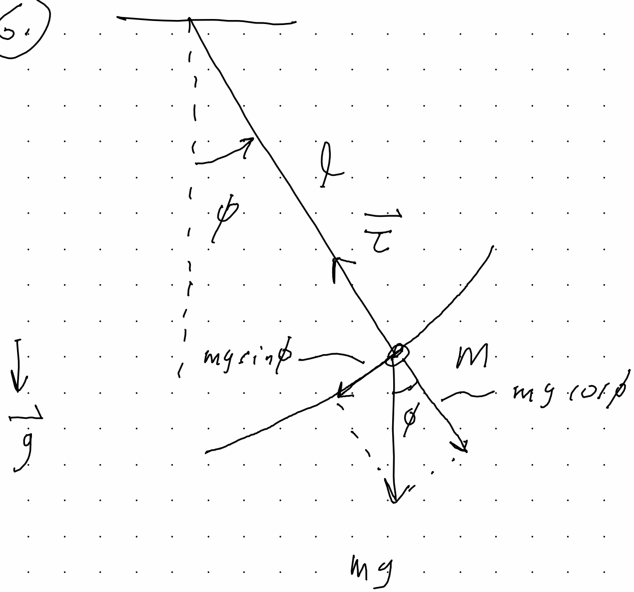
PHYS 5306 8/30

6.



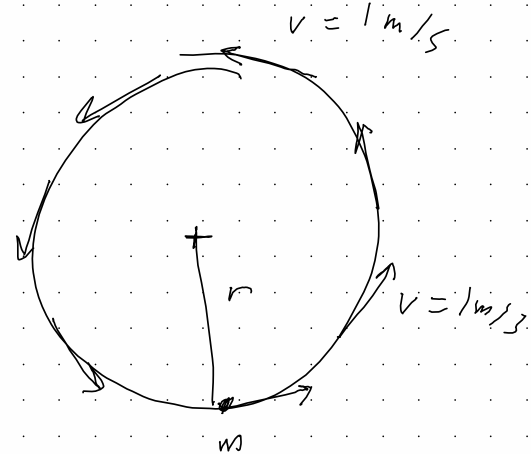
$\cancel{\tau = mg\cos\phi}$

$$\tau - mg\cos\phi = F_{centripetal}$$
$$= m\,a_{centripetal}$$
$$= m\ell\dot{\phi}^2$$

Centripetal force

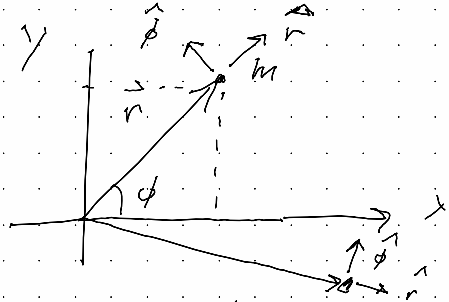
Centrifugal force

$$m\vec{a} = \vec{F}_{applied} + \vec{F}_{fictitious}$$



$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$$

$$a_{centripetal} = \frac{v^2}{r} = \omega^2 r$$



2-d motion:

2-d motion:

$$\vec r = x\hat x + y\hat y$$

$$\vec r = \vec r(t)$$

$$\vec r = r\hat r$$

$$\vec v = \dot{\vec r} = \dot x\hat x + \dot y\hat y$$

$$= \dot r\hat r + r\dot{\hat r}$$

$$\hat r = \cos\phi\,\hat x + \sin\phi\,\hat y$$

$$\hat\phi = \_\_\_\_\,\hat x \quad \_\_\_\_\,\hat y$$

$$\dot{\hat r}$$

$$\vec a = \ddot{\vec r} = (\quad)\hat r + (\quad)\hat\phi$$

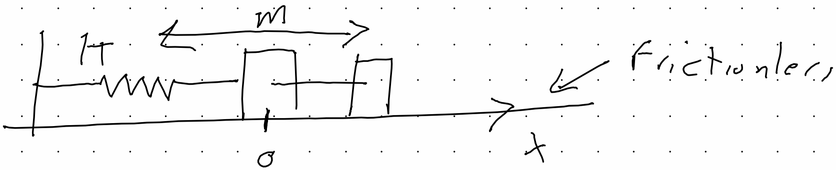
- Kinetic energy for a single mass m in Cartesian, sph. polar, and polar coordinates.

