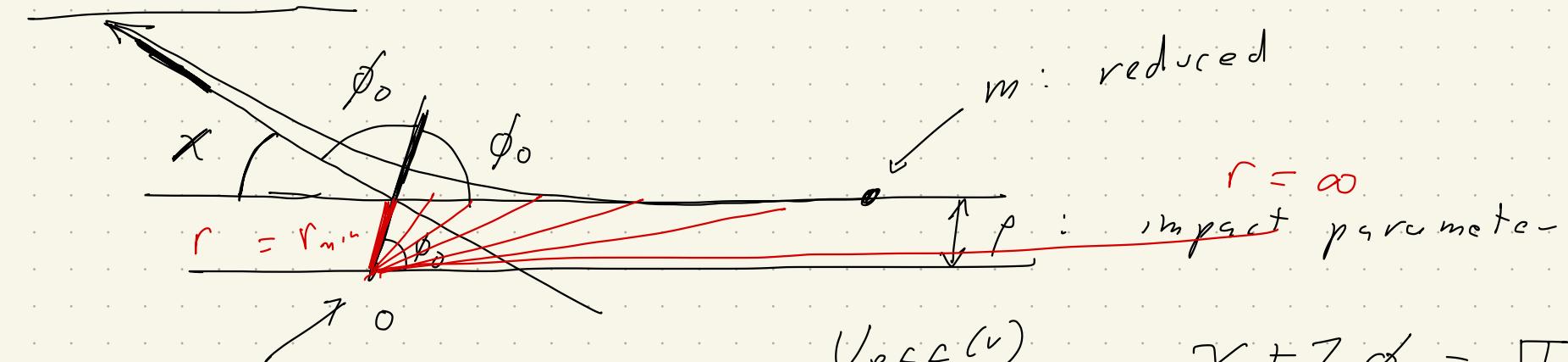


Lecture #16

Thurs 10/14



(CM
of original
system)

$$U(r) = \frac{\alpha}{r}, \quad \alpha > 0$$

(repulsive)

$$U_{\text{eff}}(r) = U(r) + \frac{M_z^2}{2mr^2}$$

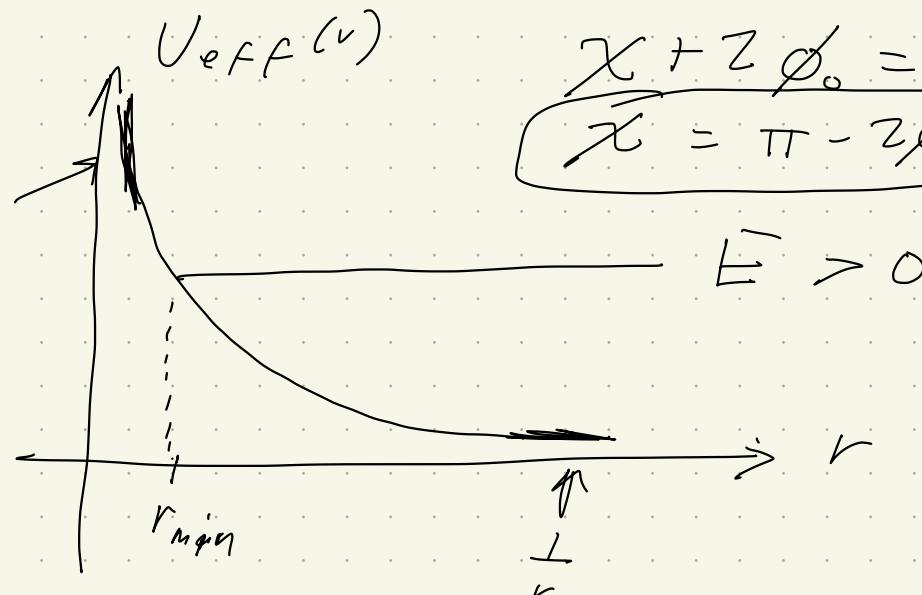
$$= \frac{\alpha}{r} + \frac{M_z^2}{2mr^2}$$

↑ ↑

m : reduced

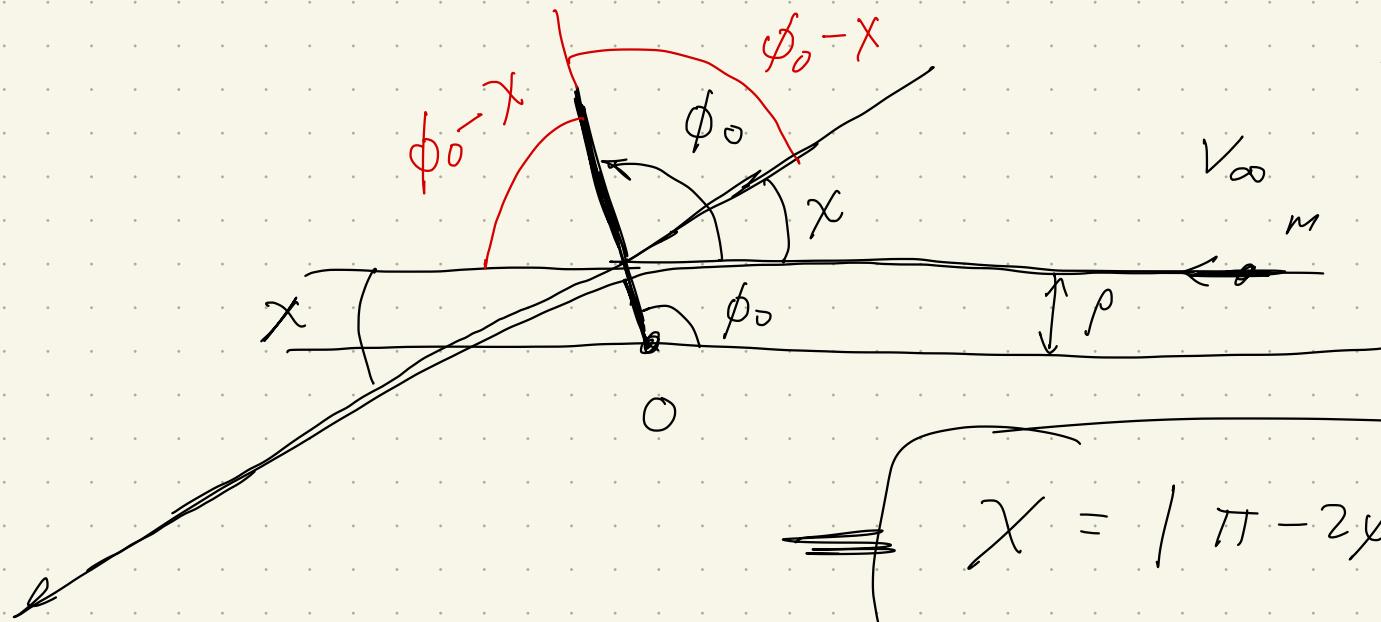
$r = \infty$

p : impact parameter



$$E = U_{\text{eff}}(r_{\min})$$

↑
turning point



$$2(\phi_0 - x) + x = \pi$$

$$2\phi_0 - x = \pi$$

$$x = 2\phi_0 - \pi$$

$$x = |\pi - 2\phi_0|$$

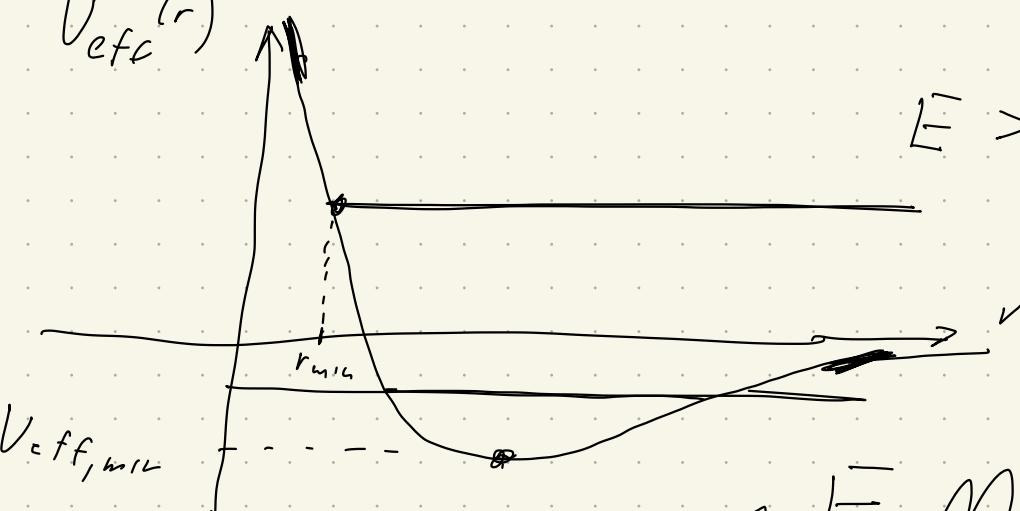
for both
repulsive &
attractive

$$U(r) = -\frac{\alpha}{r}$$

$$U_{eff}(r)$$

$$U_{eff}(r) = -\frac{\alpha}{r} + \frac{M_t^2}{2mr^2}$$

$$E > 0$$



$$E = \frac{1}{2}mv_\infty^2 \quad ; \quad M = m\rho v_\infty$$

$$E, M \\ \rho, v_\infty$$

Sect 14:

$$t = \pm \int \frac{dr}{\sqrt{\dots}} + \text{const}$$

$\Rightarrow \phi = \pm \int \frac{M dr/r^2}{\sqrt{2m(E - V(r)) - \frac{M^2}{r^2}}} + \text{const}$

