \cdot z^2 \\ \int \rho dV \cdot z^2 &= \frac{M}{\pi R^2 h} \int_0^R s^3 ds \int_0^\pi \int_{-h/2}^{h/2} z^2 dz \\ &= \frac{M}{\pi R^2 h} \cdot \frac{R^4}{4} \cdot \frac{2\pi}{3} \cdot \frac{z^3}{3} \Big|_{-h/2}^{h/2} \\ &= \frac{M}{h} \cdot \frac{2}{3} \cdot \frac{h}{8} \\ &= \frac{1}{12} M h^2 \end{aligned}$$

Thus,

$$\begin{aligned} I &= \frac{1}{2} \left( \frac{1}{2} M R^2 \right) + \frac{1}{12} M h^2 \\ &= \frac{1}{4} M R^2 + \frac{1}{12} M h^2 \\ &= \boxed{\frac{1}{4} M \left( R^2 + \frac{1}{3} h^2 \right)} \end{aligned}$$

NOTE: special limiting case,

(i) Thin rod ( $R \rightarrow 0$ )

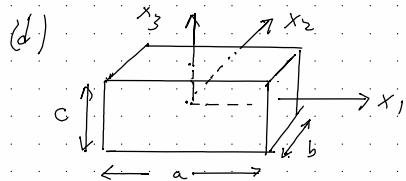
$$I_3 = 0$$

$$I_1 = I_2 = \frac{1}{12} M h^2$$

(ii) Thin disk ( $b \rightarrow 0$ )

$$I_3 = \frac{1}{2} M R^2$$

$$I_1 = I_2 = \frac{1}{4} M R^2$$



$$\rho = \frac{M}{abc}$$

$$dV = dx dy dz$$

$$I_1 = \int \rho dV (r^2 - x^2)$$

$$= \int \rho dV (y^2 + z^2)$$

$$= \frac{M}{abc} \int_{-c/2}^{c/2} dx \int_{-b/2}^{b/2} dy \int_{-a/2}^{a/2} dz (y^2 + z^2)$$

$$= \frac{M}{abc} \times \int_{-b/2}^{b/2} dy \left( y^2 z + \frac{z^3}{3} \right) \Big|_{-c/2}^{c/2}$$

$$= \frac{M}{bc} \int_{-b/2}^{b/2} dy \left( cy^2 + \frac{z^3}{3} \cdot \frac{c^3}{8} \right)$$

$$= \frac{M}{bc} \left[ c \frac{y^3}{3} \Big|_{-b/2}^{b/2} + \frac{1}{12} c^3 z \Big|_{-b/2}^{b/2} \right]$$

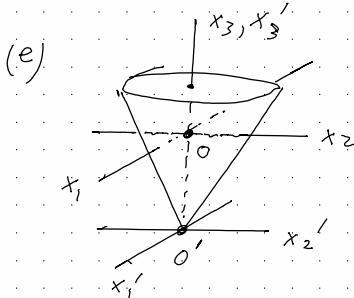
$$I_1 = \frac{\mu}{bc} \left[ c \frac{b^3}{3} + \frac{1}{12} b c^3 \right]$$

$$= \frac{M}{12 bc} \left[ c b^3 + b c^3 \right]$$

$$= \frac{M}{12} (b^2 + c^2)$$

$$I_2 = \frac{M}{12} (c^2 + a^2)$$

$$I_3 = \frac{M}{12} (a^2 + b^2)$$



First calculate

$$I_{ij}' \text{ (wrt } x_1', x_2', x_3') \text{.}$$

Then calculate  $I_{ij}$  via

$$I_{ij} = I_{ij}' - \mu (\vec{a}^2 f_{ij} - a_i a_j)$$

where  $\vec{a} = (0, 0, -d)$

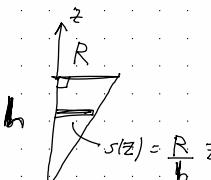
$d$ : height of com above  $O'$

Volume of cone:

$$V = \int dV$$

$$= \int_0^h dz \int_0^{2\pi} d\phi \int_0^R s ds$$

$$= \frac{1}{3}\pi \int_0^h dz \frac{s^2}{2} \Big|_0^R$$



$$V = \pi \int_0^h dz \frac{R^2 z^2}{h^2}$$

$$= \frac{\pi R^2}{h^2} \frac{z^3}{3} \Big|_0^h$$

$$= \frac{\pi R^2}{h^2} \frac{h^3}{3}$$

$$= \boxed{\frac{1}{3} \pi R^2 h}$$

$$\text{Thus, } \rho = \frac{M}{V} = \boxed{\frac{M}{\frac{1}{3} \pi R^2 h}} \quad (\text{mass density})$$

$$I_3' = \int \rho dV (r^2 - z^2)$$

$$= \int \rho dV r^2$$

$$= \rho \int_0^h dz \int_0^{2\pi} d\phi \int_0^{R^2/h} r^3 dr$$

$$= \frac{\rho}{4} \int_0^h dz r^4 \Big|_{0}^{R^2/h}$$

$$= \frac{\rho \pi}{2} \int_0^h dz \left( \frac{R^4}{h^4} \right) z^4$$

$$= \frac{\rho \pi R^4}{2 h^4} \frac{h^5}{5}$$

$$= \boxed{\frac{1}{5} \pi R^4 h^5}$$

thus,

$$I_3' = \frac{M}{\frac{1}{3} \pi R^2 h} \frac{\pi R^4 h}{10}$$

$$= \boxed{\frac{3}{10} M R^2}$$

similar to the cylinder, we have

$$I_1' = I_2' \equiv I' \text{ where}$$

$$I' = \frac{1}{2} I_3' + \int \rho dV z^2$$

$$\text{Now: } \int \rho dV z^2 = \rho \int_0^h dz z^2 \int_0^{2\pi} d\phi \int_0^{R^2/h} r^3 dr$$

$$= \frac{1}{2} \pi \rho \int_0^h dz z^2 \frac{r^5}{5} \Big|_0^{R^2/h}$$

$$= \pi \rho \frac{R^2}{h^2} \int_0^h dz z^4$$

$$= \pi \rho \frac{R^2}{h^2} \frac{h^5}{5}$$

$$= \frac{\pi}{5} \rho R^2 h^3$$

$$= \frac{\pi}{5} \left( \frac{M}{\frac{1}{3} \pi R^2 h} \right) R^2 h^3$$

$$= \boxed{\frac{3}{5} M h^2}$$

so

$$\begin{aligned} I' &= \frac{1}{2} \left( \frac{3}{10} \mu R^2 \right) + \frac{3}{5} \mu h^2 \\ &= \boxed{\frac{3}{5} \mu \left( \frac{R^2}{4} + h^2 \right)} = I_1' = I_2' \end{aligned}$$

Need to find location of COM.

$$\begin{aligned} d &= \frac{1}{\mu} \int \rho dV z \\ &= \frac{1}{\mu} \rho \int_0^h z dz \int_0^{2\pi} d\phi \int_0^{Rz/h} r dr \\ &= \frac{1}{\mu} \rho \cdot \cancel{2\pi} \int_0^h z dz \frac{1}{2} \left( \frac{R}{h} \right)^2 z^2 \\ &= \frac{\pi \rho}{\mu} \frac{R^2}{h^2} \frac{z^4}{4} \Big|_0^h \\ &= \frac{\pi \rho}{4\mu} R^2 h^2 \\ &= \frac{\pi \rho}{4\mu} \frac{R^2 h^2}{\cancel{\pi} R^2 h} \\ &= \boxed{\frac{3}{4} h} \end{aligned}$$

