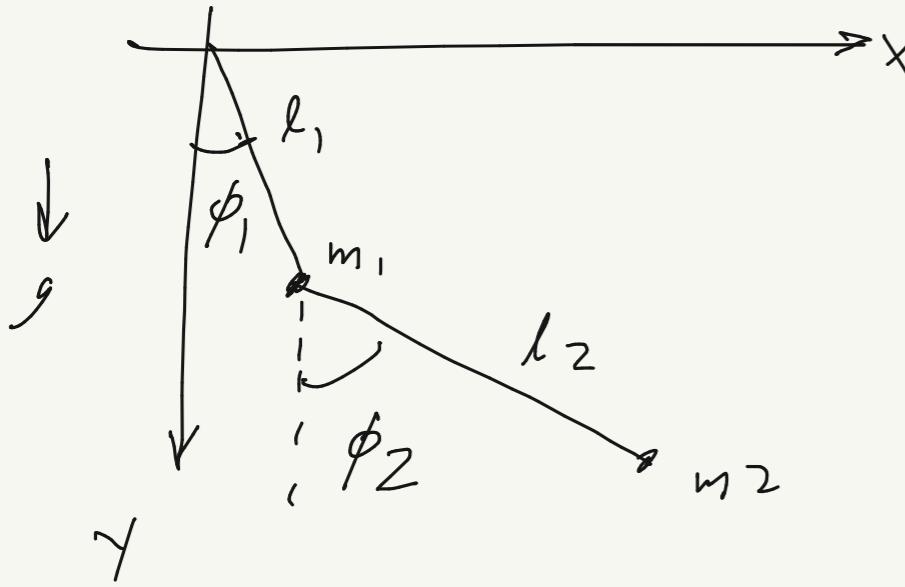


Sec 5, Prob 1 :



$$x_1 = l_1 \sin \phi_1$$

$$y_1 = l_1 \cos \phi_1$$

$$x_2 = x_1 + l_2 \sin \phi_2$$

$$y_2 = y_1 + l_2 \cos \phi_2$$

$$U = -m_1 g y_1 - m_2 g y_2$$

$$= -m_1 g l_1 \cos \phi_1 - m_2 g (l_1 \cos \phi_1 + l_2 \cos \phi_2)$$

$$= -(m_1 + m_2) g l_1 \cos \phi_1 - m_2 g l_2 \cos \phi_2$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\dot{x}_1 = l_1 \dot{\phi}_1 \cos \phi_1 \rightarrow \dot{x}_1^2 = l_1^2 \dot{\phi}_1^2 \cos^2 \phi_1$$

$$\dot{y}_1 = -l_1 \dot{\phi}_1 \sin \phi_1 \rightarrow \dot{y}_1^2 = l_1^2 \dot{\phi}_1^2 \sin^2 \phi_1$$

$$\overline{\dot{x}_1^2 + \dot{y}_1^2} = l_1^2 \dot{\phi}_1^2$$

$$\dot{x}_2 = l_1 \dot{\phi}_1 \cos \phi_1 + l_2 \dot{\phi}_2 \cos \phi_2$$

$$\dot{y}_2 = -l_1 \dot{\phi}_1 \sin \phi_1 - l_2 \dot{\phi}_2 \sin \phi_2$$

$$\rightarrow \dot{x}_2^2 = l_1^2 \dot{\phi}_1^2 \cos^2 \phi_1 + l_2^2 \dot{\phi}_2^2 \cos^2 \phi_2 + 2l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos \phi_1 \cos \phi_2$$

$$\dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 \sin^2 \phi_1 + l_2^2 \dot{\phi}_2^2 \sin^2 \phi_2 + 2l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin \phi_1 \sin \phi_2$$

$$\therefore \dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 (\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2)$$

$$= l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

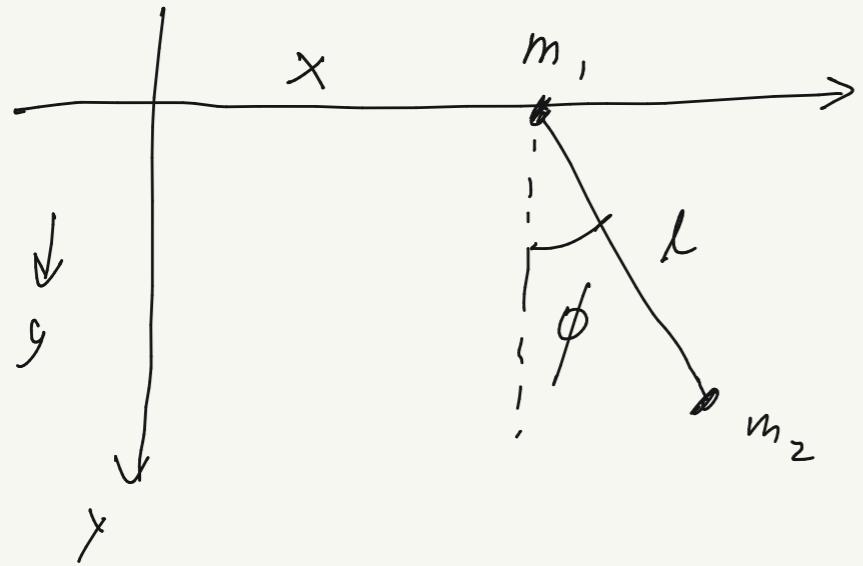
Thus,

$$\begin{aligned} T &= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \\ &= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \end{aligned}$$

$$\rightarrow L = T - U$$

$$\begin{aligned} &\approx \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \\ &\quad + (m_1 + m_2) g l_1 \cos \phi_1 + m_2 g l_2 \cos \phi_2 \end{aligned}$$

Sec 5, Prob 2:



$$(x_1, y_1) = (x, 0)$$

$$(x_2, y_2) = (x + l \cos \phi, l \sin \phi)$$

$$(\dot{x}_1, \dot{y}_1) = (\dot{x}, 0)$$

$$(\dot{x}_2, \dot{y}_2) = (\dot{x} + l \dot{\phi} \cos \phi, -l \dot{\phi} \sin \phi)$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}^2 + l^2 \dot{\phi}^2 \cos^2 \phi + 2 l \dot{x} \dot{\phi} \cos \phi + l^2 \dot{\phi}^2 \sin^2 \phi)$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \dot{x} \dot{\phi} \cos \phi$$

$$U = -m_1 g y_1 - m_2 g y_2$$

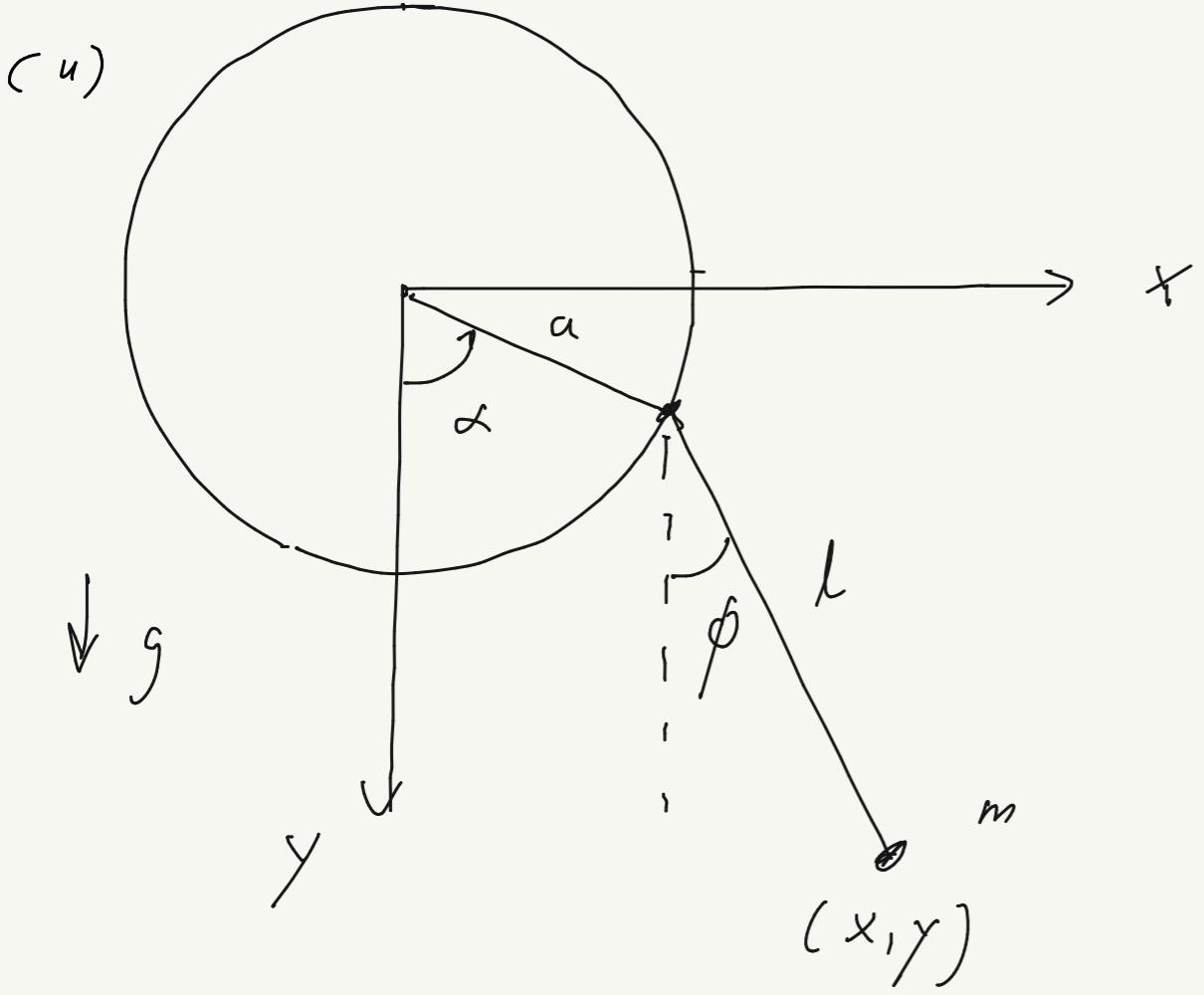
$$= -m_2 g l \cos \phi$$

$$L = T - U$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \dot{x} \dot{\phi} \cos \phi + m_2 g l \cos \phi$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 (l^2 \dot{\phi}^2 + l \dot{x} \dot{\phi} \cos \phi) + m_2 g l \cos \phi$$

Sec 5, Prob 3:



$$\alpha = \gamma t$$

$$x = a \sin \alpha + l \sin \phi$$

$$y = a \cos \alpha + l \cos \phi$$

$$U = -mg y$$

$$= -mg a \cos \alpha - mgl \cos \phi$$

prescribed function
of time (ignore)

$$= -mgl \cos \phi$$

$$\dot{x} = a \gamma \cos \alpha + l \dot{\phi} \cos \phi$$

$$\dot{y} = -a \gamma \sin \alpha - l \dot{\phi} \sin \phi$$

$$\dot{x}^2 = a^2 \gamma^2 \cos^2 \alpha + l^2 \dot{\phi}^2 \cos^2 \phi + 2al \gamma \dot{\phi} \cos \alpha \cos \phi$$

$$\dot{y}^2 = a^2 \gamma^2 \sin^2 \alpha + l^2 \dot{\phi}^2 \sin^2 \phi + 2al \gamma \dot{\phi} \sin \alpha \sin \phi$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m [a^2 \gamma^2 + l^2 \dot{\phi}^2 + 2al \gamma \dot{\phi} \cos(\alpha - \phi)]$$

$$= \underbrace{\frac{1}{2} m a^2 \gamma^2}_{\text{prescribed}} + \frac{1}{2} m l^2 \dot{\phi}^2 + mal \gamma \dot{\phi} \cos(\gamma t - \phi)$$

function of
time (ignore)

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + mal \gamma \dot{\phi} \cos(\gamma t - \phi)$$

$$L = T - U$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + mal \gamma \dot{\phi} \cos(\gamma t - \phi) + mgl \cos \phi$$

$$\text{NOTE: } \frac{d}{dt} [\gamma \dot{\phi} \cos(\gamma t - \phi)] = \frac{d}{dt} [\underbrace{-\gamma \sin(\gamma t - \phi)}_{\text{can ignore since total time derivative.}}] + \gamma^2 \cos(\gamma t - \phi)$$

Thus,

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \gamma^2 \cos(\gamma t - \phi) + m g l \cos \phi$$

NOTE:

E_{cm} should be the same for both Lagrangians:

$$(1^{st}): \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi}$$

$$\begin{aligned} & \frac{d}{dt} (m l^2 \dot{\phi} + m g l \gamma \cos(\gamma t - \phi)) \\ &= m g l \gamma \dot{\phi} \sin(\gamma t - \phi) - m g l \sin \phi \end{aligned}$$

$$\begin{aligned} m l^2 \ddot{\phi} - m g l \gamma^2 \sin(\gamma t - \phi) + \cancel{m g l \gamma \dot{\phi} \sin(\gamma t - \phi)} \\ = \cancel{m g l \gamma \dot{\phi} \sin(\gamma t - \phi)} - m g l \sin \phi \end{aligned}$$

$$\rightarrow \ddot{\phi} = \frac{a}{l} \gamma^2 \sin(\gamma t - \phi) - \frac{g}{l} \sin \phi \quad \text{ignoring}$$