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

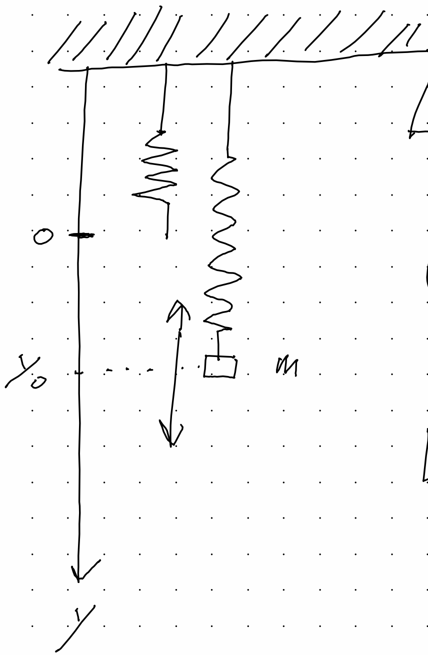
$$= \frac{1}{2} m \left(\qquad\qquad \right) \qquad (r, \theta, \phi) \text{ sph. polar}$$

$$= \frac{1}{2} m \left(\qquad\qquad \right) \qquad (\rho, \phi, z) \text{ cylindrical}$$

$x = r \sin\theta \cos\phi$, etc



$$U = \frac{1}{2} k x^2$$



$$L = T - U$$

$$T = \frac{1}{2} m \dot{y}^2$$

$$U = \underbrace{\frac{1}{2} k y^2}_{U_s} \quad \underbrace{- mgy}_{U_g}$$

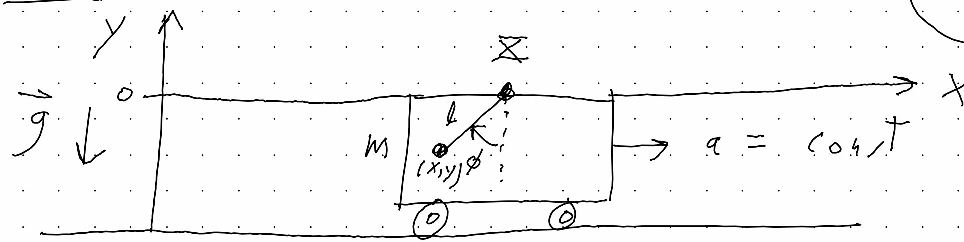
$$U_s + U_g$$

$$E = T + U$$

$$= \frac{1}{2} m \dot{y}^2 + \underbrace{\frac{1}{2} k y^2 - mgy}_{U_{eff}(y)}$$

plot this

9/1: QUIZ #1



$$X = \cancel{X_0}^{\,0} + \cancel{V_0}^{\,0} t + \frac{1}{2} a t^2$$

$$\dot{X} = at$$

$$\ddot{X} = a$$

$$L = \,? = T - U$$

$$x = X - l \sin\phi$$

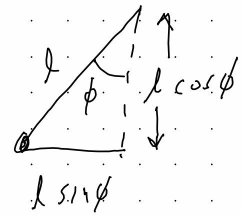
$$y = -l \cos\phi$$

$$U = mgy = -mgl\cos\phi$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{X} - l\dot{\phi}\cos\phi$$

$$\dot{y} = l\dot{\phi}\sin\phi$$



$$\vec{r} = x\hat{x} + y\hat{y}$$

$$\vec{g} = -g\hat{y}$$

$$\vec{F} = m\vec{g}$$

$$U = -m\vec{g}\cdot\vec{r}$$

$$\vec{F} = -\vec{\nabla} U = -\frac{\partial U}{\partial \vec{r}}$$

$$U = -m\vec{g}\cdot\vec{r} = mgy$$

$$\dot{x}^2 = (\dot{X} - l\dot{\phi}\cos\phi)^2$$

$$= \dot{X}^2 + l^2\dot{\phi}^2\cos^2\phi - 2l\dot{X}\dot{\phi}\cos\phi$$

$$\dot{y}^2 = l^2\dot{\phi}^2\sin^2\phi$$

$$\rightarrow \dot{x}^2 + \dot{y}^2 = \dot{X}^2 + l^2\dot{\phi}^2 - 2l\dot{X}\dot{\phi}\cos\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\dot{X}^2 + \frac{1}{2}ml^2\dot{\phi}^2 - ml\dot{X}\dot{\phi}\cos\phi$$

$$U = -mgl\cos\phi$$

$$\boxed{L = \frac{1}{2}m\dot{X}^2 + \frac{1}{2}ml^2\dot{\phi}^2 - ml\dot{X}\dot{\phi}\cos\phi + mgl\cos\phi}$$

