Notes: Thurs 8/27

1) Elliptic functions
2) Simple pendulum

go beyond small angle approx

Elliptic Functions:

$$\int \frac{dx}{\sqrt{1-x^2}} = ? = \sin^{-1}(x) + \text{const}$$

$\sin^{-1}(x) = \arcsin(x)$

$\rightarrow \sin^2\theta + \cos^2\theta = 1$

subst: $x = \sin\theta \rightarrow \sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$dx = \cos\theta \, d\theta$

$$\int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \int d\theta = \theta = \sin^{-1}(x)$$

$x = \sin\theta$

$\theta = \sin^{-1}(x)$

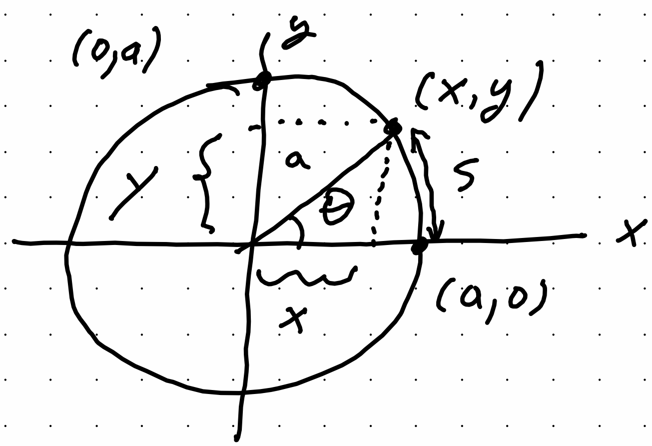
$$\frac{d}{d\theta}\sin\theta = \cos\theta \qquad \lim_{h\to 0}\left(\frac{\sin(\theta+h)-\sin\theta}{h}\right)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

circular functions..

$x^2+y^2=a^2$, a = radius



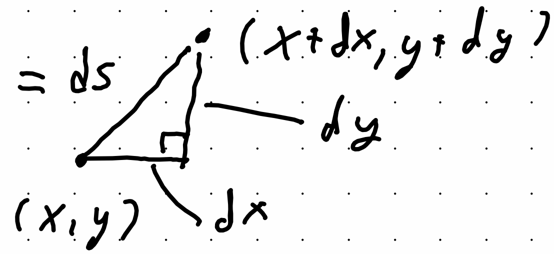
Def: $\sin\theta = \dfrac{y}{a}$, $\cos\theta = \dfrac{x}{a}$

s : arclength from $(a,0)$ to (x,y)

$s = a\theta$

$$\theta = \frac{s}{a} = \frac{1}{a}\int ds = \frac{1}{a}\int\sqrt{dx^2+dy^2} = \int d\theta$$

$\sqrt{dx^2+dy^2} = ds$



Given: $x^2+y^2=a^2$

Follows: (i) $\cancel{a^2}\cos^2\theta + \cancel{a^2}\sin^2\theta = \cancel{a^2} \rightarrow \boxed{\cos^2\theta+\sin^2\theta=1}$

(ii) $\boxed{\dfrac{d}{d\theta}\sin\theta} = \dfrac{1}{a}\dfrac{dy}{d\theta} = \dfrac{\frac{1}{\cancel{a}}\,dy}{\frac{1}{\cancel{a}}\sqrt{dx^2+dy^2}} = \dfrac{1}{\sqrt{\left(\frac{y}{x}\right)^2+1}}$

$\cancel{2}x\,dx + \cancel{2}y\,dy = 0 \rightarrow dx = -\dfrac{y}{x}\,dy$

$= \dfrac{1}{\sqrt{\dfrac{x^2+y^2}{x^2}}} = \dfrac{x}{a} = \boxed{\cos\theta}$

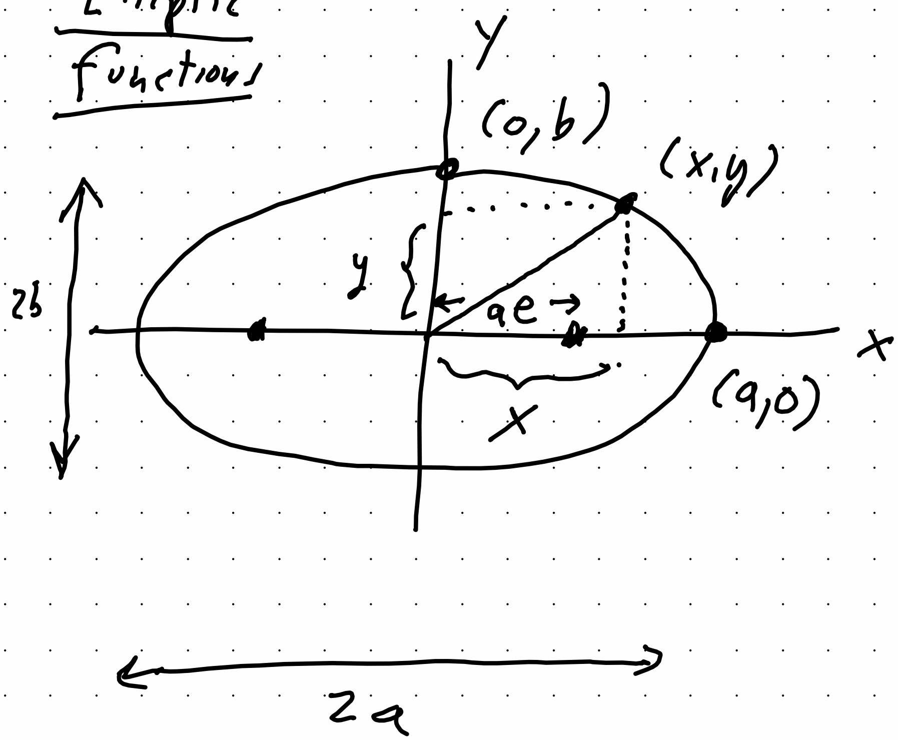
$$\frac{d(\sin\theta)}{d\theta} = \cos\theta \rightarrow \int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta$$

$$x = \sin\theta$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$$

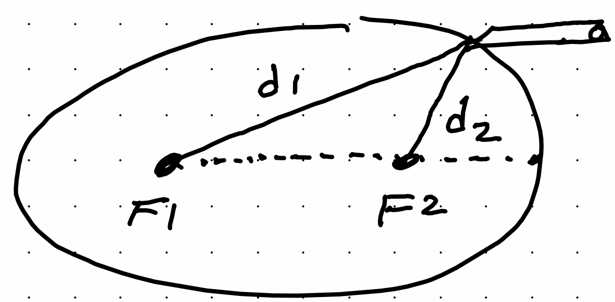
$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \theta = \sin^{-1}(x)}$$

Elliptic functions

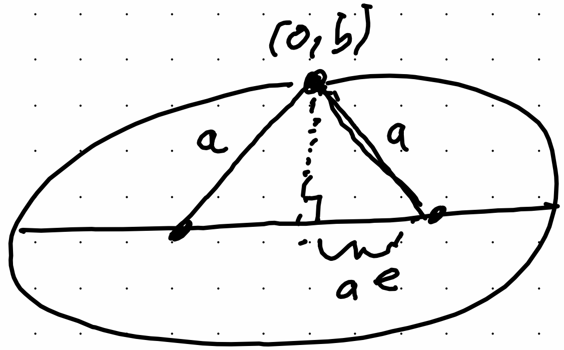


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

eccentricity : $e = 0$ (for circle)



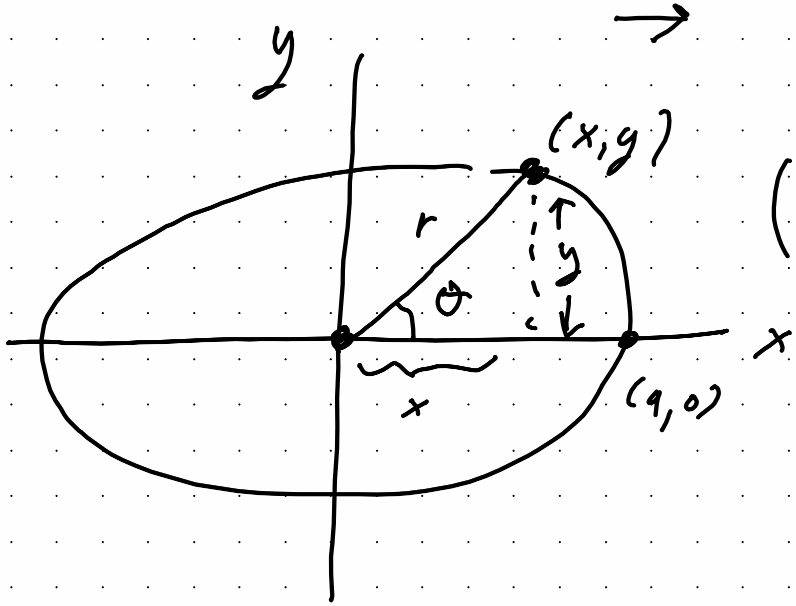
$$d_1 + d_2 = 2a$$



$$(ae)^2 + b^2 = a^2$$
$$a^2e^2 + b^2 = a^2$$
$$e^2 = \frac{a^2-b^2}{a^2} = 1-\left(\frac{b}{a}\right)^2$$

$$\rightarrow \quad e = \sqrt{1-\left(\frac{b}{a}\right)^2} \equiv k$$

notation used in elliptic function



$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

$$x^2+y^2=r^2$$
$$x=r\cos\theta, \quad y=r\sin\theta$$

Define: $\mathrm{cn}(u;k) \equiv \dfrac{x}{a}$, $\quad \mathrm{sn}(u;k) \equiv \dfrac{y}{b}$

$\mathrm{dn}(u;k) \equiv \dfrac{r}{a}$ — (=1 for a circle)

where $u = \dfrac{1}{b}\displaystyle\int_0^\theta r\,d\theta$ (=θ for a circle) $\quad y = r\sin\theta$

$$ds = \sqrt{dx^2+dy^2} = \sqrt{dr^2+r^2d\theta^2}$$

Given: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, $x^2 + y^2 = r^2$

$dn(u; k) = \frac{r}{a}$

Follows: (i) $cn^2(u;k) + sn^2(u;k) = 1$

(ii) $dn^2(u;k) + k^2 sn^2(u;k) = 1$

$u = \frac{1}{b}\int_0^\theta r\,d\theta$

(iii) $\frac{d}{du} sn(u;k) = cn(u;k)\, dn(u;k)$

$\frac{d}{du} cn(u;k) = -sn(u;k)\, dn(u;k)$

$\frac{d}{du} dn(u;k) = -k^2 sn(u;k)\, cn(u;k)$

Analogous to

$\frac{d\sin\theta}{d\theta} = \cos\theta$

$\frac{d\sin\theta}{\cos\theta} = d\theta$

$x = \sin\theta$

$\int \frac{dx}{\sqrt{1-x^2}} = \int d\theta = \theta = \sin^{-1} x$

Integrate: $\frac{d\, sn(u;k)}{du} = cn(u;k)\, dn(u;k)$

$\int \frac{d\, sn(u;k)}{cn(u;k)\, dn(u;k)} = \int du = u$

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}} = u + \text{const} = sn^{-1}(x;k) + \text{const} \qquad x \equiv sn(u;k)}$$

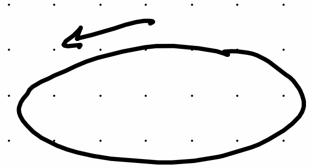
$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + \text{const}}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \equiv K(k) \longrightarrow$$ related to Period of a pendulum going beyond small-angle approximation

(complete elliptic integral of 1st kind)

$$\int_0^1 \frac{\sqrt{1-k^2x^2}\,dx}{\sqrt{1-x^2}} \equiv E(k) \longrightarrow$$ circumference around an ellipse

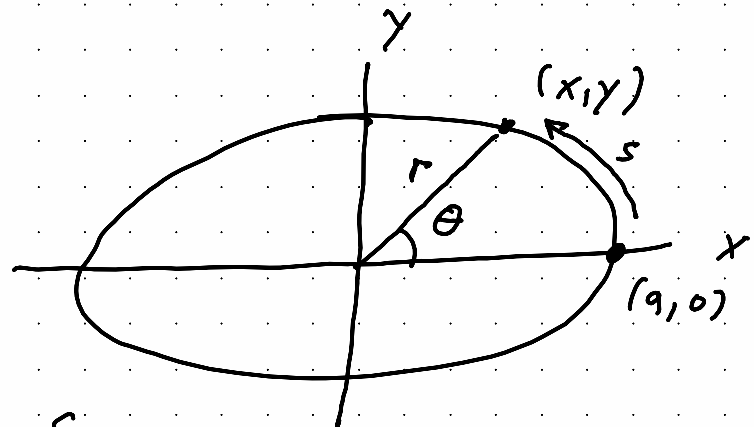
(complete elliptic integral of 2nd kind)



circle: $C = 2\pi a$

Notes: Tuesday 9/1

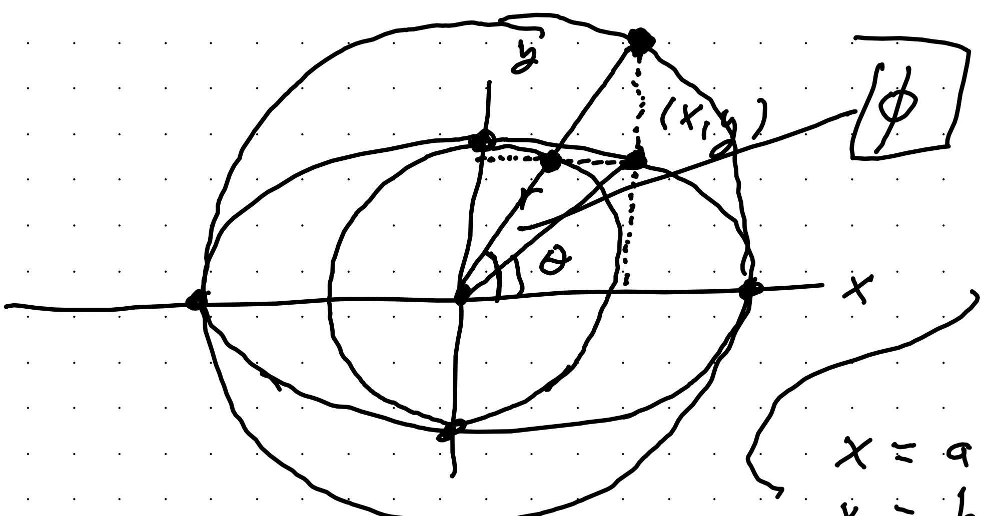
1) Review of elliptic functions
2) Simple pendulum



$$u = \frac{1}{b}\int_{(a,0)}^{(x,y)} r\,d\theta$$

$$bu = \int_0^\theta \underbrace{r\,d\theta} \leq \int_0^\theta ds = s$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\underbrace{dr^2} + r^2 d\theta^2} \geq r\,d\theta$$



$x = r\cos\theta$

$y = r\sin\theta$

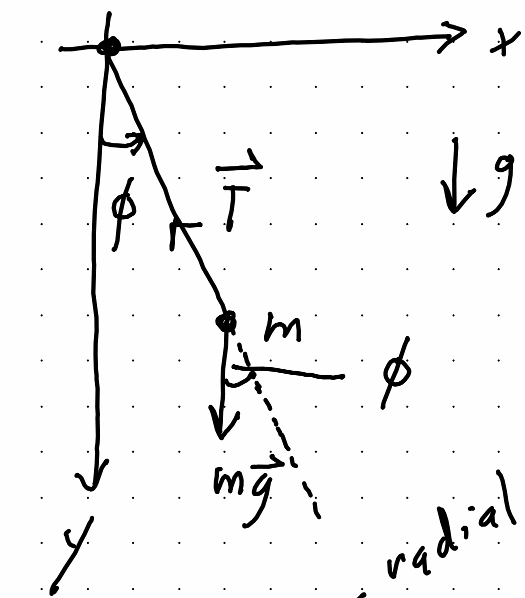
$x = a\cos\phi$

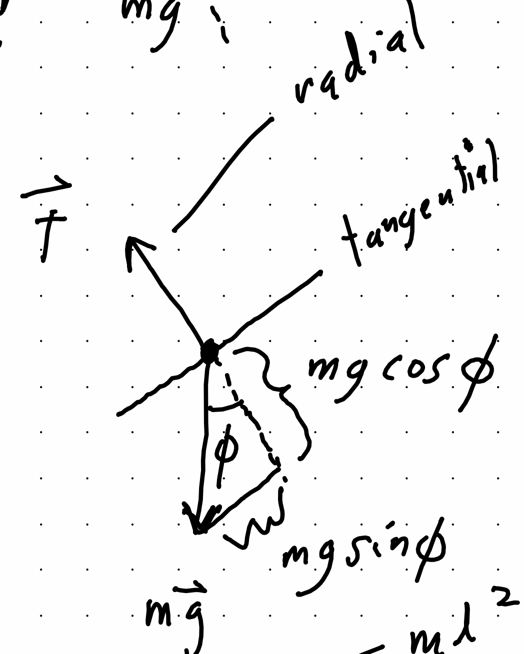
$y = b\sin\phi$

$x = a\cos\phi\,,\quad y = \underline{\underline{a}}\sin\phi$

$x = b\cos\phi \quad y = b\sin\phi$

simple pendulum:





$$\text{Torque} = \overset{ml^2}{I}\,\overset{\ddot\phi}{\alpha}$$

(i) "Freshman physics"

forces, free-body diagram

→ EOM, tension

tangential:

$$-mg\sin\phi = m a_{tangential}$$

$$-\cancel{m}g\sin\phi = \cancel{m} l \ddot\phi$$

ϕ : angular displacement [rad]

$\dot\phi$: angular velocity [rad/sec]

$\ddot\phi$: angular accel [rad/sec^2]

$$\boxed{\ddot\phi = -\frac{g}{l}\sin\phi} \quad \text{(EOM)}$$

radial: $T - mg\cos\phi = m a_{centripetal}$

$$T - mg\cos\phi = m\dot\phi^2 l$$

$$T = mg\cos\phi + m\dot\phi^2 l$$

(ii) Small angle approximation:

$$\sin\phi \approx \phi \leftarrow \phi \ll 1 \text{ radian} = 57 \text{ degrees}$$

$$\ddot{\phi} = -\frac{g}{l}\sin\phi \approx -\frac{g}{l}\phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\phi}$$

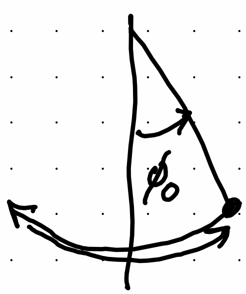
small angle approx.

$$\phi(t) = \boxed{a}\cos(\omega t) + \boxed{b}\sin(\omega t)$$

determined by initial conditions

where $\omega \equiv \sqrt{\frac{g}{l}}$

<u>ICs:</u> If $\phi(0) = \phi_0$ (at rest)



then $\boxed{\phi(t) = \phi_0 \cos(\omega t)}$

<u>Period:</u> $P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$

independent of ϕ_0 !!

(iii) Lagrangian approach

$T \equiv$ Kinetic energy

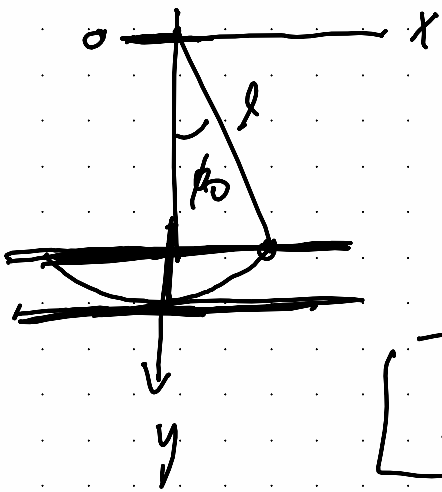
$U \equiv$ Potential energy

$$L = T - U$$

$$\boxed{T = \frac{1}{2} m l^2 \dot{\phi}^2} \quad \left(= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \right)$$

$y = l \cos\phi$

$x = l \sin\phi$



$$\boxed{U = -mgl\cos\phi + \text{const}} \leftarrow U = -mgy$$

$$U = mgl(1 - \cos\phi)$$

$$\boxed{L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi}$$

action

$$\boxed{S = \int_{t_1}^{t_2} dt\, L(\phi, \dot{\phi}, t)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \quad \text{Lagrange's equation}$$

$$\frac{d}{dt}\left(m l^2 \dot{\phi}\right) = -mgl\sin\phi$$

$$m l^2 \ddot{\phi} = -mgl\sin\phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi} \quad (\text{EOM})$$

(iv) solving $\ddot{\phi} = -\frac{g}{l}\sin\phi$ (2nd order non-linear ODE)

hard!!

$$E = \text{const}$$
$$= T + U$$
$$= \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$$

$$E = 0 - mgl\cos\phi_0$$
$$= -mgl\cos\phi_0$$

release from rest from $\phi = \phi_0$

$$-mgl\cos\phi_0 = \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$$

$$-\cancel{m}gl(\cos\phi_0 - \cos\phi) = \frac{1}{2}\cancel{m}l^2\dot{\phi}^2$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{-2\frac{g}{l}(\cos\phi_0 - \cos\phi)}$$

$|\phi| \leq \phi_0$

separable 1st order ODE

$$\boxed{\int \frac{d\phi}{\sqrt{-\frac{2g}{l}(\cos\phi_0 - \cos\phi)}} = \int dt = t + \text{const}}$$

$$t + t_0 = \int \frac{d\phi}{\sqrt{-2\frac{g}{l}\left(\cos\phi_0 - \cos\phi\right)}}$$

$$\frac{1}{\sqrt{a+bx^2}}$$

substitution:

$$\cos\phi = 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\cos\phi_0 = 1 - 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$\cos\phi = \cos\left(2\left(\frac{\phi}{2}\right)\right)$$

$$= \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\rightarrow \cos\phi_0 - \cos\phi = -2\left(\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)\right)$$

$$t + t_0 = \int \frac{d\phi}{2\sqrt{\frac{g}{l}}\sqrt{\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)}}$$

$$|\phi| \leq \phi_0$$

$$= \frac{1}{2\sqrt{\frac{g}{l}}} \int \frac{d\phi}{\sin\left(\frac{\phi_0}{2}\right)\sqrt{1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)}}}$$

let $x = \frac{\sin(\phi/2)}{\sin\left(\frac{\phi_0}{2}\right)}$

$$\boxed{x = \frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}} \rightarrow dx = \frac{1}{\sin\left(\frac{\phi_0}{2}\right)} \frac{1}{2} \cos\left(\frac{\phi}{2}\right) d\phi$$

$\sqrt{1-x^2}$

denominator

$$d\phi = \frac{2\sin\left(\frac{\phi_0}{2}\right) dx}{\cos\left(\frac{\phi}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1-\sin^2\left(\frac{\phi}{2}\right)}}$$

$$= \frac{2\sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1-\underbrace{\sin^2\left(\frac{\phi_0}{2}\right)}_{k^2} x^2}}$$

Find this out

1. $\phi(t) = \boxed{\times}$
2. Period = ??
3. Redo the analysis using Lagrange multiplier for find tension in string

$t + t_0 = \int \bigcirc$ — integrand for $sn^{-1}(x; k)$

$k = \sin\left(\frac{\phi_0}{2}\right)$

Lec #4 : Thurs 9/3

$$\phi(t) = 2\sin^{-1}\left[k\, sn\left(\omega_0\left(t+\frac{P}{4}\right); k\right)\right] \quad ☆$$

$$k \equiv \sin\left(\frac{\phi_0}{2}\right), \quad \omega_0 = \sqrt{\frac{g}{\ell}}$$

$$P = 4\sqrt{\frac{\ell}{g}}\, K(k) = 4\sqrt{\frac{\ell}{g}}\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

small angle approx

$$P_0 = 2\pi\sqrt{\frac{\ell}{g}}$$

$$\frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)} = x = sn\left[\sqrt{\frac{g}{\ell}}(t+t_0); k\right]$$

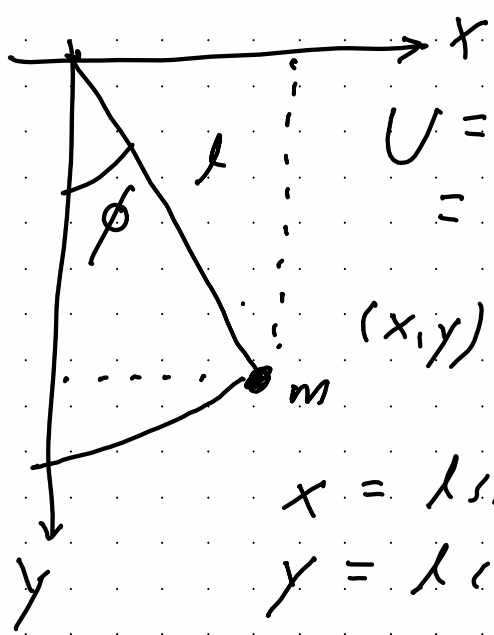
$$\sqrt{\frac{\ell}{g}}\int \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} = \sqrt{\frac{\ell}{g}}\, sn^{-1}(x; k) = t + t_0$$

$$sn^{-1}(x; k) = \sqrt{\frac{g}{\ell}}(t+t_0)$$

$$P = 4\sqrt{\frac{l}{g}}\,K(k) = \underbrace{2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{16}\phi_0^2+\dots\right)}_{\text{problem Landau 11,1}}$$

Lagrange multiplier:

$$\underbrace{\phi(x,y)}_{\text{constraint function}} = x^2+y^2-l^2 = 0$$



$$U = -mgy = -mgl\cos\phi$$

$$x = l\sin\phi$$
$$y = l\cos\phi$$

$$x^2+y^2 = l^2\sin^2\phi + l^2\cos^2\phi = l^2$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}ml^2\dot{\phi}^2$$

$$L = T - U + \lambda \varphi$$

Lagrange multiplier

$L \uparrow \quad \lambda, \cancel{\dot{\lambda}}$

$L(x, \dot{x}, y, \dot{y}, t) \quad q = (x, y) \quad \dot{q} = (\dot{x}, \dot{y})$

$L(r, \dot{r}, \phi, \dot{\phi}, t) \quad q = (r, \phi) \quad \dot{q} = (\dot{r}, \dot{\phi})$

$\lambda(t)$
$r(t)$
$\phi(t)$

$L(\phi, \dot{\phi}, t) \quad \varphi(x, y) = x^2 + y^2 - l^2 = 0 \quad \boxed{\varphi(r, \phi) = r - l = 0}$

$$\boxed{L = \frac{1}{2} m (\dot{r}^2 + \underline{\underline{r^2 \dot{\phi}^2}}) + mg \underbrace{r \cos\phi}_{y} + \lambda (r - l)}$$

$\underline{r}: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \rightarrow \boxed{\cancel{m\ddot{r}} = mr\dot{\phi}^2 + mg\cos\phi + \lambda}$

$\underline{\phi}: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \rightarrow \frac{d}{dt}(mr^2\dot{\phi}) = -mgr\sin\phi$

$\underline{\lambda}: \quad r - l = 0 \quad ✓ \qquad \cancel{2mr\dot{r}\dot{\phi}} + mr^2\ddot{\phi} = -mgr\sin\phi$

$$r - l = 0 \longrightarrow \begin{aligned} r &= l \\ \dot{r} &= 0 \\ \ddot{r} &= 0 \end{aligned}$$

$$\longrightarrow \quad \ddot{\phi} = -\frac{g}{l}\sin\dot{\phi}$$

$$ml\dot{\phi}^2 + mg\cos\phi + \lambda = 0$$

$$\boxed{\lambda = -(\underbrace{mg\cos\phi + ml\dot{\phi}^2}_{T})} \qquad \lambda = -T$$

$$L = T - U + \lambda \varphi$$

$$U = U(x, t)$$

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$\varphi(x, y, t)$$

$$U(x, y, t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}(m\dot{x}) = -\frac{\partial U}{\partial x} + \lambda \frac{\partial \varphi}{\partial x}$$

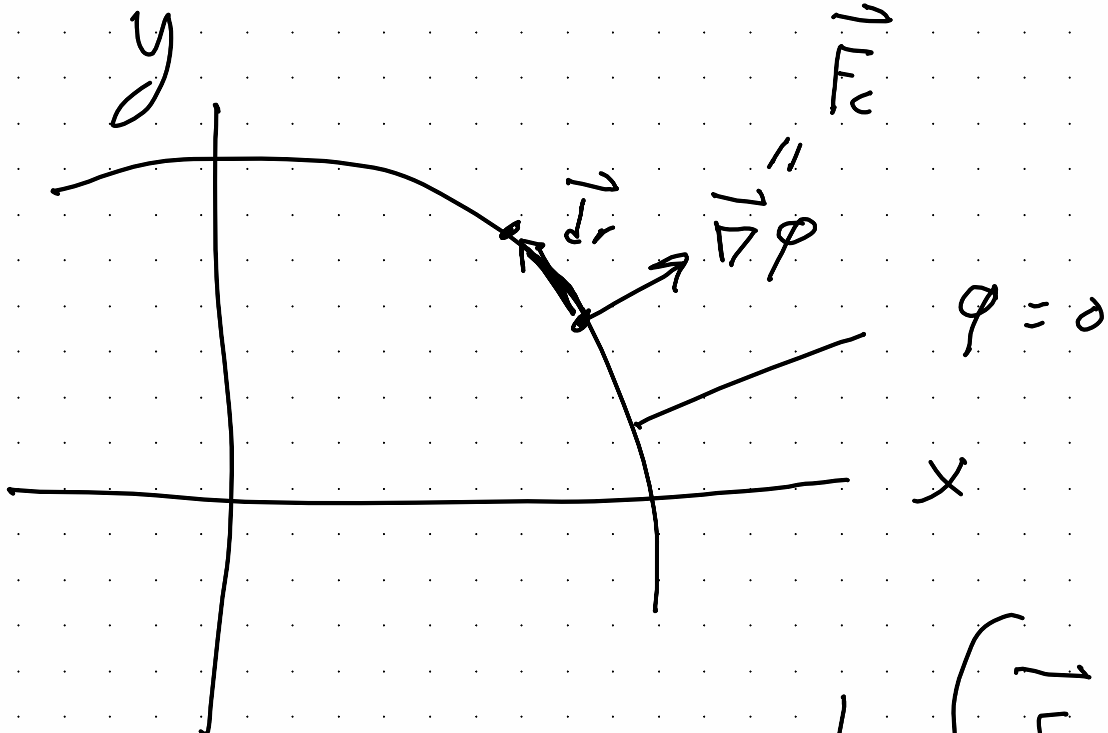
$$\frac{d}{dt}(m\dot{y}) = -\frac{\partial U}{\partial y} + \lambda \frac{\partial \varphi}{\partial y}$$

$$\boxed{\frac{d\vec{p}}{dt} = \vec{F}_{net}}$$

applied force, force constraint

$$\left.\begin{aligned} \frac{dp_x}{dt} &= F_x + \lambda \frac{\partial \varphi}{\partial x} \\ \frac{dp_y}{dt} &= F_y + \lambda \frac{\partial \varphi}{\partial y} \end{aligned}\right\} \quad \boxed{\frac{d\vec{p}}{dt} = \vec{F} + \underbrace{\lambda \vec{\nabla} \varphi}_{\text{force constraint}}}$$

$$\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) = \vec{\nabla} \varphi$$



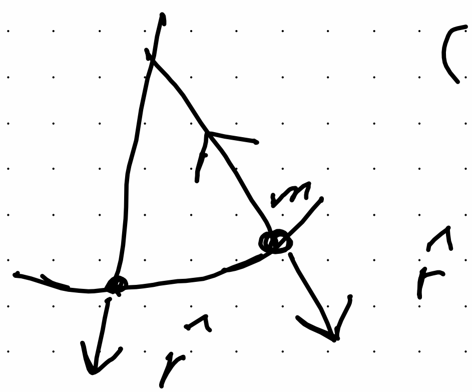
$$\int \underbrace{\vec{F}_c \cdot \vec{dr}}_{0} = \text{Work}$$

$$\vec{\nabla}\varphi \perp \varphi = \text{const}$$

$$\varphi = r - \ell$$

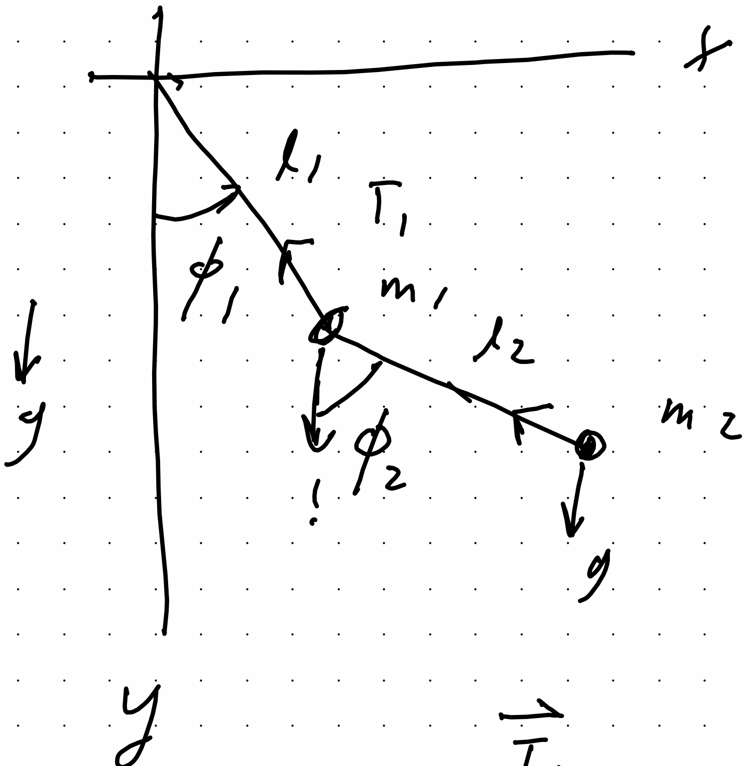
$$\vec{\nabla}\varphi = \hat{r}$$

$$\lambda \vec{\nabla}\varphi$$



(r, ϕ)

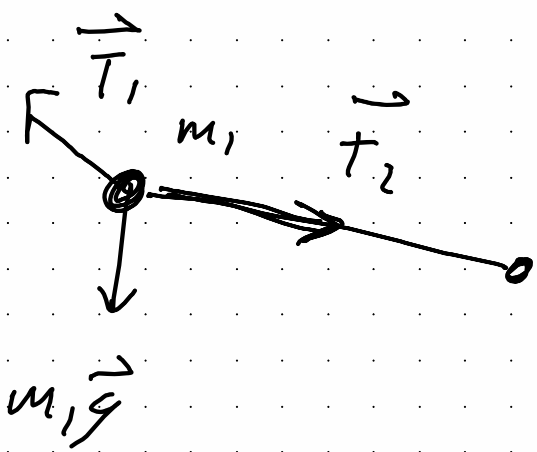
$\hat{r}, \hat{\phi}$



$$U = -m_1 g y_1 - m_2 g y_2$$

$\ddot{\phi}_1$, $\ddot{\phi}_2$

EOM



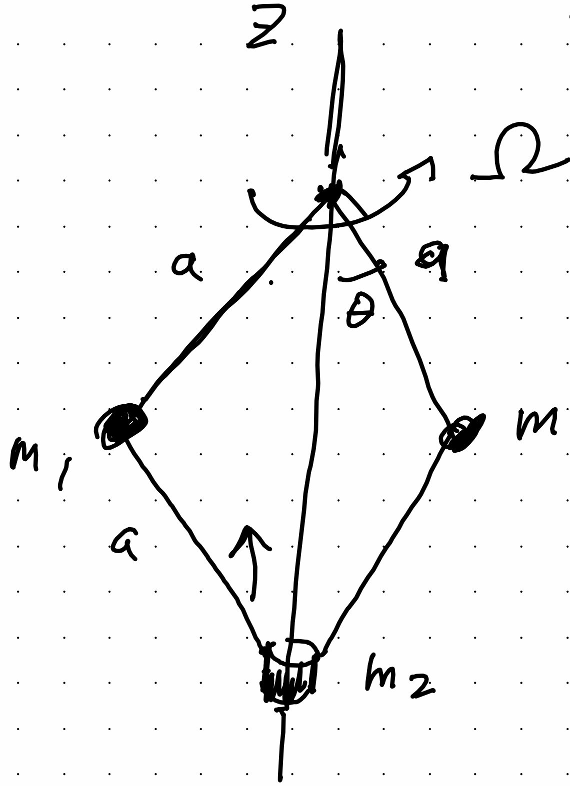
$\lambda\varphi$

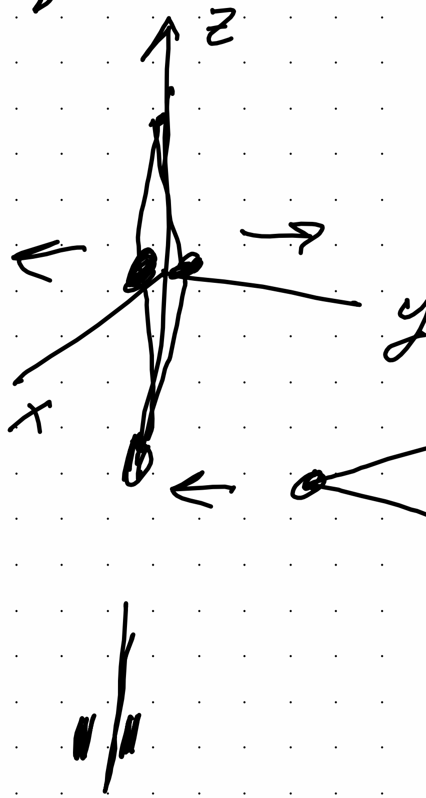
$(\lambda_1\varphi_1 + \lambda_2\varphi_2 + \dots)$

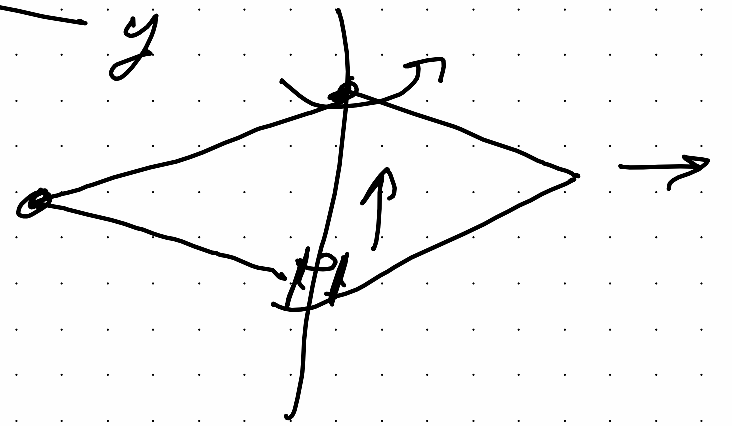
$$L \longrightarrow \text{EOM}$$

$$\underbrace{L \longrightarrow L + \frac{d}{dt}\left(f(q,t)\right)}_{L'}$$

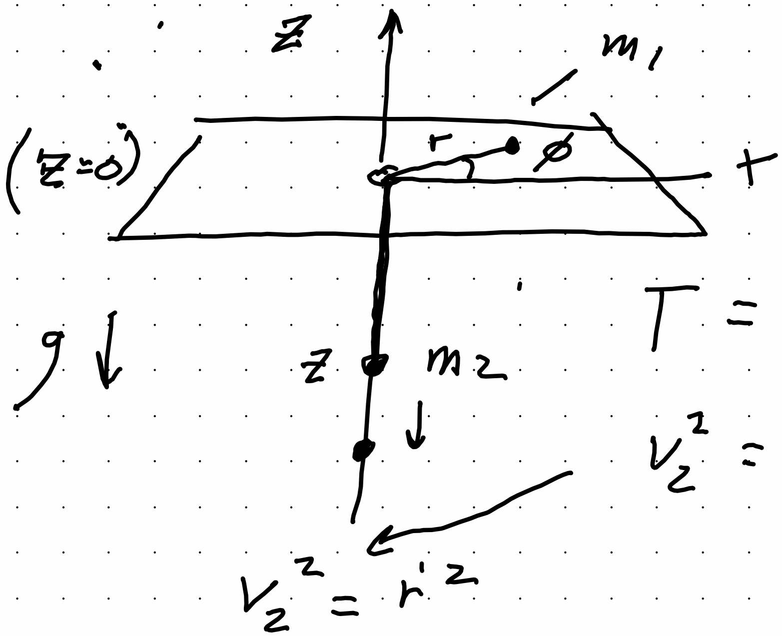
$$f(q, \cancel{\dot{q}}, t)$$







Lec #5: Tuesday 9/8



$$\boxed{r - z = \ell} = \text{length of string}$$

$$L = T - U$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$v_2^2 = \dot{z}^2, \quad v_1^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

$$\left(= \dot{x}^2 + \dot{y}^2, \quad \begin{array}{l} x = r\cos\phi \\ y = r\sin\phi \end{array}\right)$$

$$T = \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2$$

$$\boxed{= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2}$$

$$U = m_2 g z = m_2 g (r - \ell) = \underline{m_2 g r} - \underbrace{m_2 g \ell}_{\text{constant}}$$

$$\boxed{U = m_2 g r}$$

$$L = T - U = \boxed{\frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = \frac{\partial L}{\partial r}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi} \Longrightarrow$$ 2nd order EOMs

<u>No explicit t dependence:</u>

$$E = \dot r \frac{\partial L}{\partial \dot r} + \dot\phi \frac{\partial L}{\partial \dot\phi} - L = \boxed{T+U}$$

$$E = \sum_i \dot q_i \underbrace{\frac{\partial L}{\partial \dot q_i}}_{= p_i} - L$$

<u>No explicit ϕ dependence:</u>

$$p_\phi \equiv \frac{\partial L}{\partial \dot\phi} = \text{const} = M_z$$

$\vec{M}$: angular momentum (L&L notation)

$$\boxed{M_z = m_1 r^2 \dot\phi} \rightarrow \boxed{\dot\phi = \frac{M_z}{m_1 r^2}}$$

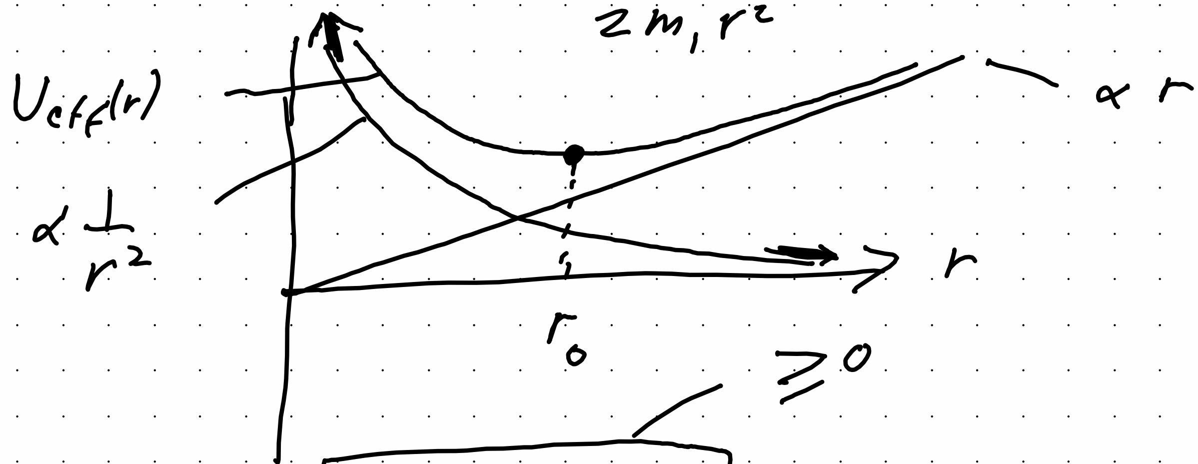
$$E = \frac{1}{2} m \dot x^2 + U(x)$$

$$\boxed{E = \frac{1}{2}(m_1+m_2)\dot r^2 + \frac{1}{2} m_1 r^2 \dot\phi^2 + m_2 g r}$$

$$\boxed{E = \frac{1}{2}(m_1+m_2)\dot r^2 + \left(\frac{M_z^2}{2m_1 r^2} + m_2 g r\right)} \leftarrow \boxed{U_{eff}(r) = U(r) + \frac{M_z^2}{2m_1 r^2}}$$

$$U_{eff}(r) = \frac{M_z^2}{2m_1 r^2} + m_2 g r$$

$$\left(U(r) = m_2 g r \right.$$



≥ 0

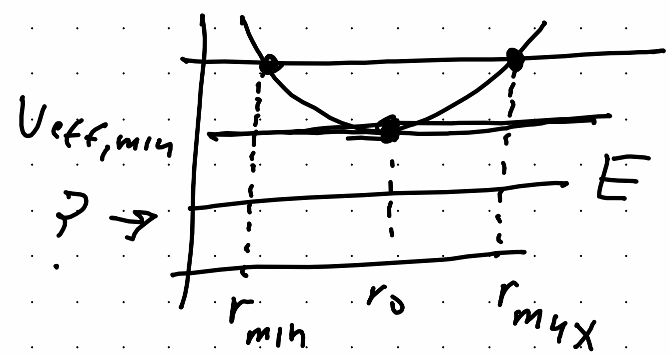
$$E = \boxed{\frac{1}{2}(m_1+m_2)\dot{r}^2} + U_{eff}(r)$$

const

r_{min}, r_{max}:

"turning points!"