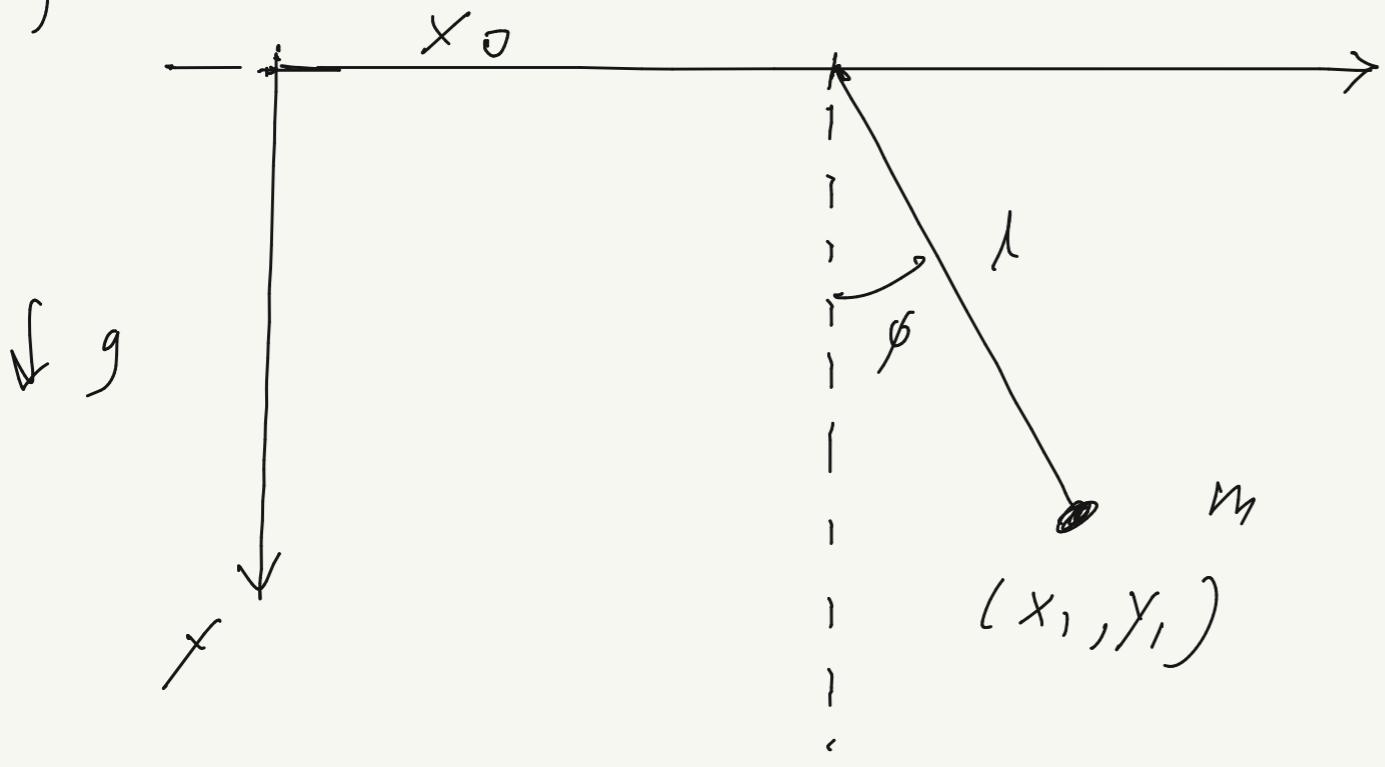
$$(2^{nd}) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi}$$

$$\frac{d}{dt} (m l^2 \dot{\phi}) = + m g l \gamma^2 \sin(\gamma t - \phi) - m g l \sin \phi$$

$$m l^2 \ddot{\phi} = m g l \gamma^2 \sin(\gamma t - \phi) - m g l \sin \phi$$

$$\rightarrow \ddot{\phi} = \frac{a}{l} \gamma^2 \sin(\gamma t - \phi) - \frac{g}{l} \sin \phi$$

(b)



$$x_0 = a \cos \phi t$$

$$\dot{x}_0 = -a \gamma \sin \phi t$$

$$x = x_0 + l \cos \phi$$

$$y = l \sin \phi$$

$$U = -mgy = -mg l \cos \phi$$

$$\dot{x} = \dot{x}_0 + l \dot{\phi} \cos \phi$$

$$= -a \gamma \sin \phi t + l \dot{\phi} \cos \phi$$

$$\dot{y} = -l \dot{\phi} \sin \phi$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m (a^2 \gamma^2 \sin^2 \phi t + l^2 \dot{\phi}^2 \cos^2 \phi - 2al\gamma \dot{\phi} \sin \phi t \cos \phi + l^2 \dot{\phi}^2 \sin^2 \phi)$$

$$= \underbrace{\frac{1}{2} m a^2 \gamma^2 \sin^2 \phi t}_{\text{prescribed function of time (ignore)}} + \frac{1}{2} m l^2 \dot{\phi}^2 - m a l \gamma \dot{\phi} \sin \phi t \cos \phi$$

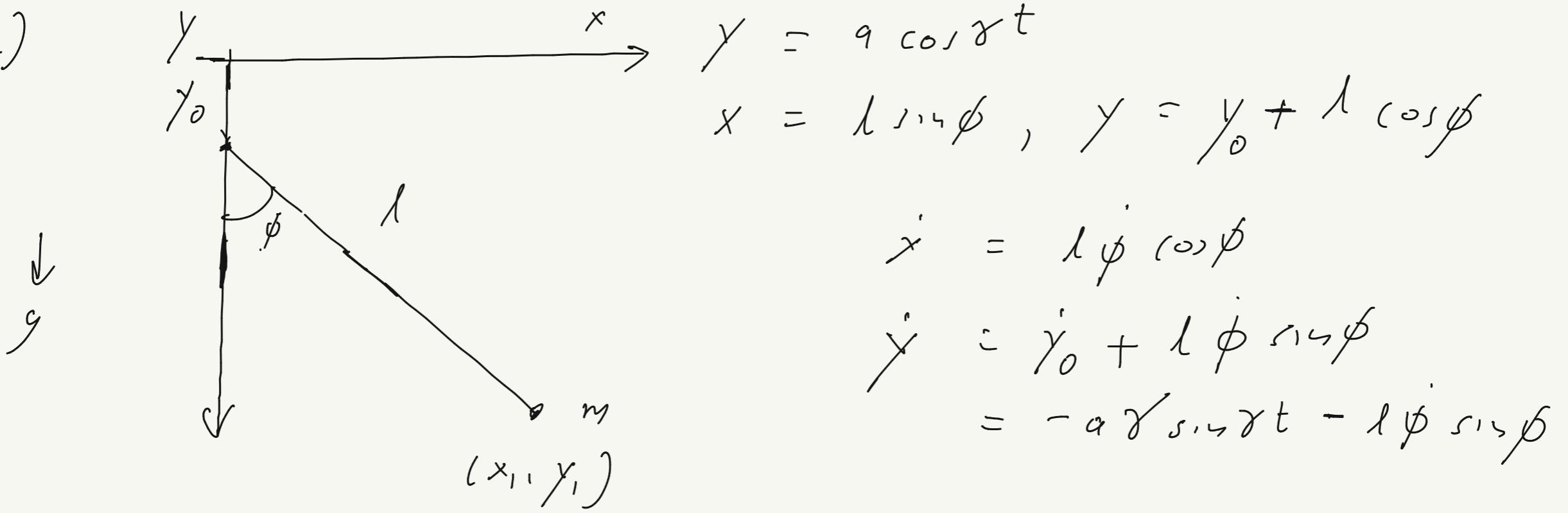
*prescribed
function of
time (ignore)*

$$= \frac{d}{dt} (\gamma \sin \phi t \sin \phi) - \gamma^2 \cos \phi t \sin \phi$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \gamma^2 \cos \phi t \sin \phi$$

$$\rightarrow L = T - U = \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \gamma^2 \cos \phi t \sin \phi + m g l \cos \phi$$

(c)



$$\dot{x}^2 = l^2 \dot{\phi}^2 \cos^2 \phi$$

$$\dot{y}^2 = g^2 \gamma^2 \sin^2 \gamma t + l^2 \dot{\phi}^2 \sin^2 \phi + 2g l \gamma \phi \sin \gamma t \sin \phi$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m (g^2 \gamma^2 \sin^2 \gamma t + l^2 \dot{\phi}^2 + 2g l \gamma \phi \sin \gamma t \sin \phi)$$

$$= \frac{1}{2} m u^2 \gamma^2 \sin^2 \gamma t + \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \gamma \phi \underbrace{\sin \gamma t \sin \phi}$$

prescribed func
of t (igno...)

$$\approx \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \gamma^2 \cos \gamma t \cos \phi$$

$$= \frac{d}{dt} (-\gamma \sin \gamma t \cos \phi) + \gamma^2 \cos \gamma t \cos \phi$$

total time
derivative
(igno...)

$$U = -mgY = -mg(y + l \cos \phi)$$

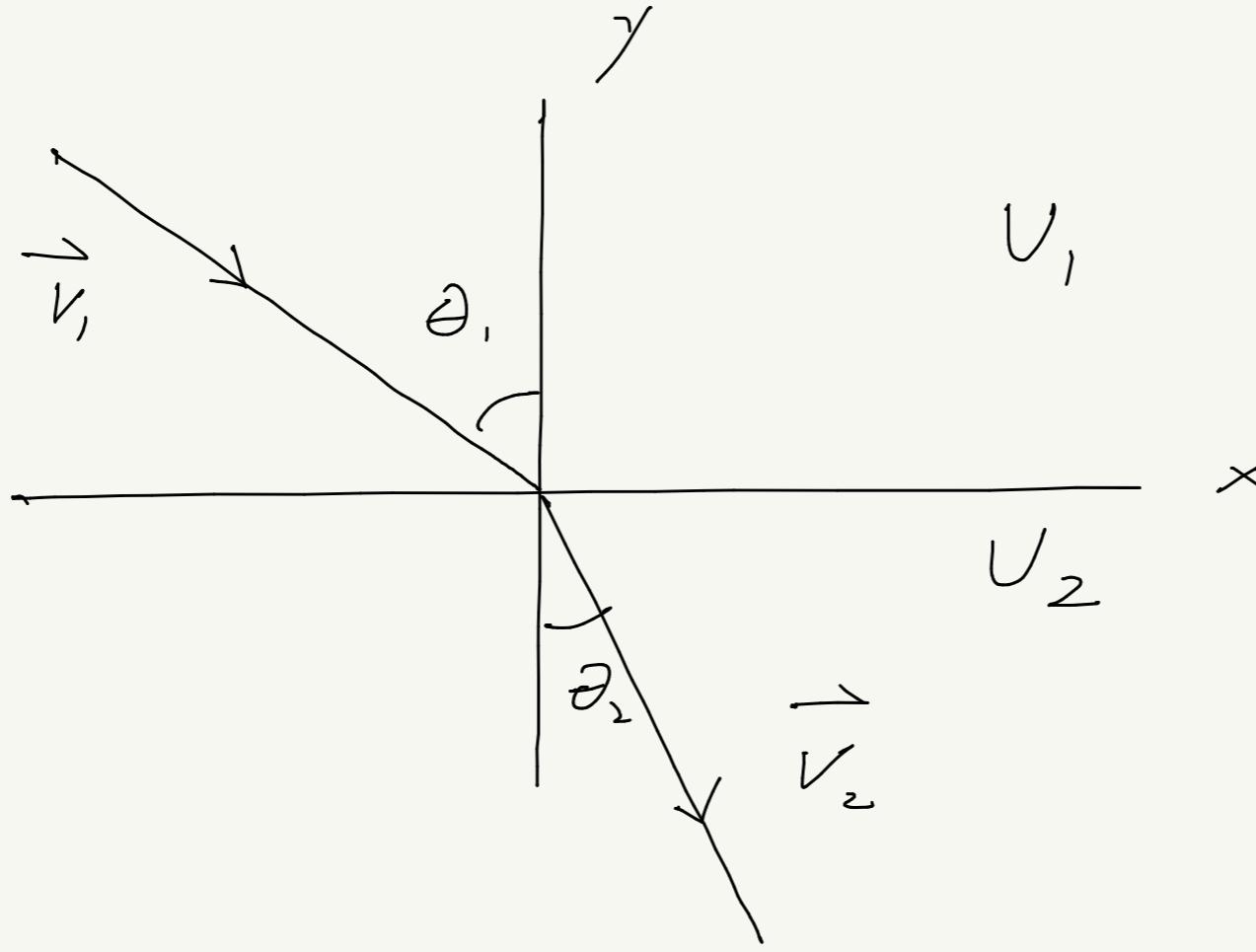
$$= -mg \underbrace{a \cos \gamma t}_{\text{prescribed func of t}} - mg l \cos \phi$$

prescribed
func of t
(igno...)

$\dot{T}_{h\nu},$

$$L = T - U = \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \gamma^2 \cos \gamma t \cos \phi + mg l \cos \phi$$

Sec 7, Prob 1 :



- Energy is conserved
- Component of linear momentum in x -direction is also conserved

$$i) E = \frac{1}{2} m v_1^2 + U_1 = \frac{1}{2} m v_2^2 + U_2$$

$$\frac{1}{2} m v_2^2 = \frac{1}{2} m v_1^2 + (U_1 - U_2)$$

$$v_2 = \sqrt{v_1^2 + \frac{2}{m} (U_1 - U_2)}$$

$$\frac{v_2}{v_1} = \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

$$ii) p_x = \cancel{m v_1 \sin \theta_1} = \cancel{m v_2 \sin \theta_2}$$

$$\rightarrow \frac{\cancel{\sin \theta_1}}{\cancel{\sin \theta_2}} = \frac{v_2}{v_1}$$

$$= \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

Sec 8, Prob 1:

$$S[\mathcal{L}] = \int_{t_1}^{t_2} dt \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$$

Let inertial frame K' move with velocity \vec{V} wrt inertial frame K .

