sec 16, prob 2

From (16.7) we have

$$p(\theta_0)\,d\theta_0 = \tfrac{1}{2}\sin\theta_0\,d\theta_0$$
$$= -\tfrac{1}{2}\,d(\cos\theta_0)$$

where θ_0 is the angle of one of the emitted particles in the com frame.

⇒ We would like to find $p(\theta)d\theta$, where θ is the angle of one of the emitted particle in the lab frame.

⇒ Since $p(\theta)d\theta = p(\theta_0)d\theta_0$

we just need to find θ_0 as a function of θ.

This is given by (16.6) which we first derive.

Proof: Given $\tan\theta = \dfrac{v_0\sin\theta_0}{v_0\cos\theta_0 + V}$

we have

$$\tan\theta\,(v_0\cos\theta_0 + V) = v_0\sqrt{1-\cos^2\theta_0}$$

square both sides

$$\tan^2\theta\,(v_0^2\cos^2\theta_0 + V^2 + 2v_0V\cos\theta_0) = v_0^2(1-\cos^2\theta_0)$$

$$\underbrace{(1+\tan^2\theta)}_{\sec^2\theta}\,v_0^2\cos^2\theta_0 + 2v_0V\tan^2\theta\cos\theta_0 + (V^2\tan^2\theta - v_0^2) = 0$$

quadratic equation for $\cos\theta_0$

$$\to \cos\theta_0 = \frac{-2v_0V\tan^2\theta \pm \sqrt{4v_0^2V^2\tan^4\theta - 4v_0^2\sec^2\theta\,(V^2\tan^2\theta - v_0^2)}}{2v_0^2\sec^2\theta}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \frac{1}{\sec^2\theta}\sqrt{\left(\frac{V}{v_0}\right)^2\tan^4\theta - \sec^2\theta\left(\left(\frac{V}{v_0}\right)^2\tan^2\theta - 1\right)}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \frac{1}{\sec\theta}\sqrt{\left(\frac{V}{v_0}\right)^2\left(\frac{\sin^4\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}\right) + 1}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{\left(\frac{V}{v_0}\right)^2\frac{\sin^2\theta(\sin^2\theta - 1)}{\cos^2\theta} + 1}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{1 - \left(\frac{V}{v_0}\right)^2\sin^2\theta}$$

(for $v_0 > V$ take $+\sqrt{\ }$, for $v_0 < V$ take both signs.)

Now differentiate both sides:

$$d(\cos\theta_0) = -\frac{V}{v_0}\,2\sin\theta\cos\theta\,d\theta \mp \sin\theta\,d\theta\sqrt{\ }$$
$$\pm\cos\theta\left(\frac{1}{\cancel{2}}\right)\frac{1}{\sqrt{\ }}(-\cancel{2})\left(\frac{V}{v_0}\right)^2\sin\theta\cos\theta\,d\theta$$

$$= \sin\theta\,d\theta\left[-2\frac{V}{v_0}\cos\theta \mp \sqrt{\ } \mp \left(\frac{V}{v_0}\right)^2\cos^2\theta\,\frac{1}{\sqrt{\ }}\right]$$

Now:

$$\sqrt{} + \left(\frac{V}{v_0}\right)^2 \cos^2\theta \; \frac{1}{\sqrt{}}$$

$$= \frac{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta + \left(\frac{V}{v_0}\right)^2 \cos^2\theta}{\sqrt{}}$$

$$= \frac{1 + \left(\frac{V}{v_0}\right)^2 \left(\cos^2\theta - \sin^2\theta\right)}{\sqrt{}}$$

$$= \frac{1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta}{\sqrt{}}$$

Thus,

$$d(\cos\theta) = \underbrace{-\sin\theta\, d\theta}_{d(\cos\theta)} \left[2\frac{V}{v_0}\cos\theta \pm \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

So:

$$p(\theta)\, d\theta = \frac{1}{2}\sin\theta\, d\theta \left[2\frac{V}{v_0}\cos\theta \pm \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

Now: for $v_0 > V$, take $+$ sign $(\theta \in [0, \pi])$

for $v_0 < V$, as θ_0 increases from 0 to π

θ increases from $0 \mapsto \theta_{max}$

θ decreases from $\theta_{max} \mapsto 0$

Thus, for $v_0 < V$ need to take the difference of the $+$ and $-$ expressions:

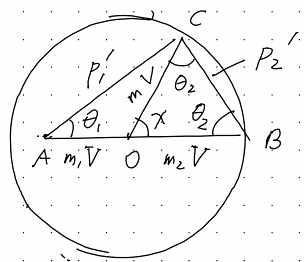
$$p(\theta)\, d\theta = \frac{1}{2}\sin\theta\, d\theta \left[2\frac{V}{v_0}\cos\theta + \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

$$- \frac{1}{2}\sin\theta\, d\theta \left[2\frac{V}{v_0}\cos\theta - \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

$$= \frac{\sin\theta\, d\theta \left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}}$$

which is valid for $0 \le \theta \le \theta_{max} = \sin^{-1}\left(\frac{v_0}{V}\right)$

Sec 17, Prob 1:



$\chi + 2\theta_2 = \pi$

$\chi = \pi - 2\theta_2$

From the above figure:

$$(m_2 v_2')^2 = 2(mV)^2 - 2(mV)^2\cos\chi$$

$$= 2m^2V^2\left(1-\cos(\pi - 2\theta_2)\right)$$

$$= 2m^2V^2\left[1 - \left(\underbrace{\cos(\pi)}_{=-1}\cos(2\theta_2) + \cancel{\sin(\pi)}^{0}\sin(2\theta_2)\right)\right]$$

$$= 2m^2V^2\left(1+\cos(2\theta_2)\right)$$

$$\rightarrow \quad v_2' = \sqrt{2}\left(\frac{m}{m_2}\right)V\sqrt{1+\left(\cos^2\theta_2 - \sin^2\theta_2\right)}$$

$$= \sqrt{2}\left(\frac{m_1}{m_1+m_2}\right)V\sqrt{2\cos^2\theta_2}$$

$$= 2\left(\frac{m_1}{m_1+m_2}\right)V\cos\theta_2$$

Thus, $\left(\frac{v_2'}{V}\right) = \left(\frac{2m_1}{m_1+m_2}\right)\cos\theta_2$

Also,

$$(mV)^2 = (m_1\bar{V})^2 + (m_1v_1')^2 - 2m_1^2v_1'\bar{V}\cos\theta_1$$

$$\rightarrow \quad (m_1v_1')^2 - 2m_1\bar{V}\cos\theta_1(m_1v_1') + m_1^2\bar{V}^2 - m^2V^2 = 0$$

Now: $\bar{V} = \frac{m_1v_1 + m_2\cancel{v_2}^{0}}{m_1+m_2} = \frac{m_1V}{m_1+m_2}$

Thus,

$$(m_1v_1')^2 - 2\left(\frac{m_1^2V}{m_1+m_2}\right)\cos\theta_1\, m_1v_1' + \frac{m_1^2m_1^2V^2}{(m_1+m_2)^2} - \frac{m_1^2m_2^2}{(m_1+m_2)^2}V^2 = 0$$

$$\rightarrow \quad (v_1')^2 - 2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1\, v_1' + \frac{m_1^2-m_2^2}{(m_1+m_2)^2}V^2 = 0$$

$$(v_1')^2 - 2\left(\frac{m_1V}{m_1+m_2}\right)\cos\theta_1\, v_1' + \left(\frac{m_1-m_2}{m_1+m_2}\right)V^2 = 0$$

$$\left(\frac{v_1'}{V}\right)^2 - 2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1\left(\frac{v_1'}{V}\right) + \left(\frac{m_1-m_2}{m_1+m_2}\right) = 0$$

Quadratic equation

$$\frac{v_1'}{v} = \frac{2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \sqrt{4\left(\frac{m_1}{m_1+m_2}\right)^2\cos^2\theta_1 - 4\left(\frac{m_1-m_2}{m_1+m_2}\right)}}{2}$$

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_1^2\cos^2\theta_1 - (m_1-m_2)(m_1+m_2)}$$

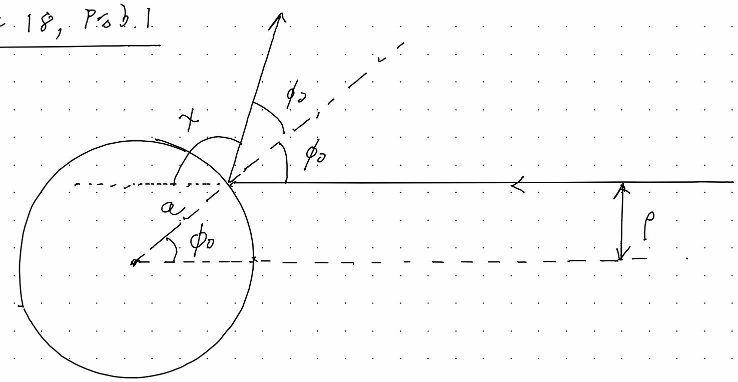
$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_1^2(\cos^2\theta_1-1) + m_2^2}$$