$(\dot{r}=0)$



i) $E = U_{eff,min} = U_{eff}(r_0)$

unif circular motion: $r = r_0$, $\dot{\phi} = \frac{M_z}{m_1 r_0^2}$

$E < U_{eff,min}$

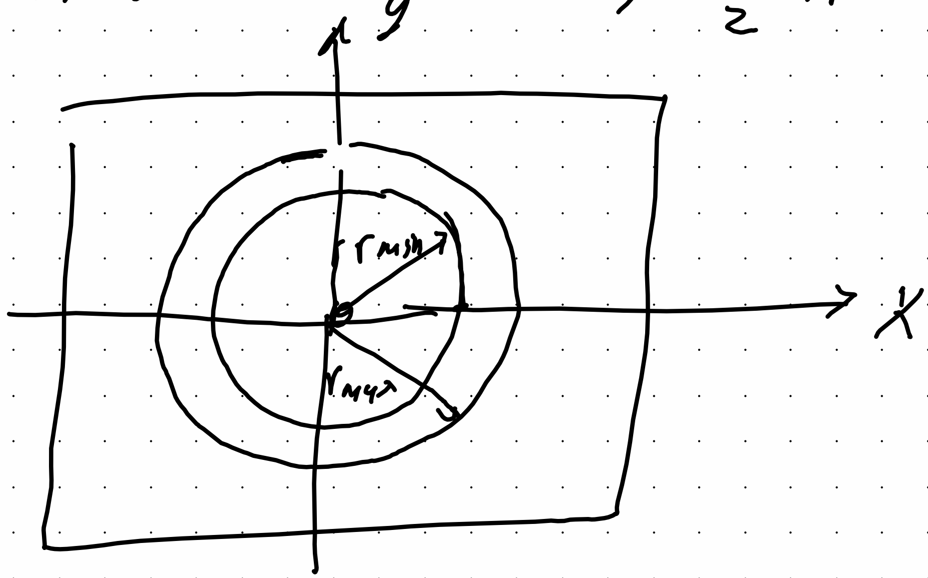
(not allowed)

ii) $E > U_{eff,min}$

$E = U_{eff}(r_{min}) = U_{eff}(r_{max})$

$E \geq U_{eff,min}$

top view



$U(r) = \frac{1}{2}kr^2$ closed bound

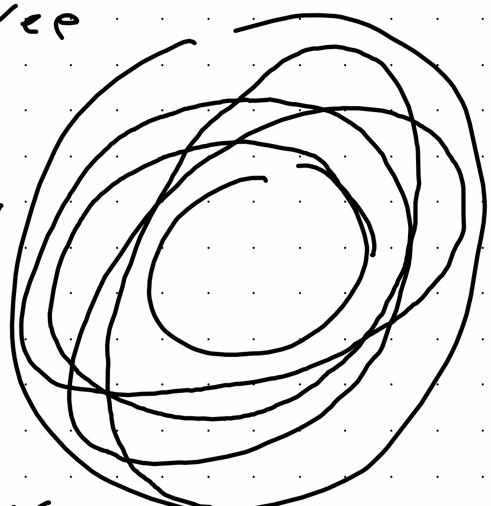
$U(r) = -\frac{G m_1 m_2}{r}$ orbits

Newtonian gravity

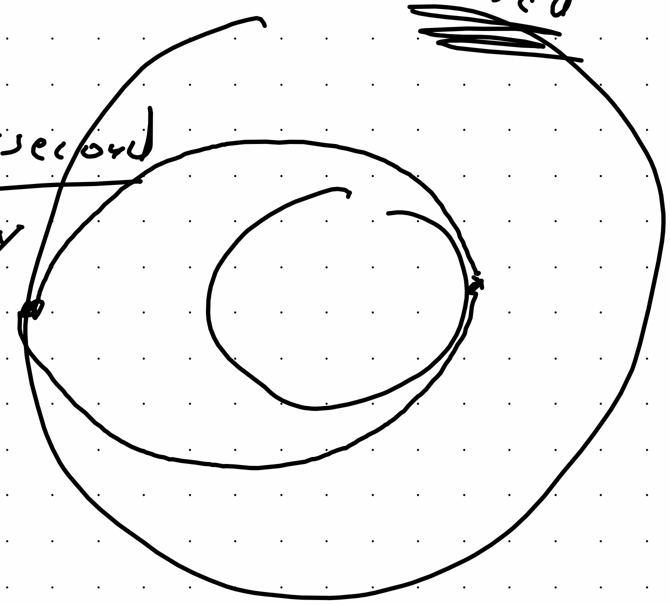
bound orbits = ellipse
are closed

1 degree
=
60 mins of arc

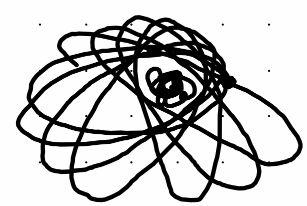
1 min of arc
=
60 sec of arc



43 arcsecond / century



perihelion precession of Mercury
closest approach to sun



$\underline{r_0:}$ $\quad \left.\dfrac{dU_{eff}}{dr}\right|_{r=r_0} = 0 \quad$ (minimum)

$$0 = \frac{d}{dr}\left(\frac{M_z^2}{2m_1 r^2} + m_2 g r\right)\Bigg|_{r=r_0}$$

$$= -\frac{M_z^2}{m_1 r_0^3} + m_2 g$$

$$\boxed{M_z^2 = m_1 m_2 g r_0^3}$$

tells you the value of M_z needed to have a specific r_0 value.

For a given M_z, this tells you what r_0 equals.

Energy equation:

$$E = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{M_z^2}{2m_1 r^2} + m_2 g r$$

$$\boxed{\dot{\phi} = \frac{M_z}{m_1 r^2}} \leftarrow \phi \text{ equation}$$

$$\frac{1}{2}(m_1+m_2)\dot{r}^2 = E - \frac{M_z^2}{2m_1 r^2} - m_2 g r$$

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}$$

$$\int \frac{dr}{\pm\sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}} = \int dt = t + \text{const}$$

$$t(r) \Longleftrightarrow r(t)$$

orbital equation:

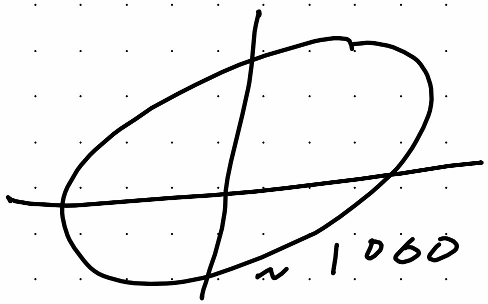
$r = r(\phi) \Longleftrightarrow \phi = \phi(r)$

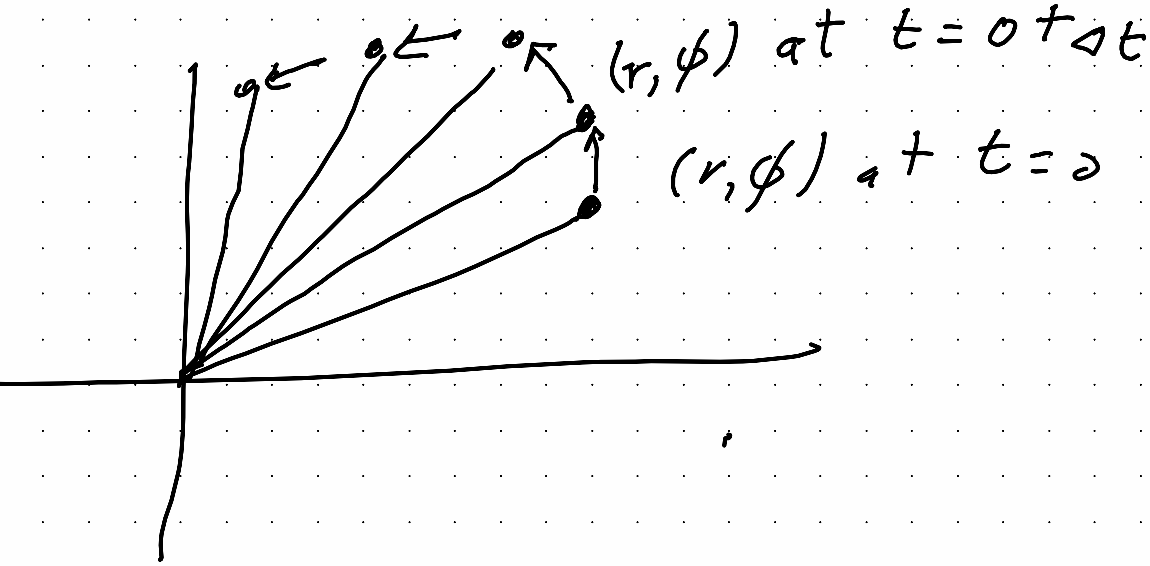
$$\frac{dr}{dt} = \dot{r} = \sqrt{\left(\frac{2}{m_1+m_2}\right)\left[E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right]}$$

$$\frac{dr}{dt} = \underbrace{\frac{dr}{d\phi}}_{}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M_z}{m_1 r^2}$$

$$\rightarrow \quad \frac{dr}{d\phi} = \frac{m_1 r^2}{M_z}\sqrt{\left(\frac{2}{m_1+m_2}\right)\left[\qquad\right]}$$

$$\int \frac{dr}{\frac{m_1 r^2}{M_z}\sqrt{(\;)[\quad]}} = \int d\phi = \phi + \text{const}$$





Know: r, ϕ at some time t

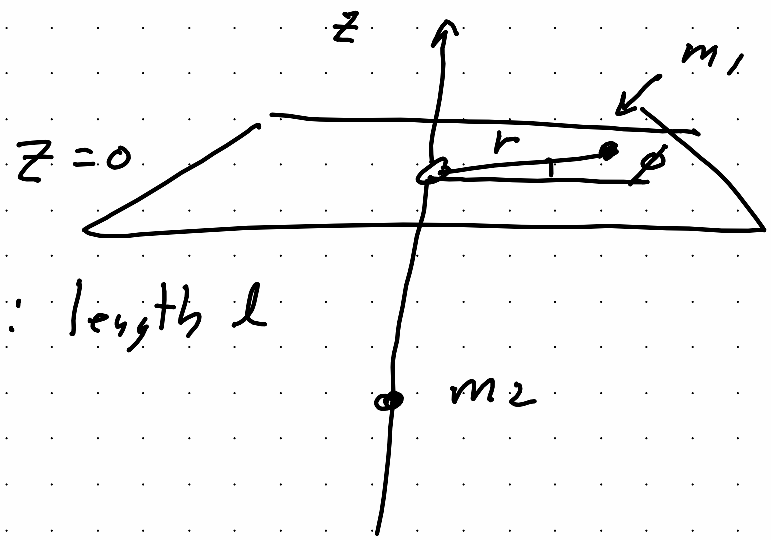
given: Δt need to know Δr and $\Delta \phi$

$$r(t+\Delta t) = r(t) + \Delta r(t) + \dots$$

$$\phi(t+\Delta t) = \phi(t) + \Delta\phi(t) + \underbrace{\dots}_{\text{ignore if } \Delta t \text{ is suff. small}}$$

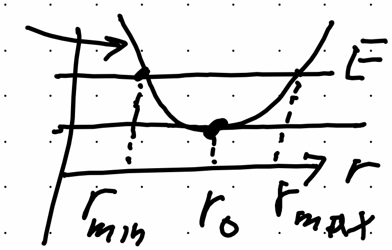
Lecture #6 : Thursday 10 Sep

1) Secs 6–10 (today), Sec 40 (next Tuesday)
2) Finish up example from last time
3) Conservation of E, $\vec{P}$, $\vec{M}$
4) Mechanical similarity
5) Quiz: last 20 minutes (1:30pm)



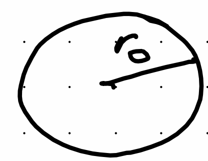
string: length l

E, M_z = const



$$\frac{d\phi}{dt} = \frac{M_z}{m_1 r^2}$$

$$\frac{dr}{dt} = \sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E - \underbrace{m_2 g r - \frac{M_z^2}{2m_1 r^2}}\right)} = \sqrt{\emptyset}$$



$$\frac{d\phi}{dt} = \frac{M_z}{m_1 r^2} \longrightarrow \Delta\phi = \Delta t \, \frac{M_z}{m_1 r^2}$$

$$\frac{dr}{dt} = \sqrt{\quad} \longrightarrow \Delta r = \Delta t \sqrt{\quad}$$

$$r(0) = r_{min}$$
$$\phi(0) = 0$$

$$\phi(\Delta t) = \phi(0) + \Delta\phi$$
$$r(\Delta t) = r(0) + \Delta r$$

$$\phi(2\cdot\Delta t) = \phi(\Delta t) + \Delta\phi$$
$$r(2\cdot\Delta t) = r(\Delta t) + \Delta r$$

$$\vdots$$

$$\phi(t_i) = \phi(t_{i-1}) + \Delta\phi$$
$$r(t_i) = r(t_{i-1}) + \Delta r$$

Cons. of $E, \vec{P}, \vec{m}$:

All of $E, \vec{P}, \vec{m}$ conserved for a closed system

no external forces

$U = U(\vec{r}_1 - \vec{r}_2, \vec{r}_2 - \vec{r}_3, \cdots)$

relative position vectors.

Even in the presence of external forces, you can still have cons. of E and some components of $\vec{P}$ and $\vec{m}$.

(i) $U = mgy$ $\quad \vec{F} = -m\vec{g}$

If U does not depend explicitly on time t, then E is conserved.

$$E \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

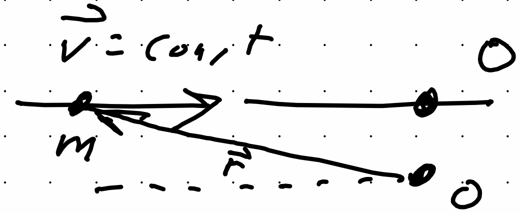
constant external field

(ii) e.g., $\downarrow \vec{F}_g = m\vec{g}$

$\longleftrightarrow$ x, y

$p_x = \text{const}$
$p_y = \text{const}$

If U is unchanged by a translation in some direction $\hat{t}$ then $\vec{P}\cdot\hat{t} = \text{const}$

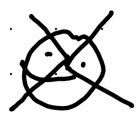


(iii) $\vec{M}$ depends on choice of origin

(a) uniform gravitational field $\vec{F} = m\vec{g}$ $\uparrow z$

$U = m_1 g z_1 + m_2 g z_2 + \dots$

If U is unchanged by a rotation about a particular axis $\hat{n}$ then $\boxed{\vec{M}\cdot\hat{n} = \text{const}}$ (e.g., $\hat{n} = \hat{z}$, $M_z = \text{const}$) provided the origin is located on the axis.

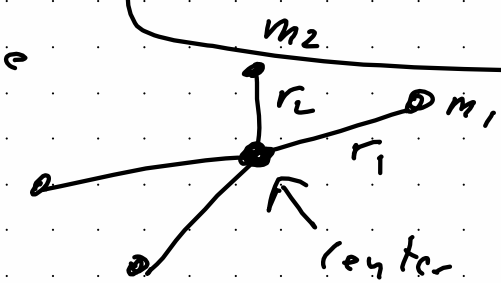


(b) central force

$\boxed{\vec{M} = \text{const}}$ $U = U(r)$

$\vec{F} = -\nabla U$

$= -\frac{dU}{dr}\hat{r}$



Mechanical similarity :

$$L \to L' = c \cdot L \qquad \text{same equations of motion}$$

suppose we rescale position vectors $\vec{r}_a \to \alpha \vec{r}_a$

$$U'(\vec{r}_1, \dots) = U(\alpha \vec{r}_1, \alpha \vec{r}_2, \dots) = \alpha^K U(\vec{r}_1, \vec{r}_2, \dots)$$

potential is homogenous of degree K w.r.t. position vector

Example: i) $U = mgy$, $K = 1$

ii) $U = \frac{1}{2} k x^2$, $K = 2$

iii) $U = -\frac{G m_1 m_2}{r}$, $K = -1$

$$U' = mg\alpha y = \alpha mgy = \alpha U$$

$$L' = T' - U' = T' - \alpha^K U = \text{const} \cdot L$$

$$= \alpha^K T - \alpha^K U$$

$$= \alpha^K (T - U) = \alpha^K L$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \frac{l^2}{t^2}$$

length

time

$$l \to l' = \alpha l$$
$$t \to t' = \beta t$$

$$\frac{l'}{l} = \alpha \qquad \frac{t'}{t} = \beta$$

$U = mgy$, $k = 1$

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{\frac{1}{2}}$$

$U = \frac{1}{2} k x^2$, $k = 2$

$t'/t = \text{const}$

$$T' = \frac{1}{2} m v'^2 = \frac{1}{2} m \frac{l'^2}{t'^2}$$

$$= \frac{1}{2} m \frac{\alpha^2 l^2}{\beta^2 t^2}$$

$$= \frac{\alpha^2}{\beta^2} T$$

$$\boxed{\frac{\alpha^2}{\beta^2} = \alpha^k} \longrightarrow \boxed{\beta = \alpha^{1-k/2}}$$

$$\beta^2 = \alpha^{2-k}$$

$$\boxed{\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-k/2}}$$

$$U = -\frac{G m_1 m_2}{r}$$

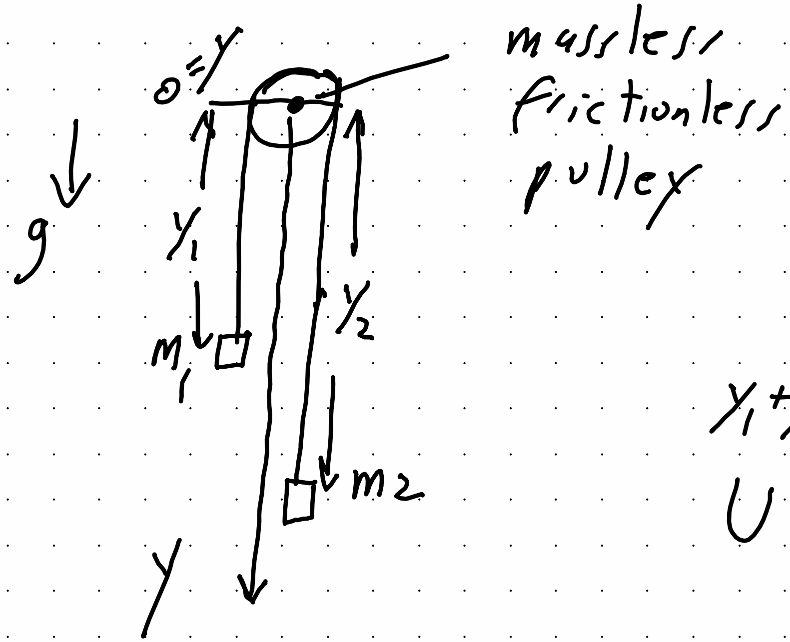
$k = -1$

$$\left(\frac{t'}{t}\right) = \left(\frac{l'}{l}\right)^{3/2}$$

$$p^2 = \text{Dist}^3$$

$$\boxed{\frac{p^2}{a^3} = \text{const}}$$

QUIZ #1 : Atwood's machine



$(m_1 > m_2)$

string: length l
(massless
inextensible
...)

i) L ?

ii) EOM

iii) solve EOM

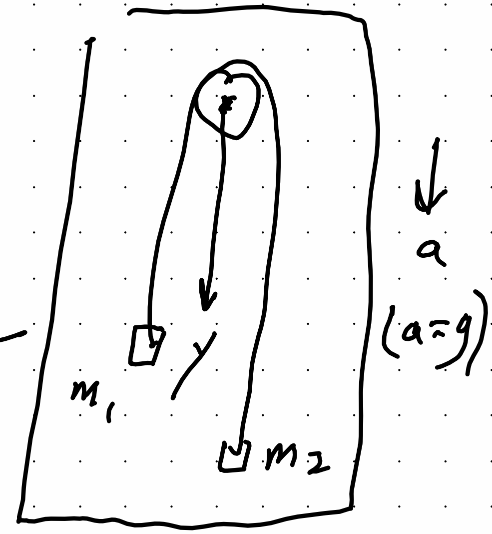
$$y_1 + y_2 = l \rightarrow y_2 = l - y_1 \rightarrow \dot{y}_2 = -\dot{y}_1$$

$$\begin{aligned} U &= -m_1 g y_1 - m_2 g y_2 \\ &= -m_1 g y_1 - m_2 g (l - y_1) \\ &= -m_1 g y_1 \underbrace{- m_2 g l}_{\text{ignore}} + m_2 g y_1 \\ &= \boxed{-(m_1 - m_2) g y_1} \end{aligned}$$

$$\begin{aligned} T &= \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \underbrace{\dot{y}_2^2}_{= \dot{y}_1^2} \\ &= \boxed{\frac{1}{2}(m_1 + m_2)\dot{y}_1^2} \end{aligned}$$

Lecture #7: Tuesday 9/15

1) Go over quiz #1
2) Modified Atwood problem
3) Finish mechanical similarity (sec 10)
4) Hamilton's equations (sec 40)

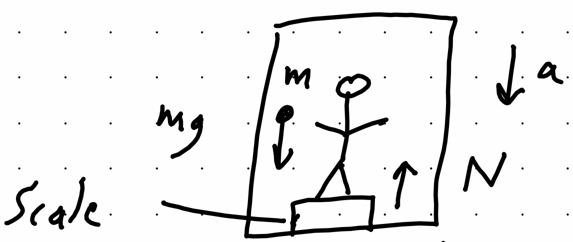


$$L = T - U$$
$$= \frac{1}{2}(m_1+m_2)\dot{y}_1^2 + (m_1-m_2)g y_1$$

EOMs: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) = \frac{\partial L}{\partial y_1} \rightarrow (m_1+m_2)\ddot{y}_1 = (m_1-m_2)g$

$$\boxed{\ddot{y}_1 = \frac{(m_1-m_2)g}{(m_1+m_2)}}$$

$$y_1(t) = y_0 + v_0 t + \frac{1}{2}\frac{(m_1-m_2)}{(m_1+m_2)} g t^2$$



$$ma = mg - N$$
$$\underbrace{N}_{w} = m(g-a)$$

apparent or effective weight

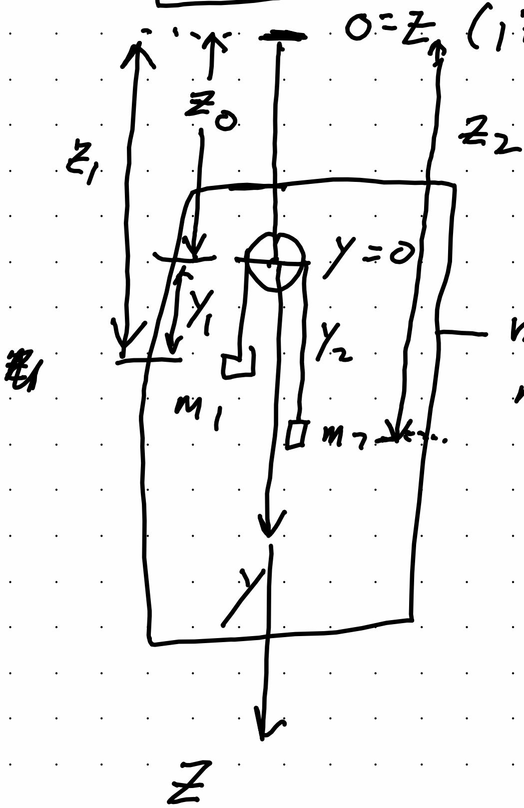
$$\boxed{\ddot{y}_1 = \frac{(m_1-m_2)}{m_1+m_2}(g-a)}$$

$\vec{F} = m\vec{a}$ (valid in an inertial ref frame)

$\vec{F} + \vec{F}_{\text{fictitious}} = m\vec{a}$ —— w.r.t. to a non-inertial ref. frame

sec 39 (L&L) $\qquad L' = L + f(t)$

$\boxed{L = T - U}$ (valid in an inertial ref frame)



non-inertial ref frame

$T = \frac{1}{2} m_1 \dot{z}_1^2 + \frac{1}{2} m_2 \dot{z}_2^2$

$U = -m_1 g z_1 - m_2 g z_2$

$z_1 = z_0 + y_1 \qquad y_2 = l - y_1$

$z_2 = z_0 + y_2$

$$U = -m_1 g (z_0 + y_1) - m_2 g (z_0 + \overbrace{y_2}^{l - y_1})$$

$$= \underbrace{-m_1 g z_0 - m_1 g y_1 - m_2 g z_0}_{\text{prescribed function of time} = \text{ignore}} \underbrace{-m_2 g l}_{= \text{const}} + m_2 g y_1$$

$$= \boxed{-(m_1 - m_2) g y_1}$$

Do this at home:

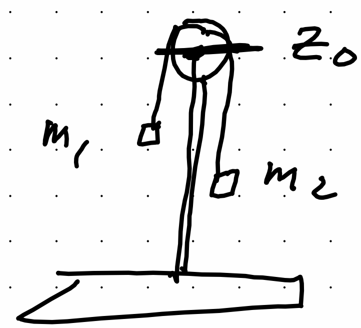
$$T = \frac{1}{2}(m_1+m_2)\dot{y}_1^2 + (m_1-m_2)\dot{z}_0\dot{y}_1$$

$$L' = L + \frac{d}{dt}\left(f(q,t)\right) \longrightarrow \text{same EOM}$$

$$(m_1-m_2)\dot{z}_0\dot{y}_1 = \underbrace{\frac{d}{dt}\Big[\underbrace{(m_1-m_2)\dot{z}_0 y_1}_{f(y_1,t)}\Big]}_{\text{ignore}} - (m_1-m_2)\underbrace{\ddot{z}_0}_{\equiv a} y_1$$

$$\boxed{T = \frac{1}{2}(m_1+m_2)\dot{y}_1^2 - (m_1-m_2)\,a\,y_1}$$

$z_0(t)$: given, not to be solved for



Hamilton's equations:

$L(q, \dot{q}, t)$

Hamiltonian:

$$E = \underbrace{\sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L}_{E(q, \dot{q}, t)}$$

$H = H(q, p)$

not here if $L = L(q, \dot{q})$

$$H = \left(\sum_i p_i \dot{q}_i - L \right) \Big|$$

$\dot{q} = \dot{q}(q, p)$

$$\boxed{p_i \equiv \frac{\partial L}{\partial \dot{q}_i}}$$

Example:

$L = \frac{1}{2} m \dot{x}^2 - U(x)$ (single particle, 1-d, const external field)

$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = \frac{p}{m}$

$H = (p\dot{x} - L) \Big|_{\dot{x} = p/m}$, $\cancel{E}$

$= \left(p\dot{x} - \frac{1}{2} m \dot{x}^2 + U(x)\right) \Big|_{\dot{x} = p/m}$ $= \frac{p^2}{m} - \frac{1}{2} m \left(\frac{p}{m}\right)^2 + U(x)$

$H = \frac{1}{2}\frac{p^2}{m} + U(x)$

EOMs: (Hamilton's equations) ((39,6) Prob 2 Sec 40)

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, \dots, \underset{\text{\# of DOF}}{s}$$

Lagrange's equation:

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}}_{} \qquad i = 1, \dots, s$$

s-equations, 2nd order ODE for q_i

2s-equations, 1st order in $\dot{q}_i, \dot{p}_i$

$\dot{x} = p/m$

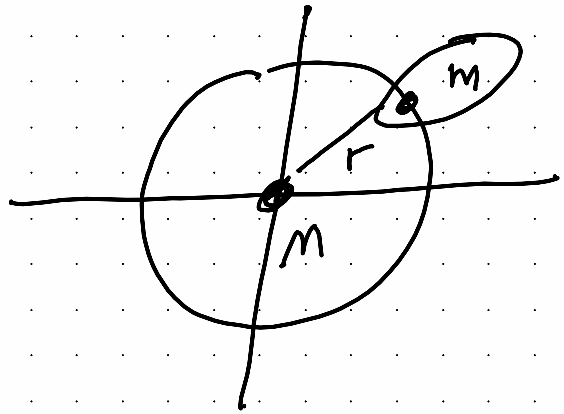
$p = m\dot{x}$

$\dot{p} = -\frac{\partial U}{\partial x}$

$$\boxed{m\ddot{x} = -\frac{\partial U}{\partial x}}$$

$$H = \frac{p^2}{2m} + U(x) \rightarrow \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$$

$$L = \frac{1}{2}m\dot{x}^2 - U(x) \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow m\ddot{x} = -\frac{\partial U}{\partial x}$$



$$U = -\frac{GMm}{r}$$

Problem: $U' = cU$

~~masses = equal~~

$r' = r$, $m' = m$

$$\frac{GM\cancel{m}}{r^2} = \frac{\cancel{m}v^2}{r}$$

$$\boxed{v = \sqrt{\frac{GM}{r}}}$$

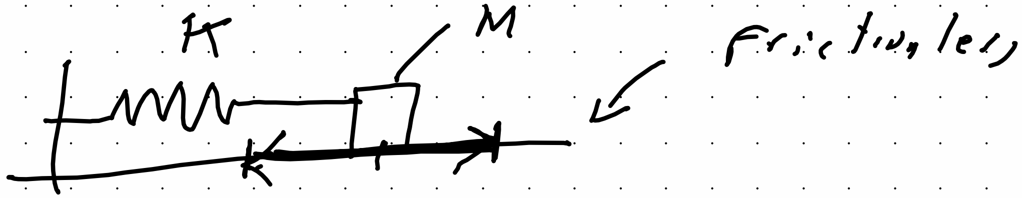
Suppose $M \to cM = M'$

$$U' = cU$$

$$\frac{2\pi r}{T'} = v' = \sqrt{\frac{cGM}{r}}$$

$$\frac{1}{T'} \propto \sqrt{c}$$

$$\sqrt{\frac{E}{E'}} = \sqrt{\frac{U'}{U}}$$



$$U = \frac{1}{2} k x^2$$

$$T = \frac{1}{2} m \dot{x}^2 \qquad L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \qquad \omega = \sqrt{\frac{k}{m}}$$

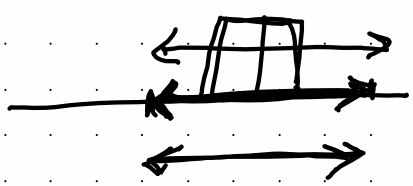
$$m\ddot{x} = -kx$$

$$\ddot{x} = -\frac{k}{m} x \rightarrow x(t) = a \cos \omega t + b \sin \omega t$$

$$U' = cU \quad (\text{for example } k' = ck)$$

$$T' = cU$$

$$\ddot{x} = -\frac{k'}{m} x, \quad \omega' = \sqrt{\frac{k'}{m}}$$



$$T = \frac{1}{2} m \dot{x}^2$$

$$T' = cT$$

$$\frac{2\pi}{P'} = \sqrt{\frac{k'}{m}}$$

$$\frac{P}{P'} = \sqrt{\frac{U'}{U}}$$

$$L = T - U$$

$$L' = cL = cT - cU$$

$$L'' = T - cU \neq cL$$

L, L' same EOMs

L, L'' different EOMs

$$\frac{1}{t'^2} = c$$

$$T = \frac{1}{2} m \dot{x}^2, \qquad U = \frac{1}{2} k x^2$$

$$L = T - U \rightarrow m\ddot{x} = -kx$$

$$L' = cL \rightarrow \cancel{c} m\ddot{x} = -\cancel{c} kx$$

$$m\ddot{x} = -kx$$

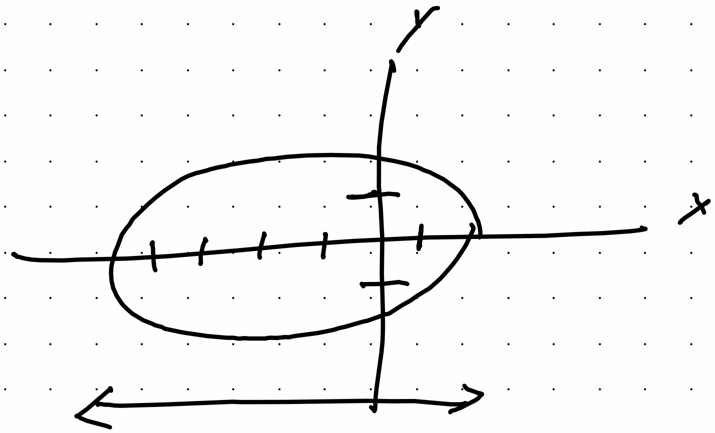
$$\ddot{x} = -\frac{k}{m} x$$

$$L'' \longrightarrow m\ddot{x} = -ckx$$

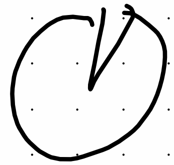
$$c x'' = \ddot{x} = -c\left(\frac{k}{m}\right) x$$

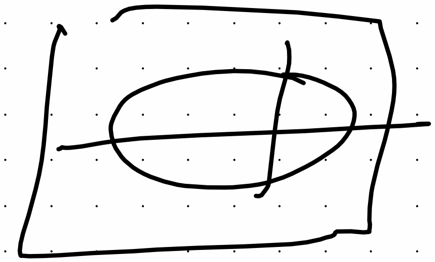
differ EOMs

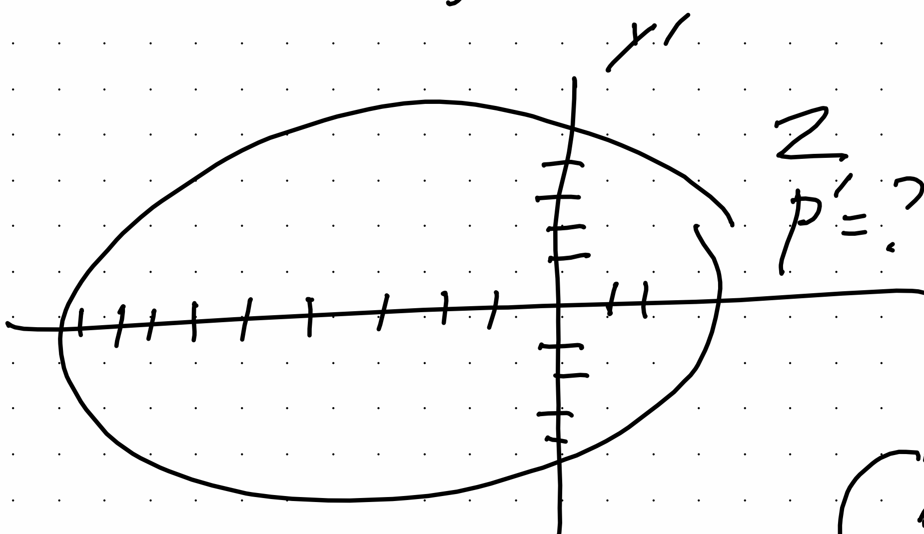
$$x = a \cos(\omega' t) + b \sin(\omega' t)$$



a, P



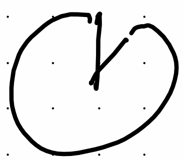




2

$P' = ?$

$\frac{P^2}{a^3} = \text{const}$



t'

a', P'

$U' = cU$

$P' = ?\, P$

$m' = m$

$l' = l$

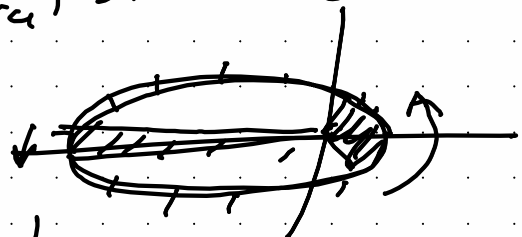
"mechanical similarity"

$L' = \text{const}\, L$

$T' - U' = (\text{const})(T - U)$

$T' - cU = c\,(T - U) \qquad cU = U'$

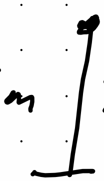
$$\frac{P'}{P} = \sqrt{\frac{U}{U'}}$$



$k \to k' = ck$

Periods

$g \to 2g$



1m $p' ?\ p$

$\frac{1}{\sqrt{2}}$

$2k$

$\sqrt{\frac{1}{2}}$

k

$T = \frac{1}{2} m \dot{x}^2$

$\cancel{T' = T}$

$U' = cU$

$\cancel{T' = cL}$

$L' = cL$

Lec #8: Thur Sep 17th

Today — 1-d motion (Sec 11)

Next two weeks — central force (Sec 13-15)

$$T = \frac{1}{2} a(q) \dot{q}^2$$

★ Midterm 1 — Tues Oct 6th

$$T = \frac{1}{2} \sum_{j,k} a_{jk}(q) \dot{q}_j \dot{q}_k$$

??

[single particle]

$j = 1, 2, 3$

$U(q_1, \dots, q_s, t)$

$$T = \frac{1}{2} m |\vec{v}|^2$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$ cartesian (x, y, z)

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2)$$ sph. polar (r, θ, ϕ)

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

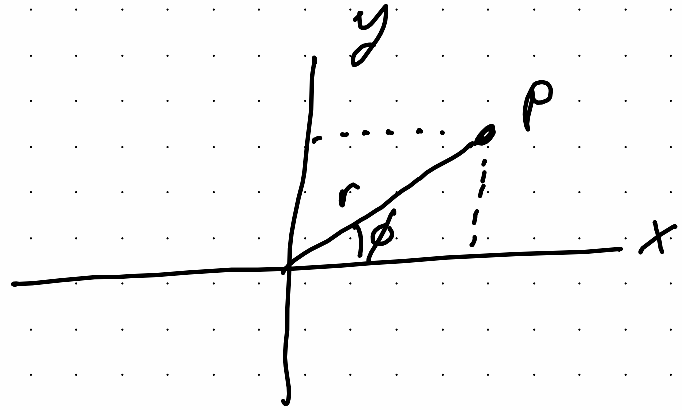
$q_1 = r$

$q_2 = \theta$

$q_3 = \phi$

$$a_{ij} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

$\underline{a_{11} = m}$, $a_{22} = m r^2$, $a_{33} = m r^2 \sin^2\theta$, $a_{12} = 0$, $a_{13} = 0$

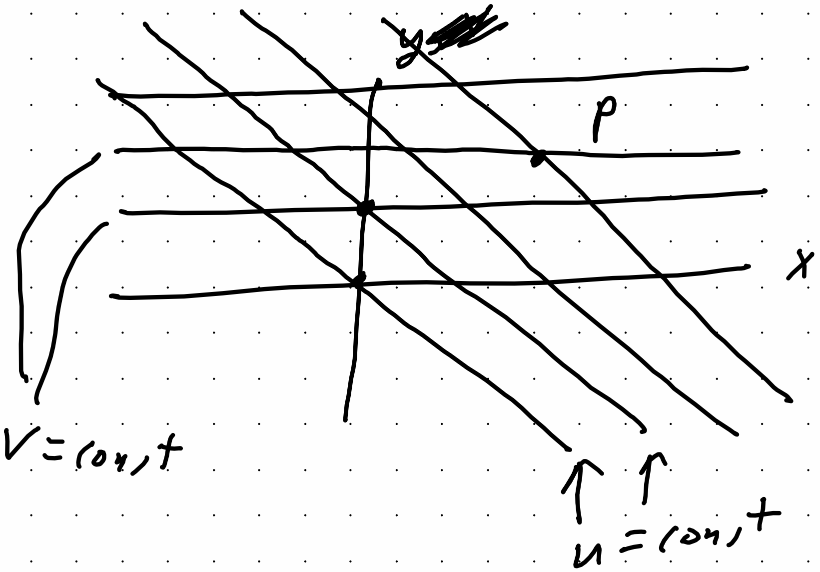


(x, y)

(r, ϕ)

$\dot{x} = \dot{u} - \dot{v}$

$\dot{y} = \dot{v}$



$u = x + y$	$x = u - v$
$\cancel{v = x - y}$	
$v = y$	$y = v$

$u = \text{const}$?

