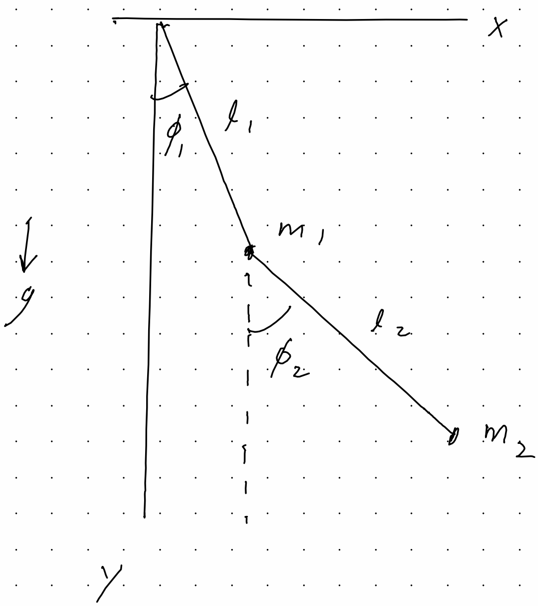
sec 5, prob 1



$x_1 = l_1 \sin\phi_1$

$y_1 = l_1 \cos\phi_1$

$x_2 = x_1 + l_2 \sin\phi_2 = l_1 \sin\phi_1 + l_2 \sin\phi_2$

$y_2 = y_1 + l_2 \cos\phi_2 = l_1 \cos\phi_1 + l_2 \cos\phi_2$

$U = -m_1 g y_1 - m_2 g y_2$

$= -m_1 g l_1 \cos\phi_1 - m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$

$= -(m_1 + m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$

$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$

$\dot{x}_1 = l_1 \cos\phi_1 \dot{\phi}_1$ $\qquad$ $\dot{y}_1 = -l_1 \sin\phi_1 \dot{\phi}_1$

$\dot{x}_1^2 = l_1^2 \cos^2\phi_1 \dot{\phi}_1^2$ $\qquad$ $\dot{y}_1^2 = l_1^2 \sin^2\phi_1 \dot{\phi}_1^2$

Thus, $\dot{x}_1^2 + \dot{y}_1^2 = l_1^2(\sin^2\phi_1 + \cos^2\phi_1)\dot{\phi}_1^2$

$$= l_1^2\dot{\phi}_1^2$$

$$\dot{x}_2 = l_1\cos\phi_1\dot{\phi}_1 + l_2\cos\phi_2\dot{\phi}_2$$

$$\rightarrow \dot{x}_2^2 = l_1^2\cos^2\phi_1\dot{\phi}_1^2 + l_2^2\cos^2\phi_2\dot{\phi}_2^2 + 2l_1l_2\cos\phi_1\cos\phi_2\dot{\phi}_1\dot{\phi}_2$$

$$\dot{y}_2 = -l_1\sin\phi_1\dot{\phi}_1 - l_2\sin\phi_2\dot{\phi}_2$$

$$\rightarrow \dot{y}_2^2 = l_1^2\sin^2\phi_1\dot{\phi}_1^2 + l_2^2\sin^2\phi_2\dot{\phi}_2^2 + 2l_1l_2\sin\phi_1\sin\phi_2\dot{\phi}_1\dot{\phi}_2$$

Thus,

$$\dot{x}_2^2 + \dot{y}_2^2 = l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2l_1l_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2)\dot{\phi}_1\dot{\phi}_2$$

$$= l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$

so $T = \frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2)$

$$= \frac{1}{2}m_1l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2$$

$$+ m_2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$

$$= \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2 + m_2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$

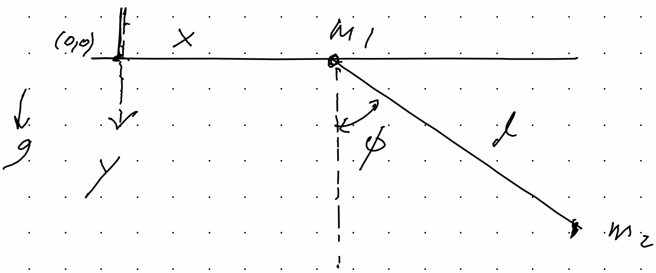
$$U = -(m_1+m_2)gl_1\cos\phi_1 - m_2gl_2\cos\phi_2$$

$$L = T - U$$

$$= \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2 + m_2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$

$$+ (m_1+m_2)gl_1\cos\phi_1 + m_2gl_2\cos\phi_2$$

Sec 5 Prob. 2



Generalised coords: x, ϕ

$$(x_1, y_1) = (x, 0)$$
$$(x_2, y_2) = (x + l\sin\phi,\ l\cos\phi)$$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_2 g l \cos\phi$$
$$T = \frac{1}{2} m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

