PHYS5320 - Lecture Notes

J.D. Romano

Fall 2012

DISCLAIMER: These notes are *not* meant to be a substitute for the text by Jackson. The goal here is simply to provide a *summary* of the key points discussed in lecture, to aid in revision for homework and exams. Many exercises are suggested for the reader. Please send comments, criticisms, suggestions, etc. to: `joseph.d.romano@gmail.com`.

Contents

1 Introduction

1.1 Lorentz force law, principle of superposition

- The goal of electrodynamics is to describe how charged particles interact with one another.
- Experiments show that a charged particle of mass m, charge q, and velocity $\mathbf{v}$ experiences a force
$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1}$$
where $\mathbf{E} \equiv \mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B} \equiv \mathbf{B}(\mathbf{r}, t)$ are the electric and magnetic fields (produced by all the other charges) evaluated at the location of the charge q.
- Using the special relativistic form of Newton's 2nd law
$$\mathbf{F} = \frac{d\mathbf{p}}{dt}\,, \qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \tag{2}$$
we obtain
$$\frac{d}{dt}\left[\frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}\right] = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{3}$$
- The problem thus reduces to determining how the fields $\mathbf{E}$ and $\mathbf{B}$ are produced by the other charges, whose positions and velocities are assumed to be known functions of time—i.e., we are *not* solving for the motion of these other charges.
- One approach to solving this problem is to take advantage of the *linearity* of electric and magnetic phenomena. This means that if a charge q_1 moving in some manner produces the fields $\mathbf{E}_1$ and $\mathbf{B}_1$, while another charge q_2 moving in some different manner produces the fields $\mathbf{E}_2$ and $\mathbf{B}_2$, then the two charges together (assuming they occupy the same positions and move the same way as they did separately) produce the combined fields
$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2\,, \qquad \mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 \tag{4}$$
- Thus, if we know what $\mathbf{E}$ and $\mathbf{B}$ are for a single charged particle Q moving in an arbitrary way, then superposition will give us the fields for an arbitrary charge distribution.
- Unfortunately, for arbitrary motion, the answer is rather complicated (Jackson, Chpt 14; Griffiths, Chpt 10; Feynman, Vol II, Chpt 21):
$$\mathbf{E}(1, t) = \frac{Q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{r}}'}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{\hat{\mathbf{r}}'}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\hat{\mathbf{r}}'\right]\,, \quad c\mathbf{B}(1, t) = \hat{\mathbf{r}}' \times \mathbf{E}(1, t) \tag{5}$$
where $\mathbf{r}'$ is the vector connecting the charge Q to the field point (1), evaluated at the retarded time t' defined implicitly by
$$c(t - t') = r' = r(t') \tag{6}$$
where c is the speed of light. (See Figs. 1 and 2.)
- Note that the values of the fields at (1) and t, depend on both the velocity and acceleration of the charge Q at the retarded time t'. This is in accord with special relativity, which says that all physical interactions propagate at speeds $\leq c$.
- Although equations (3), (5), and the principle of superposition give the *complete solution* to the problem of interacting charged particles, it is not easy to use in practice, except for some simple electrostatic problems.
- A simpler and more powerful approach is to look for solutions to the differential equations that $\mathbf{E}$ and $\mathbf{B}$ satisfy. These equations are called *Maxwell's equations*, which we write down without proof in the next subsection.

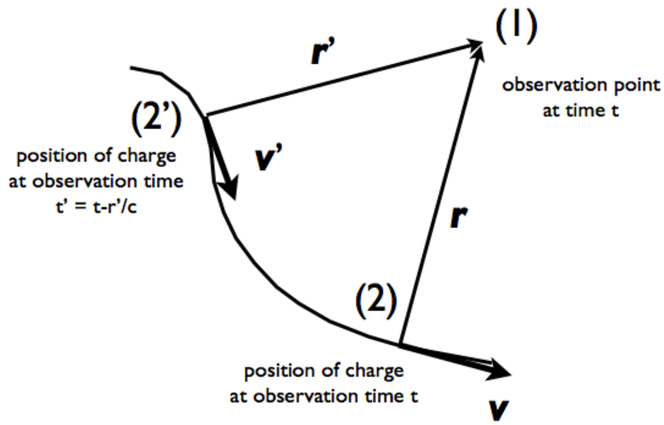


Figure 1: Trajectory of an accelerated charge Q in space. (1) is the observation point at time t; (2) and (2') are the positions of the charge Q at times t and $t' = t - r'/c$, respectively.

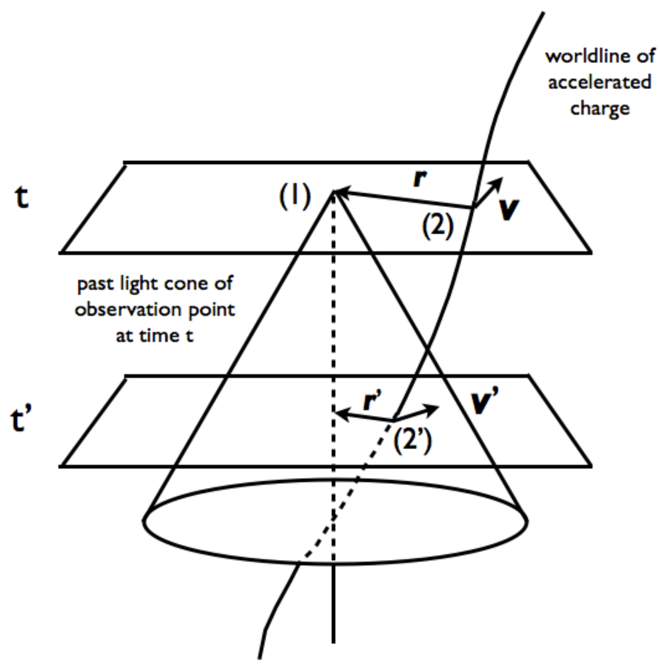


Figure 2: Worldline of an accelerated charge Q in spacetime. (1) is the observation point at time t; (2) and (2') are the positions of the charge Q at times t and $t' = t - r'/c$, respectively.

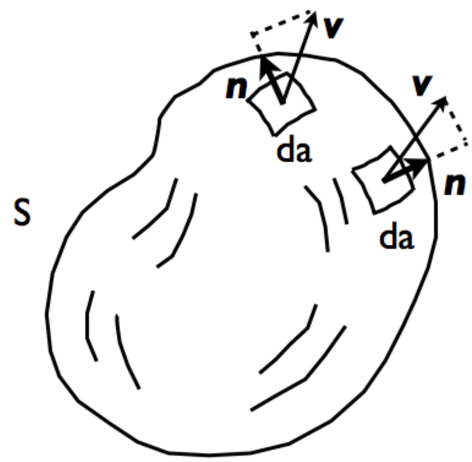


Figure 3: Components of the vector field $\mathbf{v}$ normal to the surface S.

1.2 Maxwell's equations

- Maxwell's equations are stated most simply in terms of the *flux* and *circulation* of the electric and magnetic fields, where

  (i) the flux of a vector field $\mathbf{v}$ through a surface S is given by the average normal component of the field times the surface area of S (see Fig. 3):

$$\text{flux of } \mathbf{v} \text{ through } S = \int_S \mathbf{v} \cdot \hat{\mathbf{n}}\, da \tag{7}$$

  (ii) the circulation of a vector field $\mathbf{v}$ around a closed curve C is given by the average tangential component of the field times the circumference of C (see Fig. 4):

$$\text{circulation of } \mathbf{v} \text{ around } C = \oint_C \mathbf{v} \cdot d\mathbf{s} \tag{8}$$

- Maxwell's equations are then:

  i) Gauss's law:

$$\begin{pmatrix} \text{flux of } \mathbf{E} \text{ through} \\ \text{a closed surface } S \end{pmatrix} = \frac{1}{\epsilon_0}\left(\text{ net charge inside } \right) \tag{9}$$

  ii) Faraday's law:

$$\begin{pmatrix} \text{circulation of } \mathbf{E} \text{ around} \\ \text{a closed curve } C \end{pmatrix} = -\frac{d}{dt}\begin{pmatrix} \text{flux of } \mathbf{B} \text{ through any} \\ \text{surface } S \text{ spanning } C \end{pmatrix} \tag{10}$$

  iii) No name (but means no magnetic monopoles):

$$\begin{pmatrix} \text{flux of } \mathbf{B} \text{ through} \\ \text{a closed surface } S \end{pmatrix} = 0 \tag{11}$$

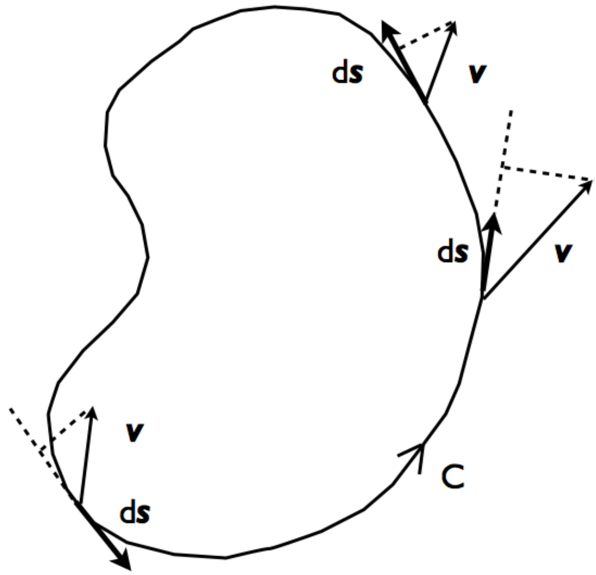


Figure 4: Components of the vector field $\mathbf{v}$ tangent to the curve C.

iv) Ampère's law (with Maxwell's correction term):

$$c^2 \begin{pmatrix} \text{circulation of } \mathbf{B} \text{ around} \\ \text{a closed curve } C \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} \text{flux of } \mathbf{E} \text{ through any} \\ \text{surface } S \text{ spanning } C \end{pmatrix} + \frac{1}{\epsilon_0} \begin{pmatrix} \text{flux of electric current through} \\ \text{any surface } S \text{ spanning } C \end{pmatrix} \tag{12}$$

- Expressed in terms of mathematical symbols:

$$\oint_S \mathbf{E} \cdot \hat{\mathbf{n}}\, da = \frac{1}{\epsilon_0} \int_V \rho\, dV \tag{13}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \hat{\mathbf{n}}\, da \tag{14}$$

$$\oint_S \mathbf{B} \cdot \hat{\mathbf{n}}\, da = 0 \tag{15}$$

$$c^2 \oint_C \mathbf{B} \cdot d\mathbf{s} = \frac{d}{dt} \int_S \mathbf{E} \cdot \hat{\mathbf{n}}\, da + \frac{1}{\epsilon_0} \int_S \mathbf{J} \cdot \hat{\mathbf{n}}\, da \tag{16}$$

where ρ is the charge density and $\mathbf{J}$ is the current density (charge per time per area).

- Applied to infinitesimal surfaces and closed curves, the above equations can be written in differential form:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{17}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{18}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{19}$$

$$c^2 \boldsymbol{\nabla} \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \frac{\mathbf{J}}{\epsilon_0} \tag{20}$$

where $\boldsymbol{\nabla}\cdot\mathbf{E}$ and $\boldsymbol{\nabla}\times\mathbf{E}$ are the *divergence* and *curl* of $\mathbf{E}$, etc. (More about this in the next section.)

- Note that Maxwell's equations are coupled, 1st-order, partial differential equations for the electric fields $\mathbf{E}$ and $\mathbf{B}$. The source terms are given by the charge density ρ and current density $\mathbf{J}$.

- Maxwell's equation imply
$$\boldsymbol{\nabla}\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \tag{21}$$
This is called the *continuity equation*; it is the mathematical statement of local charge conservation—i.e., the net flow of charge out of a any closed surface S is equal to minus the time rate of change of the net charge contained in the volume V enclosed by S.

- Exercise: Prove that the continuity equation is a consequence of Maxwell's equations.

- For *stationary* charge distributions (either all charges at rest or moving in such a way that ρ and $\mathbf{J}$ are constant in time), the time derivatives of $\mathbf{E}$ and $\mathbf{B}$ vanish. This decouples the equations, giving rise to the equations of *electrostatics*:
$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{22}$$
$$\boldsymbol{\nabla}\times\mathbf{E} = 0 \tag{23}$$
and *magnetostatics*:
$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{24}$$
$$c^2\boldsymbol{\nabla}\times\mathbf{B} = \frac{\mathbf{J}}{\epsilon_0} \tag{25}$$

- We will spend many lectures discussing the special cases of electrostatics and magnetostatics later in the semester.

- But before we get to those topics, however, we will review the mathematics of vector calculus.

2 Review of vector calculus

2.1 Vector algebra

- Dot product (also called scalar or inner product):
$$\mathbf{A}\cdot\mathbf{B} = AB\,\cos\theta \tag{26}$$
where $A \equiv |\mathbf{A}|$, $B \equiv |\mathbf{B}|$, and θ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

- Similarly, the cross product (also called vector or exterior product):
$$\mathbf{A}\times\mathbf{B} = AB\,\sin\theta\,\hat{\mathbf{n}} \tag{27}$$
where $\theta < 180°$ is the angle between $\mathbf{A}$ and $\mathbf{B}$, and $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane spanned by $\mathbf{A}$ and $\mathbf{B}$, whose direction is given by the right-hand-rule.

- Note that if $\mathbf{A}$ and $\mathbf{B}$ are parallel, then $\mathbf{A}\times\mathbf{B} = 0$, while if $\mathbf{A}$ and $\mathbf{B}$ are perpendicular, then $\mathbf{A}\cdot\mathbf{B} = 0$.

- In terms of components A_i, B_i with respect to an orthonormal basis we have

$$\mathbf{A} \cdot \mathbf{B} = \delta_{ij} A_i B_j = A_i B_i \tag{28}$$

and

$$(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk} A_j B_k \tag{29}$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{30}$$

is the Kronecker delta, and

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ even permutation of 123} \\ -1 & \text{if } ijk \text{ odd permutation of 123} \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

is the Levi-Civita symbol. A sum over repeated indices is understood (Einstein summation convention).

- The above component expressions for dot product and cross product are valid wrt *any* orthonormal basis; not just for Cartesian coordinates.

- Exercise: Prove that the geometric and component expressions for the dot product and for the cross product are equivalent to one another, choosing a convenient coordinate system to do the calculation.

- Key identity:

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \tag{32}$$

- Scalar triple product:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \tag{33}$$

- Vector triple product:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{34}$$

- Jacobi identity:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0 \tag{35}$$

- Exercise: Prove the above three identities.

2.2 Differential calculus

- Given a scalar field $T(\mathbf{r})$ and vector field $\mathbf{A}(\mathbf{r})$, one can define the following derivatives:

  (i) *Gradient:*

$$(\boldsymbol{\nabla} T) \cdot \hat{\mathbf{t}} := \lim_{\Delta s \to 0} \left[\frac{T(2) - T(1)}{\Delta s} \right] \tag{36}$$

  where $1, 2$ are the endpoints (i.e., the 'boundary') of the vector displacement $\Delta s\, \hat{\mathbf{t}}$. The gradient measures the change in T as you move in different directions from a point. $\boldsymbol{\nabla} T$ points in the direction of maximum rate of change of T, which is *perpendicular* to the contour lines $T = \text{const}$.

  (ii) *Curl*:

$$(\boldsymbol{\nabla} \times \mathbf{A}) \cdot \hat{\mathbf{n}} := \lim_{\Delta a \to 0} \left[\frac{1}{\Delta a} \oint_C \mathbf{A} \cdot d\mathbf{s} \right] \tag{37}$$

  where C is the boundary of the area $\hat{\mathbf{n}}\, \Delta a$. The curl measures the circulation of $\mathbf{A}$ around an infinitesimal closed curve.

(iii) *Divergence:*

$$\boldsymbol{\nabla}\cdot\mathbf{A} := \lim_{\Delta V\to 0}\left[\frac{1}{\Delta V}\oint_S \mathbf{A}\cdot\hat{\mathbf{n}}\,da\right] \tag{38}$$

where S is the boundary of the volume ΔV. The divergence measures the flux of $\mathbf{A}$ through the surface bounding an infinitesimal volume element.

- The above definitions are geometric and do not refer to a particular coordinate system. In the next subsection, we will write down expressions for grad, curl, and divergence in *arbitrary* orthogonal curvilinear coordinates (u, v, w).

- In Cartesian coordinates (x, y, z), the expressions turn out to be particularly simple:

$$\boldsymbol{\nabla}T = \frac{\partial T}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\,\hat{\mathbf{z}} \tag{39}$$

$$\boldsymbol{\nabla}\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}} \tag{40}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{41}$$

- In more compact form

$$(\boldsymbol{\nabla}T)_i = \partial_i T \tag{42}$$

$$(\boldsymbol{\nabla}\times\mathbf{A})_i = \epsilon_{ijk}\partial_j A_k \tag{43}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \partial_i A_i \tag{44}$$

- Product rules:

$$\boldsymbol{\nabla}(fg) = (\boldsymbol{\nabla}f)g + f(\boldsymbol{\nabla}g) \tag{45}$$

$$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} \tag{46}$$

$$\boldsymbol{\nabla}\times(f\mathbf{A}) = (\boldsymbol{\nabla}f)\times\mathbf{A} + f\boldsymbol{\nabla}\times\mathbf{A} \tag{47}$$

$$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A}) \tag{48}$$

$$\boldsymbol{\nabla}\cdot(f\mathbf{A}) = (\boldsymbol{\nabla}f)\cdot\mathbf{A} + f\boldsymbol{\nabla}\cdot\mathbf{A} \tag{49}$$

$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) \tag{50}$$

- <u>Exercise</u>: Prove the above. (Hint: Do the calculations in Cartesian coordinates where the expressions for div, grad, curl are the simplest.)

- Second derivatives:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}T \equiv \nabla^2 T \tag{51}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}T = 0 \tag{52}$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) = \text{a vector field} \tag{53}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0 \tag{54}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) \equiv \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A} \tag{55}$$

- The first line defines the Laplacian $\nabla^2 T$ of a scalar field; The last line defines the Laplacian $\nabla^2\mathbf{A}$ of a vector field.

- Thus, there are only two non-trivial second derivative operations—the Laplacian (of a scalar or vector field) and the gradient of a divergence.

- <u>Exercise</u>: Prove the second derivative expressions. (Hint: Do the calculations in Cartesian coordinates and use the fact that partial derivatives commute.)

2.3 Curvilinear coordinates

- Let $x^i = (u, v, w)$ be a set of orthogonal curvilinear coordinates, with orthonormal basis $\{\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}\}$.

- Examples include Cartesian coordinates (x, y, z), spherical polar coordinates (r, θ, ϕ), and cylindrical coordinates (ρ, ϕ, z). (Note: The cylindrical coordinate ρ should not be confused with the charge density ρ!)

- The orthonormal basis vectors $\{\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}\}$ are to be distinguished from the *coordinate* basis vectors $\{\partial/\partial\mathbf{u}, \partial/\partial\mathbf{v}, \partial/\partial\mathbf{w}\}$, which are orthogonal, but do *not* have unit norm in general. *It is only in Cartesian coordinates that all the coordinate basis vectors are also orthonormal basis vectors.*

- Note that $\partial/\partial\mathbf{u}$ denotes a coordinate basis vector; it is not the same as the partial derivative operator $\partial/\partial u$. One uses similar notation, however, since there is a 1-to-1 correspondence between vectors and directional derivative operators, which is often useful to take advantage of. For example, the relationship between the coordinate basis vectors $\{\partial/\partial\mathbf{u}, \partial/\partial\mathbf{v}, \partial/\partial\mathbf{w}\}$ and $\{\partial/\partial\mathbf{x}, \partial/\partial\mathbf{y}, \partial/\partial\mathbf{z}\}$ is

$$\frac{\partial}{\partial\mathbf{u}} = \frac{\partial x}{\partial u}\frac{\partial}{\partial\mathbf{x}} + \frac{\partial y}{\partial u}\frac{\partial}{\partial\mathbf{y}} + \frac{\partial z}{\partial u}\frac{\partial}{\partial\mathbf{z}} \tag{56}$$

$$\frac{\partial}{\partial\mathbf{v}} = \frac{\partial x}{\partial v}\frac{\partial}{\partial\mathbf{x}} + \frac{\partial y}{\partial v}\frac{\partial}{\partial\mathbf{y}} + \frac{\partial z}{\partial v}\frac{\partial}{\partial\mathbf{z}} \tag{57}$$

$$\frac{\partial}{\partial\mathbf{w}} = \frac{\partial x}{\partial w}\frac{\partial}{\partial\mathbf{x}} + \frac{\partial y}{\partial w}\frac{\partial}{\partial\mathbf{y}} + \frac{\partial z}{\partial w}\frac{\partial}{\partial\mathbf{z}} \tag{58}$$

- *NOTE: For most of the calculations in these notes, we will work with orthonormal bases!!*

- In Cartesian coordinates, the infinitesimal squared distance between two nearby points is given by

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{59}$$

- In orthogonal curvilinear coordinates (u, v, w) we have

$$ds^2 = f^2\, du^2 + g^2\, dv^2 + h^2\, dw^2 \tag{60}$$

  where f, g, and h are functions of (u, v, w), in general.

- For example

	u	v	w	f	g	h
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	$r\sin\theta$
Cylindrical	ρ	ϕ	z	1	ρ	1

- Exercise: Show that

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2\sin^2\theta\, d\phi^2 \tag{61}$$

  in spherical polar coordinates (r, θ, ϕ) using the transformation equations $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

- If the coordinates (u, v, w) were not orthogonal, ds^2 would contain additional cross terms $du\, dv$, etc.

- The infinitesimal displacement vector $d\mathbf{s}$ connecting the two nearby points is

$$d\mathbf{s} = f\, du\, \hat{\mathbf{u}} + g\, dv\, \hat{\mathbf{v}} + h\, dw\, \hat{\mathbf{w}} \tag{62}$$

  See Fig. 5.

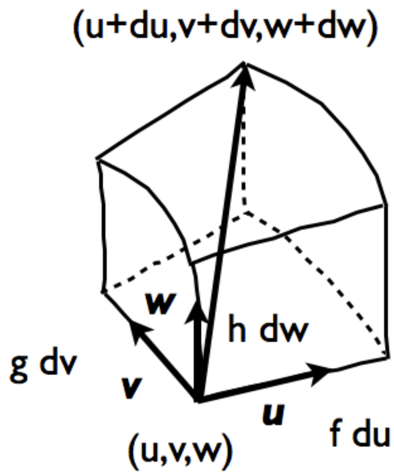


Figure 5: Infinitesimal displacement vector $d\mathbf{s}$ and volume element $dV = fgh\,du\,dv\,dw$ in general orthogonal curvilinear coordinates.

- Since we also have

$$d\mathbf{s} = du\,\frac{\partial}{\partial\mathbf{u}} + dv\,\frac{\partial}{\partial\mathbf{v}} + dw\,\frac{\partial}{\partial\mathbf{w}} \tag{63}$$

  it follows that

$$\frac{\partial}{\partial\mathbf{u}} = f\,\hat{\mathbf{u}}\,, \qquad \frac{\partial}{\partial\mathbf{v}} = g\,\hat{\mathbf{v}}\,, \qquad \frac{\partial}{\partial\mathbf{w}} = h\,\hat{\mathbf{w}} \tag{64}$$

- And since

$$\frac{\partial}{\partial\mathbf{u}}\cdot\boldsymbol{\nabla}u = 1\,, \quad \frac{\partial}{\partial\mathbf{v}}\cdot\boldsymbol{\nabla}u = 0\,, \quad \frac{\partial}{\partial\mathbf{w}}\cdot\boldsymbol{\nabla}u = 0\,, \quad \text{etc.} \tag{65}$$

  it follows that

$$\boldsymbol{\nabla}u = \frac{1}{f}\,\hat{\mathbf{u}}\,, \qquad \boldsymbol{\nabla}v = \frac{1}{g}\,\hat{\mathbf{v}}\,, \qquad \boldsymbol{\nabla}w = \frac{1}{h}\,\hat{\mathbf{w}} \tag{66}$$

- The infinitesimal volume element is

$$dV = fgh\,du\,dv\,dw \tag{67}$$

- The infinitesimal area elements are

$$\hat{\mathbf{n}}\,da = \begin{cases} \pm\hat{\mathbf{u}}\,gh\,dv\,dw \\ \pm\hat{\mathbf{v}}\,hf\,dw\,du \\ \pm\hat{\mathbf{w}}\,fg\,du\,dv \end{cases} \tag{68}$$

  with the $\pm$ depending on whether the unit normals to the area elements point in the direction of increasing (or decreasing) coordinate value. (See Fig. 6.)

- Gradient: Since

$$(\boldsymbol{\nabla}T)\cdot d\mathbf{s} = dT = \frac{\partial T}{\partial u}\,du + \frac{\partial T}{\partial v}\,dv + \frac{\partial T}{\partial w}\,dw \tag{69}$$

  it follows that

$$\boldsymbol{\nabla}T = \frac{1}{f}\frac{\partial T}{\partial u}\,\hat{\mathbf{u}} + \frac{1}{g}\frac{\partial T}{\partial v}\,\hat{\mathbf{v}} + \frac{1}{h}\frac{\partial T}{\partial w}\,\hat{\mathbf{w}} \tag{70}$$

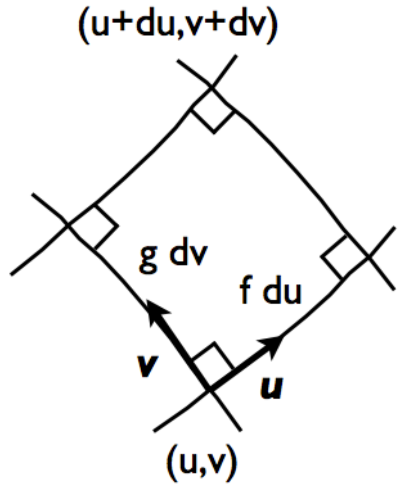


Figure 6: Infinitesimal area element $\hat{\mathbf{n}}\,da = \pm\hat{\mathbf{w}}\,fg\,du\,dv$, with unit normal pointing into or out of the page.

- Curl: Using

$$\mathbf{A}\cdot d\mathbf{s} = A_u\,f\,du + A_v\,g\,dv + A_w\,h\,dw \tag{71}$$

and the definition of the curl, it follows that

$$(\boldsymbol{\nabla}\times\mathbf{A})_u = \frac{1}{gh}\left[\frac{\partial}{\partial v}(A_w\,h) - \frac{\partial}{\partial w}(A_v\,g)\right] \tag{72}$$

$$(\boldsymbol{\nabla}\times\mathbf{A})_v = \frac{1}{hf}\left[\frac{\partial}{\partial w}(A_u\,f) - \frac{\partial}{\partial u}(A_w\,h)\right] \tag{73}$$

$$(\boldsymbol{\nabla}\times\mathbf{A})_w = \frac{1}{fg}\left[\frac{\partial}{\partial u}(A_v\,g) - \frac{\partial}{\partial v}(A_u\,f)\right] \tag{74}$$

- Divergence: Using

$$\mathbf{A}\cdot\hat{\mathbf{n}}\,da = \begin{cases} \pm A_u\,gh\,dv\,dw \\ \pm A_v\,hf\,dw\,du \\ \pm A_w\,fg\,du\,dv \end{cases} \tag{75}$$

and the defintion of the divergence, it follows that

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(A_u\,gh) + \frac{\partial}{\partial v}(A_v\,hf) + \frac{\partial}{\partial w}(A_w\,fg)\right] \tag{76}$$

- Exercise: Prove the last three statements.
- Since the Laplacian of a scalar field is defined as the divergence of the gradient, one has

$$\nabla^2 T = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial T}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{hf}{g}\frac{\partial T}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial T}{\partial w}\right)\right] \tag{77}$$

- For example

$$\boldsymbol{\nabla} T = \frac{\partial T}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\,\hat{\boldsymbol{\phi}} \tag{78}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{79}$$

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{80}$$

  in spherical polar coordinates

2.4 Integral theorems

- Using the definitions of grad, div, and curl, one can prove the following fundamental theorem of integral vector calculus:

  Fundamental theorem for gradients:

$$\int_C (\boldsymbol{\nabla} T)\cdot d\mathbf{s} = T(b) - T(a) \tag{81}$$

  where a, b are the endpoints of C.

  Stokes' theorem:

$$\int_S (\boldsymbol{\nabla}\times\mathbf{A})\cdot\hat{\mathbf{n}}\, da = \oint_C \mathbf{A}\cdot d\mathbf{s} \tag{82}$$

  where C is the closed curved bounding the surface S, with orientation given by the right-hand-rule relative to $\hat{\mathbf{n}}$.

  Divergence theorem:

$$\int_V (\boldsymbol{\nabla}\cdot\mathbf{A})\, dV = \oint_S \mathbf{A}\cdot\hat{\mathbf{n}}\, da \tag{83}$$

  where S is the closed surface bounding the volume V, with outward pointing normal.

- For infinitesimal volume elements, area elements, and path lengths, the proofs of these theorems follow trivially from the definition of divergence, curl, and gradient. For *finite* size volumes, areas, and path lengths one simply adds together the contribution from infinitesimal elements. The neighboring surfaces, edges, and endpoints of these infinitesimal elements have *oppositely-directed* normals, tangent vectors, etc. and hence yield terms that cancel out when forming the sum. (This is described in more detail below.)

- More explicitly, to prove the fundamental theorem for gradients, we note that a curve C can be divided into two shorter segments C_1 and C_2 for which

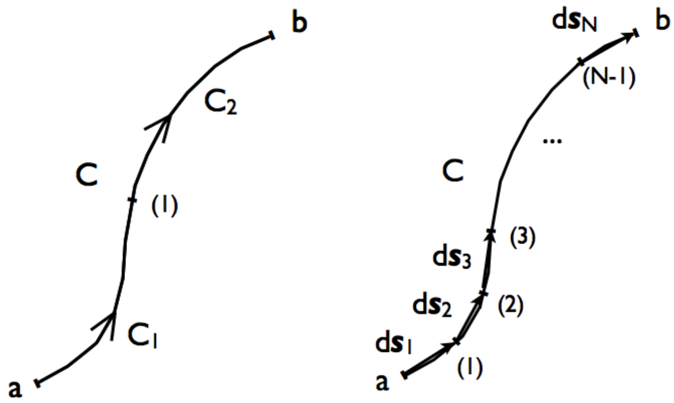
$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{C_1} \mathbf{F}\cdot d\mathbf{s} + \int_{C_2} \mathbf{F}\cdot d\mathbf{s} \tag{84}$$

  See Fig. 7, panel (a).

- We can repeat this process, leading to

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \sum_i \int_{C_i} \mathbf{F}\cdot d\mathbf{s} \tag{85}$$

  where C_i $(i = 1, 2, \cdots, N)$ are infinitesimal segments.



(a) (b)

Figure 7: Panel (a): Curve C broken up into two segments, C_1 and C_2, with common endpoint (1). Panel (b): The same curve C broken up into N infinitesimal segments, with corresponding displacement vectors $d\mathbf{s}_1, d\mathbf{s}_2, \cdots, d\mathbf{s}_N$.

- Specializing now to $\mathbf{F} = \boldsymbol{\nabla} T$, we have

$$\int_C \boldsymbol{\nabla} T \cdot d\mathbf{s} = \sum_i \int_{C_i} \boldsymbol{\nabla} T \cdot d\mathbf{s} \tag{86}$$

$$= (T(1) - T(a)) + (T(2) - T(1)) + \cdots + (T(b) - T(N-1)) \tag{87}$$

$$= T(b) - T(a) \tag{88}$$

  where the second-to-last line follows from the definition of $\boldsymbol{\nabla} T$. (See Fig. 7, panel (b).)

- Since the RHS depends only on T evaluated at the endpoints a and b, the result is *independent* of the curve C—i.e., one obtains the same result for any curve C' that starts and ends at the same points a and b.

- To prove Stokes' theorem, we note that any surface S bounded by a closed curve C can be subdivided into two smaller surfaces S_1 and S_2, bounded by the closed curves $C_1 \equiv C_a + C_{ab}$ and $C_2 \equiv C_b + C_{ab}$. (See Fig. 8.) It follows that

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \oint_{C_1} \mathbf{A} \cdot d\mathbf{s} + \oint_{C_2} \mathbf{A} \cdot d\mathbf{s} \tag{89}$$

  since the contributions from C_{ab} to the C_1 and C_2 integrals cancel out, because C_{ab} is traversed in opposite directions for the two integrals.

- This process can be repeated to yield

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \sum_i \oint_{C_i} \mathbf{A} \cdot d\mathbf{s} \tag{90}$$

  where C_i are closed curves bounding infinitesimal area elements Δa_i.

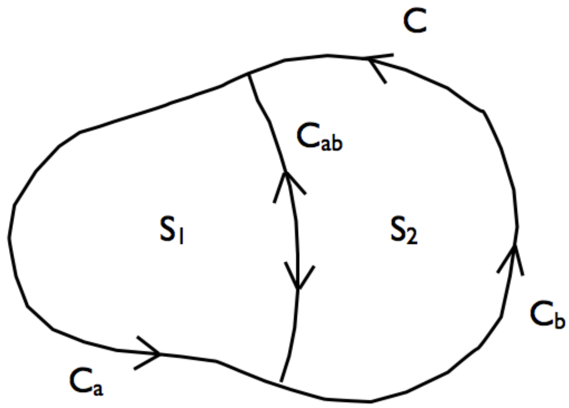


Figure 8: Surface S bounded by closed curve C is subdivided into two smaller surfaces S_1 and S_2, bounded by the closed curves $C_1 \equiv C_a + C_{ab}$ and $C_2 \equiv C_b + C_{ab}$.

- Thus,

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \sum_i \oint_{C_i} \mathbf{A} \cdot d\mathbf{s} \tag{91}$$

$$= \sum_i \left(\frac{1}{\Delta a_i} \oint_{C_i} \mathbf{A} \cdot d\mathbf{s} \right) \Delta a_i \tag{92}$$

$$= \sum_i (\boldsymbol{\nabla} \times \mathbf{A})\big|_i \cdot \hat{\mathbf{n}}_i \, \Delta a_i \tag{93}$$

$$= \int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \hat{\mathbf{n}} \, da \tag{94}$$

  where the second-to-last line follows from the definition of $\boldsymbol{\nabla} \times \mathbf{A}$.

- To prove the divergence theorem, we note that any volume V bounded by a closed surface S can be subdivided into two subvolumes V_1 and V_2, bounded by the closed surfaces $S_1 \equiv S_a + S_{ab}$ and $S_2 \equiv S_b + S_{ab}$. (See Fig. 9.) It follows that

$$\oint_S \mathbf{A} \cdot \hat{\mathbf{n}} \, da = \oint_{S_1} \mathbf{A} \cdot \hat{\mathbf{n}} \, da + \oint_{S_2} \mathbf{A} \cdot \hat{\mathbf{n}} \, da \tag{95}$$

  since the contributions from S_{ab} to the S_1 and S_2 integrals cancel out, because the normals point in opposite directions for the two integrals.

- This process can be repeated to yield

$$\oint_S \mathbf{A} \cdot \hat{\mathbf{n}} \, da = \sum_i \oint_{S_i} \mathbf{A} \cdot \hat{\mathbf{n}} \, da \tag{96}$$

  where S_i are closed surfaces bounding infinitesimal volume elements ΔV_i.

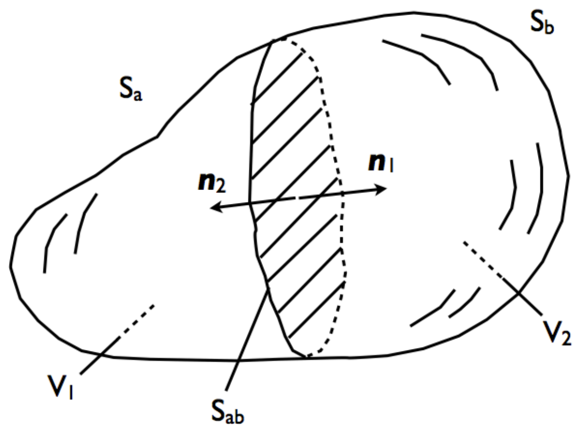


Figure 9: Volume V bounded by closed surface S is subdivided into two subvolumes V_1 and V_2, bounded by the closed surfaces $S_1 \equiv S_a + S_{ab}$ and $S_2 \equiv S_b + S_{ab}$.

- Thus,

$$\oint_S \mathbf{A} \cdot \hat{\mathbf{n}}\, da = \sum_i \oint_{S_i} \mathbf{A} \cdot \hat{\mathbf{n}}\, da \tag{97}$$

$$= \sum_i \left(\frac{1}{\Delta V_i} \oint_{S_i} \mathbf{A} \cdot \hat{\mathbf{n}}\, da \right) \Delta V_i \tag{98}$$

$$= \sum_i (\boldsymbol{\nabla} \cdot \mathbf{A})\big|_i \Delta V_i \tag{99}$$

$$= \int_V (\boldsymbol{\nabla} \cdot \mathbf{A})\, dV \tag{100}$$

  where the second-to-last line follows from the definition of $\boldsymbol{\nabla} \cdot \mathbf{A}$.

- Exercise: Use the divergence theorem to prove Green's 1st identity

$$\int_V (\boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} U + T\nabla^2 U)\, dV = \oint_S (T\boldsymbol{\nabla} U) \cdot \hat{\mathbf{n}}\, da \tag{101}$$

  and Green's theorem

$$\int_V (T\nabla^2 U - U\nabla^2 T)\, dV = \oint_S (T\boldsymbol{\nabla} U - U\boldsymbol{\nabla} T) \cdot \hat{\mathbf{n}}\, da \tag{102}$$

- Exercise: Use the divergence theorem and Stokes' theorem to prove the following integral identities (Prob 1.60, Griffiths):

$$\int_V (\boldsymbol{\nabla} T)\, dV = \oint_S T\, \hat{\mathbf{n}}\, da \tag{103}$$

$$\int_V (\boldsymbol{\nabla} \times \mathbf{A})\, dV = -\oint_S \mathbf{A} \times \hat{\mathbf{n}}\, da \tag{104}$$

$$\int_S (\boldsymbol{\nabla} T) \times \hat{\mathbf{n}}\, da = -\oint_C T\, d\mathbf{s} \tag{105}$$

  Hint: Let $\mathbf{c}$ be a constant vector field and replace $\mathbf{A}$ by $\mathbf{c}T$ in the divergence theorem to derive the first identity; replace $\mathbf{A}$ by $\mathbf{A} \times \mathbf{c}$ in the divergence theorem to derive the second; and replace $\mathbf{A}$ by $\mathbf{c}T$ in Stokes' theorem to derive the last.

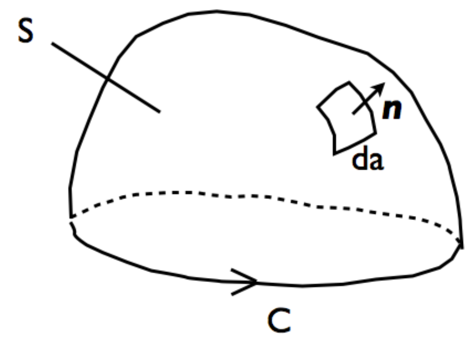


Figure 10: Surface S with boundary curve C and unit normal $\hat{\mathbf{n}}$.

- The *vector area* $\mathbf{a}$ of a surface S is defined by

$$\mathbf{a} \equiv \int_S \hat{\mathbf{n}}\, da \tag{106}$$

(See Fig. 10.)

It satisfies the following properties (Prob 1.61, Griffiths):

1) For a flat surface S, $|\mathbf{a}|$ is just the usual (scalar) area of the surface.

2) For a closed surface, $\mathbf{a} = 0$.

3) $\mathbf{a}$ is the same for all surfaces $\mathcal{S}$ sharing the same (closed) boundary curve C.

4) In terms of the boundary curve C,

$$\mathbf{a} = \frac{1}{2} \oint_C \mathbf{r} \times d\mathbf{s} \tag{107}$$

5) If $\mathbf{c}$ is a constant vector field then

$$\oint_C (\mathbf{c} \cdot \mathbf{r})\, d\mathbf{s} = \mathbf{a} \times \mathbf{c} \tag{108}$$

- <u>Exercise</u>: Prove the above using the integral theorems and integral identities given above. (Hint: For (2) use (103) with $T = 1$; for (4) use Stokes' theorem with $\mathbf{A}$ replaced by $\mathbf{c} \times \mathbf{r}$, where $\mathbf{c}$ is a constant vector field; for (5) use (105) with $T = \mathbf{c} \cdot \mathbf{r}$.)

2.5 Dirac delta function

- Using the expression for the divergence in spherical polar coordinates, one can show that

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 0\,, \quad \text{for } r \neq 0 \tag{109}$$

- For $r = 0$, consider the volume integral of $\boldsymbol{\nabla} \cdot (\hat{\mathbf{r}}/r^2)$ over a spherical volume of radius R. Using the divergence theorem, one has

$$\int_V \boldsymbol{\nabla} \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) dV = \oint_S \frac{\hat{\mathbf{r}}}{r^2} \cdot \hat{\mathbf{n}}\, da = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{1}{R^2} R^2 \sin\theta\, d\theta\, d\phi = 4\pi \tag{110}$$

independent of the radius R.

- Thus,
$$\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right)=4\pi\delta(\mathbf{r}) \tag{111}$$
where $\delta(\mathbf{r})$ is the 3-dimensional Dirac delta function.

- Since
$$\boldsymbol{\nabla}\left(\frac{1}{r}\right)=-\frac{\hat{\mathbf{r}}}{r^2} \tag{112}$$
it follows that
$$\nabla^2\left(\frac{1}{r}\right)=-4\pi\delta(\mathbf{r}) \tag{113}$$

- More generally,
$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right)=4\pi\delta(\mathbf{r}-\mathbf{r}')\,,\qquad \nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)=-4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{114}$$

- In terms of 1-d Dirac delta functions in arbitrary orthogonal curvilinear coordinates
$$\delta(\mathbf{r}-\mathbf{r}')=\frac{1}{fgh}\delta(u-u')\delta(v-v')\delta(w-w') \tag{115}$$

- In Cartesian coordinates
$$\delta(\mathbf{r}-\mathbf{r}')=\delta(x-x')\delta(y-y')\delta(z-z') \tag{116}$$

- In spherical polar coordinates
$$\delta(\mathbf{r}-\mathbf{r}')=\frac{1}{r^2\sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi') \tag{117}$$

- Exercise: Prove
$$\boldsymbol{\nabla}\cdot(r^n\hat{\mathbf{r}})=(n+2)r^{n-1}\quad\text{for } n\neq-2 \tag{118}$$
Thus, it is only for $n=-2$ that we get a Dirac delta function.

- Exercise: Prove
$$\boldsymbol{\nabla}\times(r^n\hat{\mathbf{r}})=0\quad\text{for all } n \tag{119}$$

2.6 Some theorems for vector fields

- Theorem: Any vector field $\mathbf{F}$ can be written in the form
$$\mathbf{F}=-\boldsymbol{\nabla}U+\boldsymbol{\nabla}\times\mathbf{W} \tag{120}$$

- Note that this decomposition is not unique as
$$U\rightarrow U+C\,,\quad\text{where } C=\text{const} \tag{121}$$
$$\mathbf{W}\rightarrow\mathbf{W}+\boldsymbol{\nabla}\Lambda \tag{122}$$
leaves $\mathbf{F}$ unchanged.

- Theorem:
$$\boldsymbol{\nabla}\times\mathbf{F}=0\Leftrightarrow\mathbf{F}=-\boldsymbol{\nabla}U\Leftrightarrow\oint_C\mathbf{F}\cdot d\mathbf{s}=0 \tag{123}$$

- Theorem:
$$\boldsymbol{\nabla}\cdot\mathbf{F}=0\Leftrightarrow\mathbf{F}=\boldsymbol{\nabla}\times\mathbf{W}\Leftrightarrow\oint_S\mathbf{F}\cdot\hat{\mathbf{n}}\,da=0 \tag{124}$$

- Both of the above theorems require that: (i) $\mathbf{F}$ be differentiable, and (ii) the region of interest be simply-connected (i.e., that there aren't any holes). We will assume that both of these conditions are always satisfied.

- Helmholtz theorem: If the divergence $D \equiv \boldsymbol{\nabla} \cdot \mathbf{F}$ and curl $\mathbf{C} \equiv \boldsymbol{\nabla} \times \mathbf{F}$ of a vector field $\mathbf{F}$ are specified, and if they both go to zero faster than $1/r^2$ as $r \to \infty$, and if $\mathbf{F}$ goes to zero as $r \to \infty$, then $\mathbf{F}$ is given uniquely by

$$\mathbf{F} = -\boldsymbol{\nabla}\left(\frac{1}{4\pi}\int \frac{D(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV'\right) + \boldsymbol{\nabla} \times \left(\frac{1}{4\pi}\int \frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV'\right) \tag{125}$$

  where the integrals are over all of space.

- Corollary: If $\mathbf{F}$ is differentiable and goes to zero faster than $1/r$ as $r \to \infty$, then $\mathbf{F}$ can be written uniquely in terms of its divergence and curl according to Eq. (125).

- Exercise: Prove the Helmholtz theorem. (Hint: To show that $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{C}$, you will need to change $\boldsymbol{\nabla}$ to $\boldsymbol{\nabla}'$ when differentiating the last integral, and then integrate by parts.)

- A pure *shear* field is a vector field $\mathbf{F}$ whose divergence and curl both vanish. Helmholtz's theorem implies that $\mathbf{F} = 0$ is the only pure shear field that goes to zero faster than $1/r$ as $r \to \infty$, since $\boldsymbol{\nabla} \times \mathbf{F} = 0$ iff $\mathbf{F} = -\boldsymbol{\nabla} U$ for some U, so that $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$ iff $\nabla^2 U = 0$. But the only solution U to Laplace's equation that goes to zero as $r \to \infty$ is $U = 0$. (Said another way, a non-trivial pure shear field $\mathbf{F}$ cannot vanish as $r \to \infty$.)

- Exercise: In 2-dimensions, consider the vector fields

$$\mathbf{A} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} \tag{126}$$
$$\mathbf{B} = -y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}} \tag{127}$$
$$\mathbf{C} = y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}} \tag{128}$$

  (i) Show that $\boldsymbol{\nabla} \times \mathbf{A} = 0$, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, and $\boldsymbol{\nabla} \cdot \mathbf{C} = 0$, $\boldsymbol{\nabla} \times \mathbf{C} = 0$.

  (ii) Make plots of these vector fields.

  (iii) Show that $\mathbf{A} = \hat{\rho}$ and $\mathbf{B} = \rho\,\hat{\phi}$, where (ρ, ϕ) are plane polar coordinates related to (x, y) via $x = \rho\cos\phi$ and $y = \rho\sin\phi$.

  (iv) Show that $\mathbf{C} = (1/2)\boldsymbol{\nabla} V$, where (U, V) are orthogonal hyperbolic coordinates on the plane defined by $U = x^2 - y^2$ and $V = 2xy$.

3 Electrostatics

- Electrostatics is the study of electric phenomena when the charges are *stationary*—i.e., at rest or moving in such a way that the charge density ρ and current density $\mathbf{J}$ are constant in time.

- As mentioned in Sec. 1, when ρ and $\mathbf{J}$ are constant in time, Maxwell's equations simplify. In particular, they *decouple* giving rise to separate equations for the electric and magnetic fields:

  Electrostatics:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{129}$$
$$\boldsymbol{\nabla} \times \mathbf{E} = 0 \tag{130}$$

  Magnetostatics:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{131}$$
$$c^2 \boldsymbol{\nabla} \times \mathbf{B} = \frac{\mathbf{J}}{\epsilon_0} \tag{132}$$

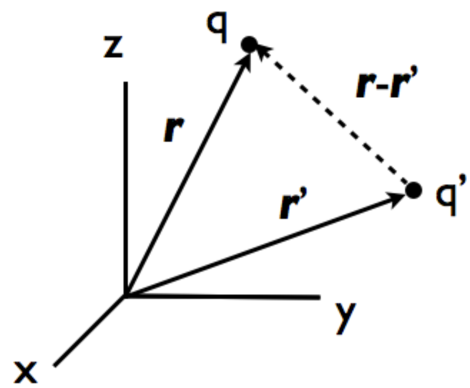


Figure 11: Two charges q and q', located at $\mathbf{r}$ and $\mathbf{r}'$, respectively.

- We will spend the next several lectures describing methods of solving the electrostatic equations. The mathematical methods that we will use (e.g., separation of variables, Green's functions, etc.) and the special functions that we will encounter (Legendre polynomials, spherical harmonics, Bessel functions, etc.) will be applicable to other fields of physics as well.
- But first, we will recall Coulomb's law and show how the equations of electrostatics follow from it. (This is more in line with the historical development of the field.)

3.1 Coulomb's law

- Consider two point charges q and q', located at positions $\mathbf{r}$ and $\mathbf{r}'$, respectively. Assume that q' is at rest; q may be moving. (See Fig. 11.)
- Coulomb's law states that the force on q exerted by q' is given by

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} q q' \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \tag{133}$$

- Note that the force is proportional to the product of the two charges, is *central* (i.e., is directed along the line connecting the two charges), and depends on the inverse *square* of the separation.
- The above form of Coulomb's law is in the SI (MKS) system of units. In these units, $\mathbf{F}$ is measured in Newtons, r in meters, and q in Coulomb. (Recall that the charge of the electron is -1.602×10^{-19} C.)
- The proportionality constant ϵ_0 is called the *permitivity of free space*. It has the value

$$\epsilon_0 = 8.85 \times 10^{-12} \, \frac{\mathrm{C}^2}{\mathrm{N} \cdot \mathrm{m}^2} \tag{134}$$

or, equivalently,

$$\frac{1}{4\pi\epsilon_0} = 10^{-7} \, c^2 \approx 9.0 \times 10^9 \, \frac{\mathrm{N} \cdot \mathrm{m}^2}{\mathrm{C}^2} \tag{135}$$

where $c = 2.998 \times 10^8$ m/s is the speed of light.

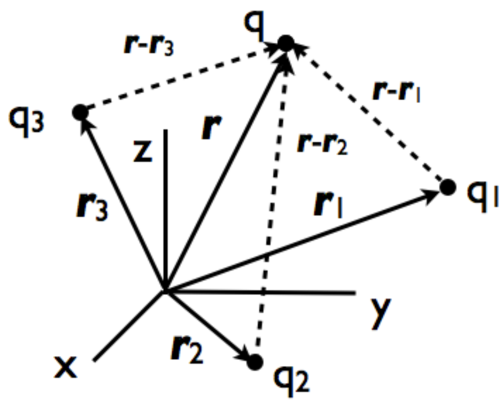


Figure 12: Illustration of the superposition principle for Coulomb's law, where the total force on q is the sum of the forces due to charges q_1, q_2, and q_3, each considered separately.

- If q' is replaced by a set of charges $q_1, q_2, \cdots, q_N$ at locations $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N$, the *superposition principle* gives

$$\mathbf{F} = q\left(\sum_{i=1}^{N} \frac{1}{4\pi\epsilon_0} q_i \frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3}\right) \tag{136}$$

  (See Fig. 12.)

- For a continuous charge distribution, the summation is replaced by an integral

$$\mathbf{F} = q\left(\frac{1}{4\pi\epsilon_0}\int dq' \, \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right) \tag{137}$$

  where

$$dq' = \begin{cases} \lambda(\mathbf{r}')\, ds' \\ \sigma(\mathbf{r}')\, da' \\ \rho(\mathbf{r}')\, dV' \end{cases} \tag{138}$$

  for line, surface, and volume charge densities.

- *This is all there is to electrostatics!* Coulomb's law and the principle of superposition contain exactly the same information as the subset of Maxwell's equations (129) and (130). There is no more and no less information in these equations.

- Note that if the positions of all of the source charges dq' are specified, then all we have to do is evaluate the above integral to get the force on q. (Then using that force in Newton's 2nd law will allow us to solve for the subsequent motion of q.)

- If the positions of the source charges are not specified a priori, as will be the case for charges in a conductor, then the situation is more complicated. The positions of the charges depend on the value of the field, which in turn depends on the positions of the charges.

- The remaining parts of this section and the next describe clever methods of calculating the forces (or, equivalently, the electric field $\mathbf{E}$) without having to do the brute force integrations.

3.2 Electric field

- It is convenient to divide the force $\mathbf{F}$ by q to obtain the *electric field*

$$\mathbf{E}(\mathbf{r}) := \frac{\mathbf{F}(\mathbf{r})}{q} = \frac{1}{4\pi\epsilon_0}\int dq' \, \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{139}$$

  which depends only on the other charges.

- $\mathbf{E}(\mathbf{r})$ is thus the force per unit charge acting on a test charge placed at $\mathbf{r}$.

- Note: By test charge we mean a charge so small that it doesn't cause the other charges to redistribute themselves in its presence.

- In terms of a volume charge density ρ:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int dV' \, \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{140}$$

3.3 Electrostatic field equations

- By explicitly taking the divergence and curl of the above expression for $\mathbf{E}(\mathbf{r})$, one finds

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{141}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = 0 \tag{142}$$

- <u>Exercise</u>: Prove these results.

- Thus, we have recovered the equations of electrostatics, (129) and (130), from Coulomb's law and the principle of superposition.

- By the Helmholtz theorem, if $\mathbf{E}$ goes to zero faster than $1/r$ as $r \to \infty$, these two equations completely and uniquely define the field.

- Equation (141) is called *Gauss's law*. It depends on the inverse-square-law nature of the electric field, since $\boldsymbol{\nabla}\cdot(r^n\hat{\mathbf{r}}) = 4\pi\,\delta(\mathbf{r})$ only for the case $n=-2$.

- Equation (142), on the other hand, would hold even if Coulomb's law was not an inverse-square-law, since $\boldsymbol{\nabla}\times(r^n\hat{\mathbf{r}}) = 0$ for any n. (But it does require the force to be central–i.e., point along the direction between the two charges.)

- The integral form of the above equations are

$$\oint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{Q_{\rm enc}}{\epsilon_0} \tag{143}$$

$$\oint_C \mathbf{E}\cdot d\mathbf{s} = 0 \tag{144}$$

  where $Q_{\rm enc}$ is the total charge enclosed by the closed surface S, and C is any closed curve.

- <u>Exercise</u>: Prove the equivalence of the differential and integral form of the electrostatic equations using the divergence theorem and Stokes' theorem.

- <u>Exercise</u>: Check that Gauss's law is valid for a point charge Q at the origin, taking S to be a sphere of any radius $r>0$ centered on Q. (See Fig. 13).

- <u>Exercise</u>: Repeat the previous problem, but this time choosing a 'potato-shaped' boundary surface S. (See Fig. 14.) Note that for a general surface

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}\,da = \cos\theta\,da \equiv da_\perp \equiv r^2\,d\Omega \equiv r^2\,\sin\theta\,d\theta\,d\phi \tag{145}$$

  (See Fig. 15.)

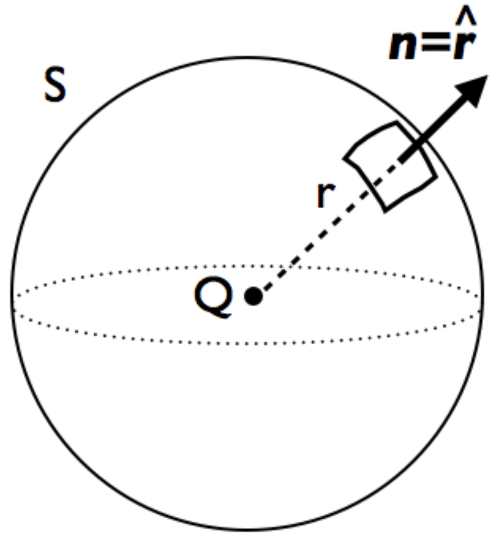


Figure 13: Spherical surface S of radius r centered on point charge Q. Note that the unit normal to the surface $\hat{\mathbf{n}}$ is directed radially away from Q, and hence points in the same direction as the electric field for the point charge $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,\hat{\mathbf{r}}$.

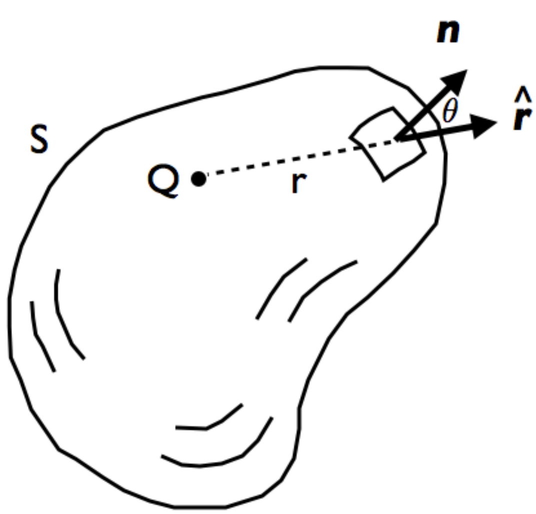


Figure 14: General 'potato-shaped' surface S enclosing a point charge Q. For this surface, the unit normals $\hat{\mathbf{n}}$ do not, in general, point radially away from Q; they make an angle θ with the radial unit vector $\hat{\mathbf{r}}$, which is the direction of $\mathbf{E}(\mathbf{r})$ for the point charge.

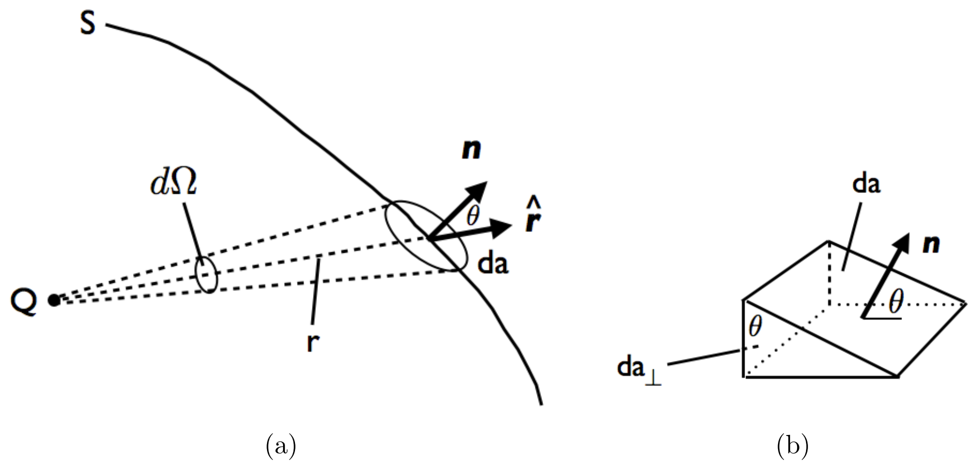


Figure 15: Relationship between the area element $\hat{\mathbf{n}}\,da$ for a general surface S, and the solid angle $d\Omega \equiv \sin\theta\,d\theta\,d\phi$ spanned by da. Panel (a): r is the distance from Q to area element da; θ is the angle between the unit normal $\hat{\mathbf{n}}$ and the radial direction $\hat{\mathbf{r}}$. Panel (b): Geometrical proof of the relation $\cos\theta\,da = da_\perp$. The connection to panel (a) is $da_\perp = r^2\,d\Omega$.

- The integral form of Gauss's law is the most efficient way to calculate the field when the geometry is sufficiently simple—i.e., spherical symmetry, cylindrical symmetry, planar symmetry.

- Exercise: Using Gauss's law, show that:

  (i) The field outside a spherically symmetric charge distribution ρ having total charge Q is given by:

  $$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2} \tag{146}$$

  (See Figure 16.)

  (ii) The field at a radial distance s from an infinitely-long straight line with uniform linear charge density λ is given by:

  $$\mathbf{E}(\mathbf{r}) = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{\mathbf{s}}}{s} \tag{147}$$

  (See Figure 17.)

  (iii) The field at a distance z from an infinite sheet in the xy-plane with uniform surface charge density σ is given by

  $$\mathbf{E}(\mathbf{r}) = \pm\frac{\sigma}{2\epsilon_0}\,\hat{\mathbf{z}} \tag{148}$$

  where the $\pm$ corresponds to $z > 0$, $z < 0$, respectively. (See Figure 18.)

- Boundary conditions:

  (i) Using the integral form of Gauss's law one can show that

  $$(\mathbf{E}_2 - \mathbf{E}_1)\cdot\hat{\mathbf{n}} = \frac{\sigma}{\epsilon_0} \tag{149}$$

  where $\hat{\mathbf{n}}$ is the unit normal pointing from region 1 to region 2. (See Fig. 19.)

  (ii) Using the integral form of $\nabla\times\mathbf{E} = 0$ one can show that

  $$(\mathbf{E}_2 - \mathbf{E}_1)\cdot\hat{\mathbf{t}} = 0 \tag{150}$$

  where $\hat{\mathbf{t}}$ is a unit tangent to the boundary separating regions 1 and 2. (See Fig. 20.)

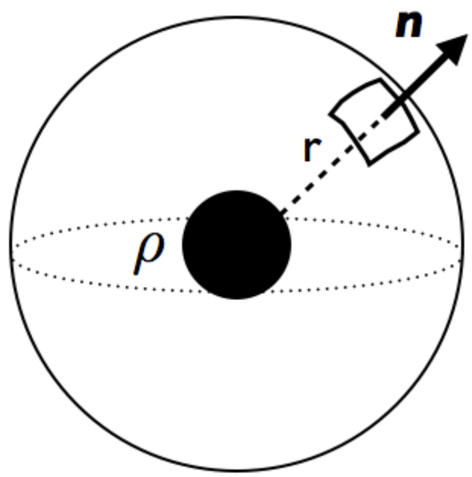


Figure 16: Geometry for Gauss's law calculation of the electric field **E** for a spherically symmetric charge distribution ρ. The boundary surface is a 2-sphere of radius r that contains the charge distribution and is concentric with it.

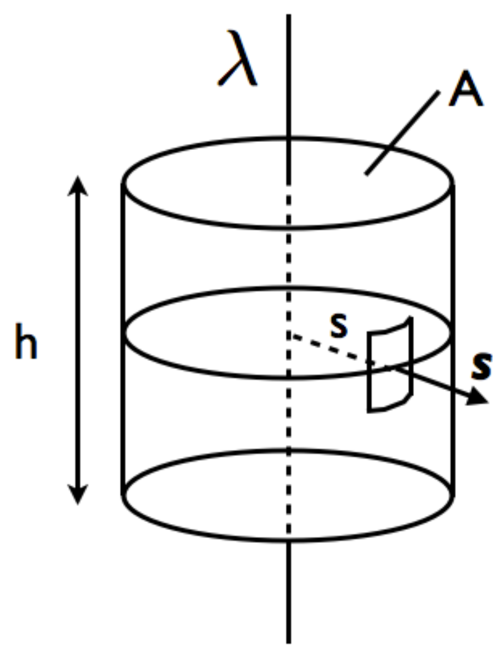


Figure 17: Geometry for Gauss's law calculation of the electric field **E** for a uniform linear charge distribution λ. The boundary surface is a cylinder of height h, cross-sectional area A, with axis coinciding with the line charge. s is the perpendicular distance from the axis to the surface of the cylinder.

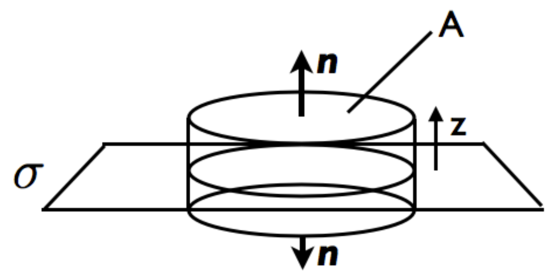


Figure 18: Geometry for Gauss's law calculation of the electric field **E** for a planar charge distribution σ. The boundary surface is a 'Gaussian pillbox' of height $2z$ centered on the charge distribution, with cross sectional area A.

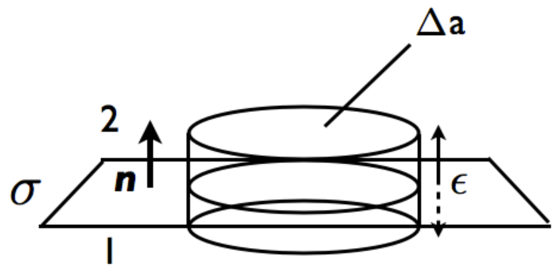


Figure 19: Gaussian pillbox of height ϵ and cross-sectional area Δa straddling a boundary surface separating regions 1 and 2. σ is the surface charge density; $\hat{\mathbf{n}}$ is the unit normal to the surface pointing from region 1 to region 2. Δa is assumed to be small enough that the electric field **E** is approximately constant over this surface. The boundary conditions on the normal components of **E** are obtained by taking the limit $\epsilon \to 0$, assuming that all fields and charge densities are finite on the surface.

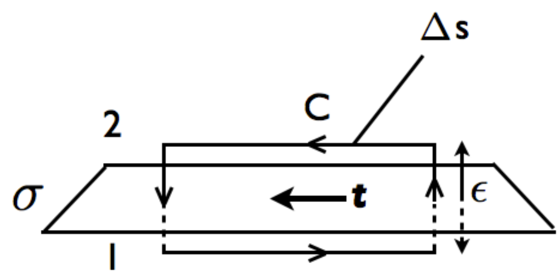


Figure 20: Rectangular closed curve C of height ϵ and width Δs straddling a boundary surface separating regions 1 and 2. σ is the surface charge density; $\hat{\mathbf{t}}$ is the unit tangent to the surface pointing in the direction of C in region 2. Δs is assumed to be small enough that the electric field **E** is approximately constant over this distance. The boundary conditions on the tangential components of **E** are obtained by taking the limit $\epsilon \to 0$, assuming that all fields and charge densities are finite on the surface.

These two conditions can be lumped into the single equation

$$\mathbf{E}_2 - \mathbf{E}_1 = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{n}} \tag{151}$$

Thus, the perpendicular component of $\mathbf{E}$ is discontinuous across a surface charge density σ, but the parallel component is continuous.

- Exercise: Prove the above statements.

3.4 Electric potential

- Since $\boldsymbol{\nabla} \times \mathbf{E} = 0$ we can write

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi \tag{152}$$

  for some scalar field $\Phi(\mathbf{r})$.

- The integral form of this equation is

$$\Phi(b) - \Phi(a) = -\int_C \mathbf{E}\cdot d\mathbf{s} \tag{153}$$

  where C is *any* curve with endpoints a and b.

- $\Phi(r)$ is called the *electric potential*. It is determined by the equation $\mathbf{E} = -\boldsymbol{\nabla}\Phi$ up to an additive constant.

- The ambiguity in the choice of constant is typically fixed by setting the potential equal to zero at infinity. For such a choice

$$\Phi(\mathbf{r}) = -\int_\infty^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{s} \tag{154}$$

- For example, for a point charge q at the origin, we can integrate $\mathbf{E}(\mathbf{r})$ to find

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{155}$$

- Of course, one can set the potential to zero at locations other than infinity.

- The SI unit of V is a volt, which equals 1 Joule/coulomb. The unit of the electric field in terms of volts, is volt/m.

- Using the superposition principle, it follows that

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int dV'\,\rho(\mathbf{r}')\,\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{156}$$

  for a general charge distribution ρ. (Note: The above potential also goes to zero as $r \to \infty$. A different expression needs to be used if $\Phi(r \to \infty) \neq 0$.)

- Using $\boldsymbol{\nabla}\cdot\mathbf{E} = \rho/\epsilon_0$, it is easy to show that the electric potential satisfies *Poisson's equation*:

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \tag{157}$$

- In regions of space where the charge distribution $\rho = 0$, the electric potential satisfies *Laplace's equation*:

$$\nabla^2\Phi = 0 \tag{158}$$

- The boundary conditions on Φ that lead to unique solutions of Laplace's equation will be described in the next section.

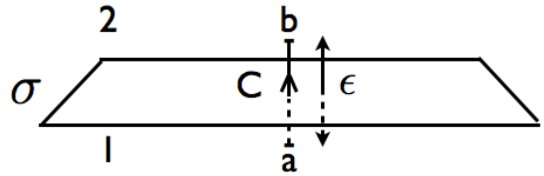


Figure 21: Curve C of length ϵ connecting point a in region 1 to point b in region 2. The boundary conditions on Φ are obtained by taking the limit $\epsilon \to 0$, assuming that all fields and charge densities are finite on the surface.

- Since the potential is a scalar field, *it is typically much easier to compute the electric field due to a charge distribution by first calculating* $\Phi(\mathbf{r})$ *and then differentiating to find* $\mathbf{E}(\mathbf{r})$. This way, one avoids doing a *vector* sum of the fields produced by the individual infinitesimal charge elements, and sums instead only the contributions to the (scalar) potential.

- Boundary conditions:

  (i) Using the integral form of $\mathbf{E} = -\boldsymbol{\nabla}\Phi$, one can show that the electric potential Φ is continuous across a boundary with surface charge density σ—i.e.,

$$\Phi_2 - \Phi_1 = 0 \tag{159}$$

  (See Fig. 21.) This assumes that the discontinuity in the electric field due to the surface charge density is finite and not given by a Dirac delta function.

  (ii) Since the normal derivative of the potential equals (minus) the normal component of the electric field, we have

$$\left(-\frac{\partial \Phi_2}{\partial n} + \frac{\partial \Phi_1}{\partial n}\right)\bigg|_S = \frac{\sigma}{\epsilon_0} \tag{160}$$

- Exercise: Prove the above two statements using $\Phi(b) - \Phi(a) = -\int_a^b \mathbf{E} \cdot d\mathbf{s}$ and the boundary conditions for the normal component of $\mathbf{E}$.

3.5 Work and energy

- The electric potential is simply related to the work one has to do *against* the electric field to move a test charge q from a to b:

$$W_{ab} = -\int_a^b \mathbf{F} \cdot d\mathbf{s} = -q\int_a^b \mathbf{E} \cdot d\mathbf{s} = q(\Phi(b) - \Phi(a)) \tag{161}$$

  This is independent of the path connecting the two points!

- If the potential is equal to zero at infinity, then the work required to bring the test charge q from infinity to $\mathbf{r}$ is given by

$$W_{\infty \to \mathbf{r}} = q\Phi(\mathbf{r}) \tag{162}$$

- Consider the work required to assemble a configuration of four point charges, as shown in Fig. 22. Calculating the work required to bring each charge successively in from infinity, one finds

$$W = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right) + \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_4}{r_{14}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}}\right) \tag{163}$$

  where $r_{ij} \equiv |\mathbf{r}_i - \mathbf{r}_j|$ is the distance between charges q_i and q_j.