Then:

$$\begin{aligned}\vec{v}_a &= \vec{v}'_a + \vec{V} \\ \vec{r}_a &= \vec{r}'_a + \vec{V} \cdot t\end{aligned}$$

$$\sum m_a \vec{v}'_a = \mu \vec{R}'$$

Thus,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} \sum_a m_a |\vec{v}'_a|^2 - U(\vec{r}_1, \vec{r}_2, \dots, t) \\ &= \frac{1}{2} \sum_a m_a |\vec{v}'_a + \vec{V}|^2 - U \\ &= \frac{1}{2} \sum_a m_a |\vec{v}'_a|^2 + \frac{1}{2} \sum_a m_a \vec{V}^2 \\ &\quad + (\sum_a m_a \vec{v}'_a) \cdot \vec{V} - U \\ &= \mathcal{L}' + \vec{p}' \cdot \vec{V} + \frac{1}{2} \mu \vec{V}^2\end{aligned}$$

$$\begin{aligned}\rightarrow S &= \int_{t_1}^{t_2} dt (\mathcal{L}' + \vec{p}' \cdot \vec{V} + \frac{1}{2} \mu \vec{V}^2) \\ &= S' + \vec{V} \cdot \sum_a m_a \vec{r}'_a \Big|_{t_1}^{t_2} + \frac{1}{2} \mu \vec{V}^2 \Big|_{t_1}^{t_2} \\ &= S' + \mu \vec{V} \cdot (\vec{R}'(t_2) - \vec{R}'(t_1)) + \frac{1}{2} \mu \vec{V}^2(t_2 - t_1)\end{aligned}$$

where \vec{R}' is com of system wrt Frame K'

Sec 9, Prob 1 :

Cylindrical coords (s, ϕ, z) :

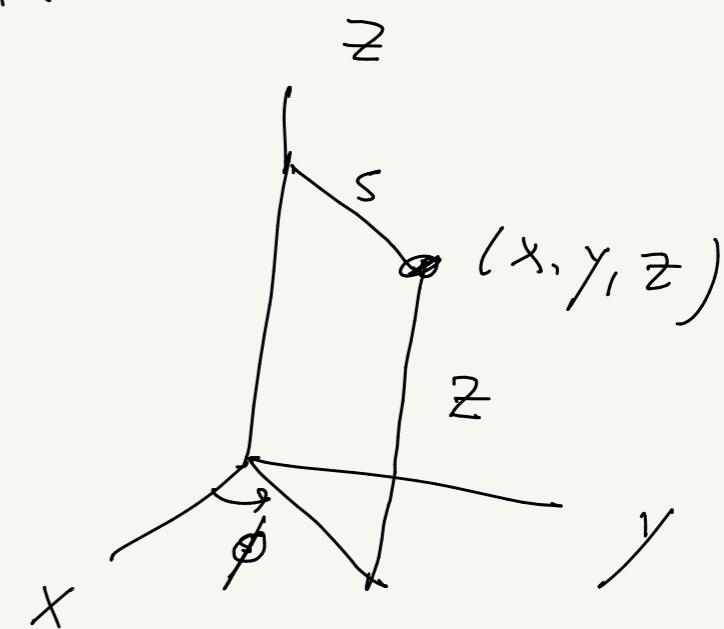
$$x = s \cos \phi, y = s \sin \phi, z = z$$

$$\vec{M} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v} = m \vec{r} \times \vec{r}$$

$$\text{Thus, } M_x = m(y\dot{z} - z\dot{y})$$

$$M_y = m(z\dot{x} - x\dot{z})$$

$$M_z = m(x\dot{y} - y\dot{x})$$



$$\dot{x} = s \cos \phi - s \phi \sin \phi$$

$$\dot{y} = s \sin \phi + s \phi \cos \phi$$

$$\dot{z} = \dot{z}$$

$$\begin{aligned} \rightarrow M_x &= m [s \sin \phi \dot{z} - \dot{z} (s \sin \phi + s \phi \cos \phi)] \\ &= m [s \sin \phi (s \dot{z} - \dot{z} s) - \dot{z} s \phi \cos \phi] \end{aligned}$$

$$\begin{aligned} M_y &= m [\dot{z} (s \cos \phi - s \phi \sin \phi) - s \cos \phi \dot{z}] \\ &= m [-\cos \phi (s \dot{z} - \dot{z} s) - \dot{z} s \phi \sin \phi] \end{aligned}$$

$$\begin{aligned} M_z &= m [s \cos \phi (s \sin \phi + s \phi \cos \phi) \\ &\quad - s \sin \phi (s \cos \phi - s \phi \sin \phi)] \\ &= m s^2 \phi \end{aligned}$$

$$\begin{aligned}
M^2 &= M_x^2 + M_y^2 + M_z^2 \\
&= m^2 \left[\sin^2 \phi (s\dot{z} - z\dot{s})^2 + z^2 s^2 \dot{\phi}^2 \cos^2 \phi \right. \\
&\quad - 2 \cancel{z s \dot{\phi} \cos \phi s \dot{\phi} (s\dot{z} - z\dot{s})} \\
&\quad + \cancel{2 z s \dot{\phi} \cos \phi s \dot{\phi} (s\dot{z} - z\dot{s})} \\
&\quad \left. + s^4 \dot{\phi}^2 \right] \\
&= m^2 \left[(s\dot{z} - z\dot{s})^2 + z^2 s^2 \dot{\phi}^2 + s^4 \dot{\phi}^2 \right] \\
&= m^2 \left[(s\dot{z} - z\dot{s})^2 + s^2 (z^2 + s^2) \dot{\phi}^2 \right]
\end{aligned}$$

Sec 9, Prob 2 :

spherical polar coords (r, θ, ϕ) :

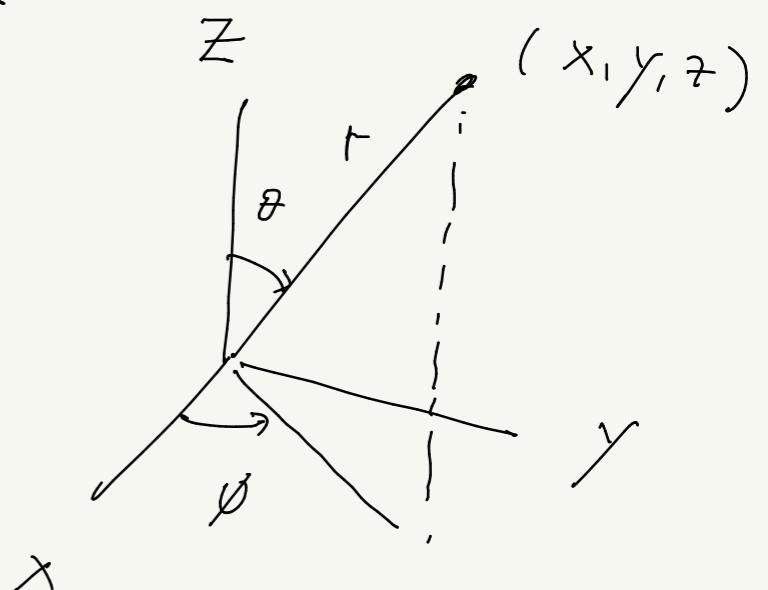
$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

$$\vec{m} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v} = m \vec{r} \times \vec{r}$$

$$\text{Thus, } M_x = m(y\dot{z} - z\dot{y})$$

$$M_y = m(z\dot{x} - x\dot{z})$$

$$M_z = m(x\dot{y} - y\dot{x})$$



$$\dot{x} = r \sin \theta \cos \phi + r \dot{\theta} \cos \theta \cos \phi - r \dot{\phi} \sin \theta \sin \phi$$

$$\dot{y} = r \sin \theta \sin \phi + r \dot{\theta} \cos \theta \sin \phi + r \dot{\phi} \sin \theta \cos \phi$$

$$\dot{z} = r \cos \theta - r \dot{\theta} \sin \theta$$

$$\begin{aligned} \rightarrow M_x &= m [r \sin \theta \sin \phi (r \cos \theta - r \dot{\theta} \sin \theta) \\ &\quad - r \cos \theta (r \sin \theta \sin \phi + r \dot{\theta} \cos \theta \sin \phi + r \dot{\phi} \sin \theta \cos \phi)] \end{aligned}$$

$$= m [-r^2 \dot{\theta} \sin \phi (\sin \theta + \cos^2 \theta) - r^2 \dot{\phi} \sin \theta \cos \theta \cos \phi]$$

$$= m [-r^2 \dot{\theta} \sin \phi - r^2 \dot{\phi} \sin \theta \cos \theta \cos \phi]$$

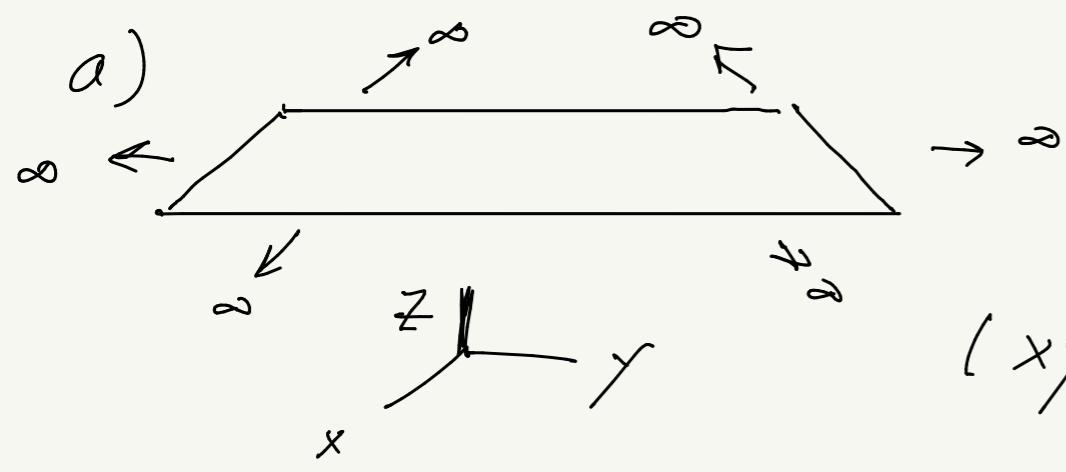
$$\begin{aligned} M_y &= m [r \cos \theta (r \sin \theta \cos \phi + r \dot{\theta} \cos \theta \cos \phi - r \dot{\phi} \sin \theta \sin \phi) \\ &\quad - r \sin \theta \cos \phi (r \cos \theta - r \dot{\theta} \sin \theta)] \end{aligned}$$

$$= m [r^2 \dot{\theta} \cos \phi - r^2 \dot{\phi} \sin \theta \cos \theta \sin \phi]$$

$$\begin{aligned}
M_2 &= m \left[r \sin \theta \cos \phi \left(\dot{r} \sin \theta \sin \phi + r \dot{\theta} \cos \theta \cos \phi + r \dot{\phi} \sin \theta \cos \phi \right) \right. \\
&\quad \left. - r \sin \theta \sin \phi \left(\dot{r} \sin \theta \cos \phi + r \dot{\theta} \cos \theta \cos \phi - r \dot{\phi} \sin \theta \sin \phi \right) \right] \\
&= m \left[r^2 \dot{\phi} \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) \right] \\
&= m r^2 \dot{\phi} \sin^2 \theta
\end{aligned}$$

$$\begin{aligned}
\overbrace{M^2} &= M_x^2 + M_y^2 + M_z^2 \\
&= m^2 \left[r^4 \dot{\theta}^2 \sin^2 \phi + r^4 \dot{\phi}^2 \sin^2 \theta \cos^2 \theta \cos^2 \phi \right. \\
&\quad + \underline{2 r^4 \dot{\theta} \dot{\phi} \sin \theta \cos \theta \sin \phi \cos \phi} \\
&\quad + r^4 \dot{\theta}^2 \cos^2 \phi + r^4 \dot{\phi}^2 \sin^2 \theta \cos^2 \theta \sin^2 \phi \\
&\quad - \underline{2 r^4 \dot{\theta} \dot{\phi} \sin \theta \cos \theta \sin \phi \cos \phi} \\
&\quad \left. + r^4 \dot{\phi}^2 \sin^4 \theta \right] \\
&= m^2 r^4 \left[\dot{\theta}^2 + \dot{\phi}^2 (\sin^2 \theta \cos^2 \theta + \sin^4 \theta) \right] \\
&= m r^4 \left[\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right]
\end{aligned}$$

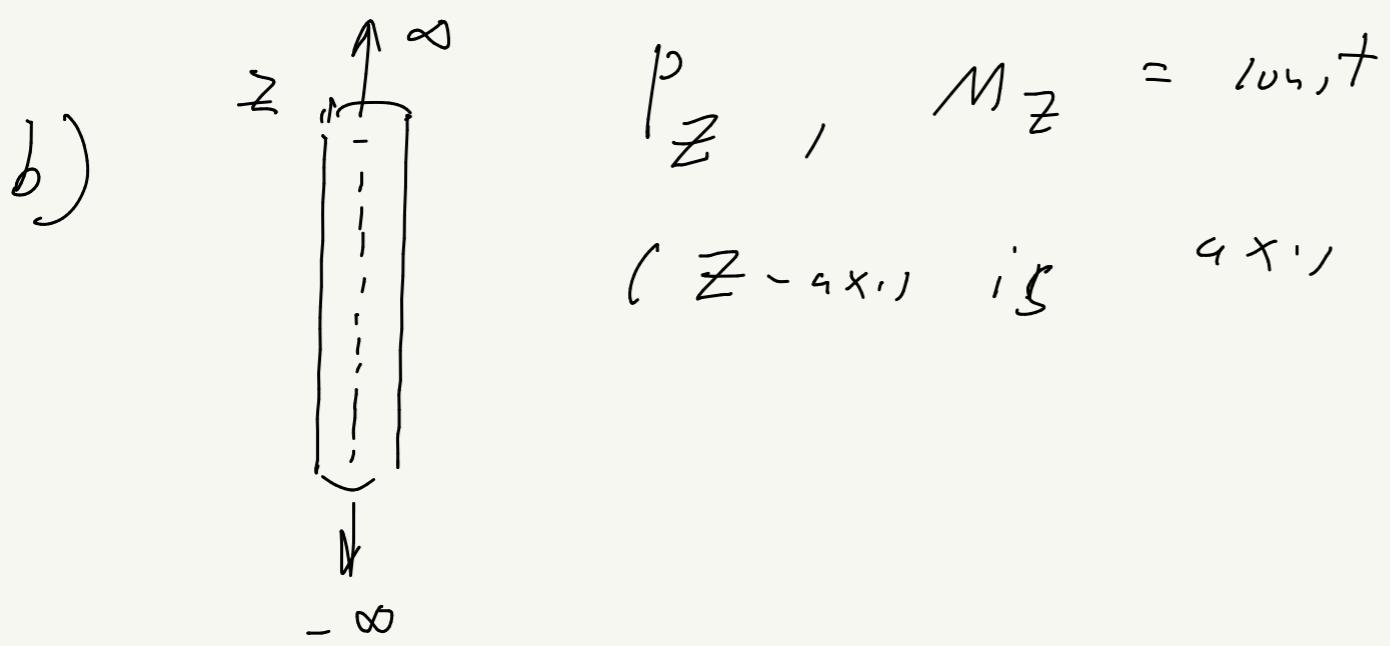
Sec 9, Prob 3:



$$P_x, P_y = \text{const}$$

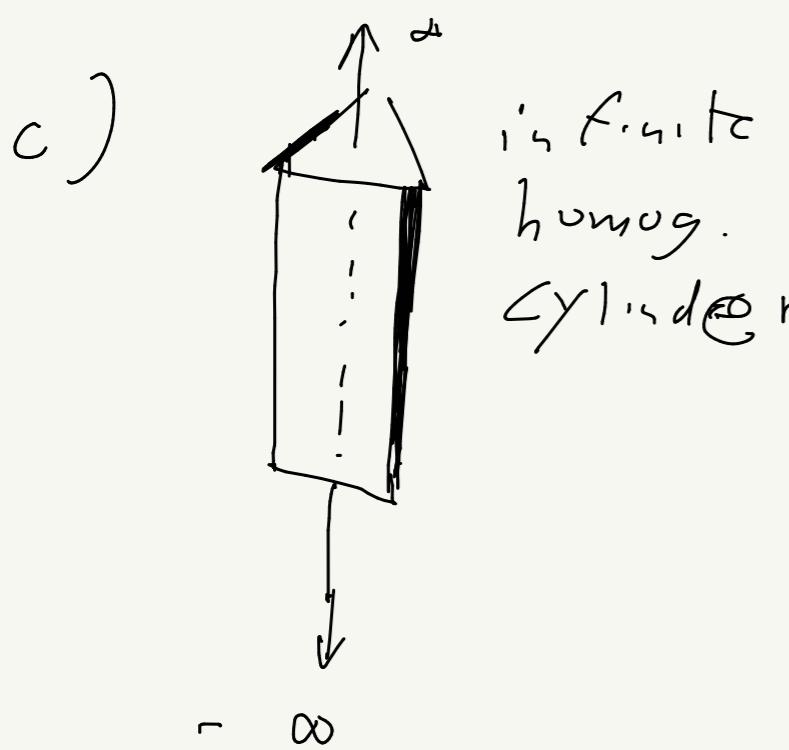
$$M_z = \text{const}$$

(xy-plane // to homog. p line)



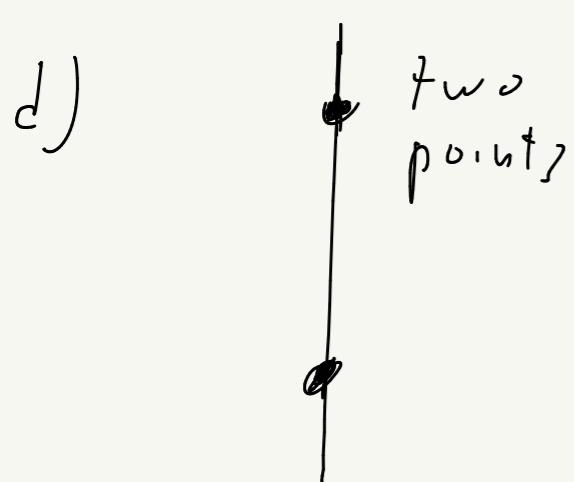
$$P_z, M_z = \text{const}$$

(z-axis is axis of cylinder)



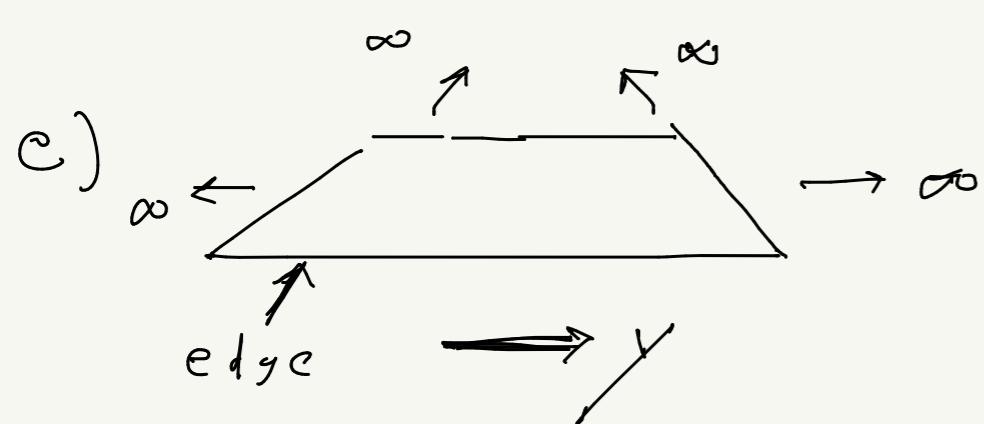
$$P_z = \text{const}$$

(z-axis is // to edge of prism)



$$M_z = \text{const}$$

(z-axis passes through the two points)

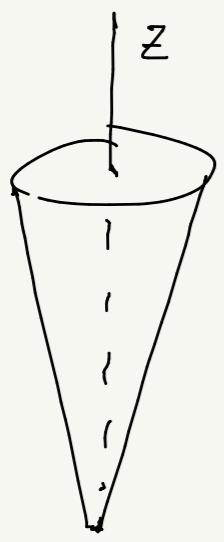


∞ homog, $\frac{1}{2}$ plane

$$P_y = \text{const}$$

(y axis is // to edge of $\frac{1}{2}$ plane)

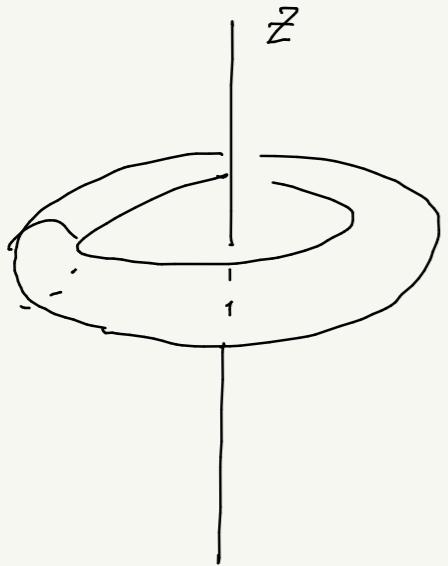
f)



$$M_z = \text{const}$$

(z-axis is axis of cone)

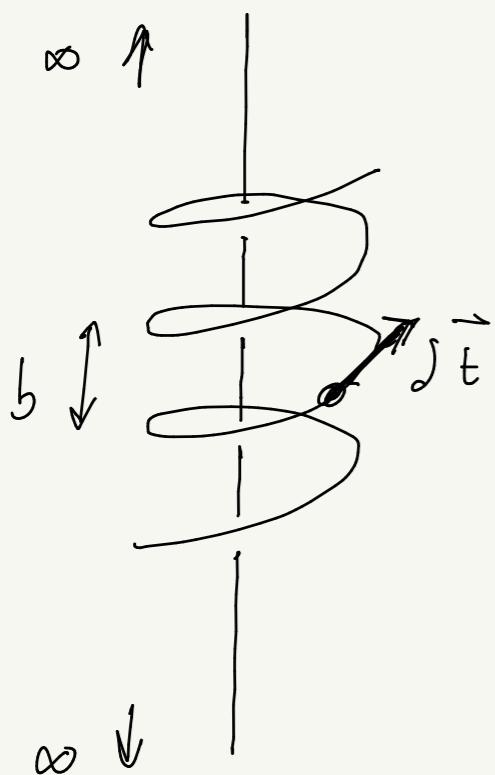
g)



$$M_z = \text{const}$$

(z-axis is axis of torus)

h)



a = radius of helix

b = height between neighboring coils of helix

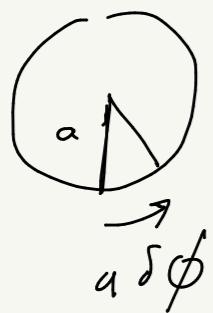
$h = b/a$ (pitch of helix)

Lagrangian, obvious w. + translation
around the helix:

$$\delta \vec{r} = a \delta \phi \hat{\phi} + \frac{b \delta \phi}{2\pi} \hat{z}$$

$$= a \delta \phi \left[\hat{\phi} + \frac{b/a}{2\pi} \hat{z} \right]$$

$$= a \delta \phi \left[\hat{\phi} + \frac{h}{2\pi} \hat{z} \right]$$



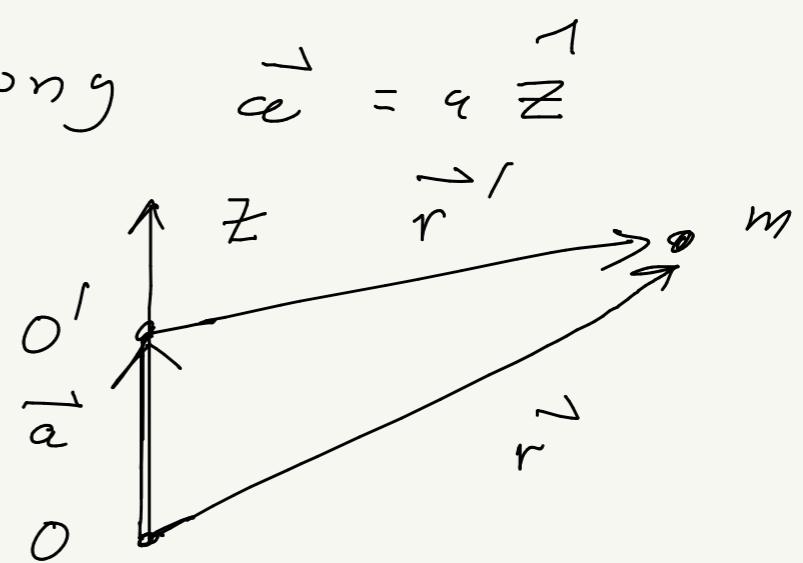
$$\begin{aligned} \delta L &= \frac{\partial L}{\partial \dot{\phi}} \delta \phi + \frac{\partial L}{\partial z} \delta z \\ &= \frac{d(\partial L)}{dt \partial \dot{\phi}} \delta \phi + \frac{d}{dt} \left(\frac{\partial L}{\partial z} \right) \frac{h}{2\pi} \delta \phi \\ &= \delta \phi \frac{d}{dt} \left[P_{\phi} + P_z \frac{h}{2\pi} \right] \xrightarrow{M_z + \frac{P_z h}{2\pi} = \text{const}} M_z \end{aligned}$$

NOTE: M_z is independent of location of origin
on Z -axis.

$$\vec{M} = \vec{r} \times \vec{p} = m \vec{r} \times \dot{\vec{r}}$$

Change origin by shifting along $\vec{a} = a \hat{z}$

$$\vec{r} = \vec{r}' + \vec{a}$$



$$\begin{aligned}\vec{M} &= m \vec{r} \times \dot{\vec{r}} \\ &= m (\vec{r}' + \vec{a}) \times \frac{d}{dt} (\vec{r}' + \vec{a}) \\ &= m \vec{r}' \times \dot{\vec{r}'} + m \vec{a} \times \dot{\vec{r}'} \\ &= \vec{M}' + \vec{a} \times \vec{p}' \quad (\text{for arbitrary } \vec{a})\end{aligned}$$

Thus,

$$\begin{aligned}M_z &= \vec{M}' \cdot \hat{z} \\ &= (\vec{M}' + \vec{a} \times \vec{p}') \cdot \hat{z} \\ &= M'_z + a (\hat{z} \times \vec{p}') \cdot \hat{z} \\ &= M'_z + a (\cancel{\hat{z} \times \hat{z}}) \cdot \vec{p}' \\ &= M'_z\end{aligned}$$

Sec 10, Prob 1:

same path, different masses, same potential energy

$$\rightarrow x' = x, m' \neq m, U' = U, t' \neq t$$

$$L = T - U = \frac{1}{2} m \dot{x}^2 - U$$

$$\begin{aligned} L' &= \frac{1}{2} m' \left(\frac{dx}{dt'} \right)^2 - U \\ &= \frac{1}{2} m' \left(\frac{t}{t'} \right)^2 \dot{x}^2 - U \end{aligned}$$

$$\text{thus } L' = L \rightarrow m' \left(\frac{t}{t'} \right)^2 = m$$

$$\left(\frac{t'}{t} \right)^2 = \frac{m'}{m}$$

$$\rightarrow \frac{t'}{t} = \sqrt{\frac{m'}{m}}$$

Sec 10, Prob 2:

same path, same mass, potential energy
differing by a constant factor ($U' = c U$)

$$\rightarrow x = x', m = m', t' \neq t$$

$$L = T - U$$
$$= \frac{1}{2} m \dot{x}^2 - U$$

$$L' = \frac{1}{2} m \left(\frac{dx}{dt'} \right)^2 - U'$$
$$= \frac{1}{2} m \left(\frac{t}{t'} \right)^2 \dot{x}^2 - c U$$

Thus, need $\left(\frac{t}{t'} \right)^2 = c$ to get same EOM,

$$\rightarrow \frac{t'}{t} = \sqrt{\frac{1}{c}}$$
$$= \sqrt{\frac{U}{U'}}$$

Sec. 40, Prob 1

Hamiltonian for a single particle

$$\begin{aligned} L &= T - U \\ &= \sum_m \frac{1}{2m} \dot{r}^2 - U(\vec{r}, t) \end{aligned}$$