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_2^2 - m_1^2\sin^2\theta_1}$$

The + sign holds for $m_1 < m_2$

+/- signs hold for $m_1 > m_2$

Sec. 18, Prob. 1



$$\chi + 2\phi_0 = \pi \quad \rightarrow \quad \phi_0 = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\rho = a\sin\phi_0$$

$$= a\sin\left(\frac{\pi}{2} - \frac{\chi}{2}\right)$$

$$= a\left(\underset{1}{\sin\left(\frac{\pi}{2}\right)}\cos\left(\frac{\chi}{2}\right) - \cancel{\cos\left(\frac{\pi}{2}\right)}^{\,0}\sin\left(\frac{\chi}{2}\right)\right)$$

$$= a\cos\left(\frac{\chi}{2}\right)$$

$$d\sigma = 2\pi\rho\, d\rho$$

$$= 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi$$

$$= \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$

where $d\Omega$ = solid angle

$$= \underset{\substack{\uparrow \\ \text{integral over } d\phi}}{2\pi}\sin\chi\, d\chi$$

$$\rho = a \cos\left(\frac{\chi}{2}\right)$$

$$d\rho = -a \cdot \frac{1}{2} \sin\left(\frac{\chi}{2}\right) d\chi$$

$$= -\frac{a}{2} \sin\left(\frac{\chi}{2}\right) d\chi$$

thus,

$$d\sigma = \frac{\rho(\chi)}{\sin\chi} \left|\frac{d\rho}{d\chi}\right| d\Omega$$

$$= \frac{a \cos(\chi/2)}{\sin\chi} \cdot \frac{a}{2} \sin\left(\frac{\chi}{2}\right) d\Omega$$

$$= \frac{a^2}{2} \frac{\sin(\chi/2)\cos(\chi/2)}{\sin\chi} d\Omega$$

$$= \boxed{\frac{a^2}{4} d\Omega} \qquad \left(\text{since } \sin\chi = 2\sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)\right)$$

total cross section

$$\sigma = \int d\sigma = \frac{a^2}{4}\int d\Omega = \frac{a^2}{4} \cdot 4\pi = \boxed{\pi a^2}$$

Now calculate differential cross section in lab frame for both m_1 and m_2

Use the result that

$$d\sigma_1 = \frac{\rho(\theta_1)}{\sin\theta_1}\left|\frac{d\rho}{d\theta_1}\right| d\Omega_1 = \rho\left|\frac{d\rho}{d(\cos\theta_1)}\right| d\Omega_1$$

Compare to:

$$d\sigma = \rho\left|\frac{d\rho}{d(\cos\chi)}\right| d\Omega$$

$$\rightarrow \frac{d\sigma_1}{d\Omega_1} = \rho\left|\frac{d\rho}{d(\cos\theta_1)}\right|$$

$$= \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| \frac{d\sigma}{d\Omega}$$

So we need to evaluate:

$$\frac{d(\cos\chi)}{d(\cos\theta_1)} \quad \text{and} \quad \frac{d(\cos\chi)}{d(\cos\theta_2)}$$

start with θ_2: (17.4)

$$\theta_2 = \frac{1}{2}(\pi - \chi) \rightarrow \boxed{\chi = \pi - 2\theta_2}$$

$$\rightarrow \cos\chi = \cos(\pi - 2\theta_2)$$

$$= \cos\pi\cos(2\theta_2) + \cancel{\sin\pi}^{\,0} \sin(2\theta_2)$$

$$= -\cos(2\theta_2)$$

$$= -(\cos^2\theta_2 - \sin^2\theta)$$

$$= -(2\cos^2\theta_2 - 1)$$

$$= -2\cos^2\theta_2 + 2$$

Thus, $\boxed{d(\cos\chi) = -4\cos\theta_2 \, d(\cos\theta_2)}$

Thus,

$$\frac{d\sigma_2}{d\Omega_2} = \frac{d\sigma}{d\Omega}\left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right|$$

$$= \frac{1}{4}a^2 \cdot |4\cos\theta_2|$$

$$= a^2|\cos\theta_2|$$

so $\boxed{d\sigma_2 = a^2|\cos\theta_2|\,d\Omega_2}$

Now consider θ_1:

From (17.4):

$$\tan\theta_1 = \frac{m_2\sin\chi}{m_1 + m_2\cos\chi}$$

Compare with

$$\tan\theta = \frac{v_0\sin\theta_0}{V + v_0\cos\theta_0} \qquad (16.5)$$

make identifications: $\theta\to\theta_1$, $v_0\to m_2$, $V\to m_1$

then we can write down from (16.6).

$$\cos\chi = -\frac{m_1}{m_2}\sin^2\theta_1 \pm \cos\theta_1\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}$$

[see also sec 16, Prob. 2 where we derived this for θ and θ_0]

We also worked out the derivative:

$$d(\cos\theta_0) = d(\cos\theta)\left[2\frac{V}{v_0}\cos\theta \pm \frac{1+\left(\frac{V}{v_0}\right)^2\cos 2\theta}{\sqrt{1-\left(\frac{V}{v_0}\right)^2\sin^2\theta}}\right]$$

So we can similarly write dow

$$d(\cos\chi) = d(\cos\theta_1)\left[2\frac{m_1}{m_2}\cos\theta_1 \pm \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\right]$$

For $m_1 < m_2$: take $+$ sign

For $m_1 > m_2$: As χ increases from 0 to π, θ_1 increases from 0 to θ_{max}, then θ_1 decreases from θ_{max} to 0. In that case

$$d(\cos\chi) = d(\cos\theta_1)[\cancel{\oplus} + \ominus] - d(\cos\theta_1)[\cancel{\oplus} - \ominus]$$

$$= 2\,d(\cos\theta_1)\,\frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

Use: (for both)

$$d\sigma_1 = \left(\frac{d\sigma}{d\Omega}\right)\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|d\Omega_1$$

$$= \frac{1}{4}a^2\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|d\Omega_1$$

Thus,

For $m_1 < m_2$:

$$d\sigma_1 = \frac{1}{4}a^2\left[2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\right]d\Omega_1$$

For $m_1 > m_2$:

$$d\sigma_1 = \frac{1}{4}a^2\cdot 2\;\frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\,d\Omega_1$$

$$= \frac{a^2}{2}\;\frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\,d\Omega_1$$

Sec. 18, Prob 2:

Hard sphere scattering again.
Calculate $d\sigma$ in terms of $d\epsilon$ where
ϵ = energy lost by scattered particle.

Now: ϵ = energy lost by scattered particle
= energy gained by m_2
$= \frac{1}{2}m_2(v_2')^2$

From Fig. 16.), we have (law of cosines):

$$(m_2v_2')^2 = (mv)^2 + (mv)^2 - 2(mv)^2\cos\chi$$
$$= 2(mv)^2[1-\cos\chi]$$
$$= 2(mv)^2\,2\sin^2\left(\frac{\chi}{2}\right)$$

so $m_2v_2' = 2\,mv\sin\left(\frac{\chi}{2}\right)$

$$\rightarrow\quad v_2' = \left(\frac{m}{m_2}\right)v\sin\left(\frac{\chi}{2}\right)$$
$$= \left(\frac{m_1}{m_1+m_2}\right)v\sin\left(\frac{\chi}{2}\right) \qquad (17.5)$$

NOTE:
$$d\sigma = \frac{1}{4}a^2\,d\Omega$$
$$= \frac{1}{4}a^2\,2\pi\sin\chi\,d\chi$$
$$= \frac{\pi a^2}{2}\,|d(\cos\chi)|$$

So we would like to related $d\epsilon$ and $d(\cos\chi)$.

Now: $\epsilon = \frac{1}{2} m_2 (v_2')^2$

$$= \frac{1}{2} m_2 \frac{4 m_1^2 v^2}{(m_1+m_2)^2} \sin^2\left(\frac{\chi}{2}\right)$$

$$= \frac{2 m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 \sin^2\left(\frac{\chi}{2}\right) \quad (\text{since } v = v_\infty)$$

$$\equiv \epsilon_{max} \sin^2\left(\frac{\chi}{2}\right)$$

where $\epsilon_{max} \equiv \frac{2 m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2$

$$= 4\left(\frac{m_1}{m_1+m_2}\right) \frac{m v_\infty^2}{2}$$

$$= 4\left(\frac{m_1}{m_1+m_2}\right) E$$

Thus, $d\epsilon = \epsilon_{max}\, 2 \sin\left(\frac{\chi}{2}\right) \cos\left(\frac{\chi}{2}\right) \frac{d\chi}{2}$

$$= \frac{1}{2} \epsilon_{max} \sin\chi \, d\chi$$

$$= \frac{1}{2} \epsilon_{max} |d(\cos\chi)|$$

So $d\sigma = \frac{\pi a^2}{2} |d(\cos\chi)|$

$$= \frac{\pi a^2}{2} \frac{2}{\epsilon_{max}} d\epsilon = \boxed{\frac{\pi a^2}{\epsilon_{max}} d\epsilon}$$

which is a uniform distribution wrt ϵ.