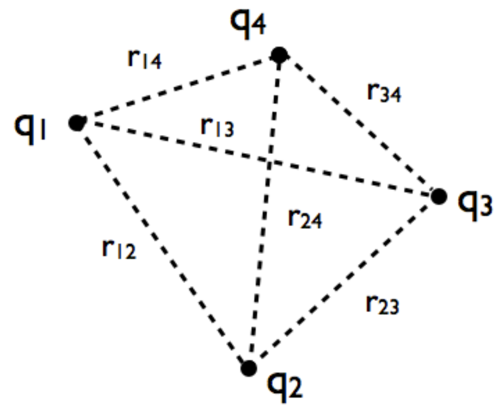


Figure 22: Configuration consisting of four point charges q_1, q_2, q_3, q_4.

- By rearranging the sum as

$$W = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}}\right) + \frac{1}{4\pi\epsilon_0}\left(\frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}}\right) + \frac{1}{4\pi\epsilon_0}\frac{q_3 q_4}{r_{34}} \tag{164}$$

  and generalizing to n discrete charges, one has

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{165}$$

  Note that the second sum is only for $j > i$.

- Equivalently,

$$W = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{166}$$

  where the restriction $j \neq i$ avoids the infinite "self-energy" terms which would occur if $i = j$. The extra factor of $1/2$ is needed to prevent double counting, since $j > i$ and $j < i$.

- This last result can also be written as

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \Phi(\mathbf{r}_i)\,, \quad \text{where} \quad \Phi(\mathbf{r}_i) = \frac{1}{4\pi\epsilon_0}\sum_{j\neq i}\frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{167}$$

  Note: $\Phi(\mathbf{r}_i)$ is the potential at $\mathbf{r}_i$ due to all the other charges, and not the charge at $\mathbf{r}_i$.

- For a continuous charge distribution, this last expression for W generalises to

$$W = \frac{1}{2}\int dV\,\rho(\mathbf{r})\Phi(\mathbf{r}) \tag{168}$$

- Using Gauss's law and integrating by parts, one can show that

$$W = \frac{\epsilon_0}{2}\int_V E^2\,dV + \frac{\epsilon_0}{2}\oint_S \Phi\mathbf{E}\cdot\hat{\mathbf{n}}\,da \tag{169}$$

  where V is any volume containing all the charge and S is its boundary surface.

- <u>Exercise</u>: Prove the above.

- If the volume is taken to be all of space, the boundary term vanishes leading to

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 \, dV \tag{170}$$

- Note:

  (i) Equation (170) suggests

$$\frac{\epsilon_0}{2} E^2 = \text{ energy density} \tag{171}$$

  implying that the energy resides in the electric field $\mathbf{E}$, rather than in the charge distribution ρ, as equation (168) might suggest. Although we cannot justify this statement at this stage, equation (171) *is* the proper interpretation of energy density for electrostatics.

  (ii) Equation (170) is manifestly *positive definite* while the point charge expression (equation (166) or (167)), may be positive or negative. The reason for this difference is that in going from equation (167) to equation (168), the potential changed from being that due to all the *other* charges (i.e., not including the charge at $\mathbf{r}$) to that due to *all* the charges (i.e., including the charge at $\mathbf{r}$). For a continuous charge distribution, these give the same potential since $dq \equiv \rho(\mathbf{r})\, dV$ vanishes as $dV \to 0$; for point charges they differ by the infinite self-energy terms.

  (iii) Equation (170) is quadratic in E, so it does *not* obey a superposition principle.

- Exercise: Show that the energy W for a point charge Q is infinite.

- Exercise: Show that the energy W for a sphere of radius R, total charge Q, and uniform charge density ρ is given by

$$W = \frac{1}{4\pi\epsilon_0} \frac{3}{5} \frac{Q^2}{R} \tag{172}$$

3.6 Conductors

- By definition, an electrical conductor contains charges that can move freely under the action of an applied electric field.

- Basic properties:

  (i) $\mathbf{E} = 0$ inside a conductor.

  (ii) The net charge density $\rho = 0$ inside a conductor.

  (iii) Any net charge must reside on the surface of a conductor.

  (iv) A conductor is an equipotential.

  (v) $\mathbf{E}$ is perpendicular to the surface, just outside the conductor.

- Exercise: Prove these properties using $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0$ and $\boldsymbol{\nabla} \times \mathbf{E} = 0$.

- When a conductor is placed in an external electric field, the free charges in the conductor move around in such a way as to create an *induced* electric field that cancels the applied external field inside the conductor.

- This induced field *shields* a hollow cavity inside a solid conductor from any external field. (See Fig. 23.)

- However, the field produced e.g., by a point charge within a hollow cavity is felt outside the conductor. (See Fig. 24.)

- Exercise: Prove these last two statements using $\boldsymbol{\nabla} \times \mathbf{E} = 0$ and Gauss's law.

- Exercise: Prove that the field due to a point charge placed *anywhere* inside a cavity of *any shape* hollowed out from a spherical conductor centered at the origin is the same as if the point charge were located at the origin in the absence of the conductor. (See Fig. 25.)

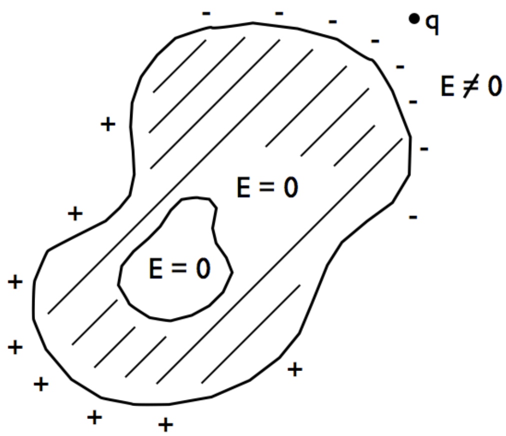


Figure 23: A point charge q produces a non-zero field $\mathbf{E} \neq 0$ outside a conductor. The induced charges on the surface of the conductor make $\mathbf{E} = 0$ inside the conductor, and inside the cavity. Note that there is no induced charge on the surface of the cavity.

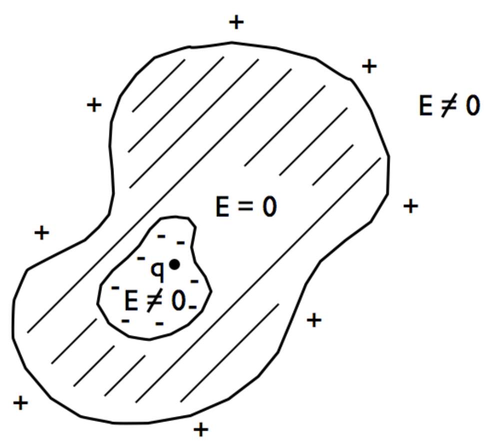


Figure 24: A point charge q is placed inside a cavity of a conductor. Although the induced charges on the surface of the cavity make $\mathbf{E} = 0$ inside the conductor, $\mathbf{E} \neq 0$ outside.

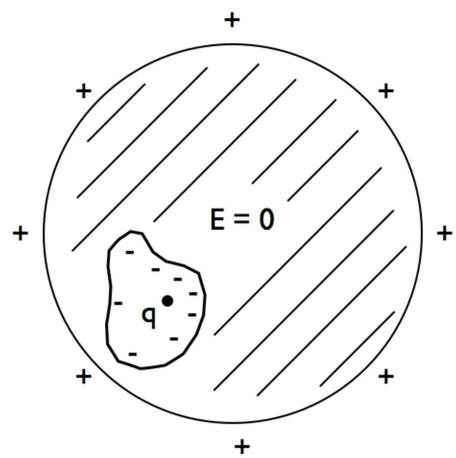


Figure 25: Charge q inside a cavity of a spherical conductor. Negative charge $-q$ is induced on the inner surface of the cavity, ensuring that $\mathbf{E} = 0$ everywhere inside the conductor. An equal amount of positive charge $+q$ is uniformly distributed on the surface of the spherical conductor, producing an electric field outside the conductor equivalent to that of a point charge q located at the center of the sphere.

- The force per unit area acting on a patch of surface charge density σ is

$$\mathbf{f} = \sigma\,\mathbf{E}_{\text{average}} = \sigma\,\frac{1}{2}(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}) \tag{173}$$

The averaging is needed to cancel the field due to the patch itself.

- Applying this general result to a conductor, with $\mathbf{E} = 0$ inside and $\mathbf{E} = (\sigma/\epsilon_0)\hat{\mathbf{n}}$ outside, we find

$$\mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\,\hat{\mathbf{n}} \tag{174}$$

- This is an outward-directed pressure, which wants to draw the conductor into the region with the non-zero electric field.

3.7 Capacitors

- A capacitor is formed by taking two conductors of arbitrary shape, and putting charge $+Q$ on one conductor and $-Q$ on the other. (See Fig. 26.)
- Capacitance is defined to be the ratio of Q to the potential difference V between the plates:

$$C := \frac{Q}{V}\,, \qquad \text{where } V := \Phi_+ - \Phi_- = -\int_{(-)}^{(+)} \mathbf{E}\cdot d\mathbf{s} \tag{175}$$

- Units: Capacitance is measured in farads (F), where 1 farad = 1 Coulomb/volt.
- Since $\mathbf{E}$ and (hence) V are both proportional to Q, the capacitance C depends only on the geometry of the conductors.
- Exercise: Show that the capacitance of a parallel plate capacitor, with plate area A and separation d (see Fig. 27) is given by

$$C = \frac{A\epsilon_0}{d} \tag{176}$$

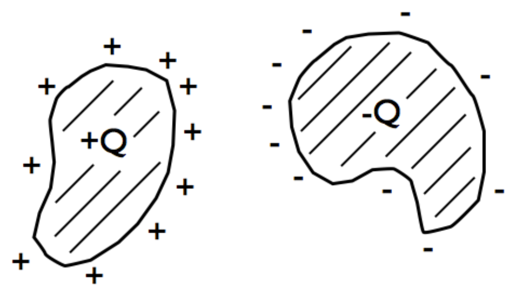


Figure 26: A capacitor can be constructed from two conductors of arbitrary shape and size, with charges $+Q$ and $-Q$, respectively.

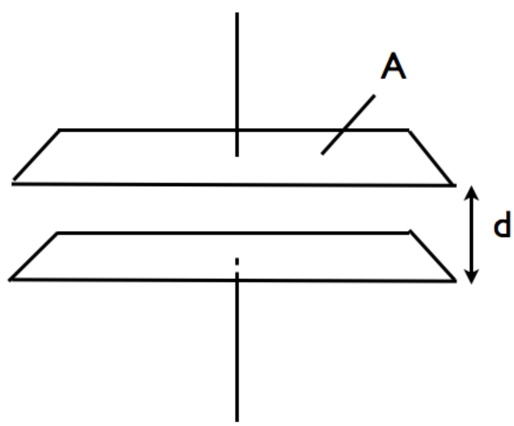


Figure 27: Geometry for a parallel plate capacitor, with plate area A and separation d.

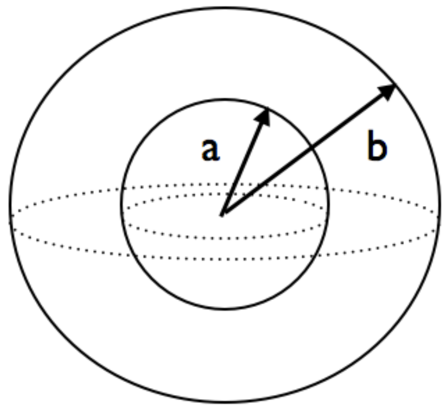


Figure 28: Geometry for a capacitor formed from two concentric spheres with radii a and b (with $b > a$).

- Exercise: Show that the capacitance of two concentric spherical shells with radii a and b (with $b > a$) (see Fig. 28) is given by

$$C = 4\pi\epsilon_0 \frac{ab}{b-a} \tag{177}$$

- Since it requires work to move charge from one conductor to the other against the field produced by the charges already present on the conductors, energy is stored in a conductor.

- This energy equals the work required to separate the charges:

$$W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 \tag{178}$$

- Exercise: Prove this result by integrating $dW = dq\,V(q) = dq\,q/C$ from $q = 0$ to $q = Q$.

3.8 Summary

- The following equations summarize the relationship between $\mathbf{E}$, Φ, and ρ in electrostatics:

  1) Relating ρ and $\mathbf{E}$:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}\,, \quad \boldsymbol{\nabla}\times\mathbf{E} = 0 \quad \Leftrightarrow \quad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V dV'\,\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{179}$$

  where V is any volume containing all the charge distribution ρ.

  2) Relating Φ and $\mathbf{E}$:

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi \quad \Leftrightarrow \quad \Phi(b) - \Phi(a) = -\int_C \mathbf{E}\cdot d\mathbf{s} \tag{180}$$

  where C is any curve with endpoints a and b.

  3) Relating ρ and Φ:

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \quad \Leftrightarrow \quad \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V dV'\,\rho(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{181}$$

  where the boundary condition on Φ is $\Phi(r\to\infty) = 0$.

4 Boundary value problems

- Goal: Solve Poisson's equation
$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \tag{182}$$
for Φ, subject to non-trivial boundary conditions.

- The following subsections present techniques for doing this.

4.1 Laplace's equation

- In regions of space where the charge density $\rho = 0$, Poission's equation reduces to Laplace's equation
$$\nabla^2\Phi = 0 \tag{183}$$

- Solutions to Laplace's equation are the 'smoothest' possible functions that satisfy the boundary conditions, having no local maxima or minima. This statements will be made more precise below.

- In 1-dimension, Laplace's equation is simply
$$\frac{d^2\Phi}{dx^2} = 0 \tag{184}$$

- The most general solution is a straight line:
$$\Phi(x) = mx + b \tag{185}$$
where the constants m and b are fixed by boundary conditions.

- Suppose the region of interest is the interval $x \in [x_1, x_2]$. Then possible boundary conditions are:

  (i) Specify Φ at both of the endpoints x_1 and x_2.

  (ii) Specify Φ and $d\Phi/dx$ at *one* of the endpoints, x_1 or x_2.

  Note that specifying $d\Phi/dx$ at *both* endpoints will either be redundant (if they have the same value) or inconsistent (if they have different values).

- Note that
$$\Phi(x) = \frac{1}{2}\left[\Phi(x+a) + \Phi(x-a)\right] \tag{186}$$
which implies that $\Phi(x)$ has no local maxima or minima. (See Figure 29.)

- In 2-dimensions, in Cartesian coordinates, Laplace's equation is
$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{187}$$

- One can show that
$$\Phi(x, y) = \frac{1}{2\pi R}\oint_C \Phi(x', y')\, ds' \tag{188}$$
where C is a circle centered at (x, y). (See Figure 30.) This result implies that $\Phi(x, y)$ has no local maxima or minima.

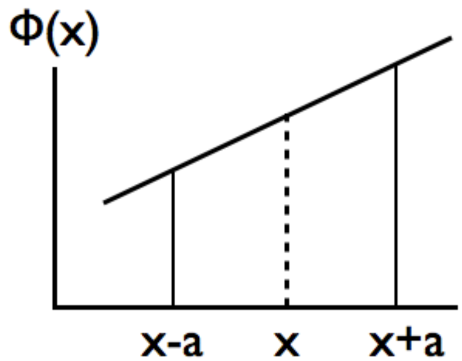


Figure 29: In 1-dimension, the most general solution to Laplace's equation $\nabla^2\Phi \equiv d^2\Phi/dx^2 = 0$ is a straight line. Note that $\Phi(x)$ is the average of its values at two equally-displaced points, $\Phi(x-a)$ and $\Phi(x+a)$.

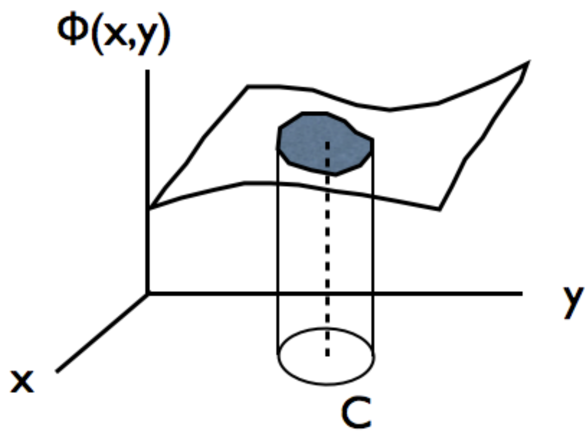


Figure 30: If Φ is a solution to Laplace's equation in 2-dimensions, then the value of Φ at any point (x, y) is the average of its values on a circle C of any radius centered at (x, y).

- Exercise: Prove the above statement using Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz \tag{189}$$

Here $f(z)$ is a complex-valued function that is *analytic* (i.e., complex differentiable) inside and on some closed curve C, and a is any point inside C. Recall that if $f(z)$ is analytic, then the real-valued functions $u(x,y)$, $v(x,y)$ defined by $f = u + i\,v$ satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{190}$$

which in turn imply that they individually satisfy Laplace's equation

$$\nabla^2 u = 0\,, \quad \nabla^2 v = 0 \tag{191}$$

Hint: Take C to be a circle of radius R centered at the point $a = (x,y)$ so that $z - a = R\, e^{i\phi}$ and $dz = iR\, e^{i\phi}\, d\phi$ for points on C. Then split Cauchy's integral formula into its real and imaginary parts.

- In 3-dimensions, one has

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_S \Phi(\mathbf{r}')\, da' \tag{192}$$

where S is a 2-sphere centered at $\mathbf{r}$. Again this implies that $\Phi(\mathbf{r})$ has no local maxima or minima.

- Exercise: Prove this last expression for $\Phi(\mathbf{r})$, assuming a point source located anywhere outside the sphere. For simplicity, you can take $\mathbf{r} = 0$ and put the point source on the z-axis, a distance d from the center of the sphere of radius R ($d > R$). (By the superposition principle, the result is then valid *any* distribution of charge outside the sphere.)

4.2 Boundary conditions and uniqueness theorems

- Theorem: A solution to Poisson's equation $\nabla^2\Phi = -\rho/\epsilon_0$ inside a volume V is uniquely determined (up to an additive constant) by specifying the charge density $\rho(\mathbf{r})$ in V and *either* the potential Φ *or* its normal derivative $\partial\Phi/\partial n$ on the closed boundary surface S.

- Specifying the potential on the boundary is called *Dirichlet* boundary conditions. The potential is uniquely determined for this case—i.e., the additive constant is zero.

- Specifying the normal derivative of the potential on the boundary is called *Neumann* boundary conditions. The additive constant may be non-zero for this case.

- For both Dirichlet and Neumann boundary conditions, the electric field—being the gradient of the potential—is uniquely determined. (The unspecified additive constant for Neumann BCs vanishes when taking the gradient.)

- One can also specify *mixed* boundary conditions, corresponding to specifying Φ on parts of S and $\partial\Phi/\partial n$ on the remaining parts of S.

- NOTE: Specifying both the potential and its normal derivative on the boundary surface *over-specifies* the problem—i.e., the potential may not be able to satisfy both of these conditions.

- Exercise: Prove the uniqueness theorem using Green's 1st identity

$$\int_V (\boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} U + T\nabla^2 U)\, dV = \oint_S (T\boldsymbol{\nabla} U)\cdot \hat{\mathbf{n}}\, da \tag{193}$$

by assuming that Φ_1 and Φ_2 are two solutions to Poisson's equation, and then showing that Φ_1 and Φ_2 differ at most by an additive constant. (Hint: Set $T = U = \Phi_1 - \Phi_2$, noting that $\nabla^2 U = 0$ and $\boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} U \geq 0$.)

- A related uniqueness theorem (see Griffiths, 3rd edition, page 118) is the following: In a volume V surrounded by *conductors* and containing a specified charge density, the electric field is uniquely determined if the total charge on each conductor is given.

4.3 Green's functions: Introduction

- Definition: A Green's function is a solution to a differential equation with a delta-function source. For Poisson's equation,

$$\nabla'^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\, \delta(\mathbf{r} - \mathbf{r}') \tag{194}$$

Note the prime on the Laplacian, meaning differentiation wrt $\mathbf{r}'$. (The factor of -4π is chosen for convenience to simplify some of the expressions for Green's functions.)

- Later we will show that $G(\mathbf{r}, \mathbf{r}')$ is *symmetric* under interchange of $\mathbf{r}$ and $\mathbf{r}'$—i.e.,

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r}) \tag{195}$$

Since the Dirac delta function is also symmetric, then it actually doesn't matter whether we are differentiating $G(\mathbf{r}, \mathbf{r}')$ with respect to $\mathbf{r}$ or $\mathbf{r}'$.

- Typically one thinks of $\mathbf{r}$ as the observation or field point and $\mathbf{r}'$ as the source point. The symmetry of the Green's function is related to the physical interchangeability of the source and observation points.

- Recall that

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi\, \delta(\mathbf{r} - \mathbf{r}') \tag{196}$$

so $1/|\mathbf{r} - \mathbf{r}'|$ is an example of a Green's function.

- The most general solution is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} + F(\mathbf{r}, \mathbf{r}') \tag{197}$$

where $F(\mathbf{r}, \mathbf{r}')$ is a solution of Laplace's equation $\nabla^2 F(\mathbf{r}, \mathbf{r}') = 0$.

- Exercise: Using Green's theorem

$$\int_V (T\nabla'^2 U - U\nabla'^2 T)\, dV' = \oint_S (T\boldsymbol{\nabla}' U - U\boldsymbol{\nabla}' T) \cdot \hat{\mathbf{n}}'\, da' \tag{198}$$

show that

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')\, dV' + \frac{1}{4\pi} \oint_S \left[G(\mathbf{r}, \mathbf{r}') \frac{\partial \Phi(\mathbf{r}')}{\partial n'} - \Phi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} \right] da' \tag{199}$$

(Hint: Set $T = \Phi(\mathbf{r}')$, $U = G(\mathbf{r}, \mathbf{r}')$.)

- The above equation for $\Phi(\mathbf{r})$ is an *integral equation* for $\Phi(\mathbf{r})$. The RHS cannot be used to calculate $\Phi(\mathbf{r})$ given *arbitrary* values for Φ and $\partial\Phi/\partial n$ on the boundary S, since that would overspecify $\Phi(\mathbf{r})$.

- However, by using the freedom in $F(\mathbf{r}, \mathbf{r}')$, one can choose $G(\mathbf{r}, \mathbf{r}')$ so that $\Phi(\mathbf{r})$ has the appropriate form for either Dirichlet or Neumann boundary conditions.

- For Dirichlet BCs, one chooses

$$G_D(\mathbf{r}, \mathbf{r}')\Big|_S = 0 \tag{200}$$

Then

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V G_D(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')\, dV' - \frac{1}{4\pi} \oint_S \Phi(\mathbf{r}') \frac{\partial G_D(\mathbf{r}, \mathbf{r}')}{\partial n'}\, da' \tag{201}$$

- For Neumann BCs, one chooses

$$\left.\frac{\partial G_N(\mathbf{r},\mathbf{r}')}{\partial n}\right|_S = -\frac{4\pi}{A} \tag{202}$$

where A is the area of S. Then

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V G_N(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,dV' + \frac{1}{4\pi}\oint_S G_N(\mathbf{r},\mathbf{r}')\frac{\partial\Phi(\mathbf{r}')}{\partial n'}\,da' + \langle\Phi\rangle_S \tag{203}$$

where $\langle\Phi\rangle_S$ is the average of Φ over the boundary surface S.

- Note that one cannot simply choose $\partial G_N/\partial n' = 0$ on S, since the divergence theorem implies

$$-4\pi = \int_V \nabla'^2 G_N(\mathbf{r},\mathbf{r}')\,dV' = \int_V \boldsymbol{\nabla}'\cdot\boldsymbol{\nabla}' G_N(\mathbf{r},\mathbf{r}')\,dV' = \oint_S \frac{\partial G_N(\mathbf{r},\mathbf{r}')}{\partial n'}\,da' \tag{204}$$

- <u>Example</u>: If S is the 'boundary' surface at $r\to\infty$, then the associated Dirichlet Green's function is:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} \qquad \text{(for } S \text{ boundary surface at } r\to\infty) \tag{205}$$

This is because the only solution to $\nabla^2 F = 0$ which vanishes as $r\to\infty$ is $F = 0$.

- <u>Exercise</u>: Using Green's theorem, prove that a Dirichlet Green's function $G_D(\mathbf{r},\mathbf{r}')$ is symmetric under interchange of $\mathbf{r}$ and $\mathbf{r}'$. Hint: Take $U(\mathbf{r}'') = G_D(\mathbf{r},\mathbf{r}'')$ and $T(\mathbf{r}'') = G_D(\mathbf{r}',\mathbf{r}'')$, where $\mathbf{r}''$ is the integration variable. (Note: Symmetry for a Neumann Green's function $G_N(\mathbf{r},\mathbf{r}')$ is *not* automatic, but can imposed as a separate requirement.)

4.4 Method of images

- <u>Basic idea</u>: Solve Poisson's equation for Φ in some region having non-trivial boundary conditions by enlarging the region to include additional ('image') charges but *no* boundaries.

- The choice of image charges is such that the potential (or its normal derivative) originally specified on the boundary is reproduced by the original charges together with the image charges.

- The method of images is an *indirect* method of solving Poisson's equation. The potential for the problem of multiple charges *without boundary* was actually solved *first*; the physically equivalent problem that is obtained by replacing one of the equipotential surfaces with a conducting surface came afterward.

- Note that the solutions we obtain using the method of images for grounded conducting surfaces (i.e., $\Phi|_S = 0$) will give us the Dirichlet Green's functions associated with those boundary surfaces.

4.4.1 Example 1: Point charge above an infinite, grounded conducting plane

- Consider a point charge q located at a distance d above an infinite, grounded conducting plane. Find the potential in the region above the plane. (See Figure 31, panel (a).)

- Choose coordinates so that the conducting plane is given by $z = 0$ and the charge q is located at $z = d$. (See Figure 31, panel (b).)

- An image charge $q_I = -q$ placed at $z = -d$ together with q produces a potential

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right) \tag{206}$$

which vanishes when $z = 0$.

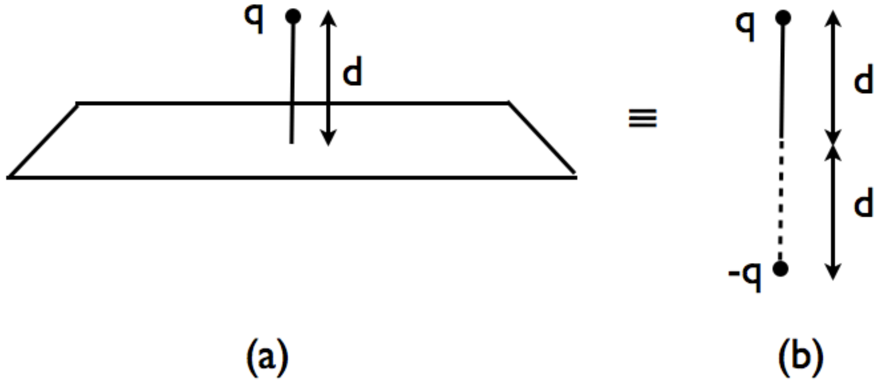


Figure 31: Panel (a): Point charge q a distance d above an infinite, grounded conducting plane. Panel (b): Equivalent problem with image charge $-q$ located a distance $2d$ from q. Note there is no conducting plane for the equivalent image problem.

- By the uniqueness theorems, the above expression is the *unique* solution to Poisson's equation in the region $z > 0$, satisfying $\Phi = 0$ on the conducting plane ($z = 0$).
- The induced surface charge on the conducting plane is given by

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial z}\bigg|_{z=0} = -\frac{qd}{2\pi} \frac{1}{(x^2 + y^2 + d^2)^{3/2}} \tag{207}$$

- The total induced charge, obtained by integrating σ over the surface, is

$$Q \equiv \int_S \sigma \, da = -q \tag{208}$$

  Note that this is the value of the image charge.
- The total force on the point charge q due to the induced charge on the conducting plane is obtained by integrating (minus) the force-per-unit-area

$$\mathbf{f} = \frac{1}{2} \frac{\sigma^2}{\epsilon_0} \hat{\mathbf{n}} \tag{209}$$

  over the surface. (Minus since $\mathbf{f}$ is the force-per-unit area *acting on a patch* of the conducting surface.)
- The result is

$$\mathbf{F}_q = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2d)^2} \hat{\mathbf{z}} \tag{210}$$

- This is the same as the force exerted on q by the image charge $q_I = -q$ at $z = -d$.
- The work required to bring the charge q in from infinity to its location a distance d above the infinite, grounded conducting plane, is given by

$$W = -\int_\infty^d \mathbf{F}_q \cdot d\mathbf{s} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{4d} \tag{211}$$

- <u>Exercise</u>: Prove this.

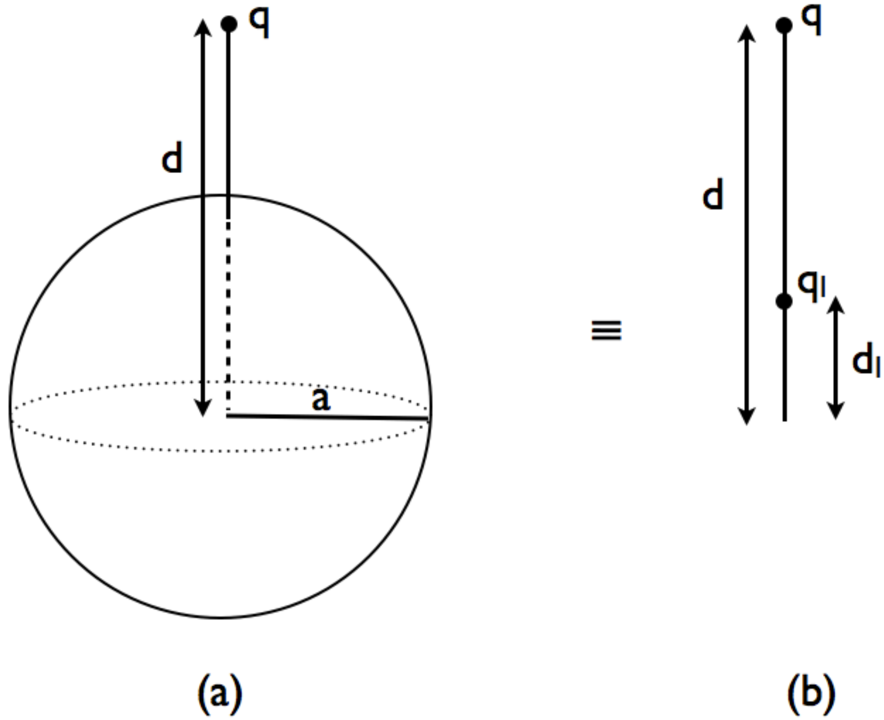


Figure 32: Panel (a): Point charge q a distance d from the center of a grounded conducting sphere of radius a. Panel (b): Equivalent problem with image charge q_I located a distance d_I from the center of the sphere. Note there is no conducting sphere for the equivalent image problem.

- This is *half* the work required to assemble the configuration consisting of the point charge q and image charge $-q$, *without* the conducting surface. The factor of $1/2$ arises since no work is done on the induced charge as it moves around on the conducting plane (an equipotential surface) in response to the point charge q being brought in from infinity.

4.4.2 Example 2: Point charge exterior to a grounded conducting sphere

- Consider a point charge q located at a distance $d > a$ from the center of grounded conducting sphere of radius a. Find the potential outside the sphere—i.e., for $r > a$. (See Figure 32, panel (a).)

- Choose spherical polar coordinates so that the center of the sphere is at the origin of coordinates and the point charge is located at $z = d$.

- By symmetry, the image charge q_I will also lie on the z axis at a distance d_I from the center. (See Figure 32, panel (b).)

- To find q_I and d_I, we note that the potential due to q and q_I can be written as

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} + \frac{q_I}{\sqrt{r^2 + d_I^2 - 2rd_I\cos\theta}}\right) \tag{212}$$

- When $|\mathbf{r}| = a$, this can be written as

$$\Phi(|\mathbf{r}| = a) = \frac{1}{4\pi\epsilon_0}\left(\frac{q/a}{\sqrt{1+(d/a)^2 - 2(d/a)\cos\theta}} + \frac{q_I/d_I}{\sqrt{1+(a/d_I)^2 - 2(a/d_I)\cos\theta}}\right) \quad (213)$$

- Exercise: Prove the last equality.
- From this expression, one sees that if

$$\frac{d}{a} = \frac{a}{d_I}\,, \qquad \frac{q_I}{d_I} = -\frac{q}{a} \quad (214)$$

then $\Phi(|\mathbf{r}| = a) = 0$, as desired.

- Thus,

$$d_I = \left(\frac{a}{d}\right) a = \left(\frac{a}{d}\right)^2 d\,, \qquad q_I = -q\,\frac{a}{d} \quad (215)$$

are the location and value of the image charge.

- Note that $d_I < a$ (since $a < d$), so the image charge is outside the region of interest ($r > a$) as it should be.
- Substituting for q_I and d_I, the potential becomes

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} - \frac{1}{\sqrt{a^2 + d^2r^2/a^2 - 2rd\cos\theta}}\right) \quad (216)$$

- The induced charge density on the spherical surface is

$$\sigma = -\epsilon_0 \frac{\partial\Phi}{\partial r}\bigg|_{r=a} = -\frac{q}{4\pi a}\frac{d^2 - a^2}{(a^2 + d^2 - 2ad\cos\theta)^{3/2}} \quad (217)$$

- The total induced charge is

$$Q \equiv \int_S \sigma\, da = -q\frac{a}{d} \quad (218)$$

As we saw for the other example, this is the value of the image charge.

- The total force on the point charge q due to the induced charge on the conducting sphere is obtained by integrating the z-component (extra factor of $\cos\theta$ in the integral) of the force-per-unit area over the sphere:

$$\mathbf{F}_q = -\int_{r=a} \frac{1}{2}\frac{\sigma^2}{\epsilon_0}\,\hat{\mathbf{r}}\,\cos\theta\, da = -\frac{1}{4\pi\epsilon_0}\frac{a}{d}\frac{q^2d^2}{(d^2 - a^2)^2}\,\hat{\mathbf{z}} \quad (219)$$

- This is the same as the force exerted on q by the image charge $q_I = -q(a/d)$ at $z = d_I = a^2/d$.
- The work required to bring the charge q in from infinity to its location a distance d from the center of the sphere, is given by

$$W = -\int_\infty^d \mathbf{F}_q \cdot d\mathbf{s} = -\frac{1}{4\pi\epsilon_0}\frac{q^2R}{2(d^2 - a^2)} \quad (220)$$

- Exercise: Prove this.
- As for the previous example, this is *half* the work required to assemble the configuration consisting of the point charge q and image charge $q_I = -q(a/d)$, *without* the conducting surface.

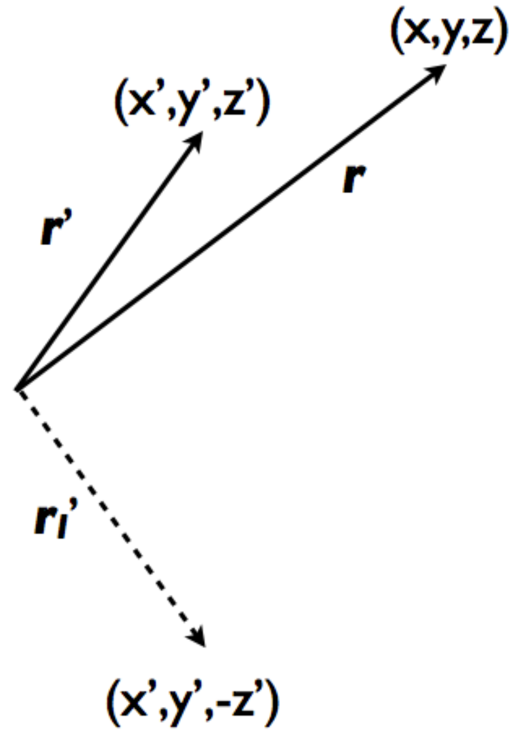


Figure 33: Relationship between the vectors $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}'_I$ for the Dirichlet Green's function for an infinite plane. These three vectors label label the field location, source location, and image charge location, respectively.

4.5 Green's function for an infinite plane

- The Dirichlet Green's function $G_D(\mathbf{r}, \mathbf{r}')$ for an infinite plane is effectively the potential for a point charge in the presence of an infinite, grounded conducting plane. We just need to write the potential in a form that is symmetric with respect to the field point $\mathbf{r}$ and point source location $\mathbf{r}'$.

- Choose coordinates so the plane is at $z = 0$. Then

$$\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}} \quad \text{and} \quad \mathbf{r}'_I = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} - z'\hat{\mathbf{z}} \tag{221}$$

denote the point source and image charge locations. (See Figure 33.)

- In terms of these quantities, the potential $\Phi(\mathbf{r})$ for the method of images problem is

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'_I|}\right) \tag{222}$$

- Expressed in terms of Cartesian coordinates,

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}\right) \tag{223}$$

The RHS is manifestly symmetric under interchange $\mathbf{r} \leftrightarrow \mathbf{r}'$, and vanishes on the plane $z = 0$.

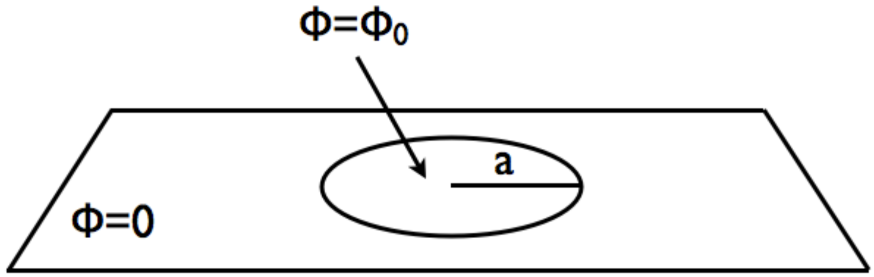


Figure 34: The potential on an infinite plane is specified to equal Φ_0 inside a circular disc of radius a, and to equal zero outside.

- Thus, the Dirichlet Green's function is

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \tag{224}$$

- In terms of $G_D(\mathbf{r},\mathbf{r}')$, the potential $\Phi(\mathbf{r})$ for $z>0$ with arbitrarily prescribed values on the plane $z=0$ is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V G_D(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,dV' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{r}')\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\,da' \tag{225}$$

- For the surface integral, we need

$$\left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\right|_S = -\left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial z'}\right|_{z'=0} = -\frac{2z}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}} \tag{226}$$

  The minus sign is because the outward pointing normal (away from the volume) is in the direction of decreasing z—i.e., $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$.

- If the charge distribution is zero (e.g, if we are interested in a solution of Laplace's equation in the region $z>0$), then the volume integral vanishes and $\Phi(\mathbf{r})$ is given simply by the surface integral.

- Exercise: (Jackson, Prob 2.7) Find the solution to Laplace's equation for $z>0$, where the potential on the plane $z=0$ is prescribed to have the value Φ_0 for a circular disc $x^2+y^2\leq a^2$, and $\Phi=\Phi_0$ otherwise. Expand the integral in a power series of a/r, where $r^2=x^2+y^2+z^2$, keeping the first few terms. (See Figure 34.)

- Answer:

$$\Phi(\mathbf{r}) = \frac{\Phi_0}{2}\left(\frac{a}{r}\right)^2\left(\frac{z}{r}\right)\left[1-\frac{3}{4}\left(\frac{a}{r}\right)^2+\frac{5}{8}\left(\frac{a}{r}\right)^4\left(1+3\,\frac{x^2+y^2}{a^2}\right)+\cdots\right] \tag{227}$$

4.6 Green's function exterior to a sphere

- Just as we saw for the infinite plane, the Dirichlet Green's function $G_D(\mathbf{r},\mathbf{r}')$ exterior to a sphere is given (up to an overall multiplicative constant) by the potential for a point charge exterior to a grounded conducting sphere. We just need to write the potential in a form that is symmetric with respect to the field point $\mathbf{r}$ and point source location $\mathbf{r}'$.

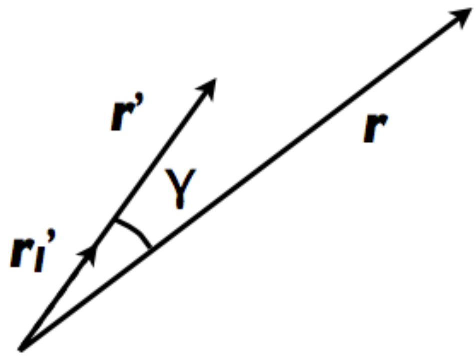


Figure 35: Relationship between the vectors $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}_I'$ for the Dirichlet Green's function exterior to a sphere of radius a. These three vectors label label the field location, source location, and image charge location, respectively. Note that both $r, r' > a$ while $r_I' < a$.

- Since $\mathbf{r}'$ now denotes the location of the point source, the value of the image charge and its location are given by

$$q_I = -q\,\frac{a}{r'}\,, \qquad \mathbf{r}_I' = \left(\frac{a}{r'}\right)^2 \mathbf{r}' \tag{228}$$

(See Figure 35.)

- In terms of these quantities, the potential $\Phi(\mathbf{r})$ for the method of images problem is

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{(a/r')}{\left|\mathbf{r}-\left(\frac{a}{r'}\right)^2\mathbf{r}'\right|}\right) \tag{229}$$

- Expressed in terms of spherical polar coordinates,

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{1}{\sqrt{a^2+\frac{r^2r'^2}{a^2}-2rr'\cos\gamma}}\right) \tag{230}$$

where

$$\cos\gamma = \cos\theta\,\cos\theta' + \sin\theta\,\sin\theta'\,\cos(\phi-\phi') \tag{231}$$

is the angle between $\mathbf{r}$ and $\mathbf{r}'$.

- Note that the RHS of the potential is manifestly symmetric under interchange $\mathbf{r}\leftrightarrow\mathbf{r}'$, and vanishes on the sphere $r = a$.

- Thus, the Dirichlet Green's function is

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{1}{\sqrt{a^2+\frac{r^2r'^2}{a^2}-2rr'\cos\gamma}} \tag{232}$$

- In terms of $G_D(\mathbf{r},\mathbf{r}')$, the potential $\Phi(\mathbf{r})$ exterior to the sphere with arbitrarily prescribed values on the surface of the sphere is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V G_D(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\,dV' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{r}')\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\,da' \tag{233}$$

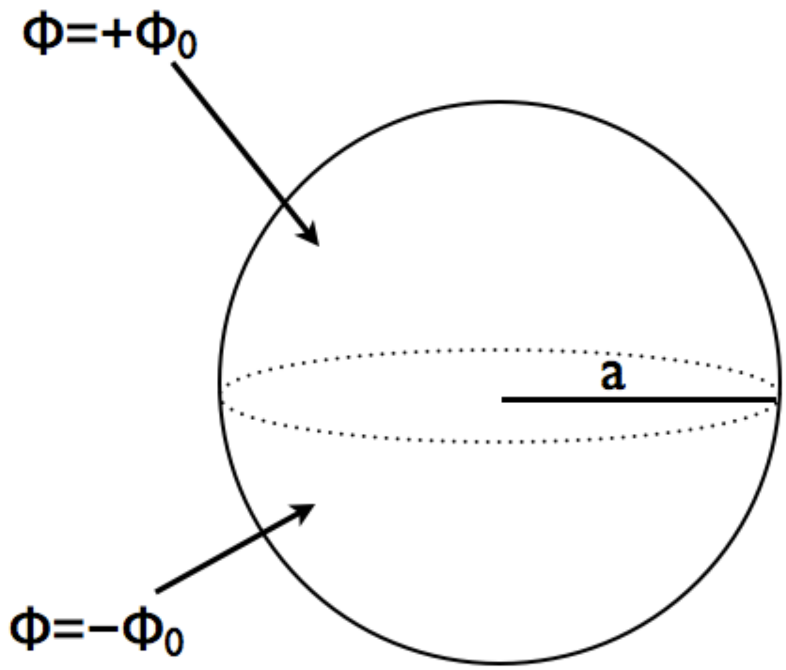


Figure 36: The potential on a sphere of radius a is specified to equal $+\Phi_0$ on the northern hemisphere and $-\Phi_0$ on the southern hemisphere, respectively.

- For the surface integral, we need

$$\left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\right|_S = -\left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial r'}\right|_{r'=a} = -\frac{(r^2-a^2)}{a(r^2+a^2-2ra\cos\gamma)^{3/2}} \tag{234}$$

  The minus sign is because the outward pointing normal (away from the volume) is in the direction of decreasing r—i.e., $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$.

- If the charge distribution is zero (e.g, if we are interested in a solution of Laplace's equation exterior to the sphere), then the volume integral vanishes and $\Phi(\mathbf{r})$ is given simply by the surface integral.

- <u>Exercise</u>: Find the solution to Laplace's equation outside a sphere of radius a with prescribed potential $\pm\Phi_0$ in the upper and lower hemispheres, respectively. Expand the integral in a power series of a/r, keeping the first few terms. (See Figure 36.)

- <u>Answer</u>:

$$\Phi(\mathbf{r}) = \frac{3\Phi_0}{2}\left(\frac{a}{r}\right)^2\left[\cos\theta - \frac{7}{12}\left(\frac{a}{r}\right)^2\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + \cdots\right] \tag{235}$$

- Note that the terms in the square brackets are proportional to the Legendre polynomials

$$P_1(x) = x\,, \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{236}$$

  with $x = \cos\theta$.

- Legendre polynomials will appear again when using separation of variables in spherical coordinates to solve Laplace's equation.

4.7 Expansions in terms of orthonormal functions

- Basic idea: Expand a square-integrable function in terms of a set of orthonormal basis functions, similar to the decomposition $\mathbf{v} = \sum_i v_i \hat{\mathbf{e}}_i$ for vectors in a finite-dimensional vector space.

- Notation: Let $\xi \in [a, b]$ and $f(\xi)$ denote any square-integrable function.

- If a discrete set of functions $\{U_n(\xi)|n = 1, 2, \cdots\}$ satisfies

$$\int_a^b d\xi \, U_n^*(\xi) U_m(\xi) = \delta_{nm} \tag{237}$$

  then the functions are said to be *orthonormal*.

- The functions $\{U_n(\xi)|n = 1, 2, \cdots\}$ form a *basis* (and are said to be *complete*) if any square-integrable function can be expanded as

$$f(\xi) = \sum_{n=1}^{\infty} A_n \, U_n(\xi) \tag{238}$$

- It follows from the orthonormality property of the $U_n(\xi)$ that

$$A_n = \int_a^b d\xi \, U_n^*(\xi) f(\xi) \tag{239}$$

- Substituting this expression for A_n back into the expansion for $f(\xi)$, one finds

$$\sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) = \delta(\xi - \xi') \tag{240}$$

  This is another way of expressing the completeness of the functions $U_n(\xi)$.

- Exercise: Prove the last two statements.

4.7.1 Fourier series

- Let $x \in [-a/2, a/2]$. Then

$$\left\{ \sqrt{\frac{2}{a}} \sin\left(\frac{n2\pi x}{a}\right) \, , \, \sqrt{\frac{2}{a}} \cos\left(\frac{n2\pi x}{a}\right) \, \middle| \, n = 1, 2, \cdots \right\} \tag{241}$$

  form an orthonormal basis for functions defined on the interval $[-a/2, a/2]$, or, equivalently, for *periodic functions* defined for $x \in (-\infty, \infty)$ with period a.

- Orthonormality:

$$\frac{2}{a} \int_{-a/2}^{a/2} dx \, \sin\left(\frac{n2\pi x}{a}\right) \sin\left(\frac{m2\pi x}{a}\right) = \delta_{nm} \tag{242}$$

$$\frac{2}{a} \int_{-a/2}^{a/2} dx \, \cos\left(\frac{n2\pi x}{a}\right) \cos\left(\frac{m2\pi x}{a}\right) = \delta_{nm} \tag{243}$$

$$\frac{2}{a} \int_{-a/2}^{a/2} dx \, \sin\left(\frac{n2\pi x}{a}\right) \cos\left(\frac{m2\pi x}{a}\right) = 0 \tag{244}$$

- Exercise: Prove the above using

$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) \tag{245}$$

$$\cos A \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right) \tag{246}$$

$$\sin A \cos B = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right) \tag{247}$$

- Completeness:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n2\pi x}{a}\right) + B_n \sin\left(\frac{n2\pi x}{a}\right)\right] \tag{248}$$

where

$$A_0 = \frac{2}{a}\int_{-a/2}^{a/2} dx\, f(x) \tag{249}$$

$$A_n = \frac{2}{a}\int_{-a/2}^{a/2} dx\, f(x)\cos\left(\frac{n2\pi x}{a}\right) \tag{250}$$

$$B_n = \frac{2}{a}\int_{-a/2}^{a/2} dx\, f(x)\sin\left(\frac{n2\pi x}{a}\right) \tag{251}$$

- In terms of complex exponentials the equations simplify:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{in2\pi x/a} \tag{252}$$

where

$$C_n = \frac{1}{a}\int_{-a/2}^{a/2} dx\, f(x) e^{-in2\pi x/a} \tag{253}$$

- The orthonormal basis functions are now:

$$\left\{ \frac{1}{\sqrt{a}} e^{\frac{in2\pi x}{a}} \,\middle|\, n = 0, \pm 1, \pm 2, \cdots \right\} \tag{254}$$

- Orthonormality:

$$\frac{1}{a}\int_{-a/2}^{a/2} dx\, e^{i(n-m)2\pi x/a} = \delta_{nm} \tag{255}$$

- Completeness:

$$\frac{1}{a}\sum_{n=-\infty}^{\infty} e^{in2\pi(x-x')/a} = \delta(x-x') \tag{256}$$

- Parseval's theorem:

$$\frac{1}{a}\int_{-a/2}^{a/2} dx\, |f(x)|^2 = \sum_{n=-\infty}^{\infty} |C_n|^2 \tag{257}$$

4.7.2 Fourier transform

- For square-integrable *non-periodic* functions defined over $x \in (-\infty, \infty)$, the Fourier series expansion generalizes to the *Fourier transform*.
- The orthonormal basis functions are now labeled by a *continuous* index $k \in (-\infty, \infty)$:

$$U_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} \tag{258}$$

- For any square-integrable function $f(x)$, we have

$$f(x) = \int_{-\infty}^{\infty} dk\, C(k)\, \frac{1}{\sqrt{2\pi}} e^{ikx} \tag{259}$$

where

$$C(k) = \int_{-\infty}^{\infty} dx\, f(x)\, \frac{1}{\sqrt{2\pi}} e^{-ikx} \tag{260}$$

- $f(x)$ and $C(k)$ are said to be a *Fourier transform pair*.
- Some authors define the expansion of $f(x)$ without the factor of $1/\sqrt{2\pi}$, but then need a factor of $1/2\pi$ in the expression for $C(k)$.
- One sometimes writes $\tilde{f}(k)$ instead of $C(k)$.
- Orthonormality:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = \delta(k - k') \tag{261}$$

- Note that the basis functions themselves are *not* square-integrable, as they are normalised in the sense of equaling a Dirac delta function.
- Completeness:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = \delta(x - x') \tag{262}$$

- Note the symmetry between the orthonormality and completeness relations.
- Parseval's theorem:

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} dk\, |C(k)|^2 \tag{263}$$

4.8 Separation of variables (rectangular coords)

- Separation of variables is an attempt to solve Laplace's equation in some region by writing $\Phi(\mathbf{r})$ as a product of functions, each of a single variable, reducing the partial differential equation to a set of ordinary differential equations, which are easier to solve. (Note that Laplace's equation is separable in 11 different coordinate systems!)
- In rectangular (i.e., Cartesian) coordinates one writes

$$\Phi(x, y, z) = X(x)Y(y)Z(z) \tag{264}$$

- In terms of X, Y, and Z, Laplace's equation

$$0 = \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \tag{265}$$

becomes

$$X''YZ + XY''Z + XYZ'' = 0 \tag{266}$$

where $'$ denotes ordinary derivative with respect to the (single) argument of the function—e.g., $X'(x) = dX/dx$.

- Dividing by $\Phi = XYZ$ yields

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0 \tag{267}$$

- Note that this is a sum of three terms, which are functions of only x, y, and z, respectively. The only way that such a sum can equal zero is for each term is equal to a *constant* (called a *separation constant*), with the sum of constants equal to zero:

$$\frac{X''}{X} = C_1\,, \quad \frac{Y''}{Y} = C_2\,, \quad \frac{Z''}{Z} = C_3 \tag{268}$$

with

$$C_1 + C_2 + C_3 = 0 \tag{269}$$

- Whether the constants are positive or negative or zero depend on the particular BCs.
- For example, suppose that we are interested in solving Laplace's equation inside a rectangular region $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, where the potential is set to zero on all faces except $z = c$, where it equals some prescribed function, $\Phi(x, y, c) = f(x, y)$.
- Then the appropriate choice of separation constants is

$$C_1 \equiv -\alpha^2 \le 0\,, \quad C_2 \equiv -\beta^2 \le 0\,, \quad C_3 \equiv \gamma^2 = \alpha^2 + \beta^2 \ge 0 \tag{270}$$

- The solutions of the individual equations for non-zero α, β, and γ are

$$X(x) = A\,\sin(\alpha x) + B\,\cos(\alpha x)\,, \tag{271}$$
$$Y(y) = C\,\sin(\beta y) + D\,\cos(\beta y)\,, \tag{272}$$
$$Z(z) = E\,\sinh(\gamma z) + F\,\cosh(\gamma z) \tag{273}$$

- The solutions of the individual equations for α, β, and γ equal to zero are

$$X(x) = A_0\,x + B_0\,, \tag{274}$$
$$Y(y) = C_0\,y + D_0\,, \tag{275}$$
$$Z(z) = E_0\,z + F_0 \tag{276}$$

- The most general solution of Laplace's equation is then a linear combination of the product solutions XYZ for the different allowed values of α and β.
- The BCs that the potential vanishes when $x = 0$, $y = 0$, and $z = 0$ imply

$$B = 0\,, \quad D = 0\,, \quad F = 0\,, \quad B_0 = 0\,, \quad D_0 = 0\,, \quad F_0 = 0\,, \tag{277}$$

- The BCs that the potential vanishes when $x = a$ and $y = b$ imply

$$\alpha = \frac{n\pi}{a}\,, \quad \beta = \frac{m\pi}{b}\,, \quad A_0 = 0\,, \quad C_0 = 0 \tag{278}$$

where n, m are positive integers.

- We need only consider n and m positive, since $n = 0$ and $m = 0$ leads to the trivial $\Phi = 0$ solution of Laplace's equation; while n and m negative introduce only an overall sign change from n and m positive, which can be absorbed in the multiplicative constants.
- Although E_0 is not constrained to vanish by the BCs, it will not enter into the final expression for Φ since $\gamma = 0$ iff $\alpha = \beta = 0$, and the X and Y solutions for $\alpha = \beta = 0$ are identically zero (we saw above that A_0, B_0, C_0, and D_0 are all constrained to vanish).

- Thus, the most general solution of Laplace's equation satisfying all of the boundary conditions except $\Phi(x,y,c) = f(x,y)$ can be written as

$$\Phi(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sinh\left(\gamma_{nm}z\right) \tag{279}$$

where

$$\gamma_{nm} = \sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}} \tag{280}$$

- Imposing the final BC at $z = c$ yields

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sinh\left(\gamma_{nm}c\right) \tag{281}$$

- Orthonomality of the sine functions on the intervals $x \in [0,a]$ and $y \in [0,b]$, lead to the solution

$$A_{nm}\sinh\left(\gamma_{nm}c\right) = \frac{4}{ab}\int_0^a dx\int_0^b dy\, f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \tag{282}$$

- To do the integration one needs to specify the explicit form of $f(x,y)$.

- To solve Laplace's equation inside the same rectangular region for more complicated BCs (e.g., where more than one face has non-zero values), we can simply superimpose the 'single-face' solutions, which all have a form similar to the above solution.

- <u>Exercise</u>: Solve the 2-dimensional Laplace equation in the region $0 \le x \le a$, $0 \le y < \infty$ subject to the BCs that the potential vanishes on the 'sides' (i.e., at $x = 0$ and $x = a$) and at the 'top' (i.e., $y \to \infty$), and is equal to a constant Φ_0 when $y = 0$.

- <u>Answer</u>:

$$\Phi(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{-n\pi y/a} \tag{283}$$

where

$$A_n = \frac{2}{a}\int_0^a dx\,\Phi_0\sin\left(\frac{n\pi x}{a}\right) = \begin{cases} \frac{4\Phi_0}{n\pi} & n = 1,3,\cdots \\ 0 & n = 2,4,\cdots \end{cases} \tag{284}$$

- <u>Exercise</u>: Using

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) = \sum_{n=\mathrm{odd}}^{\infty}\frac{z^n}{n} \tag{285}$$

show that one can explicitly evaluate the summation for $\Phi(x,y)$ yielding the analytical expression

$$\Phi(x,y) = \frac{2\Phi_0}{\pi}\tan^{-1}\left(\frac{\sin(\pi x/a)}{\sinh(\pi y/a)}\right) \tag{286}$$

4.9 Separation of variables (spherical polar coords)

- In spherical polar coordinates, Laplace's equation is

$$0 = \nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \tag{287}$$

- If one assumes the product form

$$\Phi(r, \theta, \phi) \equiv R(r)P(\theta)Q(\phi) \tag{288}$$

Laplace's equation reduces to the following ordinary differential equations:

$$Q''(\phi) = -m^2\, Q(\phi) \tag{289}$$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = l(l+1)\, R(r) \tag{290}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\, \frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right] P(\theta) = 0 \tag{291}$$

where l and m are (at this stage) arbitrary real separation constants.

- Exercise: Prove the above.

- The solutions of the ϕ-equation are

$$Q(\phi) = A_0 + B_0\phi\,, \qquad \text{for } m = 0 \tag{292}$$

$$Q(\phi) = A\, e^{im\phi} + B\, e^{-im\phi}\,, \qquad \text{for } m \neq 0 \tag{293}$$

- If ϕ can take on the full range of values $\phi \in [0, 2\pi]$, then the requirement that $Q(\phi)$ be single-valued (i.e., $Q(\phi+2\pi n) = Q(\phi)$ for integer n) implies $B_0 = 0$ and m equal an integer. (This will normally be the case for the examples that we consider.)

- The radial equation has the general solution

$$R(r) = Ar^l + Br^{-(l+1)} \tag{294}$$

- Exercise: Prove this.

- Note that if $l \geq 0$, finiteness of $\Phi(r, \theta, \phi)$ at the origin ($r = 0$) implies $B = 0$. Similarly, requiring $\Phi(r, \theta, \phi) \to 0$ as $r \to \infty$ implies $A = 0$.

- The θ-equation can be put into more standard form by making a change of variables from θ to $x = \cos\theta$:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P(x) = 0 \tag{295}$$

and then expanding the derivative,

$$(1-x^2)P''(x) - 2x\, P'(x) + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P(x) = 0 \tag{296}$$

The above equation is called the *associated Legendre's equation*.

- If $m = 0$, the above equation is called *Legendre's equation*:

$$(1-x^2)P''(x) - 2x\, P'(x) + l(l+1)\, P(x) = 0 \tag{297}$$

4.9.1 Legendre polynomials

- To find a power series solution to Legendre's equation, we first note that $x = 0$ is a regular point of the differential equation.

- Substituting

$$P(x) = \sum_{n=0}^{\infty} a_n x^n \tag{298}$$

into Legendre's equation and differentiating term by term, we obtain the recurrence relation

$$a_{n+2} = \frac{n(n+1) - l(l+1)}{(n+1)(n+2)}\, a_n \tag{299}$$

- Exercise: Prove this.

- Since the recurrence relation relates a_{n+2} to a_n, the two independent solutions to Legendre's equation are given by setting $a_0 = 1$, $a_1 = 0$ and $a_1 = 1$, $a_0 = 0$. These solutions will be *even* and *odd* functions of x, respectively.

- One can show that the power series solutions diverge at $x = \pm 1$ (corresponding to the North and South poles of the sphere) unless the series terminates after some finite value of n.

- From the recurrence relation, we see that if l is a non-negative integer, $l = 0, 1, \cdots$, one of the power series solutions terminates (the even solution if l is even, and the odd solution if l is odd). The other solution can be set to zero (by hand) by setting $a_1 = 0$ (or $a_0 = 0$).

- The finite solutions are polynomials of order l. When appropriately normalised, they are called *Legendre polynomials*, denoted $P_l(x)$.

- NOTE: If l is a negative integer, $l = -1, -2, \cdots$, one also obtains a polynomial solution. But these solutions are the *same* as those for l non-negative (e.g., $l = -2$ yields the same solution as $l = 1$) so there is no loss of generality in restricting attention to $l = 0, 1, \cdots$.

- Legendre polynomials are normalized by the condition that $P_l(1) = 1$.

- Exercise: Show that the first four Legendre polynomials are given by

$$P_0(x) = 1 \tag{300}$$
$$P_1(x) = x \tag{301}$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{302}$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{303}$$

  See Figures 37-40 for various graphical representations of these functions.

- Rodrigues' formula:

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2 - 1)^l \tag{304}$$

- Note that

$$P_l(-x) = (-1)^l P_l(x) \tag{305}$$

- Orthonormality:

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'} \tag{306}$$

  Thus, Legendre polynomials form a set of *orthogonal* polynomials.

- Exercise: Prove the above. (Hint: The proof of orthogonality is simple if you integrate Legendre's equation times $P_{l'}(x)$. The derivation of the normalization constant is harder, but can proved using mathematical induction and Rodrigues's formula for $P_l(x)$.)

- Completeness: Any square-integrable function $f(x)$ defined on the interval $x \in [-1, 1]$ can be expanded in terms of Legendre polynomials:

$$f(x) = \sum_{l=0}^{\infty} A_l\, P_l(x) \tag{307}$$

  where

$$A_l = \frac{2l+1}{2}\int_{-1}^{1} dx\, f(x)\, P_l(x) \tag{308}$$

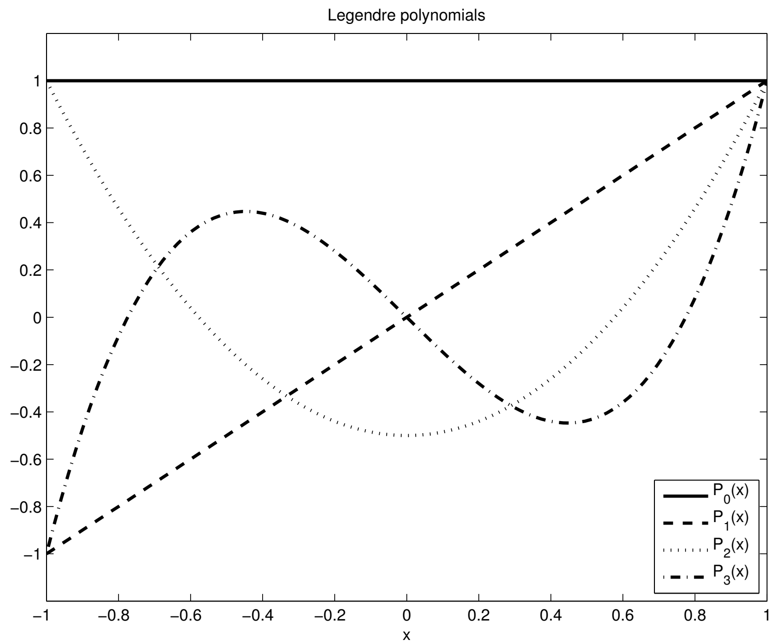


Figure 37: First few Legendre polynomials $P_l(x)$ plotted as functions of $x \in [-1, 1]$.

- Example: The function

$$f(x) = \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases} \tag{309}$$

can be expanded as

$$f(x) = \frac{3}{2}\,P_1(x) - \frac{7}{8}\,P_3(x) + \frac{11}{16}\,P_5(x) + \cdots \tag{310}$$

See Figure 41.

- Exercise: Prove this.

- Thus, the general solution to Laplace's equation for problems with *azimuthal symmetry* (i.e., no ϕ-dependence so $m = 0$) is

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(A_l\, r^l + B_l\, r^{-(l+1)} \right) P_l(\cos\theta) \tag{311}$$

- Exercise: Find the solution to Laplace's equation outside a sphere of radius a with specified potential

$$\Phi(r = a, \theta) = \begin{cases} +\Phi_0 & \text{for } 0 \le \theta < \pi/2 \\ -\Phi_0 & \text{for } \pi/2 < \theta \le \pi \end{cases} \tag{312}$$

- Answer:

$$\Phi(r, \theta) = \Phi_0 \left[\frac{3}{2} \left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8} \left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \frac{11}{16} \left(\frac{a}{r}\right)^6 P_5(\cos\theta) + \cdots \right] \tag{313}$$

NOTE: We obtained this result earlier using the Dirichlet Green's function exterior to the sphere.

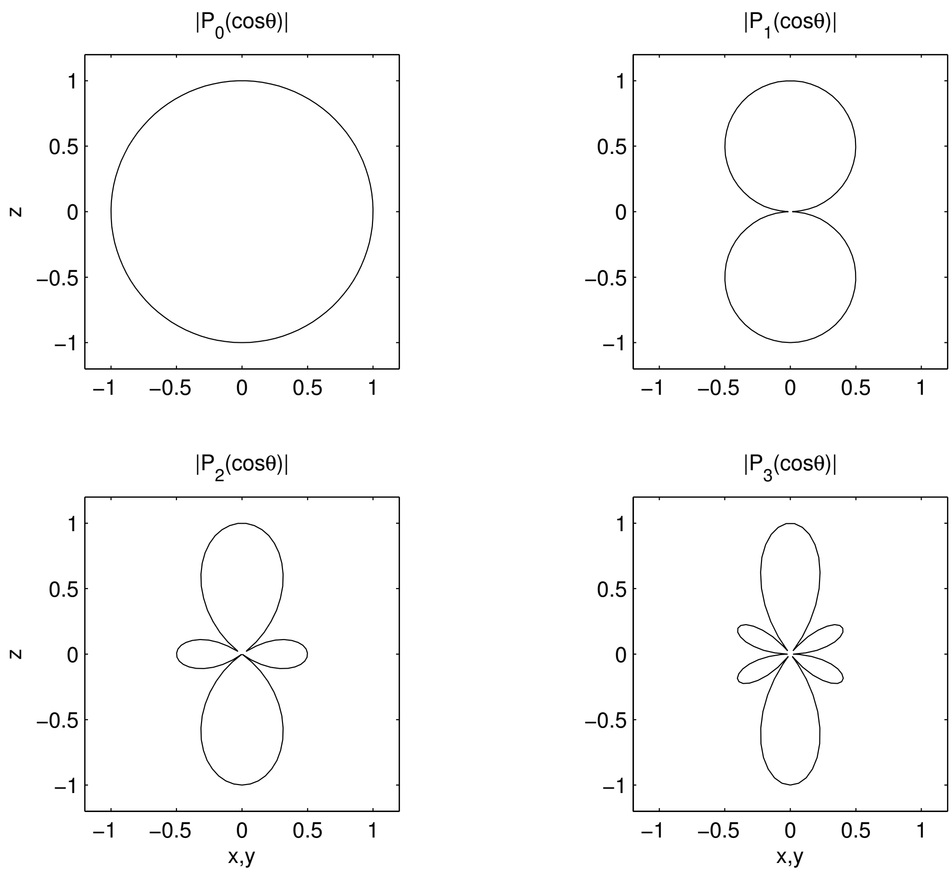


Figure 38: The *magnitude* $|P_l(\cos\theta)|$ of the first few Legendre polynomials plotted as functions of $\cos\theta$ in the x-z (or y-z) plane. The angle θ is measured wrt the positive z-axis. Note that by plotting the magnitude, information about the *sign* (i.e., $\pm$) of the Legendre polynomials $P_l(\cos\theta)$ is lost in this graphical representation.

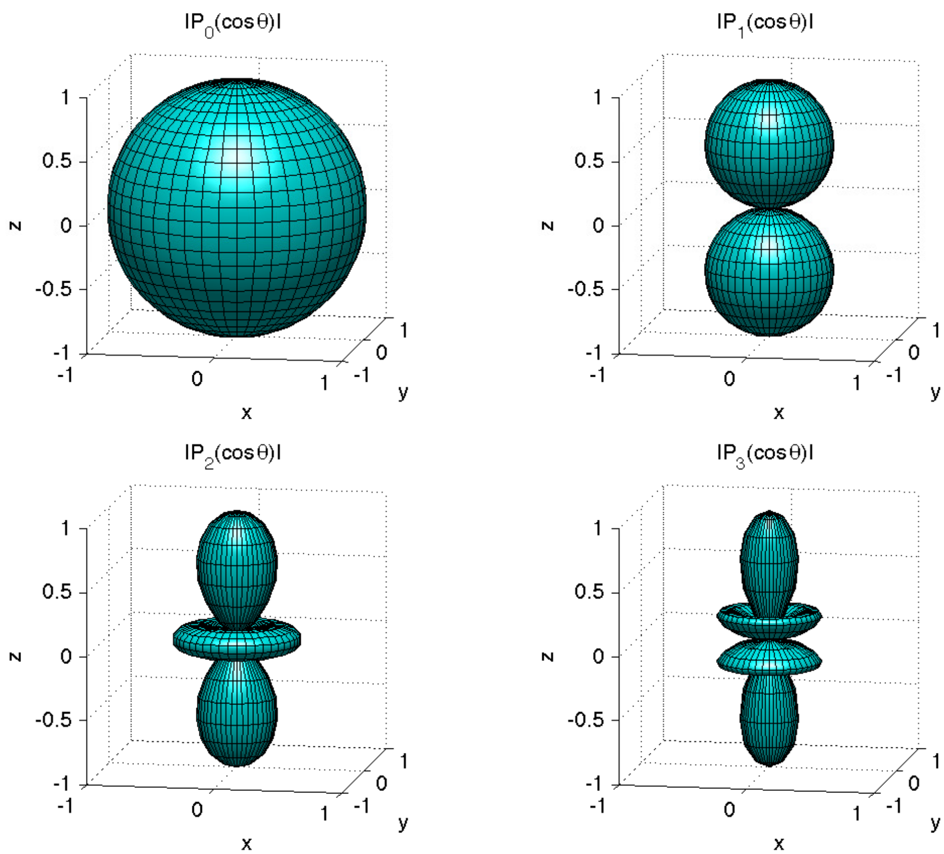


Figure 39: Same as Figure 38 but illustrated as surfaces of revolution (since there is no ϕ-dependence for the Legendre polynomials).