Now: $\dot{x}_1^2 + \dot{y}_1^2 = \dot{x}^2$

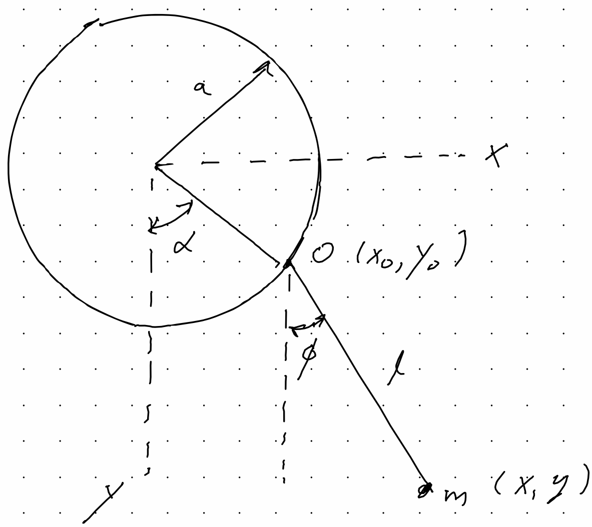
$$\dot{x}_2^2 + \dot{y}_2^2 = (\dot{x} + l\cos\phi\dot{\phi})^2 + (-l\sin\phi\dot{\phi})^2$$
$$= \dot{x}^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi} + l^2\sin^2\phi\,\dot{\phi}$$
$$= \dot{x}^2 + l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi}$$

$$\mapsto T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi})$$
$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2 l\cos\phi\,\dot{x}\dot{\phi}$$

$$L = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2 l\cos\phi\,\dot{x}\dot{\phi} + m_2 g l\cos\phi$$

Sec 5, Prob 3

(a)



point of support O moves along circle:

$x_0 = a \sin\alpha \quad , \quad y_0 = a\cos\alpha$

where $\alpha = \gamma t \quad , \quad \gamma = \text{const}$

Pendulum bob:

$(x,y): \quad x = x_0 + l\sin\phi$

$\qquad\qquad y = y_0 + l\cos\phi$

$U = -mgy = \underbrace{-mgy_0}_{\text{specified function of time}} - mgl\cos\phi$

[can ignore in L]

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$\dot{x} = \dot{x}_0 + l\cos\phi\,\dot{\phi}$

$\dot{x}^2 = \dot{x}_0^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}_0\dot{\phi}$

$$\dot{y} = \dot{y}_0 - l\sin\phi\,\dot{\phi}$$

$$\dot{y}^2 = \dot{y}_0^2 + l^2\sin^2\phi\,\dot{\phi}^2 - 2l\sin\phi\,\dot{y}_0\,\dot{\phi}$$

Thus,

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

$$= \frac{1}{2}m\left(\dot{x}_0^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}_0\,\dot{\phi} + \dot{y}_0^2 + l^2\sin^2\phi\,\dot{\phi}^2 - 2l\sin\phi\,\dot{y}_0\,\dot{\phi}\right)$$

$$= \frac{1}{2}m\left(\dot{x}_0^2+\dot{y}_0^2\right) + \frac{1}{2}ml^2\dot{\phi}^2 + ml\dot{\phi}\left(\dot{x}_0\cos\phi - \dot{y}_0\sin\phi\right)$$

NOTE: $\dot{x}_0^2 + \dot{y}_0^2 = a^2\dot{\alpha}^2 = a^2\gamma^2$

since this is a specified function of time, we can ignore it in the Lagrangian;

Thus, $$L = \frac{1}{2}ml^2\dot{\phi}^2 + ml\dot{\phi}\left(\dot{x}_0\cos\phi - \dot{y}_0\sin\phi\right) + mgl\cos\phi$$

We can rewrite the second term:

$$x_0 = a\sin\alpha \rightarrow \dot{x}_0 = a\cos\alpha\,\dot{\alpha} \qquad (\dot{\alpha} \equiv \gamma)$$

$$y_0 = a\cos\alpha \rightarrow \dot{y}_0 = -a\sin\alpha\,\dot{\alpha}$$

Thus,

$$ml\dot{\phi}(\dot{x}_0\cos\phi - \dot{y}_0\sin\phi) = ml\dot{\phi}a\gamma(\cos\alpha\cos\phi + \sin\alpha\sin\phi)$$

$$= ml\dot{\phi}a\gamma\cos(\phi-\alpha)$$

$$= ml\dot{\phi}a\gamma\cos(\phi-\gamma t)$$

Now; $\frac{d}{dt}\left[m l a \gamma \sin(\phi - \gamma t) \right]$

$$= m l a \gamma \cos(\phi - \gamma t)(\dot\phi - \gamma)$$
$$= m l a \dot\phi \gamma \cos(\phi - \gamma t) - m l a \gamma^2 \cos(\phi - \gamma t)$$

