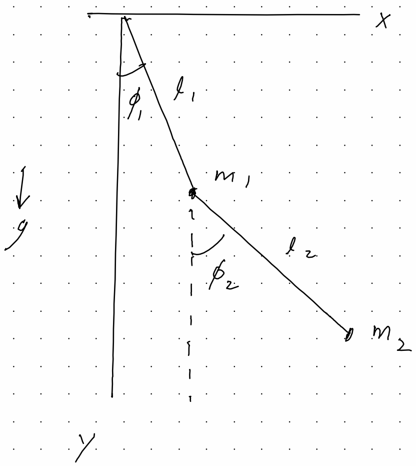
sec 5, prob 1



$x_1 = l_1 \sin\phi_1$
$y_1 = l_1 \cos\phi_1$
$x_2 = x_1 + l_2 \sin\phi_2 = l_1 \sin\phi_1 + l_2 \sin\phi_2$
$y_2 = y_1 + l_2 \cos\phi_2 = l_1 \cos\phi_1 + l_2 \cos\phi_2$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_1 g l_1 \cos\phi_1 - m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$
$$= -(m_1+m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$\dot{x}_1 = l_1 \cos\phi_1 \dot{\phi}_1$ $\qquad$ $\dot{y}_1 = -l_1 \sin\phi_1 \dot{\phi}_1$
$\dot{x}_1^2 = l_1^2 \cos^2\phi_1 \dot{\phi}_1^2$ $\qquad$ $\dot{y}_1^2 = l_1^2 \sin^2\phi_1 \dot{\phi}_1^2$

Thus, $\dot{x}_1^2 + \dot{y}_1^2 = l_1^2 (\sin^2\phi_1 + \cos^2\phi_1) \dot{\phi}_1^2$
$$= l_1^2 \dot{\phi}_1^2$$

$\dot{x}_2 = l_1 \cos\phi_1 \dot{\phi}_1 + l_2 \cos\phi_2 \dot{\phi}_2$
$\rightarrow \dot{x}_2^2 = l_1^2 \cos^2\phi_1 \dot{\phi}_1^2 + l_2^2 \cos^2\phi_2 \dot{\phi}_2^2 + 2 l_1 l_2 \cos\phi_1 \cos\phi_2 \dot{\phi}_1 \dot{\phi}_2$
$\dot{y}_2 = -l_1 \sin\phi_1 \dot{\phi}_1 - l_2 \sin\phi_2 \dot{\phi}_2$
$\rightarrow \dot{y}_2^2 = l_1^2 \sin^2\phi_1 \dot{\phi}_1^2 + l_2^2 \sin^2\phi_2 \dot{\phi}_2^2 + 2 l_1 l_2 \sin\phi_1 \sin\phi_2 \dot{\phi}_1 \dot{\phi}_2$

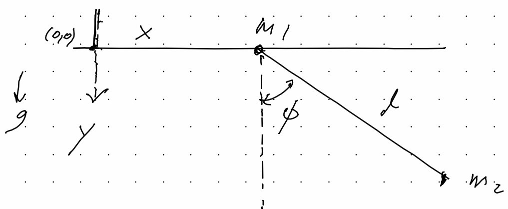
Thus,
$$\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2) \dot{\phi}_1 \dot{\phi}_2$$
$$= l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

so $T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$
$$= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2$$
$$+ m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$
$$= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

$$U = -(m_1+m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

$$L = T - U$$
$$= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$
$$+ (m_1+m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2$$

Sec 5 Prob. 2



Generalised coords: x, ϕ

$(x_1, y_1) = (x, 0)$

$(x_2, y_2) = (x + l\sin\phi,\ l\cos\phi)$

$U = -m_1 g y_1 - m_2 g y_2$

$= -m_2 g l \cos\phi$

$T = \frac{1}{2} m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2(\dot{x}_2^2 + \dot{y}_2^2)$

Now: $\dot{x}_1^2 + \dot{y}_1^2 = \dot{x}^2$

$\dot{x}_2^2 + \dot{y}_2^2 = (\dot{x} + l\cos\phi\dot{\phi})^2 + (-l\sin\phi\dot{\phi})^2$

$= \dot{x}^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi} + l^2\sin^2\phi\,\dot{\phi}$

$= \dot{x}^2 + l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi}$

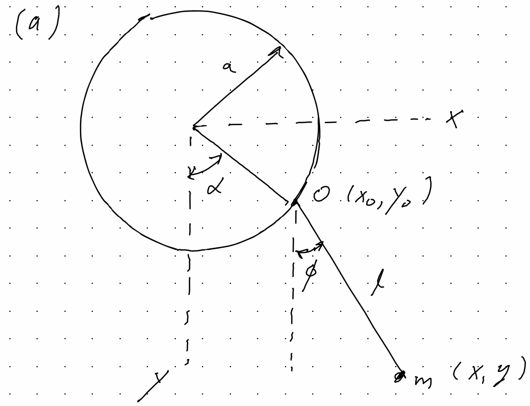
$\rightarrow T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2(\dot{x}^2 + l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi})$

$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \cos\phi\,\dot{x}\dot{\phi}$

$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2\dot{\phi}^2 + m_2 l\cos\phi\,\dot{x}\dot{\phi} + m_2 g l\cos\phi$

Sec 5, Prob 3

(a)



point of support O moves along circle:

$x_0 = a\sin\alpha$, $y_0 = a\cos\alpha$

where $\alpha = \gamma t$, $\gamma = \text{const}$

pendulum bob:

(x, y): $x = x_0 + l\sin\phi$

$y = y_0 + l\cos\phi$

$U = -mgy = \underbrace{-mgy_0}_{\text{specified function of time}} - mgl\cos\phi$

[can ignore in L]

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

$\dot{x} = \dot{x}_0 + l\cos\phi\,\dot{\phi}$

$\dot{x}^2 = \dot{x}_0^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}_0\dot{\phi}$

$$\dot y = \dot y_0 - l\sin\phi\,\dot\phi$$

$$\dot y^2 = \dot y_0^2 + l^2\sin^2\phi\,\dot\phi^2 - 2l\sin\phi\,\dot y_0\,\dot\phi$$

Thus,

$$T = \frac{1}{2}m(\dot x^2+\dot y^2)$$

$$= \frac{1}{2}m\left(\dot x_0^2 + l^2\cos^2\phi\,\dot\phi^2 + 2l\cos\phi\,\dot x_0\dot\phi + \dot y_0^2 + l^2\sin^2\phi\,\dot\phi^2 - 2l\sin\phi\,\dot y_0\dot\phi\right)$$

$$= \frac{1}{2}m\left(\dot x_0^2+\dot y_0^2\right) + \frac{1}{2}ml^2\dot\phi^2 + ml\dot\phi\left(\dot x_0\cos\phi - \dot y_0\sin\phi\right)$$

NOTE: $\dot x_0^2+\dot y_0^2 = a^2\dot\alpha^2 = a^2\gamma^2$

since this is a specified function of time, we can ignore it in the Lagrangian.

Thus, $$L = \frac{1}{2}ml^2\dot\phi^2 + ml\dot\phi\left(\dot x_0\cos\phi - \dot y_0\sin\phi\right) + mgl\cos\phi$$

We can rewrite the second term:

$$x_0 = a\sin\alpha \rightarrow \dot x_0 = a\cos\alpha\,\dot\alpha \qquad (\dot\alpha \equiv \gamma)$$

$$y_0 = a\cos\alpha \rightarrow \dot y_0 = -a\sin\alpha\,\dot\alpha$$

Thus,

$$ml\dot\phi(\dot x_0\cos\phi - \dot y_0\sin\phi) = ml\dot\phi a\gamma(\cos\alpha\cos\phi + \sin\alpha\sin\phi)$$

$$= ml\dot\phi a\gamma\cos(\phi-\alpha)$$

$$= ml\dot\phi a\gamma\cos(\phi-\gamma t)$$

Now: $$\frac{d}{dt}\left[mla\gamma\sin(\phi-\gamma t)\right]$$

$$= mla\gamma\cos(\phi-\gamma t)(\dot\phi-\gamma)$$

$$= mla\dot\phi\gamma\cos(\phi-\gamma t) - mla\gamma^2\cos(\phi-\gamma t)$$

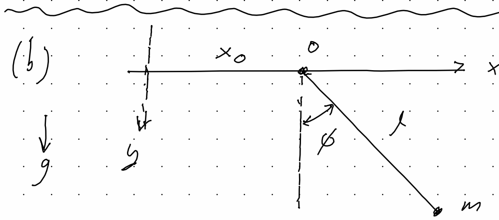
Thus,

$$mla\dot\phi\gamma\cos(\phi-\gamma t) = \frac{d}{dt}\left[mla\gamma\sin(\phi-\gamma t)\right] + mla\gamma^2\cos(\phi-\gamma t)$$

(and we can ignore the total time derivative in the Lagrangian)

$$\rightarrow L = \frac{1}{2}ml^2\dot\phi^2 + mgl\cos\phi + mla\gamma^2\cos(\phi-\gamma t)$$

(b)



point O moving according to $x_0 = a\cos\gamma t$

$$x = x_0 + l\sin\phi$$

$$y = l\cos\phi$$

$$U = -mgy = -mgl\cos\phi$$

$$T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{x}_0 + l\cos\phi\,\dot{\phi}, \qquad x_0 = a\cos\gamma t$$

$$= -a\sin(\gamma t)\,\gamma + l\cos\phi\,\dot{\phi}$$

$$\rightarrow \dot{x}^2 = a^2\gamma^2\sin^2\gamma t + l^2\cos^2\phi\,\dot{\phi}^2 - 2al\gamma\dot{\phi}\sin(\gamma t)\cos\phi$$

$$\dot{y} = -l\sin\phi\,\dot{\phi}$$

$$\rightarrow \dot{y}^2 = l^2\sin^2\phi\,\dot{\phi}^2$$

Thus, $$T = \tfrac{1}{2} m\left(a^2\gamma^2\sin^2\gamma t + l^2\cos^2\phi\,\dot{\phi}^2 - 2al\gamma\dot{\phi}\sin(\gamma t)\cos\phi + l^2\sin^2\phi\,\dot{\phi}^2\right)$$

$$= \tfrac{1}{2} m l^2\dot{\phi}^2 + \tfrac{1}{2} m a^2\gamma^2\sin^2\gamma t - mal\gamma\dot{\phi}\sin(\gamma t)\cos\phi$$

specified function of time (ignore)

$$L = \tfrac{1}{2} m l^2\dot{\phi}^2 - mal\gamma\dot{\phi}\sin(\gamma t)\cos\phi + mgl\cos\phi$$

2nd term:

$$-\frac{d}{dt}\left[mal\gamma\sin(\gamma t)\sin\phi\right] = -mal\gamma^2\cos(\gamma t)\sin\phi - mal\gamma\dot{\phi}\sin(\gamma t)\cos\phi$$

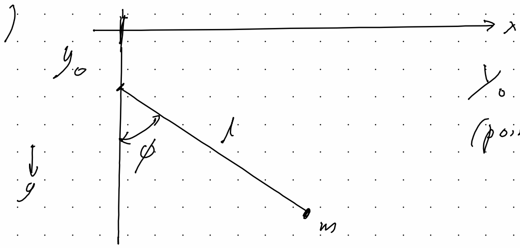
so

$$-mal\gamma\dot{\phi}\sin(\gamma t)\cos\phi = \underbrace{-\frac{d}{dt}[\ ]}_{\text{ignore}} + mal\gamma^2\cos(\gamma t)\sin\phi$$

Thus, ignoring total time derivative,

$$L = \tfrac{1}{2} m l^2\dot{\phi}^2 + mgl\cos\phi + mal\gamma^2\cos(\gamma t)\sin\phi$$

(c)



$Y_0 = a\cos\gamma t$ (point of support)

$$x = l\sin\phi$$

$$y = y_0 + l\cos\phi$$

$$= a\cos\gamma t + l\cos\phi$$

$$U = -mgy$$

$$= \underbrace{-mga\cos\gamma t}_{\text{specified function of time [can ignore]}} - mgl\cos\phi$$

$$T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = l\cos\phi\,\dot{\phi}$$

$$\dot{x}^2 = l^2\cos^2\phi\,\dot{\phi}^2$$

$$\dot{y} = -a\gamma\sin(\gamma t) - l\sin\phi\,\dot{\phi}$$

$$\dot{y}^2 = \underbrace{a^2\gamma^2\sin^2(\gamma t)}_{\text{specified function of time [can ignore]}} + l^2\sin^2\phi\,\dot{\phi}^2 + 2al\gamma\sin(\gamma t)\sin\phi\,\dot{\phi}$$

Thus, ignoring this function of time

$$T = \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \dot{\phi} \sin(\gamma t) \sin\phi$$

$$\rightarrow L = T - U$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \gamma \dot{\phi} \sin(\gamma t) \sin\phi + m g l \cos\phi$$

Rewrite 2nd term:

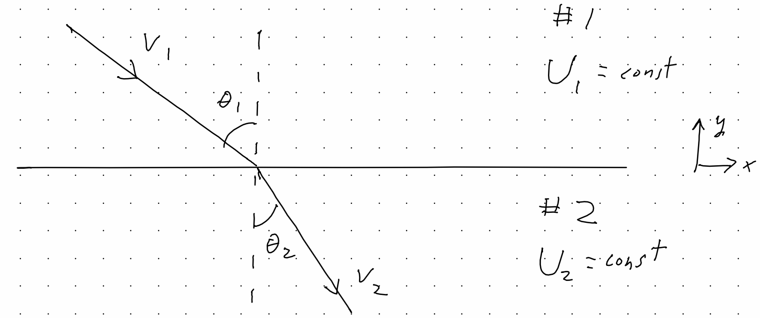
$$-\frac{d}{dt}[m a l \gamma \sin(\gamma t) \cos\phi] = -m a l \gamma^2 \cos(\gamma t) \cos\phi$$
$$+ m a l \gamma \sin(\gamma t) \sin\phi \, \dot{\phi}$$

so $m a l \gamma \dot{\phi} \sin(\gamma t) \sin\phi = -\frac{d}{dt}[\ ] + m a l \gamma^2 \cos(\gamma t) \cos\phi$

Thus, ignoring total time derivative

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \cos\phi + m a l \gamma^2 \cos(\gamma t) \cos\phi$$

Sec 7, Prob 1



- Energy conserved, since no time dependence.
- Also momentum in x-direction (// to interface) is conserved, since no x-dependence of the potential

$$U(x,y) = \begin{cases} U_1 & y \geq 0 \\ U_2 & y < 0 \end{cases}$$

v_1 : given

$$E = \frac{1}{2} m v_1^2 + U_1 = \frac{1}{2} m v_2^2 + U_2$$

$$\rightarrow \frac{1}{2} m v_2^2 = \frac{1}{2} m v_1^2 + (U_1 - U_2)$$

$$v_2^2 = v_1^2 + \frac{2(U_1 - U_2)}{m}$$

so, $v_2 = v_1 \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$

The angles θ_1, θ_2 are related by

$$p_{1x} = p_{2x}$$

$$\cancel{m} v_1 \sin\theta_1 = \cancel{m} v_2 \sin\theta_2$$

thus, $$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

Sec 8, Prob 1

Transformation of action $S = \int L\, dt$

K, K': two inertial frames

K' moves with velocity $\vec{V}$ w.r.t K

Assume that K, K' coincide at $t=0$ so $\vec{r}_a = \vec{r}_a{}'$ wrt these two frames

Now: $$\vec{v}_a = \vec{V} + \vec{v}_a{}'$$

$$L = T - U$$
$$= \sum_a \frac{1}{2} m_a |\vec{v}_a|^2 - U(\vec{r}_1, \vec{r}_2, \dots, t)$$

$$|\vec{v}_a|^2 = |\vec{V} + \vec{v}_a{}'|^2$$
$$= |\vec{V}|^2 + |\vec{v}_a{}'|^2 + 2\vec{V}\cdot\vec{v}_a{}'$$

so

$$L = \sum_a \frac{1}{2} m_a \left(|\vec{V}|^2 + |\vec{v}_a{}'|^2 + 2\vec{V}\cdot\vec{v}_a{}'\right) - U$$
$$= \frac{1}{2}\mu V^2 + T' + \vec{V}\cdot\sum_a m_a \vec{v}_a{}' - U$$

$$= T' - U + \frac{1}{2}\mu V^2 + \vec{P}'\cdot\vec{V}$$
$$= L' + \frac{1}{2}\mu V^2 + \vec{P}'\cdot\vec{V}$$

where $\vec{P}'$ = total momentum wrt K'
$\mu = \sum_a m_a$ = total mass

$$S = \int_{t_1}^{t_2} L\,dt$$

$$= \int_{t_1}^{t_2} \left(L' + \tfrac{1}{2}\mu V^2 + \vec{P}' \cdot \vec{V} \right) dt$$

$$= S' + \underbrace{\tfrac{1}{2}\mu V^2 (t_2 - t_1)}_{\text{doesn't change EOMs}} + \vec{V} \cdot \int_{t_1}^{t_2} \vec{P}'\,dt$$

Now:

$$\vec{V} \cdot \int_{t_1}^{t_2} \vec{P}'\,dt = \vec{V} \cdot \int_{t_1}^{t_2} \sum_a m_a \vec{v}_a{}'\,dt$$

$$= \vec{V} \cdot \sum_a m_a \int_{t_1}^{t_2} \left(\frac{d\vec{r}_a}{dt} \right) dt$$

$$= \vec{V} \cdot \sum_a m_a \vec{r}_a \Big|_{t_1}^{t_2}$$

$$= \vec{V} \cdot \left(\mu \vec{R}(t_2) - \mu \vec{R}(t_1) \right)$$

$$= \mu \vec{V} \cdot \underbrace{\left(\vec{R}(t_2) - \vec{R}(t_1) \right)}_{\text{difference in COM position}}$$

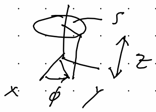
So:

$$S = S' + \tfrac{1}{2}\mu V^2 (t_2 - t_1) + \mu \vec{V} \cdot \left(\vec{R}(t_2) - \vec{R}(t_1) \right)$$

Sec 9, Prob. 1

cylindrical coordinates (s, ϕ, z)

$$s^2 = x^2 + y^2$$



$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = z$$

$$\vec{M} = \vec{r} \times \vec{p} = m\vec{r} \times \dot{\vec{r}}$$

Now: $M_x = m(y\dot{z} - z\dot{y})$

$$M_y = m(z\dot{x} - x\dot{z})$$
$$M_z = m(x\dot{y} - y\dot{x})$$
$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}$$

$$\dot{z} = \dot{z}$$
$$\dot{y} = \dot{s}\sin\phi + s\cos\phi\,\dot{\phi}$$
$$\dot{x} = \dot{s}\cos\phi - s\sin\phi\,\dot{\phi}$$

Thus, $M_x = m\left(s\sin\phi\,\dot{z} - z(\dot{s}\sin\phi + s\cos\phi\,\dot{\phi}) \right)$

$$= m(s\sin\phi\,\dot{z} - z\sin\phi\,\dot{s} - zs\cos\phi\,\dot{\phi})$$

$$M_y = m\left(z(\dot{s}\cos\phi - s\sin\phi\,\dot{\phi}) - s\cos\phi\,\dot{z} \right)$$

$$= m(z\cos\phi\,\dot{s} - zs\sin\phi\,\dot{\phi} - s\cos\phi\,\dot{z})$$

$$M_z = m\left[s\cos\phi\,(\cancel{\dot{s}\sin\phi} + s\cos\phi\,\dot{\phi}) - s\sin\phi\,(\cancel{\dot{s}\cos\phi} - s\sin\phi\,\dot{\phi}) \right]$$

$$= m s^2 \dot{\phi}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$= m^2 \Big\{ \left(\sin\phi\,(s\dot{z} - z\dot{s}) - zs\cos\phi\,\dot{\phi}\right)^2$$

$$+ \left(-\cos\phi\,(s\dot{z} - z\dot{s}) - zs\sin\phi\,\dot{\phi}\right)^2$$

$$+ \left(s^2\dot{\phi}\right)^2 \Big\}$$

$$= m^2 \Big\{ \sin^2\phi\,(s\dot{z} - z\dot{s})^2 + z^2 s^2 \cos^2\phi\,\dot{\phi}^2$$

$$\cancel{- 2zs\sin\phi\cos\phi\,\dot{\phi}\,(s\dot{z} - z\dot{s})}$$

$$+ \cos^2\phi\,(s\dot{z} - z\dot{s})^2 + z^2 s^2 \sin^2\phi\,\dot{\phi}^2$$

$$\cancel{+ 2zs\sin\phi\cos\phi\,\dot{\phi}\,(s\dot{z} - z\dot{s})}$$

$$+ s^4\dot{\phi}^2 \Big\}$$

$$= m^2 \left\{ (s\dot{z} - z\dot{s})^2 + z^2 s^2 \dot{\phi}^2 + s^4 \dot{\phi}^2 \right\}$$

$$= m^2 \left[(s\dot{z} - z\dot{s})^2 + s^2\dot{\phi}^2 (z^2 + s^2) \right]$$

Sec 9, Prob 2

repeat for spherical polar coords.

$$M_x = m(y\dot{z} - z\dot{y}) \quad , \quad \text{cyclic}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

Now:

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$\rightarrow \dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi}$$

$$\dot{y} = \dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi}$$

$$\dot{z} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

Thus,

$$M_x = m(y\dot{z} - z\dot{y})$$

$$= m\Big\{ r\sin\theta\sin\phi\,(\cancel{\dot{r}\cos\theta} - r\sin\theta\,\dot{\theta})$$

$$- r\cos\theta\,(\cancel{\dot{r}\sin\theta\sin\phi} + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi}) \Big\}$$

$$= m\left\{ -r^2\sin^2\theta\sin\phi\,\dot{\theta} - r^2\cos^2\theta\sin\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\cos\phi\,\dot{\phi} \right\}$$

$$= m\left\{ -r^2\sin\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\cos\phi\,\dot{\phi} \right\}$$

$$= -mr^2\left[\sin\phi\,\dot{\theta} + \sin\theta\cos\theta\cos\phi\,\dot{\phi} \right]$$

$$M_y = m(z\dot{x} - x\dot{z})$$
$$= m\{ r\cos\theta\,(\cancel{\dot{r}\sin\theta\cos\phi} + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi})$$
$$- r\sin\theta\cos\phi\,(\cancel{\dot{r}\cos\theta} - r\sin\theta\,\dot{\theta})\}$$

$$= m\{ r^2\cos^2\theta\cos\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\sin\phi\,\dot{\phi} + r^2\sin^2\theta\cos\phi\,\dot{\theta} \}$$

$$= m[r^2\cos\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\sin\phi\,\dot{\phi}]$$

$$= mr^2[\cos\phi\,\dot{\theta} - \sin\theta\cos\theta\sin\phi\,\dot{\phi}]$$

$$M_z = m(x\dot{y} - y\dot{x})$$
$$= m\{ r\sin\theta\cos\phi\,(\cancel{\dot{r}\sin\theta\sin\phi} + \cancel{r\cos\theta\sin\phi\,\dot{\theta}} + r\sin\theta\cos\phi\,\dot{\phi})$$
$$- r\sin\theta\sin\phi\,(\cancel{\dot{r}\sin\theta\cos\phi} + \cancel{r\cos\theta\cos\phi\,\dot{\theta}} - r\sin\theta\sin\phi\,\dot{\phi})\}$$

$$= m[r^2\sin^2\theta\cos^2\phi\,\dot{\phi} + r^2\sin^2\theta\sin^2\phi\,\dot{\phi}]$$

$$= mr^2\sin^2\theta\,\dot{\phi}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$
$$= m^2r^4[\sin\phi\,\dot{\theta} + \sin\theta\cos\theta\cos\phi\,\dot{\phi}]^2$$
$$+ m^2r^4[\cos\phi\,\dot{\theta} - \sin\theta\cos\theta\sin\phi\,\dot{\phi}]^2$$
$$+ m^2r^4\sin^4\theta\,\dot{\phi}^2$$

cross terms will cancel

$$M^2 = m^2r^4\{ \sin^2\phi\,\dot{\theta}^2 + \sin^2\theta\cos^2\theta\cos^2\phi\,\dot{\phi}^2$$
$$+ \cos^2\phi\,\dot{\theta}^2 + \sin^2\theta\cos^2\theta\sin^2\phi\,\dot{\phi}^2$$
$$+ \sin^4\theta\,\dot{\phi}^2 \}$$

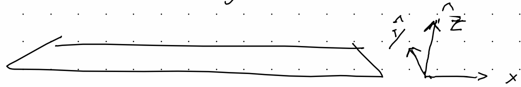
$$= m^2r^4[\dot{\theta}^2 + \sin^2\theta\cos^2\theta\,\dot{\phi}^2 + \sin^4\theta\,\dot{\phi}^2]$$

$$= m^2r^4[\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2(\cos^2\theta + \sin^2\theta)]$$

$$= m^2r^4[\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2]$$

Sec 9, Prob 3

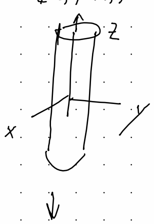
a) Infinite homogeneous plane



P_x, P_y conserved

M_z conserved, where origin is anywhere in (x,y) plane

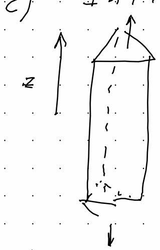
b) Infinite homogeneous cylinder



P_z conserved

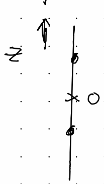
M_z conserved, origin anywhere on z-axis

c) Infinite homog prism

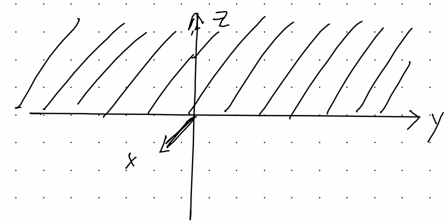


P_z conserved

d) two points : M_z conserved, origin at midpoint of line connecting the two points



e)



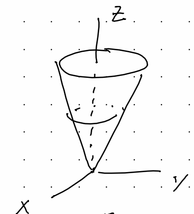
infinite, homog. half plane

$x=0,\ z>0$

$-\infty < y < \infty$

P_y conserved

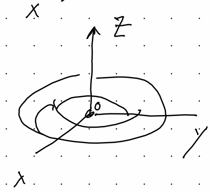
f)



homogeneous cone

M_z conserved, origin anywhere on z-axis.