$x + y = \text{const}$

$$\boxed{y = \underbrace{\text{const}}_{b} \underbrace{-}_{m} x}$$

$T = \frac{1}{2} m(\dot{u}^2 + \dot{v}^2)$??

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$ ✓.

$= \frac{1}{2} m(\dot{u}^2 + \dot{v}^2 - 2\dot{u}\dot{v} + \dot{v}^2)$

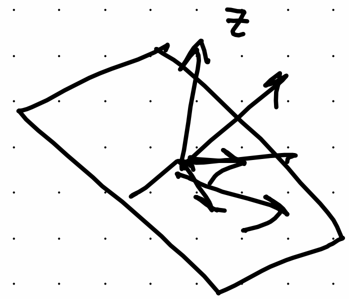
$= \frac{1}{2} m(\dot{u}^2 + 2\dot{v}^2 - 2\dot{u}\dot{v})$

$$a_{ij} = m \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$T = \frac{1}{2} m |\vec{v}|^2$$

$$\hat{e}_1 \cdot \hat{e}_1 = 1$$
$$\hat{e}_1 \cdot \hat{e}_j \neq 0$$

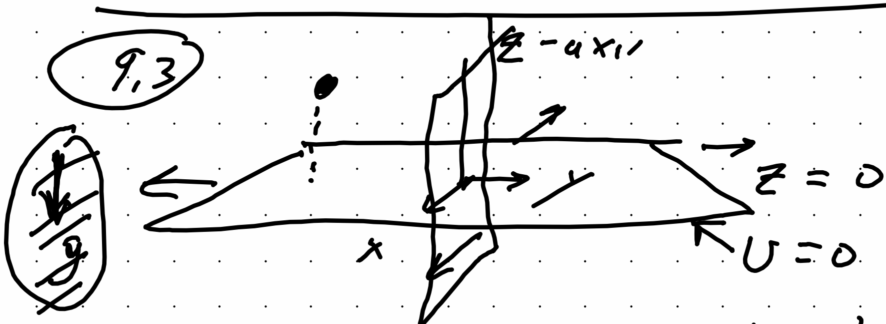
$$\vec{v} = \sum_{i=1}^{3} v_i \hat{e}_i$$



$$|\vec{v}|^2 = \vec{v} \cdot \vec{v}$$
$$= \left(\sum_i v_i \hat{e}_i\right) \cdot \left(\sum_j v_j \hat{e}_j\right)$$
$$= \sum_{i,j} v_i v_j \underbrace{\hat{e}_i \cdot \hat{e}_j}_{\text{might not } = \delta_{ij}}$$

(orthogonal)

9.3



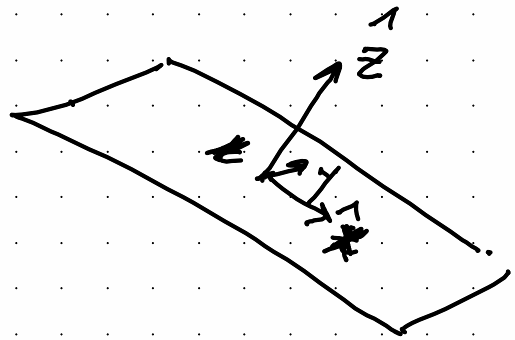
(a) $U(\cancel{x}, \cancel{y}, z)$

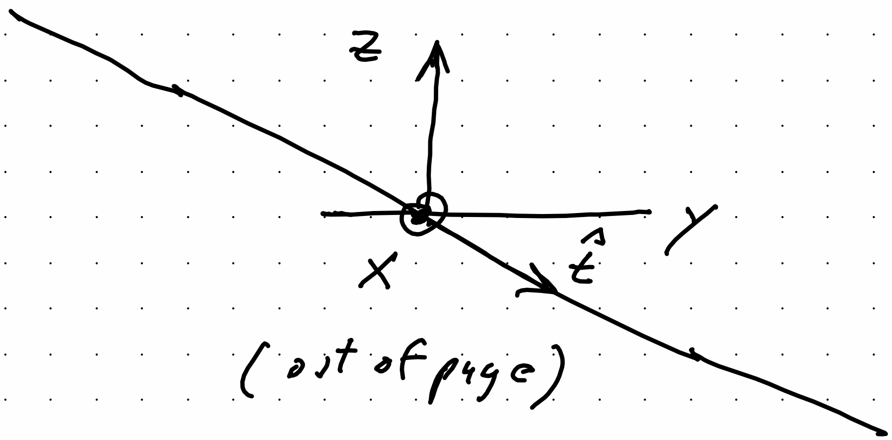
$mgz = U(z)$, $\vec{F} = -m\vec{g}$

$\delta L = 0$ if you displace in $\hat{x}, \hat{y}$ $\rightarrow$ p_x, p_y = const

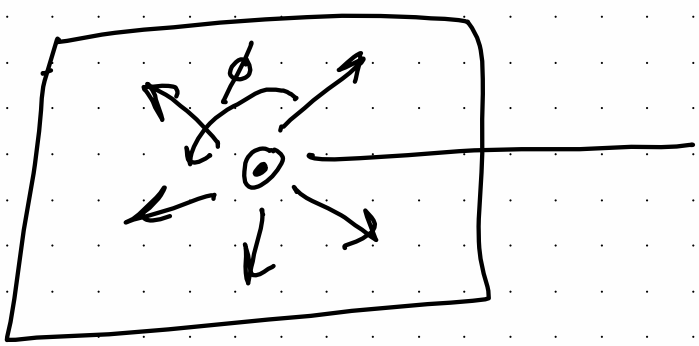
M_z = const

$$\vec{F} = -\vec{\nabla} U = -\frac{dU}{dz}\hat{z}$$



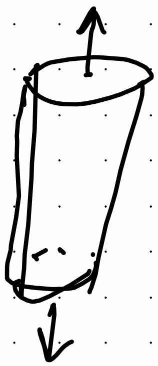


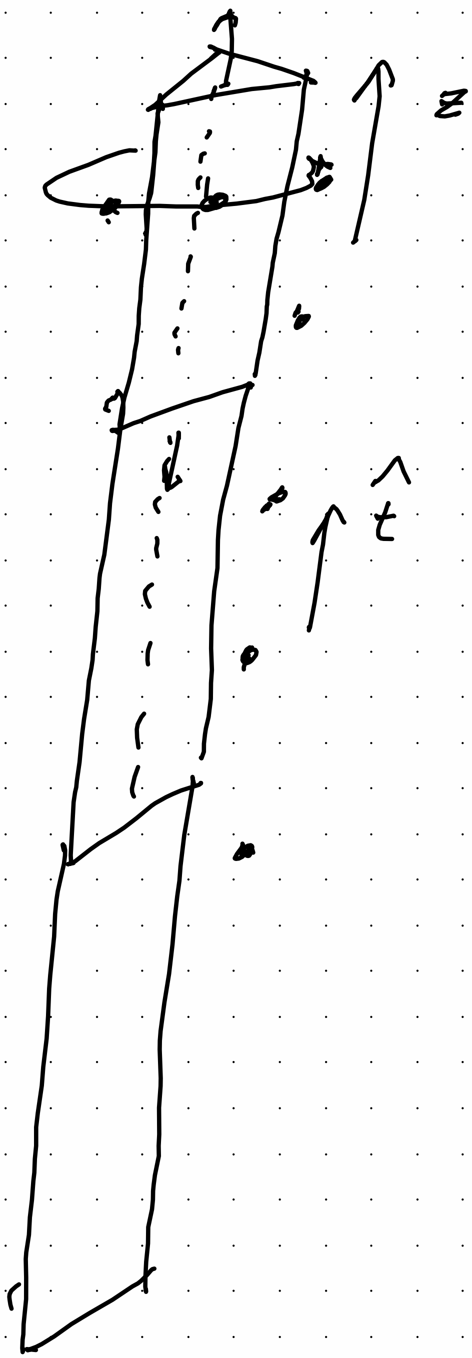
$\vec{P}\cdot\hat{t}$ = constant



$\hat{z}$ out

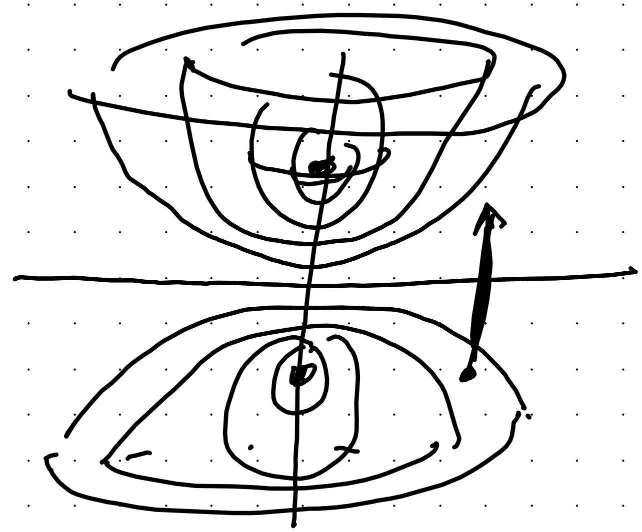
$\vec{M}\cdot\hat{z}$ = const





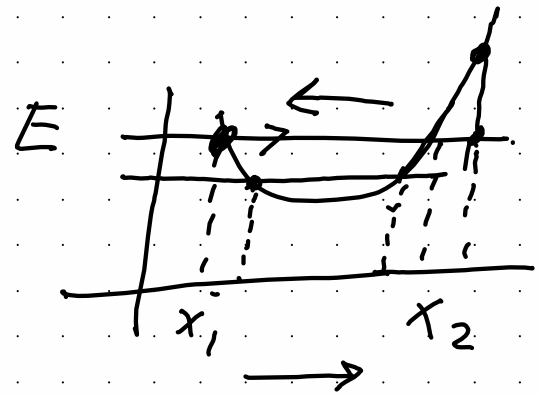
$U(x, y, \cancel{z})$

$\vec{p} \cdot \hat{t} = \text{const} = p_z$



$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$E = \underbrace{\frac{1}{2} m \dot{x}^2}_{\text{positive}} + U(x) = \text{const}$$



$x > x_2$

$$\frac{1}{2} m \dot{x}^2 = E - U(x)$$

$$\frac{dx}{dt} = \dot{x} = \pm \sqrt{\frac{2}{m}\left(E - U(x)\right)}$$

— separable 1st order ODE

$$\int \frac{dx}{\pm\sqrt{\frac{2}{m}\left(E-U(x)\right)}} = \int dt$$

$$\boxed{t = \int \frac{dx}{\sqrt{\frac{2}{m}\left(E-U(x)\right)}} + \text{const}}$$

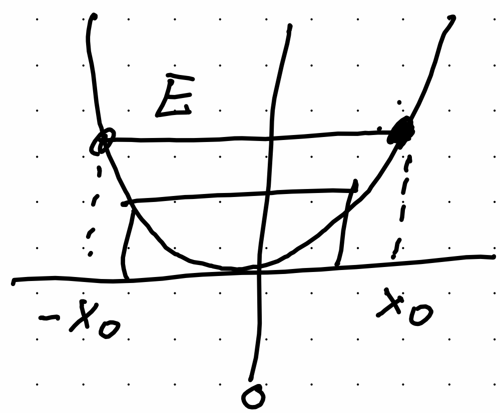
$$\Delta x = \pm \Delta t \sqrt{\frac{2}{m}\left(E - U(x)\right)}$$

$$\boxed{T(E) = 2 \int_{x_1 = x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{\frac{2}{m}\left(E-U(x)\right)}}}$$

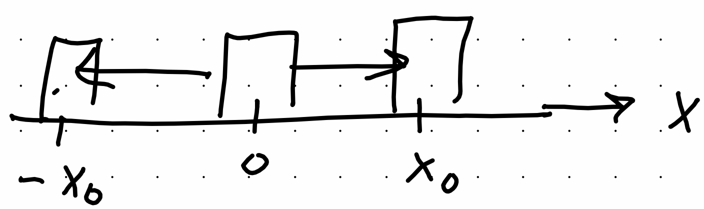
$$U(x) = \frac{1}{2} k x^2$$

$$U(\phi) = mg\cos\phi$$

$$\ddot{\phi} = -\frac{g}{l}\sin\phi$$



$$U = \frac{1}{2} k x^2$$



→ Beta Function

$$E = \frac{1}{2} k x_0^2$$

$$x_0 = \sqrt{\frac{2E}{k}}$$

Prob 2

(a) $U = A|x|^n$

(b) $U = -\dfrac{U_0}{\cosh^2(\alpha x)}$

(c) $U = U_0 \tan^2(\alpha x)$

$$\int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E - U(x)}}$$

$$\leftarrow \frac{1}{2} k x_0^2 - \frac{1}{2} k x^2$$

$$E = U(x_1) = U(x_2)$$

$$\frac{1}{2} k (x_0^2 - x^2)$$

$$\frac{dx}{\sqrt{x_0^2 - x^2}}$$

Lecture #9: Tues 9/22

1) Midterm #1: Tues Oct 6th (short answer; long problems) 40% 60%

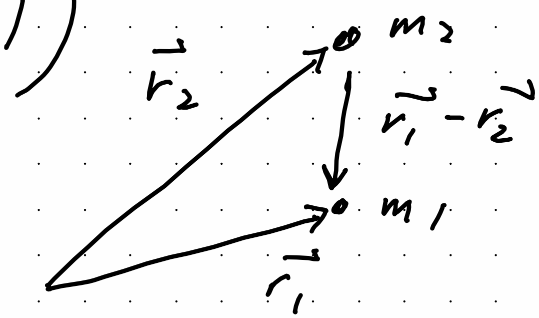
2) Next 4 lecture (central force problem)
Sec 13-15

General Formalism:

Two interacting particles (no external forces)

m_1, m_2

$U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$



$$L = T - U$$
$$= \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 - U(|\vec{r}_1 - \vec{r}_2|)$$

i) L unchanged by a translation $\vec{r}_1 \to \vec{r}_1 + \delta\vec{x}$, $\vec{r}_2 \to \vec{r}_2 + \delta\vec{x}$

$\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2 = \text{const}$

COM moves with const velocity

choose $\underset{\text{inertial}}{\wedge}$ ref frame such that COM at origin.

$\overset{|||}{\text{COM Frame}}$

$$\frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \vec{R}_{com} = 0$$

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

$$U(|\vec{r}_1 - \vec{r}_2|)$$

Define: $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$

$$U = U(r), \quad r = |\vec{r}|$$

$$\vec{r}_2 = \vec{r}_1 - \vec{r}$$

$$m_1 \vec{r}_1 + m_2 (\vec{r}_1 - \vec{r}) = 0$$

$$(m_1 + m_2) \vec{r}_1 - m_2 \vec{r} = 0$$

$$\boxed{\vec{r}_1 = \left(\frac{m_2}{m_1 + m_2}\right) \vec{r}}$$

$$\vec{r}_2 = \left(\frac{m_2}{m_1 + m_2}\right) \vec{r} - \vec{r}$$

$$\boxed{\vec{r}_2 = -\left(\frac{m_1}{m_1 + m_2}\right) \vec{r}}$$

$$T = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2$$

$$T = \frac{1}{2} m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 |\dot{\vec{r}}|^2 + \frac{1}{2} m_2 \left(\frac{-m_1}{m_1+m_2}\right)^2 |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} \frac{m_1 m_2}{(m_1+m_2)^{\cancel{2}}} |\dot{\vec{r}}|^2 \cancel{(m_2+m_1)}$$

$$= \frac{1}{2} \underbrace{\left(\frac{m_1 m_2}{m_1+m_2}\right)} |\dot{\vec{r}}|^2$$

reduced mass : m

$$= \frac{1}{2} m |\dot{\vec{r}}|^2$$

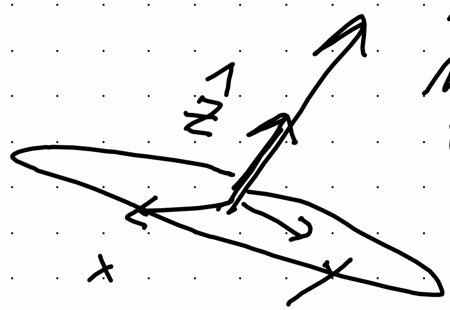
central potential

$$\boxed{L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(r)}$$

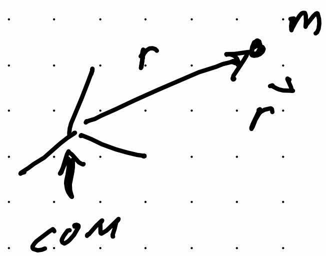
"effective" one body Lagrangian

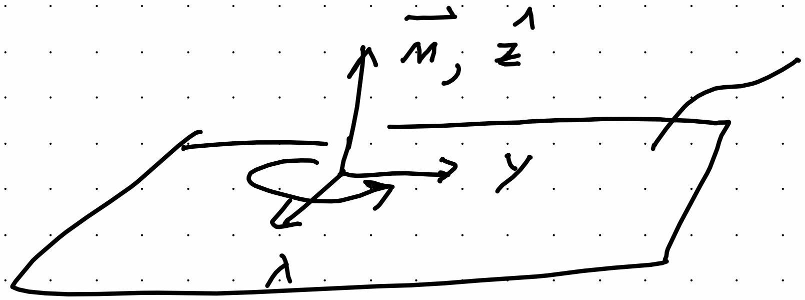
ii) L unchanged under a rotation

$\rightarrow$ $\underline{\vec{M} = \text{const}}$ $\quad (\vec{M} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v})$



$$\underbrace{\vec{M} = \vec{r}\times\vec{p}}_{\perp}$$





motion is in (x,y)

$$\boxed{\begin{aligned} L &= \frac{1}{2} m|\dot{\vec{r}}|^2 - U(r) \\ &= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r) \end{aligned}}$$

examples: $U(r) = \frac{-Gm_1m_2}{r}$ Newtonian gravity

$U(r) = \frac{1}{2}kr^2$ space oscillator

i) No explicit t-dependence

$$\boxed{E = T + U = \text{const}}$$

E, M constants of the motion

ii) No ϕ dependence

$$M_z = p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \text{const} = \boxed{mr^2\dot{\phi} \equiv M}$$

$$\begin{cases} E = \frac{1}{2}m(\dot r^2 + r^2\dot\phi^2) + U(r), \\ M = mr^2\dot\phi \longrightarrow \boxed{\dot\phi = \frac{M}{mr^2}} \end{cases}$$

$$E = \frac{1}{2}m\dot r^2 + \underbrace{\left(\frac{M^2}{2mr^2} + U(r)\right)}_{U_{eff}(r)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = \frac{\partial L}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi} = 0$$

$$\frac{dr}{dt} = \dot r = \sqrt{\frac{2}{m}\left(E - U(r) - \frac{M^2}{2mr^2}\right)}$$

$$= \sqrt{\frac{2}{m}(E-U(r)) - \frac{M^2}{m^2r^2}}$$

$$\int dt = \boxed{t = \int \frac{dr}{\sqrt{\frac{2}{m}(E-U(r)) - \frac{M^2}{m^2r^2}}} + \text{const}}$$

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M}{mr^2} = \sqrt{}$$

$$\int \frac{\frac{dr\,M}{mr^2}}{\sqrt{}} + \text{const} = \int d\phi = \phi$$

$$\boxed{\phi = \int \frac{M\,dr/r^2}{\sqrt{2m(E-U(r)) - M^2/r^2}} + \text{const}}$$

$$\phi = \phi(r) \longleftrightarrow r = r(\phi)$$

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

$$\left.\begin{aligned} r &= r(\phi) \\ r &= r(t) \\ \phi &= \phi(t) \end{aligned}\right\} \vec{r}(t)$$

Example: $U(r) = \frac{1}{2}kr^2$ (space oscillator) (r, ϕ)

(x, y)

Easy: $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}kr^2$

$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2+y^2)$

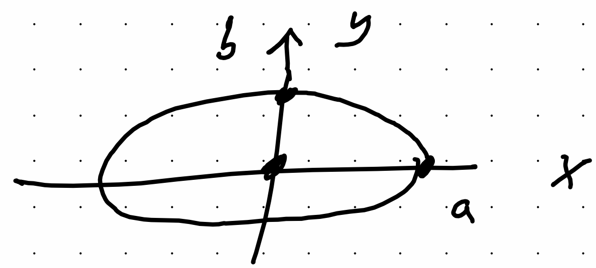
$= \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) + \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2\right)$

$x(t) = a\cos(\omega t + \cancel{\alpha}^{\,0})$, $\omega = \sqrt{\frac{k}{m}}$

$y(t) = b\sin(\omega t + \cancel{\beta}^{\,0})$,

$\longrightarrow$ closed orbit

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



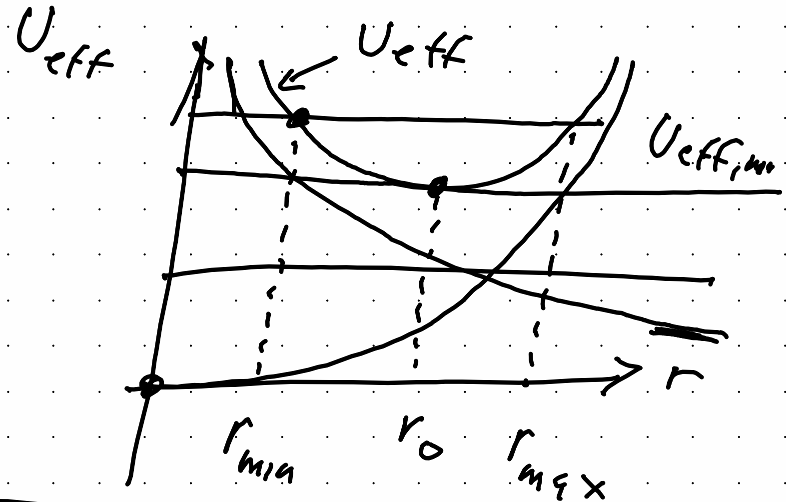
ellipse with center at origin

$U = -\frac{\alpha}{r}$, $U = \frac{1}{2}kr^2$ are only two potentials that have closed bound orbits

Harder: $U = \frac{1}{2} k r^2$

$$U_{eff} = U(r) + \frac{M^2}{2mr^2}$$

$$= \frac{1}{2} k r^2 + \frac{M^2}{2mr^2}$$



$$\phi = \int \frac{M\,dr/r^2}{\sqrt{2m\left(E - \frac{1}{2} k r^2\right) - \frac{M^2}{r^2}}} + \text{const}$$

bound orbits

substitute

$u = \frac{1}{r}$, $du = -\frac{dr}{r^2}$

$v = u^2$
$dv = 2u\,du$

$$\phi = -\int \frac{M\,du}{\sqrt{2m\left(E - \frac{1}{2}\frac{k}{u^2}\right) - M^2 u^2}} + \text{const} = -\int \frac{M u\,du}{\sqrt{2m\left(E u^2 - \frac{k}{2}\right) - M^2 u^4}} + \text{const}$$

$$\phi = -\frac{1}{2}\int \frac{dv\, M}{\sqrt{2m\left(Ev - \frac{k}{2}\right) - M^2 v^2}} + \text{const}$$

Complete the square:

$$-M^2 v^2 + 2mEv - mk$$

$$= -M^2\left(v^2 - \frac{2mEv}{M^2} + \frac{mk}{M^2}\right)$$

$$= -M^2\left(\left(v - \frac{mE}{M^2}\right)^2 - \frac{m^2E^2}{M^4} + \frac{mk}{M^2}\right)$$

$$= -M^2\left((v-A)^2 - B^2\right)$$

$$= M^2\left(B^2 - (v-A)^2\right)$$

$$A = \frac{mE}{M^2}, \quad B^2 = A^2 - \frac{mk}{M^2}$$

$$\phi = -\frac{1}{2}\int \frac{dv}{\sqrt{B^2-(v-A)^2}} + \text{const}$$

$$\int \frac{dx}{\sqrt{1-x^2}}$$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$$\int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\sqrt{1-\sin^2\theta}}}$$

$= \theta$

$= \sin^{-1}(x)$

$v - A = \pm B$

$\frac{1}{r^2} - A = \pm B$

r_{max}

$\frac{1}{r^2} = A \pm B$

3rd substitution:

$v - A = B\sin\theta$

$dv = B\cos\theta \, d\theta$

$$\phi = -\frac{1}{2}\int \frac{\cancel{B\cos\theta}\, d\theta}{\cancel{\sqrt{B^2-B^2\sin^2\theta}}} + \text{const}$$

$$= -\frac{1}{2}\theta + \text{const}$$

$$= -\frac{1}{2}\sin^{-1}\left(\frac{v-A}{B}\right) + \text{const}$$

$$= -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right) + \text{const}$$

$\frac{1}{r_{max}^2} = A - B$

$\frac{1}{r_{min}^2} = A + B$

$v = u^2$

$u = \frac{1}{r}$

$v = \frac{1}{r^2}$

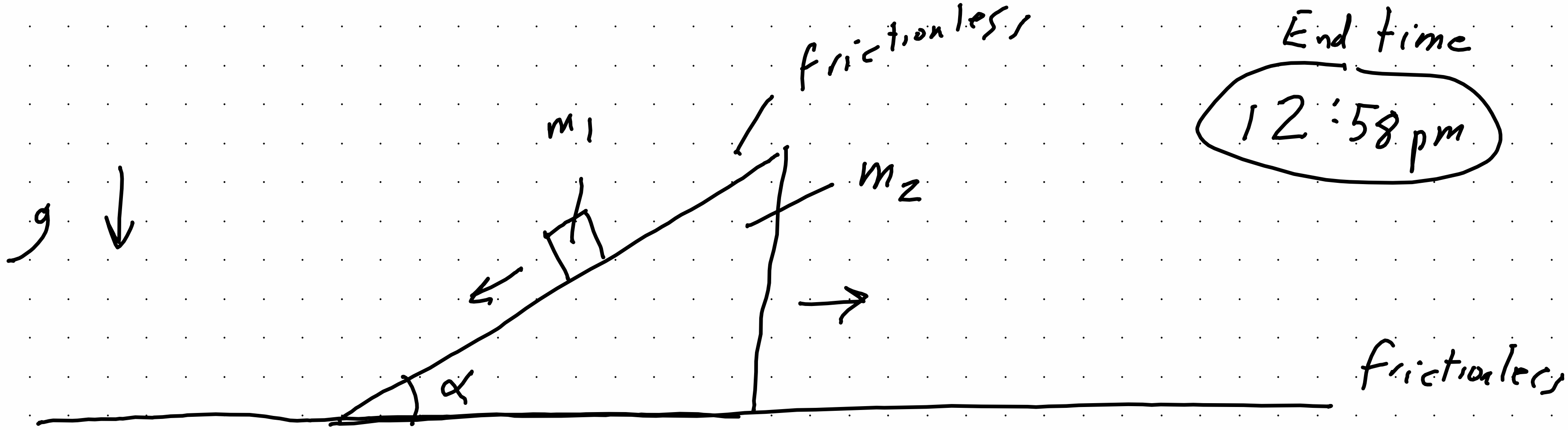
Lecture #10: 9/24

- Finish space oscillator
- Quiz #2
- Next week: Q& A

lastname_q2.pdf

End time

12:58 pm



i) generalized coords

ii) $L = T - U$

iii) EOMs

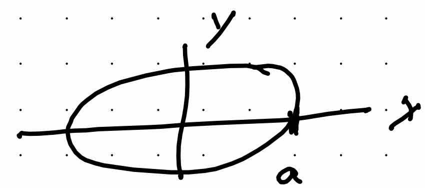
iv) special limiting cases: $\alpha = 0$, $\alpha = \pi/2$

$m_1 >> m_2$, $m_1 << m_2$

$$\phi = -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right) + \underline{\underline{\text{const}}}$$

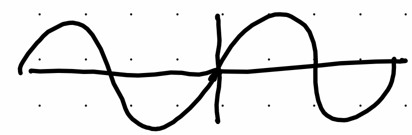
E, M

choose const so that $\phi = 0 \longleftrightarrow r = r_{max}$



$$\frac{1}{r_{max}^2} = A - B$$

$$\frac{\frac{1}{r_{max}^2} - A}{+B} = \frac{-B}{B} = -1$$



$$0 = -\frac{1}{2}\underbrace{\sin^{-1}(-1)}_{-\frac{\pi}{2}} + \text{const}$$

$$= \frac{\pi}{4} + \text{const}$$

$$\boxed{\text{const} = -\frac{\pi}{4}}$$

$$\phi = -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right) - \frac{\pi}{4}$$

$$-2\left(\phi + \frac{\pi}{4}\right) = \sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right)$$

$$-\sin\left(2\phi + \frac{\pi}{2}\right) = \left(\frac{1}{r^2} - A\right)/B$$

$$-\sin\left(2\phi+\frac{\pi}{2}\right) = \left(\frac{1}{r^2}-A\right)/B$$

$$-B\cos(2\phi) = \frac{1}{r^2} - A$$

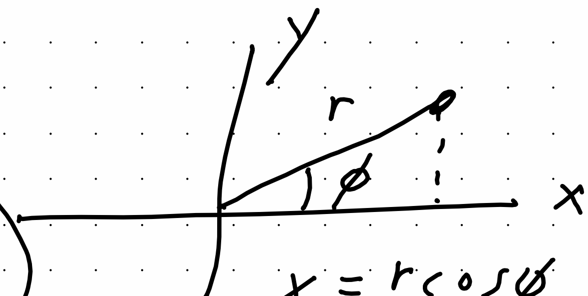
$$\boxed{\frac{1}{r^2} = A - B\cos(2\phi)}$$

$$\sin\left(2\phi+\frac{\pi}{2}\right)$$

$$= \sin(2\phi)\underbrace{\cancel{\cos\left(\frac{\pi}{2}\right)}}_{\to 0}$$

$$+ \cos(2\phi)\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=0}$$

$$= \cos(2\phi)$$



$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$\cos(2\phi) = \cos^2\phi - \sin^2\phi$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(2\phi) = \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2$$

$$\left[\frac{1}{r^2} = A - B\left(\left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2\right)\right] \times r^2$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

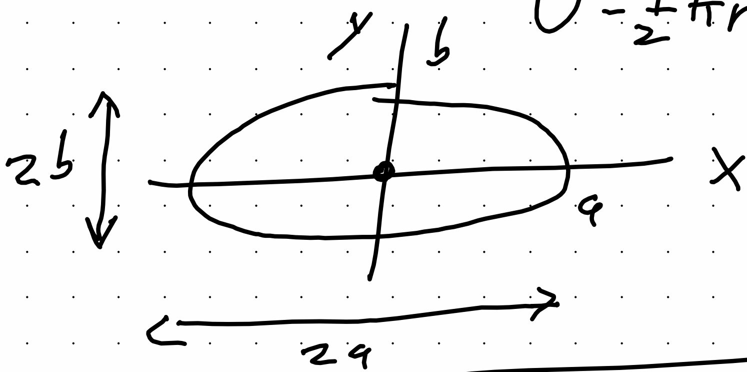
$$\frac{1}{a^2} = A - B, \quad \frac{1}{b^2} = A + B$$

$$1 = Ar^2 - B(x^2 - y^2)$$

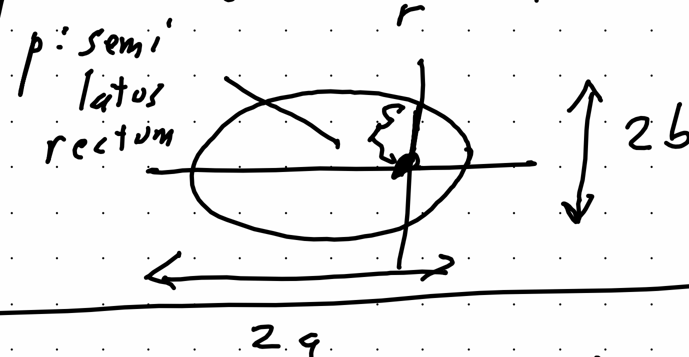
$$= A(x^2+y^2) - B(x^2-y^2) = (A-B)x^2 + (A+B)y^2 = 1$$

$$A = mE/M^2, \quad B^2 = A^2 - mk/M^2$$

$U = \frac{1}{2} k r^2$



$U = -\frac{\alpha}{r}$ (Newtonian)



$$E = \underbrace{\frac{1}{2} m \dot{r}^2}_{\substack{=0 \text{ at} \\ r=a,\ r=b}} + \frac{M^2}{2mr^2} + \frac{1}{2} k r^2 \qquad \begin{pmatrix} r=a \\ r=b \end{pmatrix}$$

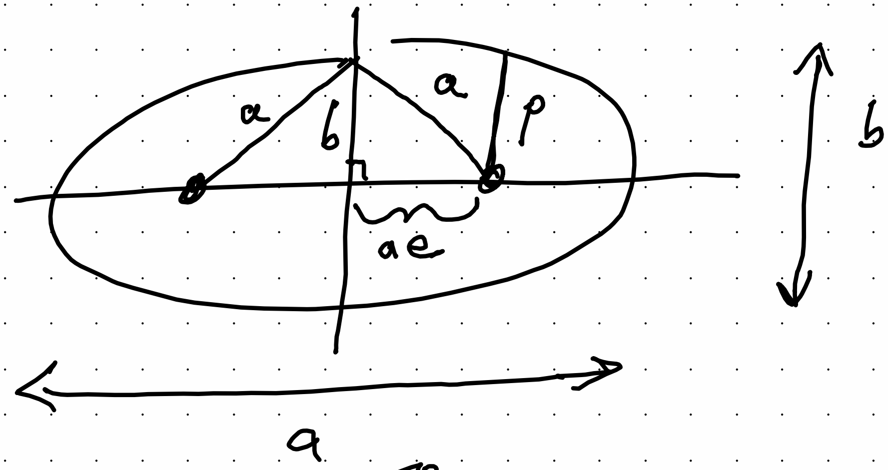
$$\boxed{\begin{aligned} E &= \frac{M^2}{2ma^2} + \frac{1}{2} k a^2 \\ E &= \frac{M^2}{2mb^2} + \frac{1}{2} k b^2 \end{aligned}}$$

$\rightarrow E a^2 = \frac{M^2}{2m} + \frac{1}{2} k a^4$

$\rightarrow E b^2 = \frac{M^2}{2m} + \frac{1}{2} k b^2$

subtract $E\cancel{(a^2 - b^2)} = \frac{1}{2} k \underbrace{(a^4 - b^4)}_{\cancel{(a^2-b^2)}(a^2+b^2)}$ $\rightarrow$ $\boxed{E = \frac{1}{2} k (a^2 + b^2)}$

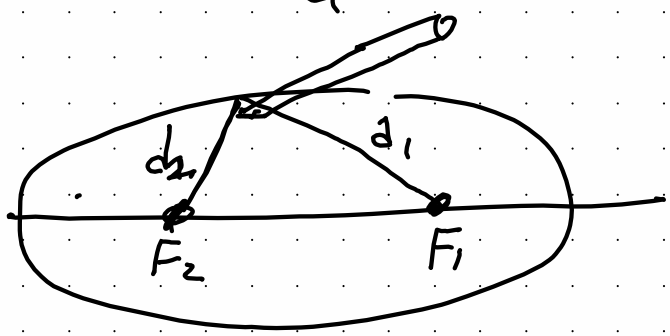
$\rightarrow$ $\boxed{M^2 = m k a^2 b^2}$

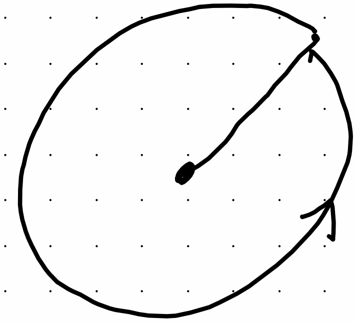


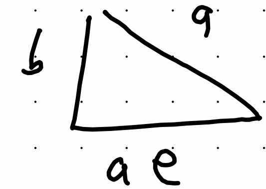
$e = 0$ circle

$e = 1$ line

dimensionless







$$a^2 = b^2 + a^2 e^2$$

$$e = \sqrt{\frac{a^2 - b^2}{a^2}}$$

$$d_1 + d_2 = 2a$$

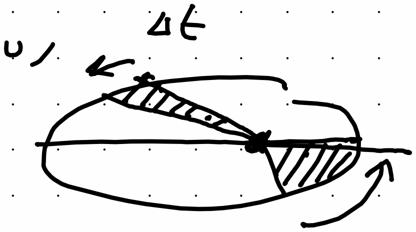
$$\boxed{b = a\sqrt{1-e^2}}$$

Kepler's laws: (for $U = -\alpha/r$)

i) elliptical orbits with sun at one focus

ii) equal areas in equal times

iii) $\frac{P^2}{a^3} = \text{const}$



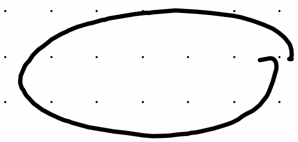
$U = \frac{1}{2} k r^2$

"Kepler's laws": (i) elliptical orbits with "Sun" at center of the ellipse

(ii) equal areas in equal times

(iii)

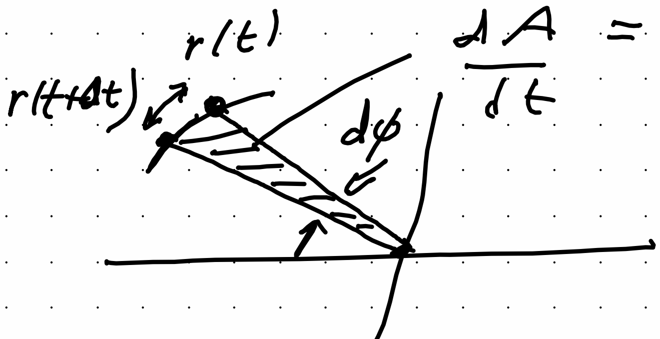
true for any central potential



$$\frac{dA}{dt} = \frac{M}{2m}$$

$$\int dA = \int \frac{M}{2m}\, dt$$

$$M = m r^2 \dot{\phi} \equiv \text{const}$$



$$\frac{dA}{dt} = \frac{1}{2} r \frac{r\, d\phi}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt}$$

height

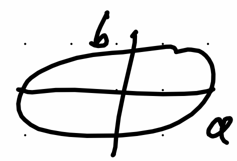
$$\frac{dA}{dt} = \text{const}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{M}{2m} = \text{const}$$

$$A = \frac{M}{2m} P$$

$A = \pi r^2$ circle

$= \pi a b$



$$\pi a b = \frac{M}{2m} P$$

$$P = \frac{2m}{M} \pi a b \quad , \quad M^2 = m k a^2 b^2$$

$$M = \sqrt{mk}\, a b$$

$$= \frac{2m \pi \cancel{ab}}{\sqrt{mk}\, \cancel{ab}}$$

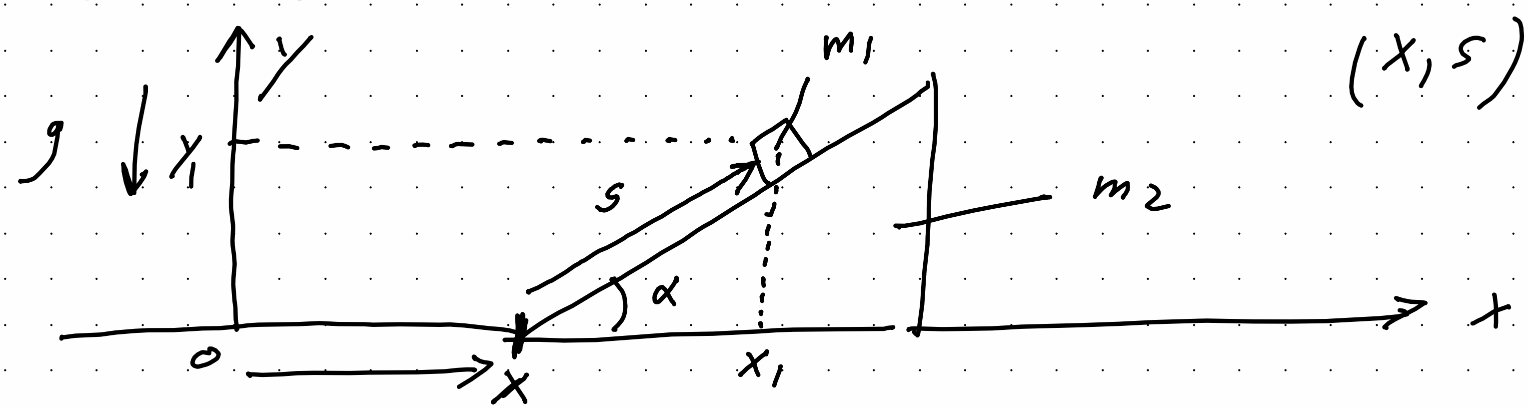
$$= \frac{2\pi}{\sqrt{k/m}}$$

$$= \frac{2\pi}{\omega} = \text{const} \quad (\text{for any } a, b)$$

$P = \text{const}$ indep of a, b !!