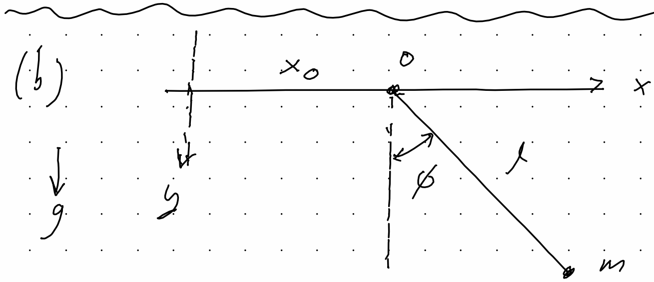
Thus,

$$m l a \dot\phi \gamma \cos(\phi - \gamma t) = \frac{d}{dt}\left[m l a \gamma \sin(\phi - \gamma t) \right] + m l a \gamma^2 \cos(\phi - \gamma t)$$

(and we can ignore the total time derivative in the Lagrangian)

$$\rightarrow L = \frac{1}{2} m l^2 \dot\phi^2 + m g l \cos\phi + m l a \gamma^2 \cos(\phi - \gamma t)$$

(b)



point O moving according to $x_0 = a \cos \gamma t$

$$x = x_0 + l \sin\phi$$
$$y = l \cos\phi$$
$$U = -mgy = -mgl\cos\phi$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{x}_0 + l \cos\phi \, \dot{\phi} \quad , \qquad x_0 = a \cos \gamma t$$

$$= -a \sin(\gamma t) \gamma + l \cos\phi \, \dot{\phi}$$

$$\rightarrow \dot{x}^2 = a^2 \gamma^2 \sin^2 \gamma t + l^2 \cos^2\phi \, \dot{\phi}^2$$

$$- 2 a l \gamma \dot{\phi} \sin(\gamma t) \cos\phi$$

$$\dot{y} = -l \sin\phi \, \dot{\phi}$$

$$\rightarrow \dot{y}^2 = l^2 \sin^2\phi \, \dot{\phi}^2$$

Thus, $T = \frac{1}{2} m \big(a^2 \gamma^2 \sin^2 \gamma t + l^2 \cos^2\phi \, \dot{\phi}^2$

$$- 2 a l \gamma \dot{\phi} \sin(\gamma t) \cos\phi + l^2 \sin^2\phi \, \dot{\phi}^2 \big)$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + \frac{1}{2} m a^2 \gamma^2 \sin^2 \gamma t$$

specified function of time (ignore)

$$- m a l \gamma \dot{\phi} \sin(\gamma t) \cos\phi$$

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 - m a l \gamma \dot{\phi} \sin(\gamma t) \cos\phi + m g l \cos\phi$$

2nd term:

$$-\frac{d}{dt}\left[m a l \gamma \sin(\gamma t) \sin\phi \right]$$

$$= - m a l \gamma^2 \cos(\gamma t) \sin\phi - m a l \gamma \dot{\phi} \sin(\gamma t) \cos\phi$$

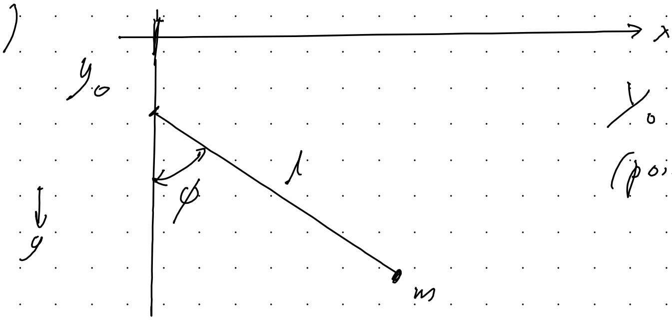
so

$$- m a l \gamma \dot{\phi} \sin(\gamma t) \cos\phi = \underbrace{-\frac{d}{dt}[\;]}_{\text{ignore}} + m a l \gamma^2 \cos(\gamma t) \sin\phi$$

Thus, ignoring total time derivatives,

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi + mal\gamma^2 \cos(\gamma t)\sin\phi$$

(c)



$y_0 = a\cos\gamma t$

(point of support)

$$x = l\sin\phi$$

$$y = y_0 + l\cos\phi$$

$$= a\cos\gamma t + l\cos\phi$$

$$U = -mgy$$

$$= \underbrace{-mga\cos\gamma t}_{} - mgl\cos\phi$$

specified function of time [can ignore]

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = l\cos\phi\,\dot{\phi}$$

$$\dot{x}^2 = l^2\cos^2\phi\,\dot{\phi}^2$$

$$\dot{y} = -a\gamma\sin(\gamma t) - l\sin\phi\,\dot{\phi}$$

$$\dot{y}^2 = \underbrace{a^2\gamma^2\sin^2(\gamma t)}_{} + l^2\sin^2\phi\,\dot{\phi}^2 + 2al\gamma\sin(\gamma t)\sin\phi\,\dot{\phi}$$

specified function of time [can ignore]