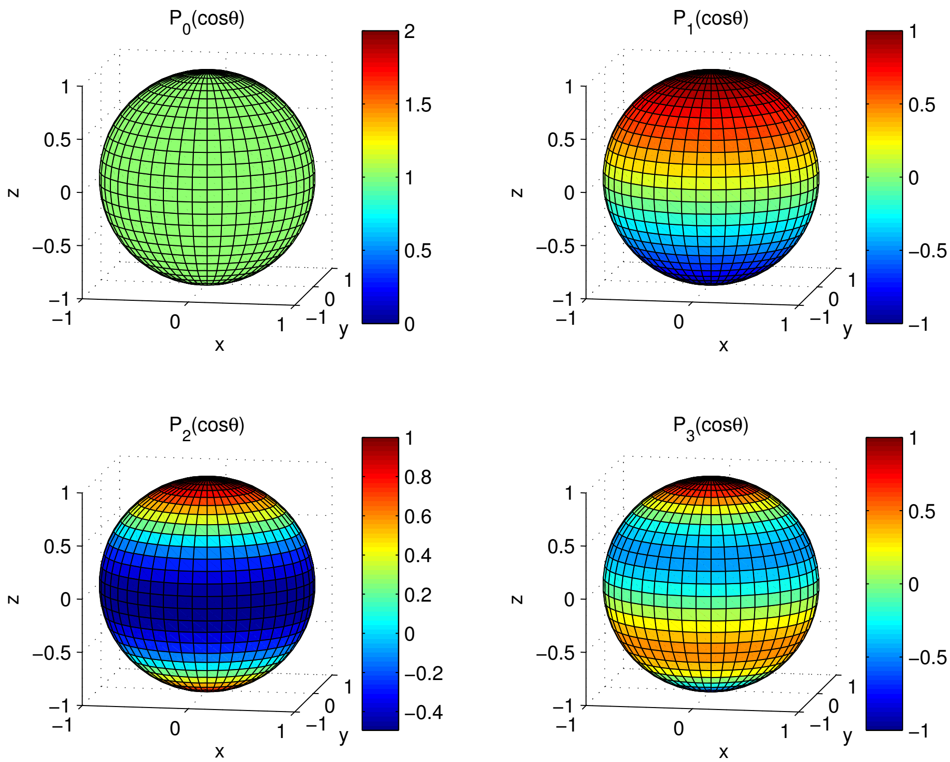


Figure 40: First few Legendre polynomials $P_l(\cos\theta)$ represented as functions on the unit 2-sphere. The color associated with each point on the sphere is the value of $P_l(\cos\theta)$ for that point (θ, ϕ). Note that in contrast to Figures 38 and 39, information about the sign of the Legendre polynomials is preserved in this graphical representation.

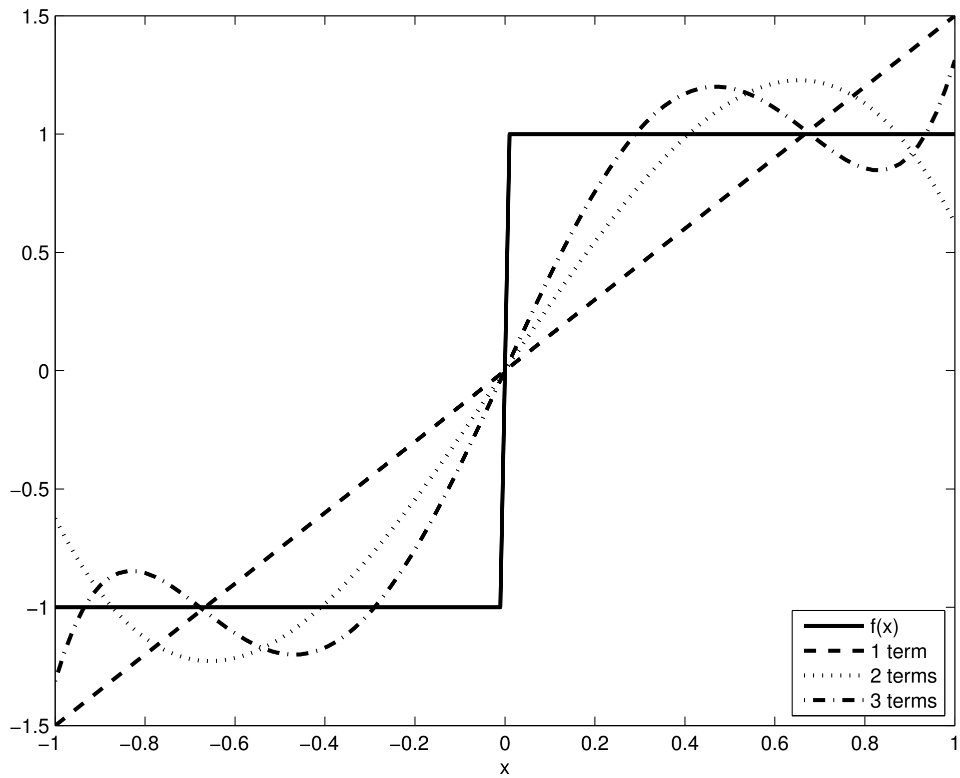


Figure 41: Expansion of the function $f(x) = \pm 1$ for $x \gtrless 0$, in terms Legendre polynomials. This plot shows how the approximation to $f(x)$ improves as more terms in the expansion are used.

- Generating function:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n \tag{314}$$

- Using the generating function, one can derive the following *recurrence relations*:

$$(n+1)\, P_{n+1} = (2n+1)x\, P_n - n\, P_{n-1} \tag{315}$$
$$P_n = P'_{n+1} - 2x\, P'_n + P'_{n-1} \tag{316}$$
$$n\, P_n = x\, P'_n - P'_{n-1} \tag{317}$$
$$(n+1)\, P_n = P'_{n+1} - x\, P'_n \tag{318}$$
$$(2n+1)\, P_n = P'_{n+1} - P'_{n-1} \tag{319}$$
$$(1-x^2)\, P'_n = n(P_{n-1} - x\, P_n) \tag{320}$$

- Note that Legendre's equation

$$(1-x^2)\, P''_n - 2x\, P'_n + n(n+1)\, P_n = 0 \tag{321}$$

can be obtained by differentiating (320) wrt x and then using (317). In addition, the normalization $P_n(1) = 1$ also follows simply from the generating function.

- Exercise: Prove the above relations by differentiating the generating function wrt t and x separately, and then combining the various expressions.

- Another important result that follows trivially from the generating function expression is an expansion of the potential of a point charge in terms of Legendre polynomials:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}}\, P_l(\cos\gamma) \tag{322}$$

where $r_<$ ($r_>$) is the smaller (larger) of r and r', and γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$:

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}' \equiv \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\, \cos(\phi-\phi') \tag{323}$$

(See Figure 42.)

- Exercise: Prove the expansion for $1/|\mathbf{r}-\mathbf{r}'|$.

4.9.2 Associated Legendre functions

- When $m \neq 0$, we need to solve the associated Legendre's equation

$$(1-x^2)\, P'' - 2x\, P' + \left[l(l+1) - \frac{m^2}{(1-x^2)}\right]\, P = 0 \tag{324}$$

- It turns out that power series solutions of this differential equation also diverge at the poles ($x = \pm 1$) unless $l = 0, 1, \cdots$ (as before) and $m = -l, -l+1, \cdots, l$.

- The finite solutions are called *associated Legendre functions* are are given by derivatives of the Legendre polynomials:

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \tag{325}$$

and

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!}\, P_l^m(x) \tag{326}$$

for $m > 0$.

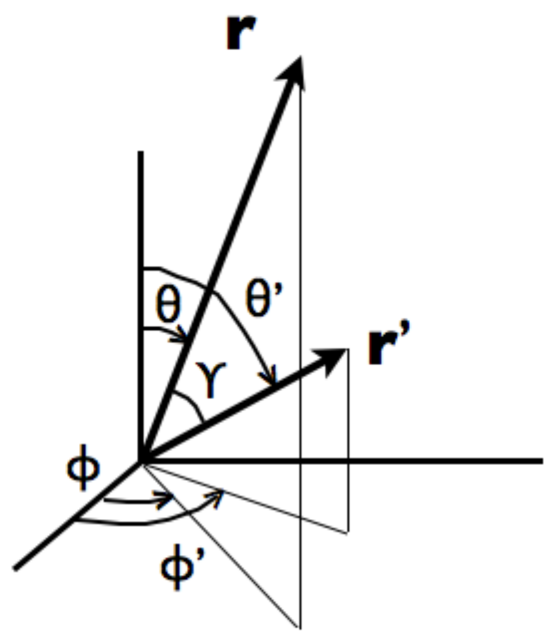


Figure 42: Position vectors $\mathbf{r}$ and $\mathbf{r}'$, and the spherical coordinates (θ, ϕ) and (θ', ϕ') specifying their directions. γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. For this example $r_> = r$ and $r_< = r'$.

- The above phase convention is that of Condon and Shortley.
- The associated Legendre functions are *not* polynomials in x on account of the square root factor $(1-x^2)^{m/2}$ for odd m. But since we are ultimately interested in the replacement $x = \cos\theta$, these non-polynomial factors are just proportional to $\sin^m\theta$. Thus, the associated Legendre functions can be written as polynomials in $\cos\theta$ if m is even, and polynomials in $\cos\theta$ multiplied by $\sin\theta$ if m is odd.
- Exercise: Show that the first few associated Legendre functions are given by:

  $l = 0$:

$$P_0^0(\cos\theta) = 1 \tag{327}$$

  $l = 1$:

$$P_1^0(\cos\theta) = \cos\theta \tag{328}$$

$$P_1^1(\cos\theta) = -\sin\theta \tag{329}$$

  $l = 2$:

$$P_2^0(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right) \tag{330}$$

$$P_2^1(\cos\theta) = -3\sin\theta\,\cos\theta \tag{331}$$

$$P_2^2(\cos\theta) = 3(1 - \cos^2\theta) \tag{332}$$

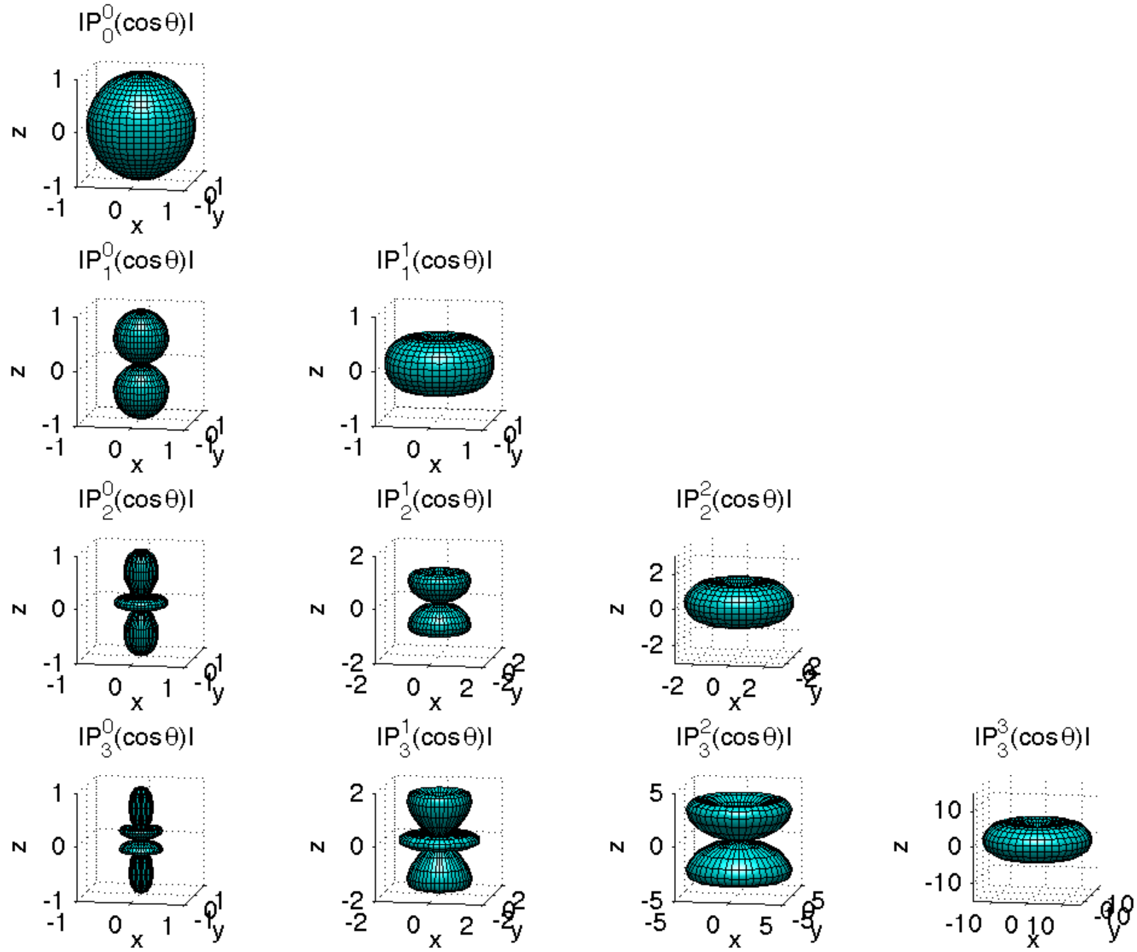


Figure 43: The *magnitude* $|P_l^m(\cos\theta)|$ of the first few associated Legendre functions plotted as surfaces of revolution. Similar to the plots in Figures 38 and 39, the sign (i.e., $\pm$) of the associated Legendre functions $P_l^m(\cos\theta)$ is lost in this graphical representation.

$\underline{l=3}$:

$$P_3^0(\cos\theta) = \frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right) \tag{333}$$

$$P_3^1(\cos\theta) = -\frac{3}{2}\sin\theta\,\left(5\cos^2\theta - 1\right) \tag{334}$$

$$P_3^2(\cos\theta) = 15\left(\cos\theta - \cos^3\theta\right) \tag{335}$$

$$P_3^3(\cos\theta) = -15\sin\theta\,\left(1 - \cos^2\theta\right) \tag{336}$$

See Figure 43 for plots of the magnitude of the first few of these functions.

- Using Rodrigues' formula, we can write down a formula valid for both positive and negative values of m:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\,\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \tag{337}$$

- Orthonormality: For each m

$$\int_{-1}^{1} dx\, P_l^m(x) P_{l'}^m(x) = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\,\delta_{ll'} \tag{338}$$

- Completeness: For each m, the associated Legendre functions form a complete set (in the index l) for square-integrable functions on $x \in [-1, 1]$:

$$f(x) = \sum_{l=0}^{\infty} A_l \, P_l^m(x) \tag{339}$$

where

$$A_l = \frac{2l+1}{2} \frac{(l-m)!}{(l+m)!} \int_{-1}^{1} dx \, f(x) \, P_l^m(x) \tag{340}$$

4.9.3 Spherical harmonics

- *Spherical harmonics* are proportional to the product of the solutions of the angular equations for Laplace's equation in spherical polar coordinates:

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi} \tag{341}$$

They are *complex* functions on the unit 2-sphere with spherical coordinates (θ, ϕ).

- The proportionality constants have been chosen so that

$$\int_{S^2} d\Omega \, Y_{lm}^*(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'} \tag{342}$$

where

$$d\Omega \equiv d(\cos\theta) \, d\phi = \sin\theta \, d\theta \, d\phi \tag{343}$$

- Note that

$$Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi) \tag{344}$$

- For the antipodal point on the sphere

$$Y_{lm}(\pi - \theta, \phi + \pi) = (-1)^l Y_{lm}(\theta, \phi) \tag{345}$$

- For $m = 0$

$$Y_{l0} = \sqrt{\frac{2l+1}{4\pi}} \, P_l(\cos\theta) \tag{346}$$

- Expressions for the first few spherical harmonics:

<u>$l = 0$</u>:

$$Y_{00}(\theta, \phi) = \sqrt{\frac{1}{4\pi}} \tag{347}$$

<u>$l = 1$</u>:

$$Y_{11}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi} \tag{348}$$

$$Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta \tag{349}$$

$$Y_{1,-1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi} \tag{350}$$

<u>$l = 2$</u>:

$$Y_{22}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi} \tag{351}$$

$$Y_{21}(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi} \tag{352}$$

$$Y_{20}(\theta,\phi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \tag{353}$$

$$Y_{2,-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i\phi} \tag{354}$$

$$Y_{2,-2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{-2i\phi} \tag{355}$$

- Since $Y_{lm}(\theta,\phi)$ differs from $P_l^m(\theta)$ by only a constant multiplicative factor and phase $e^{im\phi}$, the magnitude $|Y_l^m(\theta,\phi)|$ has the same shape as $|P_l^m(\theta)|$ (see Figure 43).

- Completeness: Any square-integrable function $f(\theta,\phi)$ on the unit 2-sphere can be expanded in terms of spherical harmonics:

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}\, Y_{lm}(\theta,\phi) \tag{356}$$

where

$$A_{lm} = \int_{S^2} d\Omega\, f(\theta,\phi)\, Y^*_{lm}(\theta,\phi) \tag{357}$$

- Equivalently, the completeness property can be written as

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) = \delta(\hat{\mathbf{\Omega}},\hat{\mathbf{\Omega}}') \tag{358}$$

where $\delta(\hat{\mathbf{\Omega}},\hat{\mathbf{\Omega}}')$ is the Dirac delta function on the 2-sphere:

$$\delta(\hat{\mathbf{\Omega}},\hat{\mathbf{\Omega}}') = \delta(\cos\theta - \cos\theta')\delta(\phi-\phi') = \frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi') \tag{359}$$

- Thus, the general solution to Laplace's equation in spherical polar coordinates is

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}\, r^l + B_{lm}\, r^{-(l+1)}\right] Y_{lm}(\theta,\phi) \tag{360}$$

- *Addition theorem*:

$$\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) = \frac{2l+1}{4\pi}\, P_l(\cos\gamma) \tag{361}$$

where

$$\cos\gamma = \hat{\mathbf{\Omega}}\cdot\hat{\mathbf{\Omega}}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') \tag{362}$$

- Completeness of the spherical harmonics and the addition theorem imply

$$\delta(\hat{\mathbf{\Omega}},\hat{\mathbf{\Omega}}') = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}P_l(\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{\Omega}}')\,, \tag{363}$$

which is an expansion of the Dirac delta function on the 2-sphere in terms of the Legendre polynomials.

- *Transformation under a rotation*:

$$Y_{lm}(R\hat{\mathbf{\Omega}}) = \sum_{m'=-l}^{l} D_{lm,m'} Y_{lm'}(\hat{\mathbf{\Omega}})\,, \tag{364}$$

  where R denotes an arbitrary rotation.

- The fact that $Y_{lm}(R\hat{\mathbf{\Omega}})$ can be written as a linear combination of the $Y_{lm'}(\hat{\mathbf{\Omega}})$ with the *same* l is a consequence of the spherical harmonics being eigenfunctions of the (rotationally-invariant) Laplacian on the unit 2-sphere with eigenvalues depending only on l:

$$^{(2)}\nabla^2 Y_{lm}(\hat{\mathbf{\Omega}}) = -l(l+1) Y_{lm}(\hat{\mathbf{\Omega}})\,. \tag{365}$$

- The coefficients $D^l_{mm'}$ are closely related to the Clebsch-Gordan coefficients. They satisfy

$$\sum_{m''=-l}^{l} D_{lm,m''} D^*_{lm',m''} = \delta_{mm'} \tag{366}$$

  as a consequence of

$$\int_{S^2} d\hat{\mathbf{\Omega}}\, Y^*_{lm}(R\hat{\mathbf{\Omega}}) Y_{l'm'}(R\hat{\mathbf{\Omega}}) = \delta_{ll'}\delta_{mm'} \tag{367}$$

- Using the addition theorem, it follows that the potential for a point source can be written as:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{r^l_<}{r^{l+1}_>}\, Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi) \tag{368}$$

  where $r_<$ ($r_>$) is the smaller (larger) of r and r'.

- This expression is fully-factorized into a product of functions of the unprimed and primed coordinates.

4.9.4 Proof of the addition theorem for spherical harmonics

- Goal: To prove the addition theorem

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi) \tag{369}$$

  where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') \tag{370}$$

- Definitions and coordinate systems:

  1) Let $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}'$ be two unit vectors, with dot product $\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{\Omega}}' = \cos\gamma$. We will keep $\hat{\mathbf{\Omega}}'$ fixed, but allow $\hat{\mathbf{\Omega}}$ to vary.

  2) Choose a coordinate system on the 2-sphere so that $\hat{\mathbf{\Omega}}$ has coordinates (θ,ϕ) and $\hat{\mathbf{\Omega}}'$ has coordinates (θ',ϕ'). (See Figure 44, panel (a).) Since $\hat{\mathbf{\Omega}}'$ is fixed, (θ',ϕ') are constants. (For example, they are *not* integrated over in any of the following expressions.) If $\gamma = 0$, then $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}'$ correspond to the same point on the 2-sphere, so that $(\theta,\phi) = (\theta',\phi')$.

  3) We can also consider a rotated coordinate system on the 2-sphere with the North Pole given by $\hat{\mathbf{\Omega}}'$. Then $\hat{\mathbf{\Omega}}$ has spherical coordinates (γ,ψ) wrt this rotated coordinate system, where ψ is an arbitrary azimuthal coordinate, since choosing $\hat{\mathbf{\Omega}}'$ as the North Pole of the rotated coordinates doesn't uniquely determine the zero of the azimuthal angle. (See Figure 44, panel (b).)

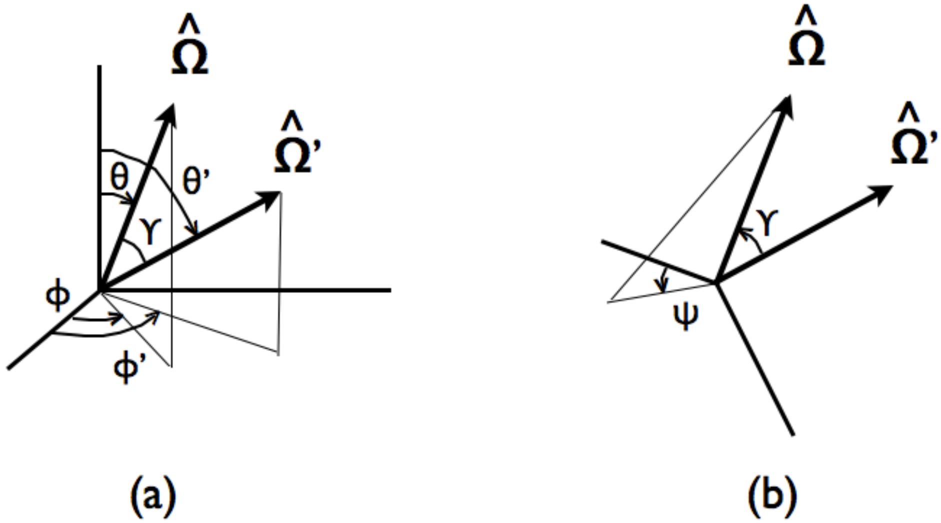


Figure 44: Panel (a): Coordinate system in which $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}'$ have coordinates (θ, ϕ) and (θ', ϕ'), respectively. γ is the angle between $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}'$. The vector $\hat{\mathbf{\Omega}}'$ is kept fixed, while $\hat{\mathbf{\Omega}}$ is allowed to vary. Panel (b): Same two unit vectors $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Omega}}'$ as in panel (a), but with respect to a coordinate system in which $\hat{\mathbf{\Omega}}'$ is the North Pole. In this coordinate system, $\hat{\mathbf{\Omega}}$ has coordinates (γ, ψ).

4) At times, we will think of γ as a function of (θ, ϕ). At other times, we will think of (θ, ϕ) as functions of (γ, ψ).

5) The Laplacian on the 2-sphere is invariant under rotations. Thus, if $f_l(\theta, \phi)$ is a eigenfunction of the (θ, ϕ)-Laplacian on the 2-sphere with eigenvalue l, then $f_l(\gamma, \psi) \equiv f_l(\theta(\gamma, \psi), \phi(\gamma, \psi))$ is an eigenfunction of the (γ, ψ)-Laplacian on the 2-sphere *with the same eigenvalue* l.

- Consider $P_l(\cos\gamma)$, and view it as a function of (θ, ϕ). Then we can write

$$P_l(\cos\gamma) = \sum_{m=-l}^{l} A_{lm} \, Y_{lm}(\theta, \phi) \tag{371}$$

- The fact that there is no sum over an l' index is a consequence of item 5 above as $P_l(\cos\gamma)$ is an eigenfunction of the (γ, ψ)-Laplacian on the 2-sphere (and hence also of the (θ, ϕ)-Laplacian) with eigenvalue l.

- Using the orthonormality of the spherical harmonics it follows that

$$A_{lm} = \int_{S^2} d\Omega_{\theta,\phi} \, P_l(\cos\gamma) Y^*_{lm}(\theta, \phi) \tag{372}$$

where

$$d\Omega_{\theta,\phi} \equiv d(\cos\theta) \, d\phi = \sin\theta \, d\theta \, d\phi \tag{373}$$

- Since

$$Y_{l0}(\gamma, \psi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\gamma) \tag{374}$$

we can also write

$$A_{lm} = \sqrt{\frac{4\pi}{2l+1}} \int_{S^2} d\Omega_{\theta,\phi}\, Y_{l0}(\gamma,\psi) Y^*_{lm}(\theta,\phi) \tag{375}$$

- By a similar argument (or by appealing to the transformation properties of the spherical harmonics under a rotation), we can write

$$Y_{lm}(\theta,\phi) = \sum_{m'=-l}^{l} B_{lm,m'} Y_{lm'}(\gamma,\psi) \tag{376}$$

where the (θ,ϕ) variables on the LHS are to be thought of a functions of (γ,ψ).

- The expansion coefficients are given by

$$B_{lm,m'} = \int_{S^2} d\Omega_{\gamma,\psi}\, Y_{lm}(\theta,\phi) Y^*_{lm'}(\gamma,\psi) \tag{377}$$

where

$$d\Omega_{\gamma,\psi} \equiv d(\cos\gamma)\, d\psi = \sin\gamma\, d\gamma\, d\psi \tag{378}$$

- If we consider $\hat{\mathbf{\Omega}}$ to point in the same direction as $\hat{\mathbf{\Omega}}'$, then $\gamma = 0$, which implies

$$Y_{lm}(\theta',\phi') = Y_{lm}(\theta,\phi)\big|_{\gamma=0} = \sum_{m'=-l}^{l} B_{lm,m'} Y_{lm'}(0,\psi) = B_{lm,0}\sqrt{\frac{2l+1}{4\pi}} \tag{379}$$

where the last equality follows from

$$Y_{lm}(0,\psi) = \begin{cases} \sqrt{\frac{2l+1}{4\pi}} & \text{if } m=0 \\ 0 & \text{otherwise} \end{cases} \tag{380}$$

- The integral expression for the $B_{lm,0}$ expansion coefficient is given by equation (377):

$$B_{lm,0} = \int_{S^2} d\Omega_{\gamma,\psi}\, Y_{lm}(\theta,\phi) Y^*_{l0}(\gamma,\psi) \tag{381}$$

- Comparing this with the integral expression for the A_{lm} expansion coefficient, we see that

$$A^*_{lm} = \sqrt{\frac{4\pi}{2l+1}} \int_{S^2} d\Omega_{\theta,\phi}\, Y^*_{l0}(\gamma,\psi) Y_{lm}(\theta,\phi) \tag{382}$$

$$= \sqrt{\frac{4\pi}{2l+1}} \int_{S^2} d\Omega_{\gamma,\psi}\, Y^*_{l0}(\gamma,\psi) Y_{lm}(\theta,\phi) \tag{383}$$

$$= \sqrt{\frac{4\pi}{2l+1}}\, B_{lm,0} \tag{384}$$

$$= \frac{4\pi}{2l+1} Y_{lm}(\theta',\phi') \tag{385}$$

where we used the rotational invariance of the area element on the 2-sphere, $d\Omega_{\theta,\phi} = d\Omega_{\gamma,\psi}$, to get the second equality.

- Thus,

$$A_{lm} = \frac{4\pi}{2l+1} Y^*_{lm}(\theta',\phi') \tag{386}$$

which gives us

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi) \tag{387}$$

as desired.

4.10 Separation of variables (cylindrical coords)

- In cylindrical polar coordinates (ρ, ϕ, z), Laplace's equation is

$$0 = \nabla^2 \Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} \tag{388}$$

- If one assumes the product form

$$\Phi(\rho, \phi, z) \equiv R(\rho) Q(\phi) Z(z) \tag{389}$$

  then Laplace's equation reduces to

$$0 = \frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{\rho^2}\frac{1}{Q}\frac{d^2 Q}{d\phi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} \tag{390}$$

  where we've divided the whole equation by $\Phi = RQZ$.

- If the separation constants are chosen so that

$$Z''(z) = k^2 Z(z)\,, \tag{391}$$

$$Q''(\phi) = -\nu^2 Q(\phi) \tag{392}$$

  then

$$R''(\rho) + \frac{1}{\rho} R'(\rho) + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R(\rho) = 0 \tag{393}$$

- If we choose the separation constants with the opposite sign for the z-equation:

$$Z''(z) = -k^2 Z(z)\,, \tag{394}$$

$$Q''(\phi) = -\nu^2 Q(\phi) \tag{395}$$

  then

$$R''(\rho) + \frac{1}{\rho} R'(\rho) - \left(k^2 + \frac{\nu^2}{\rho^2}\right) R(\rho) = 0 \tag{396}$$

- Exercise: Prove the above statements.

- The solutions to the z-equation with positive separation constant $+k^2$ are

$$Z(z) = A \sinh(kz) + B \cosh(kz) \tag{397}$$

  or, equivalently,

$$Z(z) = A'\, e^{kz} + B'\, e^{-kz} \tag{398}$$

  This choice of separation constant is needed when solving Laplace's equation with boundary condition $\Phi \to 0$ as $z \to \infty$, or if the potential is specified to have some non-zero value on the 2-d boundary surface $z =$ constant.

- The solutions to the z-equation with negative separation constant $-k^2$ are

$$Z(z) = A \sin(kz) + B \cos(kz) \tag{399}$$

  This choice of separation constant is needed when solving Laplace's equation with $\Phi = 0$ on the 2-d boundary surfaces $z = a$ and $z = b$, where a, b are finite constants.

- The solutions to the ϕ-equation are

$$Q(\phi) = C_0 + B_0 \phi\,, \quad \text{for } \nu = 0 \tag{400}$$

$$Q(\phi) = C \sin(\nu\phi) + D \cos(\nu\phi)\,, \quad \text{for } \nu \neq 0 \tag{401}$$

- If ϕ can take on the full range of values $\phi \in [0, 2\pi]$, then the requirement that $Q(\phi)$ be single-valued implies $B_0 = 0$ and ν equal an integer.

- The ρ-equations can be put into more standard form by making a change of variables $\rho \to x = k\rho$, with

$$y(x)\big|_{x=k\rho} \equiv R(\rho) \tag{402}$$

The two different equations corresponding to the different choice of sign for the separation constant $\pm k^2$ become

$$y''(x) + \frac{1}{x}\, y'(x) + \left(1 - \frac{\nu^2}{x^2}\right) y(x) = 0 \tag{403}$$

or

$$y''(x) + \frac{1}{x}\, y'(x) - \left(1 + \frac{\nu^2}{x^2}\right) y(x) = 0 \tag{404}$$

- Equation (403) is called *Bessel's equation of order* ν; equation (404) is called the *modified Bessel's equation of order* ν.

- Note that if $y(x)$ is a solution of Bessel's equation, then $\bar{y}(x) \equiv y(ix)$ is a solution of the modified Bessel's equation.

- Exercise: Prove the above.

4.10.1 Bessel functions

- To solve Bessel's equation, we note that $x = 0$ is a regular singular point of the differential equation. (The functions $p(x) \equiv 1/x$ and $q(x) \equiv (1 - \nu^2/x^2)$, which multiply $y'(x)$ and $y(x)$, respectively, are singular at $x = 0$, but $x\,p(x)$ and $x^2\,q(x)$ are both finite at $x = 0$.)

- The method of Frobenius says that such a differential equation will admit a power series solution of the form

$$y(x) = x^\sigma \sum_{n=0}^{\infty} a_n x^n \tag{405}$$

- Substituting this expansion into Bessel's equation and equating the coefficients multiplying like powers of x leads to

$$a_0(\sigma^2 - \nu^2) = 0 \tag{406}$$

$$a_1(1 + 2\sigma + \sigma^2 - \nu^2) = 0 \tag{407}$$

$$a_{n+2} = -\frac{a_n}{(n + 2 + \sigma)^2 - \nu^2} \tag{408}$$

- The first of the above equations is called the *indicial equation.* For $a_0 \neq 0$ it has the solutions:

$$\sigma = \pm\nu \tag{409}$$

- Substituting these solutions for σ into the second equation leads to

$$a_1(1 \pm 2\nu) = 0 \tag{410}$$

- For $\nu \neq \pm 1/2$, this equation implies $a_1 = 0$. But even for $\nu = \pm 1/2$, we can set $a_1 = 0$.

- Thus, $a_1 = 0$ together with the recurrence relation implies $a_n = 0$ for all *odd* values of n.

- For the even expansion coefficients, we can rewrite the recurrence relation as

$$a_{2n} = a_0\, \frac{(-1)^n \Gamma(1 + \nu)}{2^{2n} n! \Gamma(n + 1 + \nu)} \tag{411}$$

for $n = 0, 1, 2, \cdots$.

- Recall that the gamma function is defined by

$$\Gamma(z) = \int_0^\infty dx\, x^{z-1} e^{-x} \tag{412}$$

for $\mathrm{Re}(z) > 0$. The gamma function generalizes the factorial function to non-integer arguments in the sense that

$$\Gamma(n+1) = n! \quad \text{for } n = 0, 1, \cdots \tag{413}$$

$$\Gamma(z+1) = z\,\Gamma(z) \quad \text{for } \mathrm{Re}(z) > 0 \tag{414}$$

- If the normalization constant a_0 is chosen to be

$$a_0 = \frac{1}{2^\nu \Gamma(1+\nu)} \tag{415}$$

then

$$a_{2n} = \frac{(-1)^n}{2^{2n+\nu} n! \Gamma(n+1+\nu)} \tag{416}$$

- The power series solution is thus

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1+\nu)} \left(\frac{x}{2}\right)^{2n+\nu} \tag{417}$$

$J_\nu(x)$ is called *Bessel's function of the 1st kind.*

- Asymptotic form:

$$x \ll 1: \qquad J_\nu(x) \to \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \tag{418}$$

$$x \gg 1, \nu: \qquad J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{419}$$

- Thus, $J_0(0) = 1$, $J_\nu(0) = 0$ for all $\nu \neq 0$; while for large x, $J_\nu(x)$ behaves like a damped sinusoid, and has infinitely many zeros $x_{\nu n}$:

$$J(x_{\nu n}) = 0\,, \qquad \text{for } n = 1, 2, \cdots \tag{420}$$

See Figure 45.

- <u>Exercise</u>: Show that the zeros of $J_\nu(x)$ are given by

$$x_{\nu n} \simeq n\pi + \left(\nu - \frac{1}{2}\right)\frac{\pi}{2} \tag{421}$$

- If ν is not an integer, then $J_{-\nu}(x)$ is the second independent solution to Bessel's equation.
- If $\nu = m$ is an integer, then one can show that $J_{-m}(x)$ is proportional to $J_m(x)$:

$$J_{-m}(x) = (-1)^m J_m(x) \tag{422}$$

so $J_{-m}(x)$ is not an independent solution for this case.

- <u>Exercise</u>: Prove the above.
- A second solution, which is independent of $J_\nu(x)$ for all ν (integer or not), is

$$N_\nu(x) := \frac{J_\nu(x)\,\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \tag{423}$$

$N_\nu(x)$ is called a *Neumann function* or *Bessel's function of the 2nd kind*, and is sometimes denoted by $Y_\nu(x)$.

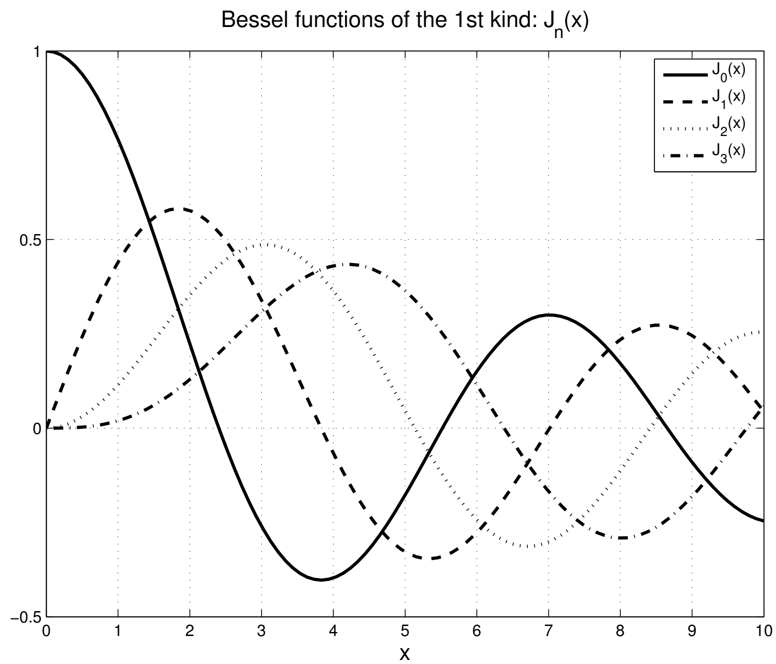


Figure 45: First few Bessel functions of the 1st kind for integer ν.

- For $\nu = m$ an integer, one needs to use L'Hopital's rule to show that the RHS of the expression defining $N_m(x)$ is well-defined.
- Asymptotic form:

$$x \ll 1: \qquad N_\nu(x) \to \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + 0.5772\cdots\right], & \nu = 0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu, & \nu \neq 0 \end{cases} \tag{424}$$

$$x \gg 1, \nu: \qquad N_\nu(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{425}$$

- Note that for all ν, $N_\nu(x) \to -\infty$ as $x \to 0$.
- As we saw for $J_\nu(x)$, for large x, $N_\nu(x)$ behaves like a damped sinusoid, 90° out of phase with $J_\nu(x)$. See Figure 46.
- Thus, the most general solution to the radial part of Laplace's equation is

$$R(\rho) = A\, J_\nu(k\rho) + B\, N_\nu(k\rho) \tag{426}$$

- Since $N_\nu(x)$ blows up at $x = 0$, if $\rho = 0$ is in the region of interest, then all of the B coefficients must vanish to yield a finite value of the potential on the axis.
- *Hankel functions* (or *Bessel functions of the 3rd kind*) are defined by

$$H_\nu^{(1)}(x) := J_\nu(x) + iN_\nu(x) \tag{427}$$

$$H_\nu^{(2)}(x) := J_\nu(x) - iN_\nu(x) \tag{428}$$

- *Modified* (or *hyperbolic*) *Bessel functions* are defined by

$$I_\nu(x) := i^{-\nu} J_\nu(ix) \tag{429}$$

$$K_\nu(x) := \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \tag{430}$$

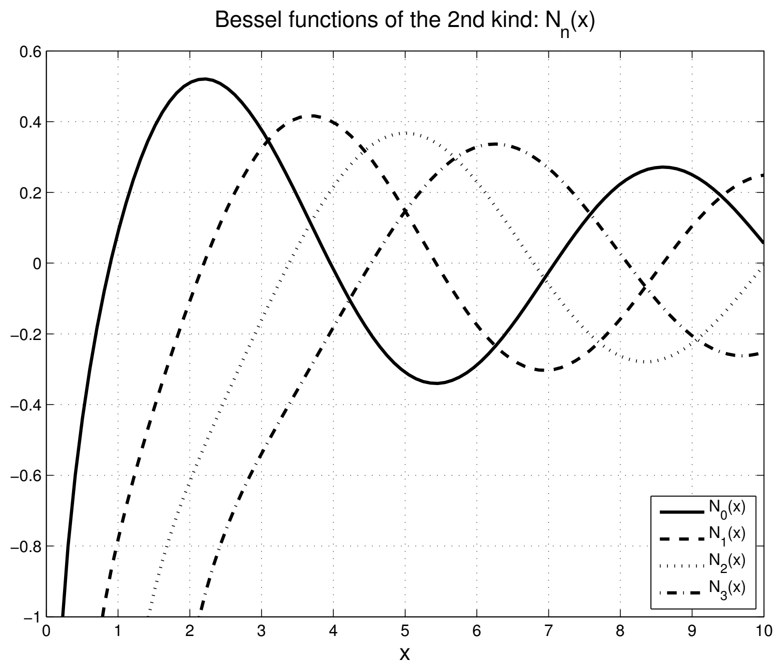


Figure 46: First few Bessel functions of the 2nd kind for integer ν.

Note the pure imaginary arguments on the RHS.

- These are two linearly independent solutions of the modified Bessel's equation (404). See Figures 47 and 48.
- Asymptotic form:

$$x \ll 1: \qquad I_\nu(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \tag{431}$$

$$K_\nu(x) \to \begin{cases} -\left[\ln\left(\frac{x}{2}\right) + 0.5772\cdots\right], & \nu = 0 \\ \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu, & \nu \neq 0 \end{cases} \tag{432}$$

$$x \gg 1, \nu: \qquad I_\nu(x) \to \frac{1}{\sqrt{2\pi x}}\, e^x \left[1 + O\left(\frac{1}{x}\right)\right] \tag{433}$$

$$K_\nu(x) \to \sqrt{\frac{\pi}{2x}}\, e^{-x} \left[1 + O\left(\frac{1}{x}\right)\right] \tag{434}$$

- Thus, $I_0(0) = 1$, $I_\nu(0) = 0$ for all $\nu \neq 0$, while $K_\nu(x) \to \infty$ as $x \to 0$ for all ν.
- For large x, $I_\nu(x) \to \infty$ while $K_\nu(x) \to 0$ for all ν.
- Thus, the most general solution to the radial part of Laplace's equation for the choice of negative separation constant $-k^2$ is

$$R(\rho) = A\, I_\nu(k\rho) + B\, K_\nu(k\rho) \tag{435}$$

- Since $K_\nu(x)$ blows up at $x = 0$, if $\rho = 0$ is in the region of interest, then all of the B coefficients must vanish to yield a finite value of the potential on the axis.
- Similarly, since $I_\nu(x)$ blows up as $x \to \infty$, if the potential is to vanish as $\rho \to \infty$, then all of the A coefficients must vanish.

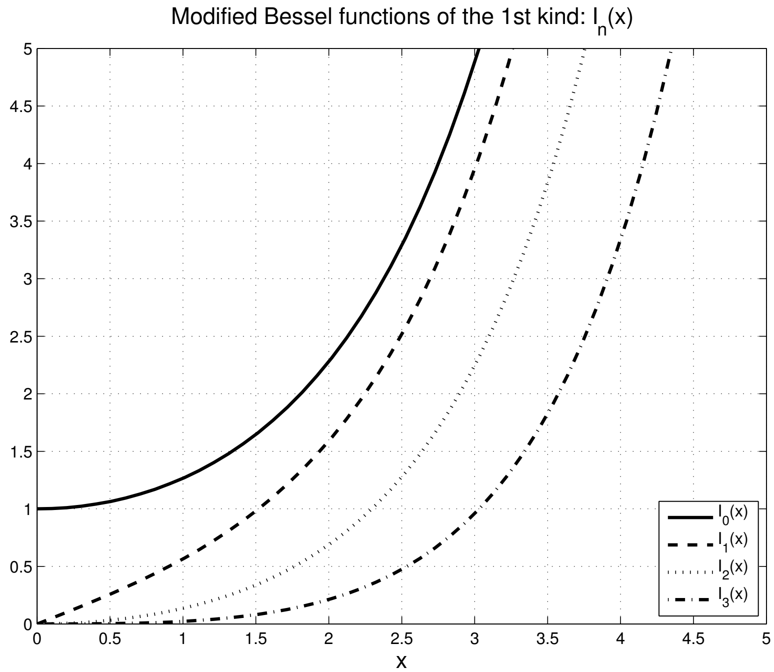


Figure 47: First few modified Bessel functions of the 1st kind for integer ν.

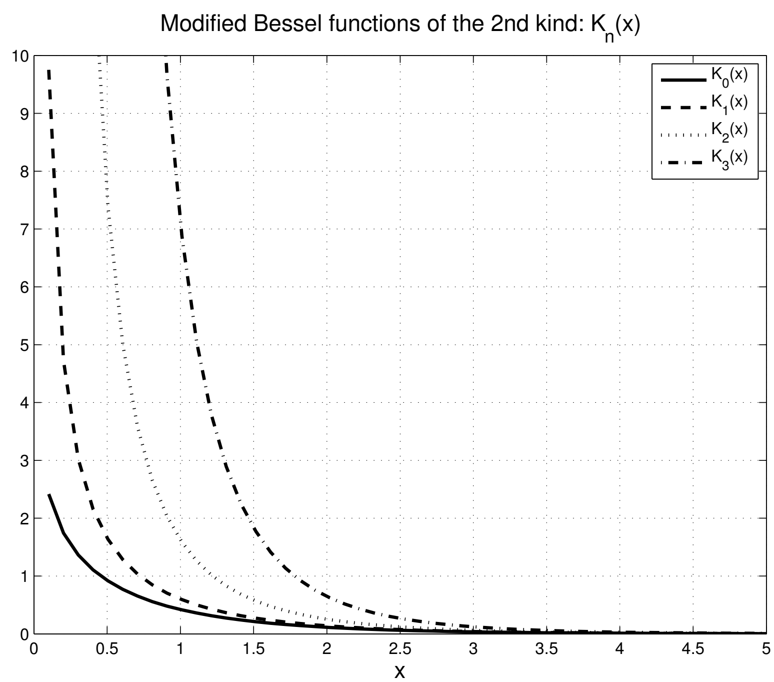


Figure 48: First few modified Bessel functions of the 2nd kind for integer ν.

- Recurrence relations:

$$\frac{d}{dx}\left(x^{\nu} J_{\nu}(x)\right) = x^{\nu} J_{\nu-1}(x) \tag{436}$$

$$\frac{d}{dx}\left(x^{-\nu} J_{\nu}(x)\right) = -x^{-\nu} J_{\nu+1}(x) \tag{437}$$

$$J'_{\nu}(x) = -\frac{\nu}{x} J_{\nu}(x) + J_{\nu-1}(x) \tag{438}$$

$$J'_{\nu}(x) = \frac{\nu}{x} J_{\nu}(x) - J_{\nu+1}(x) \tag{439}$$

$$2J'_{\nu}(x) = J_{\nu-1}(x) - J_{\nu+1}(x) \tag{440}$$

$$\frac{2\nu}{x} J_{\nu}(x) = J_{\nu-1}(x) + J_{\nu+1}(x) \tag{441}$$

- Exercise: Prove the above.

- Note that the recurrence relations also hold for $N_{\nu}(x)$, $H_{\nu}^{(1)}(x)$, $H_{\nu}^{(2)}(x)$, since they are relatively simple linear combinations of $J_{\nu}(x)$ and $J_{-\nu}(x)$.

- Orthogonality:

$$\int_a^b d\rho\, \rho J_{\nu}(k\rho) J_{\nu}(k'\rho) = 0 \quad \text{for } k \neq k' \tag{442}$$

where

$$\rho\left[J_{\nu}(k\rho)\frac{dJ_{\nu}}{d\rho}(k'\rho) - J_{\nu}(k'\rho)\frac{dJ_{\nu}}{d\rho}(k\rho)\right]\Bigg|_{\rho=a}^{b} = 0 \tag{443}$$

- Exercise: Prove this. (Hint: Let $f(\rho) = J_{\nu}(k\rho)$, $g(\rho) = J_{\nu}(k'\rho)$, write down Bessel's equation for f and g, multiply these equations by g and f, subtract, and then integrate.)

- An explicit example satifying the above boundary condition is to choose $a = 0$, rename $b = a$, and then choose k and k' so that $J_{\nu}(ka) = 0 = J_{\nu}(k'a)$. For this case k and k' take on discrete values

$$k \equiv k_{\nu n} = \frac{x_{\nu n}}{a}\,, \qquad k' \equiv k_{\nu n'} = \frac{x_{\nu n'}}{a}\,, \qquad n, n' = 1, 2, \cdots \tag{444}$$

where $x_{\nu n}$ and $x_{\nu n'}$ are the nth and n'th zeroes of $J_{\nu}(x)$.

- Note that the orthogonality of Bessel functions is wrt to different arguments $x = k\rho$ and $x' = k'\rho$ of a *single* function $J_{\nu}(x)$, and not wrt *different* functions $J_{\nu}(x)$ and $J_{\nu'}(x)$ of the same argument $x = k\rho$. (This latter case held for the Legendre polynomials $P_l(x)$ and $P_{l'}(x)$.)

- The orthogonality of Bessel functions is similar to the orthogonality of the sine functions $\sin(n2\pi x/a)$ for different values of n.

- Normalization:

$$\int_a^b d\rho\, \rho J_{\nu}(k\rho) J_{\nu}(k\rho) = \frac{1}{2}\left[\left(\rho^2 - \frac{\nu^2}{k^2}\right) J_{\nu}^2(k\rho) + \rho^2 [J'_{\nu}(k\rho)]^2\right]\Bigg|_{\rho=a}^{b} \tag{445}$$

where $J'_{\nu}(k\rho)$ denotes derivative wrt to its argument $x = k\rho$.

- Exercise: Prove the normalization condition. (Note: This is a rather tricky proof, requiring some clever integration by parts and the use of Bessel's equation to substitute for $x^2 J_{\nu}(x)$ in one of the integrals.)

- Again the RHS can be simplified for the case described above where we set $a = 0$, rename $b = a$, and take $k = x_{\nu n}/a$ for some integer n. Then

$$\int_0^a d\rho\, \rho J_\nu(x_{\nu n}\rho/a) J_\nu(x_{\nu n}\rho/a) = \frac{1}{2}a^2[J'_\nu(x_{\nu n})]^2 = \frac{1}{2}a^2 J^2_{\nu+1}(x_{\nu n}) \tag{446}$$

where a recurrence relation was used to get the last equality.

- Exercise: Prove this.

- We can put the orthogonality and normalisation equations together as a single equation:

$$\int_0^a d\rho\, \rho J_\nu(x_{\nu n}\rho/a) J_\nu(x_{\nu n'}\rho/a) = \frac{1}{2}a^2 J^2_{\nu+1}(x_{\nu n})\, \delta_{nn'} \tag{447}$$

where we have explicitly indicated the zeroes $x_{\nu n}$ and $x_{\nu n'}$ of $J_\nu(x)$.

- If the interval $[0, a]$ becomes infinite $[0, \infty)$, then the orthogonality and normalisation conditions actually become simpler

$$\int_0^\infty d\rho\, \rho J_\nu(k\rho) J_\nu(k'\rho) = \frac{1}{k}\delta(k - k') \tag{448}$$

where k now takes on a continuous range of values.

- This is similar to the transition from Fourier series (basis functions $e^{ik_n x}$ with $k_n = n2\pi/a$) to Fourier transforms (basis functions e^{ikx} with k a real variable):

$$\int_{-a/2}^{a/2} dx\, e^{i2\pi(n-n')x/a} = a\, \delta_{nn'} \longrightarrow \int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = 2\pi\, \delta(k - k') \tag{449}$$

4.10.2 Spherical Bessel functions

- *Spherical Bessel functions* are defined by

$$j_n(x) := \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x) \tag{450}$$

$$n_n(x) := \sqrt{\frac{\pi}{2x}}\, N_{n+\frac{1}{2}}(x) \tag{451}$$

where $n = 0, 1, 2, \cdots$.

- One can also define

$$h^{(1)}_n(x) := j_n(x) + i n_n(x) \tag{452}$$

$$h^{(2)}_n(x) := j_n(x) - i n_n(x) \tag{453}$$

- Given the explicit form of $J_{n+\frac{1}{2}}(x)$ one can show that

$$j_n(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right) \tag{454}$$

$$n_n(x) = -x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right) \tag{455}$$

- In particular, it follows that

$$j_0(x) = \frac{\sin x}{x}\,, \qquad n_0(x) = -\frac{\cos x}{x} \tag{456}$$

See Figures 49 and 50 for plots of the first few spherical Bessel functions.

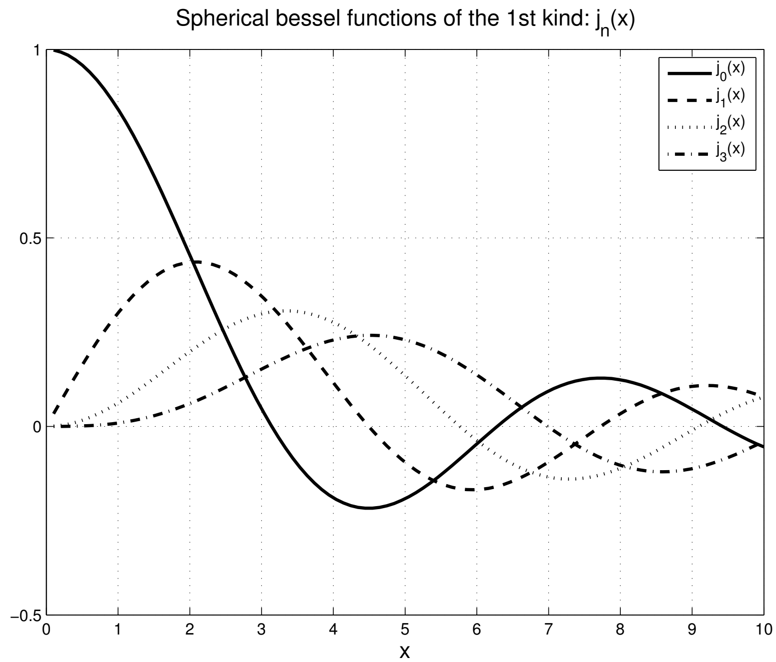


Figure 49: First few spherical Bessel functions of the 1st kind.

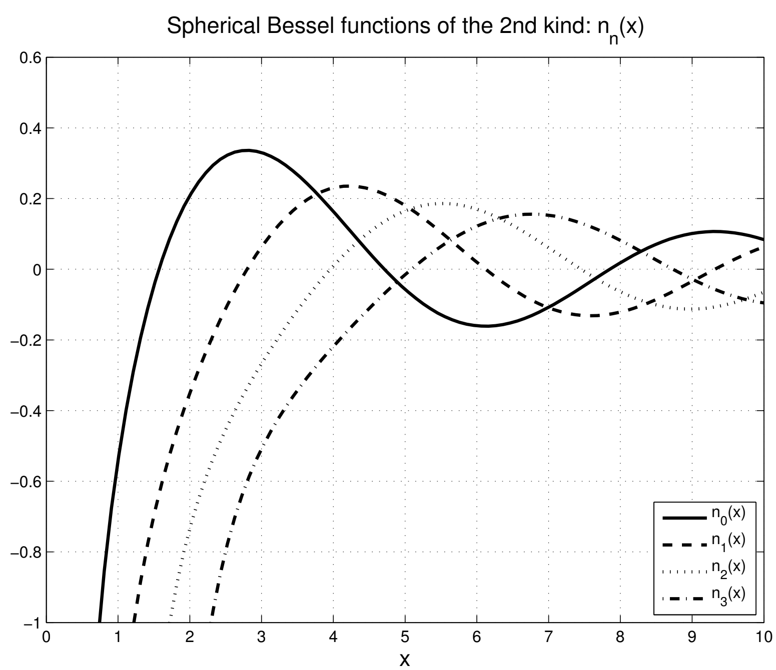


Figure 50: First few spherical Bessel functions of the 2nd kind.

- Exercise: Prove the above expression for $j_0(x)$ directly from its definition in terms of the ordinary Bessel function $J_{\frac{1}{2}}(x)$.

- Given the relationship between $j_n(x)$ and $J_{n+\frac{1}{2}}(x)$, one can show that the spherical Bessel functions satisfy the differential equation

$$j_n''(x) + \frac{2}{x}j_n'(x) + \left[1 - \frac{n(n+1)}{x^2}\right] j_n(x) = 0 \tag{457}$$

- Exercise: Prove this.

- Alternatively, one arrives at the same differential equation by using separation of variables in *spherical polar coordinates* to solve the *Helmholtz equation*:

$$\nabla^2\Phi(r,\theta,\phi) + k^2\Phi(r,\theta,\phi) = 0 \tag{458}$$

- The ϕ equation is the standard harmonic oscillator equation with separation constant $-m^2$; the θ equation is the associated Legendre's equation with separation constants l and m; and the radial equation is

$$R''(r) + \frac{2}{r}R'(r) + \left[k^2 - \frac{l(l+1)}{r^2}\right] R(r) = 0 \tag{459}$$

- Making the change of variables $x = kr$ with $y(x)|_{x=kr} = R(r)$, leads to

$$y''(x) + \frac{2}{x}y'(x) + \left[1 - \frac{l(l+1)}{x^2}\right] y(x) = 0 \tag{460}$$

  which is the differential equation (457) we found earlier with solution $y(x) = j_l(x)$.

4.10.3 Examples

- Example 1: Solve Laplace's equation interior to a cylinder of radius a and height L, with zero potential on the bottom and sides of the cylinder, and specified potential $f(\rho,\phi)$ on the top. (See Figure 51.)

- Answer:

  Choose cylindrical polar coordinates so that the axis of the cylinder coincides with the $\hat{\mathbf{z}}$ axis, and that the bottom and top of the cylinder have $z = 0$ and $z = L$ respectively.

  Note:

  1) The fact the potential has a non-zero value on the top of the cylinder implies using a positive separation constant $+k^2$ for the z-equation.

  2) The BC that the potential vanish at $z = 0$ implies that there are no $\cosh(kz)$ terms.

  3) Single-valuedness of $\Phi(\phi)$ implies that $\nu = m$ is an integrer.

  4) Finiteness at the axis ($\rho = 0$) implies that there are no $N_m(x)$ terms in the general solution to Bessel's equation.

  5) The BC that the potential vanish when $\rho = a$ implies that k take on the discrete values $k_{mn} = x_{mn}/a$, where x_{mn} is the nth zero $J_m(x)$.

  Thus,

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(x_{mn}\rho/a)\sinh(x_{mn}z/a)\left[A_{mn}\,\sin(m\phi) + B_{mn}\,\cos(m\phi)\right] \tag{461}$$

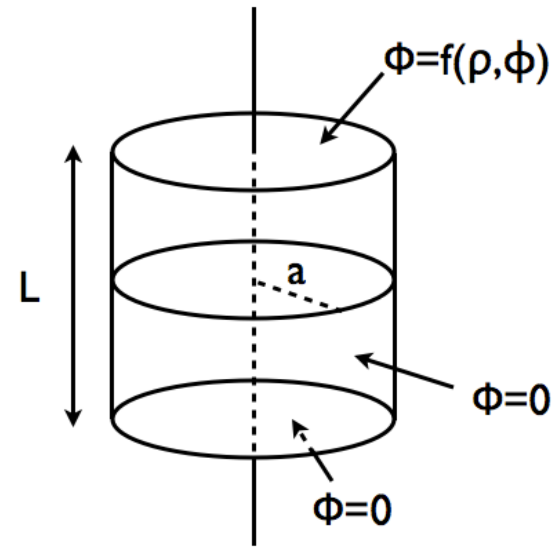


Figure 51: Cylinder of radius a and height L, with zero potential on the bottom and sides, and specified potential $\Phi = f(\rho, \phi)$ on the top.

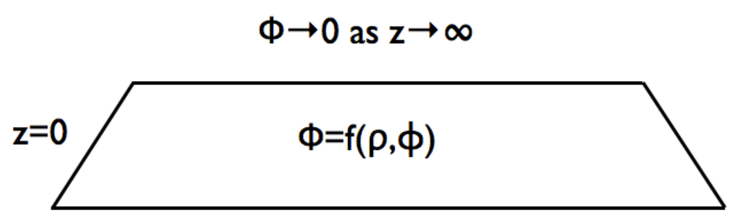


Figure 52: Infinite 2-d plane ($z = 0$) with specified potential $\Phi = f(\rho, \phi)$ on the plane and $\Phi \to 0$ as $z \to \infty$.

where the BC that $\Phi(\rho, \phi, L) = f(\rho, \phi)$ implies

$$A_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(x_{mn}) \sinh(x_{mn} L/a)} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho f(\rho, \phi) J_m(x_{mn}\rho/a) \sin(m\phi) \quad (462)$$

$$B_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(x_{mn}) \sinh(x_{mn} L/a)} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho f(\rho, \phi) J_m(x_{mn}\rho/a) \cos(m\phi) \quad (463)$$

Note that for $m = 0$, the term $B_{mn} \cos(m\phi)$ should be $B_{0n}/2$ where B_{0n} is calculated from the above integral.

- Example 2: Solve Laplace's equation above an infinite 2-d plane with specified potential $f(\rho, \phi)$ and vanishing potential infinitely far from the plane. (See Figure 52.)
- Answer:

  Choose cylindrical polar coordinates so that the 2-d plane corresponds to $z = 0$.

  Note:

1) The fact the range of z extends to ∞ implies using a positive separation constant $+k^2$ for the z-equation.

2) The BC that the potential vanishes as $z \to \infty$ implies that there are no e^{+kz} terms.

3) Single-valuedness of $\Phi(\phi)$ implies that $\nu = m$ is an integrer.

4) Finiteness at the axis ($\rho = 0$) implies that there are no $N_m(x)$ terms in the general solution to Bessel's equation.

Thus,

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, J_m(k\rho) e^{-kz} \left[A_m(k)\, \sin(m\phi) + B_m(k)\, \cos(m\phi)\right] \tag{464}$$

where the BC that $\Phi(\rho, \phi, 0) = f(\rho, \phi)$ implies

$$A_m(k) = \frac{k}{\pi} \int_0^{2\pi} d\phi \int_0^{\infty} d\rho\, \rho f(\rho, \phi) J_m(k\rho) \sin(m\phi) \tag{465}$$

$$B_m(k) = \frac{k}{\pi} \int_0^{2\pi} d\phi \int_0^{\infty} d\rho\, \rho f(\rho, \phi) J_m(k\rho) \cos(m\phi) \tag{466}$$

Note that for $m = 0$, the term $B_m(k)\cos(m\phi)$ should be $B_0(k)/2$ where $B_0(k)$ is calculated from the above integral.

- Such expansions are called *Fourier-Bessel expansions.*

4.11 Eigenfunction expansions of Green's functions

- The general form of a homogeneous, linear, second-order ordinary differential equation is

$$y''(x) + p(x)\, y'(x) + q(x)\, y(x) = 0 \tag{467}$$

where $p(x)$ and $q(x)$ are arbitrary functions of x.

- The differential equation is said to be in *self-adjoint* (or Hermitian) form if

$$\frac{d}{dx}\left[f(x)\frac{dy}{dx}\right] + g(x)\, y(x) = 0 \tag{468}$$

- Exercise: Prove that *any* homogeneous, linear, second-order ordinary differential equation can be put into self-adjoint form by multiplying the equation (467) by $\exp[\int^x p(x')dx']$. You should obtain equation (468) with

$$f(x) = \exp\left[\int^x p(x')\, dx'\right], \quad g(x) = \exp\left[\int^x p(x')\, dx'\right] q(x) \tag{469}$$

- All of the equations that we are interested in (e.g., Legendre's equation, Bessel's equation, the radial and angular parts of Laplace's equation in spherical polar coordinates, etc.) are already in self-adjoint form.

- Let $y_1(x)$ and $y_2(x)$ be two solutions of equation (467) or (468). Then $y_1(x)$ and $y_2(x)$ are linearly independent if and only if the *Wronskian*

$$W(x) \equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) \tag{470}$$

is non-zero.

- The Wronskian itself satisfies a differential equation

$$W'(x) = -p(x)\,W(x) \tag{471}$$

which can be obtained by differentiating the equation that defines W and then using equation (467). to substitute for y_1'' and y_2''.

- Exercise: Prove the above.

- From the above equation for W it follows that

$$W(x) = C\exp\left[-\int^x p(x')\,dx'\right] = \frac{C}{f(x)} \tag{472}$$

(Note: We will need to use this result later on when expanding Green's functions in cylindrical polar coordinates.)

- Consider the region $x \in [a,b]$ (where a, b could be infinite), and define an *inner product*

$$\langle y_1|y_2\rangle \equiv \int_a^b y_1^*(x)y_2(x)\,w(x)dx \tag{473}$$

Here $*$ denotes complex conjugation and $w(x) > 0$ is a (real) *weight* function.

- For most cases $w = 1$, but for Legendre polynomials and Bessel functions, the weight functions are $w(\theta) = \sin\theta$ and $w(\rho) = \rho$, respectively.

- Orthogonality and normalization of functions is defined wrt the inner product.

- A linear operator $\mathcal{L}$ is said to be *self-adjoint* (or Hermitian) wrt the inner product if and only if

$$\langle \mathcal{L}y_1|y_2\rangle = \langle y_1|\mathcal{L}y_2\rangle \tag{474}$$

or, equivalently,

$$\int_a^b [\mathcal{L}y_1(x)]^*\,y_2(x)w(x)\,dx = \int_a^b y_1^*(x)\,[\mathcal{L}y_2(x)]\,w(x)\,dx \tag{475}$$

- Consider the linear differential operator $\mathcal{L}$ defined by

$$\mathcal{L}y(x) \equiv \frac{1}{w(x)}\left\{\frac{d}{dx}\left[f(x)\frac{dy}{dx}\right] + g(x)\,y(x)\right\} \tag{476}$$

which is proportional to the LHS of the self-adjoint form (468) of a homogeneous, linear, second-order ordinary differential equation.

- One can show that $\mathcal{L}$ is Hermitian wrt the above inner product iff

$$[y_1^* f y_2' - y_2 f y_1^{*\prime}]\,\Big|_a^b = 0 \tag{477}$$

- Exercise: Prove the above result assuming $f(x)$ and $g(x)$ are real.

- In particular, $\mathcal{L}$ will be Hermitian if the functions $y_1(x)$ and $y_2(x)$ vanish on the boundary $x = a, b$. Such solutions are said to satisfy *homogeneous* boundary conditions, and we will assume such BCs when we calculate Dirichlet Green's functions. (Other boundary conditions are possible, but we will not consider them here.)

- A *Sturm-Liouville* equation has the form

$$\frac{d}{dx}\left[f(x)\frac{dy}{dx}\right] + g(x)\,y(x) - \lambda\,w(x)y(x) = 0 \tag{478}$$

where λ is some fixed constant.

- In terms of the linear operator $\mathcal{L}$, the Sturm-Liouville equation can be written as

$$\mathcal{L}y(x) = \lambda\, y(x) \tag{479}$$

which is an eigenvalue equation.

- In order that $\mathcal{L}$ be Hermitian, the homogeneous boundary conditions on the eigenfunctions restrict the allowed values of λ:

$$\mathcal{L}\psi_n(x) = \lambda_n\, \psi_n(x) \tag{480}$$

- Recall that for a Hermitian operator:

  1) the eigenvalues λ_n are real.

  2) the eigenfunctions $\psi_n(x)$ and $\psi_{n'}(x)$ corresponding to distinct eigenvalues are orthogonal.

  3) the set of eigenfunctions $\{\psi_n(x)\}$ span the space of square-integrable functions satisfying the same BCs as the eigenfunctions.

- Several examples of differential equations, their eigenfunctions, and eigenvalues are given in the next subsection, Section 4.12.

- Suppose we want to solve the *inhomogeneous* equation

$$\mathcal{L}y(x) = \frac{F(x)}{w(x)} \tag{481}$$

where $F(x)$ is some source term.

- By expanding $y(x)$ in terms of the eigenfunctions

$$y(x) = \sum_n A_n\, \psi_n(x) \tag{482}$$

one can show that

$$y(x) = \int_a^b dx' F(x') \left(\sum_n \frac{\psi_n^*(x')\psi_n(x)}{N_n \lambda_n} \right) \tag{483}$$

where

$$N_n := \int_a^b |\psi_n(x)|^2\, w(x) dx \tag{484}$$

is the normalization of the eigenfunction $\psi_n(x)$.

- <u>Exercise</u>: Prove the above.

- Recalling that a Dirichlet Green's function satisfies

$$\mathcal{L}G_D(x,x') = \delta(x-x') \Longleftrightarrow y(x) = \int_a^b dx'\, G_D(x,x')F(x') \tag{485}$$

we can conclude that

$$G_D(x,x') = \sum_n \frac{\psi_n^*(x')\psi_n(x)}{N_n \lambda_n} \tag{486}$$

- NOTES:

  1) The summation $\sum_n$ should be replaced by an integration $\int dk$ if the eigenvalues are labeled by a continuous index k.

2) The above result extends to functions of many variables:

$$G_D(\mathbf{r},\mathbf{r}') = \sum_n \frac{\psi_n^*(\mathbf{r}')\psi_n(\mathbf{r})}{N_n\lambda_n} \tag{487}$$

3) It also extends to solutions of the more general differential equation

$$\mathcal{L}y(x) - \lambda\, y(x) = \frac{F(x)}{w(x)} \tag{488}$$

were λ is a constant, not equal to any of the eigenvalues λ_n. The expression for the Dirichlet Green's function for this case is

$$G_D(x,x') = \sum_n \frac{\psi_n^*(x')\psi_n(x)}{N_n(\lambda_n - \lambda)} \tag{489}$$

- Exercise: Prove this last statement.
- Example 1: The Dirichlet Green's function for Poisson's equation in 1-dimension

$$\frac{d^2}{dx^2}G_D(x,x') = \delta(x-x') \tag{490}$$

for the finite interval $0 \le x \le a$, can be expanded in terms of sinusoids

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \tag{491}$$

with eigenvalues

$$\lambda_n = -\left(\frac{n\pi}{a}\right)^2 \tag{492}$$

Thus,

$$G_D(x,x') = \frac{2}{a}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)}{-\left(\frac{n\pi}{a}\right)^2} \tag{493}$$

- Example 2: The infinite space Dirichlet Green's function for Poisson's equation

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{494}$$

can be expanded in terms of the complex exponentials

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\, e^{i\mathbf{k}\cdot\mathbf{r}} \tag{495}$$

where $\mathbf{k}$ is a vector in 3-dimensional space, and the normalization factor has been chosen so that $N_{\mathbf{k}} = 1$. These are eigenfunctions of the 3-d Laplacian operator $\mathcal{L} = \nabla^2$ with eigenvalues

$$\lambda_{\mathbf{k}} = -k^2 \tag{496}$$

Thus,

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{4\pi}{(2\pi)^3}\int_{\text{all space}} dV_{\mathbf{k}}\, \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2} \tag{497}$$

Note that the extra factor of 4π and the absence of the minus sign in the denominator come from the definition (494) of the Green's function in 3-dimensions.