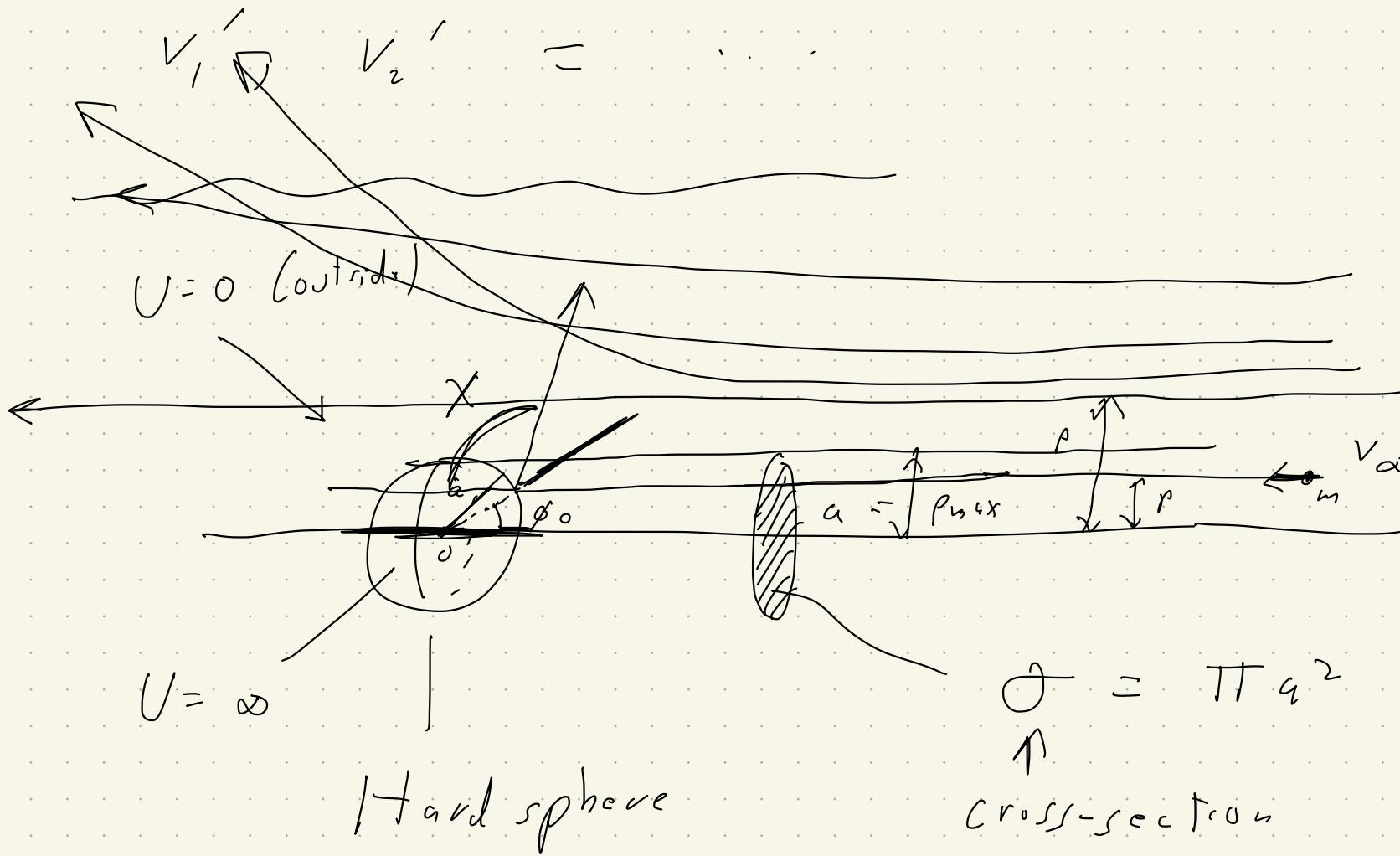
$$\begin{aligned} &= \int_{r_{min}}^{\infty} \frac{M dr/r^2}{\sqrt{2m(E - V(r)) - \frac{M^2}{r^2}}} \quad E = U_{eff}(r_{max}) \\ &= \int_{r_{min}}^{\infty} \frac{m\rho v_\infty dr/r^2}{\sqrt{2m\left(\frac{1}{2}mv_\infty^2 - V(r)\right) - \frac{m^2\rho^2 v_\infty^2}{r^2}}} \\ &= \int_{r_{min}}^{\infty} \frac{\rho dr/r^2}{\sqrt{1 - \frac{V(r)}{\frac{1}{2}mv_\infty^2} - \frac{\rho^2}{r^2}}} \end{aligned}$$

$M = m\rho v_\infty$
 $E = \frac{1}{2}mv_\infty^2$

$$\theta_2 = \frac{1}{2}(\pi - x)$$

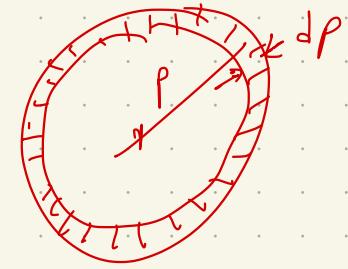
$$x = |\pi - 2\phi_0|$$

$$f_{q1}\theta_1 = \frac{m_2 \sin x}{m_1 + m_2 \cos x}$$



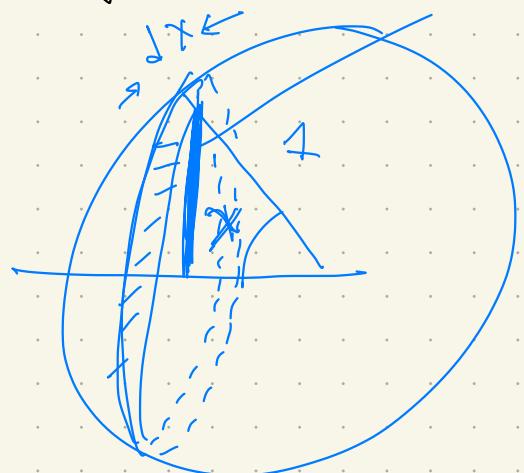
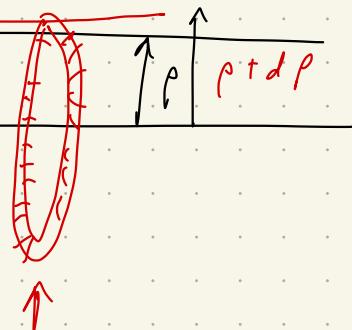
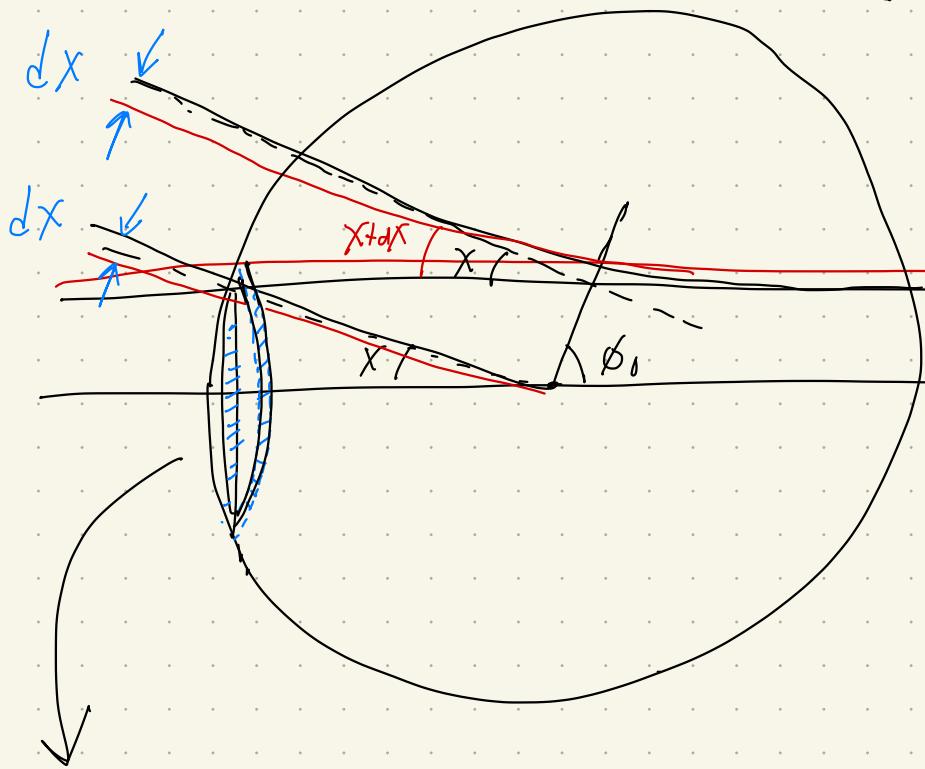
$$d\sigma = 2\pi \rho(x) \left| \frac{dp}{dx} \right| dx$$

$$d\sigma = 2\pi \rho dp$$



Differential cross sect.

