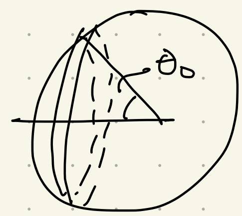
sec 16, prob 2

$$dN = \text{Fraction of particles entering } d\Omega_0$$

$$= \frac{d\Omega_0}{4\pi}$$



$$= \frac{2\pi \sin\theta_0 \, d\theta_0}{4\pi}$$

$$= \frac{1}{2} \sin\theta \, d\theta_0$$

$$= -\frac{1}{2} d(\cos\theta_0)$$

Now: (16.6)

$$\cos\theta_0 = -\frac{V}{v_0}\sin^2\theta \pm \cos\theta \sqrt{1 - \frac{V^2}{v_0^2}\sin^2\theta}$$

i) For $V < v_0$, take $+\sqrt{}$

$$d(\cos\theta_0) = -\frac{2V}{v_0}\sin\theta\cos\theta \, d\theta$$

$$- \sin\theta \, d\theta \sqrt{}$$

$$+ \frac{\cos\theta}{\sqrt{}} \frac{1}{\cancel{2}} \left(-\frac{V^2}{v_0^2}\right) \cancel{2} \sin\theta\cos\theta \, d\theta$$

$$= -\sin\theta \, d\theta \left\{ \frac{2V}{v_0}\cos\theta + \sqrt{} + \left(\frac{V}{v_0}\right)^2 \frac{\cos^2\theta}{\sqrt{}} \right\}$$

Thus,

$$dN = \frac{1}{2}\sin\theta\, d\theta \left[\frac{2V}{v_0}\cos\theta + \frac{\left(1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{}} \right\}$$

$$= \frac{1}{2}\sin\theta\, d\theta \left\{ \frac{2V}{v_0}\cos\theta + \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right\}$$

where $0 \le \theta \le \pi$

ii) For $V > v_0$, there are two solutions corresponding to the $+\sqrt{}$ and $-\sqrt{}$ in (16.6).

- For the $+\sqrt{}$ we have $d\theta/d\theta_0 > 0$
- For the $-\sqrt{}$ we have $d\theta/d\theta_0 < 0$

so we should subtact the two contributions

$$dN = dN_+ - dN_-$$

where dN_+ = above expression

$$= \frac{1}{2}\sin\theta\, d\theta \left\{ \frac{2V}{v_0}\cos\theta + \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right\}$$

and

$$dN_- = \frac{1}{2}\sin\theta\, d\theta \left\{ \frac{2V}{v_0}\cos\theta - \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right\}$$

Then,

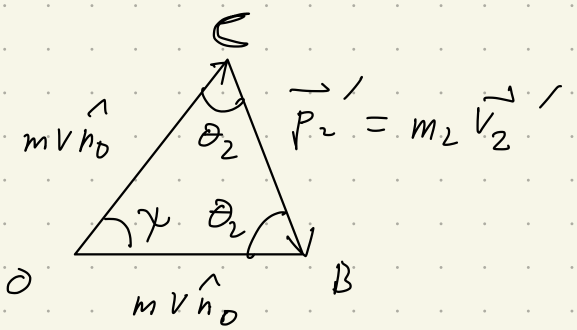
$$dN = \sin\theta \, d\theta \; \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}}$$

where $\theta \leq \theta \leq \theta_{max}$

Sec 17, Prob 1

Want to determine v_1', v_2' as functions of θ_1, θ_2 (as opposed to functions of χ)

From Fig 16, triangle OBC:



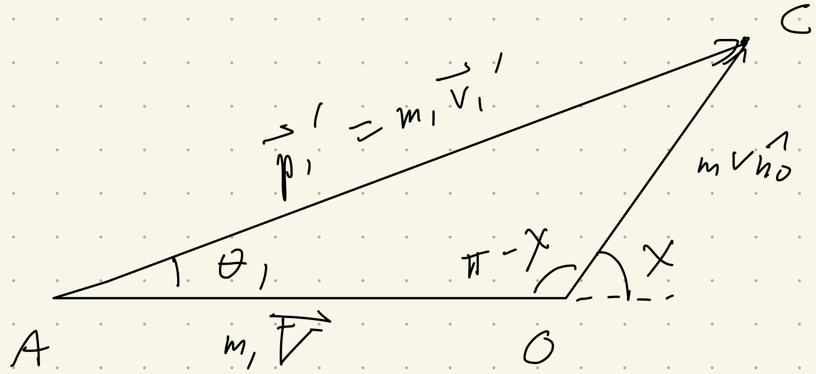
$$\chi + 2\theta_2 = \pi$$
$$\chi = \pi - 2\theta_2$$

$$(m_2 v_2')^2 = 2(mv)^2 - 2(mv)^2 \cos\chi$$
$$= 2(mv)^2 (1 - \cos\chi)$$

Now: $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$$= 1 - 2\sin^2\theta$$
$$\rightarrow 2\sin^2\theta = 1 - \cos 2\theta$$

$$\rightarrow (m_2 v_2')^2 = 4(mv)^2 \sin^2\left(\frac{\chi}{2}\right)$$
$$v_2' = 2\left(\frac{mv}{m_2}\right) \sin\left(\frac{\pi}{2} - \theta_2\right)$$
$$= 2\left(\frac{m_1}{m_1 + m_2}\right) v \cos\theta_2$$

From Fig 16, $\triangle AOC$



$$\vec{V} = \frac{m_1 \vec{v}_1 + m_2 \cancel{\vec{v}_2}^{\,0}}{m_1 + m_2} = \frac{m_1 \vec{v}_1}{m_1 + m_2} = \frac{m_1 \vec{v}}{m_1 + m_2}$$

Use law of cosines for OC:

$$OC^2 = AC^2 + AO^2 - 2\, AC \cdot AO \cos\theta_1$$

$$(mv)^2 = (m_1 v_1')^2 + (m_1 V)^2 - 2\, m_1 v_1' m_1 V \cos\theta_1$$

quadratic equation for v_1':

$$0 = (m_1 v_1')^2 - 2\, m_1^2 v_1' V \cos\theta_1 + (m_1 V)^2 - (m v)^2$$

$$= m_1^2 v_1'^2 - 2 \frac{m_1^3}{m_1 + m_2} v_1' v \cos\theta_1 + \left(\frac{m_1^4}{(m_1+m_2)^2} - \frac{m_1^2 m_2^2}{(m_1+m_2)^2} \right) v^2$$

$$= m_1^2 v^2 \left[\left(\frac{v_1'}{v}\right)^2 - 2\left(\frac{m_1}{m_1+m_2}\right)\left(\frac{v_1'}{v}\right) \cos\theta_1 + \underbrace{\frac{m_1^2 - m_2^2}{(m_1+m_2)^2}}_{= \left(\frac{m_1 - m_2}{m_1 + m_2}\right)} \right]$$

$$\rightarrow \quad \frac{v_1'}{v} = \frac{2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \sqrt{4\left(\frac{m_1}{m_1+m_2}\right)^2 \cos^2\theta_1 - 4\left(\frac{m_1 - m_2}{m_1 + m_2}\right)}}{2}$$

$$\frac{v_1'}{v} = \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{\underbrace{m_1^2\cos^2\theta_1 - (m_1^2 - m_2^2)}}$$

$$= m_1^2(\cos^2\theta_1 - 1) + m_2^2$$

$$= m_2^2 - m_1^2\sin^2\theta_1$$

thus,

$$\boxed{\frac{v_1'}{v} = \frac{m_1}{m_1+m_2}\cos\theta_1 \pm \frac{1}{m_1+m_2}\sqrt{m_2^2 - m_1^2\sin^2\theta_1}}$$

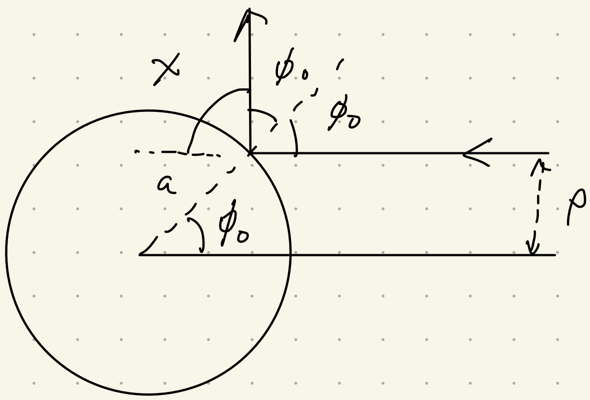
For $m_1 > m_2$, the $\sqrt{\ }$ has both $\pm$ signs

For $m_1 < m_2$, the $\sqrt{\ }$ should have the $+$ sign in order for

$$\frac{v_1'}{v} \xrightarrow[\theta_1 \to 0]{} \left(\frac{m_1}{m_1+m_2}\right) + \frac{m_2}{m_1+m_2} = 1$$

Sec. 18, Probl:

Hard sphere $\quad U = \begin{cases} 0 & r > a \\ \infty & r < a \end{cases}$



$$\chi + 2\phi_0 = \pi \quad \rightarrow \quad \phi_0 = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\sin\phi_0 = \rho/a$$

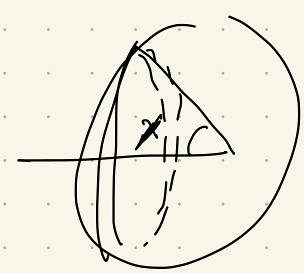
Thus, $\quad \sin\left(\frac{\pi}{2} - \frac{\chi}{2}\right) = \frac{\rho}{a}$

$$\rightarrow \quad \cos\left(\frac{\chi}{2}\right) = \frac{\rho}{a}$$

Effective cross section:

$$d\sigma = 2\pi\rho\, d\rho$$

$$d\Omega = 2\pi \sin\chi\, d\chi$$



$$\rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{\rho\, d\rho}{\sin\chi\, d\chi}$$

$$= \frac{\rho(\chi)}{\sin\chi} \left|\frac{d\rho}{d\chi}\right|$$

$$\frac{d\sigma}{d\Omega} = \frac{a\cos\left(\frac{\chi}{2}\right)}{\sin\chi}\left|\frac{d}{d\chi}\left(a\cos\left(\frac{\chi}{2}\right)\right)\right|$$

$$= \frac{a^2\cos\left(\frac{\chi}{2}\right)\frac{1}{2}\sin\left(\frac{\chi}{2}\right)}{\sin\chi}$$

$$= \frac{a^2}{2}\frac{\cancel{\cos\left(\frac{\chi}{2}\right)\sin\left(\frac{\chi}{2}\right)}}{2\cancel{\sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)}}$$

$$= \boxed{\frac{1}{4}a^2}$$

Total cross section

$$\sigma = \int\frac{d\sigma}{d\Omega}\,d\Omega, \qquad d\Omega = 2\pi\sin\chi\,d\chi$$

$$= \frac{1}{4}a^2\int_0^{\pi}2\pi\sin\chi\,d\chi$$

$$= \frac{1}{2}\pi a^2\left(-\cos\chi\right)\Big|_0^{\pi}$$

$\cos\pi = -1$

$\cos 0 = 1$

$$= \frac{1}{2}\pi a^2(-1)(-1-1)$$

$$= \boxed{\pi a^2}$$

Calculate $d\sigma$ wrt θ_1 and θ_2

$$d\sigma = 2\pi\rho\, d\rho = 2\pi\rho \left|\frac{d\rho}{d\chi}\right| d\chi$$

$$d\Omega = 2\pi \sin\chi\, d\chi \rightarrow 2\pi\, d\chi = \frac{d\Omega}{\sin\chi}$$

$$\rightarrow d\sigma = \frac{\rho}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$

Similarly $$d\sigma_1 = \frac{\rho}{\sin\theta_1}\left|\frac{d\rho}{d\theta_1}\right| d\Omega_1$$

$$d\sigma_2 = \frac{\rho}{\sin\theta_2}\left|\frac{d\rho}{d\theta_2}\right| d\Omega_2$$

$$\rightarrow \frac{d\sigma_1}{d\Omega_1} = \frac{\sin\chi\, d\chi}{\sin\theta_1\, d\theta_1}\frac{d\sigma}{d\Omega}$$

$$= \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| \frac{d\sigma}{d\Omega}$$

and $$\frac{d\sigma_2}{d\Omega_2} = \left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right| \frac{d\sigma}{d\Omega}$$

where $\frac{d\sigma}{d\Omega} = \frac{1}{4}a^2$ (for hard sphere)

Now: $2\theta_2 + \chi = \pi$ (always)

$$\chi = \pi - 2\theta_2$$

$$\rightarrow \cos\chi = \cos(\pi - 2\theta_2)$$
$$= -\cos(2\theta_2)$$
$$= -\cos^2\theta_2 + \sin^2\theta_2$$
$$= 1 - 2\cos^2\theta_2$$

$$d(\cos\chi) = -4\cos\theta_2 \, d(\cos\theta_2)$$

so $\left|\dfrac{d(\cos\chi)}{d(\cos\theta_2)}\right| = 4|\cos\theta_2|$

Thus, $$\boxed{\frac{d\sigma_2}{d\Omega_2} = \cancel{4}\cos\theta_2 \frac{1}{\cancel{4}} a^2 = a^2|\cos\theta_2|}$$

Relating θ_1 to χ:

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}$$

Want to find $\cos\chi$ in terms of θ_1

compare to (16.5), (16.6):

$$\tan\theta = \frac{v_0 \sin\theta_0}{v_0\cos\theta_0 + V}$$

$$\cos\theta_0 = -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{1 - \frac{V^2}{v_0^2}\sin^2\theta}$$

Thus,

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_2\cos\chi + m_1}$$

$$\rightarrow \boxed{\cos\chi = -\left(\frac{m_1}{m_2}\right)\sin^2\theta_1 \pm \cos\theta_1\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

where $\pm$ sign for $m_1 > m_2$ and $+$ sign for $m_2 > m_1$

(i) <u>For $m_2 > m_1$:</u> (take $+\sqrt{}$)

$$d(\cos\chi) = -2\left(\frac{m_1}{m_2}\right)\sin\theta_1\cos\theta_1\, d\theta_1 + d(\cos\theta_1)\sqrt{}$$

$$+ \frac{\cos\theta_1}{\sqrt{}}\left(\frac{1}{\cancel{2}}\right)\left(-\cancel{2}\right)\left(\frac{m_1}{m_2}\right)^2\sin\theta_1\cos\theta_1\, d\theta_1$$

<u>Now:</u> $\sin\theta_1\cos\theta_1\, d\theta_1 = -\cos\theta_1\, d(\cos\theta_1)$

$$d(\cos\chi) = d(\cos\theta_1)\left[2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \sqrt{} + \frac{\cos^2\theta_1}{\sqrt{}}\left(\frac{m_1}{m_2}\right)^2\right]$$

$$\frac{d(\cos\chi)}{d(\cos\theta_1)} = 2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1 + \cos^2\theta_1\left(\frac{m_1}{m_2}\right)^2}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

$$= 2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

$$\rightarrow \boxed{\frac{d\sigma_1}{d\Omega_1}} = \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|\frac{d\sigma}{d\Omega}$$

for $0 \le \theta_1 \le \pi$

$$= \frac{1}{4}a^2\left[2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\right]$$

(ii) For $m_1 > m_2$: contribution from both $\pm$ signs

$$\left.\begin{aligned} + \text{ sign: } & \frac{d\chi}{d\theta_1} > 0 \\ - \text{ sign: } & \frac{d\chi}{d\theta_1} < 0 \end{aligned}\right\} \text{ so need to subtract}$$

$$\frac{d(\cos\chi)}{d(\cos\theta_1)} = 2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1}{\sqrt{}}$$

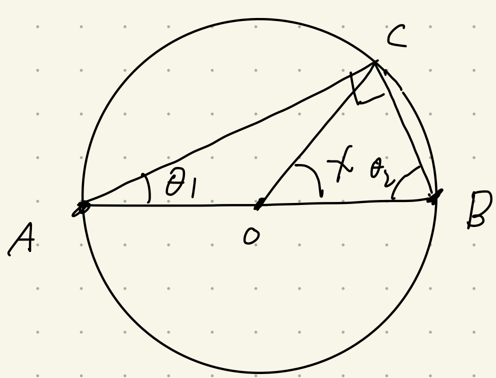
$$- \left(2\left(\frac{m_1}{m_2}\right)\cos\theta_1 - \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1}{\sqrt{}}\right)$$

$$\rightarrow \quad \frac{d(\cos\chi)}{d(\cos\theta_1)} = 2\,\frac{\left(1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1\right)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

$$\boxed{\frac{d\sigma_1}{d\Omega_1} = \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| \frac{d\sigma}{d\Omega} = \frac{1}{2}a^2\,\frac{\left(1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)\right)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}} \quad \frac{d\sigma}{d\Omega} \leftarrow \frac{1}{4}a^2$$

for $0 \le \theta_1 \le \theta_{max}$

(iii) For $m_1 = m_2$, $\theta_1 + \theta_2 = \pi/2$



$$2\theta_2 + \chi = \pi \rightarrow \theta_2 = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\rightarrow \theta_1 = \frac{\pi}{2} - \theta_2 = \frac{\pi}{2} - \left(\frac{\pi}{2} - \frac{\chi}{2}\right) = \frac{\chi}{2}$$

thus, $\dfrac{d(\cos\chi)}{d(\cos\theta_1)} = \dfrac{d(\cos(2\theta_1))}{d(\cos\theta_1)}$

But $\cos(2\theta_1) = 2\cos^2\theta_1 - 1$

$$\rightarrow \quad \frac{d(\cos\chi)}{d(\cos\theta_1)} = \frac{4\cos\theta_1 \cancel{d(\cos\theta_1)}}{\cancel{d\cos\theta_1}}$$

$$= 4\cos\theta_1$$

Thus,

$$\boxed{\frac{d\sigma_1}{d\Omega_1} = \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| \frac{d\sigma}{d\Omega} = 4|\cos\theta_1| \frac{1}{4}a^2 = a^2|\cos\theta_1|}$$

Sec 18, Prob 2:

From problem 1,
$$d\sigma = \frac{1}{4} a^2 d\Omega$$
$$= \frac{1}{4} a^2 \, 2\pi \sin\chi \, d\chi$$
$$= \frac{1}{2} \pi a^2 \sin\chi \, d\chi$$

Want to replace $\sin\chi \, d\chi$ by some function involving ϵ, the energy lost by the scattered particle.