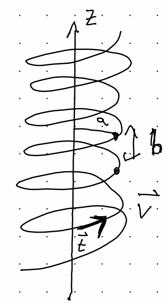
g)



homogeneous circular torus

M_z conserved with $\hat{z}$ directed as shown and origin at O.

h)



Field constant along helix:

$\Delta z = b$ when $\Delta\phi = 2\pi$

at $s^2 = x^2 + y^2 = a^2$

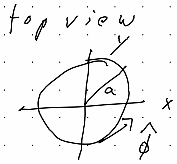
$$\frac{\Delta\phi}{2\pi} = \frac{\Delta z}{b}$$

$$\rightarrow \Delta z = \left(\frac{b}{2\pi}\right)\Delta\phi$$

$$\vec{t} = a\Delta\phi\,\hat{\phi} + \Delta z\,\hat{z}$$

$$= \cancel{a}\Delta\phi\,\underbrace{(x\hat{y} - y\hat{x})}_{\cancel{s}} + \Delta z\,\hat{z}$$

top view



$$\vec{t} = \Delta\phi\,(x\hat{y} - y\hat{x}) + \left(\frac{b}{2\pi}\right)\Delta\phi\,\hat{z}$$

$$= \Delta\phi\left[x\hat{y} - y\hat{x} + \left(\frac{b}{2\pi}\right)\hat{z}\right]$$

Field unchanged if you move along $\vec{t}$.

Thus, $\vec{P}\cdot\vec{t} = \text{const}$

$$\vec{P}\cdot\vec{t} = \Delta\phi\left[xP_y - yP_x + \left(\frac{b}{2\pi}\right)P_z\right]$$

$$= \Delta\phi\left[M_z + \frac{b}{2\pi}P_z\right]$$

so $M_z + \frac{b}{2\pi}P_z = \text{const}$

where z = axis of helix

$b = \Delta z$ for $\Delta\phi = 2\pi$ at $s = a$

Sec 10, prob 1

Different masses, same path, same potential energy

$$L_1 = \frac{1}{2}m_1 v_1^2 - U$$

$$L_2 = \frac{1}{2}m_2 v_2^2 - U$$

Thus, $m_1 v_1^2 = m_2 v_2^2$

$$\frac{m_1}{t_1^2} = \frac{m_2}{t_2^2}$$

$$\rightarrow \left(\frac{t_2}{t_1}\right)^2 = \frac{m_2}{m_1}$$

or $$\boxed{\frac{t_2}{t_1} = \sqrt{\frac{m_2}{m_1}}}$$

Sec 10, Prob 2:

Same path, mass but potential energies differing by a constant

$$L_1 = \frac{1}{2} m v_1^2 - U_1$$
$$L_2 = \frac{1}{2} m v_2^2 - U_2$$

Thus, $\frac{v_1^2}{v_2^2} = \frac{U_1}{U_2}$

$\rightarrow \frac{(l/t_1)^2}{(l/t_2)^2} = \frac{U_1}{U_2}$

so $\frac{t_2}{t_1} = \sqrt{\frac{U_1}{U_2}}$

Sec 40 - Prob 1

single particle in a constant external field

$$L = \frac{1}{2} m v^2 - U(\vec{r})$$

a) Cartesian coords (x, y, z)

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x,y,z)$$

$\rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = p_x/m$

$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \rightarrow \dot{y} = p_y/m$

$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$

$$H = \left(\sum_i p_i \dot{q}_i - L\right)\Big|_{\dot{q} = \dot{q}(q,p)}$$

$$= \left(p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(x,y,z)\right)\Big|$$

$$= p_x\left(\frac{p_x}{m}\right) + p_y\left(\frac{p_y}{m}\right) + p_z\left(\frac{p_z}{m}\right) - \frac{1}{2}m\left(\left(\frac{p_x}{m}\right)^2 + \left(\frac{p_y}{m}\right)^2 + \left(\frac{p_z}{m}\right)^2\right) + U(x,y,z)$$

$\dot{x} = p_x/m$ etc

$$= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + U(x,y,z)$$

---

b) cylindrical coords (s, ϕ, z), $s^2 = x^2 + y^2$

$$L = \frac{1}{2} m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2) - U(s, \phi, z)$$

$\rightarrow p_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s} \rightarrow \dot{s} = p_s/m$

$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ms^2\dot{\phi} \rightarrow \dot{\phi} = p_\phi/ms^2$

$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$

$$H = \left(p_s \dot{s} + p_\phi \dot{\phi} + p_z \dot{z} - \frac{1}{2} m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2) + U(s,\phi,z)\right)\Big|$$

$\dot{s} = p_s/m$ etc

$$= p_s\left(\frac{p_s}{m}\right) + p_\phi\left(\frac{p_\phi}{ms^2}\right) + p_z\left(\frac{p_z}{m}\right)$$

$$-\frac{1}{2} m\left(\left(\frac{p_s}{m}\right)^2 + s^2\left(\frac{p_\phi}{ms^2}\right)^2 + \left(\frac{p_z}{m}\right)^2\right) + U(s,\phi,z)$$

$$= \frac{1}{2m}\left(p_s^2 + \frac{p_\phi^2}{s^2} + p_z^2\right) + U(s,\phi,z)$$

---

c) spherical polar coords (r,θ,ϕ)

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - U(r,\theta,\phi)$$

$$\rightarrow p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \rightarrow \dot{r} = p_r/m$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \rightarrow \dot{\theta} = p_\theta/mr^2$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi} \rightarrow \dot{\phi} = p_\phi/mr^2\sin^2\theta$$

$$H = \left(p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) + U(r,\theta,\phi)\right)\Big|$$

$\dot{r} = p_r/m$ etc.

$$= p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{mr^2}\right) + p_\phi\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)$$

$$-\frac{1}{2}m\left(\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{mr^2}\right)^2 + r^2\sin^2\theta\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)^2\right) + U(r,\theta,\phi)$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r,\theta,\phi)$$

<u>sec 40 - prob 2</u>

$$L = \frac{1}{2}mv^2 + m\vec{v}\cdot(\vec{\Omega}\times\vec{r}) + \frac{1}{2}m|\vec{\Omega}\times\vec{r}|^2 - m\vec{W}\cdot\vec{r} - U$$

restrict to uniformly rotating frame of reference $\vec{W} = 0$, $\dot{\vec{\Omega}} = 0$

$$\rightarrow L = \frac{1}{2}mv^2 + m\vec{v}\cdot(\vec{\Omega}\times\vec{r}) + \frac{1}{2}m|\vec{\Omega}\times\vec{r}|^2 - U(\vec{r})$$

<u>Now</u>: $H = \vec{p}\cdot\vec{v} - L$

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + m\vec{\Omega}\times\vec{r} = m(\vec{v} + \vec{\Omega}\times\vec{r})$$

$$\longrightarrow \vec{v} = \frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}$$

Thus,

$$H = \vec{p}\cdot\left(\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right) - \frac{1}{2}m\left|\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right|^2$$

$$- m\left(\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right)\cdot(\vec{\Omega}\times\vec{r})$$

$$- \frac{1}{2}m|\vec{\Omega}\times\vec{r}|^2 + U(\vec{r})$$

$$= \frac{|\vec{p}|^2}{m} - \cancel{\vec{p}\cdot(\vec{\Omega}\times\vec{r})} - \frac{1}{2}m\left(\frac{|\vec{p}|^2}{m^2} + \cancel{|\vec{\Omega}\times\vec{r}|^2} - \cancel{2\frac{\vec{p}\cdot(\vec{\Omega}\times\vec{r})}{m}}\right)$$

$$- \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + m\cancel{|\vec{\Omega}\times\vec{r}|^2} - \frac{1}{2}m\cancel{|\vec{\Omega}\times\vec{r}|^2} + U(\vec{r})$$

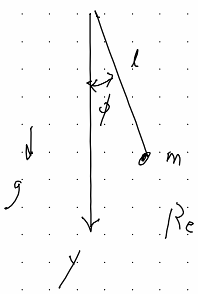
$$= \frac{|\vec{p}|^2}{2m} - \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + U(\vec{r})$$

$$= \frac{|\vec{p}|^2}{2m} - \vec{\Omega}\cdot\underbrace{(\vec{r}\times\vec{p})}_{\vec{M}\text{: angular momentum}} + U(\vec{r})$$

Sec. 11, Prob 1

Simple pendulum: $U = -mgy$

$= -mgl\cos\phi$



Also, $T = \frac{1}{2} m l^2 \dot\phi^2$

Thus, $E = T + U$

$= \frac{1}{2} m l^2 \dot\phi^2 - mgl\cos\phi$

Release from rest at $\phi = \phi_0$

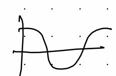
$E = -mgl\cos\phi_0$

$\rightarrow \frac{1}{2} m l^2 \dot\phi^2 - mgl\cos\phi = -mgl\cos\phi_0$

$$T(E) = 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{E - U(\phi)}}$$

$$= 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{-mgl\cos\phi_0 + mgl\cos\phi}}$$

$$= 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$



$\phi < \phi_0$

$\cos\phi > \cos\phi_0$

Now: $\cos\phi = \cos\left(2\frac{\phi}{2}\right)$

$= \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)$

$= 1 - 2\sin^2\left(\frac{\phi}{2}\right)$

Also, $\cos\phi_0 = 1 - 2\sin^2\left(\frac{\phi_0}{2}\right)$

$$T(E) = 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{2\left(\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)\right)}}$$

$$= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2\left(\frac{\phi_0}{2}\right)\left(1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)}\right)}}$$

Let: $x = \frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)} \rightarrow dx = \frac{d\phi}{2} \frac{\cos\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}$

$$= \frac{d\phi}{2} \frac{\sqrt{1 - \sin^2\left(\frac{\phi}{2}\right)}}{\sin\left(\frac{\phi_0}{2}\right)}$$

$$= \frac{d\phi}{2} \frac{\sqrt{1 - \sin^2\left(\frac{\phi_0}{2}\right)x^2}}{\sin\left(\frac{\phi_0}{2}\right)}$$

Thus,

$$T(E) = 2\sqrt{\frac{l}{g}} \int_0^1 \frac{2\,dx\, \cancel{\sin\left(\frac{\phi_0}{2}\right)}}{\sqrt{1 - \sin^2\left(\frac{\phi_0}{2}\right)x^2}\; \cancel{\sin\left(\frac{\phi_0}{2}\right)} \sqrt{1 - x^2}}$$

$$= 4\sqrt{\frac{l}{g}} \int_0^1 \frac{dx}{\sqrt{1 - x^2}\sqrt{1 - k^2 x^2}}, \quad k \equiv \sin\left(\frac{\phi_0}{2}\right)$$

$$= 4\sqrt{\frac{l}{g}}\, K(k)$$

where $K(k)$ = complete elliptic integral of the 1st kind.

Approximation:

$$T(E) = 4\sqrt{\frac{l}{g}} \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

For $k \ll 1$: $\frac{1}{\sqrt{1-k^2x^2}} \approx 1 + \frac{1}{2}k^2x^2$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \approx \int_0^1 \frac{dx}{\sqrt{1-x^2}}\left(1 + \frac{1}{2}k^2x^2\right)$$

Now: $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin(1) = \boxed{\frac{\pi}{2}}$

$$\frac{1}{2}k^2 \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^2}} = \frac{1}{2}k^2 \int_0^{\pi/2} \frac{\sin^2\theta\, \cancel{\cos\theta}\, d\theta}{\cancel{\sqrt{1-\sin^2\theta}}}$$

Let: $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$x^2 = \sin^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\rightarrow \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$\rightarrow \frac{1}{2}k^2 \int_0^1 \frac{x^2\,dx}{\sqrt{1-y^2}} = \frac{1}{2}k^2\,\frac{1}{2}\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{4}k^2\left[\frac{\pi}{2} - \frac{\sin 2\theta}{2}\Big|_0^{\pi/2}\right]$$

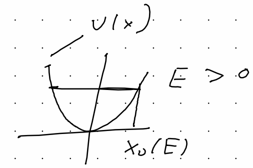
$$= \boxed{\frac{1}{8}k^2\pi}$$

$k = \sin\left(\frac{\phi_0}{2}\right) \approx \frac{\phi_0}{2}$

Thus, $T(E) \approx 4\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \frac{1}{8}k^2\pi + \ldots\right) = \boxed{2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\phi_0^2 + \ldots\right)}$

Sec 11, Prob 2:

(a) $U = A|x|^n$



$$T(E) = 2\sqrt{2m}\int_0^{x_0(E)} \frac{dx}{\sqrt{E - Ax^n}}$$

where $U(x_0) = E$

$A x_0^n = E$

$\rightarrow x_0 = \left(E/A\right)^{1/n}$

$$\sqrt{E - Ax^n} = \sqrt{E}\sqrt{1 - \left(\frac{A}{E}\right)x^n}$$

$$= \sqrt{E}\sqrt{1-t^n} \quad \text{where } t = \left(\frac{A}{E}\right)^{\frac{1}{n}} x$$

$$dt = \left(\frac{A}{E}\right)^{\frac{1}{n}} dx$$

$$t_0 = \left(\frac{A}{E}\right)^{\frac{1}{n}} x_0 = \left(\frac{A}{E}\right)^{\frac{1}{n}}\left(\frac{E}{A}\right)^{\frac{1}{n}} = 1$$

Thus,

$$T(E) = 2\frac{\sqrt{2m}}{\sqrt{E}}\int_0^1 \left(\frac{E}{A}\right)^{\frac{1}{n}} \frac{dt}{\sqrt{1-t^n}}$$

Now let $u = t^n$

$du = n t^{n-1}\,dt = n\, u^{\frac{n-1}{n}}\,dt$

$= n\, u^{1-\frac{1}{n}}\,dt$

$dt = \frac{1}{n}u^{\frac{1}{n}-1}\,du$, $\quad t = 0, 1 \rightarrow u = 0, 1$

Thus

$$T(E) = 2\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 \frac{1}{n}\, \frac{u^{\frac{1}{n}-1}\,du}{\sqrt{1-u}}$$

$$= \frac{2}{n}\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 u^{\frac{1}{n}-1}(1-u)^{-\frac{1}{2}}\,du$$

Recall: Beta Function

Gamma Function

$$B(x,y) = \int_0^1 dt\, t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

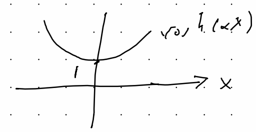
(where $Re(x)>0$, $Re(y)>0$)

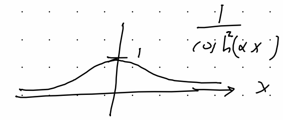
Thus,

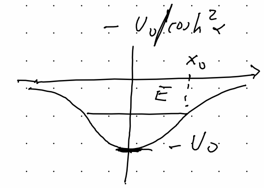
$$\boxed{T(E) = \frac{2}{n}\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}} \leftarrow \Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}$$

$$\boxed{= \frac{2}{n}\sqrt{\frac{2\pi m}{E}}\left(\frac{E}{A}\right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}}$$

(b) $U = -U_0/\cosh^2 \alpha x$







$-U_0 \le E < 0$

Thus, $E = -|E|$

$|E| < U_0$

$$T(E) = 2\sqrt{2m}\int_0^{x_0(E)} \frac{dx}{\sqrt{E-U(x)}}$$

where x_0 given by

$$E = U(x_0) = \frac{-U_0}{\cosh^2(\alpha x_0)}$$

$$\rightarrow \cosh^2(\alpha x_0) = \frac{-U_0}{E}$$

$$\cosh(\alpha x_0) = \sqrt{\frac{U_0}{|E|}}$$

$$\alpha x_0 = \cosh^{-1}\left(\sqrt{\frac{U_0}{|E|}}\right)$$

$$\boxed{x_0 = \frac{1}{\alpha}\cosh^{-1}\left(\sqrt{\frac{U_0}{|E|}}\right)}$$

$$\sqrt{E-U(x)} = \sqrt{E+\frac{U_0}{\cosh^2(\alpha x)}} \qquad ch^2 - sh^2 = 1$$

$$= \frac{1}{\cosh(\alpha x)}\sqrt{E\cosh^2(\alpha x)+U_0}$$

$$= \frac{1}{\cosh(\alpha x)}\sqrt{E\left(1+\sinh^2(\alpha x)\right)+U_0}$$

$$= \frac{1}{\cosh(\alpha x)}\sqrt{(E+U_0)+E\sinh^2(\alpha x)}$$

$$= \frac{1}{\cosh(\alpha x)}\sqrt{(U_0-|E|)-|E|\sinh^2(\alpha x)}$$

thus

$$T(E) = 2\sqrt{2m}\int_0^{x_0}\frac{dx}{\sqrt{E-U(x)}}$$

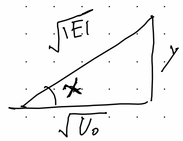
$$= 2\sqrt{2m}\int_0^{x_0}\frac{\cosh(\alpha x)\,dx}{\sqrt{(U_0-|E|)-|E|\sinh^2(\alpha x)}}$$

Let: $t = \sinh(\alpha x) \longrightarrow t_0 = \sinh(\alpha x_0)$

$dt = \alpha\cosh(\alpha x)\,dx$

so $t_0 = \sinh\left(\cosh^{-1}\sqrt{\frac{U_0}{|E|}}\right)$

$$= \sqrt{\frac{U_0-|E|}{|E|}}$$



$U_0 - y^2 = |E|$

$y = \sqrt{U_0-|E|}$

$$T(E) = 2\sqrt{2m}\int_0^{\sqrt{\frac{U_0-|E|}{|E|}}}\frac{dt}{\alpha\sqrt{(U_0-|E|)-|E|t^2}}$$

$$= 2\sqrt{2m}\int_0^{\sqrt{\frac{U_0-|E|}{|E|}}}\frac{dt}{\alpha\sqrt{U_0-|E|}\sqrt{1-\frac{|E|}{U_0-|E|}t^2}}$$

Define: $\sin u = \sqrt{\frac{|E|}{U_0-|E|}}\,t$

$du\cos u = \sqrt{\frac{|E|}{U_0-|E|}}\,dt$

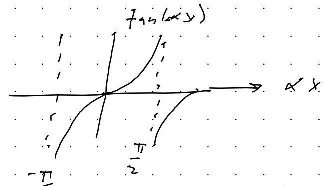
$$T(E) = \frac{2\sqrt{2m}}{\alpha\cancel{\sqrt{U_0-|E|}}}\int_0^{\pi/2}\sqrt{\frac{\cancel{U_0-|E|}}{|E|}}\frac{\cancel{\cos u}\,du}{\cancel{\sqrt{1-\sin^2 u}}}$$

$$= \frac{\cancel{2}}{\alpha}\sqrt{\frac{2m}{|E|}}\,\frac{\pi}{\cancel{2}}$$

$$= \boxed{\frac{\pi}{\alpha}\sqrt{\frac{2m}{|E|}}}$$

(c) $U = U_0 \tan^2 \alpha x$

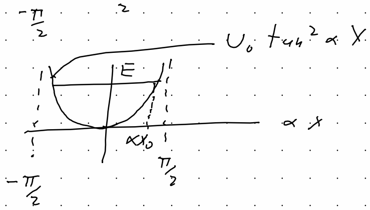
$$T = 2\sqrt{2m} \int_0^{x_0} \frac{dx}{\sqrt{E - U(x)}}$$



where x_0 determined by

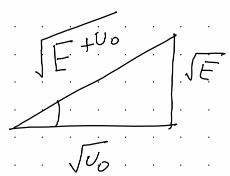
$E = U(x_0)$

$= U_0 \tan^2(\alpha x_0)$



$$\rightarrow \quad \alpha x_0 = \tan^{-1}\left(\sqrt{\frac{E}{U_0}}\right)$$

$$\boxed{x_0 = \frac{1}{\alpha} \tan^{-1}\left(\sqrt{\frac{E}{U_0}}\right)}$$



$$\sqrt{E - U(x)} = \sqrt{E - U_0 \tan^2(\alpha x)}$$

$$= \sqrt{E - U_0 \frac{\sin^2(\alpha x)}{\cos^2(\alpha x)}}$$

$$= \frac{1}{\cos \alpha x} \sqrt{E \cos^2(\alpha x) - U_0 \sin^2(\alpha x)}$$

$$= \frac{1}{\cos \alpha x} \sqrt{E\left(1 - \sin^2(\alpha x)\right) - U_0 \sin^2(\alpha x)}$$

$$= \frac{1}{\cos \alpha x} \sqrt{E - (U_0 + E)\sin^2(\alpha x)}$$

$$= \frac{\sqrt{E}}{\cos \alpha x} \sqrt{1 - \left(\frac{U_0 + E}{E}\right)\sin^2(\alpha x)}$$

Thus

$$T(E) = 2\sqrt{2m} \int_0^{x_0} \frac{\cos \alpha x \, dx}{\sqrt{E} \sqrt{1 - \left(\frac{U_0+E}{E}\right)\sin^2 \alpha x}}$$

Let $u = \sin \alpha x$

$du = \alpha \cos \alpha x \, dx$

$x = 0, x_0 \rightarrow$ $u = 0$

$u_0 = \sin(\alpha x_0)$

$= \sin\left(\tan^{-1}\sqrt{\frac{E}{U_0}}\right)$

$= \sqrt{\frac{E}{E+U_0}}$

$$\rightarrow T(E) = 2\sqrt{\frac{2m}{E}} \frac{1}{\alpha} \int_0^{u_0} \frac{du}{\sqrt{1 - \frac{U_0+E}{E} u^2}}$$

Let $\sqrt{\frac{U_0+E}{E}}\, u = \sin\theta \rightarrow \sqrt{\frac{U_0+E}{E}}\, du = \cos\theta \, d\theta$

$u = 0 \rightarrow \theta = 0$

$u = u_0 \rightarrow \sin\theta_0 = \sqrt{\frac{U_0+E}{E}} \sqrt{\frac{E}{U_0+E}} = 1 \rightarrow \theta_0 = \frac{\pi}{2}$

$$\rightarrow T(E) = 2\sqrt{\frac{2m}{E}} \frac{1}{\alpha} \int_0^{\pi/2} \frac{\cancel{\cos\theta}\, d\theta}{\sqrt{\frac{U_0+E}{E}} \cancel{\sqrt{1 - \sin^2\theta}}}$$

$$= \frac{2}{\alpha}\sqrt{\frac{2m}{E}}\sqrt{\frac{E}{U_0+E}} \frac{\pi}{2} = \boxed{\frac{\pi}{\alpha}\sqrt{\frac{2m}{E}}\sqrt{\frac{E}{U_0+E}}}$$

Sec 13, Prob 1

$\vec{X}$: position vector of M

$\vec{x}_a$: $a = 1, 2, \cdots, n$ position vector of n masses all with mass m.

Closed system $\rightarrow$ linear momentum conserved
$\rightarrow$ CoM Frame

$$M\vec{X} + m(\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_n) = 0$$

or $$M\vec{X} + m\sum_{a=1}^{n} \vec{x}_a = 0 \qquad (1)$$

Taking time derivative $\rightarrow$ $M\dot{\vec{X}} + \sum_a \dot{\vec{x}}_a = 0$. (2)

Define relative position vectors:

$\vec{r}_1 \equiv \vec{x}_1 - \vec{X}$

$\vec{r}_2 \equiv \vec{x}_2 - \vec{X}$

etc.

or $\vec{r}_a \equiv \vec{x}_a - \vec{X}$, $a = 1, 2, \cdots, n$. (3)

Summing up (3):

$$\begin{aligned}\sum \vec{r}_a &= \sum(\vec{x}_a - \vec{X}) \\ &= \sum \vec{x}_a - n\vec{X} \\ &= -\frac{M}{m}\vec{X} - n\vec{X} \\ &= -\underbrace{\frac{(M+nm)}{m}}\vec{X} = -\frac{\mu}{m}\vec{X}\end{aligned}$$

total mass

Thus, $\vec{X} = -\frac{m}{\mu}\sum_a \vec{r}_a$

and $\vec{x}_a = \vec{r}_a + \vec{X}$

give you $\vec{x}_a, \vec{X}$ in terms of $\vec{r}_a$.

