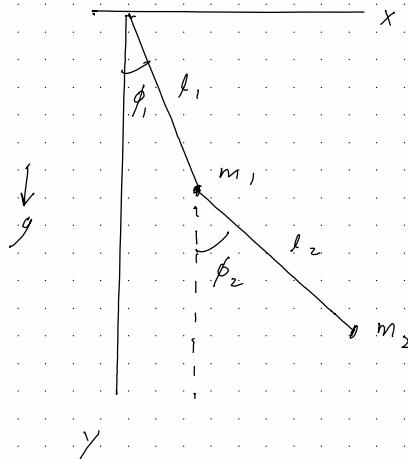


Soc 5, prob 1



$$x_1 = l_1 \sin \phi_1$$

$$y_1 = l_1 \cos \phi_1$$

$$x_2 = x_1 + l_2 \sin \phi_2 = l_1 \sin \phi_1 + l_2 \sin \phi_2$$

$$y_2 = y_1 + l_2 \cos \phi_2 = l_1 \cos \phi_1 + l_2 \cos \phi_2$$

$$U = -m_1 g y_1 - m_2 g y_2$$

$$= -m_1 g l_1 \cos \phi_1 - m_2 g (l_1 \cos \phi_1 + l_2 \cos \phi_2)$$

$$= -(m_1 + m_2) g l_1 \cos \phi_1 - m_2 g l_2 \cos \phi_2$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\dot{x}_1 = l_1 \cos \phi_1 \dot{\phi}_1 \quad \dot{y}_1 = -l_1 \sin \phi_1 \dot{\phi}_1$$

$$\dot{x}_1^2 = l_1^2 \cos^2 \phi_1 \dot{\phi}_1^2 \quad \dot{y}_1^2 = l_1^2 \sin^2 \phi_1 \dot{\phi}_1^2$$

$$\text{Thus, } \dot{x}_1^2 + \dot{y}_1^2 = l_1^2 (\sin^2 \phi_1 + \cos^2 \phi_1) \dot{\phi}_1^2$$

$$= l_1^2 \dot{\phi}_1^2$$

$$\dot{x}_2 = l_1 \cos \phi_1 \dot{\phi}_1 + l_2 \cos \phi_2 \dot{\phi}_2$$

$$\rightarrow \dot{x}_2^2 = l_1^2 \cos^2 \phi_1 \dot{\phi}_1^2 + l_2^2 \cos^2 \phi_2 \dot{\phi}_2^2 + 2l_1 l_2 \cos \phi_1 \cos \phi_2 \dot{\phi}_1 \dot{\phi}_2$$

$$\dot{y}_2 = -l_1 \sin \phi_1 \dot{\phi}_1 - l_2 \sin \phi_2 \dot{\phi}_2$$

$$\rightarrow \dot{y}_2^2 = l_1^2 \sin^2 \phi_1 \dot{\phi}_1^2 + l_2^2 \sin^2 \phi_2 \dot{\phi}_2^2 + 2l_1 l_2 \sin \phi_1 \sin \phi_2 \dot{\phi}_1 \dot{\phi}_2$$

From,

$$\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 (\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

$$= l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

$$\text{so, } T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2$$

$$+ m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

$$= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

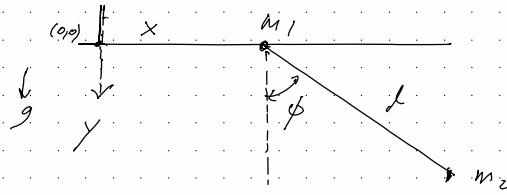
$$U = -(m_1 + m_2) g l_1 \cos \phi_1 - m_2 g l_2 \cos \phi_2$$

$$L = T - U$$

$$= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$

$$+ (m_1 + m_2) g l_1 \cos \phi_1 + m_2 g l_2 \cos \phi_2$$

### Sec 5 Prob. 2



Generalised coords:  $x, \phi$

$$(x_1, y_1) = (x, 0)$$

$$(x_2, y_2) = (x + l \sin \phi, l \cos \phi)$$

$$U = -m_1 g y_1 - m_2 g y_2$$

$$= -m_2 g l \cos \phi$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\text{Now: } \dot{x}_1^2 + \dot{y}_1^2 = \dot{x}^2$$

$$\begin{aligned}\dot{x}_2^2 + \dot{y}_2^2 &= (\dot{x} + l \cos \phi \dot{\phi})^2 + (-l \sin \phi \dot{\phi})^2 \\ &= \dot{x}^2 + l^2 \cos^2 \phi \dot{\phi}^2 + 2 l \cos \phi \dot{x} \dot{\phi}\end{aligned}$$

$$+ l^2 \sin^2 \phi \dot{\phi}$$

$$= \dot{x}^2 + l^2 \dot{\phi}^2 + 2 l \cos \phi \dot{x} \dot{\phi}$$

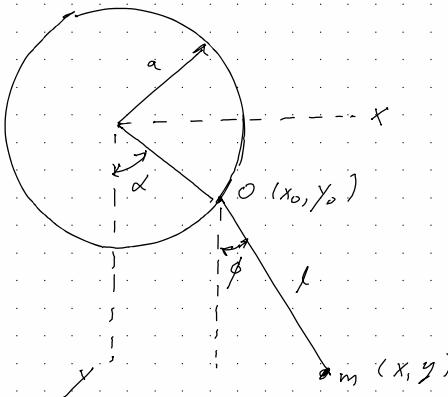
$$\rightarrow T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}^2 + l^2 \dot{\phi}^2 + 2 l \cos \phi \dot{x} \dot{\phi})$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \cos \phi \dot{x} \dot{\phi}$$

$$\begin{aligned}L &= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \cos \phi \dot{x} \dot{\phi} \\ &\quad + m_2 g l \cos \phi\end{aligned}$$

### Sec 5, Prob 3

(a)



point of support O moves along circle.

$$x_0 = a \sin \alpha \quad , \quad y_0 = a \cos \alpha$$

where  $\alpha = \omega t$ ,  $\omega = \text{const}$

Pendulum bob:

$$(x, y): \quad x = x_0 + l \sin \phi$$

$$y = y_0 + l \cos \phi$$

$$U = -m g y = -m g y_0 - m g l \cos \phi$$

Specified function of time.

[can ignore in L]

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{x}_0 + l \cos \phi \dot{\phi}$$

$$\dot{x}^2 = \dot{x}_0^2 + l^2 \cos^2 \phi \dot{\phi}^2 + 2 l \cos \phi \dot{x}_0 \dot{\phi}$$

$$\dot{y} = \dot{x}_0 - l \sin \phi$$

$$\dot{y}^2 = \dot{x}_0^2 + l^2 \sin^2 \phi \dot{\phi}^2 - 2l \sin \phi \dot{x}_0 \dot{\phi}$$

thus,

$$T = \frac{1}{2} m(\dot{x}_0^2 + \dot{y}^2)$$

$$= \frac{1}{2} m (\dot{x}_0^2 + l^2 \cos^2 \phi \dot{\phi}^2 + 2l \cos \phi \dot{x}_0 \dot{\phi})$$

$$+ \dot{y}^2 + l^2 \sin^2 \phi \dot{\phi}^2 - 2l \sin \phi \dot{x}_0 \dot{\phi})$$

$$= \frac{1}{2} m(\dot{x}_0^2 + \dot{y}_0^2) + \frac{1}{2} m l^2 \dot{\phi}^2 + m l \dot{\phi} (\dot{x}_0 \cos \phi - \dot{y}_0 \sin \phi)$$

$$\text{NOTE: } \dot{x}_0^2 + \dot{y}_0^2 = a^2 \dot{x}^2 = a^2 r^2$$

since this is a specified function of time, we can ignore it in the Lagrangian;

$$\text{thus, } L = \frac{1}{2} m l^2 \dot{\phi}^2 + m l \dot{\phi} (\dot{x}_0 \cos \phi - \dot{y}_0 \sin \phi) + m g l \cos \phi$$

We can rewrite the second term:

$$x_0 = a \sin \alpha \rightarrow \dot{x}_0 = a \cos \alpha \dot{\alpha} \quad (\alpha = r t)$$

$$y_0 = a \cos \alpha \rightarrow \dot{y}_0 = -a \sin \alpha \dot{\alpha}$$

thus,

$$\begin{aligned} m l \dot{\phi} (\dot{x}_0 \cos \phi - \dot{y}_0 \sin \phi) &= m l \dot{\phi} a \dot{\alpha} (\cos \alpha \cos \phi + \sin \alpha \sin \phi) \\ &= m l \dot{\phi} a \dot{\alpha} \cos(\phi - \alpha) \\ &= m l \dot{\phi} a r \cos(\phi - r t) \end{aligned}$$

$$\text{Now: } \frac{d}{dt} [m l a r \sin(\phi - r t)]$$

$$= m l a r \cos(\phi - r t) (\dot{\phi} - r)$$

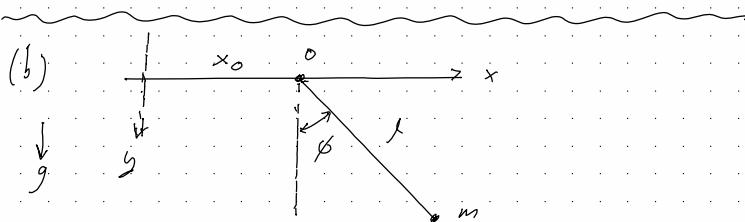
$$= m l a \dot{\phi} r \cos(\phi - r t) - m l a r^2 \cos(\phi - r t)$$

thus,

$$m l a \dot{\phi} r \cos(\phi - r t) = \frac{d}{dt} [m l a r \sin(\phi - r t)] + m l a r^2 \cos(\phi - r t)$$

(and we can ignore the total time derivative in the Lagrangian)

$$\rightarrow L = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \cos \phi + m l a r^2 \cos(\phi - r t)$$



point O moving according to  $x_0 = a \cos r t$

$$x = x_0 + l \sin \phi$$

$$y = l \cos \phi$$

$$U = -mgy = -mg l \cos \phi$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{x}_0 + l \cos \phi \dot{\phi}, \quad x_0 = a \cos \gamma t$$

$$= -a \sin(\gamma t) \gamma + l \cos \phi \dot{\phi}$$

$$\rightarrow \dot{x}^2 = a^2 \gamma^2 \sin^2(\gamma t) + l^2 \cos^2 \phi$$

$$- 2al\gamma \phi \sin(\gamma t) \cos \phi$$

$$\dot{y} = -l \sin \phi \dot{\phi}$$

$$\rightarrow \dot{y}^2 = l^2 \sin^2 \phi \dot{\phi}^2$$

Thus,  $T = \frac{1}{2} m (a^2 \gamma^2 \sin^2(\gamma t) + l^2 \cos^2 \phi \dot{\phi}^2)$

$$- 2al\gamma \phi \sin(\gamma t) \cos \phi + l^2 \sin^2 \phi \dot{\phi}^2)$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + \frac{1}{2} m a^2 \gamma^2 \sin^2(\gamma t)$$

specified function of time  
(ignore)

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 - mal\gamma \phi \sin(\gamma t) \cos \phi + mgl \cos \phi$$

2<sup>nd</sup> term:

$$- \frac{d}{dt} [ mal\gamma \sin(\gamma t) \sin \phi ]$$

$$= -mal\gamma^2 \cos(\gamma t) \sin \phi - mal\gamma \phi \sin(\gamma t) \cos \phi$$

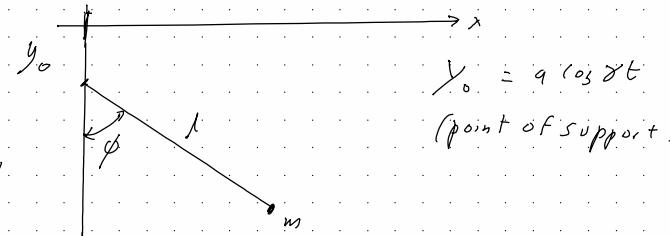
$$- mal\gamma \phi \sin(\gamma t) \cos \phi = - \frac{d}{dt} [ ] + mal\gamma^2 \cos(\gamma t) \sin \phi$$

*is no se*

Thus, ignoring total time derivatives,

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl \cos \phi + mal\gamma^2 \cos(\gamma t) \sin \phi$$

(c)



$$y_0 = a \cos \gamma t$$

(point of support)

$$x = l \sin \phi$$

$$y = y_0 + l \cos \phi$$

$$= a \cos \gamma t + l \cos \phi$$

$$U = -mgy$$

$$= -mg(a \cos \gamma t + l \cos \phi)$$

specified function of time [can ignore]

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = l \cos \phi \dot{\phi}$$

$$\dot{x}^2 = l^2 \cos^2 \phi \dot{\phi}^2$$

$$\dot{y} = -a \gamma \sin(\gamma t) - l \sin \phi \dot{\phi}$$

$$\dot{y}^2 = a^2 \gamma^2 \sin^2(\gamma t) + l^2 \sin^2 \phi \dot{\phi}^2 + 2al\gamma \sin(\gamma t) \sin \phi \dot{\phi}$$

specified function of time [can ignore]

thus, ignoring this function of time

$$T = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \dot{\phi} \sin(\gamma t) \sin \phi$$

$$\rightarrow L = T - U$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \dot{\phi} \sin(\gamma t) \sin \phi + m g l \cos \phi$$

Rewrite 2<sup>nd</sup> term:

$$-\frac{d}{dt} [m a l \dot{\phi} \sin(\gamma t) \cos \phi] = -m a l \gamma^2 \cos(\gamma t) \cos \phi$$

$$+ m a l \dot{\gamma} \sin(\gamma t) \sin \phi \dot{\phi}$$

$$\text{so } m a l \dot{\phi} \sin(\gamma t) \sin \phi = -\frac{d}{dt} [ ] + m a l \gamma^2 \cos(\gamma t) \cos \phi$$

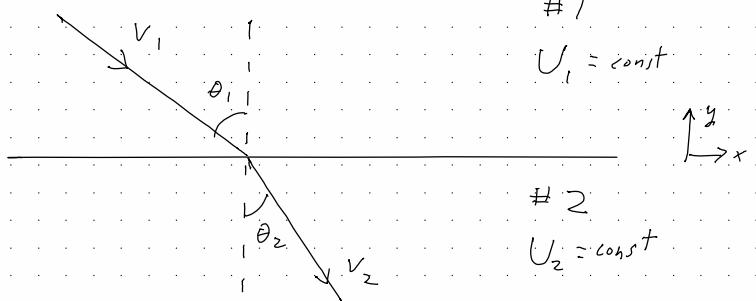
thus, ignoring total time derivative

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \cos \phi + m a l \gamma^2 \cos(\gamma t) \cos \phi$$

Sec 7, Prob 1

#1

$$U_1 = \text{const}$$



#2

$$U_2 = \text{const} +$$

- Energy conserved, since no time dependence.
- Also momentum in x-direction ( $\parallel$  to interface) is conserved, since no x-dependence of the potential

$$U(x, y) = \begin{cases} U_1 & y \geq 0 \\ U_2 & y < 0 \end{cases}$$

$V_1$  : given

$$E = \frac{1}{2} m V_1^2 + U_1 = \frac{1}{2} m V_2^2 + U_2$$

$$\rightarrow \frac{1}{2} m V_2^2 = \frac{1}{2} m V_1^2 + (U_1 - U_2)$$

$$V_2^2 = V_1^2 + \frac{2(U_1 - U_2)}{m}$$

$$\text{so, } V_2 = V_1 \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m V_1^2}}$$

The angles  $\theta_1, \theta_2$  are related by

$$p_{1x} = p_{2x}$$

$$\mu v_1 \sin \theta_1 = \mu v_2 \sin \theta_2$$

$$\text{thus, } \frac{\sin \theta_1}{\sin \theta_2} = \frac{v_2}{v_1} = \sqrt{1 + \frac{(v_1 - v_2)^2}{2m v_1^2}}$$

### Sec 8; Prob 1

Transformations of action  $S = \int L dt$

$K, K'$ : two inertial frames

$K'$  moves with velocity  $\vec{V}$  wrt  $K$

Assume that  $K, K'$  coincide at  $t=0$  so  
 $\vec{r}_a = \vec{r}'_a$  wrt these two frames

$$\text{Now: } \vec{v}_a = \vec{V} + \vec{v}'_a$$

$$L = T - U$$

$$= \sum_a \frac{1}{2} m_a |\vec{v}_a|^2 - U(\vec{r}_1, \vec{r}_2, \dots, t)$$

$$|\vec{v}_a|^2 = |\vec{V} + \vec{v}'_a|^2$$

$$= |\vec{V}|^2 + |\vec{v}'_a|^2 + 2 \vec{V} \cdot \vec{v}'_a$$

so

$$L = \sum_a \frac{1}{2} m_a (|\vec{V}|^2 + |\vec{v}'_a|^2 + 2 \vec{V} \cdot \vec{v}'_a) - U$$

$$= \frac{1}{2} \mu V^2 + T' + \vec{V} \cdot \sum_a \vec{v}'_a - U$$

$$= T' - U + \frac{1}{2} \mu V^2 + \vec{P}' \cdot \vec{V}$$

$$= L' + \frac{1}{2} \mu V^2 + \vec{P}' \cdot \vec{V}$$

where  $\vec{P}'$  = total momentum wrt  $K'$

$$\mu = \sum_a m_a \leftarrow \text{total mass}$$

$$\begin{aligned}
 S &= \int_{t_1}^{t_2} \int \rho dt \\
 &= \int_{t_1}^{t_2} (\bar{L}' + \frac{1}{2}\mu V^2 + \bar{\rho}' - \bar{V}) dt \\
 &= S' + \underbrace{\frac{1}{2}\mu V^2(t_2 - t_1)}_{\text{doesnt change}} + \bar{V} \cdot \int_{t_1}^{t_2} \bar{\rho}' dt
 \end{aligned}$$

EOMs

$$\begin{aligned}
 \bar{V} \cdot \int_{t_1}^{t_2} \bar{\rho}' dt &= \bar{V} \cdot \int_{t_1}^{t_2} \sum_m \bar{v}_a' dt \\
 &= \bar{V} \cdot \sum_a \int_{t_1}^{t_2} \left( \frac{d\bar{r}_a}{dt} \right) dt \\
 &= \bar{V} \cdot \sum_a \bar{r}_a \Big|_{t_1}^{t_2} \\
 &= \bar{V} \cdot \left( \mu \bar{R}(t_2) - \mu \bar{R}(t_1) \right) \\
 &= \mu \bar{V} \cdot \left( \bar{R}(t_2) - \bar{R}(t_1) \right)
 \end{aligned}$$

Differentiation of positions

$$\text{So: } S = S' + \frac{1}{2}\mu V^2(t_2 - t_1) + \mu \bar{V} \cdot (\bar{R}(t_2) - \bar{R}(t_1))$$

Sec 9, Prob 1.

cylindrical coordinates,  $(s, \phi, z)$

$$s^2 = x^2 + y^2$$



$$x = s \cos \phi$$

$$y = s \sin \phi$$

$$z = z$$

$$\vec{M} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{r}$$

$$\text{Now, } M_x = m(y \dot{z} - z \dot{y})$$

$$M_y = m(z \dot{x} - x \dot{z})$$

$$M_z = m(x \dot{y} - y \dot{x})$$

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}$$

$$\dot{z} = z$$

$$\dot{y} = s \sin \phi + s \cos \phi \dot{\phi}$$

$$\dot{x} = s \cos \phi - s \sin \phi \dot{\phi}$$

$$\text{Thus, } M_x = m(s \sin \phi \dot{z} - z(s \sin \phi + s \cos \phi \dot{\phi}))$$

$$= m(s \sin \phi \dot{z} - z s \sin \phi - z s \cos \phi \dot{\phi})$$

$$M_y = m(z(s \cos \phi - s \sin \phi \dot{\phi}) - s \cos \phi \dot{z})$$

$$= m(z \cos \phi \dot{s} - z s \sin \phi \dot{\phi} - s \cos \phi \dot{z})$$

$$M_z = m(s \cos \phi (s \sin \phi + s \cos \phi \dot{\phi})$$

$$- s \sin \phi (s \cos \phi - s \sin \phi \dot{\phi})]$$

$$= m s^2 \dot{\phi}$$

$$\begin{aligned}
 M^2 &= M_x^2 + M_y^2 + M_z^2 \\
 &= m^2 \left\{ (\sin \phi (sz - z's) - z s \cos \phi \dot{\phi})^2 \right. \\
 &\quad + (\cos \phi (sz - z's) - z s \sin \phi \dot{\phi})^2 \\
 &\quad \left. + (s^2 \dot{\phi})^2 \right\} \\
 &= m^2 \left\{ \sin^2 \phi (sz - z's)^2 + z^2 s^2 \cos^2 \phi \dot{\phi}^2 \right. \\
 &\quad \left. - 2 z s \sin \phi \cos \phi \dot{\phi} (sz - z's) \right. \\
 &\quad + \cos^2 \phi (sz - z's)^2 + z^2 s^2 \sin^2 \phi \dot{\phi}^2 \\
 &\quad \left. + 2 z s \sin \phi \cos \phi \dot{\phi} (sz - z's) \right. \\
 &\quad \left. + s^4 \dot{\phi}^2 \right\} \\
 &= m^2 \left\{ (sz - z's)^2 + z^2 s^2 \dot{\phi}^2 + s^4 \dot{\phi}^2 \right\} \\
 &= m^2 [ (sz - z's)^2 + s^2 \dot{\phi}^2 (z^2 + s^2) ]
 \end{aligned}$$

Sec 9, Prob 2

repeat for spherical polar coords.

$$M_x = m(yz - zy)$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$\text{Now: } x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\begin{aligned}
 \dot{x} &= r \sin \theta \cos \phi + r \cos \theta \cos \phi \dot{\theta} - r \sin \theta \sin \phi \dot{\phi} \\
 \dot{y} &= r \sin \theta \sin \phi + r \cos \theta \sin \phi \dot{\theta} + r \sin \theta \cos \phi \dot{\phi} \\
 \dot{z} &= r \cos \theta - r \sin \theta \dot{\theta}
 \end{aligned}$$

Then,

$$\begin{aligned}
 M_x &= m(yz - zy) \\
 &= m \left\{ r \sin \theta \sin \phi (r \cos \theta - r \sin \theta \dot{\theta}) \right. \\
 &\quad \left. - r \cos \theta (r \sin \theta \sin \phi + r \cos \theta \sin \phi \dot{\theta} + r \sin \theta \cos \phi \dot{\phi}) \right\} \\
 &= m \left\{ -r^2 \sin^2 \theta \sin \phi \dot{\theta} - r^2 \cos^2 \theta \sin \phi \dot{\theta} - r^2 \sin \theta \cos \theta \right. \\
 &\quad \left. \cos \phi \dot{\phi} \right\} \\
 &= m \left\{ -r^2 \sin \phi \dot{\theta} - r^2 \sin \theta \cos \theta \cos \phi \dot{\phi} \right\} \\
 &= -mr^2 [\sin \phi \dot{\theta} + \sin \theta \cos \theta \cos \phi \dot{\phi}]
 \end{aligned}$$

$$\begin{aligned}
 M_y &= m(z\dot{x} - x\dot{z}) \\
 &= m \{ r\cos\theta (r\sin\theta \cos\phi + r\cos\theta \sin\phi \dot{\theta} - r\sin\theta \sin\phi \dot{\phi}) \\
 &\quad - r\sin\theta \cos\phi (r\cos\theta - r\sin\theta \dot{\theta}) \} \\
 &= m \{ r^2 \cos^2\theta \cos\phi \dot{\theta} - r^2 \sin\theta \cos\theta \sin\phi \dot{\phi} \\
 &\quad + r^2 \sin^2\theta \cos\phi \dot{\theta} \} \\
 &= m \{ r^2 \cos\phi \dot{\theta} - r^2 \sin\theta \cos\theta \sin\phi \dot{\phi} \} \\
 &= mr^2 [\cos\phi \dot{\theta} - \sin\theta \cos\theta \sin\phi \dot{\phi}]
 \end{aligned}$$

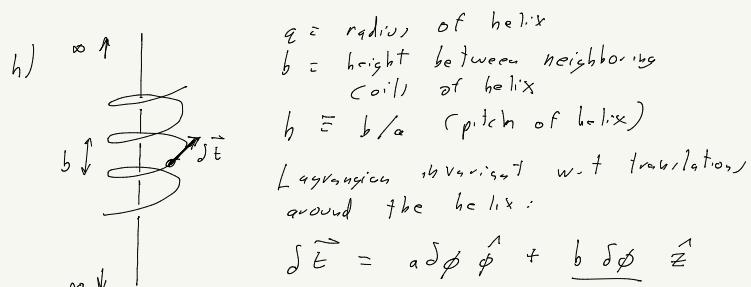
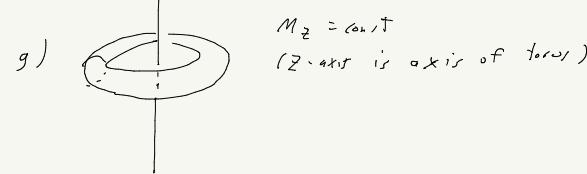
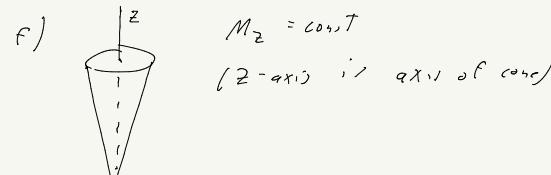
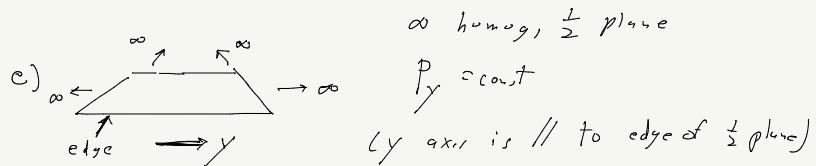
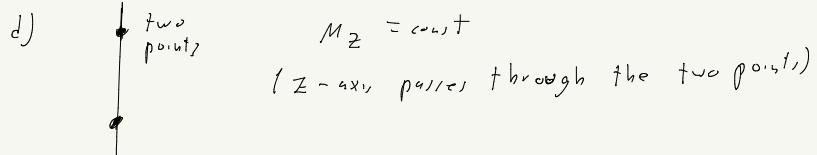
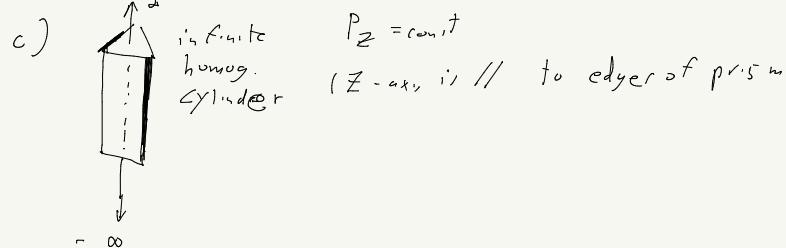
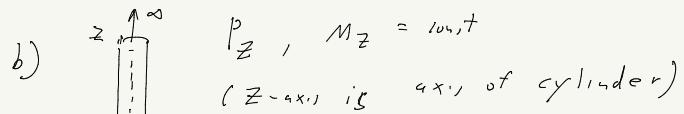
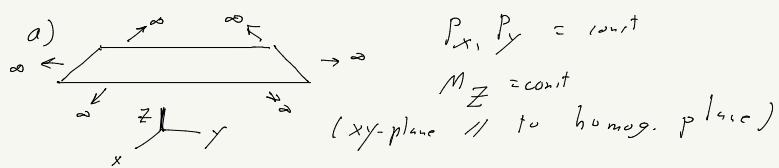
$$\begin{aligned}
 M_z &= m(xy' - yx') \\
 &= m \{ r\sin\theta \cos\phi (r\sin\theta \sin\phi + r\cos\theta \cos\phi \dot{\theta} \\
 &\quad + r\sin\theta \cos\phi) \\
 &\quad - r\sin\theta \sin\phi (r\sin\theta \cos\phi + r\cos\theta \cos\phi \dot{\theta} \\
 &\quad - r\sin\theta \sin\phi \dot{\phi}) \} \\
 &= m [r^2 \sin^2\theta \cos\phi \dot{\phi} + r^2 \sin^2\theta \sin\phi \dot{\phi}] \\
 &= mr^2 \sin^2\theta \dot{\phi}
 \end{aligned}$$

$$\begin{aligned}
 M^2 &= M_x^2 + M_y^2 + M_z^2 \\
 &= m^2 r^4 [\sin\phi \dot{\theta} + \sin\theta \cos\theta \cos\phi \dot{\phi}]^2 \\
 &\quad + m^2 r^4 [\cos\phi \dot{\theta} - \sin\theta \cos\theta \sin\phi \dot{\phi}]^2 \\
 &\quad + m^2 r^4 \sin^4\theta \dot{\phi}^2
 \end{aligned}$$

