Notes: Thurs 8/27

1) Elliptic functions
2) Simple pendulum

go beyond small angle approx

Elliptic Functions:

$$\int \frac{dx}{\sqrt{1-x^2}} = ? = \sin^{-1}(x) + \text{const}$$

$$\sin^{-1}(x) = \arcsin(x)$$

$\rightarrow \sin^2\theta + \cos^2\theta = 1$

subst: $x = \sin\theta \rightarrow \sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$dx = \cos\theta \, d\theta$

$$\int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \int d\theta = \theta = \sin^{-1}(x)$$

$x = \sin\theta$

$\theta = \sin^{-1}(x)$

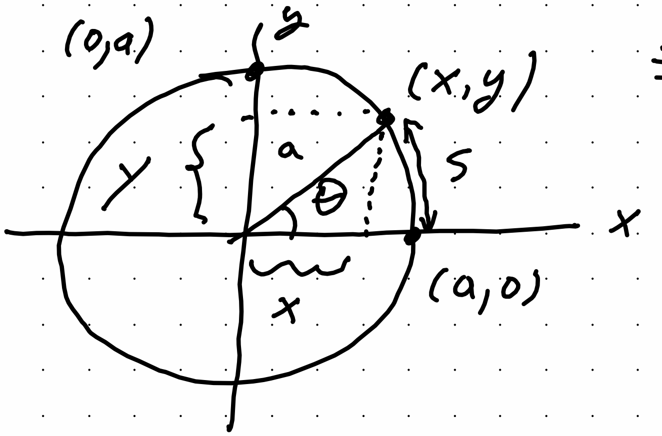
$$\frac{d}{d\theta}\sin\theta = \cos\theta \qquad \lim_{h\to 0}\left(\frac{\sin(\theta+h) - \sin\theta}{h}\right)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

circular functions..

$x^2+y^2=a^2$, a = radius



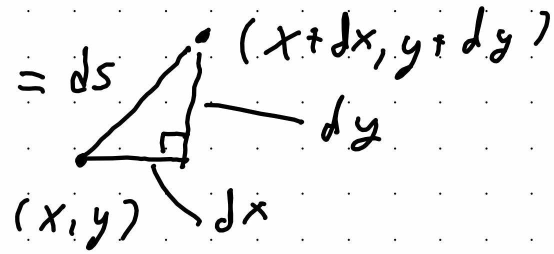
Def: $\sin\theta = \frac{y}{a}$, $\cos\theta = \frac{x}{a}$

s : arclength from $(a,0)$ to (x,y)

$s = a\theta$

$$\theta = \frac{s}{a} = \frac{1}{a}\int ds = \frac{1}{a}\int\sqrt{dx^2+dy^2} = \int d\theta$$

$\sqrt{dx^2+dy^2} = ds$



Given: $x^2+y^2=a^2$

Follows: (i) $\cancel{a^2}\cos^2\theta + \cancel{a^2}\sin^2\theta = \cancel{a^2} \rightarrow \boxed{\cos^2\theta+\sin^2\theta = 1}$

(ii) $\boxed{\frac{d}{d\theta}\sin\theta} = \frac{1}{a}\frac{dy}{d\theta} = \frac{\cancel{1}}{\cancel{a}}\frac{dy}{\frac{\cancel{1}}{\cancel{a}}\sqrt{dx^2+dy^2}} = \frac{1}{\sqrt{\left(\frac{y}{x}\right)^2+1}}$

$\cancel{2}x\,dx + \cancel{2}y\,dy = 0 \rightarrow dx = \frac{-y}{x}dy$

$= \frac{1}{\sqrt{\frac{x^2+y^2}{x^2}}} = \frac{x}{a} = \boxed{\cos\theta}$

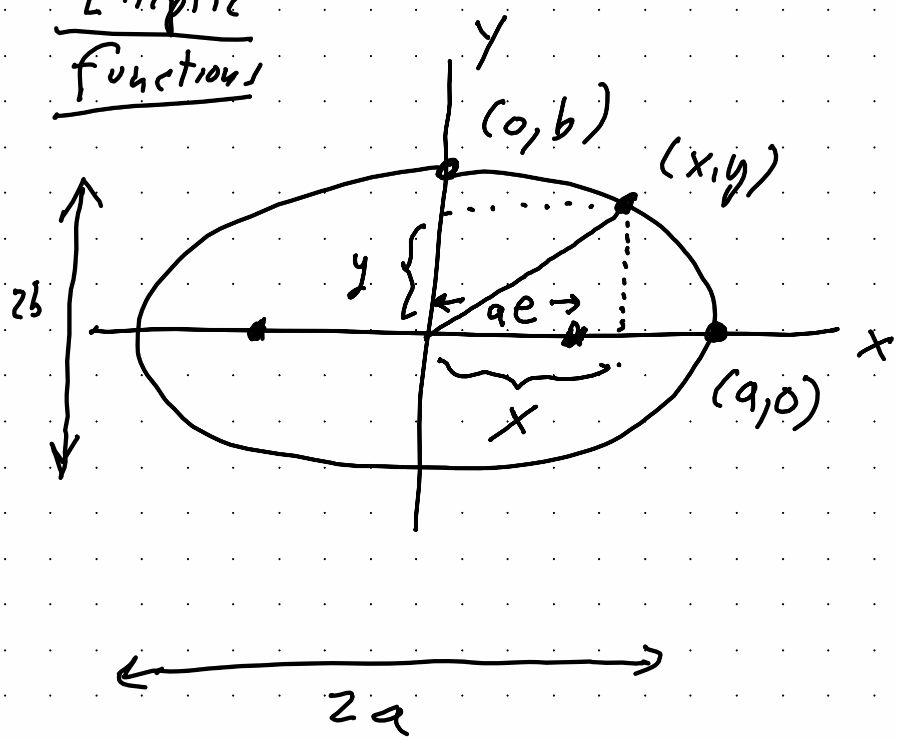
$$\frac{d(\sin\theta)}{d\theta} = \cos\theta \quad \rightarrow \quad \int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta$$

$$x = \sin\theta$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$$

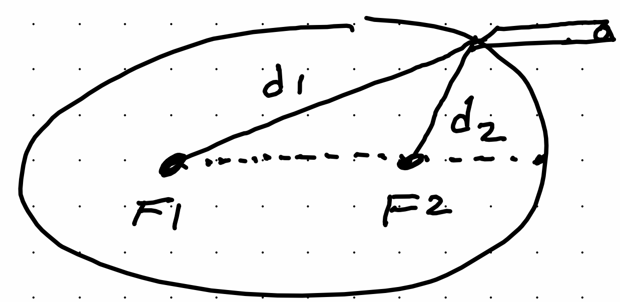
$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \theta = \sin^{-1}(x)}$$

Elliptic functions

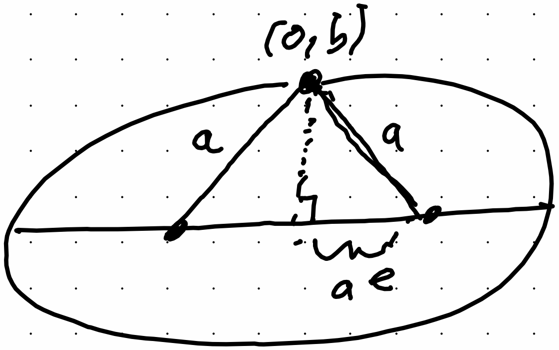


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

eccentricity : $e = 0$ (for circle)

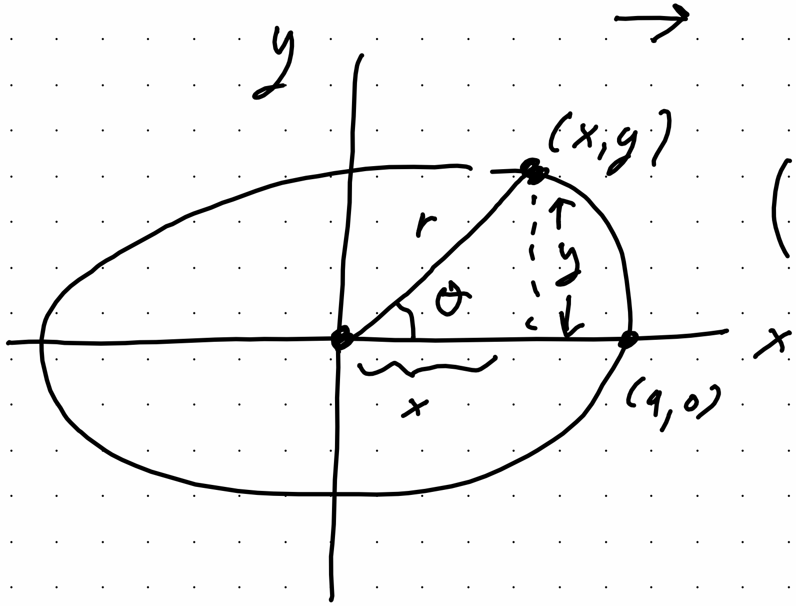


$$d_1 + d_2 = 2a$$



$$(ae)^2 + b^2 = a^2$$
$$a^2e^2 + b^2 = a^2$$
$$e^2 = \frac{a^2 - b^2}{a^2} = 1 - \left(\frac{b}{a}\right)^2$$

$$\rightarrow \quad e = \sqrt{1 - \left(\frac{b}{a}\right)^2} \equiv \underbrace{k}_{\text{notation used in elliptic function}}$$



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$x^2 + y^2 = r^2$$
$$x = r\cos\theta, \quad y = r\sin\theta$$

Define: $\quad cn(u;k) \equiv \frac{x}{a}, \quad sn(u;k) \equiv \frac{y}{b}$

$$dn(u;k) \equiv \frac{r}{a} \quad \text{— (= 1 for a circle)}$$

where $u = \frac{1}{b}\int_0^\theta r\,d\theta$ — $y = r\sin\theta$

(= θ for a circle)

$$ds = \sqrt{dx^2 + dy^2}$$
$$= \sqrt{dr^2 + r^2 d\theta^2}$$

Given: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, $\quad x^2 + y^2 = r^2$ $\qquad dn(u;k) = \frac{r}{a}$