$$L = L(\phi, \dot{\phi}, t) \longrightarrow L' = L + \frac{d}{dt}f(\phi, t)$$

$$L = \underbrace{\cancel{\tfrac{1}{2} m \dot{X}^2}}_{\text{prescribed function of time} \longrightarrow \text{ignore}} + \tfrac{1}{2} m l^2 \dot{\phi}^2 - m l \dot{X} \dot{\phi} \cos\phi + m g l \cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$$\dot{X} \dot{\phi} \cos\phi = \underbrace{\frac{d}{dt}\left(\dot{X} \sin\phi\right)}_{\ddot{X} \sin\phi + \dot{X} \dot{\phi} \cos\phi} - \ddot{X} \sin\phi$$

$$-m l \dot{X} \dot{\phi} \cos\phi = \underbrace{\frac{d}{dt}\left(-m l \dot{X} \dot{\phi} \cos\phi\right)}_{\text{ignore this in Lagrangian}} + m l \ddot{X} \sin\phi$$

$$\boxed{L = \tfrac{1}{2} m l^2 \dot{\phi}^2 + m l a \sin\phi + m g l \cos\phi}$$

i) show that both Lagrangians give the same EOMs

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi} \iff \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot\phi}\right) = \frac{\partial L'}{\partial \phi}$$

ii) $L = \frac{1}{2} m l^2 \dot\phi^2 + m a l \sin\phi + m g l \cos\phi$

does not depend explicitly on time

$$\rightarrow \quad E = h = H = \underbrace{\dot\phi \frac{\partial L}{\partial \dot\phi} - L}_{} = \text{const}$$

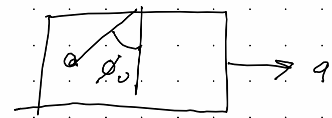
$$= \sum_i \dot q_i \underbrace{\frac{\partial L}{\partial \dot q_i}}_{p_i} - L \quad \text{(more generally)}$$

p_i = momentum conjugate to q_i

in general $\longrightarrow \neq T + U \leftarrow$ total mechanical energy

iii) Equil solution: $\dot{\phi} = 0$

$$\tan\phi_0 = \frac{a}{g}$$



Use EOMs from Lagrangian to show this

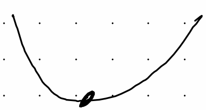
$$\boxed{E = \frac{1}{2} m l^2 \dot{\phi}^2 - m l a \sin\phi - m g l \cos\phi}$$

$$= \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - L$$

$$E = \frac{1}{2} m l^2 \dot{\phi}^2 + U_{eff}(\phi)$$

$$U_{eff}(\phi) = -ml\left(a \sin\phi + g \cos\phi\right)$$

☆ graph it



$$0 = \left.\frac{dU_{eff}}{d\phi}\right|_{\phi = \phi_0}$$

$$U_{eff}(\phi) = U_{eff}(\phi_0) + \cancelto{0}{\left.\frac{dU_{eff}}{d\phi}\right|_{\phi_0}}(\phi-\phi_0) + \frac{1}{2}\left.\frac{d^2U_{eff}}{d\phi}\right|_{\phi_0}(\phi-\phi_0)^2 + \cdots$$

$\varkappa$

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \cdots$$

$$E = T + U_{eff}(\phi)$$

$$= T + U_{eff}(\phi_0) + \frac{1}{2}\varkappa(\phi-\phi_0)^2$$

$$= \frac{1}{2}ml^2\dot{\phi}^2 + U_{eff}(\phi_0) + \frac{1}{2}\varkappa(\phi-\phi_0)^2$$

const → ignore

$$x \equiv \phi - \phi_0 \qquad |x| << 1$$

$$\dot{x} = \dot{\phi}$$

$$E = \frac{1}{2}\underbrace{ml^2}_{M}\dot{x}^2 + \frac{1}{2}\varkappa x^2, \quad \omega = \sqrt{\frac{\varkappa}{M}} = \sqrt{\frac{\varkappa}{ml^2}}$$