KE: $$\begin{aligned}T &= \frac{1}{2}\sum_a m|\dot{\vec{x}}_a|^2 + \frac{1}{2}M|\dot{\vec{X}}|^2 \\ &= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a + \dot{\vec{X}}|^2 + \frac{1}{2}M|\dot{\vec{X}}|^2 \\ &= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a|^2 + \frac{1}{2}\sum_a m\, 2\dot{\vec{r}}_a \cdot \dot{\vec{X}} \\ &\quad + \frac{1}{2}\sum_a m|\dot{\vec{X}}|^2 + \frac{1}{2}M|\dot{\vec{X}}|^2 \\ &= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a|^2 + m\left(\sum_a \dot{\vec{r}}_a\right)\cdot\dot{\vec{X}} \\ &\quad + \frac{1}{2}nm|\dot{\vec{X}}|^2 + \frac{1}{2}M|\dot{\vec{X}}|^2 \\ &= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a|^2 - \mu|\dot{\vec{X}}|^2 + \frac{1}{2}\underbrace{(nm+M)}_{M}|\dot{\vec{X}}|^2 \\ &= \frac{1}{2}m\sum_a |\dot{\vec{r}}_a|^2 - \frac{1}{2}\mu|\dot{\vec{X}}|^2\end{aligned}$$

Now rewrite last term:

$$-\frac{1}{2}\mu|\dot{\vec{X}}|^2 = -\frac{1}{2}\mu\frac{m^2}{\mu^2}\left|\sum_a \dot{\vec{r}}_a\right|^2 = -\frac{1}{2}\frac{m^2}{\mu}\left|\sum_a \dot{\vec{r}}_a\right|^2$$

Thus, $$T = \frac{1}{2}m\sum_a|\dot{\vec{r}}_a|^2 - \frac{1}{2}\frac{m^2}{\mu}\left|\sum_a \dot{\vec{r}}_a\right|^2$$

Potential energy

$$U = U\big(|\vec{x}_1-\vec{x}_2|,\ |\vec{x}_1-\vec{x}_3|,\ \dots,\ |\vec{x}_1-\vec{x}_n|,$$
$$\dots,$$
$$|\vec{x}_1-\vec{X}|,\ |\vec{x}_2-\vec{X}|,\ \dots,\ |\vec{x}_n-\vec{X}|\big)$$

$$= U\big(|\vec{r}_1-\vec{r}_2|,\ |\vec{r}_1-\vec{r}_3|,\ \dots,\ |\vec{r}_1-\vec{r}_n|,$$
$$\dots,$$
$$|\vec{r}_1|,\ |\vec{r}_2|,\ \dots,\ |\vec{r}_n|\big)$$

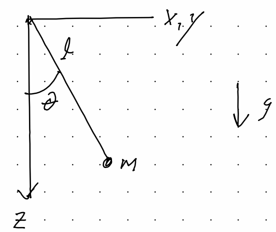
which depends only on the relative position vectors $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$

Thus,

$$L = \frac{1}{2} m \sum_a |\dot{\vec{r}}_a|^2 - \frac{1}{2}\frac{m^2}{\mu}\Big|\sum_a \dot{\vec{r}}_a\Big|^2 - U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n)$$

Sec 14, Prob. 1

Spherical pendulum:



$$T = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2)$$
$$U = -mgz$$
$$= -mgl\cos\theta$$

Thus,

$$L = T - U$$
$$= \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + mgl\cos\theta$$

No t-dependence $\rightarrow$ $E = T + U = \text{const}$

No ϕ-dependence $\rightarrow$ $\dfrac{\partial L}{\partial \dot{\phi}} = m l^2 \sin^2\theta\, \dot{\phi} \equiv M_z = \text{const}$

$$E = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) - mgl\cos\theta$$
$$= \frac{1}{2} m l^2 \left(\dot{\theta}^2 + \sin^2\theta \frac{M_z^2}{m^2 l^4 \sin^4\theta}\right) - mgl\cos\theta$$
$$= \frac{1}{2} m l^2 \dot{\theta}^2 + \underbrace{\frac{M_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta}_{U_{eff}(\theta)}$$

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{M_z^2}{2ml^2\sin^2\theta} + U(\theta)$$

$$\rightarrow \quad \frac{1}{2} m l^2 \dot{\theta}^2 = E - U(\theta) - \frac{M_z^2}{2ml^2\sin^2\theta}$$

$$\dot{\theta} = \sqrt{\frac{2}{ml^2}(E - U(\theta)) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}$$

$$\frac{d\theta}{dt} = \dot{\theta} = \sqrt{\frac{2}{ml^2}\left(E+mgl\cos\theta\right) - \frac{M_z^2}{m^2l^4\sin^2\theta}}$$

$$\rightarrow dt = \frac{d\theta}{\sqrt{\quad}}$$

$$\boxed{t = \int \frac{d\theta}{\sqrt{\frac{2}{ml^2}\left(E+mgl\cos\theta\right) - \frac{M_z^2}{m^2l^4\sin^2\theta}}} + \text{const}}$$

Path: use $M_z = ml^2\sin^2\theta\,\dot{\phi}$

Thus,
$$\frac{d\theta}{dt} = \frac{d\theta}{d\phi}\frac{d\phi}{dt} = \frac{d\theta}{d\phi}\frac{M_z}{ml^2\sin^2\theta}$$

Thus,
$$\frac{d\theta}{d\phi} = \frac{d\theta}{dt}\frac{ml^2\sin^2\theta}{M_z} = \sqrt{\quad}\;\frac{ml^2\sin^2\theta}{M_z}$$

$$d\phi = \frac{d\theta\, M_z}{\sqrt{\quad}\; ml^2\sin^2\theta}$$

$$\boxed{\phi = \int \frac{M_z\, d\theta / ml^2\sin^2\theta}{\sqrt{\frac{2}{ml^2}\left(E+mgl\cos\theta\right) - \frac{M_z^2}{m^2l^4\sin^2\theta}}} + \text{const}}$$

Turning points: (where $\dot{\theta}=0$)

$$E = U_{eff}(r) = \frac{M_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta$$

$$\rightarrow 2Eml^2\sin^2\theta = M_z^2 - 2m^2gl^3\sin^2\theta\cos\theta$$

$$2Eml^2(1-\cos^2\theta) = M_z^2 - 2m^2gl^3(1-\cos^2\theta)\cos\theta$$

$$2Eml^2 - 2Eml^2\cos^2\theta = M_z^2 - 2m^2gl^3\cos\theta + 2m^2gl^3\cos^3\theta$$

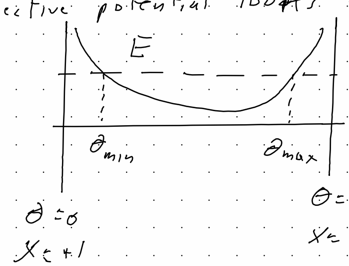
Thus,
$$2m^2gl^3\cos^3\theta + 2Eml^2\cos^2\theta - 2m^2gl^3\cos\theta + \left(M_z^2 - 2Eml^2\right) = 0$$

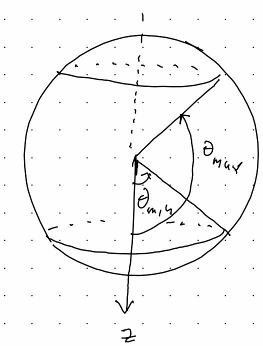
Divide by $2m^2gl^3$:

$$\rightarrow \boxed{\cos^3\theta + \frac{E}{mgl}\cos^2\theta - \cos\theta + \left(\frac{M_z^2}{2m^2gl^3} - \frac{E}{mgl}\right) = 0}$$

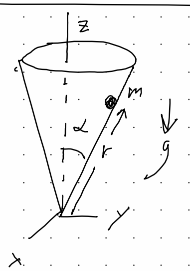
Cubic equation for $X \equiv \cos\theta$

Effective potential looks like:





Sec 14, Prob. 2



spherical polar coords (r,θ,ϕ)

Constraint $\theta=\alpha$

Generalized coords: (r,ϕ)

$$T=\frac{1}{2}m\left(\dot r^2+r^2\cancel{\dot\theta^2}+r^2\sin^2\theta\,\dot\phi^2\right)\qquad \dot\theta = 0,\ \theta=\alpha$$

$$=\frac{1}{2}m\left(\dot r^2+r^2\sin^2\alpha\,\dot\phi^2\right)$$

$$U=mgz$$
$$=mgr\cos\alpha$$

$$L=T-U$$
$$=\frac{1}{2}m\left(\dot r^2+r^2\sin^2\alpha\,\dot\phi^2\right)-mgr\cos\alpha$$

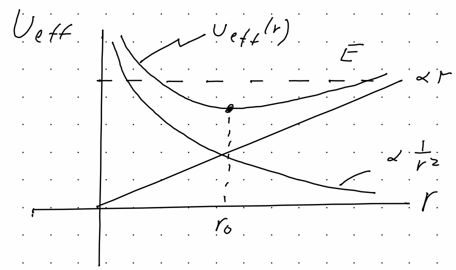
$E=\text{const}$ (since no explicit t dependence)

$$\frac{\partial L}{\partial\dot\phi}=mr^2\sin^2\alpha\,\dot\phi\equiv M_z=\text{const}$$

(since no explicit ϕ dependence)

$$E=T+U$$
$$=\frac{1}{2}m\left(\dot r^2+r^2\sin^2\alpha\,\dot\phi^2\right)+mgr\cos\alpha$$
$$=\frac{1}{2}m\dot r^2+\frac{1}{2}mr^2\sin^2\alpha\left(\frac{M_z^2}{m^2r^4\sin^4\alpha}\right)+mgr\cos\alpha$$
$$=\frac{1}{2}m\dot r^2+\frac{M_z^2}{2mr^2\sin^2\alpha}+mgr\cos\alpha$$
$$=\frac{1}{2}m\dot r^2+U_{eff}(r)$$

$$U_{eff}(r)=\frac{M_z^2}{2mr^2\sin^2\alpha}+mgr\cos\alpha$$



Bound orbits for $E>U_{eff,min}=U_{eff}(r_0)$

r_0: stable circular orbit

t-equation:

$$t=\int\frac{dr}{\sqrt{\frac{2}{m}\left(E-mgr\cos\alpha\right)-\frac{M_z^2}{m^2r^2\sin^2\alpha}}}+\text{const}$$

Using $M_z=mr^2\sin^2\alpha\,\dot\phi$

$$\rightarrow\ \frac{dr}{dt}=\frac{dr}{d\phi}\frac{d\phi}{dt}=\frac{dr}{d\phi}\frac{M_z}{mr^2\sin^2\alpha}$$

ϕ-equation:

$$\phi=\int\frac{\left(\frac{M_z}{\sin^2\alpha}\right)\frac{dr}{r^2}}{\sqrt{2m\left(E-mgr\cos\alpha\right)-\frac{M_z^2}{r^2\sin^2\alpha}}}+\text{const}$$

Turning points: $r = r_{min}, r_{max}$

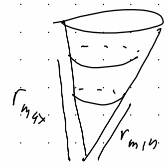
Determined by effective potential

$$E = U_{eff}(r)$$
$$= \frac{M_z^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha$$

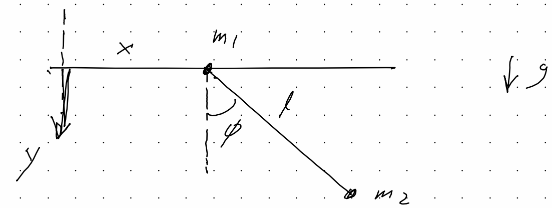
$$\rightarrow \quad 2mEr^2\sin^2\alpha = M_z^2 + 2m^2 g r^3 \sin^2\alpha\cos\alpha$$

$$0 = 2m^2 g r^3 \sin^2\alpha \cos\alpha - 2mEr^2\sin^2\alpha + M_z^2$$
$$= r^3 - \left(\frac{E}{mg\cos\alpha}\right) r^2 + \frac{M_z^2}{2m^2 g \sin^2\alpha\cos\alpha}$$

cubic equation agin



Sec 14, Prob 3



From Sec 5, Prob 2 we have

$$L = \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + \tfrac{1}{2} m_2 (l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi)$$
$$+ m_2 g l \cos\phi$$

No dependence on x:

$$\rightarrow \quad P_x \equiv \frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2 l\dot{\phi}\cos\phi$$
$$= \text{const}$$

(x-component of total momentum)

No explicit t-dependence

$$\rightarrow \quad E = T + U = \text{const}$$
$$= \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + \tfrac{1}{2} m_2 (l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi)$$
$$- m_2 g l \cos\phi$$

Work in frame where $COM_x = 0$:

$$COM_x = m_1 x + m_2(x + l\sin\phi)$$
$$= (m_1+m_2)x + m_2 l \sin\phi$$

$$COM_x = 0 \quad \rightarrow \quad x = -\left(\frac{m_2}{m_1+m_2}\right) l\sin\phi$$

$$\dot x = -\left(\frac{m_2}{m_1+m_2}\right) l \cos\phi\, \dot\phi$$

thus,

$$E = \frac12(m_1+m_2)\dot x^2 + \frac12 m_2\left(l^2\dot\phi^2 + 2l\dot x\dot\phi\cos\phi\right) - m_2 g l\cos\phi$$

$$= \frac12(m_1+m_2)\frac{m_2^2}{(m_1+m_2)^2}\, l^2\cos^2\phi\,\dot\phi^2 + \frac12 m_2 l^2\dot\phi^2 - m_2\left(\frac{m_2}{m_1+m_2}\right) l^2\cos^2\phi\,\dot\phi^2 - m_2 g l\cos\phi$$

$$= \frac12 m_2 l^2\dot\phi^2\left[1-\left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right] - m_2 g l\cos\phi$$

1-d problem:

$$\frac{E + m_2 g l\cos\phi}{\left[1-\left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right]} = \frac12 m_2 l^2\dot\phi^2$$

$$\frac{d\phi}{dt} = \dot\phi = \sqrt{\frac{\frac{2}{m_2 l^2}\left(E+m_2 g l\cos\phi\right)}{1-\left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi}}$$

$$\to dt = \frac{d\phi}{\sqrt{\ }}$$

$$= d\phi\sqrt{\frac{1-\left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi}{\frac{2}{m_2 l^2}\left(E+m_2 g l\cos\phi\right)}}$$

$$= d\phi\sqrt{\frac{m_2 l^2}{2}\,\frac{1}{m_1+m_2}}\sqrt{\frac{(m_1+m_2)-m_2\cos^2\phi}{E+m_2 g l\cos\phi}}$$

$$= d\phi\sqrt{\frac{m_2}{m_1+m_2}}\sqrt{\frac{l^2}{2}}\sqrt{\frac{m_1+m_2\sin^2\phi}{E+m_2 g l\cos\phi}}$$

so

$$\boxed{t = \sqrt{\left(\frac{m_2}{m_1+m_2}\right)\frac{l^2}{2}}\int d\phi\sqrt{\frac{m_1+m_2\sin^2\phi}{E+m_2 g l\cos\phi}} + \text{const}}$$

Now: $x_2 = x + l\sin\phi$

$y_2 = l\cos\phi$

Using $x = -\left(\frac{m_2}{m_1+m_2}\right) l\sin\phi$

$$\to x_2 = \left[-\left(\frac{m_2}{m_1+m_2}\right)l\sin\phi + l\sin\phi\right] = \left(\frac{m_1}{m_1+m_2}\right) l\sin\phi$$

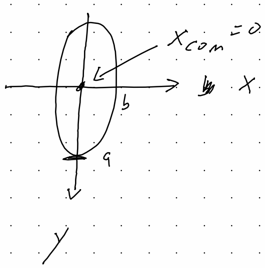
$$x_2 = \left(\frac{m_1}{m_1+m_2}\right) l \sin\phi \qquad = b \sin\phi$$

$$y_2 = l \cos\phi \qquad = a \cos\phi$$

$$\left(\frac{y_2}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = \cos^2\phi + \sin^2\phi = 1$$

which is an ellipse with semi-major and semi-minor axes:

$$a = l \ , \quad b = l\left(\frac{m_1}{m_1+m_2}\right)$$



If $m_1 >> m_2$
then $a = b \approx l$
so that m_2 moves
along a circular
arc of radius l.

Sec 15, Prob 1

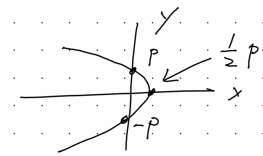
$U = -\frac{\alpha}{r}$, $E = 0$ $\longrightarrow$ $e = 1$

$$\frac{p}{r} = 1 + \cos\phi$$

when $\phi = 0$, $p = 2 \cdot r$

so $r_{min} = \frac{p}{2}$

NOTE: $p = r + r\cos\phi$
$= \sqrt{x^2+y^2} + x$



$\rightarrow (p-x)^2 = x^2+y^2$

$p^2 + \cancel{x^2} - 2px = \cancel{x^2} + y^2$

$p^2 - y^2 = 2px$

a

$\rightarrow \boxed{x = \frac{p^2 - y^2}{2p}}$ parabola

when $y = 0$, $x = \frac{1}{2}p$
$x = 0$, $y = \pm p$

Time equation:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}[E - U(r)] - \frac{M^2}{m^2 r^2}}} \longrightarrow t. \cos \phi$$

$U(r) = -\frac{\alpha}{r}$, $p = \frac{M^2}{m\alpha}$, $e=1$, $E=0$

$$\to \quad t = \int \frac{dr}{\sqrt{\frac{2}{m}\frac{\alpha}{r} - \frac{M^2}{m^2r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{2}{m}\frac{\alpha}{r} - \frac{m\alpha p}{m^2 r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{dr}{\sqrt{\frac{2}{r} - \frac{p}{r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{r\,dr}{\sqrt{2r-p}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha p}} \int \frac{r\,dr}{\sqrt{\frac{2r}{p}-1}} + \text{const}$$

Define: $\frac{2r}{p} - 1 = \xi^2 > 0 \quad (-\infty < \xi < \infty)$

$$2r = p(1+\xi^2)$$

$$\boxed{r = \frac{p}{2}(1+\xi^2)} \to \quad dr = p\xi\, d\xi$$

$$r\,dr = \frac{p^2}{2}(\xi + \xi^3)\,d\xi$$

Thus,

$$t = \sqrt{\frac{m}{\alpha p}} \int \frac{\frac{p^2}{2}(\xi+\xi^3)\,d\xi}{\sqrt{\xi^2}} + \text{const}$$

$$= \sqrt{\frac{mp^3}{\alpha}}\ \frac{1}{2} \int (1+\xi^2)\,d\xi + \text{const}$$

$$= \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}}\left(\xi + \frac{1}{3}\xi^3\right) + \text{const}$$

choose constant so that $t=0 \leftrightarrow \xi = 0$ (const $=0$)

so, $\boxed{t = \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}}\left(\xi + \frac{1}{3}\xi^3\right)}$

Now:

$$\frac{p}{r} = 1 + \cos\phi$$

$$p = r(1+\cos\phi)$$

$$p = \frac{p}{2}(1+\xi^2)(1+\cos\phi)$$

$$2 = 1 + \xi^2 + \cos\phi + \xi^2\cos\phi$$

$$1 - \xi^2 = (1+\xi^2)\cos\phi$$

$$\to \quad \boxed{\cos\phi = \frac{1-\xi^2}{1+\xi^2}}$$

$$\boxed{\begin{aligned} x &= r\cos\phi \\ &= \frac{p}{2}\left(\cancel{1+\xi^2}\right)\left(\frac{1-\xi^2}{\cancel{1+\xi^2}}\right) \\ &= \frac{p}{2}\left(1-\xi^2\right) \end{aligned}}$$

Also,

$x^2+y^2=r^2$

$$\rightarrow\ y^2 = r^2 - x^2$$

$$= \frac{p^2}{4}(1+\xi^2)^2 - \frac{p^2}{4}(1-\xi^2)^2$$

$$= \frac{p^2}{4}\left(\cancel{1}+\cancel{\xi^4}+2\xi^2-\cancel{1}-\cancel{\xi^4}+2\xi^2\right)$$

$$= p^2\xi^2$$

so $\boxed{y = p\xi}$

<u>Sec 15, Prob 3:</u>

$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U)-M^2/r^2}}$$

This is the change in ϕ as r goes from r_{min} to r_{max} and then back to r_{min}.

A closed bound orbit would have $\Delta\phi = 2\pi m/n$ for m, n integers.

<u>Consider:</u> $U = -\frac{\alpha}{r} + \delta U$ where $|\delta U| \ll |\frac{\alpha}{r}|$

For $\delta U = 0$, $\Delta\phi = 2\pi$

$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-(U+\delta U))-M^2/r^2}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U)-2m\delta U-M^2/r^2}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{\left(2m(E-U)-\frac{M^2}{r^2}\right)\left(1-\frac{2m\delta U}{2m(E-U)-M^2/r^2}\right)}}$$

$$\approx 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U)-M^2/r^2}}\left[1+\frac{m\delta U}{2m(E-U)-M^2/r^2}\right]$$

For $U = -\alpha/r$:

$$\Delta\phi \approx 2\pi + 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2 \quad m\,\delta U}{\left[2m(E-U) - M^2/r^2\right]^{3/2}}$$

$$\approx 2\pi + \delta\phi$$

where

$$\delta\phi \equiv \int_{r_{min}}^{r_{max}} \frac{2m\,\delta U\, M\,dr/r^2}{\left[2m(E-U) - M^2/r^2\right]^{3/2}}$$

$$= \frac{\partial}{\partial M}\left[\int_{r_{min}}^{r_{max}} \frac{2m\,\delta U\,dr}{\sqrt{2m(E-U) - M^2/r^2}}\right]$$

Evaluate terms in integrand along unperturbed path since δU is already small

$$\frac{p}{r} = 1 + e\cos\phi \quad , \quad p = a(1-e^2)$$

$$\rightarrow \frac{-p}{r^2}dr = -e\sin\phi\,d\phi \rightarrow \boxed{dr = \frac{r^2}{p} e\sin\phi\,d\phi}$$

$$\sqrt{} = \sqrt{2mE + \frac{2m\alpha}{r} - \frac{M^2}{r^2}}$$

$$= \sqrt{-2m|E| + \frac{4m|E|}{(1-e^2)}\frac{p}{r} - \frac{2m|E|}{(1-e^2)}\frac{p^2}{r^2}}$$

$$\sqrt{} = \sqrt{2m|E|}\sqrt{-1 + \frac{2}{(1-e^2)}(1+e\cos\phi) - \frac{1}{(1-e^2)}(1+e\cos\phi)^2}$$

$$= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}}\sqrt{-(\cancel{1}-e^2) + 2(\cancel{1}+\cancel{e\cos\phi}) - (\cancel{1}+\cancel{2e\cos\phi}+e^2\cos^2\phi)}$$

$$= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}}\sqrt{e^2(1-\cos^2\phi)}$$

$$= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}}\, e\sin\phi$$

Thus,

$$\boxed{\delta\phi = \frac{\partial}{\partial M}\left[2m\int_0^{\pi} \frac{\delta U\, r^2 \cancel{e\sin\phi}\,d\phi}{\sqrt{\frac{2m|E|}{1-e^2}}\,\cancel{e\sin\phi}\,p}\right] = \frac{\partial}{\partial M}\left[\frac{2m}{p\sqrt{\frac{2m|E|}{1-e^2}}}\int_0^{\pi} d\phi\, r^2\,\delta U\right] = \frac{\partial}{\partial M}\left[\frac{2m}{M}\int_0^{\pi} d\phi\, r^2\,\delta U\right]}$$

using (15.6): $\dfrac{M}{\sqrt{2m|E|}} = \dfrac{p}{\sqrt{1-e^2}}$

Evaluate:

$$\delta\phi = \frac{\partial}{\partial M}\left[\frac{2m}{M}\int_0^\pi d\phi\, r^2 \delta U\right]$$

for (a) $\delta U = \beta/r^2$, (b) $\delta U = \frac{\gamma}{r^3}$

(a) $$\delta\phi = \frac{\partial}{\partial M}\left[\frac{2m}{M}\int_0^\pi d\phi\, \beta\right]$$

$$= 2\pi\beta m \frac{\partial}{\partial M}\left(\frac{1}{M}\right)$$

$$= -\frac{2\pi\beta m}{M^2}$$

$$= \boxed{-\frac{2\pi\beta}{\alpha p}}$$

$\leftarrow$ Recall: $p = \frac{M^2}{m\alpha}$

$\alpha p = \frac{M^2}{m}$

(b) $$\delta\phi = \frac{\partial}{\partial M}\left[\frac{2m}{M}\int_0^\pi d\phi\, \frac{\gamma}{r}\right]$$

$$= \frac{\partial}{\partial M}\left[\frac{2m\gamma}{M}\int_0^\pi d\phi\left(\frac{1+e\cos\phi}{p}\right)\right]$$

$$= 2m\gamma \frac{\partial}{\partial M}\left[\frac{1}{Mp}\left(\pi + \cancel{e\sin\phi\Big|_0^\pi}^{\,0}\right)\right]$$

Now: $\frac{1}{Mp} = \frac{m\alpha}{M^3} \rightarrow \frac{\partial}{\partial M}\left(\frac{1}{Mp}\right) = -\frac{3m\alpha}{M^4}$

So $$\delta\phi = -6\pi\gamma\frac{m^2\alpha}{M^4}$$

$$= -6\pi\gamma\alpha\left(\frac{1}{p\alpha}\right)^2$$

$$= \boxed{\frac{-6\pi\gamma}{p^2\alpha}}$$

sec 16, prob 2

From (16.7) we have

$$p(\theta_0)d\theta_0 = \tfrac{1}{2}\sin\theta_0\, d\theta_0$$
$$= -\tfrac{1}{2}\, d(\cos\theta_0)$$

where θ_0 is the angle of one of the emitted particles in the com frame.

= We would like to find $p(\theta)d\theta$, where θ is the angle of one of the emitted particle in the lab frame.

= Since $p(\theta)d\theta = p(\theta_0)d\theta_0$

we just need to find θ_0 as a function of θ.

This is given by (16.6) which we first derive.

