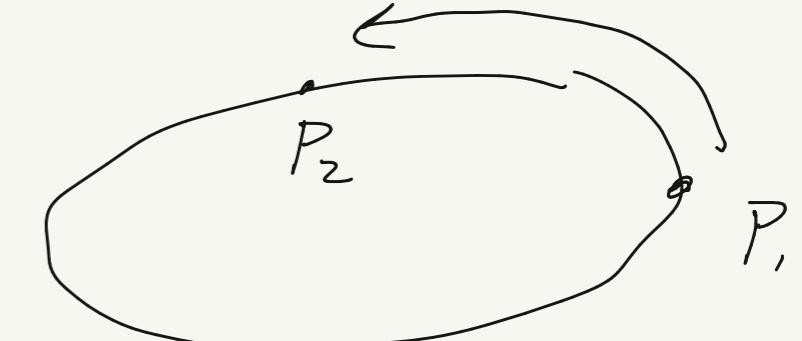


Lecture #1: Aug 24th

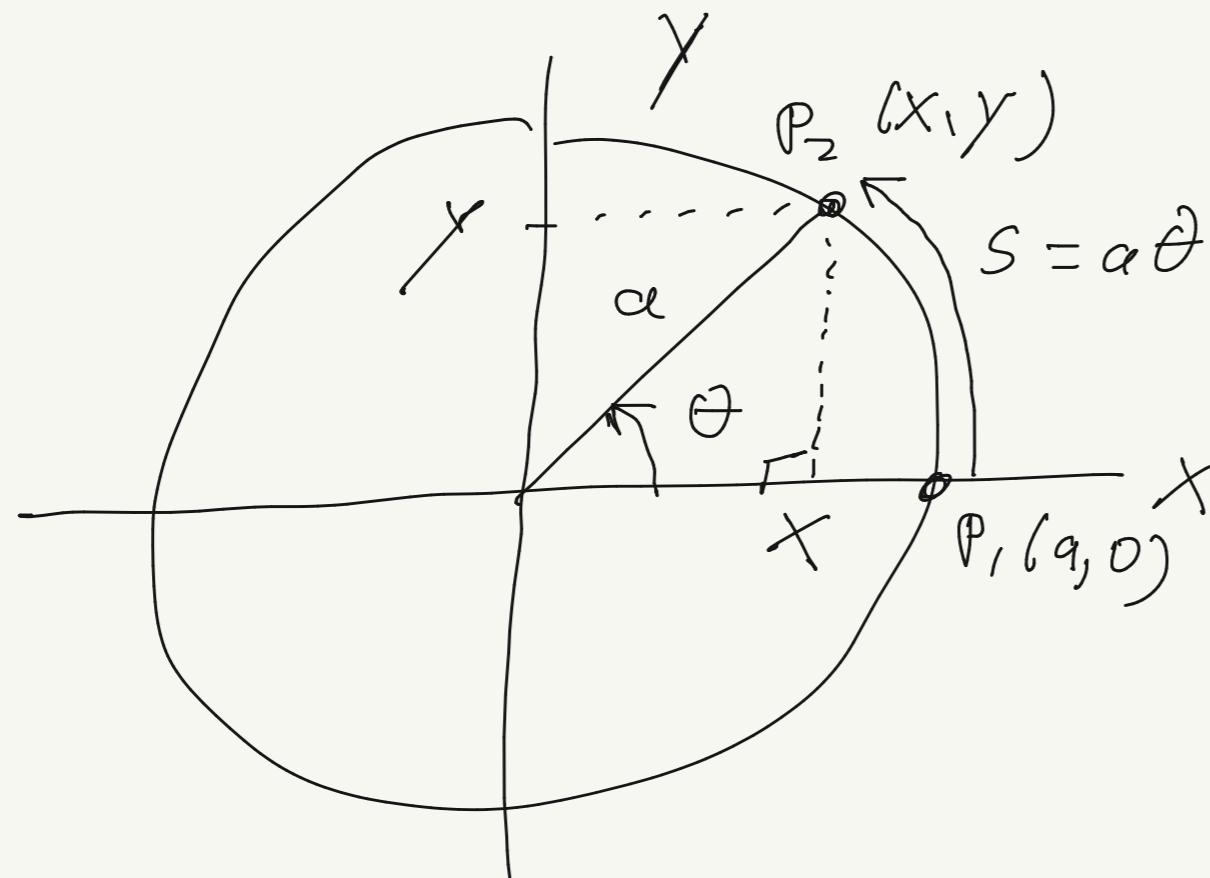
Elliptic functions / integrals:

i) circumference of an ellipse

ii) period of a simple pendulum beyond the small-angle approximation



Circular functions: sines, cosines,



$a = \text{radius}$

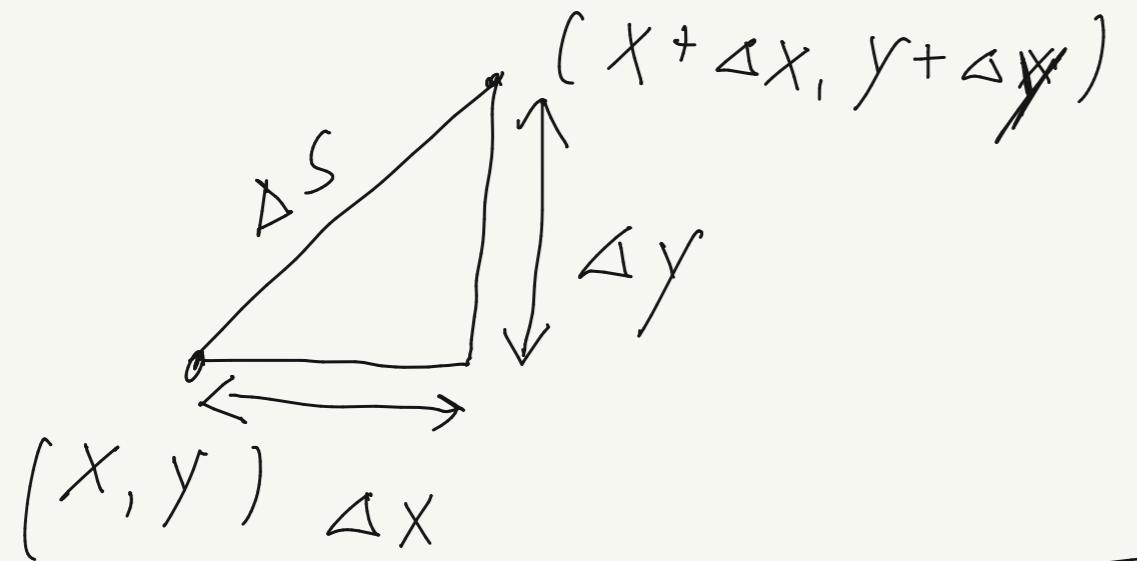
$$x^2 + y^2 = a^2$$

$$\theta = \frac{s}{a}$$

$$= \frac{1}{a} \int \sqrt{dx^2 + dy^2}$$

$$ds^2 = dx^2 + dy^2$$

$$= \int d\theta \Big|_{P_1}^{P_2}$$



$$x = a \cos \theta$$

$$y = a \sin \theta$$

$$\begin{aligned} \sin \theta &= y/a \\ \cos \theta &= x/a \end{aligned} \quad \left. \begin{array}{l} \text{def. of } \sin \theta \\ \text{def. of } \cos \theta \end{array} \right\}$$

$$\boxed{x^2 + y^2 = a^2} \rightarrow \boxed{\cancel{a^2}(\cos^2 \theta + \sin^2 \theta) = a^2} \\ \boxed{\cos^2 \theta + \sin^2 \theta = 1}$$

$$\boxed{\frac{d}{d\theta} \sin \theta = \cos \theta}$$

$$\begin{aligned} \text{Proof: } \frac{d}{d\theta} \sin \theta &= \frac{d}{d\theta} \left(\frac{y}{a} \right) \\ &= \frac{1}{a} \frac{dy}{d\theta} \\ &= \frac{dy}{\sqrt{dx^2 + dy^2}} \\ &= \frac{1}{\sqrt{dx^2 + dy^2}} \quad \leftarrow \sqrt{\left(\frac{dx}{dy} \right)^2 + 1} \end{aligned}$$

$$\begin{aligned} d\theta &= ds \\ ad\theta &= \sqrt{dx^2 + dy^2} \end{aligned}$$

$$x^2 + y^2 = a^2 \rightarrow 2x dx + 2y dy = 0$$

$$\frac{dx}{dy} = -\frac{y}{x}$$

$$\frac{d}{d\theta} \sin \theta = \frac{1}{\sqrt{\left(\frac{y}{x}\right)^2 + 1}} = \frac{1}{\sqrt{\frac{x^2}{x^2} + 1}} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{a} = \cos \theta$$

$$\boxed{\frac{d}{d\theta} \sin \theta = \cos \theta}$$

Similarly

$$\boxed{\frac{d \cos \theta}{d \theta} = -\sin \theta}$$

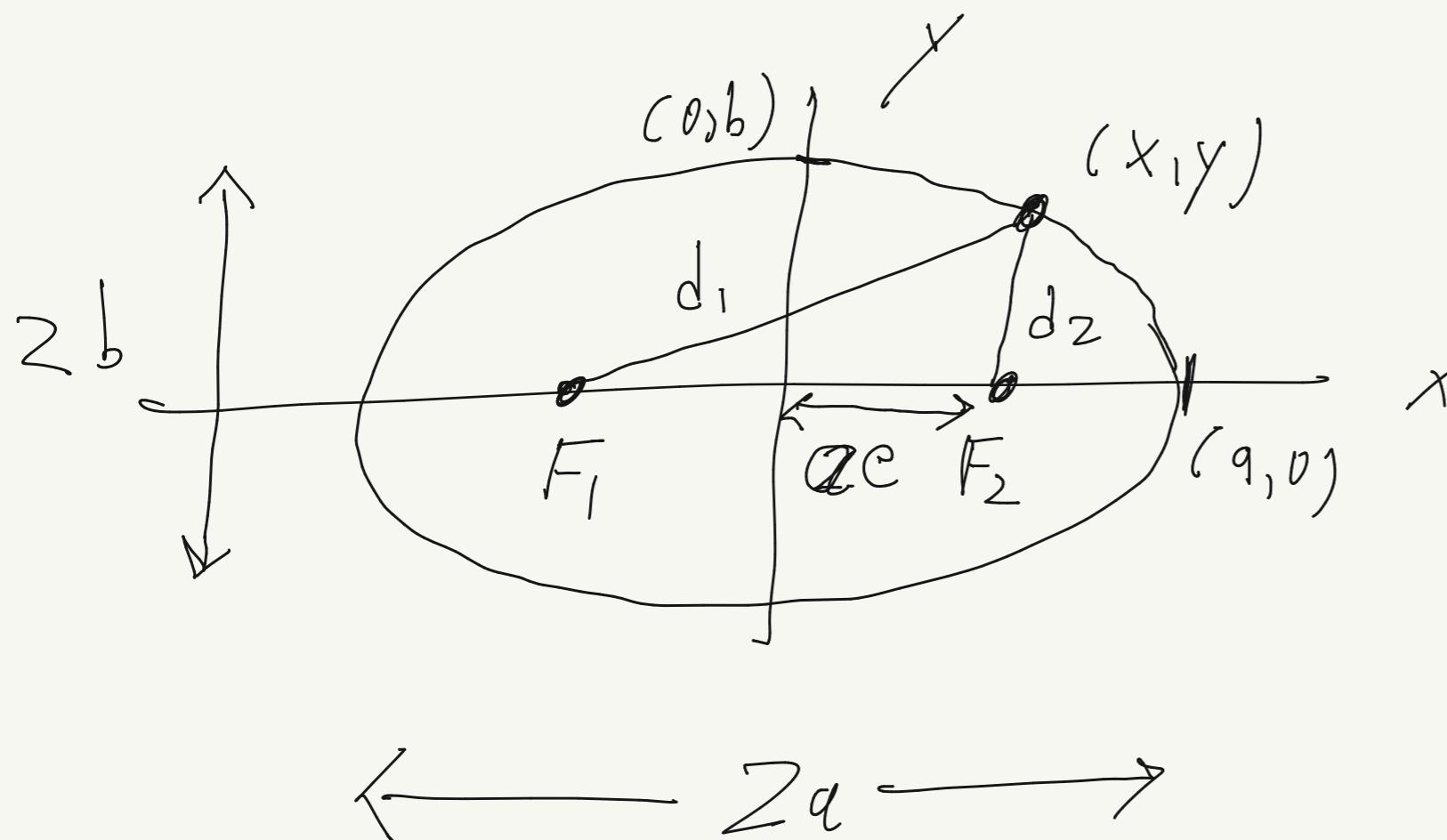
$$\int \frac{d(\sin \theta)}{\cos \theta} = \int d\theta = \theta + \text{const}$$

$$\boxed{\begin{aligned} \int \frac{dt}{\sqrt{1-t^2}} &= \theta + \text{const} \\ &= \sin^{-1} t + \text{const} \end{aligned}}$$

