Notes: Thurs 8/27

1) Elliptic functions
2) Simple pendulum

go beyond small angle approx

Elliptic Functions:

$$\int \frac{dx}{\sqrt{1-x^2}} = ? = \sin^{-1}(x) + \text{const}$$

$$\sin^{-1}(x) = \arcsin(x)$$

$\rightarrow \sin^2\theta + \cos^2\theta = 1$

subst: $x = \sin\theta \rightarrow \sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$dx = \cos\theta \, d\theta$

$$\int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \int d\theta = \theta = \sin^{-1}(x)$$

$x = \sin\theta$

$\theta = \sin^{-1}(x)$

$$\frac{d}{d\theta}\sin\theta = \cos\theta \qquad \lim_{h\to 0}\left(\frac{\sin(\theta+h)-\sin\theta}{h}\right)$$

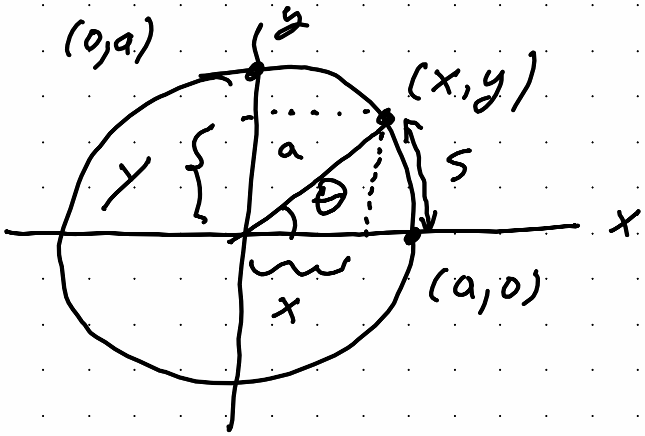
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

circular functions..

$x^2+y^2=a^2$, a = radius

Def: $\sin\theta = \frac{y}{a}$, $\cos\theta = \frac{x}{a}$



s : arclength from (a,0) to (x,y)

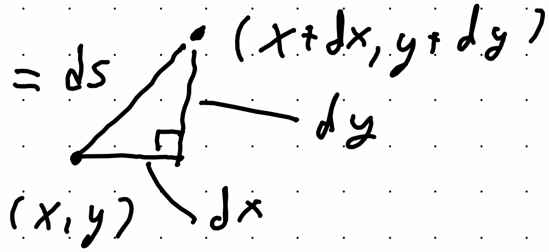
$s = a\theta$ | $\theta = \frac{s}{a}$

$= \frac{1}{a}\int ds$

$= \frac{1}{a}\int\sqrt{dx^2+dy^2}$

$= \int d\theta$

$\sqrt{dx^2+dy^2} = ds$



Given: $x^2+y^2=a^2$

Follows: (i) $\cancel{a^2}\cos^2\theta + \cancel{a^2}\sin^2\theta = \cancel{a^2} \rightarrow \boxed{\cos^2\theta+\sin^2\theta=1}$

(ii) $\boxed{\frac{d}{d\theta}\sin\theta} = \frac{1}{a}\frac{dy}{d\theta} = \frac{\cancel{\frac{1}{a}}\, dy}{\cancel{\frac{1}{a}}\sqrt{dx^2+dy^2}} = \frac{1}{\sqrt{\left(\frac{y}{x}\right)^2+1}}$

$\cancel{2}x\,dx + \cancel{2}y\,dy = 0 \rightarrow dx = -\frac{y}{x}dy$ | $= \frac{1}{\sqrt{\frac{x^2+y^2}{x^2}}} = \frac{x}{a} = \boxed{\cos\theta}$