- Example 3: The Dirichlet Green's for Poisson's equation inside a rectangular box ($0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$) can be expanded in terms of a product of sinusoids:

$$\psi_{lmn}(\mathbf{r}) = \sqrt{\frac{8}{abc}}\,\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi z}{c}\right) \tag{498}$$

These are eigenfunctions of the Laplacian with eigenvalues

$$\lambda_{lmn} = -\left[\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2\right] \tag{499}$$

Thus,

$$G_D(\mathbf{r},\mathbf{r}') = \frac{32\pi}{abc}\sum_{l,m,n=1}^{\infty}\frac{\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)}{\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2} \tag{500}$$

4.12 Summary of key eigenfunction formulas

- Cartesian coordinate:

$$-\infty < x < \infty \quad \text{(similar for } y, z\text{)} \tag{501}$$

Differential equation:

$$\frac{d^2\psi}{dx^2} = \lambda\,\psi(x) \tag{502}$$

Weight function:

$$w(x) = 1 \tag{503}$$

Eigenfunctions:

$$\psi_k(x) = e^{ikx}\,, \qquad -\infty < k < \infty \tag{504}$$

Eigenvalues:

$$\lambda_k = -k^2 \tag{505}$$

Orthonormality:

$$\int_{-\infty}^{\infty} dx\; e^{i(k-k')x} = 2\pi\,\delta(k-k') \tag{506}$$

Completeness:

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ik(x-x')} = \frac{1}{\pi}\int_0^{\infty} dk\;\cos[k(x-x')] \tag{507}$$

- Cartesian coordinate:

$$0 \le x \le a \quad \text{(similar for } y, z\text{)} \tag{508}$$

Differential equation:

$$\frac{d^2\psi}{dx^2} = \lambda\,\psi(x) \tag{509}$$

Weight function:

$$w(x) = 1 \tag{510}$$

Eigenfunctions:

$$\psi_n(x) = \sin\left(\frac{n\pi x}{a}\right)\,, \qquad n = 1, 2, \cdots \tag{511}$$

Eigenvalues:

$$\lambda_n = -\left(\frac{n\pi}{a}\right)^2 \tag{512}$$

Orthonormality:

$$\int_0^a dx\, \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n'\pi x}{a}\right) = \frac{a}{2}\,\delta_{nn'} \tag{513}$$

Completeness:

$$\delta(x - x') = \frac{2}{a} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right) \tag{514}$$

- Azimuthal angle:

$$0 \le \phi \le 2\pi \tag{515}$$

Differential equation:

$$\frac{d^2\psi}{d\phi^2} = \lambda\,\psi(\phi) \tag{516}$$

Weight function:

$$w(\phi) = 1 \tag{517}$$

Eigenfunctions:

$$\psi_m(\phi) = e^{im\phi}\,, \qquad m = 0, \pm 1, \pm 2, \cdots \tag{518}$$

Eigenvalues:

$$\lambda_m = -m^2 \tag{519}$$

Orthonormality:

$$\int_0^{2\pi} d\phi\, e^{i(m-m')\phi} = 2\pi\,\delta_{mm'} \tag{520}$$

Completeness:

$$\delta(\phi - \phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{m=1}^{\infty} \cos[m(\phi - \phi')] \tag{521}$$

- Polar angle:

$$0 \le \theta \le \pi \quad (\text{or } \; -1 \le x \le 1\,, \quad \text{where } x = \cos\theta) \tag{522}$$

Differential equation:

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dP}{d\theta}\right) = \lambda\,P(\theta) \tag{523}$$

Weight function:

$$w(\theta) = \sin\theta \tag{524}$$

Eigenfunctions:

$$P_l(\cos\theta)\,, \qquad l = 0, 1, 2, \cdots \tag{525}$$

Eigenvalues:

$$\lambda_l = -l(l+1) \tag{526}$$

Orthonormality:

$$\int_0^{\pi} d\theta\, \sin\theta P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{2}{2l+1}\delta_{ll'} \tag{527}$$

Completeness:

$$\frac{1}{\sin\theta}\delta(\theta - \theta') = \frac{2l+1}{2} \sum_{l=0}^{\infty} P_l(\cos\theta) P_l(\cos\theta') \tag{528}$$

- 2-sphere coordinates:

$$0 \le \theta \le \pi\,, \quad 0 \le \phi \le 2\pi \tag{529}$$

Differential equation:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = \lambda\, Y(\theta,\phi) \tag{530}$$

Weight function:

$$w(\theta,\phi) = \sin\theta \tag{531}$$

Eigenfunctions:

$$Y_{lm}(\theta,\phi)\,, \qquad l = 0,1,2,\cdots\,, \qquad m = -l,-l+1,\cdots,l \tag{532}$$

Eigenvalues:

$$\lambda_{lm} = -l(l+1) \tag{533}$$

Orthonormality:

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, Y_{lm}(\theta,\phi) Y^*_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \tag{534}$$

Completeness:

$$\frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \tag{535}$$

- Cylindrical radius:

$$0 \le \rho \le a \tag{536}$$

Differential equation:

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - \frac{\nu^2}{\rho^2}R(\rho) = \lambda\, R(\rho)\,, \quad \nu \text{ real (fixed)} \tag{537}$$

Weight function:

$$w(\rho) = \rho \tag{538}$$

Eigenfunctions:

$$R_n(\rho) = J_\nu(x_{\nu n}\rho/a)\,, \qquad n = 1,2,\cdots \tag{539}$$

Eigenvalues:

$$\lambda_n = -\frac{x_{\nu n}^2}{a^2} \tag{540}$$

Orthonormality:

$$\int_0^a d\rho\, \rho J_\nu(x_{\nu n}\rho/a) J_\nu(x_{\nu n'}\rho/a) = \frac{1}{2}a^2 J^2_{\nu+1}(x_{\nu n})\,\delta_{nn'} \tag{541}$$

Completeness:

$$\frac{1}{\rho}\,\delta(\rho-\rho') = \frac{2}{a^2}\sum_{n=1}^{\infty}\frac{J_\nu(x_{\nu n}\rho/a)J_\nu(x_{\nu n}\rho'/a)}{J^2_{\nu+1}(x_{\nu n})} \tag{542}$$

- Cylindrical radius:

$$0 \le \rho < \infty \tag{543}$$

Differential equation:

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - \frac{\nu^2}{\rho^2}R(\rho) = \lambda\, R(\rho)\,, \quad \nu \text{ real (fixed)} \tag{544}$$

Weight function:

$$w(\rho) = \rho \tag{545}$$

Eigenfunctions:

$$R_k(\rho) = J_\nu(k\rho)\,, \qquad k \geq 0 \tag{546}$$

Eigenvalues:

$$\lambda_k = -k^2 \tag{547}$$

Orthonormality:

$$\int_0^\infty d\rho\, \rho J_\nu(k\rho) J_\nu(k'\rho) = \frac{1}{k}\,\delta(k-k') \tag{548}$$

Completeness:

$$\frac{1}{\rho}\,\delta(\rho-\rho') = \int_0^\infty dk\, k J_\nu(k\rho) J_\nu(k\rho') \tag{549}$$

4.13 Expanding Green's functions by solving 1-d δ-function equations

- An alternative method of calculating a Green's function is to expand the equation

$$\nabla^2 G_D(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{550}$$

with respect to two of the three coordinates (e.g., x, y, or θ, ϕ, etc.) and solve the resulting ODE with a Dirac delta function source in just the remaining coordinate (e.g., z or r, etc.).

- This method is best illustrated by two examples:

- Example 1: Find an expression for the Dirichlet Green's function for Poisson's equation in 1-dimension for $0 \leq x \leq a$. (Note: For $-\infty < x < \infty$, the Dirichlet Green's function is identically zero, as the only solution to $d^2G_D/dx^2 = \delta(x-x')$ which vanishes as $|x| \to \infty$ is $G_D(x,x') = 0$.)

- Solution:

1) The solution to

$$\frac{d^2}{dx^2} G_D(x,x') = \delta(x-x') \tag{551}$$

for $x < x'$ and $x > x'$ where the RHS is zero is:

$$G_D(x,x') = \begin{cases} A(x') + B(x')\,x & \text{for } x < x' \\ C(x') + D(x')\,x & \text{for } x > x' \end{cases} \tag{552}$$

2) Applying the homogeneous BCs at $x = 0$ and $x = a$ leads to

$$A(x') = 0\,, \qquad C(x') = -D(x')\,a \tag{553}$$

so that

$$G_D(x,x') = \begin{cases} B(x')\,x & \text{for } x < x' \\ D(x')\,(x-a) & \text{for } x > x' \end{cases} \tag{554}$$

3) Applying the symmetry of $G_D(x,x')$ further reduces the freedom of the integration constants to an overall multiplicative constant:

$$B(x') = E\,(x'-a)\,, \qquad D(x') = E\,x' \tag{555}$$

so that

$$G_D(x,x') = E\,x_<(x_> - a) \tag{556}$$

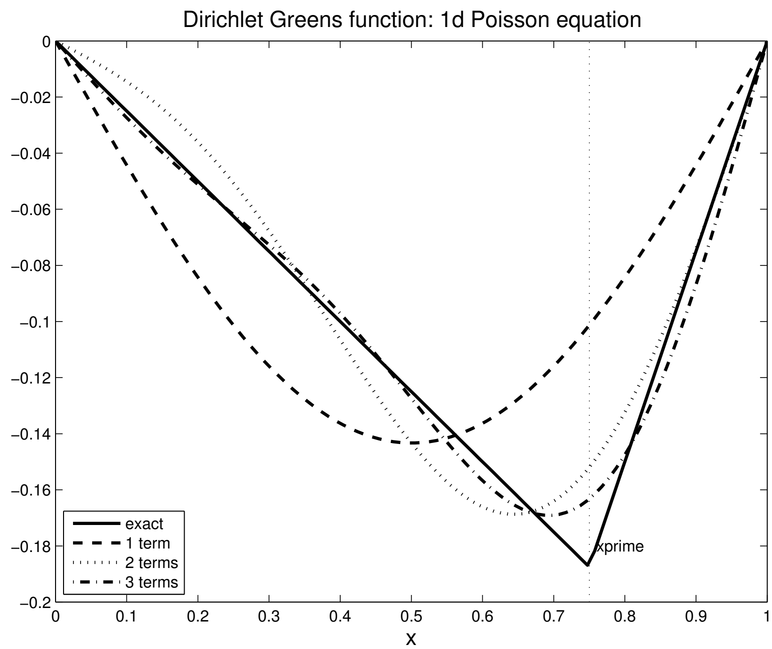


Figure 53: Dirichlet Green's function for the 1-dimensional Poisson's equation for the finite interval $0 \leq x \leq a$. The exact solution and Fourier series expansions containing one, two, and three terms are plotted, for the case $a = 1$, $x' = 0.75a$.

4) The constant E can be determined by integrating the differential equation for $G_D(x, x')$ across the delta function singularity from $x = x' - \epsilon$ to $x = x' + \epsilon$, then taking the limit $\epsilon \to 0$:

$$\lim_{\epsilon \to 0} \left\{ \frac{d}{dx} G_D(x, x') \Big|_{x=x'-\epsilon}^{x'+\epsilon} \right\} = 1 \Leftrightarrow E = \frac{1}{a} \tag{557}$$

5) Thus,

$$G_D(x, x') = -\frac{1}{a}\, x_<(a - x_>) \tag{558}$$

Note: When we used the eigenfunction method to determine the Green's function, we found (493):

$$G_D(x, x') = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right)}{-\left(\frac{n\pi}{a}\right)^2} \tag{559}$$

Thus,

$$-\frac{1}{a} x_<(a - x_>) = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right)}{-\left(\frac{n\pi}{a}\right)^2} \tag{560}$$

is the Fourier series expansion of the 1-dimensional Dirichlet Green's function. (See Figure 53.)

- Exercise: Extend the above analysis to determine the Dirichlet Green's function for the 1-dimensional simple harmonic oscillator equation

$$\frac{d^2}{dx^2} G_D(x, x') + k^2\, G_D(x, x') = \delta(x - x') \tag{561}$$

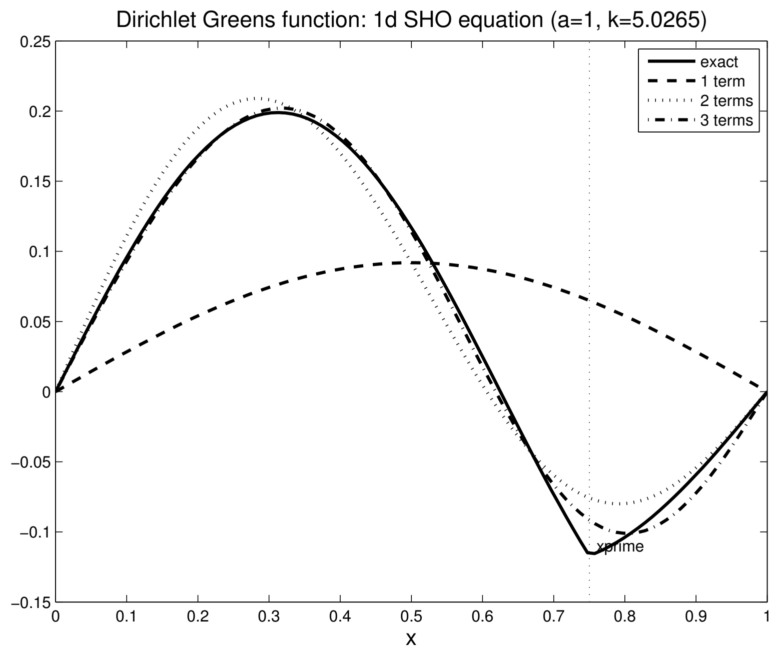


Figure 54: Dirichlet Green's function for the 1-dimensional simple harmonic oscillator equation for the finite interval $0 \leq x \leq a$. The exact solution and Fourier series expansions containing one, two, and three terms are plotted, for the case $a = 1$, $x' = 0.75a$, and $k = (4/5) \times 2\pi/a$.

for the finite interval $0 \leq x \leq a$. Assume that $k^2 \neq (n\pi/a)^2$ for any positive integer n. By solving the 1-d delta function equation, you should find

$$G_D(x, x') = -\frac{\sin(kx_<)\sin\left(k(a - x_>)\right)}{k\sin(ka)} \tag{562}$$

Alternatively, in terms of an eigenfunction expansion

$$G_D(x, x') = \frac{2}{a}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)}{k^2 - \left(\frac{n\pi}{a}\right)^2} \tag{563}$$

This last expression can be thought of as a Fourier series representation of the RHS of (562). (See Figure 54.) Note also that in the limit $k \to 0$, we recover the results (558) and (559) for the 1-d Poisson's equation, as we should.

- Example 2: Find an expression for the Dirichlet Green's function for a rectangular box ($0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$), singling out the z-coordinate for special treatment.

- Solution:

  1) Begin by expanding the Dirac delta function $\delta(\mathbf{r} - \mathbf{r}')$ in terms of the appropriate eigenfunctions for the rectangular box, leaving the $\delta(z - z')$ factor as is:

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(x - x')\delta(y - y')\delta(z - z') \tag{564}$$

$$= \delta(z - z')\frac{4}{ab}\sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right) \tag{565}$$

  2) Do the same for the Green's function, leaving the z, z' dependence to be determined:

$$G_D(\mathbf{r}, \mathbf{r}') = \frac{4}{ab}\sum_{l,m=1}^{\infty} g_{lm}(z, z')\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right) \tag{566}$$

3) Substitute the above expressions into $\nabla^2 G_D(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}')$, obtaining

$$\frac{d^2}{dz^2} g_{lm}(z,z') - k_{lm}^2 g_{lm}(z,z') = -4\pi\,\delta(z-z') \tag{567}$$

where

$$k_{lm}^2 := \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \tag{568}$$

4) Solve this equation for $z < z'$ and $z > z'$ where the RHS is zero:

$$g_{lm}(z,z') = \begin{cases} A(z')\,\sinh(k_{lm}z) + B(z')\,\cosh(k_{lm}z) & \text{for } z < z' \\ C(z')\,\sinh(k_{lm}z) + D(z')\,\cosh(k_{lm}z) & \text{for } z > z' \end{cases} \tag{569}$$

5) Apply the homogeneous BCs at $z = 0$ and $z = c$, yielding

$$B(z') = 0\,, \qquad C(z') = -D(z')\frac{\cosh(k_{lm}c)}{\sinh(k_{lm}c)} \tag{570}$$

so that

$$g_{lm}(z,z') = \begin{cases} A(z')\,\sinh(k_{lm}z) & \text{for } z < z' \\ D(z')\,\frac{\sinh[k_{lm}(c-z)]}{\sinh(k_{lm}c)} & \text{for } z > z' \end{cases} \tag{571}$$

6) Apply the symmetry of $G_D(\mathbf{r},\mathbf{r}')$ to further reduce the freedom of the integration constants to an overall multiplicative constant:

$$A(z') = E\,\frac{\sinh\left[k_{lm}(c-z')\right]}{\sinh(k_{lm}c)}\,, \qquad D(z') = E\,\sinh(k_{lm}z') \tag{572}$$

so that

$$g_{lm}(z,z') = E\,\frac{\sinh(k_{lm}z_<)\sinh\left[k_{lm}(c-z_>)\right]}{\sinh(k_{lm}c)} \tag{573}$$

7) Determine the constant E by integrating the differential equation for $g_{lm}(z,z')$ across the delta function singularity from $z = z' - \epsilon$ to $z = z' + \epsilon$, taking the limit $\epsilon \to 0$:

$$\lim_{\epsilon\to 0}\left\{ \frac{d}{dz} g_{lm}(z,z')\Big|_{z=z'-\epsilon}^{z'+\epsilon} \right\} = -4\pi \tag{574}$$

which leads to

$$E = \frac{4\pi}{k_{lm}} \tag{575}$$

8) Substitute this constant back into the formulas to obtain the solutions

$$g_{lm}(z,z') = 4\pi\frac{\sinh(k_{lm}z_<)\sinh\left[k_{lm}(c-z_>)\right]}{k_{lm}\sinh(k_{lm}c)} \tag{576}$$

and

$$\begin{aligned} G_D(\mathbf{r},\mathbf{r}') = &\frac{16\pi}{ab}\sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right) \\ &\times \frac{\sinh(k_{lm}z_<)\sinh\left[k_{lm}(c-z_>)\right]}{k_{lm}\sinh(k_{lm}c)} \end{aligned} \tag{577}$$

Note: If we compare this expression for $G_D(\mathbf{r},\mathbf{r}')$ with what we obtained earlier using the eigenfunction method:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{32\pi}{abc}\sum_{l,m,n=1}^{\infty} \frac{\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)}{\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2} \tag{578}$$

we can conclude that

$$\frac{\sinh(k_{lm}z_<)\sinh\left[k_{lm}(c-z_>)\right]}{k_{lm}\sinh(k_{lm}c)} = \frac{2}{c}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi z'}{c}\right)}{k_{lm}^2+\left(\frac{n\pi}{c}\right)^2}\sin\left(\frac{n\pi z}{c}\right) \tag{579}$$

Thus, the RHS is the Fourier series representation of the 1-dimensional Dirichlet Green's function satisfying

$$\frac{d^2}{dz^2}G_D(z,z') - k_{lm}^2\, G_D(z,z') = -\delta(z-z') \tag{580}$$

for $0 \le z \le c$. If we take the limit $k_{lm} \to 0$ of these last two equations, we recover the Fourier series expansion of the previous example with z and c replacing x and a. (One needs to apply L'Höpital's rule twice to take the limit $k_{lm} \to 0$ of the LHS of (579).)

4.14 Green's functions: More examples

- Example 1: Expand the infinite space Dirichlet Green's function $G_D(\mathbf{r},\mathbf{r}') = 1/|\mathbf{r}-\mathbf{r}'|$ in spherical polar coordinates, using the method of the previous section, singling out the r-coordinate for special treatment.

- Answer:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{581}$$

Note: Since

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma) \tag{582}$$

from the generating function for the Legendre polynomials, the above result may be thought of as an alternative derivation of the addition theorem for spherical harmonics:

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{583}$$

- Example 2: Proceeding as above, derive the Dirichlet Greens's function between two concentric spheres of radii a and b (with $a<b$).

- Answer:

$$G_D(\mathbf{r},\mathbf{r}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)}{\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left(r_<^l-\frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right) \tag{584}$$

Note: From the above result we can obtain expressions for the Dirichlet Green's functions corresponding to the following special cases:

i) Infinite space Dirichlet Green's function in spherical polar coords (set $a\to 0$, $b\to\infty$):

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l-0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{585}$$

ii) Dirichlet Green's function exterior to a sphere of radius a (set $b\to\infty$):

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{a}{r'|\mathbf{r}-\frac{a^2}{r'^2}\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}}-\frac{a^{2l+1}}{(rr')^{l+1}}\right)Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{586}$$

iii) Dirichlet Green's function interior to a sphere of radius a (set $a \to 0$, then rename $b = a$):

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{a}{r'|\mathbf{r}-\frac{a^2}{r'^2}\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{(rr')^l}{a^{2l+1}}\right) Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \tag{587}$$

- Example 3: Expand the infinite space Dirichlet Green's function $G_D(\mathbf{r},\mathbf{r}') = 1/|\mathbf{r}-\mathbf{r}'|$ in cylindrical polar coordinates, using the method of the previous section, singling out the ρ-coordinate for special treatment.

- Answer:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\ \cos\left[k(z-z')\right] e^{im(\phi-\phi')} I_m(k\rho_<)K_m(k\rho_>) \tag{588}$$

To prove the above result we note the following:

1) We have the expansions

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{\delta(\rho-\rho')}{\rho}\frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\ \cos\left[k(z-z')\right] e^{im(\phi-\phi')} \tag{589}$$

and

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, g_m(k,\rho,\rho')\cos\left[k(z-z')\right] e^{im(\phi-\phi')} \tag{590}$$

2) The differential equation satisfied by $g_m(k,\rho,\rho')$ is

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right) - \left(k^2+\frac{m^2}{\rho^2}\right) = -4\pi\,\frac{\delta(\rho-\rho')}{\rho} \tag{591}$$

For $\rho < \rho'$ and $\rho > \rho'$ this is the *modified* Bessel's equation.

3) BCs at $\rho \to 0$, $\rho \to \infty$, and symmetry of the solution imply

$$g_m(k,\rho,\rho') = E\, I_m(k\rho_<)K_m(k\rho_>) \tag{592}$$

4) Integrating the differential equation across the delta function singularity from $\rho = \rho' - \epsilon$ to $\rho = \rho' + \epsilon$ yields

$$-4\pi = E\, k\rho'\left[I_m(k\rho')K_m'(k\rho') - I_m'(k\rho')K_m(k\rho')\right] = E\, k\rho' W(k\rho') \tag{593}$$

where $W(x)$ is the Wronskian of $I_m(x)$ and $K_m(x)$, with $x = k\rho'$.

5) Since the modified Bessel's equation is in Sturm-Liouville form, we know that

$$W(x) = \frac{C}{x}\,, \quad \text{with } C = -1 \tag{594}$$

where C was evaluated using the asymptotic form of $I_m(x)$ and $K_m(x)$, as $x \to \infty$.

6) Thus, $E = 4\pi$ implying

$$g_m(k,\rho,\rho') = 4\pi\, I_m(k\rho_<)K_m(k\rho_>) \tag{595}$$

leading to the final expression for $G_D(\mathbf{r},\mathbf{r}')$.

- Example 4: Determine the Dirichlet Green's function between two infinite planes at $z = 0$ and $z = L$. (Jackson, Prob 3.17)

- Answers:
Eigenfunction method:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\int_0^\infty dk\, k\frac{e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)J_m(k\rho)J_m(k\rho')}{\left(\frac{n\pi}{L}\right)^2+k^2} \tag{596}$$

Singling out the z-coordinate for special treatment:

$$G_D(\mathbf{r},\mathbf{r}') = 2\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')\frac{\sinh(kz_<)\sinh\left[k(L-z_>)\right]}{\sinh(kL)} \tag{597}$$

Singling out the ρ-coordinate for special treatment:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)I_m\left(\frac{n\pi\rho_<}{L}\right)K_m\left(\frac{n\pi\rho_>}{L}\right) \tag{598}$$

- Example 5: Determine the Dirichlet Green's function inside a cylindrical can of height L and radius a. (Choose cylindrical polar coordinates so that the axis of the cylinder corresponds to $\rho = 0$, and the bottom and top of the can to $z = 0$ and $z = L$, respectively.) (Jackson, Prob 3.23)

- Answers:
Eigenfunction method:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{8}{La^2}\sum_{m=-\infty}^{\infty}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}\sin\left(\frac{l\pi z}{a}\right)\sin\left(\frac{l\pi z'}{a}\right)J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{l\pi}{L}\right)^2\right]J_{m+1}^2(x_{mn})} \tag{599}$$

Singling out the z-coordinate for special treatment:

$$G_D(\mathbf{r},\mathbf{r}') = \frac{4}{a}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{x_{mn}J_{m+1}^2(x_{mn})}\frac{\sinh\left(\frac{x_{mn}z_<}{a}\right)\sinh\left[\frac{x_{mn}}{a}(L-z_>)\right]}{\sinh\left(\frac{x_{mn}L}{a}\right)} \tag{600}$$

Singling out the ρ-coordinate for special treatment:

$$\begin{aligned}G_D(\mathbf{r},\mathbf{r}') =& \frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{a}\right)\sin\left(\frac{n\pi z'}{a}\right)\frac{I_m\left(\frac{n\pi\rho_<}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)}\\ &\times\left[I_m\left(\frac{n\pi a}{L}\right)K_m\left(\frac{n\pi\rho_>}{L}\right)-K_m\left(\frac{n\pi a}{L}\right)I_m\left(\frac{n\pi\rho_>}{L}\right)\right]\end{aligned} \tag{601}$$

- Example 6: Determine a closed form expression for the infinite space Dirichlet Green's function in 2-dimensions, and an expansion for it in terms of the eigenfunctions of the azimuthal coordinate ϕ. (Jackson, Prob 2.17)

- Answers:
Closed form expression:

$$G_D(\mathbf{r},\mathbf{r}') = -\ln\left(|\mathbf{r}-\mathbf{r}'|^2\right) = -\ln\left(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')\right) \tag{602}$$

Expansion:

$$G_D(\mathbf{r},\mathbf{r}') = -\ln\left(\rho_>^2\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos\left[m(\phi-\phi')\right] \tag{603}$$

Note:

1) The closed form expression can be verified by showing that

$$\nabla^2 \ln \rho = 0 \quad \text{for } \rho \neq 0 \tag{604}$$

and

$$\int_D \nabla^2 (\ln \rho)\, da = \oint_C (\boldsymbol{\nabla} \ln \rho) \cdot \hat{\mathbf{n}}\, dl = 2\pi \tag{605}$$

where D is any 2-d disk containing the origin with boundary circle C.

2) The second expression can be verified by expanding

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{\delta(\rho-\rho')}{\rho} \left\{ \frac{1}{2\pi} + \frac{1}{\pi} \sum_{m=1}^{\infty} \cos\left[m(\phi-\phi')\right] \right\} \tag{606}$$

and

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi}\, g_0(\rho,\rho') + \frac{1}{\pi} \sum_{m=1}^{\infty} g_m(\rho,\rho')\, \cos\left[m(\phi-\phi')\right] \tag{607}$$

then substituting into $\nabla^2 G_D(\mathbf{r},\mathbf{r}') = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')$ to obtain

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dg_0}{d\rho}\right) = -4\pi\, \frac{\delta(\rho-\rho')}{\rho} \tag{608}$$

and

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dg_m}{d\rho}\right) - \frac{m^2}{\rho^2}\, g_m = -4\pi\, \frac{\delta(\rho-\rho')}{\rho} \tag{609}$$

Solving these equations as we did for previous examples (with the BCs being finite at $\rho = 0$ and only logarithmic divergence as $\rho \to \infty$) leads to

$$g_0(\rho,\rho') = -4\pi\, \ln \rho_> \,, \qquad g_m(\rho,\rho') = \frac{2\pi}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \tag{610}$$

which yields the final result.

4.15 Using Green's functions to solve boundary value problems

- Given the Dirichlet Green's function for a particular geometry, the potential $\Phi(\mathbf{r})$ is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V G_D(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\, dV' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{r}')\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\, da' \tag{611}$$

- Note that

$$G_D(\mathbf{r},\mathbf{r}')\Big|_S = 0\,, \qquad \frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n}\Big|_S \equiv \hat{\mathbf{n}}' \cdot \boldsymbol{\nabla}' G_D(\mathbf{r},\mathbf{r}')\Big|_S \tag{612}$$

where $\hat{\mathbf{n}}'$ is the normal to the boundary S pointing *outward* from the volume V.

- In general, both integrals are needed to determine $\Phi(\mathbf{r})$. However, if $\rho(\mathbf{r}) = 0$ inside V, then only the surface integral is needed (i.e., solution to Laplace's equation). Also, if $\Phi(\mathbf{r})$ vanishes on the boundary (i.e., if the boundary surfaces are grounded conductors), then only the volume integral is needed.

- Example 1: Show that the following methods of solving Laplace's equation in the interior of a sphere of radius a with specified potential on the boundary ($\Phi(r=a) \equiv f(\theta,\phi)$) are equivalent: (Jackson, Prob 3.5)

i) Dirichlet Green's function method:

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi}\oint_S \Phi(\mathbf{r}')\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\,da' \tag{613}$$

where

$$G_D(\mathbf{r},\mathbf{r}') = 4\pi\sum_{l-0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{(rr')^l}{a^{2l+1}}\right)Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \tag{614}$$

ii) Separation of variables method:

$$\Phi(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}r^l\,Y_{lm}(\theta,\phi) \tag{615}$$

where

$$A_{lm} = \frac{1}{a^l}\int_{S^2} d\Omega\, f(\theta,\phi)Y^*_{lm}(\theta,\phi) \tag{616}$$

iii) Method of images:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi}\int_{S^2} d\Omega'\, f(\theta',\phi')\frac{a(a^2-r^2)}{(r^2+a^2-2ar\,\cos\gamma)^{3/2}} \tag{617}$$

- Notes:

  For (i), you should find

$$\left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial n'}\right|_S = \left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial r'}\right|_{r'=a} = -\frac{4\pi}{a^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\left(\frac{r}{a}\right)^l \tag{618}$$

  For (iii), note that

$$\frac{1}{|\mathbf{r}-\frac{a}{r'}\mathbf{r}'|} = \frac{1}{\sqrt{r^2+a^2-2ar\,\cos\gamma}} = \frac{1}{a}\sum_{l=o}^{\infty}\left(\frac{r}{a}\right)^l P_l(\cos\gamma) \tag{619}$$

  Then take the derivative of both sides wrt r to pull down a factor of l.

- Example 2: Use Dirichlet Green's function for the interior of a sphere to calculate the potential inside a grounded conducting sphere of radius b due to a ring of charge (radius $a < b$, total charge Q) in the xy-plane.

- Answer:

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}P_{2n}(\cos\theta)\left(\frac{r_<^{2n}}{r_>^{2n+1}} - \frac{(ra)^{2n}}{b^{4n+1}}\right) \tag{620}$$

  where $r_<$ ($r_>$) is the smaller (larger) of r and a.

- Hints:

$$\rho(\mathbf{r}') = \frac{Q\delta(r'-a)\delta(\cos\theta')}{2\pi a^2} \tag{621}$$

  and

$$P_{2n+1}(0) = 0\,, \qquad P_{2n}(0) = \frac{(-1)^n(2n-1)!!}{2^n n!} \tag{622}$$

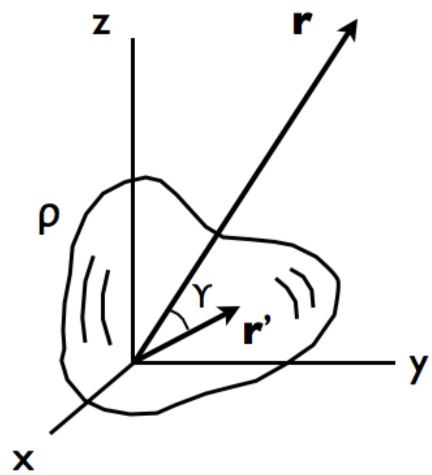


Figure 55: Localized charge distribution ρ. $\mathbf{r}$ is the field point, $\mathbf{r}'$ is the source point, and γ is the angle between them. $\mathbf{r}'$ is integrated over to find $\Phi(\mathbf{r})$.

4.16 Multipole expansions

- Goal: expand the electrostatic potential $\Phi(\mathbf{r})$ due to a localized charge distribution $\rho(\mathbf{r})$ in terms of its *multipole moments*.

- Consider a charge distribution $\rho(\mathbf{r}')$ localized to some finite volume, free of any boundary surfaces. (See Figure 55.) Then the electrostatic potential $\Phi(\mathbf{r})$ associated with this charge distribution can be written as

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} dV'\,\frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\rho(\mathbf{r}') \tag{623}$$

- Recall that the infinite space Dirichlet Green's function $1/|\mathbf{r}-\mathbf{r}'|$ can be expanded in terms of Legendre polynomials as

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\frac{(r')^l}{r^{l+1}}\,P_l(\cos\gamma) \tag{624}$$

  for $r > r'$.

- Substituting this expansion into the previous expression for $\Phi(\mathbf{r})$, and then rearranging integrals and summations, we find

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}Q^{(l)}(\theta,\phi) \tag{625}$$

  where

$$Q^{(l)}(\theta,\phi) := \int dV'\,\rho(\mathbf{r}')(r')^l P_l(\cos\gamma) \tag{626}$$

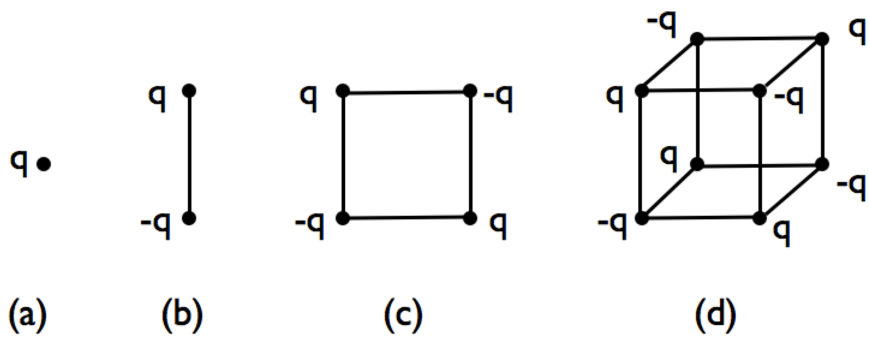


Figure 56: Charge distributions whose first non-vanishing multipole moment is the (a) monopole, (b) dipole, (c) quadrupole, and (d) octupole moment, respectively.

- Using the expression for the Legendre polynomials, we can further decompose the $Q^{(l)}$:

$$Q^{(0)}(\theta,\phi) = Q \tag{627}$$

$$Q^{(1)}(\theta,\phi) = \sum_{i=1}^{3} \hat{r}_i Q_i \tag{628}$$

$$Q^{(2)}(\theta,\phi) = \frac{1}{2}\sum_{i,j} \hat{r}_i \hat{r}_j Q_{ij} \tag{629}$$

$$Q^{(3)}(\theta,\phi) = \frac{1}{2}\sum_{i,j,k} \hat{r}_i \hat{r}_j \hat{r}_k Q_{ijk} \tag{630}$$

$$\cdots \tag{631}$$

where

$$Q \equiv \int dV' \, \rho(r') \tag{632}$$

$$Q_i \equiv \int dV' \, \rho(\mathbf{r}') r'_i \tag{633}$$

$$Q_{ij} \equiv \int dV' \, \rho(\mathbf{r}') \left[3r'_i r'_j - (r')^2 \delta_{ij}\right] \tag{634}$$

$$Q_{ijk} \equiv \int dV' \, \rho(\mathbf{r}') \left[5r'_i r'_j r'_k - (r')^2 \left(\delta_{ij} r'_k + \delta_{jk} r'_i + \delta_{ki} r'_j\right)\right] \tag{635}$$

$$\cdots \tag{636}$$

- <u>Exercise</u>: Prove the above.

- Q is called the monopole moment; $Q_i (\equiv p_i)$ the dipole moment; Q_{ij} the quadrupole moment; Q_{ijk} the octupole moment, etc. (See Figure 56.)

- Q has 1 independent component; Q_i has 3; Q_{ij} has 5 (being symmetric and trace-free in 3 dimensions); and Q_{ijk} has 7 (being symmetric with respect to interchange of any of its indices, as well as being trace-free with respect to any pair). In general, $Q^{(l)}(\theta,\phi)$, which can be written in terms of $Q_{ijk\cdots}$ with l indices, has $2l+1$ independent components.

- In terms of the multipole moments we have

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{1}{r^2}\sum_i \hat{r}_i p_i + \frac{1}{r^3}\frac{1}{2}\sum_{i,j}\hat{r}_i\hat{r}_j Q_{ij} + \frac{1}{r^4}\frac{1}{2}\sum_{i,j,k}\hat{r}_i\hat{r}_j\hat{r}_k Q_{ijk} + \cdots\right] \tag{637}$$

- Note that the (θ, ϕ) dependence of $\Phi(\mathbf{r})$ and $Q^{(l)}(\theta, \phi)$ comes from $\hat{r}_i$, etc., since

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \tag{638}$$

- The multipole moments depend in general on the choice of origin. For example, if one shifts the origin to $\mathbf{R}$, so that $r_i' = R_i + \underline{r}_i'$, then the new dipole moment $\underline{p}_i$ and new quadrupole moment $\underline{Q}_{ij}$, defined by

$$\underline{p}_i := \int dV'\,\rho(\mathbf{r}')\underline{r}_i' \tag{639}$$

$$\underline{Q}_{ij} := \int dV'\,\rho(\mathbf{r}')\,\left[3\underline{r}_i'\underline{r}_j' - (\underline{r}')^2\,\delta_{ij}\right] \tag{640}$$

are related to the original dipole moment p_i and quadrupole moment Q_{ij} by

$$\underline{p}_i = p_i - Qr_i \tag{641}$$

$$\underline{Q}_{ij} = Q_{ij} + Q(3R_iR_j - R^2\,\delta_{ij}) - 3R_ip_j - 3R_jp_i - 2\mathbf{R}\cdot\mathbf{p}\,\delta_{ij} \tag{642}$$

- <u>Exercise</u>: Prove the above.

- Note, however, that if the monopole moment Q vanishes, then the dipole moment p_i is *independent* of the choice of origin. Similarly, if the monopole moment Q and dipole moment p_i are both equal to zero, then the quadrupole moment Q_{ij} is independent of the choice of origin.

- The above extends to arbitrary l. Namely, the lth multipole moment is independent of the choice of origin if and only if all of the $l-1$ lower-order multipole moments vanish.

- We can also obtain a multipole expansion with respect to spherical harmonics, if we substitute

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\frac{(r')^l}{r^{l+1}}\left(\frac{4\pi}{2l+1}\right)\sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi')Y_{lm}(\theta, \phi) \tag{643}$$

into the original expression (623) for $\Phi(\mathbf{r})$.

- The result is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\left(\frac{4\pi}{2l+1}\right)\sum_{m=-l}^{l} q_{lm}Y_{lm}(\theta, \phi) \tag{644}$$

where

$$q_{lm} \equiv \int dV'\rho(\mathbf{r}')(r')^l\,Y_{lm}^*(\theta', \phi') \tag{645}$$

- Note that

$$q_{l,-m} = (-1)^l q_{lm}^* \tag{646}$$

similar to that for the spherical harmonics.

- Comparing the two multipole expansions, we see that

$$Q^{(l)}(\theta, \phi) = \frac{4\pi}{2l+1}\sum_{m=-1}^{l} q_{lm}Y_{lm}(\theta, \phi) \tag{647}$$

or, equivalently,

$$q_{lm} = \frac{2l+1}{4\pi}\int_{S^2} d\Omega\,Y_{lm}^*(\theta, \phi)Q^{(l)}(\theta, \phi) \tag{648}$$

- Using this formula, we find for example that

$$q_{00} = \frac{1}{\sqrt{4\pi}}\, Q \tag{649}$$

$$q_{10} = \sqrt{\frac{3}{4\pi}}\, p_z \tag{650}$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}}\, (p_x - ip_y) \tag{651}$$

$$q_{1,-1} = \sqrt{\frac{3}{8\pi}}\, (p_x + ip_y) \tag{652}$$

- Exercise: Prove these results.

- We can also expand the energy

$$W = \int dV\, \rho(\mathbf{r})\Phi(\mathbf{r}) \tag{653}$$

of a charge distribution $\rho(\mathbf{r})$ in an *external* field $\Phi(\mathbf{r})$ in terms of multipole moments.

- Assuming the potential $\Phi(\mathbf{r})$ is slowly varying in the region of space where $\rho(\mathbf{r})$ is non-negligible, we can Taylor expand the potential about an appropriate origin

$$\Phi(\mathbf{r}) = \Phi(\mathbf{0}) + \sum_i \left.\frac{\partial \Phi}{\partial r_i}\right|_{\mathbf{0}} r_i + \frac{1}{2!}\sum_{i,j} \left.\frac{\partial^2 \Phi}{\partial r_i \partial r_j}\right|_{\mathbf{0}} r_i r_j + \cdots \tag{654}$$

- Since $\mathbf{E} = -\boldsymbol{\nabla}\Phi$, it follows that

$$\Phi(\mathbf{r}) = \Phi(\mathbf{0}) - \sum_i E_i(\mathbf{0})\, r_i - \frac{1}{2!}\sum_{i,j} \left.\frac{\partial E_j}{\partial r_i}\right|_{\mathbf{0}} r_i r_j + \cdots \tag{655}$$

- In addition, since $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ for the external field, we have

$$0 = \boldsymbol{\nabla}\cdot\mathbf{E} = \sum_i \left.\frac{\partial E_i}{\partial r_i}\right|_{\mathbf{0}} = \sum_{i,j} \left.\frac{\partial E_j}{\partial r_i}\right|_{\mathbf{0}} \delta_{ij} \tag{656}$$

- Thus, we can add this term (multiplied by r^2) to the potential without changing anything so that

$$\Phi(\mathbf{r}) = \Phi(\mathbf{0}) - \sum_i E_i(\mathbf{0})\, r_i - \frac{1}{3!}\sum_{i,j} \left.\frac{\partial E_j}{\partial r_i}\right|_{\mathbf{0}} \left(3r_i r_j - r^2\, \delta_{ij}\right) + \cdots \tag{657}$$

- Substituting this into the original expression for the energy, and recalling the definitions of the multipole moments Q, $Q_i\ (= p_i)$, Q_{ij}, etc., we find that

$$W = \Phi(\mathbf{0})\, Q - \sum_i E_i(\mathbf{0})\, p_i - \frac{1}{3!}\sum_{i,j} \left.\frac{\partial E_j}{\partial r_i}\right|_{\mathbf{0}} Q_{ij} + \cdots \tag{658}$$

- Exercise: Prove this.

- Thus, we see how the charge distribution interacts with the external field: The monopole Q interacts with the potential $\Phi(\mathbf{0})$, the dipole p_i with the field $E_i(\mathbf{0})$, the quadrupole Q_{ij} with the gradient of the field $\partial E_j/\partial r_i|_{\mathbf{0}}$, etc.

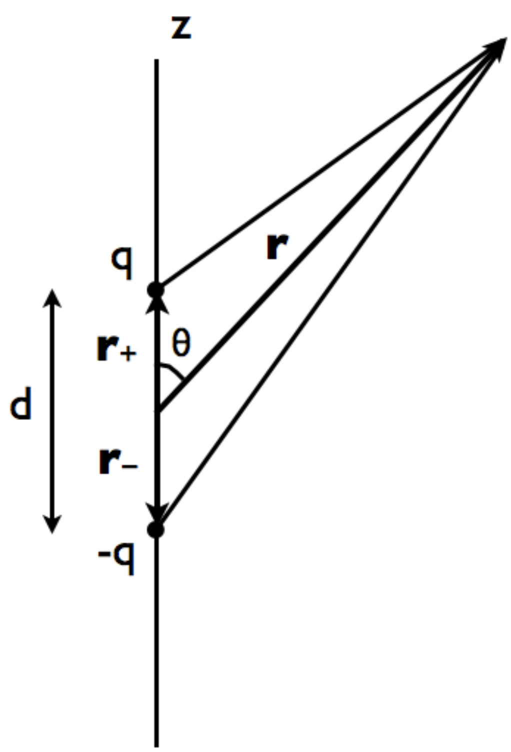


Figure 57: Physical dipole consisting of two charges q and $-q$, separated by a distance d. $\mathbf{r}$ is the field point. $\mathbf{r}_+$ and $\mathbf{r}_-$ are the position vectors for the charges q and $-q$, respectively.

4.17 Electric dipoles

- A *physical dipole* consists of two point charges $+q$ and $-q$ separated by a distance d. (See Figure 57.) It follows from the general definition that

$$\mathbf{p} \equiv \int dV\, \rho(\mathbf{r})\mathbf{r} = q\mathbf{d} \tag{659}$$

  where $\mathbf{d}$ is the vector connecting the negative charge to the positive charge.

- A *pure dipole* is the limit of a physical dipole where $d \to 0$ and $q \to \infty$, keeping $p \equiv qd$ constant.

- The potential of a pure dipole is given by

$$\Phi_{\rm dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \tag{660}$$

  The potential of a physical dipole consists of this, plus higher-order corrections terms, corresponding to odd multipole moments $l = 3, 5, 7, \cdots$.

- The electric field of a pure dipole can be obtained by taking the (negative) gradient of $\Phi_{\rm dip}(\mathbf{r})$. Assuming that $\mathbf{p}$ is located at the origin and points in the z-direction, then

$$\mathbf{E}_{\rm dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) \qquad \text{for } r \neq 0, \tag{661}$$

- <u>Exercise</u>: Prove this. (Hint: Use the expression for the gradient in spherical polar coordinates.)

- This can be recast in coordinate-independent form as

$$\mathbf{E}_{\rm dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\,\right] \qquad \text{for } r \neq 0, \tag{662}$$

- <u>Exercise</u>: Prove this.

- If we want to include the field at the origin ($r \to 0$), then we need to add a Dirac delta-function term:

$$\mathbf{E}_{\rm dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\,\right] - \frac{1}{3\epsilon_0}\mathbf{p}\,\delta(\mathbf{r}) \tag{663}$$

- This is needed to satisfy a general result for the average of the electric field over the *volume* of a sphere of radius R:

$$\mathbf{E}_{\rm ave} \equiv \frac{1}{\frac{4}{3}\pi R^3}\int_{r<R} dV\, \mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_{\rm enc}}{R^3} + \mathbf{E}_{\rm ext}(\mathbf{0}) \tag{664}$$

  where $\mathbf{E}_{\rm ext}(\mathbf{0})$ is the electric field at the center of the sphere produced by all charges *exterior* to the sphere, and $\mathbf{p}_{\rm enc}$ is the total dipole moment *enclosed* by the sphere.

- The average of the potential over the *surface* of a sphere of radius R satisfies a similar relation:

$$\Phi_{\rm ave} \equiv \frac{1}{4\pi R^2}\oint_{r=R} da\, \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q_{\rm enc}}{R} + \Phi_{\rm ext}(\mathbf{0}) \tag{665}$$

  where $\Phi_{\rm ext}(\mathbf{0})$ is the potential at the center of the sphere produced by all charges exterior to the sphere, and $Q_{\rm enc}$ is the total charge enclosed by the sphere.

- Exercise: Prove the above result for the average potential. (Hint: Consider separately the cases of a single point charge q located outside and inside the sphere at $z = r' > R$ and $z = r' < R$, respectively. Expand $1/|\mathbf{r} - \mathbf{r}'|$ in terms of powers of R/r' (or r'/R) and Legendre polynomials. Then do the integrations using the orthogonality of sines and cosines for the ϕ integration, and the orthogonality of the Legendre polynomials for the $\cos\theta$ integration. Finally, use the superposition principle to get the general result.)

- Exercise: Prove the above result for the average electric field. (Hint: First note that the volume integral over the sphere can be converted into a 2-d integral over the surface using a corollary (103) of the divergence theorem:

$$\int_V dV\, \boldsymbol{\nabla}\Phi(\mathbf{r}) = \oint_S da\, \Phi(\mathbf{r})\, \hat{\mathbf{n}} \tag{666}$$

where $\hat{\mathbf{n}}$ is the outward pointing normal, which depends on the location of the area element. Then, as before, consider separately the cases of a single point charge q located outside and inside the sphere at $z = r' > R$ and $z = r' < R$, respectively. Expand $1/|\mathbf{r} - \mathbf{r}'|$ in terms of powers of R/r' (or r'/R) and Legendre polynomials. In addition, write

$$\hat{\mathbf{n}} = \hat{\mathbf{r}} = \sin\theta\, \cos\phi\, \hat{\mathbf{x}} + \sin\theta\, \sin\phi\, \hat{\mathbf{y}} + \cos\theta\, \hat{\mathbf{z}} \tag{667}$$

Then do the integrations using the orthogonality of sines and cosines for the ϕ integration, and the orthogonality of the Legendre polynomials for the $\cos\theta$ integration. Finally, use the superposition principle to get the general result.)

- Exercise: Finally, to complete the argument, prove that the average of the dipole electric field (661) over a spherical volume *that omits a tiny sphere of radius ϵ containing the origin* is zero. (Hint: Do the angular integrals first, expanding $\hat{\mathbf{r}}$ and $\hat{\theta}$ in terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, using the orthogonality of sines and cosines to do the ϕ integrations, and noting that the $\cos\theta$ terms combine to yield a term proportional to $P_2(\cos\theta)$, which integrates to zero.) Since ϵ is arbitrary, the only way to satisfy the general result (664) for the average electric field is to include a Dirac delta-function term at the origin:

$$\mathbf{E}_{\text{dip, ave}} = \frac{1}{\frac{4}{3}\pi R^3} \int_{r<R} dV \left(\frac{-1}{3\epsilon_0}\right) \mathbf{p}\, \delta(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3} \tag{668}$$

- One can show that the energy, force, and torque on a dipole $\mathbf{p}$ in an external field $\mathbf{E}$ is given by

$$U = -\mathbf{p}\cdot\mathbf{E}\Big|_{\text{dipole}} \tag{669}$$

$$\mathbf{F} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}\big|_{\text{dipole}} \tag{670}$$

$$\mathbf{N} = \mathbf{p}\times\mathbf{E}\big|_{\text{dipole}} \tag{671}$$

where $\big|_{\text{dipole}}$ means to evaluate the field at the location of the dipole.

- We already proved the first relation for an arbitrary dipole distribution in an external field, see (658).

- Exercise: Prove the force and torque equations by considering the total force and total torque on the two charges of a physical dipole, expanding the electric field about the center of the dipole. (Note that for a pure dipole, any higher-order correction terms vanish.)

- Note: i) The minimum energy configuration is for $\mathbf{p}$ aligned with $\mathbf{E}$. ii) In order to have a non-zero net force on a dipole $\mathbf{p}$, the field $\mathbf{E}$ needs to be non-uniform, even for a pure dipole. iii) The torque on a dipole acts so as to try to rotate $\mathbf{p}$ into alignment with $\mathbf{E}$, thereby lowering the energy. (See Figure 58.)

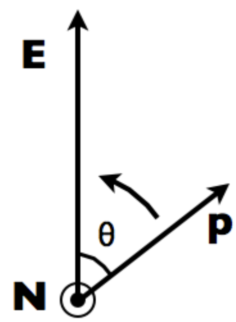


Figure 58: The torque **N** on a dipole **p** in an external electric field **E** is directed in such a way (here, out of the page) so as to try to rotate **p** into alignment with **E**.

5 Electric fields in matter

- In Sec. 3.6, we described how conductors behave in electrostatic fields. Here, we extend that discussion to *insulators* (also called *dielectrics*).
- In a conductor, electrons can move *freely* in response to an applied electric field.
- In a dielectric, the electrons are *bound* to the positive nuclei.
- In a dielectric, an applied electric field will induce an electric *dipole moment* in non-polar atoms or molecules. It will exert a torque on already existing dipole moments associated with polar atoms or molecules. (Note: CO_2 is an example of a non-polar molecule; H_2O is an example of a polar molecule.)
- The induced dipole moments in a dielectric in turn produce an electric field that *opposes* the externally applied field.
- In a conductor, the induced field *exactly* cancels the applied field in the conductor.
- In a dielectric, the cancellation is only *partial*. The amount of cancellation depends on the *dielectic constant* of the material. (More details about this later.)
- Practical example: This reduction in the applied field causes the capacitance to increase when a dielectric is inserted between the plates of a capacitor. (Again, more about this later.)

5.1 Polarization charges

- Consider a piece of material consisting of many individual dipole moments $\mathbf{p}_i$. (The dipole moments could have been induced by an external electric field, or they may be associated with polar molecules in the material.) The *polarization* $\mathbf{P}(\mathbf{r})$ is defined to be the average dipole moment per unit volume:

$$\mathbf{P}(\mathbf{r}) \equiv \frac{1}{\frac{4}{3}\pi R^3} \sum_i \mathbf{p}_i \tag{672}$$

  where the sum is over the dipole moments contained in a small spherical volume of radius R centered at $\mathbf{r}$. The volume is chosen to be large enough to contain many individual dipole moments, but is small on a macroscopic scale. (See Figure 59.)

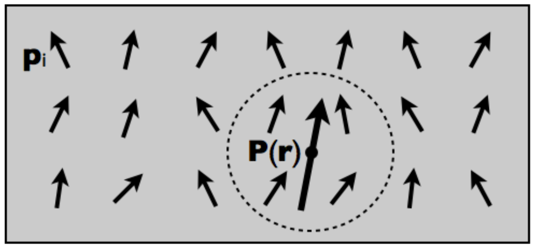


Figure 59: The polarization $\mathbf{P}(\mathbf{r})$ of a material is defined as the average dipole moment per unit volume. Here the volume is a sphere of radius R centered at $\mathbf{r}$, which is large enough to include many individual dipole moments $\mathbf{p}_i$, but is small on a macroscopic scale.

- We can calculate the potential at $\mathbf{r}$ due to the polarization $\mathbf{P}(\mathbf{r})$ by adding together the potentials (660) due to infinitesimal dipole moments $d\mathbf{p}(\mathbf{r}') = dV'\,\mathbf{P}(\mathbf{r}')$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V dV'\,\mathbf{P}(\mathbf{r}')\cdot\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{673}$$

- Noting that

$$\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{674}$$

integrating by parts, and using the divergence theorem, we obtain

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V dV'\,\frac{-\boldsymbol{\nabla}'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi\epsilon_0}\oint_S da'\,\frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathbf{n}}'}{|\mathbf{r}-\mathbf{r}'|} \tag{675}$$

- Thus, the potential is due to volume and surface charge densities

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P}\,, \qquad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \tag{676}$$

where the subscript b indicates that these are *bound* charges, attached to atoms and molecules in the dielectric.

- Nonetheless, they are *genuine* charge densities and give rise to potentials and fields like any other charge density.

- Examples of material with uniform and non-uniform polarization are shown in Figures 60 and 61.

- Note that the total bound charge is zero, as it should be, since

$$Q_{b,\mathrm{tot}} = \int_V \rho_b(\mathbf{r})\,dV + \oint_S \sigma_b(\mathbf{r})\,da = -\int_V \boldsymbol{\nabla}\cdot\mathbf{P}(\mathbf{r})\,dV + \oint_S \hat{\mathbf{n}}\cdot\mathbf{P}(\mathbf{r})\,da = 0 \tag{677}$$

where the last equality follows from the divergence theorem.

- Example: Calculate the potential and the field both inside and outside a uniformly polarized spherical volume (polarization $\mathbf{P} = P\,\hat{\mathbf{z}}$, radius R). (See Figure 62.)

- Answer:

$$\Phi(\mathbf{r}) = \begin{cases} \frac{1}{3\epsilon_0}\,Pz\,, & r < R \\ \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}\,, & r > R \end{cases} \tag{678}$$

and

$$\mathbf{E}(\mathbf{r}) = \begin{cases} -\frac{1}{3\epsilon_0}\mathbf{P}\left(= -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}\right)\,, & r < R \\ \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right]\,, & r > R \end{cases} \tag{679}$$

where $\mathbf{p} \equiv \frac{4}{3}\pi R^3\,\mathbf{P}$ is the total dipole moment of the sphere.

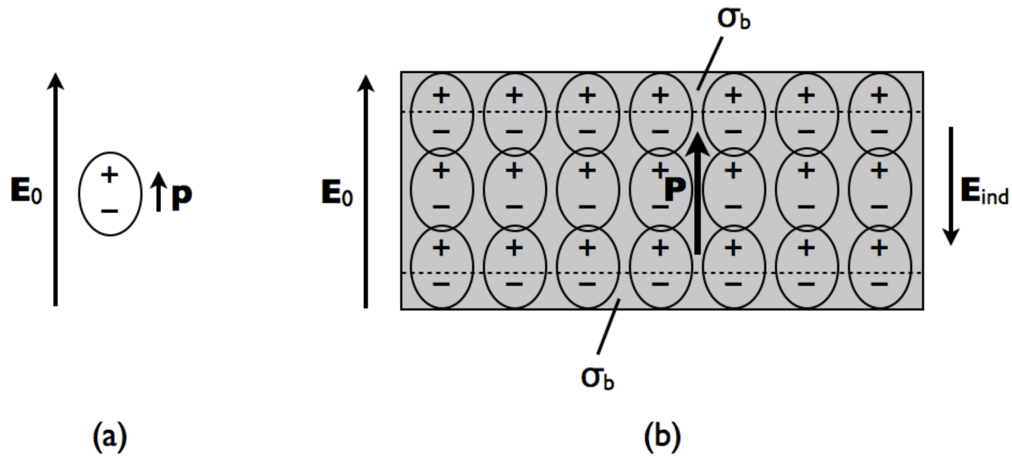


Figure 60: Panel (a): An external field $\mathbf{E}_0 = E_0\,\hat{\mathbf{z}}$ induces a dipole moment $\mathbf{p}$ in a single non-polar atom (or molecule). Panel (b): A material with uniform polarization $\mathbf{P} = P\,\hat{\mathbf{z}}$ induced by $\mathbf{E}_0$. Although the volume bound charge density $\rho_b \equiv -\boldsymbol{\nabla}\cdot\mathbf{P} = 0$, the surface charge density $\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}} \neq 0$.

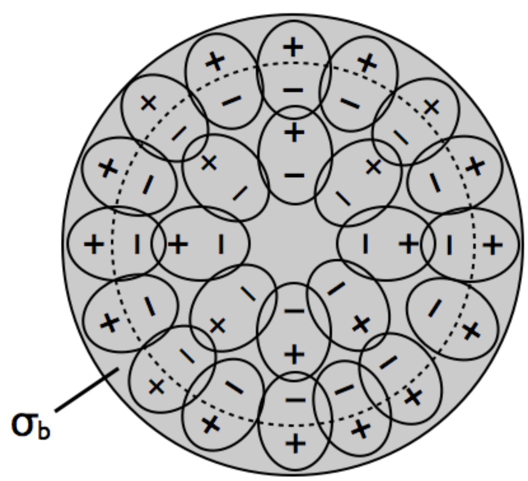


Figure 61: A spherical volume with non-uniform polarization $\mathbf{P} = P\,\hat{\mathbf{r}}$. Both ρ_b and σ_b are non-zero for this case.

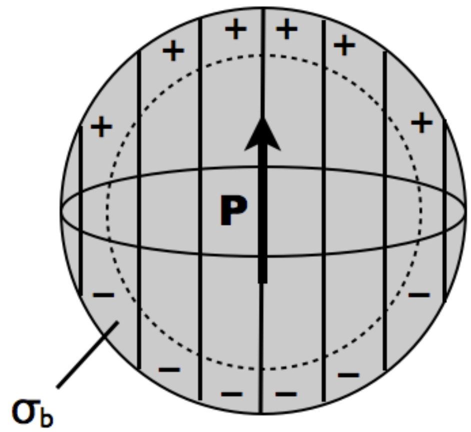


Figure 62: Uniformly polarized spherical volume. For this example, $\mathbf{P} = P\,\hat{\mathbf{z}}$, $\rho_b = 0$, and $\sigma_b = P\,\cos\theta$, which is maximum at the poles and zero at the equator.

- Exercise: Prove the above results. (Hint: First note that for $\mathbf{P} = P\,\hat{\mathbf{z}}$, $\rho_b = 0$ and $\sigma_b = P\cos\theta$. One can then do the area integral explicitly by expanding $1/|\mathbf{r}-\mathbf{r}'|$ in terms of spherical harmonics

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\, Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{680}$$

and noting that the only contribution to the ϕ' integration comes from $m=0$, and the only contribution to the $\cos\theta'$ integration comes from $l=1$.)

- Thus, the field $\mathbf{E}(\mathbf{r})$ for a uniformly polarized sphere is proportional to $\mathbf{P}$ inside the sphere (pointing in the opposite direction); outside the sphere it is the same as the field due to a single dipole at the origin, with dipole moment equal to the total dipole moment of the sphere.

5.2 Gauss's law for dielectrics

- Gauss's law in the form

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{681}$$

is always true, in vacuum or in dielectrics.

- In dielectrics, it is sometimes convenient to split the total charge density ρ into its bound and free components, $\rho = \rho_f + \rho_b$, and then use $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P}$ to rewrite Gauss's law as

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_f\,, \qquad \mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P} \tag{682}$$

- Exercise: Prove this.

- $\mathbf{D}$ is called the *electric displacement.*

- The integral form of Gauss's law for $\mathbf{D}$ is thus

$$\oint_S \mathbf{D}\cdot\hat{\mathbf{n}}\,da = Q_{f,\mathrm{enc}} \tag{683}$$

where $Q_{f,\mathrm{enc}}$ is the total enclosed free charge.

- Recall that the boundary conditions on the electric field are

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{\mathbf{n}} = \frac{\sigma}{\epsilon_0}\,, \qquad (\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{\mathbf{t}} = 0 \tag{684}$$

  where $\hat{\mathbf{n}}$ is the unit normal pointing from region 1 to region 2, and $\hat{\mathbf{t}}$ is the unit tangent to the boundary surface.

- In terms of the electric displacement, we have

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{\mathbf{n}} = \sigma_f\,, \qquad (\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{\mathbf{t}} = (\mathbf{P}_2 - \mathbf{P}_1) \cdot \hat{\mathbf{t}} \tag{685}$$

- Note that, unlike $\mathbf{E}$, the components of $\mathbf{D}$ parallel to the surface are not continuous across the boundary. This is a consequence of the fact that $\boldsymbol{\nabla} \times \mathbf{E} = 0$, but

$$\boldsymbol{\nabla} \times \mathbf{D} = \boldsymbol{\nabla} \times \mathbf{P} \tag{686}$$

  which does not equal zero, in general.

- This implies, in particular, that $\mathbf{D}$ does not admit a potential, as there is for $\mathbf{E}$.

5.3 Linear dielectrics

- In order to solve electrostatics problem in dielectric materials, we need to be given the polarization $\mathbf{P}$, or at least know how it is related to the field $\mathbf{E}$.

- For certain materials—called *linear, isotropic dielectrics*—we have

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \tag{687}$$

  where $\mathbf{E}$ is the total field, which includes the field associated with the polarization. The above relation is often a good approximation for *weak* fields $\mathbf{E}$.

- χ_e is called the *electric susceptibility* of the material.

- In general, χ_e depends on position $\mathbf{r}$. If χ_e is independent of position, then the material is said to be *homogeneous* (or *uniform*).

- Given the above relationship between $\mathbf{P}$ and $\mathbf{E}$, and the definition of $\mathbf{D}$ in terms of $\mathbf{E}$, it follows that

$$\mathbf{D} = \epsilon_0(1 + \chi_e)\mathbf{E} = k\epsilon_0\mathbf{E} = \epsilon\mathbf{E} \tag{688}$$

  where $k \equiv 1 + \chi_e$ is called the *dielectric constant*, and $\epsilon \equiv k\epsilon_0$ is called the *permitivity* of the material.

- <u>Exercise</u>: Prove this.

- The dielectric constant of dry air is 1.00054; the dielectric constant of water is 80.1.

5.4 Boundary value problems with linear dielectrics

- We will restrict attention to electrostatic boundary value problems for linear, isotropic, and *homogeneous* dielectric materials, where a number of simplifications follow from the *constant* proportionality of $\mathbf{P}$ and $\mathbf{E}$:

  1) The bound volume charge density and total charge density are proportional to ρ_f:

$$\rho_b = -\frac{\chi_e}{1 + \chi_e}\,\rho_f = -\frac{k-1}{k}\,\rho_f\,, \qquad \rho = \frac{\rho_f}{1 + \chi_e} = \frac{\rho_f}{k} \tag{689}$$

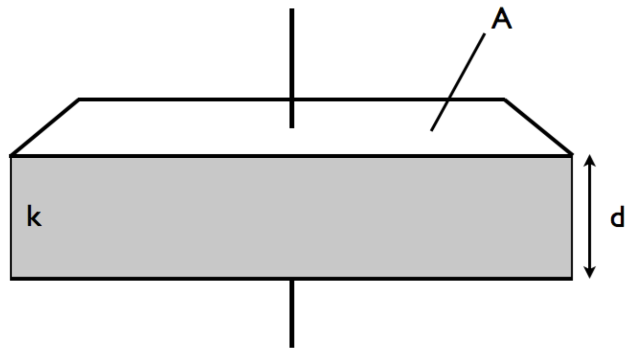


Figure 63: Parallel plate capacitor completely filled with a homogeneous, isotropic, linear dielectric (dielectric constant k).

2) The curl of both $\mathbf{D}$ and $\mathbf{P}$ vanish:

$$\nabla \times \mathbf{D} = \nabla \times \mathbf{P} = 0 \tag{690}$$

3) Gauss's law for $\mathbf{E}$ can be written simply in terms of the free charge density:

$$\nabla \cdot \mathbf{E} = \frac{\rho_f}{\epsilon} \tag{691}$$

4) The boundary conditions can be written as

$$(\epsilon_2 \mathbf{E}_2 - \epsilon_1 \mathbf{E}_1) \cdot \hat{\mathbf{n}} = \sigma_f \,, \qquad (\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{\mathbf{t}} = 0 \tag{692}$$

where $\hat{\mathbf{n}}$ is the unit normal pointing from region 1 to region 2, and $\hat{\mathbf{t}}$ is the unit tangent to the boundary surface.

- Exercise: Prove the above relations.
- Thus, all of the techniques that we learned previously for solving electrostatic boundary value problems carry over to the case of linear, isotropic, and homogeneous dielectrics with: (i) ρ replaced by ρ_f, (ii) ϵ_0 replaced by ϵ, and (iii) the perpendicular boundary condition for the electric field involving σ_f and the product of ϵ and $\mathbf{E}$.
- Example 1: Show that the capacitance of a parallel plate capacitor (separation d, plate area A) that is completely filled with a homogeneous, isotropic, linear dielectric (dielectric constant k) is

$$C = \frac{A\epsilon}{d} = k\frac{A\epsilon_0}{d} = kC_{\rm vac} \tag{693}$$

(See Figure 63.) Note that the capacitance calculated above is a factor of k times *larger* than what it is in vacuum. This is because the field $\mathbf{E}$ (and hence the potential difference V) between the capacitors plates for fixed charge Q is smaller due to the opposing electric field associated with the polarized dielectric. Hence the capacitance $C = Q/V$ (fixed Q) is larger than it is in vacuum.

- Example 2: Show that the capacitance of a parallel plate capacitor (separation d, plate area A) that is made up of two homogeneous, isotropic, linear dielectrics (each of thickness $d/2$, with permitivities ϵ_1, ϵ_2) is

$$C = \frac{A}{d}\frac{2\epsilon_1\epsilon_2}{\epsilon_1 + \epsilon_2} \tag{694}$$

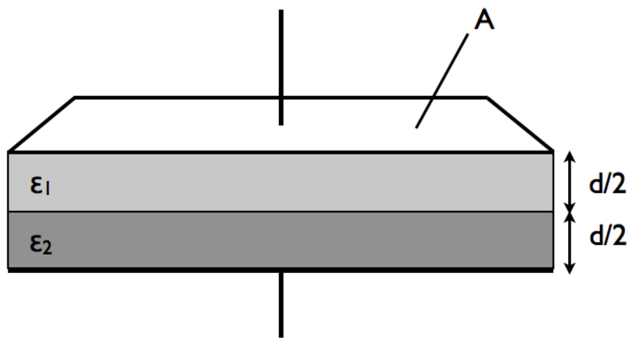


Figure 64: Parallel plate capacitor made up of two homogeneous, isotropic, linear dielectrics (each of thickness $d/2$, with permitivities ϵ_1, ϵ_2).

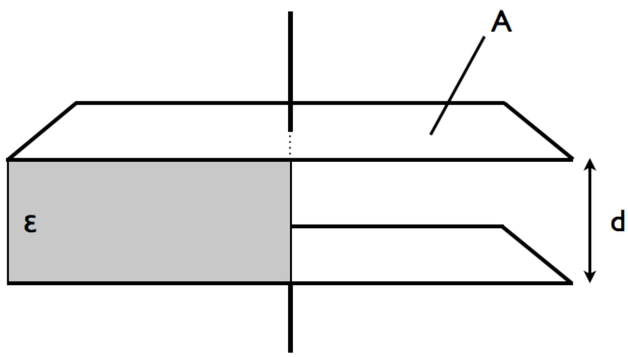


Figure 65: Parallel plate capacitor that is half-filled with a homogeneous, isotropic, linear dielectric (thickness d, area $A/2$, and dielectric constant k).

(See Figure 64.) Note that

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \tag{695}$$

where

$$C_1 = \frac{A\epsilon_1}{d/2}\,, \qquad C_2 = \frac{A\epsilon_2}{d/2} \tag{696}$$

consistent with the result that capacitors in series add like resistors in parallel.

- Example 3: Show that the capacitance of a parallel plate capacitor (separation d, plate area A) that is half-filled with a homogeneous, isotropic, linear dielectric (thickness d, area $A/2$, and dielectric constant k) is

$$C = \left(\frac{1+k}{2}\right)\frac{\epsilon_0 A}{d} \tag{697}$$

(See Figure 65.) Note that

$$C = C_1 + C_2 \tag{698}$$

where

$$C_1 = \frac{\epsilon_0 A/2}{d}\,, \qquad C_2 = \frac{k\epsilon_0 A/2}{d} \tag{699}$$

consistent with the result that capacitors in parallel (same potential difference V) add like resistors in series.