Thus, ignoring this function of time

$$T = \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \dot{\phi} \sin(\gamma t) \sin\phi$$

$$\rightarrow L = T - U$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \gamma \dot{\phi} \sin(\gamma t) \sin\phi + m g l \cos\phi$$

Rewrite 2nd term:

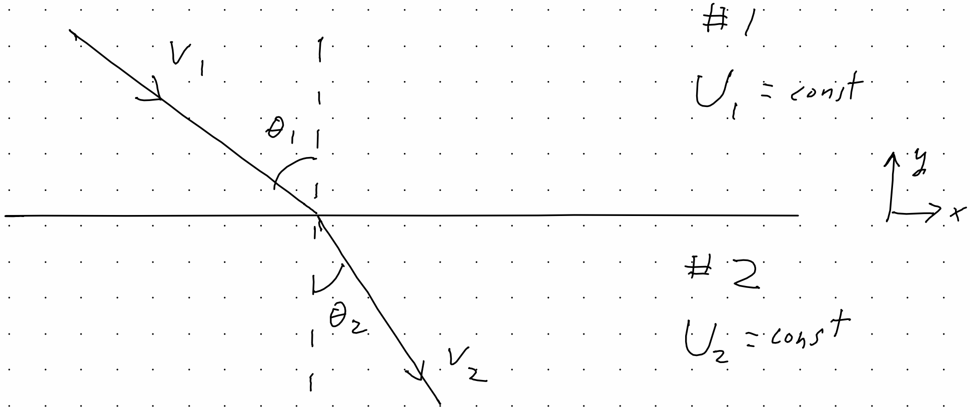
$$-\frac{d}{dt}\left[m a l \gamma \sin(\gamma t) \cos\phi \right] = -m a l \gamma^2 \cos(\gamma t) \cos\phi$$
$$+ m a l \gamma \sin(\gamma t) \sin\phi \, \dot{\phi}$$

so $m a l \gamma \dot{\phi} \sin(\gamma t) \sin\phi = -\frac{d}{dt}[\ ] + m a l \gamma^2 \cos(\gamma t) \cos\phi$

thus, ignoring total time derivative

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \cos\phi + m a l \gamma^2 \cos(\gamma t) \cos\phi$$

Sec 7, Prob 1



- Energy conserved, since no time dependence.
- Also momentum in x-direction (// to interface) is conserved, since no x-dependence of the potential

$$U(x,y) = \begin{cases} U_1 & y \geq 0 \\ U_2 & y < 0 \end{cases}$$

v_1 : given

$$E = \frac{1}{2} m v_1^2 + U_1 = \frac{1}{2} m v_2^2 + U_2$$

$$\rightarrow \quad \frac{1}{2} m v_2^2 = \frac{1}{2} m v_1^2 + (U_1 - U_2)$$

$$v_2^2 = v_1^2 + \frac{2(U_1 - U_2)}{m}$$

so, $$v_2 = v_1 \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

The angles θ_1, θ_2 are related by

$$p_{1x} = p_{2x}$$

$$\cancel{m} V_1 \sin\theta_1 = \cancel{m} V_2 \sin\theta_2$$

thus,

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

Sec 8, Prob 1

Transformation of action $S = \int L \, dt$

K, K': two inertial frames

K' moves with velocity $\vec{V}$ wrt K

Assume that K, K' coincide at $t=0$ so $\vec{r}_a = \vec{r}_a{}'$ wrt these two frames

Now: $\vec{v}_a = \vec{V} + \vec{v}_a{}'$

$$L = T - U$$
$$= \sum_a \frac{1}{2} m_a |\vec{v}_a|^2 - U(\vec{r}_1, \vec{r}_2, \ldots, t)$$

$$|\vec{v}_a|^2 = |\vec{V} + \vec{v}_a{}'|^2$$
$$= |\vec{V}|^2 + |\vec{v}_a{}'|^2 + 2\vec{V} \cdot \vec{v}_a{}'$$

so

$$L = \sum_a \frac{1}{2} m_a \left(|\vec{V}|^2 + |\vec{v}_a{}'|^2 + 2\vec{V} \cdot \vec{v}_a{}'\right) - U$$
$$= \frac{1}{2}\mu V^2 + T' + \vec{V} \cdot \sum_a m_a \vec{v}_a{}' - U$$

$$= T' - U + \frac{1}{2}\mu V^2 + \vec{P}' \cdot \vec{V}$$
$$= L' + \frac{1}{2}\mu V^2 + \vec{P}' \cdot \vec{V}$$

where $\vec{P}'$ = total momentum wrt K'

$\mu = \sum_a m_a$ = total mass

$$S = \int_{t_1}^{t_2} L\,dt$$

$$= \int_{t_1}^{t_2} \left(L' + \tfrac{1}{2}\mu V^2 + \vec{p}\,' \cdot \vec{V}\right) dt$$

$$= S' + \underbrace{\tfrac{1}{2}\mu V^2 (t_2 - t_1)}_{\text{doesn't change EOMs}} + \vec{V} \cdot \int_{t_1}^{t_2} \vec{p}\,' dt$$