(cross terms will cancel)

$$\begin{aligned}
 M^2 &= m^2 r^4 \{ \sin^2\phi \dot{\theta}^2 + \sin^2\theta \cos^2\theta \cos^2\phi \dot{\phi}^2 \\
 &\quad + \cos^2\phi \dot{\theta}^2 + \sin^2\theta \cos^2\theta \sin^2\phi \dot{\phi}^2 \\
 &\quad + \sin^4\theta \dot{\phi}^2 \} \\
 &= m^2 r^4 [\dot{\theta}^2 + \sin^2\theta \cos^2\theta \dot{\phi}^2 + \sin^4\theta \dot{\phi}^2] \\
 &= m^2 r^4 [\dot{\theta}^2 + \sin^2\theta \phi^2 / (\cos^2\theta + \sin^2\theta)] \\
 &= m^2 r^4 [\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2]
 \end{aligned}$$

Sec 9, Prob 3:



$$\begin{aligned}\delta \vec{r} &= a \delta \phi \hat{\phi} + \frac{b \delta \phi}{2\pi} \hat{z} \\ &= a \delta \phi \left[ \hat{\phi} + \frac{b}{2\pi} \hat{z} \right] \\ &= a \delta \phi \left[ \hat{\phi} + \frac{h}{2\pi} \hat{z} \right] \\ \delta L &= \frac{\partial L}{\partial \dot{\phi}} \delta \phi + \frac{\partial L}{\partial \dot{z}} \delta z \\ &= \frac{d(L/\delta \phi)}{d\phi} \delta \phi + \frac{d(L/\delta z)}{dz} \delta z \\ &= \delta \phi \frac{1}{\frac{dL}{d\phi}} \left[ \frac{p_\phi}{M_z} + p_z \frac{1}{2\pi} \right] \xrightarrow{M_z + \frac{p_z h}{2\pi} = \text{const}}\end{aligned}$$

NOTE:  $M_z$  is independent of location of origin  
on  $z$ -axis

$$\vec{M} = \vec{r} \times \vec{p} = m \vec{r} \times \dot{\vec{r}}$$

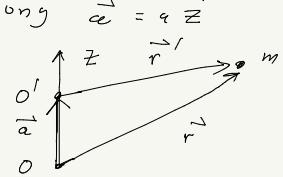
Change origin by shifting along  $\vec{a} = \hat{z}$

$$\vec{r} = \vec{r}' + \vec{a}$$

$$\begin{aligned}\vec{M} &= m \vec{r} \times \dot{\vec{r}} \\ &= m(\vec{r}' + \vec{a}) \times \frac{d}{dt}(\vec{r}' + \vec{a}) \\ &= m \vec{r}' \times \dot{\vec{r}'} + m \vec{a} \times \dot{\vec{r}'} \\ &= \vec{M}' + \vec{a} \times \vec{p}' \quad (\text{for arbitrary } \vec{a})\end{aligned}$$

Thus,

$$\begin{aligned}M_z &\equiv \vec{M} \cdot \hat{z} \\ &= (\vec{M}' + \vec{a} \times \vec{p}') \cdot \hat{z} \\ &= M'_z + a (\hat{z} \times \vec{p}') \cdot \hat{z} \\ &= M'_z + a (\hat{z} \times \hat{z}) \cdot \vec{p}' \\ &= M'_z\end{aligned}$$



Sec 10, Prob 1:

same path, different masses, same potential energy

$$\rightarrow x' = x, \quad m' \neq m, \quad U' = U, \quad t' \neq t$$

$$L = T - U = \frac{1}{2} m \dot{x}^2 - U$$

$$\begin{aligned}L' &= \frac{1}{2} m' \left( \frac{dx}{dt'} \right)^2 - U \\ &= \frac{1}{2} m' \left( \frac{t}{t'} \right)^2 \dot{x}^2 - U\end{aligned}$$

$$\text{Thus, } L' = L \rightarrow m' \left( \frac{t}{t'} \right)^2 = m$$

$$\left( \frac{t}{t'} \right)^2 = \frac{m'}{m}$$

$$\rightarrow \frac{t'}{t} = \sqrt{\frac{m'}{m}}$$

Sec 10, Prob 2:

Same path, mass but potential energies differing by a constant

$$L_1 = \frac{1}{2}mV_1^2 - U_1$$

$$L_2 = \frac{1}{2}mV_2^2 - U_2$$

$$\frac{T_{b(v)}}{V_2^2} = \frac{U_1}{U_2}$$

$$\rightarrow \frac{(1/t_1)^2}{(t/t_2)^2} = \frac{U_1}{U_2}$$

$$\text{so } \frac{t_2}{t_1} = \sqrt{\frac{U_1}{U_2}}$$

Sec 40 - Prob 1

single particle in a constant external field

$$L = \frac{1}{2}m\vec{v}^2 - U(\vec{r})$$

a) Cartesian coords  $(x, y, z)$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

$$\rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = p_x/m$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \rightarrow \dot{y} = p_y/m$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$H = \left( \frac{1}{2m} \vec{p}^2 - L \right) / \left. \frac{\partial}{\partial \dot{z}} \right|_{\dot{z} = p_z/m}$$

$$= \left( p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z) \right) / \left. \frac{\partial}{\partial \dot{z}} \right|_{\dot{z} = p_z/m}$$

$$= p_x \left( \frac{p_x}{m} \right) + p_y \left( \frac{p_y}{m} \right) + p_z \left( \frac{p_z}{m} \right) - \frac{1}{2}m \left( \left( \frac{p_x}{m} \right)^2 + \left( \frac{p_y}{m} \right)^2 + \left( \frac{p_z}{m} \right)^2 \right) + U(x, y, z)$$

$$= \frac{1}{2m} (\vec{p}^2) + U(x, y, z)$$

b) cylindrical coords  $(s, \phi, z)$ ,  $s^2 = x^2 + y^2$

$$L = \frac{1}{2}m(s^2\dot{s}^2 + \dot{z}^2) - U(s, \phi, z)$$

$$\rightarrow p_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s} \rightarrow \dot{s} = p_s/m$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ms^2\dot{\phi} \rightarrow \dot{\phi} = p_\phi/ms^2$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$\begin{aligned}
 H &= \left( p_r \dot{r} + p_\theta \dot{\theta} + p_z \dot{z} - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) + U(r, \theta, z) \right) \\
 &= p_r \left( \frac{p_r}{m} \right) + p_\theta \left( \frac{p_\theta}{mr^2} \right) + p_z \left( \frac{p_z}{m} \right) \\
 &\quad - \frac{1}{2} m \left( \left( \frac{p_r}{m} \right)^2 + r^2 \left( \frac{p_\theta}{mr^2} \right)^2 + \left( \frac{p_z}{m} \right)^2 \right) + U(r, \theta, z) \\
 &= \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2 \right) + U(r, \theta, z)
 \end{aligned}$$

c) spherical polar coords.  $(r, \theta, \phi)$

$$\begin{aligned}
 L &= \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 + \dot{r}^2) - U(r, \theta, \phi) \\
 \rightarrow p_r &= \frac{\partial L}{\partial \dot{r}} = m \dot{r} \rightarrow \dot{r} = p_r/m \\
 p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \rightarrow \dot{\theta} = p_\theta / mr^2 \\
 p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2 \theta \dot{\phi} \rightarrow \dot{\phi} = p_\phi / m r^2 \sin^2 \theta \\
 H &= \left( p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) + U(r, \theta, \phi) \right) \\
 &= p_r \left( \frac{p_r}{m} \right) + p_\theta \left( \frac{p_\theta}{mr^2} \right) + p_\phi \left( \frac{p_\phi}{mr^2 \sin^2 \theta} \right) \\
 &\quad - \frac{1}{2} m \left( \left( \frac{p_r}{m} \right)^2 + r^2 \left( \frac{p_\theta}{mr^2} \right)^2 + r^2 \sin^2 \theta \left( \frac{p_\phi}{mr^2 \sin^2 \theta} \right)^2 \right) + U(r, \theta, \phi) \\
 &= \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + U(r, \theta, \phi)
 \end{aligned}$$

sec 40 - prob 2

$$L = \frac{1}{2} m v^2 + m \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + \frac{1}{2} m |\vec{\Omega} \times \vec{r}|^2 - m \vec{W} \cdot \vec{r} - U$$

restrict to uniformly rotating frame of reference  $\vec{W} = 0$ ,  $\vec{\Omega} = \vec{\omega}$

$$\begin{aligned}
 \rightarrow L &= \frac{1}{2} m v^2 + m \vec{v} \cdot (\vec{\Omega} \times \vec{r}) \\
 &\quad + \frac{1}{2} m |\vec{\Omega} \times \vec{r}|^2 - U(r)
 \end{aligned}$$

$$\text{Now: } H = \vec{p} \cdot \vec{v} - L$$

$$\begin{aligned}
 \vec{p} &= \frac{\partial L}{\partial \vec{v}} = m \vec{v} + m \vec{\Omega} \times \vec{r} = m (\vec{v} + \vec{\Omega} \times \vec{r}) \\
 \rightarrow \vec{v} &= \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r}
 \end{aligned}$$

thus,

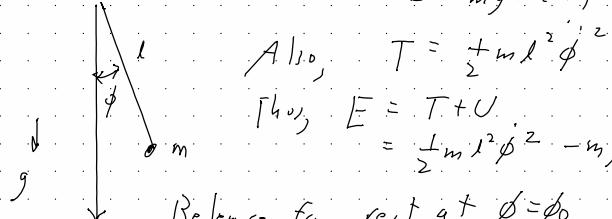
$$\begin{aligned}
 H &= \vec{p} \cdot \left( \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r} \right) - \frac{1}{2} m \left| \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r} \right|^2 \\
 &\quad - m \left( \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r} \right) \cdot \left( \vec{\Omega} \times \vec{r} \right) \\
 &\quad - \frac{1}{2} m |\vec{\Omega} \times \vec{r}|^2 + U(r)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{|\vec{p}|^2}{m} - \vec{p} \cdot \left( \vec{\Omega} \times \vec{r} \right) - \frac{1}{2} m \left( \frac{|\vec{p}|^2}{m^2} + |\vec{\Omega} \times \vec{r}|^2 \right) - \cancel{2 \vec{p} \cdot \left( \vec{\Omega} \times \vec{r} \right)} \\
 &\quad - \vec{p} \cdot \left( \vec{\Omega} \times \vec{r} \right) + m |\vec{\Omega} \times \vec{r}|^2 - \frac{1}{2} m |\vec{\Omega} \times \vec{r}|^2 + U(r) \\
 &= \frac{|\vec{p}|^2}{2m} - \vec{p} \cdot \left( \vec{\Omega} \times \vec{r} \right) + U(r) \\
 &= \frac{|\vec{p}|^2}{2m} - \vec{\Omega} \cdot \left( \vec{r} \times \vec{p} \right) + U(r)
 \end{aligned}$$

$\vec{p}$  is angular momentum

Sec. 11, Prob 1

$$\text{Simple pendulum: } U = -mgY \\ = -mgl \cos\phi$$



$$\text{Also, } T = \frac{1}{2}ml^2\dot{\phi}^2$$

$$\text{Thus, } E = T+U \\ = \frac{1}{2}ml^2\dot{\phi}^2 - mgl \cos\phi$$

Release from rest at  $\phi = \phi_0$

$$E = -mgl \cos\phi_0$$

$$\rightarrow \frac{1}{2}ml^2\dot{\phi}^2 - mgl \cos\phi = -mgl \cos\phi_0$$

$$T(E) = 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{E - U(\phi)}}$$

$$= 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{-mgl \cos\phi_0 + mgl \cos\phi}}$$

$$= 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$

$\phi < \phi_0$

$$\begin{aligned} \text{Now: } \cos\phi &= \cos\left(2 \frac{\phi}{2}\right) \\ &= \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right) \\ &= 1 - 2\sin^2\left(\frac{\phi}{2}\right) \end{aligned}$$

$$\text{Also, } \cos\phi_0 = 1 - 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$T(E) = 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{2 \left( \sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right) \right)}}$$

$$= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2\left(\frac{\phi_0}{2}\right) \left( 1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)} \right)}}$$

$$\text{Let } x = \frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)} \rightarrow dx = \frac{1}{2} \frac{\cos\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)}$$

$$= \frac{d\phi}{2} \frac{\sqrt{1 - \sin^2\left(\frac{\phi}{2}\right)}}{\sin\left(\frac{\phi_0}{2}\right)}$$

$$= \frac{d\phi}{2} \frac{\sqrt{1 - \sin^2\left(\frac{\phi_0}{2}\right)x^2}}{\sin\left(\frac{\phi_0}{2}\right)}$$

$$\begin{aligned} \text{Thus, } T(E) &= 2\sqrt{\frac{l}{g}} \int_0^1 \frac{2dx \cdot \sin\left(\frac{\phi_0}{2}\right)}{\sqrt{1 - \sin^2\left(\frac{\phi_0}{2}\right)x^2} \cdot \sin\left(\frac{\phi_0}{2}\right) \sqrt{1 - x^2}} \\ &= 4\sqrt{\frac{l}{g}} \int_0^1 \frac{dx}{\sqrt{1 - x^2} \sqrt{1 - k^2x^2}}, \quad K = \sin\left(\frac{\phi_0}{2}\right) \\ &= 4\int_{\frac{1}{2}}^1 K(k) \end{aligned}$$

where  $K(k) = \text{complete elliptic integral of the 1st kind.}$

A problem from:

$$T(E) = 4\sqrt{\frac{E}{J}} \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-H^2 x^2}}$$

For  $H \ll 1$ :  $\frac{1}{\sqrt{1-H^2 x^2}} \approx 1 + \frac{1}{2} H^2 x^2$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-H^2 x^2}} \approx \int_0^1 \frac{dx}{\sqrt{1-x^2}} \left( 1 + \frac{1}{2} H^2 x^2 \right)$$

Now:  $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin(1) = \left[ \frac{\pi}{2} \right]$

$$\frac{1}{2} H^2 \int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1}{2} H^2 \int_0^{\pi/2} \frac{\sin^2 \theta \cos \theta d\theta}{\sqrt{1-\sin^2 \theta}}$$

Let:  $x = \sin \theta$        $\cos 2\theta = 1 - 2\sin^2 \theta$   
 $dx = \cos \theta d\theta$        $\rightarrow \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$   
 $x^2 = \sin^2 \theta$

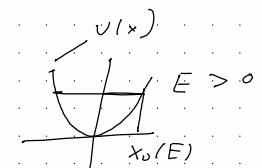
$$\begin{aligned} \rightarrow \frac{1}{2} H^2 \int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}} &= \frac{1}{2} H^2 \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2\theta) d\theta \\ &= \frac{1}{4} H^2 \left[ \frac{\pi}{2} - \frac{\sin 2\theta}{2} \Big|_0^{\pi/2} \right] \\ &= \boxed{\frac{1}{8} H^2 \pi} \quad |T = \sin \left( \frac{\phi_0}{2} \right) \approx \frac{\phi_0}{2}} \end{aligned}$$

$$Therefore, T(E) = \frac{4\sqrt{\frac{E}{J}}}{\sqrt{2}} \left( \frac{\pi}{2} + \frac{1}{8} H^2 \pi + \dots \right) = \boxed{2\sqrt{\frac{E}{J}} \left( 1 + \frac{1}{16} \phi_0^2 + \dots \right)}$$

Sec II, Prob 2:

(a)  $U = A|x|^n \quad x_0(E)$

$$T(E) = 2\sqrt{2m} \int_0^{\infty} \frac{dx}{\sqrt{E - Ax^n}}$$



where  $U(x_0) = E$

$$Ax_0^n = E$$

$$\rightarrow x_0 = \left( \frac{E}{A} \right)^{\frac{1}{n}}$$

$$\sqrt{E - Ax^n} = \sqrt{E} \sqrt{1 - \left( \frac{A}{E} \right)^n}$$

$$= \sqrt{E} \sqrt{1 - t^n} \quad \text{where } t = \left( \frac{A}{E} \right)^{\frac{1}{n}} x$$

$$dt = \left( \frac{A}{E} \right)^{\frac{1}{n}} dx$$

$$t_0 = \left( \frac{A}{E} \right)^{\frac{1}{n}} x_0 = \left( \frac{A}{E} \right)^{\frac{1}{n}} \left( \frac{E}{A} \right)^{\frac{1}{n}} = 1$$

Thus,

$$T(E) = 2\sqrt{2m} \int_0^1 \left( \frac{E}{A} \right)^{\frac{1}{n}} \frac{dt}{\sqrt{1-t^n}}$$

Now let  $u = t^n$

$$\begin{aligned} du &= n t^{n-1} dt = n u^{\frac{n-1}{n}} dt \\ &= n u^{1-\frac{1}{n}} dt \end{aligned}$$

$$dt = \frac{1}{n} u^{\frac{1}{n}-1} du, \quad t=0, 1 \rightarrow u=0, 1$$

Thus

$$T(E) = 2 \sqrt{\frac{2m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 \frac{1}{n} \frac{u^{\frac{1}{n}-1}}{\sqrt{1-u}} du$$
$$= \frac{2}{n} \sqrt{\frac{2m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 u^{\frac{1}{n}-1} (1-u)^{-\frac{1}{2}} du$$

Recall: Beta Function

$$B(x, y) = \int_0^1 dt t^{x-1} (1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

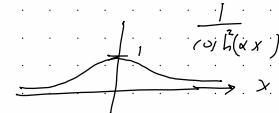
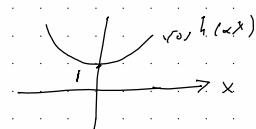
(where  $\operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0$ )

Thus,

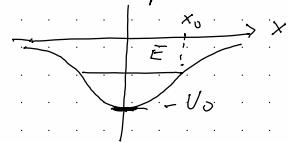
$$\boxed{T(E)} = \frac{2}{n} \sqrt{\frac{2m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$
$$= \frac{2}{n} \sqrt{\frac{2\pi m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}$$

Gamma Function

$$(b) U = -U_0 / \cosh^2 \alpha x$$



$$-U_0 / \cosh^2$$



$$-U_0 \leq E < 0$$

Thus,  $E = -|E|$

$$|E| < U_0$$

$$T(E) = 2 \sqrt{2m} \int_0^{\infty} \frac{dx}{\sqrt{E - U(x)}}$$

where  $x_0$  given by

$$E = U(x_0)$$
$$= \frac{-U_0}{\cosh^2(\alpha x_0)}$$
$$\rightarrow \cosh^2(\alpha x_0) = -\frac{U_0}{E}$$

$$\cosh(\alpha x_0) = \sqrt{\frac{U_0}{|E|}}$$

$$\alpha x_0 = \cosh^{-1}\left(\sqrt{\frac{U_0}{|E|}}\right)$$

$$\boxed{x_0 = \frac{1}{\alpha} \cosh^{-1}\left(\sqrt{\frac{U_0}{|E|}}\right)}$$

$$\begin{aligned}
 \sqrt{E - U(x)} &= \sqrt{E + \frac{U_0}{\cosh^2(\alpha x)}} \quad \cosh^2 - \sinh^2 = 1 \\
 &= \frac{1}{\cosh(\alpha x)} \sqrt{E \cosh^2(\alpha x) + U_0} \\
 &= \frac{1}{\cosh(\alpha x)} \sqrt{E(1 + \sinh^2(\alpha x)) + U_0} \\
 &= \frac{1}{\cosh(\alpha x)} \sqrt{(E+U_0) + E \sinh^2(\alpha x)} \\
 &= \frac{1}{\cosh(\alpha x)} \sqrt{(U_0 - |E|) - |E| \sinh^2(\alpha x)}
 \end{aligned}$$

thus

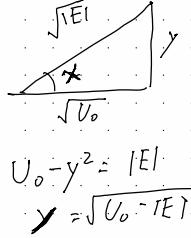
$$\begin{aligned}
 T(E) &= 2 \sqrt{2m} \int_0^{x_0} \frac{dx}{\sqrt{E - U(x)}} \\
 &= 2 \sqrt{2m} \int_0^{x_0} \frac{\cosh(\alpha x) dx}{\sqrt{(U_0 - |E|) - |E| \sinh^2(\alpha x)}}
 \end{aligned}$$

$$\text{Let: } t = \sinh(\alpha x) \rightarrow t_0 = \sinh(\alpha x_0)$$

$$dt = \alpha \cosh(\alpha x) dx$$

$$so \quad t_0 = \sinh^{-1} \left( \frac{U_0}{|E|} \right)$$

$$= \frac{U_0 - |E|}{|E|}$$



$$U_0 - y^2 = |E|$$

$$y = \sqrt{U_0 - |E|}$$

$$T(E) = 2 \sqrt{2m} \int_0^{\sqrt{\frac{U_0 - |E|}{|E|}}} dt$$

$$= 2 \sqrt{2m} \int_0^{\sqrt{\frac{U_0 - |E|}{|E|}}} \frac{dt}{\sqrt{U_0 - |E|} \sqrt{1 - \frac{|E|}{U_0 - |E|} t^2}}$$

$$\text{Def: } \sin u = \sqrt{\frac{|E|}{U_0 - |E|}} t$$

$$du \cos u = \sqrt{\frac{|E|}{U_0 - |E|}} dt$$

$$\begin{aligned}
 T(E) &= \frac{2 \sqrt{2m}}{\sqrt{U_0 - |E|}} \int_0^{\pi/2} \frac{\cos u du}{\sqrt{\frac{U_0 - |E|}{|E|}} \sqrt{1 - \frac{|E|}{U_0 - |E|} \sin^2 u}} \\
 &= \frac{\pi}{\alpha} \sqrt{\frac{2m}{|E|}} \frac{\pi}{2}
 \end{aligned}$$

$$\boxed{\frac{\pi}{\alpha} \sqrt{\frac{2m}{|E|}}}$$

$$(c) U = U_0 + \frac{1}{2} m \alpha x^2$$

$$T = 2\sqrt{2m} \int_0^{x_0} \frac{dx}{\sqrt{E - U(x)}}$$

where  $x_0$  determined by

$$E = U(x_0)$$

$$= U_0 + \frac{1}{2} m \alpha x_0^2$$

$$\Rightarrow x_0 = \tan^{-1} \left( \sqrt{\frac{E}{U_0}} \right)$$

$$\boxed{x_0 = \pm \tan^{-1} \left( \sqrt{\frac{E}{U_0}} \right)}$$

$$\sqrt{E - U(x)} = \sqrt{E - U_0 - \frac{1}{2} m \alpha x^2}$$

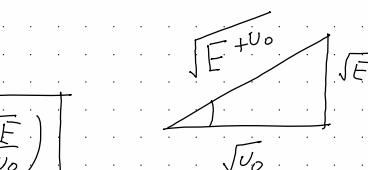
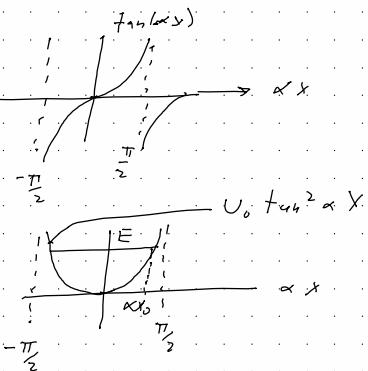
$$= \sqrt{E - U_0 - \frac{1}{2} m \alpha^2 x^2}$$

$$= \pm \sqrt{E \cos^2(\alpha x) - U_0 \sin^2(\alpha x)}$$

$$= \pm \sqrt{E (1 - \sin^2(\alpha x)) - U_0 \sin^2(\alpha x)}$$

$$= \pm \sqrt{E - (U_0 + E) \sin^2(\alpha x)}$$

$$= \frac{\sqrt{E}}{\cos \alpha x} \sqrt{1 - \left( \frac{U_0 + E}{E} \right) \sin^2(\alpha x)}$$



Thus

$$T(E) = 2\sqrt{2m} \int_0^{x_0} \frac{\cos \alpha x \, dx}{\sqrt{E - \left( U_0 + \frac{1}{2} m \alpha^2 x^2 \right)}}$$

$$\text{Let } u = \sin \alpha x$$

$$du = \alpha \cos \alpha x \, dx$$

$$x=0, x_0 \rightarrow u=0$$

$$u_0 = \sin(\alpha x_0)$$

$$= \sin \left( \tan^{-1} \sqrt{\frac{E}{U_0}} \right)$$

$$= \sqrt{\frac{E}{E+U_0}}$$

$u_0$

$$\rightarrow T(E) = 2\sqrt{2m} \frac{1}{\alpha} \int_0^{u_0} \frac{du}{\sqrt{1 - \frac{U_0+E}{E} u^2}}$$

$$\text{Let } \sqrt{\frac{U_0+E}{E}} u = \sin \theta \rightarrow \sqrt{\frac{U_0+E}{E}} du = \cos \theta \, d\theta$$

$$u=0 \rightarrow \theta=0$$

$$u=u_0 \rightarrow \sin \theta_0 = \sqrt{\frac{U_0+E}{E}} \sqrt{\frac{E}{U_0+E}} = 1 \Rightarrow \theta_0 = \frac{\pi}{2}$$

$$\rightarrow T(E) = 2\sqrt{2m} \frac{1}{\alpha} \int_0^{\pi/2} \frac{\cos \theta \, d\theta}{\sqrt{\frac{U_0+E}{E}} \sqrt{1 - \sin^2 \theta}}$$

$$= \frac{2\sqrt{2m}}{\alpha \sqrt{E+U_0}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \sin^2 \theta}} = \boxed{\frac{\pi}{\alpha} \sqrt{\frac{2m}{E+U_0}}}$$

Sec 13, Prob 1

$\vec{X}$ : position vector of  $M$

$\vec{x}_a$ :  $a=1, 2, \dots, n$  position vector of  $n$  masses  
all with mass  $m$ .