Thus,

$$I_{ij}' = I_{ij}' - \mu (\vec{a}^2 \delta_{ij} - a_i a_j)$$

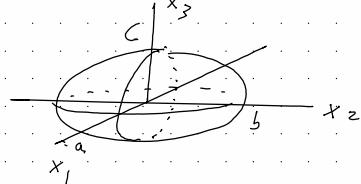
where  $\vec{a} = (0, 0, -\frac{3}{4} h) \rightarrow a^2 = \frac{9}{16} h^2$

$$\begin{aligned} \rightarrow I_1 &= I_1' - \mu a^2 \\ &= \frac{3}{5} \mu \left( \frac{R^2}{4} + h^2 \right) - \mu \frac{9}{16} h^2 \\ &= \frac{3}{20} \mu R^2 + \mu h^2 \left( \frac{3}{5} - \frac{9}{16} \right) \\ &\quad \cancel{\frac{48-45}{80}} = \frac{3}{80} \\ &= \boxed{\frac{3}{20} \mu \left( R^2 + \frac{h^2}{4} \right)} \end{aligned}$$

Also,  $I_2 = I_1$ .

$$\begin{aligned} \text{Finally, } I_3 &= I_3' - \mu (\vec{a}^2 - a^2) \\ &= I_3' \\ &= \boxed{\frac{3}{10} \mu R^2} \end{aligned}$$

(f) Ellipsoid with semi-axes  $a, b, c$



$$(a, b, c) \leftrightarrow (x_1, x_2, x_3)$$

Define rescaled coordinates:

$$(u, v, w) \equiv \left( \frac{x_1}{a}, \frac{x_2}{b}, \frac{x_3}{c} \right)$$

so flat boundary of ellipsoid

$$1 = \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 + \left(\frac{x_3}{c}\right)^2 = u^2 + v^2 + w^2$$

unit 2-sphere.

Volumes:

$$V = \int dx_1 \int dx_2 \int dx_3$$

$$= abc \int du \int dv \int dw$$

$$= abc \int d\phi \int \sin\theta d\theta \int r^2 dr$$

$$= abc \cdot 2\pi \cdot 2 \cdot \frac{r^3}{3} \Big|_0^1$$

$$= \boxed{\frac{4}{3}\pi abc}$$

$$\rightarrow \rho = \frac{\mu}{\frac{4}{3}\pi abc}$$

$$I_3 = \int \rho dV (r^2 - z^2)$$

$$= \int \rho dV (x^2 + y^2)$$

$$= \frac{\mu}{\frac{4}{3}\pi abc} \iiint dxdydz (x^2 + y^2)$$

$$= \frac{\mu}{\frac{4}{3}\pi abc} abc \iiint du dv dw (a^2 u^2 + b^2 v^2)$$

$$= \frac{\mu}{\frac{4}{3}\pi} \int r^2 dr \int_{\sin\theta}^1 \int_0^{2\pi} d\phi (a^2 r^2 \sin^2\theta \cos^2\phi + b^2 r^2 \sin^2\theta \sin^2\phi)$$

$$\text{Now: } \int_{-1}^1 \int_0^1 \int_0^{2\pi} d\phi d\theta \sin^2\theta = \int d(\cos\theta) (1 - \cos^2\theta)$$

$$= \int_{-1}^1 dx (1 - x^2)$$

$$= \left(x - \frac{x^3}{3}\right) \Big|_{-1}^1$$

$$= 2 \cdot \frac{2}{3} = \boxed{\frac{4}{3}}$$

$$\int_0^1 r^4 dr = \frac{r^5}{5} \Big|_0^1 = \boxed{\frac{1}{5}}$$

$$\int_0^{2\pi} d\phi \left\{ \frac{\sin^2 \phi}{\cos^2 \phi} \right\} \approx 2\pi \cdot \frac{1}{2} = [\pi]$$

Thus,

$$I_3 = \frac{M}{\frac{4}{5}\pi} \left( a^2 \frac{4}{3} \cdot \frac{1}{5} \cdot \pi + b^2 \frac{4}{3} \cdot \frac{1}{5} \cdot \pi \right)$$

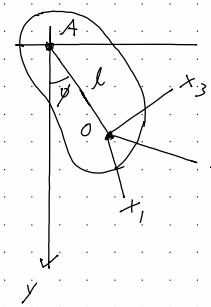
$$= \boxed{\frac{M}{5} (a^2 + b^2)}$$

Cyclically permuting  $a, b, c \rightarrow$

$$\boxed{I_1 = \frac{M}{5} (b^2 + c^2)}$$

$$\boxed{I_2 = \frac{M}{5} (c^2 + a^2)}$$

Sec 32, Prob 3:



com. at  $\vec{\theta}$   
rotation,  $\vec{x}$  is at A, out of page

$$\vec{\Omega} = \dot{\phi} \hat{n}$$

$$U = \mu g l (1 - \cos \phi)$$

$$\approx \frac{1}{2} \mu g l \dot{\phi}^2 \text{ for } \dot{\phi} \ll 1$$

$$L = T - U$$

$$T = \frac{1}{2} M V^2 + \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$$

$$\vec{\Omega} = \dot{\phi} \hat{n} \rightarrow \Omega_1 = \dot{\phi} \hat{n} \cdot \vec{x}_1 = \dot{\phi} \cos \alpha$$

$$\Omega_2 = \dot{\phi} \hat{n} \cdot \vec{x}_2 = \dot{\phi} \cos \beta$$

$$\Omega_3 = \dot{\phi} \hat{n} \cdot \vec{x}_3 = \dot{\phi} \cos \gamma$$

$$V = \mu \phi$$

thus,

$$T = \frac{1}{2} M \dot{\phi}^2 + \frac{1}{2} \dot{\phi}^2 (I_1 \cos^2 \alpha + I_2 \cos^2 \beta + I_3 \cos^2 \gamma)$$

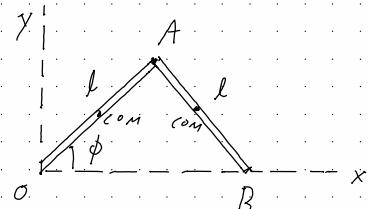
so

$$L = \frac{1}{2} \dot{\phi}^2 / M \dot{\phi}^2 + I_1 \cos^2 \alpha + I_2 \cos^2 \beta + I_3 \cos^2 \gamma$$

$$- \frac{1}{2} \mu g l \dot{\phi}^2$$

$$\rightarrow w = \sqrt{\frac{\mu g l}{M \dot{\phi}^2 + I_1 \cos^2 \alpha + I_2 \cos^2 \beta + I_3 \cos^2 \gamma}}$$