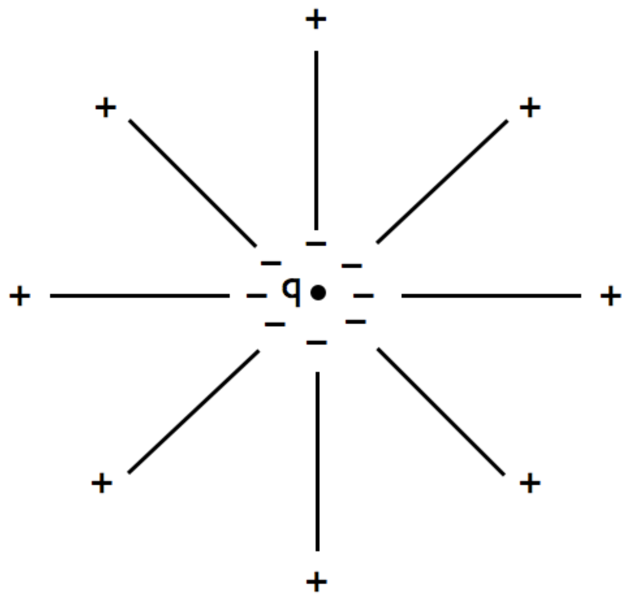


Figure 66: Positive point charge q embedded in a homogeneous, isotropic, linear dielectric (dielectric constant k). The negative charges surrounding q are *polarization charges*, which effectively reduce the magnitude of the point charge—i.e., $q \to q/k$. The positive polarization charges are located at the boundary of the dielectic.

- Example 4: Show that Coulomb's law for a point charge q embedded in a homogeneous, isotropic, linear dielectric (dielectric constant k) is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q/k}{r^2}\,\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\,\hat{\mathbf{r}} \tag{700}$$

See Figure 66.

- Example 5: Calculate the potential and field, both inside and outside a homogeneous, isotropic, linear dielectric sphere (radius R, dielectric constant k) placed in a uniform electric field $\mathbf{E}_0$. (See Figure 67.)

- Answer:

$$\Phi(\mathbf{r}) = \begin{cases} -E_0 r\cos\theta\left(\frac{3}{k+2}\right), & r<R \\ -E_0 r\cos\theta + \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, & r>R \end{cases} \tag{701}$$

and

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \frac{3}{k+2}\,\mathbf{E}_0\,, & r<R \\ \mathbf{E}_0 + \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right], & r>R \end{cases} \tag{702}$$

where

$$\mathbf{p} \equiv \frac{4}{3}\pi R^3\,\mathbf{P}\,, \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}_{\text{in}} = 3\epsilon_0\left(\frac{k-1}{k+2}\right)\mathbf{E}_0 \tag{703}$$

Note that the bound surface charge is given by

$$\sigma_b(\theta) = \mathbf{P}\cdot\hat{\mathbf{n}} = 3\epsilon_0\left(\frac{k-1}{k+2}\right)E_0\cos\theta \tag{704}$$

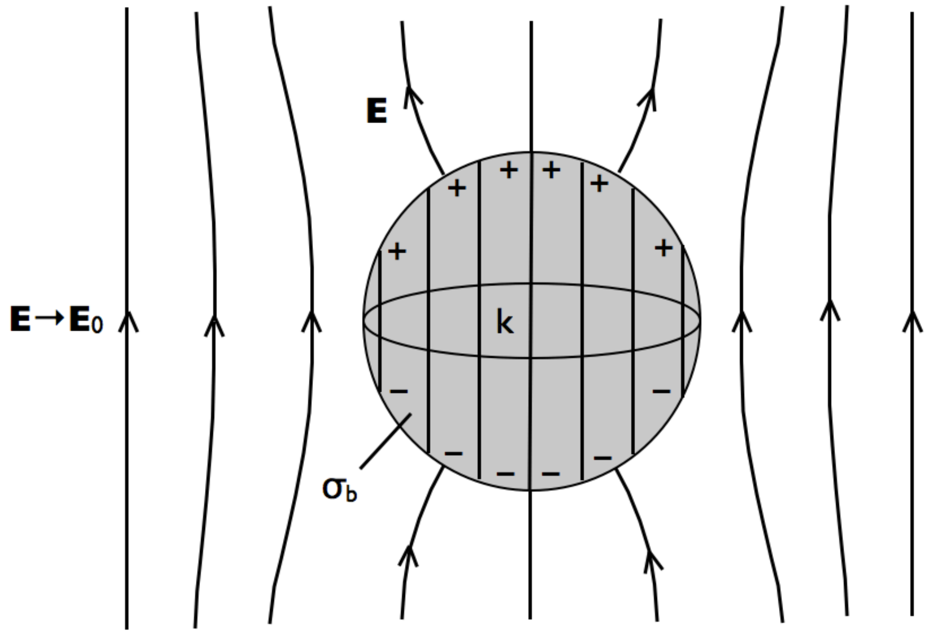


Figure 67: A homogeneous, isotropic, linear dielectric sphere (radius R, dielectric constant k) placed in a uniform electric field $\mathbf{E}_0$.

Thus, the field inside the sphere is constant (with magnitude less than E_0 due to the competing polarization field); outside the sphere, the field equals $\mathbf{E}_0$ plus the field of a dipole $\mathbf{p}$ at the origin. The polarization of the sphere $\mathbf{P}$ is constant.

The expression for the potential, field, surface charge, etc. for a *conducting* sphere of radius R in a uniform field $\mathbf{E}_0$ are obtained by letting χ_e (or k) go to infinity (Griffiths, Example 3.8).

Hint: To prove these results expand the potential $\Phi(\mathbf{r})$ both inside and outside the sphere in terms of Legendre polynomials. The expansion coefficients are determined by requiring that:

i) $\Phi(\mathbf{r})$ be finite at the origin $r = 0$

ii) $\Phi(\mathbf{r}) \to -E_0 r \cos\theta$ as $r \to \infty$

iii) Φ be continuous across the boundary $r = R$

iv) The normal derivative of $\Phi(\mathbf{r})$ be discontinuous across the boundary:

$$\left.\frac{\partial \Phi_{\rm out}}{\partial r}\right|_{r=R} = k\left.\frac{\partial \Phi_{\rm in}}{\partial r}\right|_{r=R} \tag{705}$$

Note: A quicker way to obtain the above results is to argue that the induced polarization $\mathbf{P}$ inside the spherical volume is necessarily constant, and then use equations (678) and (679) for the potential $\Phi_{\rm sphere}$ and field $\mathbf{E}_{\rm sphere}$ of a uniformly polarized sphere. The total potential and field are then $\Phi = \Phi_0 + \Phi_{\rm sphere}$ and $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_{\rm sphere}$. But be advised that this quick way works only for *ellipsoidal-shaped* dielectrics (of which a sphere is a special case); dielectrics having other, more general, shapes will *not* be uniformly polarized when placed in a uniform electric field.

- Example 6: Consider a point charge q a distance d above a semi-infinite homogeneous, isotropic, linear dielectric (dielectric constant k) with boundary surface at $z = 0$. Calculate

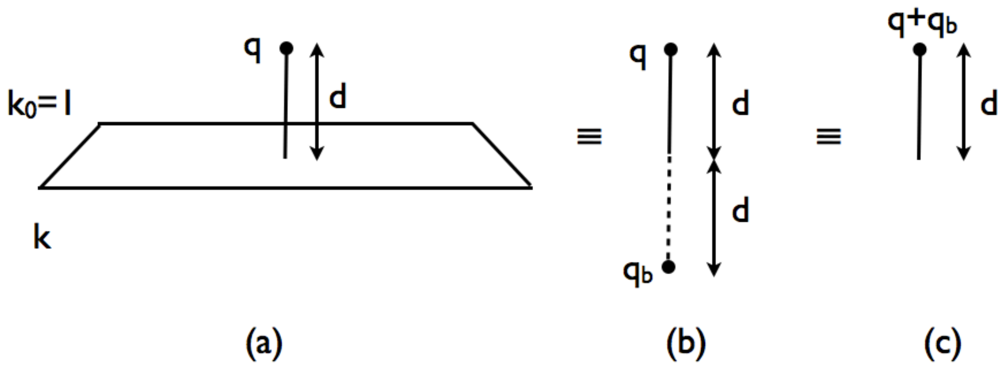


Figure 68: Panel (a): A point charge q a distance d above a semi-infinite homogeneous, isotropic, linear dielectric (dielectric constant k) with boundary surface at $z = 0$. Panel (b): Equivalent image problem for $z \geq 0$, with an image charge equal to q_b located at $z = -d$. Panel (c): Equivalent image problem for $z \leq 0$, with total image charge equal to $q + q_b$ located at $z = d$.

the potential $\Phi(\mathbf{r})$ for both $z > 0$ and $z < 0$. Also calculate the force of q due to the induced bound surface charge σ_b. (See Figure 68.)

- Answer:

$$\Phi(\mathbf{r}) = \begin{cases} \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_b}{\sqrt{x^2+y^2+(z+d)^2}}\right), & z > 0 \\ \frac{1}{4\pi\epsilon_0}\frac{q+q_b}{\sqrt{x^2+y^2+(z-d)^2}}, & z < 0 \end{cases} \tag{706}$$

and

$$\mathbf{F}_q = \frac{1}{4\pi\epsilon_0}\frac{qq_b}{(2d)^2}\,\hat{\mathbf{z}} \tag{707}$$

where

$$q_b = -\left(\frac{k-1}{k+1}\right) q \tag{708}$$

is the total (integrated) bound charge.

Proof: The simplest method of proof is to use a *method of images* argument:

1) One first calculates the bound surface charge σ_b from

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}\big|_{z=0_-} = \epsilon_0\chi_e\left[-\frac{1}{4\pi\epsilon_0}\frac{qd}{(x^2+y^2+d^2)^{3/2}} - \frac{1}{2}\frac{\sigma_b}{\epsilon_0}\right] \tag{709}$$

which yields

$$\sigma_b(x,y) = -\left(\frac{k-1}{k+1}\right)\frac{1}{2\pi}\frac{qd}{(x^2+y^2+d^2)^{3/2}} \tag{710}$$

(See Figure 69.)

2) Integrating this expression over the plane $z = 0$ gives the total bound charge q_b above.

3) For $z > 0$ we can replace the dielectric with a single bound point charge q_b at the image location $z = -d$, while for $z < 0$ we can replace the the region above the plane with a single point charge of strength $q + q_b$.

4) The potential in the two regions are then given as above. To verify that it is the correct potential, one can show that it vanishes as $r \to \infty$, is continuous across the boundary $z = 0$, and its normal derivative has the proper discontinuity σ_b/ϵ_0 across the boundary surface.

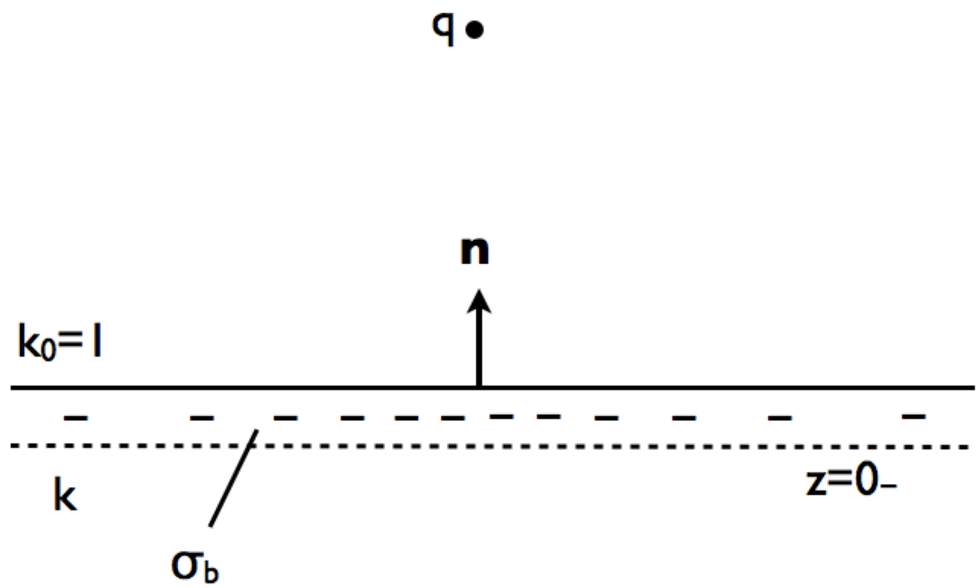


Figure 69: Definition of the unit normal $\hat{\mathbf{n}}$ to the dielectric (dielectric constant k), the location of $z = 0_-$, and the bound surface charge density σ_b for Example 6.

- Example 7: (Prob 4.25, Griffiths) Generalize the analysis of the previous problem to calculate the potential and force on a point charge q embedded in a semi-infinite homogeneous, isotropic, linear dielectric k_1, a distance d away from a plane surface ($z = 0$) that seperates the first medium from another homogeneous, isotropic, linear dielectric k_2. (See Figure 70.)

- Answer:

$$\Phi(\mathbf{r}) = \begin{cases} \frac{1}{4\pi\epsilon_0}\left(\frac{q'}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_{b,\mathrm{tot}}}{\sqrt{x^2+y^2+(z+d)^2}}\right), & z > 0 \\ \frac{1}{4\pi\epsilon_0}\frac{q'+q_{b,\mathrm{tot}}}{\sqrt{x^2+y^2+(z-d)^2}}, & z < 0 \end{cases} \tag{711}$$

and

$$\mathbf{F}_q = \frac{1}{4\pi\epsilon_0}\frac{q' q_{b,\mathrm{tot}}}{(2d)^2}\,\hat{\mathbf{z}} \tag{712}$$

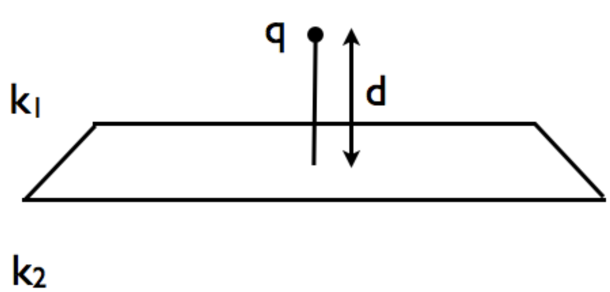


Figure 70: A point charge q embedded in a semi-infinite homogeneous, isotropic, linear dielectric k_1, a distance d away from a plane surface that seperates the first medium from another homogeneous, isotropic, linear dielectric k_2.

where

$$q' = \frac{q}{k_1}\,, \qquad q_{b,\text{tot}} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)\frac{q}{k_1} \tag{713}$$

is the total charge at $z = d$ (taking into account the bound charge induced at $z = d$) and the total (integrated) bound charge, respectively.

Proof: Use *method of images* as before:

1) First note that the free charge q is modified by the induced polarization charge q_P at $z = d$ to be

$$q' = q + q_P = \frac{q}{k_1} \tag{714}$$

2) The bound surface charges σ_{b1} and σ_{b2} at the respective boundaries of the two dielectrics can be calculated from the equations:

$$\sigma_{b1} = \mathbf{P}_1 \cdot \hat{\mathbf{n}}_1\big|_{z=0_+} = -\epsilon_0\chi_{e1}\left[-\frac{1}{4\pi\epsilon_0}\frac{q'd}{(x^2+y^2+d^2)^{3/2}} + \frac{1}{2}\frac{\sigma_{b1}}{\epsilon_0} + \frac{1}{2}\frac{\sigma_{b2}}{\epsilon_0}\right] \tag{715}$$

$$\sigma_{b2} = \mathbf{P}_2 \cdot \hat{\mathbf{n}}_2\big|_{z=0_-} = +\epsilon_0\chi_{e2}\left[-\frac{1}{4\pi\epsilon_0}\frac{q'd}{(x^2+y^2+d^2)^{3/2}} - \frac{1}{2}\frac{\sigma_{b1}}{\epsilon_0} - \frac{1}{2}\frac{\sigma_{b2}}{\epsilon_0}\right] \tag{716}$$

which yield

$$\sigma_{b1}(x,y) = +k_2\left(\frac{k_1 - 1}{k_1 + k_2}\right)\frac{1}{2\pi}\frac{q'd}{(x^2+y^2+d^2)^{3/2}} \tag{717}$$

$$\sigma_{b2}(x,y) = -k_1\left(\frac{k_2 - 1}{k_1 + k_2}\right)\frac{1}{2\pi}\frac{q'd}{(x^2+y^2+d^2)^{3/2}} \tag{718}$$

and

$$\sigma_{b,\text{tot}}(x,y) = \sigma_{b1}(x,y) + \sigma_{b2}(x,y) = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)\frac{1}{2\pi}\frac{q'd}{(x^2+y^2+d^2)^{3/2}} \tag{719}$$

(See Figure 71.)

3) Integrating this last expression over the plane $z = 0$ gives the total bound charge $q_{b,\text{tot}}$ above.

4) Replacement of the dielectrics by image charges is the same as in the previous example leading to the above expressions for the potential and force on q.

Note: All of the results of the previous example (Example 6) are special cases of this example, with $k_1 = 1$, $k_2 = k$.

5.5 Energy for dielectric systems

- Recall the energy stored in a capacitor

$$W = \frac{1}{2}CV^2 \tag{720}$$

and the general expression for the energy stored in an electrostatic system:

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2(\mathbf{r})\,dV \tag{721}$$

- The fact that the energy in a capacitor filled with a dielectric is greater than that filled with vacuum by a factor of $k = \epsilon/\epsilon_0$, suggests that W above should be replaced by

$$W = \frac{1}{2}\int_{\text{all space}} \mathbf{D}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})\,dV \tag{722}$$

for linear, isotropic dielectric systems.

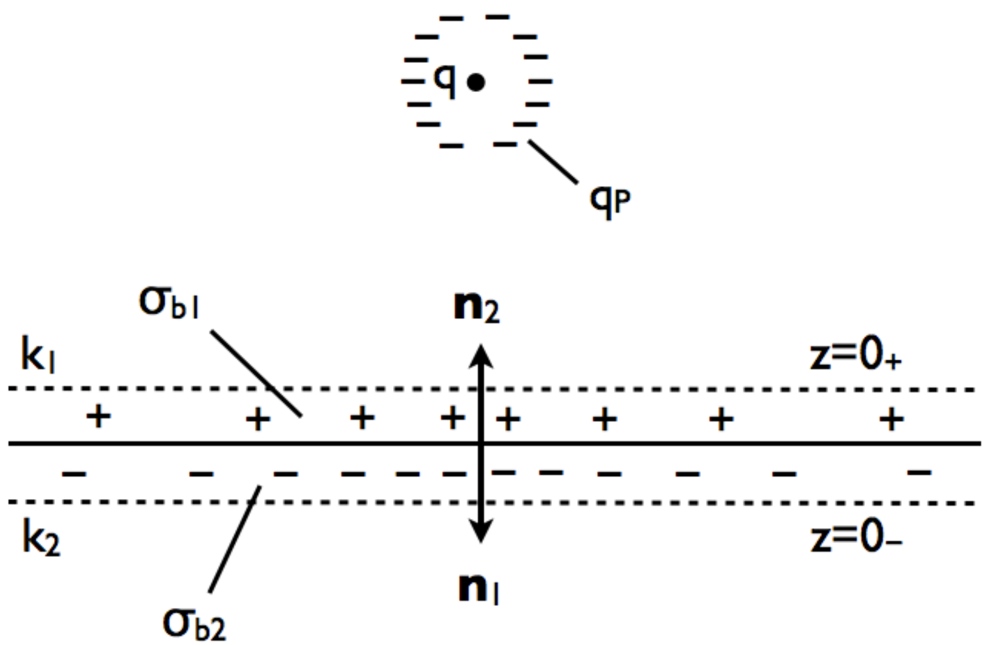


Figure 71: Definition of the unit normals $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$ to the two dielectrics (dielectric constants k_1, k_2); the location of $z = 0_-$, $z = 0_+$; the polarization charge q_P surrounding q; and the bound surface charge densities σ_{b1}, σ_{b2} for Example 7.

- Proof:

  1) Start with the expression for the energy of a general charge distribution $\rho(\mathbf{r})$ in an electrostatic potential $\Phi(\mathbf{r})$:

$$W = \int_{\text{all space}} \rho(\mathbf{r})\Phi(\mathbf{r})\, dV \tag{723}$$

  2) A change in the charge density ρ by a small amount $\Delta\rho_f$ in the free charge density leads to a corresponding change in energy:

$$\Delta W = \int_{\text{all space}} \Delta\rho_f(\mathbf{r})\Phi(\mathbf{r})\, dV \tag{724}$$

  3) Use Gauss's law for the electric displacement $\mathbf{D}$ to write

$$\Delta W = \int_{\text{all space}} \boldsymbol{\nabla} \cdot (\Delta\mathbf{D}(\mathbf{r}))\, \Phi(\mathbf{r})\, dV \tag{725}$$

  4) Then integrate by parts, throwing away the surface term at infinity:

$$\Delta W = \int_{\text{all space}} \Delta\mathbf{D}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})\, dV \tag{726}$$

  5) Now assume a linear, isotropic dielectric, for which $\mathbf{D} = \epsilon\mathbf{E}$. Since the permitivity ϵ of the dielectric is fixed, it follows that

$$\Delta(\mathbf{D} \cdot \mathbf{E}) = 2\Delta\mathbf{D} \cdot \mathbf{E} \tag{727}$$

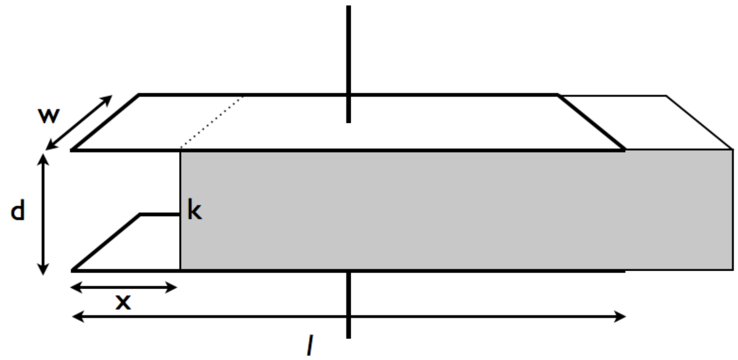


Figure 72: A linear, isotropic, homogeneous dielectric placed partailly between the plates of a parallel-plate capacitor.

6) Thus,

$$\Delta W = \frac{1}{2}\int_{\text{all space}} \Delta\left(\mathbf{D}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})\right)\,dV \tag{728}$$

or

$$W = \frac{1}{2}\int_{\text{all space}} \mathbf{D}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})\,dV \tag{729}$$

QED

- Expression (722) for W is the energy required to assemble the system of *free* charges, letting the dielectric respond to the presence of the free charges. This means that in addition to the electrostatic energy W_f of the free charges, W includes the electrostatic energy W_b of the bound polarization charges *and* the work W_{spring} required to create and rotate the dipole moments against the restoring forces of the atoms and molecules in the dielectric:

$$W = W_f + W_b + W_{\text{spring}} \tag{730}$$

- As the bound charges are always in equilibrium

$$W_b + W_{\text{spring}} = 0 \Leftrightarrow W = W_f \tag{731}$$

- The original expression (721) for the energy W includes only W_f and W_b; it does not include the "spring" energy W_{spring}, and hence represents a *different* energy for the system.

5.6 Forces on dielectrics

- Just as conductors experience a force when placed in an electrostatic field, so too do dielectrics. The reason for the force is the same—namely, the induced bound charges are primarily opposite in sign to those of the field and hence give rise to an attractive force.
- As an explicit example, we will calculate the force exerted by the electrostatic fields on a linear, isotropic, homogeneous dielectric placed partially between the plates of a parallel-plate capacitor.
- Let w denote the width of the capacitor plates, l their length, d their separation, and x the distance from the lefthand edge of the capacitor where the dielectric material starts. (See Figure 72.)

  Then one can show that

$$C = \frac{\epsilon_0 w}{d}\left[kl - (k-1)x\right] \tag{732}$$

- Exercise: Prove the above.
- Let $F_{\rm me}$ be the force that I exert on the dielectric to displace it a distance dx. Then the work that I do on the dielectric is

$$dW = F_{\rm me}\, dx \tag{733}$$

- The force exerted on the dielectric *by the field* is then

$$F = -F_{\rm me} = -\frac{dW}{dx} \tag{734}$$

- Since

$$W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} \tag{735}$$

it follows that

$$F = -\frac{dW}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx} = -\frac{\epsilon_0(k-1)w}{2d}V^2 \tag{736}$$

where I've assumed that *the charge Q on the capacitor plates is held fixed.*

- If we want to do the same calculation, but under the condition that the potential difference V between the capacitor plates is held fixed, then we must also include the *work done by the battery* to move the necessary charge to keep V constant:

$$dW = F_{\rm me}\, dx + dW_{\rm battery} = F_{\rm me}\, dx + V\, dQ \tag{737}$$

- Thus, for this case

$$F = -F_{\rm me} = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{1}{2}V^2\frac{dC}{dx} + V^2\frac{dC}{dx} = -\frac{\epsilon_0(k-1)w}{2d}V^2 \tag{738}$$

which is the same as before (as it should be).

5.7 Atomic polarizability

- Consider an atom or molecule in a linear, isotropic dielectric subject to an externally applied field $\mathbf{E}_0$, which we will assume to be uniform. The *local* electric field $\mathbf{E}_{\rm loc}$ at the location of the ith atom or molecule is the the sum of $\mathbf{E}_0$ and the fields due to all the *other* induced dipoles in the dielectric:

$$\mathbf{E}_{\rm loc} = \mathbf{E}_0 + \sum_{j\neq i}\mathbf{E}_{{\rm dip},j}(i) \tag{739}$$

where

$$\mathbf{E}_{{\rm dip},j}(i) = \frac{1}{4\pi\epsilon_0 r_{ij}^3}\left[3(\mathbf{p}_j\cdot\hat{\mathbf{r}}_{ij})\hat{\mathbf{r}}_{ij} - \mathbf{p}_j\right]\,, \quad \mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j \tag{740}$$

Note that $\mathbf{E}_{\rm loc}$ is the electric field acting on the atom or molecule, due to everything *other* than the particular atom or molecule under consideration. In general, $\mathbf{E}_{\rm loc}$ is different from the (total) macroscopic field $\mathbf{E}$ in the dielectric. $\mathbf{E}$ is the average of the microscopic electric field over a volume of radius R, which is many times larger than the size $a \sim 10^{-10}$ m of an atom or molecule.

- The *atomic polarizability* α is defined to be the proportionality constant relating the induced dipole moment $\mathbf{p}$ of the atom or molecule to the local field $\mathbf{E}_{\rm loc}$:

$$\mathbf{p} = \alpha\epsilon_0\mathbf{E}_{\rm loc} \tag{741}$$

As before, we are assuming that the field $\mathbf{E}_{\rm loc}$ is weak enough so that (741) is a good approximation (i.e., linear, isotropic dielectric assumption).

- For polar molecules, which already have built-in atomic dipole moments $\mathbf{p}$ (with constant magnitude p_0 but pointing in random directions in the absence of an external field), α is the proportionality constant relating $\langle p \rangle \equiv p_0 \langle \cos\theta \rangle$ to the magnitude $E_{\rm loc} \equiv |\mathbf{E}_{\rm loc}|$. Here, $\langle \cos\theta \rangle$ is the expected value of the cosine of the angle between $\mathbf{p}$ and $\mathbf{E}_{\rm loc}$, using Boltzmann's factor as the probability distribution to calculate the expected value. (More about this later.)

- α has dimensions of volume.

- For electronic polarization of non-polar atoms or molecules (see next subsection for details), one can obtain an order of magnitude estimate of α as follows:

  In response to $\mathbf{E}_{\rm loc}$, the centers of positive and negative charge in a non-polar atom or molecule will be displaced by a distance d. The ratio of d to the characteristic size a of an atom or molecule will be (roughly) equal to the ratio of $E_{\rm loc}$ to the typical field strength in the atom or molecule:

$$\frac{d}{a} \sim \frac{E_{\rm loc}}{\frac{q}{4\pi\epsilon_0}\frac{1}{a^2}} \tag{742}$$

  Thus,

$$p \equiv qd \sim 4\pi\epsilon_0 a^3\, E_{\rm loc} \Leftrightarrow \alpha = 4\pi a^3 \sim 10^{-29}\ {\rm m}^3\ (\equiv V_{\rm atom}) \tag{743}$$

  So α is of order the volume of an atom or molecule for electronic polarization.

- For orientation polarization associated with polar molecules, α may be much larger than $V_{\rm atom}$ if the magnitude p_0 of the built-in dipole moments is sufficiently large. An example is water (H_2O).

- The polarization $\mathbf{P}$ is related to the dipole moment $\mathbf{p}$ via

$$\mathbf{P} = N\mathbf{p} \tag{744}$$

  where N is the number density of atoms or molecules in the dielectric.

- Note that

$$N = \frac{1}{\frac{4}{3}\pi b^3} \tag{745}$$

  where b is the characteristic interatomic or intermolecular spacing. For a gas, $b \gg a$.

- For example, for a gas at STP (i.e, 0° C and 1 atm)

$$N = \frac{6.02\times 10^{23}}{22.4\times 10^{-3}\ {\rm m}^3} = 2.69\times 10^{25}\ {\rm m}^{-3} \tag{746}$$

- Since $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$ for linear, isotropic dielectrics, it follows that

$$\epsilon_0\chi_e\mathbf{E} = \mathbf{P} = N\mathbf{p} = N\alpha\epsilon_0\mathbf{E}_{\rm loc} \tag{747}$$

  Thus, to relate α to χ_e, we need to know how $\mathbf{E}_{\rm loc}$ is related to the macroscopic field $\mathbf{E}$ in the dielectric.

- For dilute materials, such as gases, it turns out that

$$\mathbf{E}_{\rm loc} \approx \mathbf{E} \quad \text{(for gases)} \tag{748}$$

  This approximation amounts to being able to ignore the field at the atom or molecule under consideration, which is produced by the induced dipole moments of the nearby molecules.

- To justify this approximation, recall the general result (664) for the average of the electric field over a spherical volume of radius R:

$$\mathbf{E}_{\rm ave} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_{\rm enc}}{R^3} + \mathbf{E}_{\rm ext}(\mathbf{0}) \tag{749}$$

where $\mathbf{p}_{\rm enc}$ is the total dipole moment enclosed by the volume, and $\mathbf{E}_{\rm ext}(\mathbf{0})$ is the field at the center of the sphere produced by all charges *exterior* to the sphere.

- In order for $\mathbf{E}_{\rm ave}$ to equal the total macroscopic field $\mathbf{E}$, we need R to be large relative to the characteristic atomic size a. For a gas, we can satisfy this requirement by taking R to equal the characteristic interatomic or intermolecular spacing b. Thus, choosing $R = b$, we have

$$\mathbf{E}_{\rm ave} = \mathbf{E}\,, \quad \mathbf{E}_{\rm ext}(\mathbf{0}) = \mathbf{E}_{\rm loc} \tag{750}$$

and

$$-\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_{\rm enc}}{R^3} = -\frac{\alpha\epsilon_0\mathbf{E}_{\rm loc}}{4\pi\epsilon_0 b^3} = -\left(\frac{a}{b}\right)^3\mathbf{E}_{\rm loc} \ll \mathbf{E}_{\rm loc} \tag{751}$$

which implies $\mathbf{E} \approx \mathbf{E}_{\rm loc}$ as claimed above.

- Thus,

$$N\alpha = \chi_e \quad \text{(for gases)} \tag{752}$$

- If the gas consists of non-polar atoms or molecules, we have (to order-of-magnitude)

$$\chi_e = N\alpha \sim 2.69\times 10^{25}\ {\rm m}^{-3} \times 10^{-29}\ {\rm m}^3 \sim 10^{-4} \quad \text{(for nonpolar gases)} \tag{753}$$

- For denser materials, such as liquids or solids, $\mathbf{E}_{\rm loc} \neq \mathbf{E}$, so the relationship between α and χ_e is necessarily more complicated. (This will be discussed in more detail in Section 5.10.)

5.8 Electronic polarization of non-polar atoms or molecules

- For a non-polar atom or molecule, the field $\mathbf{E}_{\rm loc}$ displaces the center of positive charge relative to that of the negatively-charged electron cloud by a distance d, leading to the equilibrium condition

$$qE_{\rm loc} = F_{\rm restoring} = \kappa d \tag{754}$$

where κ is a constant characterizing the strength of the restoring force.

- In terms of the mass m of the electrons and a characteristic frequency ω_0 of the atom or molecule, we have

$$\kappa = m\omega_0^2 \tag{755}$$

(The positive nucleus, being much more massive than the electrons, will effectively remain fixed in position.)

- Although there are many characteristic frequencies for an atom or molecule, for our (rough) calculations it suffices to take $\omega_0 = E_{\rm ionization}/\hbar$, where $E_{\rm ionization}$ is the first ionization energy of the atom or molecule. Then q and m will be the charge and mass of a *single* electron. (A more rigorous analysis requires a full quantum-mechanical treatment of the problem.)

- Thus,

$$p \equiv qd = \frac{q^2 E_{\rm loc}}{m\omega_0^2} \Leftrightarrow \alpha = \frac{1}{\epsilon_0}\frac{q^2}{m\omega_0^2} \tag{756}$$

This is called *electronic* polarization.

- Using the above expression for α and the relation $N\alpha = \chi_e$ for gases, we can compare theoretical estimates of the dielectric constant $k \equiv 1 + \chi_e$ with experimentally-determined values.

- For example, one can show that for hydrogen gas at STP (i.e., 0° C and 1 atm),

$$k = 1.00020 \quad (\text{versus } k_{\text{exp}} = 1.00025) \tag{757}$$

The agreement between the calculated and experimental values of dielectric constant is quite good, given the approximations that we have made. (Note, for example, that we can use the ionization energy of 13.6 eV for a single hydrogen atom, even though hydrogen gas is diatomic, i.e., H_2.)

- Exercise: Prove this last result.

- Similarly, one can show that for helium gas at STP,

$$k = 1.000062 \quad (\text{versus } k_{\text{exp}} = 1.000065) \tag{758}$$

- Exercise: Prove this. (Note: The first ionization energy for helium is 24.5 eV.)

5.9 Orientation polarization for polar molecules

- For a polar molecule with built-in dipole moment $\mathbf{p}$, we will assume that the field $\mathbf{E}_{\text{loc}}$ doesn't change the magnitude $|\mathbf{p}| \equiv p_0$, but only its *direction*, trying to rotate $\mathbf{p}$ into alignment with the field.

- Thermal fluctuations compete with this alignment; the larger the absolute temperature T, the less the alignment.

- The expected value of the alignment can be calculated using the *Boltzmann factor*

$$\exp(-u/k_B T) \tag{759}$$

where

$$u \equiv -\mathbf{p} \cdot \mathbf{E}_{\text{loc}} \tag{760}$$

is the energy of the dipole $\mathbf{p}$ in the presence of the field $\mathbf{E}_{\text{loc}}$. The Boltzmann factor is proportional to the probabilty of the dipole having energy between u and $u + du$.

- Recall that

$$k_B = 1.38 \times 10^{-23} \text{ J} \cdot \text{K}^{-1} \tag{761}$$

is Boltzmann's constant.

- The expected value of the energy is given by

$$\langle u \rangle = \frac{\int u e^{-u/k_B T}\, du}{\int e^{-u/k_B T}\, du} \tag{762}$$

where the limits of integration are $u = -p_0 E_{\text{loc}}$ and $+p_0 E_{\text{loc}}$.

- Exercise: Show that

$$\int_{-p_0 E_{\text{loc}}}^{p_0 E_{\text{loc}}} u e^{-u/k_B T}\, du = -2 k_B T\, p_0 E_{\text{loc}} \left[\cosh\left(\frac{p_0 E_{\text{loc}}}{k_B T}\right) - \frac{k_B T}{p_0 E_{\text{loc}}} \sinh\left(\frac{p_0 E_{\text{loc}}}{k_B T}\right) \right] \tag{763}$$

$$\simeq -\frac{2}{3} \frac{(p_0 E_{\text{loc}})^3}{k_B T} \tag{764}$$

and

$$\int_{-p_0 E_{\text{loc}}}^{p_0 E_{\text{loc}}} e^{-u/k_B T}\, du = 2 k_B T \sinh\left(\frac{p_0 E_{\text{loc}}}{k_B T}\right) \simeq 2 p_0 E_{\text{loc}} \tag{765}$$

where the approximate equalities $\simeq$ are for large T (i.e., $k_B T \gg p_0 E_{\text{loc}}$).

- Thus,
$$\langle u\rangle = k_BT - p_0E_{\rm loc}\coth\left(\frac{p_0E_{\rm loc}}{k_BT}\right) \simeq -\frac{1}{3}\frac{(p_0E_{\rm loc})^2}{k_BT} \tag{766}$$

- Finally, if we define
$$\langle u\rangle = -\langle \mathbf{p}\cdot\mathbf{E}_{\rm loc}\rangle = -p_0E_{\rm loc}\langle\cos\theta\rangle \equiv -\langle p\rangle E_{\rm loc} \tag{767}$$
it follows that
$$\langle p\rangle = p_0\left[\coth\left(\frac{p_0E_{\rm loc}}{k_BT}\right) - \frac{k_BT}{p_0E_{\rm loc}}\right] \simeq \frac{1}{3}\frac{p_0^2E_{\rm loc}}{k_BT} \tag{768}$$

- The equation for $\langle p\rangle$ is called the *Langevin formula*.
- Thus, the *orientation* polarization is given by
$$\alpha \simeq \frac{p_0^2}{3\epsilon_0k_BT} \tag{769}$$

- As a somewhat extreme case, consider polar molecules at 0° C with a built-in dipole moment p_0 equal to the charge of an electron times the characteristic atomic size:
$$p_0 = 1.6\times10^{-19}\ \text{C}\times10^{-10}\ \text{m} = 1.6\times10^{-29}\ \text{C}\cdot\text{m} \tag{770}$$
Then
$$\alpha \sim 10^{-27}\ \text{m}^3 \quad \text{(extreme polar molecules, } 0^\circ\text{ C)} \tag{771}$$
which is about 100 times greater than a typical value of α for electronic polarization of non-polar atoms or molecules.

- Typical values of p_0 for polar molecules will be much less than the extreme value given above. For example, water (H_2O) has a relatively large dipole moment ($p_0 = 6.1\times10^{-30}$ C$\cdot$m), but it is still a factor of 2 or 3 smaller than the extreme value.
- For a gas of polar molecules, we can use $N\alpha = \chi_e$ as before, so that
$$\chi_e \simeq \frac{Np_0^2}{3\epsilon_0k_BT} \tag{772}$$

- To check these expressions, consider water vapor at 100° C and 1 atm. Then one can show that
$$k = 1.00546 \quad (\text{versus } k_{\rm exp} = 1.00587) \tag{773}$$
which is a good approximation to the true value.

- <u>Exercise</u>: Prove this. (Note: $p_0 = 6.1\times10^{-30}$ C$\cdot$m.)

5.10 Dielectric constant for liquids and solids

- For a liquid or solid, we can no longer approximate $\mathbf{E}_{\rm loc}$ by the total macroscopic field $\mathbf{E}$. A better approximation is obtained by subtracting from $\mathbf{E}$ the field $\mathbf{E}_P$ associated with the *averaged* polarization of the nearby dipoles, and then replacing it with the actual contribution $\mathbf{E}_{\rm near}$ from the nearby dipoles:
$$\mathbf{E}_{\rm loc} = \mathbf{E} - \mathbf{E}_P + \mathbf{E}_{\rm near} \tag{774}$$

- The averaging is over a spherical volume of radius R that is much greater than the atomic size a, but small compared to the dimensions of the material. Note that for a liquid or solid, the interatomic or intermolecular spacing b will be roughly equal to the atomic or molecular size a.

- Recall that

$$\mathbf{E}_P = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_{\rm enc}}{R^3} \tag{775}$$

where $\mathbf{p}_{\rm enc}$ is the total dipole moment enclosed by the spherical volume.

- Since $b \sim a$ for a liquid or solid, the number density N is given by

$$N = \frac{1}{\frac{4}{3}\pi a^3} \tag{776}$$

- This implies, in particular, that there are $(R/a)^3$ individual atoms or molecules contained in the spherical volume of radius R.

- Using the definition of $\mathbf{p}$ in terms of $\mathbf{E}_{\rm loc}$ and the above relationship between N and a, it follows that

$$\mathbf{E}_P = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_{\rm enc}}{R^3} = -\frac{1}{4\pi\epsilon_0 R^3}\left(\frac{R}{a}\right)^3\mathbf{p} = -\frac{1}{4\pi\epsilon_0 a^3}\alpha\epsilon_0\mathbf{E}_{\rm loc} = -\frac{N\alpha}{3}\mathbf{E}_{\rm loc} \tag{777}$$

- Thus,

$$\mathbf{E}_{\rm loc} = \mathbf{E} + \frac{N\alpha}{3}\mathbf{E}_{\rm loc} + \mathbf{E}_{\rm near} \tag{778}$$

- Now, for most materials

$$\mathbf{E}_{\rm near} \approx 0 \tag{779}$$

For a cubic lattice with constant polarization, $\mathbf{E}_{\rm near}$ is *identically* zero. (See Jackson, p. 161 or Kittel, p. 457 for details.)

- Thus,

$$\mathbf{E}_{\rm loc} \approx \frac{1}{(1 - N\alpha/3)}\mathbf{E} \quad (\text{for most liquids, solids}) \tag{780}$$

Note that this equation breaks down when $N\alpha = 3$. For such cases, our previous approximation $\mathbf{E}_{\rm near} \approx 0$ is not valid.

- Since

$$\epsilon_0\chi_e\mathbf{E} = \mathbf{P} = N\mathbf{p} = N\alpha\epsilon_0\mathbf{E}_{\rm loc} \tag{781}$$

it follows that

$$\chi_e = \frac{N\alpha}{1 - N\alpha/3} \quad (\text{for most liquids, solids}) \tag{782}$$

- <u>Exercise</u>: Prove this.

- The above expression can be rewritten for the dielectric constant k in terms of α:

$$k = \frac{1 + 2N\alpha/3}{1 - N\alpha/3} \quad (\text{for most liquids, solids}) \tag{783}$$

and then inverted to find α in terms of k:

$$\alpha = \frac{3}{N}\left(\frac{k-1}{k+2}\right) \quad (\text{for most liquids, solids}) \tag{784}$$

This last equation is called the *Clausius-Mossotti* formula.

- <u>Exercise</u>: Prove the above.

- The Clausius-Mossotti equation is only *approximately valid* for liquids and solids. As mentioned above, it holds only for materials with $N\alpha < 3$.

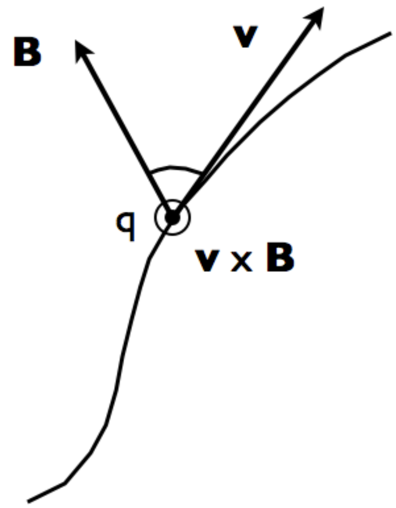


Figure 73: A point charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ experiences a force proportional to $\mathbf{v} \times \mathbf{B}$. This force is thus *perpendicular* to both $\mathbf{v}$ and $\mathbf{B}$.

- Example: (Feynman, Vol. II, Chpt. 11) Consider carbon disulfide (CS_2). In gaseous state at 0° C, the dielectric constant of CS_2 is 1.0029, which means that $N\alpha = 0.0029$. At 20° C, the density of liquid CS_2 is 381 times larger than the density of the gas at 0° C. Thus, $N\alpha$ for the liquid is $381 \times 0.0029 = 1.11$. Substituting these numbers in equation (783) yields

$$k = 2.76 \quad (\text{versus } k_{\text{exp}} = 2.64) \tag{785}$$

  so quite good agreement.

- Water at 20° C is a dielectric for which the above formulas do *not* apply. Using equation (783) yields $k = -3.05$, which is negative(!!), instead of $k_{\text{exp}} = 80.1$. Note that $N\alpha = 11.6 > 3$ for this case. (A more complete treatment is needed to theoretically calculate the dielectric constant of such materials. But I haven't found any good discussion of this to include in these notes!!)

6 Magnetostatics

6.1 Magnetic force

- Experiments show that a point charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ experiences a force

$$\mathbf{F}_{\text{mag}} = q\mathbf{v} \times \mathbf{B} \tag{786}$$

  See Figure 73.

- If there is also an electric field $\mathbf{E}$ present then the total force is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{787}$$

  This is called the *Lorentz force law.*

- Note that if $\mathbf{v} = 0$, the magnetic force is zero. Note also that the magnetic force is perpendicular to both $\mathbf{v}$ and $\mathbf{B}$.

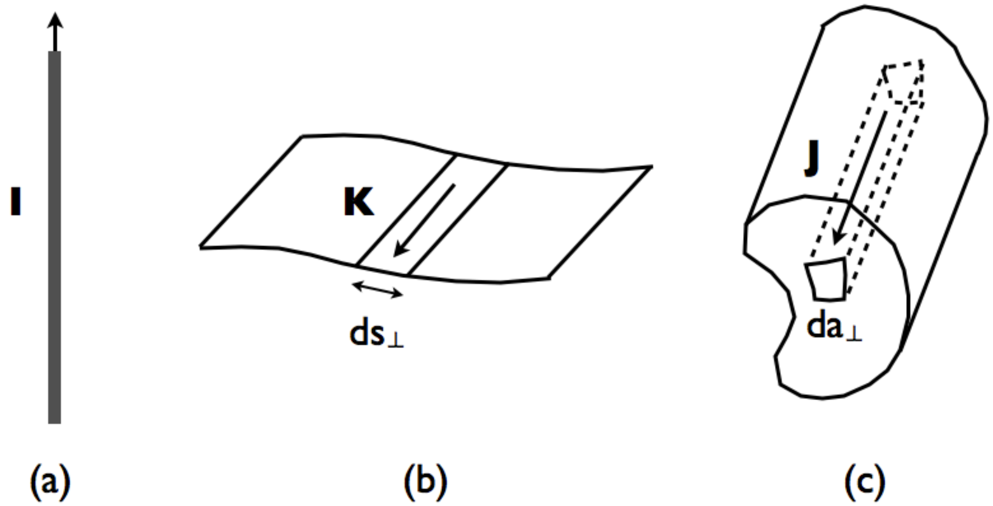


Figure 74: Examples of line, surface, and volume current densities $\mathbf{I}$, $\mathbf{K}$, and $\mathbf{J}$. Note that $\mathbf{K} = d\mathbf{I}/ds_\perp$ and $\mathbf{J} = d\mathbf{I}/da_\perp$, where $ds_\perp$ and $da_\perp$ are perpendicular to the flow of charge.

- Since $\mathbf{F}_{\rm mag}$ is perpendicular to $\mathbf{v}$, the magnetic force does *no* work on the charge:

$$dW_{\rm mag} = \mathbf{F}_{\rm mag} \cdot d\mathbf{s} = \mathbf{F}_{\rm mag} \cdot (\mathbf{v}\, dt) = 0 \tag{788}$$

- Hence the magnetic force can only change the direction of the velocity vector of the charge, but not its magnitude. (This is a consequence of the work-energy theorem which says that the work done by a force on a particle equals the change in kinetic energy of the particle.)

- If there are N point charges q_i, then the total magnetic force on the charges is given by the superposition principle:

$$\mathbf{F}_{\rm mag} = \sum_{i=1}^{N} q_i \mathbf{v}_i \times \mathbf{B}(i) \tag{789}$$

  where $\mathbf{B}(i)$ is the value of the magnetic field at the location of q_i.

- For a continous charge distribution we have

$$\mathbf{F}_{\rm mag} = \int dq\, \mathbf{v} \times \mathbf{B} \tag{790}$$

- For line, surface, and volume charge densities, we have

$$dq = \begin{cases} \lambda\, ds \\ \sigma\, da \\ \rho\, dV \end{cases}, \qquad \mathbf{v}\, dq = \begin{cases} \mathbf{I}\, ds \\ \mathbf{K}\, da \\ \mathbf{J}\, dV \end{cases} \tag{791}$$

  where

$$\mathbf{I} \equiv \lambda \mathbf{v}\,, \qquad \mathbf{K} \equiv \sigma \mathbf{v}\,, \qquad \mathbf{J} \equiv \rho \mathbf{v} \tag{792}$$

  are the line, surface, and volume *current densities*. (See Figure 74.)

- Note that $\mathbf{K}$ is the current per unit length perpendicular to the flow. Similarly, $\mathbf{J}$ is the current per unit area perpendicular to the flow.

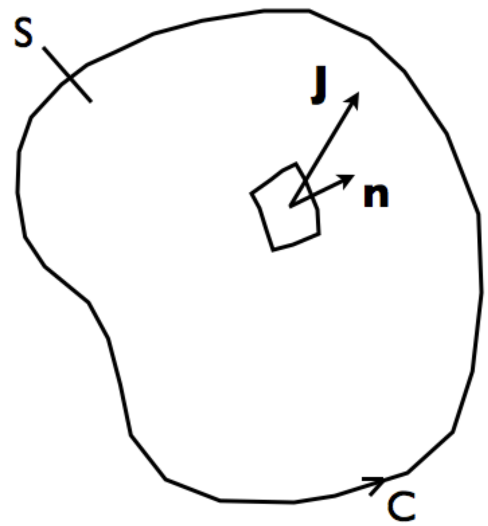


Figure 75: The total current flowing through a 2-dimensional surface S is the integral of $\mathbf{J} \cdot \hat{\mathbf{n}}$ over the surface.

- In particular, the current passing through a surface S is given by

$$I = \int_S \mathbf{J} \cdot \hat{\mathbf{n}} \, da \tag{793}$$

  See Figure 75.

- Since charge is conserved, the current flowing out of a closed surface S must equal (minus) the time rate of change of the total charge contained in the volume V enclosed by S:

$$\oint_S \mathbf{J} \cdot \hat{\mathbf{n}} \, da = -\frac{d}{dt} \left[\int_V \rho \, dV \right] \tag{794}$$

- We can use the divergence theorem to evaluate the LHS, and we can bring the time derivative inside the integral on the RHS by changing it to a partial derivative:

$$\int_V (\boldsymbol{\nabla} \cdot \mathbf{J}) \, dV = -\int_V \frac{\partial \rho}{\partial t} \, dV \tag{795}$$

  so that

$$\boldsymbol{\nabla} \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{796}$$

- The above equation is called the *continuity equation*. It is the mathematical statement of local charge conservation.

6.2 Steady currents

- Stationary charges (i.e., charges at rest) produce time-independent electric fields. This is the context of *electrostatics*.

- Similarly, *steady currents* produce time-independent magnetic fields. This is the context of *magnetostatics*.

- Note that a single charge q moving with constant velocity $\mathbf{v}$ does *not* produce a steady current.

- A volume charge density ρ gives rise to a steady current density $\mathbf{J}$ if and only if $\partial\rho/\partial t = 0$. Using the continuity equation, this implies

$$\boldsymbol{\nabla} \cdot \mathbf{J} = 0 \qquad \text{(magnetostatics)} \tag{797}$$

6.3 Biot-Savart law

- The Biot-Savart law gives the magnetic field produced by a steady current distribution:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int ds' \, \frac{\mathbf{I}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{798}$$

- There are similar equations for surface current and volume current densities $\mathbf{K}$ and $\mathbf{J}$; simply replace $\mathbf{I}(\mathbf{r}')\, ds'$ with $\mathbf{K}(\mathbf{r}')\, da'$ and $\mathbf{J}(\mathbf{r}')\, dV'$.

- The proportionality constant μ_0 is called the *permeability of free space* and has the value

$$\mu_0 = 4\pi \times 10^{-7} \; \frac{\mathrm{N}}{\mathrm{A}^2} \tag{799}$$

- The Biot-Savart law is the analog of Coulomb's law for electrostatics. But since a single charge does not give rise to a steady current, one needs to work from the very start in magnetostatics with extended current distributions.

- The MKS unit of the magnetic field is the *tesla*:

$$1 \ \mathrm{T} = 1 \ \frac{\mathrm{N}}{\mathrm{A} \cdot \mathrm{m}} \tag{800}$$

- The CGS unit of magnetic field is the *gauss*:

$$1 \ \mathrm{T} = 10^4 \ \text{gauss} \tag{801}$$

- The strength of the Earth's magnetic field is approximately 0.5 gauss.

- Note that the vector nature of the cross product in the integrand in the Biot-Savart law is such that the contribution $d\mathbf{B}$ to the magnetic field *encircles* the current element $\mathbf{I}(\mathbf{r}')\, ds'$ in accordance with the right-hand rule, with your thumb pointing in the direction of $\mathbf{I}(\mathbf{r}')$. (See Figure 76.)

- <u>Exercise</u>: Show that the magnetic field a perpendicular distance s from a segment of a straight wire has magnitude

$$B(s) = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1) \tag{802}$$

  where θ_1 and θ_2 are the angles that lines drawn to the two ends of the wire make with the vertical. (See Figure 77.)

  In agreement with our earlier general observation, $\mathbf{B}$ circles around the wire in accordance with the right-hand rule.

  For an straight wire of infinite length, $\theta_1 \to -\pi/2$ and $\theta_2 \to \pi/2$, for which

$$B(s) = \frac{\mu_0 I}{2\pi s} \tag{803}$$

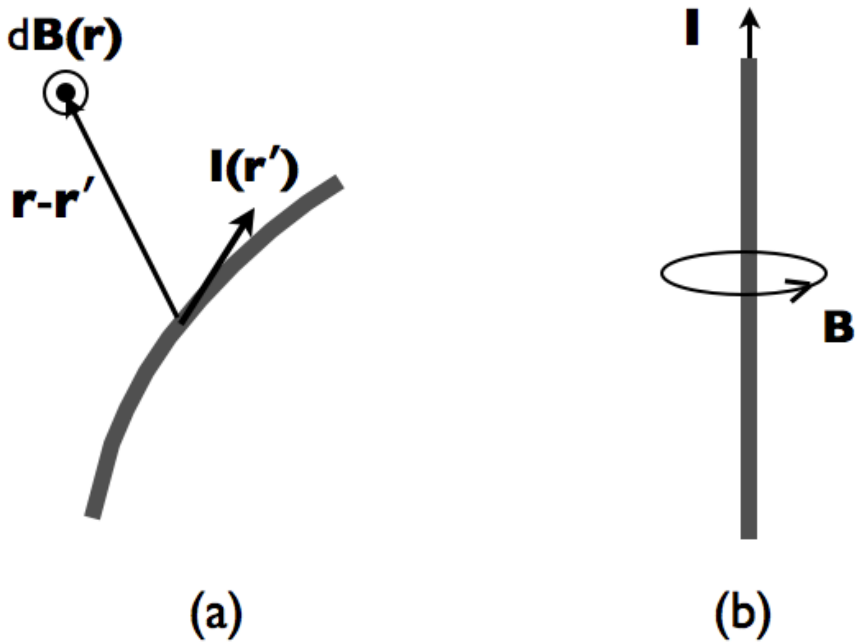


Figure 76: Magnetic fields lines **B** encircle current-carrying wires **I**, in a direction given by the right-hand rule. (a): Contribution to the magnetic field at $\mathbf{r}$ due to the current flowing at $\mathbf{r}'$. (b): Magnetic field lines for an infinitely-long, straight current-carrying wire.

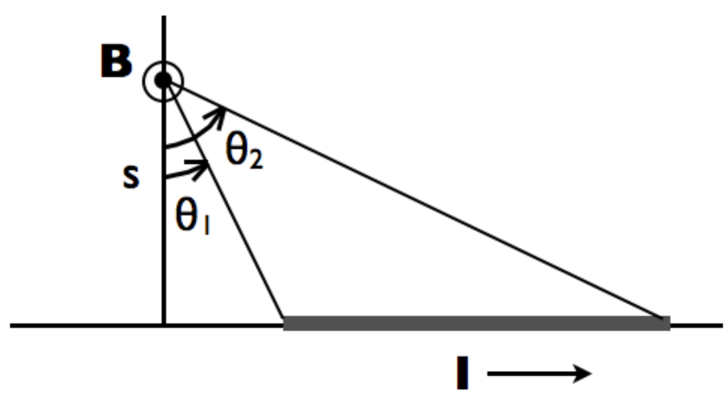


Figure 77: Definitions of s, θ_1, and θ_2 for the exercise given in the text.

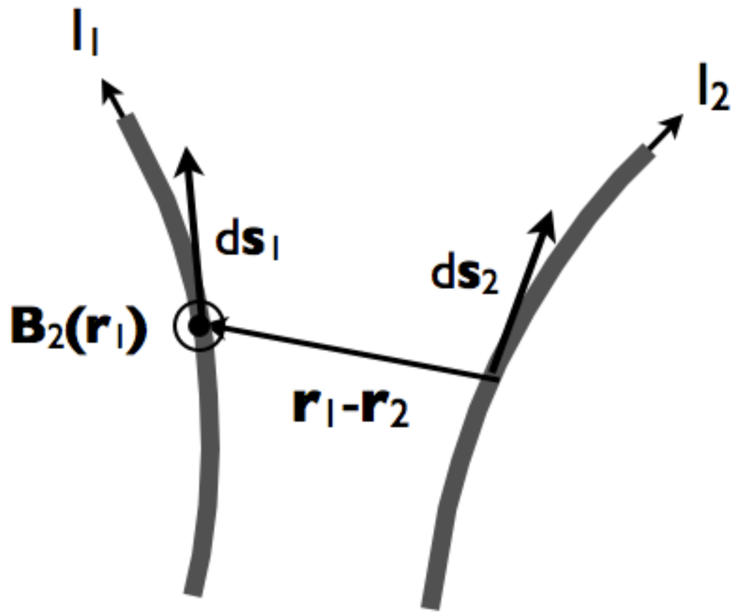


Figure 78: Definition of the quantities in Equation (804), which gives the force on wire 1 due to wire 2.

- Exercise: Using the Biot-Savart law and the Lorentz force law, show that the total force F_{12} exerted by one current-carrying wire $\mathbf{I}_2$ on another $\mathbf{I}_1$ is given by

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{d\mathbf{s}_1 \times (d\mathbf{s}_2 \times (\mathbf{r}_1 - \mathbf{r}_2))}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \tag{804}$$

  See Figure 78.

- Exercise: Using the results of the last two exercises, show that the force-per-unit-length between two straight, infinitely-long and parallel wires separated by a perpendicular distance d and carrying currents I_1 and I_2 is

$$f \equiv \frac{F}{L} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d} \tag{805}$$

  (See Figure 79.) If the currents flow in the same direction, the force is attractive; if the currents flow in the opposite direction, then the force is repulsive.

- Exercise: Show that the force between two current carrying wires (804) can be written in a more symmetric form

$$\mathbf{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint d\mathbf{s}_1 \cdot d\mathbf{s}_2 \, \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \tag{806}$$

  Hint: Use the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ and the fact that

$$d\mathbf{s}_1 \cdot \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = -d\left(\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}\right) \tag{807}$$

  which implies

$$\oint d\mathbf{s}_1 \cdot \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = 0 \tag{808}$$

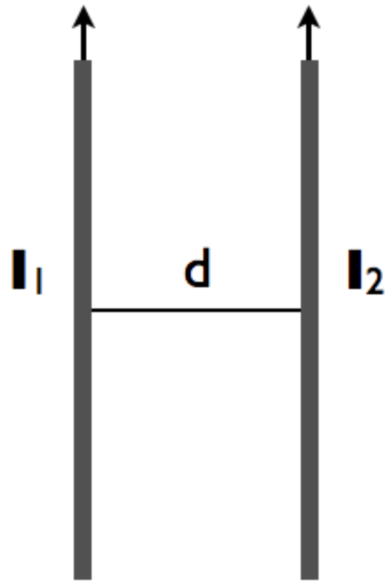


Figure 79: Two straight, infinitely-long and parallel current-carrying wires separated by a perpendicular distance d.

6.4 Magnetostatic field equations

- The Biot-Savart law for the magnetic field produced by a steady volume current density $\mathbf{J}(\mathbf{r})$ is given by:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int dV' \, \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{809}$$

- By explicitly taking the divergence and curl of the above equation, one finds

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \,, \qquad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \, \mathbf{J} \tag{810}$$

- <u>Exercise</u>: Prove these two results.

  Hint: To prove the first result, you will need to use

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) \tag{811}$$

  and

$$\boldsymbol{\nabla} \times \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) = 0 \tag{812}$$

  To prove the second result, you will need to use

$$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{A}) \tag{813}$$

  and

$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) = 4\pi \, \delta(\mathbf{r} - \mathbf{r}') \tag{814}$$

You will also need to show that

$$\int dV' \, (\mathbf{J}(\mathbf{r}') \cdot \boldsymbol{\nabla}) \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) = 0 \tag{815}$$

To do this, consider first the x-component of the LHS of the above equation. Replace $\boldsymbol{\nabla}$ by $\boldsymbol{\nabla}'$ (at the expense of a minus sign), integrate by parts, and throw away the surface term by taking the volume to go beyond the support of the current density $\mathbf{J}$. The remaining term in the integrand is proportional to $\boldsymbol{\nabla}' \cdot \mathbf{J}(\mathbf{r}')$, which vanishes for magnetostatics. Thus, the x-component is zero. The same argument shows that the y and z components are also zero.

- Using the divergence theorem, we can rewrite $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ in integral form:

$$\oint_S \mathbf{B} \cdot \hat{\mathbf{n}} \, da = 0 \tag{816}$$

- The physical interpretation of $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ is that there are no magnetic monopoles (i.e., no isolated N and S magnetic poles, in contrast to isolated $+$ and $-$ electric charges). This means that magnetic field lines can never start or stop at a point.

- Using Stokes's theorem, we can also rewrite $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$ in integral form:

$$\oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 \, I_{\rm enc} \tag{817}$$

where $I_{\rm enc}$ is the current passing through *any* 2-d surface S spanning the closed curve C. The above form of the equation is called *Ampère's law*.

- Ampère's law is the most efficient way to calculate the magnetic field when the geometry is sufficiently simple—i.e., for infinite straight lines, infinite planes, infinite selonoids, and toroidal symmetry.

- Example 1: Use Ampère's law to calculate the magnetic field a perpendicular distance s from an infinitely-long straight wire carrying a steady current I. (See Figure 80.)

- Answer:

$$B = \frac{\mu_0 I}{2\pi s} \tag{818}$$

circulating around the wire in accordance with the right-hand rule.

- Example 2: Use Ampère's law to calculate the magnetic field above and below an infinite uniform surface current $\mathbf{K}$ flowing over the plane $z = 0$ in the direction $+\hat{\mathbf{x}}$. (See Figures 81.)

- Answer:

$$\mathbf{B} = \begin{cases} -(\mu_0/2) K \, \hat{\mathbf{y}} & (\text{for } z > 0) \\ +(\mu_0/2) K \, \hat{\mathbf{y}} & (\text{for } z < 0) \end{cases} \tag{819}$$

Thus, we see that the component of the magnetic field parallel to the surface and perpendicular to the current is discontinuous across a surface current.

- Example 3: Use Ampère's law to calculate the magnetic field inside and outside an infinite solenoid with n turns per unit length and steady current I. (See Figures 82 and 83.)

- Answer:

$$\mathbf{B} = \begin{cases} +\mu_0 n I \, \hat{\mathbf{z}} & (\text{inside}) \\ 0 & (\text{outside}) \end{cases} \tag{820}$$

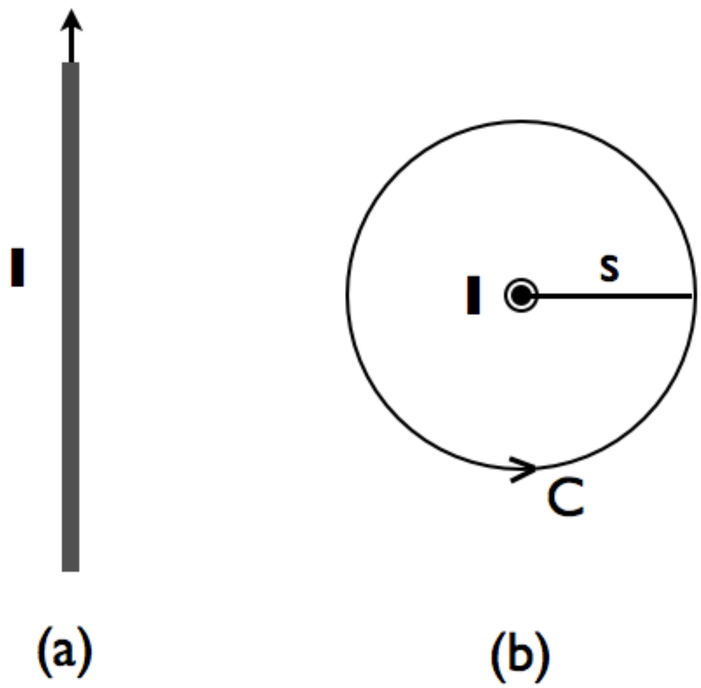
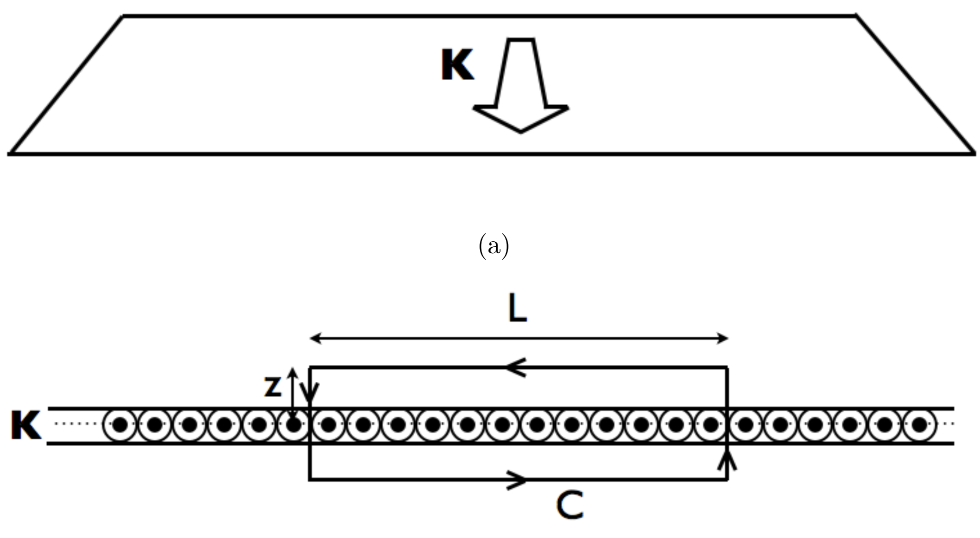


Figure 80: (a): Infinitely-long straight wire carrying steady current **I**. (b): Top view showing the Amperian loop C used to calculate the magnitude of **B** a perpendicular distance s from the wire. In this view, the current I is flowing out of the page.



(a)

(b)

Figure 81: (a): Infinite plane ($z = 0$) with surface current **K** flowing in the $+\hat{\mathbf{x}}$-direction. (b): Edge-on-view of the plane, with Amperian loop C and surface current **K** flowing out of the page.

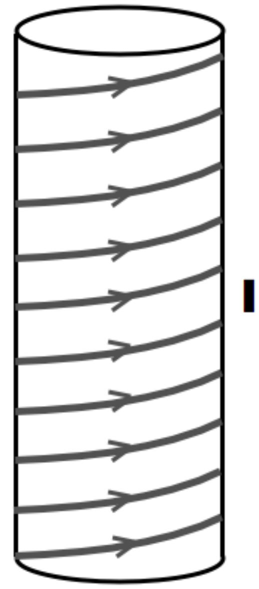


Figure 82: Infinite solenoid, with n turns per unit length and steady current I.

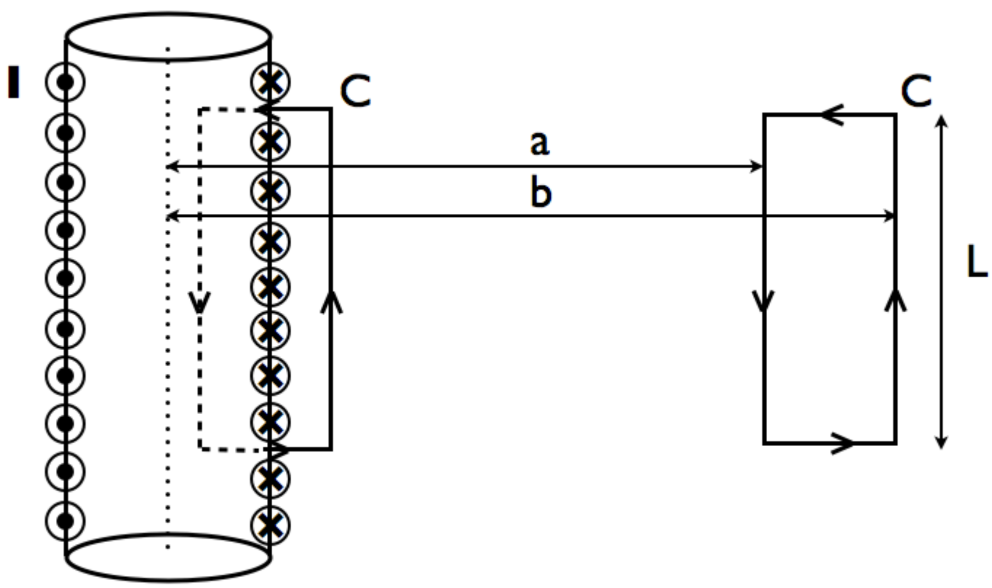


Figure 83: Two different Amperian loops C used to calculate the magnitude of the magnetic field $\mathbf{B}$ both inside and outside the solenoid.