Now:

$$\vec{V} \cdot \int_{t_1}^{t_2} \vec{p}\,' dt = \vec{V} \cdot \int_{t_1}^{t_2} \sum_a m_a \vec{v}_a\,' dt$$

$$= \vec{V} \cdot \sum_a m_a \int_{t_1}^{t_2} \left(\frac{d\vec{r}_a}{dt}\right) dt$$

$$= \vec{V} \cdot \sum_a m_a \vec{r}_a \Big|_{t_1}^{t_2}$$

$$= \vec{V} \cdot \left(\mu \vec{R}(t_2) - \mu \vec{R}(t_1)\right)$$

$$= \mu \vec{V} \cdot \Big(\underbrace{\vec{R}(t_2) - \vec{R}(t_1)}_{\text{difference in COM positions}} \Big)$$

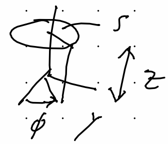
So:

$$S = S' + \tfrac{1}{2}\mu V^2 (t_2 - t_1) + \mu \vec{V} \cdot \left(\vec{R}(t_2) - \vec{R}(t_1)\right)$$

Sec 9, Prob 1

cylindrical coordinates (s, ϕ, z)

$$s^2 = x^2 + y^2$$



$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = z$$

$$\vec{M} = \vec{r} \times \vec{p} = m\vec{r} \times \dot{\vec{r}}$$

Now: $M_x = m(y\dot{z} - z\dot{y})$

$$M_y = m(z\dot{x} - x\dot{z})$$
$$M_z = m(x\dot{y} - y\dot{x})$$
$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}$$

$$\dot{z} = \dot{z}$$
$$\dot{y} = \dot{s}\sin\phi + s\cos\phi\,\dot{\phi}$$
$$\dot{x} = \dot{s}\cos\phi - s\sin\phi\,\dot{\phi}$$

Thus, $M_x = m\left(s\sin\phi\,\dot{z} - z(\dot{s}\sin\phi + s\cos\phi\,\dot{\phi})\right)$

$$= m(s\sin\phi\,\dot{z} - z\sin\phi\,\dot{s} - z s\cos\phi\,\dot{\phi})$$

$$M_y = m\left(z(\dot{s}\cos\phi - s\sin\phi\,\dot{\phi}) - s\cos\phi\,\dot{z}\right)$$

$$= m(z\cos\phi\,\dot{s} - z s\sin\phi\,\dot{\phi} - s\cos\phi\,\dot{z})$$

$$M_z = m\left[s\cos\phi\,(\cancel{\dot{s}\sin\phi} + s\cos\phi\,\dot{\phi}) - s\sin\phi\,(\cancel{\dot{s}\cos\phi} - s\sin\phi\,\dot{\phi})\right]$$

$$= ms^2\dot{\phi}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$= m^2 \Big\{ \left(\sin\phi \,(s\dot{z} - z\dot{s}) - zs\cos\phi\,\dot{\phi} \right)^2$$

$$+ \left(-\cos\phi\,(s\dot{z} - z\dot{s}) - zs\sin\phi\,\dot{\phi} \right)^2$$

$$+ \left(s^2 \dot{\phi} \right)^2 \Big\}$$

$$= m^2 \Big\{ \sin^2\phi\,(s\dot{z} - z\dot{s})^2 + z^2 s^2 \cos^2\phi\,\dot{\phi}^2$$

$$\cancel{- 2zs\sin\phi\cos\phi\,\dot{\phi}\,(s\dot{z} - z\dot{s})}$$

$$+ \cos^2\phi\,(s\dot{z} - z\dot{s})^2 + z^2 s^2 \sin^2\phi\,\dot{\phi}^2$$

$$\cancel{+ 2zs\sin\phi\cos\phi\,\dot{\phi}\,(s\dot{z} - z\dot{s})}$$

$$+ s^4 \dot{\phi}^2 \Big\}$$

$$= m^2 \left\{ (s\dot{z} - z\dot{s})^2 + z^2 s^2 \dot{\phi}^2 + s^4 \dot{\phi}^2 \right\}$$

$$= m^2 \left[(s\dot{z} - z\dot{s})^2 + s^2 \dot{\phi}^2 (z^2 + s^2) \right]$$

Sec 9, Prob 2

repeat for spherical polar coords.

$$M_x = m(y\dot{z} - z\dot{y}) \quad , \quad \text{cyclic}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