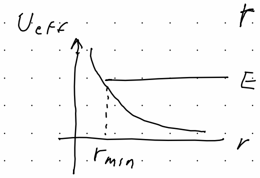
Sec 18, Prob. 4

Effective cross section to "fall" to center of $U(r) = -\alpha/r^2 \quad (\alpha > 0)$

$$U_{eff}(r) = U(r) + \frac{M^2}{2mr^2}$$

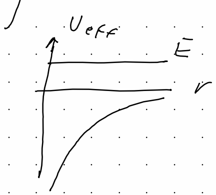
$$= -\frac{\alpha}{r^2} + \frac{M^2}{2mr^2}$$

$$= \frac{1}{r^2}\left(\frac{M^2}{2m} - \alpha\right)$$



$\frac{M^2}{2m} - \alpha > 0$

(don't fall to center since $r_{min} > 0$)



$\frac{M^2}{2m} - \alpha < 0$

Fall to center $(r=0)$

For a given $E = \frac{1}{2} m v_\infty^2 > 0$ need

$$\frac{M^2}{2m} - \alpha < 0$$

$$\frac{M^2}{2m} < \alpha$$

$$\rightarrow \quad M_{max} = \sqrt{2m\alpha}$$

Cross section: $\sigma = \pi \rho_{max}^2$, $M = \rho m v_\infty$

Thus,

$$\sigma = \pi \rho_{max}^2 \quad , \qquad \rho_{max} = \frac{M_{max}}{m\, v_\infty}$$

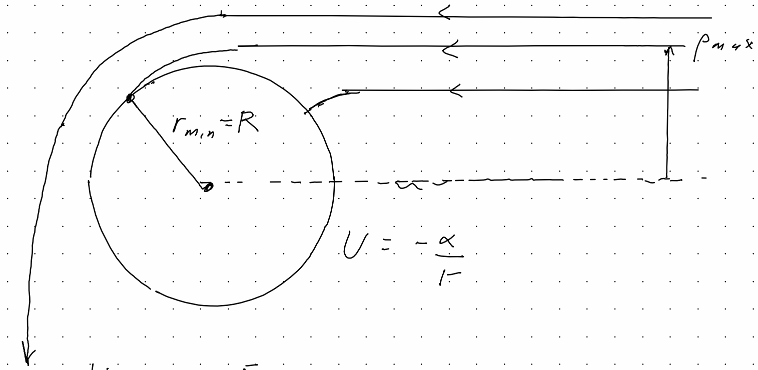
$$= \pi \frac{M_{max}^2}{m^2 v_\infty^2}$$

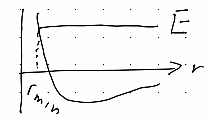
$$= \pi \frac{2 m \alpha}{m^2 v_\infty^2}$$

$$= \frac{\pi \alpha}{\frac{1}{2} m v_\infty^2}$$

$$= \boxed{\frac{\pi \alpha}{E}}$$

<u>Sec. 18, Prob 6</u>





Turning point at $r = R$

$$0 = E - U_{eff}(R)$$

$$= E - U(R) - \frac{M_{max}^2}{2 m R^2}$$

$$= E + \frac{\alpha}{R} - \frac{M_{max}^2}{2 m R^2}$$

$$\rightarrow \quad \frac{M_{max}^2}{2 m R^2} = E + \frac{\alpha}{R}$$

$$M_{max} = \rho_{max}\, m v_\infty \quad , \qquad E = \frac{1}{2} m v_\infty^2$$

Thus,

$$\sigma = \pi \rho_{max}^2$$

$$= \pi \frac{M_{max}^2}{m^2 v_\infty^2}$$

$$= \pi \frac{1}{m^2 v_\infty^2} \quad 2mR^2\left(E + \frac{\alpha}{R}\right)$$

$$= \pi R^2 \underbrace{\left(\frac{2}{m v_\infty^2}\right)}_{\frac{1}{E}}\left(E + \frac{\alpha}{R}\right)$$

$$= \boxed{\pi R^2\left(1 + \frac{\alpha}{ER}\right)}$$

where $E = \frac{1}{2} m v_\infty^2 = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right) v_\infty^2$

and $\alpha \equiv G m_1 m_2$