Cartesian:

$$L = \frac{1}{2m} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \rightarrow \dot{x} = p_x/m$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m \dot{y} \rightarrow \dot{y} = p_y/m$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m \dot{z} \rightarrow \dot{z} = p_z/m$$

$$\rightarrow H = \left(\sum_i p_i \dot{q}_i - L \right) \quad \dot{q}_i = \dot{q}_i(q, p)$$

$$\begin{aligned} &= \left(p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \sum_m \frac{1}{2m} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \right. \\ &\quad \left. + U(x, y, z, t) \right) \quad \dot{x} = p_x/m, etc \end{aligned}$$

$$= \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + U(x, y, z, t)$$

Cylindrical: (s, ϕ, z)

$$L = \frac{1}{2}m(s^2 + s^2\dot{\phi}^2 + \dot{z}^2) - U(s, \phi, z, t)$$

$$p_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s} \rightarrow \dot{s} = p_s/m$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ms^2\dot{\phi} \rightarrow \dot{\phi} = p_\phi/ms^2$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$\rightarrow H = \left(p_s \dot{s} + p_\phi \dot{\phi} + p_z \dot{z} - L \right)_{\dot{s} = p_s/m, \dot{\phi}, \dot{z}}$$

$$= \frac{p_s^2}{m} + \frac{p_\phi^2}{ms^2} + \frac{p_z^2}{m}$$

$$= \frac{1}{2}m \left[\left(\frac{p_s}{m} \right)^2 + s^2 \left(\frac{p_\phi}{ms^2} \right)^2 + \left(\frac{p_z}{m} \right)^2 \right] + U(s, \phi, z, t)$$

$$= \frac{1}{2m} \left(p_s^2 + \frac{p_\phi^2}{s^2} + p_z^2 \right) + U(s, \phi, z, t)$$

Spherical polar: (r, θ, ϕ)

$$L = \frac{1}{2}m(r^2\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - U(r, \theta, \phi, t)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \rightarrow \dot{r} = p_r/m$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \rightarrow \dot{\theta} = p_\theta/mr^2$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi} \rightarrow \dot{\phi} = p_\phi/mr^2\sin^2\theta$$

$$\rightarrow H = \left(p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - L \right) \Big|_{r = p_r/m, \epsilon t < 1}$$

$$= p_r \left(\frac{p_r}{m} \right) + p_\theta \left(\frac{p_\theta}{mr^2} \right) + p_\phi \left(\frac{p_\phi}{mr^2 \sin^2 \theta} \right)$$

$$= \frac{1}{2m} \left(\left(\frac{p_r}{m} \right)^2 + r^2 \left(\frac{p_\theta}{mr^2} \right)^2 + r^2 \sin^2 \theta \left(\frac{p_\phi}{mr^2 \sin^2 \theta} \right)^2 \right) + U(r, \theta, \phi, t)$$

$$= \frac{1}{2m} \left(\dot{p}_r^2 + \frac{\dot{p}_\theta^2}{r^2} + \frac{\dot{p}_\phi^2}{r^2 \sin^2 \theta} \right) + U(r, \theta, \phi, t)$$

Sec 40, Prob 2:

For a uniformly rotating ref. frame:

$$L = \frac{1}{2}mv^2 + \vec{m\omega} \cdot (\vec{\omega} \times \vec{r}) + \pm m|\vec{\omega} \times \vec{r}|^2 + U$$

Hamiltonian

$$H = \left(\sum_i p_i \dot{q}_i - L \right) \Big|_{\dot{q} = \dot{q}(\varepsilon, p)}$$

where

$$\vec{p} \doteq \frac{\partial L}{\partial \vec{v}}$$

$$= m\vec{v} + m(\vec{\omega} \times \vec{r})$$

$$= m[\vec{v} + \vec{\omega} \times \vec{r}] \quad \text{velocity wrt inertial frame}$$

$$\rightarrow \vec{v} = \frac{\vec{p}}{m} - \vec{\omega} \times \vec{r}$$

thus,

$$H = \left(\vec{p} \cdot \vec{v} - L \right) \Big|_{\vec{v} = \frac{\vec{p}}{m} - \vec{\omega} \times \vec{r}}$$

$$= \vec{p} \cdot \left(\frac{\vec{p}}{m} - \vec{\omega} \times \vec{r} \right) - \frac{1}{2}m \left| \frac{\vec{p}}{m} - \vec{\omega} \times \vec{r} \right|^2$$

$$= m \left(\frac{\vec{p}}{m} - \vec{\omega} \times \vec{r} \right) \cdot (\vec{\omega} \times \vec{r}) - \frac{1}{2}m|\vec{\omega} \times \vec{r}|^2 + U$$

$$= \frac{\vec{p}^2}{m} - \cancel{\vec{p} \cdot (\vec{\omega} \times \vec{r})} - \frac{1}{2}m \left(\frac{\vec{p}^2}{m^2} + |\vec{\omega} \times \vec{r}|^2 - \cancel{\frac{2}{m} \vec{p} \cdot (\vec{\omega} \times \vec{r})} \right)$$
$$= \frac{\vec{p}^2}{m} + m|\vec{\omega} \times \vec{r}|^2 - \cancel{\frac{1}{2}m|\vec{\omega} \times \vec{r}|^2} + U$$

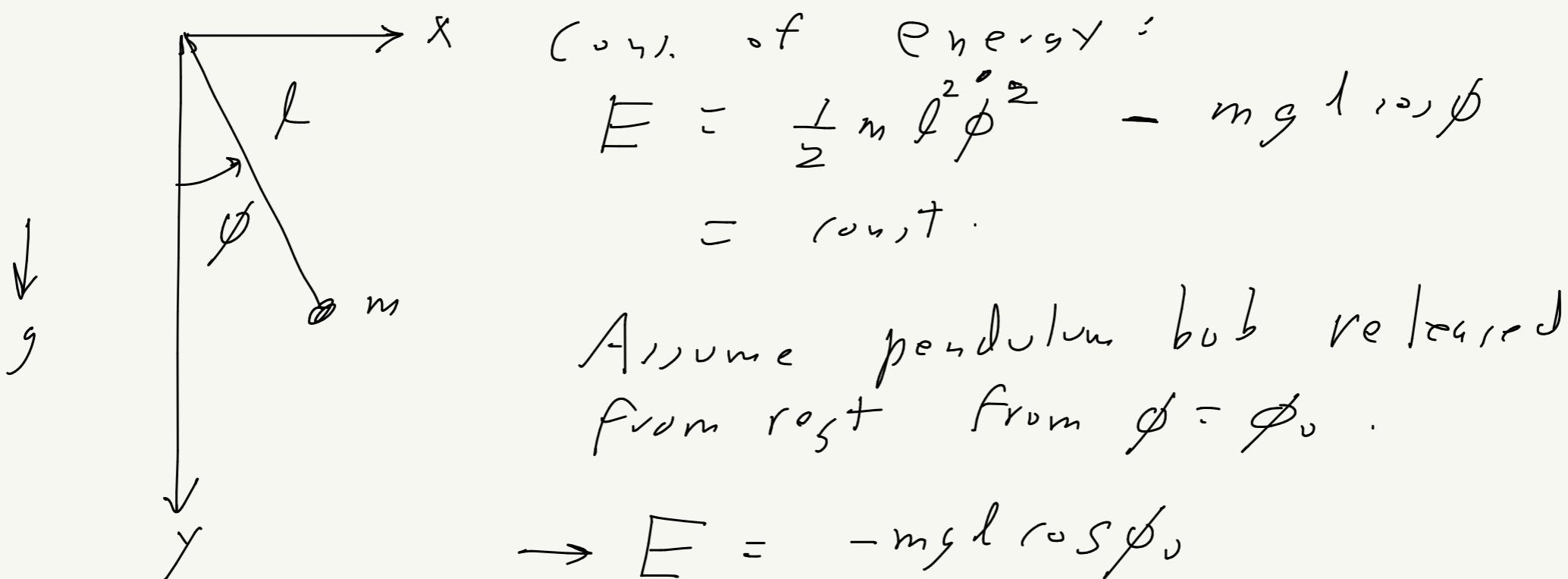
$$H = \frac{p^2}{2m} - \vec{p} \cdot (\vec{n} \times \vec{r}) + U$$

$$= \frac{p^2}{2m} - \vec{n} \cdot (\vec{r} \times \vec{p}) + U$$

$$= \frac{p^2}{2m} - \vec{n} \cdot \vec{p} + U$$

Sec 11, Prob 1:

Simple pendulum:



Assume pendulum bob released from rest from $\phi = \phi_0$.

$$\rightarrow E = -mg l \cos \phi_0$$

$$\text{thus, } -mg l \cos \phi_0 = \frac{1}{2} m l^2 \dot{\phi}^2 - mg l \cos \phi$$

$$\frac{d\phi}{dt} = \dot{\phi} = \pm \sqrt{\frac{2g}{l} (\cos \phi - \cos \phi_0)}$$

$$= \pm \sqrt{2} \omega_0 \sqrt{\cos \phi - \cos \phi_0}$$

where $\omega_0 = \sqrt{\frac{g}{l}}$ (angular freq in small-angle approximation)

Separable differential equation:

$$\sqrt{2} \omega_0 \int dt = \pm \int \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_0}}$$

$\frac{1}{4}$ period for $\phi = \phi_0 \rightarrow \phi = 0$

$$\sqrt{2} \omega_0 \frac{P}{4} = \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_0}}$$

$$\rightarrow P = \frac{1}{\omega_0} \frac{4}{\sqrt{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_0}}$$

Substitute: $\cos \phi = \cos^2(\phi_{0/2}) - \sin^2(\phi_{0/2})$

$$= 1 - 2 \sin^2(\phi_{0/2})$$

$$\cos \phi_0 = 1 - 2 \sin^2(\phi_{0/2})$$

$$\begin{aligned}\rightarrow \sqrt{ } &= \sqrt{2} \sqrt{\sin^2(\phi_{0/2}) - \sin^2(\phi_{0/2})} \\ &= \sqrt{2} \sin(\phi_{0/2}) \sqrt{1 - \frac{\sin^2(\phi_{0/2})}{\sin^2(\phi_{0/2})}} \\ &= \sqrt{2} \sin\left(\frac{\phi_0}{2}\right) \sqrt{1 - x^2}\end{aligned}$$

where $x = \frac{\sin(\phi_{0/2})}{\sin(\phi_{0/2})}$

Note: $\phi = 0, \phi_0 \rightarrow x = 0, 1$

$$\begin{aligned}dx &= \frac{1}{K} \frac{1}{2} \cos\left(\frac{\phi}{2}\right) d\phi \\ &= \frac{1}{2K} \sqrt{1 - \sin^2(\phi_{0/2})} d\phi \\ &= \frac{1}{2K} \sqrt{1 - K^2 x^2} d\phi, \quad K = \sin(\phi_{0/2})\end{aligned}$$

$$T^{ho}, \quad d\phi = \frac{2\pi dx}{\sqrt{1 - \pi^2 x^2}}$$

$$\rightarrow P = \frac{1}{\omega_0} \frac{4}{\sqrt{2}} \int_0^1 \frac{dx}{\sqrt{1 - \pi^2 x^2}} \quad \frac{1}{\sqrt{2} \sqrt{1 - x^2}}$$

$$= \frac{4}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1 - x^2} \sqrt{1 - \pi^2 x^2}}$$

$$= \frac{4}{\omega_0} K(\pi)$$

$\boxed{\text{complete elliptic integral of the 1st kind.}}$

Expand $K(\pi)$ keeping 1st non-zero correction:

$$\pi = \sin\left(\frac{\phi_0}{2}\right) \approx \frac{\phi_0}{2}$$

$$K(\pi) = \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-\pi^2 x^2}}$$

$$\approx \int_0^1 \frac{dx}{\sqrt{1-x^2}} \left(1 + \frac{1}{2} \pi^2 x^2 \right)$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2} \pi^2 \int_0^1 \frac{dx x^2}{\sqrt{1-x^2}}$$

$$= \sin^{-1}(1) + \frac{1}{2} \pi^2 \int_0^1 \frac{dx x^2}{\sqrt{1-x^2}}$$

$$\text{Now. } \sin^{-1}(1) = \frac{\pi}{2}$$

$$\int_0^1 dx \frac{x^2}{\sqrt{1-x^2}} = \int_0^{\pi/2} \frac{\cos \theta d\theta}{\sqrt{1-\sin^2 \theta}}$$

$$\left. \begin{aligned} x &= \sin \theta \\ dx &= \cos \theta d\theta \end{aligned} \right\} = \int_0^{\pi/2} \sin^2 \theta d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2} (1 - \cos 2\theta)$$

$$(using \quad \cos 2\theta = 1 - 2 \sin^2 \theta \Rightarrow \sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta))$$

$$= \frac{1}{2} \left(\frac{\pi}{2} - \frac{1}{2} \sin 2\theta \Big|_0^{\pi/2} \right)$$

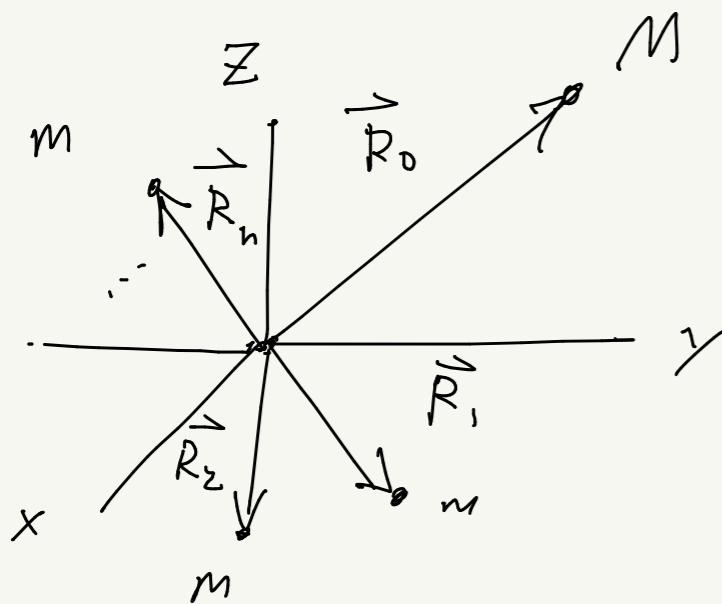
$$= \frac{\pi}{4}$$

$$T^{h_{ij}},$$

$$\begin{aligned} K(\tau) &\approx \frac{\pi}{2} + \frac{1}{2} \tau^2 \frac{\pi}{4} \\ &= \frac{\pi}{2} \left(1 + \frac{1}{4} \tau^2 \right) \quad \tau = \frac{\phi_0}{2} \\ &= \frac{\pi}{2} \left(1 + \frac{1}{16} \phi_0^2 \right) \end{aligned}$$

$$\rightarrow P = \frac{4}{\omega_0} K(\tau) \approx \frac{2\pi}{\omega_0} \left(1 + \frac{1}{16} \phi_0^2 \right)$$