Sec 32, Prob 4



two uniform rods

$$I_{com} = \frac{1}{12} M l^2$$

$$I_{end} = \frac{1}{3} M l^2$$

$$T = T_1 + T_2$$

$$\begin{aligned} T_1 &= \frac{1}{2} M \left(\frac{l}{2}\right)^2 \dot{\phi}^2 + \frac{1}{2} I_{com} \dot{\phi}^2 \\ &= \frac{1}{8} M l^2 \dot{\phi}^2 + \frac{1}{24} M l^2 \dot{\phi}^2 \\ &= \left(\frac{1}{8} + \frac{1}{24}\right) M l^2 \dot{\phi}^2 \\ &= \frac{1}{6} M l^2 \dot{\phi}^2 \end{aligned}$$

$$T_2 = \frac{1}{2} M V^2 + \frac{1}{2} I_{com} \dot{\phi}^2$$

$$\text{Now: } V^2 = \dot{x}^2 + \dot{y}^2$$

$$\dot{x} = \frac{3}{2} l \cos\phi, \quad \dot{y} = \frac{l}{2} \sin\phi$$

$$\dot{x} = -\frac{3}{2} l \sin\phi \dot{\phi}, \quad \dot{y} = \frac{l}{2} \cos\phi \dot{\phi}$$

$$\rightarrow V^2 = \frac{9}{4} l^2 \sin^2\phi \dot{\phi}^2 + \frac{l^2}{4} \cos^2\phi \dot{\phi}^2$$

$$= 2 l^2 \sin^2\phi \dot{\phi}^2 + \frac{l^2}{4} \dot{\phi}^2$$

$$= 2 l^2 \dot{\phi}^2 (\sin^2\phi + \frac{1}{4})$$

SD

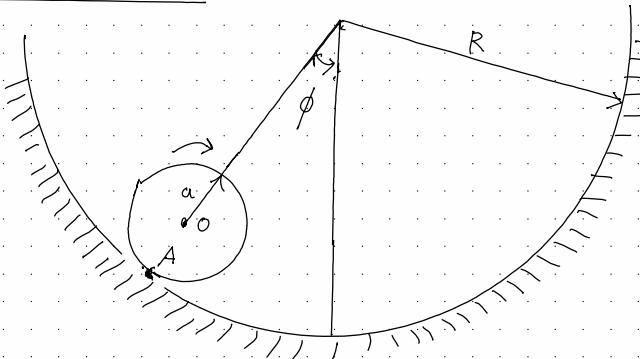
$$\begin{aligned} T_2 &= \frac{1}{2} M l^2 \dot{\phi}^2 \left( \sin^2\phi + \frac{1}{8} \right) + \frac{1}{24} M l^2 \dot{\phi}^2 \\ &= M l^2 \dot{\phi}^2 \left( \sin^2\phi + \frac{1}{8} + \frac{1}{24} \right) \\ &= M l^2 \dot{\phi}^2 \left( \sin^2\phi + \frac{1}{6} \right) \end{aligned}$$

Thus,

$$T = T_1 + T_2$$

$$\begin{aligned} &= \frac{1}{6} M l^2 \dot{\phi}^2 + M l^2 \dot{\phi}^2 \left( \sin^2\phi + \frac{1}{6} \right) \\ &= M l^2 \dot{\phi}^2 \left( \frac{1}{3} + \sin^2\phi \right) \\ &= \frac{1}{3} M l^2 \dot{\phi}^2 (1 + 3 \sin^2\phi) \end{aligned}$$

Sec 32, Prob 6:



Homogeneous cylinder of radius  $a$ , mass  $M$ :

$$I_3 = \frac{1}{2}Ma^2 \quad (\text{about com})$$

$$V = -(R-a)\dot{\phi} \quad (\text{velocity of com})$$

Instantaneous axis of rotation at A:

$$\omega = \vec{V} + \vec{\omega} + (-ah\hat{i}), \quad \vec{\omega} \text{ into page}$$

$$\text{so } V = \Omega a$$

$$\text{Thus, } \Omega a = (R-a)/\dot{\phi}$$

$$\Omega = \left(\frac{R-a}{a}\right)/\dot{\phi}$$

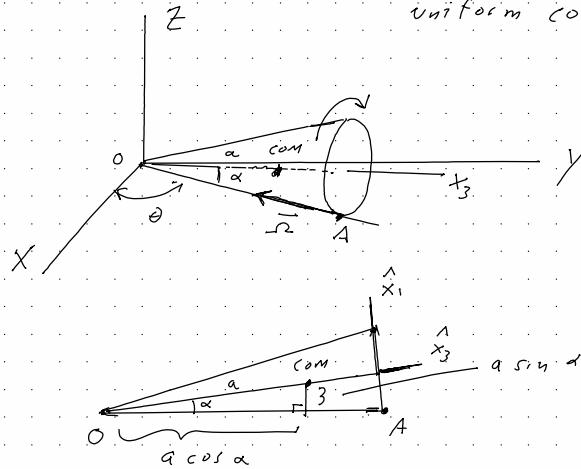
$$T = \frac{1}{2}MV^2 + \frac{1}{2}I_3\Omega^2$$

$$= \frac{1}{2}M\left(\frac{R-a}{a}\right)^2a^2\dot{\phi}^2 + \frac{1}{4}Ma^2\left(\frac{R-a}{a}\right)^2\dot{\phi}^2$$

$$= M(R-a)^2\dot{\phi}^2\left(\frac{1}{2} + \frac{1}{4}\right) = \boxed{\frac{3}{4}M(R-a)^2\dot{\phi}^2}$$

Sec 32, Prob 7.

radius  $R$ , height  $h$   
uniform cone



$$V = \text{velocity of com}$$

$$= a \cos \alpha \dot{\theta}$$

OA: instantaneous axis of rotation

$$\omega = \vec{V} + a \sin \alpha \vec{\Omega} \times \hat{h} \quad (\hat{h} = \hat{z})$$

$$\omega = V - a \sin \alpha \Omega \hat{z}$$

$$\text{thus, } a \sin \alpha \Omega = V = a \cos \alpha \dot{\theta}$$

$$\boxed{\Omega = \cot \alpha \dot{\theta}}$$

$\vec{\omega}$ : directed from A to O

$$\Omega_3 = -\Omega \cos \alpha = -\frac{\cos^2 \alpha}{\sin \alpha} \dot{\theta}$$

$$\Omega_1 = \Omega \sin \alpha = \cos \alpha \dot{\theta}$$

Thur,

$$T = \frac{1}{2} \mu V^2 + \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$$
$$= \frac{1}{2} \mu h^2 \cos^2 \alpha \dot{\theta}^2 + \frac{1}{2} \left( I_1 \cos^2 \alpha \dot{\theta}^2 + I_3 \frac{\cos^4 \alpha}{\sin^2 \alpha} \dot{\theta}^2 \right)$$

$$\text{Now: } I_1 = I_2 = \frac{3}{20} \mu / R^2 + \frac{1}{4} h^2$$

$$I_3 = \frac{3}{10} \mu R^2$$

$$\text{Also: } \alpha = \frac{3}{4} b$$

$$\tan \alpha = \frac{R}{h} \rightarrow R = h \tan \alpha$$

Thur,

$$T = \frac{1}{2} \mu \left( \frac{9}{16} \right) h^2 \cos^2 \alpha \dot{\theta}^2 + \frac{1}{2} \left[ \frac{3}{20} \mu \left( h^2 \tan^2 \alpha + \frac{1}{4} h^2 \right) \cos^2 \alpha \dot{\theta}^2 + \frac{3}{10} \mu h^2 \tan^2 \alpha \frac{\cos^4 \alpha}{\sin^2 \alpha} \dot{\theta}^2 \right]$$