ϵ = energy lost by scattered particle
= energy gained by scattering particle
$= \frac{1}{2} m_2 v_2'^2$

Now: $v_2' = \frac{2 m_1 v}{m_1 + m_2} \sin\left(\frac{\chi}{2}\right)$, $\quad v = v_1 - \cancel{v_2}^{0} = v_\infty$

$$= 2 \frac{m v_\infty}{m_2} \sin\left(\frac{\chi}{2}\right)$$

$$\rightarrow \epsilon = \frac{1}{2} m_2 \frac{4 m^2 v_\infty^2}{m_2^2} \sin^2\left(\frac{\chi}{2}\right)$$

$$= 2 \frac{m^2}{m_2} v_\infty^2 \sin^2\left(\frac{\chi}{2}\right) = 2 \frac{m_1^2 m_2 v_\infty^2}{(m_1 + m_2)^2} \sin^2\left(\frac{\chi}{2}\right)$$

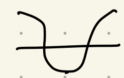
$\cos 2\theta = 1 - 2\sin^2\theta \longrightarrow \sin^2 \frac{\chi}{2} = \frac{1 - \cos\chi}{2}$

$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$$\epsilon = \frac{m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 (1-\cos\chi)$$

$$d\epsilon = \frac{m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 \sin\chi \, d\chi$$

Thus,
$$d\sigma = \frac{1}{2}\pi a^2 \sin\chi \, d\chi$$
$$= \frac{1}{2}\pi a^2 \frac{(m_1+m_2)^2}{m_1^2 m_2 v_\infty^2} d\epsilon$$

NOTE: $\epsilon = \frac{m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 (1-\cos\chi)$ 

has $\epsilon_{max} = \frac{2 m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2$ when $\chi = \pi$

$$\rightarrow \boxed{d\sigma = \frac{\pi a^2}{\epsilon_{max}} d\epsilon}$$

(Uniform distribution in ϵ between 0 and ϵ_{max})

Sec 18, Prob 4:

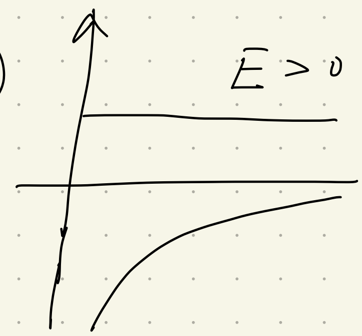
Effective cross section for particle to fall to center of potential $U = -\frac{\alpha}{r^2}$

$$U_{eff}(r) = \frac{M^2}{2mr^2} + U(r)$$

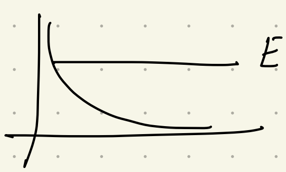
$$= \frac{M^2}{2mr^2} - \frac{\alpha}{r^2}$$

$$= \frac{1}{r^2}\left(\frac{M^2}{2m} - \alpha\right)$$

Need $\alpha \geq \frac{M^2}{2m}$ so that $\rightarrow U_{eff}(r)$



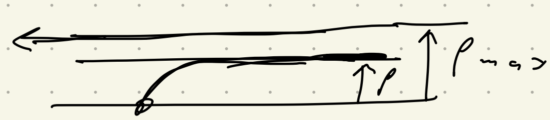
otherwise for $\alpha < \frac{M^2}{2m}$, the effective potential is repulsive



and particle can't reach the center.

$$\alpha \geq \frac{M^2}{2m} = \frac{m^2\rho^2 V_\infty^2}{2m} = \rho^2\left(\frac{1}{2} m V_\infty^2\right)$$

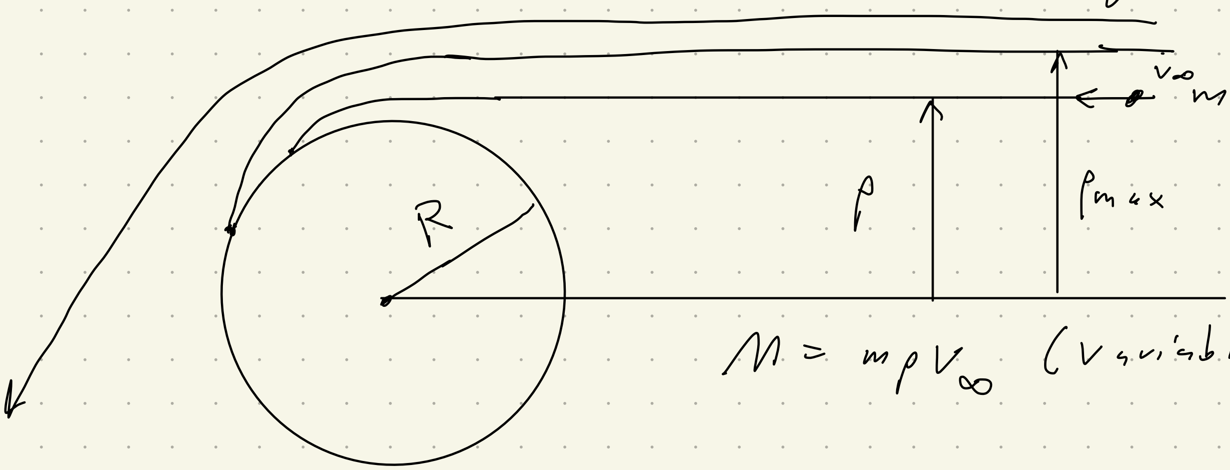
$$\rho \leq \sqrt{\frac{\alpha}{\frac{1}{2}mv_\infty^2}} = \rho_{max}$$



$$\boxed{\sigma = \pi\rho_{max}^2 = \frac{\pi\alpha}{\frac{1}{2}mv_\infty^2}}$$

Sec 18, Prob 6:

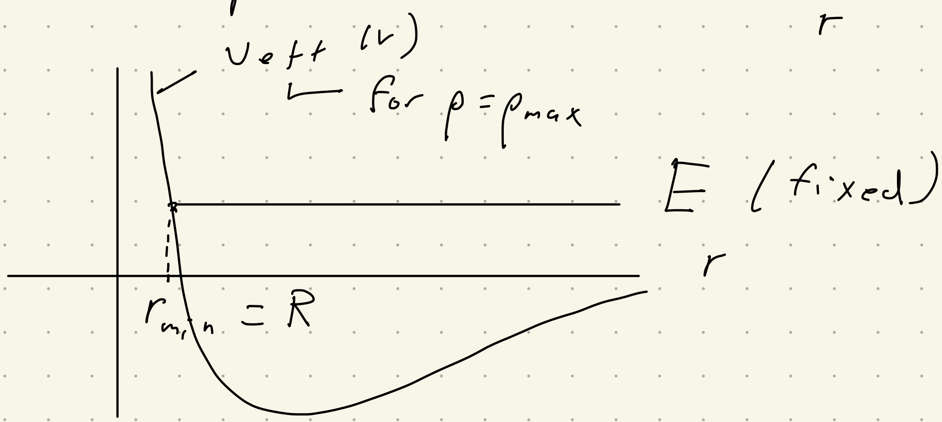
Effective cross section for particle of mass m_1 to strike a sphere of mass m_2 and radius R subject to Newtonian gravity



$E = \frac{1}{2} m v_\infty^2$ (given)

$M = m \rho v_\infty$ (variable)

outside sphere $\quad U = -\dfrac{G m_1 m_2}{r} = -\dfrac{\alpha}{r}$



$$
\begin{aligned}
\frac{1}{2} m v_\infty^2 &= U_{eff}(r_{min}) \\
&= U_{eff}(R) \\
&= -\frac{\alpha}{R} + \frac{M^2}{2 m R^2} \\
&= -\frac{\alpha}{R} + \frac{m^2 \rho_{max}^2 v_\infty^2}{2 m R^2} \\
&= -\frac{\alpha}{R} + \left(\frac{1}{2} m v_\infty^2\right) \frac{\rho_{max}^2}{R^2}
\end{aligned}
$$

Thus, $E = -\frac{\alpha}{R} + E\frac{\rho_{max}^2}{R^2}$

$$\rightarrow \frac{E + \alpha/R}{E} = \frac{\rho_{max}^2}{R^2}$$

$$\rightarrow \sigma = \pi \rho_{max}^2$$

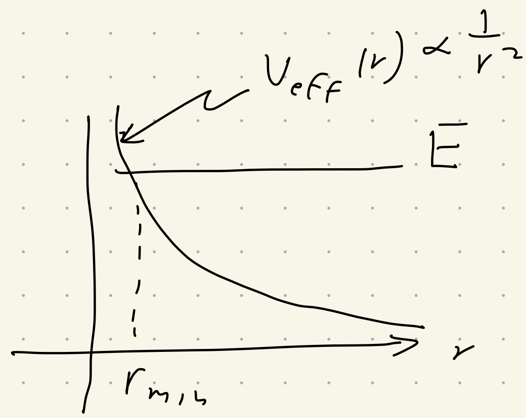
$$= \pi R^2 \left(1 + \frac{\alpha}{R} \cdot \frac{1}{E}\right)$$

$$= \pi R^2 \left(1 + \frac{G \cancel{m_1 m_2}}{R} \frac{1}{\frac{1}{2} \frac{\cancel{m_1 m_2}}{(m_1+m_2)} v_\infty^2}\right)$$

$$= \pi R^2 \left(1 + \frac{2G(m_1+m_2)}{R v_\infty^2}\right)$$

Sec 19, Prob 1: $\quad U = \dfrac{\alpha}{r^2}, \quad \alpha > 0$

$$U_{eff}(r) = U(r) + \frac{M^2}{2mr^2}$$
$$= \frac{\alpha}{r^2} + \frac{\rho^2 m^2 v_\infty^2}{2mr^2}$$
$$= \frac{1}{r^2}\left(\alpha + \frac{1}{2}mv_\infty^2 \cdot \rho^2\right)$$



$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \rho^2/r^2 - 2U/mv_\infty^2}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \left(\rho^2 + \frac{2\alpha}{mv_\infty^2}\right)\frac{1}{r^2}}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \beta^2/r^2}}$$

$$\beta^2 \equiv \rho^2 + \frac{2\alpha}{mv_\infty^2} = \rho^2 + \frac{\alpha}{E}.$$

Let: $u = \dfrac{1}{r} \rightarrow du = -\dfrac{1}{r^2}\,dr$

$$\phi_0 = \int_0^{\frac{1}{r_{min}}} \frac{\rho\, du}{\sqrt{1 - \beta^2 u^2}}, \qquad 1 - \frac{\beta^2}{r_{min}^2} = 0 \rightarrow \boxed{\beta = r_{min}}$$

$$\phi_0 = \int_0^{\frac{1}{\beta}} \frac{\rho\, du}{\sqrt{1-\beta^2 u^2}}$$

Let: $\beta u = \sin\theta \rightarrow du = \frac{1}{\beta}\cos\theta\, d\theta$

$$\sqrt{1-\beta^2 u^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$

$$u = 0 \rightarrow \theta = 0$$

$$u = \frac{1}{\beta} \rightarrow \sin\theta = 1 \rightarrow \theta = \pi/2$$

$$\rightarrow \phi_0 = \int_0^{\pi/2} \frac{\rho}{\beta} \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}}$$

$$= \frac{\pi}{2} \frac{\rho}{\beta}$$

$$= \frac{\pi}{2} \frac{\rho}{\sqrt{\rho^2 + \frac{\alpha}{E}}}$$

Square both sides

$$\left(\frac{2\phi_0}{\pi}\right)^2 = \frac{\rho^2}{\rho^2 + \frac{\alpha}{E}}$$

$$\left(\rho^2 + \frac{\alpha}{E}\right)\left(\frac{2\phi_0}{\pi}\right)^2 = \rho^2$$

$$\frac{\alpha}{E}\left(\frac{2\phi_0}{\pi}\right)^2 = \rho^2\left(1 - \left(\frac{2\phi_0}{\pi}\right)^2\right)$$

$$\rho^2 = \frac{\frac{\alpha}{E}\left(\frac{2\phi_0}{\pi}\right)^2}{1-\left(\frac{2\phi_0}{\pi}\right)^2}$$

<u>Now:</u> $2\phi_0 + \chi = \pi$ for repulsive scattering

$$\rightarrow \quad \frac{2\phi_0}{\pi} = 1 - \frac{\chi}{\pi}$$

$$\rightarrow \quad \left(\frac{2\phi_0}{\pi}\right)^2 = \left(1-\frac{\chi}{\pi}\right)^2$$

$$\rightarrow \quad 1-\left(\frac{2\phi_0}{\pi}\right)^2 = 1-\left(1+\frac{\chi^2}{\pi^2}-\frac{2\chi}{\pi}\right)$$

$$= \frac{2\chi}{\pi} - \frac{\chi^2}{\pi^2}$$

<u>Thus:</u>

$$\rho^2 = \frac{\frac{\alpha}{E}\left(1-\frac{\chi}{\pi}\right)^2}{\left(\frac{2\chi}{\pi}-\frac{\chi^2}{\pi^2}\right)}$$

$$= \frac{\frac{\alpha}{E}(\pi-\chi)^2}{2\pi\chi-\chi^2}$$

Thus,

$$\rho = \sqrt{\frac{\alpha}{E}} \; \frac{(\pi - \chi)}{\sqrt{2\pi\chi - \chi^2}}$$

Differential cross-section:

$$d\sigma = 2\pi \rho \, d\rho$$
$$= 2\pi \rho \left| \frac{d\rho}{d\chi} \right| d\chi$$

$$d\Omega = 2\pi \sin\chi \, d\chi$$

$$\rightarrow \boxed{d\sigma = \frac{\rho}{\sin\chi} \left| \frac{d\rho}{d\chi} \right| d\Omega}$$

$$\frac{d\rho}{d\chi} = \sqrt{\frac{\alpha}{E}} \left(\frac{-\sqrt{2\pi\chi - \chi^2} - \frac{1}{\cancel{2}\sqrt{}}(\cancel{2}\pi - \cancel{2}\chi)(\pi - \chi)}{2\pi\chi - \chi^2} \right)$$

$$= \sqrt{\frac{\alpha}{E}} \; \frac{-1}{(2\pi\chi - \chi^2)^{3/2}} \left(\underbrace{2\pi\chi - \chi^2 + (\pi - \chi)^2}_{\cancel{2\pi\chi} - \cancel{\chi^2} + \pi^2 + \cancel{\chi^2} - \cancel{2\pi\chi}} \right)$$

$$= -\sqrt{\frac{\alpha}{E}} \; \frac{\pi^2}{(2\pi\chi - \chi^2)^{3/2}}$$

So

$$\boxed{\left| \frac{d\rho}{d\chi} \right| = \sqrt{\frac{\alpha}{E}} \; \frac{\pi^2}{(2\pi\chi - \chi^2)^{3/2}}}$$

Thus,

$$\frac{d\sigma}{d\Omega} = \frac{1}{\sin\chi} \sqrt{\frac{\alpha}{E}} \frac{(\pi - \chi)}{\sqrt{2\pi\chi - \chi^2}} \sqrt{\frac{\alpha}{E}} \frac{\pi^2}{(2\pi\chi - \chi^2)^{3/2}}$$

$$= \frac{1}{\sin\chi} \left(\frac{\alpha}{E}\right) \frac{\pi^2 (\pi - \chi)}{(2\pi\chi - \chi^2)^2}$$

Sec 20, Prob 1:

Start with (18.4):

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1-\rho^2/r^2 - 2U/mv_\infty^2}}$$

Consider small angle scattering, where

$$\frac{2U}{mv_\infty^2} \ll 1$$

Then:

$$\frac{1}{\sqrt{1-\rho^2/r^2 - 2U/mv_\infty^2}} = \frac{1}{\sqrt{1-\rho^2/r^2}}\,\frac{1}{\sqrt{1-\frac{2U/mv_\infty^2}{1-\rho^2/r^2}}}$$

$$\approx \frac{1}{\sqrt{1-\rho^2/r^2}}\left(1+\frac{U}{mv_\infty^2}\left(\frac{1}{1-\rho^2/r^2}\right)\right)$$

$$= \frac{1}{\sqrt{1-\rho^2/r^2}} + \frac{U/mv_\infty^2}{(1-\rho^2/r^2)^{3/2}}$$

Now:

$$\int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1-\rho^2/r^2}} = \int_0^{\frac{1}{r_{min}}} \frac{\rho\, du}{\sqrt{1-\rho^2u^2}} = \int_0^{\pi/2} \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \boxed{\frac{\pi}{2}}$$

let: $u = \frac{1}{r}$

$du = -\frac{1}{r^2}\, dr$

let: $\rho u = \sin\theta$, $\rho\, du = \cos\theta\, d\theta$

$u = \frac{1}{r_{min}} \rightarrow \rho = r_{min} \rightarrow \theta = \frac{\pi}{2}$

Thus,

$$\phi_0 \approx \frac{\pi}{2} + \frac{1}{mV_\infty^2} \int_{r_{min}}^{\infty} \frac{U(r)\,\rho\, dr/r^2}{\left(1 - \rho^2/r^2\right)^{3/2}}$$

$$= \frac{\pi}{2} + \frac{1}{mV_\infty^2} \frac{\partial}{\partial \rho}\left[\int_{r_{min}}^{\infty} \frac{U(r)\, dr}{\sqrt{1-\rho^2/r^2}} \right]$$

<u>Now:</u>

$$\int_{r_{min}}^{\infty} \frac{U(r)\,dr}{\sqrt{1-\rho^2/r^2}} = \int_{r_{min}}^{\infty} \frac{U(r)\, r\, dr}{\sqrt{r^2-\rho^2}}$$

$$\approx \int_{\rho}^{\infty} \frac{U(r)\, r\, dr}{\sqrt{r^2-\rho^2}}$$

$$u = U(r) \rightarrow du = \frac{dU}{dr}\, dr$$

$$dv = \frac{r\,dr}{\sqrt{r^2-\rho^2}} = \frac{dx/2}{\sqrt{x}} \rightarrow v = \sqrt{x} = \sqrt{r^2-\rho^2}$$

$$\left(\text{let } x = r^2-\rho^2 \rightarrow dx = 2r\,dr\right)$$

Thus,

$$\int_{r_{min}}^{\infty} \frac{U(r)\,dr}{\sqrt{1-\rho^2/r^2}} \approx \cancelto{0}{\left. U(r)\sqrt{r^2-\rho^2}\right|_{\rho}^{\infty}} - \int_{\rho}^{\infty} dr\, \frac{dU}{dr}\sqrt{r^2-\rho^2}$$

assume $U(\infty) \rightarrow 0$ faster than $\frac{1}{r}$

$$\phi_0 \approx \frac{\pi}{2} - \frac{1}{m v_\infty^2} \frac{\partial}{\partial \rho}\left[\int_\rho^\infty dr \left(\frac{dU}{dr}\right)\sqrt{r^2-\rho^2}\right]$$

$$= \frac{\pi}{2} - \frac{1}{m v_\infty^2} \int_\rho^\infty dr \left(\frac{dU}{dr}\right) \frac{1}{\cancel{2}\sqrt{r^2-\rho^2}} \, {-\cancel{2}\rho} \, \Big]$$

$$= \frac{\pi}{2} + \frac{\rho}{m v_\infty^2} \int_\rho^\infty dr \left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}$$

Recall:

$$\chi + 2\phi_0 = \pi \quad \longrightarrow \quad \phi_0 - \frac{\pi}{2} = -\frac{\chi}{2}$$

Thus,

$$\chi = -2\left(\phi_0 - \frac{\pi}{2}\right)$$

$$\approx \frac{-2\rho}{m v_\infty^2} \int_\rho^\infty dr \left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}$$

Compare to:

$$\theta_1 \approx \frac{-2\rho}{m_1 v_\infty^2} \int_\rho^\infty dr \left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}} \qquad (20.3)$$

$$= \left(\frac{m_2}{m_1+m_2}\right)\left(\frac{-2\rho}{m v_\infty^2}\right) \int_\rho^\infty dr \left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}$$

$$= \left(\frac{m_2}{m_1+m_2}\right)\chi$$

(consistent with $\tan\theta_1 = \dfrac{m_2 \sin\chi}{m_1 + m_2\cos\chi} \longrightarrow \theta_1 \approx \dfrac{m_2\,\chi}{m_1+m_2}$ for $\chi \ll 1$)

Sec 21, Prob 1

$$x(t) = a\cos(\omega t + \alpha)$$

$$x_0 = a\cos\alpha$$

$$V_0 = \dot{x}(t)\Big|_{t=0}$$

$$= -a\omega\sin(\omega t + \alpha)\Big|_{t=0}$$

$$= -a\omega\sin\alpha$$

Thus, $x_0 = a\cos\alpha$

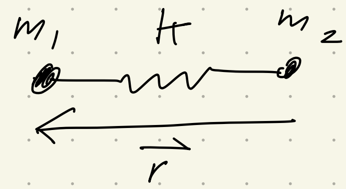
$$-\frac{V_0}{\omega} = a\sin\alpha$$

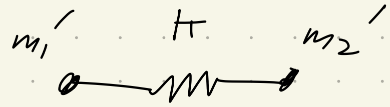
$$\rightarrow \quad a^2 = x_0^2 + \frac{V_0^2}{\omega^2}$$

$$\boxed{a = \sqrt{x_0^2 + \frac{V_0^2}{\omega^2}}}$$

Also: $\boxed{\tan\alpha = -\frac{V_0}{\omega x_0}}$

Sec 21, Prob 2





$$T = \frac{1}{2} m |\dot{\vec{r}}|^2 = \frac{1}{2} m \dot{r}^2 \quad \text{where} \quad m = \frac{m_1 m_2}{m_1 + m_2}$$

$$U = \frac{1}{2} k r^2$$

$$L = T - U$$

$$= \frac{1}{2} m \dot{r}^2 - \frac{1}{2} k r^2$$

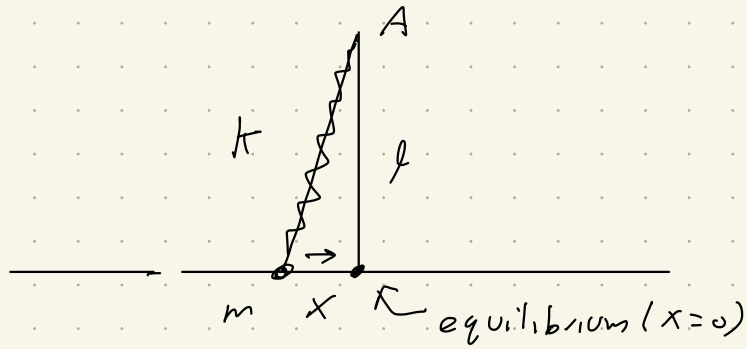
$$\omega = \sqrt{\frac{k}{m}}$$

Similarly, $\omega' = \sqrt{\frac{k}{m'}}$, $m' = \frac{m_1' m_2'}{m_1' + m_2'}$

so $\frac{\omega'}{\omega} = \sqrt{\frac{k}{m'}} \sqrt{\frac{m}{k}} = \sqrt{\frac{m}{m'}}$

$$= \sqrt{\left(\frac{m_1 m_2}{m_1 + m_2}\right)\left(\frac{m_1' + m_2'}{m_1' m_2'}\right)}$$