Closed system  $\rightarrow$  linear momentum conserved  
 $\rightarrow$  COM frame

$$M\vec{X} + m(\vec{x}_1 + \vec{x}_2 + \dots + \vec{x}_n) = 0$$

$$\text{or } M\vec{X} + m \sum_{a=1}^n \vec{x}_a = 0 \quad (1)$$

$$\text{Taking time derivative } \rightarrow M\dot{\vec{X}} + \sum_a \dot{\vec{x}}_a = 0. \quad (2)$$

Define relative position vectors:

$$\vec{r}_1 \equiv \vec{x}_1 - \vec{X}$$

$$\vec{r}_2 \equiv \vec{x}_2 - \vec{X}$$

etc.

$$\text{or } \vec{r}_a \equiv \vec{x}_a - \vec{X}, \quad a=1, 2, \dots, n. \quad (3)$$

Summing up (3):

$$\begin{aligned} \sum \vec{r}_a &= \sum (\vec{x}_a - \vec{X}) \\ &= \sum \vec{x}_a - n \vec{X} \quad \text{total mass} \\ &= -\frac{M}{m} \vec{X} - n \vec{X} \\ &= -\frac{(M+n)m}{m} \vec{X} = -\mu \vec{X} \end{aligned}$$

$$\text{Thus, } \vec{X} = -\frac{m}{\mu} \sum \vec{r}_a$$

$$\text{and } \vec{x}_a = \vec{r}_a + \vec{X}$$

give you  $\vec{x}_a, \vec{X}$  in terms of  $\vec{r}_a$ :

$$\begin{aligned} \text{Hence: } T &= \frac{1}{2} \sum_a m |\vec{x}_a|^2 + \frac{1}{2} M |\vec{X}|^2 \\ &= \frac{1}{2} m \sum_a |\vec{r}_a + \vec{X}|^2 + \frac{1}{2} M |\vec{X}|^2 \\ &= \frac{1}{2} m \sum_a |\vec{r}_a|^2 + \frac{1}{2} \sum_a m 2\vec{r}_a \cdot \vec{X} \\ &\quad + \frac{1}{2} \sum_a m |\vec{X}|^2 + \frac{1}{2} M |\vec{X}|^2 \\ &= \frac{1}{2} m \sum_a |\vec{r}_a|^2 + m \left( \sum_a \vec{r}_a \right) \cdot \vec{X} \\ &\quad + \frac{1}{2} nm |\vec{X}|^2 + \frac{1}{2} M |\vec{X}|^2 \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} m \sum_a |\vec{r}_a|^2 - \mu |\vec{X}|^2 + \frac{1}{2} (nm + M) |\vec{X}|^2 \\ &= \frac{1}{2} m \sum_a |\vec{r}_a|^2 - \frac{1}{2} \mu |\vec{X}|^2 \end{aligned}$$

Now rewrite in  $t$  form:

$$-\frac{1}{2} \mu |\vec{X}|^2 = -\frac{1}{2} \mu \frac{m^2}{\mu^2} \sum_a |\vec{r}_a|^2 = -\frac{1}{2} \frac{m^2}{\mu} \sum_a |\vec{r}_a|^2$$

$$\text{Thus, } T = \frac{1}{2} m \sum_a |\vec{r}_a|^2 - \frac{1}{2} \frac{m^2}{\mu} \sum_a |\vec{r}_a|^2$$

## Potential energy

$$U = U(|\vec{x}_1 - \vec{x}_2|, |\vec{x}_1 - \vec{x}_3|, \dots, |\vec{x}_1 - \vec{x}_n|)$$

$$|\vec{x}_1 - \vec{x}|, |\vec{x}_2 - \vec{x}|, \dots, |\vec{x}_n - \vec{x}|$$

$$= \cup \{ |\vec{r}_1 - \vec{r}_2|, |\vec{r}_1 - \vec{r}_3|, \dots, |\vec{r}_1 - \vec{r}_{n_1}| \}$$

$$|\vec{r}_1|, |\vec{r}_2|, \dots, |\vec{r}_n|)$$

which depends only on the relative position vectors  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ .

Thus,

$$L = \frac{1}{2} m \sum_a |\vec{r}_a|^2 - \frac{1}{2} \frac{m^2}{M} \left| \sum_a \vec{r}_a \right|^2 - U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n)$$

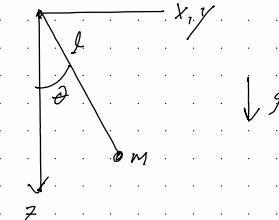
Sec 14, Prob. 1

## Spherical pendulum

$$T = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)$$

$$U = -mgz$$

$$r = -mg\cdot l \cdot \cos(\theta)$$



Thus,

$$f = T - U$$

$$= \frac{1}{2} m l^2 (\dot{\theta}^2 + m^2 \dot{\theta} \dot{\phi}^2) + m g l \cos \theta$$

$$\text{Not dependent} \rightarrow E = T + U = \text{const}$$

$$\text{No } \phi\text{-dependence} \rightarrow \frac{\partial L}{\partial \dot{\phi}} = m \ell^2 \sin^2 \theta \dot{\phi} \equiv M_2 = \cos \ell t$$

$$E = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - mg l \cos \theta$$

$$= \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2 \theta \frac{M_z^2}{m^2 l^4 \sin^2 \theta}) - m g l \cos \theta$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{M_2^2}{2m l^2 \sin^2 \theta} - mg l \cos \theta$$

$$U_{eff}(\theta)$$

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{M_2^2}{2 m l^2 \sin^2 \theta} + U(\theta)$$

$$\rightarrow \frac{1}{2} m_1^2 \dot{\theta}^2 = E - U(\theta) = \frac{M^2}{2 m_1^2 r_m^2 \theta}$$

$$\textcircled{2} = \sqrt{\frac{2}{m_e^2} (E - U(\theta))} - \frac{M_e^2}{m_e^2 e^4 s_i h^2 c}$$

$$\frac{d\theta}{dt} = \dot{\theta} = \sqrt{\frac{2}{ml^2} (E + mgl \cos\theta) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}$$

$$\rightarrow dt = \frac{d\theta}{\sqrt{\dots}}$$

$$t = \int \frac{d\theta}{\sqrt{\frac{2}{ml^2} (E + mgl \cos\theta) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}} + C_1 t$$

Path: use  $M_z = ml^2 \sin^2\theta \dot{\phi}$

$$\text{Thus, } \frac{d\theta}{dt} = \frac{d\theta}{d\phi} \frac{d\phi}{dt} \\ = \frac{d\theta}{d\phi} \frac{M_z}{ml^2 \sin^2\theta}$$

$$\text{Thus, } \frac{d\theta}{d\phi} = \frac{d\theta}{dt} \frac{ml^2 \sin^2\theta}{M_z} = \sqrt{\frac{ml^2 \sin^2\theta}{M_z}}$$

$$d\phi = \frac{d\theta M_z}{\sqrt{ml^2 \sin^2\theta}}$$

$$\phi = \int \frac{M_z d\theta / ml^2 \sin^2\theta}{\sqrt{\frac{2}{ml^2} (E + mgl \cos\theta) - \frac{M_z^2}{m^2 l^4 \sin^2\theta}}} + C_2 \phi$$

Turning points: (where  $\dot{\theta} = 0$ )

$$E = V_{\text{eff}}(r)$$

$$= \frac{M_z^2}{2ml^2 \sin^2\theta} - mgl \cos\theta$$

$$\rightarrow 2Eml^2 \sin^2\theta = M_z^2 - 2mgl^3 \sin^2\theta \cos\theta$$

$$2Eml^2 (1 - \cos^2\theta) = M_z^2 - 2mgl^3 (1 - \cos^2\theta) \cos\theta$$

$$2Eml^2 - 2Eml^2 \cos^2\theta$$

$$= M_z^2 - 2mgl^3 \cos\theta + 2mgl^3 \cos^3\theta$$

$$\text{Thus, } 2mgl^3 \cos^3\theta + 2Eml^2 \cos^2\theta - 2mgl^3 \cos\theta$$

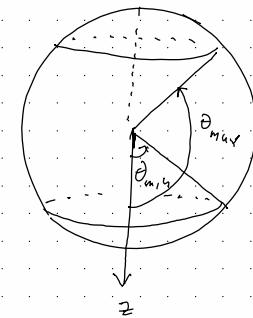
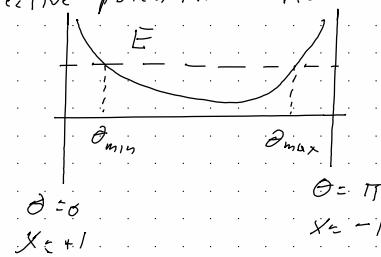
$$+ (M_z^2 - 2Eml^2) = 0$$

Divide by  $2mgl^3$ :

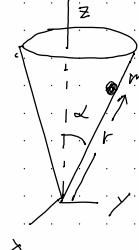
$$\rightarrow \cos^3\theta + \frac{E}{mgl} \cos^2\theta - \cos\theta + \left( \frac{M_z^2}{2mgl^3} - \frac{E}{mgl} \right) = 0$$

Cubic equation for  $x \equiv \cos\theta$ :

Effective potential looks like:



Sec 14, Prob. 2



spherical polar coords  $(r, \theta, \phi)$

constraint  $\dot{\theta} = \alpha$

Generalized coords:  $(r, \phi)$

$$T = \frac{1}{2}m(r^2 + r^2\sin^2\alpha\dot{\phi}^2)$$

$$= \frac{1}{2}m(r^2 + r^2\sin^2\alpha\dot{\phi}^2)$$

$$U = mgz$$

$$= mg r \cos \alpha$$

$$L = T - U$$

$$= \frac{1}{2}m(r^2 + r^2\sin^2\alpha\dot{\phi}^2) - mg r \cos \alpha$$

$E = \text{const}$  (since no explicit t dependence)

$$\frac{dL}{d\phi} = mr^2\sin^2\alpha\dot{\phi} = M_z = \text{const}$$

(since no explicit  $\phi$  dependence)

$$E = T + U$$

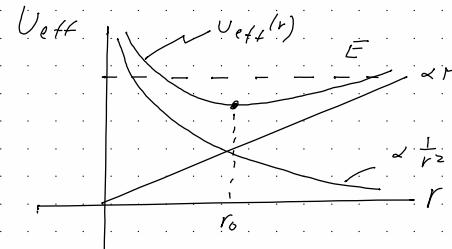
$$= \frac{1}{2}m(r^2 + r^2\sin^2\alpha\dot{\phi}^2) + mg r \cos \alpha$$

$$= \frac{1}{2}mr^2 + \frac{1}{2}mr^2\sin^2\alpha\left(\frac{M_z^2}{m^2r^4\sin^4\alpha}\right) + mg r \cos \alpha$$

$$= \frac{1}{2}mr^2 + \frac{M_z^2}{2mr^2\sin^2\alpha} + mg r \cos \alpha$$

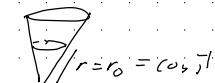
$$= \frac{1}{2}mr^2 + U_{\text{eff}}(r)$$

$$U_{\text{eff}}(r) = \frac{M_z^2}{2mr^2\sin^2\alpha} + mg r \cos \alpha$$



Bound orbits for  $E > U_{\text{eff}, \min} = U_{\text{eff}}(r_0)$

$r_0$ : stable circular orbit



t-equation:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}(E - mg r \cos \alpha) - \frac{M_z^2}{m^2 r^2 \sin^2 \alpha}}} + \text{const}$$

Using  $M_z = mr^2 \sin^2 \alpha \dot{\phi}$

$$\rightarrow \frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{M_z}{mr^2 \sin^2 \alpha}$$

$\phi$ -equation:

$$\phi = \int \frac{\left(\frac{M_z}{\sin^2 \alpha}\right) dr}{\sqrt{\frac{2m(E - mg r \cos \alpha)}{r^2 \sin^2 \alpha} - \frac{M_z^2}{r^2 \sin^2 \alpha}}} + \text{const}$$

Turning points:  $r = r_{\min}, r_{\max}$   
Determined by effective potential

$$E = U_{\text{eff}}(r)$$

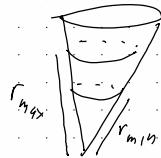
$$= \frac{M_2^2}{2m r^2 \sin^2 \alpha} + m g r \cos \alpha$$

$$\rightarrow 2m E r^2 \sin^2 \alpha = M_2^2 + 2m^2 g r^3 \sin^2 \alpha \cos \alpha$$

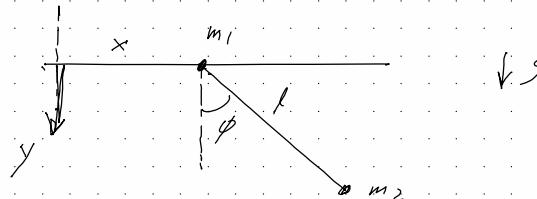
$$0 = 2m^2 g r^3 \sin^2 \alpha \cos \alpha - 2m E r^2 \sin^2 \alpha + M_2^2$$

$$= r^3 - \left( \frac{E}{m g \cos \alpha} \right) r^2 + \frac{M_2^2}{2m^2 g \sin^2 \alpha \cos \alpha}$$

cubic equation again



### Sec 14, Prob 3



From Sec 5, Prob 2 we have

$$L = \frac{1}{2}(m_1 + m_2)x^2 + \frac{1}{2}m_2 l^2 \dot{\phi}^2 + 2l \dot{x} \dot{\phi} \cos \phi + m_2 g l \cos \phi$$

No dependence on  $x$ :

$$\rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)x + m_2 l \dot{\phi} \cos \phi = \text{const}$$

( $x$ -component of total momentum)

No explicit  $t$ -dependence

$$\begin{aligned} \rightarrow E &= T + U = \text{const} \\ &= \frac{1}{2}(m_1 + m_2)x^2 + \frac{1}{2}m_2(l^2 \dot{\phi}^2 + 2l \dot{x} \dot{\phi} \cos \phi) - m_2 g l \cos \phi \end{aligned}$$

Want in frame where  $\text{com}_x = 0$ :

$$\begin{aligned} \text{com}_x &= m_1 x + m_2(x + l \sin \phi) \\ &= (m_1 + m_2)x + m_2 l \sin \phi \end{aligned}$$

$$\text{com}_x = 0 \rightarrow x = -\left(\frac{m_2}{m_1 + m_2}\right) l \sin \phi$$

$$\dot{x} = -\left(\frac{m_2}{m_1+m_2}\right) l \cos\phi \dot{\phi}$$

$t^{\text{loss}}$

$$E = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi)$$

$$- m_2 g l \cos\phi$$

$$= \frac{1}{2}(m_1+m_2) \frac{m_2^2}{(m_1+m_2)^2} l^2 \cos^2\phi \dot{\phi}^2$$

$$+ \frac{1}{2}m_2 l^2 \dot{\phi}^2 - m_2 \left(\frac{m_2}{m_1+m_2}\right) l^2 \cos^2\phi \dot{x}^2$$

$$- m_2 g l \cos\phi$$

$$= \frac{1}{2}m_2 l^2 \dot{\phi}^2 \left[ 1 - \left(\frac{m_2}{m_1+m_2}\right) \cos^2\phi \right]$$

$$- m_2 g l \cos\phi$$

1-d problem:

$$\underline{E + m_2 g l \cos\phi} = \frac{1}{2}m_2 l^2 \dot{\phi}^2$$

$$\left[ 1 - \left(\frac{m_2}{m_1+m_2}\right) \cos^2\phi \right]$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{\frac{2}{m_2 l^2} \left( E + m_2 g l \cos\phi \right)} \overline{1 - \left(\frac{m_2}{m_1+m_2}\right) \cos^2\phi}$$

$$\rightarrow dt = \frac{d\phi}{\sqrt{}}$$

$$= d\phi \sqrt{1 - \left(\frac{m_2}{m_1+m_2}\right) \cos^2\phi}$$

$$= d\phi \sqrt{\frac{m_2 l^2}{2} \frac{1}{m_1+m_2}} \sqrt{\frac{(m_1+m_2) - m_2 \cos^2\phi}{E + m_2 g l \cos\phi}}$$

$$= d\phi \sqrt{\frac{m_2}{m_1+m_2}} \sqrt{\frac{l^2}{2}} \sqrt{\frac{m_1+m_2 \sin^2\phi}{E + m_2 g l \cos\phi}}$$

so

$$t = \sqrt{\left(\frac{m_2}{m_1+m_2}\right) \frac{l^2}{2}} \int d\phi \sqrt{\frac{m_1+m_2 \sin^2\phi}{E + m_2 g l \cos\phi}} + \text{const}$$

$$\text{Now: } x_2 = x + l \sin\phi$$

$$y_2 = l \cos\phi$$

$$\text{using } x = -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi$$

$$\rightarrow x_2 = \left[ -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi + l \cos\phi \right] = \left(\frac{m_1}{m_1+m_2}\right) l \sin\phi$$

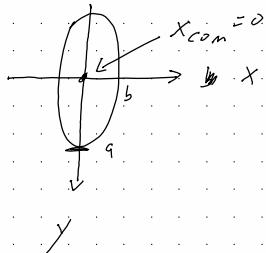
$$x_2 = \left( \frac{m_1}{m_1 + m_2} \right) l \sin \phi = b \sin \phi$$

$$y_2 = l \cos \phi = a \cos \phi$$

$$\left(\frac{y_2}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = \cos^2 \phi + \sin^2 \phi = 1$$

which is an ellipse with semi-major and semi-minor axes:

$$a = l, \quad b = l \left( \frac{m_1}{m_1 + m_2} \right)$$



If  $m_1 \gg m_2$   
then  $a = b \approx l$   
so that  $m_2$  moves  
along a circular  
arc of radius  $l$ :

Sec 15, Prob 1

$$U = -\frac{\infty}{r}, \quad E = 0 \rightarrow e = 1$$

$$\frac{p}{r} = 1 + \cos \phi$$

when  $\phi = 0, \quad p = 2 \cdot r$

$$\text{so } r_{min} = \frac{p}{2}$$

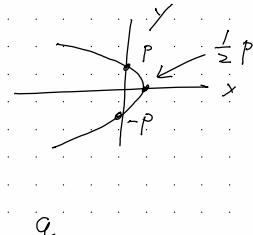
NOTE:  $p = r + r \cos \phi$   
 $= \sqrt{r^2 + y^2} + x$

$$\rightarrow (p-x)^2 = x^2 + y^2$$

$$p^2 + x^2 - 2px = x^2 + y^2$$

$$p^2 - y^2 = 2px$$

$$\rightarrow \boxed{x = \frac{p^2 - y^2}{2p}} \quad \text{parabola}$$



when  $y = 0, \quad x = \frac{1}{2}P$

$$x = 0, \quad y = \pm P$$

Time equation:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m} [E - U(r)] - \frac{m^2}{r^2}}} \quad t \text{ vs } r$$

$$U(r) = -\frac{\alpha}{r}, \quad p = \frac{mv^2}{m\alpha}, \quad e=1, \quad E=0$$

$$\rightarrow t = \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{mv^2}{m^2 r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{mv^2}{m^2 r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{dr}{\sqrt{\frac{2}{r} - \frac{p^2}{r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{r dr}{\sqrt{2r-p^2}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha}} \int \frac{r dr}{\sqrt{\frac{2r-1}{p}}} + \text{const}$$

Defn:  $\frac{2r-1}{p} = \xi^2 > 0 \quad (-\infty < \xi < \infty)$

$$2r = p(1+\xi^2)$$

$$\boxed{r = \frac{p}{2}(1+\xi^2)} \rightarrow dr = p\xi d\xi$$

$$r dr = \frac{p^2}{2}(\xi + \xi^3) d\xi$$

Thus,

$$t = \sqrt{\frac{m}{\alpha p}} \int \frac{\frac{p^2}{2}(\xi + \xi^3) d\xi}{\sqrt{\xi^2}} + \text{const}$$

$$= \sqrt{\frac{mp^3}{\alpha}} \frac{1}{2} \int (1+\xi^2) d\xi + \text{const}$$

$$= \frac{1}{2} \sqrt{\frac{mp^3}{\alpha}} \left( \xi + \frac{1}{3}\xi^3 \right) + \text{const}$$

choose const so that  $t=0 \Leftrightarrow \xi = 0$  (const=0)

$$\text{so, } \boxed{t = \frac{1}{2} \sqrt{\frac{mp^3}{\alpha}} \left( \xi + \frac{1}{3}\xi^3 \right)}$$

Now:

$$\frac{p}{r} = 1 + \cos\phi$$

$$p = r(1 + \cos\phi)$$

$$\phi = \frac{p}{2}(1+\xi^2)(1+\cos\phi)$$

$$Z = 1 + \xi^2 + \cos\phi + \xi^2 \cos\phi$$

$$1 - \xi^2 = (1+\xi^2) \cos\phi$$

$$\rightarrow \boxed{\cos\phi = \frac{1-\xi^2}{1+\xi^2}}$$

$$\begin{aligned} X &= r \cos \phi \\ &= \frac{p}{2} (1 + \xi^2) \left( \frac{1 - \xi^2}{1 + \xi^2} \right) \\ &= \frac{p}{2} (1 - \xi^2) \end{aligned}$$

Also,

$$\begin{aligned} x^2 + y^2 &= r^2 \\ \rightarrow y^2 &= r^2 - x^2 \\ &= \frac{p^2}{4} (1 + \xi^2)^2 - \frac{p^2}{4} (1 - \xi^2)^2 \\ &= \frac{p^2}{4} (x + \xi)^2 + 2\xi^2 - (x - \xi)^2 + 2\xi^2 \\ &= p^2 \xi^2 \end{aligned}$$

so  $y = p \xi$

Sec 15, Prob 3:

$$\Delta\phi = 2 \int_{r_{\min}}^{r_{\max}} \frac{M dr / r^2}{\sqrt{2m(E-U) - M^2/r^2}}$$

This is the change in  $\phi$  as  $r$  goes from  $r_{\min}$  to  $r_{\max}$  and then back to  $r_{\min}$ .

A closed bound orbit would have  $\Delta\phi = 2\pi m/n$  for  $m, n$  integers.

Consider:  $U = -\frac{\alpha}{r} + \delta U$  where  $|\delta U| \ll |\frac{\alpha}{r}|$

$$\text{For } \delta U = 0, \Delta\phi = 2\pi$$

$$\Delta\phi = 2 \int_{r_{\min}}^{r_{\max}} \frac{M dr / r^2}{\sqrt{2m(E - (U + \delta U)) - M^2/r^2}}$$

$$= 2 \int_{r_{\min}}^{r_{\max}} \frac{M dr / r^2}{\sqrt{2m(E-U) - 2m\delta U - M^2/r^2}}$$

$$= 2 \int_{r_{\min}}^{r_{\max}} \frac{M dr / r^2}{\sqrt{(2m(E-U) - M^2/r^2)} \left( 1 - \frac{2m\delta U}{2m(E-U) - M^2/r^2} \right)}$$

$$\approx 2 \int_{r_{\min}}^{r_{\max}} \frac{M dr / r^2}{\sqrt{2m(E-U) - M^2/r^2}} \left[ 1 + \frac{m\delta U}{2m(E-U) - M^2/r^2} \right]$$

For  $V = -\alpha/r$ :

$$\Delta\phi \approx 2\pi + 2 \int_{r_{min}}^{r_{max}} \frac{Md\tau/r^2}{\sqrt{2m(E-U) - M^2/r^2}} d\delta U$$

$$\approx 2\pi + \delta\phi$$

where

$$r_{max}$$

$$\delta\phi \equiv \int_{r_{min}}^{r_{max}} \frac{2m\delta U M dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}}^{3/2}$$

$$= \frac{2}{2M} \left[ \int_{r_{min}}^{r_{max}} \frac{2m\delta U dr}{\sqrt{2m(E-U) - M^2/r^2}} \right]$$

Evaluate terms in integrand along unperturbed path since  $\delta U$  is already small

$$\frac{p}{r} = 1 + e\cos\phi \quad , \quad p = q(1-e^2)$$

$$\rightarrow -\frac{p}{r^2} dr = -e\sin\phi d\phi \rightarrow \boxed{dr = \frac{r^2 e \sin\phi}{p} d\phi}$$

$$\sqrt{ } = \sqrt{2mE + 2m\alpha - \frac{M^2}{r^2}}$$

$$= \sqrt{-2m|E| + \frac{4m|E|\alpha}{r} - \frac{2m|E|p^2}{(1-e^2)^2}}$$

$$\begin{aligned} \sqrt{ } &= \sqrt{2m|E|} \sqrt{-1 + \frac{2}{(1-e^2)}(1+e\cos\phi) - \frac{1}{(1-e^2)}(1+e\cos\phi)^2} \\ &= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}} \sqrt{-1/(1-e^2) + 2/(1+e\cos\phi) - (1+2e\cos\phi + e^2\cos^2\phi)} \\ &= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}} \sqrt{e^2(1-\cos^2\phi)} \\ &= \frac{\sqrt{2m|E|}}{\sqrt{1-e^2}} e \sin\phi \end{aligned}$$

Thus,

$$\begin{aligned} |\delta\phi| &= \frac{2}{2M} \left[ 2m \int_0^\pi \frac{\delta U r^2 e \sin\phi d\phi}{\sqrt{1-e^2}} \right] \\ &= \frac{2}{2M} \left[ \frac{2m}{p \sqrt{\frac{2m|E|}{1-e^2}}} \int_0^\pi d\phi r^2 \delta U \right] \\ &= \frac{2}{2M} \left[ \frac{2m}{M} \int_0^\pi d\phi r^2 \delta U \right] \end{aligned}$$

$$\text{using (15.6): } \frac{M}{\sqrt{2m|E|}} = \frac{p}{\sqrt{1-e^2}}$$

Evaluate:

$$\delta\phi = \frac{\partial}{\partial M} \left[ \frac{2m}{M} \int_0^\pi d\phi r^2 \delta U \right]$$

For (a)  $\delta U = \beta/r^2$ , (b)  $\delta U = \gamma/r^3$

(a)  $\delta\phi = \frac{\partial}{\partial M} \left[ \frac{2m}{M} \int_0^\pi d\phi \beta \right]$

$$= 2\pi\beta m \frac{\partial}{\partial M} \left( \frac{1}{M} \right)$$

$$= -\frac{2\pi\beta m}{M^2}$$

$$= \boxed{-\frac{2\pi\beta}{\alpha p}}$$

Recall:  $J^2 = \frac{M^2}{m\alpha}$

$$\leftarrow \alpha p = \frac{M^2}{m}$$

(b)  $\delta\phi = \frac{\partial}{\partial M} \left[ \frac{2m}{M} \int_0^\pi d\phi \frac{\gamma}{r} \right]$

$$= \frac{\partial}{\partial M} \left[ \frac{2m\gamma}{M} \int_0^\pi d\phi \left( \frac{1 + e^{i\phi}}{r} \right) \right]$$

$$= 2m\gamma \frac{\partial}{\partial M} \left[ \frac{1}{mp} \left( \pi + e^{i\phi} \right) \right]$$

Now:  $\frac{1}{mp} = \frac{m\alpha}{M^3} \rightarrow \frac{\partial}{\partial M} \left( \frac{1}{mp} \right) = -\frac{3m\alpha}{M^4}$

so  $\delta\phi = -6\pi\gamma \frac{m^2\alpha}{M^4}$

$$= -6\pi\gamma\alpha \left( \frac{1}{p\alpha} \right)^2$$

$$= \boxed{-\frac{6\pi\gamma}{p^2\alpha}}$$

Sec 16, Prob 2

From (16.7) we have

$$p(\theta_0)d\theta_0 = \frac{1}{2} \sin\theta_0 d\theta_0 \\ = -\frac{1}{2} d(\cos\theta_0)$$

where  $\theta_0$  is the angle of one of the emitted particles in the com frame.