Proof: Given $\tan\theta = \dfrac{v_0 \sin\theta_0}{v_0\cos\theta_0 + V}$

we have

$$\tan\theta\,(v_0\cos\theta_0 + V) = v_0\sqrt{1-\cos^2\theta_0}$$

Square both sides

$$\tan^2\theta\,(v_0^2\cos^2\theta_0 + V^2 + 2v_0 V\cos\theta_0) = v_0^2(1-\cos^2\theta_0)$$

$$\underbrace{(1+\tan^2\theta)}_{\sec^2\theta}\, v_0^2\cos^2\theta_0 + 2v_0 V\tan^2\theta\cos\theta_0 + (V^2\tan^2\theta - v_0^2) = 0$$

quadratic equation for $\cos\theta_0$

$$\to \cos\theta_0 = \frac{-2v_0 V\tan^2\theta \pm \sqrt{4v_0^2V^2\tan^4\theta - 4v_0^2\sec^2\theta\,(V^2\tan^2\theta - v_0^2)}}{2\,v_0^2\sec^2\theta}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \frac{1}{\sec^2\theta}\sqrt{\left(\frac{V}{v_0}\right)^2\tan^4\theta - \sec^2\theta\left(\left(\frac{V}{v_0}\right)^2\tan^2\theta - 1\right)}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \frac{1}{\sec\theta}\sqrt{\left(\frac{V}{v_0}\right)^2\left(\frac{\sin^4\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}\right) + 1}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{\left(\frac{V}{v_0}\right)^2\frac{\sin^2\theta(\sin^2\theta - 1)}{\cos^2\theta} + 1}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{1 - \left(\frac{V}{v_0}\right)^2\sin^2\theta}$$

(for $v_0 > V$ take $+\sqrt{\ }$, for $v_0 < V$ take both signs.)

Now differentiate both sides:

$$d(\cos\theta_0) = -\frac{V}{v_0}\, 2\sin\theta\cos\theta\, d\theta \mp \sin\theta\, d\theta\sqrt{\ }$$
$$\pm \cos\theta\left(\frac{1}{\cancel{2}}\right)\frac{1}{\sqrt{\ }}(-\cancel{2})\left(\frac{V}{v_0}\right)^2\sin\theta\cos\theta\, d\theta$$

$$= \sin\theta\, d\theta\left[-2\frac{V}{v_0}\cos\theta \mp \sqrt{\ } \mp \left(\frac{V}{v_0}\right)^2\cos^2\theta\,\frac{1}{\sqrt{\ }}\right]$$

Now:

$$\sqrt{} + \left(\frac{V}{v_0}\right)^2 \cos^2\theta \frac{1}{\sqrt{}}$$

$$= \frac{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta + \left(\frac{V}{v_0}\right)^2 \cos^2\theta}{\sqrt{}}$$

$$= \frac{1 + \left(\frac{V}{v_0}\right)^2 (\cos^2\theta - \sin^2\theta)}{\sqrt{}}$$

$$= \frac{1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta}{\sqrt{}}$$

Thus,

$$d(\cos\theta) = \underbrace{-\sin\theta\, d\theta}_{d(\cos\theta)} \left[2\frac{V}{v_0}\cos\theta \pm \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

So:

$$p(\theta)\, d\theta = \frac{1}{2}\sin\theta\, d\theta \left[2\frac{V}{v_0}\cos\theta \pm \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

Now: for $v_0 > V$, take $+$ sign ($\theta \in [0, \pi]$)

for $v_0 < V$, as θ_0 increases from 0 to π

θ increases from $0 \mapsto \theta_{max}$

θ decreases from $\theta_{max} \mapsto 0$

Thus, for $v_0 < V$ need to take the difference of the $+$ and $-$ expressions:

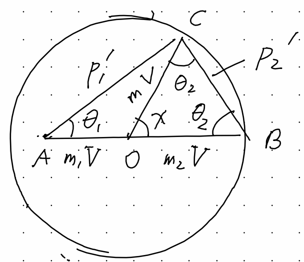
$$p(\theta)\, d\theta = \frac{1}{2}\sin\theta\, d\theta \left[2\frac{V}{v_0}\cos\theta + \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

$$- \frac{1}{2}\sin\theta\, d\theta \left[2\frac{V}{v_0}\cos\theta - \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

$$= \frac{\sin\theta\, d\theta \left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}}$$

which is valid for $0 \le \theta \le \theta_{max} = \sin^{-1}\left(\frac{v_0}{V}\right)$

Sec 17, Prob 1:



$$\chi + 2\theta_2 = \pi$$
$$\chi = \pi - 2\theta_2$$

From the above figure:

$$(m_2 v_2')^2 = 2(mv)^2 - 2(mv)^2\cos\chi$$
$$= 2m^2v^2\left(1-\cos(\pi-2\theta_2)\right)$$
$$= 2m^2v^2\left[1-\left(\underbrace{\cos(\pi)}_{-1}\cos(2\theta_2)+\cancel{\sin(\pi)}^{0}\sin(2\theta_2)\right)\right]$$
$$= 2m^2v^2\left(1+\cos(2\theta_2)\right)$$
$$\to v_2' = \sqrt{2}\left(\frac{m}{m_2}\right)v\sqrt{1+(\cos^2\theta_2-\sin^2\theta_2)}$$
$$= \sqrt{2}\left(\frac{m_1}{m_1+m_2}\right)v\sqrt{2\cos^2\theta_2}$$
$$= 2\left(\frac{m_1}{m_1+m_2}\right)V\cos\theta_2$$

Thus, $\left(\frac{v_2'}{V}\right) = \left(\frac{2m_1}{m_1+m_2}\right)\cos\theta_2$

Also,

$$(mv)^2 = (m_1 V)^2 + (m_1 v_1')^2 - 2m_1^2 v_1' V\cos\theta_1$$
$$\to (m_1v_1')^2 - 2m_1V\cos\theta_1(m_1v_1') + m_1^2V^2 - m^2v^2 = 0$$

Now: $V = \frac{m_1v_1 + m_2\cancel{v_2}^{0}}{m_1+m_2} = \frac{m_1 v}{m_1+m_2}$

Thus,

$$(m_1v_1')^2 - 2\left(\frac{m_1^2 v}{m_1+m_2}\right)\cos\theta_1\, m_1 v_1' + \frac{m_1^2 m_1^2 v^2}{(m_1+m_2)^2} - \frac{m_1^2m_2^2}{(m_1+m_2)^2}v^2 = 0$$
$$\to (v_1')^2 - 2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1\, v_1' + \frac{m_1^2-m_2^2}{(m_1+m_2)^2}v^2=0$$
$$(v_1')^2 - 2\left(\frac{m_1 v}{m_1+m_2}\right)\cos\theta_1\, v_1' + \left(\frac{m_1-m_2}{m_1+m_2}\right)v^2=0$$
$$\left(\frac{v_1'}{v}\right)^2 - 2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1\left(\frac{v_1'}{v}\right) + \left(\frac{m_1-m_2}{m_1+m_2}\right)=0$$

Quadratic equation

$$\frac{v_1'}{v} = \frac{2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \sqrt{4\left(\frac{m_1}{m_1+m_2}\right)^2\cos^2\theta_1 - 4\left(\frac{m_1-m_2}{m_1+m_2}\right)}}{2}$$

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_1^2\cos^2\theta_1 - (m_1-m_2)(m_1+m_2)}$$

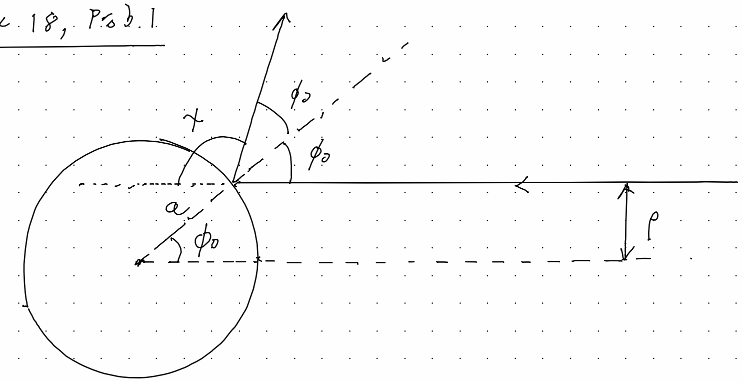
$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_1^2(\cos^2\theta_1 - 1) + m_2^2}$$

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_2^2 - m_1^2\sin^2\theta_1}$$

The + sign holds for $m_1 < m_2$

+/- signs hold for $m_1 > m_2$

Sec. 18, Prob. 1



$$\chi + 2\phi_0 = \pi \quad \rightarrow \quad \phi_0 = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\rho = a\sin\phi_0$$

$$= a\sin\left(\frac{\pi}{2} - \frac{\chi}{2}\right)$$

$$= a\left(\underset{1}{\underset{\shortparallel}{\sin\left(\frac{\pi}{2}\right)}}\cos\left(\frac{\chi}{2}\right) - \cancelto{0}{\cos\left(\frac{\pi}{2}\right)}\sin\left(\frac{\chi}{2}\right)\right)$$

$$= a\cos\left(\frac{\chi}{2}\right)$$

$$d\sigma = 2\pi\rho\,d\rho$$

$$= 2\pi\,\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi$$

$$= \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$

where $d\Omega$ = solid angle

$$= \underset{\substack{\uparrow \\ \text{integral over} \\ d\phi}}{2\pi}\,\sin\chi\,d\chi$$

$$\rho = a\cos\left(\frac{\chi}{2}\right)$$

$$d\rho = -a\,\frac{1}{2}\sin\left(\frac{\chi}{2}\right)d\chi$$

$$= -\frac{a}{2}\sin\left(\frac{\chi}{2}\right)d\chi$$

thus,

$$d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right|d\Omega$$

$$= \frac{a\cos(\chi/2)}{\sin\chi}\,\frac{a}{2}\sin\left(\frac{\chi}{2}\right)d\Omega$$

$$= \frac{a^2}{2}\,\frac{\sin(\chi/2)\cos(\chi/2)}{\sin\chi}\,d\Omega$$

$$= \boxed{\frac{a^2}{4}\,d\Omega} \qquad \left(\text{since } \sin\chi = 2\sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)\right)$$

total cross section

$$\sigma = \int d\sigma = \frac{a^2}{4}\int d\Omega = \frac{a^2}{4}\,4\pi = \boxed{\pi a^2}$$

Now calculate differential cross section in lab frame for both m_1 and m_2

Use the result that

$$d\sigma_1 = \frac{\rho(\theta_1)}{\sin\theta_1}\left|\frac{d\rho}{d\theta_1}\right|d\Omega_1 = \rho\left|\frac{d\rho}{d(\cos\theta_1)}\right|d\Omega_1$$

Compare to:

$$d\sigma = \rho\left|\frac{d\rho}{d(\cos\chi)}\right|d\Omega$$

$$\rightarrow \frac{d\sigma_1}{d\Omega_1} = \rho\left|\frac{d\rho}{d(\cos\theta_1)}\right|$$

$$= \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|\frac{d\sigma}{d\Omega}$$

So we need to evaluate:

$$\frac{d(\cos\chi)}{d(\cos\theta_1)} \quad \text{and} \quad \frac{d(\cos\chi)}{d(\cos\theta_2)}$$

Start with θ_2: (17.4)

$$\theta_2 = \frac{1}{2}(\pi - \chi) \rightarrow \boxed{\chi = \pi - 2\theta_2}$$

$$\rightarrow \cos\chi = \cos(\pi - 2\theta_2)$$

$$= \cos\pi\cos(2\theta_2) + \cancel{\sin\pi}^{\,0}\sin(2\theta_2)$$

$$= -\cos(2\theta_2)$$

$$= -\left(\cos^2\theta_2 - \sin^2\theta\right)$$

$$= -\left(2\cos^2\theta_2 - 1\right)$$

$$= -2\cos^2\theta_2 + 1$$

Thus, $\boxed{d(\cos\chi) = -4\cos\theta_2\,d(\cos\theta_2)}$

Thus,

$$\frac{d\sigma_2}{d\Omega_2} = \frac{d\sigma}{d\Omega}\left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right|$$

$$= \frac{1}{4}a^2 \cdot |4\cos\theta_2|$$

$$= a^2|\cos\theta_2|$$

so $\boxed{d\sigma_2 = a^2|\cos\theta_2|\, d\Omega_2}$

Now consider θ_1:

From (17.4):

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}$$

Compare with

$$\tan\theta = \frac{v_0\sin\theta_0}{V + v_0\cos\theta_0} \qquad (16.5)$$

Make identifications: $\theta \to \theta_1$, $v_0 \to m_2$, $V \to m_1$

Then we can write down from (16.6).

$$\cos\chi = -\frac{m_1}{m_2}\sin^2\theta_1 \pm \cos\theta_1\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}$$

[See also sec 16, Prob. 2 where we derived this for θ and θ_0]

We also worked out the derivative:

$$d(\cos\theta_0) = d(\cos\theta)\left[2\frac{V}{v_0}\cos\theta \pm \frac{1+\left(\frac{V}{v_0}\right)^2\cos 2\theta}{\sqrt{1-\left(\frac{V}{v_0}\right)^2\sin^2\theta}}\right]$$

So we can similarly write dow

$$d(\cos\chi) = d(\cos\theta_1)\left[2\frac{m_1}{m_2}\cos\theta_1 \pm \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\right]$$

For $m_1 < m_2$: take + sign

For $m_1 > m_2$: As χ increases from 0 to π, θ_1 increases from 0 to θ_{max}, then θ_1 decreases from θ_{max} to 0. In that case

$$d(\cos\chi) = d(\cos\theta_1)[\cancel{\circledcirc} + \circledcirc] - d(\cos\theta_1)[\cancel{\circledcirc} - \circledcirc]$$

$$= 2\, d(\cos\theta_1)\frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

Use: (for both)

$$d\sigma_1 = \left(\frac{d\sigma}{d\Omega}\right)\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| d\Omega_1$$

$$= \frac{1}{4}a^2\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| d\Omega_1$$

Thurs

For $m_1 < m_2$:

$$d\sigma_1 = \frac{1}{4}a^2\left[2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\right]d\Omega_1$$

For $m_1 > m_2$:

$$d\sigma_1 = \frac{1}{4}a^2 \cdot 2 \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\,d\Omega_1$$

$$= \frac{a^2}{2}\,\frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\,d\Omega_1$$

Sec. 18, Prob 2:

Hard sphere scattering again.
Calculate $d\sigma$ in terms of $d\epsilon$ where
ϵ = energy lost by scattered particle.

Now: ϵ = energy lost by scattered particle
= energy gained by m_2
$= \frac{1}{2}m_2(v_2')^2$

From Fig. 16.) we have (law of cosines):

$$(m_2 v_2')^2 = (mv)^2 + (mv)^2 - 2(mv)^2\cos\chi$$

$$= 2(mv)^2[1-\cos\chi]$$

$$= 2(mv)^2\, 2\sin^2\left(\frac{\chi}{2}\right)$$

So $m_2 v_2' = 2\, mv \sin\left(\frac{\chi}{2}\right)$

$\rightarrow$ $v_2' = \left(\frac{m}{m_2}\right) v \sin\left(\frac{\chi}{2}\right)$

$$= \left(\frac{m_1}{m_1+m_2}\right) v \sin\left(\frac{\chi}{2}\right) \qquad (17.5)$$

NOTE: $d\sigma = \frac{1}{4}a^2\, d\Omega$

$$= \frac{1}{4}a^2\, 2\pi \sin\chi\, d\chi$$

$$= \frac{\pi a^2}{2}\left|d(\cos\chi)\right|$$

So we would like to relate $d\epsilon$ and $d(\cos\chi)$.

Now: $\epsilon = \frac{1}{2} m_2 (v_2')^2$

$$= \frac{1}{2} m_2 \frac{4 m_1^2 v^2}{(m_1+m_2)^2} \sin^2\left(\frac{\chi}{2}\right)$$

$$= \frac{2 m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 \sin^2\left(\frac{\chi}{2}\right) \quad (\text{since } v = v_\infty)$$

$$\equiv \epsilon_{max} \sin^2\left(\frac{\chi}{2}\right)$$

where $\epsilon_{max} \equiv \frac{2 m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2$

$$= 4\left(\frac{m_1}{m_1+m_2}\right) \frac{m v_\infty^2}{2}$$

$$= 4\left(\frac{m_1}{m_1+m_2}\right) E$$

Thus, $d\epsilon = \epsilon_{max}\, 2 \sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)\frac{d\chi}{2}$

$$= \frac{1}{2}\epsilon_{max} \sin\chi\, d\chi$$

$$= \frac{1}{2}\epsilon_{max} |d(\cos\chi)|$$

So $d\sigma = \frac{\pi a^2}{2} |d(\cos\chi)|$

$$= \frac{\pi a^2}{2}\frac{2}{\epsilon_{max}} d\epsilon = \boxed{\frac{\pi a^2}{\epsilon_{max}} d\epsilon}$$

which is a uniform distribution wrt ϵ.

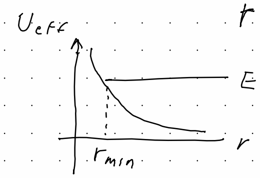
Sec 18, Prob. 4

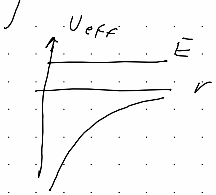
Effective cross section to "fall" to center of $U(r) = -\alpha/r^2 \quad (\alpha > 0)$

$$U_{eff}(r) = U(r) + \frac{M^2}{2mr^2}$$

$$= -\frac{\alpha}{r^2} + \frac{M^2}{2mr^2}$$

$$= \frac{1}{r^2}\left(\frac{M^2}{2m} - \alpha\right)$$





$\frac{M^2}{2m} - \alpha > 0$ (don't fall to center since $r_{min} > 0$)

$\frac{M^2}{2m} - \alpha < 0$ Fall to center ($r=0$)

For a given $E = \frac{1}{2} m v_\infty^2 > 0$ need

$$\frac{M^2}{2m} - \alpha < 0$$

$$\frac{M^2}{2m} < \alpha$$

$$\rightarrow \quad M_{max} = \sqrt{2m\alpha}$$

Cross section: $\sigma = \pi \rho_{max}^2$, $M = \rho m v_\infty$

Thus,

$$\sigma = \pi \rho_{max}^2 \quad , \qquad \rho_{max} = \frac{M_{max}}{m v_\infty}$$

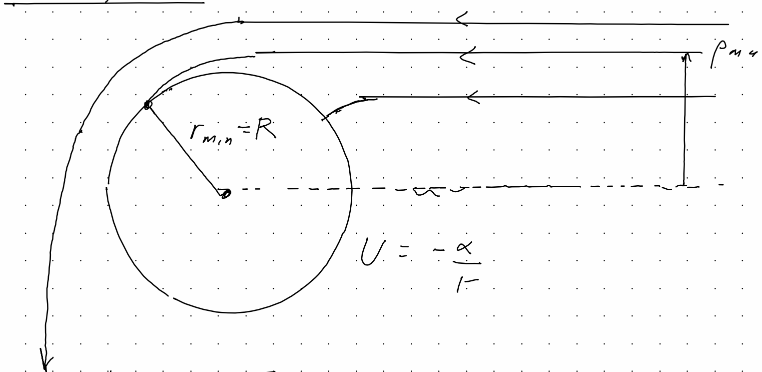
$$= \pi \frac{M_{max}^2}{m^2 v_\infty^2}$$

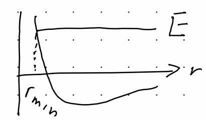
$$= \pi \frac{2 m \alpha}{m^2 v_\infty^2}$$

$$= \frac{\pi \alpha}{\frac{1}{2} m v_\infty^2}$$

$$= \boxed{\frac{\pi \alpha}{E}}$$

Sec 18, Prob 6





Turning point at $r = R$

$$0 = E - U_{eff}(R)$$

$$= E - U(R) - \frac{M_{max}^2}{2 m R^2}$$

$$= E + \frac{\alpha}{R} - \frac{M_{max}^2}{2 m R^2}$$

$$\rightarrow \frac{M_{max}^2}{2 m R^2} = E + \frac{\alpha}{R}$$

$$M_{max} = \rho_{max} m v_\infty \quad , \qquad E = \frac{1}{2} m v_\infty^2$$

Thus,

$$\sigma = \pi \rho_{max}^2$$

$$= \pi \frac{M_{max}^2}{m^2 v_\infty^2}$$

$$= \pi \frac{1}{m^2 v_\infty^2} \quad 2mR^2\left(E + \frac{\alpha}{R}\right)$$

$$= \pi R^2 \underbrace{\left(\frac{2}{m v_\infty^2}\right)}_{\frac{1}{E}}\left(E + \frac{\alpha}{R}\right)$$

$$= \boxed{\pi R^2 \left(1 + \frac{\alpha}{ER}\right)}$$

where $E = \frac{1}{2} m v_\infty^2 = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right) v_\infty^2$

and $\alpha \equiv G m_1 m_2$

Sec 19, Prob. 1:

$$U = \frac{\alpha}{r^2}, \quad \alpha > 0$$

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M\, dr/r^2}{\sqrt{2m[E - U(r)] - M^2/r^2}}$$

Substitute: $E = \frac{1}{2} m v_\infty^2$

$M = \rho m v_\infty$

$$\rightarrow \phi_0 = \int_{r_{min}}^{\infty} \frac{\rho m v_\infty\, dr/r^2}{\sqrt{2m\left[\frac{1}{2} m v_\infty^2 - U(r)\right] - \rho^2 m^2 v_\infty^2/r^2}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - 2U(r)/m v_\infty^2 - \rho^2/r^2}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \rho^2/r^2 - 2U(r)/m v_\infty^2}}$$

Substitute: $U(r) = \frac{\alpha}{r^2}$

$$\sqrt{} = \sqrt{1 - \rho^2/r^2 - \left(\frac{2\alpha}{m v_\infty^2}\right)\frac{1}{r^2}}$$

$$= \sqrt{1 - \left(\rho^2 + \frac{2\alpha}{m v_\infty^2}\right)\frac{1}{r^2}} = \sqrt{1 - \frac{A^2}{r^2}}$$

$$A^2 = \rho^2 + \frac{2\alpha}{m v_\infty^2}$$

Thus,

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \frac{A^2}{r^2}}}$$

$$= \int_{A}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \frac{A^2}{r^2}}}$$

<u>Let:</u> $u = \frac{1}{r} \rightarrow du = -\frac{1}{r^2}\,dr$

$$\frac{A^2}{r^2} = A^2 u^2$$

$$\phi_0 = -\int_{\frac{1}{A}}^{0} \frac{\rho\, du}{\sqrt{1 - A^2u^2}}$$

$$= \int_{0}^{\frac{1}{A}} \frac{\rho\, du}{\sqrt{1 - A^2u^2}}$$

<u>Let:</u> $\sin\theta = Au$

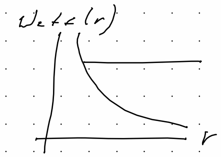
$\cos\theta\, d\theta = A\, du$

$u = 0, \frac{1}{A} \rightarrow \theta = 0, \pi/2$

$\sqrt{1 - A^2u^2} = \sqrt{1 - \sin^2\theta} = \cos\theta$

$$\rightarrow \phi_0 = \int_0^{\pi/2} \frac{\rho\, \cancel{\cos\theta}\, d\theta / A}{\cancel{\cos\theta}} = \frac{\rho}{A}\frac{\pi}{2}$$

$$\boxed{\phi_0 = \frac{\pi}{2}\frac{\rho}{\sqrt{\rho^2 + \frac{2\alpha}{m v_\infty^2}}}}$$



<u>Repulsive scattering:</u> $\chi + 2\phi_0 = \pi$

$\chi = \pi - 2\phi_0$

$$\chi = \pi - \pi\frac{\rho}{\sqrt{\rho^2 + \frac{2\alpha}{m v_\infty^2}}} = \pi\left[1 - \frac{1}{\sqrt{1 + \frac{2\alpha}{\rho^2 m v_\infty^2}}}\right]$$

$$\left(\frac{\pi\ \rho}{\sqrt{\quad}}\right)^2 = (\pi - \chi)^2$$

$$\frac{\pi^2\rho^2}{\rho^2 + \frac{2\alpha}{m v_\infty^2}} = (\pi - \chi)^2$$

$$\pi^2\rho^2 = (\pi-\chi)^2\rho^2 + (\pi-\chi)^2\frac{2\alpha}{m v_\infty^2}$$

$$\left(\pi^2 - (\pi-\chi)^2\right)\rho^2 = (\pi-\chi)^2\frac{2\alpha}{m v_\infty^2}$$

$$\left(\cancel{\pi^2} - \cancel{\pi^2} - \chi^2 + 2\pi\chi\right)\rho^2 = (\pi-\chi)^2\frac{2\alpha}{m v_\infty^2}$$

$$\rho^2 = \frac{(\pi-\chi)^2}{2\pi\chi - \chi^2}\frac{2\alpha}{m v_\infty^2}$$

$$\boxed{\rho = \frac{(\pi-\chi)}{\sqrt{2\pi\chi - \chi^2}}\sqrt{\frac{2\alpha}{m v_\infty^2}}}$$

Differential cross-section:

$$d\sigma = 2\pi\rho\, d\rho$$
$$= 2\pi\, \rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi$$
$$= \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega \quad , \quad d\Omega = 2\pi \sin\chi\, d\chi$$