Sec 21, prob 3



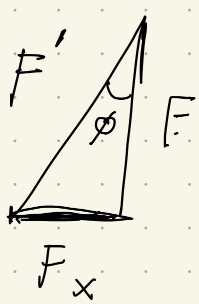
$$F = kl$$

$$F' = k\sqrt{l^2 + x^2} \qquad x \ll l$$

$$= kl\sqrt{1 + \left(\frac{x}{l}\right)^2}$$

$$= kl\left(1 + \frac{1}{2}\left(\frac{x}{l}\right)^2\right)$$

$$= F\left(1 + \frac{1}{2}\left(\frac{x}{l}\right)^2\right)$$



$$F_x = F'\sin\phi$$

$$= F'\,\frac{x}{l}$$

$$\approx F\left(1 + \frac{1}{2}\left(\frac{x}{l}\right)^2\right)\frac{x}{l}$$

$$\approx F\,\frac{x}{l}$$

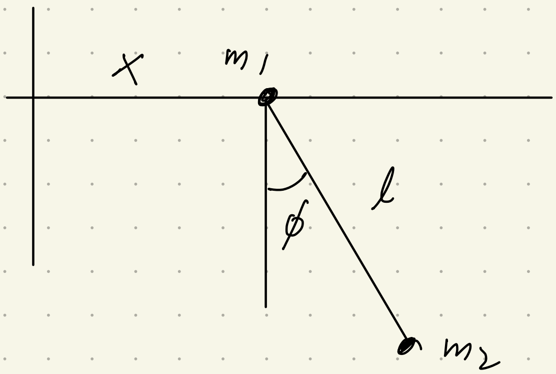
$$-F\frac{x}{l} = m\ddot{x} \;\rightarrow\; \ddot{x} = -\left(\frac{F}{ml}\right)x, \quad \boxed{\omega = \sqrt{\frac{F}{ml}}}$$

$$U = F\delta l, \quad \delta l = \sqrt{l^2 + x^2} - l \;\rightarrow\; U = \frac{1}{2}\frac{F}{l}x^2$$

$$= l\sqrt{1 + \left(\frac{x}{l}\right)^2} - l$$

$$\simeq l\,\frac{1}{2}\left(\frac{x}{l}\right)^2 = \frac{1}{2}\frac{x^2}{l}$$

Sec 21, prob 5



From Sec 14, prob 3

$$E = \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left(1 - \frac{m_2}{m_1+m_2} \cos^2\phi\right) - m_2 g l \cos\phi$$

stable equilibrium: $\phi = 0$

$$\cos\phi \approx 1 - \frac{\phi^2}{2}$$

$$\cos^2\phi \approx \left(1 - \frac{\phi^2}{2}\right)^2$$

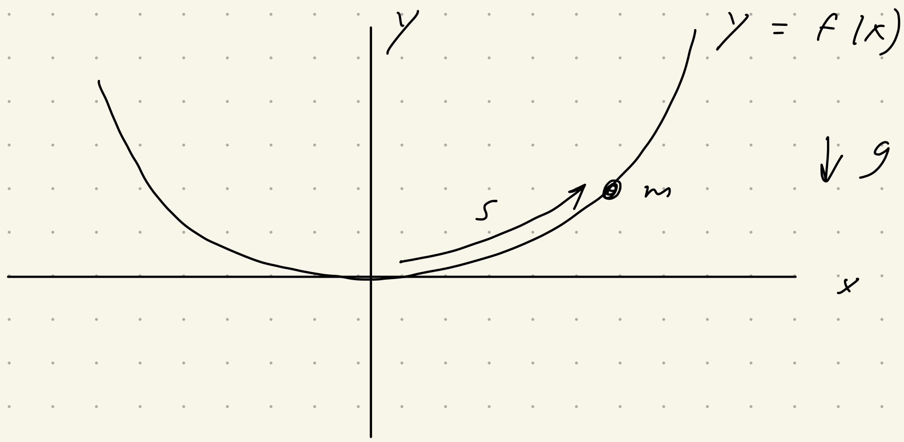
$$\approx 1 - \phi^2$$

$$E \approx \frac{1}{2} m_2 l^2 \dot{\phi}^2 \underbrace{\left(1 - \frac{m_2}{m_1+m_2}\right)}_{\frac{m_1}{m_1+m_2}} - m_2 g l \left(1 - \frac{\phi^2}{2}\right)$$

$$\rightarrow E + m_2 g l = \frac{1}{2} m l^2 \dot{\phi}^2 + \frac{1}{2} m_2 g l \phi^2$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{m_2 g l}{m l^2}} = \sqrt{\left(\frac{m_1+m_2}{m_1}\right) \frac{g}{l}}$$

Sec 21, Prob 6



$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2} m\dot{s}^2$$

$$ds = \sqrt{dx^2+dy^2}$$
$$= dx\sqrt{1+y'^2}$$

For period (or freq) to be independent of initial amplitude, we need

$U = \frac{1}{2} k s^2$ for some $k > 0$

Then $L = T - U$
$$= \frac{1}{2} m\dot{s}^2 - \frac{1}{2} k s^2$$

$$\rightarrow \quad s(t) = a\cos(\omega t + \alpha), \quad \omega = \sqrt{\frac{k}{m}}$$

$$U = \frac{1}{2} k s^2 = mgy \longrightarrow y = \frac{1}{2} A s^2$$

where $A \equiv \frac{k}{mg}$

Differentiate:
$$dy = A\, s\, ds$$
$$= A\sqrt{\frac{2y}{A}}\sqrt{dx^2+dy^2}$$

$$\cancel{d\ell} = \sqrt{2Ay}\;\cancel{dy}\sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

$$\rightarrow 1+\left(\frac{dx}{dy}\right)^2 = \frac{1}{2Ay}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{2Ay} - 1$$

$$= \frac{1-2Ay}{2Ay}$$

so $$\frac{dx}{dy} = \pm\sqrt{\frac{1-2Ay}{2Ay}}$$

$$\rightarrow \quad x = \int dy\sqrt{\frac{1-2Ay}{2Ay}} + \text{const}$$

Make a change of variables so that

$$2Ay = \frac{1}{2}(1-\cos\theta) = \sin^2\left(\frac{\theta}{2}\right)$$

$$1-2Ay = 1-\sin^2\left(\frac{\theta}{2}\right) = \cos^2\left(\frac{\theta}{2}\right)$$

$$2A\,dy = \cancel{2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\cdot\frac{d\theta}{\cancel{2}} = \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)d\theta$$

$$\cos\theta = \cos^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\theta}{2}\right) = 1-2\sin^2\left(\frac{\theta}{2}\right) = 2\cos^2\left(\frac{\theta}{2}\right)-1$$

Thus,

$$x = \int \frac{1}{2A} \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) d\theta \sqrt{\frac{\cos^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}}$$

$$= \int \frac{1}{2A} \cos^2\left(\frac{\theta}{2}\right) d\theta$$

$$= \frac{1}{4A} \int (1 + \cos\theta)\, d\theta$$

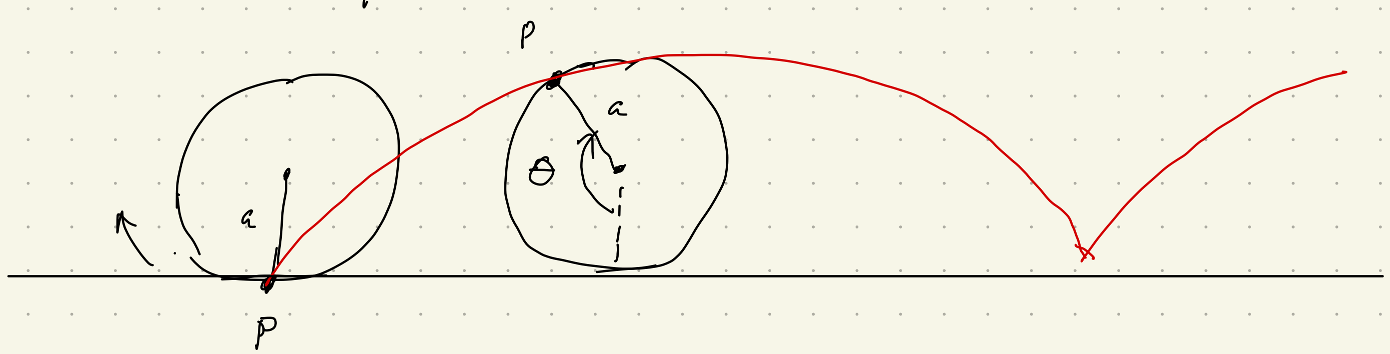
$$= \frac{1}{4A}(\theta + \sin\theta)$$

Summary:

$$x = \frac{1}{4A}(\theta + \sin\theta) = a(\theta + \sin\theta)$$

$$y = \frac{1}{4A}(1 - \cos\theta) = a(1 - \cos\theta)$$

Parametric equations for a cycloid:



Sec 22, prob 1 a, b, c :

$$\xi = e^{i\omega t}\left[\int_0^t d\bar{t}\,\frac{F(\bar{t})}{m}\,e^{-i\omega\bar{t}} + \xi_0\right]$$

$$\xi = \dot{x} + i\omega x \quad \rightarrow \quad x(t) = \frac{1}{\omega}\,\mathrm{Im}(\xi(t))$$

Assume: $x = \dot{x} = 0$ at $t = 0$

$$\rightarrow \xi_0 = 0$$

a) $F = F_0 = \text{const}$

$$\xi = e^{i\omega t}\int_0^t d\bar{t}\,\frac{F_0}{m}\,e^{-i\omega\bar{t}}$$

$$= \frac{F_0}{m}\,e^{i\omega t}\,\frac{1}{-i\omega}\,e^{-i\omega\bar{t}}\Big|_0^t$$

$$= \frac{iF_0}{m\omega}\,e^{i\omega t}\left[e^{-i\omega t} - 1\right]$$

$$= \frac{iF_0}{m\omega}\left[1 - e^{i\omega t}\right]$$

$$= \frac{iF_0}{m\omega}\left[1 - (\cos(\omega t) + i\sin(\omega t))\right]$$

$$= \frac{iF_0}{m\omega}(1 - \cos\omega t) + \frac{F_0}{m\omega}\sin\omega t$$

$$\rightarrow \boxed{x(t) = \frac{F_0}{m\omega^2}(1 - \cos\omega t)}$$

b) $F(t) = at$

$$\xi = e^{i\omega t}\int_0^t d\bar{t}\,\frac{a\bar{t}}{m}e^{-i\omega\bar{t}}$$

$$= \frac{a}{m}e^{i\omega t}\int_0^t d\bar{t}\;\bar{t}\,e^{-i\omega\bar{t}}$$

Let: $u = \bar{t}$, $du = d\bar{t}$

$$dv = d\bar{t}\,e^{-i\omega\bar{t}} \rightarrow v = \frac{1}{-i\omega}e^{-i\omega\bar{t}}$$

Thus,

$$\xi = \frac{a}{m}e^{i\omega t}\left[\frac{\bar{t}}{-i\omega}e^{-i\omega\bar{t}}\Big|_0^t - \int_0^t \frac{1}{-i\omega}e^{-i\omega\bar{t}}\,d\bar{t}\right]$$

$$= \frac{a}{m}e^{i\omega t}\left[i\frac{t}{\omega}e^{-i\omega t} + \frac{1}{i\omega}\left(\frac{1}{-i\omega}\right)e^{-i\omega\bar{t}}\Big|_0^t\right]$$

$$= \frac{a}{m}e^{i\omega t}\left[i\frac{t}{\omega}e^{-i\omega t} + \frac{1}{\omega^2}\left(e^{-i\omega t}-1\right)\right]$$

$$= i\frac{at}{m\omega} + \frac{a}{m\omega^2}\left(1 - e^{i\omega t}\right)$$

$$= i\frac{at}{m\omega} + \frac{a}{m\omega^2}\left(1 - (\cos\omega t + i\sin\omega t)\right)$$

$$= \frac{a}{m\omega^2}(1-\cos\omega t) + i\left(\frac{at}{m\omega} - \frac{a}{m\omega^2}\sin\omega t\right)$$

$$= \frac{a}{m\omega^2}(1-\cos\omega t) + i\frac{a}{m\omega^2}(\omega t - \sin\omega t)$$

$$\rightarrow \boxed{x(t) = \frac{a}{m\omega^3}(\omega t - \sin\omega t)}$$

c) $F = F_0 \exp(-\alpha t)$

$$\xi = e^{i\omega t} \int_0^t d\bar{t}\ \frac{F_0 e^{-\alpha \bar{t}}}{m}\ e^{-i\omega \bar{t}}$$

$$= \frac{F_0}{m} e^{i\omega t} \int_0^t d\bar{t}\ e^{-(i\omega+\alpha)\bar{t}}$$

$$= \frac{F_0}{m} e^{i\omega t} \frac{1}{-(i\omega+\alpha)} e^{-(i\omega+\alpha)\bar{t}} \Big|_0^t$$

$$= -\frac{F_0}{m} e^{i\omega t} \frac{1}{(i\omega+\alpha)} \left(e^{-(i\omega+\alpha)t} - 1\right)$$

$$= -\frac{F_0}{m} \frac{1}{(i\omega+\alpha)} \left(e^{-\alpha t} - e^{i\omega t}\right)$$

$$= -\frac{F_0}{m} \left(\frac{\alpha - i\omega}{\alpha^2+\omega^2}\right)\left[e^{-\alpha t} - (\cos\omega t + i \sin\omega t)\right]$$

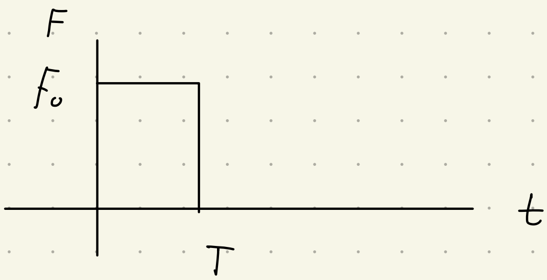
$$= -\frac{F_0}{m} \left(\frac{\alpha - i\omega}{\alpha^2+\omega^2}\right)\left[\left(e^{-\alpha t} - \cos\omega t\right) - i\sin\omega t\right]$$

$$\rightarrow x = \frac{\operatorname{Im}\xi}{\omega}$$

$$= -\frac{F_0}{m\omega}\left(\frac{1}{\alpha^2+\omega^2}\right)\left(-\omega\left(e^{-\alpha t} - \cos\omega t\right) - \alpha \sin\omega t\right)$$

$$= \boxed{\frac{F_0}{m}\left(\frac{1}{\alpha^2+\omega^2}\right)\left(e^{-\alpha t} - \cos\omega t + \frac{\alpha}{\omega}\sin\omega t\right)}$$

Sec 22, Prob 3:



$x = 0,\ \dot{x} = 0$ at $t = 0$ $\rightarrow$ $\xi_0 = 0$

$$\xi = e^{i\omega t} \int_0^t d\bar{t}\, \frac{F_0}{m} e^{-i\omega \bar{t}}$$

$$= \frac{F_0}{m} e^{i\omega t} \frac{1}{-i\omega} e^{-i\omega \bar{t}} \Big|_0^t$$

$$= \frac{-F_0}{im\omega} e^{i\omega t} \left(e^{-i\omega t} - 1 \right)$$

$$= \frac{-F_0}{im\omega} \left(1 - e^{i\omega t} \right)$$

$$= \frac{-F_0}{im\omega} \left[(1 - \cos\omega t) - i \sin\omega t \right]$$

$$= \frac{F_0}{m\omega} \left[i(1 - \cos\omega t) + \sin\omega t \right]$$

$$\rightarrow \quad x(t) = \frac{\mathrm{Im}\, \xi}{\omega}$$

$$= \frac{F_0}{m\omega^2} (1 - \cos\omega t)$$

$$\dot{x}(t) = \frac{F_0}{m\omega} \sin\omega t$$

For $t > T$, general solution is

$$x(t) = c_1 \cos(\omega(t-T)) + c_2 \sin(\omega(t-T))$$

$$\dot{x}(t) = -c_1 \omega \sin(\omega(t-T)) + c_2 \omega \cos(\omega(t-T))$$

Match x and $\dot{x}$ at $t = T$:

$$\rightarrow \quad \frac{F_0}{m\omega^2}(1 - \cos\omega T) = c_1$$

$$\rightarrow \quad \frac{F_0}{m\omega} \sin\omega T = c_2 \omega$$

Thus,
$$c_1 = \frac{F_0}{m\omega^2}(1-\cos\omega T)$$

$$c_2 = \frac{F_0}{m\omega^2} \sin\omega T$$

$$x(t) = \frac{F_0}{m\omega^2}(1-\cos\omega T)\cos(\omega(t-T)) + \frac{F_0}{m\omega^2}\sin\omega T \sin(\omega(t-T))$$

$$\rightarrow \quad a = \frac{F_0}{m\omega^2}\sqrt{(1-\cos\omega T)^2 + \sin^2\omega T}$$

$$= \frac{F_0}{m\omega^2}\sqrt{1 + \cos^2\omega T - 2\cos\omega T + \sin^2\omega T}$$

$$= \frac{F_0}{m\omega^2}\sqrt{2(1-\cos\omega T)}$$

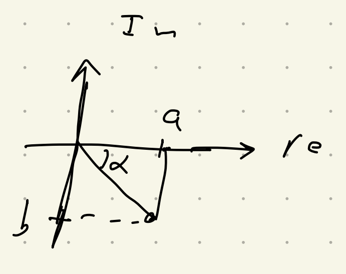
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$

$$2\sin^2\theta = 1 - \cos 2\theta$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Thus, $1 - \cos\omega T = 2\sin^2\left(\frac{\omega T}{2}\right)$

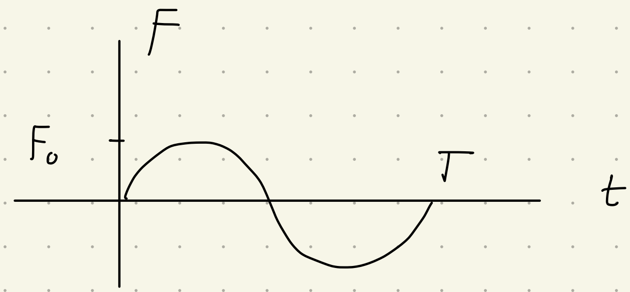
$$a = \frac{F_0}{m\omega^2}\sqrt{4\sin^2\left(\frac{\omega T}{2}\right)}$$
$$= \frac{2F_0}{m\omega^2}\sin\left(\frac{\omega T}{2}\right)$$

$a\cos\omega t + b\sin\omega t$
$$= \mathrm{Re}\left[a e^{i\omega t} - i b e^{i\omega t}\right]$$
$$= \mathrm{Re}\left[(a-ib)e^{i\omega t}\right]$$
$$= \mathrm{Re}\left[\sqrt{a^2+b^2}\, e^{i(\omega t-\alpha)}\right]$$
$$= \sqrt{a^2+b^2}\cos(\omega t-\alpha)$$



$$a - ib = \sqrt{a^2+b^2}\, e^{-i\alpha}$$
$$\tan\alpha = \frac{b}{a}$$

Sec 22, Prob 5:



$$F = F_0 \sin \omega t \quad , \qquad T = \frac{2\pi}{\omega}$$

$x=0,\ \dot{x}=0$ at $t=0 \rightarrow \xi_0 = 0$

$$\xi = e^{i\omega t} \int_0^t d\bar{t}\ \frac{F_0 \sin\omega\bar{t}}{m}\ e^{-i\omega\bar{t}}$$

$$= \frac{F_0}{m} e^{i\omega t} \int_0^t d\bar{t}\ \frac{1}{2i}\left(e^{i\omega\bar{t}} - e^{-i\omega\bar{t}}\right) e^{-i\omega\bar{t}}$$

$$= \frac{F_0}{i2m} e^{i\omega t} \int_0^t d\bar{t}\ \left(1 - e^{-2i\omega\bar{t}}\right)$$

$$= \frac{F_0}{i2m} e^{i\omega t} \left(\bar{t} + \frac{1}{i2\omega} e^{-2i\omega\bar{t}}\right)\Big|_0^t$$

$$= \frac{F_0}{i2m} e^{i\omega t} \left[t + \frac{1}{i2\omega}\left(e^{-2i\omega t} - 1\right)\right]$$

$$= \frac{F_0}{i2m} \left[t e^{i\omega t} + \frac{1}{i2\omega}\left(e^{-i\omega t} - e^{i\omega t}\right)\right]$$

$$= \frac{F_0}{i2m} \left[t e^{i\omega t} - \frac{1}{\omega} \sin\omega t\right]$$

$$= \frac{F_0}{i2m}\left[\, t\,(\cos\omega t + i\sin\omega t) - \frac{1}{\omega}\sin\omega t\right]$$

$$= \frac{F_0}{i2m}\left[\left(t\cos\omega t - \frac{1}{\omega}\sin\omega t\right) + i\,t\sin\omega t\right]$$

$$\rightarrow \quad x(t) = \frac{\operatorname{Im}\{\}}{\omega}$$

$$= \frac{-F_0}{2m\omega}\left(t\cos\omega t - \frac{1}{\omega}\sin\omega t\right)$$

$$\dot{x}(t) = \frac{-F_0}{2m\omega}\left(\cancel{\cos\omega t} - \omega t\sin\omega t - \cancel{\cos\omega t}\right)$$

$$= \frac{F_0 t}{2m}\sin\omega t$$

match with (at $t = T = 2\pi/\omega$)

$$x(t) = c_1\cos(\omega(t-T)) + c_2\sin(\omega(t-T))$$

$$\dot{x}(t) = -\omega c_1\sin(\omega(t-T)) + \omega c_2\cos(\omega(t-T))$$

thus,
$$\frac{-F_0}{2m\omega}\left(T\underbrace{\cos\omega T}_{=1} - \frac{1}{\omega}\underbrace{\sin\omega T}_{0}\right) = c_1$$

$$\boxed{c_1 = \frac{-F_0\pi}{m\omega^2}}$$

and
$$\frac{F_0 T}{2m}\underbrace{\cancel{\sin\omega T}}_{0} = \omega c_2 \rightarrow \boxed{c_2 = 0}$$