We would like to find  $p(\theta)d\theta$ , where  $\theta$  is the angle of one of the emitted particles in the lab frame.

$$\sin\theta p(\theta)d\theta = p(\theta_0)d\theta_0$$

we just need to find  $\theta_0$  as a function of  $\theta$ .

This is given by (16.6) which we first derive.

Proof: Given  $\tan\theta = \frac{v_0 \sin\theta_0}{v_0 \cos\theta_0 + V}$

we have

$$\tan\theta(v_0 \cos\theta_0 + V) = v_0 \sqrt{1 - \cos^2\theta_0}$$

Square both sides

$$\tan^2\theta(v_0^2 \cos^2\theta_0 + V^2 + 2v_0 V \cos\theta_0) = v_0^2(1 - \cos^2\theta_0)$$

$$\underbrace{(1 + \tan^2\theta)v_0^2 \cos^2\theta_0}_{\sec^2\theta} + 2v_0 V \tan^2\theta \cos\theta_0$$

$$+ (V^2 + \tan^2\theta - v_0^2) = 0$$

Quadratic equation for  $\cos\theta_0$ .

$$\rightarrow \cos\theta_0 = -2v_0 V \tan^2\theta \pm \sqrt{4v_0^2 V^2 \tan^4\theta - 4v_0^2 \sec^2\theta (V^2 \tan^2\theta - v_0^2)} \\ \geq v_0^2 \sec^2\theta$$

$$= -\frac{V}{v_0} \sin^2\theta \pm \frac{1}{\sec\theta} \sqrt{\left(\frac{V}{v_0}\right)^2 \tan^4\theta - \sec^2\theta \left(\left(\frac{V}{v_0}\right)^2 \tan^2\theta - 1\right)}$$

$$= -\frac{V}{v_0} \sin^2\theta \pm \frac{1}{\sec\theta} \sqrt{\left(\frac{V}{v_0}\right)^2 \left(\frac{\sin^4\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}\right) + 1}$$

$$= -\frac{V}{v_0} \sin^2\theta \pm \cos\theta \sqrt{\left(\frac{V}{v_0}\right)^2 \frac{\sin^2\theta(\sin^2\theta - 1)}{\cos^2\theta} + 1}$$

$$= -\frac{V}{v_0} \sin^2\theta \pm \cos\theta \sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}$$

(for  $v_0 > V$  take +, for  $v_0 < V$  take both signs.)

Now differentiate both sides:

$$d(\cos\theta_0) = -\frac{V}{v_0} 2 \sin\theta \cos\theta d\theta \mp \sin\theta \cos\theta \sqrt{\cos^2\theta \left(\frac{1}{v_0^2}\right) + \left(\frac{V}{v_0}\right)^2 \sin^2\theta \cos\theta d\theta}$$

$$= \sin\theta \cos\theta \left[ -2 \frac{V \cos\theta}{v_0} \mp \sqrt{1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta} \frac{1}{v_0} \right]$$

Now:

$$\begin{aligned} & \sqrt{1 + \left(\frac{V}{v_0}\right)^2 \cos^2 \theta} \sqrt{\frac{1}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}}} \\ &= \frac{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta + \left(\frac{V}{v_0}\right)^2 \cos^2 \theta}{\sqrt{1 + \left(\frac{V}{v_0}\right)^2 (\cos^2 \theta - \sin^2 \theta)}} \\ &= \frac{1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta}{\sqrt{1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta}} \end{aligned}$$

Thus,

$$d(1/\omega_0) = \underbrace{-\sin \theta d\theta}_{d(\cos \theta)} \left[ 2 \frac{V}{v_0} \cos \theta \pm \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}} \right]$$

So:

$$p(\theta) d\theta = \frac{1}{2} \sin \theta d\theta \left[ 2 \frac{V}{v_0} \cos \theta \pm \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}} \right]$$

Now: for  $v_0 > V$ , take + sign ( $\theta \in [0, \pi]$ )

for  $v_0 < V$ , as  $\theta_0$  increases from 0 to  $\pi$

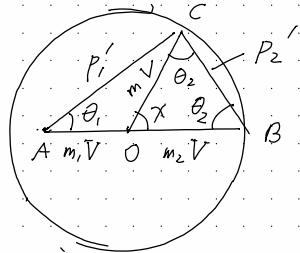
$\theta$  increases from  $0 \rightarrow \theta_{\max}$   
 $\theta$  decreases from  $\theta_{\max} \rightarrow 0$

Thus, for  $v_0 < V$  need to take the difference of the + and - expressions:

$$\begin{aligned} p(\theta) d\theta &= \frac{1}{2} \sin \theta d\theta \left[ 2 \frac{V}{v_0} \cos \theta + \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2 \theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}} \right] \\ &\quad - \frac{1}{2} \sin \theta d\theta \left[ 2 \frac{V}{v_0} \cos \theta - \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2 \theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}} \right] \\ &= \frac{\sin \theta d\theta \left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2 \theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}} \end{aligned}$$

which is valid for  $0 \leq \theta \leq \theta_{\max} = \sin^{-1}\left(\frac{v_0}{V}\right)$

Sec 17, Prob 1:



$$\begin{aligned} x + 2\theta_2 &= \pi \\ x &= \pi - 2\theta_2 \end{aligned}$$

From the above figure:

$$\begin{aligned} (m_2 v_2')^2 &= 2(mv)^2 - 2(mv)^2 \cos x \\ &= 2m^2 v^2 (1 - \cos(\pi - 2\theta_2)) \\ &= 2m^2 v^2 \left[ 1 - (\cos(\pi) \cos(2\theta_2) + \sin(\pi) \sin(2\theta_2)) \right] \\ &= 2m^2 v^2 (1 + \cos(2\theta_2)) \end{aligned}$$

$$\rightarrow v_2' = \sqrt{2} \left( \frac{m}{m_2} \right) v \sqrt{1 + (\cos^2 \theta_2 - \sin^2 \theta_2)}$$

$$= \sqrt{2} \left( \frac{m_1}{m_1 + m_2} \right) v \sqrt{2 \cos^2 \theta_2}$$

$$= 2 \left( \frac{m_1}{m_1 + m_2} \right) v \cos \theta_2$$

$$\text{Thus, } \left( \frac{v_2'}{v} \right) = \left( \frac{2m_1}{m_1 + m_2} \right) \cos \theta_2$$

Also,

$$\begin{aligned} (mv)^2 &= (m_1 v_1')^2 + (m_2 v_2')^2 - 2m_1^2 v_1' v_2' \cos \theta_1 \\ \Rightarrow (m_1 v_1')^2 &- 2m_1 v_1' v_2' \cos \theta_1 (m_2 v_2') + m_1^2 v_1'^2 - m_1^2 v^2 = 0 \end{aligned}$$

$$\text{Now: } V = \frac{m_1 v_1' + m_2 v_2'}{m_1 + m_2} = \frac{m_1 v}{m_1 + m_2}$$

Thus,

$$\begin{aligned} (m_1 v_1')^2 &- 2 \left( \frac{m_1^2 v}{m_1 + m_2} \right) \cos \theta_1 m_1 v_1' + \frac{m_1^2 m_2^2 v^2}{(m_1 + m_2)^2} \\ &- \frac{m_1^2 m_2^2}{(m_1 + m_2)^2} v^2 = 0 \end{aligned}$$

$$\rightarrow (v_1')^2 - 2 \left( \frac{m_1}{m_1 + m_2} \right) \cos \theta_1 v_1' + \frac{m_1^2 - m_2^2}{(m_1 + m_2)^2} v^2 = 0$$

$$(v_1')^2 - 2 \left( \frac{m_1 v}{m_1 + m_2} \right) \cos \theta_1 v_1' + \left( \frac{m_1 - m_2}{m_1 + m_2} \right) v^2 = 0$$

$$\left( \frac{v_1'}{v} \right)^2 - 2 \left( \frac{m_1}{m_1 + m_2} \right) \cos \theta_1 \left( \frac{v_1'}{v} \right) + \left( \frac{m_1 - m_2}{m_1 + m_2} \right) = 0$$

Quadratic equations

$$\frac{v'}{v} = \frac{2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \sqrt{4\left(\frac{m_1}{m_1+m_2}\right)^2\cos^2\theta_1 - 4\left(\frac{m_1-m_2}{m_1+m_2}\right)}}{2}$$

2

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right) \sqrt{m_1^2\cos^2\theta_1 - (m_1-m_2)(m_1+m_2)}$$

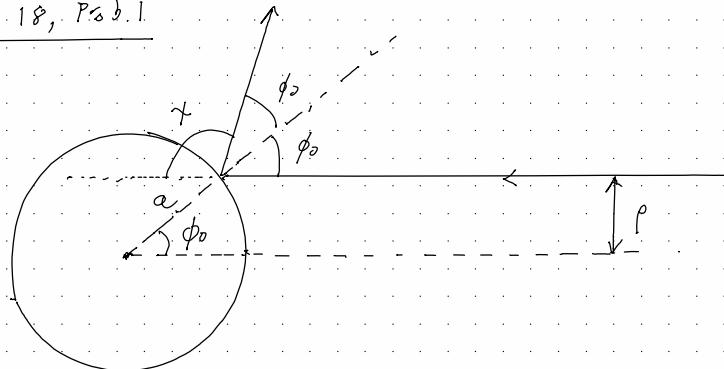
$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right) \sqrt{m_1^2(\cos^2\theta_1 - 1) + m_2^2}$$

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right) \sqrt{m_2^2 - m_1^2\sin^2\theta_1}$$

The + sign holds for  $m_1 < m_2$

+/- signs hold for  $m_1 > m_2$

Sec 18, Prob. 1



$$x + 2\phi_0 = \pi \rightarrow \phi_0 = \frac{\pi}{2} - \frac{x}{2}$$

$$r = a \sin \phi_0$$

$$= a \sin\left(\frac{\pi}{2} - \frac{x}{2}\right)$$

$$= a \left( \sin\left(\frac{\pi}{2}\right) \cos\left(\frac{x}{2}\right) - \cos\left(\frac{\pi}{2}\right) \sin\left(\frac{x}{2}\right) \right)$$

$$= a \cos\left(\frac{x}{2}\right)$$

$$d\sigma = 2\pi r d\rho$$

$$= 2\pi r(x) \left| \frac{dp}{dx} \right| dx$$

$$= \frac{p'(x)}{\sin x} \left| \frac{dp}{dx} \right| d\Omega$$

where  $d\Omega = \text{solid angle}$

$$= 2\pi \sin x dx$$

integral over  
 $d\phi$

$$\rho = a \cos\left(\frac{x}{2}\right)$$

$$\begin{aligned} d\rho &= -a \frac{1}{2} \sin\left(\frac{x}{2}\right) dx \\ &= -\frac{a}{2} \sin\left(\frac{x}{2}\right) dx \end{aligned}$$

thus,

$$\begin{aligned} d\sigma &= \frac{\rho(x)}{\sin x} \left| \frac{d\rho}{dx} \right| d\Omega \\ &= \frac{a \cos(x/2)}{\sin x} \frac{a}{2} \sin\left(\frac{x}{2}\right) d\Omega \\ &= \frac{a^2}{2} \frac{\sin(X_2) \cos(X/2)}{\sin x} d\Omega \\ &= \boxed{\frac{a^2}{4} d\Omega} \quad (\text{since } \sin x = 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)) \end{aligned}$$

Total cross section.

$$\Sigma = \int d\sigma = \frac{a^2}{4} \int d\Omega = \frac{a^2}{4} \cdot 4\pi = \boxed{4\pi a^2}$$

Now calculate differential cross section in

lab frame for both  $m_1$  and  $m_2$

Use the result that

$$d\sigma_1 = \frac{\rho(\theta_1)}{\sin \theta_1} \left| \frac{d\rho}{d\theta_1} \right| d\Omega_1 = \rho \left| \frac{d\rho}{d(\cos \theta_1)} \right| d\Omega_1$$

comprise to:

$$d\sigma = \rho \left| \frac{d\rho}{d(\cos x)} \right| d\Omega$$

$$\begin{aligned} \frac{d\sigma_1}{d\Omega_1} &= \rho \left| \frac{d\rho}{d(\cos \theta_1)} \right| \\ &= \left| \frac{d(\cos x)}{d(\cos \theta_1)} \right| \frac{d\sigma}{d\Omega} \end{aligned}$$

So we need to evaluate:

$$\frac{d(\cos x)}{d(\cos \theta_1)} \quad \text{and} \quad \frac{d(\cos x)}{d(\cos \theta_2)}$$

start with  $\theta_2$ :  $(17.4)$

$$\theta_2 = \frac{1}{2}(\pi - x) \rightarrow \boxed{x = \pi - 2\theta_2}$$

$$\Rightarrow \cos x = \cos(\pi - 2\theta_2)$$

$$= \cos \pi \cos(2\theta_2) + \sin \pi \sin(2\theta_2)$$

$$= -\cos(2\theta_2)$$

$$= -(\cos^2 \theta_2 - \sin^2 \theta_2)$$

$$= -(2\cos^2 \theta_2 - 1)$$

$$= -2\cos^2 \theta_2 + 1$$

$$\text{Thus, } \boxed{d(\cos x) = -4 \cos \theta_2 d(\cos \theta_2)}$$

Thus,

$$\begin{aligned}\frac{d\sigma_2}{d\Omega_2} &= \frac{d\sigma}{d\Omega} \left| \frac{d(\cos X)}{d(\cos \theta_2)} \right| \\ &= \frac{1}{4} a^2 \cdot |4 \cos \theta_2| \\ &= a^2 |\cos \theta_2|\end{aligned}$$

So  $d\sigma_2 = a^2 |\cos \theta_2| d\Omega_2$

Now consider  $\theta_1$ :

From (17.4):

$$t_{11} \theta_1 = \frac{m_2 \sin X}{m_1 + m_2 \cos X}$$

Compare with

$$t_{11} \theta = \frac{v_0 \sin \theta_0}{V + v_0 \cos \theta_0} \quad (16.5)$$

make identifications:  $\theta \rightarrow \theta_1$ ,  $v_0 \rightarrow m_2$ ,  $V \rightarrow m_1$

Then we can write down from (16.6).

$$\cos X = -\frac{m_1 \sin^2 \theta_1}{m_2} \pm \cos \theta_1 \sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2 \theta_1}$$

[See also sec 16, prob. 2 where we derived this for  $\theta$  and  $\theta_0$ .]

We also worked out the derivative:

$$d(\cos \theta_1) = d(\cos \theta) \left[ 2 \frac{V \cos \theta}{v_0} \pm \frac{1 + \left(\frac{V}{v_0}\right)^2 \cos^2 \theta}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2 \theta}} \right]$$