Follow: (i) $cn^2(u;k) + sn^2(u;k) = 1$

(ii) $dn^2(u;k) + k^2 sn^2(u;k) = 1$

$$u = \frac{1}{b}\int_0^\theta r\,d\theta$$

(iii) $\frac{d}{du} sn(u;k) = cn(u;k)\, dn(u;k)$

$\frac{d}{du} cn(u;k) = -sn(u;k)\, dn(u;k)$

$\frac{d}{du} dn(u;k) = -k^2 sn(u;k)\, cn(u;k)$

Analogous to

$\frac{d\sin\theta}{d\theta} = \cos\theta$

$\frac{d\sin\theta}{\cos\theta} = d\theta$

$x = \sin\theta$

$\int \frac{dx}{\sqrt{1-x^2}} = \int d\theta = \theta = \sin^{-1} x$

Integrate: $\frac{d\, sn(u;k)}{du} = cn(u;k)\, dn(u;k)$

$$\int \frac{d\, sn(u;k)}{cn(u;k)\, dn(u;k)} = \int du = u$$

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}} = u + \text{const} = sn^{-1}(x;k) + \text{const} \qquad x \equiv sn(u;k)}$$

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + \text{const}}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \equiv K(k)$$

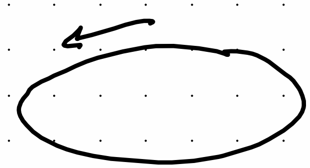
(complete elliptic integral of 1st kind)

→ related to Period of a pendulum going beyond small-angle approximation

$$\int_0^1 \frac{\sqrt{1-k^2x^2}\,dx}{\sqrt{1-x^2}} \equiv E(k)$$

(complete elliptic integral of 2nd kind

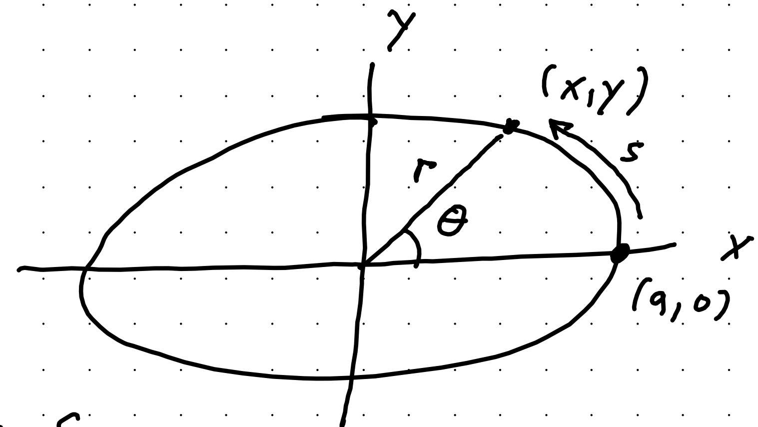
→ circumference around an ellipse



circle: $C = 2\pi a$

Notes: Tuesday 9/1

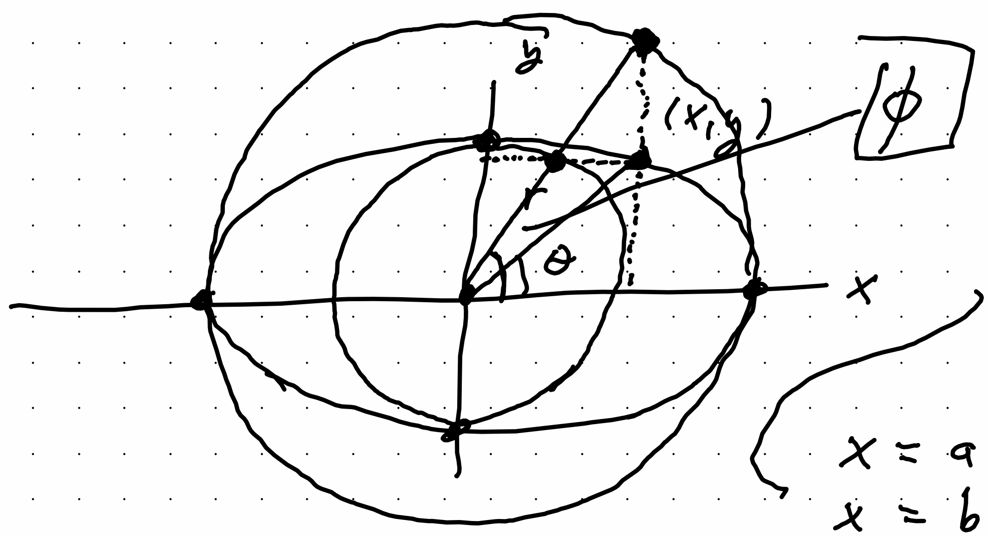
1) Review of elliptic functions
2) Simple pendulum



$$u = \frac{1}{b}\int_{(a,0)}^{(x,y)} r\,d\theta$$

$$bu = \int_0^\theta \underbrace{r\,d\theta} \le \int_0^\theta ds = s$$

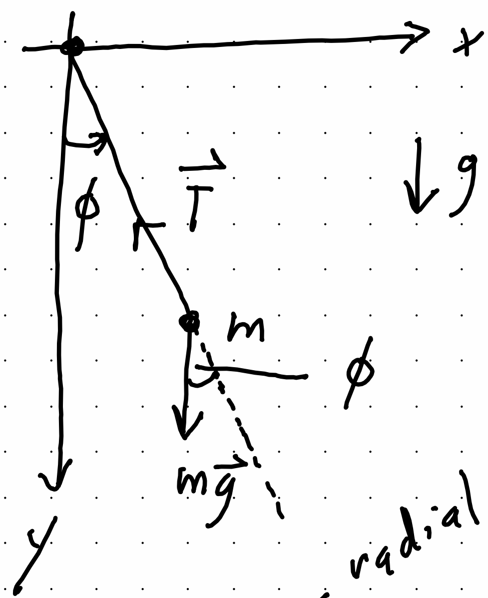
$$ds = \sqrt{dx^2+dy^2} = \sqrt{\underbrace{dr^2}+r^2d\theta^2} \ge r\,d\theta$$

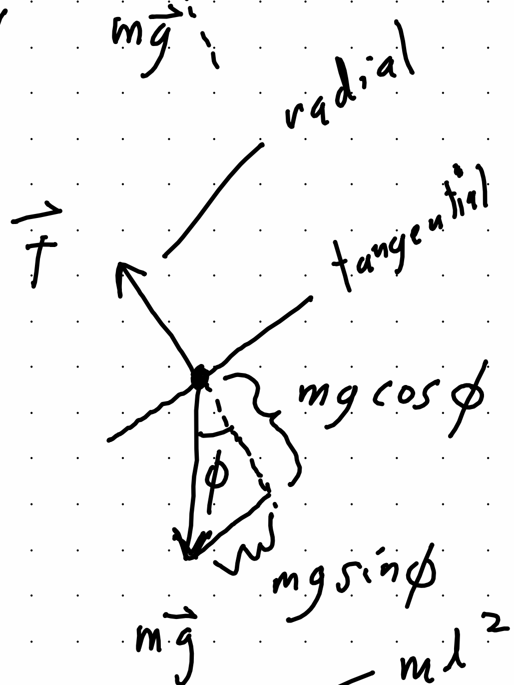


$x = r\cos\theta$
$y = r\sin\theta$
$x = a\cos\phi$
$y = b\sin\phi$

$x = a\cos\phi$, $y = \underline{\underline{a}}\sin\phi$
$x = b\cos\phi$ $y = b\sin\phi$

simple pendulum:





$$\text{Torque} = \overset{ml^2}{I}\;\overset{\ddot{\phi}}{\alpha}$$

(i) "Freshman physics"

forces, free-body diagram

$\longrightarrow$ EOM, tension

tangential:

$$-mg\sin\phi = m\,a_{\text{tangential}}$$

$$-\cancel{m}g\sin\phi = \cancel{m}l\ddot{\phi}$$

ϕ : angular displacement [rad]

$\dot{\phi}$: angular velocity [rad/sec]

$\ddot{\phi}$: angular accel [rad/sec^2]

$$\boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi} \quad \text{(EOM)}$$

radial:
$$T - mg\cos\phi = m\,a_{\text{centripetal}}$$
$$T - mg\cos\phi = m\dot{\phi}^2 l$$

$$T = mg\cos\phi + m\dot{\phi}^2 l$$

(ii) Small angle approximation:

$\sin\phi \approx \phi \longleftarrow \phi \ll 1 = 57$ degrees
radian

$$\ddot{\phi} = -\frac{g}{l}\sin\phi \approx -\frac{g}{l}\phi \longrightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\phi}$$

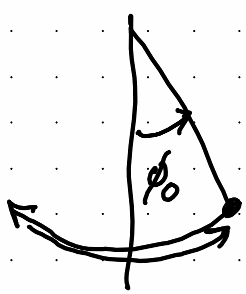
small angle approx.

$$\phi(t) = \boxed{a}\cos(\omega t) + \boxed{b}\sin(\omega t)$$

where $\omega \equiv \sqrt{\frac{g}{l}}$

determined by initial conditions

<u>ICs:</u> If $\phi(0) = \phi_0$ (at rest)



then $\boxed{\phi(t) = \phi_0\cos(\omega t)}$

<u>Period:</u> $P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$

independent of ϕ_0 !!