Thus, for $t > T$:

$$x(t) = -\frac{F_0 \pi}{m\omega^2} \cos(\omega(t-T))$$

$$\rightarrow \quad a = \frac{F_0 \pi}{m\omega^2} \quad \text{(amplitude)}$$

Sec 23, Prob 1

$$L = \frac{1}{2}(\dot{x}^2+\dot{y}^2) - \frac{1}{2}\omega_0^2(x^2+y^2) + \alpha \cancel{x}y$$

$$= \frac{1}{2}\sum_{i,k} m_{ik}\,\dot{x}_i \dot{x}_k - \frac{1}{2}\sum_{i,k} k_{ik}\,x_i x_k$$

where $m_{ik} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$k_{ik} = \begin{bmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{bmatrix}$$

$$x_i = \begin{bmatrix} x & y \end{bmatrix}$$

Characteristic equation

$$0 = \det\left(k_{ik} - \omega^2 m_{ik}\right)$$

$$= \det \begin{bmatrix} \omega_0^2 - \omega^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega^2 \end{bmatrix}$$

$$= (\omega_0^2 - \omega^2)^2 - \alpha^2$$

$$\rightarrow (\omega_0^2 - \omega^2) = \pm\alpha$$

$$\omega^2 = \omega_0^2 \mp \alpha$$

so $\boxed{\begin{aligned} \omega_+^2 &= \omega_0^2 + \alpha \\ \omega_-^2 &= \omega_0^2 - \alpha \end{aligned}}$

Eigen vectors:

ω_+^2 :
$$\begin{pmatrix} \omega_0^2-\omega_+^2 & -\alpha \\ -\alpha & \omega_0^2-\omega_+^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\alpha & -\alpha \\ -\alpha & -\alpha \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \quad A_1 = -A_2$$

so $\quad V_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

ω_-^2 :
$$\begin{pmatrix} \omega_0^2-\omega_-^2 & -\alpha \\ -\alpha & \omega_0^2-\omega_-^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \quad A_2 = A_1$$

so $\quad V_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General solution:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathrm{Re}\left(\sum_{\alpha=+,-} \overbrace{C_\alpha}^{\text{complex constants}} V_\alpha e^{i\omega_\alpha t} \right)$$

$$= \sum_{\alpha=+,-} V_\alpha \Theta_\alpha \quad , \quad \Theta_\alpha \equiv \mathrm{Re}\left(C_\alpha e^{i\omega_\alpha t}\right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = v_+ \theta_+ + v_- \theta_-$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \theta_+ + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \theta_-$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \theta_+ + \theta_- \\ -\theta_+ + \theta_- \end{bmatrix}$$

For weak coupling ($\alpha << \omega_0^2$):

$$\omega_\pm^2 = \omega_0^2 \pm \alpha$$

$$= \omega_0^2 \left(1 \pm \frac{\alpha}{\omega_0^2} \right)$$

$$\rightarrow \omega_\pm = \omega_0 \sqrt{1 \pm \frac{\alpha}{\omega_0^2}}$$

$$\approx \omega_0 \left(1 \pm \frac{1}{2} \frac{\alpha}{\omega_0^2} \right)$$

$$= \begin{cases} \omega_0 + \frac{1}{2} \frac{\alpha}{\omega_0} \\ \omega_0 - \frac{1}{2} \frac{\alpha}{\omega_0} \end{cases}$$

$$\theta_+ + \theta_- = \mathrm{Re}\left(e^{i\omega_+ t} + e^{i\omega_- t} \right)$$

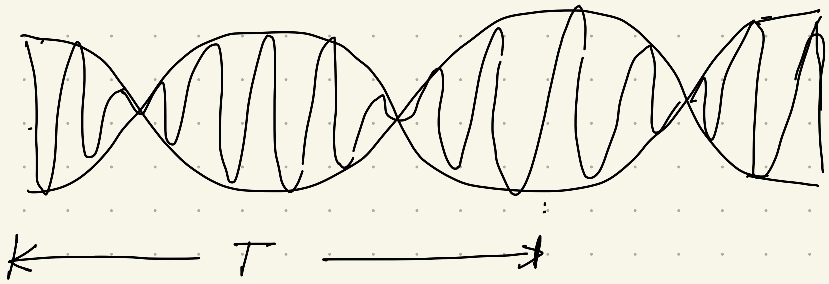
$$= \mathrm{Re}\left[e^{i\omega_0 t} \left(e^{i \frac{1}{2} \frac{\alpha}{\omega_0} t} + e^{-i \frac{1}{2} \frac{\alpha}{\omega_0} t} \right) \right]$$

$$= \mathrm{Re}\left[e^{i\omega_0 t} \, 2 \cos\left(\frac{\alpha t}{2\omega_0} \right) \right]$$

$$= 2 \cos(\omega_0 t) \cos\left(\frac{\alpha t}{2\omega_0} \right)$$

$$\theta_+ - \theta_- = -2 \sin(\omega_0 t) \sin\left(\frac{\alpha t}{2\omega_0} \right)$$

Amplitude modulation :



$$T_{beat} = \frac{1}{2} T = \frac{1}{2} \frac{2\pi}{(\alpha/2\omega_0)} = \frac{2\pi}{\alpha/\omega_0}$$

$$= \frac{2\pi}{\omega_{beat}}$$

$$\omega_{beat} = \alpha/\omega_0$$

$$= \omega_+ - \omega_-$$

Sec 23, Prob 3:

Space oscillator $U = \frac{1}{2} k r^2$

$$L = T - U$$
$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - \frac{1}{2} k r^2$$
$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k (x^2 + y^2)$$
$$= \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + \frac{1}{2} m \dot{y}^2 - \frac{1}{2} k y^2$$

General solution:

$$x = a \cos(\omega t + \alpha)$$
$$y = b \cos(\omega t + \beta)$$

where

$$\omega_x = \omega_y \equiv \omega = \sqrt{\frac{k}{m}}$$

Now: $x = a(\cos(\omega t) \cos\alpha - \sin(\omega t) \sin\alpha)$

$y = b(\cos(\omega t) \cos\beta - \sin(\omega t) \sin\beta)$

$$\rightarrow \begin{bmatrix} x/a \\ y/b \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \cos\beta & -\sin\beta \end{bmatrix}}_{M} \begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix}$$

$$\det M = -\cos\alpha \sin\beta + \sin\alpha \cos\beta$$
$$= \sin(\alpha - \beta)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus

$$\boxed{\begin{array}{c}\cos\omega t \\ \hline \sin\omega t\end{array}} = M^{-1} \boxed{\begin{array}{c} x/a \\ \hline y/b \end{array}}$$

$$= \frac{1}{\sin(\alpha-\beta)} \boxed{\begin{array}{c|c} -\sin\beta & \sin\alpha \\ \hline -\cos\beta & \cos\alpha \end{array}} \boxed{\begin{array}{c} x/a \\ \hline y/b \end{array}}$$

square and add:

$$1 = \cos^2\omega t + \sin^2\omega t$$

$$= \frac{1}{\sin^2(\alpha-\beta)}\left(-\sin\beta\left(\frac{x}{a}\right) + \sin\alpha\left(\frac{y}{b}\right)\right)^2$$

$$+ \frac{1}{\sin^2(\alpha-\beta)}\left(-\cos\beta\left(\frac{x}{a}\right) + \cos\alpha\left(\frac{y}{b}\right)\right)^2$$

$$= \frac{1}{\sin^2(\alpha-\beta)}\Bigg[\left(\frac{x}{a}\right)^2 \underbrace{\left(\sin^2\beta + \cos^2\beta\right)}_{=1}$$

$$+ \left(\frac{y}{b}\right)^2\left(\sin^2\alpha + \cos^2\alpha\right)$$

$$- 2\underbrace{\left(\sin\alpha\sin\beta + \cos\alpha\cos\beta\right)}_{\cos(\alpha-\beta)}\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\Bigg]$$

$$= \frac{1}{\sin^2(\alpha-\beta)}\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\cos(\alpha-\beta)\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\right]$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\cos(\alpha-\beta)\left(\frac{x}{a}\right)\left(\frac{y}{b}\right) = \sin^2(\alpha-\beta)$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\cos\delta = \sin^2\delta$$

where $\delta \equiv \alpha - \beta$.

<u>NOTE:</u>

(i) when $\delta = 0$

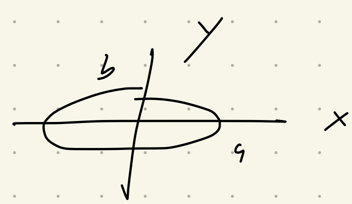
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\left(\frac{x}{a}\right)\left(\frac{y}{b}\right) = 0$$

$$\left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

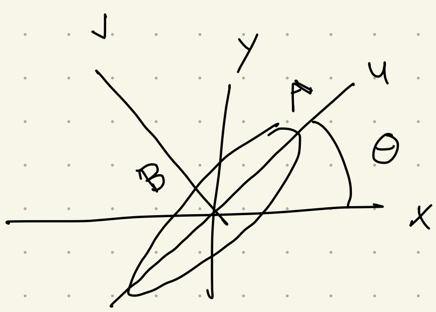
$$\frac{x}{a} = \frac{y}{b} \qquad \text{(straight line)}$$

(ii) when $\delta = \pi/2$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



(iii)



(need to determine θ, A, B in terms of δ, a, b)

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$x = u\cos\theta - v\sin\theta$$
$$y = u\sin\theta + v\cos\theta$$

$$\sin^2\delta = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\cos\delta$$

$$= \frac{1}{a^2}\left(u^2\cos^2\theta + v^2\sin^2\theta - 2uv\cos\theta\sin\theta\right)$$
$$+ \frac{1}{b^2}\left(u^2\sin^2\theta + v^2\cos^2\theta + 2uv\sin\theta\cos\theta\right)$$

$$- \frac{2\cos\delta}{ab}\left(u^2\sin\theta\cos\theta - v^2\sin\theta\cos\theta + uv\left(\cos^2\theta - \sin^2\theta\right)\right)$$

$$= u^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} - \frac{2}{ab}\cos\delta\sin\theta\cos\theta\right)$$
$$+ v^2\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} + \frac{2}{ab}\cos\delta\sin\theta\cos\theta\right)$$
$$+ 2uv\left(\left(-\frac{1}{a^2} + \frac{1}{b^2}\right)\underbrace{\sin\theta\cos\theta}_{\frac{\sin 2\theta}{2}} - \frac{\cos\delta}{ab}\cos 2\theta\right)$$

Can make the uv term vanish by choosing θ such that:

$$\frac{1}{2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\sin 2\theta - \frac{\cos\delta}{ab}\cos 2\theta = 0$$

$$\rightarrow \quad \tan 2\theta = \frac{\dfrac{\cos\delta}{ab}}{\dfrac{1}{2}\left(\dfrac{1}{b^2}-\dfrac{1}{a^2}\right)}$$

$$= \frac{2\cos\delta}{ab}\,\frac{a^2b^2}{(a^2-b^2)}$$

$$= 2\cos\delta\left(\frac{ab}{a^2-b^2}\right)$$

Thus,

$$\sin^2\delta = u^2 \quad + \quad v^2$$

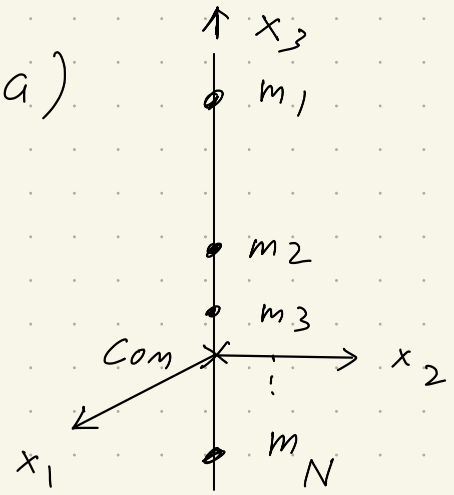
$$1 = \left(\frac{u}{A}\right)^2 + \left(\frac{v}{B}\right)^2$$

where
$$A^2 = \frac{\sin^2\delta}{\left(\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}-\dfrac{\cos\delta}{ab}\sin 2\theta\right)}$$

$$B^2 = \frac{\sin^2\delta}{\left(\dfrac{\sin^2\theta}{a^2}+\dfrac{\cos^2\theta}{b^2}+\dfrac{\cos\delta}{ab}\sin 2\theta\right)}$$

Sec 32, prob 1

a)



$I_3 = 0$ since masses only have a z-component

$$z_{com} = \frac{1}{M} \sum_a m_a z_a$$

$$I_1 = I_2 \equiv I$$

$$= \sum_a m_a \left(r_a^2 - x_{a1}^2 \right)$$

$$= \sum_a m_a \left(\cancel{x_{a2}^2}^{\,0} + x_{a3}^2 \right)$$

$$= \sum_a m_a x_{a3}^2$$

$$= \sum_a m_a \left(z_a - z_{com} \right)^2$$

Suppose we have only two masses

Then:

$$I = m_1 (z_1 - z_{com})^2 + m_2 (z_2 - z_{com})^2$$

$$z_{com} = \left(\frac{1}{m_1 + m_2} \right) (m_1 z_1 + m_2 z_2)$$

$$\rightarrow \quad z_1 - z_{com} = \frac{(\cancel{m_1} + m_2) z_1 - (\cancel{m_1 z_1} + m_2 z_2)}{m_1 + m_2}$$

$$= \frac{m_2 (z_1 - z_2)}{m_1 + m_2}$$

$$Z_2 - Z_{com} = \frac{(m_1 + \cancel{m_2}) z_2 - (m_1 z_1 + \cancel{m_2 z_2})}{m_1 + m_2}$$

$$= \frac{m_1 (z_2 - z_1)}{m_1 + m_2}$$

$$= -\frac{m_1 (z_1 - z_2)}{m_1 + m_2}$$

Thus,

$$I = m_1 \frac{m_2^2 (z_1 - z_2)^2}{(m_1 + m_2)^2} + m_2 \frac{m_1^2 (z_1 - z_2)^2}{(m_1 + m_2)^2}$$

$$= \frac{m_1 m_2}{(m_1 + m_2)^{\cancel{2}}} (z_1 - z_2)^2 \cancel{(m_2 + m_1)}$$

$$= \underbrace{\frac{m_1 m_2}{}}_{\mu} l^2 \qquad \text{where } l \equiv |z_1 - z_2|$$

$$= m l^2 \qquad \text{where } m = \text{reduced mass}$$

$$= \frac{m_1 m_2}{m_1 + m_2}$$

In general:

$$I = \sum_a m_a (z_a - z_{com})^2$$

$$= \sum_a m_a (z_a^2 - 2 z_a z_{com} + z_{com}^2)$$

$$= \sum_a m_a z_a^2 - 2\left(\sum_a m_a z_a\right) z_{com} + \mu z_{com}^2$$

$$= \sum_a m_a z_a^2 - \mu z_{com}^2$$

$$= \sum_a m_a z_a^2 - \mu \left(\frac{1}{\mu}\sum_a m_a z_a\right)\left(\frac{1}{\mu}\sum_b m_b z_b\right)$$

$$= \frac{1}{\mu}\sum_a m_a\left(\sum_b m_b\right) z_a^2 - \frac{1}{\mu}\sum_a\sum_b m_a m_b z_a z_b$$

$$= \frac{1}{\mu}\Big[\frac{1}{2}\sum_a\sum_b m_a m_b z_a^2 + \frac{1}{2}\sum_a\sum_b m_a m_b z_b^2$$

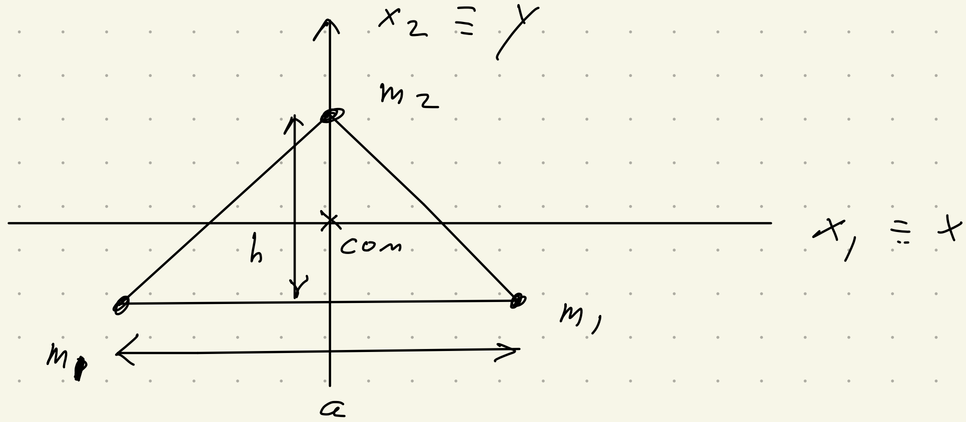
$$- \sum_a\sum_b m_a m_b z_a z_b\Big]$$

$$= \frac{1}{2\mu}\sum_a\sum_b m_a m_b \underbrace{(z_a^2 + z_b^2 - 2 z_a z_b)}_{(z_a - z_b)^2}$$

$$= \frac{1}{2\mu}\sum_a\sum_b m_a m_b l_{ab}^2$$

where $l_{ab} \equiv |z_a - z_b|$

b)



x_3 : out of page

To determine x_{12} and x_{22} we require that com be at origin.

$$2 m_1 y_1 + m_2 y_2 = 0$$

$$2 m_1 y_1 + m_2 (y_1 + h) = 0$$

$$(2 m_1 + m_2) y_1 + m_2 h = 0$$

$$\rightarrow \quad y_1 = \frac{-m_2 h}{\mu}, \quad \mu = 2m_1 + m_2$$

$$\rightarrow \quad y_2 = \frac{-2m_1 y_1}{m_2}$$

$$= -\frac{2 m_1}{\cancel{m_2}} \left(-\frac{\cancel{m_2}}{\mu} \right) h$$

$$= \frac{2 m_1 h}{\mu}$$

$$I_3 = \sum_a m_a \left(r_a^2 - z_a^2 \right)$$

$$= \sum_a m_a \left(x_a^2 + y_a^2 \right)$$

$$= 2 m_1 \left(x_1^2 + y_1^2 \right) + m_2 \left(\cancelto{0}{x_2}^2 + y_2^2 \right)$$

$$= 2 m_1 \left(\left(\frac{a}{2}\right)^2 + \frac{m_2^2 h^2}{\mu^2} \right) + m_2 \left(\frac{2 m_1 h}{\mu} \right)^2$$

$$= \frac{m_1 a^2}{2} + \frac{2 m_1 m_2^2 h^2}{\mu^2} + \frac{4 m_2 m_1^2 h^2}{\mu^2}$$

$$= \frac{m_1 a^2}{2} + \frac{2 m_1 m_2 h^2}{\mu^2} \left(m_2 + 2 m_1 \right)$$

$$= \frac{1}{2} m_1 a^2 + \frac{2 m_1 m_2 h^2}{\mu}$$

$$I_1 = \sum_a m_a \left(r_a^2 - x_a^2 \right)$$

$$= \sum_a m_a y_a^2$$

$$= 2 m_1 y_1^2 + m_2 y_2^2$$

$$= 2 m_1 \left(\frac{-m_2 h}{\mu} \right)^2 + m_2 \left(\frac{2 m_1 h}{\mu} \right)^2$$

$$= \frac{2 m_1 m_2^2 h^2}{\mu^2} + \frac{4 m_1 m_2 h^2}{\mu^2}$$

$$= \frac{2 m_1 m_2 h^2}{\mu}$$

$$I_2 = \sum_a m_a \left(r_a^2 - y_a^2 \right)$$

$$= \sum_a m_a x_a^2$$

$$= 2 m_1 \left(\frac{a}{2}\right)^2 + m_2 \cancel{x_{21}^2}^{\,0}$$

$$= \frac{1}{2} m_1 a^2$$

$\rightarrow$

NOTE: $I_3 = I_1 + I_2$

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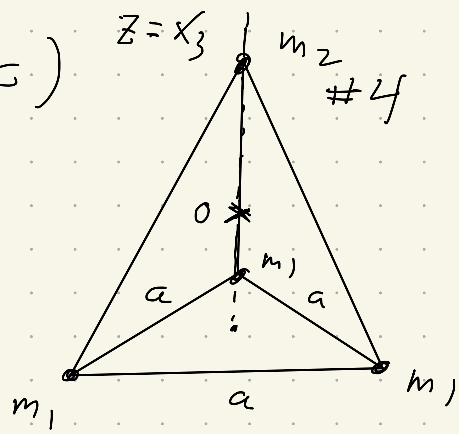
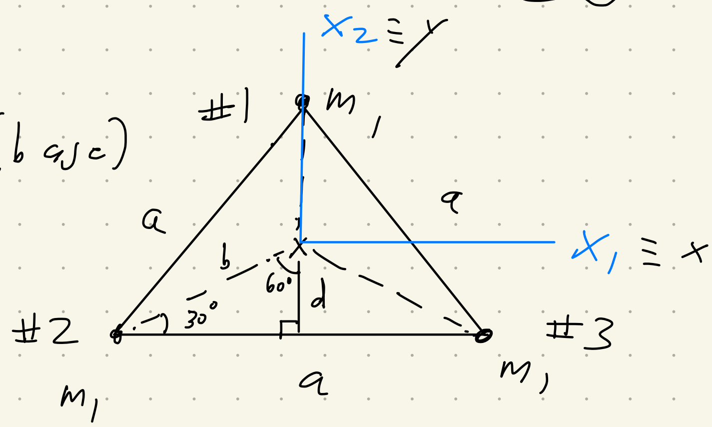
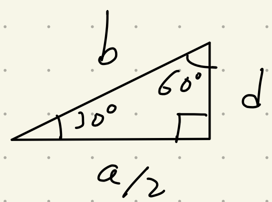
c)