Lec #11: Tuesday 9/29

- Midterm #1: Next Tuesday Oct 6^{th}
- Trial midterm on BlackBoard (do it before Thursday)
- Today: a) Quiz #2
  b) Finish $U = \frac{1}{2} kr^2$
  c) Q&A



$$L = T - U \qquad T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$x_1 = X + s\cos\alpha \rightarrow \dot{x}_1 = \dot{X} + \dot{s}\cos\alpha$

$y_1 = s \sin\alpha \rightarrow \dot{y}_1 = \dot{s} \sin\alpha$

$x_2 = X \longrightarrow \dot{x}_2 = \dot{X}$

$y_2 = 0 \longrightarrow \dot{y}_2 = 0$

$$\dot{x}_1^2 = (\dot{x} + \dot{s}\cos\alpha)^2 = \dot{x}^2 + \underline{\underline{\dot{s}^2\cos^2\alpha}} + 2\dot{x}\dot{s}\cos\alpha$$

$$\dot{y}_1^2 = \underline{\underline{\dot{s}^2\sin^2\alpha}}$$

$$T = \frac{1}{2}m_1\left(\dot{x}^2 + \dot{s}^2 + 2\dot{x}\dot{s}\cos\alpha\right) + \frac{1}{2}m_2\dot{x}^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_1\dot{s}^2 + m_1\dot{x}\dot{s}\cos\alpha$$

$$U = m_1 g y_1 = m_1 g s\sin\alpha$$

$$\boxed{L = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_1\dot{s}^2 + m_1\dot{x}\dot{s}\cos\alpha - m_1 g s\sin\alpha}$$

$E = T + U$

<u>EOMs!</u> <u>x:</u> $\frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot{x}}\right)}_{} = \cancel{\frac{\partial L}{\partial x}} \to 0$

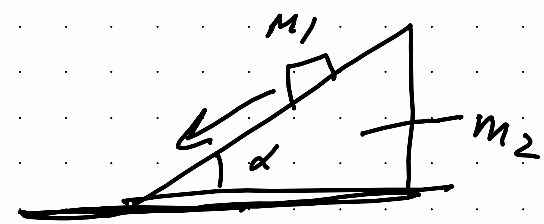
$$\boxed{P_x = (m_1+m_2)\dot{x} + m_1\dot{s}\cos\alpha = \text{const}}$$

$$\to \boxed{(m_1+m_2)\ddot{x} + m_1\ddot{s}\cos\alpha = 0}$$

<u>s:</u>

$$\frac{d}{dt}\left(\cancel{m_1}\dot{s} + \cancel{m_1}\dot{x}\cos\alpha\right) = -\cancel{m_1}g\sin\alpha \qquad \boxed{\ddot{s} + \ddot{x}\cos\alpha = -g\sin\alpha}$$

$$\ddot{x} = -\left(\frac{m_1}{m_1+m_2}\right)\ddot{s}\cos\alpha$$



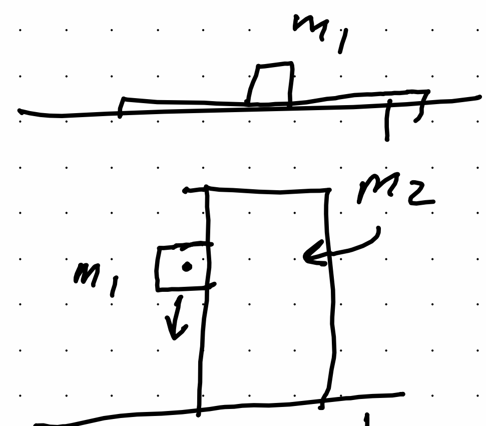
$$\ddot{s} = -g\sin\alpha\left(\frac{m_1+m_2}{m_1\sin^2\alpha+m_2}\right)$$

---

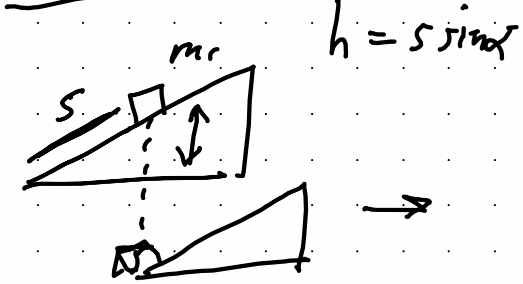
i) $m_2 \gg m_1$: $\ddot{s} \approx -g\sin\alpha$

ii) $\alpha = 0$ $\ddot{s} = 0$, $\ddot{x} = 0$

iii) $\alpha = \pi/2$ $\ddot{s} = -g$, $\ddot{x} = 0$



iv) $m_1 \gg m$: $\begin{cases} \ddot{s} \approx -\dfrac{g}{\sin\alpha} \\ \ddot{x} \approx +\dfrac{g}{\tan\alpha} \end{cases}$

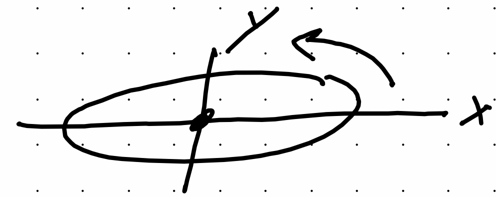


$$\ddot{h} = -g = \ddot{s}\sin\alpha \rightarrow \ddot{s} = \frac{-g}{\sin\alpha}$$

Space oscillator: $U = \frac{1}{2} k r^2$ $\quad$ $U = \frac{-\alpha}{r}$

"Kepler's" law:

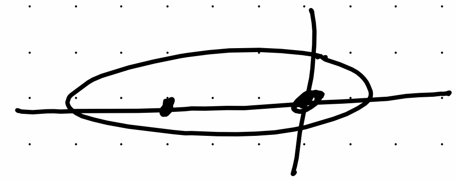
(i) ellipses with origin at center of ellipse



(ii) equal areas in equal times — applies to all central potentials

(iii) $P = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}}$ indep. of a, b

$\frac{P^2}{a^3} = \text{const}$ ordinary Kepler



$$t = -\int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2 r^2}}} + \text{const}$$

angular moment

reduced mass

$\frac{1}{2} k r^2$

want t to increase as r decreases.

$E = \frac{1}{2} k (a^2 + b^2)$

$M^2 = m k a^2 b^2$

$$t = - \int \frac{dr}{\sqrt{\frac{\cancel{2}}{m}\left(\frac{1}{\cancel{2}}k(a^2+b^2) - \frac{1}{\cancel{2}}kr^2\right) - \frac{\cancel{m}ka^2b^2}{m^{\cancel{2}}r^2}}}$$

(r, ϕ)

(x, y)

$$= - \frac{1}{\sqrt{\frac{k}{m}}} \int \frac{dr}{\sqrt{(a^2+b^2) - r^2 - \frac{a^2b^2}{r^2}}}$$

$r^2 = x^2 + y^2$

$$= - \frac{1}{\sqrt{\frac{k}{m}}} \int \frac{r\,dr}{\sqrt{r^2(a^2+b^2) - r^4 - a^2b^2}}$$

$$= - \frac{1}{\omega} \int \frac{r\,dr}{\sqrt{-\underbrace{\left(r^4 - r^2(a^2+b^2) + a^2b^2\right)}_{= (r^2-a^2)(r^2-b^2)}}}$$

substitution

$b < r < a$

$$\boxed{r^2 = a^2\cos^2\xi + b^2\sin^2\xi}$$

$\sqrt{a^2(1-\sin^2\theta)} = a\cos\theta$

$$\frac{dx}{\sqrt{a^2 - x^2}}$$

$x = a\sin\theta$

$$\boxed{x = a\cos\xi} \qquad \boxed{y = b\sin\xi}$$

$$\rightarrow \quad r^2 = a^2\cos^2\xi + b^2\sin^2\xi$$

<u>Differentiates</u> $\quad 2r\,dr = -2a^2\cos\xi\sin\xi + 2b^2\sin\xi\cos\xi$

$$= -2\sin\xi\cos\xi\,(a^2-b^2)$$

$$r\,dr = -\sin\xi\cos\xi\,(a^2-b^2)$$

$$\sqrt{-(r^2-a^2)(r^2-b^2)} = \sqrt{-(\underbrace{a^2\cos^2\xi + b^2\sin^2\xi - a^2}_{-a^2\sin^2\xi})(a^2\cos^2\xi + \underbrace{b^2\sin^2\xi - b^2}_{-b^2\cos^2\xi})}$$

$$= \sqrt{\underbrace{-(-)}_{+}(a^2-b^2)\sin^2\xi\,(a^2-b^2)\cos^2\xi}$$

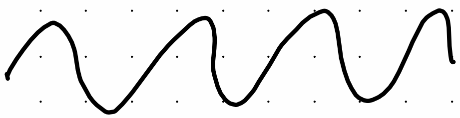
$$= \boxed{(a^2-b^2)\sin\xi\cos\xi}$$

$$t = \frac{1}{\omega}\int d\xi + \text{const} \rightarrow \boxed{\omega t = \xi} + \underbrace{\text{const}}_{=0}$$

$$\boxed{\begin{aligned} x &= a\cos(\omega t) \\ y &= b\sin(\omega t) \end{aligned}} \rightarrow r^2 = x^2 + y^2$$

$$= a^2\cos^2(\omega t) + b^2\sin^2(\omega t)$$

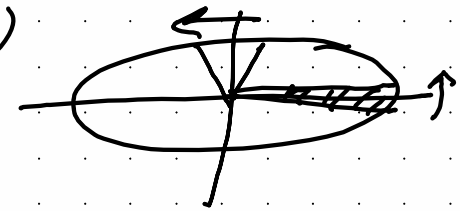
$$\underbrace{\boxed{r = \sqrt{a^2\cos^2(\omega t) + b^2\sin^2(\omega t)}}}_{r(t)}$$

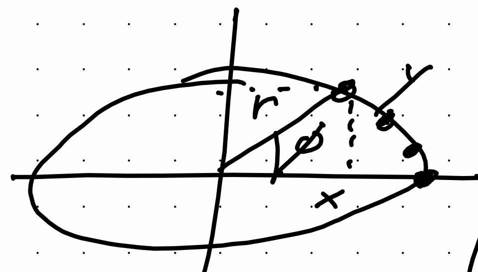


$\phi = \phi(t)$

$(x, y) \longleftrightarrow (r, \phi)$

$$\boxed{\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}}$$





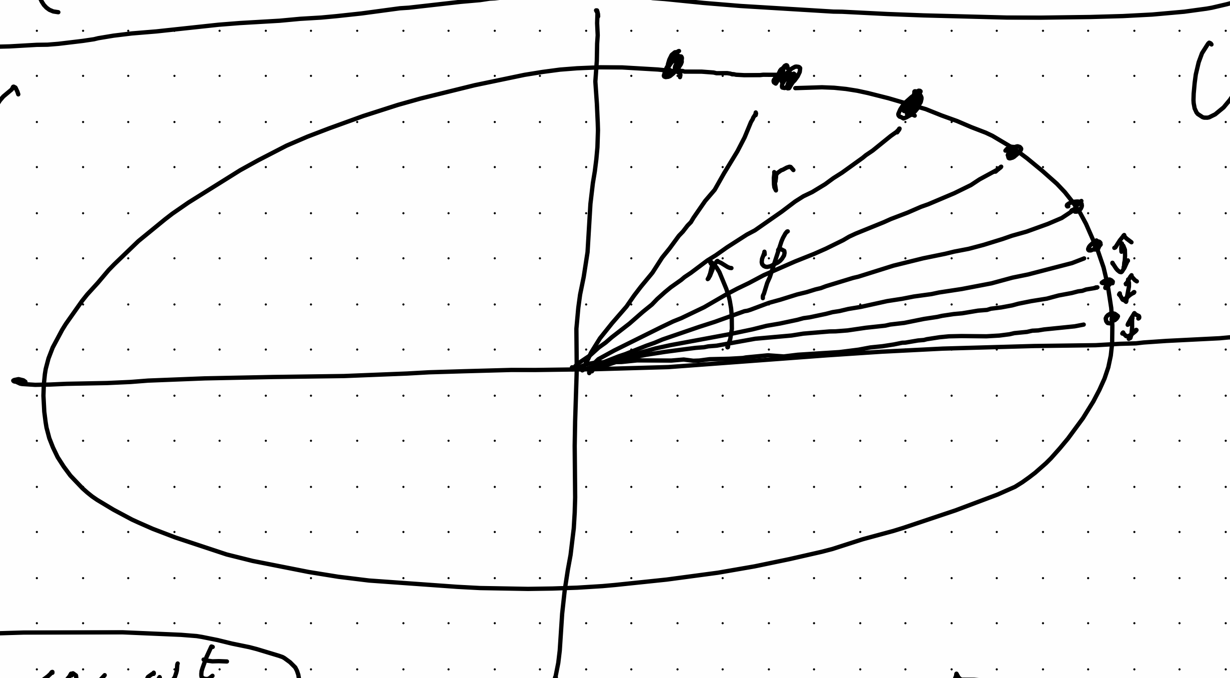
$r\cos\phi = a\cos(\omega t)$

$r\sin\phi = b\sin(\omega t)$

Divide 2nd by 1st: $\tan\phi = \frac{b}{a}\tan(\omega t) \rightarrow \boxed{\phi = \tan^{-1}\left(\frac{b}{a}\tan(\omega t)\right)}$

$\rightarrow \phi = \omega t$ if $b = a$

$$L = \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) + \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2\right)$$



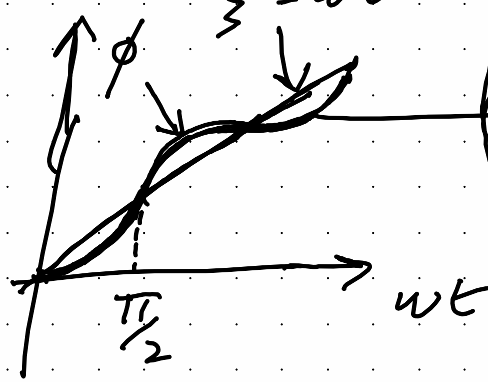
$$U = -\frac{\alpha}{r}$$

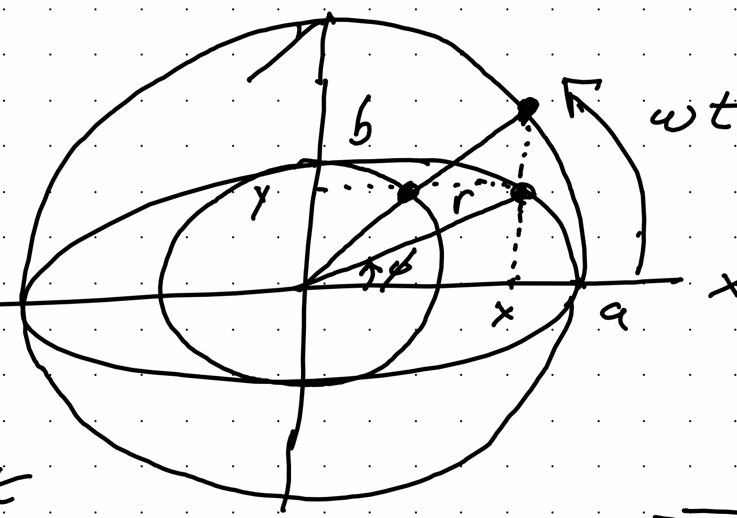
$$\boxed{+1/2}$$

$$x = a\cos\omega t$$
$$y = b\sin\omega t$$

$$x = r\cos\phi$$
$$y = r\sin\phi$$

$\xi = \omega t$





Lec #12:

- Midterm Exam 1 : Tuesday Oct 6th
- Do Trial exam by end of the day today
- Today : Q&A

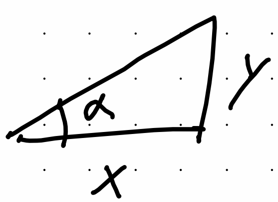
$$x^2+y^2-r_0^2=0$$

$$r-r_0=0$$

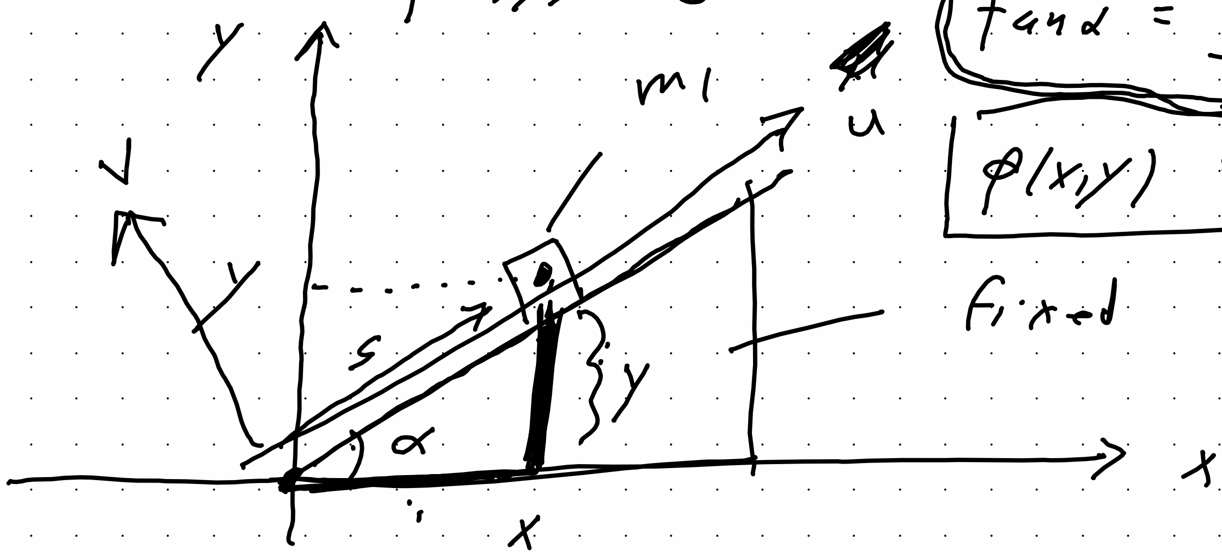
$$\varphi(x,y)=0$$

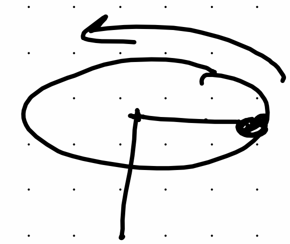
$$\tan\alpha = \frac{y}{x}$$

$$\boxed{\varphi(x,y) = y - x\tan\alpha}$$



$$\tan\alpha = \frac{y}{x}$$





$$T = \frac{1}{2} m_1 (\dot{x}^2+\dot{y}^2) \quad , \quad U = m_1 g y$$

$$L = T - U + \underline{\lambda}\,(y - x\tan\alpha)$$

(x,y)

✓ (r,ϕ)

EOMs:

x : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow$ $m_1 \ddot{x} = -\lambda \tan\alpha$ ✓

y : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} \rightarrow$ $m_1 \ddot{y} = -m_1 g + \lambda$ ✓

constraint: ✓ $\boxed{y - x\tan\alpha = 0}$

$\boxed{\ddot{y} - \ddot{x}\tan\alpha = 0}$

$\sin^2 + \cos^2 = 1$
$\tan^2 + 1 = \sec^2$

$(m_1 \ddot{x} = -\lambda \tan\alpha)\tan\alpha$
$+ m_1 \ddot{x}\tan\alpha = -m_1 g + \lambda$

$0 = -\lambda \tan^2\alpha - \lambda + m_1 g$
$= -\lambda(1 + \tan^2\alpha) + mg$

$$\boxed{L = \frac{1}{2}m_1(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(y - x\tan\alpha)}$$

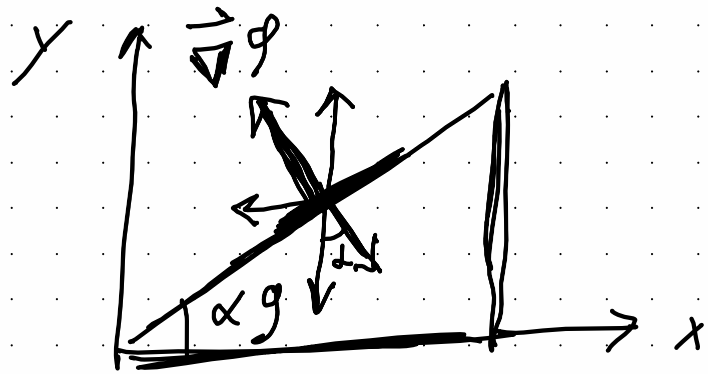
$\lambda = \frac{mg}{\sec^2\alpha} \propto mg\cos\alpha$

$$\varphi(x,y) = y - x\tan\alpha$$

$$\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{x} + \frac{\partial\varphi}{\partial y}\hat{y}$$

$$= -\tan\alpha\,\hat{x} + \hat{y} = -\frac{\sin\alpha}{\cos\alpha}\hat{x} + \hat{y}$$

$$= \frac{-\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}}{\cos\alpha}$$



$$\boxed{\begin{array}{c}\vec{N} = mg\cos\alpha \\ \perp \text{ to incline}\end{array}}$$

$$\vec{\nabla}\varphi \propto -\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}$$

$$y = x\tan\alpha$$

$$dy = \tan\alpha\,dx \rightarrow \boxed{\cos\alpha\,dy = \sin\alpha\,dx}$$

$$\vec{\nabla}\varphi\cdot d\vec{r} = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = -\cancel{\tan\alpha}\,dx + 1\,\cancel{\tan\alpha}\,dx$$

$$= 0$$

(u, v) $\qquad \varphi(u,v) = V = 0$

$$L = T - U + \lambda V$$

$V = 0$

$$\boxed{L = T - U + \lambda \varphi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}\underbrace{(m\dot{x})} = -\frac{\partial U}{\partial x}$$

$$\boxed{\frac{dp}{dt} = -\frac{\partial U}{\partial x} = F}$$

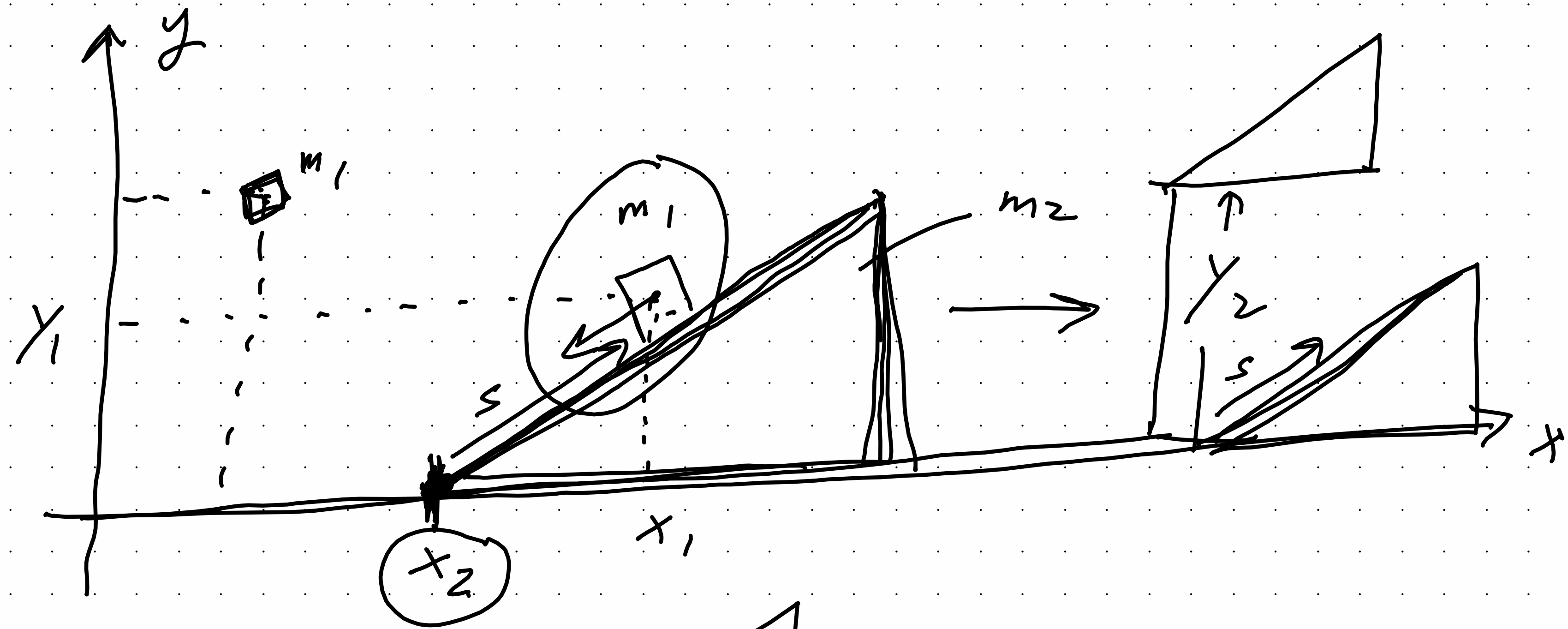
$$\frac{dp}{dt} = F$$

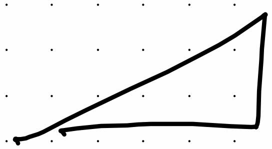
$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - U(x)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}}\right) = \frac{\partial L}{\partial \vec{r}}$$

$$\frac{d\vec{p}}{dt} = -\frac{\partial U}{\partial \vec{r}} + \lambda \frac{\partial \varphi}{\partial \vec{r}}$$

$$\frac{d\vec{p}}{dt} = -\vec{\nabla} U + \vec{\nabla}\varphi = \vec{F}_{applied} + \vec{F}_{constraint}$$





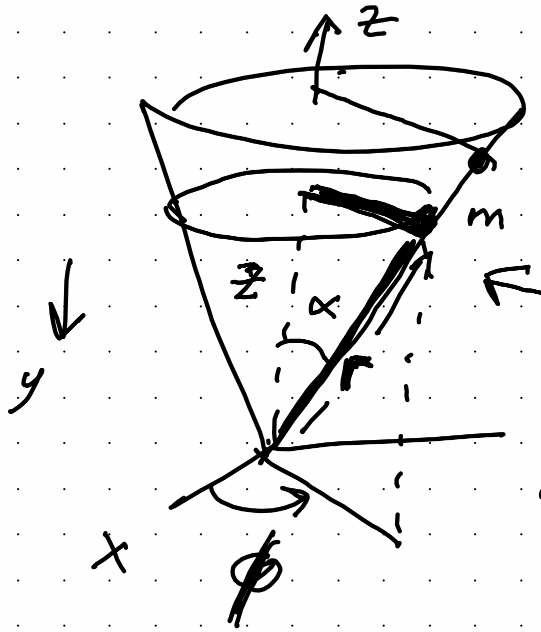
$y_2 = 0$

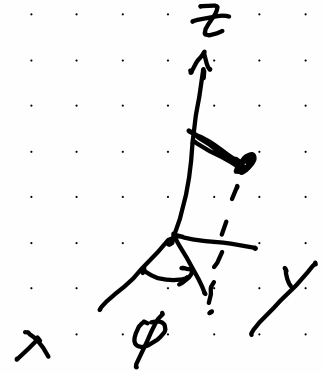
(x_1, y_1, x_2, y_2)

m_1

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \cancelto{0}{\dot{y}_2^2})$$

Sec 14, Prob 2:





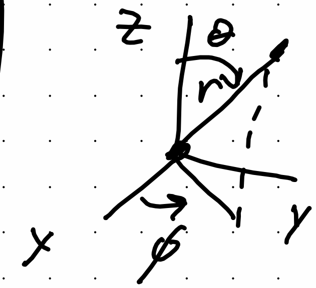
cylindrical

(r, ϕ, z)

$r^2 = x^2 + y^2$

alt. notation

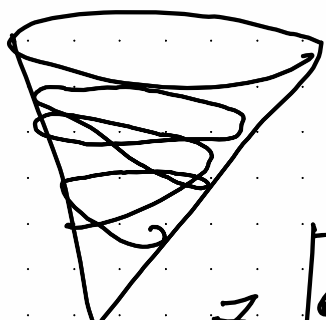
$\left\{ \begin{array}{l} s^2 = x^2 + y^2 \\ \rho^2 = x^2 + y^2 \end{array} \right.$

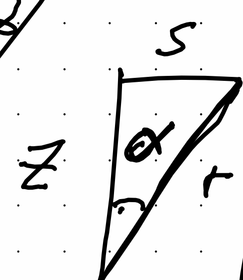


sph. polar

(r, ϕ, θ)

$r^2 = x^2 + y^2 + z^2$





$\boxed{\tan\alpha = \frac{s}{z}}$

$\theta = \alpha$

(ϕ, s) (ϕ, z)

$$T = \frac{1}{2}m\left(\dot r^2 + \underbrace{r^2\dot\theta^2}_{0} + r^2\underbrace{\sin^2\theta}_{\sin^2\alpha}\,\dot\phi^2\right)$$

sph. polar

$\theta = \alpha$

$$T = \frac{1}{2}m\left(\underbrace{\dot s^2 + s^2\dot\phi^2}_{\text{plane polar coord}} + \dot z^2\right)$$

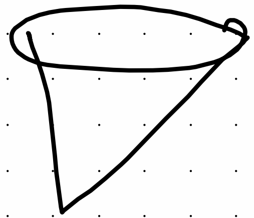
cyl. coord.

s, ϕ, z

$\tan\alpha = \dfrac{s}{z}$

$z = \dfrac{s}{\tan\alpha}$ | $s = z\tan\alpha$

$\dot z = \dfrac{\dot s}{\tan\alpha}$ | $\dot s = \dot z\tan\alpha$



$L = T - U + \lambda\varphi$

$\varphi = s - z\tan\alpha$

$$f(x,y) = e^{-\frac{1}{2}(x^2+y^2)}$$

No constraint: $\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases}$

$g(x,y,\lambda)$

$\equiv f(x,y) + \lambda \varphi(x,y)$

$\dfrac{\partial g}{\partial x} = 0$

$\dfrac{\partial g}{\partial y} = 0, \quad \dfrac{\partial g}{\partial \lambda} = 0$

suppose constraint $\quad y = \frac{1}{2}x^2 + 1$

$$F(x) = f(x,y)\Big|_{y = \frac{1}{2}x^2+1}$$

$\boxed{\dfrac{dF}{dx} = 0}$ $\quad \underbrace{\boxed{\varphi = y - \frac{1}{2}x^2 - 1}}_{=0}$

$L = T - U + \lambda \varphi$

x, y, λ

COM:

$m_1, m_2, m_3, \quad \vec{r}_1, \vec{r}_2, \vec{r}_3$

$$\vec{R}_{com} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1+m_2+m_3}$$

$$0 = \frac{\sum_a m_a \vec{r}_a}{\sum_b m_b}$$

$\vec{P} = M_{tot}\dot{\vec{R}}_{com}$

$= \text{const}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_a}\right) = 0$$

$$\vec{P} = \sum_a \frac{\partial L}{\partial \dot{\vec{r}}_a} = \text{const}$$

~~R_{com}~~ closed system: $\vec{P} = \text{const}$

$$L = \frac{1}{2}\sum_a m_a|\dot{\vec{r}}_a|^2 - U(|\vec{r}_1-\vec{r}_2|, |\vec{r}_2-\vec{r}_3|, \text{etc.})$$

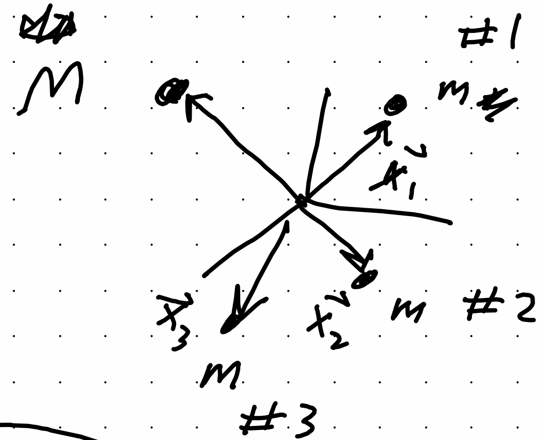
$\vec{r}_a \rightarrow \vec{r}_a + \delta\vec{x}$

$L \rightarrow L$

$$\frac{\partial L}{\partial \vec{r}} = 0$$

$$M, \quad \underbrace{m, \ldots, m}_{n}$$

$$\vec{X} \quad \vec{x}_1, \ldots, \vec{x}_n$$



$$M\vec{X} + m\sum_{a=1}^{n} \vec{x}_a = 0$$

$$\vec{r}_1 = \vec{x}_1 - \vec{X}$$
$$\vec{r}_2 = \vec{x}_2 - \vec{X}$$
$$\vdots$$
$$\vec{r}_n = \vec{x}_n - \vec{X}$$

$\vec{r}_1, \vec{r}_2$

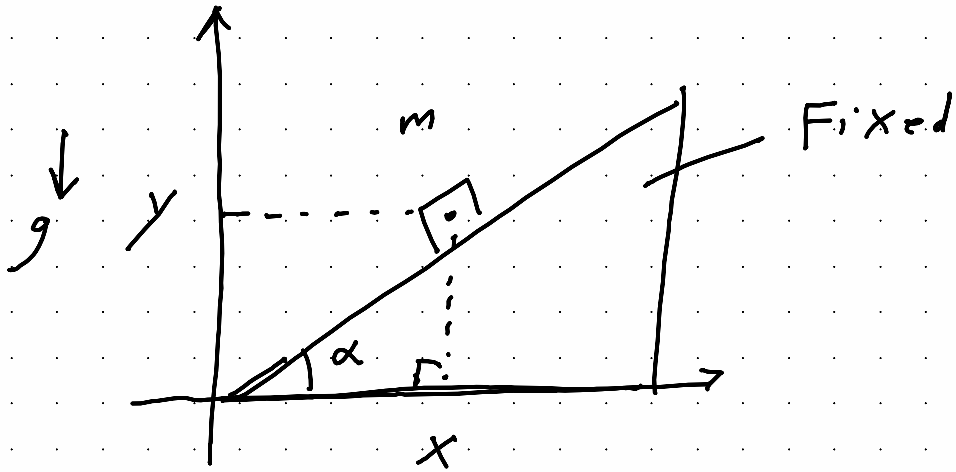
$\vec{r} = \vec{r}_1 - \vec{r}_2$

1hr + 20min

80min

120 mins

2:35pm



$\tan\alpha = \dfrac{y}{x}$

so $y = x\tan\alpha$

$$\underbrace{\varphi(x,y) = y - x\tan\alpha = 0}_{\text{constraint}}$$

$$L = T - U + \lambda\varphi$$

$$= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(y - x\tan\alpha)$$

EOMs:

x: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) = \dfrac{\partial L}{\partial x} \rightarrow m\ddot{x} = -\lambda\tan\alpha$ (1)

y: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{y}}\right) = \dfrac{\partial L}{\partial y} \rightarrow m\ddot{y} = -mg + \lambda$ (2)

λ: $y - x\tan\alpha = 0 \rightarrow y = x\tan\alpha$ (3)

Differentiate constraint twice:

$$\ddot{y} = \ddot{x}\tan\alpha$$

substitute for $\ddot{y}$, $\ddot{x}$ using (1) and (2)

$$-g + \frac{\lambda}{m} = -\frac{\lambda}{m}\tan\alpha \cdot \tan\alpha$$

$$\frac{\lambda}{m}\big(\underbrace{1+\tan^2\alpha}_{\sec^2\alpha}\big) = g$$

$$\boxed{\lambda = \frac{mg}{\sec^2\alpha}}$$

constraint Force:

$$\vec{F_c} = \lambda \vec{\nabla}\phi \qquad \text{where} \qquad \phi = y - x\tan\alpha$$

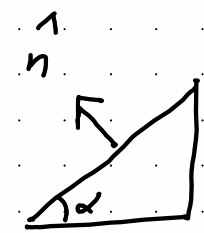
$$= \frac{mg}{\sec^2\alpha}\left(-\tan\alpha\,\hat{x} + \hat{y}\right)$$

$$= \frac{mg}{\sec^2\alpha}\left(-\frac{\sin\alpha}{\cos\alpha}\,\hat{x} + \hat{y}\right)$$

$$= mg\cos\alpha\left(-\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}\right)$$

$$= \boxed{mg\cos\alpha \;\hat{n}}$$

(where $\hat{n} = -\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}$ is $\perp$ to incline)



Return to EOMs:

$$\ddot{x} = -\frac{\lambda}{m}\tan\alpha$$

$$\ddot{y} = -g + \frac{\lambda}{m}$$

where $\lambda = \frac{mg}{\sec^2\alpha} = mg\cos^2\alpha$

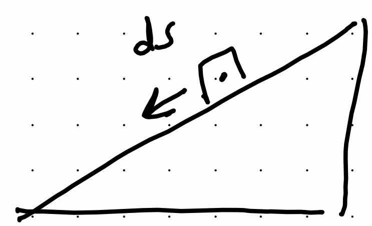
$$\rightarrow \quad \ddot{x} = -g\cos^2\alpha\tan\alpha$$

$$= -g\sin\alpha\cos\alpha$$

$$\ddot{y} = -g + g\cos^2\alpha$$

$$= -g(1-\cos^2\alpha)$$

$$= -g\sin^2\alpha$$



$ds = \sqrt{dx^2 + dy^2}$

Acceleration down the incline:

$$\ddot{s} = -\sqrt{\ddot{x}^2 + \ddot{y}^2} = -g\sin\alpha\sqrt{\cos^2\alpha + \sin^2\alpha} = \boxed{-g\sin\alpha}$$

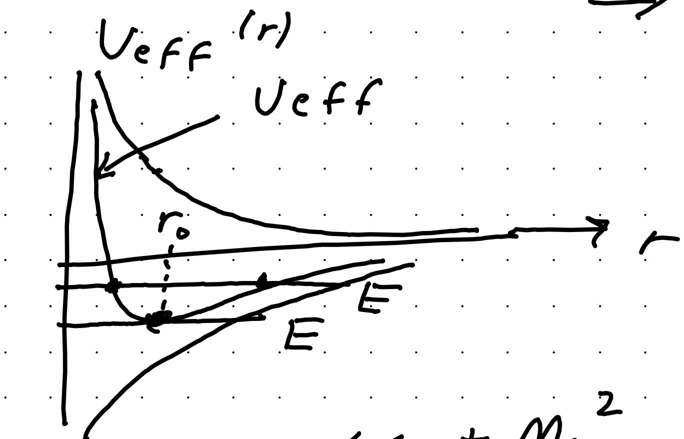
standard result

Lecture #14: Thurs Oct 8th

- midterm 1: Avg $\approx$ 14/20
- midterm 2: Nov 19th (before Thanksgiving)
- oral final

Collisions (2) & Scattering (3) (Sec 16 - 20)

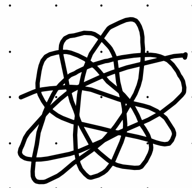
Scattering: closed system, two bodies, $\rightarrow$ central force problem

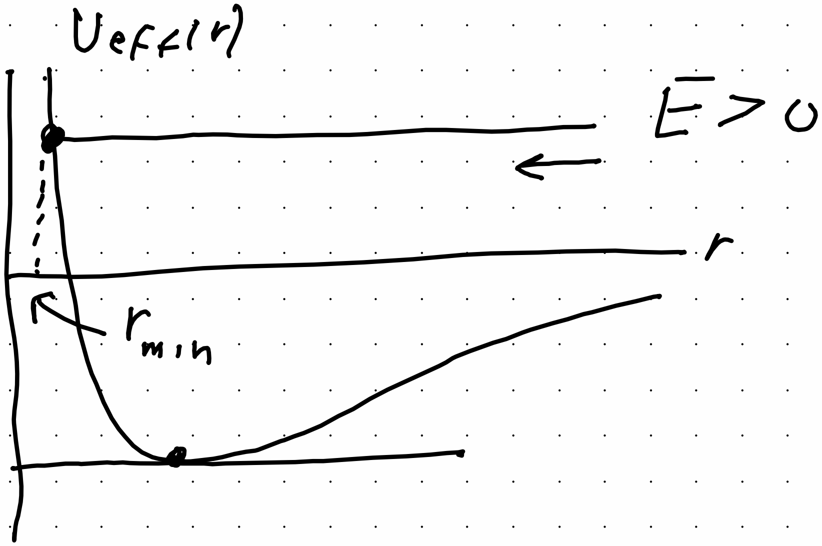


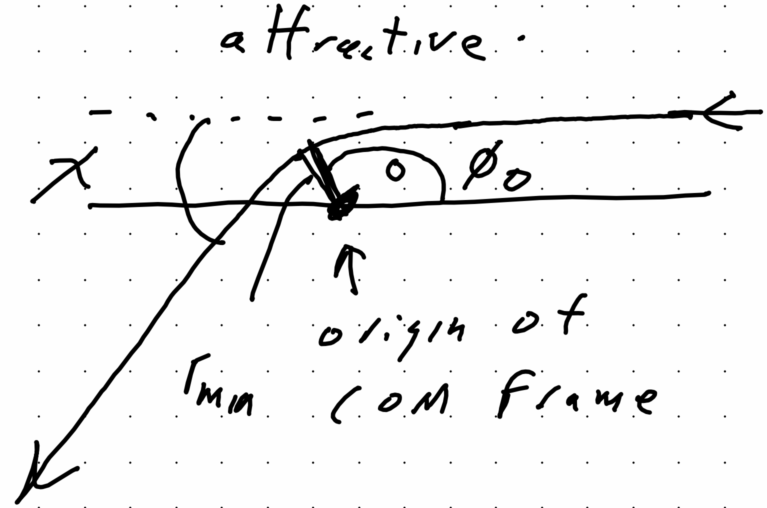
$E = U_{eff,min} \rightarrow r = r_0$ (circular)

$U_{eff,min} < E < 0 \rightarrow \underbrace{\text{bound orbit}}_{\text{ellipse for } -\alpha/r}$