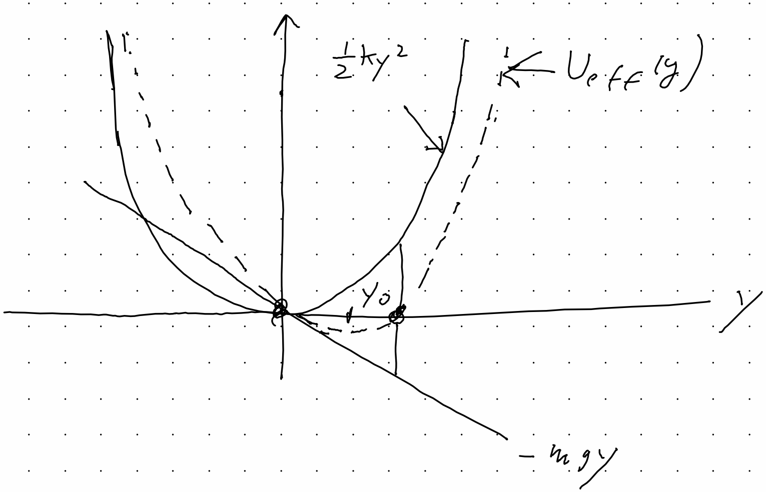
Answers to problems posed at the end of the last class

(1) $\vec{a} = (\underbrace{\ddot{r} - r\dot{\phi}^2}_{\text{centripetal acceleration}})\hat{r} + (2\dot{r}\dot{\phi} + \underbrace{r\ddot{\phi}}_{\text{tangential acceleration}})\hat{\phi}$

(2) $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$

$= \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$

(3) plot $U_{eff}(y) = \frac{1}{2}ky^2 - mgy$



$U_{eff}(y) = U_{eff}(y_0) + \frac{1}{2}\kappa(y-y_0)^2$

where y_0 is the solution to

$0 = \left.\frac{dU_{eff}}{dy}\right|_{y_0}$

and κ is given by

$\kappa \equiv \left.\frac{d^2U_{eff}}{dy^2}\right|_{y_0}$

$$0 = \left.\frac{dU_{eff}}{dy}\right|_{y_0} = ky_0 - mg \rightarrow y_0 = \frac{mg}{k}$$

$$\mathcal{H} = \left.\frac{d^2U_{eff}}{dy^2}\right|_{y_0} = k$$ (so $\mathcal{H}$ is the same as k for this problem)

$$U_{eff}(y_0) = \frac{1}{2}ky_0^2 - mgy_0 = \frac{1}{2}k\left(\frac{mg}{k}\right)^2 - mg\left(\frac{mg}{k}\right) = -\frac{1}{2}\frac{m^2g^2}{k}$$

$$\rightarrow \boxed{U_{eff}(y) = -\frac{1}{2}\frac{m^2g^2}{k} + \frac{1}{2}k\left(y - \frac{mg}{k}\right)^2}$$

You can also obtain the same expression for $U_{eff}(y)$ by completing the square:

$$\begin{aligned} U_{eff}(y) &= \frac{1}{2}ky^2 - mgy \\ &= \frac{1}{2}k\left(y^2 - \frac{2mg}{k}y\right) \\ &= \frac{1}{2}k\left[\left(y - \frac{mg}{k}\right)^2 - \frac{m^2g^2}{k^2}\right] \\ &= \frac{1}{2}k\left(y - \frac{mg}{k}\right)^2 - \frac{1}{2}\frac{m^2g^2}{k} \end{aligned}$$

9/13

Cons. of Energy:

If L does not depend explicitly on time then the energy E (h) is conserved.

my notation

implicit $q(t), \dot{q}(t)$

$L(q, \dot{q}, \cancel{t})$

$$\frac{\partial L}{\partial t} = 0$$

not the same

$$\frac{dL}{dt}$$

$$L = T - U$$

total mechanical energy $= T + U$

$$h \equiv \sum_i p_i \dot{q}_i - L = \text{const}$$

$$\boxed{\frac{dh}{dt} = 0}$$

$$= \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$$

$$H = \left(\sum_i p_i \dot{q}_i - L \right) \Bigg|_{\dot{q} = \dot{q}(q, p, t)}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = p_i(q, \dot{q}, t)$$

invert

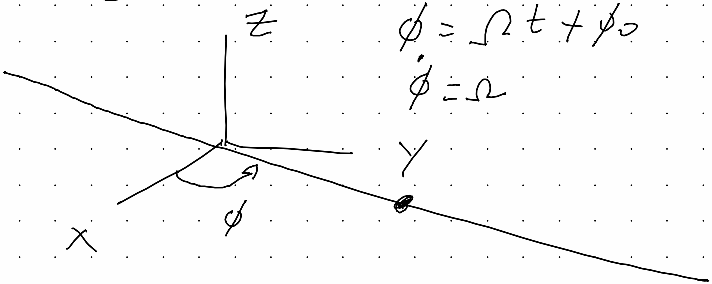
$$\dot{q}_i = \dot{q}_i(q, p, t)$$

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \longrightarrow \dot{x} = \frac{p}{m}$$

Example: bead on a ~~spin~~ rod that rotates uniformly in the xy plane.