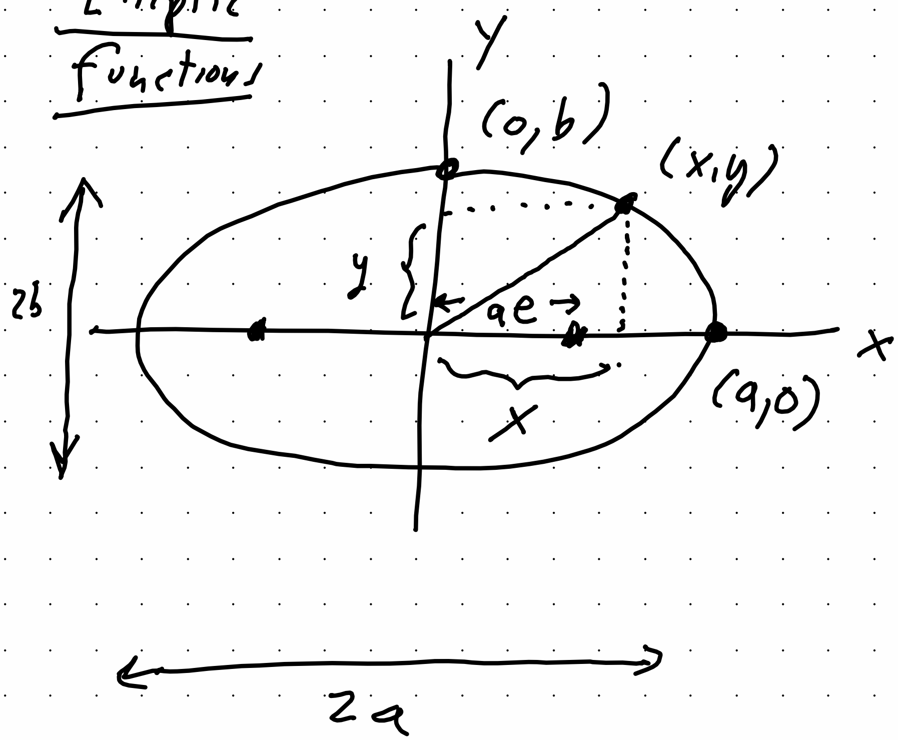
$$\frac{d(\sin\theta)}{d\theta} = \cos\theta \quad \rightarrow \quad \int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta$$

$$x = \sin\theta$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$$

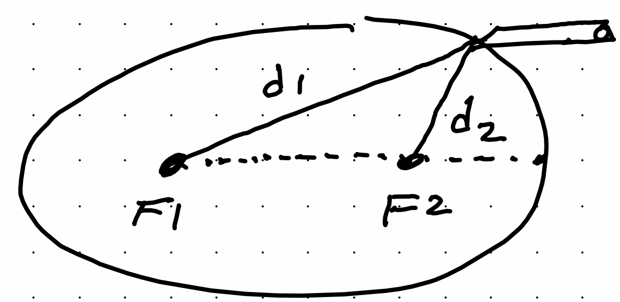
$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \theta = \sin^{-1}(x)}$$

Elliptic functions

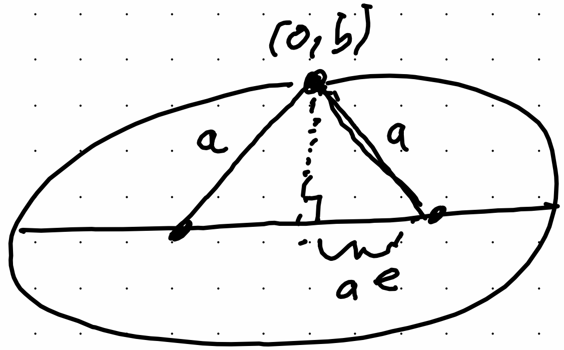


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

eccentricity : $e = 0$ (for circle)