!o



$$d\sigma = \frac{\rho(x)}{\sin x} \left| \frac{dp}{dx} \right| d\Omega$$

$$d\Omega = 2\pi \sin x dx$$

solid angle

$$\frac{d\sigma}{d\Omega} = \frac{2\pi \rho dp}{2\pi \sin x dx} = \frac{\rho(x)}{\sin x} \left| \frac{dp}{dx} \right|$$

$$\frac{d\sigma}{d\Omega} = \frac{p}{\sin x} \left| \frac{dp}{dx} \right|$$

Com frame

$$\frac{d\sigma_1}{d\Omega_1} = \frac{p}{\sin \theta_1} \left| \frac{dp}{d\theta_1} \right|$$

Lab frame
 θ_1

$$\frac{d\sigma_2}{d\Omega_2} = \frac{p}{\sin \theta_2} \left| \frac{dp}{d\theta_2} \right|$$

Lab frame
 θ_2

$$\begin{aligned} \frac{d\sigma_1}{d\Omega_1} &= \frac{\sin x}{\sin \theta_1} \left| \frac{dx}{d\theta_1} \right| \frac{d\sigma}{d\Omega} \\ &= \left| \frac{d(\cos x)}{d(\cos \theta_1)} \right| \frac{d\sigma}{d\Omega} \end{aligned}$$

calculated
for the
Com

Lecture #17: Tues Oct 19th

Next week: start small oscillations

Today / Thurs: Q & A (collisions and scattering)

Lecture #18: Thur Oct 21st

Next week: small oscillations

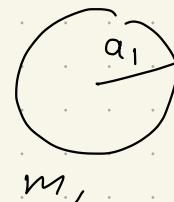
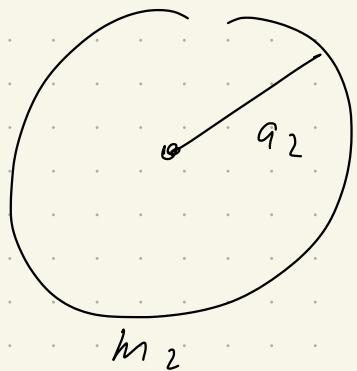
Today: Q & A, Quiz #3

(last 20 minutes)

Q3 - name.pdf

Quiz #3:

Two hard spheres with masses m_1, m_2 and radii a_1, a_2 :



$$\rho(X) = (a_1 + a_2) \cos\left(\frac{X}{2}\right)$$

a) Find $\rho = \rho(X)$ where $X = \text{scat. angle}$ w.r.t. com

b) What value of ρ will give $\theta_2 = 60^\circ$? frame

$$2\theta_2 + X = \pi \rightarrow X = \pi - 2\theta_2 \\ = 60^\circ$$

$$\rho = (a_1 + a_2) \cos(30^\circ) \\ = \frac{\sqrt{3}}{2} (a_1 + a_2)$$

Lec #19:

10/26

Small oscillations:

Sec 21

, 22, 23

more
than
1d

Free oscillations
in 1-d

Forced oscillations
in 1d

Damping : Sec 25 (not covered)

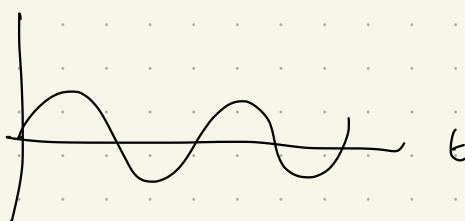
SHM, (Simple harmonic motion)

i) sinusoidal

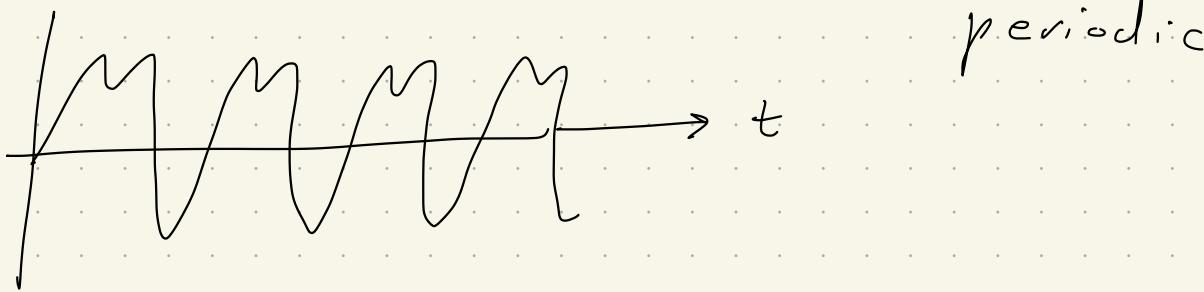
$$\omega = \frac{2\pi}{T}$$

$$x(t) = a \cos(\omega t + \alpha)$$

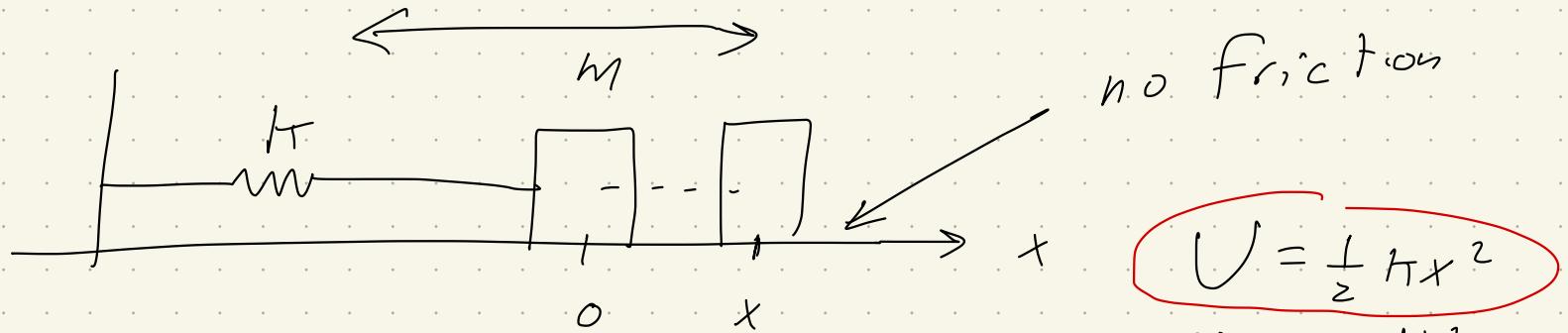
sinus



amplitude | phase
 |
 angular
 freq



Example:



$$F = -kx \quad \begin{matrix} \text{linear} \\ \text{restoring} \end{matrix}$$

$$F = m\ddot{x} = -kx \rightarrow \ddot{x} = -\frac{k}{m}x$$

$$\begin{aligned} F &= -\frac{dU}{dx} \\ &= -kx \end{aligned}$$

sol'n: $x = C_1 \cos \omega t + C_2 \sin \omega t$

$$\omega = \sqrt{\frac{k}{m}}$$

Alternatives:

$$x = a \cos(\omega t + \alpha)$$

$$x = \operatorname{Re}[A e^{i\omega t}], A = a e^{i\alpha}$$

$$T = \frac{1}{2} m \dot{x}^2$$

$$U = \frac{1}{2} k x^2$$

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \end{aligned}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \rightarrow m \ddot{x} = -kx$$

$$U(q) = q^3$$

$$q_0 = 0$$

More generally:

$$\left(\frac{d^2 U}{d q^2} \Big|_{q_0} = 0 \right)$$

pt. of inflection
(saddle)