Thus,

$$\frac{d\rho}{d\chi} = \sqrt{\frac{2\alpha}{mv_\infty^2}}\ \frac{-\sqrt{2\pi\chi-\chi^2} - \frac{1}{2\sqrt{\ }}(2\pi-2\chi)(\pi-\chi)}{2\pi\chi-\chi^2}$$

$$= -\sqrt{\frac{2\alpha}{mv_\infty^2}}\ \frac{\sqrt{2\pi\chi-\chi^2} + \frac{(\pi-\chi)^2}{\sqrt{\ }}}{2\pi\chi-\chi^2}$$

$$= -\sqrt{\frac{2\alpha}{mv_\infty^2}}\ \frac{2\pi\chi-\chi^2 + (\pi-\chi)^2}{(2\pi\chi-\chi^2)^{3/2}}$$

$$= -\sqrt{\frac{2\alpha}{mv_\infty^2}}\ \frac{\cancel{2\pi\chi} - \cancel{\chi^2} + \pi^2 + \cancel{\chi^2} - \cancel{2\pi\chi}}{(2\pi\chi-\chi^2)^{3/2}}$$

$$= -\sqrt{\frac{2\alpha}{mv_\infty^2}}\ \frac{\pi^2}{(2\pi\chi-\chi^2)^{3/2}}$$

So,

$$d\sigma = \frac{(\pi-\chi)}{\sqrt{2\pi\chi-\chi^2}}\sqrt{\frac{2\alpha}{mv_\infty^2}}\ \frac{1}{\sin\chi}\sqrt{\frac{2\alpha}{m\alpha}}\ \frac{\pi^2}{(2\pi\chi-\chi^2)^{3/2}}\, d\Omega$$

$$= \boxed{\left(\frac{2\alpha}{mv_\infty^2}\right)\frac{d\Omega}{\sin\chi}\ \frac{\pi^2(\pi-\chi)}{(2\pi\chi-\chi^2)^{3/2}}}$$

Sec 20, Prob. 1 Small-angle scattering

Start with (18.4):

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1-\rho^2/r^2 - 2U/mv_\infty^2}}$$

Assume U is weak so that $2U/mv_\infty^2 \ll 1$

$$\frac{1}{\sqrt{\ }} = \frac{1}{\sqrt{(1-\rho^2/r^2)\left(1 - \frac{2U/mv_\infty^2}{(1-\rho^2/r^2)}\right)}}$$

$$\approx \frac{1}{\sqrt{1-\rho^2/r^2}}\left(1 + \frac{1}{\cancel{2}}\frac{\cancel{2}U/mv_\infty^2}{1-\rho^2/r^2}\right)$$

$$= \frac{1}{\sqrt{1-\rho^2/r^2}} + \frac{U/mv_\infty^2}{(1-\rho^2/r^2)^{3/2}}$$

Can replace r_{min} limit by ρ:

$$\int_{\rho}^{\infty}\frac{\rho\, dr/r^2}{\sqrt{1-\rho^2/r^2}} = -\int_{\frac{1}{\rho}}^{0}\frac{\rho\, du}{\sqrt{1-\rho^2u^2}}$$

let: $u = \frac{1}{r}$

$du = -\frac{1}{r^2}dr$

$$= \int_0^{\frac{1}{\rho}}\frac{\rho\, du}{\sqrt{1-\rho^2u^2}}$$

let $\rho u = \sin\theta$

$\rho\, du = \cos\theta\, d\theta$

$u = \frac{1}{\rho} \to \theta = \pi/2$

$$= \boxed{\frac{\pi}{2}}$$

Then,

$$\phi_0 \simeq \frac{\pi}{2} + \frac{1}{m v_\infty^2} \int_\rho^\infty \frac{\rho \, dr / r^2 \, U(r)}{(1 - \rho^2/r^2)^{3/2}}$$

$$= \frac{\pi}{2} + \frac{1}{m v_\infty^2} \frac{\partial}{\partial \rho} \left[\int_\rho^\infty \frac{U(r)\, dr}{\sqrt{1 - \rho^2/r^2}} \right]$$

Now:

$$\int_\rho^\infty \frac{U(r)\,dr}{\sqrt{1-\rho^2/r^2}} = uv \Big|_\rho^\infty - \int_\rho^\infty v\, du$$

where $u = U(r)$

$$dv = \frac{dr}{\sqrt{1-\rho^2/r^2}} = \frac{r\,dr}{\sqrt{r^2-\rho^2}} \qquad x = r^2 - \rho^2$$
$$dx = 2r\,dr$$

$$= \frac{dx/2}{\sqrt{x}}$$

$$\rightarrow v = \frac{1}{2}\int \frac{dx}{\sqrt{x}} = \sqrt{x} + \text{const}$$
$$= \sqrt{r^2-\rho^2} + \text{const}$$

So:

$$\int_\rho^\infty \frac{U(r)\,dr}{\sqrt{1-\rho^2/r^2}} = \cancel{U(r)\sqrt{r^2-\rho^2}\Big|_\rho^\infty} - \int_\rho^\infty \left(\frac{dU}{dr}\right) dr \sqrt{r^2-\rho^2}$$

assuming $U(r) \rightarrow 0$ faster than $\frac{1}{r}$ as $r \rightarrow \infty$

So

$$\phi_0 = \frac{\pi}{2} + \frac{1}{m v_\infty^2} \frac{\partial}{\partial \rho}\left[- \int_\rho^\infty \frac{dU}{dr}\, dr \sqrt{r^2-\rho^2}\right]$$

$$= \frac{\pi}{2} + \frac{1}{m v_\infty^2} (-) \int_\rho^\infty \frac{dU}{dr}\, dr \frac{1}{\cancel{2}\sqrt{r^2-\rho^2}} (-\cancel{2}\rho)$$

$$= \frac{\pi}{2} + \frac{\rho}{m v_\infty^2} \int_\rho^\infty dr \frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

Scattering angle χ:

$$2\phi_0 + \chi = \pi$$
$$\chi = \pi - 2\phi_0$$

$$\rightarrow \chi = \cancel{\pi} - 2\left(\cancel{\frac{\pi}{2}} + \frac{\rho}{m v_\infty^2} \int_\rho^\infty dr \frac{dU/dr}{\sqrt{r^2-\rho^2}}\right)$$

$$= -\frac{2\rho}{m v_\infty^2} \int_\rho^\infty dr \frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

In terms of θ_1:

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2 \cos\chi} \rightarrow \theta_1 \simeq \frac{m_2 \chi}{m_1 + m_2}$$

Thus,

$$\theta_1 \simeq \left(\frac{m_2}{m_1+m_2}\right)\chi$$

$$\simeq \left(\frac{m_2}{m_1+m_2}\right)\left(\frac{-2\rho}{m v_\infty^2}\right)\int_\rho^\infty dr\, \frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

$$= \frac{-2\rho}{m_1 v_\infty^2}\int_\rho^\infty dr\, \frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

which is Eq. (20,3)

<u>Sec 21, Prob 1</u>

$x = a\cos(\omega t+\alpha)$

Express a, α in terms of x_0, v_0

$x_0 = a\cos\alpha$

$v = \dot{x} = -a\omega\sin(\omega t+\alpha)$

$\rightarrow v_0 = -a\omega\sin\alpha$

Thus, $\dfrac{v_0}{x_0} = -\omega\tan\alpha$

$$\rightarrow \boxed{\tan\alpha = \frac{-v_0}{\omega x_0}}$$

<u>Also</u>: $x_0 = a\cos\alpha$

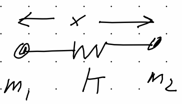
$\dfrac{v_0}{\omega} = -a\sin\alpha$

$$\rightarrow x_0^2 + \left(\frac{v_0}{\omega}\right)^2 = a^2\cos^2\alpha + a^2\sin^2\alpha$$
$$= a^2$$

so $\boxed{a = \sqrt{x_0^2 + \left(\dfrac{v_0}{\omega}\right)^2}}$

Sec 21, Prob 2:

Diatomic molecules, different isotopes m_1, m_2 and m_1', m_2'



2-body $\rightarrow$ 1-body w.r.t COM

$$T = \frac{1}{2} m \dot{x}^2 , \quad m = \frac{m_1 m_2}{m_1 + m_2}$$

$$U = \frac{1}{2} k x^2$$

Thus, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{k(m_1+m_2)}{m_1 m_2}}$

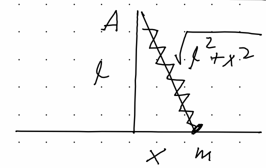
Similarly, for masses m_1', m_2' with $k' = k$:

$$\omega' = \sqrt{\frac{k}{m'}} = \sqrt{\frac{k(m_1'+m_2')}{m_1' m_2'}}$$

$$\rightarrow \frac{\omega'}{\omega} = \sqrt{\frac{\cancel{k}(m_1'+m_2')}{m_1' m_2'}} \sqrt{\frac{m_1 m_2}{\cancel{k}(m_1+m_2)}}$$

$$= \sqrt{\left(\frac{m_1'+m_2'}{m_1+m_2}\right) \frac{m_1 m_2}{m_1' m_2'}}$$

Sec 21, Prob 3:



$$U \approx F \delta\ell$$

$$\delta\ell = \sqrt{\ell^2+x^2} - \ell$$

$$= \ell \left(\sqrt{1+\left(\frac{x}{\ell}\right)^2} - 1 \right)$$

$$= \ell \left(\cancel{1} + \frac{1}{2}\left(\frac{x}{\ell}\right)^2 + \dots \cancel{-1} \right)$$

$$\approx \frac{1}{2} \frac{x^2}{\ell}$$

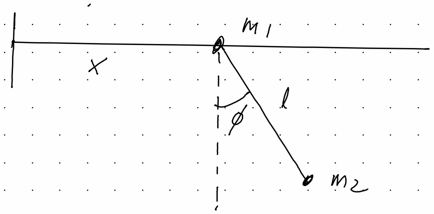
Thus, $U \approx \frac{1}{2}\left(\frac{F}{\ell}\right) x^2 = \frac{1}{2} k x^2$ for $k \equiv \frac{F}{\ell}$

Also, $T = \frac{1}{2} m \dot{x}^2$

$$\rightarrow \omega = \sqrt{\frac{F}{\ell m}}$$

Sec 21, Prob 5.

Figure 2 from Sec 5:



Recall from Sec 14, Prob 3:

$$E = \frac{1}{2} m_2 l^2 \dot\phi^2 \left(1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right) - m_2 g l \cos\phi$$

Now $\phi = 0$ corresponds to stable equilibrium.

Small oscillations: $\phi << 1$

Since $\dot\phi^2$ is small can set $\phi = 0$ in $\left(1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right)$

$$\rightarrow 1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi \rightarrow 1 - \frac{m_2}{m_1+m_2} = \frac{m_1}{m_1+m_2}$$

Then

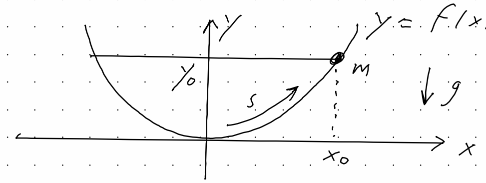
$$E = \frac{1}{2} m l^2 \dot\phi^2 - m_2 g l \left(1 - \frac{1}{2}\phi^2\right)$$

$$= \frac{1}{2} m l^2 \dot\phi^2 + \frac{1}{2} m_2 g l \phi^2 \underbrace{- m_2 g l}_{\text{const}}$$

Thus,

$$\omega = \sqrt{\frac{m_2 g l}{m l^2}} = \sqrt{\left(\frac{m_1+m_2}{m_1}\right)\frac{g}{l}} \quad \left(\rightarrow \sqrt{\frac{g}{l}} \text{ for } m_1 >> m_2\right)$$

Sec 21, Prob 6.



Find $y = f(x)$ so that period $P(y_0)$ is indep. of y_0.

Need: $E = \frac{1}{2} m v^2 + m g y$

$$= \frac{1}{2} m \left(\frac{ds}{dt}\right)^2 + \frac{1}{2} k s^2$$

where s = arc length along the curve

$$= \int_0^s ds$$

$$= \int_0^s \sqrt{dx^2 + dy^2}$$

$$= \int_0^x dx \sqrt{1 + y'^2}$$

so $m g y = \frac{1}{2} k s^2$

$$y = \frac{1}{2g} \frac{k}{m} s^2 \equiv A^2 s^2$$

$$\rightarrow \sqrt{y} = A s, \quad A = \frac{1}{\sqrt{2g}}\sqrt{\frac{k}{m}}$$

$$\frac{1}{2\sqrt{y}} \frac{dy}{dx} = A \frac{ds}{dx} = A\sqrt{1 + y'^2}$$

Thus, $\frac{1}{2}\frac{y'}{\sqrt{y}} = A\sqrt{1+y'^2}$

$$y'^2 = 4A^2 y(1+y'^2)$$

$$y'^2(1-4A^2y) = 4A^2y$$

$$\frac{dy}{dx} = y' = \sqrt{\frac{4A^2y}{1-4A^2y}}$$

so $\boxed{x = \int_0^y dy\sqrt{\frac{1-4A^2y}{4A^2y}}}$

make substitution

$$y = \frac{1}{8A^2}(1-\cos\theta)$$

$$dy = \frac{1}{8A^2}\sin\theta\, d\theta = \frac{1}{8A^2}\sqrt{1-\cos^2\theta}\, d\theta$$

$$\rightarrow \frac{1-4A^2y}{4A^2y} = \frac{1-\frac{1}{2}(1-\cos\theta)}{\frac{1}{2}(1-\cos\theta)} = \frac{\cancel{\frac{1}{2}}(1+\cos\theta)}{\cancel{\frac{1}{2}}(1-\cos\theta)}$$

$$= \frac{1+\cos\theta}{1-\cos\theta}$$

Thus,

$$x = \int \frac{1}{8A^2}\sqrt{1-\cos^2\theta}\, d\theta\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$

$$= \frac{1}{8A^2}\int d\theta\sqrt{\frac{\cancel{(1-\cos\theta)}(1+\cos\theta)(1+\cos\theta)}{\cancel{1-\cos\theta}}}$$

$$x = \frac{1}{8A^2}\int d\theta\,(1+\cos\theta)$$

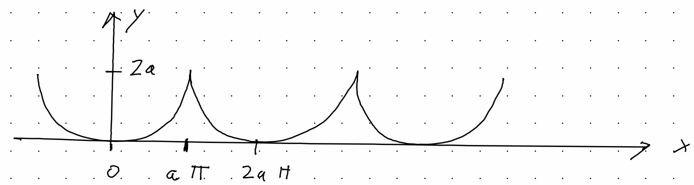
$$= \frac{1}{8A^2}(\theta+\sin\theta)$$

thus,

$$x = \frac{1}{8A^2}(\theta+\sin\theta) = a(\theta+\sin\theta)$$

$$y = \frac{1}{8A^2}(1-\cos\theta) = a(1-\cos\theta)$$

which are parametric equations for a cycloid



NOTE: $8A^2 = 8\frac{1}{2g}\frac{k}{m} = \frac{4}{g}\frac{k}{m} = \frac{4}{g}\omega^2$

Sec 22, Prob 1:

(a) $F = F_0 = \text{const}$

$x = \dot{x} = 0$ at $t=0 \rightarrow \xi_0 = 0$

$$\rightarrow \xi(t) = e^{i\omega t}\left[\cancel{\xi_0}^{\,0} + \int_0^t d\bar{t}\,\frac{F(\bar{t})}{m}\,e^{-i\omega\bar{t}}\right]$$

$$= e^{i\omega t}\,\frac{F_0}{m}\int_0^t d\bar{t}\,e^{-i\omega\bar{t}}$$

$$= e^{i\omega t}\left(\frac{F_0}{-im\omega}\right)e^{-i\omega\bar{t}}\Big|_0^t$$

$$= e^{i\omega t}\left(\frac{F_0}{-im\omega}\right)\left(e^{-i\omega t}-1\right)$$

$$= \frac{iF_0}{m\omega}\left(1-e^{i\omega t}\right)$$

$$= \frac{iF_0}{m\omega}\left(1-\cos\omega t - i\sin\omega t\right)$$

Now: $x(t) = \dfrac{\mathrm{Im}(\xi(t))}{\omega}$, $\dot{x}(t) = \mathrm{Re}(\xi(t))$

Thus,

$$\boxed{x(t) = \frac{F_0}{m\omega^2}\left(1-\cos\omega t\right)}$$

(b) $F(t) = a\,t$

$$\xi(t) = e^{i\omega t}\,\frac{a}{m}\int_0^t d\bar{t}\;\bar{t}\,e^{-i\omega\bar{t}}$$

Let $u = \bar{t} \rightarrow du = d\bar{t}$

$dv = e^{-i\omega\bar{t}}\,d\bar{t} \rightarrow v = \dfrac{1}{-i\omega}e^{-i\omega\bar{t}}$

Thus,

$$\xi(t) = e^{i\omega t}\,\frac{a}{m}\left[\frac{\bar{t}}{-i\omega}e^{-i\omega\bar{t}}\Big|_0^t + \frac{1}{i\omega}\int_0^t d\bar{t}\,e^{-i\omega\bar{t}}\right]$$

$$= e^{i\omega t}\,\frac{a}{m}\left[\frac{t}{-i\omega}e^{-i\omega t} + \frac{1}{\omega^2}\left(e^{-i\omega t}-1\right)\right]$$

$$= \frac{a}{m}\left[\left(\frac{it}{\omega}+\frac{1}{\omega^2}\right) - \frac{1}{\omega^2}e^{i\omega t}\right]$$

$$= \frac{a}{m}\left[\left(\frac{it}{\omega}+\frac{1}{\omega^2}\right) - \frac{1}{\omega^2}\left(\cos\omega t + i\sin\omega t\right)\right]$$

$$= \frac{a}{m}\left[\frac{1}{\omega^2}\left(1-\cos\omega t\right) + i\left(\frac{t}{\omega}-\frac{\sin\omega t}{\omega^2}\right)\right]$$

$$\rightarrow x(t) = \frac{\mathrm{Im}(\xi(t))}{\omega}$$

$$= \boxed{\frac{a}{m\omega^3}\left(\omega t - \sin\omega t\right)}$$

(c) $F(t) = F_0 \exp(-\alpha t)$

$$\xi(t) = e^{i\omega t}\frac{F_0}{m}\int_0^t d\bar{t}\; e^{-\alpha\bar{t}}\, e^{-i\omega\bar{t}}$$

$$= \frac{F_0}{m} e^{i\omega t}\frac{1}{-(i\omega+\alpha)}\, e^{-(i\omega+\alpha)\bar{t}}\Big|_0^t$$

$$= \frac{F_0}{m} e^{i\omega t}\frac{1}{-(i\omega+\alpha)}\left(e^{-(i\omega+\alpha)t} - 1\right)$$

NOTE: $\frac{1}{-(i\omega+\alpha)}\cdot\left(\frac{i\omega-\alpha}{i\omega-\alpha}\right) = \frac{i\omega-\alpha}{\alpha^2+\omega^2}$

$$\rightarrow \xi(t) = \frac{F_0}{m}\left(\frac{i\omega-\alpha}{\alpha^2+\omega^2}\right)\left[e^{-\alpha t} - e^{i\omega t}\right]$$

$$= \frac{F_0}{m}\left(\frac{i\omega-\alpha}{\alpha^2+\omega^2}\right)\left[e^{-\alpha t} - \cos\omega t - i\sin\omega t\right]$$

$$= \frac{F_0}{m(\alpha^2+\omega^2)}\Big[i\left(\omega e^{-\alpha t} - \omega\cos\omega t + \alpha\sin\omega t\right) - \alpha e^{-\alpha t} + \alpha\cos\omega t + \omega\sin\omega t\Big]$$

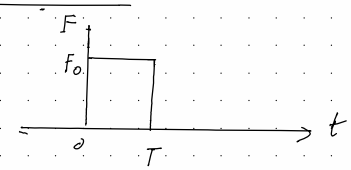
Thus,

$$x(t) = \mathrm{Im}(\xi(t))/\omega$$

$$= \frac{F_0}{m\omega(\alpha^2+\omega^2)}\left(\omega e^{-\alpha t} - \omega\cos\omega t + \alpha\sin\omega t\right)$$

$$= \boxed{\frac{F_0}{m(\alpha^2+\omega^2)}\left(e^{-\alpha t} - \cos\omega t + \frac{\alpha}{\omega}\sin\omega t\right)}$$

Sec 22, Prob 3:



Initial condition
$\xi_0 = 0$ (since $x_0 = \dot{x}_0 = 0$)

For $t > T$:

$$\xi(t) = e^{i\omega t}\int_0^T d\bar{t}\,\frac{F_0}{m}\, e^{-i\omega\bar{t}}$$

$$= e^{i\omega t}\frac{F_0}{m}\left(\frac{1}{-i\omega}\right)\left(e^{-i\omega T} - 1\right)$$

$$= \frac{iF_0}{m\omega}\left(e^{i\omega(t-T)} - e^{i\omega t}\right)$$

$$= \frac{iF_0}{m\omega}\Big[\cos(\omega(t-T)) + i\sin(\omega(t-T)) - \cos\omega t - i\sin\omega t\Big]$$

$$= \frac{iF_0}{m\omega}\left(\cos(\omega(t-T)) - \cos(\omega t)\right) - \frac{F_0}{m\omega}\left(\sin(\omega t - T) - \sin(\omega t)\right)$$

$$\rightarrow x(t) = \frac{\mathrm{Im}(\xi(t))}{\omega}$$

$$= \frac{F_0}{m\omega^2}\left(\cos(\omega(t-T)) - \cos(\omega t)\right)$$

We can also write for $t > T$:

$$x(t) = c_1 \cos(\omega(t-T)) + c_2 \sin(\omega(t-T))$$

$$\to \dot{x}(t) = -c_1 \omega \sin(\omega(t-T)) + c_2 \omega \cos(\omega(t-T))$$

Thus, $x(T) = c_1$, $\dot{x}(T) = c_2 \omega$

Match with:

$$x(t) = \frac{F_0}{m\omega^2}\left(\cos(\omega(t-T)) - \cos(\omega t)\right)$$

$$\dot{x}(t) = \frac{F_0}{m\omega}\left(-\sin(\omega(t-T)) + \sin(\omega t)\right)$$

at $t = T$,

$$\to \quad c_1 = \frac{F_0}{m\omega^2}\left(1 - \cos(\omega T)\right)$$

$$c_2 \omega = \frac{F_0}{m\omega} \sin(\omega T)$$

Thus,

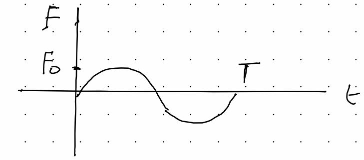
$$a \equiv \sqrt{c_1^2 + c_2^2}$$

$$= \frac{F_0}{m\omega^2}\sqrt{1 + \cos^2(\omega T) - 2\cos(\omega T) + \sin^2(\omega T)}$$

$$= \frac{F_0}{m\omega^2}\sqrt{2(1 - \cos(\omega T))}$$

$$= \boxed{\frac{2F_0}{m\omega^2}\sin\left(\frac{\omega T}{2}\right)} \quad \text{using } 1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$$

Sec. 22, Prob 5:



$$T = \frac{2\pi}{\omega}$$

$F(t) = F_0 \sin \omega t$, $\xi_0 = 0$

For $t > T$:

$$\xi(t) = e^{i\omega t} \frac{F_0}{m} \int_0^T d\bar{t} \sin \omega \bar{t}\, e^{-i\omega \bar{t}}$$

$$= \frac{F_0}{m} e^{i\omega t} \frac{1}{2i} \int_0^T d\bar{t} \left(e^{i\omega \bar{t}} - e^{-i\omega \bar{t}}\right) e^{-i\omega \bar{t}}$$

$$= \frac{F_0}{2mi} e^{i\omega t} \int_0^T d\bar{t} \left(1 - e^{-i2\omega \bar{t}}\right)$$

$$= \frac{F_0}{2mi} e^{i\omega t} \left(T + \cancelto{0}{\frac{1}{i2\omega}\left(\underbrace{e^{-i2\omega T}}_{1} - 1\right)}\right)$$

$$= \frac{F_0 T}{2mi} e^{i\omega t}$$

Thus, $x(T) = \dfrac{Im(\xi(T))}{\omega} = \dfrac{-F_0 T}{2m\omega}$ | $\dot{x}(T) = Re(\xi(T)) = 0$

We can also write for $t > T$:

$$x(t) = c_1 \cos(\omega(t-T)) + c_2 \sin(\omega(t-T))$$

$$\to \dot{x}(t) = -c_1 \omega \sin(\omega(t-T)) + c_2 \omega \cos(\omega(t-T))$$

Thus,

$$x(T) = c_1 \quad , \quad \dot{x}(T) = c_2 \omega$$

match with

$$x(T) = \frac{-F_0 T}{2m\omega} \quad , \quad \dot{x}(T) = 0$$

$$\longrightarrow \quad c_1 = \frac{-F_0 T}{2m\omega} \quad , \quad c_2 = 0$$

so

$$a = \sqrt{c_1^2 + c_2^2} = \frac{F_0 T}{2m\omega} = \frac{F_0 \cancel{2}\pi/\omega}{\cancel{2} m\omega} = \boxed{\frac{F_0 \pi}{m\omega^2}}$$

<u>Sec 23, Prob 1:</u>

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}\omega_0^2 (x^2 + y^2) + \alpha xy$$

$$= \frac{1}{2}\sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k - \frac{1}{2}\sum_{i,k} k_{ik} x_i x_k$$

where $m_{ik} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$$\sum_{i,k} k_{ik} x_i x_k = \omega_0^2 (x^2 + y^2) - 2\alpha xy$$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so $k_{ik} = \begin{bmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{bmatrix}$