(base)





$$\sin 60° = \frac{a/2}{b}$$

$$\frac{\sqrt{3}}{2} = \frac{a}{2b}$$

$$\rightarrow b = \frac{a}{\sqrt{3}}$$

$$\sin 30° = \frac{d}{b}$$

$$\frac{1}{2} = \frac{d}{a/\sqrt{3}}$$

$$d = \frac{a}{2\sqrt{3}}$$

thus, $(x_1, y_1) = \left(0, \frac{a}{\sqrt{3}}\right)$

$$(x_2, y_2) = \left(-\frac{a}{2}, -\frac{a}{2\sqrt{3}}\right)$$

$$(x_3, y_3) = \left(\frac{a}{2}, -\frac{a}{2\sqrt{3}}\right)$$

Assume origin at COM of system:

$$0 = m_1 (z_1 + z_2 + z_3) + m_2 z_4$$

$$= 3 m_1 z_1 + m_2 z_4 \quad , \qquad z_4 = h + z_1$$

h = height of tetrahedron

$$= 3 m_1 z_1 + m_2 (h + z_1)$$

$$= (3m_1 + m_2) z_1 + m_2 h$$

$$\rightarrow \quad z_1 = -\frac{m_2}{\mu} h \quad \left(= z_2 = z_3 \right)$$

$$z_4 = h + z_1 = h - \frac{m_2}{\mu} h$$

$$= \frac{h}{\mu} (\mu - m_2)$$

$$= \frac{h}{\mu} 3 m_1$$

$$= \frac{3 m_1 h}{\mu}$$

Thus,

$$(x_1, y_1, z_1) = \left(0, \frac{a}{\sqrt{3}}, -\frac{m_2 h}{\mu}\right)$$

$$(x_2, y_2, z_2) = \left(-\frac{a}{2}, \frac{-a}{2\sqrt{3}}, -\frac{m_2 h}{\mu}\right)$$

$$(x_3, y_3, z_3) = \left(\frac{a}{2}, \frac{-a}{2\sqrt{3}}, -\frac{m_2 h}{\mu}\right)$$

$$(x_4, y_4, z_4) = \left(0, 0, \frac{3 m_1 h}{\mu}\right)$$

$$I_3 = \sum_a m_a (r_a^2 - z_a^2)$$

$$= \sum_a m_a (x_a^2 + y_a^2)$$

$$= 3 m_1 h^2 + \cancel{m_2 \cdot 0^2}^{\,0}$$

$$= \cancel{3} m_1 \frac{a^2}{\cancel{3}}$$

$$= \boxed{m_1 a^2}$$

$$I_1 = \sum_a m_a (r_a^2 - x_a^2)$$

$$= \sum_a m_a (y_a^2 + z_a^2)$$

$$= m_1 \left(\left(\frac{a}{\sqrt{3}}\right)^2 + \left(\frac{-m_2 h}{\mu}\right)^2 \right)$$

$$+ m_1 \left(\left(\frac{-a}{2\sqrt{3}}\right)^2 + \left(\frac{-m_2 h}{\mu}\right)^2 \right)$$

$$+ m_1 \left(\left(\frac{-a}{2\sqrt{3}}\right)^2 + \left(\frac{-m_2 h}{\mu}\right)^2 \right)$$

$$+ m_2 \left(\cancel{0^2} + \left(\frac{3 m_1 h}{\mu}\right)^2 \right)$$

$$= m_1 \left(\frac{a^2}{3} + 2\left(\frac{a^2}{4 \cdot 3}\right) + 3 \frac{m_2^2 h^2}{\mu^2} \right) + m_2 \frac{9 m_1^2 h^2}{\mu^2}$$

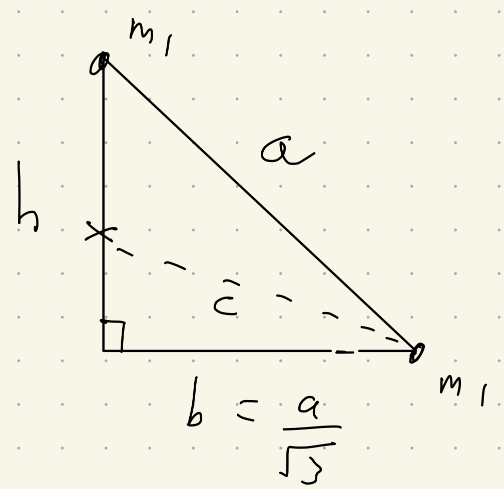
$$= m_1 \left[\frac{a^2}{3}\left(1 + \frac{1}{2}\right) + 3 \frac{m_2^2 h^2}{\mu^2} \right] + \frac{9 m_1^2 m_2}{\mu^2} h^2$$

$$I_1 = \frac{1}{2} m_1 a^2 + \frac{3 m_1 m_2}{\mu^2} h^2 (\underbrace{m_2 + 3 m_1}_{= \mu})$$

$$= \frac{1}{2} m_1 a^2 + \frac{3 m_1 m_2 h^2}{\mu}$$

$I_2 = I_1$ (since equilateral base $\rightarrow$ symmetric top)

<u>NOTE:</u> A regular tetrahedron ($m_1 = m_2 \rightarrow \mu = 4 m_1$)



$$a^2 = h^2 + \frac{a^2}{3}$$

$$\rightarrow h^2 = \frac{2}{3} a^2$$

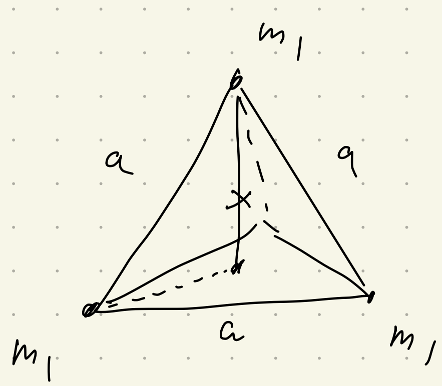
$$\text{so} \quad h = \sqrt{\frac{2}{3}}\, a$$

Thus,

$$I_1 = \frac{1}{2} m_1 a^2 + \frac{\cancel{3} m_1^2}{\mu} \frac{2}{\cancel{3}} a^2$$

$$= \frac{1}{2} m_1 a^2 + \frac{m_1^2}{4 m_1} 2 a^2$$

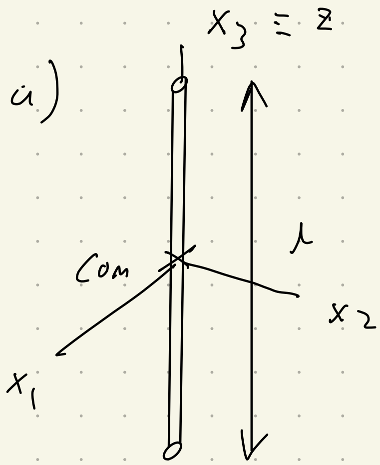
$$= m_1 a^2$$



$$\boxed{\text{so } I_1 = I_2 = I_3 = m_1 a^2}$$

Sec 32, Prob 2:

a)



$$I_3 = 0$$

$$I_1 = I_2 \equiv I$$

$$= \int \rho\, dV \left(r^2 - x^2 \right)$$

$$= \int \rho\, dV \left(\cancel{y^2}^{\,0} + z^2 \right)$$

$$= \int \rho\, dV\, z^2$$

$$\rho = \frac{\mu}{l}\, \delta(x)\, \delta(y)$$

$$I = \frac{\mu}{l} \iiint dx\,dy\,dz\; \delta(x)\,\delta(y)\; z^2$$

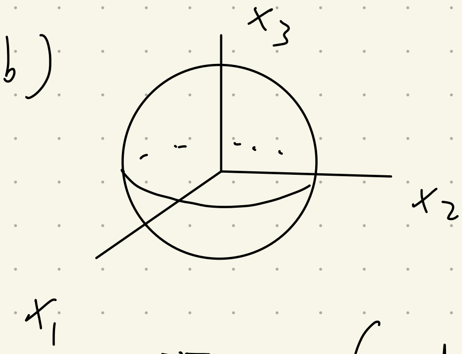
$$= \frac{\mu}{l} \int_{-l/2}^{l/2} z^2\, dz$$

$$= \frac{\mu}{l}\, \frac{z^3}{3} \Big|_{-l/2}^{l/2}$$

$$= \frac{\mu}{l}\, \frac{2}{3}\, \frac{l^3}{8}$$

$$= \frac{1}{12}\, \mu l^2 \quad = I_1 = I_2$$

b)



$$I_1 = I_2 = I_3 \equiv I$$

$$I_1 = \int \rho \, dV \, (r^2 - x^2) \quad = \int \rho \, d$$

$$= \int \rho \, dV \, (y^2 + z^2)$$

$$I_2 = \int \rho \, dV \, (x^2 + z^2)$$

$$I_3 = \int \rho \, dV \, (x^2 + y^2)$$

$$\rightarrow \quad I = \frac{1}{3} \left(I_1 + I_2 + I_3 \right)$$

$$= \frac{1}{3} \int \rho \, dV \cdot 2 \, (x^2 + y^2 + z^2)$$

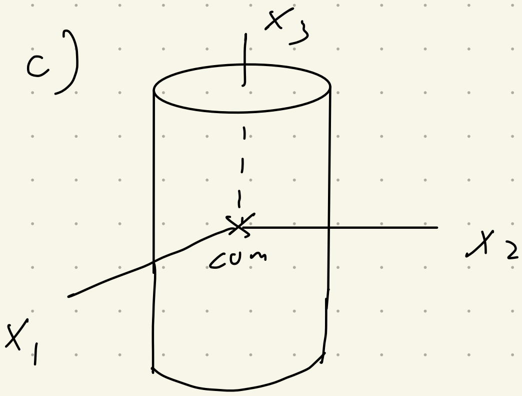
$$= \frac{2}{3} \rho \int dV \; r^2$$

$$= \frac{2}{\cancel{3}} \left(\frac{M}{\frac{4}{\cancel{3}} \pi R^3} \right) \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{R} r^2 \sin\theta \, dr \, d\theta \, d\phi \quad r^2$$

$$= \frac{M}{2 \cancel{\pi} R^3} \cdot 4 \cancel{\pi} \int_0^R r^4 \, dr$$

$$= \frac{2M}{R^3} \frac{R^5}{5} \qquad = \boxed{\frac{2}{5} M R^2}$$

c)



height : h

radius : R

$$\rho = \frac{M}{\pi R^2 h}$$

$$I_1 = I_2 \equiv I$$

$$I_3 = \int \rho\, dV \left(r^2 - z^2\right)$$

$$= \int \rho\, dV \left(x^2 + y^2\right)$$

$$= \int \rho\, dV\, s^2$$

where (s, ϕ, z) are cylindrical coords.

$$\rightarrow \quad dV = s\, ds\, d\phi\, dz$$

thus, $$I_3 = \frac{M}{\pi R^2 h} \iiint s\, ds\, d\phi\, dz\; s^2$$

$$= \frac{M}{\pi R^2 h} \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz \int_0^R ds\, s^3$$

$$= \frac{M}{\cancel{\pi} R^2 \cancel{h}} \cdot 2\cancel{\pi} \cdot \cancel{h} \cdot \frac{R^4}{4}$$

$$= \boxed{\frac{1}{2} M R^2}$$

$$I_1 = I_2 \equiv I$$

$$I = \frac{1}{2}(I_1 + I_2)$$

$$= \frac{1}{2}\left[\int \rho\, dV (y^2 + z^2) + \int \rho\, dV (x^2 + z^2)\right]$$

$$= \frac{1}{2} \int \rho\, dV (x^2 + y^2 + 2z^2)$$

$$= \frac{1}{2} \int \rho\, dV s^2 + \int \rho\, dV z^2$$

$$= \frac{1}{2} I_3 + \int \rho\, dV z^2$$

Now: $\int \rho\, dV z^2 = \rho \iiint s\, ds\, d\phi\, dz\, z^2$

$$= \frac{M}{\pi R^2 h} \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz\, z^2 \int_0^R ds\, s$$

$$= \frac{M}{\cancel{\pi} R^2 h} \cdot 2\cancel{\pi} \left.\frac{z^3}{3}\right|_{-h/2}^{h/2} \left.\frac{s^2}{2}\right|_0^R$$

$$= \frac{2M}{\cancel{R^2} h} \frac{\cancel{2}}{3}\left(\frac{h}{2}\right)^3 \frac{\cancel{R^2}}{\cancel{2}}$$

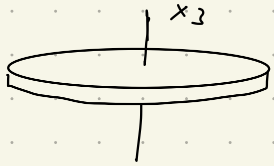
$$= \frac{1}{12} M h^2$$

$$\rightarrow I = \frac{1}{2}\left(\frac{1}{2} M R^2\right) + \frac{1}{12} M h^2$$

$$= \boxed{\frac{1}{4} M \left(R^2 + \frac{h^2}{3}\right)}$$

Limiting cases:

i)

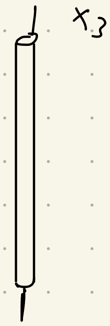


thin disk $(h \to 0)$

$$I_3 = \frac{1}{2} M R^2$$

$$I_1 = I_2 = \frac{1}{4} M R^2$$

ii)



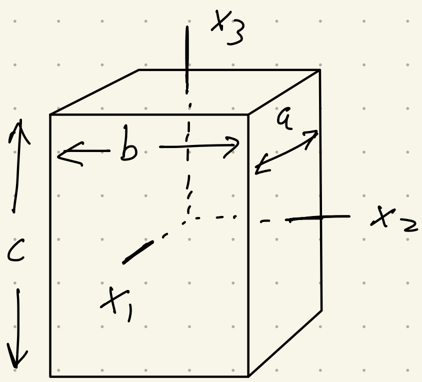
thin rod $(R \to 0)$

$$I_3 = 0$$

$$I_1 = I_2 = \frac{1}{12} M h^2$$

---

d)



$$\rho = \frac{M}{abc}$$

$$I_1 = \int \rho \, dV \, (y^2 + z^2)$$

$$= \frac{M}{abc} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz \, (y^2 + z^2)$$

$$= \frac{M}{abc} \left(ac \left. \frac{y^3}{3} \right|_{-b/2}^{b/2} + ab \left. \frac{z^3}{3} \right|_{-c/2}^{c/2} \right)$$

$$= \frac{M}{abc} \left(ac \, \frac{2}{3} \left(\frac{b}{2}\right)^3 + ab \, \frac{2}{3} \left(\frac{c}{2}\right)^3 \right)$$

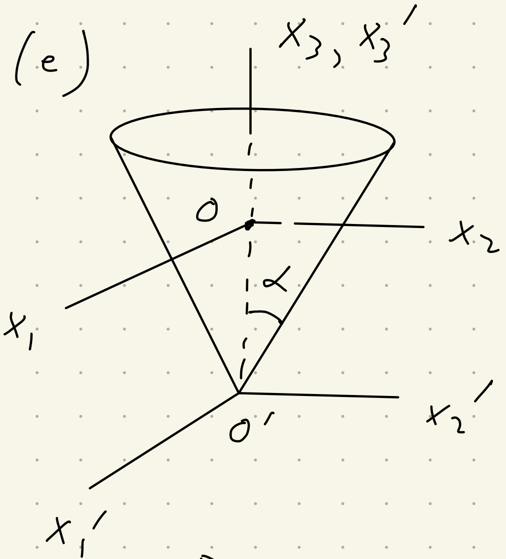
$$= \frac{1}{12} M (b^2 + c^2)$$

Similarly,

$$I_2 = \frac{1}{12} M \left(c^2 + a^2 \right)$$

$$I_3 = \frac{1}{12} M \left(a^2 + b^2 \right)$$

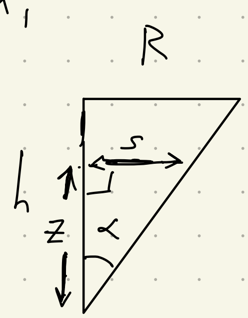
(e)



base radius : R
height : h

$$\text{vol} = \int dV$$

$$= \int_0^{2\pi} d\phi \int_0^{h} dz \int_0^{zR/h} s\, ds$$



$$\tan\alpha = \frac{s}{z} = \frac{R}{h}$$

$$\longrightarrow \quad s = z\,\frac{R}{h}$$

Thus,

$$\text{vol} = \cancel{2}\pi \int_0^h dz \left. \frac{s^2}{\cancel{2}} \right|_0^{zR/h}$$

$$= \pi \int_0^h dz\, z^2 \frac{R^2}{h^2}$$

$$= \frac{\pi R^2}{h^2} \left. \frac{z^3}{3} \right|_0^h$$

$$= \frac{1}{3} \pi R^2 h$$

$$\rho = \frac{M}{\mathrm{Vol}} = \frac{3M}{\pi R^2 h}$$

$$I_3' = \int \rho \, dV \, (x^2 + y^2)$$

$$= \rho \int s \, ds \, d\phi \, dz \; s^2$$

$$= \rho \int_0^{2\pi} d\phi \int_0^h dz \int_0^{zR/h} ds \; s^3$$

$$= \rho \; 2\pi \int_0^h dz \; \frac{s^4}{4} \Big|_0^{zR/h}$$

$$= \rho \; \frac{\pi}{2} \int_0^h dz \; z^4 \left(\frac{R}{h}\right)^4$$

$$= \rho \; \frac{\pi}{2} \left(\frac{R}{h}\right)^4 \frac{z^5}{5} \Big|_0^h$$

$$= \frac{3M}{\cancel{\pi} R^{\cancel{2}} \cancel{h}} \; \frac{\cancel{\pi}}{2} \; \frac{R^4}{\cancel{h^4}} \; \frac{\cancel{h^5}}{5}$$

$$= \boxed{\frac{3}{10} M R^2}$$

$$I_1' = I_2' \equiv I'$$

$$I' = \frac{1}{2} (I_1' + I_2')$$

$$= \frac{1}{2} I_3' + \int \rho \, dV \, z^2 \qquad \text{(like cylinder)}$$

$$\int \rho \, dV \, z^2 = \rho \int s \, ds \, d\phi \, dz \; z^2$$

$$= \rho \int_0^{2\pi} d\phi \int_0^h dz \, z^2 \int_0^{zR/h} s \, ds$$

$$= \rho \cancel{2} \pi \int_0^h dz \, z^2 \, \frac{s^2}{\cancel{2}} \Big|_0^{zR/h}$$

$$= \rho \, \pi \left(\frac{R}{h}\right)^2 \int_0^h dz \, z^4$$

$$= \rho \, \pi \left(\frac{R}{h}\right)^2 \frac{z^5}{5} \Big|_0^h$$

$$= \frac{3\mu}{\cancel{\pi} \cancel{R^2} h} \cancel{\pi} \left(\frac{\cancel{R}}{h}\right)^2 \frac{h^5}{5}$$

$$= \frac{3}{5} \mu h^2$$

$$I' = \frac{1}{2} I_3' + \frac{3}{5} \mu h^2$$

$$= \frac{1}{2} \frac{3}{10} \mu R^2 + \frac{3}{5} \mu h^2$$

$$= \boxed{\frac{3}{5} \mu \left(h^2 + \frac{R^2}{4} \right)}$$

To find location of COM:

$$z'_{com} = \frac{1}{M}\int \rho \, dV \, z$$

$$= \frac{1}{\cancel{M}} \frac{3\cancel{M}}{\pi R^2 h} \int_0^{2\pi} d\phi \int_0^h dz \, z \int_0^{zR/h} ds \, s$$

$$= \frac{3}{\cancel{\pi} R^2 h} \cancel{2\pi} \int_0^h dz \, z \, \frac{s^2}{\cancel{2}} \Big|_0^{zR/h}$$

$$= \frac{3}{R^2 h} \left(\frac{R}{h}\right)^2 \int_0^h dz \, z^3$$

$$= \frac{3}{R^2 h} \left(\frac{R}{h}\right)^2 \frac{z^4}{4} \Big|_0^h$$

$$= \frac{3}{4} \frac{1}{\cancel{R^2} h} \left(\frac{\cancel{R}}{h}\right)^2 h^4$$

$$= \boxed{\frac{3}{4} h}$$

Thus, $\vec{a} = -\frac{3}{4} h \, \hat{x}_3$

$$I_{ij} = I'_{ij} - M(\delta_{ij} a^2 - a_i a_j)$$

$$\boxed{I_3 = I'_3 - M\cancel{(a^2 - a^2)}^{0} = \frac{3}{10} M R^2}$$

$$I_1 = I_1' - M(a^2 - 0 \cdot 0)$$

$$= I_1' - M\left(\frac{3}{4}h\right)^2$$

$$= \frac{3}{5}M\left(h^2 + \frac{R^2}{4}\right) - \frac{9}{16}Mh^2$$

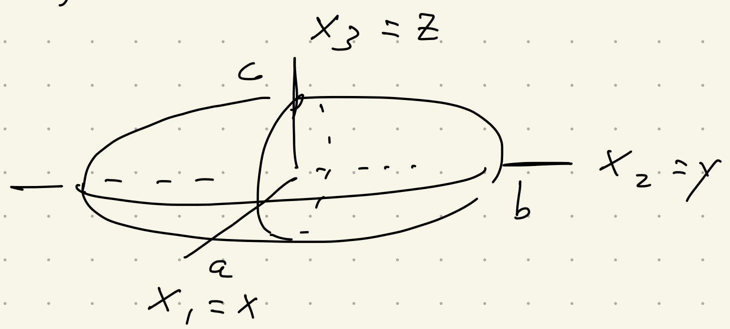
$$= M\left(\left(\frac{3}{5} - \frac{9}{16}\right)h^2 + \frac{3}{20}R^2\right)$$