$$= \mu h^2 \dot{\theta}^2 \left[ \frac{9}{32} \cos^2 \alpha + \frac{3}{40} \overbrace{\sin^4 \alpha}^{(1-\cos^2 \alpha)} + \frac{3}{160} \cos^2 \alpha \right] + \frac{3}{20} \cos^2 \alpha \dot{\theta}^2$$

$$= \mu h^2 \dot{\theta}^2 \left[ \frac{3}{40} + \cos^2 \alpha \left( \frac{9}{32} - \frac{3}{40} + \frac{3}{160} + \frac{3}{20} \right) \right]$$

$$\text{Now: } \frac{9}{32} - \frac{3}{40} + \frac{3}{160} + \frac{3}{20}$$

$$= \frac{1}{160} [45 - 12 + 3 + 24]$$

$$= \frac{60}{160}$$

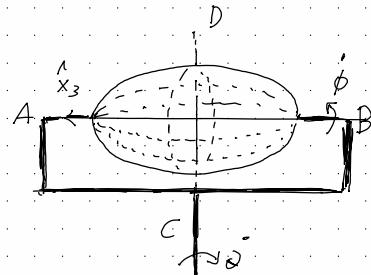
$$= \frac{15}{40}$$

Thur,

$$T = \mu h^2 \dot{\theta}^2 \left[ \frac{3}{40} + \frac{15}{40} \cos^2 \alpha \right]$$

$$= \boxed{\frac{3}{40} \mu h^2 \dot{\theta}^2 \left[ 1 + 5 \cos^2 \alpha \right]}$$

Sec 32, Prob 9.



homogeneous ellipsoid  
with principal  
moments of  
inertia  $I_1, I_2, I_3$

$$\vec{\Omega} = \dot{\phi} + \vec{\theta}$$

$$\text{Now: } \vec{\phi} = \dot{\phi} \hat{x}_3$$

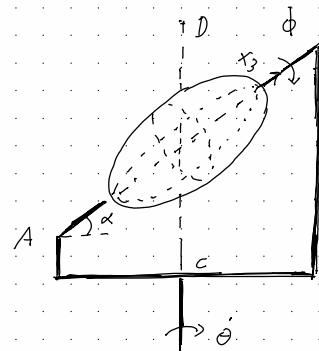
$$\vec{\theta} = \dot{\theta} [\cos \phi \hat{x}_1 + \sin \phi \hat{x}_2]$$

$$\text{so: } \vec{\Omega} = \dot{\theta} \cos \phi \hat{x}_1 + \dot{\theta} \sin \phi \hat{x}_2 + \dot{\phi} \hat{x}_3$$

$$T = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$$

$$= \frac{1}{2} [(I_1 \cos^2 \phi + I_2 \sin^2 \phi) \dot{\theta}^2 + I_3 \dot{\phi}^2]$$

Sec 32, Prob 10.



uniform ellipsoid  
with  $I_1 = I_2$   
(circular cross section)

$$\vec{\Omega} = \vec{\phi} + \vec{\theta}$$

$$\vec{\phi} = \dot{\phi} \hat{x}_3$$

$$\vec{\theta} = \dot{\theta} [\cos(\frac{\pi}{2} - \alpha) \hat{x}_3 + \sin(\frac{\pi}{2} - \alpha) (\cos \phi \hat{x}_1 + \sin \phi \hat{x}_2)]$$

$$\text{Now: } \cos(\frac{\pi}{2} - \alpha) = \cos(\frac{\pi}{2}) \cos \alpha + \sin(\frac{\pi}{2}) \sin \alpha \\ = \sin \alpha$$

$$\sin(\frac{\pi}{2} - \alpha) = \sin(\frac{\pi}{2}) \cos \alpha - \cos(\frac{\pi}{2}) \sin \alpha \\ = \cos \alpha$$

$$\text{so: } \vec{\theta} = \dot{\theta} [\sin \alpha \hat{x}_3 + \cos \alpha \cos \phi \hat{x}_1 + \cos \alpha \sin \phi \hat{x}_2]$$

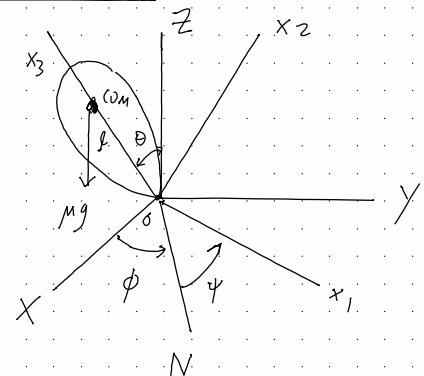
$$\rightarrow \vec{\Omega} = \dot{\theta} \cos \alpha (\cos \phi \hat{x}_1 + \sin \phi \hat{x}_2) + (\dot{\phi} + \dot{\theta} \sin \alpha) \hat{x}_3$$

$$T = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$$

$$= \frac{1}{2} [I_1 \dot{\theta}^2 \cos^2 \alpha (\cos^2 \phi + \sin^2 \phi) + I_3 (\dot{\phi} + \dot{\theta} \sin \alpha)^2]$$

$$= \frac{1}{2} [I_1 \cos^2 \alpha \dot{\theta}^2 + I_3 (\dot{\phi} + \dot{\theta} \sin \alpha)^2]$$

Sec 35, prob 1:



Symmetrical top:

$$I_1 = I_2, \quad I_3 \quad (\text{w.r.t. principal axes passing through COM})$$

$$I'_1 = I_1 + ml^2 \quad (\text{w.r.t. axes passing through}$$

$$I'_2 = I'_1 \quad O, \text{ which is displaced}$$

$$I'_3 = I_3 \quad \text{from the COM by } l \text{ in the } -x_3 \text{ direction)$$

$$L = T - U$$

$$U = mgz = mgl\cos\theta$$

$$T = \frac{1}{2} (I'_1 \Omega_1^2 + I'_2 \Omega_2^2 + I'_3 \Omega_3^2)$$

$$= \frac{1}{2} [I'_1 (\Omega_1^2 + \Omega_2^2) + I'_3 \Omega_3^2]$$

Now: (From (35.1))

$$\Omega_1 = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi$$

$$\Omega_2 = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi$$

$$\Omega_3 = \dot{\phi} \cos \theta + \dot{\psi}$$

$$\therefore \Omega_3^2 = (\dot{\phi} \cos \theta + \dot{\psi})^2$$

$$\Omega_1^2 = \dot{\phi}^2 \sin^2 \theta \sin^2 \psi + \dot{\theta}^2 \cos^2 \psi + 2\dot{\theta}\dot{\phi} \sin \theta \sin \psi \cos \psi$$

$$\Omega_2^2 = \dot{\phi}^2 \sin^2 \theta \cos^2 \psi + \dot{\theta}^2 \sin^2 \psi - 2\dot{\theta}\dot{\phi} \sin \theta \cos \psi \sin \psi$$

$$\rightarrow \Omega_1^2 + \Omega_2^2 = \dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2$$

Thus,

$$L = \frac{1}{2} [I'_1 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + I'_3 (\dot{\phi} \cos \theta + \dot{\psi})^2] - mg l \cos \theta$$