so we can similarly write down

$$d(\cos X) = d(\cos \theta_1) \left[ 2 \frac{m_1 \cos \theta_1}{m_2} \pm \frac{1 + \left(\frac{m_1}{m_2}\right)^2 \cos(2\theta_1)}{\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2 \theta_1}} \right]$$

~~~~~

For  $m_1 < m_2$ : take + sign

For  $m_1 > m_2$ : As  $X$  increases from 0 to  $\pi$ ,

$\theta_1$  increases from 0 to  $\theta_{\max}$ ; then  $\theta_1$  decreases from  $\theta_{\max}$  to 0. In that case

$$\begin{aligned}d(\cos X) &= d(\cos \theta_1) [\cancel{\theta} + \theta] - d(\cos \theta_1) [\cancel{\theta} - \theta] \\ &= 2 d(\cos \theta_1) \frac{1 + \left(\frac{m_1}{m_2}\right)^2 \cos(2\theta_1)}{\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2 \theta_1}}\end{aligned}$$

Use: (for both)

$$\begin{aligned}d\sigma_1 &= \left( \frac{d\sigma}{d\Omega} \right) \left| \frac{d(\cos X)}{d(\cos \theta_1)} \right| d\Omega_1 \\ &\approx \frac{1}{4} a^2 \left| \frac{d(\cos X)}{d(\cos \theta_1)} \right| d\Omega_1\end{aligned}$$

16 v,

For  $m_1 < m_2$ :

$$d\sigma_1 = \frac{1}{4} a^2 \left[ 2 \left( \frac{m_1}{m_2} \right) \cos \theta_1 + \frac{1 + \left( \frac{m_1}{m_2} \right)^2 \cos(2\theta_1)}{\sqrt{1 - \left( \frac{m_1}{m_2} \right)^2 \sin^2 \theta_1}} \right] d\Omega_1$$

For  $m_1 > m_2$ :

$$d\sigma_1 = \frac{1}{4} a^2 \cdot 2 \frac{1 + \left( \frac{m_1}{m_2} \right)^2 \cos(2\theta_1)}{\sqrt{1 - \left( \frac{m_1}{m_2} \right)^2 \sin^2 \theta_1}} d\Omega_1$$
$$= \frac{a^2}{2} \frac{1 + \left( \frac{m_1}{m_2} \right)^2 \cos(2\theta_1)}{\sqrt{1 - \left( \frac{m_1}{m_2} \right)^2 \sin^2 \theta_1}} d\Omega_1$$

Sec 18, Prob 2:

Hard sphere scattering again.

Calculate  $d\sigma$  in terms of  $dE$  where  
 $E$  = energy lost by scattered particle.

Now:  $E$  = energy lost by scattered particle

= energy gained by  $m_2$

$$= \frac{1}{2} m_2 (V_2')^2$$

From Fig 16., we have (law of cosines):

$$(m_2 V_2')^2 = (mV)^2 + (mV)^2 - 2(mV)^2 \cos X$$

$$= 2(mV)^2 [1 - \cos X]$$

$$= 2(mV)^2 2 \sin^2 \left( \frac{X}{2} \right)$$

$$\text{so } m_2 V_2' = 2mV \sin \left( \frac{X}{2} \right)$$

$$\rightarrow V_2' = 2 \left( \frac{m}{m_2} \right) V \sin \left( \frac{X}{2} \right)$$

$$= 2 \left( \frac{m_1}{m_1 + m_2} \right) V \sin \left( \frac{X}{2} \right) \quad (17.5)$$

$$\text{NOTE: } d\sigma = \frac{1}{4} a^2 d\Omega$$

$$= \frac{1}{4} a^2 2\pi \sin X dx$$

$$= \frac{\pi a^2}{2} \int d(\cos X) /$$

So we would like to relate  $dE$  and  $d(\cos X)$ .

$$\text{Now: } E = \frac{1}{2} m_2 (v_z')^2$$

$$= \frac{1}{2} m_2 \frac{4 m_1^2 v^2}{(m_1 + m_2)^2} \sin^2\left(\frac{\chi}{2}\right)$$

$$= \frac{Z m_1^2 m_2}{(m_1 + m_2)^2} V_\infty^2 \sin^2\left(\frac{\chi}{2}\right) \quad (\text{since } V = V_\infty)$$

$$= E_{max} \sin^2\left(\frac{\chi}{2}\right)$$

$$\text{where } E_{max} = \frac{Z m_1^2 m_2}{(m_1 + m_2)^2} V_\infty^2$$

$$= 4 \left(\frac{m_1}{m_1 + m_2}\right) \frac{m_1 V_\infty^2}{2}$$

$$= 4 \left(\frac{m_1}{m_1 + m_2}\right) E$$

$$\text{Thus, } dE = E_{max} 2 \sin\left(\frac{\chi}{2}\right) \cos\left(\frac{\chi}{2}\right) \frac{d\chi}{2}$$

$$= \frac{1}{2} E_{max} \sin X dX$$

$$= \frac{1}{2} E_{max} |d(\cos X)|$$

$$\text{So } d\sigma = \frac{\pi q^2}{2} |d(\cos X)|$$

$$= \frac{\pi q^2}{2} \frac{2}{E_{max}} dE = \boxed{\frac{\pi q^2}{E_{max}} dE}$$

which is a uniform distribution w.r.t.  $E$ .

### Sect 18, Prob. 4

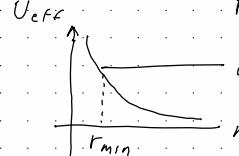
Effective cross section to "fall" to center of

$$U(r) = -\alpha/r^2 \quad (\alpha > 0)$$

$$U_{eff}(r) = U(r) + \frac{M^2}{2mr^2}$$

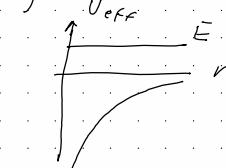
$$= -\frac{\alpha}{r^2} + \frac{M^2}{2mr^2}$$

$$= \frac{1}{r^2} \left( \frac{M^2}{2m} - \alpha \right)$$



$$\frac{M^2}{2m} - \alpha > 0$$

(don't fall to center)  
since  $r_{min} > 0$



$$\frac{M^2}{2m} - \alpha < 0$$

Fall to center ( $r=0$ )

For a given  $E = \frac{1}{2} m V_\infty^2 > 0$  need

$$\frac{M^2}{2m} - \alpha < 0$$

$$\frac{M^2}{2m} < \alpha$$

$$\rightarrow M_{max} \leq \sqrt{2m\alpha}$$

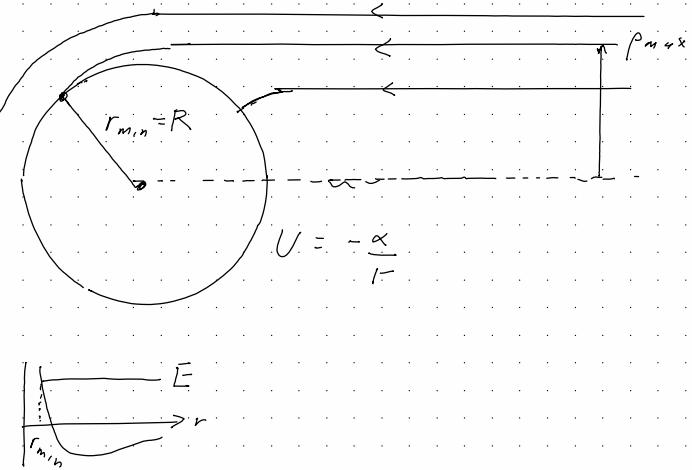
Cross section:  $\sigma = \pi M_{max}^2$ ,  $M = \rho m V_\infty$

Thus,

$$\begin{aligned}\sigma &= \pi p_{\max}^2 \\ &= \pi \frac{M_{\max}^2}{m^2 V_{\infty}^2} \\ &= \pi \frac{Z_m \alpha}{m^2 V_{\infty}^2} \\ &= \frac{\pi \alpha}{\frac{1}{2} m V_{\infty}^2} \\ &= \boxed{\frac{\pi \alpha}{E}}\end{aligned}$$

$$p_{\max} = \frac{M_{\max}}{m V_{\infty}}$$

Sec 18, Prob 6



turning point at  $r = R$

$$\begin{aligned}0 &= E - U_{eff}(R) \\ &= E - U(R) - \frac{M_{\max}^2}{2mR^2} \\ &= E + \frac{\alpha}{R} - \frac{M_{\max}^2}{2mR^2}\end{aligned}$$

$$\rightarrow \frac{M_{\max}^2}{2mR^2} = E + \frac{\alpha}{R}$$

$$M_{\max} = p_{\max} m V_{\infty}, \quad E = \frac{1}{2} m V_{\infty}^2$$

Thurj.

$$O = \pi \rho^2$$

$$= \pi \frac{M_{\max}^2}{m^2 V_\infty^2}$$

$$= \pi \frac{1}{m^2 V_\infty^2} 2mR^2 \left( E + \frac{\alpha}{R} \right)$$

$$= \pi R^2 \left( \frac{2}{m V_\infty^2} \right) \left( E + \frac{\alpha}{R} \right)$$

$\underbrace{\frac{1}{E}}$

$$= \boxed{\pi R^2 \left( 1 + \frac{\alpha}{ER} \right)}$$

where  $E = \frac{1}{2} m V_\infty^2 = \frac{1}{2} \left( \frac{m_1 m_2}{m_1 + m_2} \right) V_\infty^2$

and  $\alpha = G m_1 m_2$

Sec 19, Prob 1:

$$U = \frac{\alpha}{r^2}, \quad \alpha > 0$$

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{M dr / r^2}{\sqrt{2m [E - U(r)] - m^2 / r^2}}$$

Substitute:  $E = \frac{1}{2} m V_\infty^2$

$$M = \rho m V_\infty$$

$$\rightarrow \phi_0 = \int_{r_{\min}}^{\infty} \frac{\rho m V_\infty dr / r^2}{\sqrt{2m \left[ \frac{1}{2} m V_\infty^2 - U(r) \right] - \rho^2 m^2 V_\infty^2 / r^2}}$$

$$= \int_{r_{\min}}^{\infty} \frac{\rho dr / r^2}{\sqrt{1 - 2U(r)/m V_\infty^2 - \rho^2/r^2}}$$

$$= \int_{r_{\min}}^{\infty} \frac{\rho dr / r^2}{\sqrt{1 - \rho^2/r^2 - 2U(r)/m V_\infty^2}}$$

Substitute:  $U(r) = \frac{\alpha}{r^2}$

$$\sqrt{\quad} = \sqrt{1 - \rho^2/r^2 - \left( \frac{2\alpha}{m V_\infty^2} \right) \frac{1}{r^2}}$$

$$\therefore \sqrt{1 - \left( \rho^2 + \frac{2\alpha}{m V_\infty^2} \right) \frac{1}{r^2}} = \sqrt{1 - \frac{A^2}{r^2}}$$

$$A^2 = \rho^2 + \frac{2\alpha}{m V_\infty^2}$$

thus,

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{\rho dr / r^2}{\sqrt{1 - \frac{A^2}{r^2}}} = \int_A^{\infty} \frac{\rho dr / r^2}{\sqrt{1 - \frac{A^2}{r^2}}}$$

$$\text{Let } u = \frac{1}{r} \rightarrow du = -\frac{1}{r^2} dr$$

$$\frac{A^2}{r^2} = A^2 u^2$$

$$\phi_0 = - \int_{\frac{1}{A}}^{\frac{1}{A}} \frac{\rho du}{\sqrt{1 - A^2 u^2}}$$

$$= \int_0^{\frac{1}{A}} \frac{\rho du}{\sqrt{1 - A^2 u^2}}$$

$$\text{Let } \sin \theta = Au$$

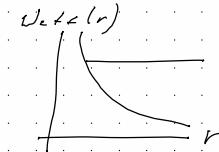
$$c \omega \theta d\theta = Adu$$

$$u=0, \frac{\pi}{A} \rightarrow \theta=0, \frac{\pi}{2}$$

$$\sqrt{1 - A^2 u^2} = \sqrt{1 - \sin^2 \theta} = \cos \theta$$

$$\rightarrow \phi_0 = \int_0^{\frac{\pi}{2}} \frac{\rho c \omega \theta d\theta / A}{\cos \theta} = \frac{\rho \frac{\pi}{2}}{A}$$

$$\boxed{\phi_0 = \frac{\pi}{2} \frac{\rho}{\sqrt{\rho^2 + \frac{2\alpha}{m V_\infty^2}}}}$$



$$\text{Repulsive scattering: } \chi + 2\phi_0 = \pi$$

$$\chi = \pi - 2\phi_0$$

$$\chi = \pi - \pi \frac{\rho}{\sqrt{\rho^2 + \frac{2\alpha}{m V_\infty^2}}} = \pi \left[ 1 - \frac{1}{\sqrt{1 + \frac{2\alpha}{\rho^2 m V_\infty^2}}} \right]$$

$$\left( \frac{\pi \rho}{\sqrt{\rho^2 + \frac{2\alpha}{m V_\infty^2}}} \right)^2 = (\pi - \chi)^2$$

$$\frac{\pi^2 \rho^2}{\rho^2 + \frac{2\alpha}{m V_\infty^2}} = (\pi - \chi)^2$$

$$\pi^2 \rho^2 = (\pi - \chi)^2 \rho^2 + (\pi - \chi)^2 \frac{2\alpha}{m V_\infty^2}$$

$$(\pi^2 - (\pi - \chi)^2) \rho^2 = (\pi - \chi)^2 \frac{2\alpha}{m V_\infty^2}$$

$$(\pi^2 - \pi^2 + 2\pi\chi - \chi^2 + 2\pi\chi) \rho^2 = (\pi - \chi)^2 \frac{2\alpha}{m V_\infty^2}$$

$$\rho^2 = \frac{(\pi - \chi)^2}{2\pi\chi - \chi^2} \frac{2\alpha}{m V_\infty^2}$$

$$\boxed{\rho = \frac{(\pi - \chi)}{\sqrt{2\pi\chi - \chi^2}} \sqrt{\frac{2\alpha}{m V_\infty^2}}}$$

Differential cross-section:

$$\begin{aligned} d\sigma &= 2\pi \rho d\rho \\ &= 2\pi \rho(x) \left| \frac{d\rho}{dx} \right| dx \\ &= \frac{\rho(x)}{\sin x} \left| \frac{d\rho}{dx} \right| d\Omega, \quad d\Omega = 2\pi \sin x dx \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d\rho}{dx} &= \frac{2\alpha}{\sqrt{mV_\infty^2}} \frac{-\sqrt{2\pi x - x^2} - 2\sqrt{(2\pi - 2x)(\pi - x)}}{2\pi x - x^2} \\ &= -\frac{\sqrt{2\alpha}}{mV_\infty^2} \frac{\sqrt{2\pi x - x^2} + \frac{(\pi - x)^2}{\sqrt{}}}{2\pi x - x^2} \\ &= -\frac{\sqrt{2\alpha}}{mV_\infty^2} \frac{2\pi x - x^2 + (\pi - x)^2}{(2\pi x - x^2)^{3/2}} \\ &= -\frac{\sqrt{2\alpha}}{mV_\infty^2} \frac{2\pi x - x^2 + \pi^2 + x^2 - 2\pi x}{(2\pi x - x^2)^{3/2}} \\ &= -\frac{\sqrt{2\alpha}}{mV_\infty^2} \frac{\pi^2}{(2\pi x - x^2)^{3/2}} \end{aligned}$$

So,

$$\begin{aligned} d\sigma &= \frac{(\pi - x)}{\sqrt{2\pi x - x^2}} \frac{\sqrt{2\alpha}}{mV_\infty^2} \frac{1}{\sin x} \frac{\sqrt{2\alpha}}{\sqrt{m\alpha}} \frac{\pi^2}{(2\pi x - x^2)^{3/2}} d\Omega \\ &\approx \left[ \left( \frac{2\alpha}{mV_\infty^2} \right) \frac{d\Omega}{\sin x} \frac{\pi^2(\pi - x)}{(2\pi x - x^2)^{3/2}} \right] 2 \end{aligned}$$

Sec 20, Prob. 1 Small-angle scattering

start with (18.4):

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{\rho dr/r^2}{\sqrt{1 - \rho^2/r^2 - 2U/mV_\infty^2}}$$

Assume  $U$  is weak so that  $2U/mV_\infty^2 \ll 1$

$$\begin{aligned} \frac{1}{\sqrt{1 - \rho^2/r^2}} &= \frac{1}{\sqrt{(1 - \rho^2/r^2)} \left( 1 - \frac{2U/mV_\infty^2}{(1 - \rho^2/r^2)} \right)} \\ &\approx \frac{1}{\sqrt{1 - \rho^2/r^2}} \left( 1 + \frac{2U/mV_\infty^2}{1 - \rho^2/r^2} \right) \\ &= \frac{1}{\sqrt{1 - \rho^2/r^2}} + \frac{U/mV_\infty^2}{(1 - \rho^2/r^2)^{3/2}} \end{aligned}$$

can replace  $r_{\min}$  limit by  $\rho$ :

$$\int_{\rho}^{\infty} \frac{\rho dr/r^2}{\sqrt{1 - \rho^2/r^2}} = - \int_{\rho}^0 \frac{\rho dy}{\sqrt{1 - \rho^2 y^2}}$$

$$\begin{aligned} \text{let } u &= \frac{y}{r} \\ dy &= -\frac{1}{r^2} dr \\ \text{let } \rho u &= \sin \theta \\ \rho du &= \cos \theta d\theta \\ u = 1 &\rightarrow \theta = \frac{\pi}{2} \\ &= \left[ \frac{\pi}{2} \right] \end{aligned}$$

Then,

$$\phi_0 \approx \frac{\pi}{2} + \frac{1}{m v_\infty^2} \int_p^\infty \frac{\rho dr / r^2 U(r)}{(1 - \rho^2/r^2)^{3/2}}$$
$$= \frac{\pi}{2} + \frac{1}{m v_\infty^2} \frac{\partial}{\partial p} \left[ \int_p^\infty \frac{U(r) dr}{\sqrt{1 - \rho^2/r^2}} \right]$$

Now:

$$\frac{\int_p^\infty U(r) dr}{r \sqrt{1 - \rho^2/r^2}} = u v \int_p^\infty - \int_p^\infty v dy$$

where  $u = U(r)$

$$dv = \frac{dr}{\sqrt{1 - \rho^2/r^2}} = \frac{r dr}{\sqrt{r^2 - \rho^2}} \quad x = r^2 - \rho^2$$
$$dx = 2r dr$$
$$= \frac{dx/2}{\sqrt{x}}$$

$$\rightarrow v = \frac{1}{2} \int \frac{dx}{\sqrt{x}} = \sqrt{x} + \text{const}$$
$$= \sqrt{r^2 - \rho^2} + \text{const}$$

so:

$$\frac{\int_p^\infty U(r) dr}{r \sqrt{1 - \rho^2/r^2}} = U(r) \cancel{\sqrt{r^2 - \rho^2}} \Big|_p^\infty - \int_p^\infty \left( \frac{dU}{dr} \right) dr \cancel{\sqrt{r^2 - \rho^2}}$$

assuming

$$U(r) \rightarrow 0 \text{ faster}$$

$$\text{than } \frac{1}{r} \text{ as } r \rightarrow \infty$$

so

$$\phi_0 = \frac{\pi}{2} + \frac{1}{m v_\infty^2} \frac{\partial}{\partial p} \left[ - \int_p^\infty \frac{dU}{dr} dr \sqrt{r^2 - \rho^2} \right]$$
$$= \frac{\pi}{2} + \frac{1}{m v_\infty^2} (-) \int_p^\infty \frac{dU}{dr} dr \frac{1}{2\sqrt{r^2 - \rho^2}} (-\cancel{\rho})$$
$$= \frac{\pi}{2} + \frac{p}{m v_\infty^2} \int_p^\infty dr \frac{dU/dr}{\sqrt{r^2 - \rho^2}}$$

Scattering angle  $X$ :

$$2\phi_0 + X = \pi$$

$$X = \pi - 2\phi_0$$

$$\rightarrow X = \pi - 2 \left( \frac{\pi}{2} + \frac{p}{m v_\infty^2} \int_p^\infty dr \frac{dU/dr}{\sqrt{r^2 - \rho^2}} \right)$$
$$= - \frac{2p}{m v_\infty^2} \int_p^\infty dr \frac{dU/dr}{\sqrt{r^2 - \rho^2}}$$

In terms of  $\Theta$ ,

$$\tan \Theta_1 = \frac{m_2 \sin X}{m_1 + m_2 \cos X} \rightarrow \Theta_1 \approx \frac{m_2 X}{m_1 + m_2}$$

Thur,

$$\begin{aligned}\Theta' &\approx \left(\frac{m_2}{m_1+m_2}\right) X \\ &\approx \left(\frac{m_2}{m_1+m_2}\right) \left(\frac{-2\rho}{m V_{\infty}^2}\right) \int_{\rho}^{\infty} dr \frac{dU/dr}{\sqrt{r^2 - \rho^2}} \\ &= \frac{-2\rho}{m_1 V_{\infty}^2} \int_{\rho}^{\infty} dr \frac{dU/dr}{\sqrt{r^2 - \rho^2}}\end{aligned}$$

which is Eq. (20,3)

Sec 21, Prob 1

$$X = a \cos(\omega t + \alpha)$$

Express  $a, \alpha$  in terms of  $x_0, v_0$

$$x_0 = a \cos \alpha$$

$$V = \dot{x} = -a \omega \sin(\omega t + \alpha)$$

$$\rightarrow v_0 = -a \omega \sin \alpha$$

$$\text{Thus, } \frac{v_0}{x_0} = -\omega \tan \alpha$$

$$\rightarrow \boxed{\tan \alpha = \frac{-v_0}{\omega x_0}}$$

$$\text{Also: } x_0 = a \cos \alpha$$

$$\frac{v_0}{\omega} = -a \sin \alpha$$

$$\rightarrow x_0^2 + \left(\frac{v_0}{\omega}\right)^2 = a^2 \cos^2 \alpha + a^2 \sin^2 \alpha \\ = a^2$$

$$\text{so } \boxed{a = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}}$$

Sec 21, Prob 2:

Diatomic molecules, different isotopes  
 $m_1, m_2$  and  $m'_1, m'_2$

$\xrightarrow{x}$       2-body  $\rightarrow$  1-body w/ com  
 $m_1 + m_2$        $T = \frac{1}{2} m \dot{x}^2, m = \frac{m_1 m_2}{m_1 + m_2}$   
 $U = \frac{1}{2} k x^2$

$$\text{Thy, } \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}$$

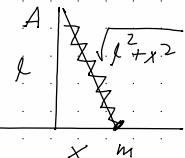
Similarly, for masses  $m'_1, m'_2$  with  $H' = H$ :

$$\omega' = \sqrt{\frac{k}{m'}} = \sqrt{\frac{k(m'_1 + m'_2)}{m'_1 m'_2}}$$

$$\Rightarrow \frac{\omega'}{\omega} = \sqrt{\frac{k(m'_1 + m'_2)}{m'_1 m'_2}} \sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}}$$

$$= \sqrt{\left(\frac{m'_1 + m'_2}{m_1 + m_2}\right) \frac{m_1 m_2}{m'_1 m'_2}}$$

Sec 21, Prob 3:



$$U \approx F l t$$

$$l t = \sqrt{x^2 + l^2} - l$$

$$= l \left( \sqrt{1 + \left(\frac{x}{l}\right)^2} - 1 \right)$$

$$= l \left( x + \frac{1}{2} \left(\frac{x}{l}\right)^2 + \dots - l \right)$$

$$\approx \frac{1}{2} \frac{x^2}{l}$$

$$\text{Thy, } U \approx \frac{1}{2} \left(\frac{F}{\omega}\right) x^2 = \frac{1}{2} k x^2 \text{ for } H \equiv \frac{F}{\omega}$$

$$\text{Ab}, \quad T = \frac{1}{2} m \dot{x}^2$$

$$\rightarrow \omega = \sqrt{\frac{F}{\lambda m}}$$

Alternatively,

$$F_x = -F \sin \alpha$$

$$\approx -F \frac{x}{l}$$

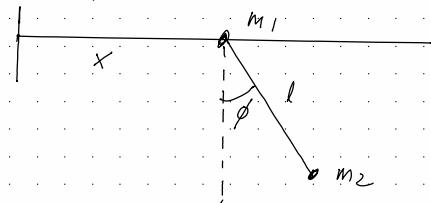
$$\sin \alpha \approx \tan \alpha = \frac{x}{l}$$

$$\text{Hence, law with } H \equiv \frac{F}{\omega}$$

$$\Rightarrow U = \frac{1}{2} \left(\frac{F}{\omega}\right) x^2 \quad (\text{as above})$$

Sec 21, Prob 5.

Figure 2 from Sec 5:



Recall from Sec 14, Prob 3:

$$E = \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left( 1 - \left( \frac{m_2}{m_1 + m_2} \right) \cos^2 \phi \right) - m_2 g l \cos \phi$$

Now  $\phi = 0$  corresponds to stable equilibrium.

Small oscillations:  $\phi \ll 1$

Since  $\dot{\phi}^2$  is small can set  $\phi = 0$  in  $\left( 1 - \left( \frac{m_2}{m_1 + m_2} \right) \cos^2 \phi \right)$

$$\rightarrow 1 - \left( \frac{m_2}{m_1 + m_2} \right) \cos^2 \phi \rightarrow 1 - \frac{m_2}{m_1 + m_2} = \frac{m_1}{m_1 + m_2}$$

Then

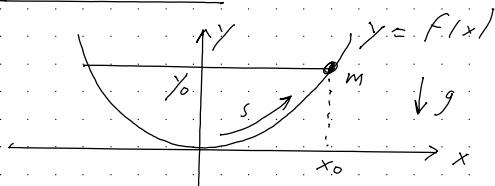
$$E = \frac{1}{2} m_1 l^2 \dot{\phi}^2 - m_2 g l \left( 1 - \frac{1}{2} \dot{\phi}^2 \right)$$

$$= \frac{1}{2} m_1 l^2 \dot{\phi}^2 + \frac{1}{2} m_2 g l \dot{\phi}^2 - \frac{m_2 g l}{m_1 + m_2}$$

Thus,

$$\omega = \sqrt{\frac{m_2 g l}{m_1 l^2}} = \sqrt{\left( \frac{m_1 + m_2}{m_1} \right) \frac{g}{l}} \quad (\rightarrow \sqrt{\frac{g}{l}} \text{ for } m_1 \gg m_2)$$

Sec 21, Prob 6



Find  $y = f(x)$  so that period  $P(y_0)$  is indep. of  $y_0$ .

$$\begin{aligned} \text{Need: } E &= \frac{1}{2} m v^2 + mgy \\ &= \frac{1}{2} m \left( \frac{ds}{dt} \right)^2 + \frac{1}{2} I s^2 \end{aligned}$$

where  $s$  = arc length along the curve

$$\begin{aligned} &= \int_0^s ds \\ &= \int_0^s \sqrt{dx^2 + dy^2} \\ &= \int_0^x dx \sqrt{1 + y'^2} \end{aligned}$$

$$\text{so } mgy = \frac{1}{2} I s^2$$

$$y = \frac{1}{2g} \frac{I}{m} s^2 \equiv A^2 s^2$$

$$\rightarrow \sqrt{y} = As, \quad A = \frac{1}{\sqrt{2g}} \sqrt{\frac{I}{m}}$$

$$\frac{1}{2\sqrt{y}} \frac{dy}{dx} = A \frac{ds}{dx} = A \sqrt{1+y'^2}$$

$$\text{Thus, } \frac{1}{2} \frac{y'}{\sqrt{y}} = A \sqrt{1+y'^2}$$

$$y'^2 = 4A^2 y (1+y'^2)$$

$$y'^2(1-4A^2y) = 4A^2y$$

$$\frac{dy}{dx} = y' = \sqrt{\frac{4A^2y}{1-4A^2y}}$$

$$\text{so } x = \int_0^y dy \sqrt{\frac{1-4A^2y}{4A^2y}}$$

multiple substitution

$$y = \frac{1}{8A^2} (1 - \cos \theta)$$

$$dy = \frac{1}{8A^2} \sin \theta d\theta = \frac{1}{8A^2} \sqrt{1 - \cos^2 \theta} d\theta$$

$$\rightarrow \frac{1-4A^2y}{4A^2y} = \frac{1 - \frac{1}{2}(1 - \cos \theta)}{\frac{1}{2}(1 - \cos \theta)} = \frac{\frac{1}{2}(1 + \cos \theta)}{\frac{1}{2}(1 - \cos \theta)} = \frac{1 + \cos \theta}{1 - \cos \theta}$$

$$\text{Thus, } x = \int \frac{1}{8A^2} \sqrt{1 - \cos^2 \theta} d\theta \sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}}$$

$$= \frac{1}{8A^2} \int d\theta \sqrt{\frac{(1 - \cos \theta)(1 + \cos \theta)(1 + \cos \theta)}{1 - \cos \theta}}$$

$$x = \frac{1}{8A^2} \int d\theta (1 + \cos \theta)$$

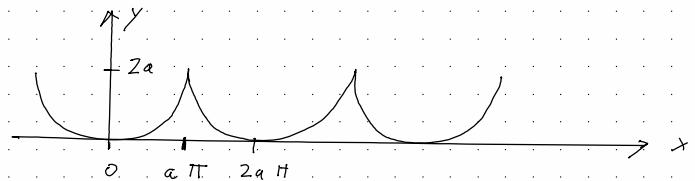
$$= \frac{1}{8A^2} (\theta + \sin \theta)$$

thus,

$$x = \frac{1}{8A^2} (\theta + \sin \theta) = a(\theta + \sin \theta)$$

$$y = \frac{1}{8A^2} (1 - \cos \theta) = a(1 - \cos \theta)$$

which are parametric equations for a cycloid



$$\text{NOTE: } 8A^2 = 8 \frac{1}{2} \frac{\pi}{m} = \frac{4}{9} \frac{\pi}{m} = \frac{4}{9} w^2$$

$$\text{Also: } \cos \theta = \cos^2 \left( \frac{\theta}{2} \right) - \sin^2 \left( \frac{\theta}{2} \right) = 1 - 2 \sin^2 \left( \frac{\theta}{2} \right)$$

$$\sin^2 \left( \frac{\theta}{2} \right) = \frac{1}{2} (1 - \cos \theta)$$