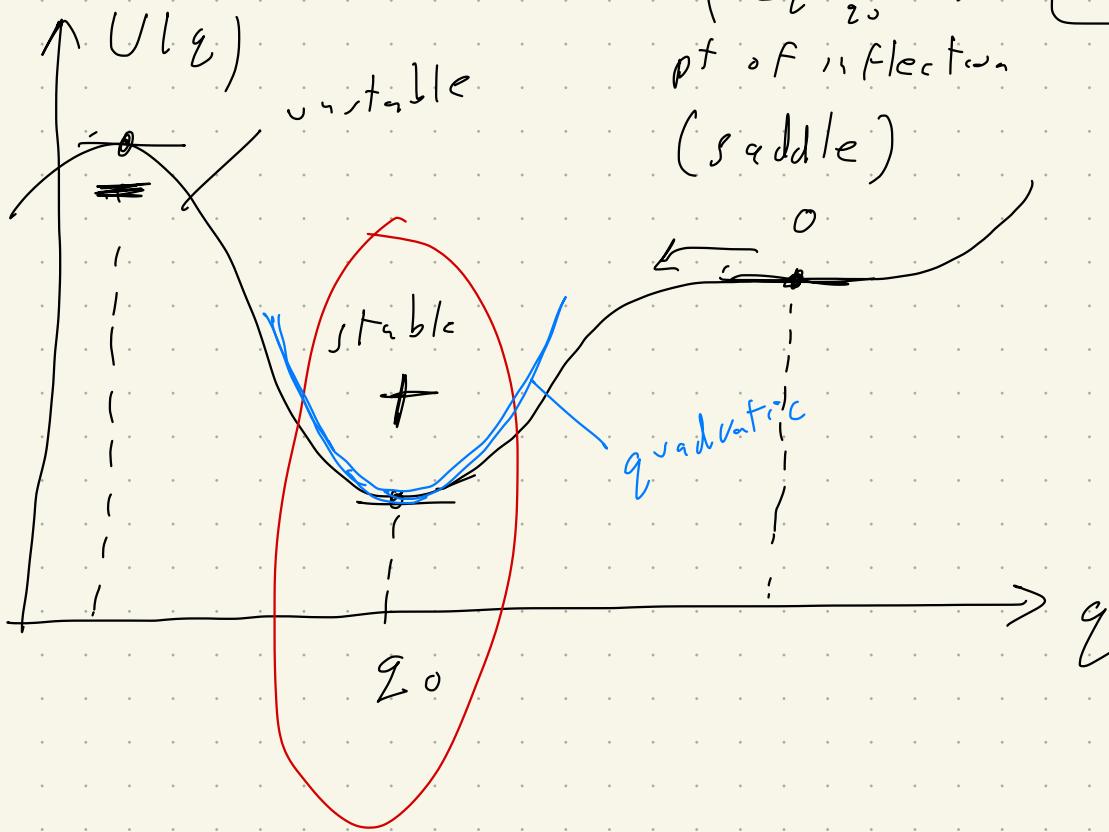
$$L = \frac{1}{2} a(q) \dot{q}^2 - U(q)$$

Equilibrium: ($q = q_0$)

$$0 = F = -\frac{dU}{dq} \Big|_{q_0}$$

stable:

$$\frac{d^2 U}{d q^2} \Big|_{q_0} \geq 0$$



$$U = \text{const}, T$$

Expansion: (about q_0)

$$U(q) = U(q_0) + \underbrace{\frac{dU}{dq} \Big|_{q_0} (q - q_0)}_{\text{const}} + \frac{1}{2} \left(\frac{d^2 U}{dq^2} \Big|_{q_0} \right) (q - q_0)^2$$

(ignore $\sim L$)

$K > 0$

$$+ \frac{1}{3!} \left(\frac{d^3 U}{dq^3} \Big|_{q_0} \right) (q - q_0)^3 + \dots$$

ignore for $q - q_0$ small

$$U(q) \approx \frac{1}{2} K x^2$$

$$T = \frac{1}{2} q(q) \dot{q}^2$$

$$= \frac{1}{2} q(q_0) \dot{x}^2$$

$$\approx \frac{1}{2} m \dot{x}^2$$

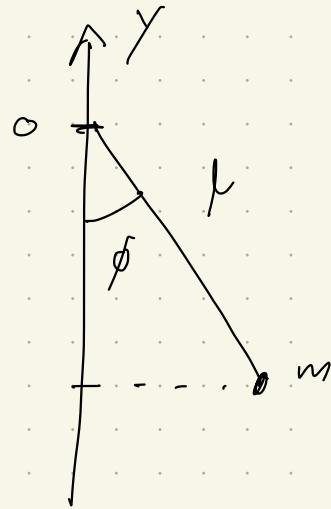
$$x = q - q_0$$

$$K = \frac{d^2 U}{dq^2} \Big|_{q_0}$$

$$L \approx \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2$$

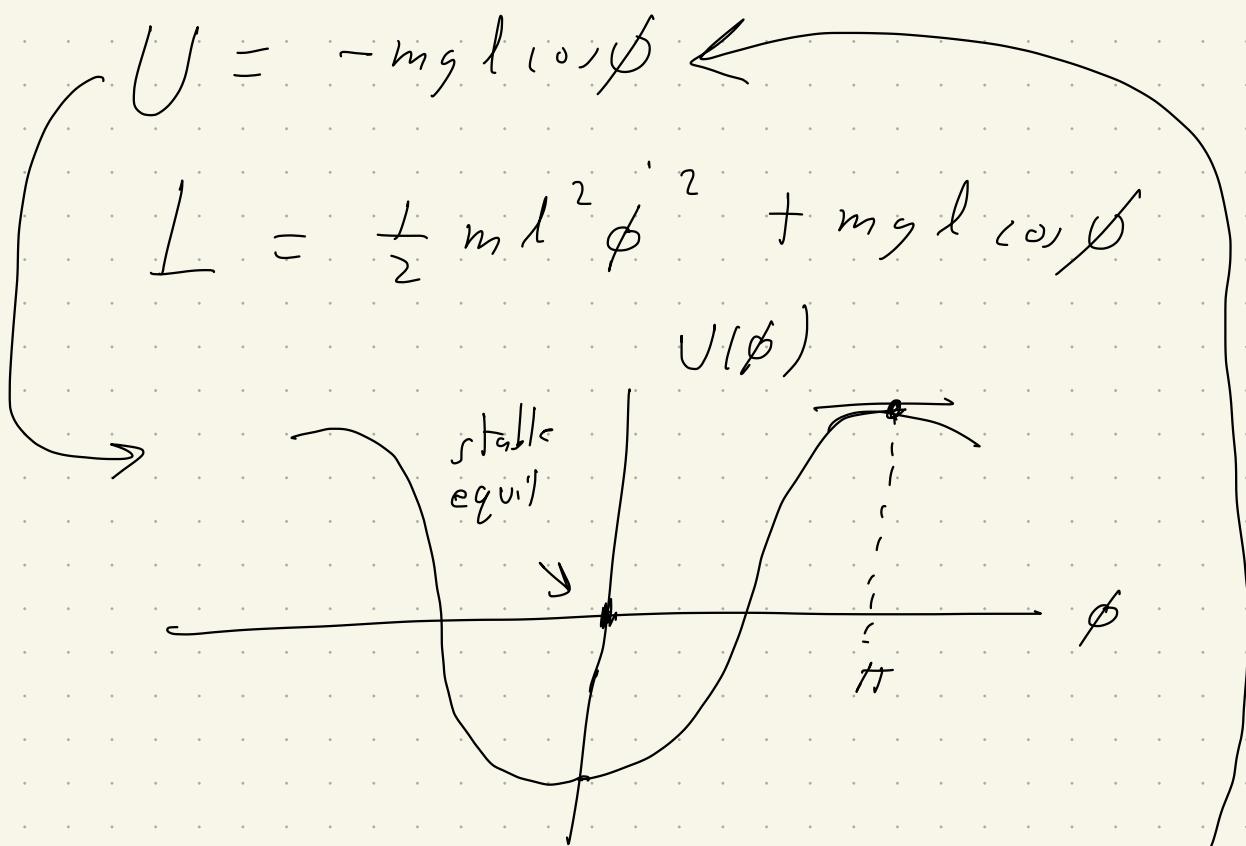
Simple pendulum:

$$T = \frac{1}{2} m l^2 \dot{\phi}^2$$

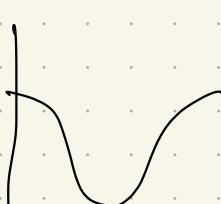


$$U = mgx$$

$$= -mgl \cos \phi$$



$$\phi_0 = 0 \text{ (equilibrium)}$$



$$\cos \phi = 1 - \frac{1}{2} \phi^2 + \dots$$

$$\cos \phi = \underbrace{\cos 0}_1 + \frac{d(\cos \phi)}{d\phi} \Big|_{\phi=0} \cdot \phi + \frac{1}{2} \frac{d^2(\cos \phi)}{d\phi^2} \Big|_{\phi=0} \phi^2$$