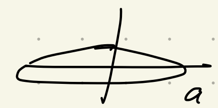
$$= M\left(\left(\frac{48-45}{80}\right)h^2 + \frac{3}{20}R^2\right)$$

$$= M\left(\frac{3}{80}h^2 + \frac{3}{20}R^2\right)$$

$$= \boxed{\frac{3}{20}M\left(R^2 + \frac{h^2}{4}\right)} = I_2$$

(f) ellipsoid of semi-axes a, b, c





$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

$$u^2 + v^2 + w^2 = 1 \qquad u = \frac{x}{a}, \quad v = \frac{y}{b}, \quad w = \frac{z}{c}$$

$$\text{Vol} = \int dV$$

$$= \iiint dx\, dy\, dz$$

$$= abc \iiint du\, dv\, dw$$

$$= abc \iiint \overbrace{r^2 \sin\theta\, dr\, d\theta\, d\phi}^{\text{unit sphere}}$$

$$= abc\, 4\pi \left.\frac{r^3}{3}\right|_0^1$$

$$= \frac{4}{3}\pi abc$$

$$\rho = \frac{M}{\frac{4}{3}\pi abc}$$

$$I_3 = \rho \int dV \,(x^2+y^2)$$

$$= \rho \int dV \,(a^2u^2 + b^2v^2)$$

$$= a^2\rho \int dV\, u^2 \quad + \quad b^2\rho\int dV\, v^2$$

$u = r\sin\theta\cos\phi$ $\qquad\qquad$ $r^2\sin\theta$

$v = r\sin\theta\sin\phi$

$$\int dV\, u^2 = abc \iiint r^2 \sin\theta\, dr\, d\theta\, d\phi\; r^2\sin^2\theta\cos^2\phi$$

$$= abc \int_0^1 r^4 dr \int_0^{2\pi} \cos^2\phi\, d\phi \int_{-1}^{1} d(\cos\theta)(1-\cos^2\theta)$$

$$= abc\, \frac{r^5}{5}\Big|_0^1 \int_0^{2\pi} \frac{1}{2}(1+\underbrace{\cancel{\cos 2\phi}}_{0})\, d\phi \int_{-1}^{1} dx\,(1-x^2)$$

$$= abc\, \frac{1}{5}\, \frac{1}{\cancel{2}} \cdot \cancel{2}\pi \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$= abc\, \frac{\pi}{5}(2)\left(1-\frac{1}{3}\right)$$

$$= abc\, \frac{4\pi}{3}\left(\frac{1}{5}\right)$$

Thus, $I_3 = \cancel{\frac{4\pi}{3}}\cancel{abc}\left(\frac{1}{5}\right)\frac{M}{\cancel{\frac{4\pi}{3}}\cancel{abc}}(a^2+b^2) = \frac{1}{5}M(a^2+b^2)$

$$\cos 2\phi = \cos^2\phi - \sin^2\phi = 2\cos^2\phi - 1$$

$$\cos^2\phi = \frac{1}{2}(1+\cos 2\phi)$$

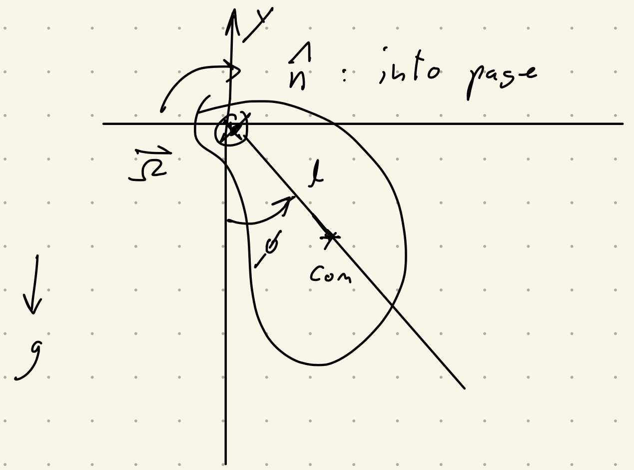
Thus,

$$I_3 = \frac{1}{5}(a^2+b^2)$$

$$I_1 = \frac{1}{5}(b^2+c^2)$$

$$I_2 = \frac{1}{5}(c^2+a^2)$$

Sec 32, Prob 3:



$U = \mu g y$, $y = l \cos\phi$

Small oscillations about $\phi = 0$:

$$y = -l\cos\phi$$
$$\cong -l\left(1 - \frac{\phi^2}{2}\right)$$

$$\rightarrow U \cong -\mu g l\left(1 - \frac{\phi^2}{2}\right)$$
$$= \frac{1}{2}\mu g l \phi^2 + \text{const}$$

$$T = \frac{1}{2}\mu|\vec{V}|^2 + \frac{1}{2} I_{ik}\Omega_i\Omega_k$$
$$= \frac{1}{2}\mu l^2\dot{\phi}^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$\vec{\Omega} = \dot{\phi}\hat{n}$$

$$\Omega_1 = \vec{\Omega}\cdot\hat{x}_1 = \dot{\phi}\,\hat{n}\cdot\hat{x}_1 = \dot{\phi}\cos\alpha$$
$$\Omega_2 = \vec{\Omega}\cdot\hat{x}_2 = \dot{\phi}\,\hat{n}\cdot\hat{x}_2 = \dot{\phi}\cos\beta$$
$$\Omega_3 = \vec{\Omega}\cdot\hat{x}_3 = \dot{\phi}\,\hat{n}\cdot\hat{x}_3 = \dot{\phi}\cos\gamma$$

Thus, $T = \frac{1}{2}\mu l^2\dot{\phi}^2 + \frac{1}{2}\dot{\phi}^2\left(I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)$

$$L = T - U$$

$$= \frac{1}{2}\dot{\phi}^2\left(ml^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)$$

$$- \frac{1}{2} mgl\,\phi^2$$

$$E = T + U$$

$$= \frac{1}{2}\dot{\phi}^2\left(ml^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)$$

$$+ \frac{1}{2} mgl\phi^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{mgl}{ml^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma}}$$

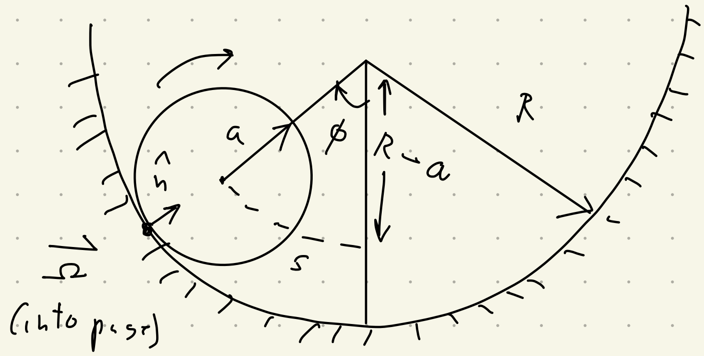
$$= \sqrt{\frac{g}{l}\,\frac{1}{\left(I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)/ml^2}}$$

Sec 32 Prob 6:

homog cylinder: μ, a, b

$$I_3 = \frac{1}{2}\mu a^2$$

$$V = \dot{s}$$
$$= (R-a)\dot{\phi}$$



$$s = (R-a)\phi$$

Point of contact with surface has $\vec{v} = 0$

$$0 = \vec{V} + \vec{\Omega}\times\vec{r}$$
$$= \vec{V} + \vec{\Omega}\times(-a\hat{n})$$

$$V = -\Omega a$$

$$\rightarrow \quad \Omega = -V/a$$
$$= -\frac{(R-a)}{a}\dot{\phi}$$

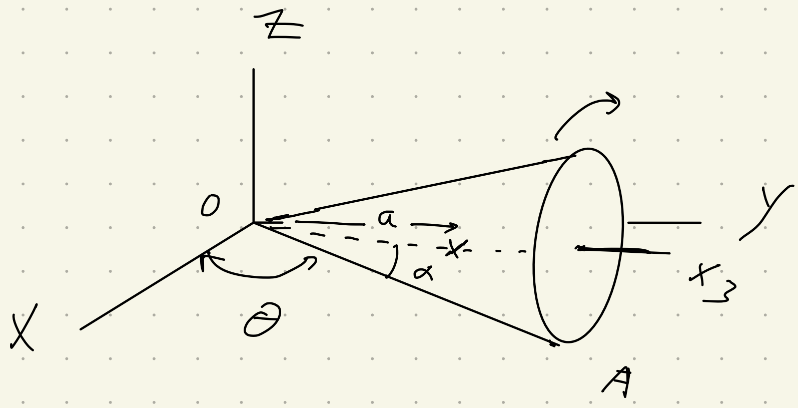
$$T = \frac{1}{2}\mu|\vec{V}|^2 + \frac{1}{2}I_3\Omega^2$$

$$= \frac{1}{2}\mu(R-a)^2\dot{\phi}^2 + \frac{1}{2}\left(\frac{1}{2}\mu a^2\right)\left(\frac{R-a}{a}\right)^2\dot{\phi}^2$$

$$= \left(\frac{1}{2}+\frac{1}{4}\right)\mu(R-a)^2\dot{\phi}^2$$

$$= \frac{3}{4}\mu(R-a)^2\dot{\phi}^2$$

Sec 32, Prob 7:



homogeneous cone:

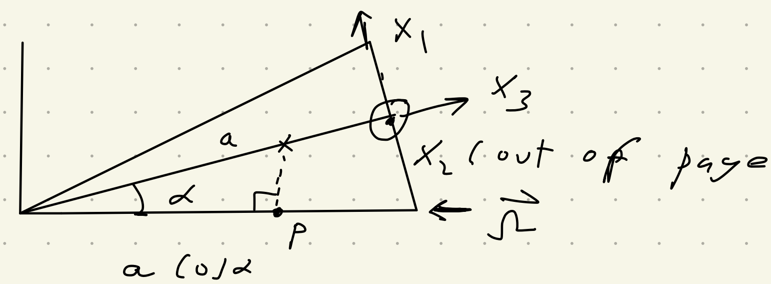
α: $\frac{1}{2}$ angle

h: height

R: radius

$$\tan\alpha = \frac{R}{h}$$

$\vec{\Omega}$: along AO



$\downarrow \vec{r} = -a\sin\alpha\,\hat{Z}$

$$s = a\cos\alpha\,\theta$$

$$V = \dot{s} = a\cos\alpha\,\dot{\theta}$$

$$\vec{\Omega} = \Omega\cos\alpha\,\hat{x}_3 - \Omega\sin\alpha\,\hat{x}_1$$

Instantaneous velocity of $P = 0$:

$$0 = \vec{v} = \vec{V} + \vec{\Omega}\times\vec{r}$$

$$0 = a\cos\alpha\,\dot{\theta} - \Omega\,a\sin\alpha$$

$$\rightarrow \quad \Omega = +\cot\alpha\,\dot{\theta}$$

$$\vec{\Omega} = -\Omega\cos\alpha\,\hat{x}_3 + \Omega\sin\alpha\,\hat{x}_1$$

$$\rightarrow \quad \Omega_3 = -\Omega\cos\alpha = -\frac{\cos^2\alpha}{\sin\alpha}\,\dot{\theta}$$

$$\Omega_1 = \Omega\sin\alpha = \cos\alpha\,\dot{\theta}$$

$$T = \frac{1}{2} M |\vec{V}|^2 + \frac{1}{2} \left(I_1 \Omega_1^2 + I_2 \Omega^2 + I_3 \Omega_3^2 \right)$$

$$= \frac{1}{2} M a^2 \cos^2\alpha\, \dot{\theta}^2$$

$$+ \frac{1}{2} I_1 \cos^2\alpha\, \dot{\theta}^2 + \frac{1}{2} I_3 \frac{\cos^4\alpha}{\sin^2\alpha} \dot{\theta}^2$$

$$= \frac{1}{2} \dot{\theta}^2 \cos^2\alpha \left[M a^2 + I_1 + I_3 \cot^2\alpha \right]$$

$$= \frac{1}{2} \dot{\theta}^2 \cos^2\alpha \left[M a^2 + \frac{3}{20} M \left(R^2 + \frac{1}{4} h^2 \right) \right.$$

$$\left. + \frac{3}{10} M R^2 \frac{\cos^2\alpha}{\sin^2\alpha} \right]$$

Recall: $\tan\alpha = \frac{R}{h}$, $a = \frac{3}{4} h$

$$\rightarrow \quad M a^2 = \frac{9}{16} M h^2$$

$$R^2 + \frac{1}{4} h^2 = h^2 \tan^2\alpha + \frac{1}{4} h^2$$

$$= h^2 \left(\frac{1}{4} + \tan^2\alpha \right)$$

$$R^2 \frac{\cos^2\alpha}{\sin^2\alpha} = h^2 \cancel{\tan^2\alpha} \cancel{\frac{\cos^2\alpha}{\sin^2\alpha}} = h^2$$

Thus,

$$[\ \ ] = \frac{9}{16} M h^2 + \frac{3}{20} M h^2 \left(\frac{1}{4} + \tan^2\alpha \right) + \frac{3}{10} M h^2$$

$$[\quad] = \mu h^2 \left(\frac{9}{16} + \frac{3}{20}\left(\frac{1}{4} + \tan^2\alpha\right) + \frac{3}{10} \right)$$

$$= \mu h^2 \left(\frac{9}{16} + \frac{3}{80} + \frac{3}{10} + \frac{3}{20} \tan^2\alpha \right)$$

$$= \mu h^2 \left(\frac{45}{80} + \frac{3}{80} + \frac{24}{80} + \frac{3}{20} \tan^2\alpha \right)$$

$$= \mu h^2 \left(\frac{9}{10} + \frac{3}{20} \tan^2\alpha \right)$$

Thus,

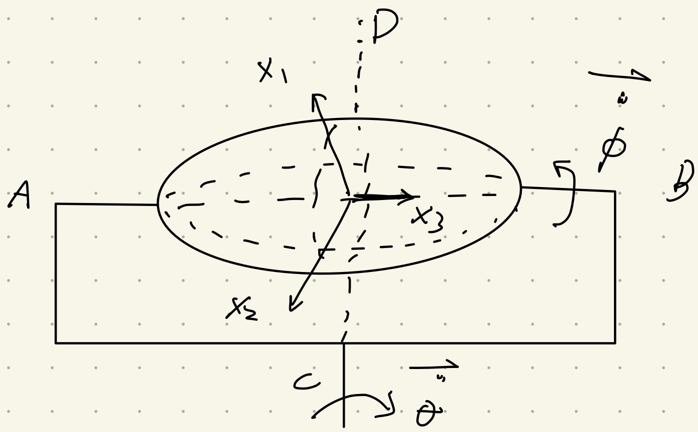
$$T = \frac{1}{2} \dot{\theta}^2 \cos^2\alpha \, \mu h^2 \left(\frac{9}{10} + \frac{3}{20} \tan^2\alpha \right)$$

$$= \frac{3}{40} \mu h^2 \dot{\theta}^2 \left(\sin^2\alpha + 6 \cos^2\alpha \right)$$

$$= \frac{3}{40} \mu h^2 \dot{\theta}^2 \left(\sin^2\alpha + \cos^2\alpha + 5 \cos^2\alpha \right)$$

$$= \frac{3}{40} \mu h^2 \dot{\theta}^2 \left(1 + 5 \cos^2\alpha \right)$$

Sec 32, Prob 9:



$(\hat{x}_1, \hat{x}_2, \hat{x}_3)$: principal axes

$$\vec{\dot\phi} = \dot\phi \, \hat{x}_3$$

$$\vec{\dot\theta} = \dot\theta \left(\cos\phi \, \hat{x}_1 + \sin\phi \, \hat{x}_2 \right)$$

$$\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta}$$

$$= \dot\theta \cos\phi \, \hat{x}_1 + \dot\theta \sin\phi \, \hat{x}_2 + \dot\phi \, \hat{x}_3$$

$$\Omega_1 = \dot\theta \cos\phi$$

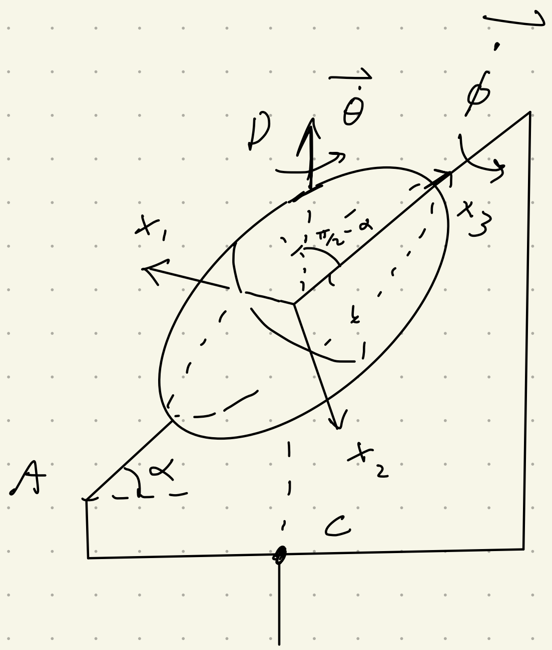
$$\Omega_2 = \dot\theta \sin\phi$$

$$\Omega_3 = \dot\phi$$

$$T = \frac{1}{2} \left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 \right)$$

$$= \frac{1}{2} \left[\left(I_1 \cos^2\phi + I_2 \sin^2\phi \right) \dot\theta^2 + I_3 \dot\phi^2 \right]$$

Sec 32, Prob 10 :



symmetric ellipsoid

$I_1 = I_2 \neq I_3$

$\vec{\Omega} = \vec{\dot\theta} + \vec{\dot\phi}$

Decompose along x_1, x_2, x_3

$\vec{\dot\phi} = \dot\phi\, \hat{x}_3$

$\vec{\dot\theta} = \dot\theta \Big(\sin\alpha\, \hat{x}_3 + \cos\alpha \cos\phi\, \hat{x}_1 + \cos\alpha \sin\phi\, \hat{x}_2 \Big)$

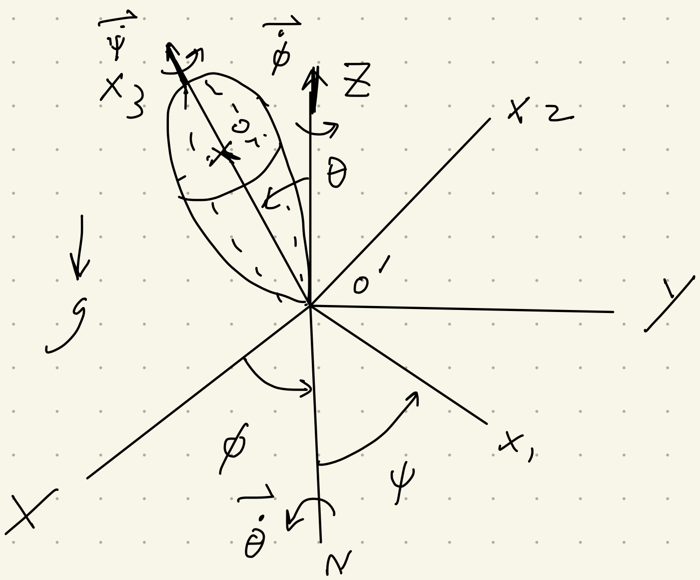
Thus,

$$\Omega_1 = \dot\theta \cos\alpha \cos\phi$$
$$\Omega_2 = \dot\theta \cos\alpha \sin\phi$$
$$\Omega_3 = \dot\theta \sin\alpha + \dot\phi$$

$$\rightarrow \quad T = \frac{1}{2}\left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2\right)$$
$$= \frac{1}{2}\left(I_1 \dot\theta^2 \cos^2\alpha \cos^2\phi + I_1 \dot\theta^2 \cos^2\alpha \sin^2\phi + I_3 (\dot\theta \sin\alpha + \dot\phi)^2\right)$$
$$= \frac{1}{2}\left(I_1 \dot\theta^2 \cos^2\alpha + I_3 (\dot\theta \sin\alpha + \dot\phi)^2\right)$$

Sec 35, Prob 1:

Heavy symmetrical top with lowest point fixed.