$$\text{Thus, } 4A^2y = \sin^2 \left( \frac{\theta}{2} \right) \rightarrow dy = \frac{1}{4A^2} \sin \left( \frac{\theta}{2} \right) \cos \frac{\theta}{2} d\theta$$

$$1 - 4A^2y = \cos^2 \left( \frac{\theta}{2} \right) \rightarrow \sqrt{1 - 4A^2y} = \frac{\cos \left( \frac{\theta}{2} \right)}{\sin \left( \frac{\theta}{2} \right)}$$

Sec 22, Prob 1:

$$(a) F = F_0 = \cos \omega t$$

$$x = \dot{x} = 0 \text{ at } t=0 \Rightarrow \ddot{x}_0 = 0$$

$$\begin{aligned}\ddot{x}(t) &= e^{i\omega t} \left[ F_0 + \int_0^t dt' \frac{F(t')}{m} e^{-i\omega t'} \right] \\ &= e^{i\omega t} \frac{F_0}{m} \int_0^t dt' e^{-i\omega t'} \\ &= e^{i\omega t} \left( \frac{F_0}{-i\omega} \right) e^{-i\omega t} \Big|_0^t \\ &= e^{i\omega t} \left( \frac{F_0}{-i\omega} \right) (e^{-i\omega t} - 1) \\ &= \frac{iF_0}{m\omega} (1 - e^{i\omega t}) \\ &= \frac{iF_0}{m\omega} (1 - \cos \omega t - i \sin \omega t)\end{aligned}$$

$$\text{Now: } x(t) = \frac{\text{Im}(\ddot{x}(t))}{\omega}, \quad \dot{x}(t) = \text{Re}(\ddot{x}(t))$$

Thus,

$$\boxed{x(t) = \frac{F_0}{m\omega^2} (1 - \cos \omega t)}$$

$$(b) F(t) = a t$$

$$\ddot{x}(t) = e^{i\omega t} \frac{a}{m} \int_0^t dt' t' e^{-i\omega t'}$$

$$\text{Let } u = t' \rightarrow du = dt'$$

$$dv = e^{-i\omega t'} dt' \rightarrow v = \frac{1}{-i\omega} e^{-i\omega t'}$$

Thus,

$$\ddot{x}(t) = e^{i\omega t} \frac{a}{m} \left[ \frac{-t}{-i\omega} e^{-i\omega t} \right]_0^t + \frac{1}{-i\omega} \int_0^t dt' e^{-i\omega t'}$$

$$= e^{i\omega t} \frac{a}{m} \left[ \frac{t}{i\omega} e^{-i\omega t} + \frac{1}{\omega^2} (e^{-i\omega t} - 1) \right]$$

$$= \frac{a}{m} \left[ \left( \frac{it}{\omega} + \frac{1}{\omega^2} \right) - \frac{1}{\omega^2} e^{i\omega t} \right]$$

$$= \frac{a}{m} \left[ \left( \frac{it}{\omega} + \frac{1}{\omega^2} \right) - \frac{1}{\omega^2} (\cos \omega t + i \sin \omega t) \right]$$

$$= \frac{a}{m} \left[ \frac{1}{\omega^2} \left( 1 - \cos \omega t \right) + i \left( \frac{t}{\omega} - \frac{\sin \omega t}{\omega^2} \right) \right]$$

$$\rightarrow x(t) = \frac{\text{Im}(\ddot{x}(t))}{\omega}$$

$$= \boxed{\frac{a}{m\omega^3} \left( \omega t - \sin \omega t \right)}$$

$$(c) F(t) = F_0 \exp(-\alpha t)$$

$$\begin{aligned}\xi(t) &= e^{i\omega t} \frac{F_0}{m} \int_0^t d\bar{t} e^{-i\omega \bar{t}} e^{-i\omega t} \\ &= \frac{F_0}{m} e^{i\omega t} \frac{1}{-(i\omega + \alpha)} e^{-(i\omega + \alpha)t} \int_0^t \\ &= \frac{F_0}{m} e^{i\omega t} \frac{1}{-(i\omega + \alpha)} (e^{-(i\omega + \alpha)t} - 1)\end{aligned}$$

NOTE:  $\frac{1}{-(i\omega + \alpha)} \cdot \left(\frac{i\omega - \alpha}{i\omega - \alpha}\right) = \frac{i\omega - \alpha}{\alpha^2 + \omega^2}$

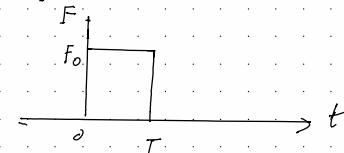
$$\begin{aligned}\rightarrow \xi(t) &= \frac{F_0}{m} \left( \frac{i\omega - \alpha}{\alpha^2 + \omega^2} \right) \left[ e^{-\alpha t} - e^{i\omega t} \right] \\ &= \frac{F_0}{m} \left( \frac{i\omega - \alpha}{\alpha^2 + \omega^2} \right) \left[ e^{-\alpha t} - \cos \omega t - i \sin \omega t \right] \\ &= \frac{F_0}{m(\alpha^2 + \omega^2)} \left[ i(w e^{-\alpha t} - \omega \cos \omega t + \omega \sin \omega t) \right. \\ &\quad \left. - \alpha e^{-\alpha t} + \alpha \cos \omega t + \omega \sin \omega t \right]\end{aligned}$$

thus,

$$x(t) = F_m(\xi(t)) / \omega$$

$$\begin{aligned}&= \frac{F_0}{m\omega(\alpha^2 + \omega^2)} \left( w e^{-\alpha t} - \omega \cos \omega t + \omega \sin \omega t \right) \\ &= \boxed{\frac{F_0}{m(\alpha^2 + \omega^2)} \left( e^{-\alpha t} - \cos \omega t + \frac{\omega}{w} \sin \omega t \right)}$$

Sec 22, Prob 3:



Initial condition  
 $\xi_0 = 0$  (since  $x_0 = \dot{x}_0 = 0$ )

For  $t > T$ :

$$\begin{aligned}\xi(t) &= e^{i\omega t} \int_0^t d\bar{t} \frac{F_0}{m} e^{-i\omega \bar{t}} \\ &= e^{i\omega t} \frac{F_0}{m} \left( \frac{1}{-i\omega} \right) (e^{-i\omega T} - 1) \\ &= \frac{iF_0}{m\omega} (e^{i\omega(t-T)} - e^{i\omega t}) \\ &= \frac{iF_0}{m\omega} \left[ \cos(\omega(t-T)) + i \sin(\omega(t-T)) \right. \\ &\quad \left. - \cos \omega t - i \sin \omega t \right] \\ &= \frac{iF_0}{m\omega} \left( \cos(\omega(t-T)) - \cos(\omega t) \right) \\ &\quad - \frac{F_0}{m\omega} (\sin(\omega t - T) - \sin(\omega t))\end{aligned}$$

$$\rightarrow x(t) = \frac{I_m(\xi(t))}{\omega}$$

$$= \frac{F_0}{m\omega^2} (\cos(\omega(t-T)) - \cos(\omega t))$$

We can also write for  $t > T$ :

$$x(t) = c_1 \cos(\omega(t-T)) + c_2 \sin(\omega(t-T))$$

$$\rightarrow \dot{x}(t) = -c_1 \omega \sin(\omega(t-T)) + c_2 \omega \cos(\omega(t-T))$$

$$\text{Thus, } x(T) = c_1, \quad \dot{x}(T) = c_2 \omega$$

Match with:

$$x(t) = \frac{F_0}{m\omega^2} (\cos(\omega(t-T)) - \cos(\omega t))$$

$$\dot{x}(t) = \frac{F_0}{m\omega} (-\sin(\omega(t-T)) + \sin(\omega t))$$

at  $t = T$ ,

$$\rightarrow c_1 = \frac{F_0}{m\omega^2} (1 - \cos(\omega T))$$

$$c_2 \omega = \frac{F_0}{m\omega} \sin(\omega T)$$

Thus,

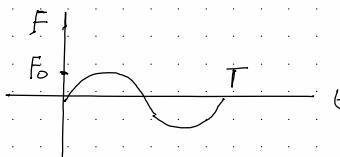
$$\alpha = \sqrt{c_1^2 + c_2^2}$$

$$= \frac{F_0}{m\omega^2} \sqrt{1 + \cos^2(\omega T) - 2 \cos(\omega T) + \sin^2(\omega T)}$$

$$= \frac{F_0}{m\omega^2} \sqrt{2(1 - \cos(\omega T))}$$

$$= \boxed{\frac{2F_0}{m\omega^2} \sin\left(\frac{\omega T}{2}\right)} \quad \text{using } 1 - \cos\theta = 2\sin^2(\frac{\theta}{2})$$

Sec 22, Prob 5:



$$T = \frac{2\pi}{\omega}$$

$$f(t) = F_0 \sin \omega t, \quad \xi_0 = 0$$

For  $t > T$ :

$$\xi(t) = e^{i\omega t} \frac{F_0}{m} \int_0^T dt' \sin \omega t' e^{-i\omega t'}$$

$$= \frac{F_0}{m} e^{i\omega t} \frac{1}{2i} \int_0^T dt' (e^{i\omega t'} - e^{-i\omega t'}) e^{-i\omega t'}$$

$$= \frac{F_0}{2mi} e^{i\omega t} \int_0^T dt' (1 - e^{-i2\omega t'})$$

$$= \frac{F_0}{2mi} e^{i\omega t} \left( T + \frac{1}{i2\omega} (e^{-i2\omega T} - 1) \right)$$

$$= \frac{F_0 T}{2mi} e^{i\omega t}$$

$$\text{Thus, } x(T) = \frac{\text{Im}(\xi(T))}{\omega} \quad \left| \begin{array}{l} \dot{x}(T) = \text{Re}(\xi(T)) \\ = 0 \end{array} \right.$$

$$= -\frac{F_0 T}{2mw}$$

We can also write for  $t > T$ :

$$x(t) = c_1 \cos(\omega(t-T)) + c_2 \sin(\omega(t-T))$$

$$\rightarrow \dot{x}(t) = -c_1 \omega \sin(\omega(t-T)) + c_2 \omega \cos(\omega(t-T))$$

$T^{\text{tors}}$

$$x(T) = c_1, \quad \dot{x}(T) = c_2 \omega$$

match w.t.

$$x(T) = \frac{-F_0 T}{2m\omega}, \quad \dot{x}(T) = 0$$

$$\rightarrow c_1 = \frac{-F_0 T}{2m\omega}, \quad c_2 = 0$$

$$\begin{aligned} \text{so } a &= \sqrt{c_1^2 + c_2^2} \\ &= \frac{F_0 T}{2m\omega} \end{aligned}$$

$$= \frac{F_0 \pi T / \omega}{2m\omega}$$

$$= \frac{F_0 \pi}{m\omega^2}$$

Sec 23, Prob 1:

$$\begin{aligned} L &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}\omega_0^2(x^2 + y^2) + \alpha xy \\ &= \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k - \frac{1}{2} \sum_{i,k} K_{ik} x_i x_k \end{aligned}$$

where  $m_{ik} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$  and

$$\begin{aligned} \sum_{i,k} K_{ik} x_i x_k &= \omega_0^2(x^2 + y^2) - 2\alpha xy \\ &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

$$\text{so } K_{ik} = \begin{pmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{pmatrix}$$

EOMs:

$$\frac{d}{dt} \sum_k m_{ik} \dot{x}_k = - \sum_k K_{ik} x_k$$

$$\sum_k m_{ik} \ddot{x}_k = - \sum_k K_{ik} x_k$$

$$\sum_k (m_{ik} \ddot{x}_k + K_{ik} x_k) = 0$$

Trial solution:  $x_k = A_k e^{i\omega t}$

$$\rightarrow \sum_k (-\omega^2 m_{ik} + K_{ik}) A_k e^{i\omega t} = 0$$

Thus,

$$\sum (k_{ii} - \omega^2 m_{ii}) A_{ii} = 0$$

$$\rightarrow \det(k_i - \omega^2 m_{ii}) = 0$$

$$0 = \det \begin{vmatrix} \omega_0^2 - \omega^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega^2 \end{vmatrix}$$

$$= (\omega_0^2 - \omega^2)^2 - \alpha^2$$

$$= \omega^4 - 2\omega^2\omega_0^2 + (\omega_0^4 - \alpha^2)$$

Quadratic equation:

$$\omega_t^2 = \frac{2\omega_0^2 \pm \sqrt{4\omega_0^4 - 4(\omega_0^4 - \alpha^2)}}{2}$$

$$= \frac{2\omega_0^2 \pm 2\alpha}{2}$$

$$= [\omega_0^2 \pm \alpha] \text{ (normal mode freqs)}$$

/Normal mode vectors:

$$\underline{\omega_t^2 = \omega_0^2 + \alpha}$$

$$\sum_k (k_{ii} - \omega_t^2 m_{ii}) A_{ii} = 0$$

$$\begin{vmatrix} \omega_0^2 - \omega_t^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega_t^2 \end{vmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} -\alpha & -\alpha \\ -\alpha & -\alpha \end{vmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Thus, } -\alpha(A_1 + A_2) = 0$$

$$A_2 = -A_1$$

$$\text{so Eigenvector is } \begin{Bmatrix} 1 & 1 \\ \sqrt{2} & -1 \end{Bmatrix} = V_+$$

$$\omega_0^2 = \omega_-^2 :$$

$$\begin{vmatrix} \omega_0^2 - \omega_-^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega_-^2 \end{vmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{vmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Thus, } A_2 = A_1$$

$$\rightarrow \begin{Bmatrix} 1 & 1 \\ \sqrt{2} & 1 \end{Bmatrix} = V_-$$

$V_+$  corresponds to out-of-phase motion:

$$V_+ e^{i\omega t} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{i\sqrt{\omega_0^2 + \alpha} t}$$

$V_-$  corresponds to in-phase motion:

$$V_- e^{i\omega t} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{i\sqrt{\omega_0^2 - \alpha} t}$$

NOTE:

For weak coupling,  $\alpha \ll \omega_0^2$ , we have

$$\begin{aligned}\omega_{\pm}^2 &= \omega_0^2 \pm \alpha \\ &= \omega_0^2 \left( 1 \pm \frac{\alpha}{\omega_0^2} \right)\end{aligned}$$

$$\begin{aligned}\omega_{\pm} &= \omega_0 \sqrt{1 \pm \frac{\alpha}{\omega_0^2}} \\ &\approx \omega_0 \left( 1 \pm \frac{\alpha}{2\omega_0^2} \right) \quad \text{nearly equal freq}\end{aligned}$$

Bent freq:

$$\begin{aligned}\omega_+ - \omega_- &\approx \omega_0 \left( 1 + \frac{\alpha}{2\omega_0^2} \right) - \omega_0 \left( 1 - \frac{\alpha}{2\omega_0^2} \right) \\ &= \boxed{\frac{\alpha}{\omega_0}}\end{aligned}$$

Sec 23, Prob 3:

$$U = \frac{1}{2} k r^2, \quad \text{Simple oscillator}$$

central potential  $\rightarrow$  motion in a plane ( $x, y$ )  
or ( $r, \phi$ )

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k (x^2 + y^2)$$

$$= \left( \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \right) + \left( \frac{1}{2} m \dot{y}^2 - \frac{1}{2} k y^2 \right)$$

EOMs:

$$m \ddot{x} = -kx \rightarrow x = a \cos(\omega t + \alpha)$$

$$m \ddot{y} = -ky \rightarrow y = b \cos(\omega t + \beta)$$

$$\text{where } \omega = \sqrt{\frac{k}{m}}$$

$$\begin{aligned}\text{Write } \omega t + \beta &= \omega t + \alpha + (\beta - \alpha) \\ &= \phi + \delta\end{aligned}$$

$$\text{where } \phi \equiv \omega t + \alpha, \quad \delta \equiv \beta - \alpha$$

$$\text{Then } x = a \cos \phi \rightarrow \boxed{\cos \phi = \frac{x}{a}}$$

$$\begin{aligned}y &= b \cos(\phi + \delta) \\ &= b (\cos \phi \cos \delta - \sin \phi \sin \delta)\end{aligned}$$

$$\begin{aligned}\rightarrow \frac{y}{b} &= \cos \phi \cos \delta - \sin \phi \sin \delta \\ &= \left( \frac{x}{a} \right) \cos \delta - \sin \phi \sin \delta\end{aligned}$$

$$\text{so } \boxed{\sin \phi = \frac{1}{\sin \delta} \left[ \left( \frac{x}{a} \right) \cos \delta - \frac{y}{b} \right]}$$

Square and add two boxed expressions

$$\cos^2 \phi = \left(\frac{x}{a}\right)^2$$

$$+ \sin^2 \phi = \frac{1}{\sin^2 \delta} \left[ \left(\frac{x}{a}\right)^2 \cos^2 \delta + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab} \cos \delta \right]$$

$$1 = \left(\frac{x}{a}\right)^2 \left(1 + \frac{\cos^2 \delta}{\sin^2 \delta}\right) + \left(\frac{y}{b}\right)^2 \frac{1}{\sin^2 \delta} - \frac{2xy}{ab} \frac{\cos \delta}{\sin^2 \delta}$$

$$\rightarrow \sin^2 \delta = \left(\frac{x}{a}\right)^2 \underbrace{\left(\sin^2 \delta + \cos^2 \delta\right)}_{=1} + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab} \cos \delta$$

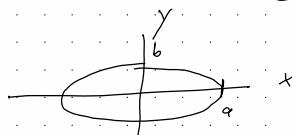
$$= \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab} \cos \delta$$

$$\text{Thus, } \boxed{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab} \cos \delta = \sin^2 \delta}$$

Suppose  $\delta = \frac{\pi}{2}$

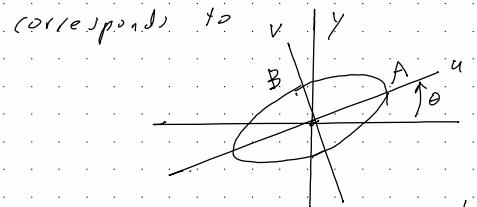
$$\text{Then } \cos \delta = 0, \sin \delta = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad (\text{ellipse centered at origin with semi-major axis } a, \text{ semi-minor axis } b)$$



In general

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab} \cos \delta = \sin^2 \delta$$



Find angle  $\theta$  such that the above expression maps to

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$$

$$\text{Now: } \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\rightarrow \sin^2 \delta = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab} \cos \delta$$

$$= \frac{1}{a^2} (\cos \theta u - \sin \theta v)^2 + \frac{1}{b^2} (\sin \theta u + \cos \theta v)^2 - \frac{2(\cos \theta u - \sin \theta v)(\sin \theta u + \cos \theta v)}{ab} \cos \delta$$

$$\begin{aligned}
&= \frac{1}{a^2} (u^2 \cos^2 \theta + v^2 \sin^2 \theta - 2uv \sin \theta \cos \theta) \\
&+ \frac{1}{b^2} (u^2 \sin^2 \theta + v^2 \cos^2 \theta + 2uv \sin \theta \cos \theta) \\
&- \frac{2}{ab} \cos \delta (u^2 \sin \theta \cos \theta - v^2 \sin \theta \cos \theta + uv \cos 2\theta) \\
&= u^2 \left( \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} - 2 \frac{\sin \theta \cos \theta \cos \delta}{ab} \right) \\
&+ v^2 \left( \frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{b^2} + 2 \frac{\sin \theta \cos \theta \cos \delta}{ab} \right) \\
&+ 2uv \left( -\frac{\sin \theta \cos \theta}{a^2} + \frac{\sin \theta \cos \theta}{b^2} - \frac{\cos 2\theta \cos \delta}{ab} \right)
\end{aligned}$$

We can make the factor multiplying  $4v$  equal to zero by choosing  $\theta$  appropriately:

$$-\sin \theta \cos \theta \left( \frac{1}{a^2} - \frac{1}{b^2} \right) = \frac{\cos 2\theta \cos \delta}{ab}$$

$$\frac{1}{2} \sin 2\theta \left( \frac{a^2 - b^2}{a^2 b^2} \right) = \frac{\cos 2\theta \cos \delta}{ab}$$

$$\boxed{\tan 2\theta = \left( \frac{2ab}{a^2 - b^2} \right) \cos \delta}$$

NOTE: if  $a=b$  then  $\theta = \cos 2\theta$

$$\rightarrow \theta = \frac{\pi}{4}$$

so choosing  $\theta$  as above, we have

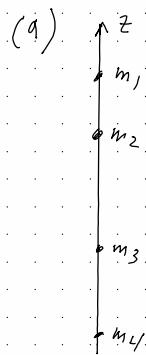
$$\begin{aligned}
\sin^2 \delta &= u^2 \textcircled{1} + v^2 \textcircled{2} \\
1 &= u^2 \textcircled{1} + v^2 \frac{\textcircled{2}}{\sin^2 \delta}
\end{aligned}$$

$$= \left( \frac{u}{A} \right)^2 + \left( \frac{v}{B} \right)^2$$

$$\text{where } A^2 = \frac{\sin^2 \delta}{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} - 2 \frac{\sin \theta \cos \theta \cos \delta}{ab}}$$

$$B^2 = \frac{\sin^2 \delta}{\left( \frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{b^2} + 2 \frac{\sin \theta \cos \theta \cos \delta}{ab} \right)}$$

Sec 32, Prob 1:



$$I_3 = 0 \text{ since } x_a = y_a = 0 \\ \text{For all masses}$$

$$I_1 = \sum_a m_a (r_a^2 - \ell_a^2) = \sum_a m_a z_a^2$$

$$I_2 = \sum_a m_a (r_a^2 - \ell_a^2) = \sum_a m_a z_a^2$$

$$\rightarrow I_1 = I_2 = I$$

$$= \sum_a m_a z_a^2$$

(assuming COM at  $z=0$ )

If COM is not at  $z=0$ , but at  $z_{\text{COM}}$ , then:

$$I = \sum_a m_a (z_a - z_{\text{COM}})^2, \quad z_{\text{COM}} = \frac{1}{m} \sum_b m_b z_b \\ = \sum_a m_a (z_a^2 + z_{\text{COM}}^2 - 2z_{\text{COM}} z_a) \\ = \sum_a m_a z_a^2 + m z_{\text{COM}}^2 - 2z_{\text{COM}} \sum_a m_a z_a \\ = \sum_a m_a z_a^2 - m z_{\text{COM}}^2$$

This last expression can be written in terms of  $\ell_{ab} = |z_a - z_b|$  as follows:

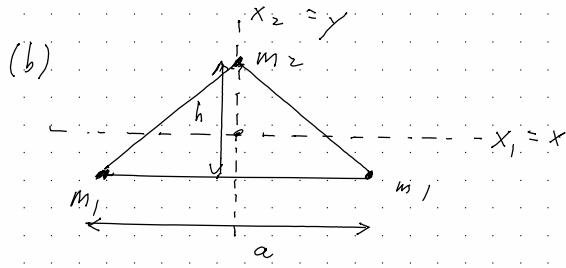
$$I = \frac{1}{2} \sum_a m_a z_a^2 + \frac{1}{2} \sum_b m_b z_b^2 - \frac{1}{m} \left( \sum_a m_a z_a \right) \left( \sum_b m_b z_b \right) \\ = \frac{1}{2m} \sum_{a,b} m_a m_b z_a^2 + \frac{1}{2m} \sum_{a,b} m_a m_b z_b^2 - \frac{1}{m} \sum_a m_a z_a \sum_b m_b z_b$$

Thus

$$I = \frac{1}{2m} \sum_a \sum_b m_a m_b (z_a^2 + z_b^2 - 2z_a z_b) \\ = \frac{1}{2m} \sum_a \sum_b m_a m_b (z_a - z_b)^2 \\ = \frac{1}{2m} \sum_a \sum_b m_a m_b \ell_{ab}^2$$

NOTE: For just two masses:

$$I = \frac{1}{2m} (m_1 m_2 \ell^2 + m_2 m_1 \ell^2) \\ = \frac{m_1 m_2}{m} \ell^2 \\ = m \ell^2 \quad \text{where } m = \frac{m_1 m_2}{m_1 + m_2} \\ \ell = |z_1 - z_2|$$



Assume COM at  $(x_1, x_2) = (x, y) = (0, 0)$

$$\text{Then } 2m_1y_1 + m_2y_2 = 0$$

$$\text{where } y_2 - y_1 = h$$

$$\text{thus, } 2m_1y_1 + m_2(h + y_1) = 0$$

$$(2m_1 + m_2)y_1 + m_2h = 0$$

$$y_1 = \frac{-m_2h}{\mu}, \quad \mu = 2m_1 + m_2 \\ = \text{total mass}$$

$$\text{and } y_2 = y_1 + h \\ = \frac{-m_2h}{\mu} + h \\ = \frac{(\mu - m_2)h}{\mu} \\ = \frac{2m_1h}{\mu}$$

All moments about COM = 0

$$\text{Thus, } I_3 = \sum_a m_a(r_a^2 - z_a^2) = \sum_a m_a(x_a^2 + y_a^2)$$

$$I_1 = \sum_a m_a(r_a^2 - x_a^2) = \sum_a m_a y_a^2$$

$$I_2 = \sum_a m_a(r_a^2 - x_a^2) \approx \sum_a m_a x_a^2$$

$$\text{Thus, } I_3 = I_1 + I_2$$

so need to calculate  $I_1, I_2$

$$I_1 = \sum_a m_a y_a^2$$

$$= 2m_1 y_1^2 + m_2 y_2^2$$

$$= 2m_1 \frac{m_2^2 h^2}{\mu^2} + m_2 \frac{4m_1^2 h^2}{\mu^2}$$

$$= \frac{2m_1 m_2 h^2 (m_2 + 2m_1)}{\mu^2}$$

$$= \boxed{\frac{2m_1 m_2 h^2}{\mu}}$$

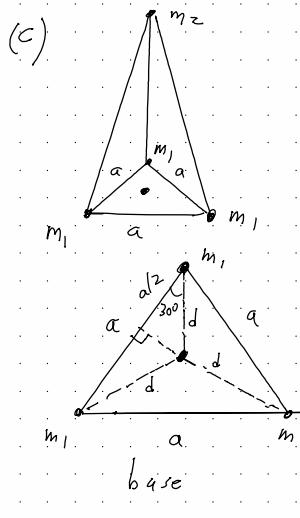
$$I_2 = \sum_a m_a x_a^2$$

$$= m_1 \left(\frac{a}{2}\right)^2 + m_2 \left(-\frac{a}{2}\right)^2$$

$$= \boxed{\frac{m_1 a^2}{2}}$$

$$I_3 = I_1 + I_2$$

$$= \boxed{\frac{2m_1 m_2 h^2 + m_1 a^2}{\mu}}$$



tetrahedron, height  $h$   
base: equilateral  $\triangle$  with  
side length  $a$

$$\begin{aligned} d &= \text{distance to } \alpha\text{-axis} \\ &\cos 30^\circ = \frac{a}{2d} = \frac{\sqrt{3}}{2} \\ \rightarrow d &= \frac{a}{\sqrt{3}} \end{aligned}$$

COM lies on axis of symmetry ( $Z$ -axis)

Assume COM has  $Z=0$

$$\begin{aligned} \text{Then } O &= m_2 Z_2 + 3m_1 Z_1, \quad Z_2 - Z_1 = h \\ &= m_2 (Z_1 + h) + 3m_1 Z_1 \\ &= (3m_1 + m_2) Z_1 + m_2 h \\ \rightarrow Z_1 &= \frac{-m_2 h}{3m_1 + m_2} = -\frac{m_2 h}{\mu} \end{aligned}$$