$U_{eff} = -\alpha/r + \frac{M_z^2}{2mr^2}$

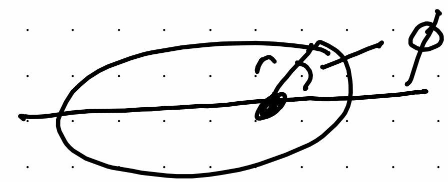


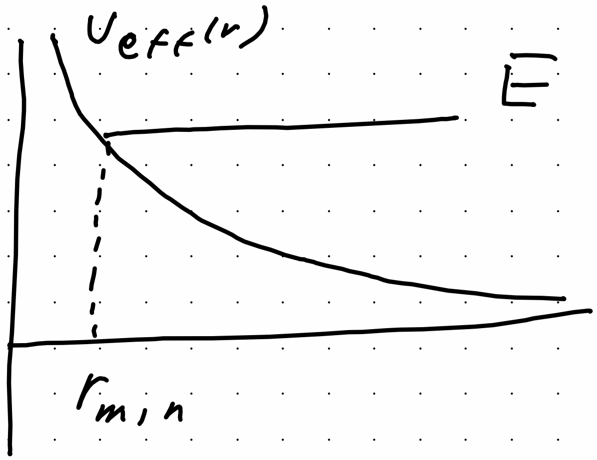


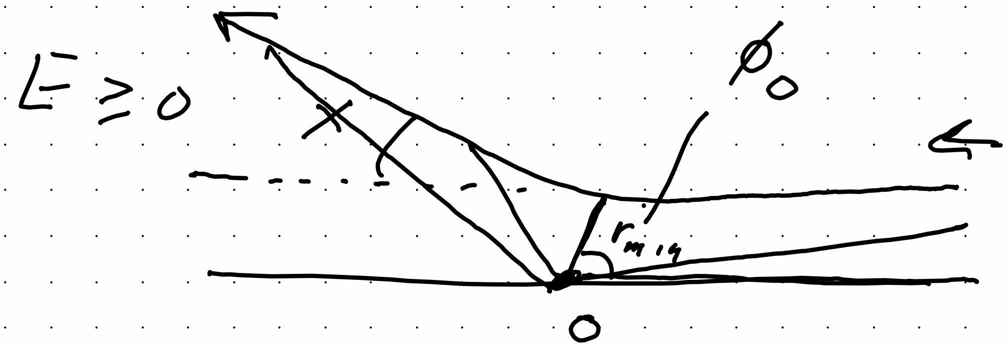


$U(r) = +\alpha/r$

$U_{eff}(r) = \alpha/r + M_z^2/2mr^2$







$$\int_0^{\phi_0} d\phi = \int_{r_{min}}^{\infty} \frac{M_z\, dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M_z^2}{r^2}}}$$

(14.7) L & L

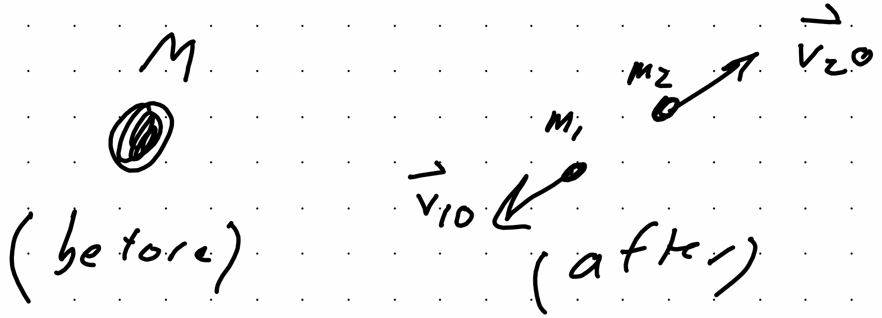
$$\boxed{\phi_0 = \int_{r_{min}}^{\infty} \frac{M_z\, dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M_z^2}{r^2}}}}$$

$E, M_z, U(r)$

Fig 18 in L & L

Collisions: (Sec 16, 17)

- Elastic collisions of two particles
- Spontaneous disintegration of a single particle



Analyze in COM Frame (to begin with)

$$\vec{p}_{tot,0} = \vec{0}$$

(tot, 0: COM)

cons of momentum

$\vec{p}_0$: momentum of mass m_1 after disintegration

$-\vec{p}_0$: " m_2

cons. of energy

$$E_i = E_{1i} + T_{10} + E_{2i} + T_{20}$$

E_i: internal energy of mass $\underline{M} = m_1 + m_2$

E_{1i}: internal energy of m_1

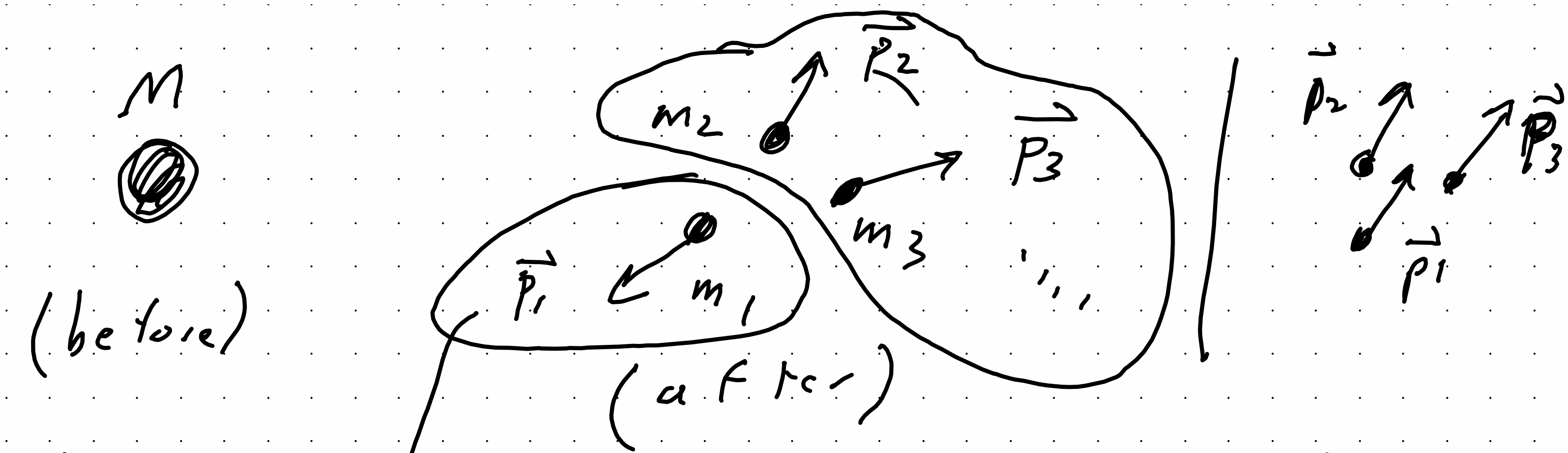
T_{10}: KE in COM frame $= \frac{1}{2} m_1 |\vec{v}_{10}|^2 = \frac{|\vec{p}_0|^2}{2m_1}$

$$E_i = E_{1i} + \frac{|\vec{p}_0|^2}{2m_1} + E_{2i} + \frac{|\vec{p}_0|^2}{2m_2}$$

$$\underbrace{E_i - E_{1i} - E_{2i}}_{\epsilon \,=\, \text{disintegration energy} \,\geq 0} = \frac{p_0^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

$$= \frac{p_0^2}{2}\left(\frac{m_1 + m_2}{m_1 m_2}\right) = \frac{p_0^2}{2m}$$

$$p_0 = \sqrt{2m\epsilon} \rightarrow v_{10} = p_0/m_1 \,,\; v_{20} = p_0/m_2$$



$\vec{p}_{tot} = 0$ $\qquad$ $0 = \vec{p}_{tot} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots$

$\vec{p}_1 = \vec{p}_0$

$T_{1,0} = \dfrac{p_0^2}{2m}$

Q: what condition is there on the velocities of $m_2, m_3, \ldots$ such that KE of m_1 is largest?

Consider COM of $m_2, m_3, \ldots$ $\qquad$ E_i': internal energy of $m_2, m_3, \ldots$

$\vec{p}_2 + \vec{p}_3 + \cdots = -\vec{p}_0$ $\qquad$ $KE = \dfrac{p_0^2}{2(\underbrace{M - m_1}_{(m_2 + m_3 + \cdots)})}$

cons. of energy:

$$\underbrace{E_i}_{\substack{\text{int. energy}\\ \text{of } M}} = \underbrace{E_{1i} + \frac{p_0^2}{2m_1}}_{\substack{\text{int. + KE}\\ \text{of } m_1}} + \underbrace{E_i' + \frac{p_0^2}{2(M-m_1)}}_{\substack{\text{int + KE}\\ \text{of } m_2,\, m_3,\, \ldots}}$$

$$E_i - E_{1i} - E_i' = \frac{p_0^2}{2}\left(\frac{1}{m_1} + \frac{1}{M-m_1}\right)$$

$$E_i - E_{1i} - E_i' = \frac{p_0^2}{2}\,\frac{M}{m_1(M-m_1)}$$

$$E_{2i} + E_{3i} + \ldots$$

$$\boxed{T_{10} = \frac{p_0^2}{2m_1} = \left(\frac{M-m_1}{M}\right)\left(E_i - E_{1i} - E_i'\right)}$$

T_{10} ~~to be~~ maximum when E_i' is minimum.

$$E_i' + \frac{p_0^2}{2(M-m_1)} = \underbrace{E_{2i} + \frac{p_2^2}{2m_2} + E_{3i} + \frac{p_3^2}{2m_3} + \cdots}_{\text{int energy + KE of masses } m_2,\, m_3,\, \ldots}$$

$\vec{p}_2 = m_2\vec{v}_2$, etc.

$$= E_{2i} + \tfrac{1}{2} m_2 |\vec{v}_2|^2 + E_{3i} + \tfrac{1}{2} m_2 |\vec{v}_3|^2 + \cdots$$

$$E_i' = (E_{2i} + E_{3i} + \cdots) + \left(\tfrac{1}{2} m_2 |\vec{v}_2|^2 + \tfrac{1}{2} m_3 |\vec{v}_3|^2 + \cdots\right) - \frac{|\vec{p}_2 + \vec{p}_3 + \cdots|^2}{2(m_2+m_3+\cdots)}$$

$= |\vec{v}_0|^2$

minimized

$$= (E_{2i} + E_{3i} + \cdots) + \left(\tfrac{1}{2} m_2 |\vec{v}_2|^2 + \tfrac{1}{2} m_3 |\vec{v}_3|^2 + \cdots\right) - \frac{1}{2}\frac{|m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots|^2}{m_2+m_3+\cdots}$$

$$|\vec{v}_2 + \vec{v}_3|^2 = |\vec{v}_2|^2 + |\vec{v}_3|^2 + 2\vec{v}_2\cdot\vec{v}_3$$

IF $\vec{v}_2 = \vec{v}_3 = \cdots \equiv \vec{v}_0$

then:

$$\frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \cdots$$

$$= \frac{1}{2}(m_2 + m_3 + \cdots)|\vec{v}_0|^2$$

and

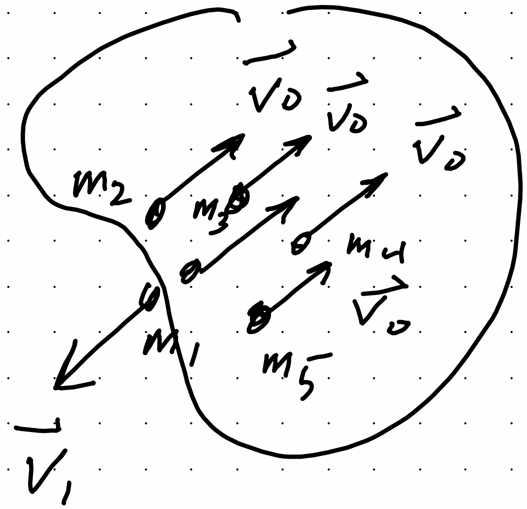
$$\frac{1}{2} \frac{(m_2 \vec{v}_2 + m_3 \vec{v}_3 + \cdots)^2}{m_2 + m_3 + \cdots}$$

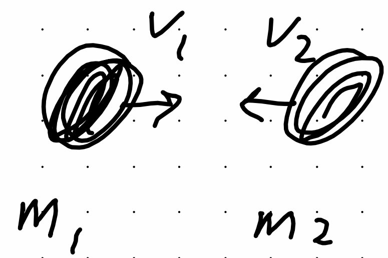
$$= \frac{1}{2} |\vec{v}_0|^2 \frac{(m_2 + m_3 + \cdots)^{\cancel{2}}}{\cancel{m_2 + m_3 + \cdots}}$$

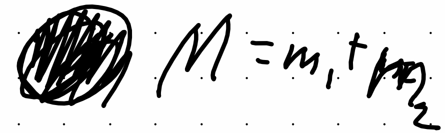
$$= \frac{1}{2} |\vec{v}_0|^2 (m_2 + m_3 + \cdots)$$

$=$

$$T_{10,max} = \left(\frac{M - m_1}{M}\right) \epsilon$$







CM

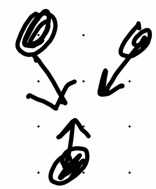
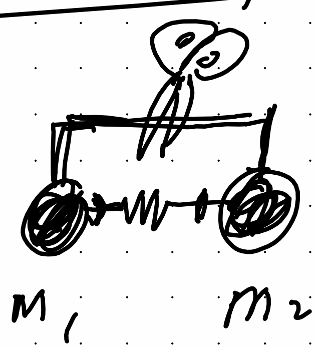
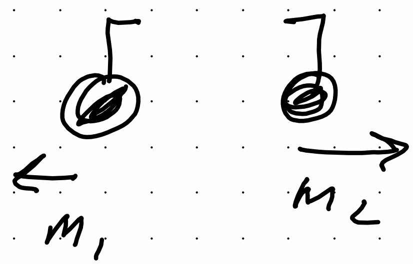


Fig. 14	Figs 15,16





Lec #15: Tues 10/13

— solutions to midterm #1 posted

— next two weeks: Sec 16-20
Collisions and scattering

Figures 14, 15, 16

Elastic collision of two particles

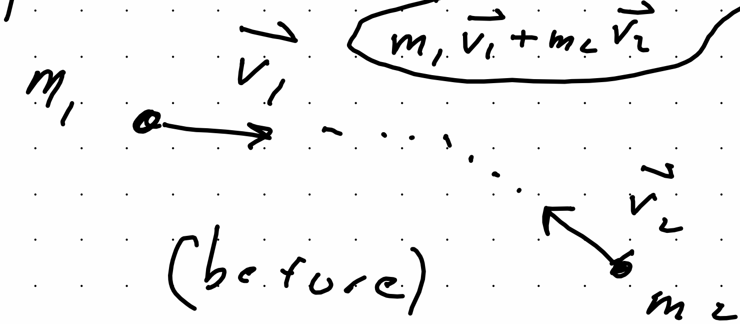
closed system: E, $\vec{P}$, $\vec{M}$ are conserved

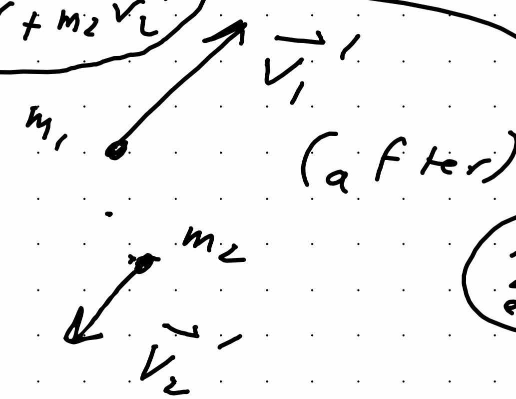
✓ i) KE = const [ignore internal energies]

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 (v_1')^2 + \frac{1}{2}m_2 (v_2')^2 \quad (1)$$

✓ ii) $\vec{P}$ = const

$$\vec{P} = \vec{P}' \;\Rightarrow\; m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1{}' + m_2\vec{v}_2{}' \quad (2 \text{ equations})$$





After: $\vec{v}_1{}'$, $\vec{v}_2{}'$: $\boxed{\begin{array}{l}4 \text{ DOF} \\ 3 \text{ equations}\end{array}}$

Elastic collision:

$\vec{v} \equiv \vec{v}_1 - \vec{v}_2 = \vec{v}_{10} - \vec{v}_{20}$ relative velocity vector

$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 = \vec{r}_{10} - \vec{r}_{20}$ relative position vector

$\vec{v} = \dot{\vec{r}}$

$\vec{v}_1 = \vec{V} + \vec{v}_{10}$

$\vec{v}_2 = \vec{V} + \vec{v}_{20}$

$\vec{r}_1 = \vec{R} + \vec{r}_{10}$

$\vec{r}_2 = \vec{R} + \vec{r}_{20}$

$\vec{v}_1$: velocity of particle 1 wrt lab frame before collision

$\vec{v}_{10}$: " wrt COM frame before collision

$\vec{v}_1{}'$, $\vec{v}_{10}{}'$, $\vec{v}_2{}'$, $\vec{v}_{20}{}'$: velocities after collision

$$T_0 = \frac{1}{2} m_1 |\dot{\vec{r}}_{10}|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_{20}|^2$$

$$\vec{V}_{10} = \dot{\vec{r}}_{10} = \left(\frac{m_2}{m_1+m_2}\right) \dot{\vec{r}} \qquad \dot{\vec{r}} = \vec{V} \qquad \vec{r} \equiv \vec{r}_{10} - \vec{r}_{20}$$

$$\vec{V}_{20} = \dot{\vec{r}}_{20} = -\left(\frac{m_1}{m_1+m_2}\right) \dot{\vec{r}}$$

$$T_0 = \frac{1}{2} m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 |\dot{\vec{r}}|^2 + \frac{1}{2} m_2 \left(\frac{-m_1}{m_1+m_2}\right)^2 |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} \left(\frac{m_1 m_2}{m_1+m_2}\right) |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} m |\dot{\vec{r}}|^2 = \frac{1}{2} m |\vec{V}|^2$$

$$T_0 = T_0'$$

$$\cancel{\tfrac{1}{2}} \cancel{m} |\vec{V}|^2 = \cancel{\tfrac{1}{2}} \cancel{m} |\vec{V}'|^2$$

$$\boxed{V = V'}$$

~~relative~~ magnitude of relative velocity vector = const

Before collision

$$\vec{v}_{10} = \left(\frac{m_2}{m_1+m_2}\right)\vec{v}, \qquad \vec{v}_{20} = \left(\frac{-m_1}{m_1+m_2}\right)\vec{v}$$

$$\vec{v} = \vec{v}_1 - \cancel{\vec{v}_2} = \vec{v}_1$$

$\vec{v}_1 = \vec{v}$

$\vec{v}_2 = 0$

$\underbrace{}_{\text{lab frame}}$

After collision:

$$\vec{v}_{10}' = \left(\frac{m_2}{m_1+m_2}\right) v\,\hat{n}_0$$

v: magnitude of $\vec{v}$

$\hat{n}_0$: unit vector pointing in some direction (angle $=\chi$ wrt COM frame)

$$\vec{v}_{20}' = \left(\frac{-m_1}{m_1+m_2}\right) v\,\hat{n}_0$$

$$\boxed{\vec{v}_1' = \vec{v}_{10}' + \vec{V} = \left(\frac{m_2}{m_1+m_2}\right) v\,\hat{n}_0 + \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}}$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \rightarrow \vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}$$

$$\vec{V}_1{}' = \vec{v}_{10}{}' + \vec{V} = \left(\frac{m_2}{m_1+m_2}\right) v\hat{n}_0 + \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} = \vec{V}_1{}'$$

$$\vec{V}_2{}' = \vec{v}_{20}{}' + \vec{V} = \left(\frac{-m_1}{m_1+m_2}\right) v\hat{n}_0 + \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} = \vec{V}_2{}'$$

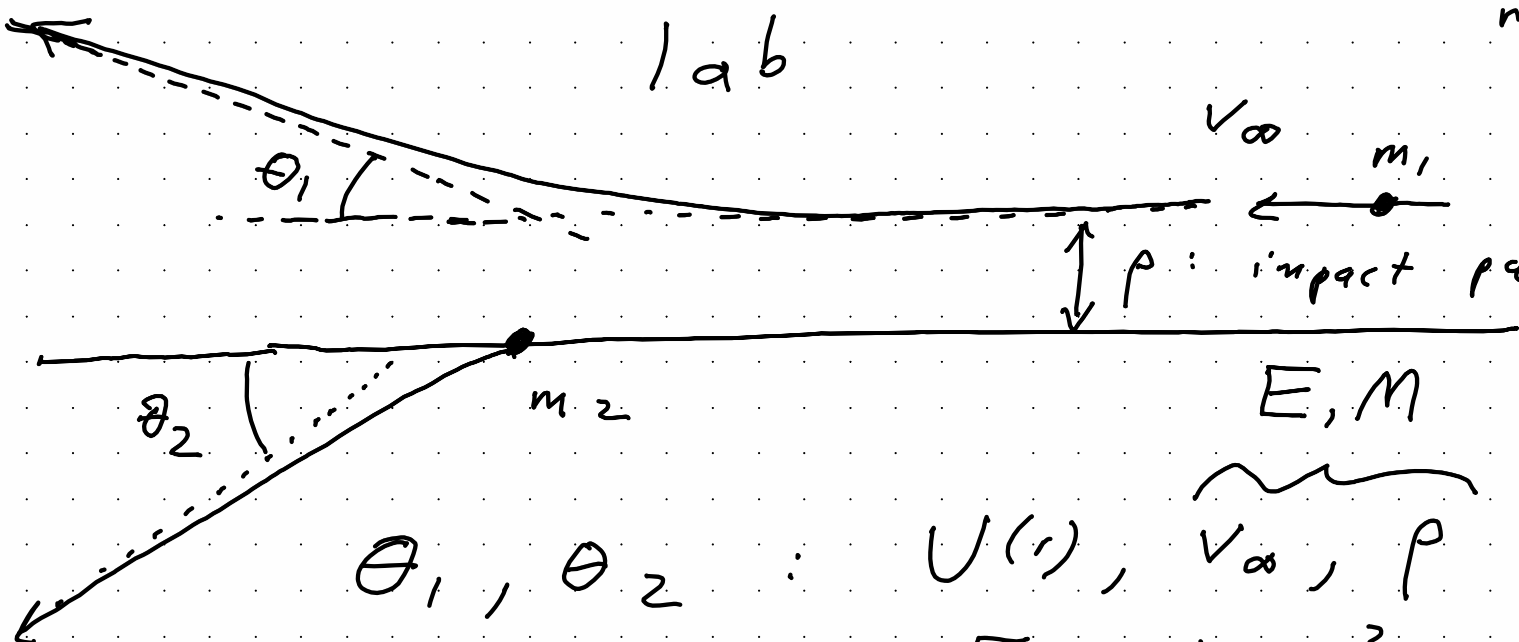
$$\vec{p}_1{}' = m_1\vec{v}_1{}' = m v\hat{n}_0 + m_1\underbrace{\left(\frac{\vec{p}_1+\vec{p}_2}{m_1+m_2}\right)}_{\vec{V}}$$

$$\vec{p}_2{}' = m_2\vec{v}_2{}' = -m v\hat{n}_0 + m_2\left(\frac{\vec{p}_1+\vec{p}_2}{m_1+m_2}\right)$$

Lec #16: Thurs 10/15

- Quiz #3 next week
- General remarks about scattering
- Example: Prob 18.1 (Hard-sphere)

$$m = \frac{m_1 m_2}{m_1 + m_2}$$



$U \to 0$
$r \to \infty$

$$\theta_1, \theta_2 : \quad U(r), \overbrace{v_\infty, \rho}^{E, M}$$

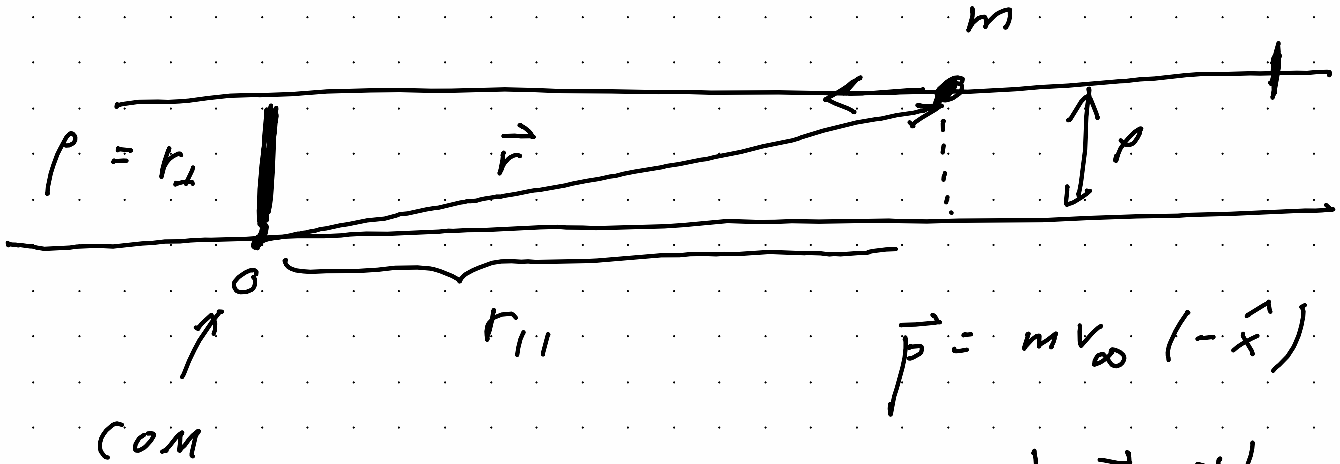
χ : scattering angle w/t COM frame

$$E = \frac{1}{2} m v_\infty^2$$

$$M = \rho m v_\infty$$

$r_\perp$

$|\vec{r} \times \vec{p}| = M$

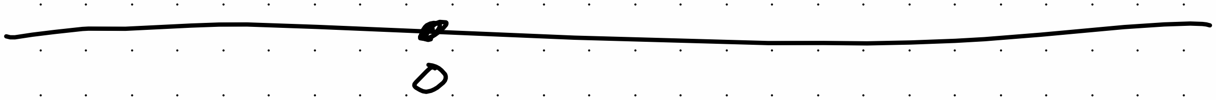


$$\vec{p} = m v_\infty (-\hat{x})$$

$$\boxed{\begin{aligned} E &= \tfrac{1}{2} m v_\infty \\ M &= \rho m v_\infty \end{aligned}}$$

$$\begin{aligned} M &= |\vec{r} \times \vec{p}| \\ &= r_\perp m v_\infty \\ &= \rho m v_\infty \end{aligned}$$

$$d\rho \longleftrightarrow d\chi \qquad d\sigma = 2\pi \rho \, d\rho$$



$$d\sigma = 2\pi\, \rho(\chi) \left|\frac{d\rho}{d\chi}\right| d\chi$$

$$d\sigma = \frac{\rho(\chi)}{\sin\chi} \left|\frac{d\rho}{d\chi}\right| d\Omega = \rho \left|\frac{d\rho}{d(\cos\chi)}\right| d\Omega$$

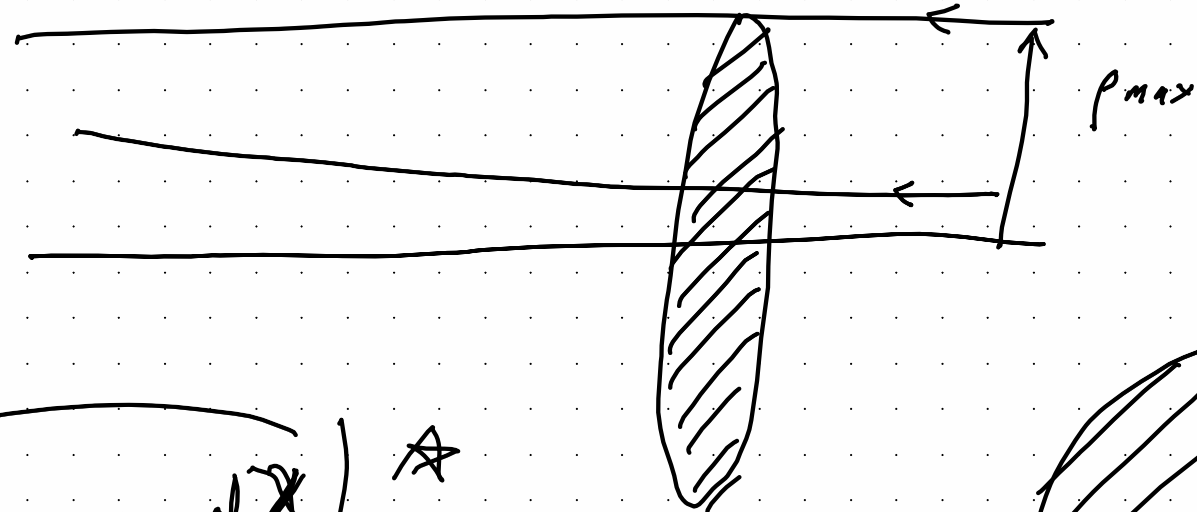
$$\boxed{\frac{d\sigma}{d\Omega} = \rho \left|\frac{d\rho}{d(\cos\chi)}\right|}$$ in COM frame

$$\boxed{\frac{d\sigma_1}{d\Omega_1} = \rho \left|\frac{d\rho}{d(\cos\theta_1)}\right|}$$ in lab frame θ_1

$$\boxed{= \frac{d\sigma}{d\Omega} \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|}$$

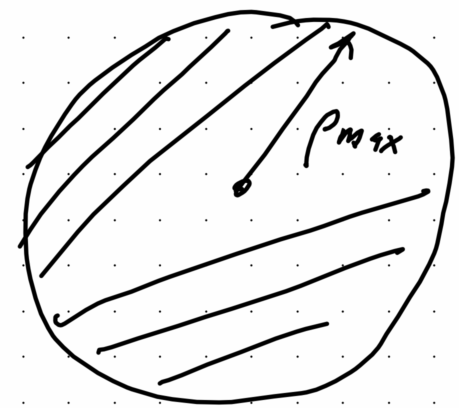
$$\boxed{\frac{d\sigma_2}{d\Omega_2} = \frac{d\sigma}{d\Omega} \left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right|}$$

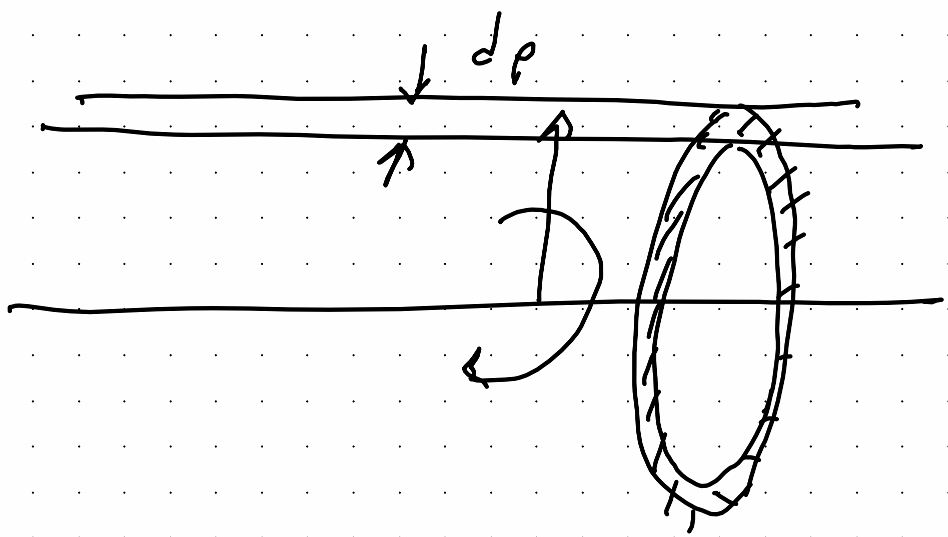
wrt lab frame (θ_1, θ_2)



$\boxed{d\rho \leftrightarrow d\chi}$

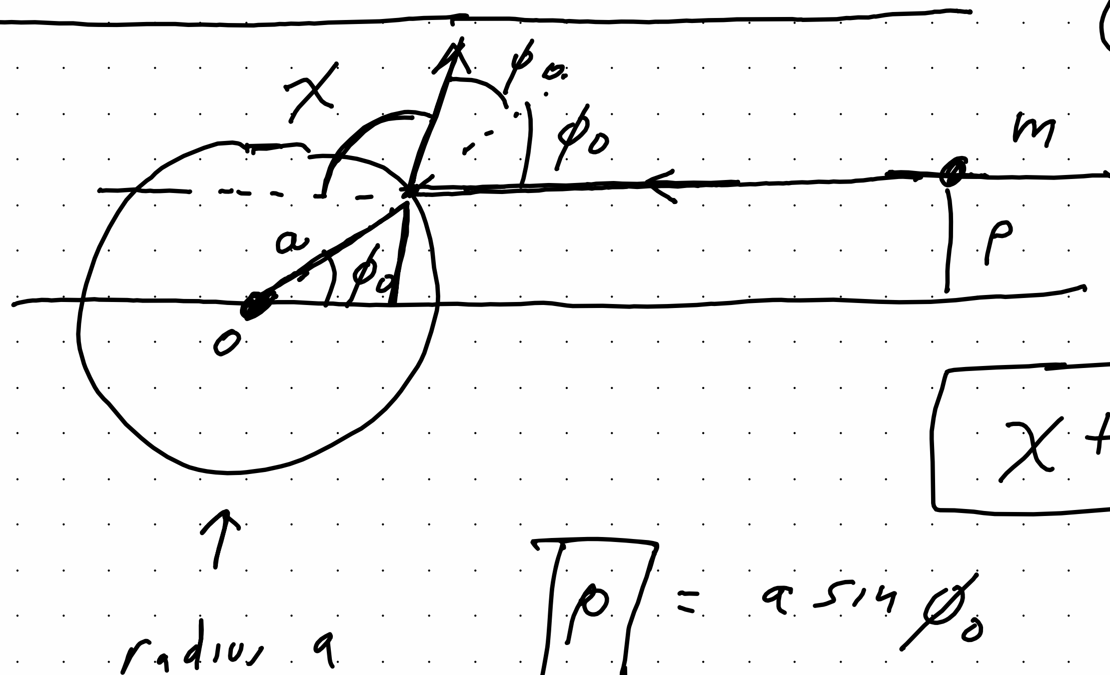
perspective





$\sigma = \pi \rho_{max}^2$

Hard sphere: prob 18.1 $U(r) = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases}$



$\boxed{\chi \longleftrightarrow \rho}$

$\boxed{\chi + 2\phi_0 = \pi}$

$\phi_0 = \frac{\pi}{2} - \frac{\chi}{2}$

$$\rho = a \sin\phi_0 = a\sin\left(\frac{\pi}{2} - \frac{\chi}{2}\right) = a\cos\left(\frac{\chi}{2}\right)$$

$$\frac{d\rho}{d\chi} = -\frac{a}{2}\sin\left(\frac{\chi}{2}\right)$$

$$d\sigma = 2\pi\rho\, d\rho = 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi$$

$$d\sigma = \cancel{2}\pi \; a \cos\left(\frac{\chi}{2}\right) \frac{a}{\cancel{2}} \sin\left(\frac{\chi}{2}\right) d\chi$$

$$= \pi a^2 \sin\left(\frac{\chi}{2}\right) \cos\left(\frac{\chi}{2}\right) d\chi$$

$$= \boxed{\frac{\pi a^2}{2} \sin\chi \, d\chi}$$

$$d\Omega = 2\pi \sin\chi \, d\chi \quad \rightarrow \quad \sin\chi \, d\chi = \frac{1}{2\pi} d\Omega$$

$$d\sigma = \frac{\cancel{\pi} a^2}{2} \cdot \frac{1}{2\cancel{\pi}} d\Omega$$

$$= \boxed{\frac{1}{4} a^2 \, d\Omega}$$ — uniform distribution

---

Total cross-section:

$$\sigma_{tot} \equiv \sigma = \int_{\text{unit sphere}} d\sigma = \frac{1}{4} a^2 \underbrace{\int_{\text{sphere}} d\Omega}_{= 4\pi} = \boxed{\pi a^2}$$

$$\frac{d\sigma_1}{d\Omega_1} = \frac{d\sigma}{d\Omega} \left| \frac{d(\cos\chi)}{d(\cos\theta_1)} \right|$$

$\frac{1}{4} a^2$

$$\frac{d\sigma_2}{d\Omega_2} = \frac{d\sigma}{d\Omega} \left| \frac{d(\cos\chi)}{d(\cos\theta_2)} \right|$$

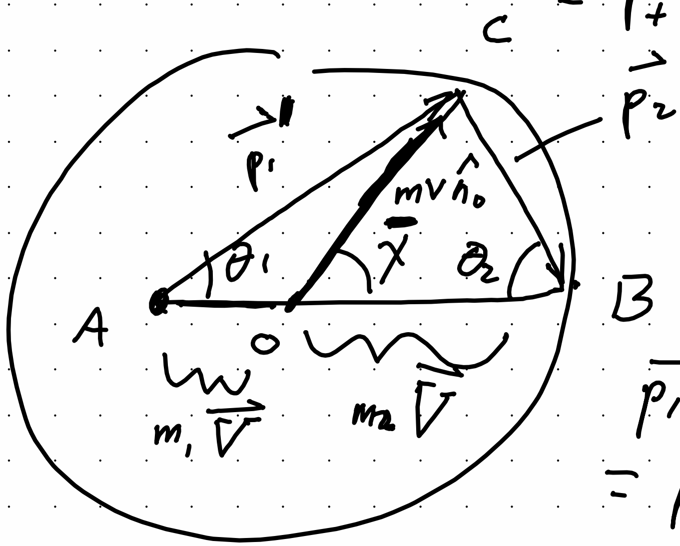
$$\boxed{\begin{aligned} &\chi = \pi - 2\theta_2 \\ &\cos\chi = -\frac{m_1}{m_2}\sin^2\theta_1 \pm \cos\theta_1 \sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1} \end{aligned}}$$

$$\sin\theta_1 \, d\theta_1 = -d(\cos\theta_1)$$

Lecture #17 Tue, 10/20

- Quiz #3 — this week
- Q&A : $\vec{p_1}' + \vec{p_2}' = \vec{P}_{tot}'$

$= \vec{P}_{tot}$



$\vec{p_1} + \cancel{\vec{p_2}}^{\to 0}$

$= \vec{p_1}$

$m_1 \vec{V} + m_2 \vec{V}$

$= (m_1 + m_2)\vec{V} = \vec{P}_{tot}$

$\vec{V} = \dfrac{\vec{p_1} + \cancel{\vec{p_2}}^{\to 0}}{m_1 + m_2}$

com velocity

$= \dfrac{\vec{p_1}}{m_1 + m_2}$

$= \dfrac{m_1 \vec{v_1}}{m_1 + m_2}$

$m_1 \vec{V} = m_1 \left(\dfrac{m_1 \vec{v_1}}{m_1 + m_2} \right)$

$m_1 < m_2$, $v_2 = 0$

$\vec{v_2} = 0 \to \boxed{\vec{p_2} = 0}$

$= \left(\dfrac{m_1^2}{m_1 + m_2} \right) \vec{v_1}$

m_1 m_2

$$m_1 \vec{r_1} + m_2 \vec{r_2} = 0$$

$$\vec{r} = \vec{r_1} - \vec{r_2}$$

$$T = \frac{1}{2} m_1 |\dot{\vec{r_1}}|^2 + \frac{1}{2} m_2 |\dot{\vec{r_2}}|^2$$

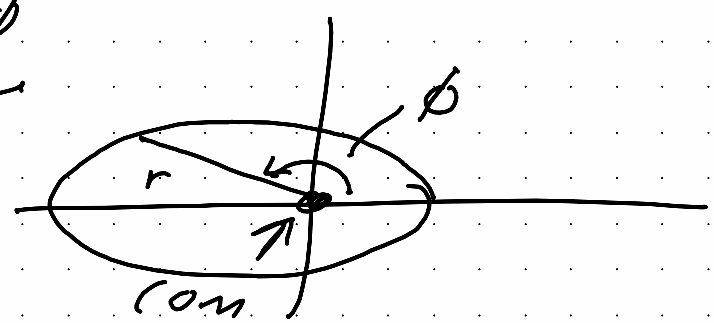
$$= \frac{1}{2} m |\dot{\vec{r}}|^2 \qquad m = \frac{m_1 m_2}{m_1 + m_2}$$