$$t = \sin \theta$$

$$\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - t^2}$$

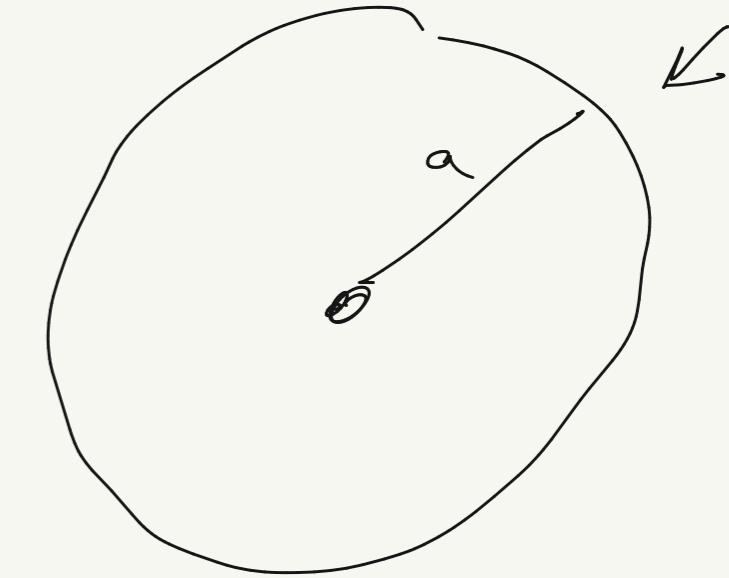
Lec #2 : Aug 26th



$$d_1 + d_2 = 2a$$

$$x^2 + y^2 = a^2$$

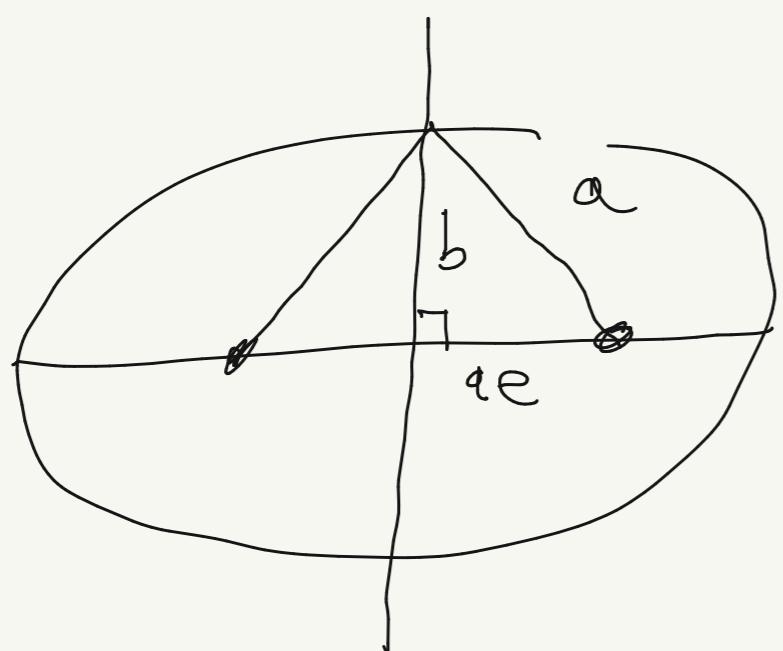
$$e \neq \frac{b}{a}$$



$$e \neq 1 - \frac{b}{a}$$

$$e^2 = \frac{b^2}{1-a^2}$$

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

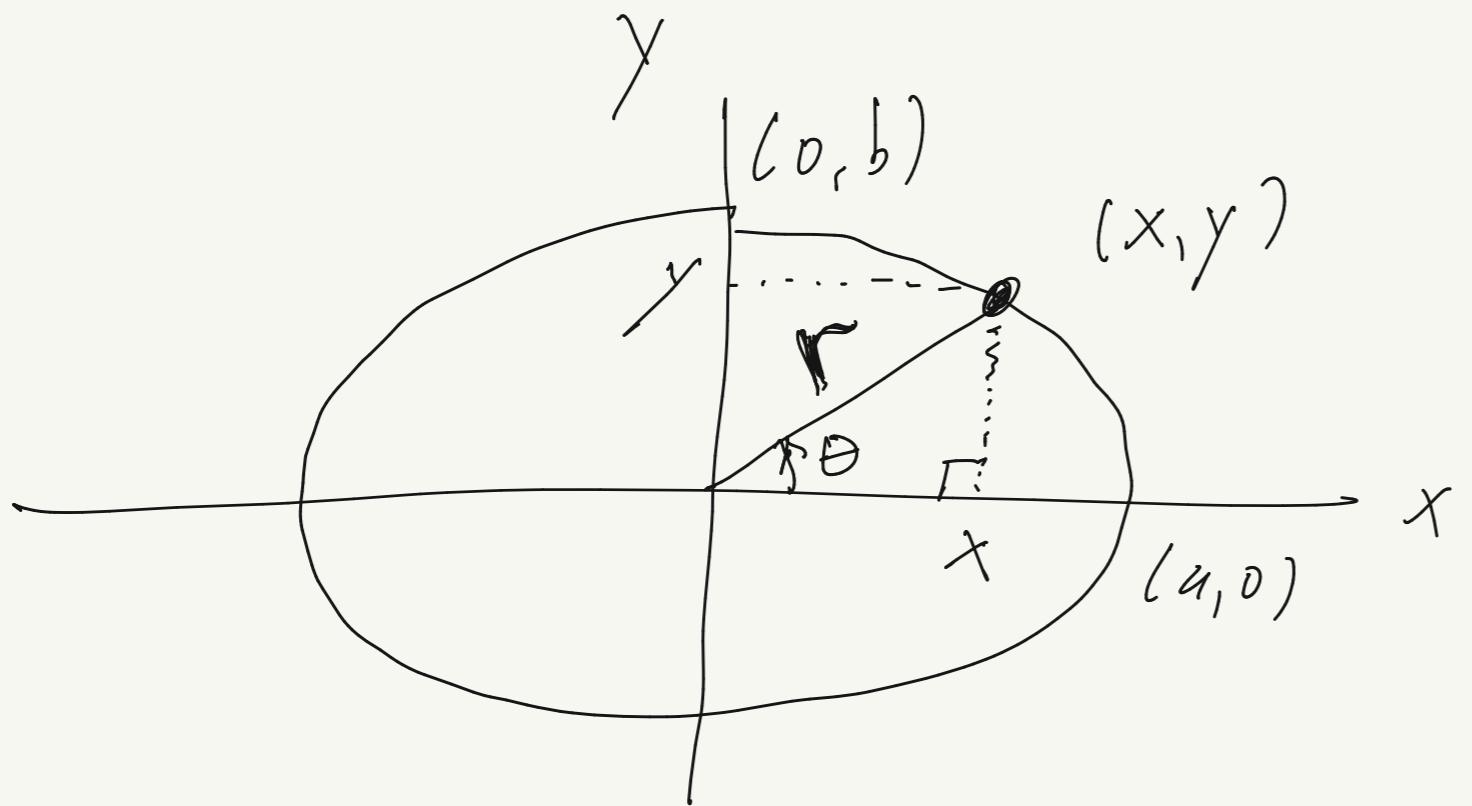


$$PF: a^2 = b^2 + (ae)^2$$

$$a^2(1-e^2) = b^2$$

$$1-e^2 = \left(\frac{b}{a}\right)^2$$

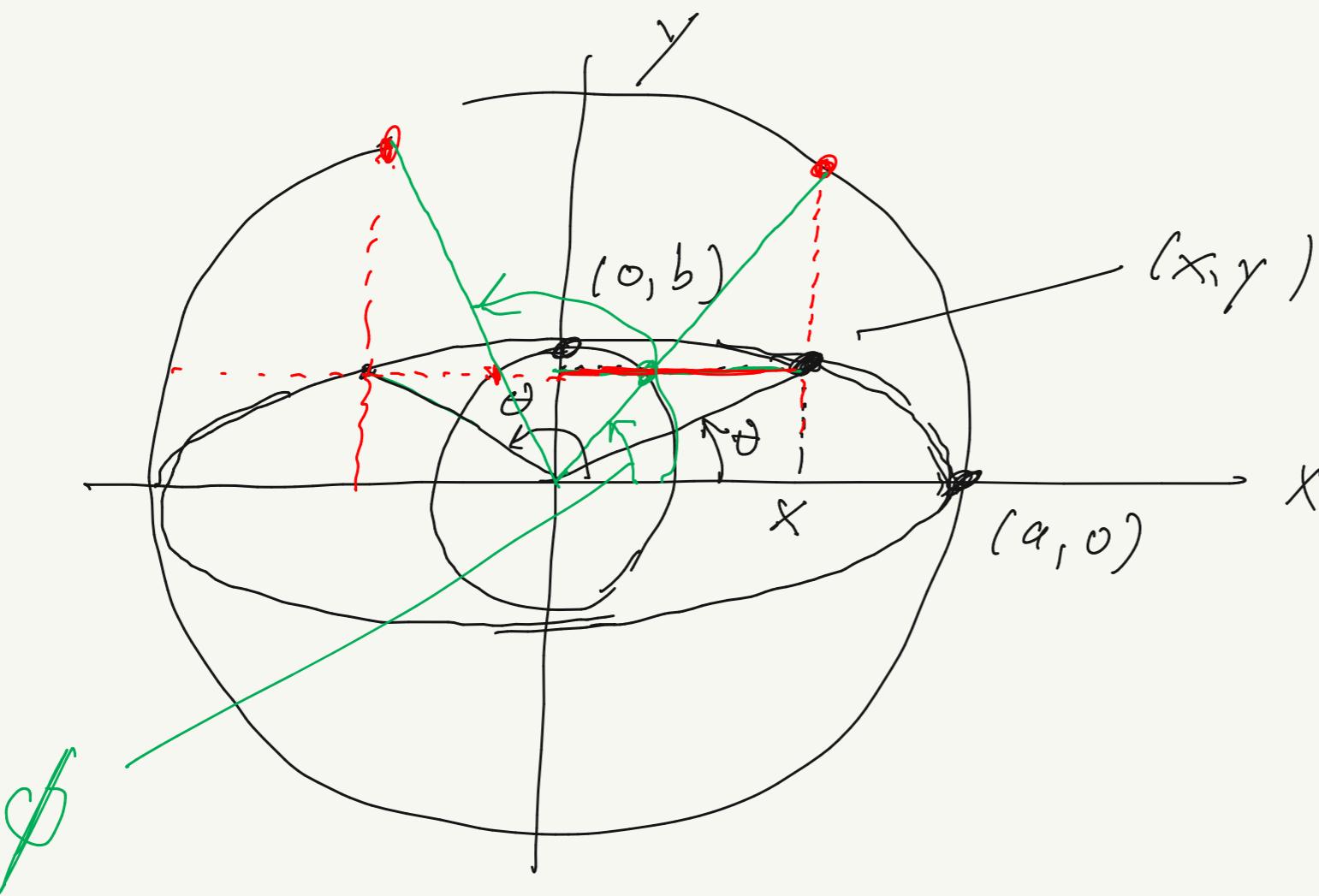
$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \leftarrow$$

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ r &\text{ changes} \end{aligned}$$

$$\begin{cases} x = a \cos \phi \\ y = b \sin \phi \end{cases}$$



$$0 < e < 1$$

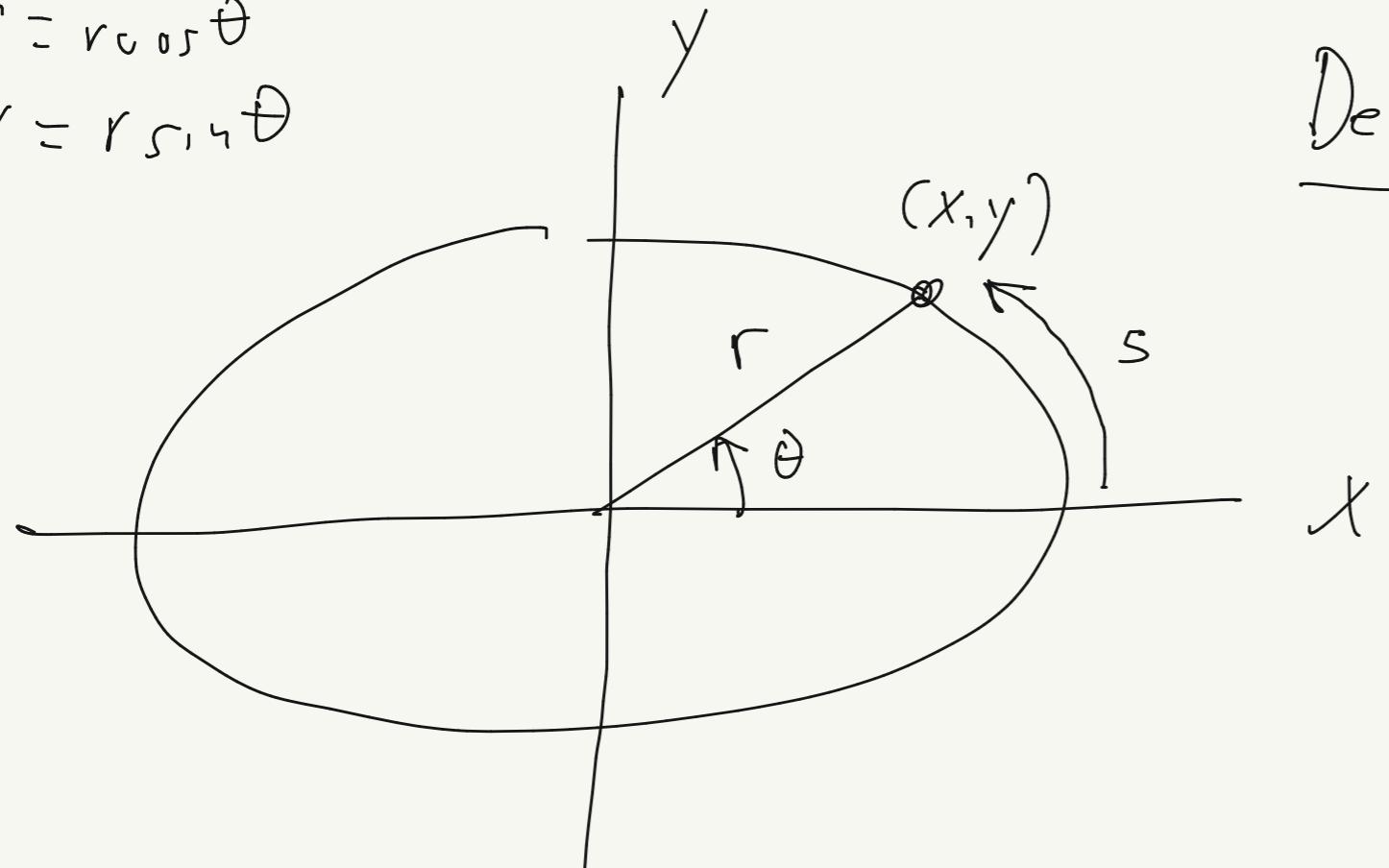
$e = 0$ circle

$e \approx 1$ parabola

$e > 1$ hyperbola

$$x = r \cos \theta$$

$$y = r \sin \theta$$



Defn:

$$\begin{cases} \operatorname{cn}(u; k) = x/a \\ \operatorname{sn}(u; k) = y/b \\ \operatorname{dn}(u; k) = r/a \end{cases}$$

$$k = e, \quad 0 < k < 1$$

\curvearrowright
Modulus

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{dr^2 + r^2 d\theta^2} \end{aligned}$$

$$\oint u = \int_0^\theta r d\theta < \# S$$

$$u \equiv \frac{1}{b} \int_0^\theta r d\theta$$

\uparrow
not θ , not arc length

$$\begin{aligned} x &= r \cos \theta, \quad y = r \sin \theta \\ dx &= dr \cos \theta - r \sin \theta d\theta \\ dy &= dr \sin \theta + r \cos \theta d\theta \end{aligned}$$

$\operatorname{cn}(u)$ $\operatorname{sn}(u)$	pendulum $k = \sin\left(\frac{\phi_0}{2}\right)$
--------------------------------------------------	-----------------------------------------------------

Property:

$$\boxed{Cn^2 u + Sn^2 u = 1} \quad \leftarrow \quad \left(\frac{x}{a} \right)^2 + \left(\frac{y}{b} \right)^2 = 1$$
$$\leftarrow \quad x^2 + y^2 = r^2$$

$$\begin{aligned} d n^2 u &= Cn^2 u + \left(\frac{b}{a} \right)^2 Sn^2 u \\ &= 1 - Sn^2 u + \left(\frac{b}{a} \right)^2 Sn^2 u \\ &= 1 - Sn^2 u \left(1 - \left(\frac{b}{a} \right)^2 \right) \\ &= 1 - H^2 Sn^2 u \end{aligned}$$

$$\boxed{d n^2 u + H^2 Sn^2 u = 1}$$

$$\frac{d}{du} \sin u = \frac{1}{b} \frac{dy}{du}$$

$$= \frac{dy}{r d\theta}$$

$$u = \int_0^\theta (r d\theta)$$

$$du = \frac{r d\theta}{b} \rightarrow b du = r d\theta$$



$$x = r \cos \theta, y = r \sin \theta$$

$$\begin{aligned} dx &= dr \cos \theta - r \sin \theta d\theta & \rightarrow \sin \theta dx = \sin \theta \cos \theta dr - r \sin^2 \theta d\theta \\ dy &= dr \sin \theta + r \cos \theta d\theta & \rightarrow -\cos \theta dy = -\cos \theta \sin \theta dr - r \cos^2 \theta d\theta \end{aligned}$$

add

$$\sin \theta dx - \cos \theta dy = -r d\theta$$

$$\frac{y}{r} dx - \frac{x}{r} dy = -r d\theta$$

$$\rightarrow \boxed{rd\theta = \frac{-y}{r} dx + \frac{x}{r} dy}$$

$$\frac{d}{du} \sin u = \frac{dy}{-\frac{y}{r} dx + \frac{x}{r} dy}$$

$$= \frac{r}{-\frac{y}{r} \frac{dx}{dy} + x}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \cancel{\int \frac{x dx}{a^2} + \cancel{\int y dy}}_{\text{= 0}} = 0$$

$$\frac{dx}{dy} = -\frac{y}{x} \frac{a^2}{b^2}$$

$$\frac{d \sin u}{du} = \frac{r}{-y \left(\frac{-y}{x}\right) \frac{a^2}{b^2} + x} = \frac{r}{y^2 \left(\frac{a}{b}\right)^2 + x^2}$$

$$= \frac{r}{a} \frac{x}{a} \left(\frac{1}{\left(\frac{y}{b}\right)^2 + \left(\frac{x}{a}\right)^2} \right)$$

$$= \frac{\sin u \cdot \cos u}{1}$$

$$\boxed{\frac{d}{du} \sin u = \cos u \cdot \frac{d}{du} u}$$

$$\frac{d}{du} \boxed{\operatorname{cn} u} = -\operatorname{sn} u \cdot \operatorname{dn} u$$

$$\frac{d}{du} \operatorname{dn} u = -\pi^2 \operatorname{sn} u \cdot \operatorname{cn} u$$

$$\frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \cdot \operatorname{dn} u$$