Sec 13, Prob 1:



\vec{R}_0 : position vector for mass M
 $\vec{R}_i, i = 1, 2, \dots, n$: position vectors for n masses all with mass m

Com frame: $\vec{O} = M \vec{R}_0 + m \sum_i \vec{R}_i$

Relative position vectors:

$$\vec{r}_i = \vec{R}_i - \vec{R}_0$$

Thus, $\vec{O} = M \vec{R}_0 + m \sum_i (\vec{R}_0 + \vec{r}_i) = (M + nm) \underbrace{\vec{R}_0}_{\text{total mass } M} + m \sum_i \vec{r}_i$

$$\rightarrow \vec{R}_0 = - \frac{m}{M} \sum_i \vec{r}_i$$

Potential energy:

$$U = U(|\vec{R}_1 - \vec{R}_2|, \dots, |\vec{R}_1 - \vec{R}_0|, \dots, |\vec{R}_n - \vec{R}_0|)$$

$$= U(|\vec{r}_1 - \vec{r}_2|, \dots, |\vec{r}_1|, \dots, |\vec{r}_n|)$$



depends only on the relative

position vectors $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$

T_{net,Σ} energy:

$$T = \frac{1}{2} M |\vec{R}_0|^2 + \frac{1}{2} m \sum_i |\vec{r}_i|^2$$

Now: $\vec{R}_i = \vec{r}_i + \vec{R}_0$

$$\rightarrow |\vec{R}_i|^2 = |\vec{r}_i|^2 + |\vec{R}_0|^2 + 2 \vec{r}_i \cdot \vec{R}_0$$

and $\vec{R}_0 = -\frac{m}{M} \sum_i \vec{r}_i$

$$\rightarrow |\vec{R}_0|^2 = \frac{m^2}{M^2} \left| \sum_i \vec{r}_i \right|^2$$

+ b)

$$T = \frac{1}{2} M \frac{m^2}{M^2} \left| \sum_i \vec{r}_i \right|^2 + \frac{1}{2} m \sum_i |\vec{r}_i|^2$$

$$+ \frac{1}{2} m n |\vec{R}_0|^2 + m \sum_i \vec{r}_i \cdot \vec{R}_0$$

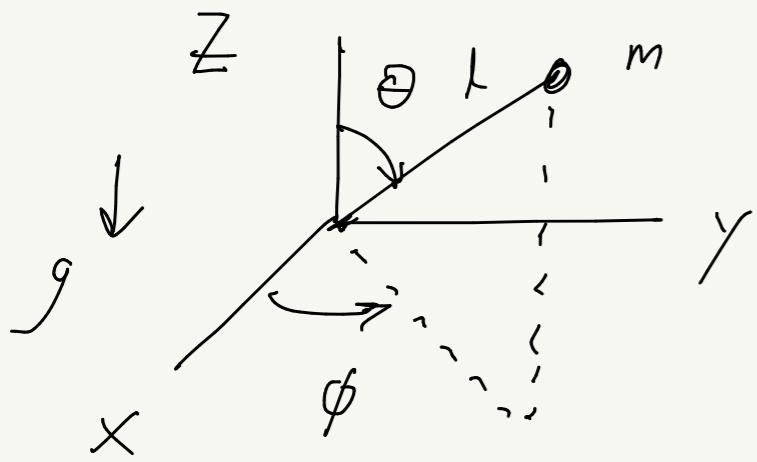
$$= \frac{1}{2} m \sum_i |\vec{r}_i|^2 + \frac{1}{2} \frac{m^2}{M^2} \left| \sum_i \vec{r}_i \right|^2$$

$$+ \frac{1}{2} m n \frac{m^2}{M^2} \left| \sum_i \vec{r}_i \right|^2 - \frac{m^2}{M} \left| \sum_i \vec{r}_i \right|^2$$

$$= \frac{1}{2} m \sum_i |\vec{r}_i|^2 + \frac{1}{2} \frac{m^2}{M^2} \left| \sum_i \vec{r}_i \right|^2 \underbrace{(M + mn - 2m)}_M$$

$$= \frac{1}{2} m \sum_i |\vec{r}_i|^2 - \frac{1}{2} \frac{m^2}{M} \left| \sum_i \vec{r}_i \right|^2$$

$$\rightarrow L = \frac{1}{2} m \sum_i |\vec{r}_i|^2 - \frac{1}{2} \frac{m^2}{M} \left| \sum_i \vec{r}_i \right|^2 - U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n)$$



$$\begin{aligned}x &= l \sin \theta \cos \phi \\y &= l \sin \theta \sin \phi \\z &= l \cos \theta\end{aligned}$$

$$U = mgz = mg l \cos \theta$$

$$\begin{aligned}T &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\&= \frac{1}{2} m (l^2 \dot{\theta}^2 + l^2 \sin^2 \theta \dot{\phi}^2)\end{aligned}$$

$$\begin{aligned}L &= T - U \\&= \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - mg l \cos \theta\end{aligned}$$

No explicit z dependence or ϕ dependence

$$\rightarrow E \equiv T + U = \text{const}, \quad M_z \equiv p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \text{const}$$

$$E = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + mg l \cos \theta$$

$$M_z = ml^2 \sin^2 \theta \dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{ml^2 \sin^2 \theta}$$

Thus,

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m l^2 \sin^2 \theta \left(\frac{M_z^2}{m^2 l^4 \sin^4 \theta} \right) + mg l \cos \theta$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{M_z^2}{2 m l^2 \sin^2 \theta} + mg l \cos \theta$$

$\underbrace{\quad}_{\text{effective potential} \equiv V_{\text{eff}}(\theta)}$

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + U_{eff}(\theta)$$

$$\rightarrow \dot{\theta} = \pm \sqrt{\frac{2}{ml^2} (E - U_{eff}(\theta))}$$

so

$$\frac{d\theta}{\pm \sqrt{\frac{2}{ml^2} (E - U_{eff}(\theta))}} = dt$$

$$\rightarrow t = \int \frac{d\theta}{\sqrt{\frac{2}{ml^2} (E - U_{eff}(\theta))}} + \text{const}$$

$\rightarrow t = t(\theta)$

~~Trajectory :~~

To find $\theta = \theta(\phi)$ write

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d\theta}{d\phi} \dot{\phi} = \frac{d\theta}{d\phi} \frac{M_2}{ml^2 \sin^2 \theta}$$

so

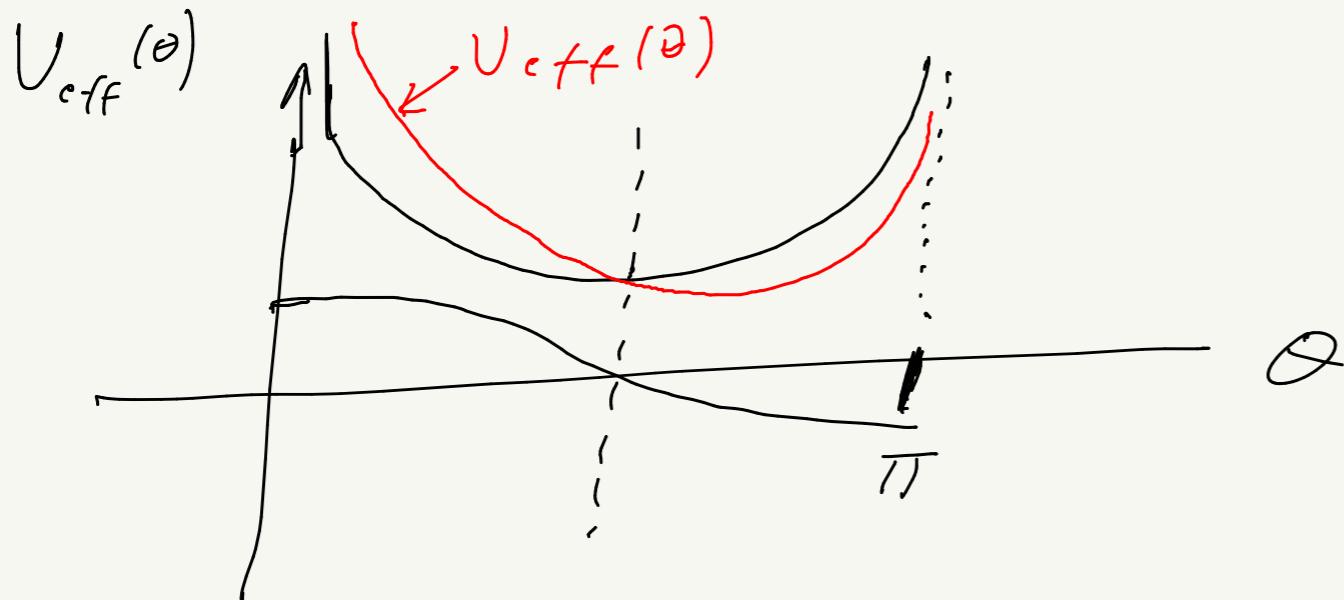
$$\frac{d\theta}{d\phi} \frac{M_2}{ml^2 \sin^2 \theta} = \pm \sqrt{\frac{2}{ml^2} (E - U_{eff}(\theta))}$$

$$\frac{d\theta / \sin^2 \theta}{\pm \frac{ml^2}{M_2} \sqrt{\frac{2}{ml^2} (E - U_{eff}(\theta))}} = d\phi$$

$$\rightarrow \phi = \int \frac{d\theta / \sin^2 \theta}{\sqrt{\frac{2 ml^2}{M_2^2} (E - U_{eff}(\theta))}} + \text{const}$$

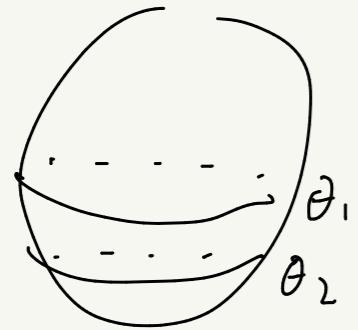
$$U_{\text{eff}}(\theta) = \frac{M_z^2}{2ml^2 \sin^2 \theta} + mg l \cos \theta$$

$$= \frac{M_z^2}{2ml^2 (1 - \cos^2 \theta)} + mg l \cos \theta$$



Turning points: $\theta = \theta_1, \theta_2$ for which

$$E - U_{\text{eff}}(\theta) = 0$$



$$E = U_{\text{eff}}(\theta)$$

$$= \frac{M_z^2}{2ml^2 (1 - \cos^2 \theta)} + mg l \cos \theta$$

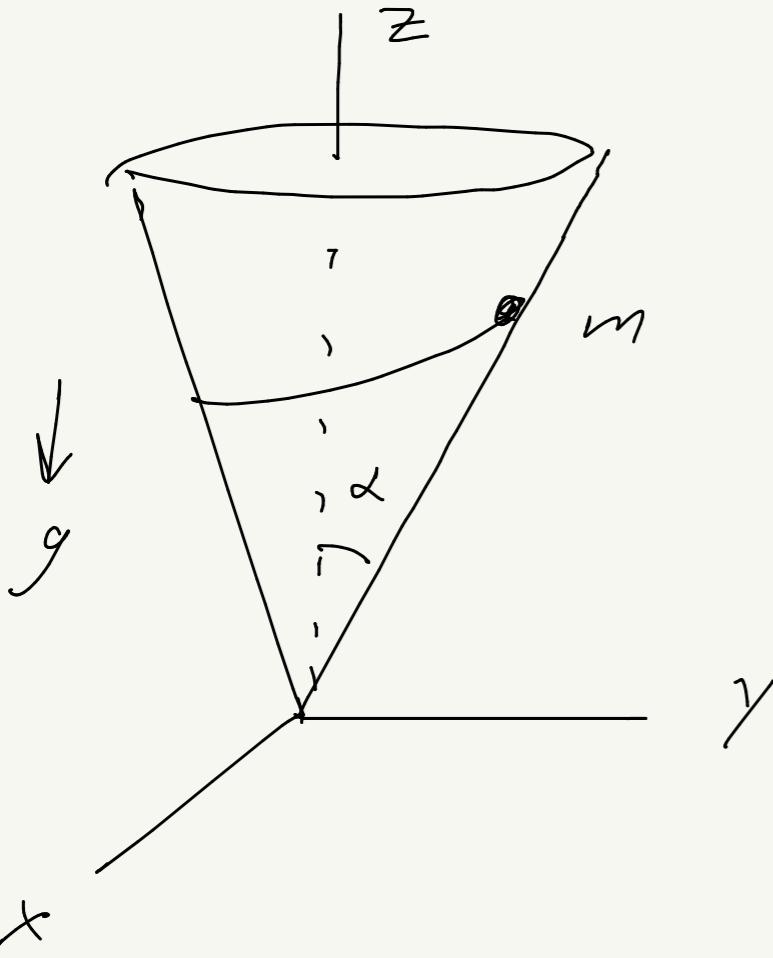
$$2ml^2 E (1 - \cos^2 \theta) = M_z^2 + 2m^2 g l^3 \cos \theta (1 - \cos^2 \theta)$$

$$2ml^2 E - 2ml^2 E \cos^2 \theta = M_z^2 + 2m^2 g l^3 \cos \theta - 2m^2 g l^3 \cos^3 \theta$$

$$\rightarrow (2ml^2 E - M_z^2) - 2m^2 g l^3 \cos \theta - 2ml^2 E \cos^2 \theta + 2m^2 g l^3 \cos^3 \theta = 0$$