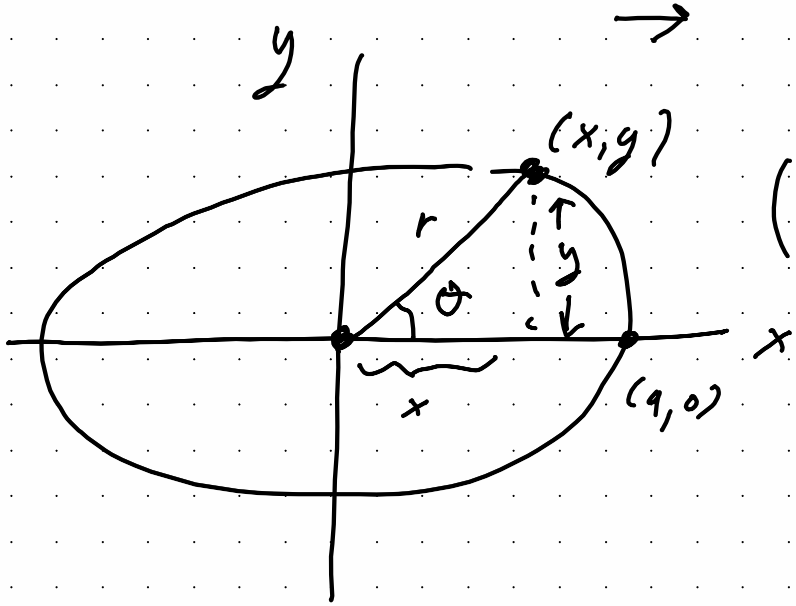
$$d_1 + d_2 = 2a$$



$$(ae)^2 + b^2 = a^2$$
$$a^2e^2 + b^2 = a^2$$
$$e^2 = \frac{a^2-b^2}{a^2} = 1-\left(\frac{b}{a}\right)^2$$

$$\rightarrow \quad e = \sqrt{1-\left(\frac{b}{a}\right)^2} \equiv k$$

notation used in elliptic function



$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

$$x^2+y^2=r^2$$
$$x = r\cos\theta, \quad y = r\sin\theta$$

Define: $\mathrm{cn}(u;k) \equiv \dfrac{x}{a}$, $\quad \mathrm{sn}(u;k) \equiv \dfrac{y}{b}$

$\mathrm{dn}(u;k) \equiv \dfrac{r}{a}$ — (=1 for a circle)

where $u = \dfrac{1}{b}\displaystyle\int_0^\theta r\,d\theta$ — $y = r\sin\theta$

(=θ for a circle)

$$ds = \sqrt{dx^2+dy^2}$$
$$= \sqrt{dr^2+r^2d\theta^2}$$

Given: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, $x^2 + y^2 = r^2$

$dn(u;k) = \frac{r}{a}$

Follow: (i) $cn^2(u;k) + sn^2(u;k) = 1$

(ii) $dn^2(u;k) + k^2 sn^2(u;k) = 1$

$$u = \frac{1}{b}\int_0^\theta r\, d\theta$$

(iii) $\frac{d}{du} sn(u;k) = cn(u;k)\, dn(u;k)$

$\frac{d}{du} cn(u;k) = -sn(u;k)\, dn(u;k)$

$\frac{d}{du} dn(u;k) = -k^2 sn(u;k)\, cn(u;k)$

Analogous to

$\frac{d\sin\theta}{d\theta} = \cos\theta$

$\frac{d\sin\theta}{\cos\theta} = d\theta$

$x = \sin\theta$

$\int \frac{dx}{\sqrt{1-x^2}} = \int d\theta = \theta = \sin^{-1} x$

Integrate: $\frac{d\, sn(u;k)}{du} = cn(u;k)\, dn(u;k)$

$$\int \frac{d\, sn(u;k)}{cn(u;k)\, dn(u;k)} = \int du = u$$

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}} = u + \text{const} = sn^{-1}(x;k) + \text{const} \qquad x \equiv sn(u;k)}$$

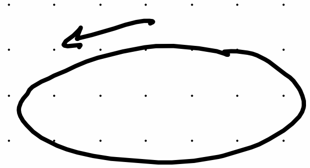
$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + \text{const}}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \equiv K(k) \longrightarrow$$ related to Period of a pendulum going beyond small-angle approximation

(complete elliptic integral of 1st kind)

$$\int_0^1 \frac{\sqrt{1-k^2x^2}\,dx}{\sqrt{1-x^2}} \equiv E(k) \longrightarrow$$ circumference around an ellipse

(complete elliptic integral of 2nd kind)