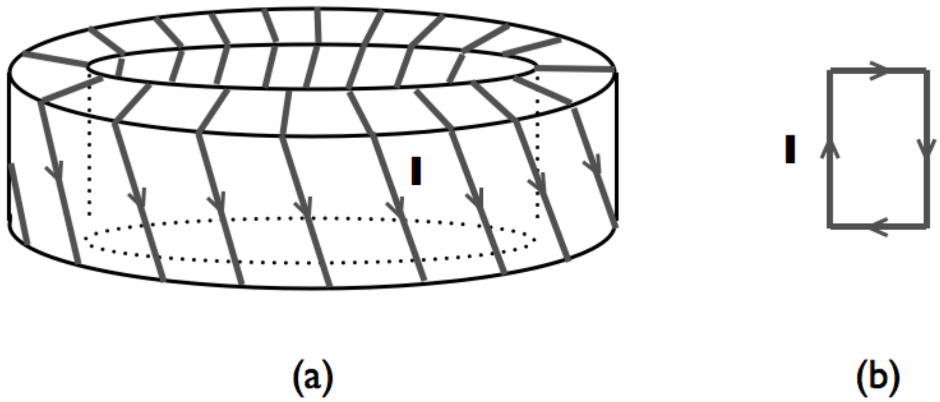


Figure 84: (a): Example of a solenoid in the shape of a circular torus, with N turns and steady current I. (b): This particular torus has a rectangular cross-section, but the results of Example 4 hold for *any* uniform cross-section.

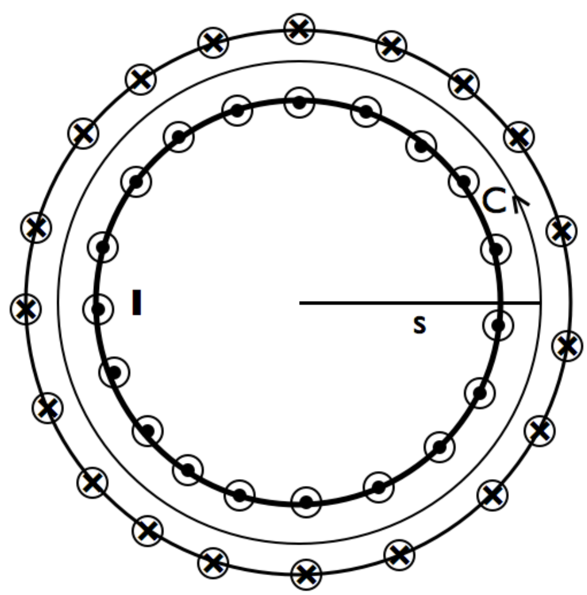


Figure 85: Amperian loop C used to calculate the magnitude of the magnetic field $\mathbf{B}$ inside the torus. Outside the torus, $\mathbf{B}(\mathbf{r}) = 0$.

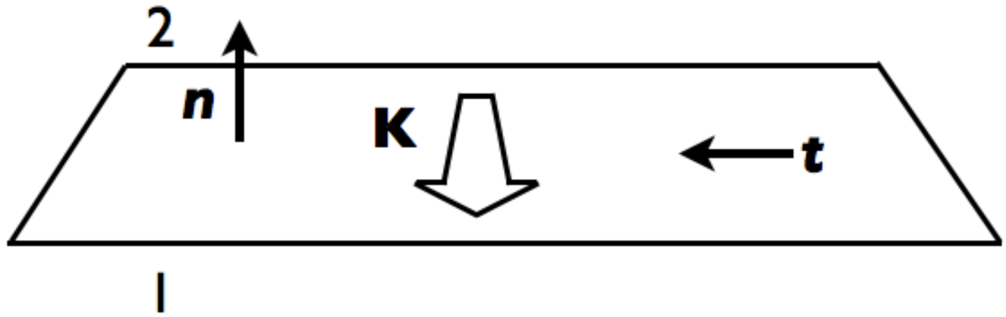


Figure 86: Definitions of the unit normal $\hat{\mathbf{n}}$, unit tangent $\hat{\mathbf{t}}$, and surface current $\mathbf{K}$, at the boundary between two regions 1 and 2.

.

- Example 4: Use Ampère's law to calculate the magnetic field inside and outside a circular ring (torus) that has N turns and steady current I. The torus can have *any* cross-section (i.e., it need not be a circle) provided it is uniform. (See Figures 84 and 85.)

- Answer:
$$\mathbf{B} = \begin{cases} \frac{\mu_0 N I}{2\pi s}\,\hat{\boldsymbol{\phi}} & \text{(inside)} \\ 0 & \text{(outside)} \end{cases} \tag{821}$$
(Hint: First show that the direction of the field is circumferential, then apply Ampère's law.)

- Thus, in magnetostatics
$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0\,, \qquad \boldsymbol{\nabla}\times\mathbf{B} = \mu_0\,\mathbf{J} \tag{822}$$

- By the Helmholtz theorem, if $\mathbf{B}$ goes to zero faster than $1/r$ as $r\to\infty$, these two equations completely define the field.

- Boundary conditions: (See Figure 86.)
(i) Using the integral form of $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, one can show that
$$(\mathbf{B}_2 - \mathbf{B}_1)\cdot\hat{\mathbf{n}} = 0 \tag{823}$$
where $\hat{\mathbf{n}}$ is the unit normal pointing from region 1 to region 2.
(ii) Using the integral form of $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\,\mathbf{J}$, one can show that
$$(\mathbf{B}_2 - \mathbf{B}_1)\cdot\hat{\mathbf{t}} = \mu_0\,(\mathbf{K}\times\hat{\mathbf{n}})\cdot\hat{\mathbf{t}} \tag{824}$$
where $\hat{\mathbf{t}}$ is a unit tangent to the boundary separating regions 1 and 2.
These two conditions can be lumped into the single equation
$$\mathbf{B}_2 - \mathbf{B}_1 = \mu_0\,\mathbf{K}\times\hat{\mathbf{n}} \tag{825}$$
Thus, the perpendicular component of $\mathbf{B}$ is continuous across a surface current density $\mathbf{K}$, but the parallel component of $\mathbf{B}$ pointing in the direction perpendicular to $\mathbf{K}$ is discontinuous.

- Exercise: Prove the above statements.

6.5 Vector potential

- Since $\boldsymbol{\nabla}\cdot\mathbf{B}=0$ we can write
$$\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A} \tag{826}$$
for some vector field $\mathbf{A}$.

- $\mathbf{A}$ is called the *magnetic vector potential* or simply *vector potential*.

- Note that $\mathbf{A}$ is not uniquely determined by the equation $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ as
$$\mathbf{A}':=\mathbf{A}+\boldsymbol{\nabla}\Lambda \tag{827}$$
has the same curl as $\mathbf{A}$, and hence corresponds to the same magnetic field $\mathbf{B}$.

- Exercise: Prove this.

- Such a transformation $\mathbf{A}\mapsto\mathbf{A}'$ is called a *gauge transformation*.

- This freedom of adding the gradient of a potential to $\mathbf{A}$ corresponds to the freedom of specifying the divergence of $\mathbf{A}$.

- In magnetostatics it is often convenient to set
$$\boldsymbol{\nabla}\cdot\mathbf{A}=0 \tag{828}$$
This is sometimes called the *Coulomb gauge*.

- Exercise: Show that this is always possible. (Hint: If $\boldsymbol{\nabla}\cdot\mathbf{A}\neq 0$ make a gauge transformation (827) to $\mathbf{A}'$, and show that the condition $\boldsymbol{\nabla}\cdot\mathbf{A}'=0$ becomes Poisson's equation for Λ, which is guaranteed to have a solution.)

- The only remaining freedom in $\mathbf{A}$ is an additive constant which we typically fix by requiring that $\mathbf{A}\to 0$ as $r\to\infty$.

- In terms of $\mathbf{A}$, Ampère's law becomes
$$\mu_0\,\mathbf{J}=\boldsymbol{\nabla}\times\mathbf{B}=\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})-\nabla^2\mathbf{A} \tag{829}$$

- Assuming $\boldsymbol{\nabla}\cdot\mathbf{A}=0$, Ampère's law for the vector potential is simply
$$\nabla^2\mathbf{A}=-\mu_0\,\mathbf{J} \tag{830}$$

- This is a set of three Poisson's equation for the cartesian components A_x, A_y, A_z of $\mathbf{A}$. Hence, standard techniques for solving Poisson's equation (which we learned in the context of electrostatics) can be used here as well.

- In particular, if there are no boundary surfaces and the current density $\mathbf{J}\to 0$ as $r\to\infty$, then we can immediately write down the (unique) solution
$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int dV'\,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{831}$$

- In terms of surface and line current densities $\mathbf{K}$ and $\mathbf{I}$, we have
$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int da'\,\frac{\mathbf{K}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,,\qquad \mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int dl'\,\frac{\mathbf{I}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,, \tag{832}$$

- Example 1: Find the vector potential inside and outside a spherical shell of radius R that carries a uniform surface charge density σ and is spinning with constant angular velocity $\boldsymbol{\omega}$. (See Figure 87.)

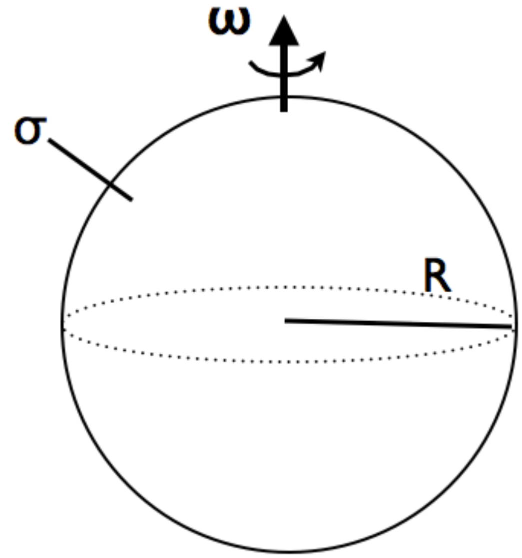


Figure 87: Spherical shell of radius R with uniform surface charge density σ, spinning with constant angular velocity $\boldsymbol{\omega}$.

.

- Answer:

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{1}{3}\mu_0 \sigma R\, \boldsymbol{\omega} \times \mathbf{r} & \text{(inside)} \\ \frac{1}{3}\mu_0 \sigma R^4\, \frac{\boldsymbol{\omega}\times\hat{\mathbf{r}}}{r^2} & \text{(outside)} \end{cases} \tag{833}$$

Direct calculation shows that the magnetic field inside the sphere is uniform:

$$\mathbf{B}(\mathbf{r}) = \frac{2}{3}\mu_0 \sigma R\, \boldsymbol{\omega} \qquad \text{(inside)} \tag{834}$$

Outside the sphere, the magnetic field is given by

$$\mathbf{B}(\mathbf{r}) = \frac{1}{3}\mu_0 \sigma \omega \frac{R^4}{r^3}\left[2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}\right] \qquad \text{(outside)} \tag{835}$$

where we've put $\hat{\mathbf{z}}$ along $\boldsymbol{\omega}$ and used spherical polar coordinates (r, θ, ϕ).

Note that up to proportionality constants, the above expression for $\mathbf{B}(\mathbf{r})$ has the same form as the electric field $\mathbf{E}(\mathbf{r})$ due to an electric dipole moment $\mathbf{p}$.

In fact, we will show later on that this is also true for a magnetic dipole moment $\mathbf{m}$. Thus, outside the sphere

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_{\rm dipole}(\mathbf{r}) \equiv \frac{\mu_0}{4\pi}\frac{m}{r^3}\left[2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}\right] \qquad \text{for } \mathbf{m} \equiv \frac{4}{3}\pi R^3\, \sigma R \boldsymbol{\omega} \tag{836}$$

- Example 2: Calculate the vector potential of an infinite solenoid with radius R, current I, and n turns per unit length (c.f., Figure 82).

- Answer:

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{1}{2}\mu_0 n I s\, \hat{\boldsymbol{\phi}} & \text{(inside)} \\ \frac{1}{2}\mu_0 n I \frac{R^2}{s}\, \hat{\boldsymbol{\phi}} & \text{(outside)} \end{cases} \tag{837}$$

Hint: You cannot use the integral expression for $\mathbf{A}(\mathbf{r})$ since the solenoid and the current density extend to infinity. You can, however, use Stokes's theorem to relate the line integral of $\mathbf{A}$ around a closed loop C to the magnetic flux Φ_B passing through *any* 2-dimensional surface S spanning the closed curve C:

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \hat{\mathbf{n}} \, da = \int_S \mathbf{B} \cdot \hat{\mathbf{n}} \, da \equiv \Phi_B \tag{838}$$

- Boundary conditions:

  Assuming that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ it follows that:

  (i) $\mathbf{A}$ is continuous across a surface current:

$$\mathbf{A}_1 - \mathbf{A}_2 = 0 \tag{839}$$

  (ii) Its normal derivative inherits the discontinuity of the tangential component of $\mathbf{B}$:

$$\left(\frac{\partial \mathbf{A}_2}{\partial n} - \frac{\partial \mathbf{A}_1}{\partial n} \right) \bigg|_S = -\mu_0 \, \mathbf{K} \tag{840}$$

- Exercise: Prove the above two statements.

  Hint: For (i) use the integral forms of $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ and $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$ to show that the normal and tangential components of $\mathbf{A}$ are continuous across the boundary surface. For (ii), write out the components of $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ for the case where the surface current $\mathbf{K}$ flows in the plane $z = 0$ in the $+\hat{\mathbf{x}}$ direction. Note that $\partial \mathbf{A}/\partial x$ and $\partial \mathbf{A}/\partial y$ are continuous across the boundary (since the components of $\mathbf{A}$ are continuous), as is $\partial A_z/\partial z$ (since $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ everywhere).

6.6 Multipole expansion of the vector potential

- Consider a current loop $\mathbf{I}(\mathbf{r}')$ localised to some finite volume, free of any boundary surfaces. Then the vector potential $\mathbf{A}(\mathbf{r})$ associated with this current distribution can be written as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint d\mathbf{s}' \, \frac{1}{|\mathbf{r} - \mathbf{r}'|} \tag{841}$$

- By expanding the Dirichlet Green's function $1/|\mathbf{r} - \mathbf{r}'|$ in terms of Legendre polynomials:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{(r')^l}{r^{l+1}} \, P_l(\cos\gamma) \, , \qquad \text{for } r > r' \tag{842}$$

  we can write

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \left[\frac{1}{r} \oint d\mathbf{s}' \, + \frac{1}{r^2} \oint d\mathbf{s}' \, r' \, \cos\gamma + \cdots \right] \tag{843}$$

  where $\cos\gamma \equiv \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'$. The successive terms in the expansion are called the monopole, dipole, quadrupole, ... contributions to the vector potential.

- Note that the first integral equals zero, since the current loop is closed. Thus, the monopole contribution to the potential is zero. (This is consistent with our earlier statement that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ means that there are no magnetic monopoles.)

- The second integral can be rewritten as

$$\oint d\mathbf{s}' \, r' \, \cos\gamma = \oint d\mathbf{s}' \, \hat{\mathbf{r}} \cdot \mathbf{r}' = \mathbf{a} \times \hat{\mathbf{r}} \tag{844}$$

  where $\mathbf{a} \equiv \int da' \, \hat{\mathbf{n}}'$ is the *vector area* of the loop. (The last equality follows from (108).)

- Thus, the leading order term in the expansion of the vector potential is the dipole term

$$\mathbf{A}_{\rm dip}(\mathbf{r}) \equiv \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \tag{845}$$

where

$$\mathbf{m} \equiv I\mathbf{a} \tag{846}$$

is the *magnetic dipole moment* associated with the current loop.

- Exercise: Show that one obtains the same results for the expansion of the vector potential if we start with a localized volume current density $\mathbf{J}(\mathbf{r})$ instead of $\mathbf{I}(\mathbf{r})$.

  Hint: You will first need to derive the identity

$$0 = \int dV\ \left[f(\boldsymbol{\nabla} g)\cdot\mathbf{J} + g(\boldsymbol{\nabla} f)\cdot\mathbf{J}\right] \tag{847}$$

  which follows from the divergence theorem, assuming that the product $fg\mathbf{J}$ goes to zero sufficiently fast as $r\to\infty$, and $\boldsymbol{\nabla}\cdot\mathbf{J}=0$. Taking $f=1$ and $g=r_i$ gives

$$0 = \int dV\ J_i(\mathbf{r}) \tag{848}$$

  which is needed to conclude that the monopole contribution vanishes:

$$\mathbf{A}_{\rm mon}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r}\int dV'\,\mathbf{J}(\mathbf{r}') = 0 \tag{849}$$

  Taking $f=r_i$ and $g=r_j$ yields

$$0 = \int dV\ \left[r_i J_j(\mathbf{r}) + r_j J_i(\mathbf{r})\right] \tag{850}$$

  which is needed to rewrite the dipole contribution in standard form:

$$\mathbf{A}_{\rm dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^2}\int dV'\,\mathbf{J}(\mathbf{r}')\,\hat{\mathbf{r}}\cdot\mathbf{r}' = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \tag{851}$$

  where

$$\mathbf{m} \equiv \frac{1}{2}\int dV'\,\mathbf{r}'\times\mathbf{J}(\mathbf{r}') \tag{852}$$

  The integrand of the above expression is defined to be the *magnetization*:

$$\mathbf{M}(\mathbf{r}) \equiv \frac{1}{2}\mathbf{r}\times\mathbf{J}(\mathbf{r}) \tag{853}$$

  which is the magnetic dipole moment per unit volume. Finally, by replacing $dV'\,\mathbf{J}(\mathbf{r}')$ by $ds'\,\mathbf{I}(\mathbf{r}')$ in the integral expression for $\mathbf{m}$, you can show that

$$\mathbf{m} = I\,\frac{1}{2}\oint \mathbf{r}'\times d\mathbf{s}' = I\mathbf{a} \tag{854}$$

  which is our original expression for the magnetic dipole moment.

6.7 Magnetic dipoles

- The magnetic dipole moment of a flat circular current loop of radius R in the xy-plane carrying CCW current I is

$$\mathbf{m} = I\pi R^2\,\hat{\mathbf{z}} \tag{855}$$

  This is an example of a physical magnetic dipole. (See Figure 88.)

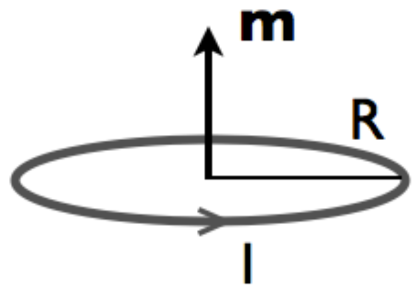


Figure 88: Perspective view of a physical magnetic dipole $\mathbf{m}$ defined by a circular current loop (radius R, current I).

- A physical magnetic dipole will have higher-order multipole moments, in addition to the leading-order dipole contribution.
- A pure magnetic dipole, defined by the limit $a \to 0$, $I \to \infty$ such that $m \equiv Ia = \text{const}$, will have only a dipole contribution to its vector potential:

$$\mathbf{A}(\mathbf{r}) = \mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \tag{856}$$

- If we put a pure magnetic dipole at the origin with $\mathbf{m}$ directed along $+\hat{\mathbf{z}}$, we can write

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}} \tag{857}$$

  in spherical polar coordinates (r, θ, ϕ).
- The associated magnetic field is

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) \qquad \text{for } r \neq 0, \tag{858}$$

- Exercise: Prove this.
- Note that the RHS has the same form as the electric field $\mathbf{E}(\mathbf{r})$ for an electric dipole moment $\mathbf{p}$.
- It can thus be recast in coordinate-independent form in the same manner as for $\mathbf{E}_{\text{dip}}(\mathbf{r})$:

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right] \qquad \text{for } r \neq 0, \tag{859}$$

- If we want to include the field at the origin ($\mathbf{r} = \mathbf{0}$), then we need to add a Dirac delta-function term:

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right] + \frac{2\mu_0}{3}\mathbf{m}\,\delta(\mathbf{r}) \tag{860}$$

- This is needed to satisfy a general result for the average of the magnetic field over the *volume* of a sphere of radius R:

$$\mathbf{B}_{\text{ave}} \equiv \frac{1}{\frac{4}{3}\pi R^3}\int_{r<R} dV\,\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{2\mathbf{m}_{\text{enc}}}{R^3} + \mathbf{B}_{\text{ext}}(\mathbf{0}) \tag{861}$$

  where $\mathbf{B}_{\text{ext}}(\mathbf{0})$ is the magnetic field at the center of the sphere produced by all steady currents *exterior* to the sphere, and $\mathbf{m}_{\text{enc}}$ is the total magnetic dipole moment *enclosed* by the sphere. (This is almost identical to the expression (664) for the average electric field over a spherical volume, except that the first term here differs from that for $\mathbf{E}_{\rm ave}$ by a factor of -2.)

- <u>Exercise</u>: Prove the above result for the average magnetic field.

  Hint: First note that the volume integral over the sphere can be converted into a 2-d integral over the surface using a corollary (104) of the divergence theorem:

$$\int_V (\boldsymbol{\nabla} \times \mathbf{A})\, dV \;=-\oint_S A(\mathbf{r}) \times \hat{\mathbf{n}}\, da \tag{862}$$

  where $\hat{\mathbf{n}}$ is the outward pointing normal, which depends on the location of the area element. Thus,

$$\int_{r<R} \mathbf{B}(\mathbf{r})\, dV \;=-\oint_{r=R} \mathbf{A}(\mathbf{r}) \times \hat{\mathbf{n}}\, da \;=-\frac{\mu_0}{4\pi}\int_{r<R} dV'\, \mathbf{J}(\mathbf{r}') \times \oint_{r=R} \frac{\hat{\mathbf{n}}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}\, da \tag{863}$$

  You can then do the surface integral by choosing coordinates (r,θ,ϕ) so that $\mathbf{r}'$ lies on the z-axis and

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta) \tag{864}$$

  where $r_<$ ($r_>$) is the lesser (greater) or r' and R. In addition,

$$\hat{\mathbf{n}} = \hat{\mathbf{r}} = \sin\theta\,\cos\phi\,\hat{\mathbf{x}} + \sin\theta\,\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \tag{865}$$

  and

$$da = R^2\, d(\cos\theta)\, d\phi \tag{866}$$

  Then one can show that

$$\oint_{r=R} \frac{\hat{\mathbf{n}}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}\, da = \begin{cases} \frac{4\pi}{3}\mathbf{r}' & (\text{for } r' < R) \\ \frac{4}{3}\pi R^3\, \frac{\mathbf{r}'}{r'^3} & (\text{for } r' > R) \end{cases} \tag{867}$$

  Substitution of this result back into (863) leads to the final result (861).

- One can show that the force and torque on a magnetic dipole $\mathbf{m}$ in an external magnetic field $\mathbf{B}$ is given by

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B})\big|_{\rm dipole} \tag{868}$$

$$\mathbf{N} = \mathbf{m} \times \mathbf{B}\big|_{\rm dipole} \tag{869}$$

  where $\big|_{\rm dipole}$ means to evaluate the field at the location of the dipole.

- <u>Exercise</u>: Prove the force and torque equations by considering the total force and total torque on a physical magnetic dipole consisting of a rectangular current loop, expanding the magnetic field about the center of the loop. (Note that for a pure dipole, any higher-order corrections vanish.) The proof can be extended to an *arbitrary* magnetic dipole moment $\mathbf{m}$, associated with any localised current distribution $\mathbf{I}$ or $\mathbf{J}$.

- Since $\mathbf{F} = -\boldsymbol{\nabla} U$, we also have

$$U = -\mathbf{m}\cdot\mathbf{B}\big|_{\rm dipole} \tag{870}$$

  which is similar to $U = -\mathbf{p}\cdot\mathbf{E}|_{\rm dipole}$ for the energy of an electric dipole moment $\mathbf{p}$ in an electric field $\mathbf{E}$. In both cases, the minimum energy configuration is for the dipole aligned with the field. (Note that one can also obtain this expression for the energy by calculating the work required to move a magnetic dipole in from infinity and then rotating it into its final position, ignoring any energy required to keep the dipole current constant.)

6.8 Magnetic dipoles and angular momentum

- All magnetic phenomena can ultimately be attributed to electric charges in motion. For example, the magnetic properties of a material can be explained in terms of the orbital and spinning motion of electrons in atoms. (See the next section for more details.)

- In this subsection, we derive three results:

  i) The angular momentum and magnetic dipole moment of an orbiting (or spinning) point charge q are *proportional* to one another.

  ii) Magnetic dipoles *precess* in an external magnetic field.

  iii) The introduction of a magnetic field induces a magnetic dipole moment, which is directed *opposite* to the applied field.

- Item i): Consider a point charge q moving in a circular orbit of radius R with constant speed v. If we let m_q denote the mass of the charge and choose coordinates so that the orbit is in the xy-plane with center at the origin, then the (orbital) angular momentum of the motion is given by

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} = R m_q v \, \hat{\mathbf{z}} \tag{871}$$

- Although the motion of a single point charge does not give rise to a steady current I, if the period $T = 2\pi R/v$ is short enough, then the motion will *look* like a steady current with

$$I \equiv \frac{\Delta q}{\Delta t} = \frac{q}{T} = \frac{qv}{2\pi R} \tag{872}$$

- We can then associate with this motion a magnetic dipole moment

$$\mathbf{m} = I \, \pi R^2 \, \hat{\mathbf{z}} = \frac{qvR}{2} \, \hat{\mathbf{z}} \tag{873}$$

- Comparing this with the above expression (871) for the orbital angular momentum $\mathbf{L}$, we see that

$$\mathbf{m} = \frac{q}{2m_q} \mathbf{L} \quad \text{(orbital angular momentum)} \tag{874}$$

- NOTE: Although we derived the above result classically, it turns out that it is also true quantum mechanically—e.g., for the orbital motion of an electron in an atom.

  For a spinning point charge with spin angular momentum $\mathbf{S}$, a quantum mechanical calculation yields

$$\mathbf{m} = \frac{q}{m_q} \mathbf{S} \quad \text{(spin angular momentum)} \tag{875}$$

  (A classical calculation for a spinning uniform sphere of charge using (852) yields (874) which is off by a factor of 2.)

  For a general combination of orbital and spin angular momentum $\mathbf{J} \equiv \mathbf{L} + \mathbf{S}$ (not to be confused with the current density $\mathbf{J}$), we have

$$\mathbf{m} = g \left(\frac{q}{2m_q} \right) \mathbf{J} \tag{876}$$

  where g is the Lande' g-factor, which is a dimensionless quantity of order unity that depends on the detailed quantum-mechanical state of the charge.

  In what follows, we will restrict attention to orbital angular momentum and classical calculations.

- Item ii): Given (874) and the fact that an external magnetic field $\mathbf{B}$ exerts a torque on a magnetic dipole $\mathbf{m}$, it follows that

$$\frac{d\mathbf{L}}{dt} = \mathbf{N} = \mathbf{m} \times \mathbf{B} = \frac{q}{2m_q} \mathbf{L} \times \mathbf{B} \tag{877}$$

- Thus, $\mathbf{d}L$ is perpendicular to the plane spanned by $\mathbf{L}$ and $\mathbf{B}$.

- This implies that $\mathbf{L}$ and hence $\mathbf{m}$ (by (874)) will *precess* about $\mathbf{B}$ with an angular velocity given by
$$\omega_L = \frac{qB}{2m_q} \tag{878}$$
ω_L is called the *Lamor frequency.*

- Exercise: Prove this. (Hint: Take $\mathbf{B}$ to point along the $\hat{\mathbf{z}}$-axis, and let θ be the angle between $\mathbf{L}$ and $\mathbf{B}$. Then
$$d\mathbf{L} = L\sin\theta\, d\phi\, \hat{\boldsymbol{\phi}} \tag{879}$$
where
$$d\phi = \omega_L\, dt \tag{880}$$
is the angle turned through in the time interval dt. Equating the above expression for $d\mathbf{L}/dt$ with the RHS of (877) yields the desired result.)

- Item iii): We will show in Section 8 that a changing magnetic field gives rise to an electric field according to Faraday's law:
$$\oint_C \mathbf{E}\cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \tag{881}$$
where
$$\Phi_B \equiv \int_S \mathbf{B}\cdot\hat{\mathbf{n}}\, da \tag{882}$$
is the *magnetic flux* through any surface S spanning the closed curve C.

- The minus sign on the RHS of Faraday's law is sometimes called *Lenz's law*. It means that the induced electric field drives a current around C that *opposes* the change in the magnetic flux.

- To see what Faraday's law has to say about magnetic dipoles in changing magnetic fields, consider the situation described earlier of a point charge q moving in a circular orbit of radius R in the xy-plane with constant speed v.

- If we now apply a magnetic field $\mathbf{B}$ in the $\hat{\mathbf{z}}$-direction, Faraday's law implies
$$E\,2\pi R = -\frac{d}{dt}\left(B\,\pi R^2\right) \Leftrightarrow \mathbf{E} = -\frac{R}{2}\frac{dB}{dt}\,\hat{\boldsymbol{\phi}} \tag{883}$$
where we chose C to be the circular path of the point charge, traversed in the CCW as seen from above.

- Exercise: Prove this.

- The induced electric field exerts a torque on q given by
$$\mathbf{N} \equiv \mathbf{r}\times\mathbf{F} = \mathbf{r}\times q\mathbf{E} = RqE\,\hat{\mathbf{z}} = -\frac{1}{2}R^2 q\frac{dB}{dt}\,\hat{\mathbf{z}} \tag{884}$$

- But since $\mathbf{N} = d\mathbf{L}/dt$, we have
$$\frac{d\mathbf{L}}{dt} = -\frac{1}{2}R^2 q\frac{dB}{dt}\,\hat{\mathbf{z}} \tag{885}$$
which integrates to
$$\Delta\mathbf{L} = -\frac{1}{2}R^2 q\mathbf{B} \tag{886}$$

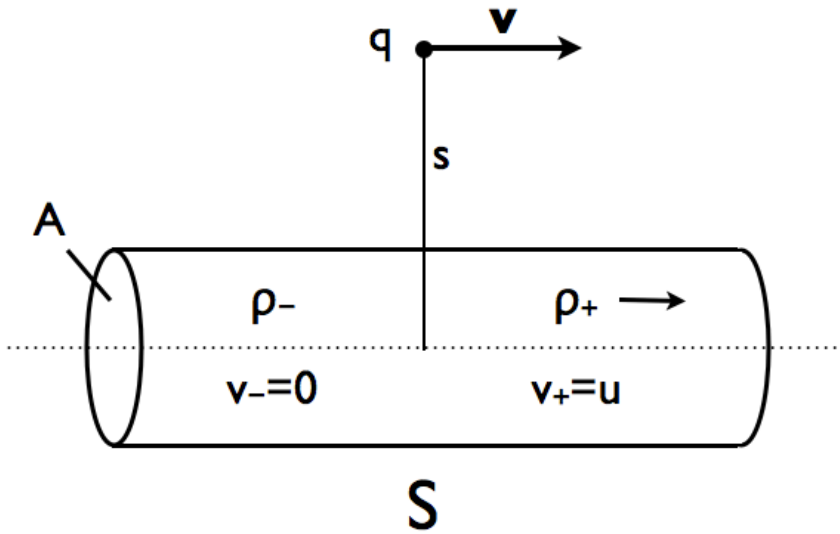


Figure 89: Point charge q moving with velocity $\mathbf{v}$, parallel to an infinitely-long, straight current carrying wire. Here s is the perpendicular distance of q from the wire, and $\rho_\pm$ are the charge densities of the positive and negative charges that make-up the wire. In this reference frame, the wire is electrically neutral ($\rho_+ + \rho_- = 0$) and the current I is due solely to the motion of the positive charges ($I = \rho_+ Au$, where A is the cross-sectional area of the wire and $v_+ = u$ is the velocity of the positive charges).

- Finally, if we use (874) to relate $\mathbf{m}$ and $\mathbf{L}$, we can conclude that

$$\Delta\mathbf{m} = \frac{q}{2m_q}\,\Delta\mathbf{L} = -\frac{q^2R^2}{4m_q}\,\mathbf{B} \tag{887}$$

- Thus, the introduction of a magnetic field $\mathbf{B}$ induces a magnetic dipole moment $\Delta\mathbf{m}$ in the direction *opposite* to the applied field $\mathbf{B}$. This is the phenomenon of *diamagnetism*, which we describe in more detail in Section 7.

6.9 Relativity of magnetic and electric fields

- Consider a point charge q moving with velocity $\mathbf{v}$, parallel to an infinitely-long, straight current carrying wire I as shown in Figure 89.

- Assuming that the wire is electrically neutral—i.e.,

$$\rho \equiv \rho_+ + \rho_- = 0 \tag{888}$$

it follows that the only force on q is the magnetic force

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B} \tag{889}$$

- Since the magnetic field a perpendicular distance s from a current-carrying wire has magnitude

$$B(s) = \frac{\mu_o I}{2\pi s} \tag{890}$$

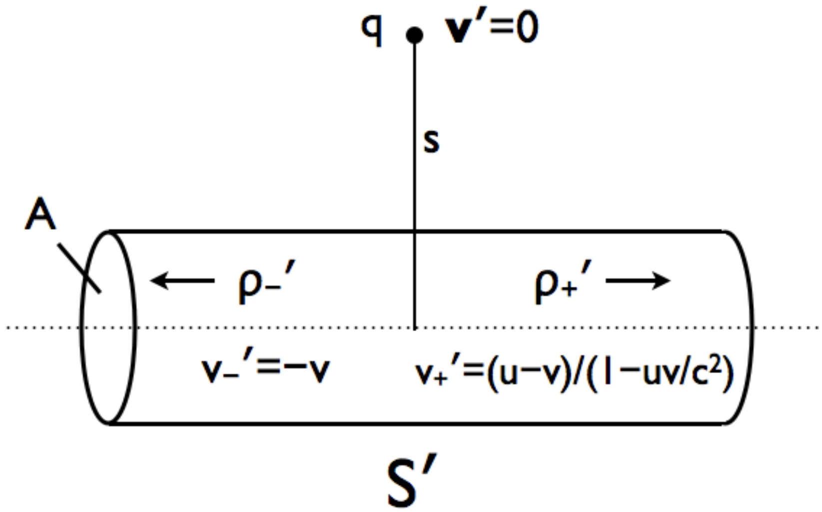


Figure 90: Same situation as in Figure 89, but viewed in a reference frame S' in which the point charge q is at rest. In this frame, the wire is no-longer electrically neutral, as described in the text.

and circles around the wire, it follows that

$$\mathbf{F} = -qv\frac{\mu_0 I}{2\pi s}\,\hat{\mathbf{y}} \tag{891}$$

- If we assume that only the positive charges are responsible for the current ($v_+ = u$, $v_- = 0$), then $I = \rho_+ A u$ (where A is the cross-sectional area of the wire), and

$$\mathbf{F} = -\frac{\mu_0 q}{2\pi s}\rho_+ A u v\,\hat{\mathbf{y}} \tag{892}$$

- Thus, for positive q, the force is directed toward the wire, so the point charge accelerates in the $-\hat{\mathbf{y}}$ direction.
- Consider the same situation but seen with respect to a frame of reference *moving* with velocity $\mathbf{v}$. (See Figure 90.) In this reference frame, q is instantaneously at *rest*—i.e., $\mathbf{v}' = 0$. (We will denote the reference frame moving with velocity $\mathbf{v}$ by S' and the original reference frame by S. We will use primes to distinguish the values of various quantities measured in S' from those measured in S.)
- In S' the magnetic force is necessarily zero. Hence, the charge must feel an *electrical* force in S', if it is to accelerate downward as calculated above.
- To prove that this is the case, first note that the velocities of the positive and negative charges wrt S' are given by the composition of velocities formula

$$v_+' = \frac{u - v}{1 - uv/c^2}\,, \quad v_-' = -v \tag{893}$$

  This is how velocities "add" in special relativity.
- Note also that the charge densities in $\rho_\pm'$ measured in S' differ from those measured in S because of relativistic *length contraction*. (Recall that a rod of length L_0 as measured in its rest frame has length

$$L = \sqrt{1 - v^2/c^2}\,L_0 \tag{894}$$

as measured in a frame moving with speed v parallel to the direction of the rod.) Although the volume changes, the value and number of electrical charges in the volume is the *same* in both reference frames.

- Thus,

$$\rho'_- = \frac{1}{\sqrt{1 - {v'_-}^2/c^2}} \rho_{-,0} = \frac{1}{\sqrt{1 - v^2/c^2}} \rho_- \tag{895}$$

  where $\rho_{-,0} = \rho_-$ is the rest charge density.

- For the positive charge things are slightly more complicated:

$$\rho'_+ = \frac{1}{\sqrt{1 - {v'_+}^2/c^2}} \rho_{+,0} \,, \quad \rho_+ = \frac{1}{\sqrt{1 - u^2/c^2}} \rho_{+,0} \tag{896}$$

  so that

$$\rho'_+ = \frac{\sqrt{1 - u^2/c^2}}{\sqrt{1 - {v'_+}^2/c^2}} \rho_+ \tag{897}$$

- Now one can show that

$$\frac{1}{\sqrt{1 - {v'_+}^2/c^2}} = \frac{1 - uv/c^2}{\sqrt{1 - u^2/c^2}\sqrt{1 - v^2/c^2}} \tag{898}$$

- Thus,

$$\rho' \equiv \rho'_+ + \rho'_- = -\frac{uv/c^2}{\sqrt{1 - v^2/c^2}}\, \rho_+ \tag{899}$$

  So the wire is electrically-charged when view in S'.

- <u>Exercise</u>: Prove these last two statements.

- Now, the electric field a perpendicular distance s from the a line charge λ is directed radially and has magnitude

$$E(s) = \frac{\lambda}{2\pi\epsilon_0 s} \tag{900}$$

- Thus, the force on q as seen in S' is

$$\mathbf{F}' = q\mathbf{E} = q\frac{\rho' A}{2\pi\epsilon_0 s}\, \hat{\mathbf{y}}' = -\frac{q}{2\pi\epsilon_0 s} \frac{uv/c^2}{\sqrt{1 - v^2/c^2}}\, \rho_+ A\, \hat{\mathbf{y}}' \tag{901}$$

- Comparing this with the earlier calculation (892), we see that

$$F_{y'} = \frac{1}{\epsilon_0\mu_0} \frac{1}{c^2} \frac{F_y}{\sqrt{1 - v^2/c^2}} = \frac{F_y}{\sqrt{1 - v^2/c^2}} \tag{902}$$

  where the last line follows from the relation $\epsilon_0\mu_0 = 1/c^2$.

- But this is precisely the way the transverse components of force transform in special relativity (see below). *Thus, the magnetic force becomes an electrical force when viewed in a reference frame where the velocity of the charge q is zero.*

- Proof of transformation property for transverse force components:

  Recall that

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \,, \quad \text{where } \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \quad \text{(relativistic momentum)} \tag{903}$$

Also

$$dt = \frac{dt'}{\sqrt{1 - v^2/c^2}} \tag{904}$$

which is time dilation for the point charge q (rest frame of q is S'). Thus,

$$dp_{y'} = dp_y \Leftrightarrow F_{y'}dt' = F_y dt \Leftrightarrow F_{y'} = \frac{F_y}{\sqrt{1 - v^2/c^2}} \tag{905}$$

QED

7 Magnetic fields in matter

7.1 Paramagnetism, diamagnetism, ferromagnetism

- There are 3 types of magnetic materials: *paramagnetic*, *diamagnetic*, and *ferromagnetic*. A proper treatment of each of these types of materials requires quantum mechanics (which we will not go into); classical arguments can only be taken so far before they fail. (See Feynman, Vol II, chpts. 34-37 for details.)

- Paramagnetism:

  In a paramagnetic material, an applied magnetic field $\mathbf{B}_0$ induces a magnetization $\mathbf{M}$ ($\equiv$ average magnetic dipole moment per unit volume, see next subsection) in the *same* direction as $\mathbf{B}_0$. (We are ignoring anisotropic materials here.)

  This is similar to the induced polarization $\mathbf{P}$ in dielectrics being in the same direction as the applied electric field $\mathbf{E}_0$, although for dielectrics, the induced electric field is directed *opposite* the induced polarization $\mathbf{P}$ and hence *opposite* $\mathbf{E}_0$.

  Paramagnetism is associated with the magnetic dipole moments of the *spin* angular momenta of the electrons in an atom. Since the Pauli exclusion principle pairs together electrons with opposite spin, paramagnetic behavior typically occurs only in atoms with an *odd* number of electrons, so that each atom has a non-zero magnetic dipole moment to begin with, similar to dielectrics that have *polar* molecules.

  The magnetic dipoles in the material, which are randomly aligned before the external field is applied, experience a torque $\mathbf{N} = \mathbf{m} \times \mathbf{B}_{\rm loc}$ that tends to align them in the direction of the field. (Here $\mathbf{B}_{\rm loc}$ is the magnetic field acting on the atom due to everything other than the particular dipole under consideration.)

  Thermal fluctuations compete with this alignment, similar to the polar molecule case for dielectrics. One needs to use the Boltzmann factor $\exp(-u/k_BT)$, where $u \equiv -\mathbf{m} \cdot \mathbf{B}_{\rm loc}$, to determine the expected value of the orientation of the magnetic dipoles.

- Diamagnetism:

  In a diamagnetic material, the induced magnetization $\mathbf{M}$ is in the direction *opposite* to the applied field $\mathbf{B}_0$.

  This is similar to dielectrics in the sense that the induced electric field in the dielectric is in the direction *opposite* to the applied electric field $\mathbf{E}_0$, even though the induced polarization $\mathbf{P}$ is the *same* direction as $\mathbf{E}_0$.

  Due to the repulsion between the applied field and the induced magnetization, diamagnets can be leviated in a magnetic field—e.g., a levitating frog.

  As explained in Section 6, diamagnetism is assoicated with the magnetic moments of the *orbital* motion of the electrons around the nucleus of an atom. Faraday's law implies that the introduction of the external field $\mathbf{B}_0$ induces an electric field tangent to the orbit, which either speeds up or slows down the electrons in such a way as to change $\mathbf{m}$ in the direction opposite to $\mathbf{B}_0$.

Diamagnetism is typically much weaker than paramagnetism, and hence only occurs in atoms with an *even* number of electrons. This is similar to dielectrics that have *non-polar* atoms or molecules.

Diamagnetism is generally independent of temperature, in contrast to paramagnetism.

Examples of diamagnetic materials are water, copper, gold, and silver. Bismuth is the strongest diamagnetic material.

- Ferromagnetism:

  Ferromagnets are extreme examples of paramagnetic materials whose magnetization persists even after the applied field is removed.

  There is an effective force between the magnetic dipole moments of the individual atoms that tends to keep the magnetic dipoles aligned with one another. This force is about ten thousand times larger than the direct magnetic force between neighboring dipoles, and can only be explained by quantum mechanics.

  Below a critical temperature T_c (called the *Curie point*), the magnetic dipoles in a ferromagnet will sponateously align themselves forming domains of uniform magnetization. (The domains are macroscopically small, but microscopically large, containing billons of atoms or molecules whose magnetic dipoles all point in the same direction.) Above T_c, thermal randomization prevents alignment of the dipoles. (The Curie point for iron is 770° C.)

  In an unmagnetized ferromagnet below the Curie point, the magnetic dipoles in each domain are aligned, but in a direction which is *different* from those in neighboring domains. The average magnetization for such a material is therefore zero.

  However, in the presence of an external field $\mathbf{B}_0$, the size and shape of the domains will change; domains with magnetic dipoles aligned with the applied field tend to grow, while domains with dipoles pointing in other directions tend to shrink.

  The actual value of the magnetization depends on the past history of the material (*hysteresis curve*).

  Examples of ferromagnetic materials are iron, nickel, and cobalt.

7.2 Bound currents

- Consider a piece of material containing many individual magnetic dipoles moments $\mathbf{m}_i$. The magnetic dipole moments could be due to the intrinsic spin of unpaired electrons or to orbital motion of the electrons around the nucleii of the atoms. (See Figure 91 for different graphical representations of a magnetic dipole $\mathbf{m}$.) The *magnetization* $\mathbf{M}(\mathbf{r})$ is defined to be the average dipole moment per unit volume:

$$\mathbf{M}(\mathbf{r}) \equiv \frac{1}{\frac{4}{3}\pi R^3} \sum_i \mathbf{m}_i \tag{906}$$

  where the sum is over the dipole moments contained in a spherical volume of radius R centered at $\mathbf{r}$. The volume is chosen to be large enough to contain many individual dipole moments, but is small on a macroscopic scale. (See Figure 92.)

- We can calculate the vector potential at $\mathbf{r}$ due to the polarization by adding together the potentials due to infinitesimal magnetic dipoles $d\mathbf{m}(\mathbf{r}') = \mathbf{M}(\mathbf{r}')\, dV'$:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V dV'\, \mathbf{M}(\mathbf{r}') \times \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{907}$$

- Noting that

$$\boldsymbol{\nabla}' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \tag{908}$$

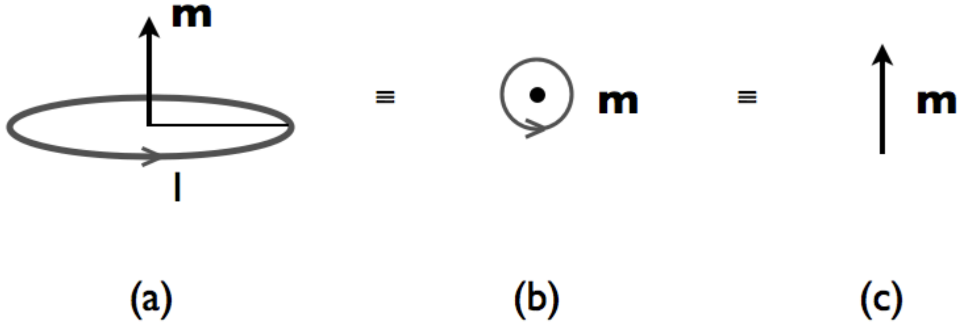


Figure 91: Different graphical representations of a magnetic dipole moment $\mathbf{m}$: Panel (a): Perspective view of a current loop. Panel (b): Current loop seen from above with $\mathbf{m}$ pointing out of the page. Panel (c): Representation as a simple vector, with the direction of the current loop understood (determined by the right-hand rule).

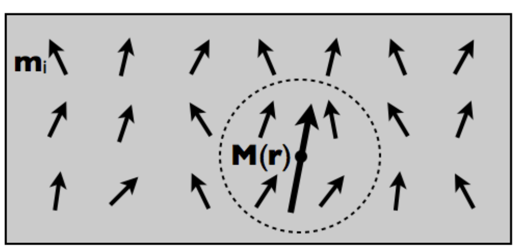


Figure 92: The magnetization $\mathbf{M}(\mathbf{r})$ of a material is defined as the average magnetic dipole moment per unit volume. Here the volume is a sphere of radius R centered at $\mathbf{r}$, which is large enough to include many individual dipole moments $\mathbf{m}_i$, but is small on a macroscopic scale.

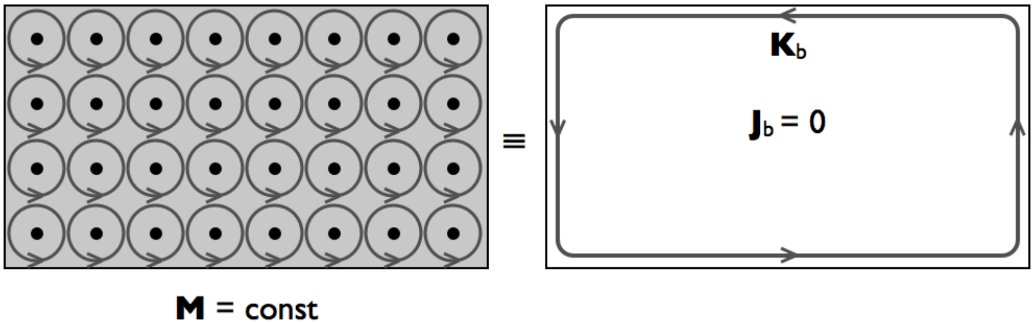


Figure 93: Material with uniform magnetization $\mathbf{M}$, and an equivalent representation of the material in terms of bound current densities $\mathbf{J}_b$ and $\mathbf{K}_b$. Since $\mathbf{M}$ = const, it follows that $\mathbf{J}_b \equiv \boldsymbol{\nabla} \times \mathbf{M} = 0$ but $\mathbf{K}_b \equiv \mathbf{M} \times \hat{\mathbf{n}} \neq 0$.

integrating by parts using the product rule

$$\boldsymbol{\nabla} \times (f\mathbf{A}) = f\boldsymbol{\nabla} \times \mathbf{A} - \mathbf{A} \times \boldsymbol{\nabla} f \tag{909}$$

and using a corollary (104) of the divergence theorem, we obtain

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V dV' \, \frac{\boldsymbol{\nabla}' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mu_0}{4\pi} \oint_S da' \, \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}'}{|\mathbf{r} - \mathbf{r}'|} \tag{910}$$

- Thus, the vector potential is due to volume and surface current densities

$$\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M} \,, \qquad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \tag{911}$$

  where the subscript b indicates that these are *bound* current densities, attached to current loops fixed in the material.

- Nonetheless, they are *genuine* current densities and give rise to potentials and fields like any other current density. An example of material with uniform magnetization is shown in Figure 93.

- Note that

$$\boldsymbol{\nabla} \cdot \mathbf{J}_b = 0 \tag{912}$$

  as it should be for a volume current density in magnetostatics.

- Note also that the total bound current is zero if we consider a surface sufficently large to include all of $\mathbf{J}_b$ and $\mathbf{K}_b$.

- Example: Prove this last statement. (Hint: Recall that $I = \int_S \mathbf{J} \cdot \hat{\mathbf{n}} \, da$ for a volume current density $\mathbf{J}$, while $I = \int_C (\mathbf{K} \times \hat{\mathbf{n}}) \cdot d\mathbf{s}$ for a surface current density $\mathbf{K}$. In the first expression, $\hat{\mathbf{n}}$ is the unit (outward pointing) normal to S; in the second expression, $\hat{\mathbf{n}}$ is the unit (outward pointing) normal to the surface in which $\mathbf{K}$ and $d\mathbf{s}$ live.)

- Example: Calculate the vector potential and magnetic field both inside and outside a uniformly magnetized spherical volume (magnetization $\mathbf{M} = M\,\hat{\mathbf{z}}$, radius R). (See Figure 94.)

- Answer:

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{1}{3}\mu_0 \, \mathbf{M} \times \mathbf{r} & \text{(inside)} \\ \frac{1}{3}\mu_0 R^3 \, \frac{\mathbf{M} \times \hat{\mathbf{r}}}{r^2} & \text{(outside)} \end{cases} \tag{913}$$

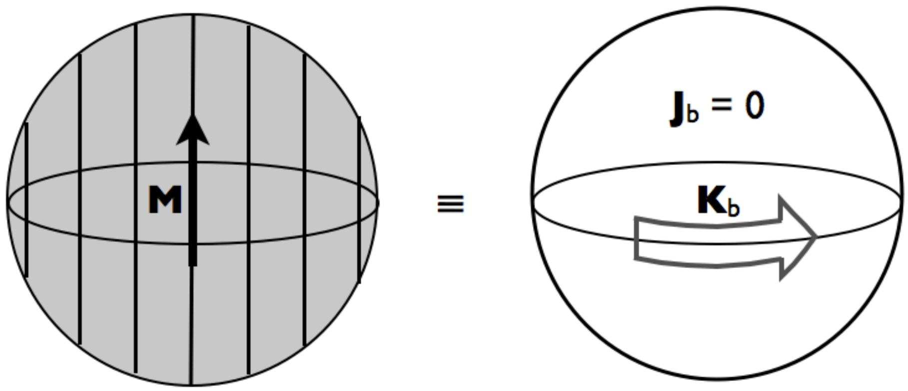


Figure 94: Uniformly magnetized spherical volume. For this example, $\mathbf{M} = M\,\hat{\mathbf{z}}$, $\mathbf{J}_b = 0$, and $\mathbf{K}_b = M \sin\theta\,\hat{\phi}$, which is zero at the poles and maximum at the equator.

and

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \frac{2}{3}\mu_0 \mathbf{M} \left(= \frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3}\right) & \text{(inside)} \\ \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right] & \text{(outside)} \end{cases} \tag{914}$$

where $\mathbf{m} = \frac{4}{3}\pi R^3\,\mathbf{M}$ is the total magnetic dipole moment of the sphere.

- Exercise: Prove the above results. (Hint: First note that for $\mathbf{M} = M\,\hat{\mathbf{z}}$, $\mathbf{J}_b = 0$ and $\mathbf{K}_b = M\sin\theta\,\hat{\boldsymbol{\phi}}$. Thus, the surface current is equivalent to that of a uniformly rotating spherical shell of uniform charge density σ with angular velocity $\boldsymbol{\omega}$, with the identification $\mathbf{M} = \sigma R\boldsymbol{\omega}$. You can then take over the results for $\mathbf{A}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ from the rotating spherical shell example.)
- Thus, the field $\mathbf{B}(\mathbf{r})$ for a uniformly polarized sphere is proportional to $\mathbf{M}$ inside the sphere; outside the sphere it is the same as the field due to a single magnetic dipole at the origin, with dipole moment equal to the total magnetic dipole moment of the sphere.
- The results are almost identical to those for the electric field inside and outside a uniformly polarized spherical volume. The only difference is a factor of -2 on the RHS of the expression for the magnetic field inside the sphere.

7.3 Ampère's law for magnetic materials

- Ampère's law in the form

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\,\mathbf{J} \tag{915}$$

is always true in magnetostatics, in vacuum or in a magnetized piece of material.

- In magnetic materials, it is sometimes convenient to split the total current density $\mathbf{J}$ into its bound and free components, $\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b$, and then use $\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M}$ to rewrite Ampère's law as

$$\boldsymbol{\nabla}\times\mathbf{H} = \mathbf{J}_f\,, \qquad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \tag{916}$$

- Exercise: Prove this.
- $\mathbf{H}$ is a sometimes called the *auxiliary field.*
- Experimenters tend to refer to $\mathbf{H}$ more often than to $\mathbf{B}$, since it is fairly easy to measure and control the amount of free current $\mathbf{J}_f$. Similarly, experimenters refer to $\mathbf{E}$ more often than to $\mathbf{D}$, since it easier to measure and control the potential difference on conductors than to measure the amount of free charge.

- The integral form of Ampère's law for $\mathbf{H}$ is

$$\oint_C \mathbf{H} \cdot d\mathbf{s} = I_{f,\text{enc}} \tag{917}$$

where $I_{f,\text{enc}}$ is the total enclosed free current.

- Recall that the boundary conditions on the magnetic field are

$$\mathbf{B}_2 - \mathbf{B}_1 = \mu_0 \, \mathbf{K} \times \hat{\mathbf{n}} \tag{918}$$

or, equivalently,

$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{\mathbf{n}} = 0 \,, \qquad (\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{\mathbf{t}} = \mu_0 \, (\mathbf{K} \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{t}} \tag{919}$$

where $\hat{\mathbf{n}}$ is the unit normal pointing from region 1 to region 2, $\hat{\mathbf{t}}$ is a unit tangent to the boundary, and $\mathbf{K}$ is a surface current density. (See Figure 86 from Section 6.)

- In terms of the auxiliary field $\mathbf{H}$, we have

$$(\mathbf{H}_2 - \mathbf{H}_1) \cdot \hat{\mathbf{n}} = -(\mathbf{M}_2 - \mathbf{M}_1) \cdot \hat{\mathbf{n}} \,, \qquad (\mathbf{H}_2 - \mathbf{H}_1) \cdot \hat{\mathbf{t}} = (\mathbf{K}_f \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{t}} \tag{920}$$

- Note that, unlike $\mathbf{B}$, the components of $\mathbf{H}$ normal to the surface are not continuous across the boundary. This is a consequence of the fact that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, but

$$\boldsymbol{\nabla} \cdot \mathbf{H} = -\boldsymbol{\nabla} \cdot \mathbf{M} \tag{921}$$

which does not equal zero, in general.

- The fact that $\boldsymbol{\nabla} \cdot \mathbf{H} \neq 0$ in general means that simply solving Ampère's law in the form $\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_f$ does not completely determine $\mathbf{H}$. (Recall that a vector field is specified by *both* its divergence and curl.) Thus, if $\mathbf{J}_f = 0$, we cannot necessarily conclude that $\mathbf{H} = 0$. This will only be the case if $\boldsymbol{\nabla} \cdot \mathbf{M} = 0$, so $\boldsymbol{\nabla} \cdot \mathbf{H} = 0$ as well.

- If $\mathbf{J}_f = 0$ everywhere, then

$$\boldsymbol{\nabla} \times \mathbf{H} = 0 \Leftrightarrow \mathbf{H} = -\boldsymbol{\nabla} W \tag{922}$$

W is a scalar potential for the auxiliary field $\mathbf{H}$, just like Φ for the electrostatic field $\mathbf{E}$.

- Since $\boldsymbol{\nabla} \cdot \mathbf{H} = -\boldsymbol{\nabla} \cdot \mathbf{M}$, it follows that

$$\nabla^2 W = \boldsymbol{\nabla} \cdot \mathbf{M} \tag{923}$$

Thus, W satisfies Poisson's equation with source term $\boldsymbol{\nabla} \cdot \mathbf{M}$.

- All of the techniques that we learned to solve Poisson's equation for $\Phi(\mathbf{r})$ in electrostatics can be carried over to solving for $W(\mathbf{r})$ in magnetostatics.

- <u>Exercise</u>: Use $W(\mathbf{r})$ to find the magnetic field $\mathbf{B}(\mathbf{r})$ both inside and outside a uniformly magnetized spherical volume (magnetization $\mathbf{M} = M\hat{\mathbf{z}}$, radius R).

  Hint: Since $\boldsymbol{\nabla} \cdot \mathbf{M} = 0$ both inside and outside the sphere, Poisson's equation reduces to Laplace's equation in these two regions. The solutions for $W(\mathbf{r})$ in each region can be expanded in terms of Legendre polynomials and solutions to the radial equation. The boundary conditions are:

  i) W should be finite at the origin ($r = 0$) and vanish as $r \to \infty$.

  ii) W should be continuous across the boundary $r = R$. (This condition follows from integrating $\boldsymbol{\nabla} W = -\mathbf{H}$ across the boundary.)

iii) The normal derivative of W is discontinuous, since it inherts the discontinuity in the normal component of $\mathbf{H}$:

$$\left(-\frac{\partial W_{\rm out}}{\partial r}+\frac{\partial W_{\rm in}}{\partial r}\right)\Bigg|_{r=R} = \hat{\mathbf{n}}\cdot(\mathbf{H}_{\rm out}-\mathbf{H}_{\rm in})\Big|_{r=R} \tag{924}$$

$$= \hat{\mathbf{n}}\cdot(-\mathbf{M}_{\rm out}+\mathbf{M}_{\rm in})\Big|_{r=R} \tag{925}$$

$$= M\cos\theta \tag{926}$$

At the end of the calculation, you should find

$$W(\mathbf{r}) = \begin{cases} \frac{1}{3}Mr\cos\theta & \text{(inside)} \\ \frac{1}{3}M\frac{R^3}{r^2}\cos\theta & \text{(outside)} \end{cases} \tag{927}$$

which leads to the same expression, Eq. (914), that we found for $\mathbf{B}(\mathbf{r})$ when working this problem in the previous subsection.

7.4 Linear materials

- In order to solve magnetostatics problem in magnetic materials, we need to be given the magnetization $\mathbf{M}$, or at least know how it is related to the field $\mathbf{B}$ or $\mathbf{H}$.
- For certain materials—called *linear, isotropic materials*—we have

$$\mathbf{M} = \chi_m\mathbf{H} \tag{928}$$

  where $\mathbf{H}$ is the total auxiliary field.
- The above relation is often a good approximation for *weak* fields $\mathbf{H}$.
- Note the difference from the linear dielectric case where $\mathbf{P} = \epsilon_0\chi_e\,\mathbf{E}$. Consistency with the linear dielectric definition suggests the definition $\mathbf{M} = (\chi_m/\mu_0)\mathbf{B}$, which is *not* used.
- χ_m is called the *magnetic susceptibility* of the material. It is positive for *paramagnetic* materials, and negative for *diamagnetic* materials.
- In general, χ_m depends on position $\mathbf{r}$. If χ_m is independent of position, then the material is said to be *homogeneous* (or *uniform*).
- Given the above relationship between $\mathbf{M}$ and $\mathbf{H}$, and the definition of $\mathbf{H}$ in terms of $\mathbf{B}$ and $\mathbf{M}$, it follows that

$$\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H} = \mu\mathbf{H} \tag{929}$$

  where $\mu \equiv (1+\chi_m)\mu_0$ is called the *permeability* of the material.
- Exercise: Prove this.
- Compare $\mathbf{B} = \mu\mathbf{H}$ to the linear dielectric case where $\mathbf{D} = \epsilon\mathbf{E}$.

7.5 Boundary value problems in linear media

- We will restrict attention to magnetostatic problems in linear, isotropic, and *homogeneous* magnetic materials, where a number of simplifications follow from the *constant* proportionality of $\mathbf{M}$ and $\mathbf{H}$:

  1) The bound and total volume current densities are proportional to $\mathbf{J}_f$:

$$\mathbf{J}_b = \chi_m\,\mathbf{J}_f\,, \qquad \mathbf{J} = (1+\chi_m)\,\mathbf{J}_f \tag{930}$$

  2) The divergence of both $\mathbf{H}$ and $\mathbf{M}$ vanish:

$$\boldsymbol{\nabla}\cdot\mathbf{H} = -\boldsymbol{\nabla}\cdot\mathbf{M} = 0 \tag{931}$$

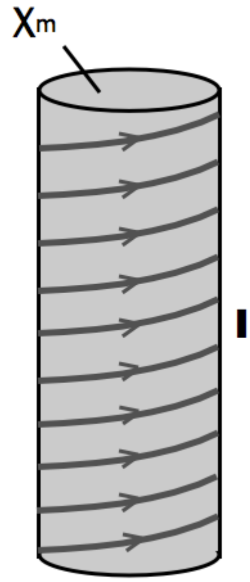


Figure 95: Infinite solenoid filled with a homogeneous, isotropic, linear material with magnetic susceptibility χ_m.

3) Ampère's law for **B** can be written simply in terms of the free current density:

$$\nabla \times \mathbf{B} = \mu \, \mathbf{J}_f \tag{932}$$

4) The boundary conditions can be written as

$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{\mathbf{n}} = 0 \,, \qquad \left(\frac{\mathbf{B}_2}{\mu_2} - \frac{\mathbf{B}_1}{\mu_1} \right) \cdot \hat{\mathbf{t}} = (\mathbf{K}_f \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{t}} \tag{933}$$

where $\hat{\mathbf{n}}$ is the unit normal pointing from region 1 to region 2, and $\hat{\mathbf{t}}$ is the unit tangent to the boundary surface.

- Exercise: Prove the above relations.
- Thus, all of the techniques that we learned previously for solving magnetostatic problems carry over to the case of linear, isotropic, and homogeneous materials with: (i) **J** replaced by $\mathbf{J}_f$, (ii) μ_0 replaced by μ, and (iii) the parallel boundary conditions for the magnetic field involving $\mathbf{K}_f$ and the ratio of **B** and μ.
- Example 1: An infinite solenoid (current I, n turns per unit length) is filled with a homogeneous, isotropic, linear material with magnetic susceptibility χ_m. Find $\mathbf{B}(\mathbf{r})$ both inside and outside the solenoid. (See Figure 95.)
- Answer:

$$\mathbf{B}(s) = \begin{cases} \mu I n \, \hat{\mathbf{z}} & \text{(inside)} \\ 0 & \text{(outside)} \end{cases} \tag{934}$$

Note that the only difference from the vacuum case is the replacement of μ_0 by μ.

- Example 2: Calculate $\mathbf{B}(\mathbf{r})$ outside an infinitely long, straight current-carrying wire (current I) embedded in homogeneous, isotropic, linear material with magnetic susceptibility χ_m. (See Figure 96.)

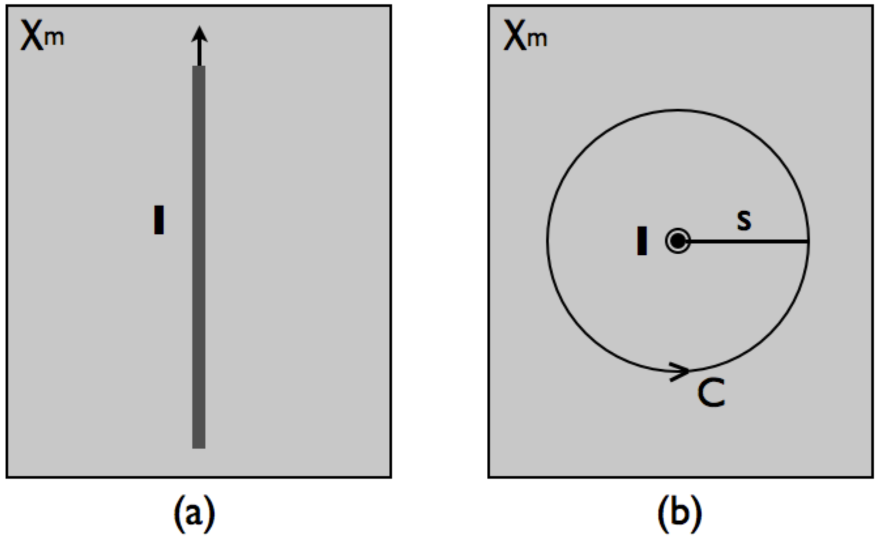


Figure 96: Panel (a): Infinitely long, straight current-carrying wire embedded in a homogeneous, isotropic, linear material with magnetic susceptibility χ_m. Panel (b): Top view of the wire (current coming out of the page), and an Ampèrian loop C for calculating $\mathbf{B}(s)$ at a perpendicular distance s from the wire.

- Answer:

$$\mathbf{B}(s) = \frac{\mu I}{2\pi s}\,\hat{\boldsymbol{\phi}} \tag{935}$$

Again the only difference from the vacuum case is the replacement of μ_0 by μ.

- Example 3: A current I flows down an infinitely long, straight wire of radius a. The wire is made of homogeneous, isotropic, linear material with magnetic susceptibility χ_m. Assume that I is distributed uniformly across the cross-sectional area of the wire. Find $\mathbf{B}(\mathbf{r})$ both inside and outside the wire. (See Figure 97.)

- Answer:

$$\mathbf{B}(s) = \begin{cases} \frac{\mu I s}{2\pi a^2}\,\hat{\boldsymbol{\phi}} & \text{(inside)} \\ \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}} & \text{(outside)} \end{cases} \tag{936}$$

Note that outside the wire, the field agrees with that for the vacuum case.

- Example 4: Calculate the field $\mathbf{B}(\mathbf{r})$ both inside and outside a spherical volume of homogeneous, isotropic, linear material (radius R, magnetic susceptibility χ_m) placed in a uniform magnetic field $\mathbf{B}_0$. (See Figure 98.)

- Answer:

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \left(\frac{1+\chi_m}{1+\chi_m/3}\right)\mathbf{B}_0 & \text{(inside)} \\ \mathbf{B}_0 + \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right] & \text{(outside)} \end{cases} \tag{937}$$

where

$$\mathbf{m} \equiv \frac{4}{3}\pi R^3\,\mathbf{M}\,, \qquad \mathbf{M} = \chi_m \mathbf{H}_{\text{in}} = \left(\frac{\chi_m}{1+\chi_m/3}\right)\frac{\mathbf{B}_0}{\mu_0} \tag{938}$$

These results are very similar to what we found for a spherical volume of homogeneous, isotropic, linear dielectric material in a uniform electric field $\mathbf{E}_0$.

Hint: Since there are no free currents, $\boldsymbol{\nabla}\times\mathbf{H} = 0$ implying $\mathbf{H} = -\boldsymbol{\nabla}W$, where W satisfies Poisson's equation $\nabla^2 W = \boldsymbol{\nabla}\cdot\mathbf{M}$. But since $\boldsymbol{\nabla}\cdot\mathbf{M} = 0$ both inside and outside the sphere,

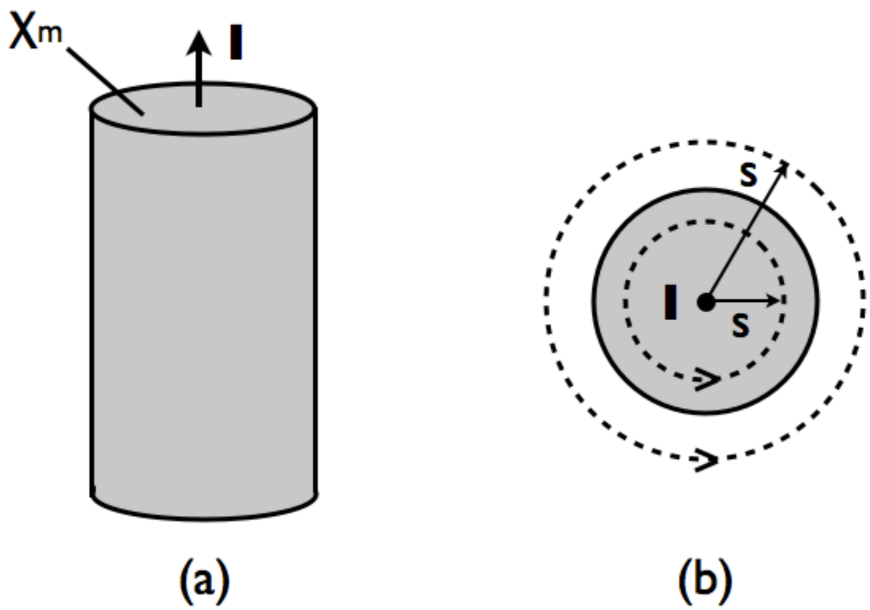


Figure 97: Panel (a): An infinitely long, straight current-carrying wire of radius a made of a homogeneous, isotropic, linear material with magnetic susceptibility χ_m. Panel (b): Top view of the wire (current $\mathbf{I}$ coming out of the page), and two different Ampèrian loops C, one inside and one outside the wire, for calculating $\mathbf{B}(s)$ at a perpendicular distance s from the wire.