1) No explicit  $t$ -dependence

$$E = T + U = \text{const}$$

2) No explicit  $\phi$  dependence:

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = I'_1 \sin^2 \theta \dot{\phi} + I'_3 (\dot{\phi} \cos \theta + \dot{\psi}) \cos \theta = \text{const}$$

3) No explicit  $\psi$  dependence:

$$p_\psi \equiv \frac{\partial L}{\partial \dot{\psi}} = I'_3 (\dot{\phi} \cos \theta + \dot{\psi}) = \text{const}$$

We can solve  $P_\phi$ ,  $P_\psi$  for  $\dot{\phi}$ ,  $\dot{\psi}$ :

$$P_\phi = I_1' \sin^2 \theta \dot{\phi} + I_3 (\phi \cos \theta + \psi) \cos \theta$$

$$P_\psi = I_3 (\dot{\phi} \cos \theta + \dot{\psi})$$

$$\rightarrow P_\phi = I_1' \sin^2 \theta \dot{\phi} + P_\psi \cos \theta$$

$$\text{so } I_1' \sin^2 \theta \dot{\phi} = P_\phi - P_\psi \cos \theta$$

$$\boxed{\dot{\phi} = \frac{P_\phi - P_\psi \cos \theta}{I_1' \sin^2 \theta}}$$

A4J:

$$\frac{P_\psi}{I_3} = \phi \cos \theta + \psi$$

$$\rightarrow \boxed{\dot{\psi} = \frac{P_\psi}{I_3} - \dot{\phi} \cos \theta}$$

$$= \frac{P_\psi}{I_3} - \left( \frac{P_\phi - P_\psi \cos \theta}{I_1' \sin^2 \theta} \right) \cos \theta$$

A10:

$$E = T + U$$

$$= \frac{1}{2} [ I_1' (\sin^2 \theta \dot{\phi}^2 + \dot{\theta}^2) + I_3 (\dot{\phi} \cos \theta + \dot{\psi})^2 ]$$

$$+ \mu g l \cos \theta$$

can be rewr. Then as

$$E = \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2} I_1' \sin^2 \theta \dot{\phi}^2$$

$$+ \frac{1}{2} I_3 \left( \frac{P_\psi}{I_3} \right)^2 + \mu g l \cos \theta$$

$$= \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2} I_1' \sin^2 \theta \frac{(P_\phi - P_\psi \cos \theta)^2}{I_1' \sin^4 \theta}$$

$$+ \frac{1}{2} \frac{P_\psi^2}{I_3} + \mu g l \cos \theta$$

$$= \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2} \frac{P_\psi^2}{I_3} + \frac{1}{2} \frac{(P_\phi - P_\psi \cos \theta)^2}{I_1' \sin^2 \theta}$$

$$+ \mu g l \cos \theta$$

$$\underline{\text{Now: }} \frac{1}{2} \frac{P_\psi^2}{I_3} = \text{const}$$

$$\underline{\text{Also: }} \mu g l \cos \theta = -\mu g l (1 - \cos \theta) + \mu g l$$

so

$$E - \frac{1}{2} \frac{P_\psi^2}{I_3} - \mu g l = \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2} \frac{(P_\phi - P_\psi \cos \theta)^2}{I_1' \sin^2 \theta}$$

$$- \mu g l (1 - \cos \theta)$$

$$\rightarrow E' = \frac{1}{2} I_1' \dot{\theta}^2 + U_{\text{eff}}(\theta)$$

$$\text{where } \boxed{U_{\text{eff}}(\theta) = \frac{1}{2} \frac{(P_\phi - P_\psi \cos \theta)^2}{I_1' \sin^2 \theta} - \mu g l (1 - \cos \theta)}$$

EOM:

$$\dot{E}' = \frac{1}{2} I_1' \dot{\theta}^2 + U_{\text{eff}}(\theta)$$

$$\dot{\theta}^2 = \frac{2}{I_1'} (E' - U_{\text{eff}}(\theta))$$

$$\frac{d\theta}{dt} = \sqrt{\frac{2}{I_1'} (E' - U_{\text{eff}}(\theta))}$$

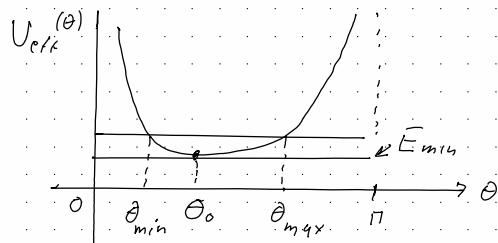
$$\rightarrow dt = \int \frac{d\theta}{\sqrt{\frac{2}{I_1'} (E' - U_{\text{eff}}(\theta))}}$$

$$t = \int \frac{d\theta}{\sqrt{\frac{2}{I_1'} (E' - U_{\text{eff}}(\theta))}} + \text{const}$$

Effective potential:

$$U_{\text{eff}}(\theta) = \frac{1}{2} \frac{(P_\phi - P_\gamma \cos \theta)^2}{I_1' \sin^2 \theta} - mgI(1 - \cos \theta)$$

For  $P_\phi \neq P_\gamma$ ,  $U_{\text{eff}}(\theta) \rightarrow \infty$  as  $\theta \rightarrow 0, \pi$



To find  $\theta_0$ :

$$\dot{\theta} = \frac{dU_{\text{eff}}}{d\theta} \Big|_{\theta_0}$$

$$= \frac{(P_\phi - P_\gamma \cos \theta_0) P_\gamma \sin \theta_0 - (P_\phi - P_\gamma \cos \theta_0) \omega \theta_0}{I_1' \sin^3 \theta_0}$$

$$- \mu g l \sin \theta_0$$

$$= \beta \frac{P_\gamma}{I_1' \sin \theta_0} - \beta^2 \frac{\omega \theta_0}{I_1' \sin^3 \theta_0} - \mu g l \sin \theta_0$$

Multiply through by  $-I_1' \sin^3 \theta_0$ :

$$\dot{\theta} = \beta^2 \cos \theta_0 - \beta P_\gamma \sin^2 \theta_0 + \mu g l I_1' \sin^4 \theta_0$$

Quadratic equation for  $\dot{\theta}$ :

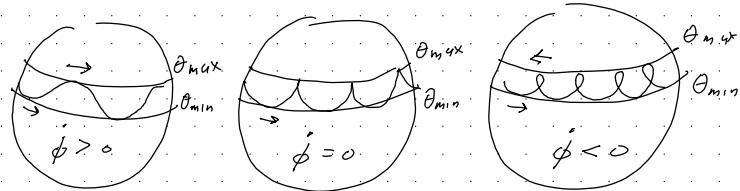
$$\beta \dot{\theta} = \frac{P_\gamma \sin^2 \theta_0 \pm \sqrt{P_\gamma^2 \sin^4 \theta_0 - 4 \mu g l I_1' \sin^4 \theta_0 \cos \theta_0}}{2 \cos \theta_0}$$

$$= \frac{P_\gamma \sin^2 \theta_0}{2 \cos \theta_0} \left( 1 \pm \sqrt{1 - \frac{4 \mu g l I_1' \cos \theta_0}{P_\gamma^2}} \right)$$

need to be  $\geq 0$  for a real solution.