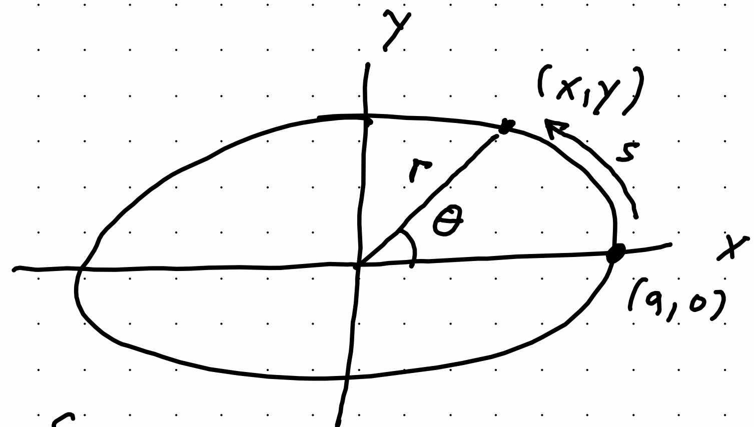


circle: $C = 2\pi a$

Notes: Tuesday 9/1

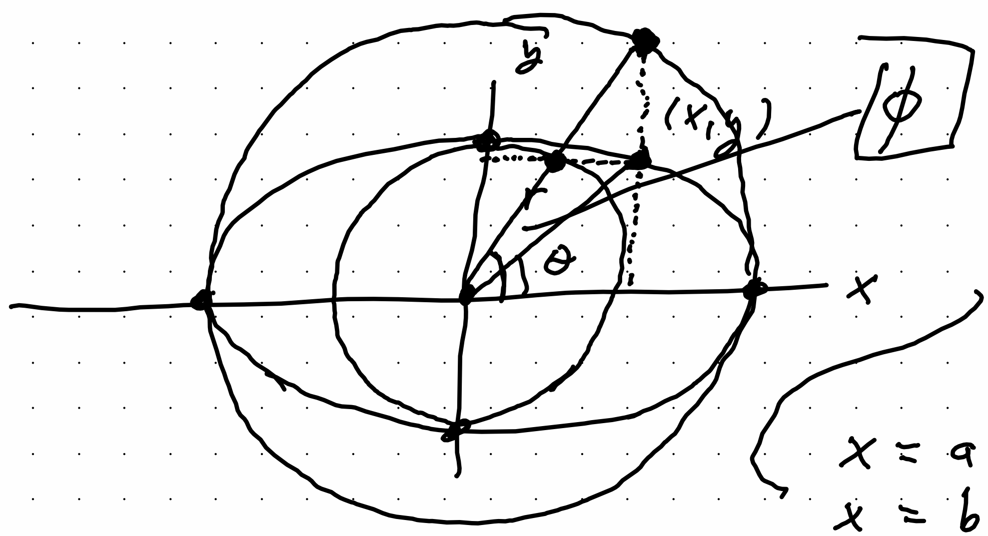
1) Review of elliptic functions
2) Simple pendulum

$$u = \frac{1}{b}\int_{(a,0)}^{(x,y)} r\,d\theta$$



$$bu = \int_0^\theta r\,d\theta \le \int_0^\theta ds = s$$

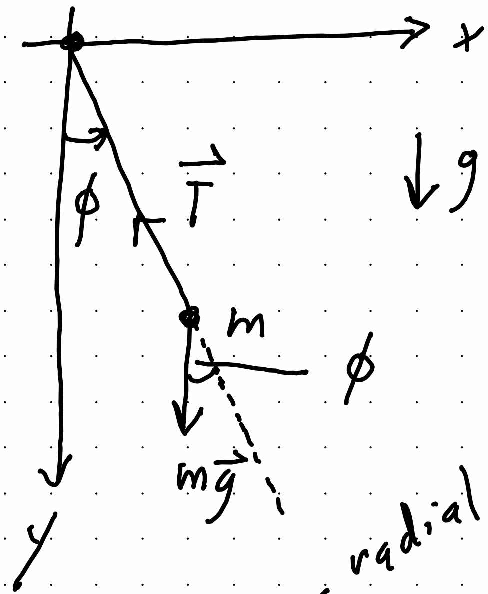
$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\theta^2} \ge r\,d\theta$$