(iii) Lagrangian approach $\quad T \equiv$ Kinetic energy

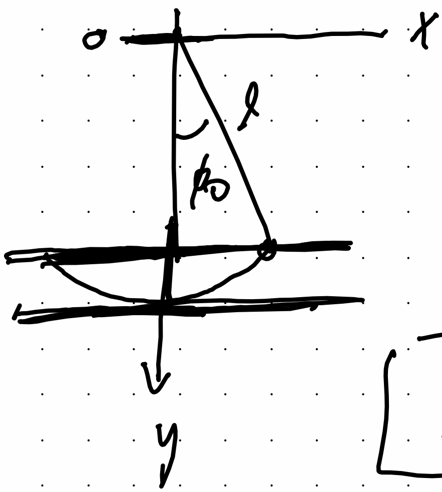
$U \equiv$ Potential energy

$$L = T - U$$

$$T = \frac{1}{2} m l^2 \dot{\phi}^2 \quad \left(= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \right)$$

$y = l \cos\phi$

$x = l \sin\phi$



$$U = -mgl\cos\phi + \text{const} \leftarrow U = -mgy$$

$$U = mgl(1 - \cos\phi)$$

$$\boxed{L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi}$$

action

$$S = \int_{t_1}^{t_2} dt\, L(\phi, \dot{\phi}, t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \qquad \text{Lagrange's equation}$$

$$\frac{d}{dt}\left(m l^2 \dot{\phi}\right) = -mgl\sin\phi$$

$$m l^2 \ddot{\phi} = -mgl\sin\phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi} \quad \text{(EOM)}$$

(iv) solving $\ddot{\phi} = -\frac{g}{l}\sin\phi$ (2nd order non-linear ODE)

hard!!

$$E = \text{const} = T + U = \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$$

$$E = 0 - mgl\cos\phi_0 = -mgl\cos\phi_0$$

release from rest from $\phi = \phi_0$

$$-mgl\cos\phi_0 = \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$$

$$-\cancel{m}gl(\cos\phi_0 - \cos\phi) = \frac{1}{2}\cancel{m}l^2\dot{\phi}^2$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{-2\frac{g}{l}(\cos\phi_0 - \cos\phi)}$$

$|\phi| \leq \phi_0$

Separable 1st order ODE

$$\int \frac{d\phi}{\sqrt{-\frac{2g}{l}(\cos\phi_0 - \cos\phi)}} = \int dt = t + \text{const}$$

$$t + t_0 = \int \frac{d\phi}{\sqrt{-2\frac{g}{l}(\cos\phi_0 - \cos\phi)}}$$

$$\frac{1}{\sqrt{a + bx^2}}$$

substitution:

$$\cos\phi = 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\cos\phi_0 = 1 - 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$\cos\phi = \cos\left(2\left(\frac{\phi}{2}\right)\right)$$

$$= \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\rightarrow \cos\phi_0 - \cos\phi = -2\left(\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)\right)$$

$$t + t_0 = \int \frac{d\phi}{2\sqrt{\frac{g}{l}}\sqrt{\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)}}$$

$$|\phi| \leq \phi_0$$

$$= \frac{1}{2\sqrt{\frac{g}{l}}} \int \frac{d\phi}{\sin\left(\frac{\phi_0}{2}\right)\sqrt{1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)}}}$$

let $x = \frac{\sin(\phi/2)}{\sin\left(\frac{\phi_0}{2}\right)}$

$$\boxed{x = \frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}} \longrightarrow dx = \frac{1}{\sin\left(\frac{\phi_0}{2}\right)} \frac{1}{2} \cos\left(\frac{\phi}{2}\right) d\phi$$

$\sqrt{1-x^2}$

denominator

$$d\phi = \frac{2\sin\left(\frac{\phi_0}{2}\right) dx}{\cos\left(\frac{\phi}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1-\sin^2\left(\frac{\phi}{2}\right)}}$$

$$= \frac{2\sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1-\underbrace{\sin^2\left(\frac{\phi_0}{2}\right)}_{k^2} x^2}}$$

Find this out

① $\phi(t) = \boxed{\cancel{}}$

② Period = ??

③ Redo the analysis using Lagrange multiplier for find tension in string

$t + t_0 = \int \bigcirc$ — integrand for $sn^{-1}(x; k)$

$k = \sin\left(\frac{\phi_0}{2}\right)$

Lec #4 : Thurs 9/3

$$\phi(t) = 2\sin^{-1}\left[k\,\mathrm{sn}\left(\omega_0\left(t+\frac{P}{4}\right); k\right)\right] \star$$

$$k \equiv \sin\left(\frac{\phi_0}{2}\right), \quad \omega_0 = \sqrt{\frac{g}{\ell}}$$

$$P = 4\sqrt{\frac{\ell}{g}}\, K(k) = 4\sqrt{\frac{\ell}{g}} \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

small angle approx

$$P_0 = 2\pi\sqrt{\frac{\ell}{g}}$$

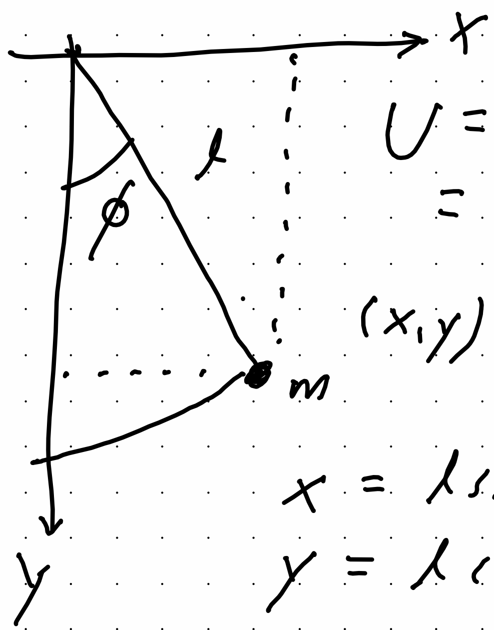
$$\frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)} = x = \mathrm{sn}\left[\sqrt{\frac{g}{\ell}}(t+t_0); k\right]$$

$$\sqrt{\frac{\ell}{g}} \int \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} = \sqrt{\frac{\ell}{g}}\,\mathrm{sn}^{-1}(x; k) = t + t_0$$

$$\mathrm{sn}^{-1}(x; k) = \sqrt{\frac{g}{\ell}}(t+t_0)$$

$$P = 4\sqrt{\frac{l}{g}}\,K(k) = \underbrace{2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{16}\phi_0^2+\dots\right)}_{\text{problem Landau 11,1}}$$

Lagrange multiplier:



$$U = -mgy$$
$$= -mgl\cos\phi$$

$$x = l\sin\phi$$
$$y = l\cos\phi$$

$$\underbrace{\phi(x,y)}_{\text{constraint function}} = x^2+y^2-l^2 = 0$$

$$x^2+y^2 = l^2\sin^2\phi + l^2\cos^2\phi$$
$$= l^2$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$
$$= \frac{1}{2}ml^2\dot{\phi}^2$$

$$L = T - U + \lambda \varphi$$

Lagrange multiplier

$\lambda, \cancel{\dot{\lambda}}$

$L(x, \dot{x}, y, \dot{y}, t)$ $\quad q = (x, y)$ $\quad \dot{q} = (\dot{x}, \dot{y})$

$L(r, \dot{r}, \phi, \dot{\phi}, t)$ $\quad q = (r, \phi)$ $\quad \dot{q} = (\dot{r}, \dot{\phi})$

$\lambda(t)$
$r(t)$
$\phi(t)$

$L(\phi, \dot{\phi}, t)$ $\quad \varphi(x, y) = x^2 + y^2 - l^2 = 0$ $\quad \boxed{\varphi(r, \phi) = r - l = 0}$

$$\boxed{L = \frac{1}{2} m(\dot{r}^2 + \underline{\underline{r^2 \dot{\phi}^2}}) + mg \underbrace{r\cos\phi}_{y} + \lambda (r - l)}$$

$\underline{r}$: $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \rightarrow \boxed{\cancel{m\ddot{r}} = mr\dot{\phi}^2 + mg\cos\phi + \lambda}$

$\underline{\phi}$: $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \rightarrow \underline{\underline{\frac{d}{dt}}}(mr^2\dot{\phi}) = -mgr\sin\phi$

$\underline{\lambda}$: $\quad r - l = 0$ ✓ $\quad \cancel{2m r\dot{r}\dot{\phi}} + mr^2\ddot{\phi} = -mgr\sin\phi$

$$r - l = 0 \longrightarrow r = l$$

$$\dot{r} = 0$$

$$\ddot{r} = 0$$

$$\rightarrow \quad \ddot{\phi} = -\frac{g}{l}\sin\phi$$

$$ml\dot{\phi}^2 + mg\cos\phi + \lambda = 0$$

$$\boxed{\lambda = -\underbrace{(mg\cos\phi + ml\dot{\phi}^2)}_{T}}$$

$$\lambda = -T$$

$$L = T - U + \lambda \varphi$$

$$U = U(x,t)$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\varphi(x,y,t)$$

$$U(x,y,t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}(m\dot{x}) = -\frac{\partial U}{\partial x} + \lambda \frac{\partial \varphi}{\partial x}$$

$$\frac{d}{dt}(m\dot{y}) = -\frac{\partial U}{\partial y} + \lambda \frac{\partial \varphi}{\partial y}$$

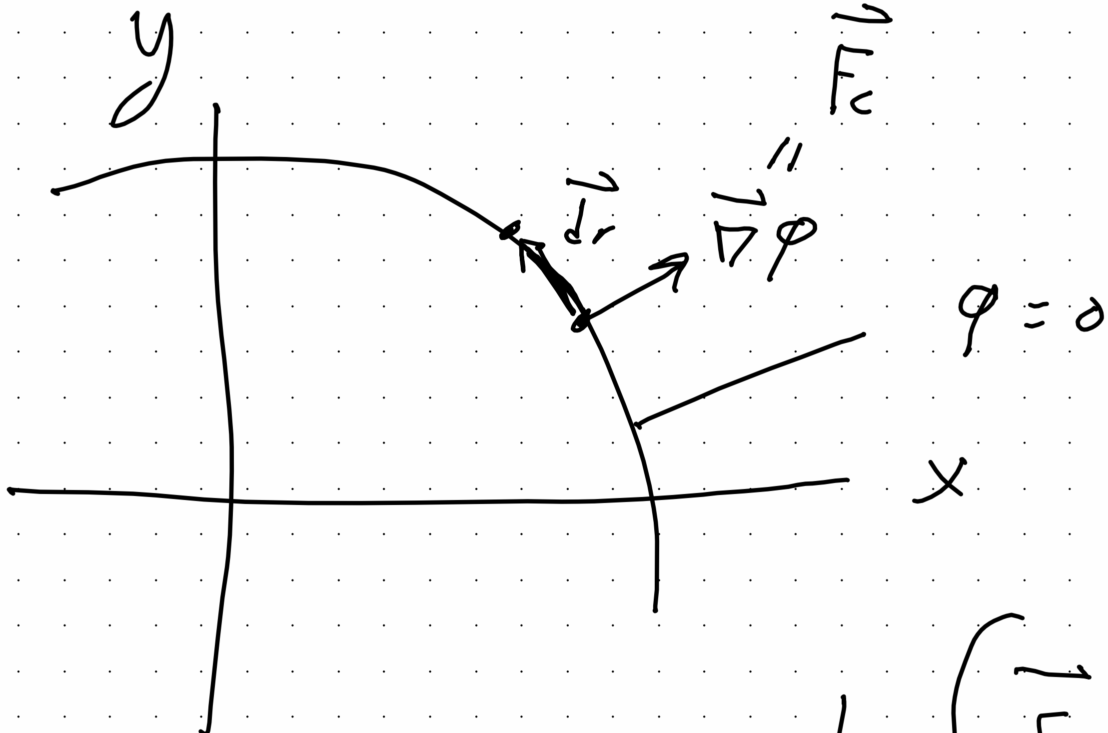
$$\boxed{\frac{d\vec{p}}{dt} = \vec{F}_{net}}$$

$$\frac{dp_x}{dt} = F_x + \lambda \frac{\partial \varphi}{\partial x}$$

$$\frac{dp_y}{dt} = F_y + \lambda \frac{\partial \varphi}{\partial y}$$

$$\boxed{\frac{d\vec{p}}{dt} = \overbrace{\vec{F}}^{\text{applied force}} + \overbrace{\lambda \vec{\nabla}\varphi}^{\text{force constraint}}}$$

$$\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) = \vec{\nabla}\varphi$$