<u>EOMs:</u>

$$\frac{d}{dt}\sum_k m_{ik} \dot{x}_k = -\sum_k k_{ik} x_k$$

$$\sum_k m_{ik} \ddot{x}_k = -\sum_k k_{ik} x_k$$

$$\sum_k (m_{ik} \ddot{x}_k + k_{ik} x_k) = 0$$

<u>Trial solution:</u> $x_k = A_k e^{i\omega t}$

$$\rightarrow \sum_k (-\omega^2 m_{ik} + k_{ik}) A_k e^{i\omega t} = 0$$

Thus,

$$\sum_K (K_{iK} - \omega^2 m_{iK}) A_K = 0$$

$$\rightarrow \quad \det(K_{iK} - \omega^2 m_{iK}) = 0$$

$$0 = \det \begin{bmatrix} \omega_0^2 - \omega^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega^2 \end{bmatrix}$$

$$= (\omega_0^2 - \omega^2)^2 - \alpha^2$$

$$= \omega^4 - 2\omega^2\omega_0^2 + (\omega_0^4 - \alpha^2)$$

Quadratic equation:

$$\omega_\pm^2 = \frac{2\omega_0^2 \pm \sqrt{\cancel{4\omega_0^4} - 4(\cancel{\omega_0^4} - \alpha^2)}}{2}$$

$$= \frac{2\omega_0^2 \pm 2\alpha}{2}$$

$$= \boxed{\omega_0^2 \pm \alpha}$$ (normal mode freqs)

Normal mode vectors:

$\omega_+^2 = \omega_0^2 + \alpha$:

$$\sum_K (K_{iK} - \omega_+^2 m_{iK}) A_K = 0$$

$$\begin{bmatrix} \omega_0^2 - \omega_+^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega_+^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\alpha & -\alpha \\ -\alpha & -\alpha \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $-\alpha(A_1 + A_2) = 0$

$$A_2 = -A_1$$

so eigenvector is $\boxed{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = V_+}$

$\omega_0^2 = \omega_-^2$:

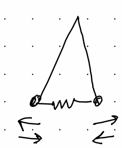
$$\begin{bmatrix} \omega_0^2 - \omega_-^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega_-^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $A_2 = A_1$

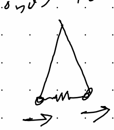
$$\rightarrow \quad \boxed{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = V_-}$$

V_+ corresponds to out-of-phase motion:



$$V_+ e^{i\omega_+ t} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\sqrt{\omega_0^2 + \alpha}\, t}$$

V_- corresponds to in-phase motion:



$$V_- e^{i\omega_- t} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\sqrt{\omega_0^2 - \alpha}\, t}$$

NOTE:

For weak coupling, $\alpha \ll \omega_0^2$, we have

$$\omega_\pm^2 = \omega_0^2 \pm \alpha$$
$$= \omega_0^2\left(1 \pm \frac{\alpha}{\omega_0^2}\right)$$

$$\rightarrow \omega_\pm = \omega_0\sqrt{1 \pm \frac{\alpha}{\omega_0^2}}$$

$$\simeq \omega_0\left(1 \pm \frac{1}{2}\frac{\alpha}{\omega_0^2}\right)$$ nearly equal Freqs

Beat Freq:

$$\omega_+ - \omega_- \simeq \omega_0\left(\cancel{1} + \frac{1}{2}\frac{\alpha}{\omega_0^2}\right) - \omega_0\left(\cancel{1} - \frac{1}{2}\frac{\alpha}{\omega_0^2}\right)$$
$$= \boxed{\frac{\alpha}{\omega_0}}$$

Sec 23, Prob 3:

$U = \frac{1}{2}kr^2$, space oscillator

central potential $\rightarrow$ motion in a plane (x,y) or (r,ϕ)

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{1}{2}k(x^2+y^2)$$
$$= \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) + \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2\right)$$

EOMs:

$$m\ddot{x} = -kx \quad \rightarrow \quad x = a\cos(\omega t + \alpha)$$
$$m\ddot{y} = -ky \quad \rightarrow \quad y = b\cos(\omega t + \beta)$$

where $\omega \equiv \sqrt{\frac{k}{m}}$

Write $\omega t + \beta = \omega t + \alpha + (\beta - \alpha)$
$= \phi + \delta$

where $\phi \equiv \omega t + \alpha$, $\delta \equiv \beta - \alpha$

Then $x = a\cos\phi \longrightarrow \boxed{\cos\phi = \frac{x}{a}}$

$$y = b\cos(\phi + \delta)$$
$$= b(\cos\phi\cos\delta - \sin\phi\sin\delta)$$

$$\rightarrow \frac{y}{b} = \cos\phi\cos\delta - \sin\phi\sin\delta$$
$$= \left(\frac{x}{a}\right)\cos\delta - \sin\phi\sin\delta$$

so $$\boxed{\sin\phi = \frac{1}{\sin\delta}\left[\left(\frac{x}{a}\right)\cos\delta - \frac{y}{b}\right]}$$

Square and add two boxed expressions

$$\cos^2\phi = \left(\frac{x}{a}\right)^2$$

$$+\ \sin^2\phi = \frac{1}{\sin^2\delta}\left[\left(\frac{x}{a}\right)^2\cos^2\delta + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab}\cos\delta\right]$$

$$1 = \left(\frac{x}{a}\right)^2\left(1+\frac{\cos^2\delta}{\sin^2\delta}\right) + \left(\frac{y}{b}\right)^2\frac{1}{\sin^2\delta} - \frac{2xy}{ab}\frac{\cos\delta}{\sin^2\delta}$$

$$\rightarrow \sin^2\delta = \left(\frac{x}{a}\right)^2\underbrace{\left(\sin^2\delta+\cos^2\delta\right)}_{=1} + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab}\cos\delta$$

$$= \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab}\cos\delta$$

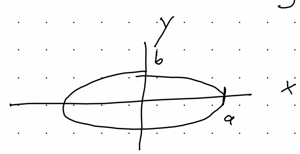
Thus, $$\boxed{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab}\cos\delta = \sin^2\delta}$$

Suppose $\delta = \frac{\pi}{2}$:

Then $\cos\delta = 0$, $\sin\delta = 1$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

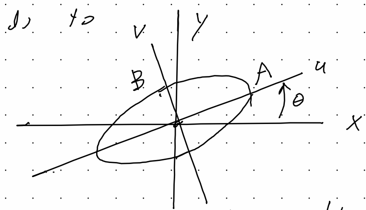
(ellipse centered at origin with semi-major axis $= a$ semi-minor axis $= b$.)



In general

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab}\cos\delta = \sin^2\delta$$

corresponds to



Find angle θ such that the above expression maps to

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$$

Now: $$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$\rightarrow \sin^2\delta = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab}\cos\delta$$

$$= \frac{1}{a^2}\left(\cos\theta\, u - \sin\theta\, v\right)^2 + \frac{1}{b^2}\left(\sin\theta\, u + \cos\theta\, v\right)^2 - \frac{2\left(\cos\theta\, u - \sin\theta\, v\right)\left(\sin\theta\, u + \cos\theta\, v\right)}{ab}\cos\delta$$

$$= \frac{1}{a^2}\left(u^2\cos^2\theta + v^2\sin^2\theta - 2uv\sin\theta\cos\theta\right)$$

$$+ \frac{1}{b^2}\left(u^2\sin^2\theta + v^2\cos^2\theta + 2uv\sin\theta\cos\theta\right)$$

$$- \frac{2}{ab}\cos\delta\left(u^2\sin\theta\cos\theta - v^2\sin\theta\cos\theta + uv\cos 2\theta\right)$$

$$= u^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} - \frac{2\sin\theta\cos\theta\cos\delta}{ab}\right)$$

$$+ v^2\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} + \frac{2\sin\theta\cos\theta\cos\delta}{ab}\right)$$

$$+ 2uv\left(-\frac{\sin\theta\cos\theta}{a^2} + \frac{\sin\theta\cos\theta}{b^2} - \frac{\cos 2\theta\cos\delta}{ab}\right)$$

We can make the factor multiplying uv equal to zero by choosing θ appropriately:

$$-\sin\theta\cos\theta\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{\cos 2\theta\cos\delta}{ab}$$

$$\frac{1}{2}\sin 2\theta\left(\frac{a^2-b^2}{a^2b^2}\right) = \frac{\cos 2\theta\cos\delta}{ab}$$

$$\boxed{\tan 2\theta = \left(\frac{2ab}{a^2-b^2}\right)\cos\delta}$$

<u>NOTE:</u> if $a=b$ then $0 = \cos 2\theta$

$$\rightarrow \theta = \frac{\pi}{4}$$

So choosing θ as above, we have

$$\sin^2\delta = u^2 \oslash + v^2 \oslash$$

$$1 = u^2\frac{\oslash}{\sin^2\delta} + v^2\frac{\oslash}{\sin^2\delta}$$

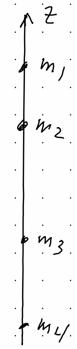
$$= \left(\frac{u}{A}\right)^2 + \left(\frac{v}{B}\right)^2$$

where
$$A^2 = \frac{\sin^2\delta}{\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} - \frac{2\sin\theta\cos\theta\cos\delta}{ab}\right)}$$

$$B^2 = \frac{\sin^2\delta}{\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} + 2\frac{\sin\theta\cos\theta\cos\delta}{ab}\right)}$$

Sec 32, Prob 1:

(a)



$I_3 = 0$ since $x_a = y_a = 0$ for all masses

$$I_1 = \sum_a m_a (r_a^2 - \cancel{x_a^2}^{\,0}) = \sum_a m_a z_a^2$$

$$I_2 = \sum_a m_a (r_a^2 - \cancel{y_a^2}^{\,0}) = \sum_a m_a z_a^2$$

$$\rightarrow I_1 = I_2 \equiv I$$
$$= \sum_a m_a z_a^2$$

...

(assuming COM at $z=0$)

If COM is not at $z=0$, but at z_{com}, then:

$$I = \sum_a m_a (z_a - z_{com})^2 \,, \qquad z_{com} = \frac{1}{\mu} \sum_b m_b z_b$$
$$= \sum_a m_a (z_a^2 + z_{com}^2 - 2 z_{com} z_a)$$
$$= \sum_a m_a z_a^2 + \mu z_{com}^2 - 2 z_{com} \underbrace{\sum_a m_a z_a}_{\mu z_{com}}$$
$$= \sum_a m_a z_a^2 - \mu z_{com}^2$$

This last expression can be rewritten in terms of $l_{ab} \equiv |z_a - z_b|$ as follows:

$$I = \frac{1}{2} \sum_a m_a z_a^2 + \frac{1}{2} \sum_b m_b z_b^2 - \frac{1}{\mu} \left(\sum_a m_a z_a \right) \left(\sum_b m_b z_b \right)$$
$$= \frac{1}{2\mu} \sum_a \sum_b m_b m_a z_a^2 + \frac{1}{2\mu} \sum_a \sum_b m_a m_b z_b^2 - \frac{1}{\mu} \sum_a \sum_b m_a m_b z_a z_b$$

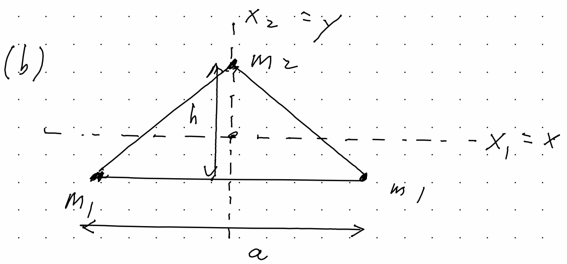
Thus

$$I = \frac{1}{2\mu} \sum_a \sum_b m_a m_b (z_a^2 + z_b^2 - 2 z_a z_b)$$
$$= \frac{1}{2\mu} \sum_a \sum_b m_a m_b (z_a - z_b)^2$$
$$= \frac{1}{2\mu} \sum_a \sum_b m_a m_b l_{ab}^2$$

NOTE: For just two masses:

$$I = \frac{1}{2\mu} (m_1 m_2 l^2 + m_2 m_1 l^2)$$
$$= \frac{m_1 m_2}{\mu} l^2$$
$$= m l^2 \quad \text{where} \quad m \equiv \frac{m_1 m_2}{m_1 + m_2}$$
$$l = |z_1 - z_2|$$

(b)



Assume COM at $(x_1, x_2) = (x, y) = (0, 0)$

Then $2m_1 y_1 + m_2 y_2 = 0$

where $y_2 - y_1 = h$

thus, $2m_1 y_1 + m_2(h + y_1) = 0$

$$(2m_1 + m_2) y_1 + m_2 h = 0$$

$$y_1 = \frac{-m_2 h}{\mu}, \quad \mu = 2m_1 + m_2 = \text{total mass}$$

and
$$y_2 = y_1 + h = \frac{-m_2 h}{\mu} + h = \frac{(\mu - m_2) h}{\mu} = \frac{2m_1 h}{\mu}$$

All masses have $z_a = 0$

Thus, $I_3 = \sum_a m_a (r_a^2 - z_a^2) = \sum_a m_a (x_a^2 + y_a^2)$

$I_1 = \sum_a m_a (r_a^2 - x_a^2) = \sum_a m_a y_a^2$

$I_2 = \sum_a m_a (r_a^2 - y_a^2) = \sum_a m_a x_a^2$

Thus, $I_3 = I_1 + I_2$

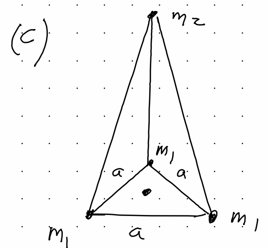
so need to calculate I_1, I_2.

$$I_1 = \sum_a m_a y_a^2 = 2m_1 y_1^2 + m_2 y_2^2 = 2m_1 \frac{m_2^2 h^2}{\mu^2} + m_2 \frac{4m_1^2 h^2}{\mu^2} = \frac{2 m_1 m_2 h^2}{\mu^2}(m_2 + 2m_1) = \boxed{\frac{2 m_1 m_2}{\mu} h^2}$$

$$I_2 = \sum_a m_a x_a^2 = m_1 \left(\frac{a}{2}\right)^2 + m_1 \left(\frac{-a}{2}\right)^2 = \boxed{\frac{m_1 a^2}{2}}$$

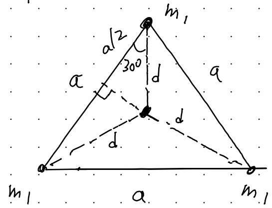
$$I_3 = I_1 + I_2 = \boxed{\frac{2 m_1 m_2}{\mu} h^2 + \frac{m_1 a^2}{2}}$$

(c)

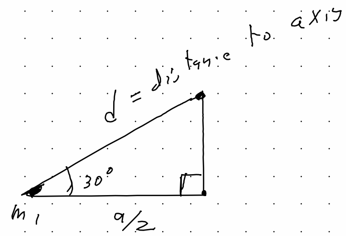


tetrahedron, height h

base: equilateral Δ with side length a



base



$$\cos 30^\circ = \frac{a}{2d} = \frac{\sqrt{3}}{2}$$

$$\rightarrow \quad d = \frac{a}{\sqrt{3}}$$

COM lies on axis of symmetry (z-axis)

Assume COM has $z=0$

Then $0 = m_2 z_2 + 3m_1 z_1$, $\quad z_2 - z_1 = h$

$$= m_2(z_1+h) + 3m_1 z_1$$

$$= (3m_1+m_2) z_1 + m_2 h$$

$$\rightarrow \quad z_1 = \frac{-m_2 h}{3m_1+m_2} = -\frac{m_2 h}{\mu}$$

$$z_2 = h + z_1$$

$$= h - \frac{m_2 h}{\mu}$$

$$= \frac{3m_1 h}{\mu}$$

Since a tetrahedron has 3-fold rotational symmetry, the x_1 principal axes can be chosen arbitrarily in the plane $\perp$ to the symmetry axis ($x_3 \equiv z$). [x_2 is $\perp$ to x_1, x_3]

Thus, $I_1 = I_2 \equiv I$

$$I_3 = \sum_a m_a (r_a^2 - z_a^2)$$

$$= \sum_a m_a s_a^2 \qquad \text{where } s^2 = r^2 - z^2$$

$$\left.\begin{aligned} I_1 &= \sum_a m_a (r_a^2 - x_a^2) \\ I_2 &= \sum_a m_a (r_a^2 - y_a^2) \end{aligned}\right\} \text{equal } (I_1 = I_2 \equiv I)$$

$$2I = I_1 + I_2$$

$$= \sum_a m_a (2r_a^2 - x_a^2 - y_a^2)$$

$$= \sum_a m_a (2(s_a^2 + z_a^2) - s_a^2)$$

$$= \sum_a m_a s_a^2 + 2\sum_a m_a z_a^2$$

$$= I_3 + 2\sum_a m_a z_a^2$$

Thus,

$$I = \frac{1}{2} I_3 + \sum_a m_a z_a^2$$

Now

$$I_3 = \sum_a m_a s_a^2$$

$$= 3 m_1 d^2$$

$$= \cancel{3} m_1 \frac{a^2}{\cancel{3}} = \boxed{m_1 a^2}$$

Also,

$$\sum_a m_a z_a^2 = 3m_1 z_1^2 + m_2 z_2^2$$

$$= 3m_1\left(-\frac{m_2 h}{\mu}\right)^2 + m_2\left(\frac{3m_1 h}{\mu}\right)^2$$

$$= \frac{3m_1 m_2^2 h^2}{\mu^2} + 9\,\frac{m_1^2 m_2 h^2}{\mu}$$

$$= \frac{3m_1 m_2 h^2}{\mu^2}\underbrace{(m_2 + 3m_1)}_{\mu}$$

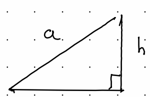
$$= \frac{3m_1 m_2 h^2}{\mu}$$

Thus,

$$I = \frac{1}{2} I_3 + \sum_a m_a z_a^2$$

$$= \boxed{\frac{1}{2} m_1 a^2 + \frac{3m_1 m_2 h^2}{\mu}}$$

Regular tetrahedron: $m_1 = m_2$



$$h^2 + \frac{a^2}{3} = a^2 \rightarrow h = \sqrt{\frac{2}{3}}\,a$$

$$\mu = 4m_1$$

$$I_3 = m_1 a^2$$

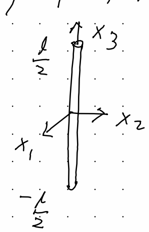
$$I = \frac{1}{2}m_1 a^2 + \frac{\cancel{3}\cancel{m_1} m_1}{4\cancel{m_1}}\left(\frac{2}{\cancel{3}}\right)a^2$$

$$= m_1 a^2 \quad (= I_1 = I_2)$$

so $I_1 = I_2 = I_3 = m_1 a^2$

Sec 32, Prob 2:

(a) Thin rod of length ℓ:



$I_3 = \boxed{0}$

and $I_1 = I_2 \equiv I$

$$I = \int \rho\, dV\,(r^2 - x^2)$$

$$= \int \rho\, dV\, z^2$$

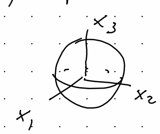
$$= \int_{-\ell/2}^{\ell/2} dz \left(\frac{\mu}{\ell}\right) z^2$$

$$= \frac{\mu}{\ell}\,\frac{z^3}{3}\Big|_{-\ell/2}^{\ell/2}$$

$$= \frac{\mu}{\ell}\,\frac{2}{3}\,\frac{\ell^3}{8}$$

$$= \boxed{\frac{1}{12}\mu \ell^2}$$

(b) Sphere of radius R:



$$I_1 = I_2 = I_3 = I$$

$$I = \frac{1}{3}(I_1 + I_2 + I_3)$$

$$= \frac{1}{3}\left[\int \rho\, dV\,(r^2 - x^2) + \int \rho\, dV\,(r^2 - y^2) + \int \rho\, dV\,(r^2 - z^2)\right]$$

$$= \frac{1}{3}\int \rho\, dV\,[3r^2 - x^2 - y^2 - y^2]$$

$$= \frac{2}{3}\int \rho\, dV\, r^2$$

$$I = \frac{2}{3}\int \rho\, dV\, r^2 \qquad \rho = \frac{M}{\frac{4}{3}\pi R^3}$$

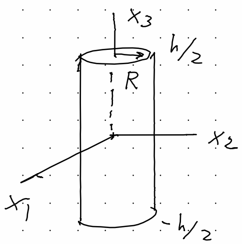
$$= \frac{2}{3}\frac{M}{\frac{4}{3}\pi R^3}\int_0^R r^4 dr \underbrace{\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi}_{4\pi}$$

$$= \frac{M}{2\pi R^3}\cdot 4\pi \int_0^R r^4 dr$$

$$= \frac{2M}{R^3}\,\frac{R^5}{5}$$

$$= \boxed{\frac{2}{5} M R^2}$$

(c)



$$\rho = \frac{M}{\pi R^2 \cdot h}$$

$$dV = ds\, s\, d\phi\, dz$$

where $s^2 = x^2 + y^2$

$$I_1 = I_2 \equiv I$$

$$2I = I_1 + I_2$$

$$= \int \rho\, dV (r^2 - x^2) + \int \rho\, dV (r^2 - y^2)$$

$$= \int \rho\, dV (2r^2 - s^2)$$

$$= \int \rho\, dV (2(s^2+z^2) - s^2)$$

$$= \int \rho\, dV\, s^2 + 2\int \rho\, dV\, z^2$$

$$\rightarrow I = \frac{1}{2} I_3 + \int \rho\, dV \cdot z^2$$

$$I_3 = \int \rho\, dV\, s^2$$

$$= \frac{M}{\pi R^2 h}\int_0^R s^3 ds \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz$$

$$= \frac{M}{\pi R^2 \cancel{h}}\,\frac{R^4}{4}\cdot 2\pi \cdot \cancel{h}$$

$$= \boxed{\frac{1}{2} M R^2}$$

$$I = \frac{1}{2} I_3 + \int \rho\, dV\, z^2$$

$$\int \rho\, dV\, z^2 = \frac{M}{\pi R^2 h}\int_0^R s\, ds \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} z^2 dz$$

$$= \frac{M}{\cancel{\pi R^2} h}\,\frac{\cancel{R^2}}{\cancel{2}}\cdot \cancel{2\pi}\cdot \frac{z^3}{3}\Big|_{-h/2}^{h/2}$$

$$= \frac{M}{h}\,\frac{2}{3}\,\frac{h^3}{8}$$

$$= \frac{1}{12} M h^2$$

Thus,

$$I = \frac{1}{2}\left(\frac{1}{2} M R^2\right) + \frac{1}{12} M h^2$$

$$= \frac{1}{4} M R^2 + \frac{1}{12} M h^2$$

$$= \boxed{\frac{1}{4} M \left(R^2 + \frac{1}{3} h^2\right)}$$

NOTE: special limiting case:

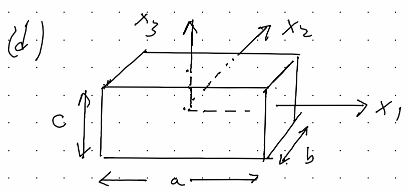
(i) Thin rod $(R \to 0)$

$$I_3 = 0$$
$$I_1 = I_2 = \frac{1}{12} M h^2$$

(ii) Thin disk $(h \to 0)$

$$I_3 = \frac{1}{2} M R^2$$
$$I_1 = I_2 = \frac{1}{4} M R^2$$

(d)



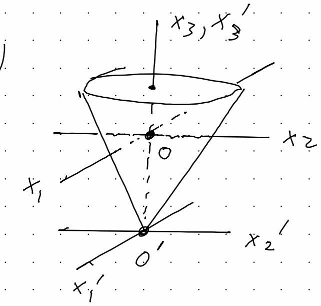
$$\rho = \frac{M}{abc}$$

$$dV = dx\,dy\,dz$$

$$I_1 = \int \rho\, dV\,(r^2 - x^2)$$
$$= \int \rho\, dV\,(y^2 + z^2)$$
$$= \frac{M}{abc} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz\,(y^2+z^2)$$
$$= \frac{M}{\cancel{a}bc}\,\cancel{a} \int_{-b/2}^{b/2} dy\,\left(y^2 z + \frac{z^3}{3}\right)\Big|_{-c/2}^{c/2}$$
$$= \frac{M}{bc} \int_{-b/2}^{b/2} dy\left(c y^2 + \frac{2}{3}\frac{c^3}{8}\right)$$
$$= \frac{M}{bc}\left[c\frac{y^3}{3}\Big|_{-b/2}^{b/2} + \frac{1}{12}c^3 y\Big|_{-b/2}^{b/2}\right]$$

$$I_1 = \frac{M}{bc}\left[c\,\frac{2}{3}\,\frac{b^3}{8} + \frac{1}{12} b c^3\right]$$
$$= \frac{M}{12bc}\left[c b^3 + b c^3\right]$$
$$= \frac{M}{12}(b^2 + c^2)$$
$$I_2 = \frac{M}{12}(c^2 + a^2)$$
$$I_3 = \frac{M}{12}(a^2 + b^2)$$

(e)



First calculate I'_{ij} (w.r.t. x_1', x_2', x_3'). Then calculate I_{ij} via

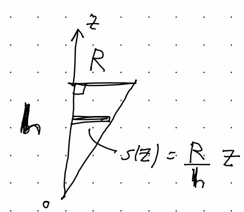
$$I_{ij} = I'_{ij} - M(a^2\delta_{ij} - a_i a_j)$$

where $\vec{a} = (0, 0, -d)$

d: height of com above O'