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1$$

$$\operatorname{dn}^2 u + \pi^2 \operatorname{sn}^2 u = 1$$

$$\int \frac{d(\operatorname{cn} \theta)}{\operatorname{cos} \theta} = \int d\theta = \theta$$

$$\int \frac{dt}{\sqrt{1-t^2}} = \operatorname{sn}^{-1} t \quad \theta = \operatorname{cn}^{-1} t$$

Integrate!

$$\int \frac{d(\operatorname{sn} u)}{\operatorname{cn} u \cdot \operatorname{dn} u} = \int du = u + \operatorname{const}$$

$$t = \operatorname{sn} u$$

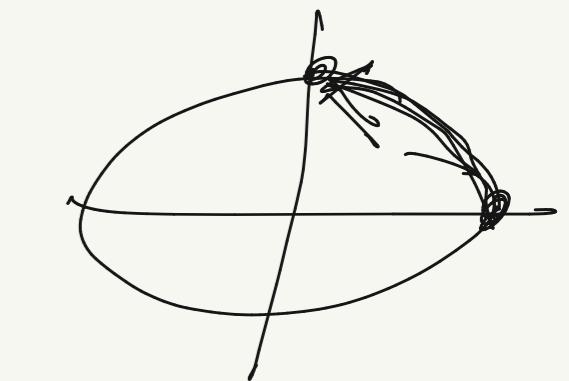
$$\int \frac{dt}{\sqrt{1-t^2} \sqrt{1-\pi^2 t^2}} = \operatorname{sn}^{-1}(t; \pi) + \operatorname{const}$$

$$F(\phi, k) = \int_0^{\sin \phi} \frac{dt}{\sqrt{1-t^2} \sqrt{1-k^2 t^2}}$$

$$\left. \begin{aligned} \sin u &= \sin \phi \\ \frac{y}{b} &= \sin \phi \\ \cnu u &= \cos \phi \end{aligned} \right\}$$

incomplete elliptic integral of 1st kind (angular dependent, period of a simple pendulum)

$$E(\phi, k) = \int_0^{\sin \phi} \frac{dt}{\sqrt{1-t^2} \sqrt{1-k^2 t^2}}$$



incomplete elliptic integral of 2nd kind (arc length along ellipse)

$$\phi = \frac{\pi}{2}$$

$$\int_0^1 \frac{dt}{\sqrt{1-t^2} \sqrt{1-k^2 t^2}} = K(k)$$

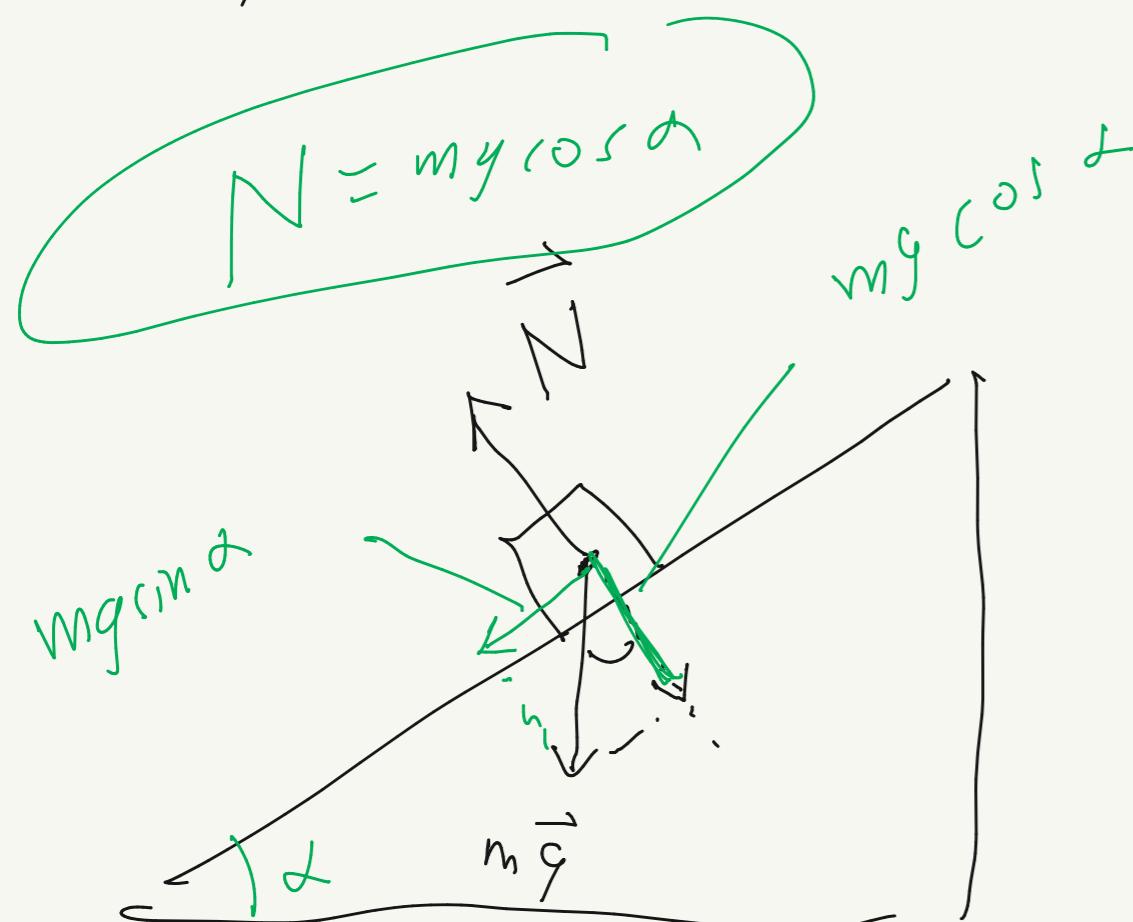
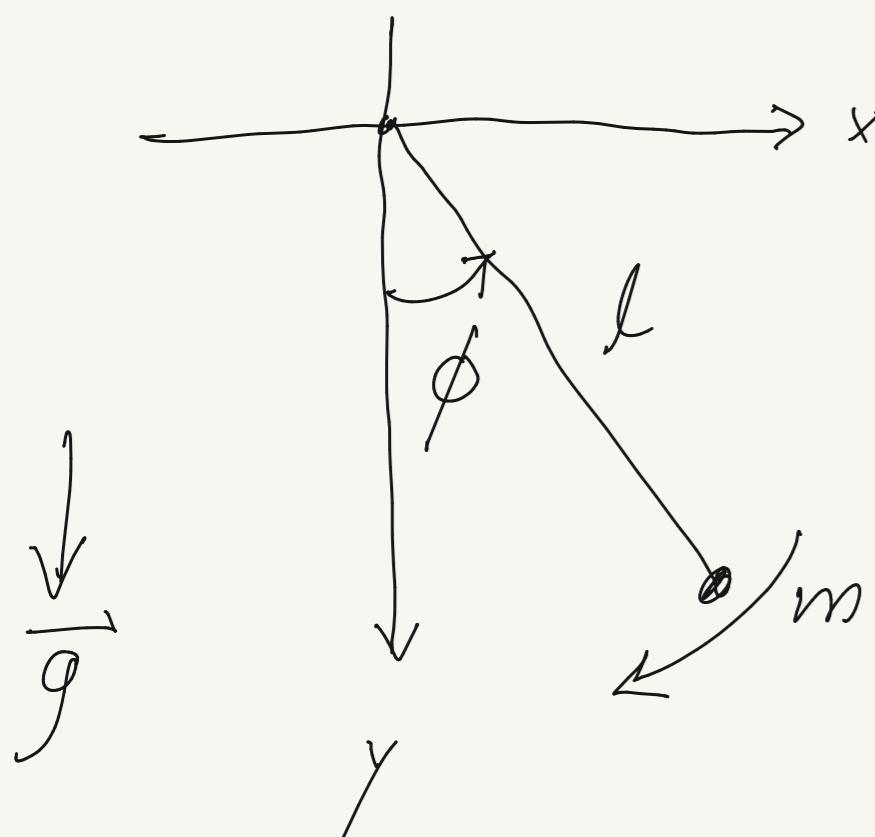
$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} \sqrt{1-k^2 t^2} = E(k)$$

complete elliptic integrals
of 1st and
2nd kind

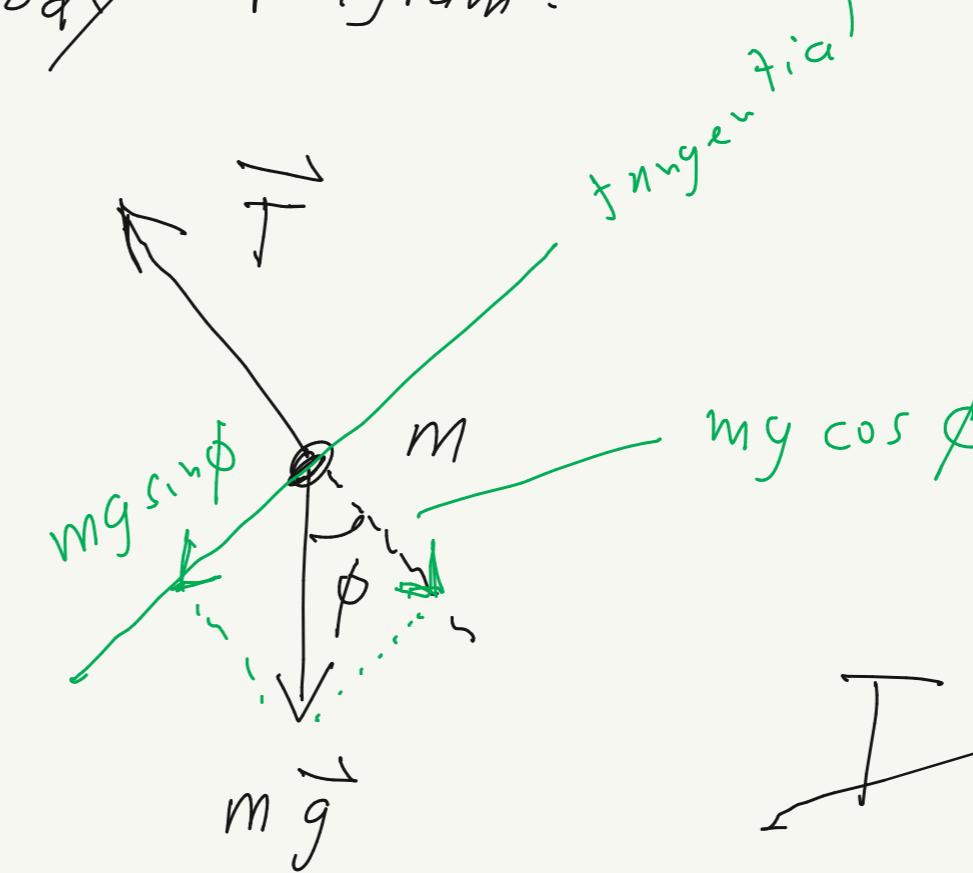
Lec #3: Aug 31st

Simpl. pendulum:

("Freshman physics")



Free-body diagram:



$$1) T - mg \cos \phi = m \dot{\phi}^2 l$$

$$[T = m \dot{\phi}^2 l + mg \cos \phi]$$

$$2) mg \sin \phi = -m \alpha_{\text{tangential}}$$

$$\ddot{\phi} = -\frac{g}{l} \sin \phi$$

EOM

$$a_c = \frac{v^2}{r} = \omega^2 r$$

$$a_t = \alpha r$$

$$\alpha = \ddot{\phi}, \omega = \dot{\phi}$$

$$T = mg \cos \phi$$

cent. petal acceleration

$$\ddot{\phi} = -\frac{g}{l} \sin \phi \quad \left(\text{hard to solve; } 2^{\text{nd}} \text{ order non-linear ODE} \right)$$

Small-angle approx: $\phi \ll 1$ rad $\approx 57^\circ$ (π radians = 180°)

$$\rightarrow \ddot{\phi} = -\frac{g}{l} \phi \quad \omega_0 = \sqrt{\frac{g}{l}}$$

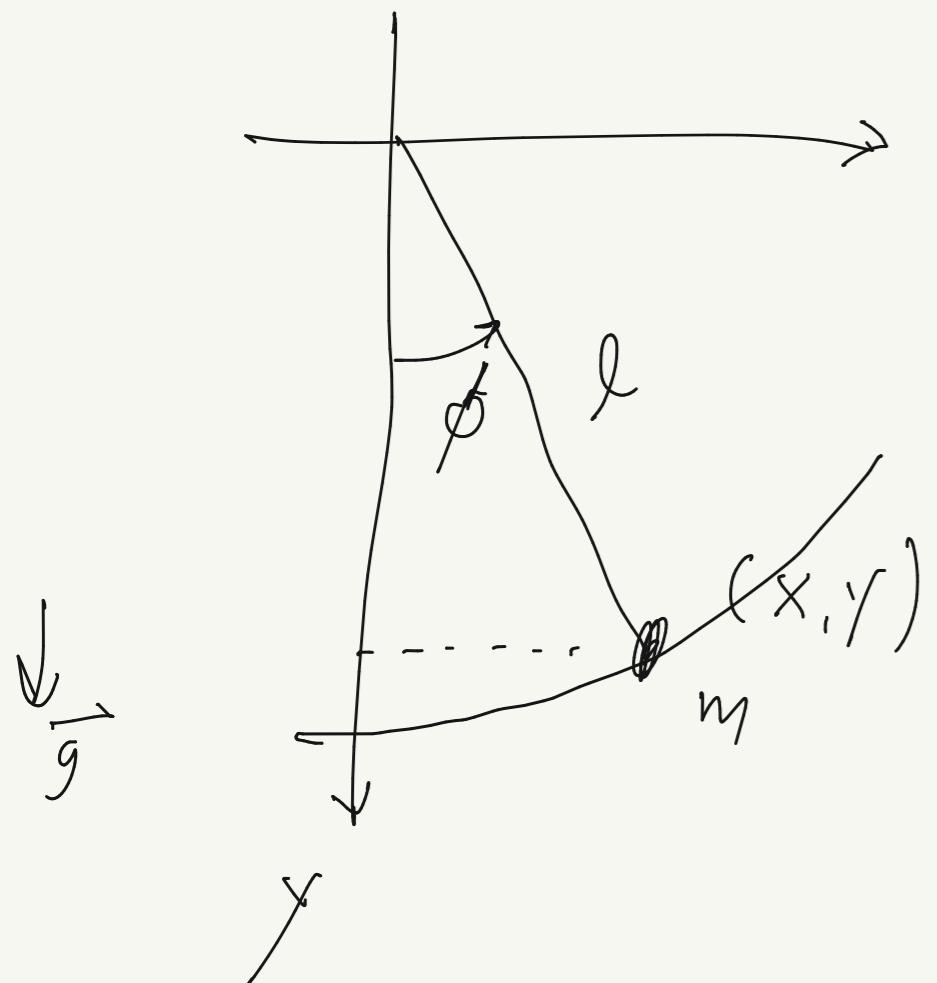
$$\phi(t) = A e^{-i \sqrt{\frac{g}{l}} t}$$

Complex

$$= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

$$= a \cos(\omega_0 t + \alpha)$$

Lagrangian formulation:



$$L = T - U$$

$$U = -mgx \\ = -mgl \cos\phi$$

$$y = l \cos\phi \\ x = l \sin\phi$$

$$T = \frac{1}{2}m(x^2 + y^2) \\ = \frac{1}{2}m l^2 \dot{\phi}^2$$

$$\dot{x} = l\dot{\phi} \cos\phi \\ \dot{y} = -l\dot{\phi} \sin\phi$$

$$L = \frac{1}{2}m l^2 \dot{\phi}^2 + mgl \cos\phi$$

~~Final~~

$$(x, y) \quad x = r \cos\phi \\ (r, \phi) \quad y = r \sin\phi$$

$$T = \frac{1}{2}m(r^2 + r^2\dot{\phi}^2)$$

Lagrange's equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi}$$

$$\frac{d}{dt}(ml^2\dot{\phi}) = -mgl \sin\phi$$

$$ml^2\ddot{\phi} = -mgl \sin\phi$$

$$\ddot{\phi} = -\frac{g}{l} \sin\phi$$

L does not depend explicitly on time $t \rightarrow E$ is conserved.