The last equation is a transcendental equation for  $\theta_0$  since  $P = P\phi - P\psi \cos \theta$

For  $E > E_{\min}$ ,  $\theta$  varies between  $\theta_{\min}$  and  $\theta_{\max}$ . The motion of the  $x_3$ -axis of the top can have the following three forms depending on the sign of  $\dot{\phi}$  when  $\theta = \theta_{\max}$ .



This motion is called nutation.

### Sec 3.5, Prob 2

For rotation of a top around a vertical axis, to be stable, we need  $\frac{d^2U_{eff}}{d\theta^2} |_{\theta=0} > 0$

Now:

$$U_{eff}(\theta) = \frac{(P\phi - P\psi \cos \theta)^2}{2 I_1' \sin^2 \theta} - \mu g l (1 - \cos \theta)$$

Also:

$$P\phi = I_1' \sin^2 \theta \dot{\phi} + I_3 (\dot{\phi} \cos \theta + \dot{\psi})_{co, \theta}$$

$$P\psi = I_3 (\dot{\phi} \cos \theta + \dot{\psi})$$

In the limit  $\theta \rightarrow 0$

$$P\phi \approx I_3 (\dot{\phi} + \dot{\psi}) \quad \text{so they are equal}$$

$$P\psi \approx I_3 (\dot{\phi} + \dot{\psi}) \quad \text{in this limit}$$

thus,

$$U_{eff}(\theta) \approx \frac{P\phi^2 (1 - \cos \theta)^2}{2 I_1' \sin^2 \theta} - \mu g l (1 - \cos \theta)$$

$$\approx \frac{P\phi^2 \left(\frac{\theta^2}{2}\right)^2}{2 I_1' \theta^2} - \mu g l \frac{\theta^2}{2}$$

$$\approx \left( \frac{1}{8} \frac{P\phi^2}{I_1'} - \frac{1}{2} \mu g l \right) \theta^2$$

Thur

$$U_{\text{eff}}(\theta) \approx \left( \frac{1}{8} \frac{P\phi}{I_1} - \frac{1}{2} M g l \right) \theta^2$$

$$\rightarrow U_{\text{eff}}(0) = 0$$

$$\frac{dU_{\text{eff}}}{d\theta} \Big|_{\theta=0} = 2 \left( \frac{1}{8} \frac{P\phi}{I_1} - \frac{1}{2} M g l \right) \theta \Big|_{\theta=0} = 0$$

$$\frac{d^2U_{\text{eff}}}{d\theta^2} \Big|_{\theta=0} = 2 \left( \frac{1}{8} \frac{P\phi}{I_1} - \frac{1}{2} M g l \right)$$

Need  $\frac{d^2U_{\text{eff}}}{d\theta^2} \Big|_{\theta=0} > 0$  for stable rotation:

$$\frac{1}{8} \frac{P\phi}{I_1} - \frac{1}{2} M g l > 0$$

$$\boxed{\frac{P\phi}{I_1} > \frac{4 M g l}{5}}$$

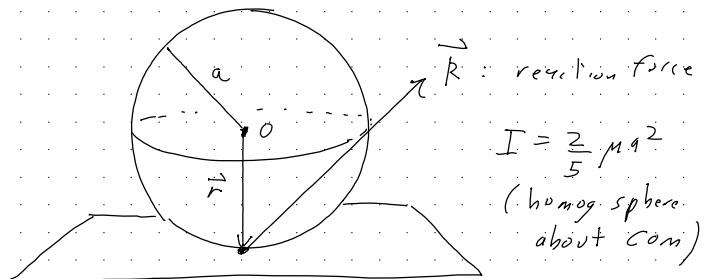
Since  $P\phi = P\psi = I_3 \Omega_3$ , we can also write this condition as

$$\frac{I_3^2 \Omega_3^2}{I_1^2} > \frac{4 M g l I_1'}{5}$$

$$\boxed{\Omega_3^2 > \frac{4 M g l I_1'}{I_3^2}}$$

Sec 38, Prob 1:

Homogeneous sphere (radius  $a$ ) rolling without slipping on a horizontal surface, subject to applied force  $\vec{F}$  and torque  $\vec{T}$ :



$$\frac{d\vec{P}}{dt} = \vec{F} + \vec{R} \rightarrow m \frac{d\vec{V}}{dt} = \vec{F} + \vec{R}$$

$$\frac{d\vec{m}}{dt} = \vec{T} + \vec{F} \times \vec{R} \rightarrow I \frac{d\vec{\Omega}}{dt} = \vec{T} - a \vec{z} \times \vec{R}$$

Rolling without slipping:

$$\vec{O} = \vec{V} + \vec{\Omega} \times \vec{r}$$

$$= \vec{V} - a \vec{\Omega} \times \vec{z}$$

$$\text{so } \vec{V} = a \vec{\Omega} \times \vec{z}$$

Using  $(\vec{A} \times \vec{B})_i = A_2 B_3 - A_3 B_2$ , etc

$$\boxed{\begin{aligned} V_x &= a \Omega_y \\ V_y &= -a \Omega_x \\ V_z &= 0 \end{aligned}}$$

no motion off surface

Combined

$$\mu \frac{d\vec{V}}{dt} = \vec{F} + \vec{R} \quad (1)$$

$$I \frac{d\vec{\alpha}}{dt} = \vec{R} - a \hat{z} \times \vec{R} \quad (2)$$

$$V_x = a \Omega_y, V_y = -a \Omega_x, V_z = 0 \quad (\text{constraint})$$

Take time derivative of constraint equation:

$$\frac{dV_x}{dt} = a \frac{d\Omega_y}{dt}, \quad \frac{dV_y}{dt} = -a \frac{d\Omega_x}{dt}$$

Substitute from (1), (2) into these two equations

$$\frac{1}{\mu} (F_x + R_x) = \frac{a}{I} (K_y - a R_x)$$

$$\frac{1}{\mu} (F_y + R_y) = -\frac{a}{I} (I K_x + a R_y)$$

Thus,

$$F_x + R_x = \frac{ma}{I} K_y - \frac{ma^2}{I} R_x$$

$$R_x \left( \frac{I + ma^2}{I} \right) = \frac{ma}{I} K_y - F_x$$

$$R_x \frac{\frac{3}{5} ma^2}{\frac{2}{5} ma^2} = \frac{ma}{\frac{2}{5} ma^2} K_y - F_x$$

$$\rightarrow \boxed{R_x = \frac{5}{7} \frac{K_y}{a} - \frac{2}{7} F_x}$$

Similarly,

$$F_y + R_y = -\frac{ma}{I} K_x - \frac{ma^2}{I} R_y$$

$$R_y \left( \frac{I + ma^2}{I} \right) = -\frac{ma}{I} K_x - F_y$$

$$R_y \frac{\frac{3}{5} ma^2}{\frac{2}{5} ma^2} = -\frac{ma}{\frac{2}{5} ma^2} K_x - F_y$$

$$\rightarrow \boxed{R_y = -\frac{5}{7} \frac{K_x}{a} - \frac{2}{7} F_y}$$

Also,

$$\mu \frac{dV_z}{dt} = F_z + R_z$$

$$\rightarrow \boxed{R_z = -F_z}$$

Using these expression for  $R_x, R_y$  we can write down EOMs for  $V_x, V_y$

$$\begin{aligned} \boxed{\mu \frac{dV_x}{dt}} &= F_x + R_x \\ &= F_x + \frac{5}{7} \frac{K_y}{a} - \frac{2}{7} F_x \\ &= \frac{5}{7} (F_x + \frac{K_y}{a}) \end{aligned}$$