$$\begin{aligned} Z_2 &= h + Z_1 \\ &= h - \frac{m_2 h}{\mu} \\ &= \frac{m_1 h}{\mu} \end{aligned}$$

Since a tetrahedron has 3-fold rotational symmetry,  
the  $x_1$  principal axes can be chosen  
arbitrarily in the plane  $\perp$  to the  
symmetry axis ( $x_3 \equiv Z$ ). [ $x_2$  is  $\perp$  to  $x_1, x_3$ ]

thus,  $I_1 = I_2 \equiv I$

$$\begin{aligned} I_3 &= \sum_a m_a (r_a^2 - z_a^2) \\ &= \sum_a m_a s_a^2 \quad \text{where } s^2 = r^2 - z^2 \end{aligned}$$

$$\begin{aligned} I_1 &= \sum_a m_a (r_a^2 - x_a^2) \\ I_2 &= \sum_a m_a (r_a^2 - y_a^2) \quad \Rightarrow \text{equal } (I_1 = I_2 \equiv I) \end{aligned}$$

$$2I = I_1 + I_2$$

$$\begin{aligned} &= \sum_a m_a (2r_a^2 - x_a^2 - y_a^2) \\ &= \sum_a m_a (2(s_a^2 + z_a^2) - s_a^2) \\ &= \sum_a m_a s_a^2 + 2 \sum_a m_a z_a^2 \\ &= I_3 + 2 \sum_a m_a z_a^2 \end{aligned}$$

thus,

$$I = \frac{1}{2} I_3 + \sum_a m_a z_a^2$$

Now

$$\begin{aligned} I_3 &= \sum_a m_a s_a^2 \\ &= 3m_1 d^2 \\ &= 3m_1 \frac{a^2}{12} = \boxed{m_1 a^2} \end{aligned}$$

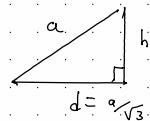
A 1/20,

$$\begin{aligned}\sum_q m_q z_q^2 &= 3m_1 z_1^2 + m_2 z_2^2 \\&= 3m_1 \left(-\frac{m_2 h}{M}\right)^2 + m_2 \left(\frac{3m_1 h}{M}\right)^2 \\&= \frac{3m_1 m_2^2 h^2}{M^2} + \frac{9m_1^2 m_2 h^2}{M} \\&= \frac{3m_1 m_2 h^2}{M^2} \underbrace{(m_2 + 3m_1)}_{M} \\&= \frac{3m_1 m_2 h^2}{M}\end{aligned}$$

Thus,

$$\begin{aligned}I &= \frac{l}{2} I_3 + \sum_q m_q z_q^2 \\&= \boxed{\frac{1}{2} m_1 q^2 + \frac{3m_1 m_2 h^2}{M}}\end{aligned}$$

Regular tetrahedron:  $m_1 = m_2$

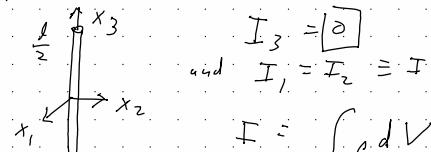


$$\begin{aligned}h^2 + \frac{a^2}{3} &= a^2 \rightarrow h = \sqrt{\frac{2}{3}} a \\M &= 4m_1\end{aligned}$$

$$\begin{aligned}I_3 &= m_1 q^2 \\I &= \frac{1}{2} m_1 q^2 + \frac{3m_1 m_1}{4m_1} \left(\frac{\sqrt{2}}{3} a\right)^2 \\&= m_1 q^2 (= I_1 = I_2)\end{aligned}\quad \left.\right\} \text{ so } I_1 = I_2 = I_3 = m_1 q^2$$

Sec 32, Prob 2:

(a) Thin rod of length  $l$ :



$$I = \int \rho dV (r^2 - x^2)$$

$$= \int \rho dV z^2$$

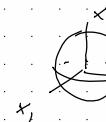
$$= \int dz \left(\frac{M}{l}\right) z^2$$

$$= \frac{M}{l} \frac{z^3}{3} \Big|_{-l/2}^{l/2}$$

$$= \frac{M}{l} \frac{2}{3} \frac{l^3}{8}$$

$$= \boxed{\frac{1}{12} M l^2}$$

(b) Sphere of radius  $R$ :



$$I_1 = I_2 = I_3 = I$$

$$I = \frac{1}{3}(I_1 + I_2 + I_3)$$

$$= \frac{1}{3} \left[ \int \rho dV (r^2 - x^2) + \int \rho dV (r^2 - y^2) \right]$$

$$+ \int \rho dV (r^2 - z^2) \right]$$

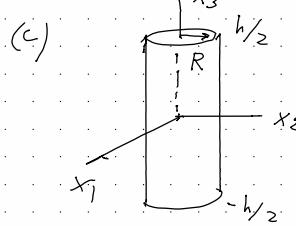
$$= \frac{1}{3} \int \rho dV [3r^2 - x^2 - y^2 - z^2]$$

$$= \frac{2}{3} \int \rho dV r^2$$

$$\begin{aligned} I &= \frac{2}{3} \int \rho dV \cdot r^2 \\ &= \frac{2}{3} \frac{M}{4\pi R^3} \int_0^R r^4 dr \int_0^\pi \int_0^{2\pi} d\theta d\phi \\ &= \frac{M}{2\pi R^3} \cdot \frac{4\pi}{3} \int_0^R r^4 dr \end{aligned}$$

$$= \frac{2M}{R^3} \cdot \frac{R^5}{5}$$

$$= \boxed{\frac{2}{5} M R^2}$$



$$\rho = \frac{M}{\pi R^2 \cdot h}$$

$$dV = ds \, d\phi \, dz$$

where  $s^2 = x^2 + y^2$

$$I_1 = I_2 = I$$

$$2I = I_1 + I_2$$

$$= \int \rho dV (r^2 - x^2) + \int \rho dV (r^2 - y^2)$$

$$= \int \rho dV (zr^2 - s^2)$$

$$= \int \rho dV (2(s^2 + z^2) - s^2)$$

$$= \int \rho dV s^2 + 2 \int \rho dV z^2$$

$$\rightarrow I = \frac{1}{2} I_3 + \int \rho dV \cdot z^2$$

$$\rho = \frac{M}{\frac{4}{3} \pi R^3}$$

$$\begin{aligned} I_3 &= \int \rho dV \cdot z^2 \\ &= \frac{M}{\pi R^2 h} \int_0^R s^3 ds \int_0^\pi d\phi \int_{-h/2}^{h/2} dz \\ &= \frac{M}{\pi R^2 h} \cdot \frac{R^4}{4} \cdot 2\pi \cdot h \\ &= \boxed{\frac{1}{2} M R^2} \end{aligned}$$

$$\begin{aligned} I &= \frac{1}{2} I_3 + \int \rho dV \cdot z^2 \\ \int \rho dV \cdot z^2 &= \frac{M}{\pi R^2 h} \int_0^R s^3 ds \int_0^\pi d\phi \int_{-h/2}^{h/2} z^2 dz \\ &= \frac{M}{\pi R^2 h} \cdot \frac{R^4}{4} \cdot \frac{2\pi}{3} \cdot \frac{z^3}{3} \Big|_{-h/2}^{h/2} \\ &= \frac{M}{h} \cdot \frac{2}{3} \cdot \frac{h}{8} \\ &= \frac{1}{12} M h^2 \end{aligned}$$

Thus,

$$\begin{aligned} I &= \frac{1}{2} \left( \frac{1}{2} M R^2 \right) + \frac{1}{12} M h^2 \\ &= \frac{1}{4} M R^2 + \frac{1}{12} M h^2 \\ &= \boxed{\frac{1}{4} M \left( R^2 + \frac{1}{3} h^2 \right)} \end{aligned}$$

NOTE: special limiting case,

(i) Thin rod ( $R \rightarrow 0$ )

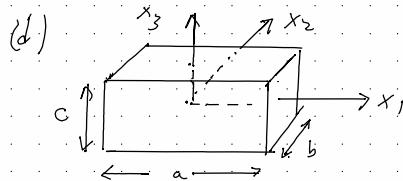
$$I_3 = 0$$

$$I_1 = I_2 = \frac{1}{12} M h^2$$

(ii) Thin disk ( $b \rightarrow 0$ )

$$I_3 = \frac{1}{2} M R^2$$

$$I_1 = I_2 = \frac{1}{4} M R^2$$



$$\rho = \frac{M}{abc}$$

$$dV = dx dy dz$$

$$I_1 = \int \rho dV (r^2 - x^2)$$

$$= \int \rho dV (y^2 + z^2)$$

$$= \frac{M}{abc} \int_{-c/2}^{c/2} dx \int_{-b/2}^{b/2} dy \int_{-a/2}^{a/2} dz (y^2 + z^2)$$

$$= \frac{M}{abc} \times \int_{-b/2}^{b/2} dy \left( y^2 z + \frac{z^3}{3} \right) \Big|_{-c/2}^{c/2}$$

$$= \frac{M}{bc} \int_{-b/2}^{b/2} dy \left( cy^2 + \frac{z^3}{3} \cdot \frac{c^3}{8} \right)$$

$$= \frac{M}{bc} \left[ c \frac{y^3}{3} \Big|_{-b/2}^{b/2} + \frac{1}{12} c^3 z \Big|_{-b/2}^{b/2} \right]$$

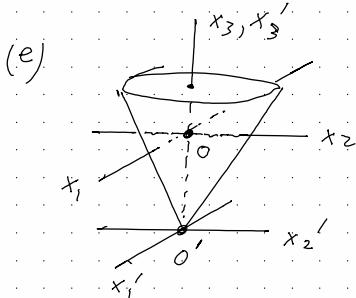
$$I_1 = \frac{\mu}{bc} \left[ c \frac{b^3}{3} + \frac{1}{12} b c^3 \right]$$

$$= \frac{M}{12 bc} \left[ c b^3 + b c^3 \right]$$

$$= \frac{M}{12} (b^2 + c^2)$$

$$I_2 = \frac{M}{12} (c^2 + a^2)$$

$$I_3 = \frac{M}{12} (a^2 + b^2)$$



First calculate

$$I_{ij}' \text{ (wrt } x_1', x_2', x_3') \text{.}$$

Then calculate  $I_{ij}$  via

$$I_{ij} = I_{ij}' - \mu (\vec{a}^2 f_{ij} - a_i a_j)$$

where  $\vec{a} = (0, 0, -d)$

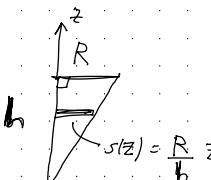
$d$ : height of com above  $O'$

Volume of cone:

$$V = \int dV$$

$$= \int_0^h dz \int_0^{2\pi} d\phi \int_0^R s ds$$

$$= \frac{1}{3}\pi \int_0^h dz \frac{s^2}{2} \Big|_0^R$$



$$V = \pi \int_0^h dz \frac{R^2 z^2}{h^2}$$

$$= \frac{\pi R^2}{h^2} \frac{z^3}{3} \Big|_0^h$$

$$= \frac{\pi R^2}{h^2} \frac{h^3}{3}$$

$$= \boxed{\frac{1}{3} \pi R^2 h}$$

$$\text{Thus, } \rho = \frac{M}{V} = \boxed{\frac{M}{\frac{1}{3} \pi R^2 h}} \quad (\text{mass density})$$

$$I_3' = \int \rho dV (r^2 - z^2)$$

$$= \int \rho dV r^2$$

$$= \rho \int_0^h dz \int_0^{2\pi} d\phi \int_0^{R^2/h} r^3 dr$$

$$= \frac{\rho}{4} \int_0^h dz r^4 \Big|_{0}^{R^2/h}$$

$$= \frac{\rho \pi}{2} \int_0^h dz \left( \frac{R^4}{h^4} \right) z^4$$

$$= \frac{\rho \pi R^4}{2 h^4} \frac{h^5}{5}$$

$$= \frac{1}{5} \pi R^4 h$$

thus,

$$I_3' = \frac{M}{\frac{1}{3} \pi R^2 h} \frac{\pi R^4 h}{10}$$

$$= \boxed{\frac{3}{10} M R^2}$$

similar to the cylinder, we have

$$I_1' = I_2' \equiv I' \text{ where}$$

$$I' = \frac{1}{2} I_3' + \int \rho dV z^2$$

$$\text{Now: } \int \rho dV z^2 = \rho \int_0^h dz z^2 \int_0^{2\pi} d\phi \int_0^{R^2/h} r^3 dr$$

$$= \frac{1}{2} \pi \rho \int_0^h dz z^2 \frac{r^5}{5} \Big|_0^{R^2/h}$$

$$= \pi \rho \frac{R^2}{h^2} \int_0^h dz z^4$$

$$= \pi \rho \frac{R^2}{h^2} \frac{h^5}{5}$$

$$= \frac{\pi}{5} \rho R^2 h^3$$

$$= \frac{\pi}{5} \left( \frac{M}{\frac{1}{3} \pi R^2 h} \right) R^2 h^3$$

$$= \frac{3}{5} M h^2$$

so

$$\begin{aligned} I' &= \frac{1}{2} \left( \frac{3}{10} \mu R^2 \right) + \frac{3}{5} \mu h^2 \\ &= \boxed{\frac{3}{5} \mu \left( \frac{R^2}{4} + h^2 \right)} = I_1' = I_2' \end{aligned}$$

Need to find location of COM.

$$\begin{aligned} d &= \frac{1}{\mu} \int \rho dV z \\ &= \frac{1}{\mu} \rho \int_0^h z dz \int_0^{2\pi} d\phi \int_0^{Rz/h} r dr \\ &= \frac{1}{\mu} \rho \cdot \cancel{2\pi} \int_0^h z dz \frac{1}{2} \left( \frac{R}{h} \right)^2 z^2 \\ &= \frac{\pi \rho}{\mu} \frac{R^2}{h^2} \frac{z^4}{4} \Big|_0^h \\ &= \frac{\pi \rho}{4\mu} R^2 h^2 \\ &= \frac{\pi \rho}{4\mu} \frac{R^2 h^2}{\cancel{\pi} R^2 h} \\ &= \boxed{\frac{3}{4} h} \end{aligned}$$

Thus,

$$I_{ij} = I_{ij}' - \mu (\vec{a}^2 \delta_{ij} - a_i a_j)$$

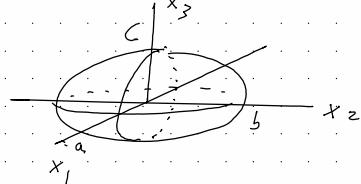
where  $\vec{a} = (0, 0, -\frac{3}{4} h) \rightarrow a^2 = \frac{9}{16} h^2$

$$\begin{aligned} \rightarrow I_1 &= I_1' - \mu a^2 \\ &= \frac{3}{5} \mu \left( \frac{R^2}{4} + h^2 \right) - \mu \frac{9}{16} h^2 \\ &= \frac{3}{20} \mu R^2 + \mu h^2 \left( \frac{3}{5} - \frac{9}{16} \right) \\ &\quad \cancel{\frac{48-45}{80}} = \frac{3}{80} \\ &= \boxed{\frac{3}{20} \mu \left( R^2 + \frac{h^2}{4} \right)} \end{aligned}$$

Also,  $I_2 = I_1$ .

$$\begin{aligned} \text{Finally, } I_3 &= I_3' - \mu (\vec{a}^2 - a^2) \\ &= I_3' \\ &= \boxed{\frac{3}{10} \mu R^2} \end{aligned}$$

(f) Ellipsoid with semi-axes  $a, b, c$



$$(a, b, c) \leftrightarrow (x_1, x_2, x_3)$$

Define rescaled coordinates:

$$(u, v, w) \equiv \left( \frac{x_1}{a}, \frac{x_2}{b}, \frac{x_3}{c} \right)$$

so flat boundary of ellipsoid

$$1 = \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 + \left(\frac{x_3}{c}\right)^2 = u^2 + v^2 + w^2$$

unit  $\mathbb{Z}$ -sphere.

Volumes:

$$V = \int dx_1 \int dx_2 \int dx_3$$

$$= abc \int du \int dv \int dw$$

$$= abc \int d\phi \int \sin\theta d\theta \int r^2 dr$$

$$= abc \cdot 2\pi \cdot 2 \cdot \frac{r^3}{3} \Big|_0^1$$

$$= \boxed{\frac{4}{3}\pi abc}$$

$$\rightarrow \rho = \frac{\mu}{\frac{4}{3}\pi abc}$$

$$I_3 = \int \rho dV (r^2 - z^2)$$

$$= \int \rho dV (x^2 + y^2)$$

$$= \frac{\mu}{\frac{4}{3}\pi abc} \iiint dxdydz (x^2 + y^2)$$

$$= \frac{\mu}{\frac{4}{3}\pi abc} abc \iiint du dv dw (a^2 u^2 + b^2 v^2)$$

$$= \frac{\mu}{\frac{4}{3}\pi} \int r^2 dr \int_{\sin\theta}^1 \int_0^{2\pi} d\phi (a^2 r^2 \sin^2\theta \cos^2\phi + b^2 r^2 \sin^2\theta \sin^2\phi)$$

$$\text{Now: } \int_{\sin\theta}^1 \int_0^{2\pi} d\theta \sin^2\theta = \int_0^1 d(\cos\theta) (1 - \cos^2\theta)$$

$$= \int_{-1}^1 dx (1 - x^2)$$

$$= \left(x - \frac{x^3}{3}\right) \Big|_{-1}^1$$

$$= 2 \cdot \frac{2}{3} = \boxed{\frac{4}{3}}$$

$$\int_0^1 r^4 dr = \frac{r^5}{5} \Big|_0^1 = \boxed{\frac{1}{5}}$$

$$\int_0^{2\pi} d\phi \left\{ \frac{\sin^2 \phi}{\cos^2 \phi} \right\} \approx 2\pi \cdot \frac{1}{2} = [\pi]$$

Thus,

$$I_3 = \frac{M}{\frac{4}{5}\pi} \left( a^2 \frac{4}{3} \cdot \frac{1}{5} \cdot \pi + b^2 \frac{4}{3} \cdot \frac{1}{5} \cdot \pi \right)$$

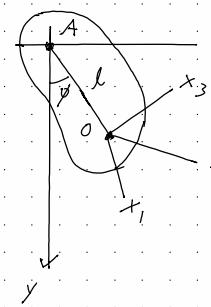
$$= \boxed{\frac{M}{5} (a^2 + b^2)}$$

Cyclically permuting  $a, b, c \rightarrow$

$$\boxed{I_1 = \frac{M}{5} (b^2 + c^2)}$$

$$\boxed{I_2 = \frac{M}{5} (c^2 + a^2)}$$

Sec 32, Prob 3:



com. at  $\vec{\theta}$   
rotation,  $+x$  is at  $A$ , out of page

$$\vec{\Omega} = \dot{\phi} \hat{n}$$

$$U = \mu g l (1 - \cos \phi)$$

$$\approx \frac{1}{2} \mu g l \dot{\phi}^2 \text{ for } \dot{\phi} \ll 1$$

$$L = T - U$$

$$T = \frac{1}{2} M V^2 + \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$$

$$\vec{\Omega} = \dot{\phi} \hat{n} \rightarrow \Omega_1 = \dot{\phi} \hat{n} \cdot \hat{x}_1 = \dot{\phi} \cos \alpha$$

$$\Omega_2 = \dot{\phi} \hat{n} \cdot \hat{x}_2 = \dot{\phi} \cos \beta$$

$$\Omega_3 = \dot{\phi} \hat{n} \cdot \hat{x}_3 = \dot{\phi} \cos \gamma$$

$$V = \mu g l \phi$$

thus,

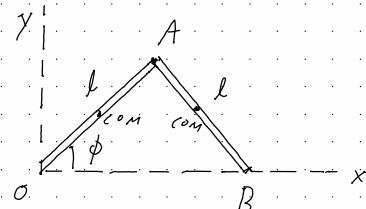
$$T = \frac{1}{2} M l^2 \dot{\phi}^2 + \frac{1}{2} \dot{\phi}^2 (I_1 \cos^2 \alpha + I_2 \cos^2 \beta + I_3 \cos^2 \gamma)$$

so

$$L = \frac{1}{2} \dot{\phi}^2 (M l^2 + I_1 \cos^2 \alpha + I_2 \cos^2 \beta + I_3 \cos^2 \gamma) - \frac{1}{2} \mu g l \dot{\phi}^2$$

$$\rightarrow w = \sqrt{\frac{\mu g l}{M l^2 + I_1 \cos^2 \alpha + I_2 \cos^2 \beta + I_3 \cos^2 \gamma}}$$

Sec 32, Prob 4



two uniform rods

$$I_{com} = \frac{1}{12} M l^2$$

$$I_{end} = \frac{1}{3} M l^2$$

$$T = T_1 + T_2$$

$$\begin{aligned} T_1 &= \frac{1}{2} M \left(\frac{l}{2}\right)^2 \dot{\phi}^2 + \frac{1}{2} I_{com} \dot{\phi}^2 \\ &= \frac{1}{8} M l^2 \dot{\phi}^2 + \frac{1}{24} M l^2 \dot{\phi}^2 \\ &= \left(\frac{1}{8} + \frac{1}{24}\right) M l^2 \dot{\phi}^2 \\ &= \frac{1}{6} M l^2 \dot{\phi}^2 \end{aligned}$$

$$T_2 = \frac{1}{2} M V^2 + \frac{1}{2} I_{com} \dot{\phi}^2$$

$$\text{Now: } V^2 = \dot{x}^2 + \dot{y}^2$$

$$\dot{x} = \frac{3}{2} l \cos\phi, \quad \dot{y} = \frac{l}{2} \sin\phi$$

$$\dot{x} = -\frac{3}{2} l \sin\phi \dot{\phi}, \quad \dot{y} = \frac{l}{2} \cos\phi \dot{\phi}$$

$$\rightarrow V^2 = \frac{9}{4} l^2 \sin^2\phi \dot{\phi}^2 + \frac{l^2}{4} \cos^2\phi \dot{\phi}^2$$

$$= 2 l^2 \sin^2\phi \dot{\phi}^2 + \frac{l^2}{4} \dot{\phi}^2$$

$$= 2 l^2 \dot{\phi}^2 (\sin^2\phi + \frac{1}{4})$$

SD

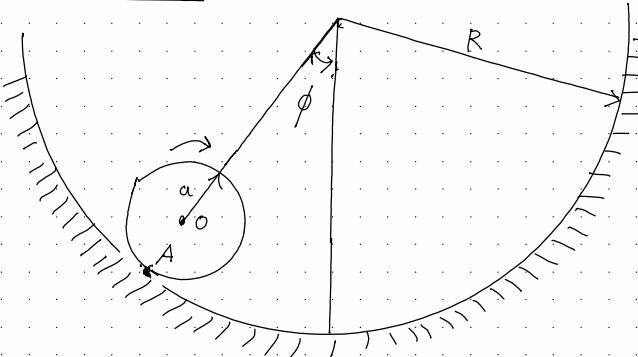
$$\begin{aligned} T_2 &= \frac{1}{2} M l^2 \dot{\phi}^2 \left( \sin^2\phi + \frac{1}{8} \right) + \frac{1}{24} M l^2 \dot{\phi}^2 \\ &= M l^2 \dot{\phi}^2 \left( \sin^2\phi + \frac{1}{8} + \frac{1}{24} \right) \\ &= M l^2 \dot{\phi}^2 \left( \sin^2\phi + \frac{1}{6} \right) \end{aligned}$$

Thus,

$$T = T_1 + T_2$$

$$\begin{aligned} &= \frac{1}{6} M l^2 \dot{\phi}^2 + M l^2 \dot{\phi}^2 \left( \sin^2\phi + \frac{1}{6} \right) \\ &= M l^2 \dot{\phi}^2 \left( \frac{1}{3} + \sin^2\phi \right) \\ &= \frac{1}{3} M l^2 \dot{\phi}^2 (1 + 3 \sin^2\phi) \end{aligned}$$

Sec 32, Prob 6:



Homogeneous cylinder of radius  $a$ , mass  $M$ :

$$I_3 = \frac{1}{2}Ma^2 \quad (\text{about com})$$

$$V = (R-a)\dot{\phi} \quad (\text{velocity of com})$$

Instantaneous axis of rotation at A:

$$\omega = \vec{V} + \vec{\omega} + (-ah\hat{i}), \quad \vec{\omega} \text{ into page}$$

$$\text{so } V = \Omega a$$

$$\text{Thus, } \Omega a = (R-a)/\dot{\phi}$$

$$\Omega = \left(\frac{R-a}{a}\right)/\dot{\phi}$$

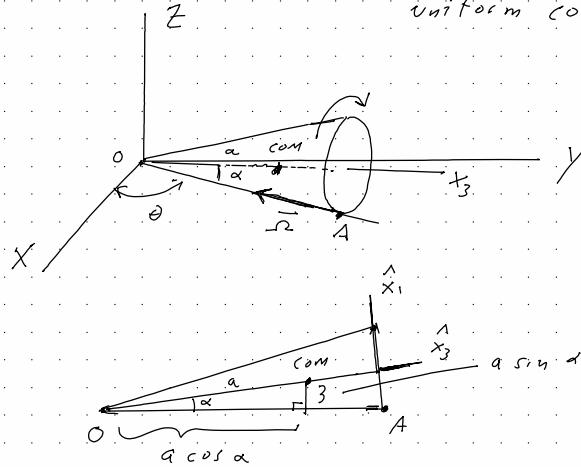
$$T = \frac{1}{2}MV^2 + \frac{1}{2}I_3\Omega^2$$

$$= \frac{1}{2}M\left(\frac{R-a}{a}\right)^2a^2\dot{\phi}^2 + \frac{1}{4}Ma^2\left(\frac{R-a}{a}\right)^2\dot{\phi}^2$$

$$= M(R-a)^2\dot{\phi}^2\left(\frac{1}{2} + \frac{1}{4}\right) = \boxed{\frac{3}{4}M(R-a)^2\dot{\phi}^2}$$

Sec 32, Prob 7.

radius  $R$ , height  $h$   
uniform cone



$$V = \text{velocity of com}$$

$$= a \cos \alpha \dot{\theta}$$

OA: instantaneous axis of rotation

$$\omega = \vec{V} + a \sin \alpha \vec{\Omega} \times \hat{h} \quad (\hat{h} = \hat{z})$$

$$\omega = V - a \sin \alpha \Omega \hat{z}$$

$$\text{thus, } a \sin \alpha \Omega = V = a \cos \alpha \dot{\theta}$$

$$\boxed{\Omega = \cot \alpha \dot{\theta}}$$

$\vec{\omega}$ : directed from A to O

$$\Omega_3 = -\Omega \cos \alpha = -\frac{\cos^2 \alpha}{\sin \alpha} \dot{\theta}$$

$$\Omega_1 = \Omega \sin \alpha = \cos \alpha \dot{\theta}$$

Thur,

$$T = \frac{1}{2} \mu V^2 + \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$$
$$= \frac{1}{2} \mu h^2 \cos^2 \alpha \dot{\theta}^2 + \frac{1}{2} \left( I_1 \cos^2 \alpha \dot{\theta}^2 + I_3 \frac{\cos^4 \alpha}{\sin^2 \alpha} \dot{\theta}^2 \right)$$

Now:  $I_1 = I_2 = \frac{3}{20} \mu / R^2 + \frac{1}{4} b^2$

$$I_3 = \frac{3}{10} \mu R^2$$

Also:  $\alpha = \frac{3}{4} b$

$$\tan \alpha = \frac{R}{h} \rightarrow R = h \tan \alpha$$

Thur,

$$T = \frac{1}{2} \mu \left( \frac{9}{16} \right) h^2 \cos^2 \alpha \dot{\theta}^2 + \frac{1}{2} \left[ \frac{3}{20} \mu \left( h^2 \tan^2 \alpha + \frac{1}{4} h^2 \right) \cos^2 \alpha \dot{\theta}^2 + \frac{3}{10} \mu h^2 \tan^2 \alpha \frac{\cos^4 \alpha}{\sin^2 \alpha} \dot{\theta}^2 \right]$$

$$= \mu h^2 \dot{\theta}^2 \left[ \frac{9}{32} \cos^2 \alpha + \frac{3}{40} \overbrace{\sin^4 \alpha}^{(1-\cos^2 \alpha)} + \frac{3}{160} \cos^2 \alpha \right] + \frac{3}{20} \cos^2 \alpha \dot{\theta}^2$$

$$= \mu h^2 \dot{\theta}^2 \left[ \frac{3}{40} + \cos^2 \alpha \left( \frac{9}{32} - \frac{3}{40} + \frac{3}{160} + \frac{3}{20} \right) \right]$$

Now:  $\frac{9}{32} - \frac{3}{40} + \frac{3}{160} + \frac{3}{20}$

$$= \frac{1}{160} [45 - 12 + 3 + 24]$$

$$= \frac{60}{160}$$

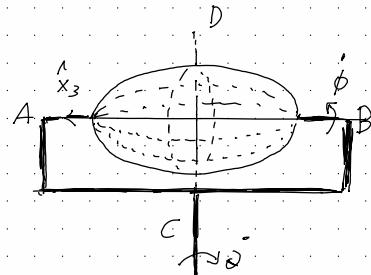
$$= \frac{15}{40}$$

Thur,

$$T = \mu h^2 \dot{\theta}^2 \left[ \frac{3}{40} + \frac{15}{40} \cos^2 \alpha \right]$$

$$= \boxed{\frac{3}{40} \mu h^2 \dot{\theta}^2 \left[ 1 + 5 \cos^2 \alpha \right]}$$

Sec 32, Prob 9.



homogeneous ellipsoid  
with principal  
moments of  
inertia  $I_1, I_2, I_3$

$$\vec{\Omega} = \dot{\phi} + \vec{\theta}$$

$$\text{Now: } \vec{\phi} = \dot{\phi} \hat{x}_3$$

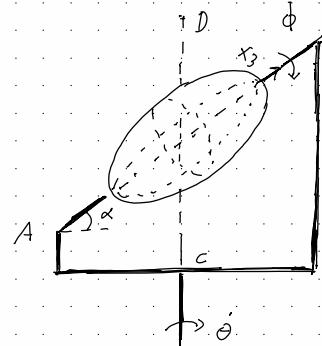
$$\vec{\theta} = \dot{\theta} [\cos \phi \hat{x}_1 + \sin \phi \hat{x}_2]$$

$$\text{so: } \vec{\Omega} = \dot{\theta} \cos \phi \hat{x}_1 + \dot{\theta} \sin \phi \hat{x}_2 + \dot{\phi} \hat{x}_3$$

$$T = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$$

$$= \frac{1}{2} [(I_1 \cos^2 \phi + I_2 \sin^2 \phi) \dot{\theta}^2 + I_3 \dot{\phi}^2]$$

Sec 32, Prob 10.



uniform ellipsoid  
with  $I_1 = I_2$   
(circular cross section)

$$\vec{\Omega} = \vec{\phi} + \vec{\theta}$$

$$\vec{\phi} = \dot{\phi} \hat{x}_3$$

$$\vec{\theta} = \dot{\theta} [\cos(\frac{\pi}{2} - \alpha) \hat{x}_3 + \sin(\frac{\pi}{2} - \alpha) (\cos \phi \hat{x}_1 + \sin \phi \hat{x}_2)]$$

$$\text{Now: } \cos(\frac{\pi}{2} - \alpha) = \cos(\frac{\pi}{2}) \cos \alpha + \sin(\frac{\pi}{2}) \sin \alpha \\ = \sin \alpha$$

$$\sin(\frac{\pi}{2} - \alpha) = \sin(\frac{\pi}{2}) \cos \alpha - \cos(\frac{\pi}{2}) \sin \alpha \\ = \cos \alpha$$

$$\text{so: } \vec{\theta} = \dot{\theta} [\sin \alpha \hat{x}_3 + \cos \alpha \cos \phi \hat{x}_1 + \cos \alpha \sin \phi \hat{x}_2]$$

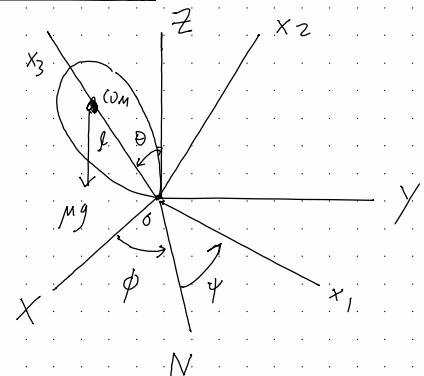
$$\rightarrow \vec{\Omega} = \dot{\theta} \cos \alpha (\cos \phi \hat{x}_1 + \sin \phi \hat{x}_2) + (\dot{\phi} + \dot{\theta} \sin \alpha) \hat{x}_3$$

$$T = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$$

$$= \frac{1}{2} [I_1 \dot{\theta}^2 \cos^2 \alpha (\cos^2 \phi + \sin^2 \phi) + I_3 (\dot{\phi} + \dot{\theta} \sin \alpha)^2]$$

$$= \frac{1}{2} [I_1 \cos^2 \alpha \dot{\theta}^2 + I_3 (\dot{\phi} + \dot{\theta} \sin \alpha)^2]$$

Sec 35, prob 1:



Symmetrical top:

$$I_1 = I_2, \quad I_3 \quad (\text{wrt principal axes passing through COM})$$

$$I'_1 = I_1 + ml^2 \quad (\text{wrt axes passing through}$$