$\downarrow \sin \phi$

$$= 1 - \frac{1}{2} \phi^2 + \dots$$

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl \left(1 - \frac{1}{2} \phi^2 \right)$$

$$= \frac{1}{2} \cancel{m l^2} \dot{\phi}^2 - \frac{1}{2} \cancel{mgl} \phi^2 + mgl$$

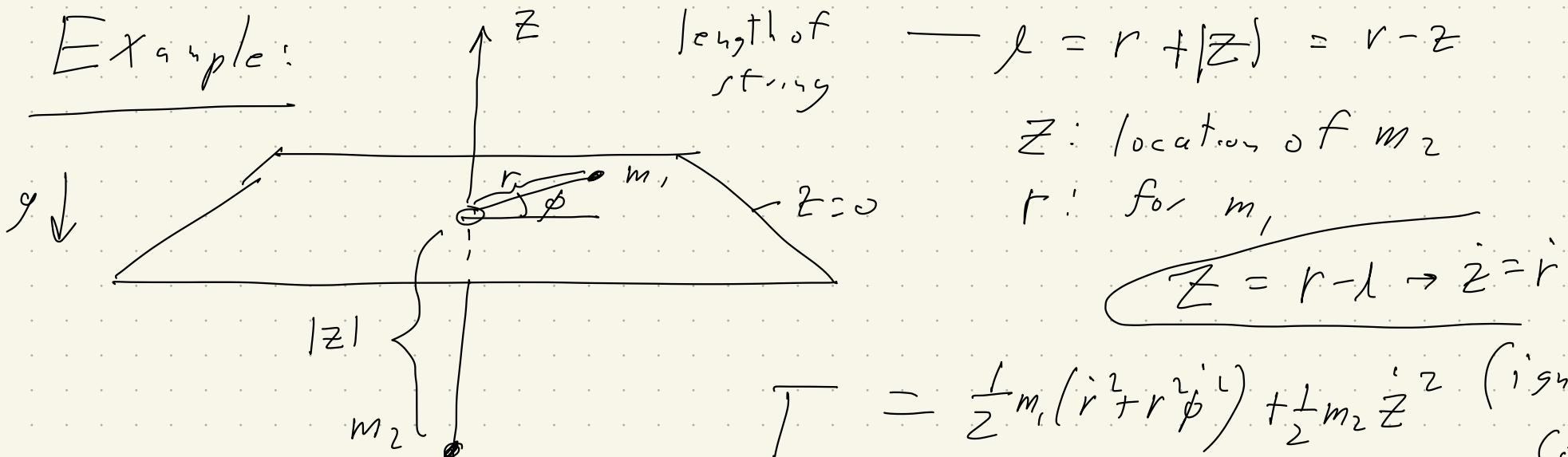
M H $l g u v e$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} \cancel{m} \dot{\phi}^2$$

→

$$\omega = \sqrt{\frac{H}{m}} = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}$$

Example:



$$T = \frac{1}{2}m_1(r^2 + r^2\dot{\phi}^2) + \frac{1}{2}m_2z^2 \quad (\text{ignoring centripetal force})$$

$$U = m_2gz = m_2g(r - \ell)$$

$$= m_2gr - m_2gl$$

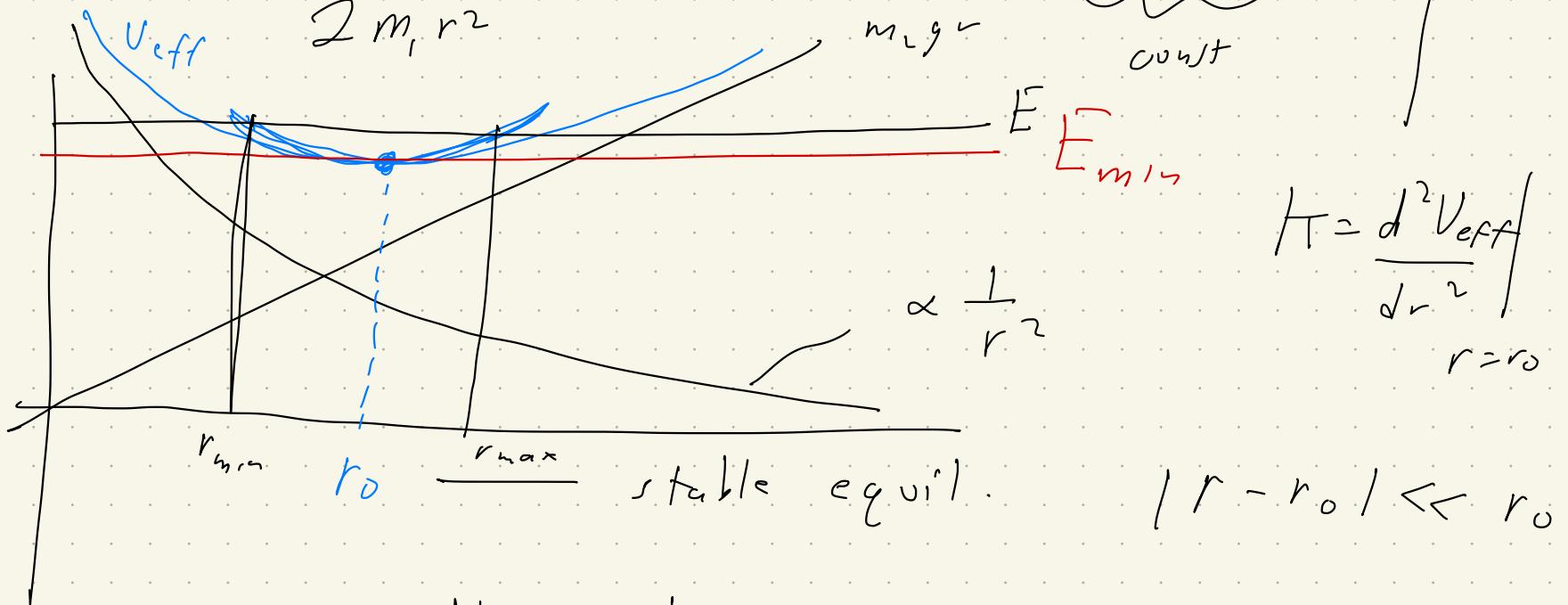
$$L = \frac{1}{2}(m_1 + m_2)r^2 + \frac{1}{2}m_1r^2\dot{\phi}^2 - m_2gr$$

$$\text{i)} \quad M_z = \frac{\partial L}{\partial \dot{\phi}} = m_1r^2\dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{m_1r^2}$$

$$\text{ii)} \quad E = T + U = \frac{1}{2}(m_1 + m_2)r^2 + \underbrace{\frac{1}{2}m_1r^2\dot{\phi}^2}_{\frac{M_z^2}{2m_1r^2}} + m_2gr$$

$$E = \frac{1}{2} \cancel{(m_1 + m_2)} r^2 + U_{\text{eff}}(r)$$

$$U_{\text{eff}}(r) = \frac{M_z^2}{2m_1 r^2} + m_2 g r = U(r_0) + \frac{1}{2} k(r - r_0)^2$$