Volume of cone:

$$V = \int dV$$
$$= \int_0^h dz \int_0^{2\pi} d\phi \int_0^{\frac{R}{h}z} s\,ds$$
$$= \cancel{2}\pi \int_0^h dz\,\frac{s^2}{\cancel{2}}\Big|_0^{Rz/h}$$



$$V = \pi \int_0^h dz \, \frac{R^2 z^2}{h^2}$$

$$= \frac{\pi R^2}{h^2} \left. \frac{z^3}{3} \right|_0^h$$

$$= \frac{\pi R^2}{h^2} \frac{h^3}{3}$$

$$= \boxed{\frac{1}{3} \pi R^2 h}$$

Thus, $\rho = \frac{M}{V} = \boxed{\frac{M}{\frac{1}{3}\pi R^2 h}}$ (mass density)

$$I_3' = \int \rho \, dV \, (r^2 - z^2)$$

$$= \int \rho \, dV \, s^2$$

$$= \rho \int_0^h dz \int_0^{2\pi} d\phi \int_0^{Rz/h} s^3 \, ds$$

$$= \frac{\rho}{4} \cdot 2\pi \int_0^h dz \left. s^4 \right|_0^{\frac{Rz}{h}}$$

$$= \frac{\rho \pi}{2} \int_0^h dz \left(\frac{R^4}{h^4} \right) z^4$$

$$= \frac{\rho \pi R^4}{2 h^4} \frac{h^5}{5}$$

$$= \frac{\rho \pi}{10} R^4 h$$

Thus,

$$I_3' = \frac{M}{\frac{1}{3}\cancel{\pi} R^2 \cancel{h}} \frac{\pi}{10} R^4 \cancel{h}$$

$$= \boxed{\frac{3}{10} M R^2}$$

Similar to the cylinder, we have $I_1' = I_2' \equiv I'$ where

$$I' = \frac{1}{2} I_3' + \int \rho \, dV \, z^2$$

Now:

$$\int \rho \, dV z^2 = \rho \int_0^h dz \, z^2 \int_0^{2\pi} d\phi \int_0^{Rz/h} s \, ds$$

$$= \cancel{2} \pi \rho \int_0^h dz \, z^2 \left. \frac{s^2}{\cancel{2}} \right|_0^{Rz/h}$$

$$= \pi \rho \frac{R^2}{h^2} \int_0^h dz \, z^4$$

$$= \pi \rho \frac{R^2}{h^2} \frac{h^5}{5}$$

$$= \frac{\pi}{5} \rho R^2 h^3$$

$$= \frac{\pi}{5} \left(\frac{M}{\frac{1}{3} \pi \cancel{R^2} h} \right) \cancel{R^2} h^3$$

$$= \frac{3}{5} M h^2$$

So

$$I' = \frac{1}{2}\left(\frac{3}{10}\mu R^2\right) + \frac{3}{5}\mu h^2$$

$$= \boxed{\frac{3}{5}\mu\left(\frac{R^2}{4} + h^2\right)} = I_1' = I_2'$$

Need to find location of COM:

$$d = \frac{1}{\mu}\int \rho\, dV\, z$$

$$= \frac{1}{\mu}\rho\int_0^h z\,dz \int_0^{2\pi} d\phi \int_0^{Rz/h} s\,ds$$

$$= \frac{1}{\mu}\rho \cdot \cancel{2}\pi \int_0^h z\,dz\; \frac{1}{\cancel{2}}\left(\frac{R}{h}\right)^2 z^2$$

$$= \frac{\pi\rho}{\mu}\frac{R^2}{h^2}\frac{z^4}{4}\Big|_0^h$$

$$= \frac{\pi\rho}{4\mu} R^2 h^2$$

$$= \frac{\cancel{\pi}}{4\cancel{\mu}} \cdot \frac{\cancel{\mu}}{\frac{1}{3}\cancel{\pi}\cancel{R^2} h} \cancel{R^2} h^2$$

$$= \boxed{\frac{3}{4}h}$$

Thus,

$$I_{ij} = I_{ij}' - \mu(a^2\delta_{ij} - a_i a_j)$$

where $\vec{a} = (0, 0, -\frac{3}{4}h) \rightarrow a^2 = \frac{9}{16}h^2$

$$\rightarrow I_1 = I_1' - \mu a^2$$

$$= \frac{3}{5}\mu\left(\frac{R^2}{4} + h^2\right) - \mu\frac{9}{16}h^2$$

$$= \frac{3}{20}\mu R^2 + \mu h^2 \underbrace{\left(\frac{3}{5} - \frac{9}{16}\right)}_{}$$

$$\frac{48 - 45}{80} = \frac{3}{80}$$

$$= \boxed{\frac{3}{20}\mu\left(R^2 + \frac{h^2}{4}\right)}$$

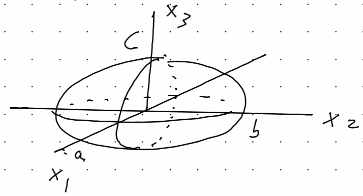
Also, $I_2 = I_1$

Finally, $I_3 = I_3' - \mu(\cancel{a^2 - a^2}) \rightarrow 0$

$$= I_3'$$

$$= \boxed{\frac{3}{10}\mu R^2}$$

(f) Ellipsoid with semi-axes a, b, c



$(a,b,c) \leftrightarrow (x_1, x_2, x_3)$

Define rescaled coordinates:

$$(u, v, w) \equiv \left(\frac{x_1}{a}, \frac{x_2}{b}, \frac{x_3}{c}\right)$$

So that boundary of ellipsoid

$$1 = \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 + \left(\frac{x_3}{c}\right)^2 = \underbrace{u^2 + v^2 + w^2}_{\text{unit 2-sphere}}$$

Volume:

$$V = \int dx_1 \int dx_2 \int dx_3$$

$$= abc \int du \int dv \int dw$$

$$= abc \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^1 r^2 dr$$

$$= abc \; 2\pi \cdot 2 \cdot \frac{r^3}{3}\Big|_0^1$$

$$= \boxed{\frac{4}{3}\pi abc}$$

$$\longrightarrow \rho = \frac{M}{\frac{4}{3}\pi abc}$$

$$I_3 = \int \rho \, dV (r^2 - z^2)$$

$$= \int \rho \, dV (x^2 + y^2)$$

$$= \frac{M}{\frac{4}{3}\pi abc} \iiint dx\, dy\, dz \, (x^2 + y^2)$$

$$= \frac{M}{\frac{4}{3}\pi \cancel{abc}} \cancel{abc} \iiint du\, dv\, dw \left(a^2 u^2 + b^2 v^2\right)$$

$$= \frac{M}{\frac{4}{3}\pi} \int_0^1 r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \left(a^2 r^2 \sin^2\theta \cos^2\phi + b^2 r^2 \sin^2\theta \sin^2\phi\right)$$

Now:

$$\int_0^{\pi} \sin\theta \, d\theta \, \sin^2\theta = \int_{-1}^{1} d(\cos\theta)\left(1 - \cos^2\theta\right)$$

$$= \int_{-1}^{1} dx \, (1 - x^2)$$

$$= \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$= 2 \cdot \frac{2}{3} = \boxed{\frac{4}{3}}$$

$$\int_0^1 r^4 dr = \frac{r^5}{5}\Big|_0^1 = \boxed{\frac{1}{5}}$$

$$\int_0^{2\pi} d\phi \begin{Bmatrix}\sin^2\phi \\ \cos^2\phi\end{Bmatrix} = 2\pi\cdot\frac{1}{2} = \boxed{\pi}$$

Thus,

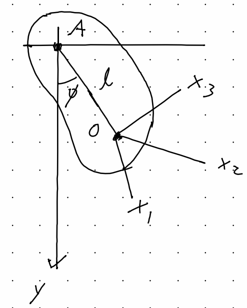
$$I_3 = \frac{M}{\cancel{\frac{4}{3}}\cancel{\pi}}\left(a^2\,\cancel{\frac{4}{3}}\cdot\frac{1}{5}\cdot\cancel{\pi} + b^2\,\cancel{\frac{4}{3}}\cdot\frac{1}{5}\cdot\cancel{\pi}\right)$$

$$= \boxed{\frac{M}{5}\left(a^2+b^2\right)}$$

cyclically permuting a, b, c $\rightarrow$

$$\boxed{\begin{aligned} I_1 &= \frac{M}{5}\left(b^2+c^2\right) \\ I_2 &= \frac{M}{5}\left(c^2+a^2\right)\end{aligned}}$$

Sec 32, Prob 3:



COM at O

rotation axis at A, out of page

$$\vec{\Omega} = \dot{\phi}\,\hat{n}$$

$$U = \mu g l\left(1-\cos\phi\right)$$

$$\approx \frac{1}{2}\mu g l\phi^2 \quad \text{for } \phi \ll 1$$

$$L = T - U$$

$$T = \frac{1}{2}\mu V^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$\vec{\Omega} = \dot{\phi}\,\hat{n} \rightarrow \begin{aligned} \Omega_1 &= \dot{\phi}\,\hat{n}\cdot\hat{x}_1 = \dot{\phi}\cos\alpha \\ \Omega_2 &= \dot{\phi}\,\hat{n}\cdot\hat{x}_2 = \dot{\phi}\cos\beta \\ \Omega_3 &= \dot{\phi}\,\hat{n}\cdot x_3 = \dot{\phi}\cos\gamma \end{aligned}$$

$$V = l\dot{\phi}$$

Thus,

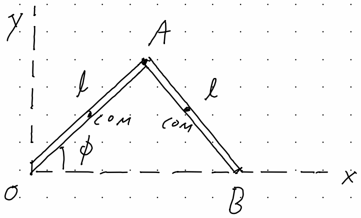
$$T = \frac{1}{2}\mu l^2\dot{\phi}^2 + \frac{1}{2}\dot{\phi}^2\left(I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)$$

so

$$L = \frac{1}{2}\dot{\phi}^2\left(\mu l^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right) - \frac{1}{2}\mu g l\phi^2$$

$$\rightarrow \omega = \sqrt{\frac{\mu g l}{\mu l^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma}}$$

<u>Sec. 32, Prob 4</u>



two uniform rods

$I_{com} = \frac{1}{12} M l^2$

$I_{end} = \frac{1}{3} M l^2$

$T = T_1 + T_2$

$T_1 = \frac{1}{2} M \left(\frac{l}{2}\right)^2 \dot{\phi}^2 + \frac{1}{2} I_{com} \dot{\phi}^2$

$= \frac{1}{8} M l^2 \dot{\phi}^2 + \frac{1}{24} M l^2 \dot{\phi}^2$

$= \left(\frac{1}{8} + \frac{1}{24}\right) M l^2 \dot{\phi}^2$

$= \frac{1}{6} M l^2 \dot{\phi}^2$

$T_2 = \frac{1}{2} M V^2 + \frac{1}{2} I_{com} \dot{\phi}^2$

<u>Now:</u> $V^2 = \dot{x}^2 + \dot{y}^2$

$x = \frac{3}{2} l \cos\phi \quad , \quad y = \frac{l}{2} \sin\phi$

$\dot{x} = -\frac{3}{2} l \sin\phi\, \dot{\phi} \quad , \quad \dot{y} = \frac{l}{2} \cos\phi\, \dot{\phi}$

$\rightarrow V^2 = \frac{9}{4} l^2 \sin^2\phi\, \dot{\phi}^2 + \frac{l^2}{4} \cos^2\phi\, \dot{\phi}^2$

$= 2 l^2 \sin^2\phi\, \dot{\phi}^2 + \frac{l^2}{4} \dot{\phi}^2$

$= 2 l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{8}\right)$

so

$T_2 = \frac{1}{\cancel{2}} M \cancel{2} l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{8}\right) + \frac{1}{24} M l^2 \dot{\phi}^2$

$= M l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{8} + \frac{1}{24}\right)$

$= M l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{6}\right)$

Thus,

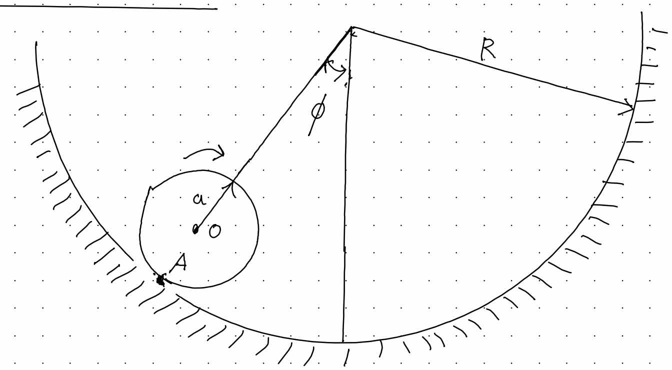
$T = T_1 + T_2$

$= \frac{1}{6} M l^2 \dot{\phi}^2 + M l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{6}\right)$

$= M l^2 \dot{\phi}^2 \left(\frac{1}{3} + \sin^2\phi\right)$

$= \frac{1}{3} M l^2 \dot{\phi}^2 \left(1 + 3 \sin^2\phi\right)$

Sec 32, Prob 6:



Homogeneous cylinder of radius a, mass μ:

$$I_3 = \frac{1}{2}\mu a^2 \quad \text{(about com)}$$

$$V = -(R-a)\dot\phi \quad \text{(velocity of com)}$$

Instantaneous axis of rotation at A:

$$0 = \vec V + \vec\Omega + (-a\hat n), \quad \vec\Omega \text{ into page}$$

so $V = \Omega a$

Thus, $\Omega a = (R-a)|\dot\phi|$

$$\Omega = \left(\frac{R-a}{a}\right)|\dot\phi|$$

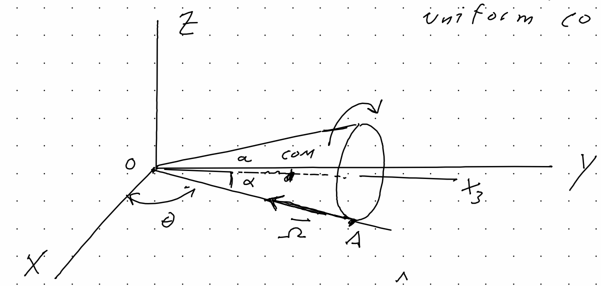
$$T = \frac{1}{2}\mu V^2 + \frac{1}{2} I_3 \Omega^2$$

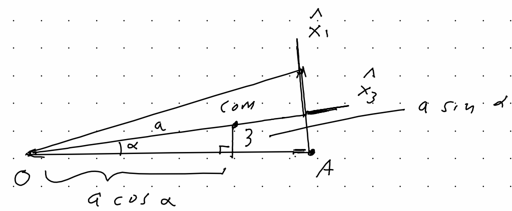
$$= \frac{1}{2}\mu\left(\frac{R-a}{\cancel{a}}\right)^2\cancel{a^2}\dot\phi^2 + \frac{1}{4}\mu\cancel{a^2}\left(\frac{R-a}{\cancel{a}}\right)^2\dot\phi^2$$

$$= \mu(R-a)^2\dot\phi^2\left(\frac{1}{2}+\frac{1}{4}\right) = \boxed{\frac{3}{4}\mu(R-a)^2\dot\phi^2}$$

Sec 32, Prob 7.

radius R, height h
uniform cone





V = velocity of COM
$= a\cos\alpha\,\dot\theta$

OA: instantaneous axis of rotation

$$0 = \vec V - a\sin\alpha\,\vec\Omega \times \hat n \qquad (\hat n = \hat z)$$

$$0 = V - a\sin\alpha\,\Omega$$

Thus, $a\sin\alpha\,\Omega = V = a\cos\alpha\,\dot\theta$

$$\boxed{\Omega = \cot\alpha\,\dot\theta}$$

$\vec\Omega$: directed from A to O

$$\Omega_3 = -\Omega\cos\alpha = -\frac{\cos^2\alpha}{\sin\alpha}\dot\theta$$

$$\Omega_1 = \Omega\sin\alpha = \cos\alpha\,\dot\theta$$

Thus,

$$T = \frac{1}{2}\mu v^2 + \frac{1}{2}\left(I_1\Omega_1^2 + \overset{(I_2 = I_1)}{I_1}\,\Omega_2^2 + I_3\Omega_3^2\right)$$

$$= \frac{1}{2}\mu\, a^2\cos^2\alpha\,\dot\theta^2 + \frac{1}{2}\left(I_1\cos^2\alpha\,\dot\theta^2 + I_3\frac{\cos^4\alpha}{\sin^2\alpha}\dot\theta^2\right)$$

Now: $I_1 = I_2 = \frac{3}{20}\mu\left(R^2 + \frac{1}{4}h^2\right)$

$I_3 = \frac{3}{10}\mu R^2$

Also: $a = \frac{3}{4}h$

$\tan\alpha = \frac{R}{h} \rightarrow R = h\tan\alpha$

Thus,

$$T = \frac{1}{2}\mu\left(\frac{9}{16}\right)h^2\cos^2\alpha\,\dot\theta^2$$

$$+ \frac{1}{2}\left[\frac{3}{20}\mu\left(h^2\tan^2\alpha + \frac{1}{4}h^2\right)\cos^2\alpha\,\dot\theta^2 + \frac{3}{10}\mu\, h^2\tan^2\alpha\frac{\cos^4\alpha}{\sin^2\alpha}\dot\theta^2\right]$$

$$= \mu h^2\dot\theta^2\left[\frac{9}{32}\cos^2\alpha + \frac{3}{40}\overset{(1-\cos^2\alpha)}{\sin^2\alpha} + \frac{3}{160}\cos^2\alpha + \frac{3}{20}\cos^2\alpha\right]$$

$$= \mu h^2\dot\theta^2\left[\frac{3}{40} + \cos^2\alpha\left(\frac{9}{32} - \frac{3}{40} + \frac{3}{160} + \frac{3}{20}\right)\right]$$

Now: $\frac{9}{32} - \frac{3}{40} + \frac{3}{160} + \frac{3}{20}$

$$= \frac{1}{160}\left[45 - 12 + 3 + 24\right]$$

$$= \frac{60}{160}$$

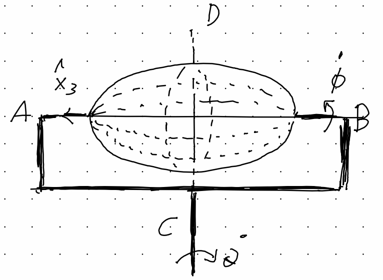
$$= \frac{15}{40}$$

Thus,

$$T = \mu h^2\dot\theta^2\left[\frac{3}{40} + \frac{15}{40}\cos^2\alpha\right]$$

$$= \boxed{\frac{3}{40}\mu h^2\dot\theta^2\left[1 + 5\cos^2\alpha\right]}$$

Sec 32, Prob 9.



homogeneous ellipsoid with principal moments of inertia I_1, I_2, I_3

$$\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta}$$

Now: $\vec{\dot\phi} = \dot\phi \hat{x}_3$

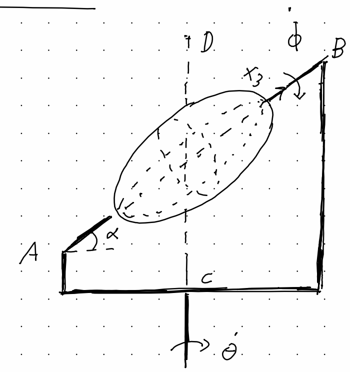
$$\vec{\dot\theta} = \dot\theta \left[\cos\phi \, \hat{x}_1 + \sin\phi \, \hat{x}_2 \right]$$

so $\vec{\Omega} = \dot\theta \cos\phi \, \hat{x}_1 + \dot\theta \sin\phi \, \hat{x}_2 + \dot\phi \, \hat{x}_3$

$$T = \frac{1}{2}\left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 \right)$$

$$= \frac{1}{2}\left[\left(I_1 \cos^2\phi + I_2 \sin^2\phi \right) \dot\theta^2 + I_3 \dot\phi^2 \right]$$

Sec 32, Prob 10.



Uniform ellipsoid with $I_1 = I_2$ (circular cross section)

$$\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta}$$

$$\vec{\dot\phi} = \dot\phi \hat{x}_3$$

$$\vec{\dot\theta} = \dot\theta \left[\cos\left(\tfrac{\pi}{2} - \alpha\right) \hat{x}_3 + \sin\left(\tfrac{\pi}{2}-\alpha\right) \left(\cos\phi \, \hat{x}_1 + \sin\phi \, \hat{x}_2 \right) \right]$$

Now! $\cos\left(\frac{\pi}{2} - \alpha\right) = \cancelto{0}{\cos\left(\frac{\pi}{2}\right)} \cos\alpha + \sin\left(\frac{\pi}{2}\right) \sin\alpha$

$$= \sin\alpha$$

$\sin\left(\frac{\pi}{2} - \alpha\right) = \sin\frac{\pi}{2} \cos\alpha - \cancelto{0}{\cos\left(\frac{\pi}{2}\right)} \sin\alpha$

$$= \cos\alpha$$

so $\vec{\dot\theta} = \dot\theta \left[\sin\alpha \, \hat{x}_3 + \cos\alpha \cos\phi \, \hat{x}_1 + \cos\alpha \sin\phi \, \hat{x}_2 \right]$

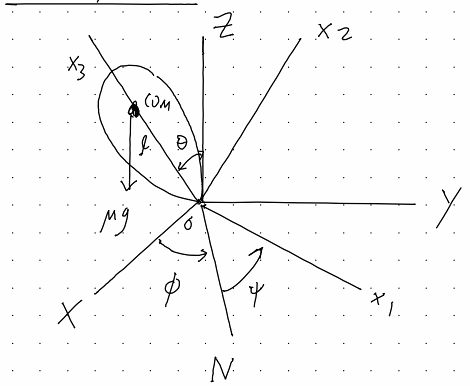
$\rightarrow \vec{\Omega} = \dot\theta \cos\alpha \left(\cos\phi \, \hat{x}_1 + \sin\phi \, \hat{x}_2 \right) + \left(\dot\phi + \dot\theta \sin\alpha \right) \hat{x}_3$

$$T = \frac{1}{2}\left(I_1 \Omega_1^2 + I_1 \Omega_2^2 + I_3 \Omega_3^2 \right)$$

$$= \frac{1}{2}\left[I_1 \dot\theta^2 \cos^2\alpha \left(\cos^2\phi + \sin^2\phi \right) + I_3 \left(\dot\phi + \dot\theta \sin\alpha \right)^2 \right]$$

$$= \frac{1}{2}\left[I_1 \cos^2\alpha \, \dot\theta^2 + I_3 \left(\dot\phi + \dot\theta \sin\alpha \right)^2 \right]$$

Sec 35, prob 1:



symmetrical top:

$I_1 = I_2$, I_3 (wrt principal axes passing through COM)

$I_1' = I_1 + Ml^2$ (wrt axes passing through O, which is displaced from the COM by l in the $-x_3$ direction)

$I_2' = I_1'$

$I_3' = I_3$

$L = T - U$

$U = mgZ = mgl\cos\theta$

$$T = \frac{1}{2}\left(I_1'\Omega_1^2 + I_2'\Omega_2^2 + I_3'\Omega_3^2\right)$$

$$= \frac{1}{2}\left[I_1'(\Omega_1^2 + \Omega_2^2) + I_3\Omega_3^2\right]$$

Now: (from (35.1))

$$\Omega_1 = \dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi$$

$$\Omega_2 = \dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi$$

$$\Omega_3 = \dot\phi\cos\theta + \dot\psi$$

so $\Omega_3^2 = \left(\dot\phi\cos\theta + \dot\psi\right)^2$

$$\Omega_1^2 = \dot\phi^2\sin^2\theta\sin^2\psi + \dot\theta^2\cos^2\psi + 2\dot\theta\dot\psi\sin\theta\sin\psi\cos\psi$$

$$\Omega_2^2 = \dot\phi^2\sin^2\theta\cos^2\psi + \dot\theta^2\sin^2\psi - 2\dot\theta\dot\psi\sin\theta\cos\psi\sin\psi$$

$$\rightarrow \Omega_1^2 + \Omega_2^2 = \dot\phi^2\sin^2\theta + \dot\theta^2$$

Thus,

$$L = \frac{1}{2}\left[I_1'(\dot\phi^2\sin^2\theta + \dot\theta^2) + I_3(\dot\phi\cos\theta + \dot\psi)^2\right] - Mgl\cos\theta$$

1) No explicit t-dependence

$$E = T + U = \text{const}$$

2) No explicit ϕ dependence:

$$p_\phi \equiv \frac{\partial L}{\partial\dot\phi} = I_1'\sin^2\theta\,\dot\phi + I_3(\dot\phi\cos\theta + \dot\psi)\cos\theta = \text{const}$$

3) No explicit ψ dependence:

$$p_\psi \equiv \frac{\partial L}{\partial\dot\psi} = I_3(\dot\phi\cos\theta + \dot\psi) = \text{const}$$

We can solve p_ϕ, p_ψ for $\dot\phi$, $\dot\psi$:

$$p_\phi = I_1' \sin^2\theta\,\dot\phi + I_3(\dot\phi\cos\theta + \dot\psi)\cos\theta$$

$$p_\psi = I_3(\dot\phi\cos\theta + \dot\psi)$$

$$\rightarrow \quad p_\phi = I_1'\sin^2\theta\,\dot\phi + p_\psi\cos\theta$$

so $I_1'\sin^2\theta\,\dot\phi = p_\phi - p_\psi\cos\theta$

$$\boxed{\dot\phi = \frac{p_\phi - p_\psi\cos\theta}{I_1'\sin^2\theta}}$$

And:

$$\frac{p_\psi}{I_3} = \dot\phi\cos\theta + \dot\psi$$

$$\rightarrow \quad \boxed{\dot\psi = \frac{p_\psi}{I_3} - \dot\phi\cos\theta = \frac{p_\psi}{I_3} - \left(\frac{p_\phi - p_\psi\cos\theta}{I_1'\sin^2\theta}\right)\cos\theta}$$