$I_1 = I_2$, μ = total mass

l = distance from O to COM

$$U = \mu g l \cos\theta$$

$$T = \frac{1}{2}\mu V^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$\Omega_1 = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$

$$\Omega_2 = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi \qquad (35.1)$$

$$\Omega_3 = \dot\phi \cos\theta + \dot\psi$$

pt O at rest

$$0 = \vec{V} + \vec{\Omega} \times \vec{r}$$

$$= \vec{V} - \vec{\Omega} \times l\,\hat{x}_3$$

$$\vec{A}\times\vec{B} = \hat{x}_1 (A_2B_3 - A_3B_2) + \cdots$$

$$\rightarrow \vec{V} = l\vec{\Omega}\times\hat{x}_3$$

$$= l\,(\Omega_2 \hat{x}_1 - \Omega_1 \hat{x}_2)$$

$$|\vec{V}|^2 = l^2(\Omega_2^2 + \Omega_1^2)$$

$$\Omega_1^2 + \Omega_2^2 = \dot\phi^2 \sin^2\theta \sin^2\psi + \dot\theta^2 \cos^2\psi$$
$$+ \cancel{2\dot\phi\dot\theta \sin\theta \sin\psi \cos\psi}$$
$$+ \dot\phi^2 \sin^2\theta \cos^2\psi + \dot\theta^2 \sin^2\psi$$
$$- \cancel{2\dot\phi\dot\theta \sin\theta \sin\psi \cos\psi}$$

$$= \dot\phi^2 \sin^2\theta + \dot\theta^2$$

Thus,

$$L = T - U$$
$$= \frac{1}{2}\mu l^2 (\dot\phi^2 \sin^2\theta + \dot\theta^2)$$
$$+ \frac{1}{2} I_1 (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3 (\dot\phi \cos\theta + \dot\psi)^2$$
$$- \mu g l \cos\theta$$

$$= \frac{1}{2}(I_1 + \mu l^2)(\dot\phi^2 \sin^2\theta + \dot\theta^2)$$
$$+ \frac{1}{2} I_3 (\dot\phi \cos\theta + \dot\psi)^2 - \mu g l \cos\theta$$

$$= \frac{1}{2} I_1' (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3' (\dot\phi \cos\theta + \dot\psi)^2$$
$$- \mu g l \cos\theta$$

where $\left.\begin{aligned} I_1' &= I_2' = I_1 + \mu l^2 \\ I_3' &= I_3 \end{aligned}\right\}$ moments of inertia w.r.t. origin at O′

E, p_ϕ, p_ψ conserved since no explicit t dependence, ϕ dependence, or ψ dependence

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1' \sin^2\theta\, \dot{\phi} + I_3' (\dot{\phi}\cos\theta + \dot{\psi})\cos\theta$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3' (\dot{\phi}\cos\theta + \dot{\psi})$$

NOTE: $p_\phi = M_z \quad , \quad p_\psi = M_3 = I_3'\Omega_3$

proof: $M_z = \vec{M} \cdot \hat{z}$

$$= (M_1 \hat{x}_1 + M_2 \hat{x}_2 + M_3 \hat{x}_3) \cdot \hat{z}$$

Now: $\hat{x}_3 \cdot \hat{z} = \cos\theta$

$\hat{x}_2 \cdot \hat{z} = \sin\theta \cos\psi$

$\hat{x}_1 \cdot \hat{z} = \sin\theta \sin\psi$

$$\rightarrow M_z = M_1 \sin\theta \sin\psi + M_2 \sin\theta\cos\psi + M_3 \cos\theta$$

$$= I_1'\Omega_1 \sin\theta\sin\psi + I_1'\Omega_2 \sin\theta\cos\psi + I_3'\Omega_3\cos\theta$$

$$M_z = I_1'(\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi) \sin\theta \sin\psi$$
$$+ I_1'(\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi) \sin\theta \cos\psi$$
$$+ I_3'(\dot\phi \cos\theta + \dot\psi) \cos\theta$$

$$= I_1' \dot\phi \sin^2\theta + I_3'(\dot\phi \cos\theta + \dot\psi) \cos\theta$$

$$= p_\phi$$

$$E = T + U$$

$$= \frac{1}{2} I_1' (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3' (\dot\phi \cos\theta + \dot\psi)^2$$
$$+ mgl\cos\theta$$

Can invert

$$M_3 = I_3'(\dot\phi \cos\theta + \dot\psi)$$
$$M_z = I_1' \dot\phi \sin^2\theta + I_3'(\dot\phi \cos\theta + \dot\psi)\cos\theta$$

to express $\dot\phi$, $\dot\psi$ in terms of M_3, M_z.

$$\rightarrow M_z = I_1' \dot\phi \sin^2\theta + M_3 \cos\theta$$

$$\boxed{\dot\phi = \frac{M_z - M_3\cos\theta}{I_1' \sin^2\theta}}$$

$$\frac{M_3}{I_3'} = \dot{\phi}\cos\theta + \dot{\psi}$$

$$\rightarrow \boxed{\dot{\psi} = \frac{M_3}{I_3'} - \dot{\phi}\cos\theta = \frac{M_3}{I_3'} - \left(\frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}\right)\cos\theta}$$

Thus,

$$E = \frac{1}{2} I_1' \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2} I_3' \frac{M_3^2}{(I_3')^2} + \mu g l \cos\theta$$

$$= \frac{1}{2} I_1' \left(\left(\frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}\right)^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2}\frac{M_3^2}{I_3'} + \mu g l \cos\theta$$

$$= \frac{1}{2} I_1' \left(\frac{(M_z - M_3\cos\theta)^2}{(I_1')^2 \sin^2\theta} + \dot{\theta}^2 \right) + \frac{1}{2}\frac{M_3^2}{I_3'} + \mu g l \cos\theta$$

$$= \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2}\frac{(M_z - M_3\cos\theta)^2}{I_1'\sin^2\theta} + \underbrace{\frac{1}{2}\frac{M_3^2}{I_3'}}_{\text{const}} - \mu g l (1 - \cos\theta) + \underbrace{\mu g l}_{\text{const}}$$

$$\underbrace{E - \mu g l - \frac{1}{2}\frac{M_3^2}{I_3'}}_{E'} = \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2}\frac{(M_z - M_3\cos\theta)^2}{I_1' \sin^2\theta} - \mu g l (1 - \cos\theta)$$

$$E' = \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2}\frac{(M_z - M_3\cos\theta)^2}{I_1' \sin^2\theta} - \mu g l (1 - \cos\theta)$$

$$= \frac{1}{2} I_1' \dot{\theta}^2 + V_{eff}(\theta)$$

where

$$V_{eff}(\theta) = \frac{(M_z - M_3\cos\theta)^2}{2 I_1' \sin^2\theta} - \mu g l (1 - \cos\theta)$$

Thus, $\pm\sqrt{\frac{2}{I_1'}\left(E' - V_{eff}(\theta)\right)} = \dot{\theta} = \frac{d\theta}{dt}$

$$\boxed{t = \pm \int \frac{d\theta}{\sqrt{\frac{2}{I_1'}\left(E' - V_{eff}(\theta)\right)}} + \text{const}}$$

(solution via quadrature)

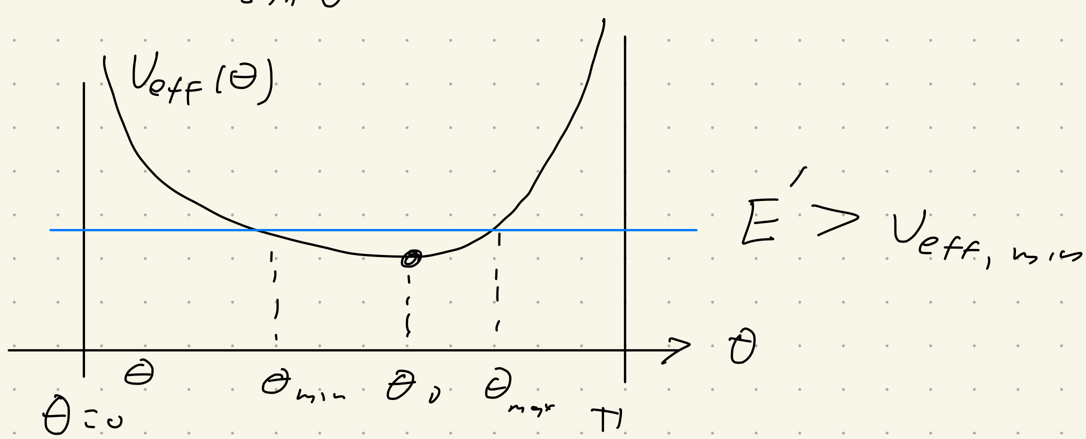
$$U_{eff}(\theta) = \frac{(M_z - M_3\cos\theta)^2}{2I_1'\sin^2\theta} - \mu g l(1-\cos\theta)$$

$$= \frac{1}{2I_1'\sin^2\theta}\Big[(M_z - M_3\cos\theta)^2 - 2\mu g l\, I_1'\sin^2\theta\,(1-\cos\theta)\Big]$$

At $\theta = 0, \pi$: $\quad 1-\cos\theta = 0$

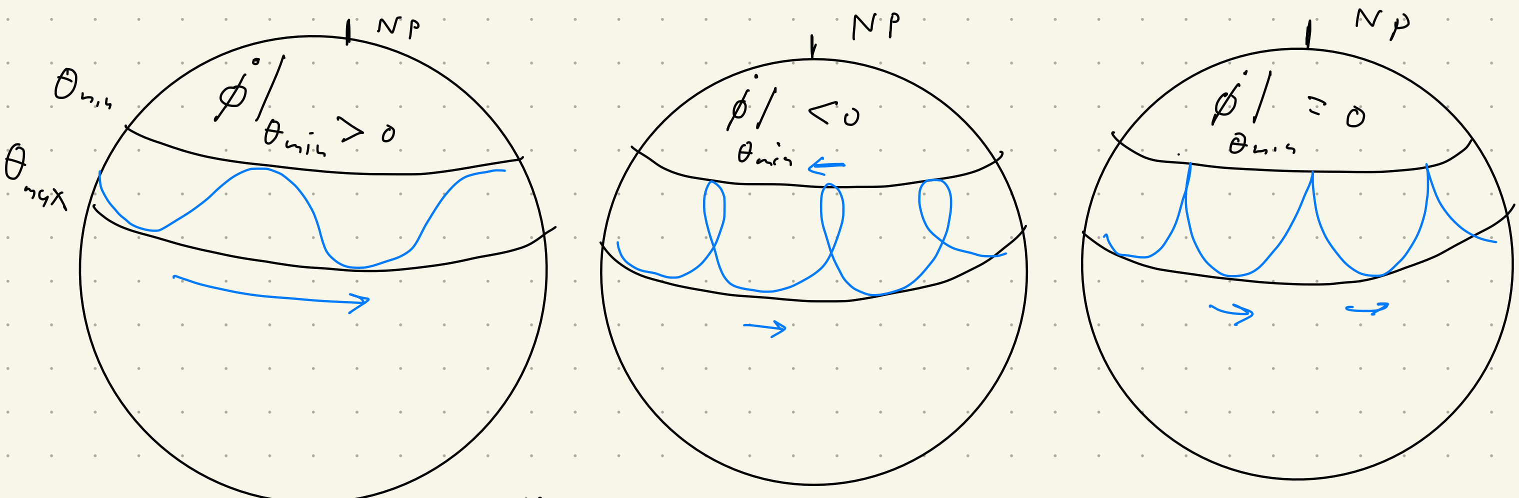
but $U_{eff}(\theta) \to \infty$ if $M_z \neq M_3$

due to $\frac{1}{\sin^2\theta}$ factor



$$\dot\phi = \frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}$$

(form of motion depends on the sign of $M_z - M_3\cos\theta_{min}$)



"nutation"

Sec 35, prob 2:

For stability of the top's motion about the vertical, need $\theta \to 0$ to be a minimum of $V_{eff}(\theta)$. So $\left.\dfrac{d^2V_{eff}(\theta)}{d\theta^2}\right|_{\theta=0} > 0$

$$V_{eff}(\theta) = \frac{(M_z - M_3\cos\theta)^2}{2I_1'\sin^2\theta} - \mu g l(1-\cos\theta)$$

Also:

$$M_z = I_1'\sin^2\theta\,\dot\phi + I_3'(\dot\phi\cos\theta + \dot\psi)\cos\theta$$

$$M_3 = I_3'(\dot\phi\cos\theta + \dot\psi)$$

In limit $\theta \to 0$:

$$M_z \simeq I_3'(\dot\phi + \dot\psi)$$

$$M_3 \simeq I_3'(\dot\phi + \dot\psi)$$

So $M_z = M_3$ in this limit

$$\to V_{eff}(\theta) \simeq \left.\left(\frac{M_3^2(1-\cos\theta)^2}{2I_1'\sin^2\theta} - \mu g l(1-\cos\theta)\right)\right|_{\theta\to 0}$$

$$\simeq \frac{M_3^2(\theta^2/2)^2}{2I_1'\theta^2} - \mu g l\frac{\theta^2}{2}$$

$$U_{eff}(\theta) \simeq \frac{1}{8}\frac{M_3^2}{I_1'}\theta^2 - \frac{1}{2}\mu g l\,\theta^2$$

$$= \frac{1}{2}\underbrace{\left[\frac{M_3^2}{4I_1'} - \mu g l\right]}_{\equiv K}\theta^2$$

Need $K \equiv \left.\frac{d^2 U_{eff}}{d\theta^2}\right|_{\theta=0} > 0$

Thus, $\frac{M_3^2}{4I_1'} - \mu g l > 0$

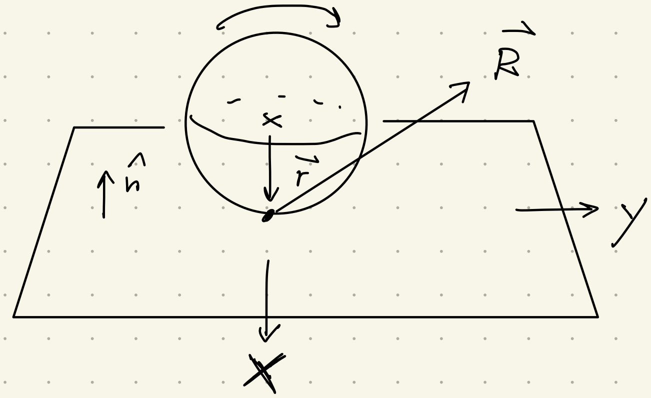
$$M_3^2 > 4I_1'\mu g l$$

$$\boxed{M_3 > 2\sqrt{I_1'\mu g l}}$$

since $M_3 = I_3 \Omega_3$

$$\longrightarrow \quad \Omega_3 > 2\sqrt{\frac{I_1'\mu g l}{(I_3')^2}}$$

Sec 38, Prob 1:



Find EOM of a homogeneous sphere (radius a, mass μ) rolling without slipping on a horizontal surface under external force $\vec{F}$ and torque $\vec{K}$.

Rolling without slipping

$$0 = \vec{V} + \vec{\Omega} \times \vec{r} \qquad (\hat{n} = \hat{z})$$
$$= \vec{V} - a\vec{\Omega} \times \hat{n}$$

$\vec{\Omega}$: angular velocity vector (in XY plane)

$\vec{R}$: reaction force (in arbitrary direction for rolling without slipping)

$$\frac{d\vec{P}}{dt} = \sum \vec{f} = \vec{F} + \vec{R}$$

$$\frac{d\vec{M}}{dt} = \sum \vec{r} \times \vec{f} = \vec{K} - a\hat{n} \times \vec{R}$$

$\vec{K}$: applied torque around COM

$\vec{F}$: applied force

$$\vec{M} = I\vec{\Omega}, \qquad I = \frac{2}{5}\mu a^2$$

$V_z = 0$ (since COM has $z = a = $ const)

$$\mu \frac{d\vec{V}}{dt} = \vec{F} + \vec{R} \qquad (1)$$

$$I \frac{d\vec{\Omega}}{dt} = \vec{K} - a\hat{z} \times \vec{R} \qquad (2)$$

$$\vec{V} = a\vec{\Omega} \times \hat{z} \qquad \text{(constraint)}$$

$$V_x = a(\vec{\Omega} \times \hat{z})_x = a\Omega_y$$

$$V_y = a(\vec{\Omega} \times \hat{z})_y = -a\Omega_x$$

$$0 = a(\vec{\Omega} \times \hat{z})_z = 0 \qquad \text{(no new information)}$$

Thus, $\boxed{\Omega_x = \frac{-V_y}{a}}$, $\boxed{\Omega_y = \frac{V_x}{a}}$, $\boxed{V_z = 0}$

Take time derivative of constraint equation:

$$\frac{d\vec{V}}{dt} = a\frac{d\vec{\Omega}}{dt} \times \hat{z}$$

$$\rightarrow \mu \frac{d\vec{V}}{dt} = \mu a \frac{d\vec{\Omega}}{dt} \times \hat{z}$$

$$\rightarrow \vec{F} + \vec{R} = \mu a \left(\frac{1}{I}\vec{K} - \frac{a}{I}\hat{z} \times \vec{R} \right) \times \hat{z}$$

Thus

$$F_x + R_x = \mu a\left(\frac{1}{I}K_y - \frac{a}{I}(\hat{z}\times\vec{R})_y\right)$$

$$= \frac{\mu a}{I}K_y - \frac{\mu a^2}{I}R_x$$

$$= \frac{\mu a}{\frac{2}{5}\mu a^2}K_y - \frac{\mu a^2}{\frac{2}{5}\mu a^2}R_x$$

$$= \frac{5}{2}\frac{K_y}{a} - \frac{5}{2}R_x$$

$$\rightarrow \quad \frac{7}{2}R_x = \frac{5}{2}\frac{K_y}{a} - F_x$$

$$\boxed{R_x = \frac{5}{7}\frac{K_y}{a} - \frac{2}{7}F_x}$$

Similarly,

$$F_y + R_y = \mu a\left(-\frac{K_x}{I} + \frac{a}{I}(\hat{z}\times\vec{R})_x\right)$$

$$= -\frac{\mu a K_x}{I} - \frac{\mu a^2}{I}R_y$$

$$= -\frac{5}{2}\frac{K_x}{a} - \frac{5}{2}R_y$$

$$\rightarrow \quad \frac{7}{2}R_y = -\frac{5}{2}\frac{K_x}{a} - F_y$$

$$\boxed{R_y = -\frac{5}{7}\frac{K_x}{a} - \frac{2}{7}F_y}$$

$$F_z + R_z = 0$$ (since no z-component of any $\vec{A} \times \hat{z}$)

Thus, $$\boxed{R_z = -F_z}$$

Summary:

$$R_x = \frac{5}{7}\frac{K_y}{a} - \frac{2}{7} F_x$$

$$R_y = -\frac{5}{7}\frac{K_x}{a} - \frac{2}{7} F_y$$

$$R_z = -F_z$$

$$\mu \frac{dV_x}{dt} = F_x + R_x = \frac{5}{7} F_x + \frac{5}{7}\frac{K_y}{a}$$

$$\mu \frac{dV_y}{dt} = F_y + R_y = \frac{5}{7} F_y - \frac{5}{7}\frac{K_x}{a}$$

$$V_z = 0$$

$$\Omega_x = -\frac{V_y}{a}$$

$$\Omega_y = +\frac{V_x}{a}$$

$$I \frac{d\Omega_z}{dt} = K_z$$

Example:

suppose: $\vec{F} = \underbrace{-\mu g\,\hat{z}}_{\text{gravity}} + F_0\,\hat{x}$ $\quad F_0$ const

$$\vec{K} = a\,G_0\,\hat{z}$$

Then $K_x = K_y = 0$, $\quad K_z = a\,G_0$

$$F_x = F_0\,,\quad F_y = 0\,,\quad F_z = -\mu g$$

$$\mu \frac{dV_x}{dt} = \frac{5}{7}\left(F_x + \cancel{\frac{K_y}{a}}^{\,0}\right) = \frac{5}{7} F_0$$

$$\rightarrow \boxed{V_x = V_{x0} + \frac{5}{7}\frac{F_0}{\mu}t}$$

$$\mu \frac{dV_y}{dt} = \frac{5}{7}\left(\cancel{F_y}^{\,0} - \cancel{\frac{K_x}{a}}^{\,0}\right) = 0 \rightarrow \boxed{V_y = V_{y0}}$$

$$\boxed{V_z = 0}$$

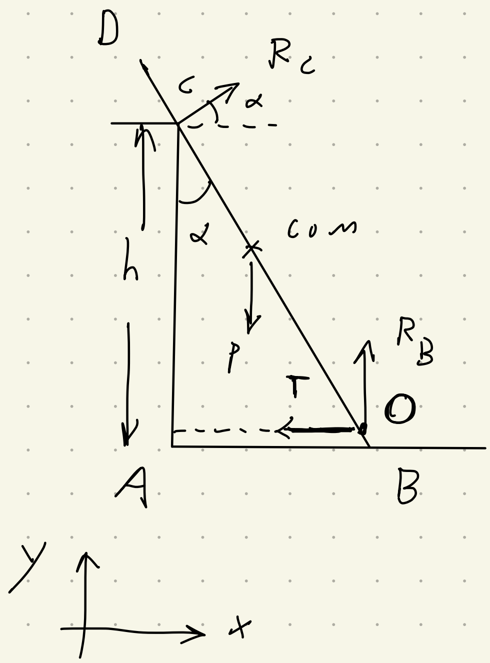
$$I\,\frac{d\Omega_z}{dt} = a\,G_0 \rightarrow \Omega_z = \Omega_{z0} + \frac{a\,G_0}{I}\,t$$

$$\boxed{\Omega_z = \Omega_{z0} + \frac{5}{2}\frac{G_0}{\mu a}\,t}$$

$$\boxed{\Omega_x = -\frac{V_y}{a}\,,\quad \Omega_y = \frac{V_x}{a}}$$

$$\boxed{R_x = -\frac{2}{7}F_0}\,,\quad \boxed{R_y = 0}\,,\quad \boxed{R_z = \mu g} \leftarrow \text{normal force}$$

Sec 38, Prob 2:



Uniform thin rod, weight P, length l

$$I = \frac{1}{12} m l^2 \quad \text{(about com)}$$

$$= \frac{1}{12} \frac{P}{g} l^2$$

EOMs:

$$\sum \vec{F} = 0$$

$$\sum \vec{r} \times \vec{F} = 0$$

including reaction forces

1) $0 = -T + R_c \cos\alpha$ (x-component of forces)

2) $0 = R_c \sin\alpha - P + R_B$ (y-component of force)