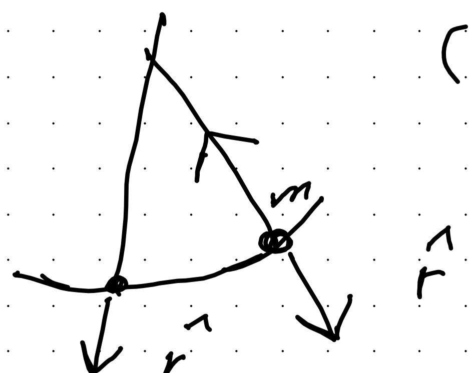
$$\underbrace{\int \vec{F}_c \cdot d\vec{r}}_{0} = \text{Work}$$

$$\vec{\nabla}\varphi \perp \varphi = \text{const}$$

$$\varphi = r - \ell$$

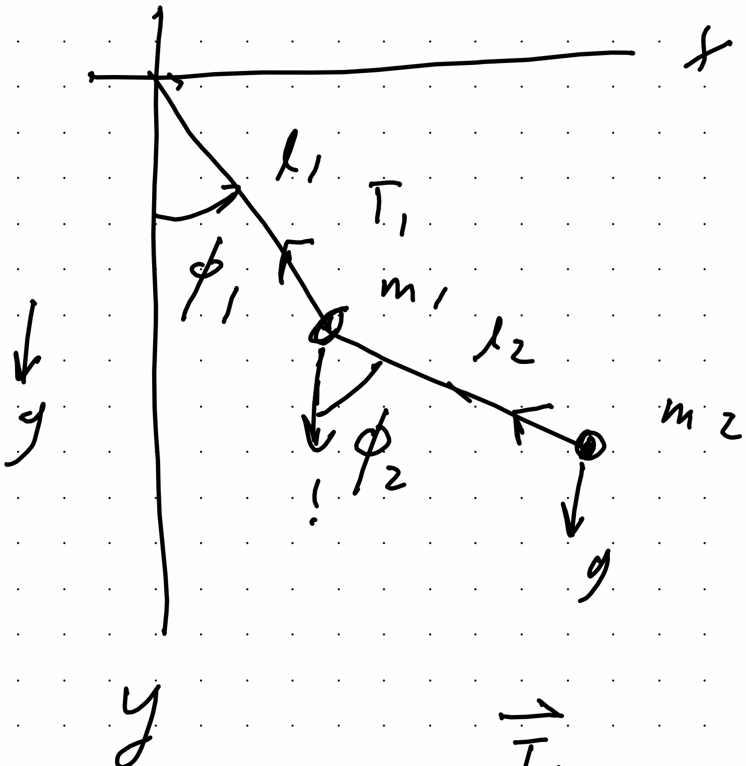
$$\vec{\nabla}\varphi = \hat{r}$$

$$\lambda \vec{\nabla}\varphi$$



(r, ϕ)

$\hat{r}, \hat{\phi}$



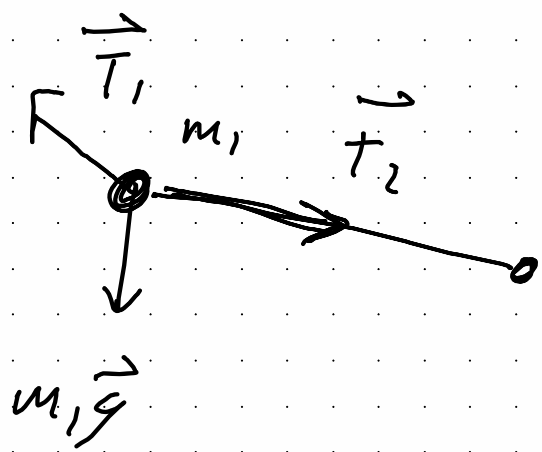
$$U = -m_1 g y_1 - m_2 g y_2$$

$$\ddot{\phi}_1 , \ddot{\phi}_2$$

EOM

$$\lambda \varphi$$

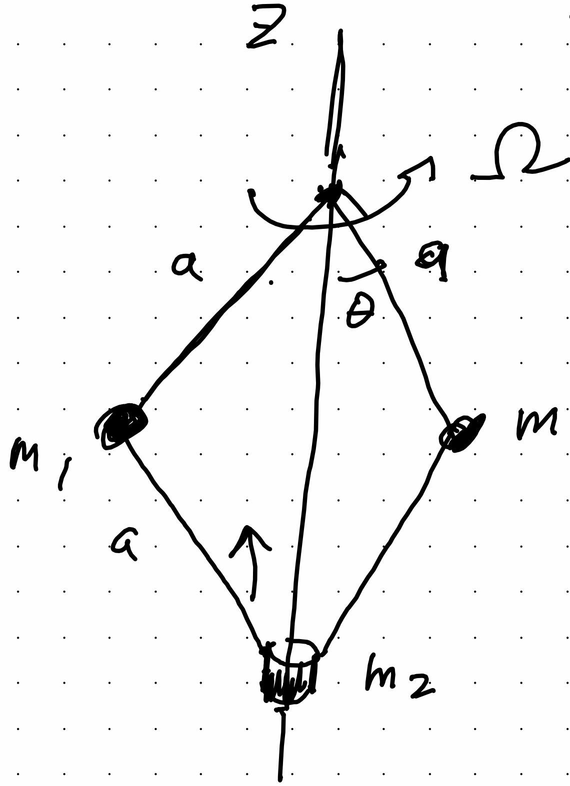
$$(\lambda_1 \varphi_1 + \lambda_2 \varphi_2 + \cdots)$$

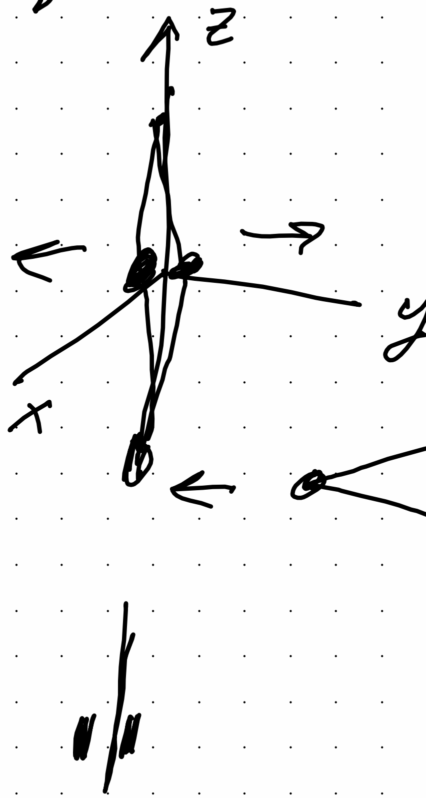


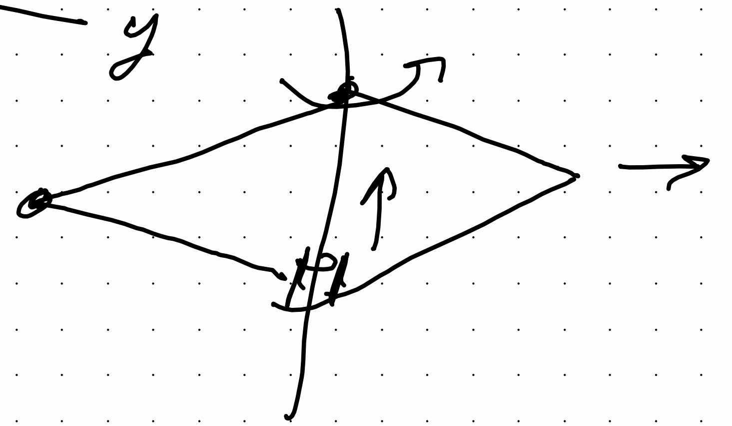
$$L \longrightarrow \text{EOM}$$

$$L \longrightarrow \underbrace{L + \frac{d}{dt}\left(f(q,t)\right)}_{L'}$$

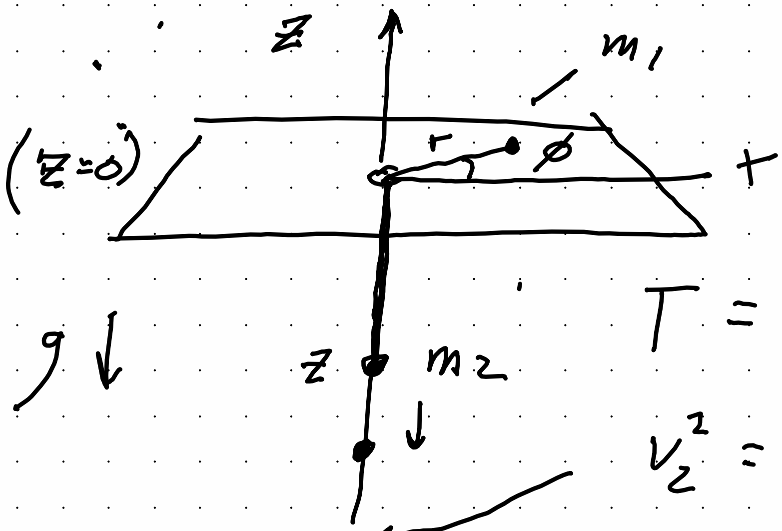
$f(q, \cancel{\dot{q}}, t)$







Lec #5: Tuesday 9/8



$$\boxed{r - z = l} = \text{length of string}$$

$$L = T - U$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$v_2^2 = \dot{z}^2, \quad v_1^2 = \dot{r}^2 + r^2 \dot{\phi}^2$$

$$\left(= \dot{x}^2 + \dot{y}^2, \quad \begin{matrix} x = r\cos\phi \\ y = r\sin\phi \end{matrix} \right)$$

$$v_2^2 = \dot{r}^2$$

$$\boxed{\begin{aligned} T &= \frac{1}{2} m_1 (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2 \\ &= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 \end{aligned}}$$

$$U = m_2 g z = m_2 g (r - l) = \underline{m_2 g r} - \underbrace{m_2 g l}_{\text{constant}}$$

$$\boxed{U = m_2 g r}$$

$$L = T - U = \boxed{\frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = \frac{\partial L}{\partial r}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi} \implies$$ 2nd order EOMs