$$I'_2 = I'_1 \quad O, \text{ which is displaced}$$

$$I'_3 = I_3 \quad \text{from the COM by } l \text{ in the}$$

$-x_3$  direction)

$$L = T - U$$

$$U = mgz = mgl \cos \theta$$

$$T = \frac{1}{2} (I'_1 \Omega_1^2 + I'_2 \Omega_2^2 + I'_3 \Omega_3^2)$$

$$= \frac{1}{2} [I'_1 (\Omega_1^2 + \Omega_2^2) + I'_3 \Omega_3^2]$$

Now: (From (35.1))

$$\Omega_1 = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi$$

$$\Omega_2 = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi$$

$$\Omega_3 = \dot{\phi} \cos \theta + \dot{\psi}$$

$$\therefore \Omega_3^2 = (\dot{\phi} \cos \theta + \dot{\psi})^2$$

$$\Omega_1^2 = \dot{\phi}^2 \sin^2 \theta \sin^2 \psi + \dot{\theta}^2 \cos^2 \psi + 2\dot{\theta}\dot{\psi} \sin \theta \sin \psi \cos \psi$$

$$\Omega_2^2 = \dot{\phi}^2 \sin^2 \theta \cos^2 \psi + \dot{\theta}^2 \sin^2 \psi - 2\dot{\theta}\dot{\psi} \sin \theta \cos \psi \sin \psi$$

$$\rightarrow \Omega_1^2 + \Omega_2^2 = \dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2$$

Thus,

$$L = \frac{1}{2} [I'_1 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + I'_3 (\dot{\phi} \cos \theta + \dot{\psi})^2] - mg l \cos \theta$$

1) No explicit  $t$ -dependence

$$E = T + U = \text{const}$$

2) No explicit  $\phi$  dependence:

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = I'_1 \sin^2 \theta \dot{\phi} + I'_3 (\dot{\phi} \cos \theta + \dot{\psi}) \cos \theta = \text{const}$$

3) No explicit  $\psi$  dependence: (see next page)  $= M_z$

$$p_\psi \equiv \frac{\partial L}{\partial \dot{\psi}} = I'_3 (\dot{\phi} \cos \theta + \dot{\psi}) = \text{const} = M_3$$

To show that  $p_z = M_z$ :

$$\begin{aligned} M_z &= \vec{M} \cdot \hat{\vec{z}} \\ &= (M_1 \hat{x}_1 + M_2 \hat{x}_2 + M_3 \hat{x}_3) \cdot \hat{\vec{z}} \\ &= M_1 \hat{x}_1 \cdot \hat{\vec{z}} + M_2 \hat{x}_2 \cdot \hat{\vec{z}} + M_3 \hat{x}_3 \cdot \hat{\vec{z}} \end{aligned}$$

Now:  $\hat{x}_3 \cdot \hat{\vec{z}} = \cos\theta$

$$\hat{x}_1 \cdot \hat{\vec{z}} = \sin\theta \sin\psi$$

$$\hat{x}_2 \cdot \hat{\vec{z}} = \sin\theta \cos\psi$$

Thus,

$$\begin{aligned} M_z &= M_1 \sin\theta \sin\psi + M_2 \sin\theta \cos\psi + M_3 \cos\theta \\ &= I_1' \Omega_1 \sin\theta \sin\psi + I_1' \Omega_2 \sin\theta \cos\psi \\ &\quad + I_3 \Omega_3 \cos\theta \\ &= I_1' (\sin^2\theta \sin^2\psi \dot{\phi} + \cancel{\sin\theta \sin\psi \cos\psi \dot{\theta}}) \\ &\quad + I_1' (\sin^2\theta \cos^2\psi \dot{\phi} - \cancel{\sin\theta \cos\psi \sin\psi \dot{\theta}}) \\ &\quad + I_3 (\phi \cos\theta + \dot{\psi}) \cos\theta \\ &= I_1' \sin^2\theta \dot{\phi} + I_3 \cos\theta (\phi \cos\theta + \dot{\psi}) \end{aligned}$$

We can solve  $M_z, M_3$  for  $\dot{\phi}, \dot{\psi}$ :

$$M_z = I_1' \sin^2\theta \dot{\phi} + I_3 (\phi \cos\theta + \dot{\psi}) \cos\theta$$

$$M_3 = I_3 (\phi \cos\theta + \dot{\psi})$$

$$\rightarrow M_z = I_1' \sin^2\theta \dot{\phi} + M_3 \cos\theta$$

$$\text{so } I_1' \sin^2\theta \dot{\phi} = M_z - M_3 \cos\theta$$

$$\boxed{\dot{\phi} = \frac{M_z - M_3 \cos\theta}{I_1' \sin^2\theta}}$$

Also:

$$\frac{M_3}{I_3} = \phi \cos\theta + \dot{\psi}$$

$$\rightarrow \boxed{\dot{\psi} = \frac{M_3}{I_3} - \phi \cos\theta}$$

$$= \frac{M_3}{I_3} - \left( \frac{M_z - M_3 \cos\theta}{I_1' \sin^2\theta} \right) \cos\theta$$

Also:

$$E = T + U$$

$$\begin{aligned} &= \frac{1}{2} [ I_1' (\sin^2\theta \dot{\phi}^2 + \dot{\theta}^2) + I_3 (\phi \cos\theta + \dot{\psi})^2 ] \\ &\quad + \mu g l \cos\theta \end{aligned}$$

can be re-written as

$$\begin{aligned}
 E &= \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2} I_1' \sin^2 \theta \dot{\phi}^2 \\
 &\quad + \frac{1}{2} I_3 \left( \frac{M_3}{I_3} \right)^2 + \mu g l \cos \phi \\
 &= \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2} I_1' \sin^2 \theta \frac{(M_2 - M_3 \cos \theta)^2}{I_1' \sin^4 \theta} \\
 &\quad + \frac{1}{2} \frac{M_3^2}{I_3} + \mu g l \cos \phi \\
 &= \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2} \frac{M_3^2}{I_3} + \frac{1}{2} \frac{(M_2 - M_3 \cos \theta)^2}{I_1' \sin^2 \theta} \\
 &\quad + \mu g l \cos \theta
 \end{aligned}$$

$$\text{Now: } \frac{1}{2} \frac{M_3^2}{I_3} = \text{const}$$

$$\text{Also: } \mu g l \cos \theta = -\mu g l (1 - \cos \theta) + \mu g l$$

so

$$\underbrace{E - \frac{1}{2} \frac{M_3^2}{I_3} - \mu g l}_{E'} = \frac{1}{2} I_1' \dot{\theta}^2 + \frac{1}{2} \frac{(M_2 - M_3 \cos \theta)^2}{I_1' \sin^2 \theta} - \mu g l (1 - \cos \theta)$$

$$E' = \frac{1}{2} I_1' \dot{\theta}^2 + U_{\text{eff}}(\theta)$$

$$\text{where } U_{\text{eff}}(\theta) = \frac{1}{2} \frac{(M_2 - M_3 \cos \theta)^2}{I_1' \sin^2 \theta} - \mu g l (1 - \cos \theta)$$

EOM:

$$\ddot{\theta}^2 = \frac{2}{I_1'} (E' - U_{\text{eff}}(\theta))$$

$$\frac{d\theta}{dt} = \sqrt{\frac{2}{I_1'} (E' - U_{\text{eff}}(\theta))}$$

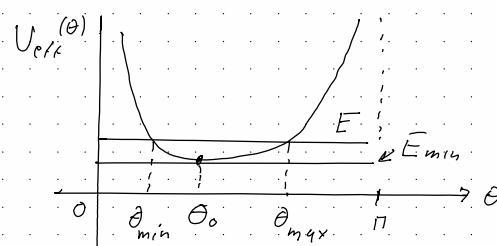
$$\rightarrow \int dt = \int \frac{d\theta}{\sqrt{\frac{2}{I_1'} (E' - U_{\text{eff}}(\theta))}}$$

$$t = \int \frac{d\theta}{\sqrt{\frac{2}{I_1'} (E' - U_{\text{eff}}(\theta))}} + \text{const}$$

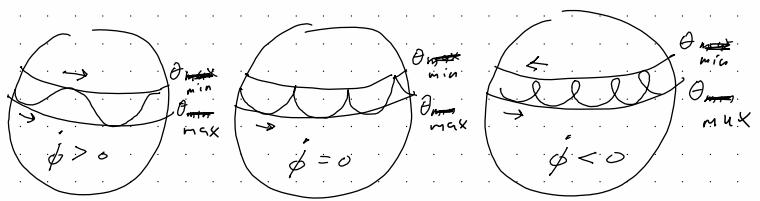
Effective potential:

$$U_{\text{eff}}(\theta) = \frac{1}{2} \frac{(M_2 - M_3 \cos \theta)^2}{I_1' \sin^2 \theta} - \mu g l (1 - \cos \theta)$$

For  $M_2 \neq M_3$ ,  $U_{\text{eff}}(\theta) \rightarrow \infty$  as  $\theta \rightarrow 0, \pi$ .



For  $E > E_{\min}$ ,  $\theta$  varies between  $\theta_{\min}$  and  $\theta_{\max}$ . The motion of the  $x_3 - x_{12}$  of the top can have the following three forms depending on the sign of  $\dot{\phi}$  when  $\theta = \theta_{\min}$ :



This motion is called nutation.

### Sec 3.5, Prob 2

For rotation of a top around a vertical axis, to be stable, we need  $\frac{d^2 U_{eff}}{d\theta^2} |_{\theta=0} > 0$

Now:

$$U_{eff}(\theta) = \frac{(M_z - M_3 \cos \theta)^2}{2 I_1 \sin^2 \theta} - \mu g l (1 - \cos \theta)$$

Also:

$$M_z = I_1' \sin^2 \theta \dot{\phi} + I_3 (\dot{\phi} \cos \theta + \dot{\psi})_{co, \theta}$$

$$M_3 = I_3 (\dot{\phi} \cos \theta + \dot{\psi})$$

In the limit  $\theta \rightarrow 0$

$$M_z \approx I_3 (\dot{\phi} + \dot{\psi}) \quad \text{so they are equal}$$

$$M_3 \approx I_3 (\dot{\phi} + \dot{\psi}) \quad \text{in this limit}$$

Thus,

$$U_{eff}(\theta) \approx \frac{M_3^2 (1 - \cos \theta)^2}{2 I_1' \sin^2 \theta} - \mu g l (1 - \cos \theta)$$

$$\approx \frac{M_3^2 \left(\frac{\theta^2}{2}\right)^2}{2 I_1' \theta^2} - \mu g l \frac{\theta^2}{2}$$

$$\approx \left( \frac{1}{8} \frac{M_3^2}{I_1'} - \frac{1}{2} \mu g l \right) \theta^2$$

Thur

$$U_{\text{eff}}(\theta) \approx \left( \frac{1}{8} \frac{M_3}{I_1} - \frac{1}{2} M g l \right) \theta^2$$

$$\rightarrow U_{\text{eff}}(0) = 0$$

$$\frac{dU_{\text{eff}}}{d\theta} \Big|_{\theta=0} = 2 \left( \frac{1}{8} \frac{M_3}{I_1} - \frac{1}{2} M g l \right) \theta \Big|_{\theta=0} = 0$$

$$\frac{d^2U_{\text{eff}}}{d\theta^2} \Big|_{\theta=0} = 2 \left( \frac{1}{8} \frac{M_3}{I_1} - \frac{1}{2} M g l \right)$$

Need  $\frac{d^2U_{\text{eff}}}{d\theta^2} \Big|_{\theta=0} > 0$  for stable rotation:

$$\frac{1}{8} \frac{M_3^2}{I_1} - \frac{1}{2} M g l > 0$$

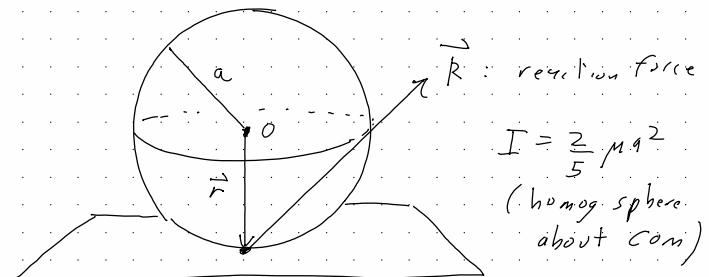
$$\boxed{M_3^2 > 4 M g l I_1}$$

Equivalently, since  $M_3 = I_3 \omega_3$ :

$$\boxed{\omega_3^2 > \frac{4 M g l I_1}{F_3^2}}$$

Sec 38, Prob 1:

Homogeneous sphere (radius  $a$ ) rolling without slipping on a horizontal surface, subject to applied force  $\vec{F}$  and torque  $\vec{T}$ :



$$I = \frac{2}{5} M a^2$$

(homog sphere  
about com)

$$\frac{d\vec{P}}{dt} = \vec{F} + \vec{R} \rightarrow m \frac{d\vec{V}}{dt} = \vec{F} + \vec{R}$$

$$\frac{d\vec{m}}{dt} = \vec{F} + F_x \vec{R} \rightarrow I \frac{d\vec{\Omega}}{dt} = \vec{F} - a \vec{z} \times \vec{R}$$

Rolling without slipping:

$$\begin{aligned} \vec{O} &= \vec{V} + \vec{\Omega} \times \vec{r} \\ &= \vec{V} - a \vec{\Omega} \times \vec{z} \end{aligned}$$

$$\text{so } \vec{V} = a \vec{\Omega} \times \vec{z}$$

using  $(\vec{A} \times \vec{B})_i = A_2 B_3 - A_3 B_2$ , etc

$$\boxed{\begin{aligned} V_x &= a \Omega_y \\ V_y &= -a \Omega_x \\ V_z &= 0 \end{aligned}}$$

no motion off surface

Combined

$$\mu \frac{d\vec{V}}{dt} = \vec{F} + \vec{R} \quad (1)$$

$$I \frac{d\vec{\alpha}}{dt} = \vec{R} - a \hat{z} \times \vec{R} \quad (2)$$

$$V_x = a \Omega_y, V_y = -a \Omega_x, V_z = 0 \quad (\text{constraint})$$

Take time derivative of constraint equation:

$$\frac{dV_x}{dt} = a \frac{d\Omega_y}{dt}, \quad \frac{dV_y}{dt} = -a \frac{d\Omega_x}{dt}$$

Substitute from (1), (2) into these two equations

$$\frac{1}{\mu} (F_x + R_x) = \frac{a}{I} (K_y - a R_x)$$

$$\frac{1}{\mu} (F_y + R_y) = -\frac{a}{I} (I K_x + a R_y)$$

Thus,

$$F_x + R_x = \frac{ma}{I} K_y - \frac{ma^2}{I} R_x$$

$$R_x \left( \frac{I + ma^2}{I} \right) = \frac{ma}{I} K_y - F_x$$

$$R_x \frac{\frac{3}{5}ma^2}{\frac{2}{5}ma^2} = \frac{ma}{\frac{2}{5}ma^2} K_y - F_x$$

$$\rightarrow \boxed{R_x = \frac{5}{7} \frac{K_y}{a} - \frac{2}{7} F_x}$$

Similarly,

$$F_y + R_y = -\frac{ma}{I} K_x - \frac{ma^2}{I} R_y$$

$$R_y \left( \frac{I + ma^2}{I} \right) = -\frac{ma}{I} K_x - F_y$$

$$R_y \frac{\frac{3}{5}ma^2}{\frac{2}{5}ma^2} = -\frac{ma}{\frac{2}{5}ma^2} K_x - F_y$$

$$\rightarrow \boxed{R_y = -\frac{5}{7} \frac{K_x}{a} - \frac{2}{7} F_y}$$

Also,

$$\mu \frac{dV_z}{dt} = F_z + R_z$$

$$\rightarrow \boxed{R_z = -F_z}$$

Using these expression for  $R_x, R_y$  we can write down EOMs for  $V_x, V_y$

$$\begin{aligned} \boxed{\mu \frac{dV_x}{dt}} &= F_x + R_x \\ &= F_x + \frac{5}{7} \frac{K_y}{a} - \frac{2}{7} F_x \\ &= \frac{5}{7} (F_x + \frac{K_y}{a}) \end{aligned}$$

Similarly

$$\begin{aligned} \boxed{m \frac{dV_x}{dt}} &= F_y + R_y \\ &= F_y - \frac{5}{7} \frac{K_x}{a} - \frac{2}{7} F_x \\ &= \frac{5}{7} \left( F_y - \frac{K_x}{a} \right) \end{aligned}$$

Summary:

$$R_x = \frac{5}{7} \frac{K_x}{a} - \frac{2}{7} F_x$$

$$R_y = -\frac{5}{7} \frac{K_x}{a} - \frac{2}{7} F_y$$

$$R_z = -F_z$$

$$m \frac{dV_x}{dt} = \frac{5}{7} \left( F_y + \frac{K_x}{a} \right)$$

$$m \frac{dV_y}{dt} = \frac{5}{7} \left( F_y - \frac{K_x}{a} \right)$$

$$V_z = 0$$

$$\Omega_x = -\frac{V_y}{a}$$

$$\Omega_y = \frac{V_x}{a}$$

$$I \frac{d\Omega_z}{dt} = K_z$$

Example:

$$\text{Suppose: } \vec{F} = -mg\hat{z} + F_0\hat{x}$$

$$K = 0$$

$$\text{Then: } F_x = F_0, F_y = 0, F_z = -mg$$

Solution from previous page goes here.

$$R_x = -\frac{2}{7} F_0$$

$$R_y = 0$$

$$R_z = mg \quad (\text{normal force upward})$$

$$m \frac{dV_x}{dt} = \frac{5}{7} F_0 \rightarrow V_x = \frac{5}{7} \frac{F_0}{m} t + V_{x0}$$

$$m \frac{dV_y}{dt} = 0 \rightarrow V_y = \text{const} = V_{y0}$$

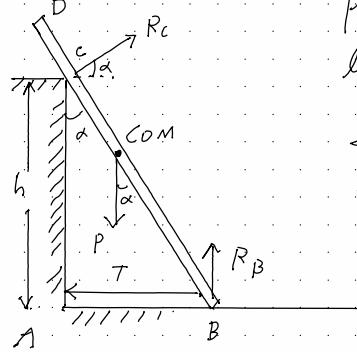
$$V_z = 0$$

$$\Omega_x = -\frac{V_{y0}}{a}$$

$$\Omega_y = \frac{V_x}{a} = \frac{5}{7} \frac{F_0}{ma} t + \frac{V_{x0}}{a}$$

$$I \frac{d\Omega_z}{dt} = 0 \rightarrow \Omega_z = \text{const} = \Omega_{z0}$$

Sec 38, Prob 2



$$P = \text{weight} = \mu g$$
$$l = \text{length of uniform rod}$$

$$\sum \vec{F} = 0$$

$$\sum \vec{r} \times \vec{F} = 0$$

including reaction Force  
 $T, R_C, R_B$

horizontal direction

$$-T + R_C \cos \alpha = 0 \quad (1)$$

vertical direction

$$R_C \sin \alpha - P + R_B = 0 \quad (2)$$

torques around B:

$$\frac{l}{2} \sin \alpha P - \frac{h}{\cos \alpha} R_C = 0 \quad (3)$$

Thus,

$$\frac{1}{\cos \alpha} R_C = \frac{l \sin \alpha}{2} P$$

$$R_C = \frac{l P \sin \alpha \cos \alpha}{2h}$$

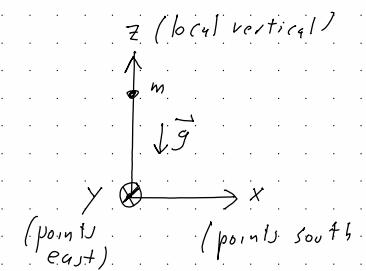
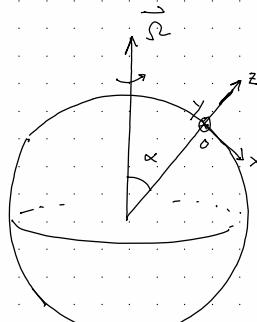
$$T = R_C \cos \alpha$$

$$R_B = P - R_C \sin \alpha$$



$$\cos \alpha = \frac{l}{d}$$
$$\Rightarrow d = \frac{h}{\cos \alpha}$$

Sec 39, Prob 1:



( $\alpha$ : location on Earth relative to North pole)

Newton's 2nd law in rotating frame

$$m \vec{a} = m \vec{g} - m \vec{W} - 2m \vec{\Omega} \times \vec{v} - m \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$\text{where } \vec{W} = \vec{\Omega} \times (\vec{\Omega} \times \vec{R})$$

= acceleration of origin of rotating frame

Thus,

$$m \vec{a} = m \vec{g} - m \vec{\Omega} \times (\vec{\Omega} \times (\vec{r} + \vec{R})) - 2m \vec{\Omega} \times \vec{v}$$

2nd order in  $\vec{\Omega}$   
[ignore]

$$\approx m \vec{g} - 2m \vec{\Omega} \times \vec{v}$$

$$\text{Now: } \vec{g} = -g \vec{z}$$

$$\vec{\Omega} = \sqrt{\cos \alpha} \vec{z} + \sqrt{\sin \alpha} \vec{x}$$

$$(\vec{r} \times \vec{v})_x = \cancel{r y^2 v_z} - r z^2 y \\ = -r z y$$

$$(\vec{\Omega} \times \vec{v})_y = \Omega_z v_x - \Omega_x v_z$$

$$= \Omega_z x - \Omega_x z$$

$$(\vec{u} \times \vec{v})_z = u_x v_y - u_y v_x$$

$$= u_x y$$

Fhus,

$$x = +2 \Omega_z y$$

$$y = -2(\Omega_z x - \Omega_x z)$$

$$Z = -g + \Omega_x y$$

Want to solve these equations to 1st order in  $\Omega$

The  $0^{th}$  order solution is

$$\left. \begin{array}{l} x_0(t) = 0 \\ y_0(t) = 0 \\ z_0(t) = h - \frac{1}{2}gt^2 \end{array} \right\} \begin{array}{l} \text{drop from} \\ \text{vertical} \\ \text{height } h \end{array}$$

Write  $x(t) = x_0(t) + x_1(t)$ , etc.

Then

$$x_i = 2 \alpha z y_i \approx 0$$

$$y_1 = -2(\alpha_z x_1 - \alpha_x(-\gamma t + z_{11})) \approx -2\alpha_x \gamma t$$

$$-f + z_1 = -g + \Omega x y_1 \approx 0$$

Thus, only need to solve

$$y_1 = -2 \omega_x g t$$

$$\rightarrow x_1 = -\Omega_x g t^2$$

$$y_1 = -\frac{1}{3} \sin x + t^3$$

$$\text{Now: } \Omega_x = -\Omega \sin \alpha$$

$$T = \sqrt{\frac{2h}{g}} \quad (0^{\text{th}} \text{ order solution})$$

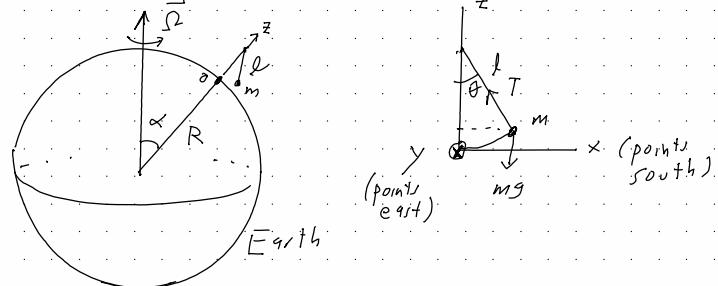
so object hits the ground at

$$Y = Y_1 = \left( \frac{2h}{g} \right)^{3/2}$$

indicates that the object hit the ground to the East of the dropping location.

Sec 3.9, Prob 3.

Foucault's pendulum small oscillations



Newton's 2nd law in rotating frame:

$$m\vec{a} = \vec{T} + m\vec{g} - m\vec{W} - 2m\vec{\Omega} \times \vec{v} - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

Now  $\vec{W}$  = acceleration of origin O

$$= \vec{\Omega} \times (\vec{\Omega} \times \vec{R})$$

Then,

$$m\vec{a} = \vec{T} + m\vec{g} - m\vec{\Omega} \times (\vec{\Omega} \times (\vec{r} + \vec{R})) - 2m\vec{\Omega} \times \vec{v}$$

$\underbrace{\quad}_{\text{2nd order in } \vec{\Omega}}$

[ignore]

$$\approx \vec{T} + m\vec{g} - 2m\vec{\Omega} \times \vec{v}$$

$$\text{Now: } \vec{T} = T \cos \theta \hat{z} - T \sin \theta \cos \phi \hat{x} - T \sin \theta \sin \phi \hat{y}$$

$$\approx T \hat{z} - T \left(\frac{x}{D}\right) \hat{x} - T \left(\frac{y}{D}\right) \hat{y}$$

assuming  $\theta \ll 1$  (small oscillations)

$$\vec{g} = -g \hat{z}$$

$$(\vec{\Omega} \times \vec{v})_x = \Omega_y v_z - \Omega_z v_y \\ = \Omega_y \hat{z} - \Omega_z \hat{y} \\ \approx -\Omega_z \hat{y}$$

$$(\vec{\Omega} \times \vec{v})_y = \Omega_z v_x - \Omega_x v_z \\ = \Omega_z \hat{x} - \Omega_x \hat{z} \\ \approx \Omega_z \hat{x}$$

$$(\vec{\Omega} \times \vec{v})_z = \Omega_x v_y - \Omega_y v_x \\ = \Omega_x \hat{y}$$

$$\vec{m}\vec{a} = \vec{T} + m\vec{g} - 2m\vec{\Omega} \times \vec{v}$$

becomes

$$m\vec{x}' = -T \left(\frac{x}{D}\right) + 2m\Omega_z \hat{y}$$

$$m\vec{y}' = -T \left(\frac{y}{D}\right) - 2m\Omega_z \hat{x}$$

$$0 = m\vec{z}' = T - mg - 2m\Omega_x \hat{y}$$

$$\text{Now: } \Omega_x \hat{y} \approx \omega \frac{D}{P}$$

where  $D = \text{max displacement in } xy\text{-plane}$   
 $P = \text{period} = 2\pi \sqrt{\frac{D}{g}} = \frac{2\pi}{\omega}$

$$\frac{\Omega D}{P} = \frac{\Omega D}{2\pi} \omega$$

$$<< \frac{\ell \omega^2}{2\pi}$$

$$= \frac{k g}{2\pi \ell}$$

$$= \frac{g}{2\pi}$$

$$\text{thus, } m\Omega_x \dot{y} \ll mg$$

so we can ignore these terms relative to  $mg$

$$\rightarrow 0 \approx T - mg \rightarrow [T \approx mg]$$

$$mx'' \approx -mg \frac{x}{\ell} + 2m\Omega_z \dot{y}$$

$$[x'' \approx -w^2 x + 2\Omega_z \dot{y}]$$

$$my'' \approx -mg \frac{y}{\ell} - 2m\Omega_z \dot{x}$$

$$[y'' \approx -w^2 y - 2\Omega_z \dot{x}]$$



since  $\Omega \ll \omega$   
 $D \ll \ell$  (small oscillations)

standard "trick":

Define  $\zeta = x + iy$  (complex valued)

then  $\ddot{\zeta} = \ddot{x} + i\ddot{y}$

$$\ddot{\zeta} = x'' + iy''$$

just need  
to solve  
these two  
equations

$$\text{so } \ddot{\zeta} = -w^2 \zeta + 2i\Omega_z (\dot{y} - i\dot{x})$$

$$= -w^2 \zeta - 2i\Omega_z (\dot{x} + i\dot{y})$$

$$= -w^2 \zeta - 2i\Omega_z \zeta$$

thus,

$$\ddot{\zeta} + 2i\Omega_z \dot{\zeta} + w^2 \zeta = 0$$

trial solution:  $e^{i\lambda t}$

$$\zeta(t) = e^{i\lambda t}$$

$$\rightarrow -\lambda^2 - 2i\Omega_z \lambda + w^2 = 0$$

$$\lambda^2 + 2i\Omega_z \lambda - w^2 = 0$$

solve quadratic:

$$\lambda_{\pm} = \frac{-2\Omega_z \pm \sqrt{4\Omega_z^2 + 4w^2}}{2}$$

$$\approx -\Omega_z \pm \omega \quad (\text{since } \Omega \ll \omega)$$

General solution:

$$\zeta(t) = A e^{-i(\Omega_z - \omega)t} + B e^{-i(\Omega_z + \omega)t}$$

$$= e^{-i\Omega_z t} (A e^{i\omega t} + B e^{-i\omega t})$$

$$= e^{-i\Omega_z t} \zeta_0(t)$$

where  $\zeta_0(t)$  = general solution of small  
oscillation problem in an inertial frame.

The factor  $e^{-i\Omega_z t}$  causes the plane of oscillation of the pendulum to precess with angular frequency

$$\Omega_z = \sqrt{\omega} \cos \alpha$$

angle of location  
on Earth  
 $(\alpha = 0 \leftrightarrow \text{NPole})$   
 $(\alpha = \pi/2 \leftrightarrow \text{Equator})$

Precession period:

$$P_{\text{precession}} = \frac{2\pi}{\Omega_z}$$

$$= \frac{2\pi}{\sqrt{\omega} \cos \alpha}$$

$$\Omega = \text{1 revolution / 24 hrs}$$

$$= \frac{2\pi \text{ rad}}{24 \text{ hrs}}$$

$$\text{so } P_{\text{precession}} = \frac{24 \text{ hrs}}{\cos \alpha}$$

$$= \begin{cases} 24 \text{ hrs at N Pole} \\ \infty \text{ at equator (so no precession)} \end{cases}$$