$$T = \left. \frac{d^2 U_{\text{eff}}}{dr^2} \right|_{r=r_0}$$

$$r_0 = ?$$

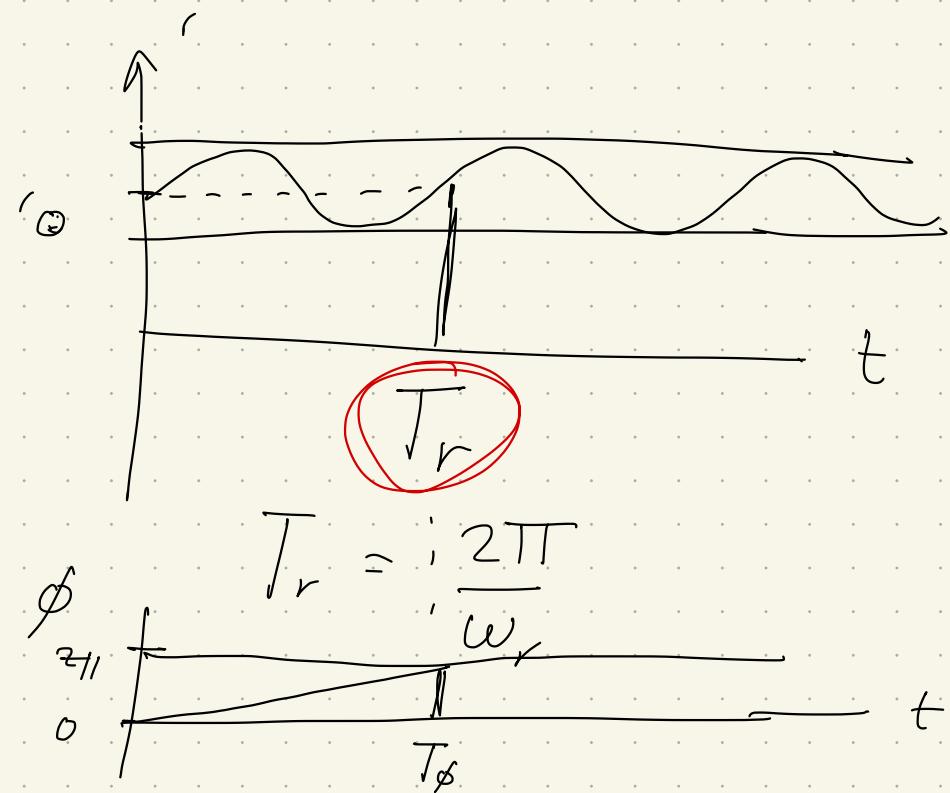
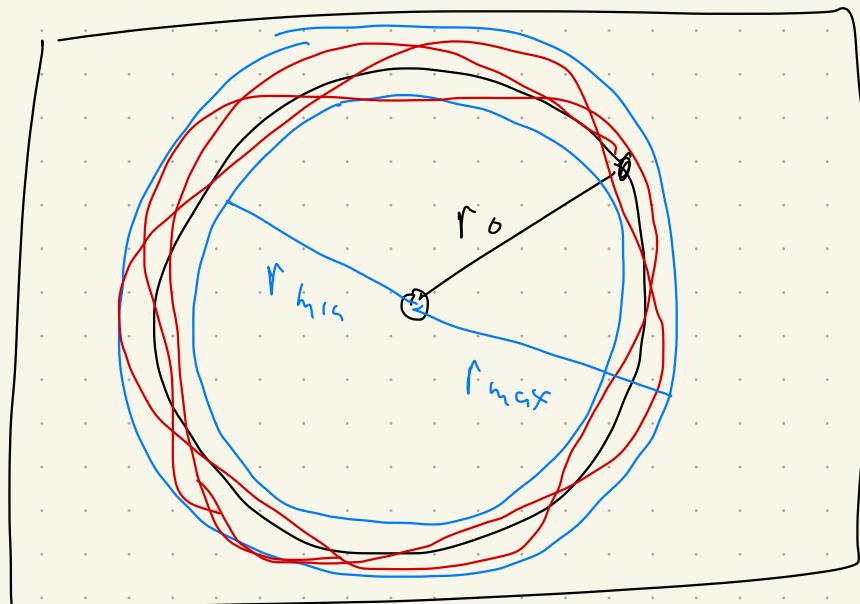
$$0 = \left. \frac{d U_{\text{eff}}}{dr} \right|_{r=r_0}$$

$$= - \frac{M_z^2}{m_1 r_0^3} + m_2 g$$



$$\boxed{M_z^2 = m_1 m_2 g r_0^3}$$

$$\begin{aligned}
 \frac{\int_0^r U_{\text{eff}}}{\int_{r_0}^r} &= \frac{3 M_2^2}{m_1 r_0^4} \\
 &= \frac{3 m_1 m_2 g r_0^3}{m_1 r_0^4} \\
 &= \frac{3 m_2 g}{r_0} = k
 \end{aligned}$$



$$T_\phi = \frac{2\pi}{\omega_\phi} = \frac{2\pi}{\dot{\phi}/r_0}$$

$$\dot{\phi} = \frac{M_z}{m_1 r^2}$$

$$\omega_\phi = \dot{\phi} \Big|_{r=r_0} = \frac{M_z}{m_1 r_0^2} = \frac{\sqrt{m_1 m_2 g r_0^3}}{m_1 r_0^2}$$

$$\omega_\phi = \sqrt{\frac{m_2 g}{m_1 r_0}}$$

$$\omega_r = \sqrt{\frac{3 m_2 g}{(m_1 + m_2) r_0}}$$

Compare to

$$f \quad \frac{3}{m_1 + m_2} = \frac{1}{m_1} \quad \text{then} \quad \omega_r = \omega_\phi$$

$$\frac{3m_1}{m_1 + m_2} = m_1 + m_2$$

$$\boxed{2m_1 = m_2}$$



Lec #20: 10/28

Forced oscillations:

$$m\ddot{x} = -kx + F(t)$$

$$\ddot{x} + \frac{k}{m}x = \frac{F(t)}{m}$$

$$\boxed{\ddot{x} + \omega^2 x = \frac{F(t)}{m}}, \quad \omega = \sqrt{\frac{k}{m}}$$

general sol'n:

$$x(t) = x_h(t) + x_p(t)$$

↑
homogeneous
($F(t)=0$)

particular

(any solution for $F(t)$)

$$\boxed{x_h(t) = a \cos(\omega t + \alpha)}$$

a, α : two constants
(initial conditions)

Suppose: $F(t) = f \cos(\gamma t + \beta)$

$$x_p'' + \omega^2 x_p = \frac{f}{m} \cos(\gamma t + \beta)$$

Guess !

$$x_p(t) = b \cos(\gamma t + \beta)$$

$$\rightarrow -b\gamma^2 \cancel{\cos(\gamma t + \beta)} + \omega^2 b \cancel{\cos(\gamma t + \beta)} = \frac{f}{m} \cancel{\cos(\gamma t + \beta)}$$

$$b(\omega^2 - \gamma^2) = \frac{f}{m}$$

$$\rightarrow b = \frac{f}{m(\omega^2 - \gamma^2)}$$

+

$\text{10 cos}(\omega t)$

$$x_p(t) = \frac{f}{m(\omega^2 - \gamma^2)} \cos(\gamma t + \beta)$$

$$x(t) = a \cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)} \cos(\gamma t + \beta)$$

$$x(t) = a \cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)} [\cos(\gamma t + \beta) - \cos(\omega t + \beta)]$$



L'Hopital's

$$= \frac{\frac{d}{d\gamma} (num)}{\frac{d}{d\gamma} (den)} \Big|_{\gamma \rightarrow \omega}$$

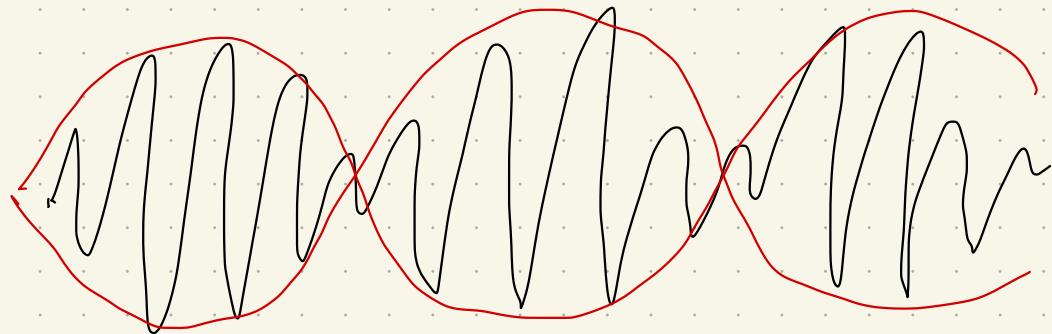
$$= \frac{+ f t \sin(\omega t + \beta)}{+ 2 m \omega}$$

at resonance

$$\downarrow (\gamma = \omega)$$

$$= \frac{ft}{2m\omega} \sin(\omega t + \beta)$$

$$x(t) = a \cos(\omega t + \alpha) + \frac{ft}{2m\omega} \sin(\omega t + \beta)$$



$$\frac{\omega + \gamma}{2}$$

$$|\omega - \gamma| = \omega_{\text{beat}}$$

General: for arbitrary $F(t)$

$$F(t) = \operatorname{Re} \int_{-\infty}^{\infty} d\gamma \tilde{F}(\gamma) e^{i\gamma t}$$