<u>No explicit t dependence:</u>

$$E = \dot r \frac{\partial L}{\partial \dot r} + \dot\phi \frac{\partial L}{\partial \dot\phi} - L = \boxed{T+U}$$

$$E = \sum_i \dot q_i \underbrace{\frac{\partial L}{\partial \dot q_i}}_{= p_i} - L$$

<u>No explicit ϕ dependence:</u>

$$p_\phi \equiv \frac{\partial L}{\partial \dot\phi} = \text{const} = M_z$$

$\vec M$: angular momentum (L&L notation)

$$\boxed{M_z = m_1 r^2 \dot\phi} \rightarrow \boxed{\dot\phi = \frac{M_z}{m_1 r^2}}$$

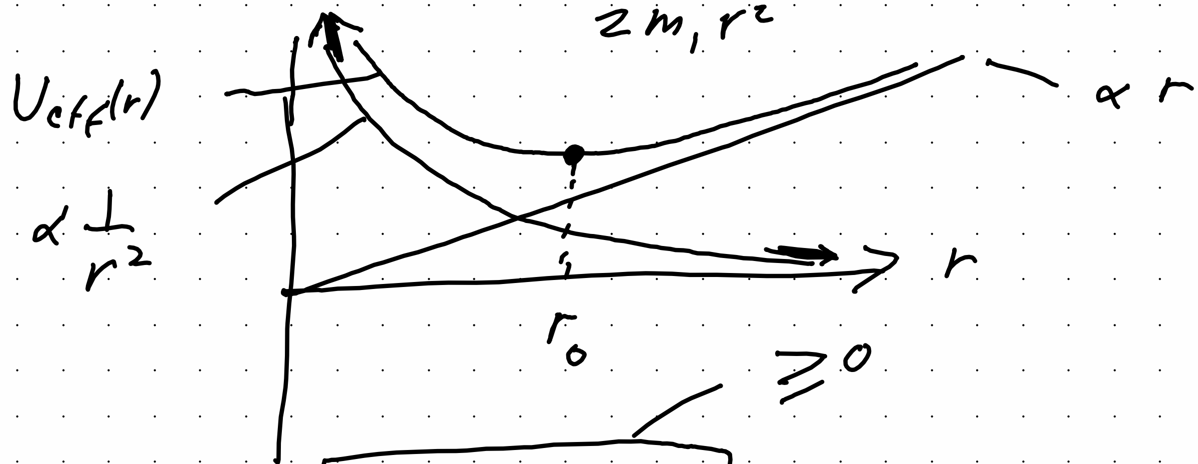
$$E = \tfrac{1}{2} m \dot x^2 + U(x)$$

$$\boxed{E = \tfrac{1}{2}(m_1+m_2)\dot r^2 + \tfrac{1}{2} m_1 r^2 \dot\phi^2 + m_2 g r}$$

$$\boxed{E = \tfrac{1}{2}(m_1+m_2)\dot r^2 + \left(\frac{M_z^2}{2 m_1 r^2} + m_2 g r\right)} \leftarrow \boxed{U_{eff}(r) = U(r) + \frac{M_z^2}{2 m_1 r^2}}$$

$$U_{eff}(r) = \frac{M_z^2}{2m_1 r^2} + m_2 g r$$

$$U(r) = m_2 g r$$

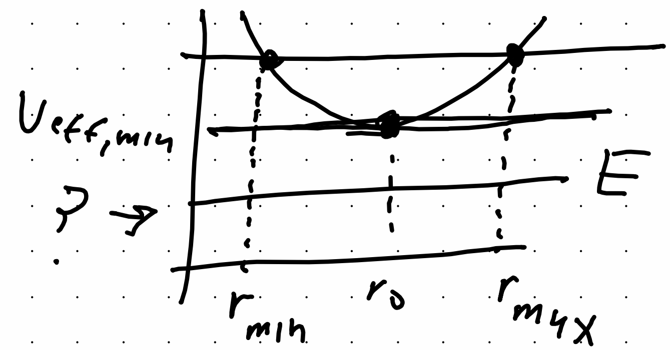


$$E = \overbrace{\boxed{\frac{1}{2}(m_1+m_2)\dot{r}^2}}^{\geq 0} + U_{eff}(r)$$

const

r_{min}, r_{max}: "turning points"!

($\dot{r} = 0$)



i) $E = U_{eff,min} = U_{eff}(r_0)$

unif circular motion: $r = r_0$, $\dot{\phi} = \frac{M_z}{m_1 r_0^2}$

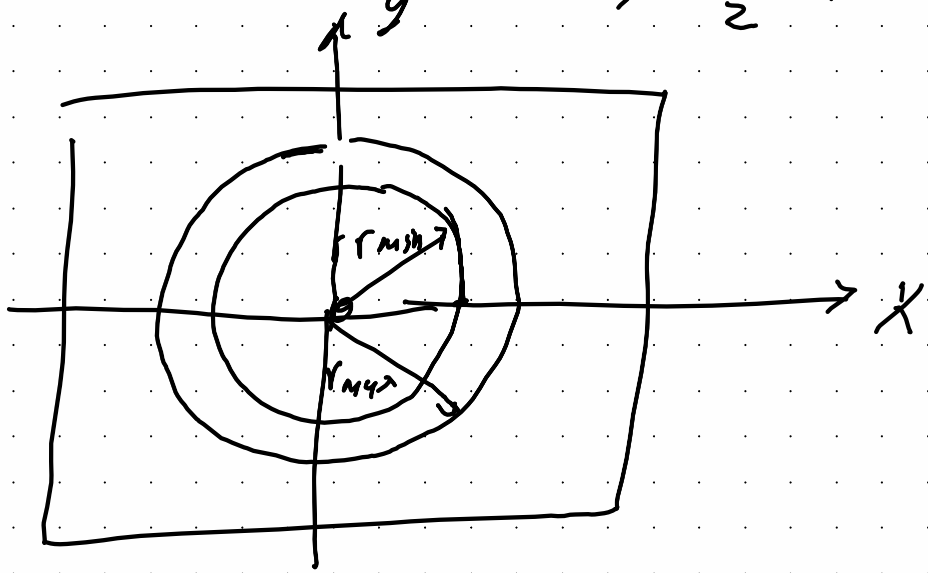
ii) $E > U_{eff,min}$

$E = U_{eff}(r_{min}) = U_{eff}(r_{max})$

$E < U_{eff,min}$ (not allowed)

$E \geq U_{eff,min}$

top view



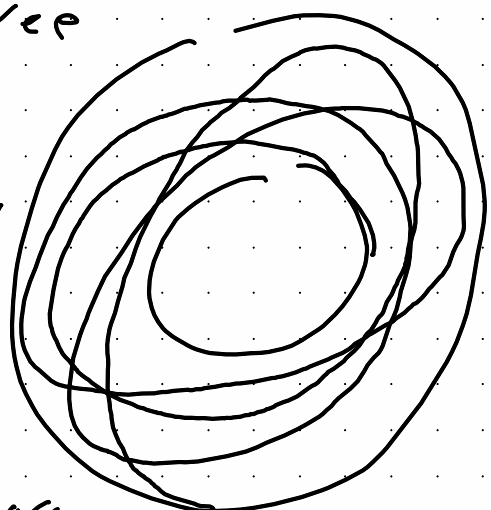
$U(r) = \frac{1}{2} k r^2$ — closed bound orbits

$$\underbrace{U(r) = -\frac{G m_1 m_2}{r}}_{\text{Newtonian gravity}}$$

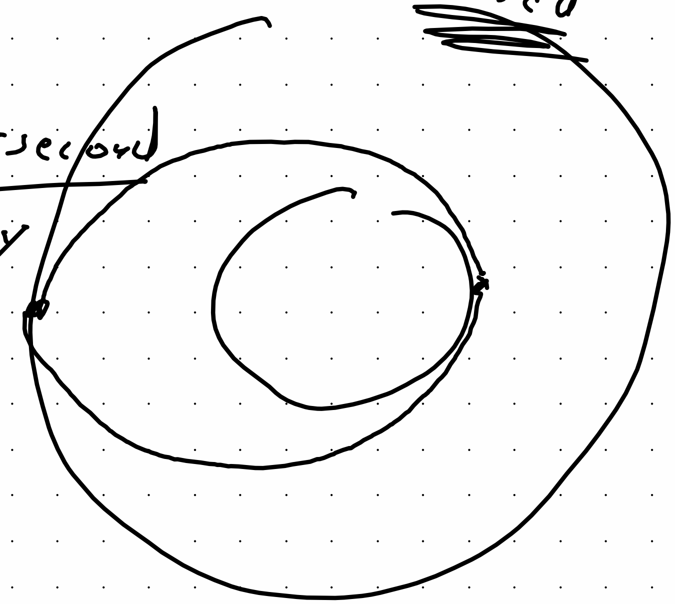
bound orbits = ellipse
are closed

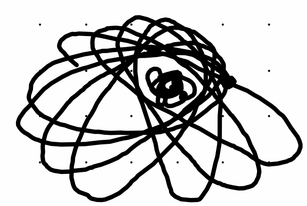
1 degree
=
60 mins of arc

1 min of arc
=
60 sec of arc



43 arcsecond / century





perihelion precession of Mercury

└ closest approach to sun

$\underline{r_0}$: $\left.\dfrac{dU_{eff}}{dr}\right|_{r=r_0} = 0$ (minimum)

$$0 = \frac{d}{dr}\left(\frac{M_z^2}{2m_1 r^2} + m_2 g r\right)\Bigg|_{r=r_0}$$

$$= -\frac{M_z^2}{m_1 r_0^3} + m_2 g$$

$$\boxed{M_z^2 = m_1 m_2 g r_0^3}$$

tells you the value of M_z needed to have a specific r_0 value.

For a given M_z, this tells you what r_0 equals.