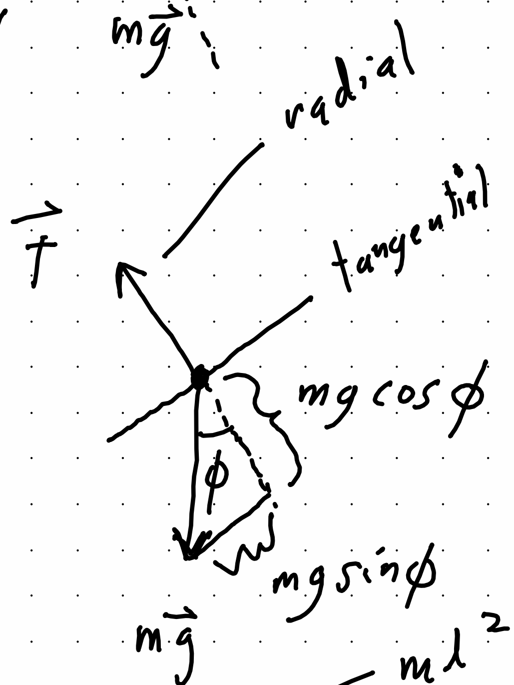


$x = r\cos\theta$
$y = r\sin\theta$
$x = a\cos\phi$
$y = b\sin\phi$

$x = a\cos\phi\,,\ y = \underline{a}\sin\phi$
$x = b\cos\phi \quad y = b\sin\phi$

simple pendulum:





$$\text{Torque} = \overset{ml^2}{I}\,\overset{\ddot\phi}{\alpha}$$

(i) "Freshman physics"

forces, free-body diagram

$\longrightarrow$ EOM, tension

tangential:

$$-mg\sin\phi = m a_{\text{tangential}}$$

$$-\cancel{m}g\sin\phi = \cancel{m}\, l\,\ddot\phi$$

ϕ : angular displacement [rad]

$\dot\phi$: angular velocity [rad/sec]

$\ddot\phi$: angular accel [rad/sec^2]

$$\boxed{\ddot\phi = -\frac{g}{l}\sin\phi} \quad \text{(EOM)}$$

radial: $T - mg\cos\phi = m a_{\text{centripetal}}$

$$T - mg\cos\phi = m\dot\phi^2 l$$

$$T = mg\cos\phi + m\dot\phi^2 l$$

(ii) Small angle approximation:

$$\sin\phi \approx \phi \longleftarrow \phi \ll 1 \text{ radian} = 57 \text{ degrees}$$

$$\ddot{\phi} = -\frac{g}{l}\sin\phi \approx -\frac{g}{l}\phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\phi}$$

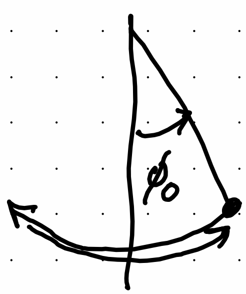
small angle approx.

$$\phi(t) = \boxed{a}\cos(\omega t) + \boxed{b}\sin(\omega t)$$

determined by initial conditions

where $\omega \equiv \sqrt{\frac{g}{l}}$

<u>ICs:</u> If $\phi(0) = \phi_0$ (at rest)



then $\boxed{\phi(t) = \phi_0 \cos(\omega t)}$

<u>Period:</u> $P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$

independent of ϕ_0 !!