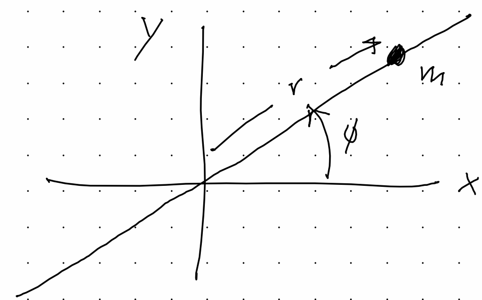
$\boxed{U=0}$



$\phi = \Omega t + \phi_0$

$\dot{\phi} = \Omega$

(top view)



$$L = T - \cancel{U}$$
$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \Omega^2 \right)$$

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right)$$
$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$\phi = \Omega t$$

$$h = \dot{r} \frac{\partial L}{\partial \dot{r}} - L = \text{const}$$
$$= \dot{r}\, m \dot{r} - \frac{1}{2} m \left(\dot{r}^2 + r^2 \Omega^2 \right)$$
$$= \frac{1}{2} m \dot{r}^2 - \frac{1}{2} m r^2 \Omega^2$$
$$= \frac{1}{2} m \left(\dot{r}^2 - r^2 \Omega^2 \right)$$

EOM's

Solve EOM

determine the constraint force !

$$\phi = \Omega t$$

$$C = \phi - \Omega t = 0$$

constraint

constraint force

$$\vec{F}_{constraint} = \lambda \vec{\nabla} C = \lambda \frac{\partial C}{\partial \vec{r}}$$

$$L = T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} + \lambda \frac{\partial C}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} + \lambda \frac{\partial C}{\partial \phi}$$

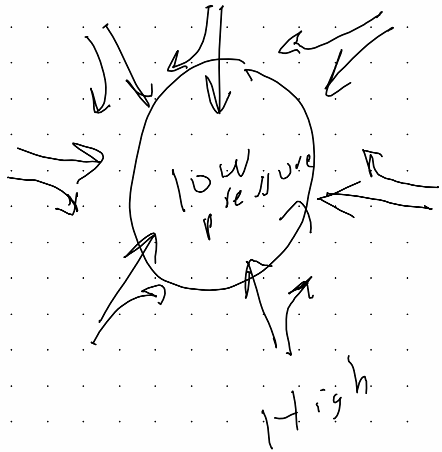
$$C = \phi - \Omega t = 0$$

$\rightarrow$ 3 equations

3 unknowns

$r(t)$, $\phi(t)$

$\lambda(t)$



$\vec{\Omega} \times \vec{v}$

velocity in the non-inertial

$$H(q,p,t) = \left[\sum_i p_i \dot{q}_i - L(q,\dot{q},t) \right] \Bigg|_{\dot{q} = \dot{q}(q,p,t)}$$

<u>Hamilton's equations:</u>

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, 2, \cdots, n$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad i = 1, 2, \cdots, n$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \qquad i = 1, 2, \cdots, n$$

<u>Example:</u> $L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x,y,z)$

i) Find $H(q,p,t)$

ii) Write down Hamilton's EOMs

iii) Show Hamilton's EOMs are equivalent to Lagrange equations

$$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial t} dt$$

$$= \dot{p}\, dq + p\, d\dot{q} + \frac{\partial L}{\partial t} dt$$

$$= \dot{p}\, dq + d(p\dot{q}) - \dot{q}\, dp + \frac{\partial L}{\partial t} dt$$

$$\underbrace{d(p\dot{q}) - dL}_{= d(p\dot{q} - L) = dH} = \dot{q}\, dp - \dot{p}\, dq - \frac{\partial L}{\partial t} dt$$

$$\boxed{dH = -\dot{p}\, dq + \dot{q}\, dp - \frac{\partial L}{\partial t} dt}$$

$$f\, dg = d(fg) - g\, df$$

$$d(fg) = f\, dg + g\, df$$

$$\boxed{\begin{aligned} \frac{\partial H}{\partial q} &= -\dot{p} \\ \frac{\partial H}{\partial p} &= \dot{q} \\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} \end{aligned}}$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$