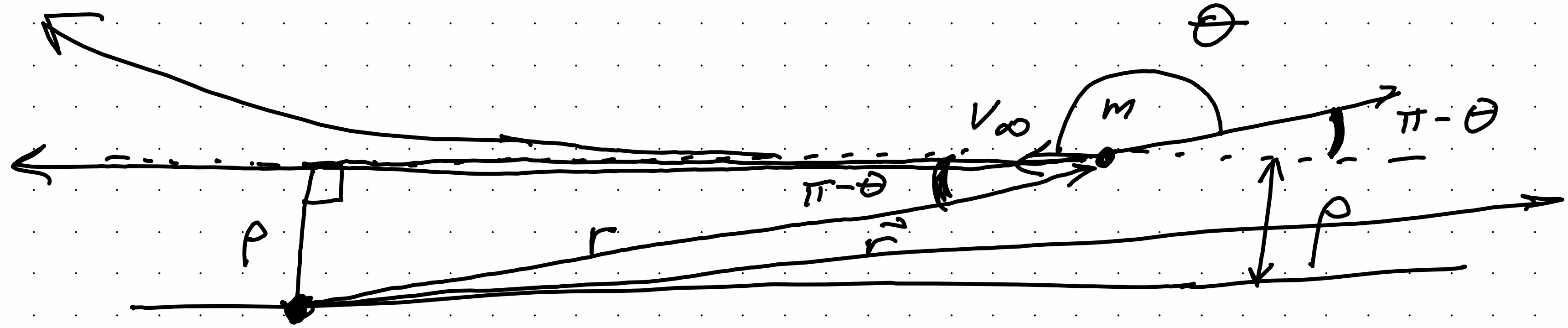
---

r vs ϕ

$$\vec{r_1} = \frac{m_2}{m_1 + m_2} \vec{r}$$

$$\vec{r_2} = \frac{-m_1}{m_1 + m_2} \vec{r}$$





$U = 0$

$M = \rho m V_\infty$

$\vec{M} = \vec{r} \times \vec{p}$ $\quad\quad \vec{p} = m\vec{V_\infty}$

$= m\vec{r} \times \vec{V_\infty}$

$M = |\vec{M}| = m \underbrace{r V_\infty \sin\theta}$

$= m V_\infty \underbrace{r \sin(\pi-\theta)}$

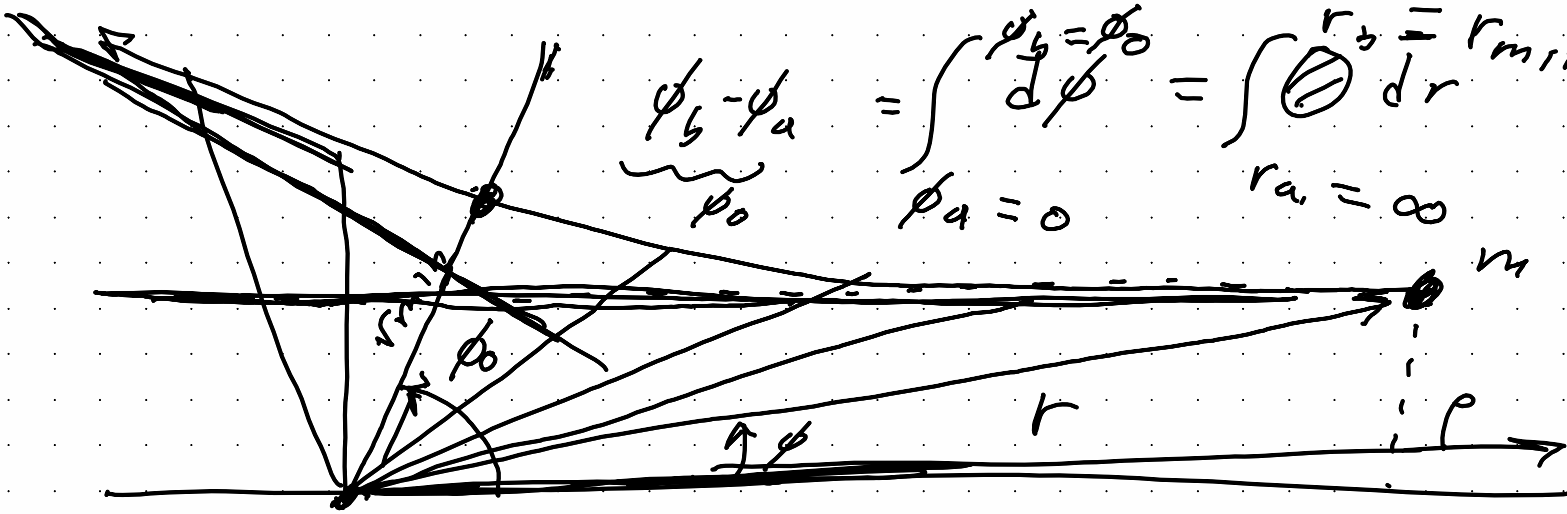
$= \rho m V_\infty \quad \rho$

$\sin(\pi-\theta)$

$= \cancel{\sin\pi}\cos\theta$

$- \cos\pi \sin\theta$

$= \sin\theta$



$$\underbrace{\phi_b - \phi_a}_{\phi_0} = \int_{\phi_a = 0}^{\phi_b = \phi_0} d\phi = \int_{r_a = \infty}^{r_b = r_{min}} \oslash \, dr$$

$$U(r) = \frac{\alpha}{r}$$

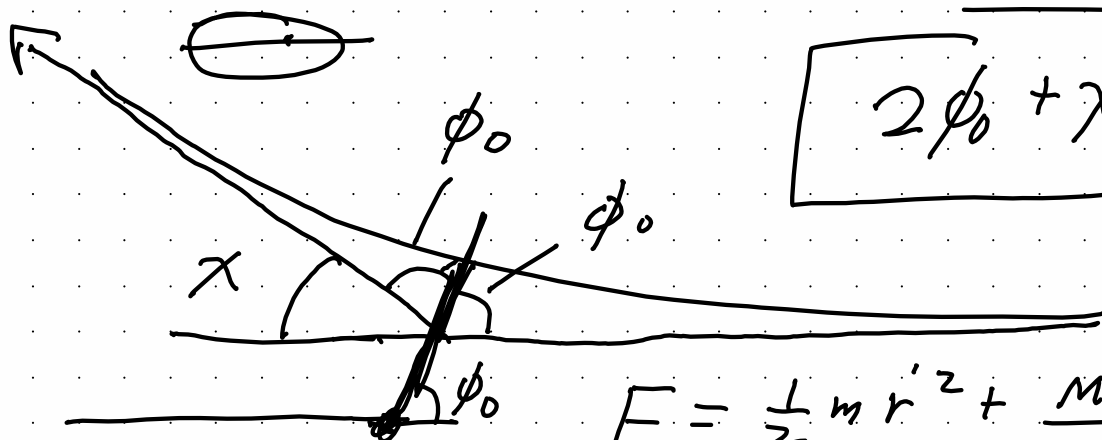
$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M\,dr/r^2}{\pm\sqrt{\underbrace{2m(E-U) - M^2/r^2}_{>0}}}$$

$$M = \rho m v_\infty$$

$$E = \tfrac{1}{2} m v_\infty^2$$

$$\boxed{2\phi_0 + \chi = \pi}$$

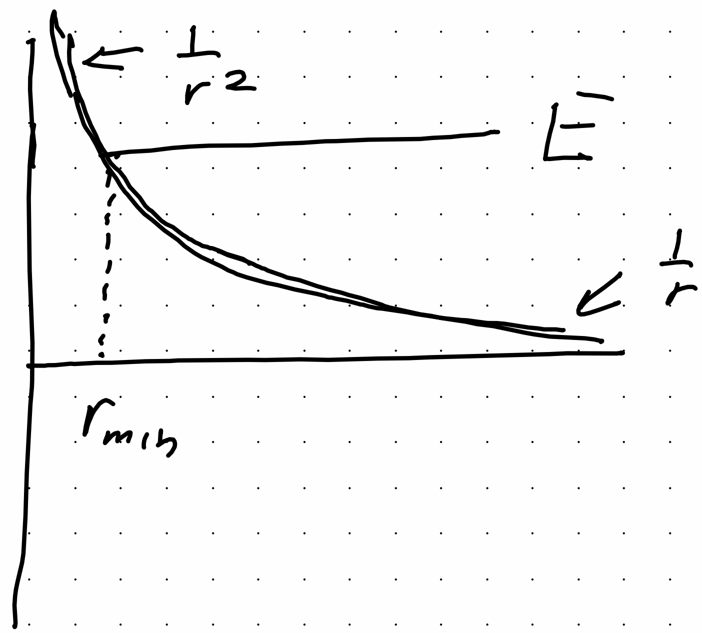
$$\boxed{\frac{v_\infty}{\rho}}$$



$$\chi \longleftrightarrow \rho$$

$$E = \tfrac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} + U(r)$$

$$\dot{r} = \pm\sqrt{}$$



$U = \frac{\alpha}{r}$

$\alpha > 0$

coulomb scattering

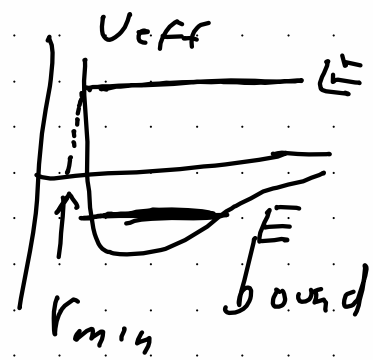
$+ \; +$

$- \; -$

$$U_{eff} = U(r) + \frac{M^2}{2mr^2}$$

$$E = U_{eff}(r_{min})$$

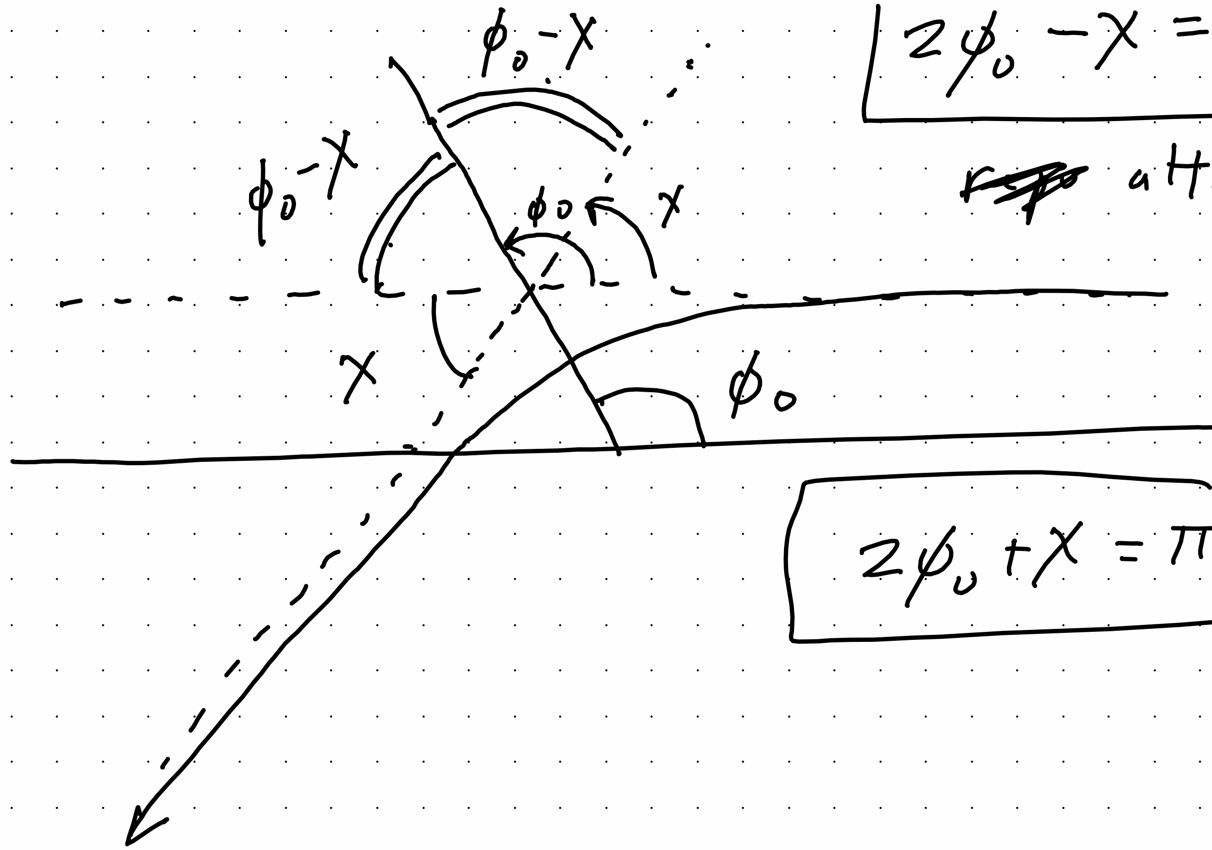
$U = -\frac{\alpha}{r}$



$$2(\phi_0-\chi)+\chi=\pi$$

$$\boxed{2\phi_0-\chi=\pi}$$

~~rep~~ attractive

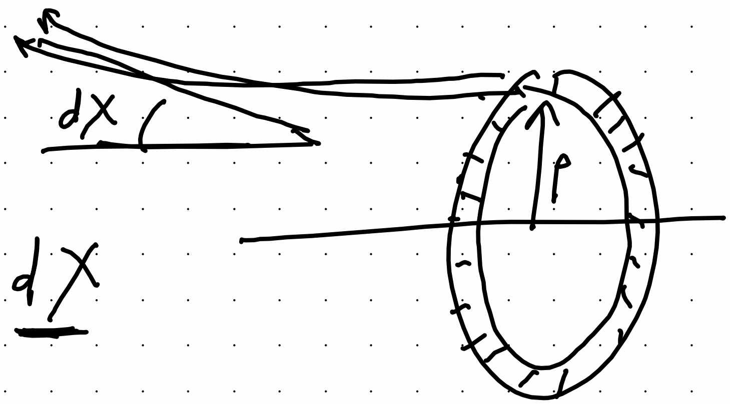


$$\boxed{2\phi_0+\chi=\pi}$$

$$d\sigma = 2\pi \rho \, d\rho$$

$$= 2\pi \rho(\chi) \left|\frac{d\rho}{d\chi}\right| d\chi$$

$$= \frac{\rho(\chi)}{\sin\chi} \left|\frac{d\rho}{d\chi}\right| d\Omega$$



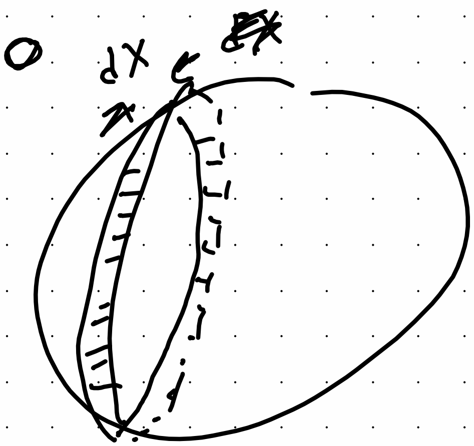
$d\rho > 0$

$d\chi < 0$

$$\boxed{d\Omega = 2\pi \sin\chi \, d\chi}$$

$||$

$d\sigma$



wrt COM frame

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to go to lab frame

$\chi \rightarrow \theta_1$

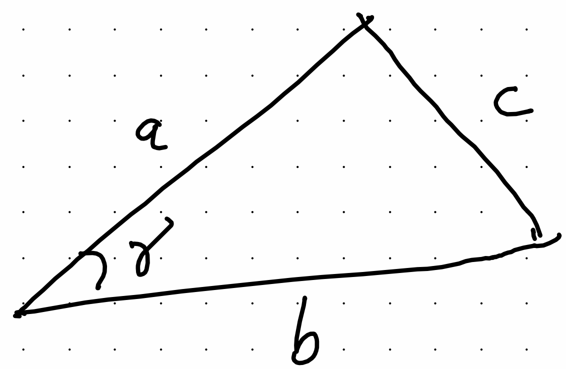
or θ_2

$d\sigma_1 , \; d\sigma_2$

$d\theta_1, d\theta_2, d\Omega_1, d\Omega_2$

Lecture #18:   Thurs 10/22

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$



$$\cos^2 x + \sin^2 x = 1$$

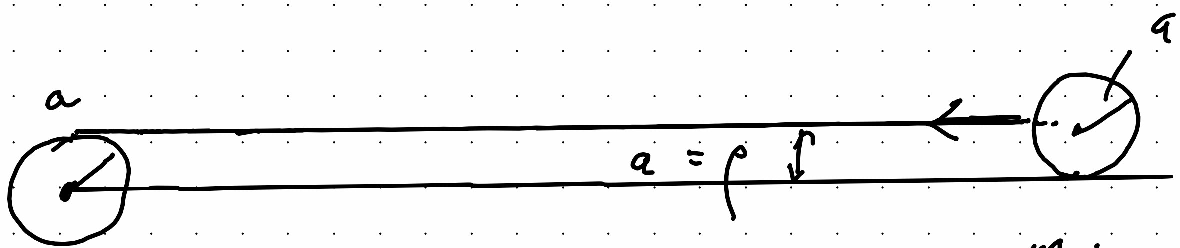
$$\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

QUIZ #3:

name_q3.pdf

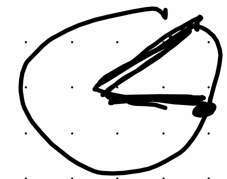
lab Frame



m_2
$v_2 = 0$

$(m_1 = m_2)$

m_1
$v_1 = v_0$



(hard sphere scattering)

1) calculate χ (scattering angle of the reduced mass w.r.t COM frame)

2) calculate θ_1, θ_2 (scattering angles w.r.t lab frame)

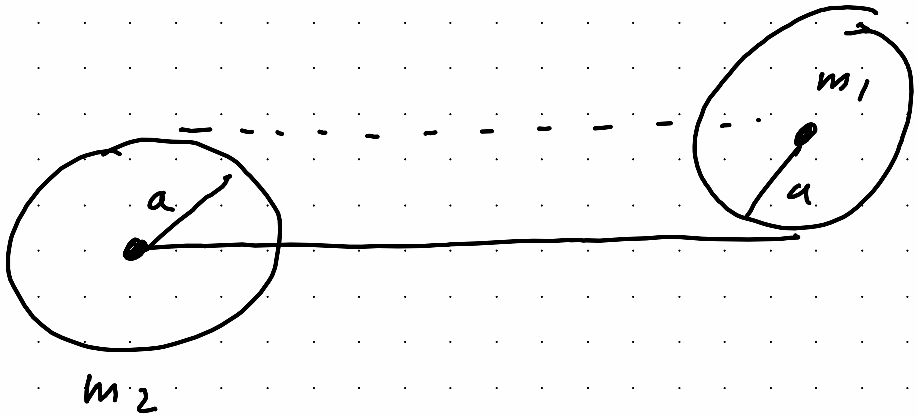
$$\boxed{\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}}\,, \quad \boxed{\theta_2 = \frac{1}{2}(\pi - \chi)}$$

elastic scattering

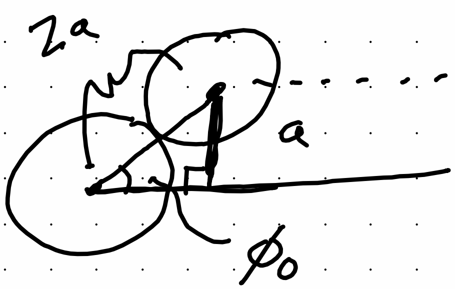
m_2 initially at rest

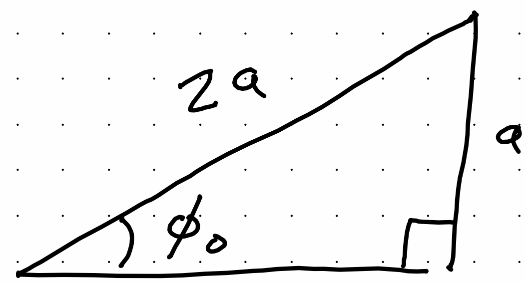
3) How do θ_1, θ_2 change if masses m_1 and m_2 change? $(m_1 >> m_2\ ;\ m_1 << m_2)$

$$U(r) = \begin{cases} \infty & r < 2a \\ 0 & r > 2a \end{cases}$$



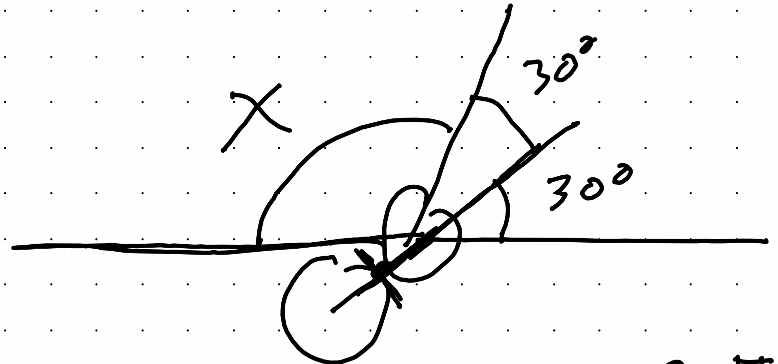
$$\vec{r} = \vec{r_1} - \vec{r_2}$$





$$\sin\phi_0 = \frac{a}{2a} = \frac{1}{2}$$

$$\boxed{\phi_0 = \pi/6 = 30^\circ}$$



$$\boxed{\chi = 120^\circ} = \frac{2\pi}{3}$$

independent of m_1 and m_2

$$\frac{2\pi}{3} = 120^\circ$$

2) $$\theta_2 = \frac{1}{2}(\pi - \chi) = \frac{1}{2}(180^\circ - 120^\circ) = \boxed{30^\circ}$$

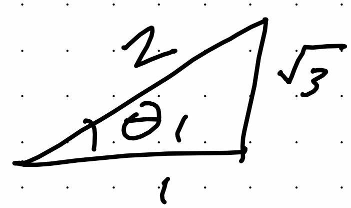
$$\boxed{\theta_2} = 30^\circ$$

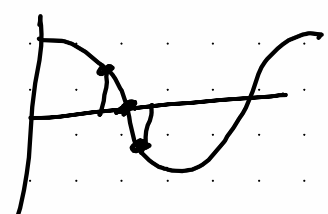
$$\tan\theta_1 = \frac{\cancel{m_2}\sin(120^\circ)}{\cancel{m_1} + \cancel{m_2}\cos(120^\circ)} = \frac{\sin(120^\circ)}{1 + \cos(120^\circ)}$$

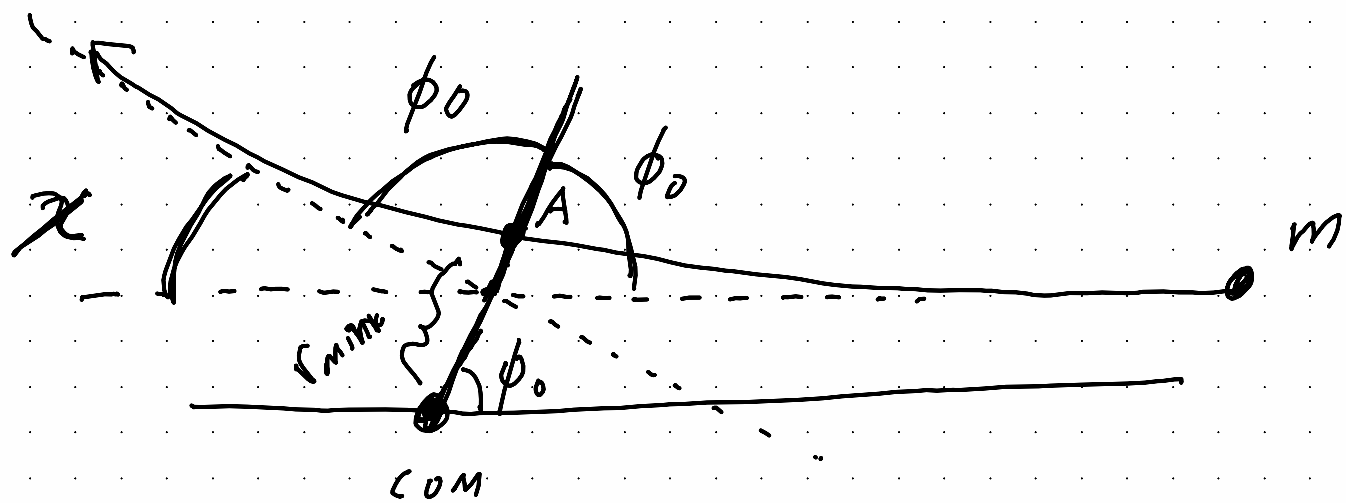
$$= \frac{\sqrt{3}/2}{1 - 1/2}$$

$$= \sqrt{3}$$

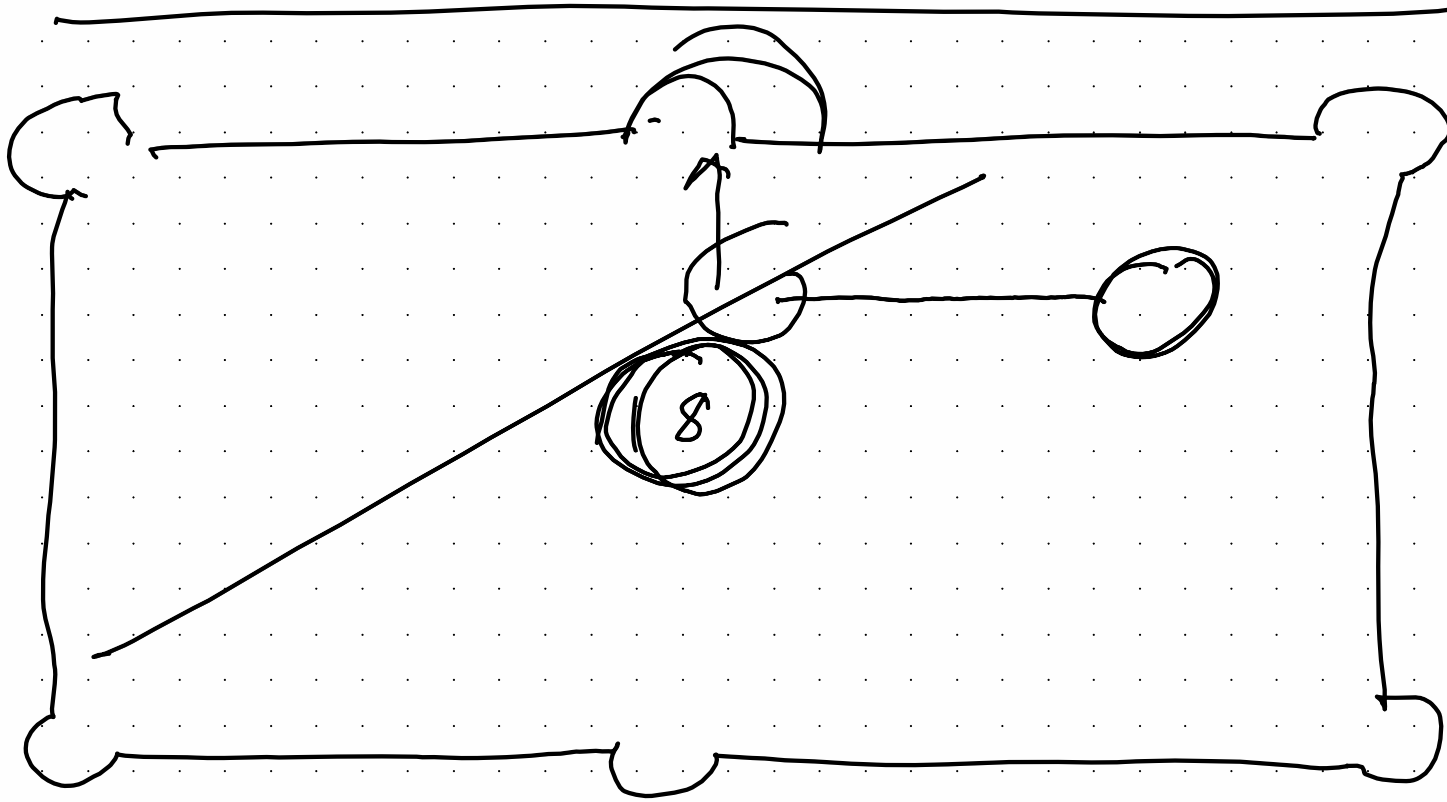
$$\boxed{\theta_1 = 60^\circ}$$







ϕ_0
ϕ_0
χ
A
m
r_{min}
ϕ_0
COM



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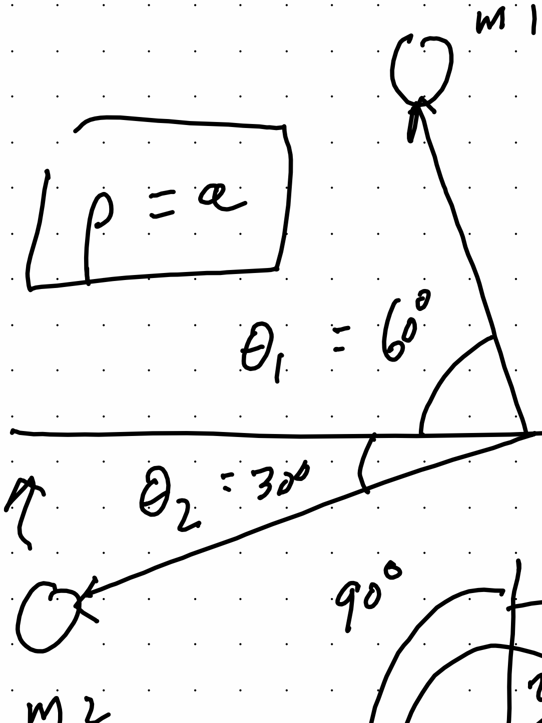
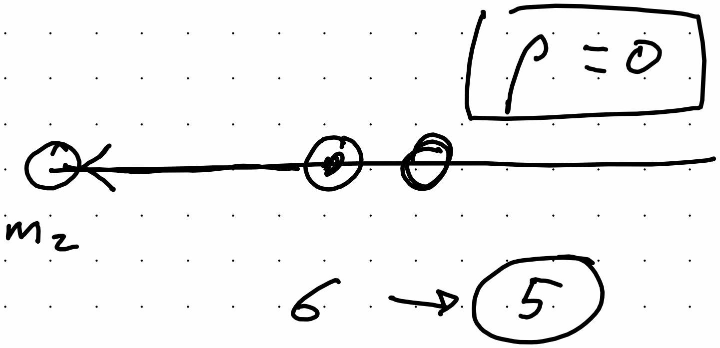
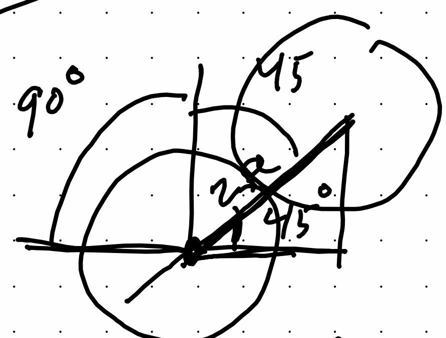
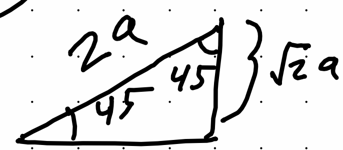


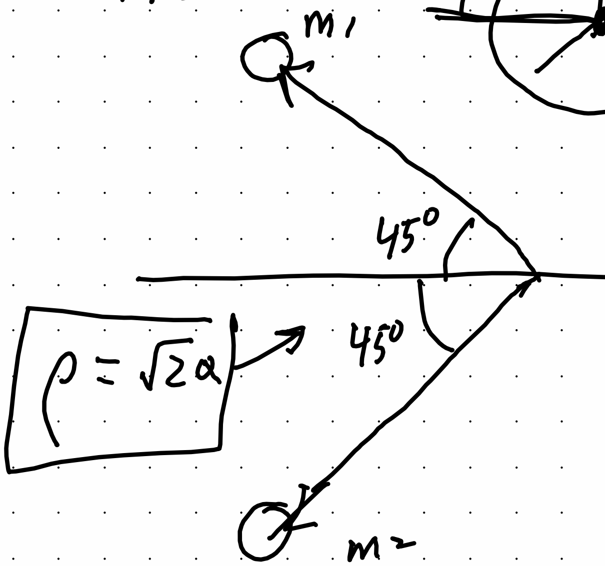




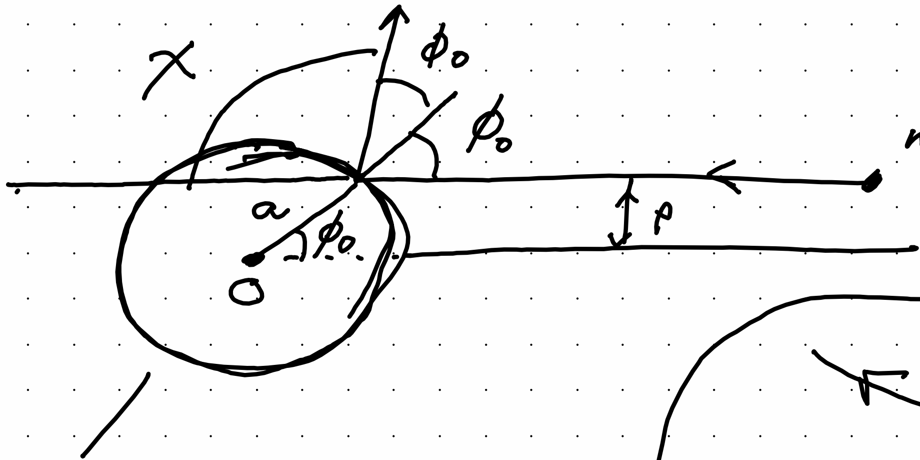
↑ increase : 1
decreases : 3



$\chi = \pi/2$



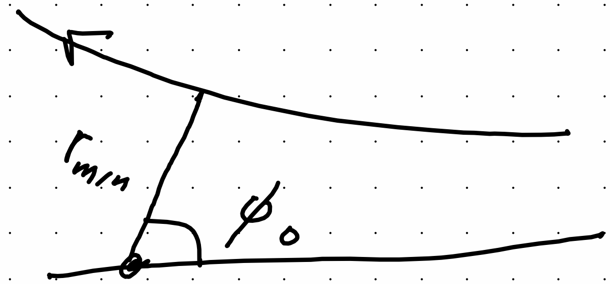
$$\theta_2 = \frac{1}{2}(\pi - \chi)$$
$$= \frac{1}{2}\left(\pi - \frac{\pi}{2}\right) = \pi/4$$
$$\theta_1 = \pi/4$$



radius a

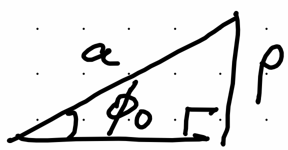
$2\phi_0 + \chi = \pi$

$$\boxed{\chi = \pi - 2\phi_0}$$



$$U(r) = \begin{cases} 0 & r > a \\ \infty & r < a \end{cases}$$

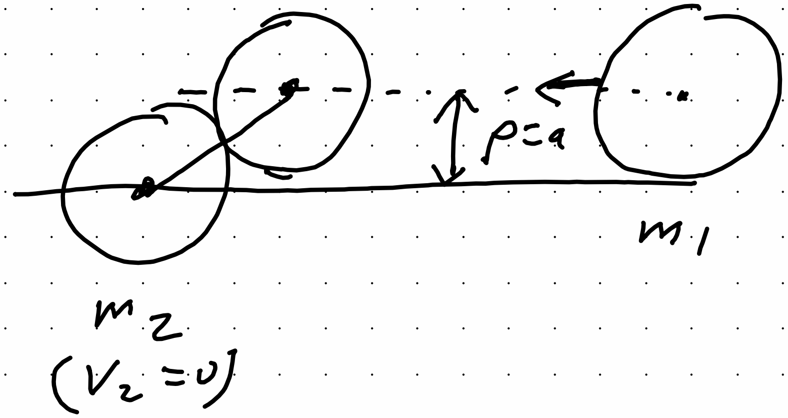
$$\boxed{\begin{aligned} \tan\theta_1 &= \frac{m_2 \sin\chi}{m_1 + m_2 \cos\chi} \\ \theta_2 &= \frac{1}{2}(\pi - \chi) \end{aligned}}$$



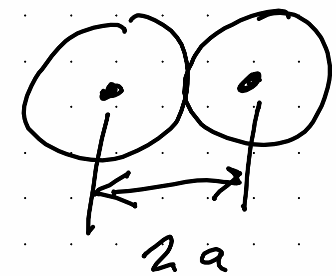
$$\sin\phi_0 = \frac{\rho}{a}$$

$$\phi_0 = \sin^{-1}(\rho/a)$$

Prob 18.1



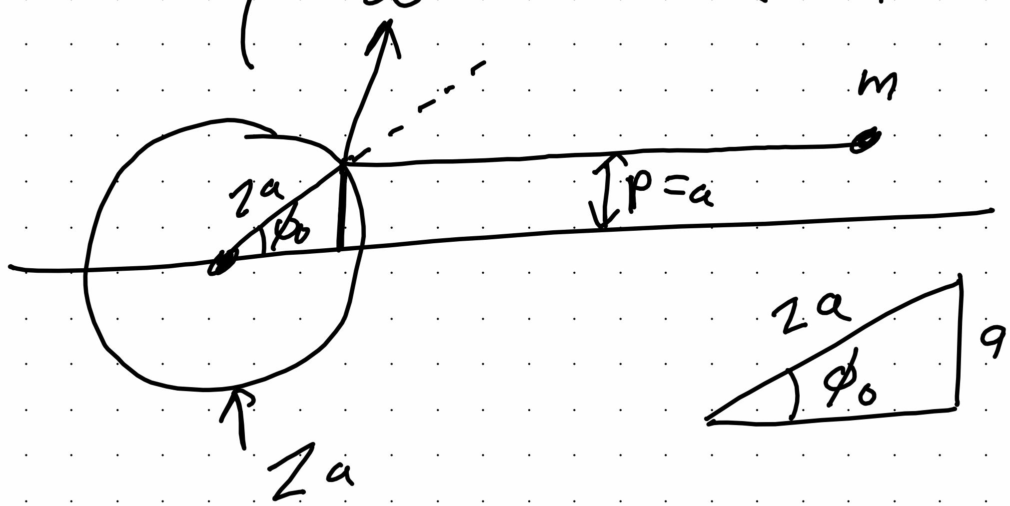
lab Frame



$m_1 = m_2$

$\rho = a$

$$U(r) = \begin{cases} 0 & r > 2a \\ \infty & r < 2a \end{cases}$$



COM Fram

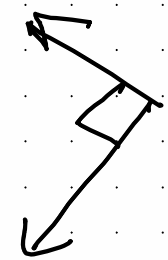
$\sin\phi_0 = \frac{1}{2}$

$\phi_0 = 30^\circ$

$\chi = \pi - 2\phi_0 \quad = 180^\circ - 2 \cdot 30^\circ = \boxed{120^\circ}$

independent of m_1, m_2

$$\boxed{\begin{aligned} \tan\theta_1 &= \frac{m_2 \sin\chi}{m_1 + m_2 \cos\chi} \\ \theta_2 &= \frac{1}{2}(\pi - \chi) \end{aligned}}$$



$\theta_2 = \frac{1}{2}(180^\circ - 120^\circ) = \boxed{30^\circ}$ (indep. of m_1, m_2)

$$\tan\theta_1 = \frac{\sin(120^\circ)}{1 + \cos(120^\circ)} = \frac{\sqrt{3}/2}{1 - 1/2} = \sqrt{3}$$

$\boxed{\theta_1 = 60^\circ}$

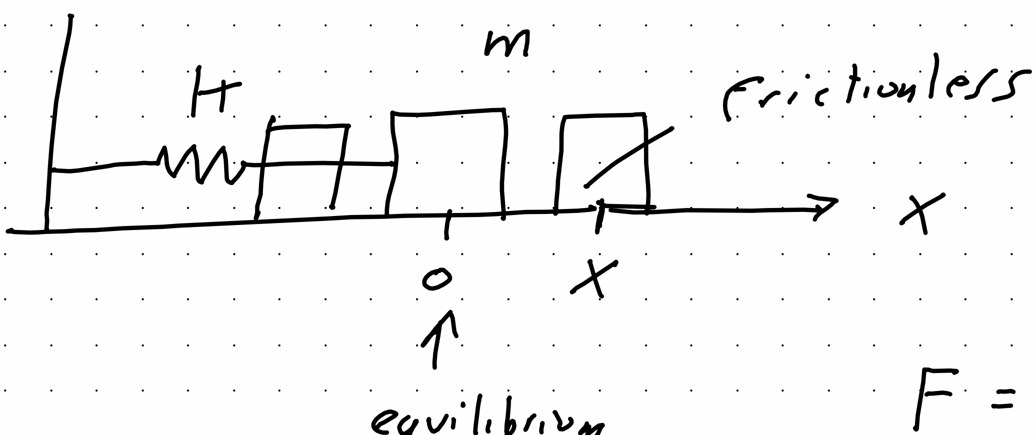
$m_1 << m_2 \rightarrow \tan\theta_1 \approx \tan\chi \longrightarrow \boxed{\theta_1 = 120^\circ}$

$m_1 >> m_2 \rightarrow \tan\theta_1 \approx \frac{m_2}{m_1}\sin\chi \longrightarrow \boxed{\theta_1 \approx 0^\circ}$

Lec #19: Tuesday 10/27

— Small oscillation (next 3 classes)

sec 21 (Free oscillation in 1-d), 22 (Forced oscillation in 1-d), 23 — several dimensions



$F = -kx$

k — spring constant

$F = m\ddot{x}$

$-k\cancel{x} = m\ddot{x}$

$\rightarrow \quad \ddot{x} = -\frac{k}{m}x \longrightarrow x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$

$\omega = \sqrt{\frac{k}{m}}$ (angular freq)

$$x(t) = \underbrace{c_1 \cos \omega t + c_2 \sin \omega t}_{}$$

determined by ICs.

$$x(t) = a \cos(\omega t + \alpha)$$

a: amplitude

α: initial phase

$$x(t) = \mathrm{Re}\left[A e^{i\omega t} \right], \quad A = a e^{i\alpha}$$

A: complex

$$\mathrm{Re}\left[a e^{i\alpha} e^{i\omega t} \right] = \mathrm{Re}\left[a e^{i(\omega t + \alpha)} \right]$$

$$= \mathrm{Re}\left[a \left(\cos(\omega t + \alpha) + i \sin(\omega t + \alpha) \right) \right]$$

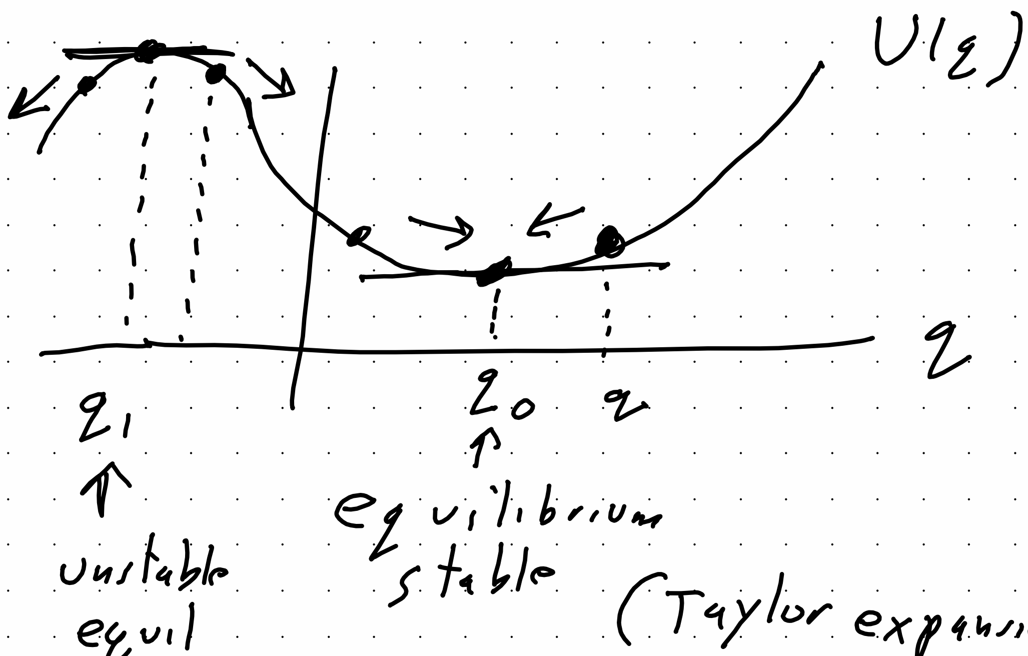
$$= a \cos(\omega t + \alpha)$$

Small oscillation:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots$$

1-d system: q generalized coord

$$L = \frac{1}{2} a(q)\, \dot{q}^2 - U(q)$$

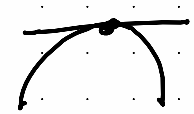


$$F = -\frac{dU}{dq}$$

$$F(q_0) = 0$$

$$\left.\frac{dU}{dq}\right|_{q_0} = 0$$

(Taylor expansion)



$$U(q) = \underline{U(q_0)} + \underbrace{\cancel{\left.\frac{dU}{dq}\right|_{q_0}}}_{0}(q - q_0) + \frac{1}{2}\left.\frac{d^2U}{dq^2}\right|_{q_0}(q - q_0)^2 + \dots$$