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$$

$$= T + U$$

Example:

$$\boxed{E} = \frac{\partial L}{\partial \dot{\phi}} \phi - L$$

$$= ml^2 \dot{\phi} \ddot{\phi} - \left(\frac{1}{2} ml^2 \dot{\phi}^2 + mgl \cos \phi \right)$$

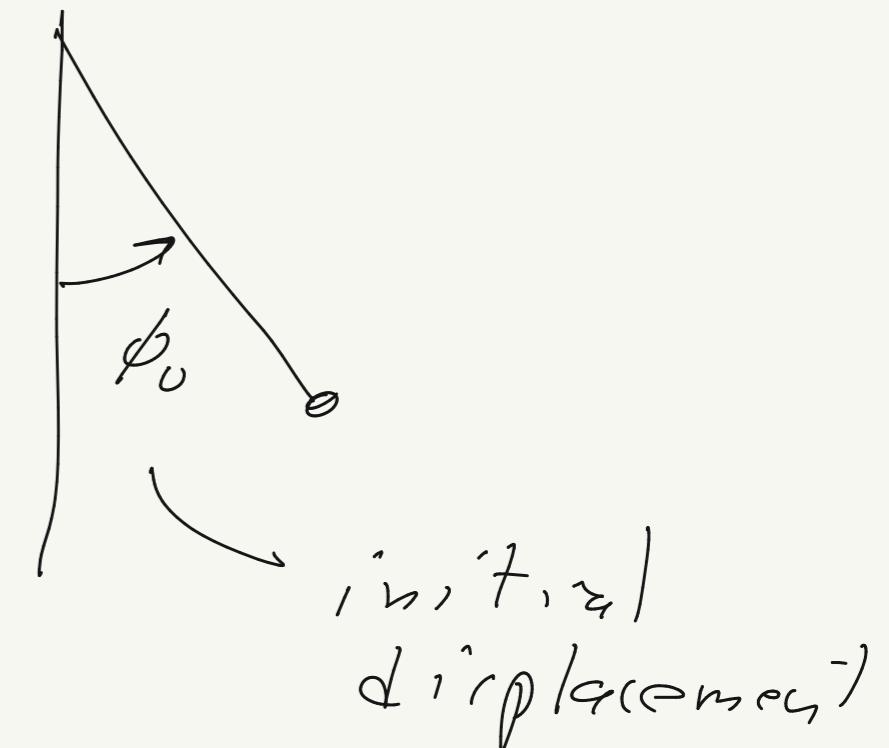
$$= \frac{1}{2} ml^2 \dot{\phi}^2 - mgl \cos \phi$$

$$= T + U$$

$$E = -mgl \cos \phi_0$$

$$-mgl \cos \phi_0 = \frac{1}{2} ml^2 \dot{\phi}^2 - mgl \cos \phi$$

$$\dot{\phi}^2 = \frac{2}{l^2} (gl \cos \phi - gl \cos \phi_0)$$



L & L 11.1

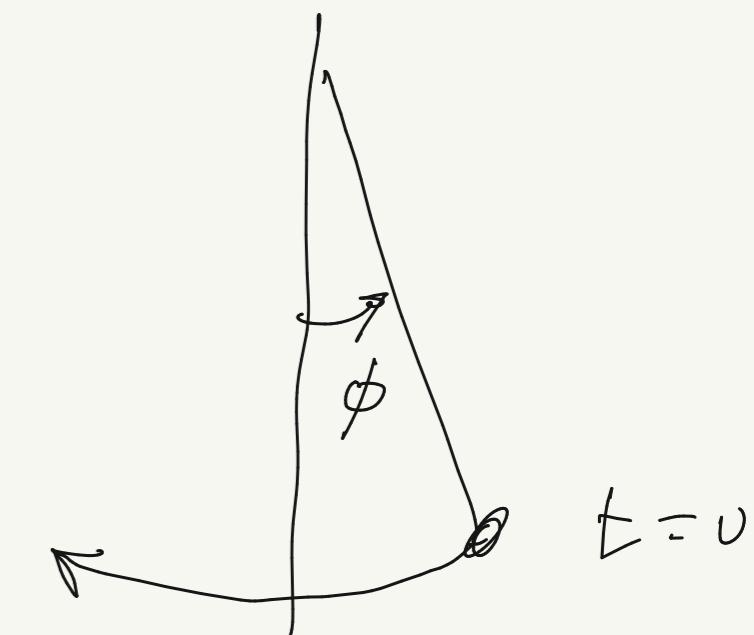
$$\dot{\phi}^2 = 2 \frac{g}{l} (\cos \phi - \cos \phi_0)$$

$$\frac{d\phi}{dt} = \dot{\phi} = \pm \sqrt{2 \frac{g}{l} \sqrt{\cos \phi - \cos \phi_0}}$$

$$\int_{\phi_0}^{\phi} dt = \int_{0}^{t} \frac{-d\phi}{\sqrt{2 \omega_0 \sqrt{\cos \phi - \cos \phi_0}}}$$

$$\omega_0 t + \text{const} = \pm \int \frac{d\phi}{\sqrt{2 \sqrt{\cos \phi - \cos \phi_0}}}$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} =$$



$$\cos \phi = \cos \left(2 \cdot \frac{\phi}{2} \right) = \cos^2 \left(\frac{\phi}{2} \right) - \sin^2 \left(\frac{\phi}{2} \right) = 1 - 2 \sin^2 \left(\frac{\phi}{2} \right)$$

$$\cos \phi_0 = 1 - 2 \sin^2 \left(\frac{\phi_0}{2} \right)$$

$$\Rightarrow \sqrt{\cos \phi - \cos \phi_0} = \sqrt{2} \sqrt{\sin^2 \left(\frac{\phi_0}{2} \right) - \sin^2 \left(\frac{\phi}{2} \right)} = \sqrt{2} \left| \sin \left(\frac{\phi_0}{2} \right) \right| \sqrt{1 - \frac{\sin^2 \left(\frac{\phi_0}{2} \right)}{\sin^2 \left(\frac{\phi}{2} \right)}}$$

$$\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t + \text{const}$$

$$k = \left| \sin\left(\frac{\phi_0}{2}\right) \right| \quad (0 < \tau < 1)$$

$$x = \frac{\sin\left(\frac{\phi}{2}\right)}{\left| \sin\left(\frac{\phi_0}{2}\right) \right|} = \frac{\sin\left(\frac{\phi}{2}\right)}{\tau} \rightarrow dx = \frac{1}{\tau} \frac{1}{2} \cos\left(\frac{\phi}{2}\right) d\phi$$

$$\sqrt{2} \sqrt{\cos\phi - \cos\phi_0} = 2\tau \sqrt{1-x^2}$$

$$dx = \frac{1}{2\tau} \sqrt{1-\sin^2\left(\frac{\phi}{2}\right)} d\phi$$

$$d\phi = \frac{2\tau dx}{\sqrt{1-\tau^2 x^2}}$$

$$w_0 t + \text{const} = \pm \int \frac{2\tau dx}{\sqrt{1-\tau^2 x^2}} \rightarrow 2\tau \int \frac{dx}{\sqrt{1-x^2}}$$

$$= \int_0^x \frac{dx}{\sqrt{1-x^2} \sqrt{1-\tau^2 x^2}}$$

$$= \sin^{-1}(x; \tau)$$

$$\left\{ \begin{array}{l} x = \sin\left(\frac{\phi}{2}\right) \\ \tau = \left| \sin\left(\frac{\phi_0}{2}\right) \right| \end{array} \right.$$

$$t = 0 \iff \phi = \phi_0$$

$$\text{const} = \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-H^2 x^2}} = K(H)$$

complete elliptic
integral of the
1st kind

$$\omega_0 t + E(K) = \sin^{-1} \left(\frac{\sin(\frac{\phi}{2})}{K} \right); \quad | \quad K \equiv \sin(\frac{\phi_0}{2})$$

$$\sin(\omega_0 t + E(K); K) = \frac{1}{K} \sin\left(\frac{\phi}{2}\right)$$

→ $\phi(t) = 2 \arcsin \left(K \sin(\omega_0 t + E(K); K) \right)$

$$P = \frac{4}{\omega_0} E(K) \quad \rightarrow \quad \frac{\omega_0 P}{4} = E(K)$$

Lec # 4:

2 Sep 2021

$$\omega_0 \int_0^t dt = - \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\theta}}$$

$$\omega_0 t = - \left[\int_{\phi_0}^0 + \int_0^{\phi} \right] \frac{d\phi}{\sqrt{\theta}}$$

$$= + \int_0^{\phi_0} \frac{d\phi}{\sqrt{\theta}} - \int_0^{\phi} \frac{d\phi}{\sqrt{\theta}}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-\frac{\theta}{\theta_0}x^2}} - \int_0^x \frac{dx}{\sqrt{1-x^2} \sqrt{1-\frac{\theta}{\theta_0}x^2}}$$

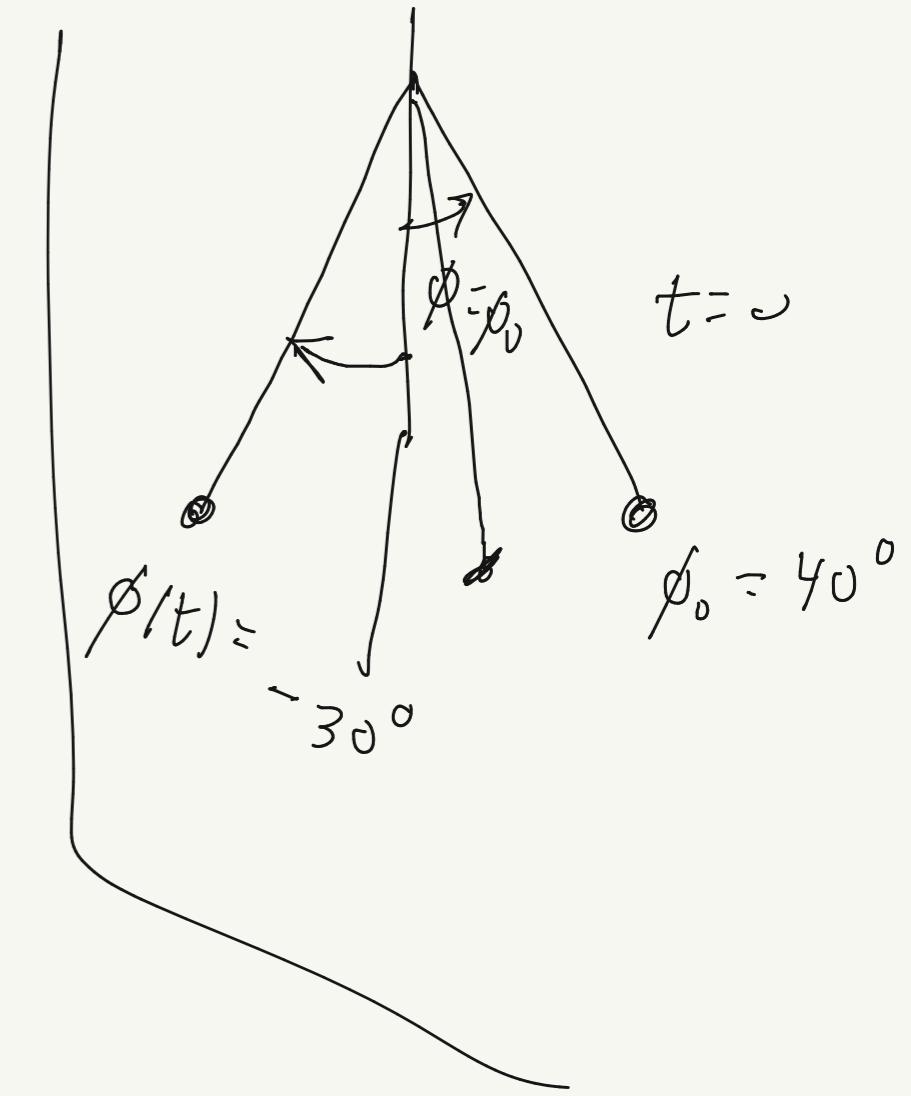
$$= \bar{E}(k) - \operatorname{sn}^{-1}(x; k)$$

$$\operatorname{sn}^{-1}(x; k) = \bar{E}(k) - \omega_0 t$$

$$\frac{\operatorname{sn}(\phi)}{k} = x = \operatorname{sn}[\bar{E}/k - \omega_0 t; k] = \operatorname{cn}(\omega_0 t; k)$$