Energy equation:

$$E = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{M_z^2}{2m_1 r^2} + m_2 g r$$

$$\boxed{\dot{\phi} = \frac{M_z}{m_1 r^2}} \leftarrow \phi \text{ equation}$$

$$\frac{1}{2}(m_1 + m_2)\dot{r}^2 = E - \frac{M_z^2}{2m_1 r^2} - m_2 g r$$

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\left(\frac{2}{m_1 + m_2}\right)\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}$$

$$\int \frac{dr}{\pm\sqrt{\left(\frac{2}{m_1 + m_2}\right)\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}} = \int dt = t + \text{const}$$

$$t(r) \longleftrightarrow r(t)$$

orbital equation:

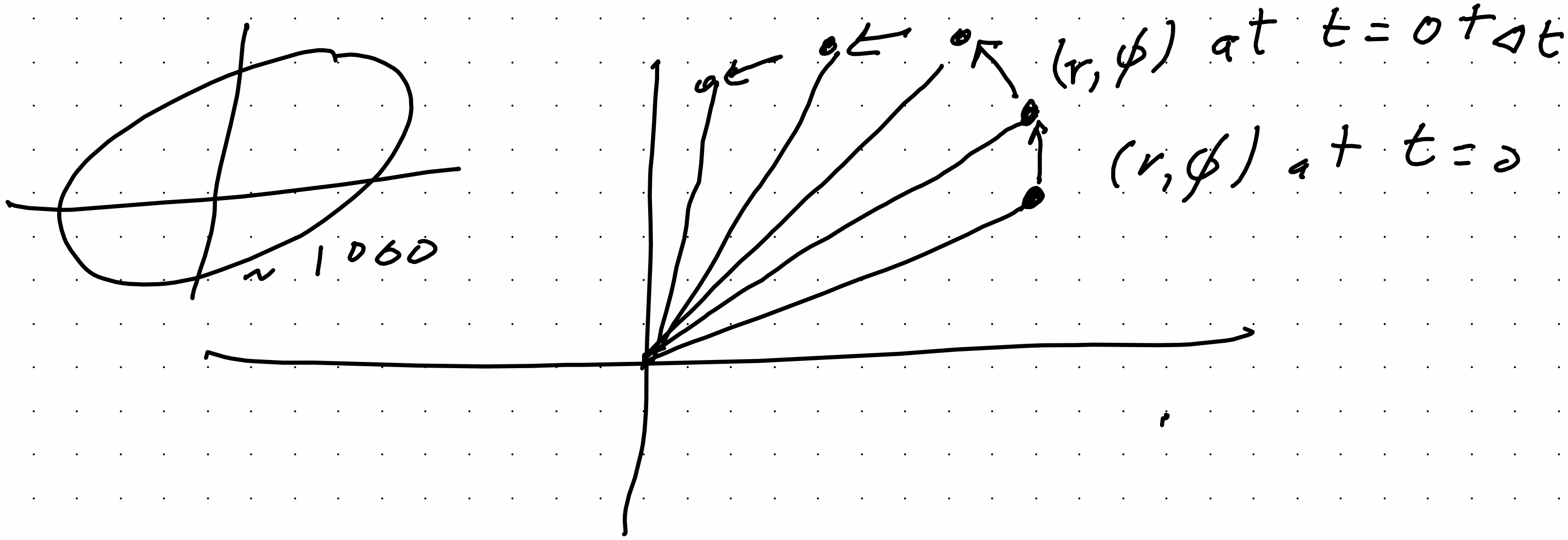
$$r = r(\phi) \Longleftrightarrow \phi = \phi(r)$$

$$\frac{dr}{dt} = \dot{r} = \sqrt{\left(\frac{2}{m_1+m_2}\right)\left[E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right]}$$

$$\frac{dr}{dt} = \underbrace{\frac{dr}{d\phi}\frac{d\phi}{dt}}_{} = \frac{dr}{d\phi}\frac{M_z}{m_1 r^2}$$

$$\rightarrow \frac{dr}{d\phi} = \frac{m_1 r^2}{M_z}\sqrt{\left(\frac{2}{m_1+m_2}\right)\left[\qquad\right]}$$

$$\int \frac{dr}{\frac{m_1 r^2}{M_z}\sqrt{(\,)[\quad]}} = \int d\phi = \phi + \text{const}$$



Know: r, ϕ at some time t

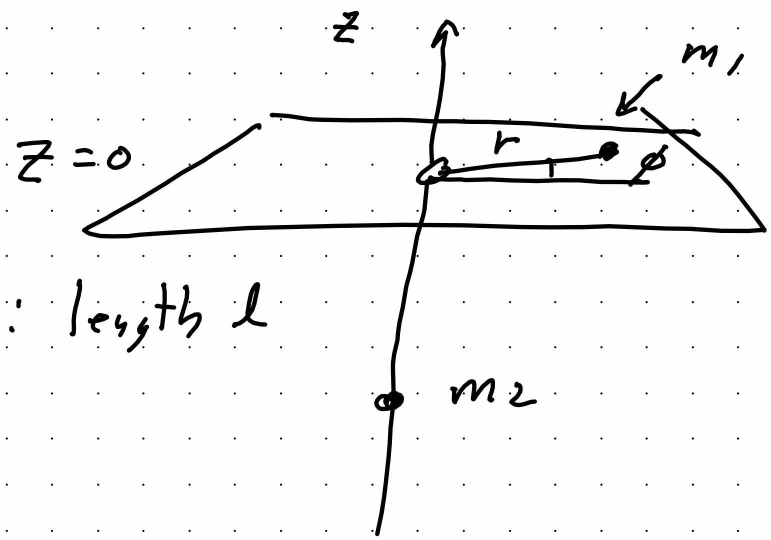
given: Δt need to know Δr and $\Delta \phi$

$$r(t+\Delta t) = r(t) + \Delta r(t) + \dots$$

$$\phi(t+\Delta t) = \phi(t) + \Delta \phi(t) + \underbrace{\dots}_{\text{ignore if } \Delta t \text{ is suff. small}}$$

Lecture #6 : Thursday 10 Sep

1) Secs 6–10 (today), Sec 40 (next Tuesday)
2) Finish up example from last time
3) Conservation of $E, \vec{P}, \vec{M}$
4) Mechanical similarity
5) Quiz: last 20 minutes (1:30pm)



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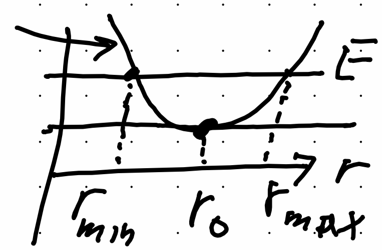
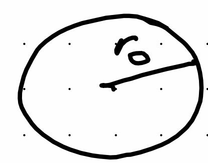
E, M_z = const

U_{eff}



$$\frac{d\phi}{dt} = \frac{M_z}{m_1 r^2}$$



$$\frac{dr}{dt} = \sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E \underbrace{- m_2 g r - \frac{M_z^2}{2m_1 r^2}}\right)} = \sqrt{0}$$

$$\frac{d\phi}{dt} = \frac{M_z}{m_1 r^2} \longrightarrow \Delta\phi = \Delta t \, \frac{M_z}{m_1 r^2}$$

$$\frac{dr}{dt} = \sqrt{\quad} \longrightarrow \Delta r = \Delta t \sqrt{\quad}$$

$$r(0) = r_{min}$$
$$\phi(0) = 0$$

$$\phi(\Delta t) = \phi(0) + \Delta\phi$$
$$r(\Delta t) = r(0) + \Delta r$$

$$\phi(2\cdot\Delta t) = \phi(\Delta t) + \Delta\phi$$
$$r(2\cdot\Delta t) = r(\Delta t) + \Delta r$$

$$\vdots$$

$$\phi(t_i) = \phi(t_{i-1}) + \Delta\phi$$
$$r(t_i) = r(t_{i-1}) + \Delta r$$

Cons. of $E, \vec{P}, \vec{M}$:

All of $E, \vec{P}, \vec{M}$ conserved for a closed system

no external forces

$U = U(\vec{r}_1 - \vec{r}_2, \vec{r}_2 - \vec{r}_3, \cdots)$

relative position vectors.

Even in the presence of external forces, you can still have cons. of E and some components of $\vec{P}$ and $\vec{M}$.

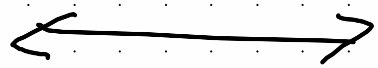
(i) $U = mgy$ $\quad \vec{F} = -m\vec{g}$

If U does not depend explicitly on time t, then E is conserved.

$$E \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

constant external field

(ii) e.g., $\downarrow \vec{F}_g = m\vec{g}$



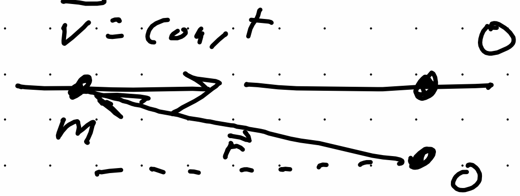
x, y

$P_x = \text{const}$

$P_y = \text{const}$

If U is unchanged by a translation in some direction $\hat{t}$ then $\vec{P} \cdot \hat{t} = \text{const}$

$\vec{v} = \text{const}$

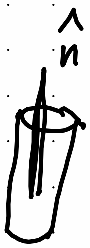


(iii) $\vec{M}$ depends on choice of origin

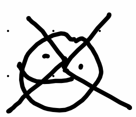
(a) uniform gravitational field $\vec{F} = m\vec{g}$

$\uparrow z$

$U = m_1 g z_1 + m_2 g z_2 + \ldots$



If U is unchanged by a rotation about a particular axis $\hat{n}$ then $\boxed{\vec{M} \cdot \hat{n} = \text{const}}$ (e.g., $\hat{n} = \hat{z}$, $M_z = \text{const}$) provided the origin is located on the axis.