Similarly

$$\begin{aligned} \boxed{m \frac{dV_x}{dt}} &= F_y + R_y \\ &= F_y - \frac{5}{7} \frac{K_x}{a} - \frac{2}{7} F_x \\ &= \frac{5}{7} \left( F_y - \frac{K_x}{a} \right) \end{aligned}$$

~~~~~

Summary:

$$R_x = \frac{5}{7} \frac{K_x}{a} - \frac{2}{7} F_x$$

$$R_y = -\frac{5}{7} \frac{K_x}{a} - \frac{2}{7} F_y$$

$$R_z = -F_z$$

$$m \frac{dV_x}{dt} = \frac{5}{7} \left( F_y + \frac{K_x}{a} \right)$$

$$m \frac{dV_y}{dt} = \frac{5}{7} \left( F_y - \frac{K_x}{a} \right)$$

$$V_z = 0$$

$$\Omega_x = -\frac{V_y}{a}$$

$$\Omega_y = \frac{V_x}{a}$$

$$I \frac{d\Omega_z}{dt} = K_z$$

Example:

$$\text{Suppose: } \vec{F} = -mg\hat{z} + F_0\hat{x}$$

$$K = 0$$

$$\text{Then: } F_x = F_0, F_y = 0, F_z = -mg$$

Solution from previous page goes here.

$$R_x = -\frac{2}{7} F_0$$

$$R_y = 0$$

$$R_z = mg \quad (\text{normal force upward})$$

$$m \frac{dV_x}{dt} = \frac{5}{7} F_0 \rightarrow V_x = \frac{5}{7} \frac{F_0}{m} t + V_{x0}$$

$$m \frac{dV_y}{dt} = 0 \rightarrow V_y = \text{const} = V_{y0}$$

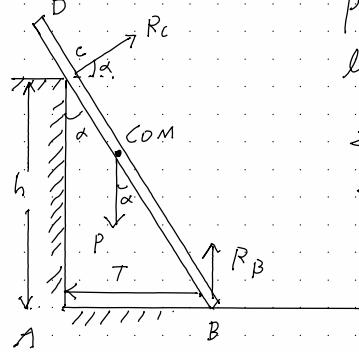
$$V_z = 0$$

$$\Omega_x = -\frac{V_{y0}}{a}$$

$$\Omega_y = \frac{V_x}{a} = \frac{5}{7} \frac{F_0}{ma} t + \frac{V_{x0}}{a}$$

$$I \frac{d\Omega_z}{dt} = 0 \rightarrow \Omega_z = \text{const} = \Omega_{z0}$$

Sec 38, Prob 2



$$P = \text{weight} = \mu g$$

$\ell$ : length of uniform rod

$$\sum \vec{F} = 0$$

$$\sum \vec{r} \times \vec{F} = 0$$

including reaction Force  
 $T, R_C, R_B$

horizontal direction

$$-T + R_C \cos \alpha = 0 \quad (1)$$

vertical direction

$$R_C \sin \alpha - P + R_B = 0 \quad (2)$$

torques around B:

$$\frac{l}{2} \sin \alpha P - \frac{h}{\cos \alpha} R_C = 0 \quad (3)$$

Thus,

$$\frac{1}{\cos \alpha} R_C = \frac{l \sin \alpha}{2} P$$

$$R_C = \frac{l P \sin \alpha \cos \alpha}{2h}$$

$$T = R_C \cos \alpha$$

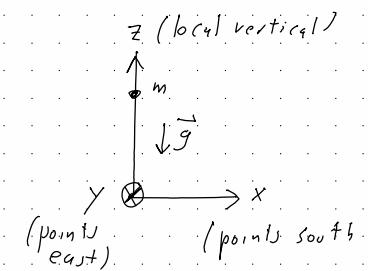
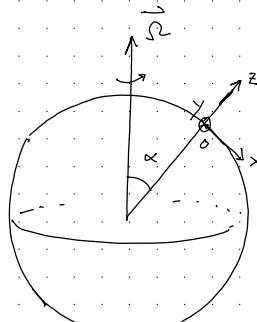
$$R_B = P - R_C \sin \alpha$$



$$\cos \alpha = \frac{l}{d}$$

$$\Rightarrow d = \frac{h}{\cos \alpha}$$

Sec 39, Prob 1:



( $\alpha$ : location on Earth relative to North pole)

Newton's 2nd law in rotating frame

$$m \vec{a} = m \vec{g} - m \vec{W} - 2m \vec{\Omega} \times \vec{v} - m \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$\text{where } \vec{W} = \vec{\Omega} \times (\vec{\Omega} \times \vec{R})$$

= acceleration of origin of rotating frame

Thus,

$$m \vec{a} = m \vec{g} - m \vec{\Omega} \times (\vec{\Omega} \times (\vec{r} + \vec{R})) - 2m \vec{\Omega} \times \vec{v}$$

2nd order in  $\vec{\Omega}$   
[ignore]

$$\approx m \vec{g} - 2m \vec{\Omega} \times \vec{v}$$

$$\text{Now: } \vec{g} = -g \vec{z}$$

$$\vec{\Omega} = \Omega \cos \alpha \vec{z} + \Omega \sin \alpha \vec{x}$$

$$(\vec{\omega} \times \vec{v})_x = \cancel{\omega_y v_z} - \omega_z \dot{y} \\ = -\omega_z \dot{y}$$

$$(\vec{\omega} \times \vec{v})_y = \omega_z v_x - \omega_x v_z \\ = \omega_z \dot{x} - \omega_x \dot{z} \\ (\vec{\omega} \times \vec{v})_z = \omega_x v_y - \cancel{\omega_y v_x} \\ = \omega_x \dot{y}$$

Thus,

$$\ddot{x} = +2\omega_z \dot{y}$$

$$\ddot{y} = -2(\omega_z \dot{x} - \omega_x \dot{z})$$

$$\ddot{z} = -g + \omega_x \dot{y}$$

Want to solve these equations to 1<sup>st</sup> order in  $\omega$

The 0<sup>th</sup> order solution is

$$\left. \begin{aligned} x_0(t) &= 0 \\ y_0(t) &= 0 \\ z_0(t) &= h - \frac{1}{2}gt^2 \end{aligned} \right\} \begin{array}{l} \text{drop from} \\ \text{vertical} \\ \text{height } h \end{array}$$

Write  $x(t) = x_0(t) + x_1(t)$ , etc.

Then

$$\ddot{x}_1 = 2\omega_z \dot{y}_1 \approx 0$$

$$\ddot{y}_1 = -2(\omega_z \dot{x}_1 - \omega_x(-gt + z_1)) \approx -2\omega_x g t$$

$$\cancel{\omega_z \dot{z}_1} = -g + \omega_x \dot{y}_1 \approx 0$$

Thus, only need to solve

$$\ddot{y}_1 = -2\omega_x g t$$

$$\rightarrow \dot{y}_1 = -\omega_x g t^2$$

$$y_1 = -\frac{1}{3}\omega_x g t^3$$

$$\text{Now: } \omega_x = -\omega \sin \alpha$$

$$T = \sqrt{\frac{2h}{g}} \quad (0^{\text{th}} \text{ order solution to} \\ \theta = h - \frac{1}{2}g T^2)$$

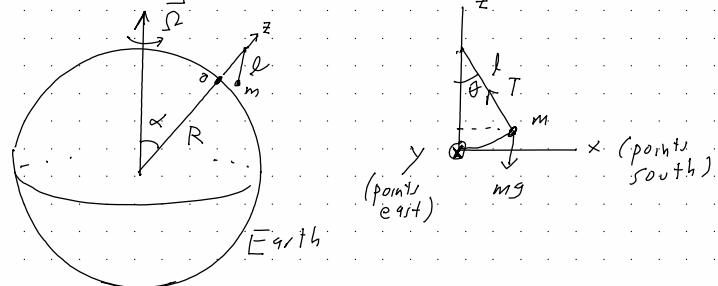
so object hits the ground at

$$y = Y_1 = +\frac{1}{3}\omega \sin \alpha \left( \frac{2h}{g} \right)^{3/2}$$