Now: $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$\rightarrow \dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi}$$

$$\dot{y} = \dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi}$$

$$\dot{z} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

Thus,

$$M_x = m(y\dot{z} - z\dot{y})$$

$$= m\Big\{ r\sin\theta\sin\phi\,(\cancel{\dot{r}\cos\theta} - r\sin\theta\,\dot{\theta})$$

$$- r\cos\theta\,(\cancel{\dot{r}\sin\theta\sin\phi} + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi})\Big\}$$

$$= m\Big\{ -r^2\sin^2\theta\sin\phi\,\dot{\theta} - r^2\cos^2\theta\sin\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\cos\phi\,\dot{\phi}\Big\}$$

$$= m\Big\{ -r^2\sin\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\cos\phi\,\dot{\phi}\Big\}$$

$$= -mr^2\left[\sin\phi\,\dot{\theta} + \sin\theta\cos\theta\cos\phi\,\dot{\phi}\right]$$

$$
\begin{aligned}
M_y &= m(z\dot{x} - x\dot{z}) \\
&= m\Big\{ r\cos\theta \left(\cancel{\dot{r}\sin\theta\cos\phi} + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi} \right) \\
&\qquad - r\sin\theta\cos\phi \left(\cancel{\dot{r}\cos\theta} - r\sin\theta\,\dot{\theta} \right) \Big\} \\
&= m\Big\{ r^2\cos^2\theta\cos\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\sin\phi\,\dot{\phi} \\
&\qquad + r^2\sin^2\theta\cos\phi\,\dot{\theta} \Big\} \\
&= m\left[r^2\cos\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\sin\phi\,\dot{\phi} \right] \\
&= mr^2\left[\cos\phi\,\dot{\theta} - \sin\theta\cos\theta\sin\phi\,\dot{\phi} \right]
\end{aligned}
$$

$$
\begin{aligned}
M_z &= m(x\dot{y} - y\dot{x}) \\
&= m\Big\{ r\sin\theta\cos\phi \left(\dot{r}\,\cancel{\sin\theta\sin\phi} + \cancel{r\cos\theta\sin\phi\,\dot{\theta}} + r\sin\theta\cos\phi\,\dot{\phi} \right) \\
&\qquad - r\sin\theta\sin\phi \left(\dot{r}\,\cancel{\sin\theta\cos\phi} + \cancel{r\cos\theta\cos\phi\,\dot{\theta}} - r\sin\theta\sin\phi\,\dot{\phi} \right) \Big\} \\
&= m\left[r^2\sin^2\theta\cos^2\phi\,\dot{\phi} + r^2\sin^2\theta\sin^2\phi\,\dot{\phi} \right] \\
&= mr^2\sin^2\theta\,\dot{\phi}
\end{aligned}
$$

$$
\begin{aligned}
M^2 &= M_x^2 + M_y^2 + M_z^2 \\
&= m^2r^4\left[\sin\phi\,\dot{\theta} + \sin\theta\cos\theta\cos\phi\,\dot{\phi} \right]^2 \\
&\quad + m^2r^4\left[\cos\phi\,\dot{\theta} - \sin\theta\cos\theta\sin\phi\,\dot{\phi} \right]^2 \\
&\quad + m^2r^4\sin^4\theta\,\dot{\phi}^2
\end{aligned}
$$

cross terms will cancel

$$M^2 = m^2 r^4 \left\{ \sin^2\phi\, \dot\theta^2 + \sin^2\theta \cos^2\theta \cos^2\phi\, \dot\phi^2 \right.$$
$$+ \cos^2\phi\, \dot\theta^2 + \sin^2\theta \cos^2\theta \sin^2\phi\, \dot\phi^2$$
$$\left. + \sin^4\theta\, \dot\phi^2 \right\}$$

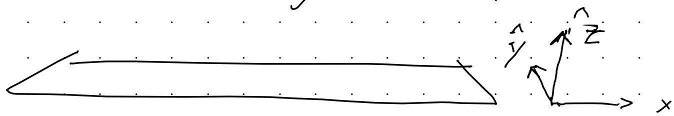
$$= m^2 r^4 \left[\dot\theta^2 + \sin^2\theta \cos^2\theta\, \dot\phi^2 + \sin^4\theta\, \dot\phi^2 \right]$$

$$= m^2 r^4 \left[\dot\theta^2 + \sin^2\theta\, \dot\phi^2 \left(\cos^2\theta + \sin^2\theta \right) \right]$$

$$= m^2 r^4 \left[\dot\theta^2 + \sin^2\theta\, \dot\phi^2 \right]$$

Sec 9, Prob 3

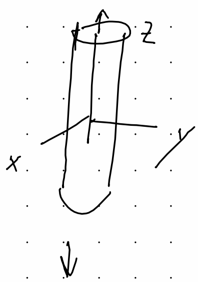
a) Infinite homogeneous plane



P_x, P_y conserved

M_z conserved, where origin is anywhere in (x,y) plane

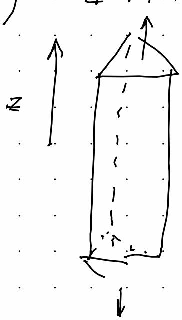
b) Infinite homogeneous cylinder



P_z conserved

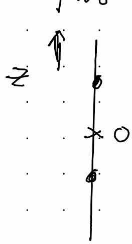
M_z conserved, origin anywhere on z-axis