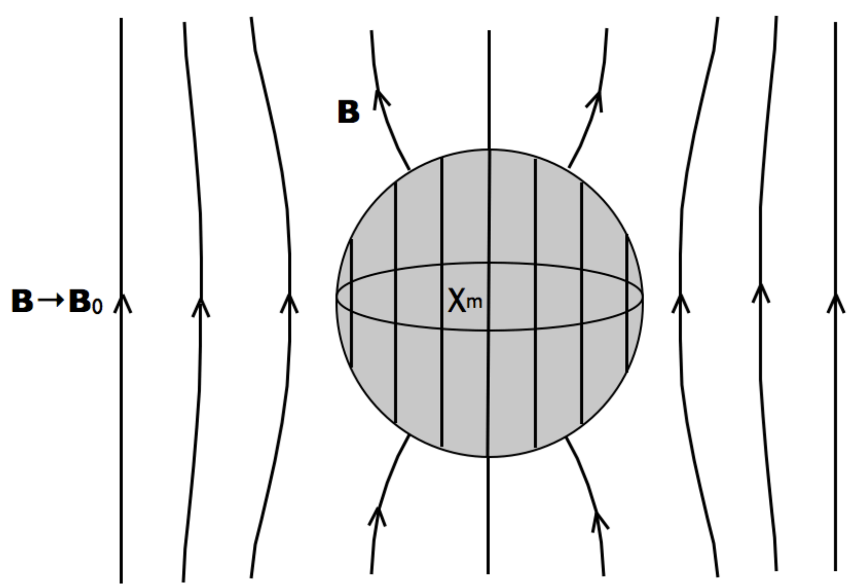


Figure 98: A homogeneous, isotropic, linear spherical volume (radius R, magnetic susceptibility χ_m) placed in a uniform magnetic field $\mathbf{B}_0$.

you need only solve Laplace's equation in these two regions. The solutions for $W(\mathbf{r})$ in each region can be expanded in terms of Legendre polynomials and solutions to the radial equation. Taking the z-axis to lie along the direction of $\mathbf{B}_0$, the boundary conditions are:

i) W should be finite at the origin ($r = 0$) and should approach $-(B_0/\mu_0) r\cos\theta$ as $r \to \infty$.

ii) W should be continuous across the boundary $r = R$.

iii) The normal derivative of W is discontinuous across the boundary, satisfying

$$\left(-\mu_0 \frac{\partial W_{\rm out}}{\partial r} + \mu \frac{\partial W_{\rm in}}{\partial r}\right)\Bigg|_{r=R} = 0 \tag{939}$$

This last condition is equivalent to the boundary condition $(\mathbf{B}_{\rm out} - \mathbf{B}_{\rm in})\cdot\hat{\mathbf{n}} = 0$ written in terms of W, using $\mathbf{B} = \mu\mathbf{H} = -\mu\boldsymbol{\nabla} W$.

At the end of the calculation, you should find

$$W(\mathbf{r}) = \begin{cases} -\left(\frac{1}{1+\chi_m/3}\right)\frac{B_0}{\mu_0}\, r\cos\theta & \text{(inside)} \\ -\left[1 - \left(\frac{R}{r}\right)^3\left(\frac{\chi_m/3}{1+\chi_m/3}\right)\right]\frac{B_0}{\mu_0}\, r\cos\theta & \text{(outside)} \end{cases} \tag{940}$$

which leads to the above expression for $\mathbf{B}(\mathbf{r})$.

Note: A quicker way to obtain the above results is to argue that the induced magnetization $\mathbf{M}$ inside the spherical volume is necessarily constant, and then use equation (914) for the field $\mathbf{B}_{\rm sphere}$ of a uniformly magnetized sphere. The total field is then $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_{\rm sphere}$. But be advised that this quick way works only for *ellipsoidal-shaped* materials (of which a sphere is a special case); linear, isotropic, and homogoneous materials having other, more general, shapes will *not* be uniformly magnetized when placed in a uniform magnetic field.

8 Electrodynamics

8.1 Ohm's law

- To drive a current $\mathbf{J}$ in a wire, you need to apply a force (per unit charge) $\mathbf{f}$:

$$\mathbf{J} = \sigma\mathbf{f} \tag{941}$$

  where σ is the *conductivity* of the material, not to be confused with surface charge density. ($\rho \equiv 1/\sigma$ is the *resistivity*, not to be confused with volume charge density.)

- The conductivity of conductors is more than 20 orders of magnitude greater than that of insulators. For most metals, we can take $\sigma = \infty$ (a *perfect conductor*). For a perfect conductor, $\mathbf{E} = \mathbf{J}/\sigma = 0$ even for non-zero $\mathbf{J}$.

- For electromagnetic forces

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \approx \sigma\mathbf{E} \tag{942}$$

  where we've neglected the magnetic contribution to get the last line. For most materials, this is a good approximation, but not for plasmas.

- $\mathbf{J} = \sigma\mathbf{E}$ implies a linear relation between the current I and voltage V:

$$V = IR \tag{943}$$

  R is called the *resistance* of the material; it depends on the size, shape, and type of material. The above equation is called *Ohm's law*.

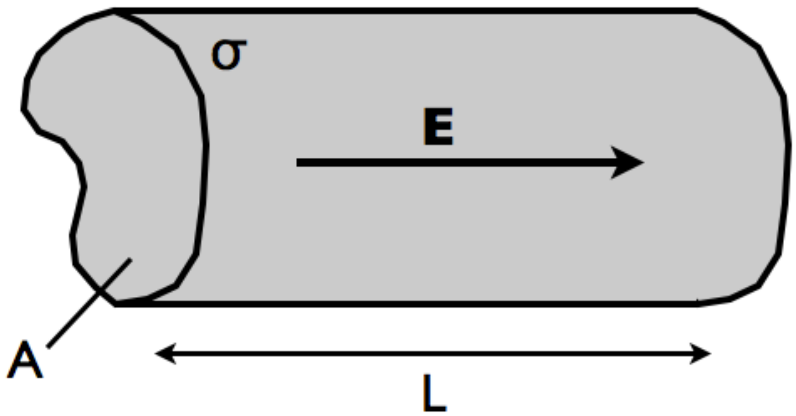


Figure 99: Cylindrical resistor with uniform cross-sectional area A, length L, and conductivity σ. The electric field $\mathbf{E}$ is uniform throughout the resistor.

- Example 1: (7.1, Griffiths) Show that the resistance of a cylindrical resistor of cross-sectional area A, length L, and conductivity σ is

$$R = \frac{L}{\sigma A} \tag{944}$$

  Hint: Calculate the current I and the potential difference V between the ends of the resistor. The field $\mathbf{E}$ (and $\mathbf{J}$) is uniform throughout the resistor. (See Figure 99.)

- Example 2: (7.2, Griffiths) Two long cylinders (radii a and b, $a < b$) are separated by a material of conductivity σ. If they are held at a potential difference V, show that the current I that flows from one cylinder to the other (in a length L) is given by (See Figure 100.)

$$I = \frac{2\pi\sigma L}{\ln(b/a)} V \Leftrightarrow R = \frac{\ln(b/a)}{2\pi\sigma L} \tag{945}$$

8.2 Electromotive force

- *Electromotive force* (emf) is defined to be the line integral

$$\mathcal{E} = \oint_C \mathbf{f} \cdot d\mathbf{s} \tag{946}$$

  where $\mathbf{f}$ is some force per unit charge and C is a closed curve.

- The closed curve C could correspond to an actual electrical circuit, but it doesn't have to.

- The integration is done over the shape of the loop C *at a fixed instant of time* t. Whether the loop is stationary or moving as a whole is irrelevant for the integration.

- $\mathbf{f}$ might be an electric field $\mathbf{E}$, the magnetic force on a moving charge, $\mathbf{f}_{\text{mag}} \equiv \mathbf{v} \times \mathbf{B}$, the chemical force inside a battery, $\mathbf{f}_s$, or any force that can drive a current around a closed loop.

- The MKS unit of emf is the volt (work per unit charge).

- In a electrical circuit

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E} \tag{947}$$

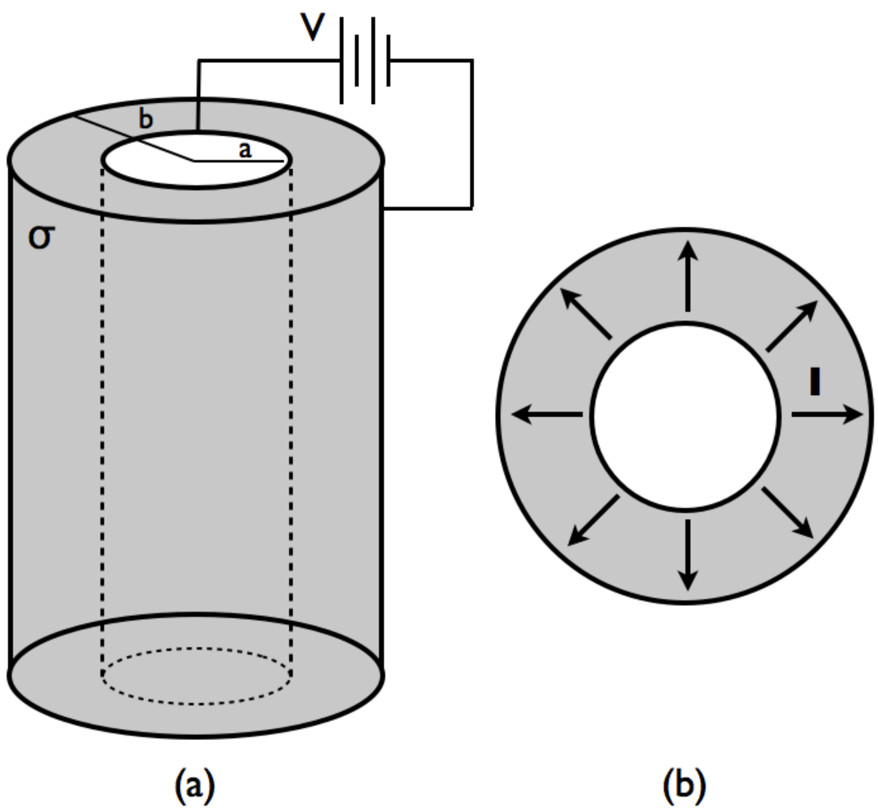


Figure 100: Panel (a): Two cylinders (radii a and b, $a < b$) are separated by a material with conductivity σ. They are held at a potential difference V. Panel (b): Top view. Current I flows radially outward from the inner cylinder to the outer cylinder.

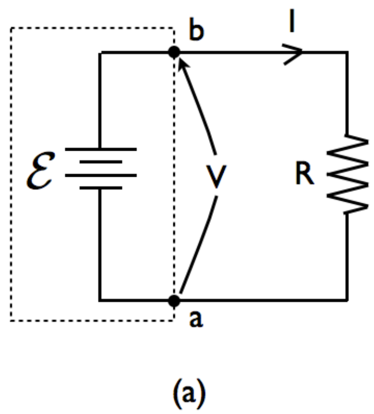


Figure 101: Panel (a): Ideal battery with zero internal resistance. The potential difference V at the terminals of the battery equals the emf $\mathcal{E}$. Panel (b): Real battery with non-zero internal resistance r. The potential difference V at the terminals of the battery equals the emf $\mathcal{E} - Ir$, where I is the current flowing in the circuit.

where $\mathbf{f}_s$ is the force per unit charge associated with the source (e.g., a battery) and $\mathbf{E}$ is the electrostatic field that prevents the piling-up of charge in any portion of the circuit. This is why a lamp lights almost instantaneously, even though the individual electrons move at a snail's pace down the wire ($v_{\text{drift}} \sim .1$mm/s for 1 Amp of current in a copper wire with a diameter of 1 mm).

- For an ideal source (e.g., a battery with no internal resistance), $\mathbf{f}_s = -\mathbf{E}$ inside the source. This implies

$$V \equiv -\int_a^b \mathbf{E} \cdot d\mathbf{s} = \int_a^b \mathbf{f}_s \cdot d\mathbf{s} = \oint_C \mathbf{f}_s \cdot d\mathbf{s} = \oint_C \mathbf{f} \cdot d\mathbf{s} = \mathcal{E} \tag{948}$$

where a and b denote the terminals of the battery, and the second-to-last equality follows from $\oint \mathbf{E} \cdot d\mathbf{s} = 0$. Thus, the potential difference between the terminals of an ideal battery equals the emf. (See Figure 101, Panel (a).)

- For a real battery with internal resistance r, the potential difference between the terminals is $V = \mathcal{E} - Ir$, where I is the current flowing in the circuit. (See Figure 101, Panel (b).)

8.3 Electromagnetic induction

- Another way to generate an emf is to move a circuit through a magnetic field. This is called *motional* emf; it is the principle underlying the operation of an electric generator.
- Although the magnetic field is responsible for setting up the emf, it does *not* do any work on the charges. (Magnetic fields never do any work.) It is whatever (or whoever) is moving the circuit that is doing the work.
- Exercise: A rectangular loop of wire (height h) is pulled with constant velocity $\mathbf{v}$ through a region of uniform magnetic field $\mathbf{B}$ (with $\mathbf{B}$ perpendicular to $\mathbf{v}$). (See Figure 102.) Show that the induced emf around the circuit is given by

$$\mathcal{E} = vBh \tag{949}$$

- The above result can be rewritten as

$$\mathcal{E} = -\frac{dx}{dt}Bh = -\frac{d}{dt}(Bhx) = -\frac{d\Phi_B}{dt} \tag{950}$$

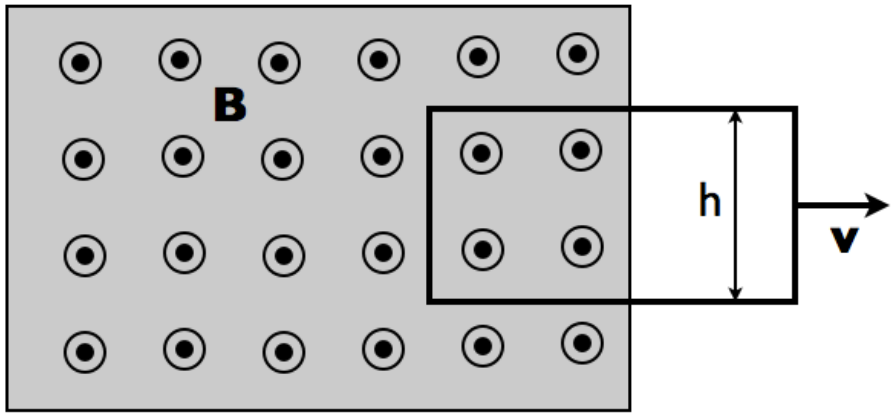


Figure 102: Rectangular loop of wire (height h) is pulled with constant velocity $\mathbf{v}$ through a region of uniform magnetic field $\mathbf{B}$. Here $\mathbf{B}$ is pointing out of the page.

where

$$\Phi_B \equiv \int_S \mathbf{B} \cdot \hat{\mathbf{n}}\, da \tag{951}$$

is the magnetic flux passing through any surface S spanning the circuit C.

- The equation

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{952}$$

is called the *flux rule* for electromagnetic induction. It is a general result that holds for arbitrarily shaped loops moving in arbitrary directions in non-uniform magnetic fields.

- The minus sign on the RHS of the flux rule is sometimes called *Lenz's law*. It means that the induced emf is in the direction which *opposes* the change in magnetic flux—i.e., it drives a current that tries to keep the magnetic flux constant.

- A nice demonstration of electromagnetic induction and Lenz's law is to drop a magnet down a cylindrical tube made of conducting material. The magnet falls very slowly, as induced currents around the tube generate magnetic fields that oppose the changing flux of the falling magnet.

8.4 Faraday's law

- In 1831, Faraday showed that:

  1) a current flows in a circuit that is moved into or out of a stationary magnetic field.

  2) a current flows through a stationary circuit if a magnetic field is moved toward or away from the circuit.

  3) a current flows through a stationary circuit even if the magnet is also held stationary, provided the strength of the (electro)magnet is changing. (See Figure 103.)

- Experiment 1) is an example of motional emf, discussed in the previous subsection.

- Experiment 2) and 3) show that a changing magnetic field produces an electric field which is responsible for the emf.

- The agreement of experiments 1) and 2) is consistent with the prinicipal of relativity.

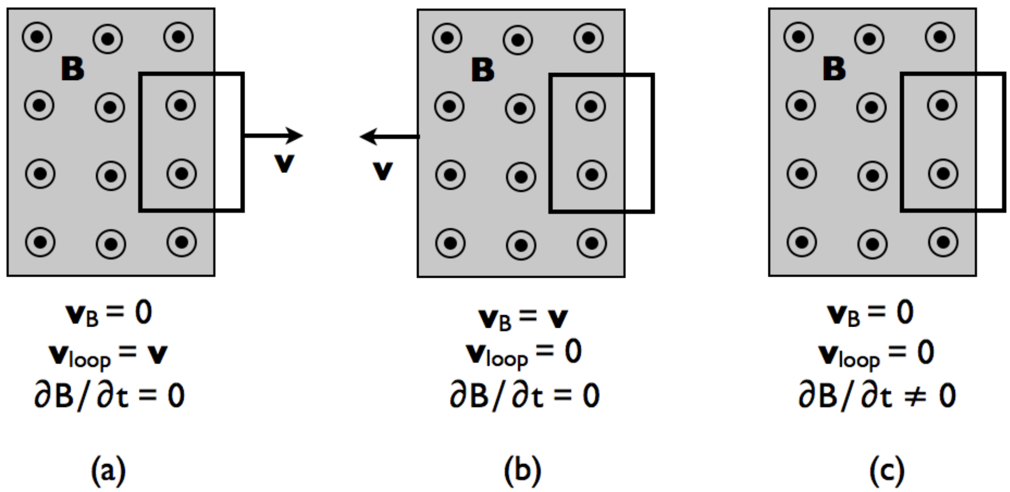


Figure 103: Panel (a): A rectangular loop of wire is pulled with non-zero velocity $\mathbf{v}$ through a stationary magnetic field $\mathbf{B}$. Panel (b): The rectangular loop of wire is at rest, but the source of the magnetic field is now moving with non-zero velocity $\mathbf{v}$. Panel (c): The rectangular loop of wire and the source of the magnetic field are both at rest, but the strength of the magnetic field is changing—i.e., $\partial B/\partial t \neq 0$. In *all* three cases, a current flows through the wire.

- All three of the above experiments can be explained using the flux rule

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{953}$$

- Expanding the LHS of the flux rule as

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{s} = \int_S (\boldsymbol{\nabla} \times \mathbf{E}) \cdot \hat{\mathbf{n}}\, da \tag{954}$$

  and the RHS as

$$-\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left(\int_S \mathbf{B} \cdot \hat{\mathbf{n}}\, da\right) = -\int_S \left(\frac{\partial \mathbf{B}}{\partial t}\right) \cdot \hat{\mathbf{n}}\, da \tag{955}$$

  we obtain

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{956}$$

  This is the differential form of Faraday's law.

- The integral form of Faraday's law is

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \tag{957}$$

- Note that a pure Faraday-induced electric field satisfies $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ in addition to (956); hence, it is determined by $-\partial \mathbf{B}/\partial t$ in the same way that $\mathbf{B}$ is determined by $\mu_0 \mathbf{J}$ in magnetostatics.

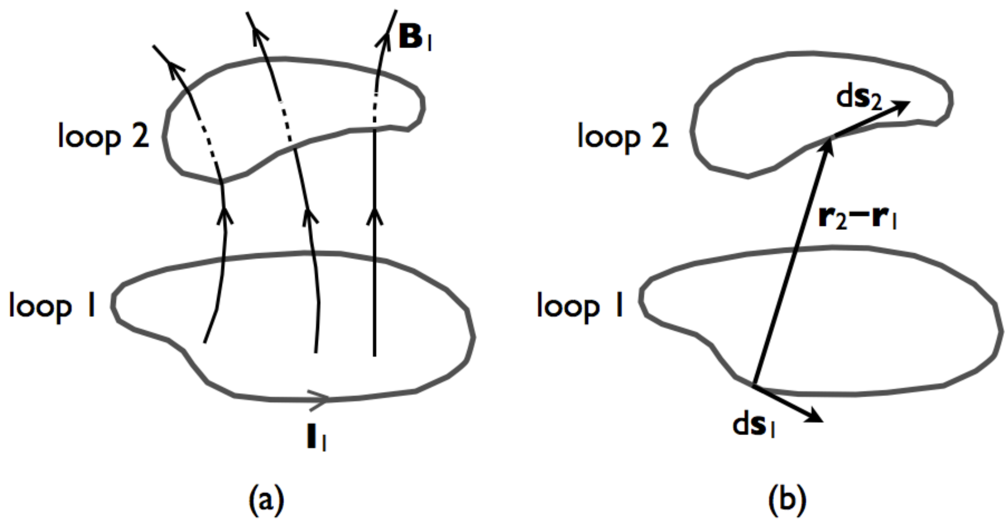


Figure 104: Panel (a): A current I_1 flowing in loop 1 produces a magnetic field $\mathbf{B}_1$, which gives rise to a magnetic flux through loop 2. Panel (b): Definitions of $d\mathbf{s}_1$, $d\mathbf{s}_2$, and $\mathbf{r}_2 - \mathbf{r}_1$, which appear in the Neumann formula (961).

8.5 Inductors

- Consider two closed loops 1 and 2. A current I_1 in loop 1 produces a magnetic field $\mathbf{B}_1$, which gives rise to a magnetic flux through loop 2:

$$\Phi_{B,2} = \int_{S_2} \mathbf{B}_1 \cdot \hat{\mathbf{n}}_2 \, da_2 \tag{958}$$

  See Figure 104.

- Since $\Phi_{B,2}$ is proportional to $\mathbf{B}_1$, which in turn is proportional to I_1, we can write

$$\Phi_{B,2} = M_{21} I_1 \tag{959}$$

  where M_{21} is a constant, called the *mutual inductance* of the two loops. M_{21} depends only on the geometry and relative positions of the two loops.

- By expanding the expression for $\Phi_{B,2}$ in terms of $\mathbf{B}_1 = \boldsymbol{\nabla} \times \mathbf{A}_1$, and then using Stokes' theorem and the expression

$$\mathbf{A}_1(\mathbf{r}_2) = \frac{\mu_0 I}{4\pi} \oint_{C_1} \frac{d\mathbf{s}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \tag{960}$$

  one can show that

$$M_{21} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{s}_1 \cdot d\mathbf{s}_2}{|\mathbf{r}_2 - \mathbf{r}_1|} \tag{961}$$

  This is called the *Neumann formula*.

- <u>Exercise</u>: Prove the above.

- Note that the double integral above is symmetric under interchange of 1 and 2, so

$$M_{21} = M_{12} \equiv M \tag{962}$$

  The equality of M_{21} and M_{12} means that if a current I in loop 1 produces a flux $\Phi_{B,2} = M_{21} I$ passing through loop 2, then the *same* current I in loop 2 will produce the *same* flux $\Phi_{B,1} = M_{12} I = M_{21} I = \Phi_{B,2}$ passing through loop 1.

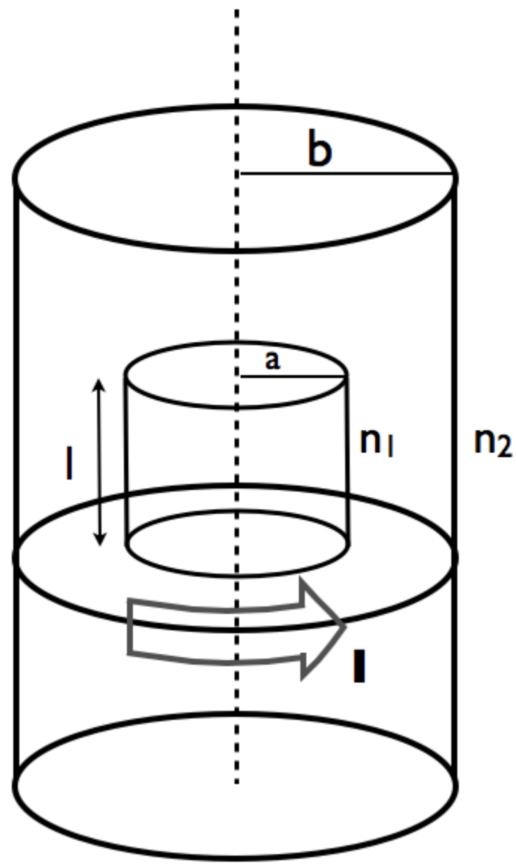


Figure 105: A short solenoid (radius a, length l, n_1 turns per unit length) lies on the axis of a very long solenoid (radius $b > a$, n_2 turns per unit length). To calculate the mutual inductance, it is simplest to let current I flow around the long solenoid, and then calculate the flux through the short solenoid.

- Example: (7.10, Griffiths) A short solenoid (radius a, length l, n_1 turns per unit length) lies on the axis of a very long solenoid (radius $b > a$, n_2 turns per unit length). (See Figure 105.) Find an expression for the mutual inductance M.

- Answer:
$$M = \mu_0 \pi a^2 \, n_1 n_2 l \tag{963}$$
Hint: Let current I flow around the long solenoid, and then calculate the flux through the short solenoid. Remember that the flux depends on the total number of turns linked by the magnetic field.

- A single current loop I produces a field $\mathbf{B}$, which gives rise to a flux Φ_B through the loop. (See Figure 106.) Since Φ_B is proportional to $\mathbf{B}$, which in turn is proportional to I, we can write
$$\Phi_B = LI \tag{964}$$
where L is a constant, called the *self inductance* (or simply inductance). Like the mutual inductance M, L depends only on the geometry of the loop.

- The unit of inductance is the *henry*. (1 henry equals 1 volt·sec/Amp).

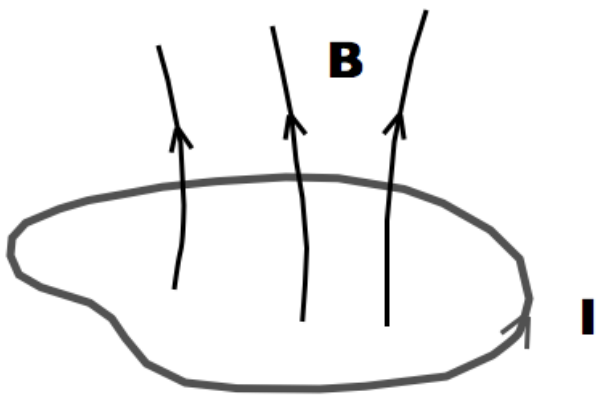


Figure 106: Current I flowing in single loop produces a magnetic field $\mathbf{B}$, which gives rise to a magnetic flux through the loop.

- For a changing current dI/dt, we have a changing flux

$$\frac{d\Phi_B}{dt} = L\frac{dI}{dt} \tag{965}$$

Thus, from the flux rule, we can conclude that

$$\mathcal{E} = -L\frac{dI}{dt} \tag{966}$$

so there is an emf (called a *back* emf) induced in the loop.

- The minus sign in the above formula means that the back emf is in the direction which opposes the the change in current.

- Inductance is similar to mass in the sense that it opposes any change in the amount of current flowing through a circuit, in the same way that mass opposes any change in its state of motion (i.e., in its velocity).

- The back emf is responsible for sparks that we sometimes see when we unplug a fan from an outlet, without first turning it off at the switch. (The back emf tries to keep the current flowing.)

- Example: (7.11, Griffiths) Show that the inductance L for a toroidal coil with N total turns and rectangular cross section (inner radius a, outer radius b, height h) is given by

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right) \tag{967}$$

See Figure 107.

Hint: For current I, the magnetic field inside the coil is given by $\mathbf{B}(s) = (\mu_0 N I)/(2\pi s)\,\hat{\boldsymbol{\phi}}$, in cylindrical coordinates.

8.6 Energy in magnetic fields

- Work is required to get a current flowing in a circuit, since one has to fight against the back emf $\mathcal{E} = -L\,dI/dt$, where L is the inductance of the circuit.

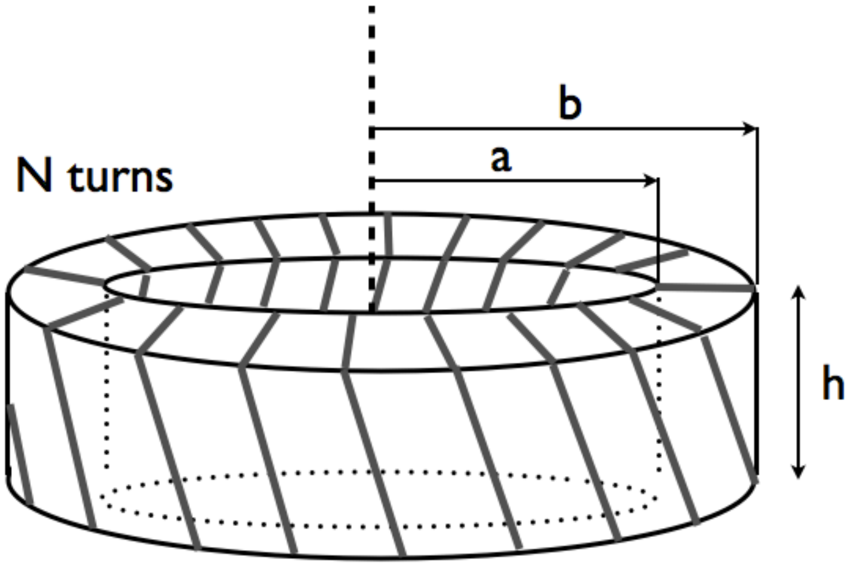


Figure 107: A toroidal coil with N total turns and rectangular cross section (inner radius a, outer radius b, height h).

- The work required to move a unit charge once around the circuit *against* the back emf is $-\mathcal{E}$. Since $I = dQ/dt$, the rate of doing work against the back emf is

$$\frac{dW}{dt} = -\mathcal{E}I = +L\frac{dI}{dt}\,I = \frac{d}{dt}\left(\frac{1}{2}LI^2\right) \tag{968}$$

Thus,

$$W = \frac{1}{2}LI^2 \tag{969}$$

- This is the work required to get a current I flowing in the circuit, starting with zero current. This energy is *stored* in the circuit, and can be recovered when the current is turned off.

- One can rewrite the above expression for the work W in a more general form by recalling that $LI = \Phi_B$, and writing Φ_B in terms of the vector potential $\mathbf{A}$:

$$W = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi_B = \frac{1}{2}I\int_S \mathbf{B}\cdot\hat{\mathbf{n}}\,da = \frac{1}{2}I\int_S (\boldsymbol{\nabla}\times\mathbf{A})\cdot\hat{\mathbf{n}}\,da = \frac{1}{2}\oint_C \mathbf{A}\cdot\mathbf{I}\,ds \tag{970}$$

- The last expression can now easily be generalized to a volume current density $\mathbf{J}$:

$$W = \frac{1}{2}\int_V \mathbf{A}\cdot\mathbf{J}\,dV \tag{971}$$

Note the similarity with the electrostatic expression $W = \frac{1}{2}\int_V \Phi\rho\,dV$.

- Using Ampére's law and a vector identity for the divergence of a curl, one can show that

$$W = \frac{1}{2\mu_0}\left[\int_V B^2\,dV - \oint_S (\mathbf{A}\times\mathbf{B})\cdot\hat{\mathbf{n}}\,da\right] \tag{972}$$

This is a general result, which is valid for *any* volume V.

- <u>Exercise</u>: Prove the above.

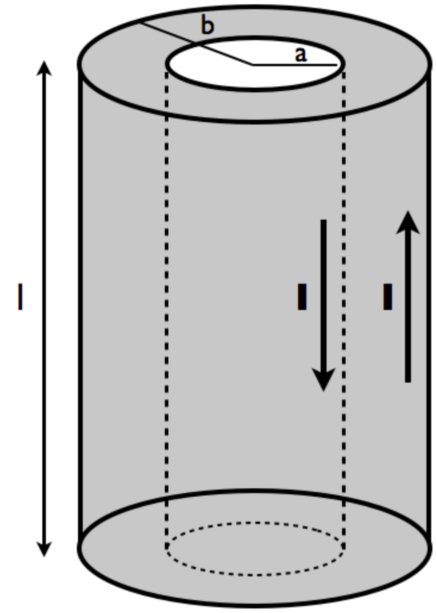


Figure 108: A section l of a coaxial cable (radii a, b, with $a < b$) with current I flowing down the inner cylinder and back up along the outer cylinder.

- If one now takes V to be all of space, the surface integral vanishes, and

$$W = \frac{1}{2\mu_0} \int_{\text{all space}} B^2 \, dV \tag{973}$$

Note the similarity with the electrostatic expression $W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 \, dV$.

- As we shall see in a later subsection, the above expressions for W in terms of B^2 and E^2 are actually valid for time-varying fields as well. The earlier expressions for W in terms of $\mathbf{A}$ and $\mathbf{J}$ and V and ρ are valid only for stationary charges and steady currents.

- The above result suggest the identification

$$\frac{1}{2\mu_0} B^2 = \text{energy density} \tag{974}$$

- Example: (7.13, Griffiths) Calculate the energy stored in a section l of a coaxial cable (radii a, b, with $a < b$) with current I flowing down the inner cylinder and back up along the outer cylinder. (See Figure 108.)

- Answer:

$$W = \frac{\mu_0 I^2 l}{4\pi} \ln\left(\frac{b}{a}\right) \tag{975}$$

Hint: The magnetic field inside the solenoid is $\mathbf{B}(s) = (\mu_0 I)/(2\pi s)\,\hat{\phi}$, in cylindrical coordinates.

8.7 Maxwell's equations

- The equations of electrodynamics before Maxwell were

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{(Gauss's law)} \tag{976}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(no magnetic monopoles)} \tag{977}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law)} \tag{978}$$

$$\nabla \times \mathbf{B} = \mu_0\, \mathbf{J} \qquad \text{(Ampere's law)} \tag{979}$$

- There is a problem, however, with Ampère's law, since it implies that $\nabla \cdot \mathbf{J} = 0$, which is valid for magnetostatics, but should be

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{980}$$

in electrodynamics. This is the continuity equation; it is the mathematical statement of local charge conservation.

- Maxwell fixed this problem by modifying the RHS of Ampère's law so that it is consistent with the continuity equation:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{(modified Ampere's law)} \tag{981}$$

- Exercise: Show that the continuity equation follows from the above equation.
- Maxwell's correction term to Ampère's law is sometimes called the *displacement current*:

$$\mathbf{J}_D \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{982}$$

- Since the displacement current is proportional to $\partial \mathbf{E}/\partial t$, we see that *changing electric fields give rise to magnetic fields*, similar to Faraday's finding that changing magnetic fields give rise to electric fields.
- The displacement current is needed for consistency of the integral form of Ampère's law

$$\oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d}{dt}\left(\int_S \mathbf{E} \cdot \hat{\mathbf{n}}\, da \right) \tag{983}$$

when applied to the case of a current charging a parallel-plate capacitor C. The 2-dimensional surface S spanning the Amperian loop C can either cut through the wire (so that the RHS is simply $\mu_0 I_{\text{enc}} = \mu_0 I$) or it can extend between the plates of the capacitor, where the RHS is given by the second term

$$\mu_0 \epsilon_0 \frac{d}{dt}(EA) = \mu_0 \epsilon_0 \frac{d}{dt}\left(\frac{V}{d} A\right) = \mu_0 \frac{d}{dt}\left(V \frac{\epsilon_0 A}{d}\right) = \mu_0 \frac{d}{dt}(CV) = \mu_0 \frac{dQ}{dt} = \mu_0 I \tag{984}$$

See Figure 109.

- The complete set of Maxwell's equation are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{985}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{986}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{987}$$

$$\nabla \times \mathbf{B} = \mu_0\, \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{988}$$

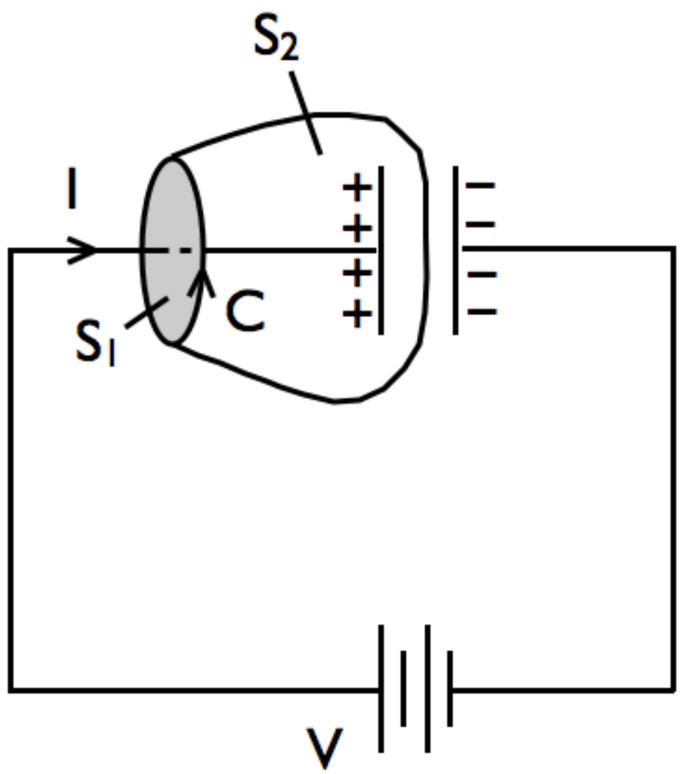


Figure 109: Current I flows in a circuit charging a parallel-plate capacitor. Two different 2-dimensional surfaces spanning the Ampérian loop C are shown; S_1 cuts through the wire, so that $I_{\text{enc}} = I$, while S_2 extends between the plates of the capacitor, where $I_{\text{enc}} = 0$. Nonetheless, the line integral of $\mathbf{B}$ around C is the same for both surfaces provided one includes the *displacement* current, which is proportional to the time rate-of-change of the electric flux through the surface.

- These equations, together with the Lorentz force law

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{989}$$

and Newton's 2nd law ($\mathbf{F} = d\mathbf{p}/dt$, where $\mathbf{p} \equiv m\mathbf{v}/\sqrt{1 - v^2/c^2}$) is a complete description of classical electrodynamics.

- It is sometimes convenient to split the equations into source-free equations:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0\,, \qquad \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{990}$$

and equations with sources:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}\,, \qquad \boldsymbol{\nabla} \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0\, \mathbf{J} \tag{991}$$

- Maxwell's equations are general, and hold equally well in vacuum or matter. But when solving the equations in matter, it is often convenient to work with the free charge and free current densities, ρ_f and $\mathbf{J}_f$.

  The source-free equations are unchanged:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0\,, \qquad \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{992}$$

but the equations with sources become:

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f\,, \qquad \boldsymbol{\nabla} \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_f \tag{993}$$

where

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}\,, \qquad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \tag{994}$$

- Exercise: Prove the above form of Maxwell's equations in matter.

  Hint: Split the total charge and current densities into their free and bound components:

$$\rho = \rho_f + \rho_b\,, \qquad \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_P \tag{995}$$

where

$$\rho_b \equiv -\boldsymbol{\nabla} \cdot \mathbf{P}\,, \qquad \mathbf{J}_b \equiv \boldsymbol{\nabla} \times \mathbf{M}\,, \qquad \mathbf{J}_P \equiv \frac{\partial \mathbf{P}}{\partial t} \tag{996}$$

The *polarization* current $\mathbf{J}_P$ is needed when the polarization $\mathbf{P}$ is time-dependent.

- Boundary conditions:

  In terms of $\mathbf{E}$ and $\mathbf{B}$:

$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{\mathbf{n}} = 0 \tag{997}$$
$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{\mathbf{t}} = 0 \tag{998}$$
$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{\mathbf{n}} = \frac{\sigma}{\epsilon_0} \tag{999}$$
$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{\mathbf{t}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{t}} \tag{1000}$$

In terms of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, $\mathbf{H}$:

$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{\mathbf{n}} = 0 \tag{1001}$$
$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{\mathbf{t}} = 0 \tag{1002}$$
$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{\mathbf{n}} = \sigma_f \tag{1003}$$
$$(\mathbf{H}_2 - \mathbf{H}_1) \cdot \hat{\mathbf{t}} = (\mathbf{K}_f \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{t}} \tag{1004}$$

where $\hat{\mathbf{n}}$ is the unit normal pointing from region 1 to region 2, and $\hat{\mathbf{t}}$ is a unit tangent vector to the boundary surface.

- Exercise: Show that these equations follow from the general set of Maxwell's equations, and from Maxwell's equation in matter. (Hint: Write the equations in integral form, and then integrate over a Gaussian pillbox or Amperian loop, in the limit where the height of the pillbox and the height of the loop go to zero.)

8.8 Scalar and vector potentials

- The source-free Maxwell's equations can be solved by writing $\mathbf{E}$ and $\mathbf{B}$ in terms of scalar and vector potentials Φ and $\mathbf{A}$:

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{1005}$$

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{1006}$$

- Exercise: Prove this.

- Note that the relationship between $\mathbf{B}$ and $\mathbf{A}$ is the same as for the magnetostatic case; the expression for $\mathbf{E}$ requires the additional term $-\partial\mathbf{A}/\partial t$ in order to satisfy Faraday's law.

- The above equations relating $\mathbf{B}$ and $\mathbf{E}$ to $\mathbf{A}$ and Φ do *not* uniquely determine $\mathbf{A}$ and Φ. One can change $\mathbf{A}$ and Φ to new potentials

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\Lambda \tag{1007}$$

$$\Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \tag{1008}$$

  without changing $\mathbf{B}$ and $\mathbf{E}$.

- Exercise: Prove this.

- The above set of transformation equations is called a *gauge transformation.*

- One can write the remaining two Maxwell's equations with sources in terms of Φ and $\mathbf{A}$:

$$\Box^2\Phi = -\frac{1}{\epsilon_0}\,\rho - \frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t}\right) \tag{1009}$$

$$\Box^2\mathbf{A} = -\mu_0\,\mathbf{J} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t}\right) \tag{1010}$$

  where

$$\Box^2 = \nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2} \tag{1011}$$

  is the *D'Alembertian* or wave operator.

- Exercise: Prove the above.

- It is always possible to choose the potentials Φ and $\mathbf{A}$ so that

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t} = 0 \tag{1012}$$

  This is called the *Lorentz gauge.*

- Exercise: Prove this.

  Hint: Suppose that

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t} = \alpha \tag{1013}$$

Make a gauge transformation (1007) and (1008) to new potentials $\mathbf{A}'$ and Φ'. Show that the requirement

$$\nabla \cdot \mathbf{A}' + \mu_0 \epsilon_0 \frac{\partial \Phi'}{\partial t} = 0 \tag{1014}$$

is equivalent to

$$\Box^2 \Lambda = -\alpha \tag{1015}$$

Since this is just the wave equation with a source term, it is always possible to solve this equation, and hence satisfy the Lorentz gauge condition.

- Note that the Lorentz gauge still does not fix the potentials completely, as an additional gauge transformation by Λ' with $\Box^2 \Lambda' = 0$ respects the Lorentz gauge.

- In the Lorentz gauge, the equations for Φ and $\mathbf{A}$ are simply

$$\Box^2 \Phi = -\frac{1}{\epsilon_0}\,\rho\,, \qquad \Box^2 \mathbf{A} = -\mu_0\,\mathbf{J} \tag{1016}$$

These are just wave equations for Φ and the components of $\mathbf{A}$, with source terms $-\rho/\epsilon_0$ and $-\mu_0 \mathbf{J}$, respectively.

8.9 Poynting's theorem

- Previous calculations of the energy stored in electrostatic and magnetic fields suggest that

$$U_{\text{em}} \equiv \int_{\text{all space}} \frac{1}{2}\left(\epsilon_0\,E^2 + \frac{1}{\mu_0}\,B^2\right) dV \tag{1017}$$

is the total energy stored in the electromagnetic field.

- Here we prove this statement systematically, by deriving a work-energy theorem for charged particles moving in an electromagnetic field:

1) Start with the expression for the work done on a single charge q by the electromagnetic field:

$$dW = \mathbf{F} \cdot d\mathbf{s} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s} = q\mathbf{E} \cdot d\mathbf{s} = q\mathbf{E} \cdot \mathbf{v}\,dt \tag{1018}$$

Note that the magnetic force does no work on the charge.

2) For a volume current density $\mathbf{J}$, we can replace $q\mathbf{v}$ by $\rho\mathbf{v}\,dV = \mathbf{J}\,dV$, and then integrate over a volume V:

$$\frac{dW}{dt} = \int_V \mathbf{E} \cdot \mathbf{J}\,dV \tag{1019}$$

3) Using $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial \mathbf{E}/\partial t$, we can substitute for $\mathbf{J}$ obtaining

$$\frac{dW}{dt} = \int_V \left(\frac{1}{\mu_0}\,\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \frac{\epsilon_0}{2}\,\frac{\partial E^2}{\partial t}\right) dV \tag{1020}$$

4) Using the vector identity for the divergence of a curl, we can rewrite the first term as

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{B}) \tag{1021}$$

5) Finally, using Faraday's law to replace $\nabla \times \mathbf{E}$ by $-\partial \mathbf{B}/\partial t$, and the divergence theorem to write the volume integral of $\nabla \cdot (\mathbf{E} \times \mathbf{B})$ as a surface integral, we obtain *Poynting's theorem*:

$$\frac{dW}{dt} = -\frac{d}{dt} \int_V \frac{1}{2}\left(\epsilon_0\,E^2 + \frac{1}{\mu_0}\,B^2\right) dV - \frac{1}{\mu_0} \oint_S (\mathbf{E} \times \mathbf{B}) \cdot \hat{\mathbf{n}}\,da \tag{1022}$$

or, more compactly,

$$\frac{dW}{dt} = -\frac{dU_{\rm em}}{dt} - \oint_S \mathbf{S}\cdot\hat{\mathbf{n}}\,da \tag{1023}$$

where

$$\mathbf{S} \equiv \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{1024}$$

is called the *Poynting vector*. (It represents the energy per unit area per unit time carried by the electromagnetic field.)

- Thus, in a given time interval, the work done on the charges by the elecromagnetic field is equal to the decrease in the energy stored in the fields minus the energy in the fields that flows out of the volume V through the surface S.
- If we denote the energy density in the electromagnetic fields by

$$u_{\rm em} \equiv \frac{1}{2}\left(\epsilon_0\, E^2 + \frac{1}{\mu_0}\, B^2\right) \tag{1025}$$

and the density of *mechanical energy* of the charges by $u_{\rm mech}$, so that

$$W = \int_V u_{\rm mech}\, dV \tag{1026}$$

we can write the work-energy theorem for electromagnetism as

$$\frac{d}{dt}\int_V \left(u_{\rm mech} + u_{\rm em}\right)\, dV = -\oint_S \mathbf{S}\cdot\hat{\mathbf{n}}\,da \tag{1027}$$

- Using the divergence theorem, we can rewrite the above equation in differential form

$$\frac{\partial}{\partial t}\left(u_{\rm mech} + u_{\rm em}\right) = -\boldsymbol{\nabla}\cdot\mathbf{S} \tag{1028}$$

This is the statement of local energy conservation in electrodynamics. (Note the similarity of this equation to the continuity equation $\partial\rho/\partial t = -\boldsymbol{\nabla}\cdot\mathbf{J}$ for local charge conservation.)

8.10 Conservation of momentum

- A similar analysis applies to conservation of momentum.
- The net force on the charges and currents due to the electromagnetic field is

$$\mathbf{F} = \int_V \left(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right) dV \tag{1029}$$

- According to Newton's 2nd law, the net force is also equal to the time rate of change of the momentum of the charges

$$\mathbf{F} = \frac{d\mathbf{P}_{\rm mech}}{dt} \tag{1030}$$

so that

$$\frac{d\mathbf{P}_{\rm mech}}{dt} = \int_V \left(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right) dV \tag{1031}$$

- Using Maxwell's equations to replace ρ and $\mathbf{J}$ on the RHS of the above equation in terms of $\mathbf{E}$ and $\mathbf{B}$, we obtain

$$\frac{d\mathbf{P}_{\rm mech}}{dt} = -\int_V \epsilon_0\mathbf{E}\times\mathbf{B}\,dV + \oint_S \mathbf{T}\cdot\hat{\mathbf{n}}\,da \tag{1032}$$

where $\mathbf{T}$ is the *Maxwell stress tensor*

$$T_{ij} \equiv \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2}\delta_{ij}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \tag{1033}$$

whose components represent the pressures and shears in the field (force/area).

- This above equation for $d\mathbf{P}_{\text{mech}}/dt$ can be written in a more suggestive form if we define

$$\mathbf{P}_{\text{em}} \equiv \int_V \epsilon_0 \mathbf{E} \times \mathbf{B}\, dV \tag{1034}$$

as the total momentum in the electromagnetic field. Then

$$\frac{d\mathbf{P}_{\text{mech}}}{dt} = -\frac{d\mathbf{P}_{\text{em}}}{dt} + \oint_S \mathbf{T} \cdot \hat{\mathbf{n}}\, da \tag{1035}$$

- Thus, the time rate of change of the momentum of the charges is equal to the decrease of the momentum stored in the fields minus the momentum in the fields that flows out of the volume V through the surface S. This is the integral form of *conservation of momentum* in electrodynamics.

- If we further define the momentum densities:

$$\mathbf{g} \equiv \epsilon_0 \mathbf{E} \times \mathbf{B} = \frac{1}{c^2}\mathbf{S} \tag{1036}$$

for the fields, and

$$\int_V \mathbf{p}_{\text{mech}}\, dV \equiv \mathbf{P}_{\text{mech}} \tag{1037}$$

for the charges, it follows that

$$\frac{\partial}{\partial t}\left(\mathbf{p}_{\text{mech}} + \mathbf{g}\right) = \boldsymbol{\nabla} \cdot \mathbf{T} \tag{1038}$$

This is the differential form of conservation of momentum in electrodynamics.

8.11 Conservation of angular momentum

- A similar analysis applies to conservation of angular momentum.

- The net torque on the charges and currents due to the electromagnetic field is

$$\mathbf{N} = \int_V \mathbf{r} \times (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})\, dV \tag{1039}$$

- According to Newton's 2nd law, the net torque is also equal to the time rate of change of the angular momentum of the charges

$$\mathbf{N} = \frac{d\mathbf{L}_{\text{mech}}}{dt} \tag{1040}$$

so that

$$\frac{d\mathbf{L}_{\text{mech}}}{dt} = \int_V \mathbf{r} \times (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})\, dV \tag{1041}$$

- Using Maxwell's equations to replace ρ and $\mathbf{J}$ on the RHS of the above equation in terms of $\mathbf{E}$ and $\mathbf{B}$, we obtain

$$\frac{d\mathbf{L}_{\text{mech}}}{dt} = -\int_V \epsilon_0 \mathbf{r} \times (\mathbf{E} \times \mathbf{B})\, dV + \oint_S (\mathbf{r} \times \mathbf{T}) \cdot \hat{\mathbf{n}}\, da \tag{1042}$$

where $\mathbf{T}$ is the Maxwell stress tensor as before.

- This above equation for $d\mathbf{L}_{\text{mech}}/dt$ can be written in a more suggestive form if we define

$$\mathbf{L}_{\text{em}} \equiv \int_V \epsilon_0 \mathbf{r} \times (\mathbf{E} \times \mathbf{B})\, dV \tag{1043}$$

as the total angular momentum in the electromagnetic field. Then

$$\frac{d\mathbf{L}_{\text{mech}}}{dt} = -\frac{d\mathbf{L}_{\text{em}}}{dt} + \oint_S (\mathbf{r} \times \mathbf{T}) \cdot \hat{\mathbf{n}}\, da \tag{1044}$$

- Thus, the time rate of change of the angular momentum of the charges is equal to the decrease of the angular momentum stored in the fields minus the angular momentum in the fields that flows out of the volume V through the surface S. This is the integral form of *conservation of angular momentum* in electrodynamics.

- If we further define the angular momentum densities:

$$\mathbf{h} \equiv \epsilon_0 \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) = \mathbf{r} \times \mathbf{g} \tag{1045}$$

for the fields, and

$$\int_V \boldsymbol{\ell}_{\text{mech}}\, dV \equiv \mathbf{L}_{\text{mech}} \tag{1046}$$

for the charges, it follows that

$$\frac{\partial}{\partial t}\left(\boldsymbol{\ell}_{\text{mech}} + \mathbf{h}\right) = \boldsymbol{\nabla} \cdot (\mathbf{r} \times \mathbf{T}) \tag{1047}$$

This is the differential form of conservation of angular momentum in electrodynamics.

9 Electromagnetic waves

9.1 Electromagnetic waves in vacuum

- Maxwell's equations:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0\,, \qquad \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{1048}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}\,, \qquad \boldsymbol{\nabla} \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \tag{1049}$$

- Taking $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E})$ and $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B})$ and using the $\boldsymbol{\nabla} \cdot \mathbf{E}$ and $\boldsymbol{\nabla} \cdot \mathbf{B}$ equations, one obtains (inhomogeneous) wave equations for the fields

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial \mathbf{J}}{\partial t} + \frac{1}{\epsilon_0} \boldsymbol{\nabla} \rho \tag{1050}$$

$$\nabla^2 \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \boldsymbol{\nabla} \times \mathbf{J} \tag{1051}$$

- In the absence of charges and currents, we have the (homogeneous) wave equations:

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{1052}$$

$$\nabla^2 \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \tag{1053}$$

with wave velocity:

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \equiv c = 2.998 \times 10^8 \text{m/s} \tag{1054}$$

- Monochromatic plane wave solutions (complex):

$$\tilde{\mathbf{E}}(\mathbf{r},t) = (\tilde{E}_1\,\hat{\boldsymbol{\epsilon}}_1 + \tilde{E}_2\,\hat{\boldsymbol{\epsilon}}_2)\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,, \tag{1055}$$

$$\tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{c}\,\hat{\mathbf{k}}\times\tilde{\mathbf{E}}(\mathbf{r},t) \tag{1056}$$

where $\hat{\boldsymbol{\epsilon}}_{1,2}$ are two linearly independent polarization vectors, orthogonal to the wave vector (or propagation vector) $\mathbf{k}$. The angular frequency and wavelength of the radiation are given by

$$\omega = kc\,,\quad \lambda = \frac{2\pi}{k}\,,\quad k = |\mathbf{k}| \geq 0 \tag{1057}$$

and $\tilde{E}_{1,2}$ are complex amplitudes which we can write as the product of a real amplitude and a phase:

$$\tilde{E}_1 = E_1\,e^{i\delta_1}\,,\quad \tilde{E}_2 = E_2\,e^{i\delta_2} \tag{1058}$$

- The real solutions for $\mathbf{E}$ and $\mathbf{B}$ are obtained by taking the real parts of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$:

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}[\tilde{\mathbf{E}}(\mathbf{r},t)]\,,\quad \mathbf{B}(\mathbf{r},t) = \mathrm{Re}[\tilde{\mathbf{B}}(\mathbf{r},t)] \tag{1059}$$

- Note that $\mathbf{E}$ and $\mathbf{B}$ are transverse (i.e., perpendicular) to the direction of propagation $\hat{\mathbf{k}}$. This is a consequence of Maxwell's equations $\boldsymbol{\nabla}\cdot\mathbf{E}=0$ and $\boldsymbol{\nabla}\cdot\mathbf{B}=0$.
- In addition, $\mathbf{E}$ and $\mathbf{B}$ are perpendicular to one another and have magnitudes related by $B = E/c$. This is consequence of Maxwell's equation $\boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ or $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\epsilon_0\partial\mathbf{E}/\partial t$. (Only one of these equations is needed; the other gives the same information.)
- Thus, $\mathbf{E}$, $\mathbf{B}$, $\mathbf{k}$ form a right-handed system of orthogonal vectors.
- There is no loss of generality in assuming monochromatic plane wave solutions, since the most general solution to the homogeneous wave equation is a linear combination of such solutions for different frequencies $\omega = kc$ and directions of propagation $\hat{\mathbf{k}}$.

9.2 Polarization

- If the phases δ_1 and δ_2 are equal (i.e., $\delta_1 = \delta_2 \equiv \delta$), the wave is *linearly polarized* with a polarization vector that makes an angle $\theta \equiv \mathrm{atan}(E_2/E_1)$ with respect to $\hat{\boldsymbol{\epsilon}}_1$ and has magnitude $E_0 \equiv \sqrt{E_1^2+E_2^2}$.

  Example:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0\,\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta)\,,\quad \text{where}\quad \mathbf{E}_0 \equiv E_1\,\hat{\boldsymbol{\epsilon}}_1 + E_2\,\hat{\boldsymbol{\epsilon}}_2 \tag{1060}$$

- If $\delta_2 - \delta_1 = \pi/2$ and $E_1 = E_2 \equiv E_0$, the wave is *left-circularly polarized*. The electric field vector has constant magnitude and rotates in a CCW circle at fixed $\mathbf{r}$ when looking into the wave (i.e., in the $-\hat{\mathbf{k}}$ direction). If $\delta_2 - \delta_1 = -\pi/2$ and $E_1 = E_2$, the wave is *right-circularly polarized*.

  Example: (left-circularly polarized)

$$\mathbf{E}(\mathbf{r},t) = E_0\,[\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_1)\,\hat{\boldsymbol{\epsilon}}_1 - \sin(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_1)\,\hat{\boldsymbol{\epsilon}}_2] \tag{1061}$$

  Example: (right-circularly polarized)

$$\mathbf{E}(\mathbf{r},t) = E_0\,[\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_1)\,\hat{\boldsymbol{\epsilon}}_1 + \sin(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_1)\,\hat{\boldsymbol{\epsilon}}_2] \tag{1062}$$

- Note that

$$\hat{\boldsymbol{\epsilon}}_\pm \equiv \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{\epsilon}}_1 \pm \hat{\boldsymbol{\epsilon}}_2\right) \tag{1063}$$

are linearly-independent (complex) basis vectors for left- and right-circularly polarized waves, respectively

$$\hat{\boldsymbol{\epsilon}}_\pm \cdot \hat{\mathbf{k}} = 0\,, \quad \hat{\boldsymbol{\epsilon}}^*_\pm \cdot \hat{\boldsymbol{\epsilon}}_\pm = 1\,, \quad \hat{\boldsymbol{\epsilon}}^*_\pm \cdot \hat{\boldsymbol{\epsilon}}_\mp = 0 \tag{1064}$$

They can be used in place of $\hat{\boldsymbol{\epsilon}}_{1,2}$ when writing down the general monochromatic plane wave solution:

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \left(\tilde{E}_+\,\hat{\boldsymbol{\epsilon}}_+ + \tilde{E}_-\,\hat{\boldsymbol{\epsilon}}_-\right) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,, \tag{1065}$$

$$\tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{c}\,\hat{\mathbf{k}}\times\tilde{\mathbf{E}}(\mathbf{r},t) \tag{1066}$$

- In other words, the most general monochromatic plane wave solution of the vacuum Maxwell equations can be written as a linear superposition of two linearly polarized monochromatic plane waves or two circularly polarized monochromatic plane waves (one left-circularly polarized, the other right-circularly polarized).

- If $\delta_1 \neq \delta_2$ and $|\delta_1 - \delta_2| \neq \pi/2$, then $\mathbf{E}(\mathbf{r},t)$ is *elliptically polarized.* At fixed $\mathbf{r}$ the tip of the electric field vector traces out an ellipse (over time) in the plane transverse to $\hat{\mathbf{k}}$.

- Note that the ellipse will, in general, be rotated by an angle $\alpha/2$ with respect the $\hat{\boldsymbol{\epsilon}}_1$ axis, and have semi-major and semi-minor axes $a \equiv 1+\Delta$ and $b \equiv 1-\Delta$, where α and $0 \leq \Delta \leq 1$ are determined by $\tilde{E}_\pm = E_\pm\, e^{i\delta_\pm}$ via:

$$\alpha \equiv \delta_- - \delta_+\,, \quad \Delta \equiv \frac{E_-}{E_+} \tag{1067}$$

(We are assuming here that $E_+ > E_-$; otherwise we need to interchange the pluses and minuses in the above formulas.)

<u>Example</u>:

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{\sqrt{2}}E_+\bigg\{\left[\cos\left(\mathbf{k}\cdot\mathbf{r}-\omega t - \frac{\alpha}{2}+\delta\right) + \Delta\cos\left(\mathbf{k}\cdot\mathbf{r}-\omega t + \frac{\alpha}{2}+\delta\right)\right]\hat{\boldsymbol{\epsilon}}_1 \tag{1068}$$

$$- \left[\sin\left(\mathbf{k}\cdot\mathbf{r}-\omega t - \frac{\alpha}{2}+\delta\right) - \Delta\sin\left(\mathbf{k}\cdot\mathbf{r}-\omega t + \frac{\alpha}{2}+\delta\right)\right]\hat{\boldsymbol{\epsilon}}_2\bigg\} \tag{1069}$$

where $\delta \equiv \delta_+ + \alpha/2$.

- $\Delta = 0$ corresponds to circular polarization and $\Delta = 1$ to linear polarization.

- The (time-averaged) intensity of a linearly-polarized monochromatic plane wave

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0\,\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta)\,, \quad \mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) \tag{1070}$$

is given by

$$I \equiv \frac{d^2E}{da\,dt} = \langle \mathbf{S}\cdot\hat{\mathbf{n}}\rangle = \frac{1}{2}\frac{E_0^2}{\mu_0 c}\cos\theta = \frac{1}{2}c\epsilon_0 E_0^2\cos\theta \tag{1071}$$

where $\mathbf{S} \equiv \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ is the Poynting vector and θ is the angle between the wave vector $\hat{\mathbf{k}}$ and the unit normal $\hat{\mathbf{n}}$ to the area element da.

9.3 Electromagnetic waves in linear homogeneous media

- For linear homogeneous media

$$\mathbf{D} = \epsilon \mathbf{E}\,, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \tag{1072}$$

where ϵ and μ are independent of position and time, but might depend on frequency (see e.g., the Sec. 9.5 on dispersion).

- In the absence of free charges and free currents, the homogeneous wave equations for $\mathbf{E}$ and $\mathbf{B}$ are the same as in vacuum with the ϵ_0, μ_0 replaced by ϵ, μ.

- The wave velocity is now

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}\,, \quad \text{where} \quad n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \quad \text{(index of refraction)} \tag{1073}$$

- For most materials at optical frequencies,

$$\mu \cong \mu_0 \quad \Rightarrow \quad n \cong \sqrt{\epsilon_r} \tag{1074}$$

where $\epsilon_r \equiv \epsilon/\epsilon_0$ is the dielectic constant.

- Note that $n = 1$ in vacuum and $n > 1$ in other materials. Examples: $n = 1.000293$ for air at $0°$ C, 1atm; $n \cong 1.33$ for water at $20°$ C; $n \cong 1.5$ for crown glass.

- Boundary conditions for two different linear homogeneous media in the absence of free charges and free currents:

$$\text{(i) } \epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp}\,, \qquad \text{(ii) } B_{1\perp} = B_{2\perp}\,, \tag{1075}$$

$$\text{(iii) } \mathbf{E}_{1\parallel} = \mathbf{E}_{2\parallel}\,, \qquad \text{(iv) } \frac{1}{\mu_1}\mathbf{B}_{1\parallel} = \frac{1}{\mu_2}\mathbf{B}_{2\parallel} \tag{1076}$$

where 1 and 2 denote the two different media, and $\perp$ and $\parallel$ denote the vector components perpendicular and tangent to the boundary surface between the two media. Note that (iii) and (iv) are actually two conditions each.

- Boundary conditions (i) and (iv) are special cases of $D_{1\perp} - D_{2\perp} = \sigma_f$ and $\mathbf{H}_{1\parallel} - \mathbf{H}_{2\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}$ for linear homogeneous media when there are no free charges and no free currents. (Here $\hat{\mathbf{n}}$ is the unit normal pointing from medium 2 to medium 1.)

- In the following two subsections, we will analyze the propagation of a linearly polarized monochromatic plane wave from one dielectric material to another (e.g., from air to glass). For all cases, we will choose our coordinate system so that the boundary surface is given by the plane $z = 0$. ($z < 0$ corresponds to medium 1; $z > 0$ to medium 2.) For visualization purposes, we will have $\hat{\mathbf{z}}$ point to the right and $\hat{\mathbf{x}}$ pointing upward; $\hat{\mathbf{y}}$ points out of the page.

9.3.1 Reflection and transmission at normal incidence

- Consider a linearly polarized monochromatic plane wave propagating in the $\hat{\mathbf{z}}$ direction, with angular frequency ω and polarization direction $\hat{\mathbf{x}}$:

$$\tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0I}\, e^{i(k_1 z - \omega t)}\, \hat{\mathbf{x}}\,, \tag{1077}$$

$$\tilde{\mathbf{B}}_I(z,t) = \frac{\tilde{E}_{0I}}{v_1}\, e^{i(k_1 z - \omega t)}\, \hat{\mathbf{y}} \tag{1078}$$

- We are interested in determining the reflection and transmission of the incident wave for the case of normal incidence on the plane interface between two linear homogeneous dielectrics with indices of refraction n_1 and n_2 ($n_1 \equiv c/v_1$, $n_2 \equiv c/v_2$).

- Since the frequency ω is the same for the incident, reflected, and transmitted waves

$$\omega = k_1 v_1 = k_2 v_2 \quad \Leftrightarrow \quad k_2 = \frac{v_1}{v_2}\, k_1 = \frac{n_2}{n_1}\, k_1 \tag{1079}$$

- It turns out that:

  i) the reflected and transmitted waves propagate in the $\mp\hat{\mathbf{z}}$ directions, respectively (consequence of Snell's law; see next section).

  ii) the polarization directions of the reflected and transmitted waves are also in the $\hat{\mathbf{x}}$ direction (consequence of boundary conditions).

- Thus, we can write for the reflected wave

$$\tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0R}\, e^{i(-k_1 z - \omega t)}\, \hat{\mathbf{x}}\,, \tag{1080}$$

$$\tilde{\mathbf{B}}_R(z,t) = -\frac{\tilde{E}_{0R}}{v_1}\, e^{i(-k_1 z - \omega t)}\, \hat{\mathbf{y}} \tag{1081}$$

  and for the transmitted wave

$$\tilde{\mathbf{E}}_T(z,t) = \tilde{E}_{0T}\, e^{i(k_2 z - \omega t)}\, \hat{\mathbf{x}}\,, \tag{1082}$$

$$\tilde{\mathbf{B}}_T(z,t) = \frac{\tilde{E}_{0T}}{v_2}\, e^{i(k_2 z - \omega t)}\, \hat{\mathbf{y}} \tag{1083}$$

- Application of the boundary conditions at $z = 0$ for all t yields

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}\,, \quad \tilde{E}_{0I} - \tilde{E}_{0R} = \beta \tilde{E}_{0T} \tag{1084}$$

  where

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \tag{1085}$$

- Solving for $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$:

$$\tilde{E}_{0R} = \left(\frac{1-\beta}{1+\beta}\right) \tilde{E}_{0I}\,, \quad \tilde{E}_{0T} = \left(\frac{2}{1+\beta}\right) \tilde{E}_{0I} \tag{1086}$$

- Note that the transmitted and incident waves are always in phase, while the reflected and incident waves are in phase if $\beta < 1$ and $180°$ out of phase if $\beta > 1$.

- Reflection and transmission coefficients:

$$R \equiv \frac{I_R}{I_I} = \left(\frac{1-\beta}{1+\beta}\right)^2\,, \quad T \equiv \frac{I_T}{I_I} = \frac{4\beta}{(1+\beta)^2} \tag{1087}$$

- It follows that $R + T = 1$ as it should from conservation of energy.

- If $\mu \cong \mu_0$, which is valid for most materials at optical frequencies, then $\beta \cong v_1/v_2$, which implies

$$\tilde{E}_{0R} \cong \left(\frac{v_2 - v_1}{v_1 + v_2}\right) \tilde{E}_{0I}\,, \quad \tilde{E}_{0T} \cong \left(\frac{2v_2}{v_1 + v_2}\right) \tilde{E}_{0I} \tag{1088}$$

  and

$$R \cong \left(\frac{v_2 - v_1}{v_1 + v_2}\right)^2\,, \quad T \cong \frac{4 v_1 v_2}{(v_1 + v_2)^2} \tag{1089}$$

- The above formulae in terms of v_1 and v_2 are identical to those for waves on a string.

9.3.2 Reflection and transmission at oblique incidence

- Consider a linearly polarized monochromatic plane wave propagating in the $\hat{\mathbf{k}}_I$ direction, incident obliquely on the plane interface between two linear homogoneous dielectrics, with indices of refraction n_1 and n_2, respectively:

$$\tilde{\mathbf{E}}_I(\mathbf{r},t) = \tilde{\mathbf{E}}_{0I}\, e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}\,, \tag{1090}$$

$$\tilde{\mathbf{B}}_I(\mathbf{r},t) = \frac{1}{v_1}\hat{\mathbf{k}}_I \times \tilde{\mathbf{E}}_I(\mathbf{r},t) \tag{1091}$$

- Since ω is the same for the incident, reflected, and transmitted waves

$$\omega = k_I v_1 = k_R v_1 = k_T v_2 \quad \Leftrightarrow \quad k_I = k_R\,, \quad k_T = \frac{v_1}{v_2}k_I = \frac{n_2}{n_1}k_I \tag{1092}$$

- The boundary conditions at the plane interface ($z=0$) between the two dielectrics have the form:

$$\mathbf{A}e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)} + \mathbf{B}e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)} = \mathbf{C}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)} \quad (\text{for } \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) \tag{1093}$$

- Since these conditions must hold for all t and for all points in the xy-plane, it follows that (Prob. 9.15, Griffiths)

$$\mathbf{A}+\mathbf{B} = \mathbf{C}\,, \quad \text{and} \quad \mathbf{k}_I\cdot\mathbf{r} = \mathbf{k}_R\cdot\mathbf{r} = \mathbf{k}_T\cdot\mathbf{r} \quad (\text{for } \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) \tag{1094}$$

  which implies

$$k_{Ix} = k_{Rx} = k_{Tx}\,, \quad k_{Iy} = k_{Ry} = k_{Ty}\,, \tag{1095}$$

- For convenience we will choose our x and y axes so that $k_{Iy}=0$.

- Thus,

$$k_{Iy} = 0\,, \quad k_{Ry} = 0\,, \quad k_{Ty} = 0 \tag{1096}$$

  which means that the wave vectors $\mathbf{k}_I$, $\mathbf{k}_R$, $\mathbf{k}_T$ all lie in the xz-plane (called the *plane of incidence*), which also includes the normal to the boundary surface (i.e., $\hat{\mathbf{z}}$).

- In addition,

$$k_I \sin\theta_I = k_R\sin\theta_R = k_T\sin\theta_T \tag{1097}$$

  where θ_I, θ_R, θ_T are the angles that the wave vectors make with the normal to the boundary surface. Since $k_I = k_R$, it follows that

$$\theta_I = \theta_R \tag{1098}$$

  which is the *law of reflection.*

- Using $k_T = k_I v_1/v_2 = k_I n_2/n_1$ it follows that

$$\frac{\sin\theta_I}{v_1} = \frac{\sin\theta_T}{v_2} \quad \text{or} \quad n_1\sin\theta_I = n_2\sin\theta_T \tag{1099}$$

  which is the *law of refraction* or *Snell's law.*

- Note that the above results followed from very general considerations of the boundary conditions and were not tied to the specific form of Maxwell's equations. Thus, the result that all the wave vectors lie in the plane of incidence and the laws of reflection and refraction hold for *any type of wave motion*, provided they can be approximated as plane waves, with wave vector perpendicular to the planes of constant phase. (This is called the *geometrical optics* approximation.)