Cubic equation for $\cos \theta$

Sec 14, Prob 2



spherical coord: (r, θ, ϕ)

$$\theta = \alpha$$

$$\rightarrow x = r \sin \alpha \cos \phi$$

$$y = r \sin \alpha \sin \phi$$

$$z = r \cos \alpha$$

$$\begin{aligned} T &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \frac{1}{2} m (r^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \\ &= \frac{1}{2} m (r^2 + r^2 \sin^2 \alpha \dot{\phi}^2) \end{aligned}$$

$$U = mgz = mg r \cos \alpha$$

$$L = T - U$$

No explicit t, ϕ dependence \rightarrow

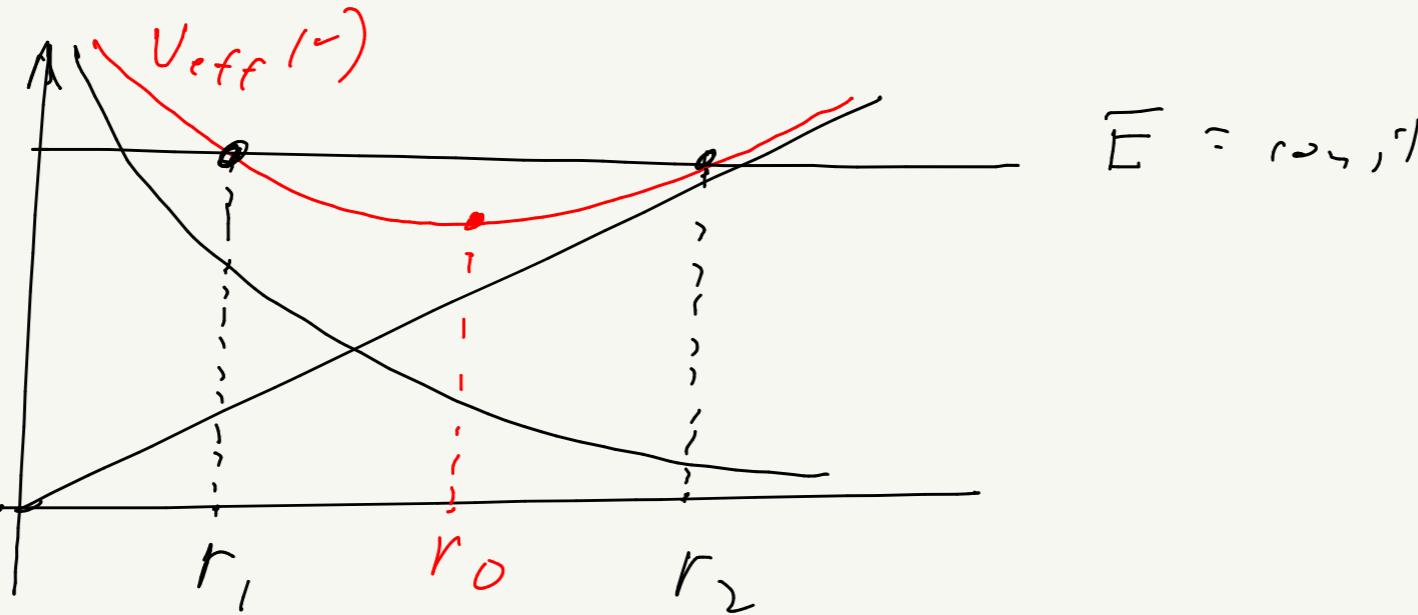
$$E = T + U = \text{const}$$

$$M_Z \equiv p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2 \alpha \dot{\phi} = \text{const}$$

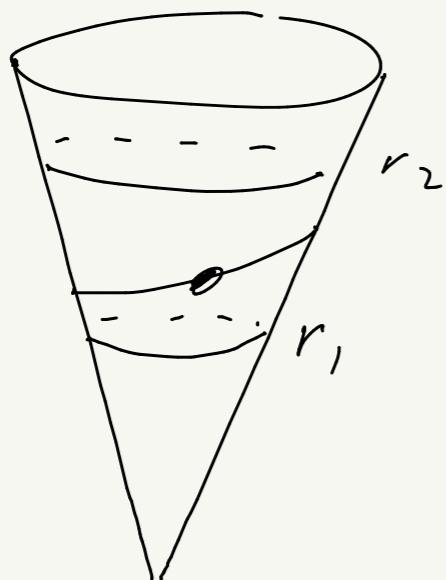
$$\rightarrow \dot{\phi} = \frac{M_Z}{m r^2 \sin^2 \alpha}$$

$$\begin{aligned} E &= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \sin^2 \alpha \left(\frac{M_Z}{m r^2 \sin^2 \alpha} \right)^2 + mg r \cos \alpha \\ &= \frac{1}{2} m \dot{r}^2 + \frac{M_Z^2}{2 m r^2 \sin^2 \alpha} + mg r \cos \alpha \\ &= \frac{1}{2} m \dot{r}^2 + V_{eff}(r) \end{aligned}$$

$$U_{\text{eff}}(r) = \frac{M_2^2}{2m r^2 \sin^2 \alpha} + m g r \cos \alpha$$



Turning points; $r = r_1, r_2$ when $E = U_{\text{eff}}(r)$



$$E = \frac{M_2^2}{2m r^2 \sin^2 \alpha} + m g r \cos \alpha$$

$$2mE r^2 \sin^2 \alpha = M_2^2 + 2m^2 g r^3 \sin^2 \alpha \cos \alpha$$

$$\underbrace{0 = M_2^2 - 2mE r^2 \sin^2 \alpha - 2m^2 g r^3 \sin^2 \alpha \cos \alpha}_{\text{cubic equation for } r}$$

$$r_1 \leq r \leq r_2$$

Integrals for $t = t(r)$, $\phi = \phi(r)$:

$$E = \frac{1}{2} m r^2 + U_{\text{eff}}(r)$$

$$\pm \sqrt{\frac{2}{m} (E - U_{\text{eff}}(r))} = \frac{dr}{dt}$$

$$\rightarrow t = \left[\int \frac{dr}{\sqrt{\frac{2}{m} (E - U_{\text{eff}}(r))}} + \text{const} \right]$$

Orbit equations:

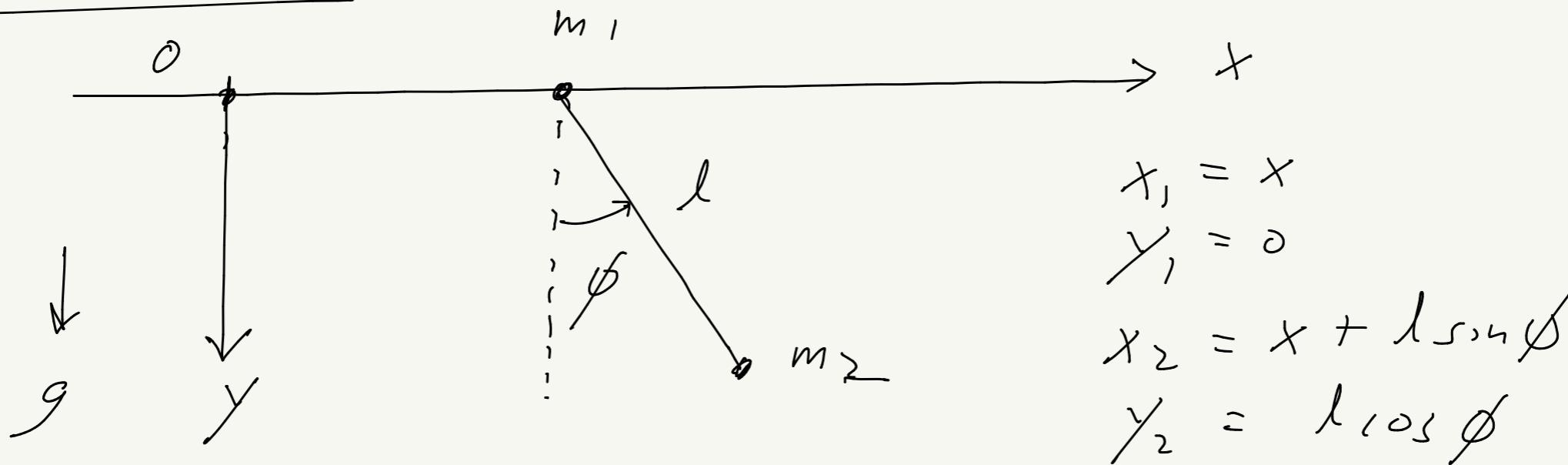
$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{M_z}{mr^2 \sin^2 \alpha}$$

$$so \quad \frac{dr}{d\phi} \frac{M_z}{mr^2 \sin^2 \alpha} = \pm \sqrt{\frac{2}{m}(E - V_{eff}(r))}$$

$$\frac{dr/r^2}{\frac{M_z \sin^2 \alpha}{mr^2}} = \pm d\phi$$

$$\rightarrow \phi = \frac{\pm M_z}{\sin^2 \alpha} \int \frac{dr/r^2}{\sqrt{\frac{2m(E - V_{eff}(r))}{m}}} + c_{const}$$

Sec 14, P., b 3 :



$$x_1 = x$$

$$y_1 = 0$$

$$x_2 = x + l \sin \phi$$

$$y_2 = l \cos \phi$$

$$U = -m g y_2$$

$$= -m g l \cos \phi$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left[(\dot{x} + l \dot{\phi} \cos \phi)^2 + l^2 \dot{\phi}^2 \sin^2 \phi \right]$$

$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left[\dot{x}^2 + 2 l \dot{x} \dot{\phi} \cos \phi + l^2 \dot{\phi}^2 \cos^2 \phi + l^2 \dot{\phi}^2 \sin^2 \phi \right]$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \dot{x} \dot{\phi} \cos \phi$$

$$L = T - U$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \dot{x} \dot{\phi} \cos \phi + m g l \cos \phi$$

No explicit t , x dependence \rightarrow

$$E = T + U = \text{const}$$

$$P_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2) \dot{x} + m_2 l \dot{\phi} \cos \phi = \text{const}$$

Since $p_x = \text{const}$, we can work in the x -com frame where $x_{\text{com}} = 0$ and $p_x = 0$

In this frame:

$$\begin{aligned} 0 &= x_{\text{com}} \\ &= \frac{m_1 x + m_2 (x + l \sin \phi)}{m_1 + m_2} \\ &= \frac{(m_1 + m_2) x + m_2 l \sin \phi}{m_1 + m_2} \\ &= x + \left(\frac{m_2}{m_1 + m_2} \right) l \sin \phi \\ \rightarrow &\boxed{x = - \left(\frac{m_2}{m_1 + m_2} \right) l \sin \phi} \end{aligned}$$

$$\text{Thus, } x = - \left(\frac{m_2}{m_1 + m_2} \right) l \dot{\phi} \cos \phi$$

$$\begin{aligned} \rightarrow E &= \frac{1}{2} (m_1 + m_2) \frac{\frac{m_2^2}{(m_1 + m_2)^2} l^2 \dot{\phi}^2 \cos^2 \phi + \frac{1}{2} m_2 l^2 \dot{\phi}^2}{-\frac{m_2^2 l^2}{m_1 + m_2} \dot{\phi}^2 \cos^2 \phi} - m_2 g l \cos \phi \\ &= \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left(1 - \left(\frac{m_2}{m_1 + m_2} \right) \cos^2 \phi \right) - m_2 g l \cos \phi \\ &= \frac{1}{2} \left(\frac{m_2}{m_1 + m_2} \right) l^2 \dot{\phi}^2 \left(m_1 + m_2 \sin^2 \phi \right) - m_2 g l \cos \phi \end{aligned}$$

Solu. for $\dot{\phi}$:

$$E + m_2 g l \cos \phi = \frac{1}{2} \left(\frac{m_2}{m_1 + m_2} \right) l^2 \dot{\phi}^2 (m_1 + m_2 \sin^2 \phi)$$

$$\frac{\pm \sqrt{\frac{2}{l^2} \left(\frac{m_1 + m_2}{m_2} \right) (E + m_2 g l \cos \phi)}}{m_1 + m_2 \sin^2 \phi} = \frac{d\phi}{dt}$$

$$\rightarrow t = \pm \sqrt{\frac{l^2}{2} \left(\frac{m_2}{m_1 + m_2} \right)} \int d\phi \sqrt{\frac{m_1 + m_2 \sin^2 \phi}{E + m_2 g l \cos \phi}} + c_{\text{const}}$$

NOTE: In the x -com frame $x = -\left(\frac{m_2}{m_1 + m_2}\right) l \sin \phi$

$$\begin{aligned} x_2 &= x + l \sin \phi \\ &= -\left(\frac{m_2}{m_1 + m_2}\right) l \sin \phi + l \sin \phi \\ &= \left(\frac{m_1}{m_1 + m_2}\right) l \sin \phi \\ &\equiv b \sin \phi \end{aligned}$$

$$y_2 = l \cos \phi \equiv a \cos \phi$$

Thus,

$$\left(\frac{x_2}{b}\right)^2 + \left(\frac{y_2}{a}\right)^2 = \sin^2\phi + \cos^2\phi = 1$$

So m_2 traces out an ellipse in the x -com
frame.