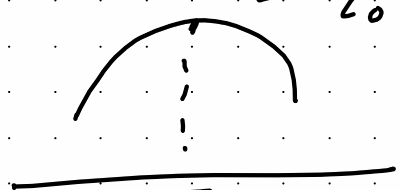
Small displacement away from equilibrium

$$\frac{|q-q_0|}{q_0} \ll 1 \quad (\text{ignore } O(3))$$

$$\boxed{U(q) \cong U(q_0) + \frac{1}{2}\frac{d^2U}{dq^2}\Big|_{q_0}(q-q_0)^2}$$

$$\frac{d^2U}{dq^2}\Big|_{q_0} > 0$$

$$\frac{d^2U}{dq^2}\Big|_{q_0} < 0$$



$$k = \frac{d^2U}{dq^2}\Big|_{q_0}$$

$$U(q) \cong \underbrace{U(q_0)}_{\text{const}} + \frac{1}{2}k\underbrace{(q-q_0)^2}_{x^2}$$

$$F = -\frac{dU}{dq}$$

$$= -k(q-q_0)$$

$$= -kx$$

$$U = \frac{1}{2}kx^2$$

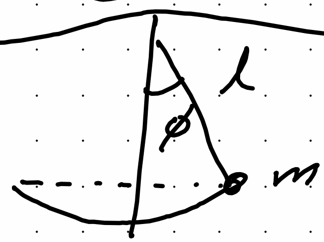
$$L = \frac{1}{2} a(q)\, \dot q^{2} - U(q)$$

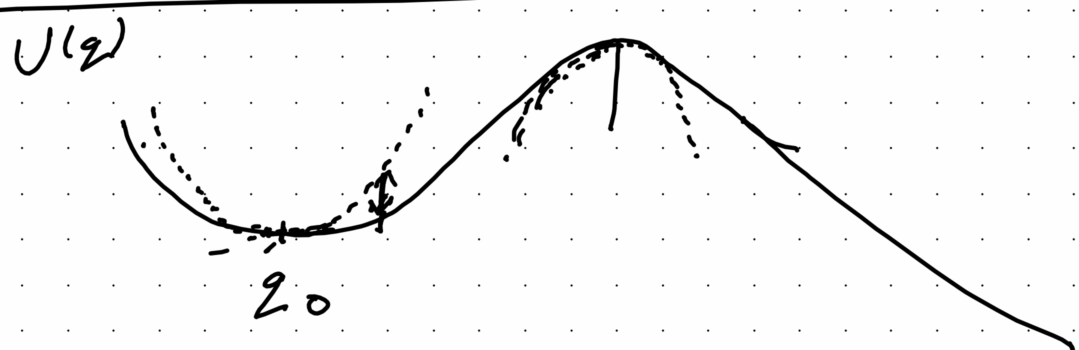
$$= \frac{1}{2} \underset{m}{\underset{\shortparallel}{a(q_0)}}\, \dot x^{2} - \underbrace{U(q_0)}_{\text{const}} - \frac{1}{2} k x^{2}$$

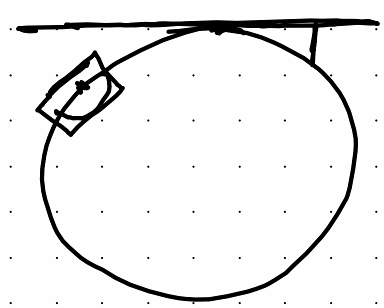
$$x = q - q_0 \rightarrow \dot x = \dot q$$

$$L = \frac{1}{2} m \dot x^{2} - \frac{1}{2} k x^{2} \xrightarrow[\text{EOM}]{} m \ddot x = -kx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) = \frac{\partial L}{\partial x}$$

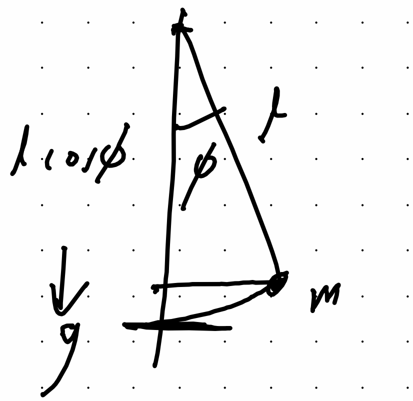






Example:

simple pendulum



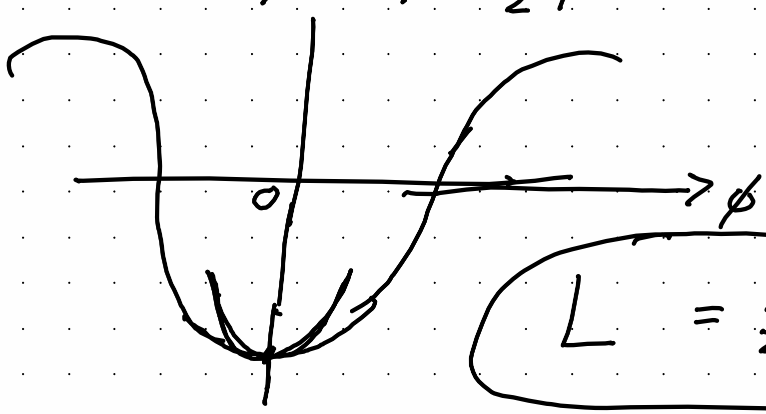
$\uparrow y$

mgy

$$L = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1-\cos\phi)$$
$$= \frac{1}{2}ml^2\dot{\phi}^2 + \underline{mgl\cos\phi}$$
$$= T - U$$

$$y = l - l\cos\phi$$
$$= l(1-\cos\phi)$$

$$U(\phi) = -mgl\cos\phi \Leftarrow -mgl\left(1-\frac{1}{2}\phi^2\right)$$
$$\cos\phi = 1 - \frac{1}{2}\phi^2 + \dots$$
$$= \underbrace{-mgl}_{\text{const}} + \frac{1}{2}mgl\phi^2$$

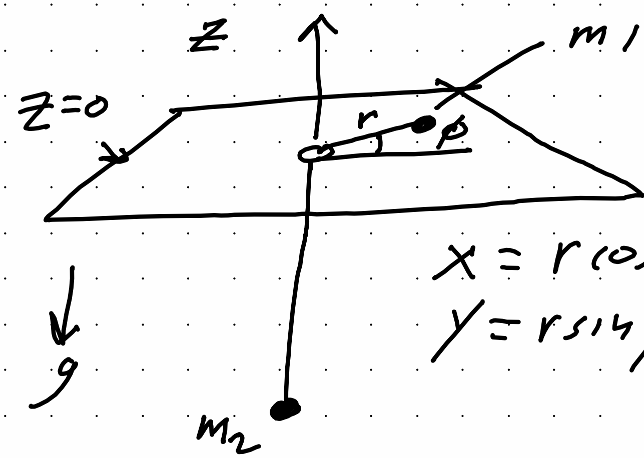


$\phi = 0$

$$L = \frac{1}{2}\underbrace{ml^2}_{= M}\dot{\phi}^2 - \frac{1}{2}\underbrace{mgl}_{K}\phi^2$$

$$\omega = \sqrt{\frac{K}{M}} = \sqrt{\frac{\cancel{m}gl}{\cancel{m}l^2}} = \boxed{\sqrt{\frac{g}{l}}}$$

Example: 2-d problem



$x = r\cos\phi$

$y = r\sin\phi$

$$T = \frac{1}{2} m_1 \overbrace{\left(\dot r^2 + r^2 \dot\phi^2\right)}^{\dot x^2 + \dot y^2} + \frac{1}{2} m_2 \underbrace{\dot z^2}_{\dot r^2}$$

$$\boxed{T = \frac{1}{2}(m_1 + m_2)\dot r^2 + \frac{1}{2} m_1 r^2 \dot\phi^2}$$

string: length l

$r + |z| = l$

$r - z = l$

$z = r - l$

$\rightarrow \dot z = \dot r$

$$\boxed{U = m_2 g z} = m_2 g (\underbrace{r - l}_{l \text{ const}})$$

$$L = T - U$$

$$\boxed{L = \frac{1}{2}(m_1 + m_2)\dot r^2 + \frac{1}{2} m_1 r^2 \dot\phi^2 - m_2 g r}$$

No ϕ dependence

$$\dot\phi = \frac{M_z}{m_1 r^2} \leftarrow M_z = \frac{\partial L}{\partial \dot\phi} = m_1 r^2 \dot\phi = \text{const}$$

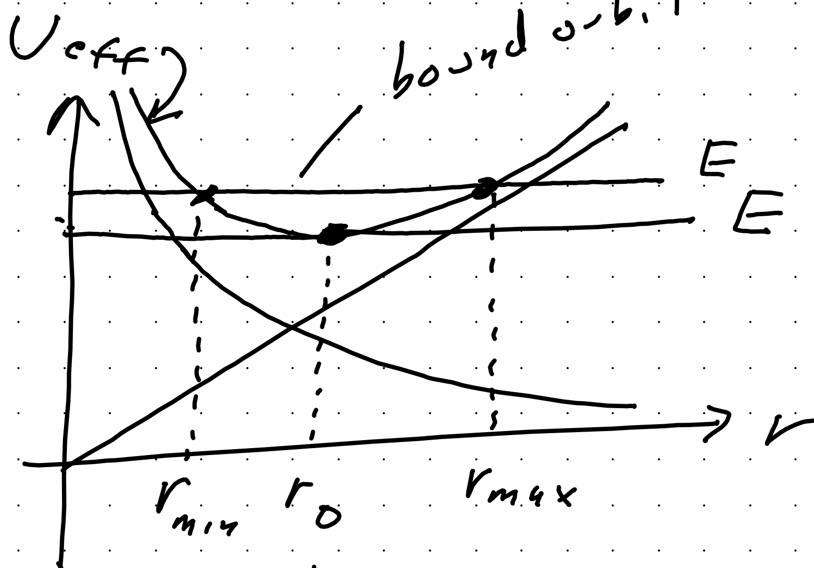
No explicit timedependence $\rightarrow$ $E = \text{const}$

$$E = T + U$$

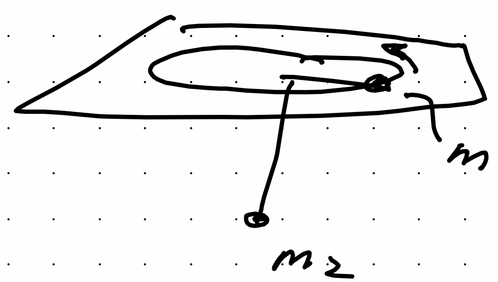
$$= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\phi}^2 + m_2 g r$$

$$= \boxed{\frac{1}{2}(m_1+m_2)\dot{r}^2} + \underbrace{\frac{M_z^2}{2m_1 r^2} + m_2 g r}_{U_{eff}(r)} - m_2 g \ell$$

not effect EOMs



stable equilibrium

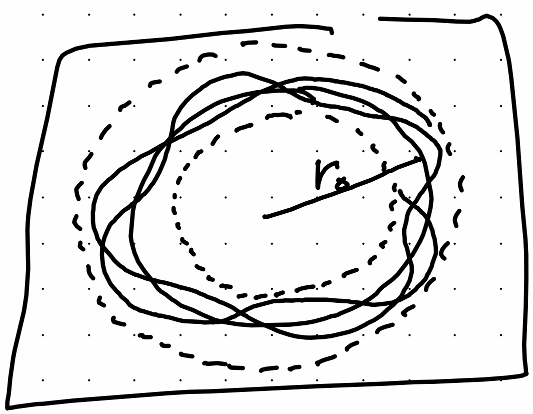


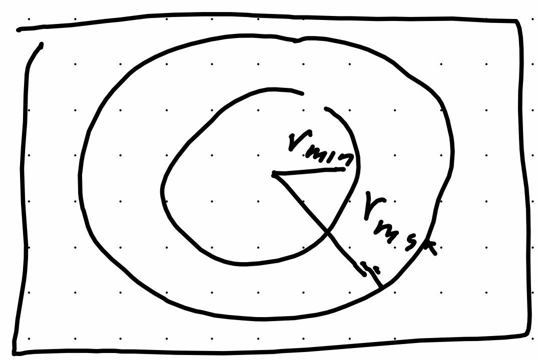
$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0}, \quad U_{eff}(r) = \frac{M_z^2}{2m_1 r^2} + m_2 g r$$

$$= \frac{-M_z^2}{m_1 r_0^3} + m_2 g$$

$$\boxed{M_z^2 = m_1 m_2 g r_0^3}$$

Small oscillation around r_0: $\frac{|r - r_0|}{r_0} \ll 1$





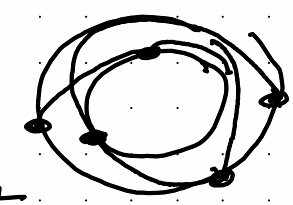
top view

$$\frac{dU_{eff}}{dr} = -\frac{M_z^2}{m_1 r^3} + m_2 g$$

$$\rightarrow \left.\frac{d^2 U_{eff}}{dr^2}\right|_{r_0} = \frac{3 M_z^2}{m_1 r_0^4} \quad , \quad M_z^2 = m_1 m_2 g r_0^3$$

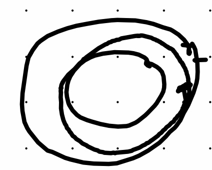
$$K = \frac{3 \cancel{m_1} m_2 g r_0^3}{\cancel{m_1} r_0^4} = \boxed{\frac{3 m_2 g}{r_0}}$$

$$E = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \underbrace{U_{eff}(r_0)}_{const} + \frac{1}{2}K(r - r_0)^2$$



$$x = r - r_0$$

$$E = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}K x^2 + const$$



$$\omega_r = \sqrt{\frac{K}{m_1 + m_2}} = \boxed{\sqrt{\frac{3 m_2 g}{(m_1 + m_2) r_0}}}$$

$$\omega_\phi \equiv \dot{\phi}\Big|_{r_0} \qquad M_z = m_1 r^2 \dot{\phi}$$

$$= \frac{M_z}{m_1 r_0^2}$$

$$= \frac{\sqrt{m_1 m_2 g r_0^3}}{m_1 r_0^2} = \frac{\sqrt{m_1 m_2 g r_0^3}}{\sqrt{m_1^2 r_0^4}}$$

$$= \boxed{\sqrt{\frac{m_2 g}{m_1 r_0}}} \qquad 0 \to 2\pi$$

$$\boxed{\omega_r = \sqrt{\frac{3 m_2 g}{(m_1 + m_2) r_0}}}$$

not equal in general!

For $\omega_r = \omega_\phi$

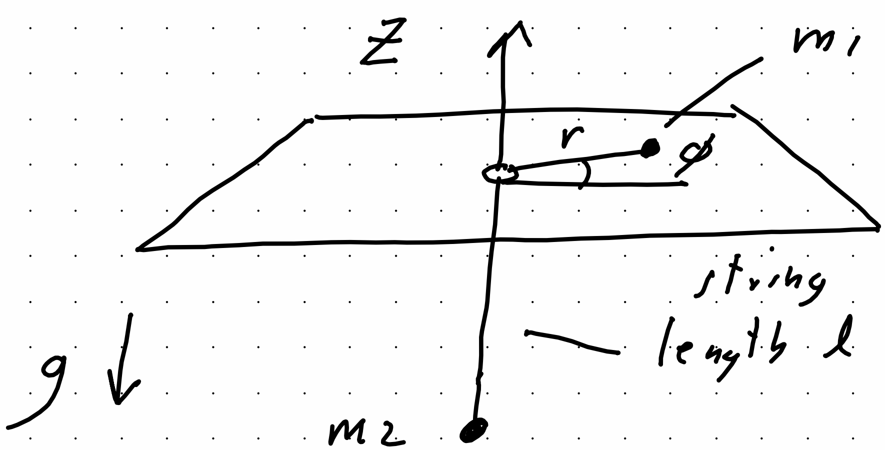
$$\frac{3}{m_1 + m_2} = \frac{1}{m_1}$$

$$m_1 + m_2 = 3 m_1$$

closed bound orbits

Lecture #20

- Quiz #4 – Tuesday
- Today: (i) Finish up example from last time
  (ii) Forced oscillations



$r = r_0$ (stable circular orbit)

$\omega_r = ?$

small oscillation

$|r - r_0| \ll r_0$

$$\omega_r = \sqrt{\frac{3 m_2 g}{(m_1 + m_2) r_0}}, \quad \omega_\phi = \sqrt{\frac{m_2 g}{m_1 r_0}} = \dot{\phi}\Big|_{r_0}$$

$\omega_r = \omega_\phi \rightarrow$ closed orbit $\frac{3}{m_1 + m_2} = \frac{1}{m_1} \rightarrow 3m_1 = m_1 + m_2$

$m_1 = \frac{1}{2} m_2$

Forced oscillations: (1-d)

$$m\ddot{x} = -kx + F(t)$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}, \quad \omega \equiv \sqrt{\frac{k}{m}}$$

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2 + \underbrace{x F(t)}$$

General soln

$$x(t) = \underbrace{x_h(t)}_{\text{general soln to } F(t)=0} + \underbrace{x_p(t)}_{\text{particular soln}}$$

general soln to $F(t)=0$: $a\cos(\omega t + \alpha)$

particular soln: any solution to the equation with RHS $= F(t)$

Example:

$$F(t) = f\cos(\gamma t + \beta)$$

$$x_p(t) = b\cos(\gamma t + \beta) = \frac{f}{m}\left(\frac{1}{\omega^2 - \gamma^2}\right)$$

$$-b\gamma^2 \cancel{\cos(\gamma t+\beta)} + \omega^2 b \cancel{\cos(\gamma t + \beta)} = \frac{f}{m}\cancel{\cos(\gamma t+\beta)}$$

$$b = \frac{f}{m}\left(\frac{1}{\omega^2 - \gamma^2}\right)$$

Resonance: $\gamma = \omega$

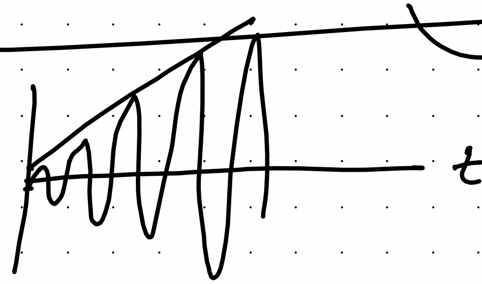
$$x(t) = a\cos(\omega t+\alpha) + \underbrace{\frac{f}{m(\omega^2-\gamma^2)}\left(\cos(\gamma t+\beta)-\cos(\omega t+\beta)\right)}_{\text{particular solution}}$$

$0 \rightarrow \quad 0 \downarrow$

$$\frac{0}{0} \qquad \text{L'Hopital's}$$

$$\frac{\left.\frac{d}{d\gamma}(\text{num})\right|_{\gamma=\omega}}{\left.\frac{d}{d\gamma}(\text{den})\right|_{\gamma=\omega}} = \frac{+f\,t\sin(\omega t+\beta)}{+2m\omega}$$

$$\boxed{x(t) = a\cos(\omega t+\alpha) + \underbrace{\frac{f\,t}{2m\omega}\sin(\omega t+\beta)}_{}}$$



after some time t, this term no longer

$\longrightarrow$ invalidates small osc. approx.

General $F(t)$:

$$F(t) = \int_{-\infty}^{\infty} d\gamma \underbrace{\tilde{F}(\gamma)}_{\text{complex}} e^{i\gamma t}$$

Examples:

$$F(t) = f_2 \sin(2\omega t) + f_3 \sin(3\omega t) + f_4 \sin(4\omega t)$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m} \quad (2^{nd} \text{ order})$$

$$\left.\begin{aligned} \xi &= \dot{x} + i\omega x \\ \dot{\xi} &= \ddot{x} + i\omega \dot{x} \end{aligned}\right\} \rightarrow \boxed{\dot{\xi} - i\omega\xi = \frac{F(t)}{m}} \quad \left(\begin{matrix}1^{st}\\ \text{order}\end{matrix}\right)$$

$$y' + P(x)\, y = Q(x)$$

$$y' = \frac{dy}{dx}$$

by Boas "Math method"

$$\dot{\xi} - i\omega\xi = \frac{F(t)}{m}$$

$$\boxed{dy + \left(P(x)y - Q(x)\right)dx = 0}$$ — not exact!

"Exact" differential:

If exact $\quad 1\, dy + \left(P(x)y - Q(x)\right)dx = dU(x,y)$

$$dU(x,y) = \left(\frac{\partial U}{\partial x}\right)dx + \left(\frac{\partial U}{\partial y}\right)dy$$

$$\rightarrow \frac{\partial^2 U}{\partial y\,\partial x} = \frac{\partial^2 U}{\partial x\,\partial y}$$

$$\cancel{\frac{\partial 1}{\partial x}} \overset{?}{=} P(x)$$

0

$$0 \neq P(x)$$

Claim:

$$\underbrace{\mu(x)}_{\text{integrating factor}}\Big[\overbrace{dy + (P(x)y - Q(x))dx}^{=0 \text{ (original equation)}}\Big] = dU$$

$$\text{RHS} = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

$$\mu(x) = \frac{\partial U}{\partial y}$$

$$\mu(x)\big(P(x)y - Q(x)\big) = \frac{\partial U}{\partial x}$$

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x} \rightarrow \mu'(x) = \mu(x)P(x)$$

$$\int \frac{\mu'}{\mu} = \int P(x)dx$$

$$\ln \mu = \int P(x)dx \rightarrow \mu = e^{\int P(x)dx}$$

$$I(x) = \int P(x)dx$$

$$\mu(x) = e^{I(x)}, \quad I(x) = \int dx\, P(x)$$

$$\frac{\partial U}{\partial y} = \mu(x)$$

$$U(x,y) = \mu(x)\, y + g(x) = C$$

const

$$\frac{\partial U}{\partial x} = \mu(x)\big(P(x)\, y - Q(x)\big)$$

$$\cancel{\mu'(x)\, y} + g'(x) = \mu(x)\big(\cancel{P(x)\, y} - Q(x)\big)$$

$$g'(x) = -\mu(x)\, Q(x)$$

$$\rightarrow \quad g(x) = -\int dx\, Q(x)\mu(x) = -\int dx\, Q(x) e^{I(x)}$$

$$\boxed{y = \frac{1}{\mu(x)}\left(C - g(x)\right) = e^{-I(x)}\left(C + \int dx\, Q(x)\, e^{I(x)}\right)}, \quad \mu(x) = e^{I(x)}, \quad I(x) = \int dx\, P(x)$$

$$y' + P(x)y = Q(x)$$

$$\boxed{\begin{aligned} x(t) &= \mathrm{Im}\left(\frac{\xi(t)}{\omega}\right) \\ \dot{x}(t) &= \mathrm{Re}(\xi(t)) \end{aligned}}$$

$$\dot{\xi} - i\omega\xi = \frac{F(t)}{m}$$

$$\xi = \dot{x} + i\omega x$$

$$y(x) \to \xi(t)$$

$$P(x) \to -i\omega$$

$$I(x) \to I(t) = -i\omega t$$

$$Q(x) \to F(t)/m$$

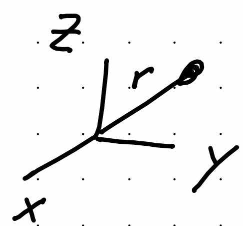
$$\boxed{\xi(t) = e^{i\omega t}\left[\xi_0 + \int_0^t d\bar{t}\; e^{-i\omega\bar{t}}\, F(\bar{t})/m\right]}$$

Lecture #21: Tues 11/3

Q4: on Thursday

Thursday — rigid body motion / non-inertial reference frames

Sec 23: Free oscillations in 2-d or higher



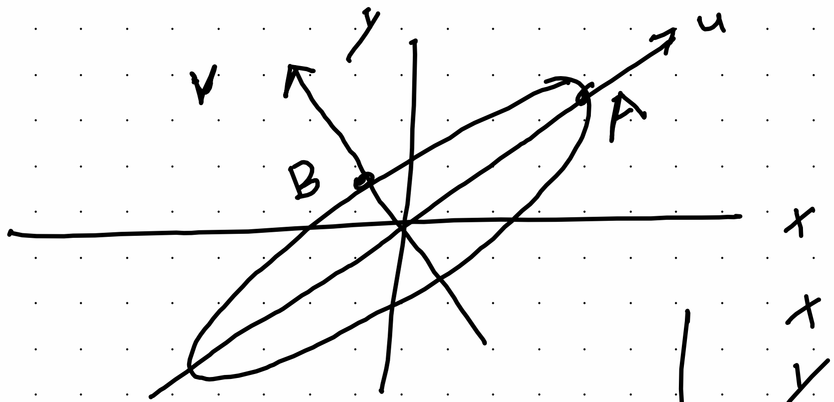
Prob 3, Sec 23: $U = \frac{1}{2} k r^2$ — space oscillator

motion is in 2-d plane (x,y)

$$L = T - U$$
$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k (x^2 + y^2)$$
$$= \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2\right) + \left(\frac{1}{2} m\dot{y}^2 - \frac{1}{2} k y^2\right)$$

2 independent oscillation in the x, y-directions

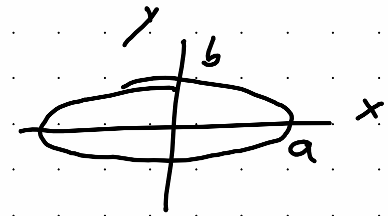
Ang freq: $\omega_x = \sqrt{\frac{k}{m}}$, $\omega_y = \sqrt{\frac{k}{m}}$ $\rightarrow$ $\omega_x = \omega_y$



most general motion

$$x = a\cos(\omega t + \alpha)$$
$$y = b\sin(\omega t + \beta)$$

$\alpha = \beta = 0$

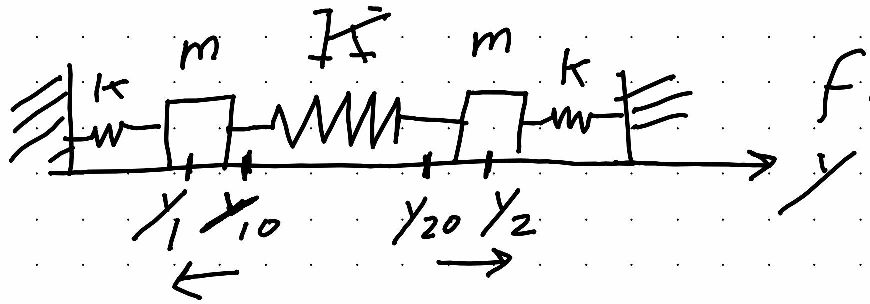


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Normal coordinates:

N independent oscillations with frequency $\omega_1, \omega_2, \ldots, \omega_N$

Example:



frictionless

$i, k = 1, 2, \cdots N$

$N = 2$

$$\left.\begin{aligned} x_1 &= y_1 - y_{10} \\ x_2 &= y_2 - y_{20} \end{aligned}\right\}$$ small deviations away from equilibrium

$$(y_2 - y_1) - (y_{20} - y_{10})$$
$$= (y_2 - y_{20}) - (y_1 - y_{10})$$
$$= x_2 - x_1$$

$$L = T - U$$

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$
$$= \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) = \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k$$

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} K (x_2 - x_1)^2$$

$$(x_1^2 + x_2^2 - 2 x_1 x_2)$$

$$= \frac{1}{2} (k + K) x_1^2 + \frac{1}{2} (k + K) x_2^2 - K x_1 x_2$$

$$= \frac{1}{2} \sum_{i,k} k_{ik} x_i x_k$$

$$T = \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$= \frac{1}{2} \left[m_{11} \dot{x}_1^2 + m_{12} \dot{x}_1 \dot{x}_2 + m_{21} \dot{x}_2 \dot{x}_1 + m_{22} \dot{x}_2^2 \right]$$

$$= \frac{1}{2} \left[m_{11} \dot{x}_1^2 + m_{22} \dot{x}_2^2 + 2 m_{12} \dot{x}_1 \dot{x}_2 \right]$$

$$\rightarrow m_{11} = m, \quad m_{22} = m, \quad m_{12} = 0 = m_{21}$$

$$\boxed{m_{ik} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}$$

$$U = \frac{1}{2} \sum_{i,k} k_{ik} x_i x_k = \frac{1}{2} (k + K) x_1^2 + \frac{1}{2} (k + K) x_2^2 - 2 K x_1 x_2$$

$$\boxed{K_{ik} = \begin{bmatrix} k + K & -K \\ -K & k + K \end{bmatrix}}$$

$$L = T - U$$

$$= \frac{1}{2}\sum_{j,k} m_{jk}\dot{x}_j \dot{x}_k - \frac{1}{2}\sum_{j,k} k_{jk} x_j x_k$$

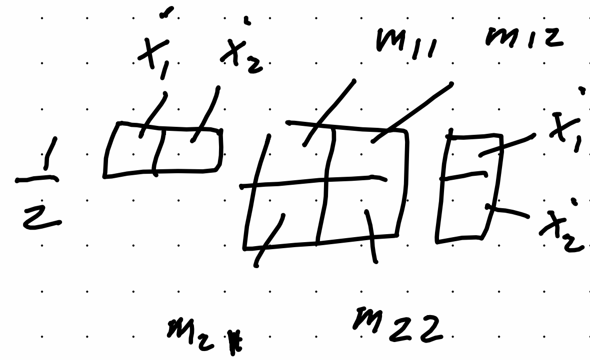
δ_{ji}

δ_{ki}

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \qquad (i = 1, 2, \cdots N)$$

$$\boxed{\underbrace{\sum_k m_{ik}\ddot{x}_k}_{} = -\sum_k k_{ik} x_k}$$

$$\underset{m}{\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}} \underset{\dot{x}}{\begin{pmatrix} \\ \\ \end{pmatrix}} \begin{matrix} \dot{x}_1 \\ \dot{x}_2 \end{matrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$$



$$\|$$

$$\frac{1}{2}\sum_{i,k} m_{ik}\dot{x}_i \dot{x}_k$$

$$\frac{1}{2}\sum_{j,l} m_{jl}\dot{x}_j \dot{x}_l$$

$$\frac{\partial}{\partial x_1}\left(k_{11}x_1^2 + k_{22}x_2^2 + 2k_{12}x_1x_2\right)\left(-\frac{1}{2}\right)$$

$$= -\left(k_{11}x_1 + k_{12}x_2\right) = -\sum_k k_{1k} x_k$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = \frac{\partial L}{\partial x_2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$

$i = 1, 2$

$$\sum_k m_{ik}\, \ddot{x}_k = -\sum_k k_{ik}\, x_k$$

$$\sum_k \left(m_{ik}\, \ddot{x}_k + k_{ik}\, x_k \right) = 0$$

<u>Trial solution:</u>

$$x_k = A_k e^{i\omega t}$$

$$\ddot{x}_k = -\omega^2 A_k e^{i\omega t}$$

0 vector

$$\sum_k \big(\underbrace{-m_{ik}\,\omega^2 + k_{ik}}_{\text{matrix}}\big) A_k e^{i\omega t} = 0$$

$$\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix} \begin{bmatrix} \; \\ \; \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$M^{-1}\left(M\,\underline{v} = \begin{bmatrix}0\\0\end{bmatrix}\right)$$

$$\underline{v} = M^{-1}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$K_{ik} - \omega^2 m_{ik}$ must be a singular matrix

$$\boxed{\det(K_{ik} - \omega^2 m_{ik}) = 0}$$

characteristic equation for angular freq

eigenvalues $\rightarrow$ $\omega_1^2, \omega_2^2, \dots, \omega_N^2$

(normal mode freqs)

eigenvector $\rightarrow$ $\underline{v_1}, \underline{v_2}, \dots, \underline{v_N}$

(normal mode oscillation)

$$\det\left(\begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \right) = 0$$

$$\det \begin{bmatrix} (k+K)-\omega^2 m & -K \\ -K & (k+K)-\omega^2 m \end{bmatrix} = 0$$

$$\oslash\, \omega^4 + \oslash\, \omega^2 + \oslash = 0$$

$\longrightarrow$ quadratic equation for ω^2 $(\equiv \lambda)$

$$\oslash\, \lambda^2 + \oslash\, \lambda + \oslash = 0$$

solve quadratic equation: $\quad \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\boxed{\begin{aligned} \omega_+^2 &= \frac{k+2K}{m} \\ \omega_-^2 &= \frac{k}{m} \end{aligned}}$$

$\underline{\omega_+^2}$:

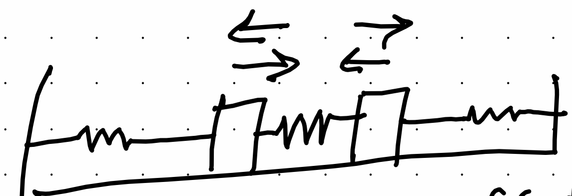
$$\begin{pmatrix} k+K-m\omega_+^2 & -K \\ -K & k+K-m\omega_+^2 \end{pmatrix} \underbrace{\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}}_{} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

V_+ : eigenvector assoc. with ω_+

$A_2 = -A_1$

$$\boxed{V_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Normal mode oscillation



unaffected

$\underline{\omega_-^2}$:

$$\boxed{V_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

complex constant determined by initial conditions

General solution:

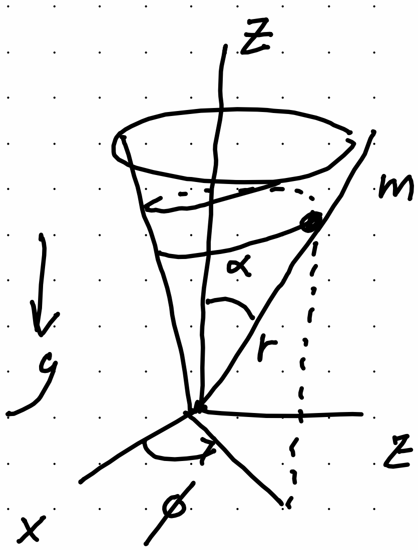
$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \mathrm{Re}\left(C_+ V_+ e^{i\omega_+ t} + C_- V_- e^{i\omega_- t} \right)$$

Lec 22: Nov 5^{th}

- QUIZ #4 today (QUIZ #5 - next thursday
  QUIZ #6 - last day of class)

- Rigid body motion: Sec 31-36
  Non-inertial ref frames: Sec 38, 39

Q4: name_q4.pdf



a) write down the Lagrangian

b) Find r_0 for a stable circular orbit for fixed angular momentum M_z.

c) Find angular freq of small radial oscillations about r_0.