- Having taken care of the $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ part of the boundary condition, we can now deal with the vector parts:

$$\text{(i)} \quad \epsilon_1(\tilde{E}_{0Iz} + \tilde{E}_{0Rz}) = \epsilon_2 \tilde{E}_{0Tz} \; , \tag{1100}$$

$$\text{(ii)} \quad \tilde{B}_{0Iz} + \tilde{B}_{0Rz} = \tilde{B}_{0Tz} \; , \tag{1101}$$

$$\text{(iii)} \quad \tilde{E}_{0Ix,y} + \tilde{E}_{0Rx,y} = \tilde{E}_{0Tx,y} \; , \tag{1102}$$

$$\text{(iv)} \quad \frac{1}{\mu_1}(\tilde{B}_{0Ix,y} + \tilde{B}_{0Rx,y}) = \frac{1}{\mu_1}\tilde{B}_{0Tx,y} \tag{1103}$$

- To proceed further, we consider two special cases: (i) polarization direction in the plane of incidence, (ii) polarization direction perpendicular to the plane of incidence. The general case is a superpostion of these two.

9.3.3 Polarization in the plane of incidence

- Note the polarization directions for the reflected and transmitted waves will also be in the plane of incidence.
- Application of the boundary conditions for this case yields

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \alpha \tilde{E}_{0T} \; , \quad \tilde{E}_{0I} - \tilde{E}_{0R} = \beta \tilde{E}_{0T} \tag{1104}$$

where

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I} \; , \quad \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \tag{1105}$$

- Note that α is a function of the incident angle

$$\alpha = \frac{\sqrt{n_2^2 - n_1^2 \sin^2\theta_I}}{n_2 \cos\theta_I} \tag{1106}$$

where we used Snell's law to relate θ_T to θ_I.

- Solving for $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$:

$$\tilde{E}_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) \tilde{E}_{0I} \; , \quad \tilde{E}_{0T} = \left(\frac{2}{\alpha + \beta}\right) \tilde{E}_{0I} \tag{1107}$$

- Explicitly,

$$\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{n_1\sqrt{n_2^2 - n_1^2\sin^2\theta_I} - \frac{\mu_1}{\mu_2} n_2^2 \cos\theta_I}{n_1\sqrt{n_2^2 - n_1^2\sin^2\theta_I} + \frac{\mu_1}{\mu_2} n_2^2 \cos\theta_I} \; , \tag{1108}$$

$$\frac{\tilde{E}_{0T}}{\tilde{E}_{0I}} = \frac{2 n_1 n_2 \cos\theta_I}{n_1\sqrt{n_2^2 - n_1^2\sin^2\theta_I} + \frac{\mu_1}{\mu_2} n_2^2 \cos\theta_I} \tag{1109}$$

- These are *Fresnel's equations* for polarization in the plane of incidence.
- The transmitted and incident waves are always in phase, while the reflected and incident waves are in phase if $\alpha > \beta$ and $180°$ out of phase if $\alpha < \beta$.
- Reflection and transmission coefficients:

$$R \equiv \frac{I_R}{I_I} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2 \; , \quad T \equiv \frac{I_T}{I_I} = \frac{4\alpha\beta}{(\alpha + \beta)^2} \tag{1110}$$

which sum to 1 as they should.

- For $\theta_I = 0$, all of the above expressions reduce to those that we found earlier for normal incidence.
- *Brewster's angle*: incident angle for which there is no reflected wave (i.e., $\tilde{E}_{0R} = 0$).
- This holds for $\alpha = \beta$. Solving for $\theta_I \equiv \theta_B$ leads to

$$\sin^2 \theta_B = \frac{1 - \beta^2}{(n_1/n_2)^2 - \beta^2} \tag{1111}$$

- For the typical case where $\mu_{1,2} \cong \mu_0$, the above equation simplifies to

$$\tan \theta_B \cong \frac{n_2}{n_1} \tag{1112}$$

- We will see in the next subsection that Brewster's angle does not exist when the polarization direction is perpendicular to the plane of incidence. This means that an electromagnetic wave obliquely incident at $\theta_I = \theta_B$ on the boundary surface between two dielectrics will have a reflected wave that is polarized *perpendicular* to the plane of incidence, which is *parallel* to the boundary surface.
- This is the reason why polarized sunglasses with the transmission axis in the vertical direction reduce the glare of sunlight reflected off a horizontal surface like the surface of a lake.

9.3.4 Polarization perpendicular to the plane of incidence

- Proceeding as in the previous subsection, one can show that

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}\,, \quad \tilde{E}_{0I} - \tilde{E}_{0R} = \alpha\beta\tilde{E}_{0T} \tag{1113}$$

where

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}\,, \quad \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \tag{1114}$$

as before.

- Solving for $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$:

$$\tilde{E}_{0R} = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right)\tilde{E}_{0I}\,, \quad \tilde{E}_{0T} = \left(\frac{2}{1 + \alpha\beta}\right)\tilde{E}_{0I} \tag{1115}$$

- Explicitly,

$$\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{n_1\cos\theta_I - \frac{\mu_1}{\mu_2}\sqrt{n_2^2 - n_1^2\sin^2\theta_I}}{n_1\cos\theta_I + \frac{\mu_1}{\mu_2}\sqrt{n_2^2 - n_1^2\sin^2\theta_I}}\,, \tag{1116}$$

$$\frac{\tilde{E}_{0T}}{\tilde{E}_{0I}} = \frac{2n_1\cos\theta_I}{n_1\cos\theta_I + \frac{\mu_1}{\mu_2}\sqrt{n_2^2 - n_1^2\sin^2\theta_I}} \tag{1117}$$

- These are *Fresnel's equations* for polarization perpendicular to the plane of incidence.
- The transmitted and incident waves are always in phase, while the reflected and incident waves are in phase if $\alpha\beta < 1$ and $180°$ out of phase if $\alpha\beta > 1$.
- Reflection and transmission coefficients:

$$R \equiv \frac{I_R}{I_I} = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right)^2\,, \quad T \equiv \frac{I_T}{I_I} = \frac{4\alpha\beta}{(1 + \alpha\beta)^2} \tag{1118}$$

which sum to 1 as they should.

- One can show that there is no Brewster's angle when the polarization vector is perpendicular to plane of incidence. For the typical case where $\mu_{1,2} \cong \mu_0$, it follows that $\tilde{E}_{0R} = 0$ iff $n_1 \cong n_2$, which implies that the two dielectrics are optically indistinguishable. But for optically indistinguishable materials, there would be no reflection for *any* choice of incident angle θ_I.

9.4 Electromagnetic waves in conductors

- To handle the case of electromagnetic waves in conductors, we have to generalize the previous analyses to allow for the presence of free charge and free current densities.

- Maxwells equations in a linear homogeneous material ($\mathbf{D} = \epsilon E$, $\mathbf{H} = \mathbf{B}/\mu$) with free charge and free current densities:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0\,, \qquad \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{1119}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho_f}{\epsilon}\,, \qquad \boldsymbol{\nabla} \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} = \mu \mathbf{J}_f \tag{1120}$$

  Note that these equations are identical in form to the vacuum Maxwell equations with ϵ_0, μ_0 replaced by ϵ, μ, and σ, $\mathbf{J}$ replaced by σ_f, $\mathbf{J}_f$.

- For an *ohmic conductor*

$$\mathbf{J}_f = \sigma \mathbf{E} \tag{1121}$$

  where σ is the conductivity. (Note: $\sigma = \infty$ for a *perfect* conductor).

- The continuity equation for an ohmic conductor is

$$-\frac{\partial \rho_f}{\partial t} = \boldsymbol{\nabla} \cdot \mathbf{J}_f = \boldsymbol{\nabla} \cdot (\sigma \mathbf{E}) = \sigma \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\sigma}{\epsilon} \rho_f \tag{1122}$$

  which can be solved for $\rho_f(t)$:

$$\rho_f(t) = \rho_f(0)\, e^{-(\sigma/\epsilon)t} \tag{1123}$$

- Provided we wait for times that are long compared to the *characteristic decay time* $\tau \equiv \epsilon/\sigma$ (which is zero for a perfect conductor), the free charge density ρ_f will have decayed to zero implying

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \tag{1124}$$

- In addition, for an ohmic conductor, the last of Maxwell's equations can be written as

$$\boldsymbol{\nabla} \times \mathbf{B} - \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} = \mu\sigma \mathbf{E} \tag{1125}$$

- Taking $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E})$ and $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B})$ and using the $\boldsymbol{\nabla} \cdot \mathbf{E}$ and $\boldsymbol{\nabla} \cdot \mathbf{B}$ equations, one obtains the following modified wave equations:

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t} \tag{1126}$$

$$\nabla^2 \mathbf{B} - \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = \mu\sigma \frac{\partial \mathbf{B}}{\partial t} \tag{1127}$$

- As we shall see below, the extra terms of the RHS of the above equations introduce *dissipation.*

- The modified wave equations still admit linearly polarized monochromatic plane wave solutions

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{\mathbf{E}}_0\, e^{i(\tilde{\mathbf{k}}\cdot\mathbf{r}-\omega t)}\,, \tag{1128}$$

$$\tilde{\mathbf{B}}(\mathbf{r},t) = \tilde{\mathbf{B}}_0\, e^{i(\tilde{\mathbf{k}}\cdot\mathbf{r}-\omega t)} \tag{1129}$$

but now with a *complex-valued* wave vector $\tilde{\mathbf{k}} = \tilde{k}\,\hat{\mathbf{k}}$ satisfying

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \tag{1130}$$

- As before, Maxwell's equations imply that the electric and magnetic fields are *transverse* to the direction of propagation and are *orthogonal* to one another:

$$\tilde{\mathbf{E}}_0\cdot\hat{\mathbf{k}} = 0\,, \quad \tilde{\mathbf{B}}_0\cdot\hat{\mathbf{k}} = 0\,, \quad \tilde{\mathbf{B}}_0 = \frac{\tilde{\mathbf{k}}}{\omega}\times\tilde{\mathbf{E}}_0 \tag{1131}$$

But since $\tilde{\mathbf{k}}$ is complex, $\mathbf{E}$ and $\mathbf{B}$ are *no longer in phase* with one another.

- If we define

$$\tilde{k} \equiv Ke^{i\phi} \quad \text{and} \quad \tilde{E}_0 \equiv B_0\, e^{i\delta_E}\,, \quad \tilde{B}_0 \equiv B_0\, e^{i\delta_B} \tag{1132}$$

it follows that

$$\delta_B = \delta_E + \phi \quad \text{and} \quad \frac{B_0}{E_0} = \frac{K}{\omega} \tag{1133}$$

So the magnetic field *lags* behind the electric field by ϕ.

- If we define

$$\tilde{k} \equiv k + i\kappa \tag{1134}$$

and use the result that the square-root of a complex number $a+ib$ is given by

$$\sqrt{a+ib} = \frac{1}{\sqrt{2}}\left[\left(\sqrt{a^2+b^2}+a\right)^{1/2} + i\left(\sqrt{a^2+b^2}-a\right)^{1/2}\right] \tag{1135}$$

then

$$k = \omega\sqrt{\frac{\mu\epsilon}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right)^{1/2}\,, \quad \kappa = \omega\sqrt{\frac{\mu\epsilon}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right)^{1/2} \tag{1136}$$

- From this it follows that

$$K = \sqrt{k^2+\kappa^2} = \omega\sqrt{\mu\epsilon}\left[1+\left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/4} \tag{1137}$$

$$\phi = \text{atan}\left(\frac{\kappa}{k}\right) = \text{atan}\left[\frac{\left(\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right)^{1/2}}{\left(\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right)^{1/2}}\right] \tag{1138}$$

- Thus,

$$\frac{B_0}{E_0} = \sqrt{\mu\epsilon}\left[1+\left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/4} \tag{1139}$$

- The solutions for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ are damped sinusoids—i.e., there is attenuation of the wave as it propagates into the conductor. If we take $\hat{\mathbf{k}} = \hat{\mathbf{z}}$, then

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{\mathbf{E}}_0\, e^{-\kappa z}e^{i(kz-\omega t)}\,, \tag{1140}$$

$$\tilde{\mathbf{B}}(\mathbf{r},t) = \tilde{\mathbf{B}}_0\, e^{-\kappa z}e^{i(kz-\omega t)} \tag{1141}$$

- The *skin depth* is the distance d at which the amplitude of the wave has decayed to $e^{-1}(=0.37)$ of its value at $z=0$:
$$d \equiv \frac{1}{\kappa} \tag{1142}$$

- From k we can calculate the wave velocity, wavelength, and index of refraction from the standard formulae:
$$v = \frac{\omega}{k}\,, \quad \lambda = \frac{2\pi}{k}\,, \quad n = \frac{c}{v} = \frac{ck}{\omega} \tag{1143}$$

- Since k depends on ω, all of the above quantities are *frequency-dependent.* Such a material is said to be *dispersive.* (See Sec. 9.5 for more details.)

9.4.1 Reflection and transmission at a conducting surface, normal incidence

- To analyze reflection and transmission at a conducting surface, one needs to apply the boundary conditions for two linear homogeneous media in the presence of free charges and free currents:
$$\text{(i) } \epsilon_1 E_{1\perp} - \epsilon_2 E_{2\perp} = \sigma_f\,, \qquad \text{(ii) } B_{1\perp} = B_{2\perp}\,, \tag{1144}$$
$$\text{(iii) } \mathbf{E}_{1\parallel} = \mathbf{E}_{2\parallel}\,, \qquad \text{(iv) } \frac{1}{\mu_1}\mathbf{B}_{1\parallel} - \frac{1}{\mu_2}\mathbf{B}_{2\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}} \tag{1145}$$
where $\hat{\mathbf{n}}$ is the unit normal pointing from medium 2 to medium 1.

- If we restrict attention to ohmic conductors ($\mathbf{J}_f = \sigma\mathbf{E}$) and normal incidence ($E_{1\perp} = E_{2\perp} = 0$), then $\mathbf{K}_f = 0$ and $\sigma_f = 0$ as before.

- Applying these boundary conditions for normal incidence at a conducting surface, one can show that
$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}\,, \quad \tilde{E}_{0I} - \tilde{E}_{0R} = \tilde{\beta}\tilde{E}_{0T} \tag{1146}$$
where
$$\tilde{\beta} \equiv \frac{\mu_1 v_1}{\mu_2(\omega/\tilde{k}_2)} \tag{1147}$$

- Note that $\tilde{\beta}$ is complex since the wave vector $\tilde{\mathbf{k}}_2$ is complex in the conducting medium.

- Solving for $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$:
$$\tilde{E}_{0R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0I}\,, \quad \tilde{E}_{0T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0I} \tag{1148}$$

- For a perfect conductor there is 100% reflection with a 180° phase shift:
$$\tilde{E}_{0R} = -\tilde{E}_{0I}\,, \quad \tilde{E}_{0T} = 0 \tag{1149}$$
(<u>Proof</u>: $\sigma = \infty$ implies $\tilde{k}_2 = \infty$ and thus $\tilde{\beta} = \infty$.)

- Application: A typical mirror is just a pane glass with a thin metal coating on the back.

9.5 Dispersion

- By definition, a *dispersive* material is one for which the wave velocity v is a function of the frequency of the wave. Since $v = c/n$ where n is the index of refraction, a dispersive material has an index of refraction which is a function of frequency.

- For typical materials $\mu \cong \mu_0$, which implies $n \cong \sqrt{\epsilon_r}$, where ϵ_r is the dielectric constant ($\epsilon_r \equiv \epsilon/\epsilon_0$). Thus, a dispersive material has a dielectric constant which is a function of frequency.

- Experimentally, it is observed that the index of refraction n decreases with increasing wavelength (i.e., increases with increasing frequency). This is responsible for a prism splitting white light into its constituent colors, or raindrops splitting sunlight into a rainbow. Blue light is bent more than red light, since it has a higher frequency and hence a correspondingly larger index of refraction.

- Wave velocity (sometimes called phase velocity):

$$v = \frac{\omega}{k} \tag{1150}$$

  This gives the velocity of each sinusoidal component of a wave.

- Group velocity:

$$v_g = \frac{d\omega}{dk} \tag{1151}$$

  This is the velocity of a wave packet (i.e., a localized wave made up by superimposing monochromatic waves of different frequencies).

- NOTE! It is possible for either v or v_g to exceed c in some cases. However, it is usually the case that the energy carried by a wave travels with $v_g \leq c$.

- One can model the response of electrons in a dielectric to an electromagnetic wave as a damped harmonic oscillator driven by the electric field of the wave. This simple model give rise to a *complex-valued* dielectric constant $\tilde{\epsilon}_r$ and frequency-dependent index of refraction n, similar to what is observed experimentally.

- Model:

$$m\frac{d^2x}{dt^2} = -m\omega_0^2 x - m\gamma\frac{dx}{dt} + qE_0\cos(\omega t) \tag{1152}$$

  where $\omega_0 \equiv \sqrt{k/m}$ is the natural frequency of the electron bound to a molecule ($F_{\rm binding} = -kx$), γ is the damping factor, and ω is the driving frequency of the electric field associated with a linearly polarized monochromatic plane wave.

- Complex solution: ($E_0\cos(\omega t)$ replaced by $E_0 e^{-i\omega t}$)

$$\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t} \quad \text{with} \quad \tilde{x}_0 = \frac{qE_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \tag{1153}$$

- Complex dipole moment:

$$\tilde{p}(t) \equiv q\tilde{x}(t) = \frac{q^2}{m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}\tilde{E}(t) \tag{1154}$$

- Complex dipole moment per unit volume:

$$\tilde{\mathbf{P}}(t) = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\tilde{\mathbf{E}}(t) \tag{1155}$$

  where N is the number of molecules per unit volume, and f_j is the number of electrons having natural frequency ω_j and damping γ_j. The complex proportionality factor between $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{E}}$ means that $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{E}}$ are *not* in phase.

- Comlex electric susceptibility:

$$\tilde{\mathbf{P}}(t) = \epsilon_0\tilde{\chi}_e\tilde{\mathbf{E}}(t) \tag{1156}$$

- Complex dielectric constant:

$$\tilde{\epsilon}_r \equiv 1 + \tilde{\chi}_e = 1 + \frac{Nq^2}{m\epsilon_0}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right) \tag{1157}$$

- Wave equation in a dispersive media (with no free charges and no free currents):

$$\nabla^2\tilde{\mathbf{E}} - \tilde{\epsilon}\mu_0\frac{\partial^2\tilde{\mathbf{E}}}{\partial t^2} = 0\,, \quad \nabla^2\tilde{\mathbf{B}} - \tilde{\epsilon}\mu_0\frac{\partial^2\tilde{\mathbf{B}}}{\partial t^2} = 0 \tag{1158}$$

  where $\tilde{\epsilon} \equiv \tilde{\epsilon}_r\epsilon_0$, and we are asuuming that $\mu \cong \mu_0$, which is valid for many materials at optical frequencies.

- Linearly polarized monochromatic plane wave solutions:

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{\mathbf{k}}\cdot\mathbf{r}-\omega t)}\,, \quad \tilde{\mathbf{B}}(\mathbf{r},t) = \frac{\tilde{\mathbf{k}}}{\omega}\times\tilde{\mathbf{E}}(\mathbf{r},t) \tag{1159}$$

  with complex wave vector $\tilde{\mathbf{k}} \equiv \tilde{k}\,\hat{\mathbf{k}}$ satisfying

$$\tilde{k} = \omega\sqrt{\tilde{\epsilon}\mu_0} \tag{1160}$$

- If we define

$$\tilde{k} \equiv k + i\kappa \tag{1161}$$

  then one can show that

$$k \cong \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2-\omega^2)^2 + \gamma_j^2\omega^2}\right] \tag{1162}$$

$$\kappa \cong \frac{Nq^2\omega^2}{2m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2-\omega^2)^2 + \gamma_j^2\omega^2} \tag{1163}$$

  where $\cong$ means that we assumed $|\tilde{\chi}_e| \ll 1$.

- From k we can calculate the wave velocity, wavelength, and group velocity using the standard formulae:

$$v = \frac{\omega}{k}\,, \quad \lambda = \frac{2\pi}{k}\,, \quad v_g \equiv \frac{d\omega}{dk} = \frac{1}{dk/d\omega} \tag{1164}$$

- Assuming negligible damping (i.e., $\gamma_j = 0$), one can show (Prob. 9.25, Griffiths)

$$v_g = c\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j(\omega_j^2+\omega^2)}{(\omega_j^2-\omega^2)^2}\right]^{-1} \tag{1165}$$

  Note that $v_g < c$ always.

- In terms of k and κ, the solutions are damped sinusoids, which are attenuated as they propagate through the material.

  Example ($\hat{\mathbf{k}} = \hat{\mathbf{z}}$):

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)} \tag{1166}$$

  Since the intensity is proportional to $|\mathbf{E}|^2$, the absorption goes like $e^{-2\kappa z}$.

- Absorption coefficient:

$$\alpha \equiv 2\kappa \cong \frac{Nq^2\omega^2}{m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2-\omega^2)^2 + \gamma_j^2\omega^2} \tag{1167}$$

- Index of refraction:

$$n = \frac{ck}{\omega} \cong 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2-\omega^2)^2 + \gamma_j^2\omega^2} \tag{1168}$$

- At a resonance ($\omega = \omega_j$) the absorption coefficient α has a local maximum, and the index of refraction n goes from $n > 1$ (when $\omega < \omega_j$) to $n < 1$ (when $\omega > \omega_j$). (Recall that the wave velocity $v > c$ when $n < 1$.)
- A large value of α corresponds to the material being *opaque* to electromagnetic radiation at that particular frequency.
- The rapid drop in the index of refraction when ω passes through a resonance is called *anomalous dispersion*.
- Away from resonances (so that we can ignore the γ_j term) and assuming that $\omega \ll \omega_j$ (which is valid for most gases at optical frequencies), the expression for n simplifies to

$$n \cong 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right) = 1 + A\left(1 + \frac{B}{\lambda^2}\right) \tag{1169}$$

where $\lambda = 2\pi v/\omega$.

- The above expression is called *Cauchy's formula*. (A is called the coefficient of refraction, and B the coefficient of dispersion.)
- Cauchy's formula shows that away from resonances ($\omega \ll \omega_j$) the index of refraction decreases with increasing wavelength as observed experimentally.

10 Electromagnetic radiation

10.1 Retarded potentials

- For radiation, it is simplest to solve the equations for the potentials Φ and $\mathbf{A}$ and then differentiate to obtain the fields $\mathbf{E}$ and $\mathbf{B}$.
- Recall that in the Lorentz gauge

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t} = 0 \tag{1170}$$

Maxwell's equations in vacuum can be written in terms of the potentials as

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{1}{\epsilon_0}\,\rho\,, \qquad \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\,\mathbf{J} \tag{1171}$$

- These equations have the general form:

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi\, f(\mathbf{r}, t) \tag{1172}$$

- Solution:

$$\Psi(\mathbf{r}, t) = \int dV' \int dt'\, G^{(+)}(\mathbf{r}, \mathbf{r}', t, t') f(\mathbf{r}', t') \tag{1173}$$

where

$$G^{(+)}(\mathbf{r}, \mathbf{r}', t, t') = G^{(+)}(\mathbf{R}, \tau) = \frac{\delta\left(\tau - \frac{R}{c}\right)}{R} = \frac{\delta\left(t' - \left[t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\right)}{|\mathbf{r}-\mathbf{r}'|} \tag{1174}$$

is the *retarded Green's function.* It is a solution of the equation

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') \tag{1175}$$

- Thus,

$$\begin{aligned}\Psi(\mathbf{r},t) &= \int dV' \int dt'\, \frac{f(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\,\delta\left(t' - \left[t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\right) && (1176)\\ &= \int dV'\, \frac{f(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} && (1177)\\ &= \int dV'\, \frac{[f(\mathbf{r}',t')]_{\rm ret}}{|\mathbf{r}-\mathbf{r}'|} && (1178)\end{aligned}$$

where the second line follows trivially by integrating over t' using the delta function. The notation $[\cdots]_{\rm ret}$ means to evaluate t' at the retarded time

$$t_r \equiv t - \frac{|\mathbf{r}-\mathbf{r}'|}{c} \tag{1179}$$

- Retarded solutions for Φ, $\mathbf{A}$:

$$\begin{aligned}\Phi(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int dV' \int dt'\, \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\,\delta\left(t' - \left[t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\right) && (1180)\\ &= \frac{1}{4\pi\epsilon_0}\int dV'\, \frac{\rho(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} && (1181)\end{aligned}$$

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int dV' \int dt'\, \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\,\delta\left(t' - \left[t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\right) && (1182)\\ &= \frac{\mu_0}{4\pi}\int dV'\, \frac{\mathbf{J}(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} && (1183)\end{aligned}$$

where we did the trivial integrations over t' as before.

10.2 Lienard-Wiechert potentials for an accelerated point charge

- Particle trajectory: $\mathbf{r}' = \mathbf{r}_0(t')$
- Charge and current densities:

$$\begin{aligned}\rho(\mathbf{r}',t') &= q\,\delta(\mathbf{r}'-\mathbf{r}_0(t')) && (1184)\\ \mathbf{J}(\mathbf{r}',t') &= q\mathbf{v}(t')\,\delta(\mathbf{r}'-\mathbf{r}_0(t')) && (1185)\end{aligned}$$

where $\mathbf{v}(t') = \dot{\mathbf{r}}_0(t')$.

- Integral expressions for Φ, $\mathbf{A}$:

$$\begin{aligned}\Phi(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int dV' \int dt'\, \frac{q\,\delta(\mathbf{r}'-\mathbf{r}_0(t'))}{|\mathbf{r}-\mathbf{r}'|}\,\delta\left(t' - \left[t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\right) && (1186)\\ &= \frac{1}{4\pi\epsilon_0}\int dt'\, \frac{q}{|\mathbf{r}-\mathbf{r}_0(t')|}\,\delta\left(t' - \left[t - \frac{|\mathbf{r}-\mathbf{r}_0(t')|}{c}\right]\right) && (1187)\end{aligned}$$

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int dV' \int dt'\, \frac{q\mathbf{v}(t')\,\delta(\mathbf{r}'-\mathbf{r}_0(t'))}{|\mathbf{r}-\mathbf{r}'|}\,\delta\left(t' - \left[t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\right) && (1188)\\ &= \frac{\mu_0}{4\pi}\int dt'\, \frac{q\mathbf{v}(t')}{|\mathbf{r}-\mathbf{r}_0(t')|}\,\delta\left(t' - \left[t - \frac{|\mathbf{r}-\mathbf{r}_0(t')|}{c}\right]\right) && (1189)\end{aligned}$$

where we did the trivial integrations over $\mathbf{r}'$.

- The t' integration is non-trivial now, since the argument of the delta function is a non-trivial function of t':

$$f(t') \equiv t' - \left[t - \frac{|\mathbf{r} - \mathbf{r}_0(t')|}{c}\right] \tag{1190}$$

- Thus, we will need to use the general result

$$\delta(f(t')) = \sum_i \frac{\delta(t - t_i)}{|f'(t_i)|} \tag{1191}$$

where the sum is over the zeros of f.

- For $f(t')$ given above, there is only one zero

$$f(t_r) = 0 \Leftrightarrow t_r \equiv t - \frac{|\mathbf{r} - \mathbf{r}_0(t_r)|}{c} \tag{1192}$$

Note that the above equation for t_r is an *implicit* expression for the retarded time (since it appears on both sides of the equation) in contrast to the previous definition in Eq. (1179).

- Derivative:

$$f'(t') = 1 - \boldsymbol{\beta}(t') \cdot \hat{\mathbf{R}}(t')\,, \quad \text{where} \quad \boldsymbol{\beta}(t') \equiv \frac{\mathbf{v}(t')}{c}\,, \quad \mathbf{R}(t') \equiv \mathbf{r} - \mathbf{r}_0(t') \tag{1193}$$

- Jacobian of the transformation:

$$|f'(t_r)| = 1 - \boldsymbol{\beta}(t_r) \cdot \hat{\mathbf{R}}(t_r) \tag{1194}$$

which is positive since $|\mathbf{v}(t')| < c$.

- Thus,

$$\delta\left(t' - \left[t - \frac{|\mathbf{r} - \mathbf{r}_0(t')|}{c}\right]\right) = \frac{\delta(t' - t_r)}{1 - \boldsymbol{\beta}(t_r) \cdot \hat{\mathbf{R}}(t_r)} \tag{1195}$$

which implies

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{R - \boldsymbol{\beta} \cdot \mathbf{R}}\right]_{\rm ret}\,, \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\left[\frac{q\mathbf{v}}{R - \boldsymbol{\beta} \cdot \mathbf{R}}\right]_{\rm ret} \tag{1196}$$

where $[\cdots]_{\rm ret}$ means evaluate all functions of time at the retarded time t_r. These are the Lienard-Wiechert potentials.

10.3 Electric and magnetic fields for an accelerated point charge

- Given the Lienard-Wiechert potentials Φ and $\mathbf{A}$, we can now differentiate them with respect to $\mathbf{r}$ and t to obtain the electric and magnetic fields:

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t}\,, \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{1197}$$

- Note that the derivative of t_r with respect to t and the gradient of t_r with respect to $\mathbf{r}$ are non-trivial, due to the implicit definition of t_r:

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{r}_0(t_r)|}{c} \tag{1198}$$

For example,

$$\frac{\partial t_r}{\partial t} = \frac{1}{1 - \boldsymbol{\beta}(t_r) \cdot \hat{\mathbf{R}}(t_r)}\,, \qquad \boldsymbol{\nabla} t_r = \frac{-\hat{\mathbf{R}}(t_r)/c}{1 - \boldsymbol{\beta}(t_r) \cdot \hat{\mathbf{R}}(t_r)} \tag{1199}$$

- Thus, by taking such derivatives of Φ and $\mathbf{A}$, one can show that

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{R}}(t_r)}{R^2(t_r)} + \frac{R(t_r)}{c}\frac{\partial}{\partial t}\left(\frac{\hat{\mathbf{R}}(t_r)}{R^2(t_r)}\right) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\hat{\mathbf{R}}(t_r)\right)\right] \tag{1200}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\mathbf{R}}(t_r) \times \mathbf{E}(\mathbf{r},t) \tag{1201}$$

These are Feynman's expressions for the fields of an accelerated charge.

- The simplest way to "derive" these results is to evaluate the derivatives in the Feynman's expression for $\mathbf{E}$ above, do the same for $\mathbf{E} = -\boldsymbol{\nabla}\Phi - \partial\mathbf{A}/\partial t$, and then show that all the terms agree. Similarly for $\mathbf{B}$.

- An alternative expression for the fields, which more clearly separates the radiative and non-radiative parts of the field, is

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{(1-\beta^2)(\hat{\mathbf{R}}-\boldsymbol{\beta})}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}})^3 R^2} + \frac{\hat{\mathbf{R}} \times \left([\hat{\mathbf{R}}-\boldsymbol{\beta}] \times \mathbf{a}\right)}{c^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}})^3 R}\right]_{\text{ret}} \tag{1202}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\mathbf{R}}(t_r) \times \mathbf{E}(\mathbf{r},t) \tag{1203}$$

- The radiative term is the second term in $\mathbf{E}$; it goes like $1/R$ and is proportional to the acceleration of the charge at the retarded time, $\mathbf{a}(t_r)$.

10.4 Field for a point charge moving with constant velocity

- For a point charge moving with constant velocity (i.e., $\mathbf{a} = 0$), the second term in Eq. (1202) for the electric field is zero. Thus,

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{(1-\beta^2)(\hat{\mathbf{R}}-\boldsymbol{\beta})}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}})^3 R^2}\right]_{\text{ret}} \tag{1204}$$

where $\boldsymbol{\beta} \equiv \mathbf{v}/c = \text{const}$.

- For constant velocity, one can also show that

$$\mathbf{R}(t_r) - \frac{\mathbf{v}}{c}R(t_r) = \mathbf{R}(t) \tag{1205}$$

which implies

$$[\hat{\mathbf{R}} - \boldsymbol{\beta}]_{\text{ret}} = \frac{\mathbf{R}(t)}{R(t_r)} \tag{1206}$$

- This means that the electric field at $(\mathbf{r},t)$ points along a line from the *present* position of the particle $\mathbf{r}_0(t)$, even though what's responsible for the field now was produced by the particle when it was located at $\mathbf{r}_0(t_r)$.

- If we further assume that the particle is at the origin when $t = 0$, so that

$$\mathbf{r}_0(t) = \mathbf{v}\,t \tag{1207}$$

it follows that

$$c(t-t_r) = R(t_r) = |\mathbf{r} - \mathbf{v}\,t_r| = \sqrt{r^2 + v^2 t_r^2 - 2t_r\mathbf{v}\cdot\mathbf{r}} \tag{1208}$$

- Squaring the above expression and solving the quadratic equation for the retarded time t_r yields

$$t_r = \frac{(c^2 t - \mathbf{v}\cdot\mathbf{r}) \pm \sqrt{(c^2 t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2 t^2 - r^2)}}{c^2 - v^2} \tag{1209}$$

- Of the two solutions, we need to take the one with the minus sign, since for $\mathbf{v} = 0$, we know that t_r should equal $t - r/c$.
- Using the explicit expression for t_r in terms of t, one can show

$$[R - \boldsymbol{\beta} \cdot \mathbf{R}]_{\text{ret}} = \sqrt{(ct - \boldsymbol{\beta} \cdot \mathbf{r})^2 + (1 - \beta^2)(r^2 - c^2 t^2)} \tag{1210}$$

- Putting all these results together, we find

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{R}(t)\left(1 - \beta^2\right)}{\left[(ct - \boldsymbol{\beta} \cdot \mathbf{r})^2 + (1 - \beta^2)(r^2 - c^2 t^2)\right]^{3/2}} \tag{1211}$$

- Alternatively, we can write

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}}(t)}{R^2(t)} \frac{\left(1 - \beta^2\right)}{\left(1 - \beta^2 \sin^2\theta\right)^{3/2}} \tag{1212}$$

  where θ is the angle between $\mathbf{v}$ and $\mathbf{R}(t)$.
- For $\theta = 0$ (i.e., when the observation point is along the line of motion of the charge), $\mathbf{E}(\mathbf{r}, t)$ is reduced from its static Coulomb value by a factor of $(1 - \beta^2)$. For $\theta = \pi/2$ (i.e., when the observation point is perpendicular to the line of motion of the charge), $\mathbf{E}(\mathbf{r}, t)$ is enhanced over its static Coulomb value by a factor of $1/(1 - \beta^2)^{1/2}$.
- Using the above expression for $\mathbf{E}(\mathbf{r}, t)$ and Eq. (1203) for the magnetic field, it follows that

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \boldsymbol{\beta} \times \mathbf{E}(\mathbf{r}, t) \tag{1213}$$

10.5 Power radiated by an accelerated point charge

- Radiative components of the fields (keep only $1/R$ terms):

$$\mathbf{E}(\mathbf{r}, t) \cong \frac{q}{4\pi\epsilon_0} \left[\frac{\hat{\mathbf{R}} \times \left([\hat{\mathbf{R}} - \boldsymbol{\beta}] \times \mathbf{a} \right)}{c^2 (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3 R} \right]_{\text{ret}} \tag{1214}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \hat{\mathbf{R}}(t_r) \times \mathbf{E}(\mathbf{r}, t) \tag{1215}$$

- Thus, $\mathbf{E}$, $\mathbf{B}$, and $\hat{\mathbf{R}}(t_r)$ form a right-handed system of orthogonal vectors.
- Poynting vector:

$$\mathbf{S}(\mathbf{r}, t) = \frac{1}{\mu_0} \mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \tag{1216}$$

$$= \frac{1}{\mu_0 c} \mathbf{E}(\mathbf{r}, t) \times \left(\hat{\mathbf{R}}(t_r) \times \mathbf{E}(\mathbf{r}, t) \right) \tag{1217}$$

$$= \frac{1}{\mu_0 c} \left[\hat{\mathbf{R}}(t_r) E^2(\mathbf{r}, t) - \mathbf{E}(\mathbf{r}, t)(\mathbf{E}(\mathbf{r}, t) \cdot \hat{\mathbf{R}}(t_r)) \right] \tag{1218}$$

$$= \frac{1}{\mu_0 c} E^2(\mathbf{r}, t)\, \hat{\mathbf{R}}(t_r) \tag{1219}$$

- The amount of energy radiated through an infinitesimal area element $\hat{\mathbf{n}}\, da$ at $(\mathbf{r}, t)$ during a infinitesimal time interval t to $t + dt$ is given by

$$d^2 W = \mathbf{S}(\mathbf{r}, t) \cdot \hat{\mathbf{n}}\, da\ dt \tag{1220}$$

- Thus, the rate at which the radiated energy passes through the area element is

$$\frac{d^2W}{dt} = \mathbf{S}(\mathbf{r},t)\cdot\hat{\mathbf{n}}\,da \tag{1221}$$

- This rate differs, however, from the rate at which the energy left the particle at the retarded time t_r, due to the motion of the charge (Doppler effect). Denoting this rate by d^2W/dt_r, we have

$$\frac{d^2W}{dt_r} = \frac{d^2W}{dt}\frac{dt}{dt_r} = \frac{d^2W}{dt}\Big(1-\boldsymbol{\beta}(t_r)\cdot\hat{\mathbf{R}}(t_r)\Big) \tag{1222}$$

- The total power emitted by the point charge is then the integral of d^2W/dt_r over a 2-sphere as $R\to\infty$, where $R\equiv|\mathbf{R}(t_r)| = |\mathbf{r}-\mathbf{r}_0(t_r)|$. (Note that the 2-sphere is centered on the position of the point charge at the retarded time t_r.) Thus,

$$P \equiv \lim_{R\to\infty}\oint\Big(1-\boldsymbol{\beta}(t_r)\cdot\hat{\mathbf{R}}(t_r)\Big)\,\mathbf{S}(\mathbf{r},t)\cdot\hat{\mathbf{n}}\,da \tag{1223}$$

  where $\hat{\mathbf{n}}\,da = \hat{\mathbf{R}}(t_r)\,R^2(t_r)d\Omega$.

- Note that the power radiated into the infinitesimal solid angle $d\Omega = d(\cos\theta)\,d\phi$ centered at $(\mathbf{r},t)$ is

$$\frac{dP}{d\Omega} = \Big(1-\boldsymbol{\beta}(t_r)\cdot\hat{\mathbf{R}}(t_r)\Big)\,\mathbf{S}(\mathbf{r},t)\cdot\hat{\mathbf{R}}(t_r)R^2(t_r) \tag{1224}$$

- Substituting for $\mathbf{S}$ using previous formulas, we have

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c}\left[\frac{\left|\hat{\mathbf{R}}\times([\hat{\mathbf{R}}-\boldsymbol{\beta}]\times\mathbf{a})\right|^2}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{R}})^5}\right]_{\rm ret} \tag{1225}$$

- Rather than try to do the integral over $d\Omega$ directly, we consider the following special cases:

- <u>Particle instantaneously at rest at t_r:</u>

  For this case $\boldsymbol{\beta}(t_r)=0$, which implies

$$\mathbf{E}(\mathbf{r},t) \cong \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{R}}\times\Big(\hat{\mathbf{R}}\times\mathbf{a}\Big)}{c^2R}\right]_{\rm ret} \tag{1226}$$

  and

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c}\,\left[a^2\sin^2\theta\right]_{\rm ret} \tag{1227}$$

  where θ is the angle between $\mathbf{a}(t_r)$ and $\hat{\mathbf{R}}(t_r)$. The $\sin^2\theta$ factor corresponds to a 'donut' radiation pattern—i.e., no radiation is emitted parallel to $\mathbf{a}(t_r)$, while maximum emission is in the plane perpendicular to $\mathbf{a}(t_r)$.

  Larmor formula (total power radiated):

$$P = \left[\frac{\mu_0 q^2 a^2}{6\pi c}\right]_{\rm ret} \tag{1228}$$

- <u>Lienard's generalization of the Larmor formula for arbitrary velocity:</u>

  To generalize Larmor's formula to a point charge moving with arbitrary instantaneous velocity $\mathbf{v}(t_r)$, it is simplest to note that:

  i) Total radiated power P transforms as a scalar under Lorentz transformations.

ii) One can always choose an inertial reference frame that is instantaneously co-moving with the particle (so $\mathbf{v}(t_r) = 0$ in this frame).

iii) Thus, if we can find a Lorentz invariant form for P that reduces to the Larmor formula when $\mathbf{v}(t_r) = 0$, we will have an expression for P that is valid for arbitrary $\mathbf{v}(t_r)$.

Lienard's generalization:

$$P = \left[\frac{\mu_0 q^2 \eta_{\alpha\beta} a^\alpha a^\beta}{6\pi c}\right]_{\text{ret}} \tag{1229}$$

which amounts to simply replacing the 3-acceleration $\mathbf{a}$ with the 4-acceleration a^α of the point charge.

Using the fact that in an arbitrary frame

$$a^\alpha \equiv \frac{du^\alpha}{d\tau} = \gamma^4 \left(\boldsymbol{\beta}\cdot\mathbf{a},\, \mathbf{a} + \boldsymbol{\beta}\times(\boldsymbol{\beta}\times\mathbf{a})\,\right) \quad \text{where} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \tag{1230}$$

it follows that

$$P = \left[\frac{\mu_0 q^2 \gamma^6}{6\pi c}\,\left(|\mathbf{a}|^2 - |\boldsymbol{\beta}\times\mathbf{a}|^2\right)\right]_{\text{ret}} \tag{1231}$$

- <u>$\mathbf{v}(t_r)$ and $\mathbf{a}(t_r)$ instantaneously parallel to one another</u>:

  For this case $\boldsymbol{\beta}(t_r)\times\mathbf{a}(t_r) = 0$, which implies

  $$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c}\left[\frac{a^2\sin^2\theta}{(1-\beta\cos\theta)^5}\right]_{\text{ret}} \quad \text{and} \quad P = \left[\frac{\mu_0 q^2\gamma^6 a^2}{6\pi c}\right]_{\text{ret}} \tag{1232}$$

  where θ is the angle between $\mathbf{v}(t_r)$ and $\hat{\mathbf{R}}(t_r)$.

  Note that for $v \to 0$, the above expression reduces to the 'donut' radiation pattern, proportional to $\sin^2\theta$. For $v \to c$, the lobes of the 'donut' are pushed forward and point in the direction of propagation $\mathbf{v}(t_r)$.

  The above angular distribution is the same for both accelerating and decelerating charged particles. The power radiated by a rapidly decelerated electron is called *bremsstrahlung*, which means "braking radiation."

- <u>$\mathbf{v}(t_r)$ and $\mathbf{a}(t_r)$ instantaneously perpendicular to one another</u>:

  For this case $\boldsymbol{\beta}(t_r)\cdot\mathbf{a}(t_r) = 0$, which implies (Prob 11.16, Griffiths):

  $$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left[\frac{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^5}\right]_{\text{ret}} \tag{1233}$$

  and

  $$P = \left[\frac{\mu_0 q^2\gamma^4 a^2}{6\pi c}\right]_{\text{ret}} \tag{1234}$$

  where θ is the angle between $\mathbf{v}(t_r)$ and $\hat{\mathbf{R}}(t_r)$.

  The radiation has its maximum in the direction of motion of the charge.

  Uniform circular motion is an example where $\mathbf{v}(t_r)$ and $\mathbf{a}(t_r)$ are perpendicular to one another. Such radiation is called *synchrotron* radiation, named after the circular-shaped particle accelerators.

  <u>Example</u>: Synchrotron radiation is emitted by an electron spiralling around magnetic field lines.

10.6 Electric dipole radiation

- Consider two tiny metal spheres located at $z = \pm d/2$, connected by a wire such that the charges on the two spheres are $\pm q(t)$ where

$$q(t) = q_0 \cos(\omega t) \tag{1235}$$

- The current in the wire is given by

$$I(t)\,\hat{\mathbf{z}} = \frac{dq}{dt}\,\hat{\mathbf{z}} = -q_0\omega \sin(\omega t)\,\hat{\mathbf{z}} \tag{1236}$$

- There is a time-varying electric dipole moment given by

$$\mathbf{p}(t) \equiv q(t) d\,\hat{\mathbf{z}} = p_0 \cos(\omega t)\,\hat{\mathbf{z}} \tag{1237}$$

- Assumptions:

  i) Large-r approximation (radiation zone):

$$r \gg d \quad \text{and} \quad r \gg \lambda \equiv \frac{2\pi c}{\omega} \tag{1238}$$

  ii) Long-wavelength approximation (slow-motion sources):

$$\lambda \gg d \quad \Leftrightarrow \quad \frac{\omega d}{c} \ll 1 \tag{1239}$$

- Keeping only the leading order terms, one finds

$$\Phi(\mathbf{r}, t) \cong -\frac{p_0\omega}{4\pi\epsilon_0 c}\frac{\cos\theta}{r}\,\sin\left(\omega t_0\right) \tag{1240}$$

$$\mathbf{A}(\mathbf{r}, t) \cong -\frac{\mu_0 p_0 \omega}{4\pi r}\,\sin\left(\omega t_0\right)\,\hat{\mathbf{z}} \tag{1241}$$

  where

$$t_0 \equiv t - \frac{r}{c} \tag{1242}$$

  is the retarded time at the origin.

- Radiative components of the fields:

$$\mathbf{E}(\mathbf{r}, t) \cong -\frac{\mu_0 p_0 \omega^2}{4\pi}\frac{\sin\theta}{r}\,\cos\left(\omega t_0\right)\,\hat{\boldsymbol{\theta}} \tag{1243}$$

$$\mathbf{B}(\mathbf{r}, t) \cong -\frac{\mu_0 p_0 \omega^2}{4\pi c}\frac{\sin\theta}{r}\,\cos\left(\omega t_0\right)\,\hat{\boldsymbol{\phi}} \tag{1244}$$

- Note that

$$\mathbf{B} \cong \frac{1}{c}\hat{\mathbf{r}} \times \mathbf{E} \tag{1245}$$

- Poynting vector:

$$\mathbf{S}(\mathbf{r}, t) \cong \frac{\mu_0 p_0^2 \omega^4}{16\pi^2 c r^2}\sin^2\theta\,\cos^2\left(\omega t_0\right)\,\hat{\mathbf{r}} \tag{1246}$$

- Thus, $\mathbf{E}$, $\mathbf{B}$, and $\hat{\mathbf{r}}$ form a right-handed system of orthogonal vectors.

- Angular distribution of radiated power:

$$\frac{dP}{d\Omega} = \mathbf{S}(\mathbf{r}, t) \cdot \hat{\mathbf{r}}\, r^2 \cong \frac{\mu_0 p_0^2 \omega^4}{16\pi^2 c} \sin^2\theta \, \cos^2(\omega t_0) \tag{1247}$$

  The $\sin^2\theta$ factor means that the radiation pattern has the general shape of a 'donut', with no radiation in the direction of the oscillating electric dipole moment.

- Radiated power:

$$P(t) \cong \frac{\mu_0 p_0^2 \omega^4}{6\pi c} \cos^2(\omega t_0) \tag{1248}$$

- Time-averaged Poynting vector:

$$\langle \mathbf{S}(\mathbf{r}) \rangle \cong \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c r^2} \sin^2\theta \, \hat{\mathbf{r}} \tag{1249}$$

- Time-averaged radiated power:

$$\langle P \rangle \cong \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \tag{1250}$$

- Example: Sunlight stimulates atoms in the atmosphere to oscillate as tiny electric dipoles. The blue color of the sky overhead is due to the reradiated power being proportional to ω^4, and hence dominated by the larger frequencies (i.e., the blue end of the spectrum). This also explains the red color of the sky seen at sunset, since the light which enters our eyes from the horizon is the light that didn't get reradiate away in other direction.

- *Radiation resistance* is the resistance that gives the same power loss to heat as the oscillating dipole gives out in electromagnetic radiation:

$$\langle P \rangle = \langle I^2 \rangle R_{\text{rad}} \tag{1251}$$

  For electric dipole radiation, this can be shown to equal

$$R_{\text{rad}} = 790\,(d/\lambda)^2 \text{ ohms} \tag{1252}$$

  Since $d \ll \lambda$ in the long-wavelength approximation, radiation resistance is typically negligible compared to other resistances in ordinary electric circuits.

- The above expression for the potentials, fields, etc. can be expressed in coordinate-independent form:

$$\Phi(\mathbf{r}, t) \cong -\frac{1}{4\pi\epsilon_0 r c} \hat{\mathbf{r}} \cdot \dot{\mathbf{p}}(t_0) \tag{1253}$$

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r} \dot{\mathbf{p}}(t_0) \tag{1254}$$

$$\mathbf{E}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0)) \tag{1255}$$

$$\mathbf{B}(\mathbf{r}, t) \cong -\frac{\mu_0}{4\pi r c} \hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0) \tag{1256}$$

$$\mathbf{S}(\mathbf{r}, t) \cong \frac{\mu_0}{16\pi^2 c} \frac{|\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0)|^2}{r^2} \hat{\mathbf{r}} \tag{1257}$$

$$P(t) \cong \frac{\mu_0 |\ddot{\mathbf{p}}(t_0)|^2}{6\pi c} \tag{1258}$$

  We'll see these are exactly the expressions from the leading order terms for radiation from an arbitrary localized source.

- Note that
$$-\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{V}) = \mathbf{V}-(\mathbf{V}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} \equiv \mathbf{V}_\perp \tag{1259}$$
and
$$\hat{\mathbf{r}}\times\mathbf{V} = \hat{\mathbf{r}}\times\mathbf{V}_\perp \tag{1260}$$
where $\mathbf{V}_\perp$ is the component of $\mathbf{V}$ perpendicular to the unit vector $\hat{\mathbf{r}}$. Thus,
$$\mathbf{E}(\mathbf{r},t) \cong -\frac{\mu_0}{4\pi r}\ddot{\mathbf{p}}_\perp(t_0) \tag{1261}$$
$$\mathbf{B}(\mathbf{r},t) \cong -\frac{\mu_0}{4\pi rc}\,\hat{\mathbf{r}}\times\ddot{\mathbf{p}}_\perp(t_0) \tag{1262}$$
$$\text{etc.} \tag{1263}$$

- This means that the radiation in a particular direction is due to just the component of $\ddot{\mathbf{p}}(t_0)$ *perpendicular* to that direction.

10.7 Magnetic dipole radiation

- Consider a loop of wire of radius a in the xy-plane with current
$$\mathbf{I}(t) = I_0\cos(\omega t)\,\hat{\boldsymbol{\phi}} \tag{1264}$$

- This corresponds to a time-varying magnetic dipole with dipole moment
$$\mathbf{m}(t) = \pi a^2\, I_0\cos(\omega t)\,\hat{\mathbf{z}} \equiv m_0\cos(\omega t)\,\hat{\mathbf{z}} \tag{1265}$$

- Since $\rho = 0$, the scalar potential $\Phi(\mathbf{r},t) = 0$.

- If we make the same large-r, long-wavelength approximations that we made for electric dipole radiation (with a replacing d), we find
$$\mathbf{A}(\mathbf{r},t) \cong -\frac{\mu_0 m_0\omega}{4\pi c}\frac{\sin\theta}{r}\sin(\omega t_0)\ \hat{\boldsymbol{\phi}} \tag{1266}$$
where
$$t_0 \equiv t-\frac{r}{c} \tag{1267}$$

- Radiative components of the fields:
$$\mathbf{E}(\mathbf{r},t) \cong \frac{\mu_0 m_0\omega^2}{4\pi c}\frac{\sin\theta}{r}\ \cos(\omega t_0)\ \hat{\boldsymbol{\phi}} \tag{1268}$$
$$\mathbf{B}(\mathbf{r},t) \cong -\frac{\mu_0 m_0\omega^2}{4\pi c^2}\frac{\sin\theta}{r}\ \cos(\omega t_0)\ \hat{\boldsymbol{\theta}} = \frac{1}{c}\hat{\mathbf{r}}\times\mathbf{E}(\mathbf{r},t) \tag{1269}$$

- Poynting vector and radiated power:
$$\mathbf{S}(\mathbf{r},t) \cong \frac{\mu_0 m_0^2\omega^4}{16\pi^2c^3r^2}\sin^2\theta\,\cos^2(\omega t_0)\ \hat{\mathbf{r}} \tag{1270}$$
$$P(t) \cong \frac{\mu_0 m_0^2\omega^4}{6\pi c^3}\ \cos^2(\omega t_0) \tag{1271}$$

- As before, the radiated power has the shape of a donut, being zero in the direction of the oscillating magnetic dipole moment.

- Time-averaged quantities:
$$\langle\mathbf{S}(\mathbf{r})\rangle \cong \frac{\mu_0 m_0^2\omega^4}{32\pi^2c^3r^2}\sin^2\theta\,\hat{\mathbf{r}}\,,\qquad \langle P\rangle \cong \frac{\mu_0 m_0^2\omega^4}{12\pi c^3} \tag{1272}$$

- In coordinate-independent form:

$$\Phi(\mathbf{r},t) \cong 0 \tag{1273}$$
$$\mathbf{A}(\mathbf{r},t) \cong -\frac{\mu_0}{4\pi r c}\,\hat{\mathbf{r}} \times \dot{\mathbf{m}}(t_0) \tag{1274}$$
$$\mathbf{E}(\mathbf{r},t) \cong \frac{\mu_0}{4\pi r c}\,\hat{\mathbf{r}} \times \ddot{\mathbf{m}}(t_0) \tag{1275}$$
$$\mathbf{B}(\mathbf{r},t) \cong \frac{\mu_0}{4\pi r c^2}\,\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{m}}(t_0)) \tag{1276}$$
$$\mathbf{S}(\mathbf{r},t) \cong \frac{\mu_0}{16\pi^2 c^3}\frac{|\hat{\mathbf{r}} \times \ddot{\mathbf{m}}(t_0)|^2}{r^2}\,\hat{\mathbf{r}} \tag{1277}$$
$$P(t) \cong \frac{\mu_0 |\ddot{\mathbf{m}}(t_0)|^2}{6\pi c^3} \tag{1278}$$

- Radiation resistance:

$$R_{\rm rad} = 3 \times 10^5\,(a/\lambda)^4\,\text{ohms} \tag{1279}$$

This again turns out to be negligible for ordinary electric circuits that satisfy the long-wavelength approximation.

- Note that for a system where both electric and magnetic dipole radiation are present

$$\frac{\langle P_{\rm magnetic}\rangle}{\langle P_{\rm electric}\rangle} = \frac{m_0^2}{p_0^2 c^2} \simeq \left(\frac{\omega a}{c}\right)^2 \ll 1 \tag{1280}$$

assuming $a \sim d$ (comparable size) and $(\omega a)/c \ll 1$ (long-wavelength approximation). Thus, electric dipole radiation dominates over magnetic dipole radiation.

10.8 Radiation from an arbitrary localized source

- For an arbitrary localized source, we will assume that

$$r \gg d\,,\lambda \quad \text{(radiation zone)} \tag{1281}$$

where d and λ are representative of the overall size of the source and wavelength of the radiation. (We will place the origin of our coordinate system somewhere within the source so d is the radius of a sphere that just completely encloses the source.)

- Since we are ultimately interested in the radiation emitted by the source, we will drop any terms in the potentials or fields that fall-off faster than $1/r$.

- In this section, we will also assume that

$$\lambda \gg d \quad \text{(long wavelength approximation)} \tag{1282}$$

and consider only the leading term when Taylor expanding the sources terms ρ, $\mathbf{J}$ in the expressions for the retarded potentials Φ, $\mathbf{A}$.

- As we shall see below, these assumptions correspond to *electric dipole* radiation.

- In the next section, we will also keep the second-order terms, which correspond to magnetic dipole and electric quadrupole radiation.

- Retarded potentials:

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int dV'\,\frac{\rho(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} \tag{1283}$$
$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int dV'\,\frac{\mathbf{J}(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} \tag{1284}$$

- In the radiation zone, one has
$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \cong \frac{1}{r} \tag{1285}$$
so that
$$\Phi(\mathbf{r},t) \cong \frac{1}{4\pi\epsilon_0 r}\int dV'\,\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c) \tag{1286}$$
$$\mathbf{A}(\mathbf{r},t) \cong \frac{\mu_0}{4\pi r}\int dV'\,\mathbf{J}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c) \tag{1287}$$

- To evaluate the integrals, we need to Taylor expand the source terms. First, we write
$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2+r'^2-2\mathbf{r}\cdot\mathbf{r}'} = r\left[1+\frac{r'^2}{r^2}-2\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r}\right]^{1/2} \cong r-\hat{\mathbf{r}}\cdot\mathbf{r}' \tag{1288}$$
where we have ignored terms of order $O(1/r)$ and higher. This implies
$$t-\frac{|\mathbf{r}-\mathbf{r}'|}{c} \cong t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c} \equiv t_0+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c} \tag{1289}$$
where
$$t_0 \equiv t-\frac{r}{c} \quad \text{(retarded time at origin)} \tag{1290}$$
Thus,
$$\rho\left(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c\right) = \rho\left(\mathbf{r}',t_0\right)+\dot{\rho}\left(\mathbf{r}',t_0\right)\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}+\frac{1}{2!}\ddot{\rho}\left(\mathbf{r}',t_0\right)\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)^2+\cdots \tag{1291}$$
and similarly for $\mathbf{J}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)$.

- For an oscillatory source of frequency ω,
$$\dot{\rho} \simeq \omega\rho\,, \quad \ddot{\rho} \simeq \omega^2\rho\,, \quad \text{etc.} \tag{1292}$$
implying
$$\rho\left(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c\right) \simeq \rho\left(\mathbf{r}',t_0\right)\left[1+\left(\frac{\omega d}{c}\right)+\frac{1}{2!}\left(\frac{\omega d}{c}\right)^2+\cdots\right] \tag{1293}$$
and similarly for $\mathbf{J}$. Here we've made the substitution $\hat{\mathbf{r}}\cdot\mathbf{r}' \simeq d$. Thus, the correction terms correspond to increasing powers of d/λ, where $\lambda = 2\pi c/\omega \simeq c/\omega$.

- In the long-wavelength approximation, we keep only the first non-trivial terms in the expansion for ρ and $\mathbf{J}$.

- For $\Phi(\mathbf{r},t)$ we need
$$\rho\left(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c\right) \cong \rho\left(\mathbf{r}',t_0\right)+\dot{\rho}\left(\mathbf{r}',t_0\right)\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c} \tag{1294}$$
which leads to
$$\Phi(\mathbf{r},t) \cong \frac{1}{4\pi\epsilon_0 r}Q(t_0)+\frac{1}{4\pi\epsilon_0 rc}\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0) \tag{1295}$$
where
$$Q(t_0) \equiv \int dV'\,\rho(\mathbf{r}',t_0)\,, \quad \mathbf{p}(t_0) \equiv \int dV'\,\mathbf{r}'\rho(\mathbf{r}',t_0) \tag{1296}$$

- Note that the first term of Φ is not radiative since the continuity equation implies conservation of electric charge:
$$\frac{\partial\rho}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{J} \quad \Rightarrow \quad Q(t_0) \equiv Q = \text{const} \tag{1297}$$
This means that the *electric monopole term does not contribute to the radiation.*

- For $\mathbf{A}(\mathbf{r}, t)$ we need just

$$\mathbf{J}\left(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c\right) \cong \mathbf{J}\left(\mathbf{r}', t_0\right) \tag{1298}$$

which leads to

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r}\,\dot{\mathbf{p}}(t_0) \tag{1299}$$

- To obtain this last result, we used the continuity equation to show that

$$\int dV'\,\mathbf{J}(\mathbf{r}', t_0) = \dot{\mathbf{p}}(t_0) \tag{1300}$$

[Proof: Multiply both sides of the continuity equation by r_i and integrate over a volume containing the source.]

- Ignoring the electric monopole term, the above expressions for Φ and $\mathbf{A}$ agree with the coordinate-independent expressions that we found previously for electric dipole radiation. Thus, the radiative parts of the fields are given by

$$\mathbf{E}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r}\,\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0)) \tag{1301}$$

$$\mathbf{B}(\mathbf{r}, t) \cong -\frac{\mu_0}{4\pi r c}\,\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0) \tag{1302}$$

and Poynting's vector and the radiated power are

$$\mathbf{S}(\mathbf{r}, t) \cong \frac{\mu_0}{16\pi^2 c}\frac{|\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0)|^2}{r^2}\,\hat{\mathbf{r}} \tag{1303}$$

$$P(t) \cong \frac{\mu_0 |\ddot{\mathbf{p}}(t_0)|^2}{6\pi c} \tag{1304}$$

- Note that for a single point charge q, the electric dipole moment is $\mathbf{p}(t) = q\,\mathbf{d}(t)$, where $\mathbf{d}(t)$ is the position of the charge relative to some origin. Thus, the radiated power is

$$P(t) = \frac{\mu_0 q^2 |\mathbf{a}(t_0)|^2}{6\pi c} \tag{1305}$$

where $\mathbf{a}(t) = \ddot{\mathbf{d}}(t)$. This is *Larmor's formula* for the power radiated by a non-relavistic particle ($v \ll c$).

10.9 Electric quadrupole radiation

- In the previous section, we saw that keeping the 1st-order term in the Taylor expansion for $\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)$ and the 0th-order term for $\mathbf{J}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)$ correspond to *electric dipole* radiation from an arbitrary localized source.

- In this section, we keep the next higher-order terms in the Taylor expansions for ρ and $\mathbf{J}$. This leads to *magnetic dipole* and *electric quadrupole* radiation.

- Expanding

$$\rho\left(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c\right) \cong \rho\left(\mathbf{r}', t_0\right) + \dot{\rho}\left(\mathbf{r}', t_0\right)\frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c} + \ddot{\rho}\left(\mathbf{r}', t_0\right)\left(\frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right)^2 \tag{1306}$$

$$\mathbf{J}\left(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c\right) \cong \mathbf{J}\left(\mathbf{r}', t_0\right) + \dot{\mathbf{J}}\left(\mathbf{r}', t_0\right)\frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c} \tag{1307}$$

leads to

$$\Phi(\mathbf{r}, t) \cong \frac{1}{4\pi\epsilon_0 r}\left[Q + \frac{1}{c}\,\hat{\mathbf{r}} \cdot \dot{\mathbf{p}}(t_0) + \frac{1}{2c^2}\,\hat{r}_i \hat{r}_j\,\ddot{\mathcal{Q}}_{ij}(t_0)\right] \tag{1308}$$

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r}\left[\dot{\mathbf{p}}(t_0) - \frac{1}{c}\,\hat{\mathbf{r}} \times \dot{\mathbf{m}}(t_0) + \frac{1}{2c}\,\hat{\mathbf{r}} \cdot \ddot{\boldsymbol{\mathcal{Q}}}(t_0)\right] \tag{1309}$$

where Q and $\mathbf{p}(t_0)$ are the total electric charge and electric dipole moment as before, and

$$\mathbf{m}(t_0) \equiv \frac{1}{2}\int dV'\, \mathbf{r}' \times \mathbf{J}(\mathbf{r}', t_0) \tag{1310}$$

is the magnetic dipole moment, and

$$\boldsymbol{\mathcal{Q}}_{ij}(t_0) \equiv \int dV'\, \rho(\mathbf{r}', t_0)\, r_i' r_j' \tag{1311}$$

is the second moment of the electric charge density.

- Note that $\mathcal{Q}_{ij}$ is related to the (trace-free) electric quadrupole moment tensor

$$Q_{ij}(t_0) \equiv \int dV'\, \rho(\mathbf{r}', t_0)\left[3r_i' r_j' - (r')^2\,\delta_{ij}\right] \tag{1312}$$

via

$$Q_{ij} = 3\mathcal{Q}_{ij} - \delta_{ij}\,\mathcal{Q}_{kk} \tag{1313}$$

- To prove the above result for $\mathbf{A}(\mathbf{r}, t)$, we need to evaluate the integral

$$\left[\int dV'\, \dot{\mathbf{J}}(\mathbf{r}', t_0)\,\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right]_i = \hat{r}_j \frac{1}{c}\frac{d}{dt}\left[\int dV'\, J_i(\mathbf{r}', t_0)\, r_j'\right] \tag{1314}$$

- The integral on the RHS can written as the sum of a symmetric and anti-symmetric part:

$$\int dV'\, J_i\, r_j' = \frac{1}{2}\int dV'\, \left[J_i\, r_j' + J_j\, r_i'\right] + \frac{1}{2}\int dV'\, \left[J_i\, r_j' - J_j\, r_i'\right] \tag{1315}$$

- The anti-symmetric part is just

$$\frac{1}{2}\int dV'\, \left[J_i(\mathbf{r}', t_0)\, r_j' - J_j(\mathbf{r}', t_0)\, r_i'\right] = -\epsilon_{ijk} m_k(t_0) \tag{1316}$$

while the symmetric part is

$$\frac{1}{2}\int dV'\, \left[J_i(\mathbf{r}', t_0)\, r_j' + J_j(\mathbf{r}', t_0)\, r_i'\right] = \frac{1}{2}\dot{\mathcal{Q}}_{ij}(t_0) \tag{1317}$$

[Proof of the symmetric part: Multiply the continuity equation by $r_i r_j$ and integrate over a volume containing the source.]

- Radiative components of the fields:

$$\mathbf{E}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r}\left[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0)) + \frac{1}{c}\hat{\mathbf{r}} \times \ddot{\mathbf{m}}(t_0) + \frac{1}{6c}\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times [\hat{\mathbf{r}} \cdot \dddot{\mathbf{Q}}(t_0)]\right)\right] \tag{1318}$$

$$\mathbf{B}(\mathbf{r}, t) \cong -\frac{\mu_0}{4\pi r c}\left[\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0) - \frac{1}{c}\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{m}}(t_0)) + \frac{1}{6c}\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times [\hat{\mathbf{r}} \cdot \dddot{\mathbf{Q}}(t_0)]\right)\right] \tag{1319}$$

where we used

$$\hat{\mathbf{r}} \times [\hat{\mathbf{r}} \cdot \dddot{\boldsymbol{\mathcal{Q}}}(t_0)] = \frac{1}{3}\hat{\mathbf{r}} \times [\hat{\mathbf{r}} \cdot \dddot{\mathbf{Q}}(t_0)] \tag{1320}$$

to write the above expressions in terms of $\mathbf{Q}$ instead of $\boldsymbol{\mathcal{Q}}$.

- As usual,

$$\mathbf{B} \cong \frac{1}{c}\hat{\mathbf{r}} \times \mathbf{E} \quad \text{and} \quad \mathbf{S} \cong \frac{1}{\mu_0 c}E^2\,\hat{\mathbf{r}} \tag{1321}$$

- Note that both $\mathbf{E}$ and $\mathbf{B}$ are the sum of three parts: an electric dipole piece (involving $\ddot{\mathbf{p}}(t_0)$), an magnetic dipole piece (involving $\ddot{\mathbf{m}}(t_0)$), and an electric quadrupole piece (involving $\dddot{\mathbf{Q}}(t_0)$).

- To calculate the total radiated power P, we need to integrate the normal component of $\mathbf{S}$ over a 2-sphere of radius $r \to \infty$. Somewhat remarkably, there are no-cross between the three components when calculating P. Using

$$\oint d\Omega\,\hat{r}_i = 0\,, \quad \oint d\Omega\,\hat{r}_i\hat{r}_j\hat{r}_k = 0 \tag{1322}$$

and

$$\oint d\Omega\,\hat{r}_i\hat{r}_j = \frac{4\pi}{3}\,\delta_{ij}\,, \quad \oint d\Omega\,\hat{r}_i\hat{r}_j\hat{r}_k\hat{r}_l = \frac{4\pi}{15}\,(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{1323}$$

one can show that

$$P \cong P_{\text{electric dipole}} + P_{\text{magnetic dipole}} + P_{\text{electric quadrupole}} \tag{1324}$$

$$= \frac{\mu_0}{6\pi c}\left(|\ddot{\mathbf{p}}(t_0)|^2 + \frac{1}{c^2}|\ddot{\mathbf{m}}(t_0)|^2 + \frac{1}{120c^2}\dddot{Q}_{ij}(t_0)\dddot{Q}_{ij}(t_0)\right) \tag{1325}$$

- Example: Suppose a system has a cylindrically-symmetric electric quadrupole moment

$$Q_{ij}(t) = Q_0(t)\begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1326}$$

Then

$$\frac{dP}{d\Omega} = \frac{\mu_0\dddot{Q}_0^{\,2}(t_0)}{256\pi^2c^3}\sin^2\theta\,\cos^2\theta\,, \quad P = \frac{\mu_0\dddot{Q}_0^{\,2}(t_0)}{480\pi c^3} \tag{1327}$$

where θ is the angle measured with respect to the z-axis (i.e., the axis of symmetry).

- Example: (Prob 11.11, Griffiths)

Consider two oppositely directed electric dipoles

$$\mathbf{p}_\pm(t) = \pm p_0\,\cos(\omega t)\,\hat{\mathbf{z}} \tag{1328}$$

located at $z = \pm d/2$. Then

$$\Phi(\mathbf{r},t) \cong -\frac{\mu_0 p_0\omega^2 d}{4\pi\epsilon_0 r}\cos^2\theta\,\cos(\omega t_0) \tag{1329}$$

$$\mathbf{A}(\mathbf{r},t) \cong -\frac{\mu_0 p_0\omega^2 d}{4\pi rc}\cos\theta\,\cos(\omega t_0)\,\hat{\mathbf{z}} \tag{1330}$$

$$\mathbf{E}(\mathbf{r},t) \cong \frac{\mu_0 p_0\omega^3 d}{4\pi rc}\sin\theta\cos\theta\,\sin(\omega t_0)\,\hat{\boldsymbol{\theta}} \tag{1331}$$

$$\mathbf{B}(\mathbf{r},t) \cong \frac{\mu_0 p_0\omega^3 d}{4\pi rc^2}\sin\theta\cos\theta\,\sin(\omega t_0)\,\hat{\boldsymbol{\phi}} \tag{1332}$$

$$\mathbf{S}(\mathbf{r},t) \cong \frac{\mu_0 p_0^2\omega^6 d^2}{16\pi^2r^2c^3}\sin^2\theta\cos^2\theta\,\sin^2(\omega t_0)\,\hat{\mathbf{r}} \tag{1333}$$

with time-averaged power

$$\langle P\rangle \cong \frac{\mu_0 p_0^2\omega^6 d^2}{60\pi c^3} \tag{1334}$$

Note the ω^6 dependence for electric quadrupole radiation, compared to ω^4 for electric and magnetic dipole radiation.