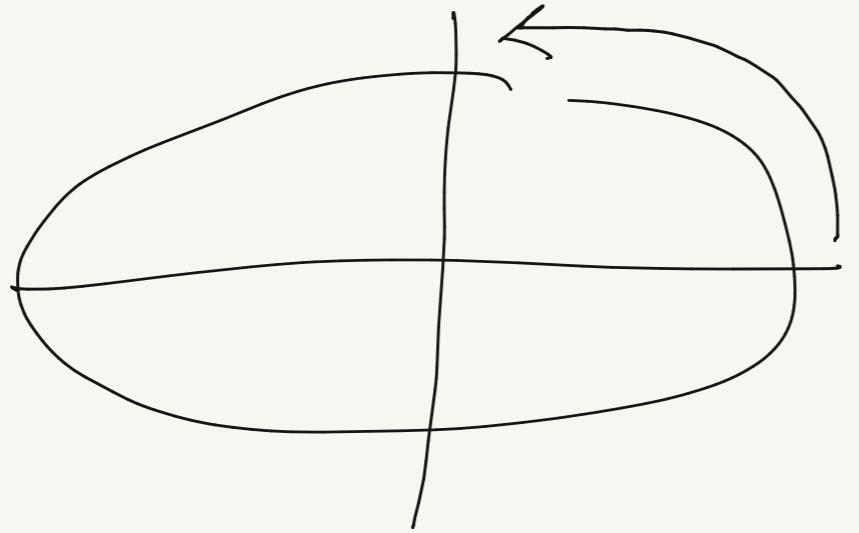
$w_0 = \sqrt{\frac{g}{l}}$

$\rightarrow \boxed{\phi(t) = 2 \arcsin \left(k \operatorname{sn}(\bar{E}/k - \omega_0 t; k) \right)}$



$$\begin{aligned} x &= \frac{\operatorname{sn}\left(\frac{\phi}{z}\right)}{\operatorname{sn}\left(\frac{\phi_0}{z}\right)} \\ k &= \sqrt{\operatorname{sn}\left(\frac{\phi_0}{z}\right)} \end{aligned}$$

$$k = \sqrt{\operatorname{sn}\left(\frac{\phi_0}{z}\right)}$$



$$\frac{1}{\sqrt{1-\pi^2 x^2}} \approx 1 + \frac{1}{2} \pi^2 x^2$$

$$(1+\epsilon)^P \approx 1 + P\epsilon$$

$$\omega_0 \frac{P}{g} = K(\pi)$$

$$\rightarrow \boxed{P = \frac{4}{\omega_0} E(\pi)}$$

$$K(\pi) = \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-\pi^2 x^2}}$$

$\pi=0$: $K(0) = \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} 1 = \frac{\pi}{2}$

$$P = \frac{4}{\omega_0} \frac{\pi}{2} = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{k}{g}}$$

$$0 < \pi < < 1$$

$$K(\pi) = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \left(1 + \frac{1}{2} \pi^2 x^2 \right) = \frac{\pi}{2} + \int_0^1 \frac{dx}{\sqrt{1-x^2}} \frac{1}{2} \pi^2 x^2$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

Pf: $\sin\left(\frac{\pi}{2} - \theta\right) = \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} \cos \theta - \cancel{\cos\left(\frac{\pi}{2}\right)} \sin \theta$

$$= \cos \theta$$

$$\sin(-\theta) = -\sin \theta, \quad \cos(-\theta) = \cos \theta$$

$$\sin(\pi - x; \pi) = \sin(x)$$

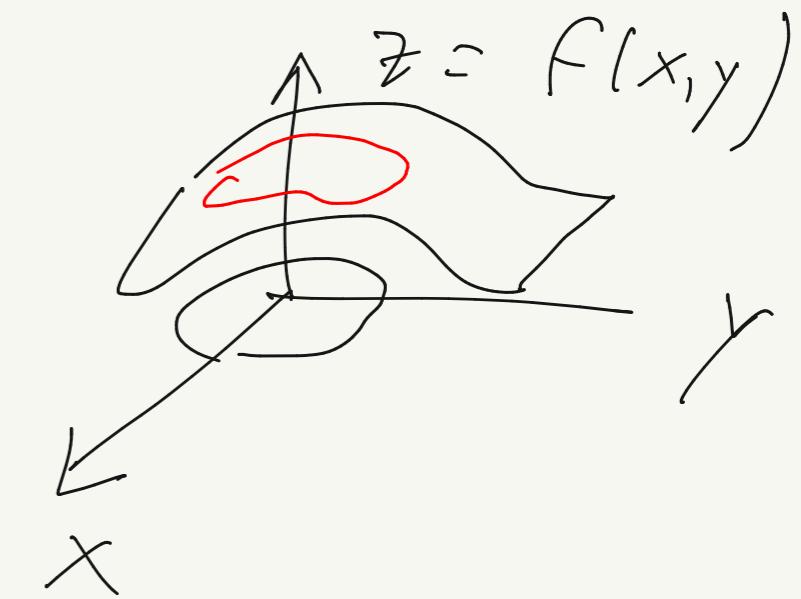
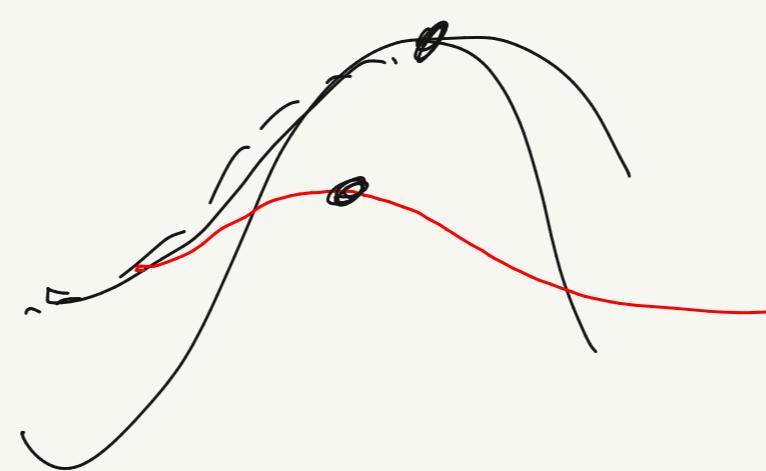
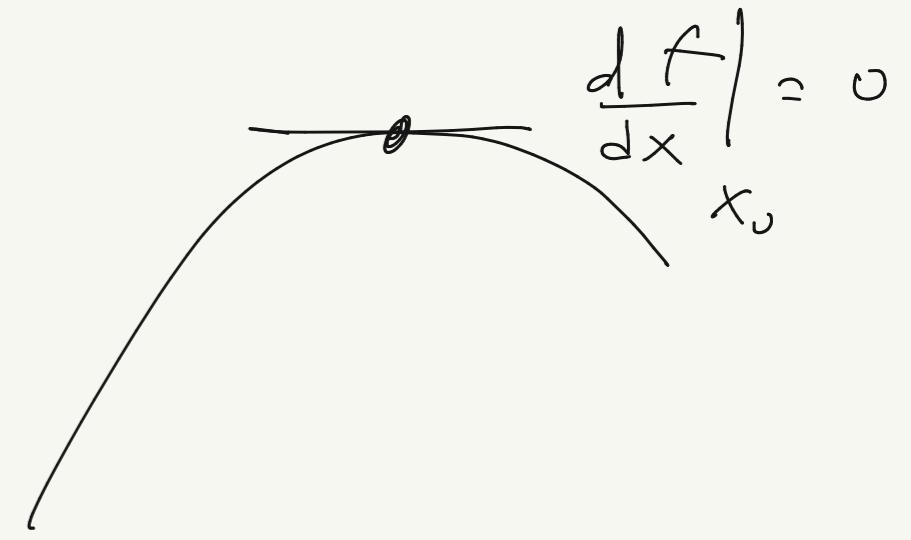
Lagrange multipliers

$f(x) : \max \text{ or } \min ?$

$f(x, y) :$ If

$$\frac{df}{dx} = 0$$

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$



$f(x, y) : \text{Find extreme value subject to constraint } g(x, y) = 0 ?$

1) "reduced square method":

solve constraint $g(x, y) = 0 \rightarrow y = g(x)$

$$F(x) = f(x, y)$$

e.g., $x^2 + y^2 = 1$

$$2x dx + 2y dy = 0$$

2) "method of Lagrange multipliers"

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

$$\begin{cases} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial \lambda} = 0 \end{cases}$$

$$dx = -\frac{y}{x} dy$$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0$$

$$F(x, y, \lambda) = f(x, y) + \lambda \varphi(x, y)$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0$$

$$\frac{\partial F}{\partial \lambda} = \varphi(x, y) = 0$$

$$\varphi(r, \phi) = 0 = r - l$$

$$L(f(r, \phi, \dot{r}, \dot{\phi}, t)) + \lambda(r - l) = L'$$

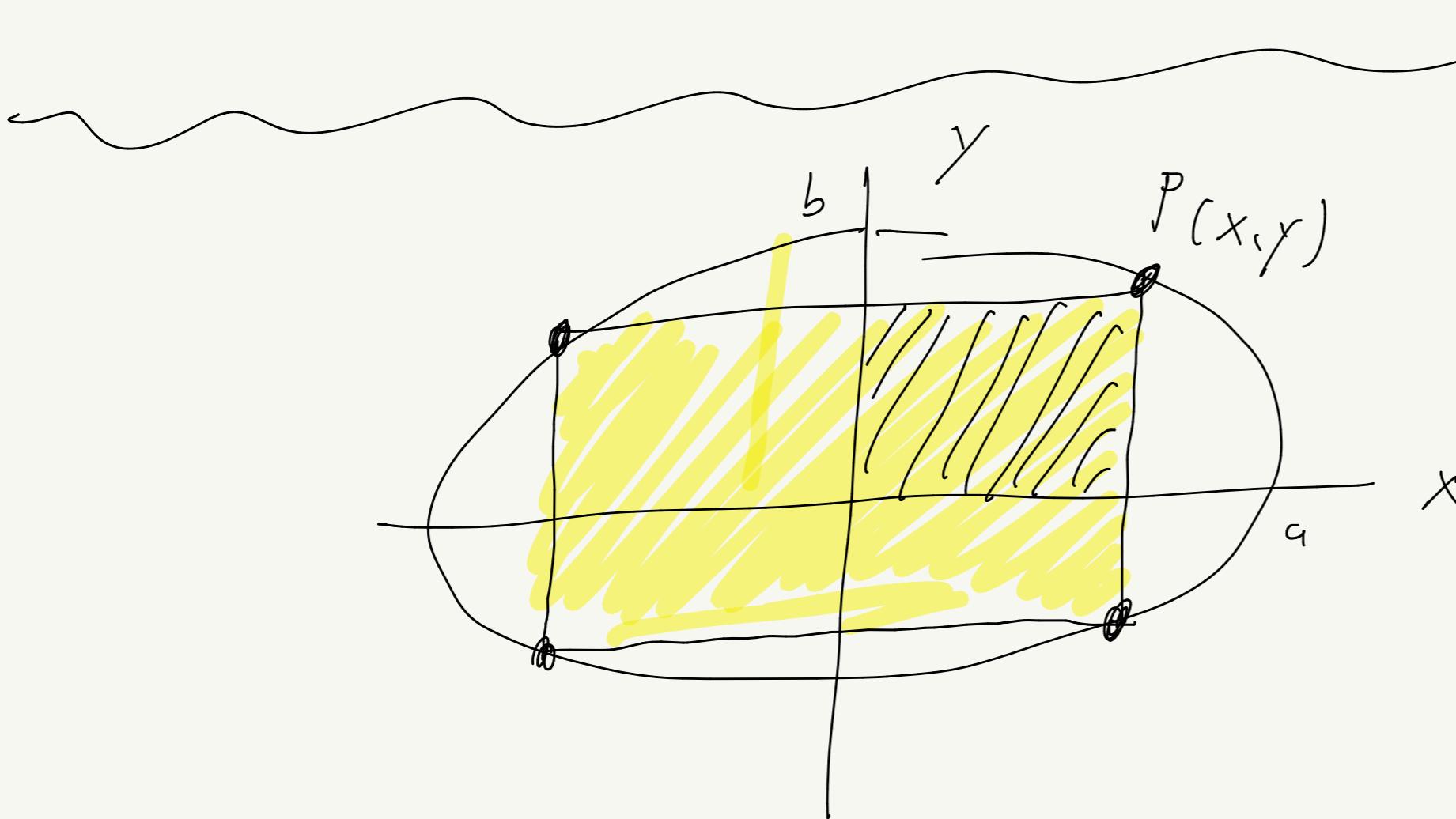
$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{q}} \right) = \frac{\partial L'}{\partial q}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q} + \lambda \frac{\partial \varphi}{\partial q}$$

$$\frac{dP}{dt} = - \frac{\partial U}{\partial q} + \lambda \frac{\partial \varphi}{\partial q} = F_{pp} + F_{constraint}$$

1) Lagrange multipliers

2) Example: 2-d oscillating orbit



Maximize the area of a rectangle whose corners lie on the ellipse

$$\rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$A(x, y) = 4xy$$

$$\varphi(x, y) = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 0$$

$$F(x, y, \lambda) = 4xy + \lambda \varphi(x, y)$$

$\frac{\partial F}{\partial x} = 4y + \lambda \left(-\frac{2x}{a^2}\right) = 0$
$\frac{\partial F}{\partial y} = 4x + \lambda \left(-\frac{2y}{b^2}\right) = 0$
$\frac{\partial F}{\partial \lambda} = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 0$

$$\frac{\partial F}{\partial x} = 4x + \lambda \left(-\frac{2x}{a^2} \right) = 0$$

$$4y^2a^2 - 2\lambda xy = 0$$

~~cancel~~

$$\frac{\partial F}{\partial y} = 4x + \lambda \left(-\frac{2y}{b^2} \right) = 0$$

$$4x^2b^2 - 2\lambda yx = 0$$

~~cancel~~

$$\frac{\partial F}{\partial \lambda} = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 0 \quad \leftarrow$$

subtract

$$4y^2a^2 - 4x^2b^2 = 0$$

$$x^2a^2 = x^2b^2$$

$$\frac{x}{a} = \pm \frac{x}{b}$$

~~cancel~~

$$0 = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$$

$$= 1 - 2\left(\frac{x}{a}\right)^2$$

$$\frac{x}{a} = \pm \frac{1}{\sqrt{2}}$$

$$x = \frac{a}{\sqrt{2}}$$

$$y = \frac{b}{\sqrt{2}}$$

$$A_{max} = 4xy \\ = 4 \frac{a}{\sqrt{2}} \frac{b}{\sqrt{2}}$$

$$= 2ab$$

Reduced Space method

$$F(x) = 4xy \quad | \\ y = b\sqrt{1 - \left(\frac{x}{a}\right)^2}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \\ y = b\sqrt{1 - \left(\frac{x}{a}\right)^2}$$

$$= 4b \times \sqrt{1 - \left(\frac{x}{a}\right)^2}$$

maximize:

$$0 = F'(x) = 4b \sqrt{\quad} + 4b \times \left(\frac{1}{\cancel{2}}\right) \frac{1}{\sqrt{\quad}} \left(-\frac{\cancel{2}x}{a^2} \right)$$

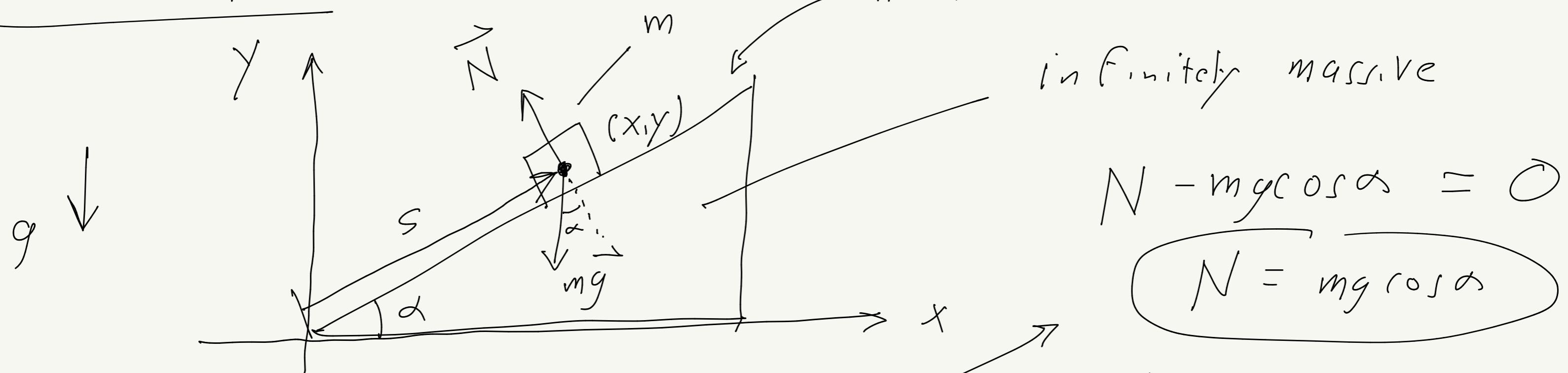
$$= 4b \left(\sqrt{\quad} = \left(\frac{x}{a}\right)^2 \frac{1}{\sqrt{\quad}} \right)$$

$$= \frac{4b}{\sqrt{\quad}} \left(1 - \left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^2 \right)$$

$$= \frac{4b}{\sqrt{\quad}} \left(1 - 2 \left(\frac{x}{a}\right)^2 \right) \rightarrow \left(\frac{x}{a}\right)^2 = \frac{1}{2}$$

$$x = \pm \frac{a}{\sqrt{2}}$$

Mechanics problem:



Freshman physics analysis:

$$m s' = -mg \sin \alpha$$

$$s' = -g \sin \alpha$$

Lagrangian analysis:

$$x = s \cos \alpha, y = s \sin \alpha$$

$$\frac{y}{x} = \tan \alpha$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$y = x \tan \alpha$$

$$U = mg y$$

$$\varphi(x, y) = y - x \tan \alpha = 0$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mg y$$

$$L' = L + \lambda \varphi$$

$$F = A + \lambda \varphi$$

$$= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgx + \lambda(y - x \tan \alpha)$$

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{x}} \right) = \frac{\partial L'}{\partial x}$$

$$\rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} + \lambda \frac{\partial \varphi}{\partial x}$$

$$\frac{d p_x}{dt} = F_x + \lambda \frac{\partial \varphi}{\partial x}$$

$$\frac{d p_y}{dt} = F_y + \lambda \frac{\partial \varphi}{\partial y}$$

$$\frac{d \vec{p}}{dt} = \vec{F} + \lambda \vec{\nabla} \varphi$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) = \frac{\partial L}{\partial y} + \lambda \frac{\partial \varphi}{\partial y}$$

$$\varphi(x, y) = y - x \tan \alpha = 0$$

$$\frac{d}{dt}(m \dot{x}) = 0 + \lambda(-\tan \alpha)$$

$$\rightarrow m \ddot{x} = -\lambda \tan \alpha$$

$$\frac{d}{dt}(m \dot{y}) = -mg + \lambda$$

$$\ddot{x} = -\frac{1}{m} \tan \alpha$$

$$\ddot{y} = -g + \frac{\lambda}{m}$$

$$y - x \tan \alpha = 0$$

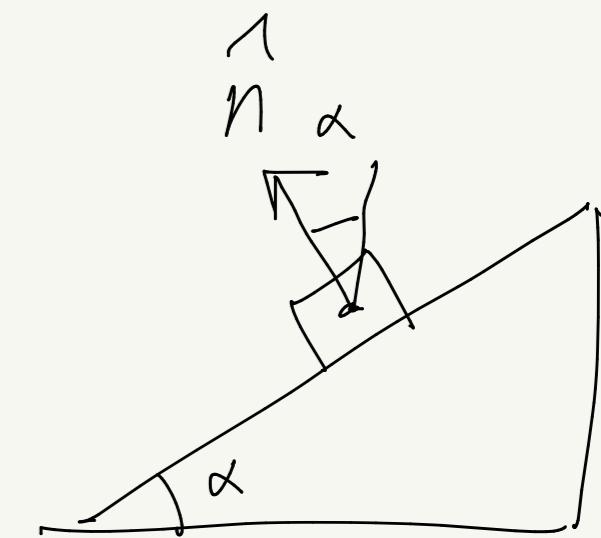
$$y - \dot{x} \tan \alpha = 0 \rightarrow \dot{y} = \dot{x} \tan \alpha$$

$$\left(-g + \frac{\lambda}{m} \right) = -\frac{\lambda}{m} \tan \alpha \cdot \tan \alpha$$

$$\phi = y - x \tan \alpha$$

$$\frac{\lambda}{m} (1 + \tan^2 \alpha) = g$$

$$\sec^2 \alpha = \frac{1}{\cos^2 \alpha}$$



$$\frac{\lambda}{m} = g \cos^2 \alpha$$

$$\lambda = mg \cos^2 \alpha$$

$$\vec{F}_c = \lambda \vec{\nabla} \phi$$

$$= \lambda (\vec{y} - \vec{x} \tan \alpha)$$

unit

$$\vec{F}_c = \lambda \left(\vec{y} - \vec{x} \frac{\sin \alpha}{\cos \alpha} \right)$$

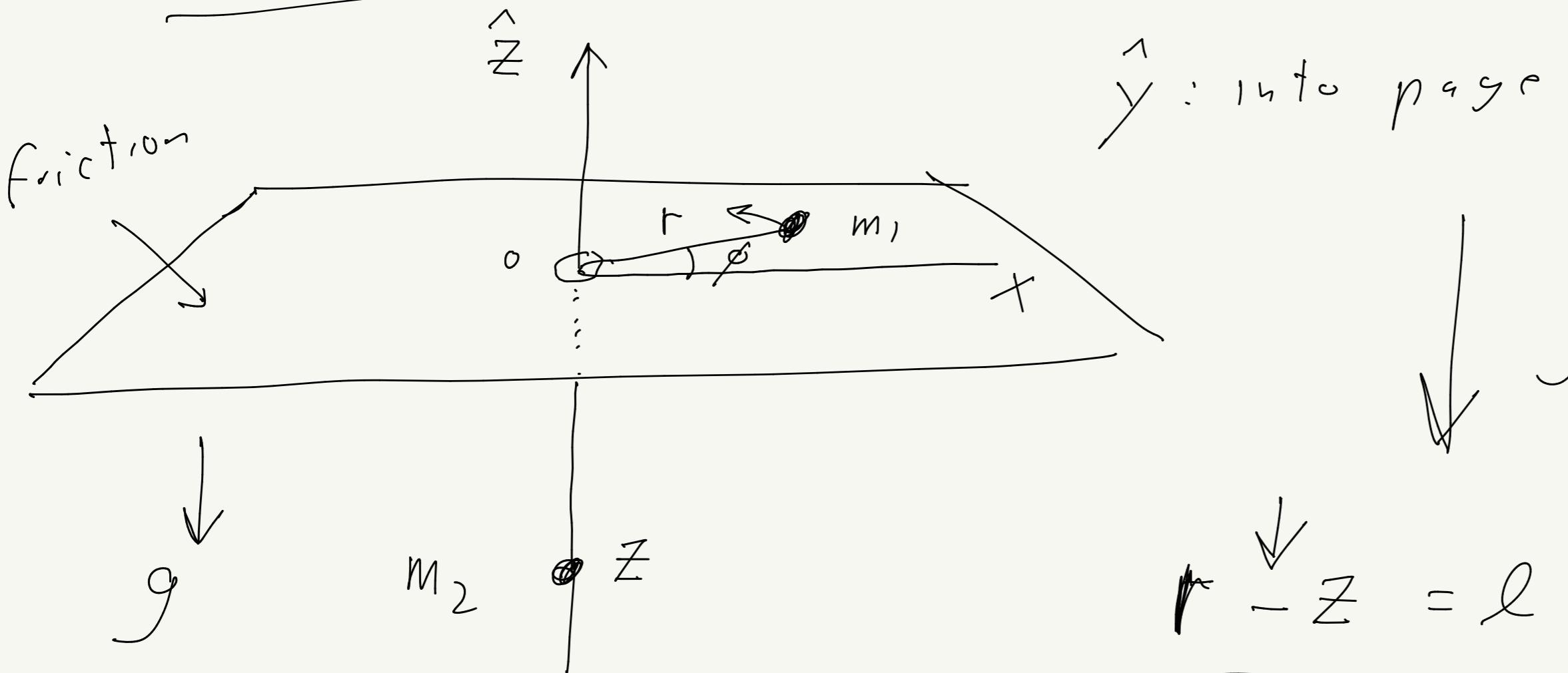
$$= \frac{\lambda}{\cos \alpha} \left(\cos \alpha \vec{y} - \vec{x} \sin \alpha \right)$$

$$= \frac{\lambda}{\cos \alpha} \vec{n} = \boxed{mg \cos \alpha \vec{n}}$$

Q.

2-d Example:

string length = ℓ



$$x = r \cos \phi$$

$$y = r \sin \phi$$

$$r - z = \ell$$

$$z = r - \ell$$

$$\begin{aligned} T &= \frac{1}{2} m_1 (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_2 \dot{z}^2 \\ &= \frac{1}{2} m_1 (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2 \\ &= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 \end{aligned}$$

$$U = m_2 g z = m_2 g (r - \ell) = m_2 g r$$

ignore
1 (con, +)

$$-m_2 g \ell$$

$$L = T - U$$

$$= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r$$

$$(r, \phi) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \cancel{\frac{\partial L}{\partial \phi}} = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = \text{const} = \boxed{m_1 r^2 \dot{\phi} = M_2}$$

No explicit time-dependence:

$$E = \text{const} = \underbrace{\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i}_{\sim} - L = \overbrace{T + U}$$

~~scribble~~

$$E = \frac{1}{2}(m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 + m_2 g r$$

const.

$M_z = m_1 r^2 \dot{\phi}$ \rightarrow $\dot{\phi} = \frac{M_z}{m_1 r^2}$

$$\rightarrow E = \frac{1}{2}(m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \left(\frac{M_z}{m_1 r^2} \right)^2 + m_2 g r$$

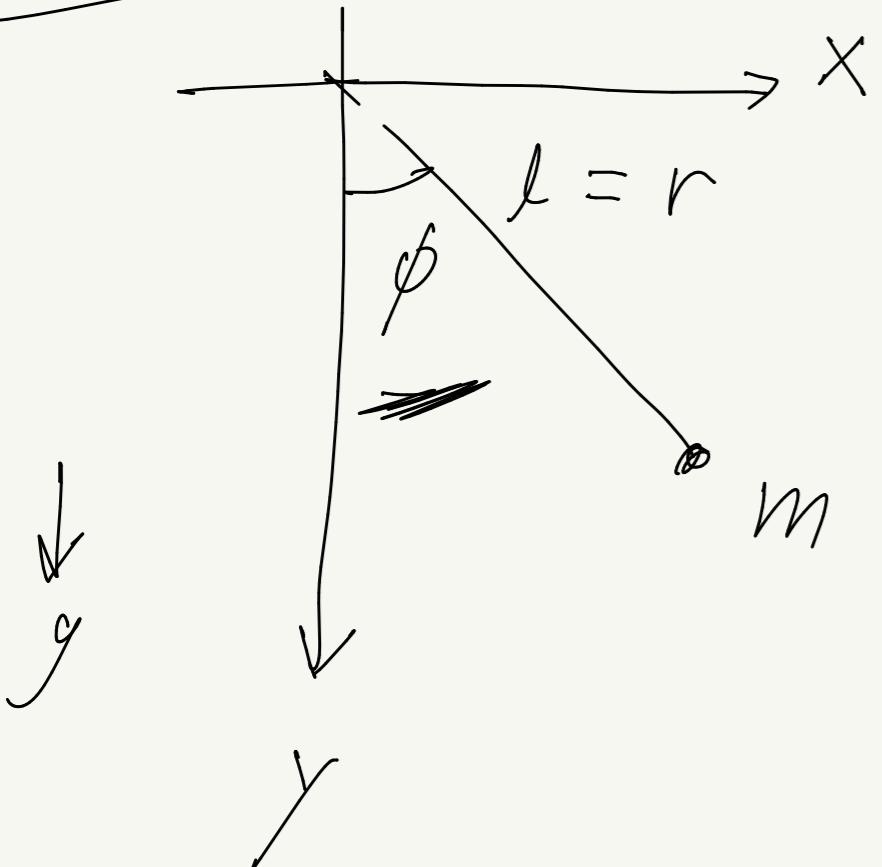
$$= \frac{1}{2}(m_1 + m_2) \dot{r}^2 + \frac{M_z^2}{2 m_1 r^2} + m_2 g r$$

$\underbrace{\qquad\qquad\qquad}_{U_{eff}(r)}$

Lec #6:

Sept. 9th

Quiz #1



Calculate tension in the string
using the method of
Lagrange multipliers.

joseph.d.romano@ttu.edu

constraint:

$$\underline{\phi} = \ell - r = 0$$

$$U = -mg y$$

$$= -mg r \cos \phi$$

$$\vec{F}_c = \lambda \vec{\nabla} \phi$$

$$L = T - U$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + m g r \cos \phi$$

Eric
Niramay
Muhammad

$$(1) \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} + \lambda \frac{\partial \phi}{\partial r}$$

$$(2) \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} + \lambda \frac{\partial \phi}{\partial \phi}$$

$$\begin{cases} \phi = 0 \\ \rightarrow r = \ell \end{cases} \quad (3)$$

$$\frac{d}{dt}(mr) = mr\dot{\phi}^2 + mg \cos\phi - \lambda$$

$$mr'' = mr\dot{\phi}^2 + mg \cos\phi - \lambda$$

$$(1) ml^2\ddot{\phi} = -mgls\sin\phi$$

$$\frac{d}{dt}(mr^2\dot{\phi}) = -mg r s\sin\phi$$

$$2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} = -mg r s\sin\phi$$

$$(2) \ddot{\phi} = \frac{-g s\sin\phi}{l}$$

$$\boxed{r=l} \rightarrow \boxed{\dot{r}=0}, \boxed{\ddot{r}=0}$$

(3)

$$\varphi = l - r$$

$$0 = ml\dot{\phi}^2 + mg \cos\phi - \lambda$$

$$\cancel{\lambda} = ml\dot{\phi}^2 + mg \cos\phi$$

$$\nabla\varphi = -\vec{r}$$

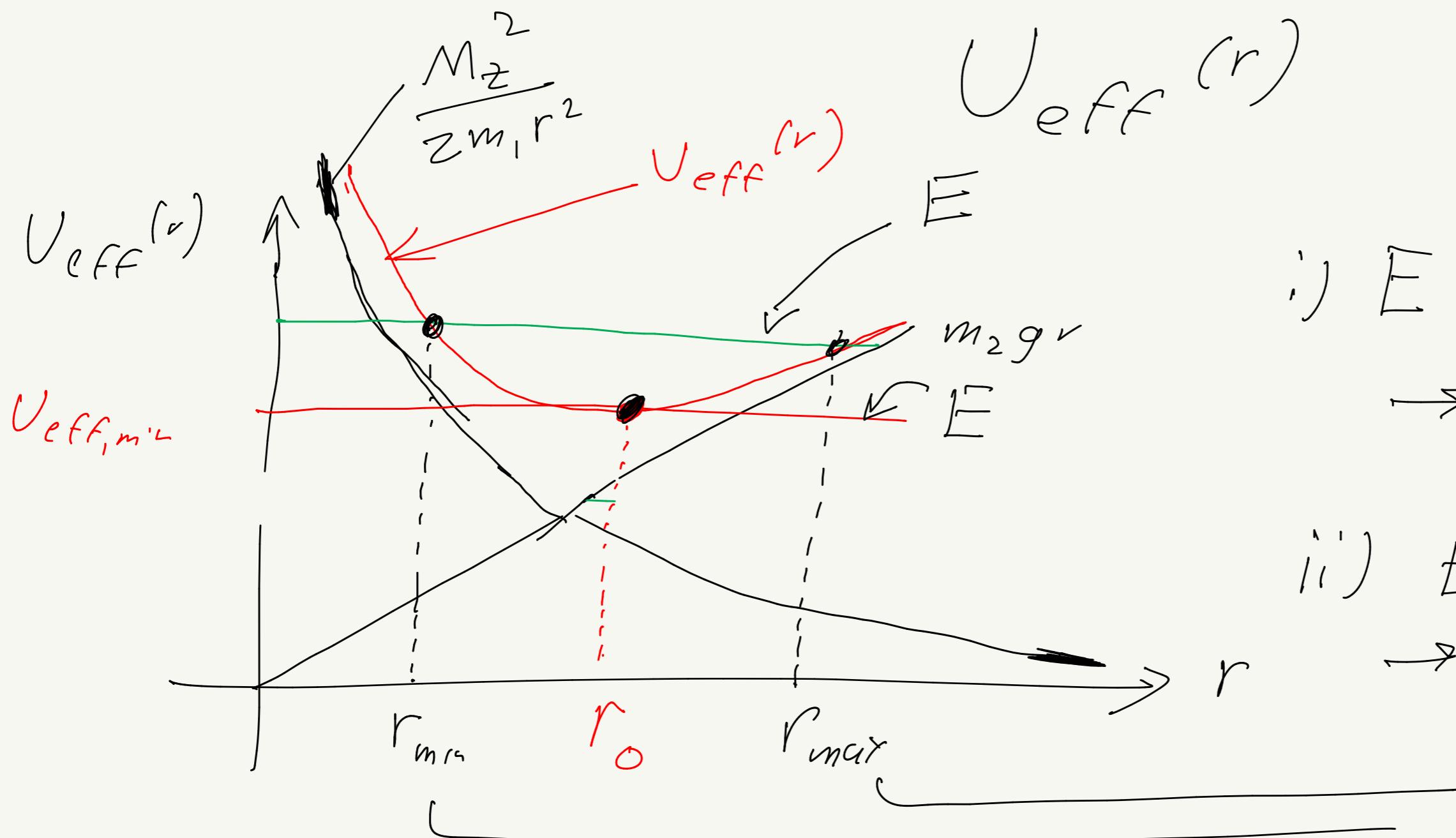
$$\begin{aligned} \cancel{\lambda - mg \cos\phi} \\ = ml\dot{\phi}^2 \end{aligned}$$