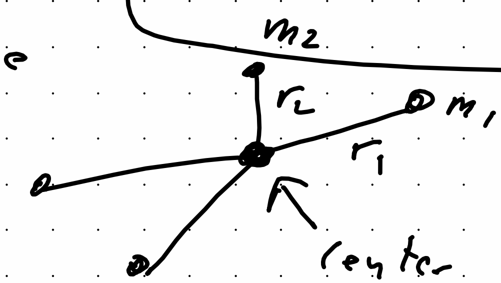
(b) central force

$U = U(r)$

$\vec{F} = -\nabla U$

$= -\frac{dU}{dr}\hat{r}$

$\boxed{\vec{M} = \text{const}}$



Mechanical similarity:

$$L \to L' = c \cdot L$$ same equations of motion

suppose we rescale position vectors $\vec{r}_a \to \alpha \vec{r}_a$

$$U'(\vec{r}_1, \dots) = U(\alpha \vec{r}_1, \alpha \vec{r}_2, \dots) = \alpha^k U(\vec{r}_1, \vec{r}_2, \dots)$$

potential is homogenous of degree k w.r.t position vector

Example: i) $U = mgy$, $k = 1$

ii) $U = \frac{1}{2}kx^2$, $k = 2$

iii) $U = -\frac{Gm_1 m_2}{r}$, $k = -1$

$$U' = mg\,\alpha y = \alpha\, mgy = \alpha U$$

$$L' = T' - U' = T' - \alpha^k U = \text{const} \cdot L$$

$$= \alpha^k T - \alpha^k U$$

$$= \alpha^k (T - U) = \alpha^k L$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \frac{l^2}{t^2}$$

length

time

$$l \rightarrow l' = \alpha l$$
$$t \rightarrow t' = \beta t$$

$$\frac{l'}{l} = \alpha$$
$$\frac{t'}{t} = \beta$$

$U = mgy$, $k = 1$

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{\frac{1}{2}}$$

$U = \frac{1}{2} k x^2$, $k = 2$

$t'/t = \text{const}$

$$T' = \frac{1}{2} m v'^2 = \frac{1}{2} m \frac{l'^2}{t'^2}$$
$$= \frac{1}{2} m \frac{\alpha^2 l^2}{\beta^2 t^2}$$
$$= \frac{\alpha^2}{\beta^2} T$$

$$\boxed{\frac{\alpha^2}{\beta^2} = \alpha^k} \rightarrow \boxed{\beta = \alpha^{1-k/2}}$$

$$\beta^2 = \alpha^{2-k}$$

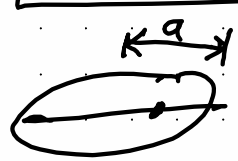
$$\boxed{\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-k/2}}$$

$$U = -\frac{G m_1 m_2}{r}$$

$k = -1$

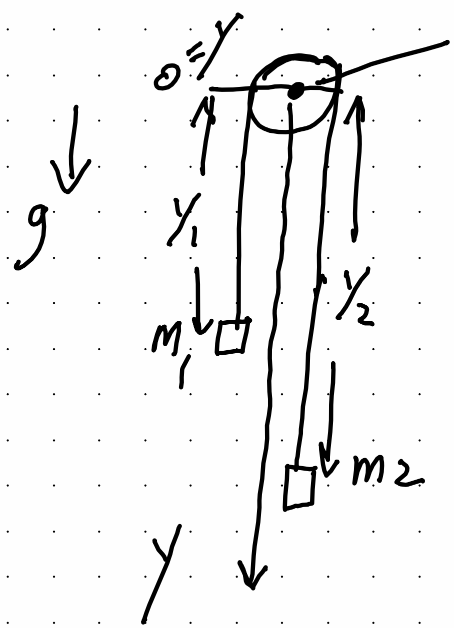
$$\left(\frac{t'}{t}\right) = \left(\frac{l'}{l}\right)^{3/2}$$

$$p^2 = \text{Dist}^3$$



$$\boxed{\frac{p^2}{a^3} = \text{const}}$$

QUIZ #1 : Atwood's machine



massless frictionless pulley

$(m_1 > m_2)$

string: length ℓ
(massless
inextensible
...)

i) L ?

ii) EOM

iii) solve EOM

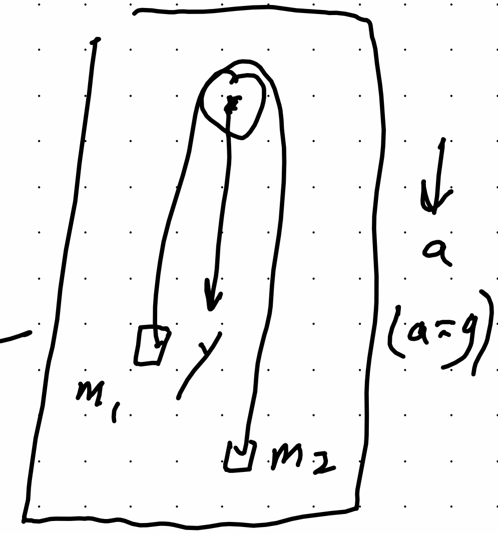
$$y_1 + y_2 = \ell \rightarrow y_2 = \ell - y_1 \rightarrow \dot{y}_2 = -\dot{y}_1$$

$$\begin{aligned} U &= -m_1 g y_1 - m_2 g y_2 \\ &= -m_1 g y_1 - m_2 g (\ell - y_1) \\ &= -m_1 g y_1 \underbrace{- m_2 g \ell}_{\text{ignore}} + m_2 g y_1 \\ &= \boxed{-(m_1 - m_2) g y_1} \end{aligned}$$

$$\begin{aligned} T &= \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \underbrace{\dot{y}_2^2}_{= \dot{y}_1^2} \\ &= \boxed{\frac{1}{2}(m_1 + m_2) \dot{y}_1^2} \end{aligned}$$

Lecture # 7: Tuesday 9/15

1) Go over quiz #1
2) Modified Atwood problem
3) Finish mechanical similarity (sec 10)
4) Hamilton's equations (sec 40)



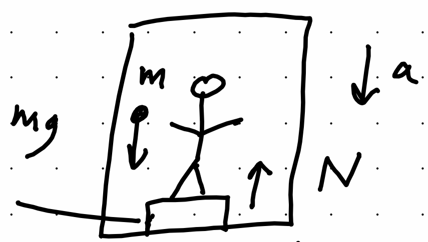
$$L = T - U$$

$$= \frac{1}{2}(m_1+m_2)\dot{y}_1^2 + (m_1-m_2)gy_1$$

EOMs: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) = \frac{\partial L}{\partial y_1} \rightarrow (m_1+m_2)\ddot{y}_1 = (m_1-m_2)g$

$$\boxed{\ddot{y}_1 = \frac{(m_1-m_2)g}{(m_1+m_2)}}$$

$$y_1(t) = y_0 + v_0 t + \frac{1}{2}\frac{(m_1-m_2)}{(m_1+m_2)}g t^2$$



$$ma = mg - N$$

$$\underbrace{N}_{w} = m(g-a)$$

apparent or effective weight

$$\boxed{\ddot{y}_1 = \frac{(m_1-m_2)}{m_1+m_2}(g-a)}$$

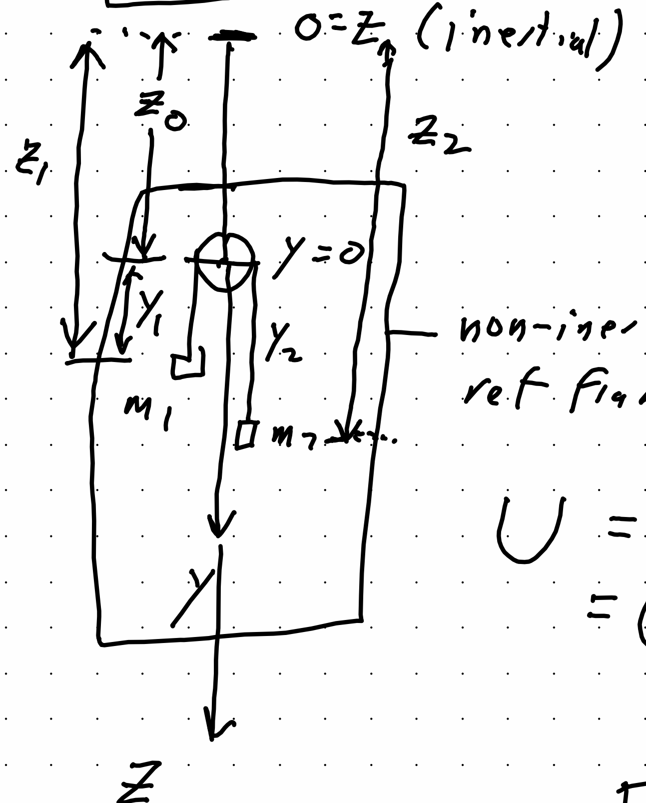
$\vec{F} = m\vec{a}$ (valid in an inertial ref frame)

$\vec{F} + \vec{F}_{fictitious} = m\vec{a}$ —— w.r.t. to a non-inertial ref. frame

<u>sec 39</u> (L&L)

$L' = L + f(t)$

$\boxed{L = T - U}$ (valid in an inertial ref frame)



$T = \frac{1}{2} m_1 \dot{z}_1^2 + \frac{1}{2} m_2 \dot{z}_2^2$

$U = -m_1 g z_1 - m_2 g z_2$

$z_1 = z_0 + y_1$ | $y_2 = l - y_1$

$z_2 = z_0 + y_2$

$U = -m_1 g (z_0 + y_1) - m_2 g (z_0 + \underbrace{y_2}_{l - y_1})$

$= -m_1 g z_0 - m_1 g y_1 - m_2 g z_0 \underbrace{- m_2 g l}_{\text{const}} + m_2 g y_1$

prescribed function of time = ignore

$= \boxed{-(m_1 - m_2) g y_1}$

Do this at home:

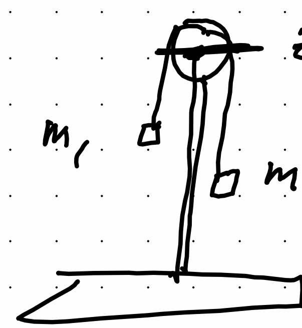
$$T = \frac{1}{2}(m_1+m_2)\dot{y}_1^2 + (m_1-m_2)\dot{z}_0\dot{y}_1$$

$$L' = L + \frac{d}{dt}\big(f(q,t)\big) \longrightarrow \text{same EOM}$$

$$(m_1-m_2)\dot{z}_0\dot{y}_1 = \underbrace{\frac{d}{dt}\Big[\underbrace{(m_1-m_2)\dot{z}_0 y_1}_{f(y_1,t)}\Big]}_{\text{ignore}} - (m_1-m_2)\underbrace{\ddot{z}_0}_{\equiv a} y_1$$

$$\boxed{T = \frac{1}{2}(m_1+m_2)\dot{y}_1^2 - (m_1-m_2)\,a\,y_1}$$

$z_0(t)$: given, not to be solved for



$L(q, \dot{q}, t)$

Hamilton's equations:

Hamiltonian:

$$E = \underbrace{\sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L}_{E(q, \dot{q}, t)}$$

$$H = H(q, p) \longleftrightarrow E$$

not here if $L = L(q, \dot{q})$

$$H = \left(\sum_i p_i \dot{q}_i - L \right) \Big|_{\dot{q} = \dot{q}(q, p)}$$

$$\boxed{p_i \equiv \frac{\partial L}{\partial \dot{q}_i}}$$

Example:

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

(single particle, 1-d, const external field)

$$p = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \longrightarrow \dot{x} = \frac{p}{m}$$

$$H = (p\dot{x} - L) \Big|_{\dot{x} = p/m}$$

$$= \left(p\dot{x} - \frac{1}{2} m \dot{x}^2 + U(x)\right) \Big|_{\dot{x} = p/m}$$

$$= \frac{p^2}{m} - \frac{1}{2} m \left(\frac{p}{m}\right)^2 + U(x)$$

$$H = \frac{1}{2} \frac{p^2}{m} + U(x)$$

EOMs: (Hamilton's equations) ((39,6) Prob 2 Sec 40)

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, \ldots, s$$

s = # of DOF

Lagrange's equation:

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}}_{s\text{-equations, } 2^{nd} \text{ order ODE for } q_i} \qquad i = 1, \ldots, s$$

$\rightarrow$ 2s-equations, 1^{st} order in $\dot{q}_i, \dot{p}_i$

$\dot{x} = p/m$

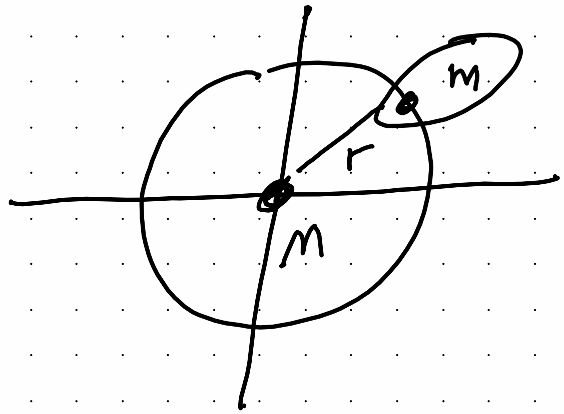
$p = m\dot{x}$

$\dot{p} = -\frac{\partial U}{\partial x}$

$$\boxed{m\ddot{x} = -\frac{\partial U}{\partial x}}$$

$$H = \frac{p^2}{2m} + U(x) \rightarrow \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$$

$$L = \frac{1}{2}m\dot{x}^2 - U(x) \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow m\ddot{x} = -\frac{\partial U}{\partial x}$$



$$U = -\frac{GMm}{r}$$

Problem: $U' = cU$

~~masses = equal~~

$r' = r, \; m' = m$

$$\frac{GM\cancel{m}}{r^2} = \frac{\cancel{m}v^2}{r}$$

$$\boxed{v = \sqrt{\frac{GM}{r}}}$$

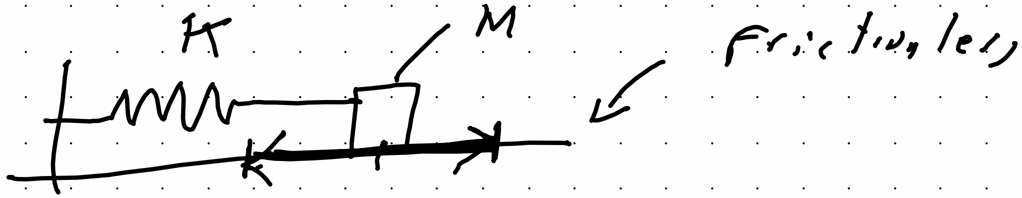
Suppose $M \to cM = M'$

$U' = cU$

$$\frac{2\pi r}{T'} = v' = \sqrt{\frac{c\,GM}{r}}$$

$$\frac{1}{T'} \propto \sqrt{c}$$

$$\sqrt{\frac{E}{E'}} = \sqrt{\frac{U'}{U}}$$



$U = \frac{1}{2} k x^2$

$T = \frac{1}{2} m \dot{x}^2$

$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$

$\omega = \sqrt{\frac{k}{m}}$

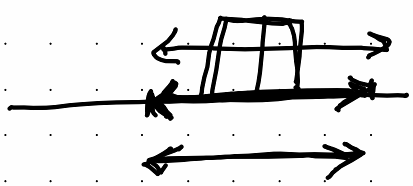
$m \ddot{x} = -kx$

$\ddot{x} = -\frac{k}{m} x \rightarrow x(t) = a \cos \omega t + b \sin \omega t$

$U' = cU$ (for example $k' = ck$)

$T' = cU$

$\ddot{x} = -\frac{k'}{m} x$, $\omega' = \sqrt{\frac{k'}{m}}$



$T = \frac{1}{2} m \dot{x}^2$

$T' = cT$

$\frac{2\pi}{P'} = \sqrt{\frac{k'}{m}}$

$\frac{P}{P'} = \sqrt{\frac{U'}{U}}$

$L = T - U$

$L' = cL = cT - cU$

$L'' = T - cU \neq cL$

L, L' same EOMs

L, L'' different EOMs

$\frac{1}{t'^2} = c$

$T = \frac{1}{2}m\dot{x}^2, \quad U = \frac{1}{2}kx^2$

$L = T - U \rightarrow m\ddot{x} = -kx$

$L' = cL \rightarrow \cancel{c}\,m\ddot{x} = -\cancel{c}\,kx$

$m\ddot{x} = -kx$

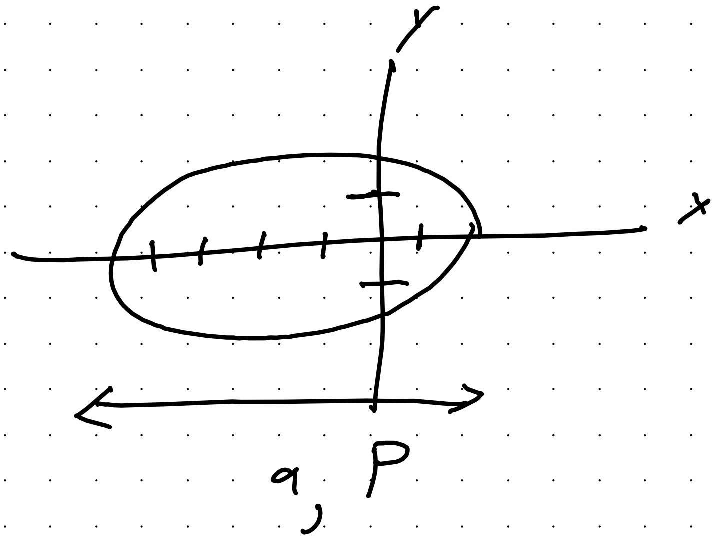
$\ddot{x} = -\frac{k}{m}x$

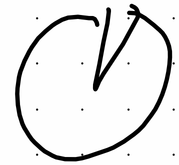
$L'' \longrightarrow m\ddot{x} = -ckx$

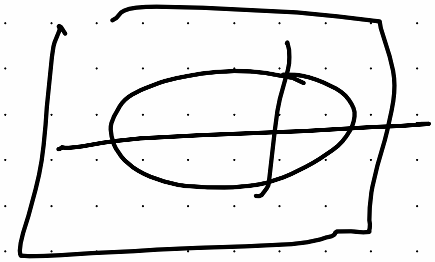
$cx'' = \ddot{x} = -c\left(\frac{k}{m}\right)x$

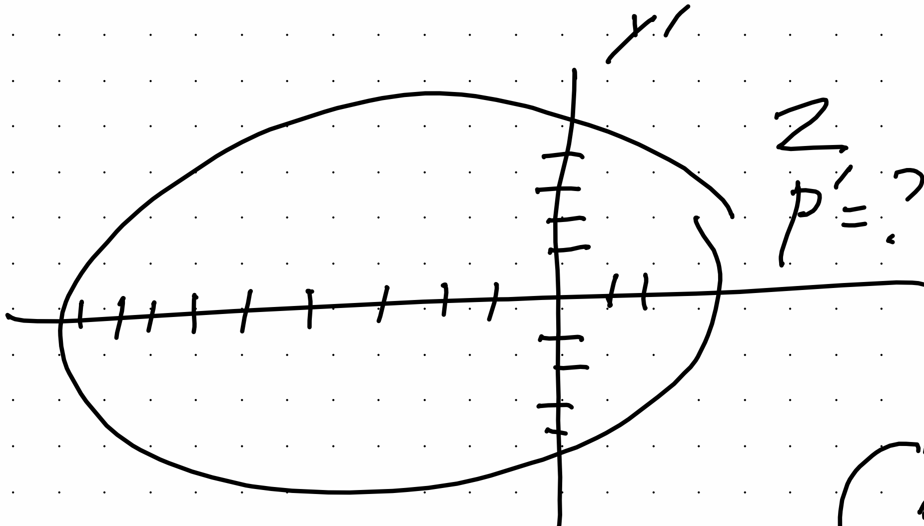
differ EOMs

$x = a\cos(\omega' t) + b\sin(\omega' t)$





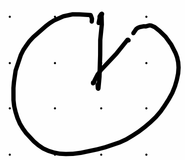




2

$P' = ?$

$\frac{P^2}{a^3}$ = const



t'

$$U' = cU$$

$$P' = \ ?\ P$$

$$m' = m$$

$$l' = l$$

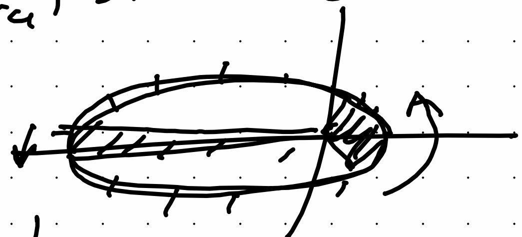
"mechanical similarity"

$$L' = \text{const}\ L$$

$$T' - U' = (\text{const})(T - U)$$

$$T' - cU = c\,(T - U) \qquad cU = U'$$

$$\frac{P'}{P} = \sqrt{\frac{U}{U'}}$$



$$k \to k' = ck$$

Periods

$2k$

$\sqrt{\frac{1}{2}}$

k

$$T = \frac{1}{2} m \dot{x}^2$$

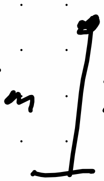
$$\cancel{T' = T}$$

$$U' = cU$$

$$\cancel{L' \neq cL}$$

$$L' = cL$$

$$g \to 2g$$



1m P' ? P

$$\frac{1}{\sqrt{2}}$$