(iii) Lagrangian approach

$T \equiv$ Kinetic energy

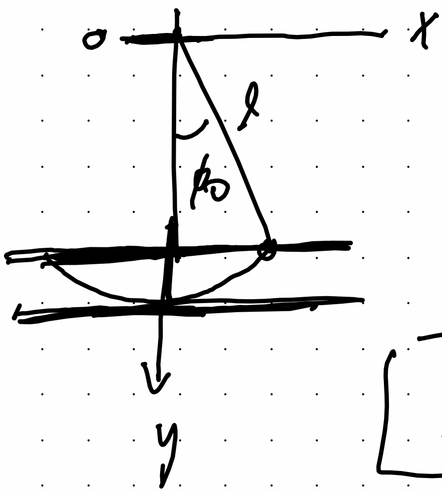
$U \equiv$ Potential energy

$$L = T - U$$

$$\boxed{T = \frac{1}{2} m l^2 \dot{\phi}^2} \quad \left(= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \right)$$

$y = l \cos\phi$

$x = l \sin\phi$



$$\boxed{U = -mgl\cos\phi + \text{const}} \leftarrow U = -mgy$$

$$U = mgl(1 - \cos\phi)$$

$$\boxed{L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi}$$

action

$$\boxed{S = \int_{t_1}^{t_2} dt\, L(\phi, \dot{\phi}, t)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \qquad \text{Lagrange's equation}$$

$$\frac{d}{dt}\left(m l^2 \dot{\phi}\right) = -mgl\sin\phi$$

$$m l^2 \ddot{\phi} = -mgl\sin\phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi} \quad \text{(EoM)}$$

(iv) solving $\ddot{\phi} = -\frac{g}{l}\sin\phi$ (2^{nd} order non-linear ODE)

$\uparrow$ hard!!

$$E = \text{const}$$
$$= T + U$$
$$= \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$$

$$E = 0 - mgl\cos\phi_0$$
$$= -mgl\cos\phi_0$$

release from rest from $\phi = \phi_0$

$$-mgl\cos\phi_0 = \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$$
$$-\cancel{m}gl(\cos\phi_0 - \cos\phi) = \frac{1}{2}\cancel{m}l^2\dot{\phi}^2$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{-2\frac{g}{l}(\cos\phi_0 - \cos\phi)}$$

$|\phi| \leq \phi_0$

Separable 1st order ODE

$$\int \frac{d\phi}{\sqrt{-\frac{2g}{l}(\cos\phi_0 - \cos\phi)}} = \int dt = t + \text{const}$$

$$t + t_0 = \int \frac{d\phi}{\sqrt{-2\frac{g}{l}(\cos\phi_0 - \cos\phi)}}$$

$$\frac{1}{\sqrt{a+bx^2}}$$

substitution:

$$\cos\phi = 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\cos\phi_0 = 1 - 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$\cos\phi = \cos\left(2\left(\frac{\phi}{2}\right)\right)$$

$$= \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\rightarrow \cos\phi_0 - \cos\phi = -2\left(\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)\right)$$

$$t + t_0 = \int \frac{d\phi}{2\sqrt{\frac{g}{l}}\sqrt{\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)}}$$

$$|\phi| \leq \phi_0$$

$$= \frac{1}{2\sqrt{\frac{g}{l}}} \int \frac{d\phi}{\sin\left(\frac{\phi_0}{2}\right)\sqrt{1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)}}}$$

let $x = \frac{\sin(\phi/2)}{\sin\left(\frac{\phi_0}{2}\right)}$

$$\boxed{x = \frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}} \longrightarrow dx = \frac{1}{\sin\left(\frac{\phi_0}{2}\right)} \frac{1}{2} \cos\left(\frac{\phi}{2}\right) d\phi$$

$$\sqrt{1-x^2}$$

denominator

$$d\phi = \frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\cos\left(\frac{\phi}{2}\right)}$$

$$= \frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1 - \sin^2\left(\frac{\phi}{2}\right)}}$$

$$= \boxed{\frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1 - \underbrace{\sin^2\left(\frac{\phi_0}{2}\right)}_{k^2} x^2}}}$$

Find this out

1. $\phi(t) = \boxed{\times}$
2. Period = ??
3. Redo the analysis using Lagrange multiplier for find tension in string

$$t + t_0 = \int \bigcirc$$ — integrand for $sn^{-1}(x; k)$

$$k = \sin\left(\frac{\phi_0}{2}\right)$$