Sec 15, Prob 1 :

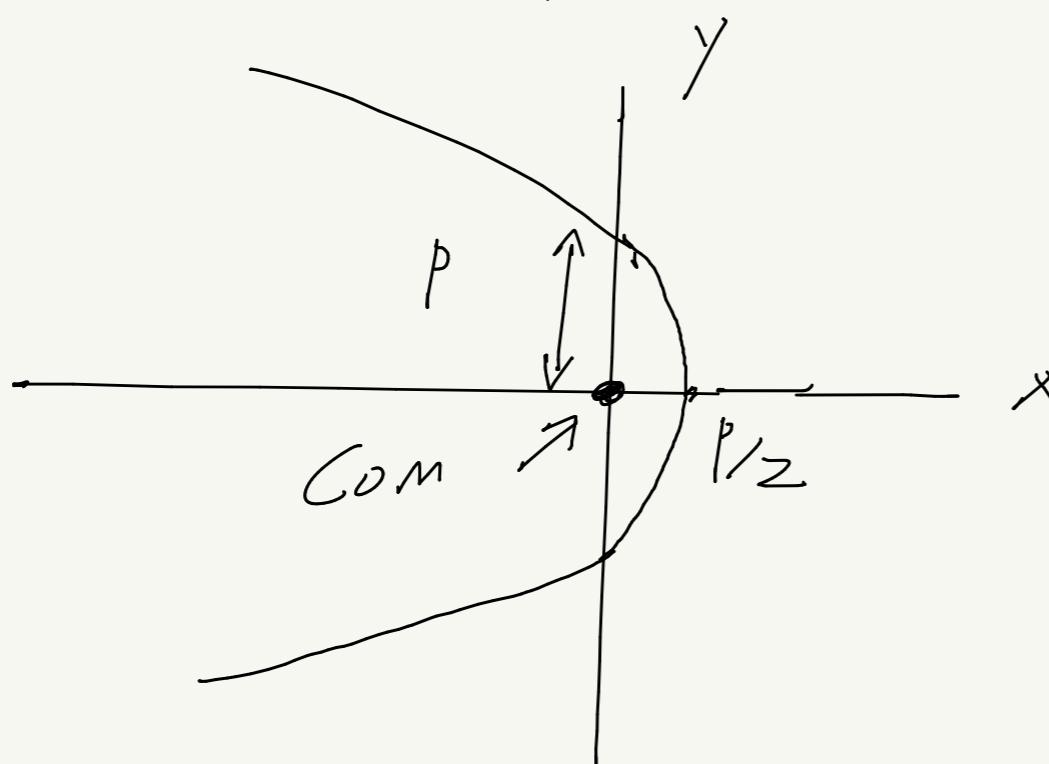
$$E = 0 \quad (\text{parabola}), \quad e = 1$$

$$U = -\alpha/r$$

$$\frac{p}{r} = 1 + e \cos \phi$$

$$= 1 + \cos \phi$$

$$\phi = 0 \rightarrow \frac{p}{r} = 2 \rightarrow r_{min} = \frac{p}{2}$$



Time dependence:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{m^2}{m^2 r^2}}} + C_1$$

$$= \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{m^2}{m^2 r^2}}} + C_2 t$$

$$= \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{m^2}{m^2 r^2}}} + C_3 t$$

$$\text{Recall: } p = \frac{m^2}{m\alpha} \rightarrow m^2 = m\alpha p$$

$$\rightarrow t = \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{m\alpha p}{m^2 r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{\alpha}{m}} \sqrt{\frac{2}{r} - \frac{p}{r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha p}} \int \frac{r dr}{\sqrt{\frac{2r}{p} - 1}} + \text{const}$$

$$\underline{\text{Let: }} \frac{2r}{p} - 1 = \xi^2 \quad (\xi = 0 \rightarrow \frac{2r}{p} - 1 = 0) \\ \rightarrow r = \frac{p}{2}\xi$$

$$\text{Then, } \frac{dr}{p} = \xi d\xi$$

$$dr = p\xi d\xi$$

$$\underline{\text{Also: }} \frac{2r}{p} = 1 + \xi^2 \rightarrow r = \frac{p}{2}(1 + \xi^2)$$

$$\rightarrow t = \sqrt{\frac{m}{\alpha p}} \int \frac{\frac{p}{2}(1 + \xi^2) p \xi d\xi}{\sqrt{1 + \xi^2}} + \text{const}$$

$$= \frac{p^2}{2} \sqrt{\frac{m}{\alpha p}} \int (1 + \xi^2) d\xi + \text{const}$$

$$S \vee t = \frac{1}{2} \sqrt{\frac{mp^3}{\alpha}} \left(\zeta + \frac{\zeta^3}{3} \right) + \text{const}$$

choose const = 0 so that at $t=0 \iff \zeta = 0$

thus,

$$\begin{cases} r = \frac{p}{2} (1 + \zeta^2) \\ t = \frac{1}{2} \sqrt{\frac{mp^3}{\alpha}} \left(\zeta + \frac{1}{3} \zeta^3 \right) \end{cases}$$

Recall: $\frac{p}{r} = 1 + \cos \phi$

$$S \quad 1 + \cos \phi = \frac{2}{1 + \zeta^2}$$

$$\begin{cases} \cos \phi = \frac{2}{1 + \zeta^2} - 1 \\ = \frac{1 - \zeta^2}{1 + \zeta^2} \end{cases}$$

Cartesian: $x = r \cos \phi$

$$= \frac{p}{2} (1 + \zeta^2) \left(\frac{1 - \zeta^2}{1 + \zeta^2} \right)$$

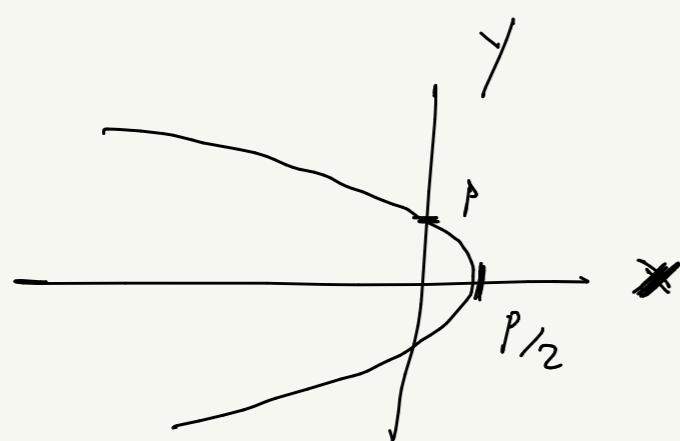
$$= \frac{p}{2} (1 - \zeta^2)$$

$$y = r \sin \phi = \frac{p}{2} (1 + \zeta^2) \sqrt{1 - \left(\frac{1 - \zeta^2}{1 + \zeta^2} \right)^2}$$

$$\begin{aligned}
 Y &= \frac{P}{2} \left(1 + \frac{1}{z^2} \right) \frac{1}{\sqrt{\left(1 + \frac{1}{z^2} \right)^2 - \left(1 - \frac{1}{z^2} \right)^2}} \\
 &= \frac{P}{2} \sqrt{1 + \frac{2}{z^2} + \frac{1}{z^4} - \left(1 - \frac{2}{z^2} + \frac{1}{z^4} \right)} \\
 &= \frac{P}{2} \sqrt{4z^2} \\
 &= Pz
 \end{aligned}$$

Thus,

$$\boxed{
 \begin{aligned}
 X &= \frac{P}{2} (1 - z^2) \\
 Y &= Pz
 \end{aligned}
 }$$



NOTE:

$$\begin{aligned}
 X &= \frac{P}{2} - \frac{P}{2} z^2 \\
 &= \frac{P}{2} - \frac{y^2}{2P}
 \end{aligned}$$

Parabola

$$X = y^2$$

$$\underline{Y=0}: \quad X = \frac{P}{2}$$

$$\underline{X=0}: \quad Y = \pm P$$

Sec 15, Prob 3:

$$\Delta\phi = 2 \int_{r_{min}}^{r_{max}} \frac{M dr / r^2}{\sqrt{2m(E-U) - M^2/r^2}} \quad (14.10)$$

Consider a small perturbation $\delta U(r)$ to the potential energy:

$$\rightarrow \Delta\phi = 2 \int_{r_{min}}^{r_{max}} \frac{M dr / r^2}{\sqrt{2m(E-U-\delta U) - M^2/r^2}}$$

$$= 2 \int_{r_{min}}^{r_{max}} \frac{M dr / r^2}{\sqrt{2m(E-U) - M^2/r^2} - 2m\delta U}$$

$$= 2 \int_{r_{min}}^{r_{max}} \frac{M dr / r^2}{\sqrt{2m(E-U) - M^2/r^2} \sqrt{1 - \frac{2m\delta U}{2m(E-U) - M^2/r^2}}}$$

$$\approx 2 \int_{r_{min}}^{r_{max}} \frac{M dr / r^2}{\sqrt{2m(E-U) - M^2/r^2}} \left(1 + \frac{m\delta U}{2m(E-U) - M^2/r^2} \right)$$

$$= 2 \int_{r_{min}}^{r_{max}} \frac{M dr / r^2}{\sqrt{2m(E-U) - M^2/r^2}} + 2 \int_{r_{min}}^{r_{max}} \frac{M m \delta U dr / r^2}{(2m(E-U) - M^2/r^2)^{3/2}}$$

$\Delta\phi_0$ $\delta\phi$

$$\frac{N_{Dw}}{\partial M} \left(\frac{1}{\sqrt{2m(E-U) - m^2/r^2}} \right) = -\frac{1}{2} \frac{(1-2m/r^2)}{\left(2m(E-U) - m^2/r^2 \right)^{3/2}}$$

$$= \frac{M/r^2}{\left(2m(E-U) - m^2/r^2 \right)^{3/2}}$$

Thus,

$$\delta\phi = \frac{\partial}{\partial M} \left(\int_{r_{min}}^{r_{max}} \frac{2m \delta U dr}{\sqrt{2m(E-U) - m^2/r^2}} \right)$$

Consider the case $U(r) = -\alpha/r$

Then: $\Delta\phi_0 = 2\pi$ (since a bound orbit is an ellipse, which is closed)

$$\text{Also: } \delta\phi = \frac{\partial}{\partial M} \left(\int_{r_{min}}^{r_{max}} \frac{2m \delta U dr}{\sqrt{2m(E + \alpha/r) - m^2/r^2}} \right)$$

In the integral, we can use the solution for the unperturbed motion:

$$\frac{P}{r} = 1 + e \cos\phi \rightarrow -\frac{P}{r^2} dr = -e \sin\phi d\phi$$

$$\text{so } dr = \frac{e}{P} r^2 \sin\phi d\phi$$

$$r = r_{min}, r_{max} \leftrightarrow \phi = 0, \pi$$

$$\begin{aligned} \sqrt{\quad} &= \sqrt{2m(E + \frac{\alpha}{r}) - \frac{m^2}{r^2}} \\ &= \sqrt{2mE + \frac{2m\alpha(1+e^{i\omega\phi})}{P} - m^2 \frac{(1+e^{i\omega\phi})^2}{P^2}} \\ &= \frac{1}{P} \sqrt{2mEp^2 + 2m\alpha p(1+e^{i\omega\phi}) - m^2(1+e^{i\omega\phi})^2} \end{aligned}$$

Re (q11): $P = \frac{M^2}{m\alpha}$

$$e = \sqrt{1 + \frac{2EM^2}{m\alpha^2}}$$

Then, $2mEp^2 = 2mE \frac{M^4}{m^2\alpha^2} = \left(\frac{2EM^2}{m\alpha^2}\right) M^2$

$$2m\alpha p = 2M^2$$

$$\begin{aligned} \rightarrow \sqrt{\quad} &= \frac{m\alpha}{M^2} \sqrt{\left(\frac{2EM^2}{m\alpha^2}\right) h^2 + 2M^2(1+e^{i\omega\phi}) - M^2(1+e^{2i\omega\phi}) + 2e^{i\omega\phi}} \\ &= \frac{m\alpha}{M} \sqrt{\underbrace{\frac{2EM^2}{m\alpha^2} + 1}_{e^2} - e^{2i\omega\phi}} \\ &= \frac{m\alpha e}{M} \sin \phi \end{aligned}$$

Theo,

$$\delta\phi = \frac{\partial}{\partial M} \left(\int_0^{\pi} \frac{2\mu \delta U r^2 \frac{e}{p} \sin\phi d\phi}{\max_{M} \sin\phi} \right)$$

$$= \frac{\partial}{\partial M} \left(\frac{2M}{\alpha p} \int_0^{\pi} r^2 \delta U d\phi \right)$$

$$= \frac{\partial}{\partial M} \left(\frac{2m}{M} \int_0^{\pi} r^2 \delta U d\phi \right) \quad \left(\begin{array}{l} \text{using} \\ p = \frac{M^2}{m \alpha} \end{array} \right)$$

where $\frac{p}{r} = 1 + e \cos\phi$

$$(a) \delta U = \frac{\beta}{r^2}$$

$$\rightarrow \delta\phi = \frac{\partial}{\partial M} \left(\frac{2m}{M} \int_0^{\pi} r^2 \frac{\beta}{r^2} d\phi \right)$$

$$= \frac{\partial}{\partial M} \left(\frac{2m}{M} \beta \pi \right)$$

$$= -\frac{2\pi m \beta}{M^2} \quad \boxed{= -\frac{2\pi \beta}{\alpha p}}$$

$$(b) \quad \delta U = \frac{\gamma}{r^3}$$

$$\begin{aligned}
\rightarrow \delta \phi &= \frac{\partial}{\partial M} \left(\frac{2m}{M} \int_0^\pi r^2 \frac{\gamma}{r^3} d\phi \right) \\
&= \frac{\partial}{\partial M} \left(\frac{2m\gamma}{M} \int_0^\pi \frac{d\phi}{r} \right) \\
&= \frac{\partial}{\partial M} \left(\frac{2m\gamma}{M} \frac{1}{p} \int_0^\pi (1 + e^{i\omega_0 \phi}) d\phi \right) \\
&= \frac{\partial}{\partial M} \left(\frac{2m\gamma}{Mp} \left[\pi + e^{i\omega_0 \phi} \Big|_0^\pi \right] \right) \\
&= \frac{\partial}{\partial M} \left(\frac{2\pi m\gamma}{Mp} \right) \\
&= \frac{\partial}{\partial M} \left(\frac{2\pi m^2 \gamma \alpha}{M^3} \right) \quad \xleftarrow{\text{using}} \quad p = \frac{M^2}{m \alpha} \\
&= -\frac{6\pi m^2 \gamma \alpha}{M^4} \\
&= -\frac{6\pi m^2 \gamma \alpha}{p^2 m^2 \alpha^2} \\
&= \boxed{-\frac{6\pi \gamma}{p^2 \alpha}}
\end{aligned}$$