$$\vec{F}_c = -(ml\dot{\phi}^2 + mg \cos\phi)\vec{r}$$

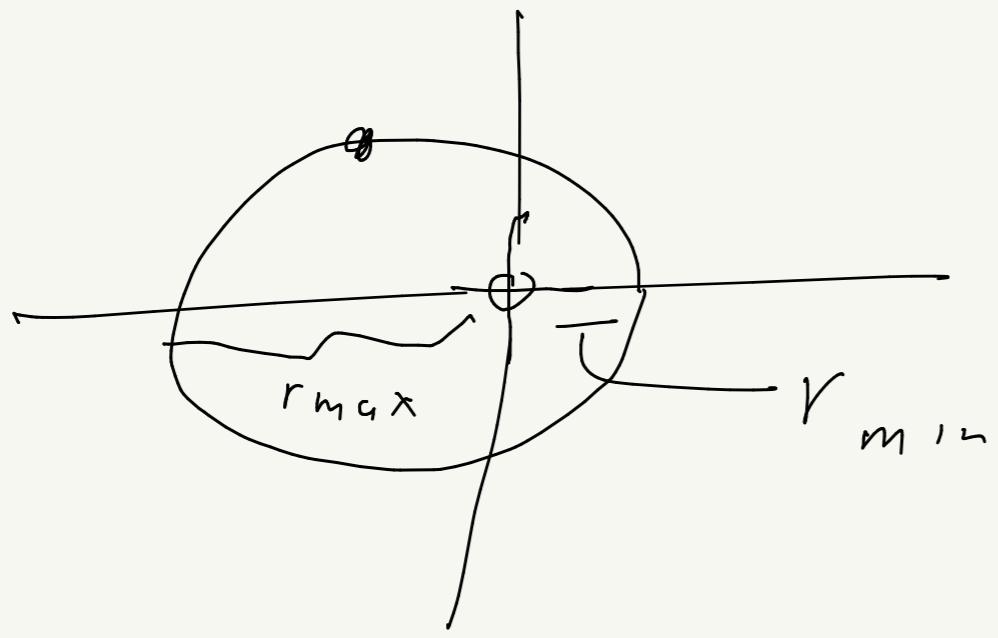
Revisit: oscillating orb. t example

$$E = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \left(\frac{M_2}{m_1 r^2} \right)^2 + m_2 g r$$

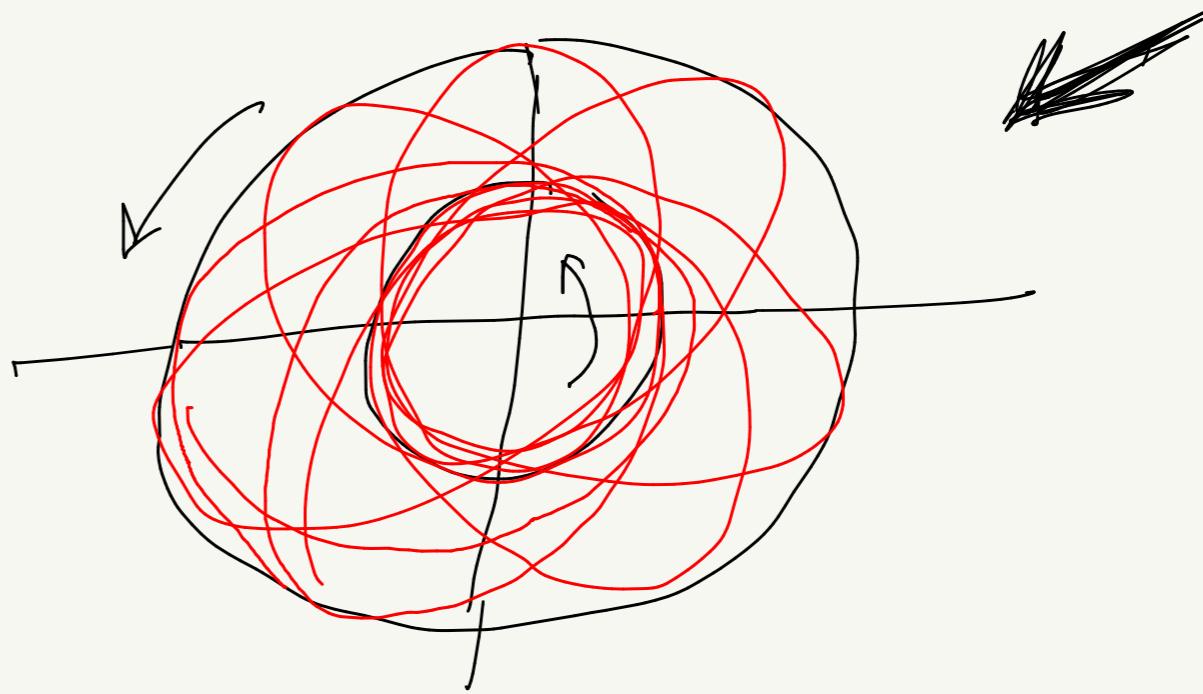
$$= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{\frac{M_2^2}{2 m_1 r^2}}{+ m_2 g r}$$



- $E \geq U_{\text{eff}, \text{min}}$
- i) $E = U_{\text{eff}, \text{min}}$
→ stable circular orbit $r = r_0$
 - ii) $E > U_{\text{eff}, \text{min}}$
→ bounded motion between r_{min} and r_{max} turning points



closed bound orbit



bound orbit
that's not closed

r_0 : minimum of the effective potential

$$\Omega = \left. \frac{dU_{\text{eff}}}{dr} \right|_{r=r_0} = \left. \left(m_2 g - \frac{M_2^2}{m_1 r^3} \right) \right|_{r=r_0}$$

$$\Omega = m_2 g - \frac{M_2^2}{m_1 r_0^3}$$

$$M_2^2 = m_1 m_2 g r_0^3$$

$$E = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + V_{\text{eff}}(r),$$

$$E - V_{\text{eff}}(r) = \frac{1}{2} (m_1 + m_2) \dot{r}^2$$

$$V_{\text{eff}}(r) = m_2 gr + \frac{M_2^2}{2m_1 r^2}$$

$$\frac{dr}{dt} = \dot{r} = \pm \sqrt{\left(\frac{2}{m_1 + m_2}\right) (E - V_{\text{eff}}(r))}$$

$$\int dt = \pm \int \frac{dr}{\sqrt{\left(\frac{2}{m_1 + m_2}\right)(E - V_{\text{eff}}(r))}} \rightarrow t = t(r)$$

$$M_2 = m_1 r^2 \dot{\phi}$$

$$\dot{\phi} = \frac{M_2}{m_1 r^2}$$

$$\frac{dr}{dt} = \pm \sqrt{\textcircled{1}}$$

$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} =$$

$$\frac{M_2}{m_1 r^2} \frac{dr}{d\phi}$$

$$\frac{M_2}{m_1 r^2} \frac{dr}{d\phi} = \pm \sqrt{\ell}$$

orb. t eqn_o

$$\int \frac{M_2 dr}{m_1 r^2 \sqrt{\ell}} = \int \pm d\phi$$

$$\rightarrow \phi = \phi(r)$$

$$r = r(\phi)$$

$$\frac{dr}{dt} = \pm \sqrt{\ell}$$

$$\frac{d\phi}{dt} = \frac{M_2}{m_1 r^2}$$

$$\ell = \frac{2}{m_1 + m_2} (E - U_{eff}(r))$$

~~start~~ choose E, M_2, m_1, m_2 , etc.

start system off at some values of r and ϕ at $t=0$

step 1: ~~r_1, ϕ_1~~
step 2: $r_2 = r_1 + \Delta r, \phi_2 = \phi_1 + \Delta \phi$
 (repeat)

Lec #7:

Sep 14th

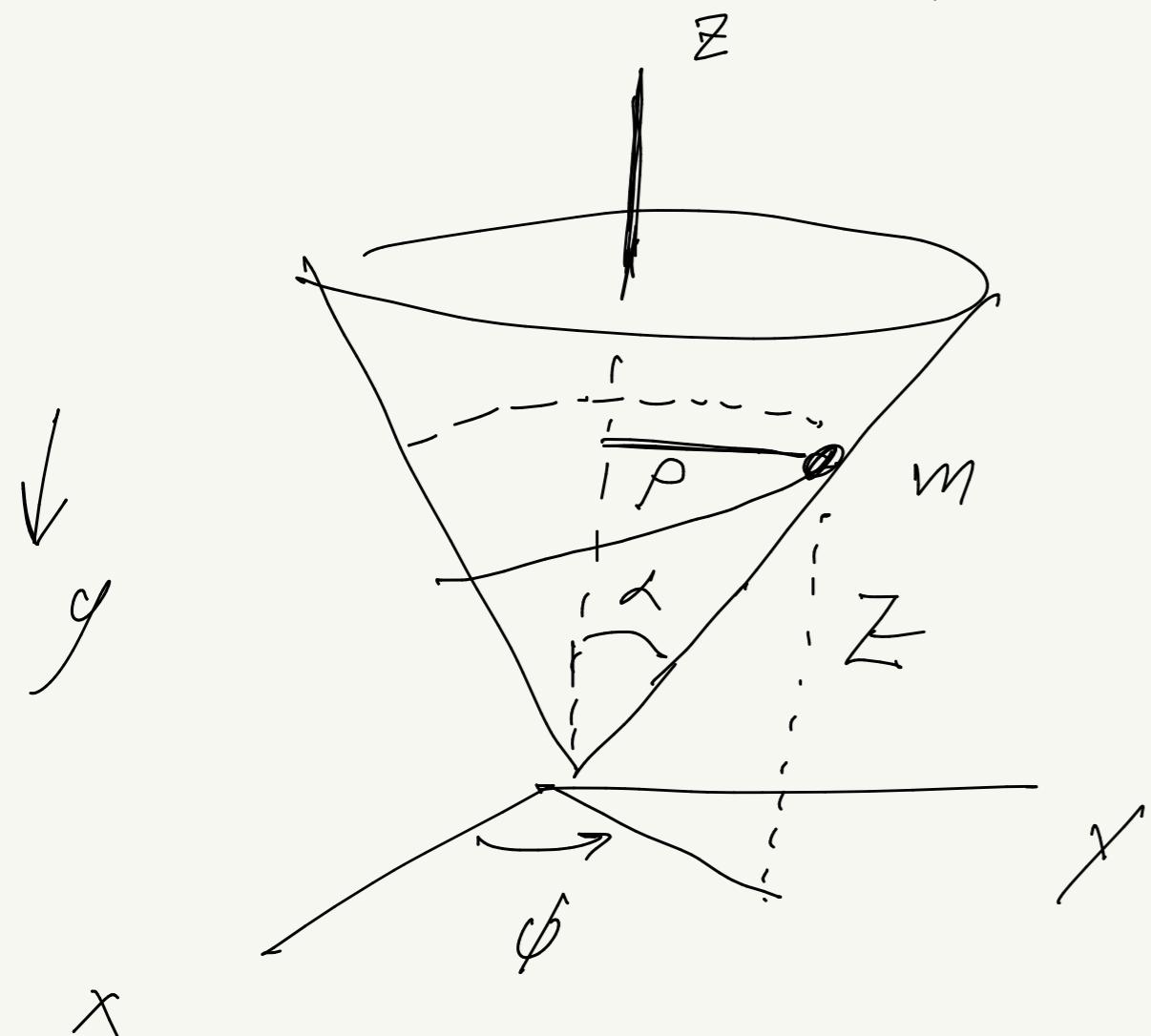
Lec #8:

Sep 16th

$$\phi(\rho, \phi, z) = \rho - z \tan \alpha = 0$$

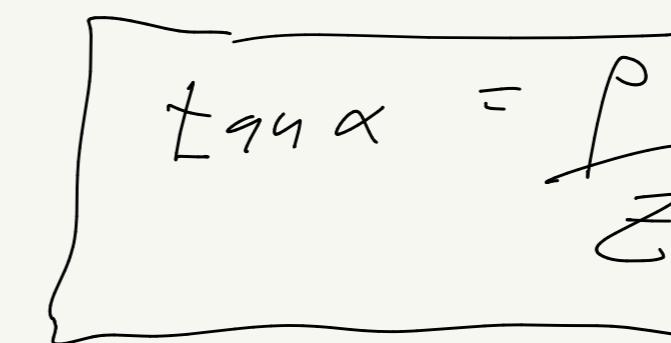
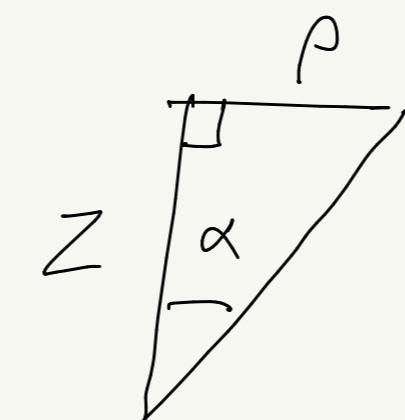
(Q) & A:

Determine constraint force vary
method of Lagrange multiplier



$$U = mgz$$

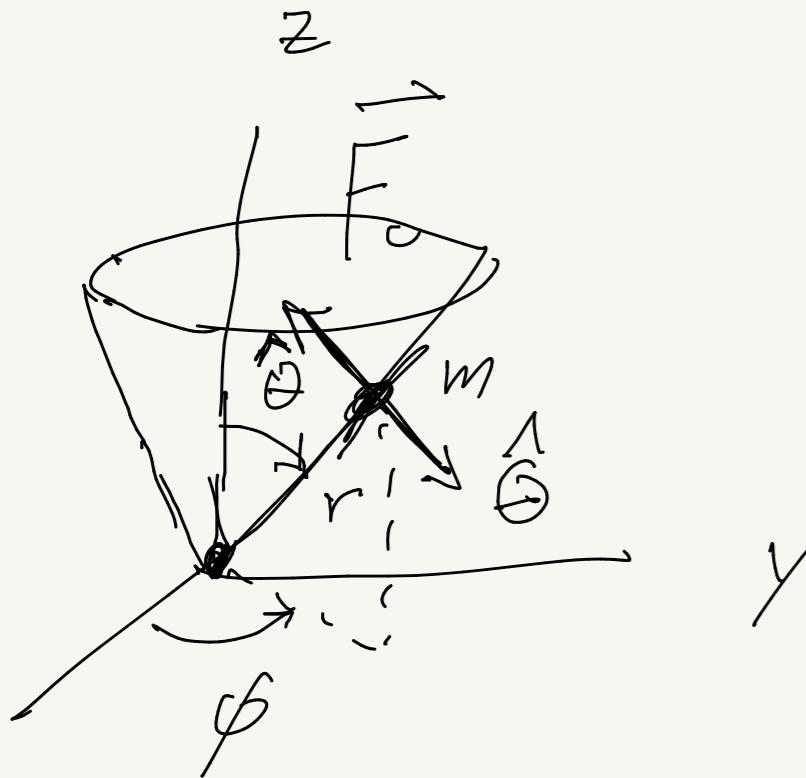
cylindrical coord (ρ, ϕ, z)



$$T = \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2)$$

$$= \frac{1}{2} m (z^2 \tan^2 \alpha + z^2 \tan^2 \alpha \dot{\phi}^2 + \dot{z}^2)$$

$$= \frac{1}{2} m (z^2 (1 + \tan^2 \alpha) + z^2 \tan^2 \alpha \dot{\phi}^2 + \dot{z}^2)$$



$$T = \frac{1}{2}m(r^2 + r^2\dot{\phi}^2 + r^2 \sin^2\theta \dot{\phi}^2)$$

$$\boxed{\theta = \alpha} = \text{const}$$

$$T = \frac{1}{2}m(r^2 + r^2 \sin^2\alpha \dot{\phi}^2)$$

$$U = mgZ = mgr \cos\alpha$$

x

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

$$\phi(r, \theta, \phi) = \alpha - \theta = \text{const}$$

constraint

$$L' = \left(\frac{1}{2}m(r^2 + r^2\dot{\phi}^2 + r^2 \sin^2\theta \dot{\phi}^2) - mgr \cos\theta \right) + \lambda(\alpha - \theta)$$

(r, phi), phi, r = l

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{r}} \right) = \frac{\partial L'}{\partial r}$$

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{\theta}} \right) = \frac{\partial L'}{\partial \theta}$$

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{\phi}} \right) = \frac{\partial L'}{\partial \phi}$$

$$\varphi = (\alpha - \theta) = \phi \rightarrow \theta = \alpha$$

Differentiate constraint w/r/t time

$$\dot{\varphi} = 0, \quad \ddot{\varphi} = 0$$

$$\dot{\theta} = \omega$$

$$F_C^L = \lambda \vec{\nabla} \varphi = -\lambda \vec{\omega}$$

$$L' = L + \lambda \varphi$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} + \lambda \frac{\partial \varphi}{\partial r}$$

etc.

w/r/t time

solve for λ
(algebraic)

Hamiltonian:

$$L = T - U$$

$$H = \left(\sum_i p_i \dot{q}_i - L \right) \quad \left(\frac{dE}{dt} = 0 \text{ if } \frac{\partial L}{\partial t} = 0 \right)$$

$$H(q, p, t)$$

DOF

$q \equiv q_i$ generalized coord,

$$i = 1, \dots, s$$

$p \equiv p_i$ generalized momenta

$$i = 1, \dots, s$$

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \quad (\text{def'n}) = f(q, \dot{q})$$

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

Energy conserved?

$$E = \frac{1}{2} m \dot{x}^2 + U(x)$$

Yes because $\frac{\partial L}{\partial t} = 0$

Momentum conjugate to x ?

$$p = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \rightarrow \dot{x} = \frac{p}{m}$$

p_x, p_y, p_z

$$\begin{aligned} F &= (p \dot{x} - L)_{\dot{x}} = p/m \\ &= (p \dot{x} - (\frac{1}{2} m \dot{x}^2 - U(x))) \\ &= p \cdot \frac{p}{m} - \frac{1}{2} m \left(\frac{p}{m} \right)^2 + U(x) \\ &= \frac{p^2}{2m} + U(x) \end{aligned}$$

Hamilton's equations:

$$\left\{ \begin{array}{l} \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad i=1, \dots, s \\ \dot{q}_i = \frac{\partial H}{\partial p_i} \end{array} \right. \quad i=1, \dots, s$$

(2s) 1st-order ordinary
differential equations,
(coupled)

Euler-Lagrange
equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$$

$$(i=1, 2, \dots, s)$$

$$L = L(\dot{q}, q)$$

s 2nd order ordinary
diff equations,
(coupled)

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$H = \frac{p^2}{2m} + U(x)$$

EL equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

$$m \ddot{x} = - \frac{\partial U}{\partial x}$$

$$(m\ddot{x} = F)$$

-Hamilton's equations:

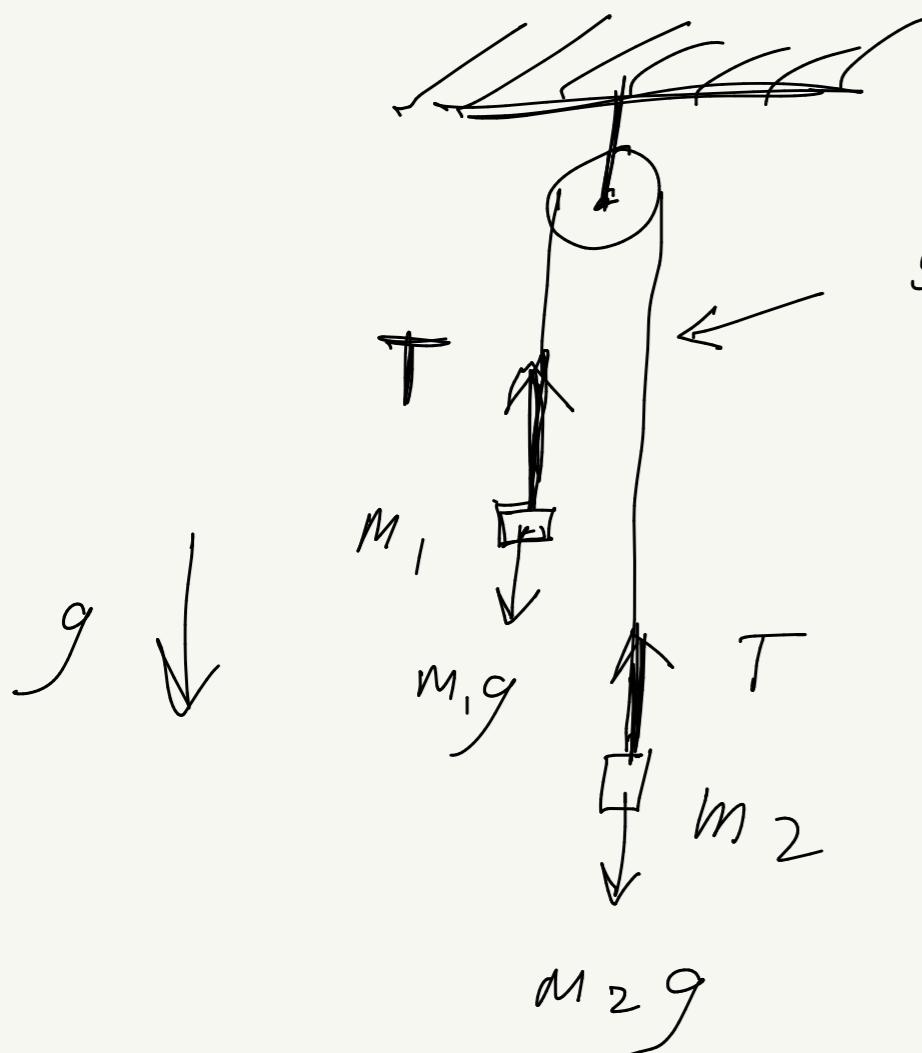
$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = - \frac{\partial H}{\partial x} = - \frac{\partial U}{\partial x}$$

$$(p = m\dot{x})$$

$$m \ddot{x} = - \frac{\partial U}{\partial x}$$

A 'Twood' machine:



Net force down:

$$m_1 g - T = m_1 \alpha$$

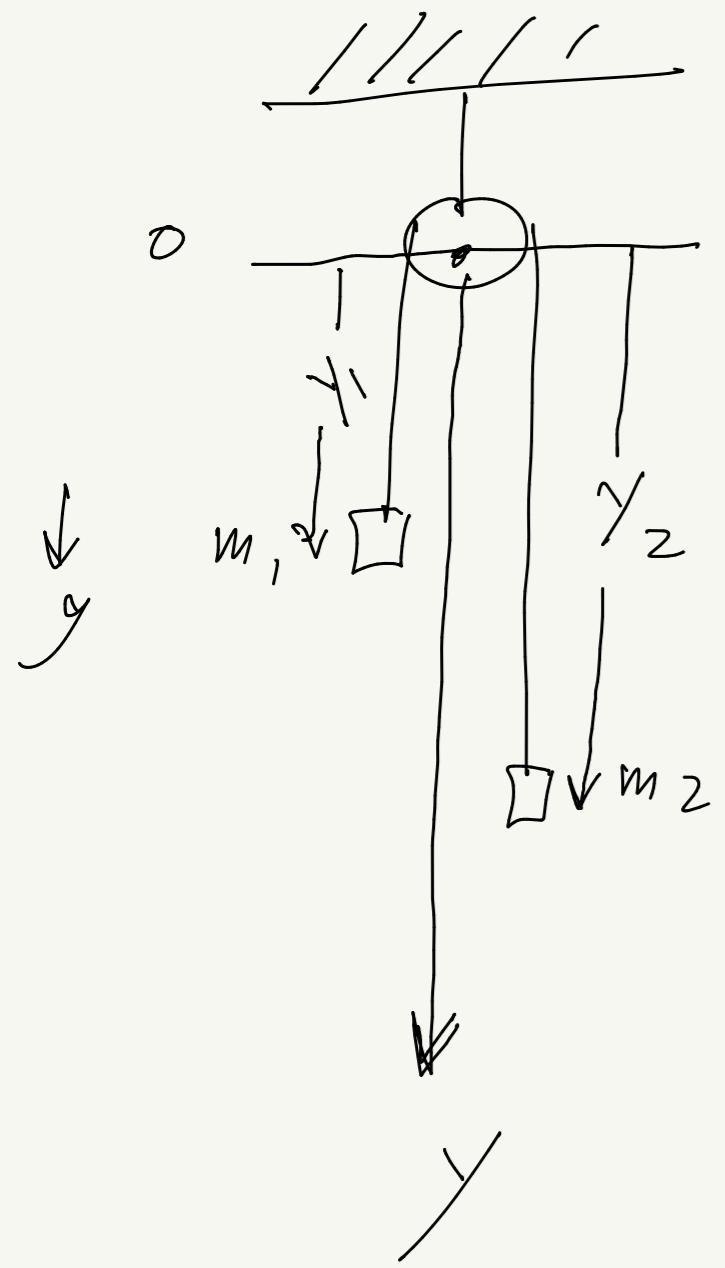
$$m_2 g - T = -m_2 \alpha$$

subtract $(m_1 - m_2)g = (m_1 + m_2)\alpha$

$$\boxed{\alpha = \frac{(m_1 - m_2)g}{m_1 + m_2}}$$

$$\alpha = \frac{g(m_1 - m_2)}{m_1 + m_2}$$

(down)



$$L = T - U$$

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2$$

$$\begin{aligned} T &= \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_1^2 \\ &= \frac{1}{2} (m_1 + m_2) \dot{y}_1^2 \end{aligned}$$

$$y_2 = l - y_1$$

$$P = l - (y_1 + y_2)$$

$$\begin{aligned} U &= -m_1 g y_1 - m_2 g y_2 \\ &= -m_1 g y_1 - m_2 g (l - y_1) \\ &= -(m_1 - m_2) g y_1 \underbrace{- m_2 g l}_{\text{const}} \end{aligned}$$

$$y_1 + y_2 = l$$

$$y_2 = l - y_1$$

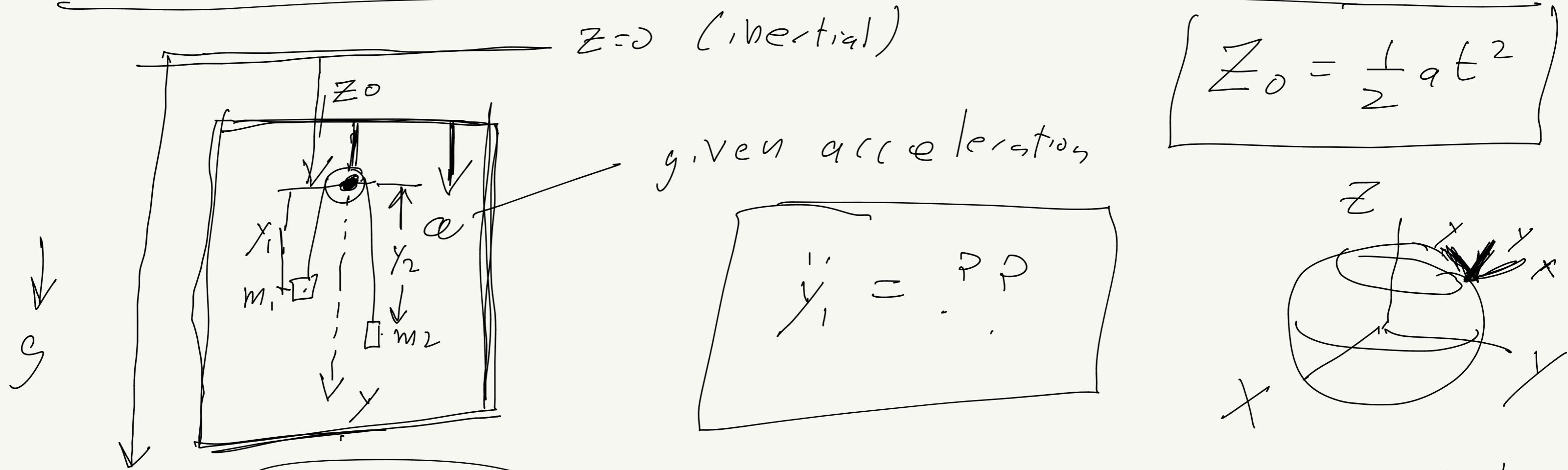
$$L = \frac{1}{2} (m_1 + m_2) \dot{y}_1^2 + (m_1 - m_2) g y_1 \quad \cancel{(m_1 - m_2) g y_1}$$

$$\ddot{y}_2 = -\ddot{y}_1$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}_1} \right) = \frac{\partial L}{\partial y_1} \rightarrow (m_1 + m_2) \ddot{y}_1 = (m_1 - m_2) g$$

$$\ddot{y}_1 = \frac{(m_1 - m_2) g}{m_1 + m_2}$$

Analyze Atwood's machine in an accelerating reference frame.



$L = T - U$ is valid only wrt an inertial reference frame

$$\vec{F} = m \vec{a} \quad (\text{only wrt inertial frame})$$