3) $0 = P \frac{l}{2} \sin\alpha - R_c \frac{h}{\cos\alpha}$ (ccw torque about O)

Using $\cos\alpha = \frac{h}{d}$

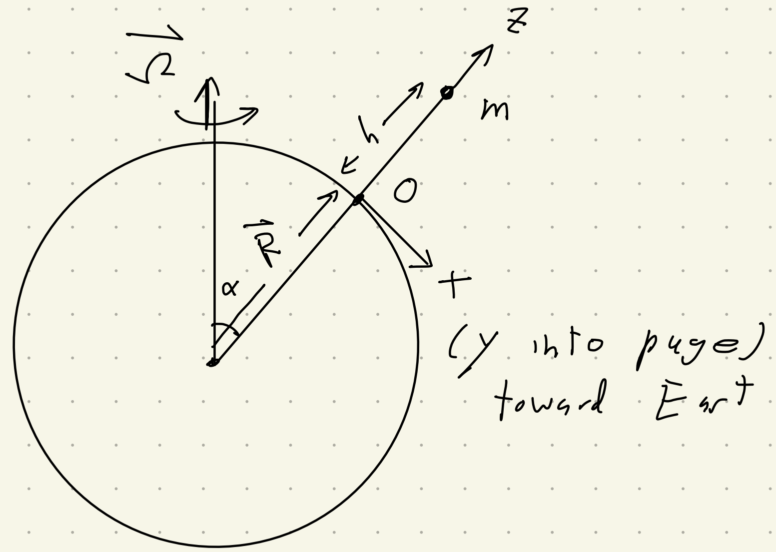
Thus,

$$R_c = P \frac{l}{2} \frac{\sin\alpha \cos\alpha}{h} = \frac{Pl}{4h} \sin 2\alpha$$

$$T = R_c \cos\alpha$$

$$R_B = P - R_c \sin\alpha$$

Sec 39, Prob 1:



$h << R$

$\vec{r}$ measured wrt O

$$m\vec{a} = m\vec{g} - m\vec{W} - 2m\vec{\Omega}\times\vec{v} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\vec{W} = \vec{\Omega}\times(\vec{\Omega}\times\vec{R})$$ (acceleration of O wrt inertial frame)

Ignore terms that are $O(\Omega^2)$

$$\rightarrow \quad m\vec{a} \approx m\vec{g} - 2m\vec{\Omega}\times\vec{v}$$

$$\rightarrow \quad \vec{a} = \vec{g} - 2\vec{\Omega}\times\vec{v}$$, $\vec{g}$, $\vec{\Omega}$ are constants

$$\frac{d\vec{v}}{dt} = \vec{g} - 2\vec{\Omega}\times\vec{v}$$

Write $\vec{v} \equiv \vec{v}_1 + \vec{v}_2$ where $\vec{v}_2$ is perturbation.

i.e, $\dfrac{d\vec{v}_1}{dt} = \vec{g} \quad \rightarrow \quad \vec{v}_1 = \vec{v}_0 + \vec{g}t$

initial velocity

$$\rightarrow \quad \frac{d\vec{v}_2}{dt} = -2\vec{\Omega}\times(\vec{v}_1 + \vec{v}_2)$$

$$\approx -2\vec{\Omega}\times\vec{v}_1$$ (ignore the smaller $\vec{v}_2$ term)

$$= -2\vec{\Omega}\times(\vec{v}_0 + \vec{g}t)$$

Thus, $\vec{v}_2 \approx -2\vec{\Omega}\times\vec{v}_0\, t - \vec{\Omega}\times\vec{g}\, t^2$

$\rightarrow$ $\vec{v} = \vec{v}_0 + \vec{g}\,t - 2\vec{\Omega}\times\left(\vec{v}_0\, t + \frac{1}{2}\vec{g}\,t^2\right)$

Also $\vec{v} = \dfrac{d\vec{r}}{dt}$

$$\rightarrow \boxed{\begin{aligned}\vec{r} = \vec{h} + \vec{v}_0\, t + \frac{1}{2}\vec{g}\,t^2 &- \\ - \vec{\Omega}\times\vec{v}_0\, t^2 - \frac{1}{3}\left(\vec{\Omega}\times\vec{g}\right)t^3&\end{aligned}}$$

Take $\vec{h} = h\hat{z}$

$\vec{v}_0 = 0$

$\vec{g} = -g\hat{z}$

$\vec{\Omega} = \Omega\cos\alpha\,\hat{z} - \Omega\sin\alpha\,\hat{x}$

Then

$$\vec{r} = h\hat{z} - \frac{1}{2}g\hat{z}\,t^2 + \frac{1}{3}g\Omega\sin\alpha\,\hat{y}\,t^3$$

When the body hits the ground

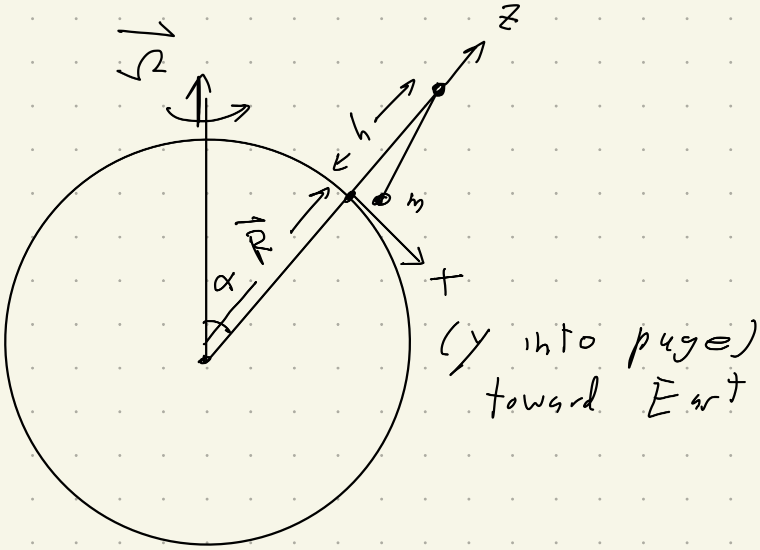
$h - \frac{1}{2}gT^2 = 0 \quad \rightarrow \quad T = \sqrt{\dfrac{2h}{g}}$

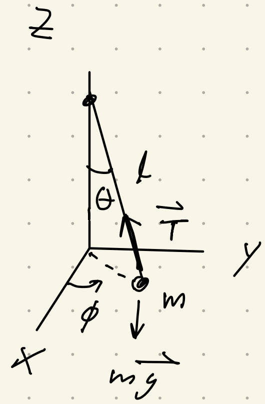
$$\rightarrow \quad \delta\vec{r} = \frac{1}{3}g\Omega\sin\alpha\left(\frac{2h}{g}\right)^{3/2}\hat{y}$$

$$= \frac{\sqrt{8}}{3}\sqrt{\frac{h^3}{g}}\,\Omega\sin\alpha\,\hat{y}$$

(deflection to the East)

Sec 39, Prob 3:



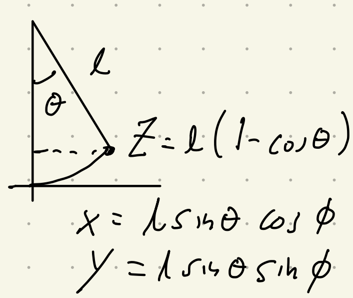


Foucault's pendulum: small oscillations $(\theta << 1)$

Again ignore terms that are 2^{nd} order in $\vec{\Omega}$.

$$m\vec{a} = \vec{T} + m\vec{g} - 2m(\vec{\Omega} \times \vec{v})$$

Now:



$$x = l\sin\theta\cos\phi \approx l\theta\cos\phi$$

$$y = l\sin\theta\sin\phi \approx l\theta\sin\phi$$

$$z = l(1-\cos\theta) \approx l\frac{\theta^2}{2} \approx \boxed{0}$$

$$\boxed{\ddot{z} = 0}$$

consider motion in xy plane

$$T_z = -T\cos\theta \approx -T$$

$$T_x = -T\sin\theta\cos\phi = -T\frac{x}{l}$$

$$T_y = -T\sin\theta\sin\phi = -T\frac{y}{l}$$

$$\vec{g} = -g\hat{z}$$

$$\vec{\Omega} = \Omega\cos\alpha\,\hat{z} - \Omega\sin\alpha\,\hat{x} \equiv \Omega_z\hat{z} + \Omega_x\hat{x}$$

$$\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \cancel{\dot{z}\hat{z}}^{=0} \approx \dot{x}\hat{x} + \dot{y}\hat{y}$$

$$(\vec{\Omega}\times\vec{v})_x = \Omega_y \cancel{\dot{z}}^{\,0} - \Omega_z \dot{y} = -\Omega_z \dot{y}$$

$$(\vec{\Omega}\times\vec{v})_y = \Omega_z \dot{x} - \cancel{\Omega_x \dot{z}}^{\,0} = \Omega_z \dot{x}$$

$$(\vec{\Omega}\times\vec{v})_z = \Omega_x \dot{y} - \cancel{\Omega_y}^{\,0} \dot{x} = \Omega_x \dot{y}$$

Thus,

$$m\ddot{x} \approx -T\frac{x}{\ell} + 2m\,\Omega_z \dot{y}$$

$$m\ddot{y} \approx -T\frac{y}{\ell} - 2m\,\Omega_z \dot{x}$$

$$\underbrace{0 \approx m\ddot{z}}_{} = -T - mg - 2m\,\Omega_x \dot{y}$$

<u>Now</u>: $\Omega_x \dot{y}^{\,\circ} \sim \Omega \dfrac{D}{P}$ D: displacement in xy plane ($D \ll \ell$)

$$= \frac{\Omega D\,\omega}{2\pi}$$

$$\ll \frac{\omega^2 \ell}{2\pi}$$

P: period of oscillation $= \dfrac{2\pi}{\omega}$, $\omega = \sqrt{\dfrac{g}{\ell}}$

$(\Omega \ll \omega)$

$$= \frac{g}{\cancel{\ell}}\frac{\cancel{\ell}}{2\pi}$$

$$= g/2\pi$$

Thus $m\Omega_x \dot{y} \ll mg$

$$\rightarrow 0 \approx -T - mg \rightarrow \boxed{T \approx mg}$$

Rewrite equations:

$$m\ddot{x} \approx -mg\frac{x}{l} + 2m\Omega_z\dot{y}$$

$$\rightarrow \ddot{x} \approx -\frac{g}{l}x + 2\Omega_z\dot{y}$$

$$\boxed{\ddot{x} = -\omega^2 x + 2\Omega_z\dot{y}}$$

similarly,

$$\boxed{\ddot{y} = -\omega^2 y - 2\Omega_z\dot{x}}$$

Define: $\xi = x + iy$

$$\rightarrow \dot{\xi} = \dot{x} + i\dot{y}$$

$$\rightarrow \ddot{\xi} = \ddot{x} + i\ddot{y}$$

Thus, $\ddot{\xi} = -\omega^2\xi + 2\Omega_z(\dot{y} - i\dot{x})$

$$\ddot{\xi} = -\omega^2\xi - 2i\Omega_z\dot{\xi}$$

so $\ddot{\xi} + 2i\Omega_z\dot{\xi} + \omega^2\xi = 0$

Guess: $\xi = Ae^{i\lambda t}$

$$\rightarrow -\lambda^2\cancel{\xi} + 2i\Omega_z i\lambda\cancel{\xi} + \omega^2\cancel{\xi} = 0$$

$$\lambda^2 + 2\Omega_z\lambda - \omega^2 = 0$$

$$\rightarrow \lambda_\pm = \frac{-2\Omega_z \pm \sqrt{4\Omega_z^2 + 4\omega^2}}{2}$$

Use $\Omega^2 << \omega^2$ to approximate

$$\sqrt{4\Omega_z^2 + 4\omega^2} \approx 2\omega$$

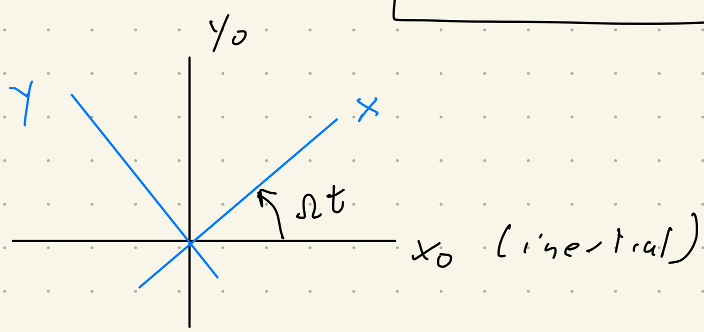
Thus, $$\lambda_\pm \approx \frac{-2\Omega_z \pm 2\omega}{2} = -\Omega_z \pm \omega$$

General solution:

$$\xi(t) = A e^{i\lambda_+ t} + B e^{i\lambda_- t}$$
$$= e^{-i\Omega_z t}\left(A e^{i\omega t} + B e^{-i\omega t}\right)$$

Note:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\Omega_z t & \sin\Omega_z t \\ -\sin\Omega_z t & \cos\Omega_z t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$



$$\rightarrow \quad \xi = x + iy$$
$$= x_0 \cos(\Omega_z t) + y_0 \sin(\Omega_z t)$$
$$+ i\left(-x_0 \sin(\Omega_z t) + y_0 \cos(\Omega_z t)\right)$$
$$= (x_0 + iy_0)(\cos\Omega_z t - i \sin\Omega_z t)$$
$$= \xi_0 e^{-i\Omega_z t}$$

Thus, motion in inertial frame is

$$\xi_0(t) = x_0(t) + i y_0(t)$$
$$= \underbrace{A e^{i\omega t} + B e^{-i\omega t}}$$

complex, determined by initial conditions.

<u>Example:</u>

Suppose
$$x(0) = D$$
$$y(0) = 0$$
$$\left.\begin{array}{l} \dot{x}(0) = 0 \\ \dot{y}(0) = 0 \end{array}\right\} \text{ released from rest}$$

Then $\xi(0) = D$, $\dot{\xi}(0) = 0$

$$\xi(t) = e^{-i\Omega_z t}\left(A e^{i\omega t} + B e^{-i\omega t}\right)$$

$$\rightarrow \xi(0) = \boxed{A + B = D}$$

$$\dot{\xi}(t) = -i\Omega_z e^{-i\Omega_z t}\left(A e^{i\omega t} + B e^{-i\omega t}\right)$$
$$+ e^{-i\Omega_z t}\left(i\omega A e^{i\omega t} - i\omega B e^{-i\omega t}\right)$$

$$\rightarrow \dot{\xi}(0) = -i\Omega_z(A + B) + i\omega(A - B) = 0$$

$$\rightarrow \boxed{A + B = \frac{\omega}{\Omega_z}(A - B)}$$

$$A = a_1 + i a_2$$
$$B = b_1 + i b_2$$

$$A + B = \underset{\substack{\uparrow \\ \text{real}}}{D} \quad \longrightarrow \quad \begin{aligned} a_1 + b_1 &= D \\ a_2 + b_2 &= 0 \end{aligned}$$

$$A + B = \frac{\omega}{\Omega_z}(A - B) \quad \longrightarrow \quad \begin{aligned} D &= \frac{\omega}{\Omega_z}(a_1 - b_1) \\ 0 &= \frac{\omega}{\Omega_z}(a_2 - b_2) \end{aligned}$$

$$\left.\begin{aligned} \rightarrow \quad & a_2 - b_2 = 0 \\ \text{and} \quad & a_2 + b_2 = 0 \end{aligned}\right\} \quad \begin{aligned} &\boxed{a_2 = 0} \\ &\boxed{b_2 = 0} \end{aligned}$$

$$\left.\begin{aligned} a_1 + b_1 &= D \\ a_1 - b_1 &= D\frac{\Omega_z}{\omega} \end{aligned}\right\} \quad \boxed{a_1 = \frac{D}{2}\left(1 + \frac{\Omega_z}{\omega}\right)}$$

$$b_1 = D - \frac{D}{2}\left(1 + \frac{\Omega_z}{\omega}\right)$$

$$\boxed{b_1 = \frac{D}{2}\left(1 - \frac{\Omega_z}{\omega}\right)}$$

Thus, $\xi_0(t) = x_0(t) + i y_0(t)$

$$= a_1 e^{i\omega t} + b_1 e^{-i\omega t}$$

$$= \frac{D}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) + \frac{D}{2}\frac{\Omega_z}{\omega}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

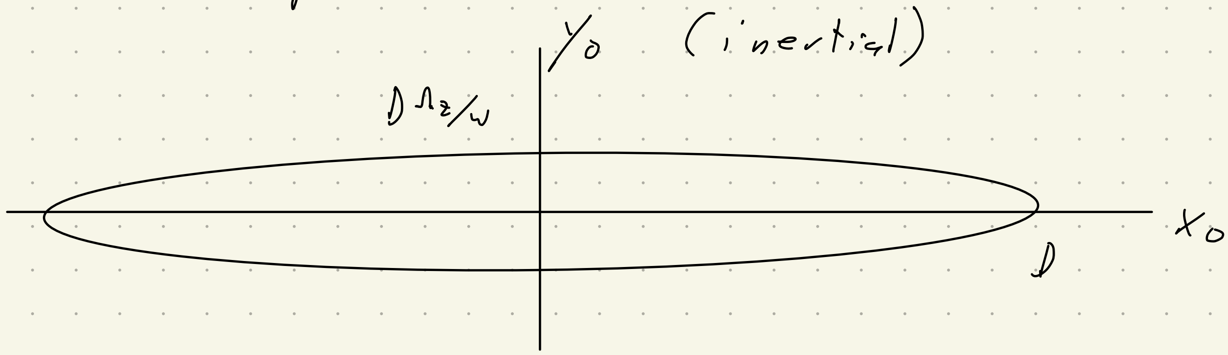
$$\xi_0(t) = D\cos\omega t + i\,D\left(\frac{\Omega_z}{\omega}\right)\sin\omega t$$

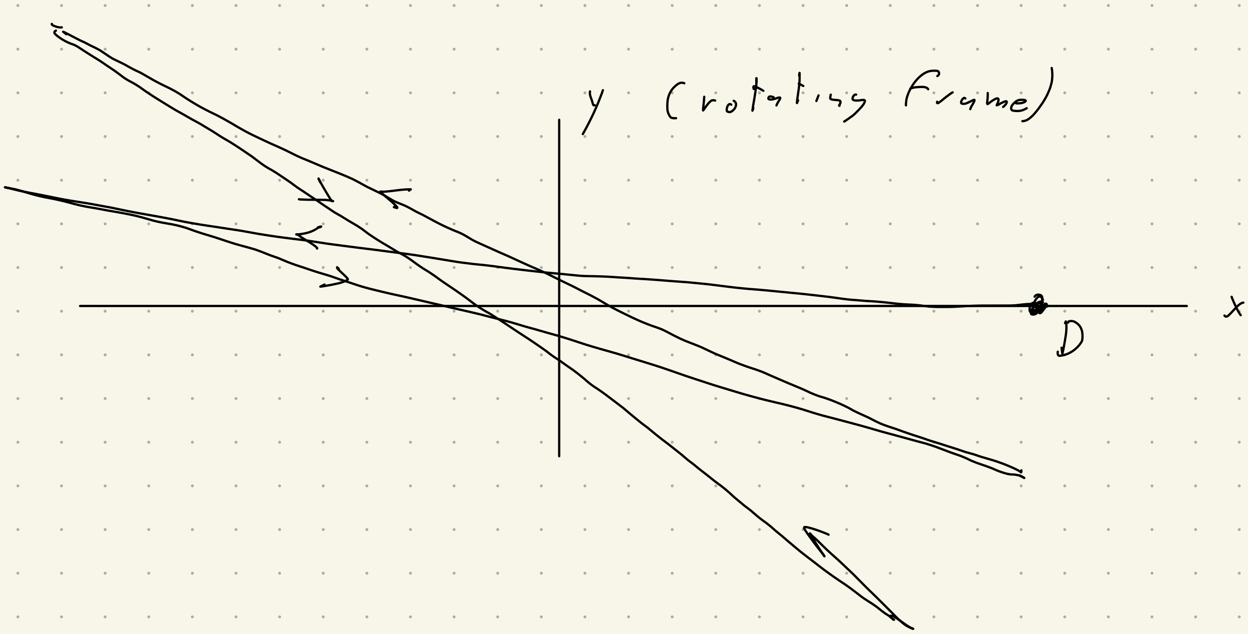
$$= x_0(t) + i y_0(t)$$

Thus, $x_0(t) = D\cos\omega t$

$y_0(t) = D\left(\frac{\Omega_z}{\omega}\right)\sin\omega t$

$$\underbrace{\frac{x_0^2}{D^2} + \frac{y_0^2}{\left(D\left(\frac{\Omega_z}{\omega}\right)\right)^2} = 1}_{\text{ellipse}}$$





Precessional Frequency:

$$\Omega_z = \Omega \cos\alpha$$

$$\text{Period} = \frac{2\pi}{\Omega_z}$$

$$= \frac{2\pi}{\Omega \cos\alpha}$$

$$= \frac{\cancel{2\pi}}{\left(\frac{\cancel{2\pi}}{24\,\text{hrs}}\right)\cos\alpha}$$

$$= \frac{24\ \text{hrs}}{\cos\alpha}$$

$\alpha = 0$ (NP) $\rightarrow$ Period $= 24$ hrs

$\alpha = \frac{\pi}{2}$ (equator) $\rightarrow$ Period $= \infty$

(no precession)