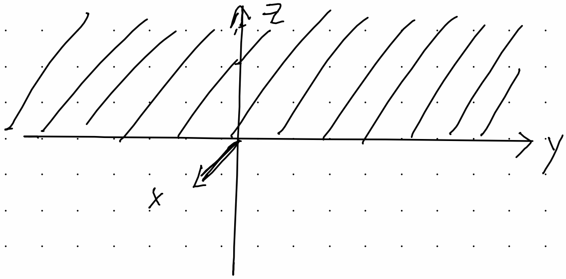
c) Infinite homog. prism



P_z conserved

d) two points : M_z conserved, origin at midpoint of line connecting the two points



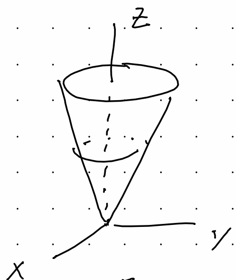
e) 

infinite, homog. half plane

$x = 0, \; z > 0$

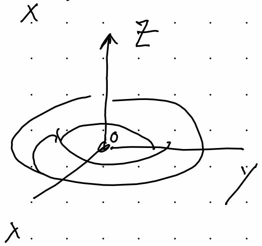
$-\infty < y < \infty$

p_y conserved

f) 

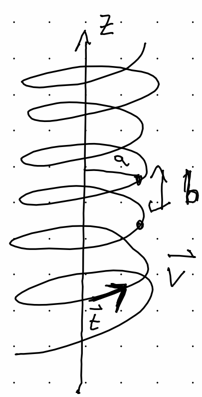
homogeneous cone

M_z conserved, origin anywhere on z-axis.

g) 

homogeneous circular torus

M_z conserved with $\hat{z}$ directed as shown and origin at O.

h) 

Field constant along helix:

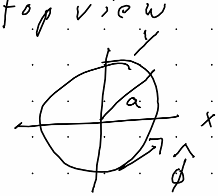
$\Delta z = b$ when $\Delta\phi = 2\pi$

at $s^2 = x^2 + y^2 = a^2$

$$\frac{\Delta\phi}{2\pi} = \frac{\Delta z}{b}$$

$$\rightarrow \quad \Delta z = \left(\frac{b}{2\pi}\right)\Delta\phi$$

top view



$$\vec{t} = a\Delta\phi\,\hat{\phi} + \Delta z\,\hat{z}$$

$$= \cancel{a}\Delta\phi\,\underbrace{\left(\frac{x\hat{y} - y\hat{x}}{\cancel{s}}\right)} + \Delta z\,\hat{z}$$

$$\vec{t} = \Delta\phi\,(x\hat{y} - y\hat{x}) + \left(\frac{b}{2\pi}\right)\Delta\phi\,\hat{z}$$

$$= \Delta\phi\left[x\hat{y} - y\hat{x} + \left(\frac{b}{2\pi}\right)\hat{z}\right]$$

Field unchanged if you move along $\vec{t}$.

thus, $\vec{P}\cdot\vec{t} = \text{const}$

$$\vec{P}\cdot\vec{t} = \Delta\phi\left[xP_y - yP_x + \left(\frac{b}{2\pi}\right)P_z\right]$$

$$= \Delta\phi\left[M_z + \frac{b}{2\pi}P_z\right]$$

so $M_z + \frac{b}{2\pi}P_z = \text{const}$

where z = axis of helix

$b = \Delta z$ for $\Delta\phi = 2\pi$ at $s = a$

Sec 10, prob 1

Different masses, same path, same potential energy

$$L_1 = \frac{1}{2} m_1 v_1^2 - U$$

$$L_2 = \frac{1}{2} m_2 v_2^2 - U$$

Thus, $m_1 v_1^2 = m_2 v_2^2$

$$\frac{m_1}{t_1^2} = \frac{m_2}{t_2^2}$$

$$\longrightarrow \left(\frac{t_2}{t_1}\right)^2 = \frac{m_2}{m_1}$$

or $$\boxed{\frac{t_2}{t_1} = \sqrt{\frac{m_2}{m_1}}}$$

Sec 10, Prob 2:

Same path, mass but potential energies differing by a constant

$$L_1 = \frac{1}{2} m v_1^2 - U_1$$
$$L_2 = \frac{1}{2} m v_2^2 - U_2$$

Thus, $$\frac{v_1^2}{v_2^2} = \frac{U_1}{U_2}$$

$$\rightarrow \frac{(l/t_1)^2}{(l/t_2)^2} = \frac{U_1}{U_2}$$

so $$\frac{t_2}{t_1} = \sqrt{\frac{U_1}{U_2}}$$

sec 40 - Prob 1

single particle in a constant external field

$$L = \frac{1}{2} m v^2 - U(\vec{r})$$

a) Cartesian coords (x, y, z)