Also:

$$E = T + U$$

$$= \frac{1}{2}\left[I_1'(\sin^2\theta\,\dot\phi^2 + \dot\theta^2) + I_3(\dot\phi\cos\theta + \dot\psi)^2\right] + \mu g l\cos\theta$$

can be rewritten as

$$E = \frac{1}{2}I_1'\dot\theta^2 + \frac{1}{2}I_1'\sin^2\theta\,\dot\phi^2 + \frac{1}{2}I_3\left(\frac{p_\psi}{I_3}\right)^2 + \mu g l\cos\phi$$

$$= \frac{1}{2}I_1'\dot\theta^2 + \frac{1}{2}I_1'\sin^2\theta\,\frac{(p_\phi - p_\psi\cos\theta)^2}{I_1'^2\sin^4\theta} + \frac{1}{2}\frac{p_\psi^2}{I_3} + \mu g l\cos\phi$$

$$= \frac{1}{2}I_1'\dot\theta^2 + \frac{1}{2}\frac{p_\psi^2}{I_3} + \frac{1}{2}\frac{(p_\phi - p_\psi\cos\theta)^2}{I_1'\sin^2\theta} + \mu g l\cos\theta$$

Now: $\frac{1}{2}\frac{p_\psi^2}{I_3} = \text{const}$

Also: $\mu g l\cos\theta = -\mu g l(1-\cos\theta) + \mu g l$

so

$$\underbrace{E - \frac{1}{2}\frac{p_\psi^2}{I_3} - \mu g l}_{E'} = \frac{1}{2}I_1'\dot\theta^2 + \frac{1}{2}\frac{(p_\phi - p_\psi\cos\theta)^2}{I_1'\sin^2\theta} - \mu g l(1-\cos\theta)$$

$$\rightarrow \quad E' = \frac{1}{2}I_1'\dot\theta^2 + U_{eff}(\theta)$$

where $\boxed{U_{eff}(\theta) = \frac{1}{2}\frac{(p_\phi - p_\psi\cos\theta)^2}{I_1'\sin^2\theta} - \mu g l(1-\cos\theta)}$

EOM:

$$E' = \frac{1}{2} I_1' \dot{\theta}^2 + U_{eff}(\theta)$$

$$\dot{\theta}^2 = \frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)$$

$$\frac{d\theta}{dt} = \sqrt{\frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)}$$

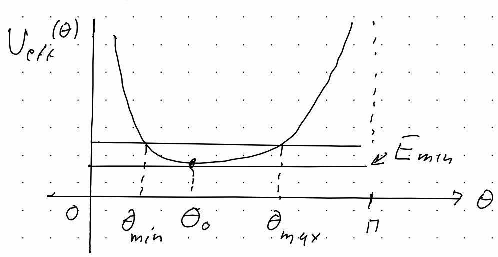
$$\rightarrow \int dt = \int \frac{d\theta}{\sqrt{\frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)}}$$

$$\boxed{t = \int \frac{d\theta}{\sqrt{\frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)}} + \text{const}}$$

Effective potential:

$$U_{eff}(\theta) = \frac{1}{2}\frac{(p_\phi - p_\psi \cos\theta)^2}{I_1' \sin^2\theta} - \mu g l (1 - \cos\theta)$$

For $p_\phi \neq p_\psi$, $U_{eff}(\theta) \to \infty$ as $\theta \to 0, \pi$



To find θ_0:

$$0 = \left.\frac{dU_{eff}}{d\theta}\right|_{\theta_0}$$

$$= \frac{(p_\phi - p_\psi \cos\theta_0)\, p_\psi \sin\theta_0}{I_1' \sin^2\theta_0} - \frac{(p_\phi - p_\psi\cos\theta_0)^2 \cos\theta_0}{I_1' \sin^3\theta_0}$$

$$- \mu g l \sin\theta_0$$

$$= \beta \frac{p_\psi}{I_1' \sin\theta_0} - \beta^2 \frac{\cos\theta_0}{I_1' \sin^3\theta_0} - \mu g l \sin\theta_0$$

Multiply through by $-I_1' \sin^3\theta_0$:

$$0 = \beta^2 \cos\theta_0 - \beta p_\psi \sin^2\theta_0 + \mu g l I_1' \sin^4\theta_0$$

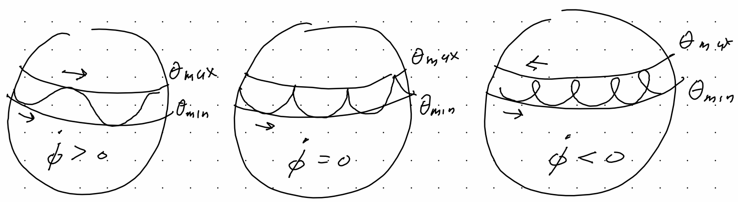
Quadratic equation for β:

$$\beta_\pm = \frac{p_\psi \sin^2\theta_0 \pm \sqrt{p_\psi^2 \sin^4\theta_0 - 4\mu g l I_1' \sin^4\theta_0 \cos\theta_0}}{2\cos\theta_0}$$

$$= \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \sqrt{\underbrace{1 - \frac{4\mu g l I_1' \cos\theta_0}{p_\psi^2}}_{\text{needs to be } \geq 0 \text{ for a real solution.}}}\right)$$

The last equation is a transcendental equation for θ_0 since $\beta = p_\phi - p_\psi \cos\theta$

For $E > E_{min}$, θ varies between θ_{min} and θ_{max}. The motion of the x_3-axis of the top can have the following three forms depending on the sign of $\dot\phi$ when $\theta = \theta_{max}$.



This motion is called nutation.

Sec 3.5, Prob 2

For rotation of a top around a vertical axis to be stable, we need $\left.\dfrac{d^2 U_{eff}}{d\theta^2}\right|_{\theta=0} > 0$

Now:

$$U_{eff}(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1' \sin^2\theta} - Mgl(1-\cos\theta)$$

Also:

$$p_\phi = I_1' \sin^2\theta\, \dot\phi + I_3(\dot\phi \cos\theta + \dot\psi)\cos\theta$$

$$p_\psi = I_3(\dot\phi \cos\theta + \dot\psi)$$

In the limit $\theta \to 0$

$$\left.\begin{aligned} p_\phi &\simeq I_3(\dot\phi + \dot\psi) \\ p_\psi &\simeq I_3(\dot\phi + \dot\psi) \end{aligned}\right\}$$ so they are equal in this limit

Thus,

$$U_{eff}(\theta) \simeq \frac{p_\phi^2 (1-\cos\theta)^2}{2 I_1' \sin^2\theta} - Mgl(1-\cos\theta)$$

$$\simeq \frac{p_\phi^2 \left(\frac{\theta^2}{2}\right)^2}{2 I_1' \theta^2} - Mgl\frac{\theta^2}{2}$$

$$= \left(\frac{1}{8}\frac{p_\phi^2}{I_1'} - \frac{1}{2}Mgl\right)\theta^2$$

Thus

$$U_{eff}(\theta) \simeq \left(\frac{1}{8}\frac{p_\phi}{I_1'} - \frac{1}{2}\mu g l\right)\theta^2$$

$$\longrightarrow \quad U_{eff}(0) = 0$$

$$\left.\frac{dU_{eff}}{d\theta}\right|_{\theta=0} = 2\left(\frac{1}{8}\frac{p_\phi}{I_1'} - \frac{1}{2}\mu g l\right)\theta\Big|_{\theta=0} = 0$$

$$\left.\frac{d^2U_{eff}}{d\theta^2}\right|_{\theta=0} = 2\left(\frac{1}{8}\frac{p_\phi}{I_1'} - \frac{1}{2}\mu g l\right)$$

Need $\left.\frac{d^2U_{eff}}{d\theta^2}\right|_{\theta=0} > 0$ for stable rotation!

$$\frac{1}{8}\frac{p_\phi^2}{I_1'} - \frac{1}{2}\mu g l > 0$$

$$\boxed{p_\phi^2 > 4\mu g l I_1'}$$

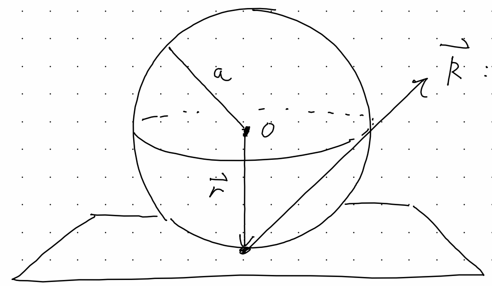
Since $p_\phi = p_\psi = I_3\Omega_3$, we can also write this condition as

$$I_3^2\Omega_3^2 > 4\mu g l I_1'$$

$$\boxed{\Omega_3^2 > \frac{4\mu g l I_1'}{I_3^2}}$$

Sec. 38, Prob 1:

Homogeneous sphere (radius a) rolling without slipping on a horizontal surface, subject to applied force $\vec{F}$ and torque $\vec{K}$.



$\vec{R}$: reaction force

$I = \frac{2}{5}\mu a^2$ (homog. sphere about com)

$$\frac{d\vec{P}}{dt} = \vec{F} + \vec{R} \longrightarrow \mu\frac{d\vec{V}}{dt} = \vec{F} + \vec{R}$$

$$\frac{d\vec{M}}{dt} = \vec{K} + \vec{r}\times\vec{R} \rightarrow I\frac{d\vec{\Omega}}{dt} = \vec{K} - a\hat{z}\times\vec{R}$$

Rolling without slipping:

$$0 = \vec{V} + \vec{\Omega}\times\vec{r}$$
$$= \vec{V} - a\vec{\Omega}\times\hat{z}$$

So $\vec{V} = a\vec{\Omega}\times\hat{z}$

Using $(\vec{A}\times\vec{B})_1 = A_2B_3 - A_3B_2$, etc

$$\boxed{\begin{aligned} V_x &= a\Omega_y \\ V_y &= -a\Omega_x \\ V_z &= 0 \end{aligned}}$$

no motion off surface

Combine:

$$\mu \frac{d\vec{V}}{dt} = \vec{F} + \vec{R} \qquad (1)$$

$$I \frac{d\vec{\Omega}}{dt} = \vec{K} - a\hat{z}\times\vec{R} \qquad (2)$$

$V_x = a\Omega_y$, $V_y = -a\Omega_x$, $V_z = 0$ (constraint)

Take time derivative of constraint equation:

$$\frac{dV_x}{dt} = a\frac{d\Omega_y}{dt}, \quad \frac{dV_y}{dt} = -a\frac{d\Omega_x}{dt}$$

substitute from (1), (2) into these two equations

$$\frac{1}{\mu}(F_x + R_x) = \frac{a}{I}(K_y - aR_x)$$

$$\frac{1}{\mu}(F_y + R_y) = \frac{-a}{I}(K_x + aR_y)$$

Thus,

$$F_x + R_x = \frac{\mu a}{I}K_y - \frac{\mu a^2}{I}R_x$$

$$R_x\left(\frac{I + \mu a^2}{I}\right) = \frac{\mu a}{I}K_y - F_x$$

$$R_x \frac{\frac{7}{5}\cancel{\mu a^2}}{\frac{2}{5}\cancel{\mu a^2}} = \frac{\cancel{\mu a}}{\frac{2}{5}\mu a^2}K_y - F_x$$

$$\rightarrow \boxed{R_x = \frac{5}{7}\frac{K_y}{a} - \frac{2}{7}F_x}$$

Similarly,

$$F_y + R_y = -\frac{\mu a}{I}K_x - \frac{\mu a^2}{I}R_y$$

$$R_y\left(\frac{I + \mu a^2}{I}\right) = -\frac{\mu a}{I}K_x - F_y$$

$$R_y \frac{\frac{7}{5}\cancel{\mu a^2}}{\frac{2}{5}\cancel{\mu a^2}} = -\frac{\cancel{\mu a}}{\frac{2}{5}\mu a^2}K_x - F_y$$

$$\rightarrow \boxed{R_y = -\frac{5}{7}\frac{K_x}{a} - \frac{2}{7}F_y}$$

Also,

$$\mu\frac{d\cancel{V_z}^{\,0}}{dt} = F_z + R_z$$

$$\rightarrow \boxed{R_z = -F_z}$$

Using these expression for R_x, R_y we can write down EOMs for V_x, V_y

$$\boxed{\begin{aligned}\mu\frac{dV_x}{dt} &= F_x + R_x \\ &= F_x + \frac{5}{7}\frac{K_y}{a} - \frac{2}{7}F_x \\ &= \frac{5}{7}\left(F_x + \frac{K_y}{a}\right)\end{aligned}}$$

Similarly

$$\boxed{\mu\frac{dV_y}{dt}} = F_y + R_y$$

$$= F_y - \frac{5}{7}\frac{K_x}{a} - \frac{2}{7}F_y$$

$$= \boxed{\frac{5}{7}\left(F_y - \frac{K_x}{a}\right)}$$

Summary:

$$R_x = \frac{5}{7}\frac{K_y}{a} - \frac{2}{7}F_x$$

$$R_y = -\frac{5}{7}\frac{K_x}{a} - \frac{2}{7}F_y$$

$$R_z = -F_z$$

$$\mu\frac{dV_x}{dt} = \frac{5}{7}\left(F_x + \frac{K_y}{a}\right)$$

$$\mu\frac{dV_y}{dt} = \frac{5}{7}\left(F_y - \frac{K_x}{a}\right)$$

$$V_z = 0$$

$$\Omega_x = -\frac{V_y}{a}$$

$$\Omega_y = \frac{V_x}{a}$$

$$I\frac{d\Omega_z}{dt} = K_z$$

Example:

Suppose: $\vec{F} = -\mu g\hat{z} + F_0\hat{x}$

$\vec{K} = 0$

Then: $F_x = F_0$, $F_y = 0$, $F_z = -\mu g$

Solution from previous page gives:

$$R_x = -\frac{2}{7}F_0$$

$$R_y = 0$$

$R_z = \mu g$ (normal force upward)

$$\mu\frac{dV_x}{dt} = \frac{5}{7}F_0 \rightarrow V_x = \frac{5}{7}\frac{F_0}{\mu}t + V_{x0}$$

$$\mu\frac{dV_y}{dt} = 0 \rightarrow V_y = \text{const} = V_{y0}$$

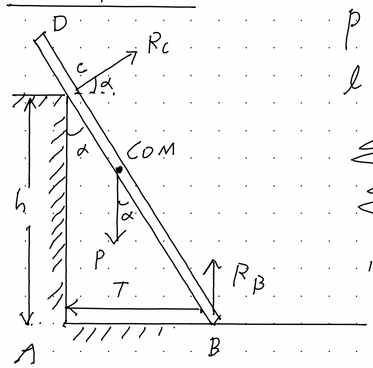
$$V_z = 0$$

$$\Omega_x = -\frac{V_{y0}}{a}$$

$$\Omega_y = \frac{V_x}{a} = \frac{5}{7}\frac{F_0}{\mu a}t + \frac{V_{x0}}{a}$$

$$I\frac{d\Omega_z}{dt} = 0 \rightarrow \Omega_z = \text{const} = \Omega_{z0}$$

Sec 38, Prob 2



P = weight $= \mu g$

l : length of uniform rod

$\sum \vec{F} = 0$

$\sum \vec{r} \times \vec{F} = 0$

including reaction force T, R_C, R_B

horizontal direction

$$-T + R_C \cos\alpha = 0 \quad (1)$$

vertical direction

$$R_C \sin\alpha - P + R_B = 0 \quad (2)$$

torques around D:

$$\frac{l}{2} \sin\alpha \, P - \frac{h}{\cos\alpha} R_C = 0 \quad (3)$$

$\cos\alpha = \frac{h}{d}$

$\rightarrow d = \frac{h}{\cos\alpha}$

Thus,

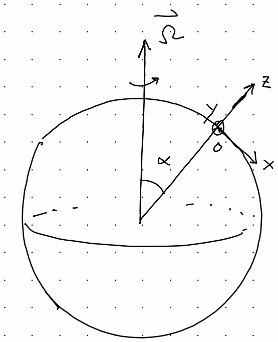
$$\frac{h}{\cos\alpha} R_C = \frac{l}{2} \sin\alpha \, P$$

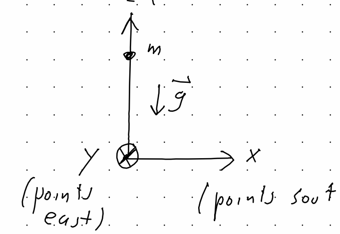
$$\boxed{R_C = \frac{lP}{2h} \sin\alpha \cos\alpha}$$

$$\boxed{T = R_C \cos\alpha}$$

$$\boxed{R_B = P - R_C \sin\alpha}$$

Sec 39, Prob 1:





(α: location on Earth relative to North pole)

Newton's 2nd law in rotating frame

$$m\vec{a} = m\vec{g} - m\vec{W} - 2m\vec{\Omega}\times\vec{v} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

where $\vec{W} = \vec{\Omega}\times(\vec{\Omega}\times\vec{R})$

= acceleration of origin of rotating frame

Thus,

$$m\vec{a} = m\vec{g} - \underbrace{m\vec{\Omega}\times(\vec{\Omega}\times(\vec{r}+\vec{R}))}_{\substack{\text{2nd order in } \vec{\Omega} \\ \text{[ignore]}}} - 2m\vec{\Omega}\times\vec{v}$$

$$\approx m\vec{g} - 2m\vec{\Omega}\times\vec{v}$$

Now: $\vec{g} = -g\hat{z}$

$\vec{\Omega} = \Omega\cos\alpha\,\hat{z} - \Omega\sin\alpha\,\hat{x}$

$$(\vec{\Omega}\times\vec{v})_x = \cancel{\Omega_y}^{\to 0} v_z - \Omega_z \dot{y}$$
$$= -\Omega_z \dot{y}$$
$$(\vec{\Omega}\times\vec{v})_y = \Omega_z v_x - \Omega_x v_z$$
$$= \Omega_z \dot{x} - \Omega_x \dot{z}$$
$$(\vec{\Omega}\times\vec{v})_z = \Omega_x v_y - \cancel{\Omega_y}^{\to 0} v_x$$
$$= \Omega_x \dot{y}$$

Thus,

$$\ddot{x} = +2\Omega_z \dot{y}$$
$$\ddot{y} = -2(\Omega_z \dot{x} - \Omega_x \dot{z})$$
$$\ddot{z} = -g + \Omega_x \dot{y}$$

Want to solve these equations to 1^{st} order in Ω

The 0^{th} order solution is

$$\left.\begin{aligned} x_0(t) &= 0 \\ y_0(t) &= 0 \\ z_0(t) &= h - \tfrac{1}{2}gt^2 \end{aligned}\right\} \text{drop from vertical height } h$$

Write $x(t) = x_0(t) + x_1(t)$, etc.

Then

$$\ddot{x}_1 = 2\Omega_z \dot{y}_1 \approx 0$$
$$\ddot{y}_1 = -2(\Omega_z \dot{x}_1 - \Omega_x(-gt + \dot{z}_1)) \approx -2\Omega_x g t$$
$$\cancel{-g} + \ddot{z}_1 = \cancel{-g} + \Omega_x \dot{y}_1 \approx 0$$

Thus, only need to solve

$$\ddot{y}_1 = -2\Omega_x g t$$
$$\to \dot{y}_1 = -\Omega_x g t^2$$
$$y_1 = -\tfrac{1}{3}\Omega_x g t^3$$

<u>Now:</u> $\Omega_x = -\Omega \sin\alpha$

$T = \sqrt{\frac{2h}{g}}$ (0^{th} order solution to $0 = h - \frac{1}{2}gT^2$

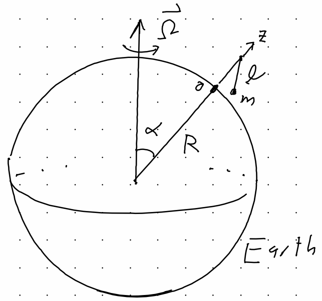
so object hits the ground at

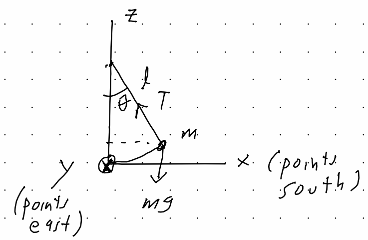
$$y = y_1 = +\tfrac{1}{3}\Omega \sin\alpha \left(\frac{2h}{g}\right)^{3/2}$$

indicates that the object hits the ground to the East of the dropping location.

Sec 39, Prob 3.

Foucault's pendulum, small oscillations





Newton's 2nd law in rotating frame:

$$m\vec a = \vec T + m\vec g - m\vec W - 2m\vec\Omega\times\vec v - m\vec\Omega\times(\vec\Omega\times\vec r)$$

Now $\vec W$ = acceleration of origin O

$$= \vec\Omega\times(\vec\Omega\times\vec R)$$

Thus,

$$m\vec a = \vec T + m\vec g \underbrace{- m\vec\Omega\times(\vec\Omega\times(\vec r+\vec R))}_{\substack{2^{nd}\text{ order in }\Omega \\ [\text{ignore}]}} - 2m\vec\Omega\times\vec v$$

$$\approx \vec T + m\vec g - 2m\vec\Omega\times\vec v$$

Now: $\vec T = T\cos\theta\,\hat z - T\sin\theta\cos\phi\,\hat x - T\sin\theta\sin\phi\,\hat y$

$$\approx T\hat z - T\left(\frac{x}{l}\right)\hat x - T\left(\frac{y}{l}\right)\hat y$$

assuming $\theta << 1$ (small oscillations)

$\vec g = -g\hat z$

$$(\vec\Omega\times\vec v)_x = \Omega_y v_z - \Omega_z v_y$$
$$= \cancelto{0}{\Omega_y}\dot z - \Omega_z\dot y$$
$$\approx -\Omega_z\dot y$$

$$(\vec\Omega\times\vec v)_y = \Omega_z v_x - \Omega_x v_z$$
$$= \Omega_z\dot x - \Omega_x\dot z$$
$$\approx \Omega_z\dot x$$

$$(\vec\Omega\times\vec v)_z = \Omega_x v_y - \cancelto{0}{\Omega_y} v_x$$
$$= \Omega_x\dot y$$

Thus,

$$m\vec a = \vec T + m\vec g - 2m\vec\Omega\times\vec v$$

becomes

$$m\ddot x = -T\left(\frac{x}{l}\right) + 2m\Omega_z\dot y$$
$$m\ddot y = -T\left(\frac{y}{l}\right) - 2m\Omega_z\dot x$$
$$0 = m\ddot z = T - mg - 2m\Omega_x\dot y$$

Now: $\Omega_x\dot y \sim \Omega\frac{D}{P}$

where D = max displacement in xy-plane

P = period $= 2\pi\sqrt{\frac{l}{g}} = \frac{2\pi}{\omega}$

Now $\dfrac{\Omega D}{P} = \dfrac{\Omega D}{2\pi}\,\omega$

$<< \dfrac{l\,\omega^2}{2\pi}$ ← since $\Omega << \omega$, $D << l$ (small oscillation)

$= \dfrac{l}{2\pi}\,\dfrac{g}{l}$

$= \dfrac{g}{2\pi}$

Thus, $m\Omega_x \dot{y} << mg$

so we can ignore these terms relative to mg

$\rightarrow \; 0 \approx T - mg \;\rightarrow\; \boxed{T \approx mg}$

$m\ddot{x} \approx -mg\dfrac{x}{l} + 2m\Omega_z \dot{y}$

$$\boxed{\ddot{x} \approx -\omega^2 x + 2\Omega_z \dot{y}}$$

$m\ddot{y} \approx -mg\dfrac{y}{l} - 2m\Omega_z \dot{x}$

$$\boxed{\ddot{y} \approx -\omega^2 x - 2\Omega_z \dot{x}}$$

just need to solve these two equation

Standard "trick":

Define $\zeta = x + iy$ (complex valued)

then $\dot{\zeta} = \dot{x} + i\dot{y}$

$\ddot{\zeta} = \ddot{x} + i\ddot{y}$

so $\ddot{\zeta} = -\omega^2\zeta + 2\Omega_z(\dot{y} - i\dot{x})$

$= -\omega^2\zeta - 2i\Omega_z(\dot{x} + i\dot{y})$

$= -\omega^2\zeta - 2i\Omega_z\dot{\zeta}$

thus,

$$\ddot{\zeta} + 2i\Omega_z\dot{\zeta} + \omega^2\zeta = 0$$

Trial solution:

$$\zeta(t) = e^{i\lambda t}$$

$\rightarrow \; -\lambda^2 - 2\lambda\Omega_z + \omega^2 = 0$

$\lambda^2 + 2\lambda\Omega_z - \omega^2 = 0$

solve quadratic:

$$\lambda_\pm = \frac{-2\Omega_z \pm \sqrt{4\Omega_z^2 + 4\omega^2}}{2}$$

$\approx -\Omega_z \pm \omega$ (since $\Omega << \omega$)

General solution:

$$\zeta(t) = Ae^{-i(\Omega_z - \omega)t} + Be^{-i(\Omega_z + \omega)t}$$

$$= e^{-i\Omega_z t}\left(Ae^{i\omega t} + Be^{-i\omega t}\right)$$

$$= e^{-i\Omega_z t}\,\zeta_0(t)$$

where $\zeta_0(t)$ = general solution of small oscillation problem in an <u>inertial</u> frame.

The factor $e^{-i\Omega_z t}$ causes the plane of oscillation of the pendulum to precess with angular frequency

$$\Omega_z = \Omega \cos\alpha$$

angle of location on Earth
($\alpha = 0 \leftrightarrow$ N Pole
$\alpha = \pi/2 \leftrightarrow$ Equator)

Precession period:

$$P_{precession} = \frac{2\pi}{\Omega_z} = \frac{2\pi}{\Omega\cos\alpha}$$

$$\Omega = 1\,\text{revolution}/24\,\text{hrs} = \frac{2\pi\,\text{rad}}{24\,\text{hrs}}$$

so $P_{precession} = \dfrac{24\,\text{hrs}}{\cos\alpha}$

$$= \begin{cases} 24\,\text{hrs} & \text{at N Pole} \\ \infty & \text{at equator (so no precession)} \end{cases}$$