$z = r\cos\alpha$, $T = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2 + \cancel{r^2\dot{\theta}^2})$

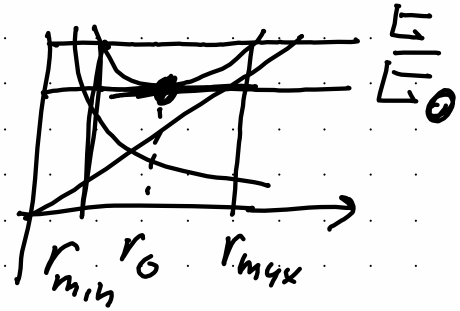
$$E = \frac{1}{2} m\dot{r}^2 + \underbrace{U_{eff}(r)}$$

$$\frac{M_z^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha$$

$M_z^2 = m^2 g r_0^3 \cos\alpha \sin^2\alpha$

$M_z = mr^2\sin^2\alpha\,\dot\phi$

U_{eff}



$\omega = \sqrt{\frac{K}{m}}$

$x = q - q_0$

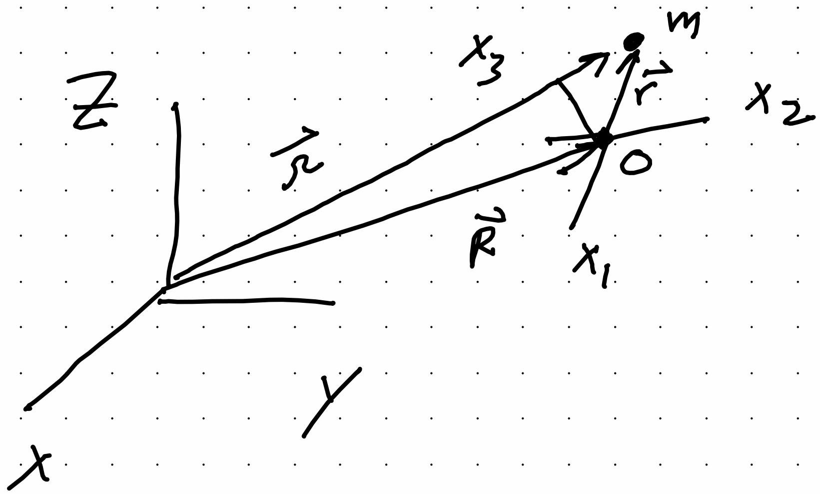
$$K \equiv \left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} = \frac{3mg\cos\alpha}{r_0}$$

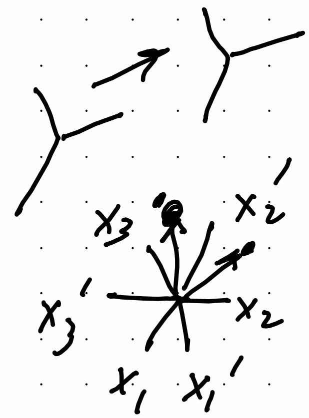
r_0

$$\omega_r = \sqrt{\frac{K}{m}} = \boxed{\sqrt{\frac{3g\cos\alpha}{r_0}}}$$

$$\dot\phi = \frac{M_z}{mr^2\sin^2\alpha} = \frac{M_z}{mr_0^2\sin^2\alpha} = \frac{\sqrt{\cancel{m^2}\, g r_0^3 \cos\alpha \sin^2\alpha}}{\sqrt{\cancel{m^2}\, r_0^4 \sin^4\alpha}}$$

$$\omega_\phi = \sqrt{\frac{g}{r_0}\frac{\cos\alpha}{\sin^2\alpha}} \neq \omega_r$$





$\vec V$, $\vec\Omega$

linear velocity — depend on O

ang velocity — does not depend on O

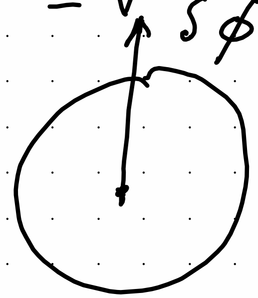
angular velocity vector

$\vec\rho = \vec R + \vec r$

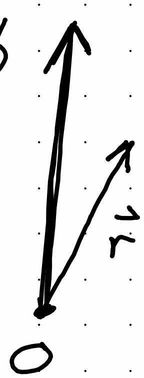
$$\boxed{\vec v = \frac{d\vec\rho}{dt} = \vec V + \underbrace{\vec\Omega \times \vec r}_{\text{rotational velocity}}}$$

translational velocity of O

$\vec R \to \vec R + d\vec R$, $\dfrac{d\vec R}{dt} = \vec V \, \delta\vec\phi$

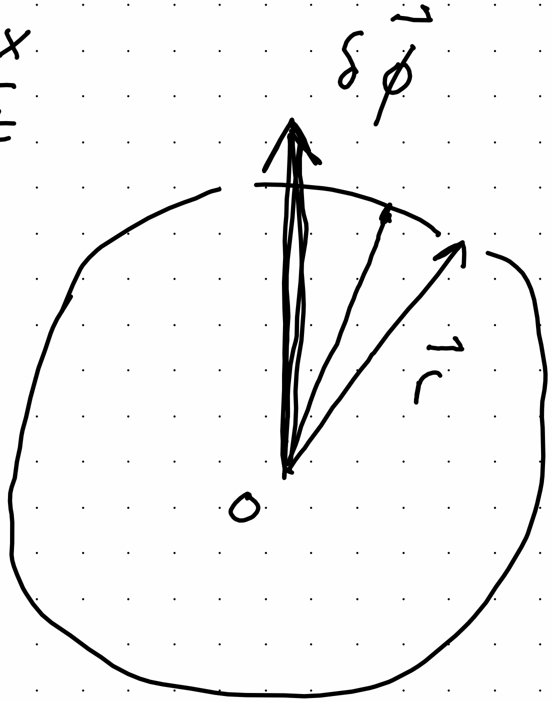


$\delta\vec\phi$

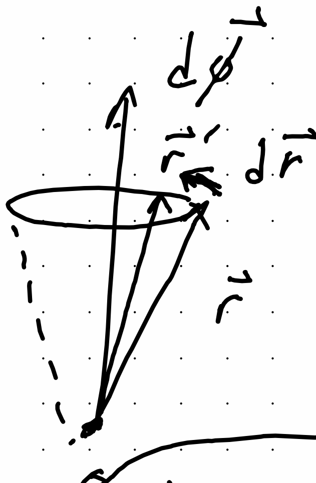


infinitesimal rotation about an axis through an infinitesimal angle $\delta\phi$

$\boxed{V} = \frac{dx}{dt}$

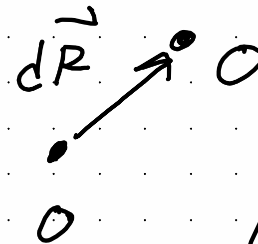


$\frac{\to 0}{\to 0}$



$$\boxed{\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}}$$

$\vec{V} = \frac{d\vec{R}}{dt}$ — 3 DOF associated with Origin of body frame



(ϕ, θ, ψ)

$d\vec{R}$ $\quad \vec{\phi}$: 3 DOF assoc with orientation of body frame wrt inertial frame

$d\vec{r} = d\vec{\phi} \times \vec{r}$

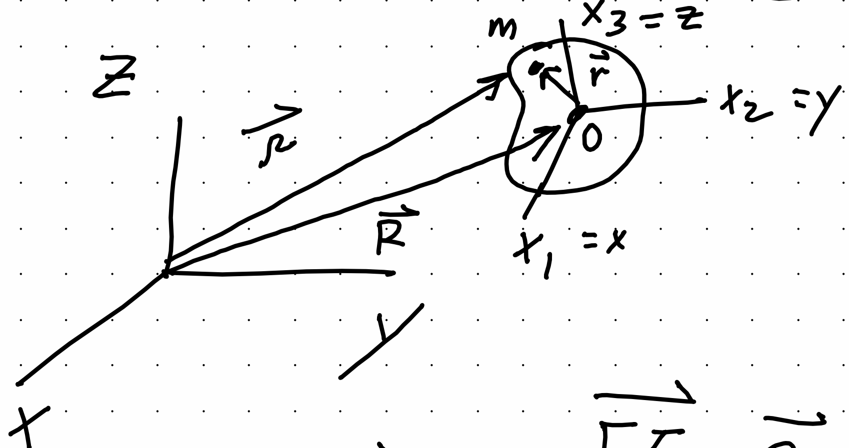
$\frac{d\vec{r}}{dt} = \frac{d\vec{\phi}}{dt} \times \vec{r}$

ω

$\vec{\Omega}$

Lec #23: Tuesday Nov 10th

- Quiz #5: Thursday
- Midterm #2: Next Thursday 11/19 (scattering, small oscillations, some rigid body)
- Today: Rigid body motion
  (Sec 31–36, 38, 39)
  - 38: static equilibrium
  - 39: non-inertial



O: at COM (usually)

x_1, x_2, x_3: fixed in RB

$$\vec{v} = \frac{d\vec{\mathfrak{r}}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}$$

$\vec{\Omega}$: angular velocity vector

$$\left.\begin{aligned} \vec{V} &= \frac{d\vec{R}}{dt} \\ \vec{\Omega} &= \frac{d\vec{\phi}}{dt} \end{aligned}\right\} \dot{q}_i$$

$(\vec{R}, \vec{\phi})$: 6 DOF $= q_i$

$$T = \frac{1}{2} m v^2 \quad , \qquad T_{rot} = \frac{1}{2} I \Omega^2$$

$$\vec{p} = m \vec{v} \qquad \vec{M} = I \vec{\Omega} \rightarrow M_i = \sum_j I_{ij} \Omega_j$$

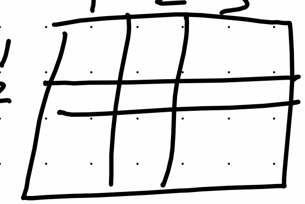
rotational quantities

I : moment of inertia

$\longrightarrow$ I_{ij} : inertia tensor

$1 \quad 2 \quad 3 \rightarrow j$

$1, 2, 3 \downarrow i$



$M_i \qquad I_{ij} \qquad \Omega_j$

KE:

$$\vec{v}_a = \vec{V} + \vec{\Omega}\times\vec{r}_a$$

$$T = \frac{1}{2}\sum_a m_a |\vec{v}_a|^2,$$

$$= \frac{1}{2}\sum_a m_a |\vec{V} + \vec{\Omega}\times\vec{r}_a|^2$$

$$|\vec{A}+\vec{B}|^2 = A^2 + B^2 + 2\vec{A}\cdot\vec{B}$$

$$= \frac{1}{2}\sum_a m_a \left(|\vec{V}|^2 + |\vec{\Omega}\times\vec{r}_a|^2 + 2\vec{V}\cdot(\vec{\Omega}\times\vec{r}_a) \right)$$

$$1 = \frac{1}{2}\sum_a m_a |\vec{V}|^2 = \boxed{\frac{1}{2}\mu V^2}$$

μ total mass

$$3 = \sum_a m_a \vec{V}\cdot(\vec{\Omega}\times\vec{r}_a)$$

$$= \left(\sum_a m_a \vec{r}_a\right)\cdot(\vec{V}\times\vec{\Omega})$$

$$= \underbrace{\mu \vec{R}_{com}}_{=0 \text{ for}} \cdot (\vec{V}\times\vec{\Omega})$$

$\boxed{O \text{ at COM}}$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

(2) $= \frac{1}{2} \sum_a m_a \left| \vec{\Omega} \times \vec{r_a} \right|^2$

$$= \frac{1}{2} \sum_a m_a \left(\vec{\Omega} \times \vec{r_a} \right) \cdot \left(\vec{\Omega} \times \vec{r_a} \right)$$

$$= \frac{1}{2} \sum_a m_a \; \vec{\Omega} \cdot \left(\vec{r_a} \times \left(\vec{\Omega} \times \vec{r_a} \right) \right)$$

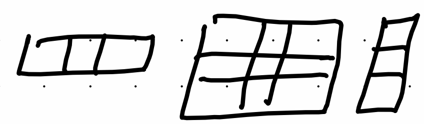
$$= \frac{1}{2} \sum_a m_a \; \vec{\Omega} \cdot \left(\vec{\Omega} \, r_a^2 - \vec{r_a} \left(\vec{r_a} \cdot \vec{\Omega} \right) \right)$$

$$= \frac{1}{2} \sum_a m_a \left(\Omega^2 r_a^2 - \left(\vec{\Omega} \cdot \vec{r_a} \right)^2 \right)$$

$$= \frac{1}{2} \sum_a m_a \left(\sum_{i,j} \Omega_i \Omega_j \delta_{ij} \, r_a^2 - \sum_{i,j} \Omega_i r_{ai} \Omega_j r_{aj} \right)$$

$$= \frac{1}{2} \sum_{i,j} \left(\sum_a m_a \left(r_a^2 \delta_{ij} - r_{ai} r_{aj} \right) \right) \Omega_i \Omega_j$$

$$= \boxed{\frac{1}{2} \sum_{i,j} I_{ij} \, \Omega_i \Omega_j} \quad \left(= \frac{1}{2} \right.$$



$)$

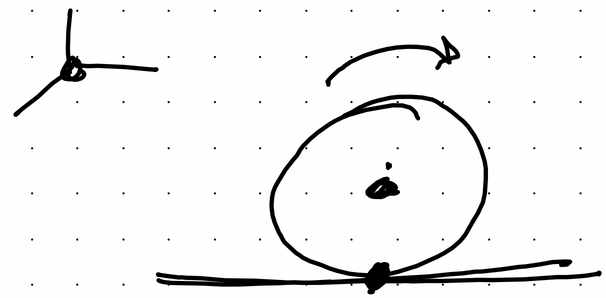
$$\vec{A} \times \left(\vec{B} \times \vec{C} \right) = \vec{B} \left(\vec{A} \cdot \vec{C} \right) - \vec{C} \left(\vec{A} \cdot \vec{B} \right)$$

$$\boxed{T = \underbrace{\tfrac{1}{2}\mu V^2}_{\text{trans}} + \underbrace{\tfrac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\Omega_j}_{\text{rotational}}} \quad +$$

$\left(\frac{1}{2} I \Omega^2\right)$ Freshman physics

for COM at origin
OF RB Frame

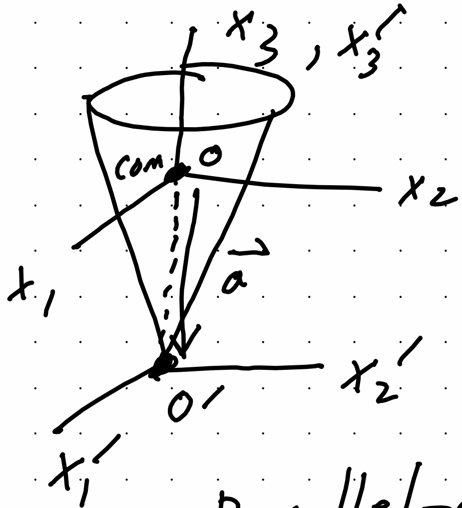
$\vec{M}$: wrt COM of body



$$\vec{M} = \sum_a \vec{r_a} \times \vec{p_a}$$
$$= \sum_a m_a \, \vec{r_a} \times \vec{v_a}$$
$$= \sum_a m_a \, \vec{r_a} \times \left(\vec{V} + \vec{\Omega} \times \vec{r_a} \right)$$
$$= \sum_a m_a \, \vec{r_a} \times \left(\vec{\Omega} \times \vec{r_a} \right)$$
$$= \sum_a m_a \left(\vec{\Omega}\, r_a^2 - \vec{r_a}\,(\vec{r_a} \cdot \vec{\Omega}) \right)$$

$$\boxed{M_i = \sum_j I_{ij}\,\Omega_j}$$

$\vec{M} = I\vec{\Omega}$ (Fresh. physics)



parallel-axis theorem

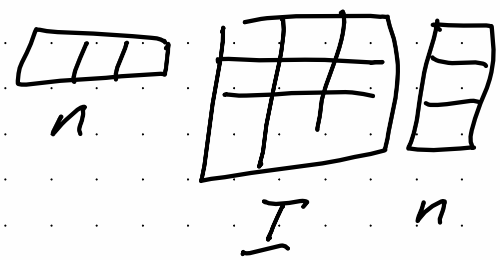
$$\underbrace{I'_{ij}}_{\text{wrt } O'}, \quad \underbrace{I_{ij}}_{\text{wrt } O \text{ (com)}}$$

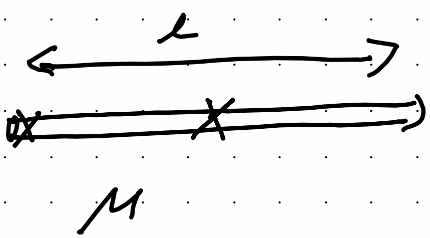
$$I'_{ij} = I_{ij} + \mu(a^2\delta_{ij} - a_i a_j)$$

$\vec{a}$: vector from O to O'

$$\underbrace{I(\hat{n})}_{\text{moment of inertia}} = \sum_{i,j} I_{ij} \underbrace{n_i n_j}$$

$\hat{n}$: axis of rotation





$$\boxed{\begin{aligned} I_{com} &= \frac{1}{12}\mu l^2 \\ I_{end} &= \frac{1}{3}\mu l^2 \end{aligned}}$$

$$\boxed{\underbrace{I_{ij}}_{\substack{3\times 3 \text{ real}\\ \text{symmetric}}} = \sum_a m_a \left(r_a^2 \delta_{ij} - r_{ai} r_{aj} \right) = \int \underbrace{\rho\, dV}_{dm} \left(r^2 \delta_{ij} - r_i r_j \right)}$$

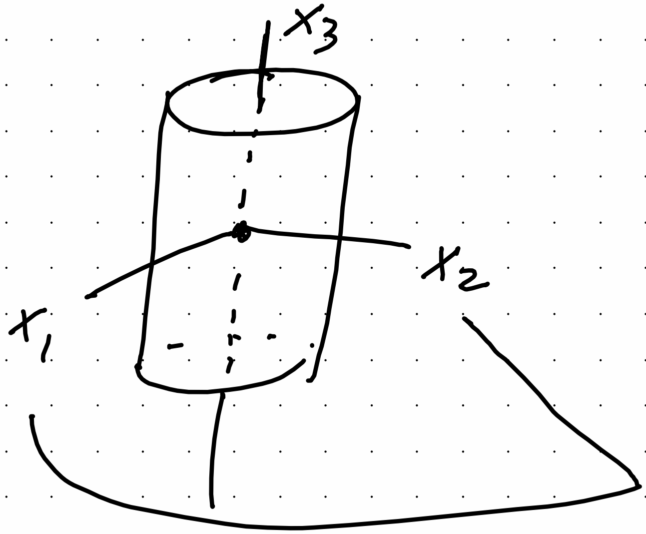
can always be diagonalized

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

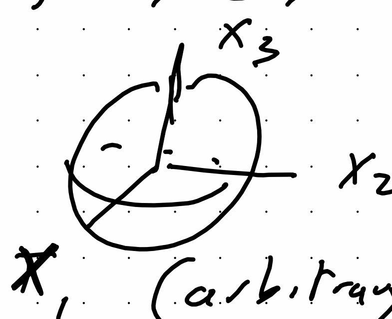
$$I_{ij} = I_i \delta_{ij}$$

$$T_{rot} = \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j = \frac{1}{2} \left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 \right)$$

<u>principle axes:</u> (x_1, x_2, x_3)



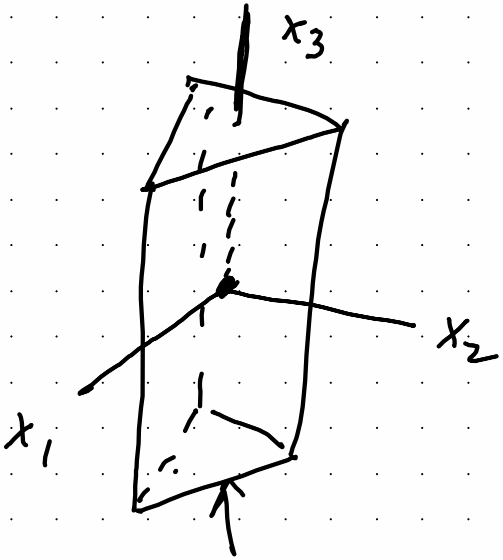
arbitrary



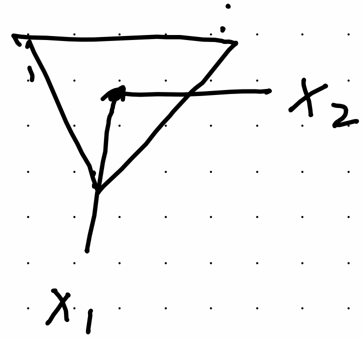
(arbitrary)

$$M_i = \sum_j I_{ij} \Omega_j$$

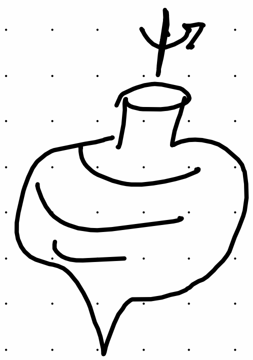
$$M_1 = I_1 \Omega_1, \quad M_2 = I_2 \Omega_2, \quad M_3 = I_3 \Omega_3$$

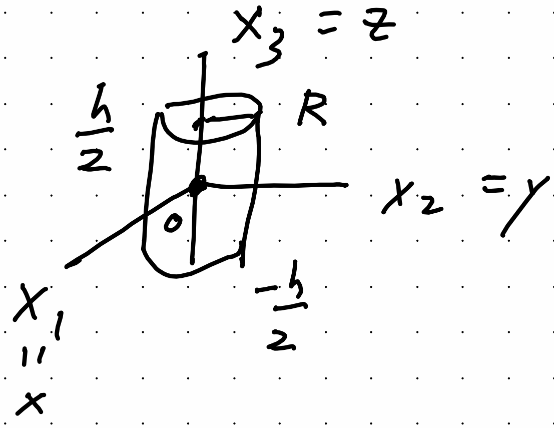


equilateral triang



$$\underbrace{I_1 = I_2 \neq I_3}_{\text{symmetrical top}}$$





total mass M

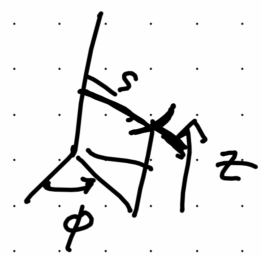
$$\rho = \frac{M}{\text{volume}} = \frac{M}{\pi R^2 h}$$

$$I_3 = I_{33} = \int \rho \, dV \left(r^2 \underbrace{\delta_{33}}_{1} - \underbrace{r_3 r_3}_{z^2} \right)$$

$$= \int \rho \, dV \left(r^2 - z^2 \right)$$

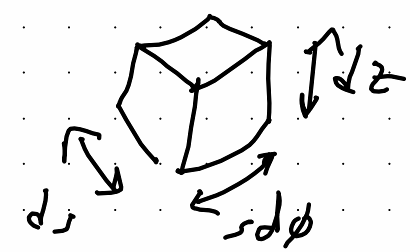
$$= \int \rho \, dV \left(x^2 + y^2 \right) = \int \rho \, dV s^2$$

<u>cylindrical:</u> s, ϕ, z $\quad s^2 = x^2 + y^2$



$$dV = ds \; s\,d\phi \; dz$$

$$= s\,ds\,d\phi\,dz$$

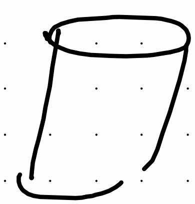


$$I_3 = \int \rho \, dV \, s^2$$

$$= \frac{M}{\pi R^2 h} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-h/2}^{h/2} dz}_{h} \underbrace{\int_0^{R} s^3 \, ds}_{\frac{R^4}{4}}$$

$$= \frac{M}{\cancel{\pi} \cancel{R^2} \cancel{h}} \cdot 2\cancel{\pi}\,\cancel{h}\,\frac{R^{\cancel{4}\,2}}{4}$$

$$= \boxed{\frac{1}{2} M R^2}$$



$$I_1 = I_2 \equiv I$$

$$I_1 = \int \rho \, dV (r^2 - x^2)$$

$$+ \; I_2 = \int \rho \, dV (r^2 - y^2)$$

$$2I = \int \rho \, dV \, (2r^2 - x^2 - y^2)$$

$$\left(\begin{aligned} r^2 &= s^2 + z^2 \\ x^2 + y^2 &= s^2 \end{aligned} \right)$$

$$2I = \int \rho \, dV \, (s^2 + 2z^2)$$

$$\boxed{I = \frac{1}{2} \underbrace{\int \rho \, dV \, s^2}_{I_3} + \int \rho \, dV \, z^2 = \frac{1}{2} I_3 + \int \rho \, dV \, z^2}$$

easier to evaluate

$$\int \rho \, dV \, z^2 = \frac{M}{\pi R^2 h} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-h/2}^{h/2} dz \cdot z^2}_{\left.\frac{z^3}{3}\right|_{-h/2}^{h/2}} \underbrace{\int_0^R s\,ds}_{\frac{R^2}{2}}$$

$$\left.\frac{z^3}{3}\right|_{-h/2}^{h/2} = \frac{2}{3}\left(\frac{h}{2}\right)^3 = \frac{h^3}{12}$$

$$= \frac{M}{\cancel{\pi} \cancel{R^2} h} \cdot \cancel{2}\cancel{\pi} \cdot \frac{h^3}{12} \cdot \frac{\cancel{R^2}}{\cancel{2}}$$

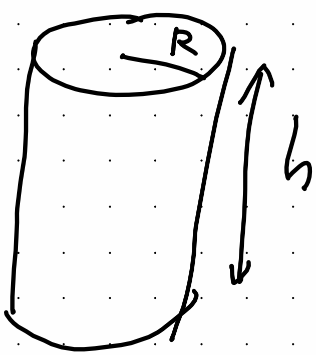
$$= \boxed{\frac{M h^2}{12}}$$

$$I = \frac{1}{2}\left(\frac{1}{2}\mu R^2\right) + \frac{\mu h^2}{12}$$

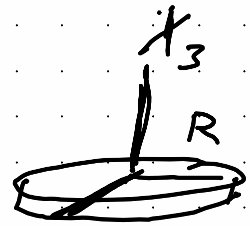
$$= \frac{1}{4}\mu R^2 + \frac{1}{12}\mu h^2$$

$$= \frac{1}{4}\mu\left(R^2 + \frac{1}{3}h^2\right) = I_1, I_2$$

$$I_3 = \frac{1}{2}\mu R^2$$



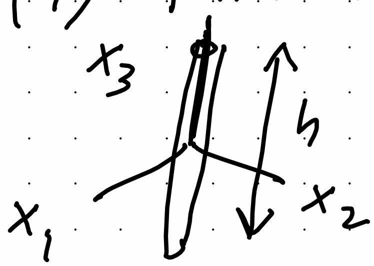
<u>Limiting cases</u>: (i) Disk ($h \to 0$)



$I_3 = \frac{1}{2}\mu R^2$

$I_1, I_2 = \frac{1}{4}\mu R^2$

(i) thin rod ($R \to 0$)



$I_3 = 0$

$I_1 = I_2 = \frac{1}{12}\mu h^2$

$$L = T - U$$

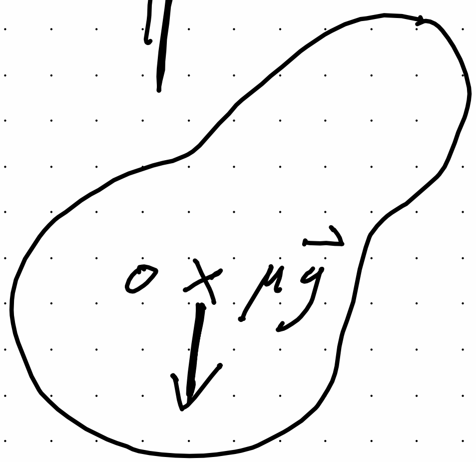
$$= \frac{1}{2}\mu V^2 + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\Omega_j - U$$

$$\vec{V} = \dot{\vec{R}} \qquad L(\vec{R}, \vec{\phi}, \dot{\vec{R}}, \dot{\vec{\phi}})$$

$$\vec{\Omega} = \dot{\vec{\phi}}$$

$\vec{g} \downarrow$

uniform
Field

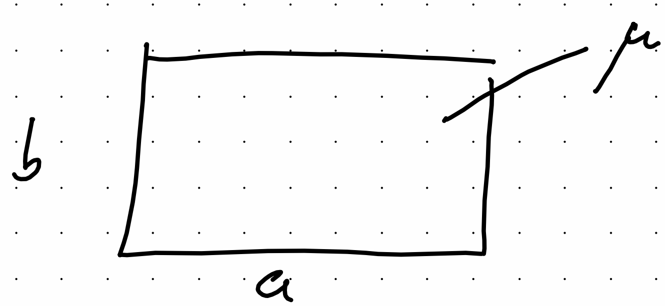


Lecture #24: Thursday 11/12

- Quiz #5 (today)
- Midterm #2 (next Thursday) (scattering, small oscillations, RB motion)
- Today's topics:
  (1) RB EOMs
  (2) Euler's equations
  (3) Euler angles

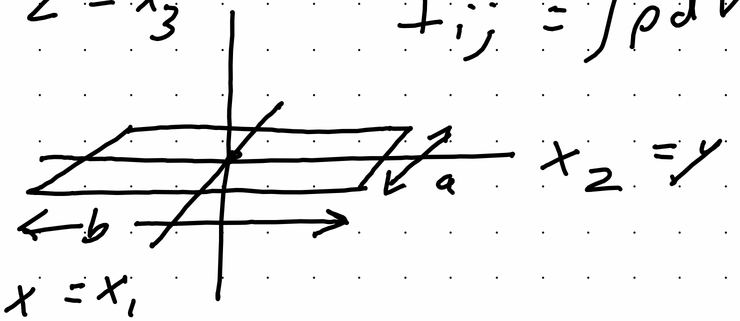
Q5: Calculate the principal moments of inertia for a 2-d rectangle with sidelengths a, b.

uniform



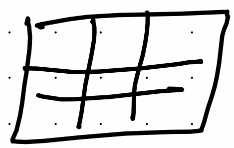
name_q5.pdf

$z = x_3$



$$I_{ij} = \int \rho\, dV \left(r^2 \delta_{ij} - r_i r_j \right)$$

$$r^2 = x^2 + y^2 + z^2$$



$$dm = \rho\, dV = \sigma\, dx\, dy = \frac{M}{ab}\, dx\, dy$$

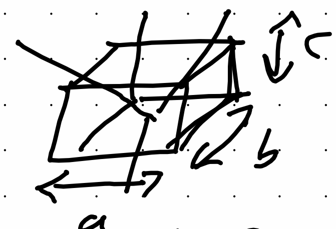
$$I_1 = \int \rho\, \underline{dV} \left(r^2 - x^2 \right)$$

$\parallel$

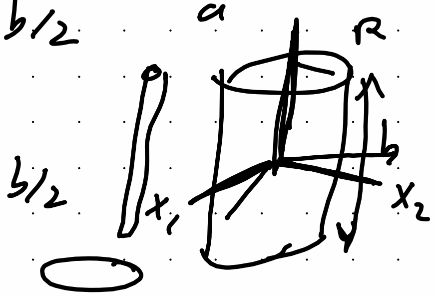
I_{11}

$$= \int \rho\, dV\, y^2$$

$$= \frac{M}{ab} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\, y^2$$



$$= \frac{M}{\cancel{a}b}\, \cancel{a}\, \left. \frac{y^3}{3} \right|_{-b/2}^{b/2}$$



$$= \frac{M}{b} \frac{2}{3} \left(\frac{b}{2} \right)^3$$

$$= \frac{M}{b} \frac{2}{3} \frac{b^3}{8} = \boxed{\frac{1}{12} M b^2}$$

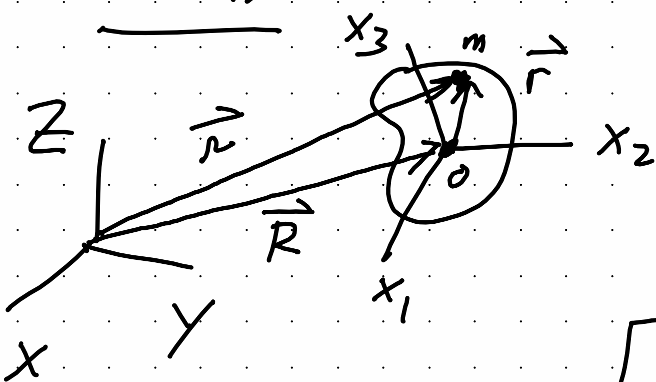
$$\boxed{I_2 = \frac{1}{12} M a^2}$$

$$I_3 = \int \sigma\, dx\, dy \left(\underline{x^2} + \underline{y^2} \right)$$

$$= I_1 + I_2$$

$$\boxed{I_3 = \frac{1}{12} M \left(a^2 + b^2 \right)}$$

EOMs:

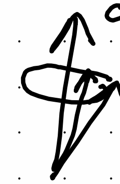


Final result:

$$\frac{d\vec P}{dt} = \vec F = \sum \vec f$$

$$\frac{d\vec M}{dt} = \vec K = \sum \vec r \times \vec f$$

$$\vec \rho = \vec R + \vec r \, , \quad \boxed{\delta \vec \rho = \delta \vec R + \underbrace{\delta \vec \phi}_{\delta \Omega_k} \times \vec r}$$



$$L = T - U$$

$$= \frac{1}{2}\mu |\vec V|^2 + \frac{1}{2}\sum_{i,j} I_{ij}\Omega_i \Omega_j - U(\vec \rho)$$

$$\delta L = \mu \vec V \cdot \delta \vec V + \sum_{i,j} I_{ij}\Omega_j \delta\Omega_i \boxed{- \sum \frac{\partial U}{\partial \vec \rho} \cdot \delta \vec \rho}$$

$$- \sum \frac{\partial U}{\partial \vec \rho} \cdot \left(\delta \vec R + \delta \vec \phi \times \vec r\right)$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \vec V}\right) = \frac{\partial L}{\partial \vec R}} \, , \quad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \vec \Omega}\right) = \frac{\partial L}{\partial \vec \phi}}$$

$$\frac{1}{2}\sum_{i,j} I_{ij}\,\underline{\delta\Omega_i}\,\Omega_j + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\delta\Omega_j$$

$$= \quad '' \quad + \frac{1}{2}\sum_{j,i} I_{ji}\,\Omega_j\,\underline{\delta\Omega_i}$$

$$= \quad '' \quad + \frac{1}{2}\underbrace{\sum_{j,i}}_{\text{swap}} I_{ij}\,\Omega_j\,\delta\Omega_i$$

$$= \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i$$

$$= \boxed{\sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i}$$

$$\delta L = m\vec{V}\cdot\underline{\delta\vec{V}} + \sum_{ij} I_{ij}\,\Omega_j\,\delta\Omega_i$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) \;/\; \vec{B}\cdot(\vec{C}\times\vec{A})$$

$$-\left(\sum \frac{\partial U}{\partial\vec{r}}\right)\cdot\delta\vec{R} \; - \underbrace{\sum \frac{\partial U}{\partial\vec{r}}\cdot(\delta\vec{\phi}\times\vec{r})}$$

$$= -\sum \delta\vec{\phi}\cdot\left(\vec{r}\times\frac{\partial U}{\partial\vec{r}}\right)$$

$$= -\delta\vec{\phi}\cdot\left(\sum \vec{r}\times\frac{\partial U}{\partial\vec{r}}\right)$$

$$\sum_i M_i\,\delta\Omega_i = \vec{M}\cdot\delta\vec{\Omega}$$

$$dF = \,\,dx + \,\,dy$$

$$\boxed{\delta L = \vec{P}\cdot\delta\vec{V} + \vec{M}\cdot\delta\vec{\Omega} + \left(\sum\vec{f}\right)\cdot\delta\vec{R} + \delta\vec{\phi}\cdot\left(\sum\vec{r}\times\vec{f}\right)}$$

$$\vec{p} = \frac{\partial L}{\partial\vec{V}}, \quad \vec{M} = \frac{\partial L}{\partial\vec{\Omega}}, \quad \sum\vec{f} = \frac{\partial L}{\partial\vec{R}}, \quad \sum\vec{r}\times\vec{f} = \frac{\partial L}{\partial\vec{\phi}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\vec{V}}\right) = \frac{\partial L}{\partial\vec{R}} \rightarrow \boxed{\frac{d\vec{p}}{dt} = \sum\vec{f} \equiv \vec{F}}$$

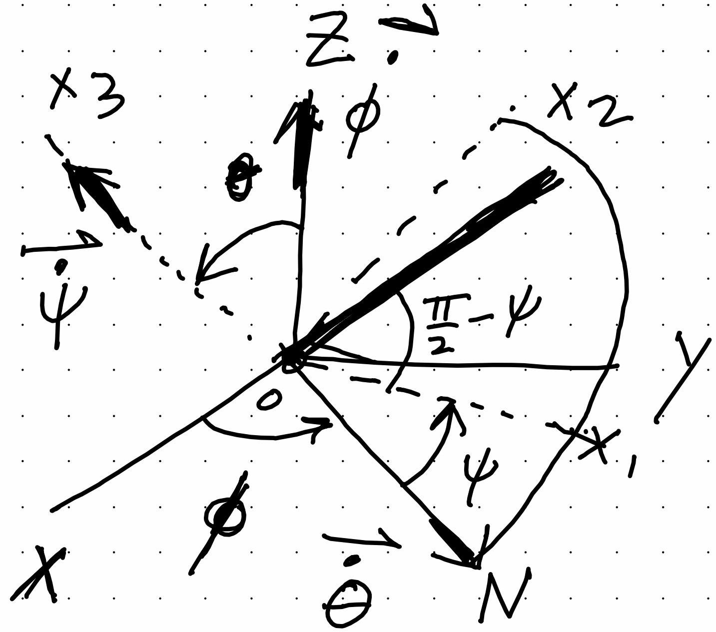
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\vec{\Omega}}\right) = \frac{\partial L}{\partial\vec{\phi}} \rightarrow \boxed{\frac{d\vec{M}}{dt} = \sum\vec{r}\times\vec{f} = \vec{K}}$$

Euler's equations / Euler angles ☆

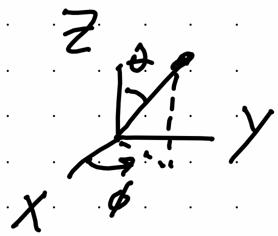
$\vec{R}, \vec{\phi}$

Ω_i : ith component of $\vec{\Omega}$

L wrt $\hat{x}_i$



(line of node)



$$\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta} + \vec{\dot\psi}$$

$$\vec{\dot\psi} = \dot\psi\, \hat{x}_3$$

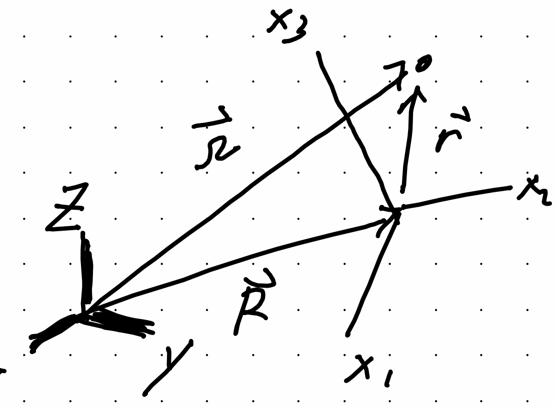
$$\vec{\dot\theta} = \dot\theta \cos\psi\, \hat{x}_1 - \dot\theta \sin\psi\, \hat{x}_2$$

$$\vec{\dot\phi} = \dot\phi \cos\theta\, \hat{x}_3 + \dot\phi \sin\theta \left(\underbrace{\sin\psi}_{\cos(\frac{\pi}{2}-\psi)} \hat{x}_1 + \underbrace{\cos\psi}_{\sin(\frac{\pi}{2}-\psi)} \hat{x}_2 \right)$$

Announcements

- Midterm II is this thursday
- Today:
  - i) Euler angles
  - ii) Euler's equation for RB motion
  - iii) Free rotation with $\vec{\Omega} = \text{const}$
  - iv) " of a symmetric top ($I_1 = I_2$)
  - v) Heavy symmetrical top with one point fixed [prob 35.1]

$$\vec{\Omega} = (\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi)\, \hat{x}_1 + (\dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi)\, \hat{x}_2 + (\dot{\psi} + \dot{\phi} \cos\theta)\, \hat{x}_3$$



$$\boxed{\begin{aligned} \Omega_1 &= \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\ \Omega_2 &= \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\ \Omega_3 &= \dot{\psi} + \dot{\phi} \cos\theta \end{aligned}}$$

$$\vec{v} = \frac{d\vec{\mathcal{R}}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r} \quad \cancel{+ \frac{d\vec{r}}{dt}}$$

$$\frac{d'\vec{r}}{dt}$$ wrt RB

Euler's equations:

$$\frac{d\vec{P}}{dt} = \sum \vec{f} = \vec{F}$$

$$\frac{d\vec{M}}{dt} = \sum \vec{r} \times \vec{f} = \vec{K}$$

(wrt RB axes) $\vec{A}$: any vector

$$\boxed{\underbrace{\frac{d\vec{A}}{dt}}_{\text{wrt inertial frame}} = \underbrace{\frac{d'\vec{A}}{dt}}_{\text{wrt rotating frame}} + \underbrace{\vec{\Omega}}_{\text{angular velocity of the rotating frame}} \times \vec{A}}$$

$$\left(\frac{d'\vec{A}}{dt}\right)_i = \frac{dA_i}{dt} \equiv \dot{A}_i$$

cartesian components wrt rotating frame

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$$

$$\cancel{= \ddot{r}\hat{r}}$$

$$\vec{A} = \sum_i \underline{\underline{A_i}}\ \underline{\underline{\hat{x}_i}}$$

$$\frac{d\vec{A}}{dt} = \underbrace{\sum_i \left(\frac{dA_i}{dt}\right)\hat{x}_i}_{\frac{d'\vec{A}}{dt}} + \underbrace{\sum_i A_i \frac{d\hat{x}_i}{dt}}_{\vec{\Omega}\times\vec{A}}$$

$$\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\Omega}\times\vec{A} \quad , \quad \left(\frac{d'\vec{A}}{dt}\right)_i = \dot{A}_i$$

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d'\vec{P}}{dt} + \vec{\Omega}\times\vec{P}$$

$$\vec{P} = \mu\vec{V}$$

$$\rightarrow \boxed{F_1} = \dot{P}_1 + (\vec{\Omega}\times\vec{P})_1$$

$$= \dot{P}_1 + \Omega_2 P_3 - \Omega_3 P_2$$

$$\boxed{= \mu(\dot{V}_1 + \Omega_2 V_3 - \Omega_3 V_2)}$$

similar equations for F_2, F_3

$$\boxed{\vec{K} = \frac{d\vec{M}}{dt}} = \frac{d'\vec{M}}{dt} + \vec{\Omega}\times\vec{M}$$

$$M_i = I_i \Omega_i$$

$$K_x = \dot{M}_x$$

$$\boxed{K_1} = \dot{M}_1 + \Omega_2 M_3 - \Omega_3 M_2$$

$$= I_1\dot{\Omega}_1 + \Omega_2 I_3 \Omega_3 - \Omega_3 I_2 \Omega_2$$

$$\boxed{= I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_2)}$$

*(similar equations for K_2, K_3)

Free rotation: $K_i = 0, \quad F_i = 0$

$$0 = I_1 \dot{\Omega}_1 + \Omega_2 \Omega_3 (I_3 - I_2), \ \cancel{etc.}$$
$$0 = I_2 \dot{\Omega}_2 + \Omega_3 \Omega_1 (I_1 - I_3)$$
$$0 = I_3 \dot{\Omega}_3 + \Omega_1 \Omega_2 (I_2 - I_1)$$

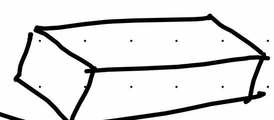
Free rotation with $\vec{\Omega}$ = const:

$$0 = \Omega_2 \Omega_3 (I_3 - I_2)$$
$$0 = \Omega_3 \Omega_1 (I_1 - I_3)$$
$$0 = \Omega_1 \Omega_2 (I_2 - I_1)$$

$$\frac{d\vec{\Omega}}{dt} = 0$$
$$\frac{d'\vec{\Omega}}{dt} = 0$$
$$\boxed{\frac{d\vec{\Omega}}{dt} = \frac{d'\vec{\Omega}}{dt}} + \cancel{\vec{\Omega} \times \vec{\Omega}}$$
$$\frac{d\vec{M}}{dt} \neq \frac{d'\vec{M}}{dt}$$

$\boxed{\Omega_1 = \text{const}, \ \Omega_2 = 0, \ \Omega_3 = 0}$ ← stable

$I_1 < I_2 < I_3$



unstable

$\boxed{\Omega_2 = \text{const}, \ \Omega_1 = 0, \ \Omega_3 = 0}$

$\boxed{\Omega_3 = \text{const}, \ \Omega_1 = 0, \ \Omega_2 = 0}$

$\Omega_1 = \text{const}, \quad \Omega_2 = 0, \quad \Omega_3 = 0 \quad$: exact

$$\boxed{\begin{aligned} \Omega_1 &= \text{const} + \epsilon_1 \\ \Omega_2 &= \epsilon_2 \\ \Omega_3 &= \epsilon_3 \end{aligned}}$$

$\epsilon_{1,2,3}$: small time dependent perturbations

Keep 0th and 1st order terms.
Ignore 2nd order, e.g. $\epsilon_2 \epsilon_3$

$$0 = I_1 \frac{d}{dt}(\underset{\to 0}{\text{const}} + \epsilon_1) + \underbrace{\epsilon_2 \epsilon_3 (I_3 - I_2)}_{\text{2nd order} \to \text{ignore}}$$

$$\approx I \dot{\epsilon}_1$$

$$\to \epsilon_1 = \text{const} \to \Omega_1 = \text{const}$$

$$\left.\begin{aligned} 0 &= I_2 \dot{\epsilon}_2 + \epsilon_3 \Omega_1 (I_1 - I_3) \\ 0 &= I_3 \dot{\epsilon}_3 + \Omega_1 \epsilon_2 (I_2 - I_1) \end{aligned}\right\}$$

coupled 1st order diff. equations

Differentiate

$$0 = I_2 \ddot{\epsilon}_2 + \dot{\epsilon}_3 \Omega_1 (I_1 - I_3)$$

$$= I_2 \ddot{\epsilon}_2 - \frac{\Omega_1 \epsilon_2 (I_2 - I_1) \Omega_1 (I_1 - I_3)}{I_3}$$

$$I_1 < I_2 < I_3$$

$$0 = I_2 \ddot{\epsilon}_2 + \dot{\epsilon}_3 \Omega_1 (I_1 - I_3)$$

$$= I_2 \ddot{\epsilon}_2 - \frac{\Omega_1 \epsilon_2 (I_2 - I_1)}{I_3} \Omega_1 (I_1 - I_3)$$

$$= \ddot{\epsilon}_2 + \underbrace{\frac{\Omega_1^2 (I_2 - I_1)(I_3 - I_1)}{I_2 I_3}}_{\omega^2} \epsilon_2$$

$$\boxed{\begin{matrix}(I_3 - I_2) \\ (I_1 - I_2)\end{matrix}}$$

$$= \ddot{\epsilon}_2 + \omega^2 \epsilon_2$$

$$\ddot{\epsilon}_2 = -\omega^2 \epsilon_2 \longrightarrow \text{SHM}$$

$$\epsilon_2 = A \cos \omega t + B \sin \omega t = C \cos(\omega t + \alpha)$$

Similarly,

$$\ddot{\epsilon}_3 = -\omega^2 \epsilon_3 \longrightarrow \epsilon_3 = D \cos(\omega t + \beta)$$

(ϵ_2, ϵ_3 are bound by their initial deviations away from 0)

For $\Omega_2 = \text{const}$ solution

perturbations

$$\ddot{\epsilon}_1 = +\omega^2 \epsilon_1$$

$$\ddot{\epsilon}_3 = +\omega^2 \epsilon_3$$

$$\rightarrow \quad \epsilon_1(t) = A \underbrace{e^{\omega t}}_{\text{grows exponentially}} + B \underbrace{e^{-i\omega t}}_{\text{damped exponentially}}$$

$\rightarrow$ instability