indicates that the object  
hits the ground to the East  
of the dropping location.

Sec 3.9, Prob 3.

Foucault's pendulum small oscillations



Newton's 2nd law in rotating frame:

$$m\vec{a} = \vec{T} + m\vec{g} - m\vec{W} - 2m\vec{\Omega} \times \vec{v} - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

Now  $\vec{W}$  = acceleration of origin O

$$= \vec{\Omega} \times (\vec{\Omega} \times \vec{R})$$

Then,

$$m\vec{a} = \vec{T} + m\vec{g} - m\vec{\Omega} \times (\vec{\Omega} \times (\vec{r} + \vec{R})) - 2m\vec{\Omega} \times \vec{v}$$

$\underbrace{\quad}_{\text{2nd order in } \vec{\Omega}}$

[ignore]

$$\approx \vec{T} + m\vec{g} - 2m\vec{\Omega} \times \vec{v}$$

$$\text{Now: } \vec{T} = T \cos \theta \hat{z} - T \sin \theta \cos \phi \hat{x} - T \sin \theta \sin \phi \hat{y}$$

$$\approx T \hat{z} - T \left(\frac{x}{D}\right) \hat{x} - T \left(\frac{y}{D}\right) \hat{y}$$

assuming  $\theta \ll 1$  (small oscillations)

$$\vec{g} = -g \hat{z}$$

$$(\vec{\Omega} \times \vec{v})_x = \Omega_y v_z - \Omega_z v_y \\ = \Omega_y \hat{z} - \Omega_z \hat{y} \\ \approx -\Omega_z \hat{y}$$

$$(\vec{\Omega} \times \vec{v})_y = \Omega_z v_x - \Omega_x v_z \\ = \Omega_z \hat{x} - \Omega_x \hat{z} \\ \approx \Omega_z \hat{x}$$

$$(\vec{\Omega} \times \vec{v})_z = \Omega_x v_y - \Omega_y v_x \\ = \Omega_x \hat{y}$$

Then,

$$m\vec{a} = \vec{T} + m\vec{g} - 2m\vec{\Omega} \times \vec{v}$$

becomes

$$m\vec{x}' = -T \left(\frac{x}{D}\right) + 2m\vec{\Omega}_z \hat{y}$$

$$m\vec{y}' = -T \left(\frac{y}{D}\right) - 2m\vec{\Omega}_z \hat{x}$$

$$0 = m\vec{z}' = T - mg - 2m\vec{\Omega}_x \hat{y}$$

Now:  $\vec{\Omega}_x \hat{y} \approx \omega \frac{D}{P}$

where  $D = \text{max displacement in } xy\text{-plane}$   
 $P = \text{period} = 2\pi \sqrt{\frac{D}{g}} = \frac{2\pi}{\omega}$

$$\frac{\Omega D}{P} = \frac{\Omega D}{2\pi} \omega$$

$$<< \frac{\ell \omega^2}{2\pi}$$

$$= \frac{k g}{2\pi \ell}$$

$$= \frac{g}{2\pi}$$

$$\text{thus, } m\Omega_x \dot{y} \ll mg$$

so we can ignore these terms relative to  $mg$

$$\rightarrow 0 \approx T - mg \rightarrow [T \approx mg]$$

$$mx'' \approx -mg \frac{x}{\ell} + 2m\Omega_z \dot{y}$$

$$[x'' \approx -w^2 x + 2\Omega_z \dot{y}]$$

$$my'' \approx -mg \frac{y}{\ell} - 2m\Omega_z \dot{x}$$

$$[y'' \approx -w^2 y - 2\Omega_z \dot{x}]$$

just need  
to solve  
these two  
equations

standard "trick":

Define  $\xi = x + iy$  (complex valued)

then  $\ddot{\xi} = \ddot{x} + i\ddot{y}$

$$\ddot{\xi} = \ddot{x} + i\ddot{y}$$



since  $\Omega \ll \omega$   
 $D \ll \ell$  (small oscillations)

$$\text{so } \ddot{\xi} = -w^2 \xi + 2i\Omega_z (\dot{y} - i\dot{x})$$

$$= -w^2 \xi - 2i\Omega_z (\dot{x} + i\dot{y})$$

$$= -w^2 \xi - 2i\Omega_z \xi$$

thus,

$$\ddot{\xi} + 2i\Omega_z \dot{\xi} + w^2 \xi = 0$$

trial solution:  $\xi(t) = e^{i\lambda t}$

$$\xi(t) = e^{i\lambda t}$$

$$\rightarrow -\lambda^2 - 2i\Omega_z \lambda + w^2 = 0$$

$$\lambda^2 + 2i\Omega_z \lambda - w^2 = 0$$

solve quadratic:

$$\lambda_{\pm} = \frac{-2\Omega_z \pm \sqrt{4\Omega_z^2 + 4w^2}}{2}$$

$$\approx -\Omega_z \pm \omega \quad (\text{since } \Omega \ll \omega)$$

General solution:

$$\xi(t) = A e^{-i(\Omega_z + \omega)t} + B e^{-i(\Omega_z - \omega)t}$$

$$= e^{-i\Omega_z t} (A e^{i\omega t} + B e^{-i\omega t})$$

$$= e^{-i\Omega_z t} \xi_0(t)$$

where  $\xi_0(t)$  = general solution of small oscillation problem in an inertial frame.

The factor  $e^{-i\Omega_z t}$  causes the plane of oscillation of the pendulum to precess with angular frequency

$$\Omega_z = \sqrt{\omega} \cos \alpha$$

angle of location  
on Earth

$(\alpha = 0 \leftrightarrow \text{NPole})$   
 $(\alpha = \pi/2 \leftrightarrow \text{Equator})$

Precession period:

$$P_{\text{precession}} = \frac{2\pi}{\Omega_z}$$

$$= \frac{2\pi}{\sqrt{\omega} \cos \alpha}$$

$$\Omega = \text{1 revolution / 24 hrs}$$

$$= \frac{2\pi \text{ rad}}{24 \text{ hrs}}$$

$$\text{so } P_{\text{precession}} = \frac{24 \text{ hrs}}{\cos \alpha}$$

$$= \begin{cases} 24 \text{ hrs at N Pole} \\ \infty \text{ at equator (so no precession)} \end{cases}$$