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

$$\rightarrow \quad p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = p_x/m$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \rightarrow \dot{y} = p_y/m$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$H = \left(\sum_i p_i \dot{q}_i - L \right) \Big|_{\dot{q} = \dot{q}(q, p)}$$

$$= \left(p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(x, y, z) \right) \Big|_{\dot{x} = p_x/m \text{ etc}}$$

$$= p_x \left(\frac{p_x}{m} \right) + p_y \left(\frac{p_y}{m} \right) + p_z \left(\frac{p_z}{m} \right)$$

$$- \frac{1}{2} m \left(\left(\frac{p_x}{m} \right)^2 + \left(\frac{p_y}{m} \right)^2 + \left(\frac{p_z}{m} \right)^2 \right) + U(x, y, z)$$

$$= \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + U(x, y, z)$$

---

b) cylindrical coords (s, ϕ, z), $s^2 = x^2 + y^2$

$$L = \frac{1}{2} m (\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2) - U(s, \phi, z)$$

$$\rightarrow \quad p_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s} \rightarrow \dot{s} = p_s/m$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m s^2 \dot{\phi} \rightarrow \dot{\phi} = p_\phi / m s^2$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$H = \left(p_s \dot{s} + p_\phi \dot{\phi} + p_z \dot{z} - \frac{1}{2} m (\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2) + U(s, \phi, z)\right)\Big|$$

$\dot{s} = p_s/m$ etc

$$= p_s\left(\frac{p_s}{m}\right) + p_\phi\left(\frac{p_\phi}{ms^2}\right) + p_z\left(\frac{p_z}{m}\right)$$

$$- \frac{1}{2} m \left(\left(\frac{p_s}{m}\right)^2 + s^2\left(\frac{p_\phi}{ms^2}\right)^2 + \left(\frac{p_z}{m}\right)^2\right) + U(s, \phi, z)$$

$$= \frac{1}{2m}\left(p_s^2 + \frac{p_\phi^2}{s^2} + p_z^2\right) + U(s, \phi, z)$$

---

c) spherical polar coords (r, θ, ϕ)

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2) - U(r, \theta, \phi)$$

$$\leftrightarrow \quad p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \rightarrow \dot{r} = p_r/m$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \rightarrow \dot{\theta} = p_\theta / mr^2$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2\theta \, \dot{\phi} \rightarrow \dot{\phi} = p_\phi / mr^2 \sin^2\theta$$

$$H = \left(p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2) + U(r, \theta, \phi)\right)\Big|$$

$\dot{r} = p_r/m$ etc.

$$= p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{mr^2}\right) + p_\phi\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)$$

$$- \frac{1}{2}m\left(\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{mr^2}\right)^2 + r^2\sin^2\theta\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)^2\right) + U(r, \theta, \phi)$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r, \theta, \phi)$$

sec 40 - prob 2

$$L = \frac{1}{2}mv^2 + m\vec{v}\cdot(\vec{\Omega}\times\vec{r}) + \frac{1}{2}m|\vec{\Omega}\times\vec{r}|^2 - m\vec{W}\cdot\vec{r} - U$$

restrict to uniformly rotating frame of reference $\vec{W} = 0$, $\dot{\vec{\Omega}} = 0$

$$\rightarrow \quad L = \frac{1}{2}mv^2 + m\vec{v}\cdot(\vec{\Omega}\times\vec{r}) + \frac{1}{2}m|\vec{\Omega}\times\vec{r}|^2 - U(\vec{r})$$

Now: $H = \vec{p}\cdot\vec{v} - L$

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + m\vec{\Omega}\times\vec{r} = m(\vec{v} + \vec{\Omega}\times\vec{r})$$

$$\longrightarrow \quad \vec{v} = \frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}$$

Thus,

$$H = \vec{p}\cdot\left(\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right) - \frac{1}{2}m\left|\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right|^2 - m\left(\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right)\cdot(\vec{\Omega}\times\vec{r}) - \frac{1}{2}m|\vec{\Omega}\times\vec{r}|^2 + U(\vec{r})$$

$$= \frac{|\vec{p}|^2}{m} - \cancel{\vec{p}\cdot(\vec{\Omega}\times\vec{r})} - \frac{1}{2}m\left(\frac{|\vec{p}|^2}{m^2} + \cancel{|\vec{\Omega}\times\vec{r}|^2} - \cancel{\frac{2\vec{p}\cdot(\vec{\Omega}\times\vec{r})}{m}}\right) - \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + m\cancel{|\vec{\Omega}\times\vec{r}|^2} - \frac{1}{2}m\cancel{|\vec{\Omega}\times\vec{r}|^2} + U(\vec{r})$$

$$= \frac{|\vec{p}|^2}{2m} - \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + U(\vec{r})$$

$$= \frac{|\vec{p}|^2}{2m} - \vec{\Omega}\cdot\underbrace{(\vec{r}\times\vec{p})}_{\vec{M}\text{ : angular momentum}} + U(\vec{r})$$