$\underbrace{\hspace{10em}}$

Fourier transform

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}$$

$$y' + P(x)y = Q(x)$$

Math methods

$$Y = Y(x)$$

Let:

$$\xi = \dot{x} + i\omega x$$

complex

$$\begin{aligned}\xi &= \dot{x} + i\omega x \\ &= \ddot{x} + i\omega (\dot{x} - i\omega x) \\ &= \ddot{x} + i\omega \dot{x} + \omega^2 x\end{aligned}$$

$$\rightarrow \xi' - i\omega \xi = \ddot{x} + \omega^2 x = \frac{F(t)}{m}$$

Homog: $\xi_h' - i\omega \xi_h = 0 \rightarrow \xi_h(t) = A e^{i\omega t}$

Guess: $\xi_p(t) = A(t) e^{i\omega t}$

replace
Complex constant

$$\boxed{\ddot{\xi}_p - i\omega \dot{\xi}_p = \frac{F(t)}{m}}, \quad \xi_p = A(t) e^{i\omega t}$$

$$\rightarrow \dot{A} e^{i\omega t} + \cancel{iAw e^{i\omega t}} - \cancel{i\omega A e^{i\omega t}} = \frac{F(t)}{m}$$

$$\dot{A} = \frac{e^{-i\omega t} F(t)}{m}$$

$$A(t) = \int dt \frac{F(t)}{m} e^{-i\omega t} + \text{const}$$

$$\boxed{\xi(t) = e^{i\omega t} \left[\int_0^t d\bar{t} \frac{F(\bar{t})}{m} e^{-i\omega \bar{t}} + \xi_0 \right]}$$

$$\boxed{\xi = x + i\omega x \rightarrow x(t) = \frac{1}{\omega} \text{Im}(\xi(t))}$$

Complex
constant
(I.C.)

$$y' + P(x)y = Q(x) \quad f(x)dx = g(y)dy$$

$$\frac{dy}{dx} + P(x)y - Q(x) = 0$$

$$\boxed{1 dy + (P(x)y - Q(x))dx = 0} \neq dU$$

$$dU(x,y) = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

$$\frac{\partial^2 U}{\partial y \partial x}$$

$$=$$

$$\frac{\partial^2 U}{\partial x \partial y}$$

M. Boas
~~Mathematica~~

$$\mu(x) [dy + (P(x)y - Q(x))dx] = dU$$

$$\frac{d\mu}{dx} = \frac{\partial}{\partial y} (P(x)y - Q(x)) \mu(x)$$

$$\frac{d\mu}{dx} = P(x) \mu(x)$$

$$\int \frac{d\mu}{\mu} = \int P(x) dx$$

$$\ln \mu = \int P(x) dx$$
$$\mu(x) = e^{\int p(x) dx}$$

Lec #21: Nov 2nd

Today: Sec 23 Free oscillations in 2 or more dimensions.

Thurs: Rigid body motion

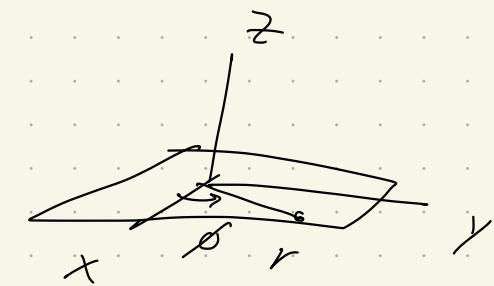
~~Exam~~ Exam 2: Nov 18th → Nov 23rd

Quiz #4: Today (at end of class)

Example: space oscillator

$$U = \frac{1}{2} kr^2 = \frac{1}{2} k(x^2 + y^2)$$

$$\begin{aligned} T &= \frac{1}{2} m(r^2 + r^2\dot{\phi}^2) \\ &= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) \end{aligned}$$



$$L = T - U$$

$$= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k(x^2 + y^2)$$

$$= \underbrace{\left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \right)}_{w_x = \sqrt{\frac{k}{m}}} + \underbrace{\left(\frac{1}{2} m \dot{y}^2 - \frac{1}{2} k y^2 \right)}_{w_y = \sqrt{\frac{k}{m}}}$$

$$w_x = \sqrt{\frac{k}{m}}$$

$$w_y = \sqrt{\frac{k}{m}}$$

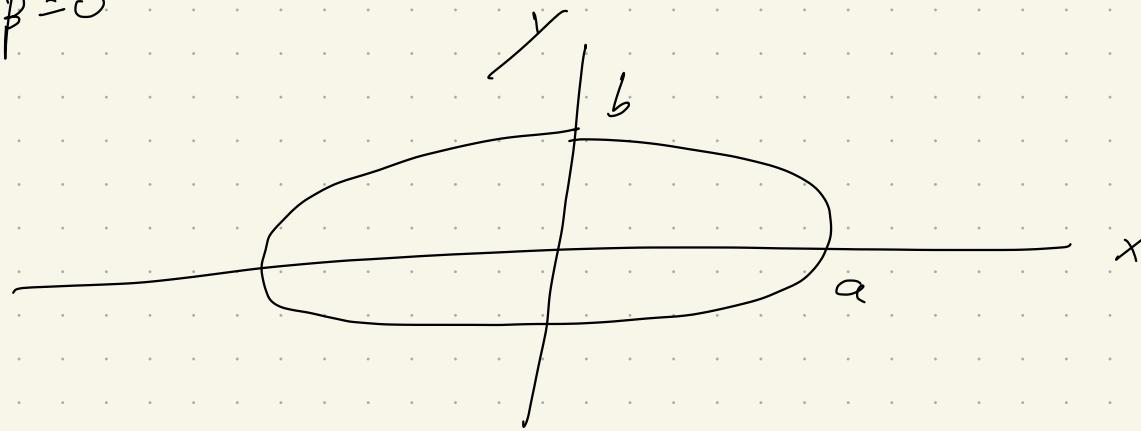
$$\omega = \sqrt{\frac{k}{m}}$$

$$x = a \cos(\omega t + \alpha)$$

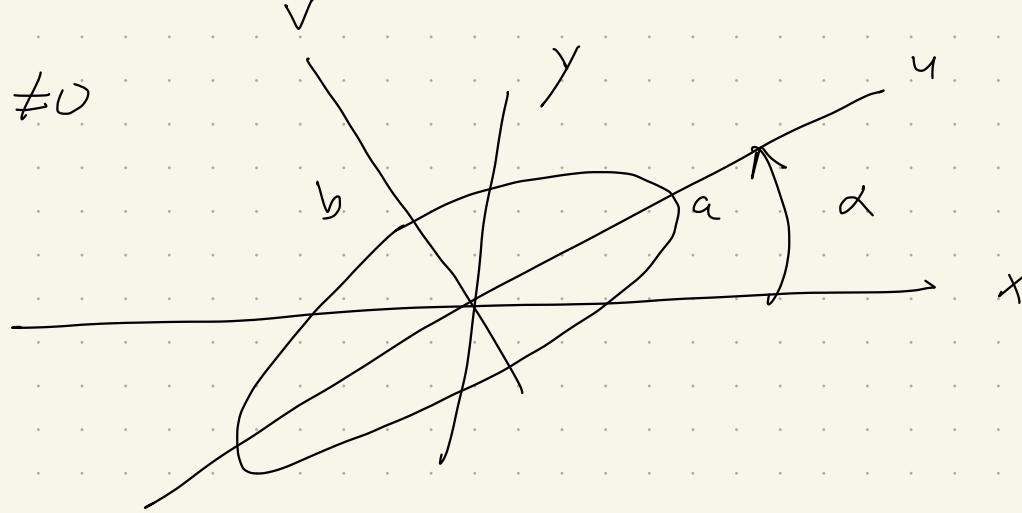
$$y = b \sin(\omega t + \beta)$$

general solutions

$$\alpha = \beta = 0$$

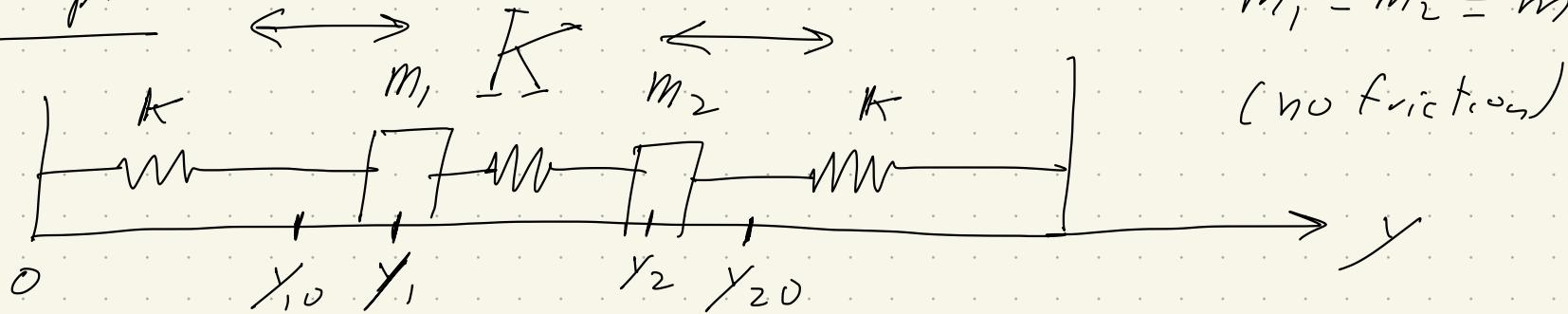


$$\alpha = \beta \neq 0$$



$\alpha \neq \beta$ — ellipse

Example:



$$m_1 = m_2 = m$$

(no friction)

$$L = T - U$$

$$x_1 = y_1 - y_{10}$$

$$x_2 = y_2 - y_{20}$$

$$T = \frac{1}{2} m \dot{y}_1^2 + \frac{1}{2} m \dot{y}_2^2 = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$U = \frac{1}{2} \kappa x_1^2 + \frac{1}{2} \kappa x_2^2 + \frac{1}{2} K (x_2 - x_1)^2$$

$$= \frac{1}{2} \kappa x_1^2 + \frac{1}{2} \kappa x_2^2 + \frac{1}{2} K (x_2^2 + x_1^2 - 2x_1 x_2)$$

$$= \frac{1}{2} (\kappa + K) x_1^2 + \frac{1}{2} (\kappa + K) x_2^2 - \cancel{\frac{1}{2} K x_1 x_2} - \cancel{\frac{1}{2} K x_1 x_2} - \cancel{\frac{1}{2} K x_2 x_1}$$

$$T = \frac{1}{2} \sum_{i=1}^n m_i \kappa \dot{x}_i^2$$

$$U = \frac{1}{2} \sum_{i,j} K_{ij} x_i x_j$$

$$K_{ij} = \begin{bmatrix} -K & -K \\ -K & K+2K \end{bmatrix}$$

m	0
0	m

$$L = \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k - \frac{1}{2} \sum_{i,k} t_{ijk} \dot{x}_i \dot{x}_k$$

geht auf)

$$\cancel{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_j} \right)} = \frac{\partial L}{\partial x_j} \quad j = 1, 2, \dots$$

$$\cancel{\frac{d}{dt} \left(\sum_k m_{jk} \dot{x}_k \right)} = - \sum_k t_{jk} \dot{x}_k$$

$$\cancel{\frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k} = \frac{1}{2} (m_{11} \dot{x}_1^2 + m_{12} \dot{x}_1 \dot{x}_2 + m_{13} \dot{x}_1 \dot{x}_3 + m_{21} \dot{x}_2 \dot{x}_1 + m_{22} \dot{x}_2^2 + m_{23} \dot{x}_2 \dot{x}_3 + m_{31} \dot{x}_3 \dot{x}_1 + m_{32} \dot{x}_3 \dot{x}_2 + m_{33} \dot{x}_3^2)$$

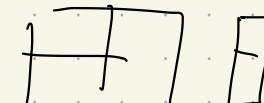
$$\begin{aligned} \frac{\partial L}{\partial \dot{x}_2} &= \frac{1}{2} (m_{12} \dot{x}_1 + m_{21} \dot{x}_1 + 2m_{22} \dot{x}_2 + m_{23} \dot{x}_3 + m_{32} \dot{x}_3) \\ &= \frac{1}{2} (m_{21} \dot{x}_1 + 2m_{22} \dot{x}_2 + 2m_{23} \dot{x}_3) \\ &= \sum_k m_{2k} \dot{x}_k \end{aligned}$$

$$\sum_{\pi} m_{j\pi} x_{\pi} = - \sum_{\pi} \tau_{j\pi} x_{\pi}$$

Guess: $x_{\pi} = A_{\pi} e^{i\omega t}$

$$\rightarrow \dot{x}_{\pi} = -\omega^2 A_{\pi} e^{i\omega t}$$

$$\sum_{\pi} -\omega^2 m_{j\pi} A_{\pi} e^{i\omega t} = - \sum_{\pi} \tau_{j\pi} A_{\pi} e^{i\omega t}$$



$$\sum_{\pi} (\tau_{j\pi} - \omega^2 m_{j\pi}) A_{\pi} = 0 \quad \text{0 vector}$$

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

TF invertible

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0 = \det(K_j H - \omega^2 m_j H)$$

characteristic equations

polynomial $(\omega^2)^N$
equation \rightarrow

Normal mode

freqs,

characteristic freqs,

eigen freqs

ω^2

label
the

N diff. eigs
freqs

$$0 = \det \left(\begin{bmatrix} K+K & -K \\ -K & K+K \end{bmatrix} - \omega^2 \begin{bmatrix} m_0 & 0 \\ 0 & m_0 \end{bmatrix} \right)$$

$$= \det \begin{vmatrix} K+K-\omega^2 m_0 & -K \\ -K & K+K-\omega^2 m_0 \end{vmatrix}$$

$$= ((K+K)-\omega^2 m_0)^2 - K^2$$

$$((k + K) - \omega^2 m)^2 = K^2$$

$$(k + K) - \omega^2 m = \pm K$$

$$\rightarrow \omega^2 = \frac{(k + K) \pm K}{m}$$

$$\boxed{\omega_+^2 = \frac{k + 2K}{m}, \quad \omega_-^2 = \frac{K}{m}}$$

eigen
freqs

solve for eigenvectors:



$$\omega_+^2 = \frac{K+2K}{m}$$

$$V_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

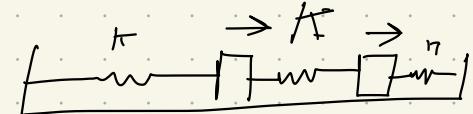
$$\omega_-^2 = \frac{K}{m}$$

$$V_- = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

normal mode
oscillation

$$\Delta \pi_\alpha = \frac{1}{2} \begin{bmatrix} V_+ & V_- \end{bmatrix} \begin{bmatrix} K \\ K \end{bmatrix}$$

Π matrix of eigenvectors



$$x_H = \sum_{\alpha} \Delta \pi_{\alpha} \theta_{\alpha}$$

$$\theta_{\alpha} = \operatorname{Re} [c_{\alpha} e^{i \omega_{\alpha} t}]$$

Complex const
determined by
 $\pm C_s$

$$L = T - U$$

$$= \frac{1}{2} \sum_{i, \pi} m_i \pi x_i \cdot x_\pi - \frac{1}{2} \sum_{i, \pi} K_{i, \pi} x_i \cdot x_\pi$$

$$= \frac{1}{2} \vec{x}^T M \vec{x} - \frac{1}{2} \vec{x}^T K \vec{x}$$

$$= \frac{1}{2} \vec{\theta}^T (\Delta^T_m \Delta) \vec{\theta} - \frac{1}{2} \vec{\theta}^T (\Delta^T K \Delta) \vec{\theta}$$

$$= \frac{1}{2} \sum_{\alpha} M_{\alpha} \vec{\theta}_{\alpha}^2 - \frac{1}{2} \sum_{\alpha} J_{\alpha} \vec{\theta}_{\alpha}^2$$

★ ★

where $M = \text{diagonal matrix} = \Delta^T_m \Delta$
 $J = \text{diag}$

$$= \Delta^T K \Delta$$

Eigenvectors
 diagonalize
 both $m_{i, \pi}$ and
 $K_{i, \pi}$

CHECK:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{m}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{m}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow M_+ = m, M_- = \omega$$

$$\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} K+K & -K \\ -K & K+K \end{vmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} K+2K & K \\ -K-2K & K \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 2K+4K & 0 \\ 0 & 2K \end{vmatrix}$$

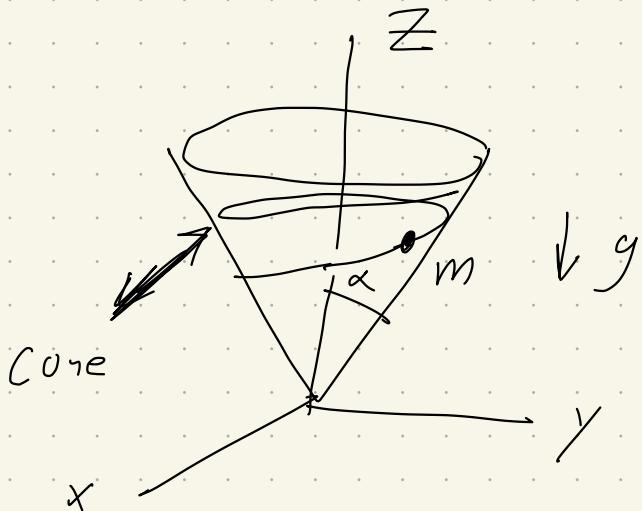
$$= \begin{vmatrix} K+2K & 0 \\ 0 & K \end{vmatrix}$$

$$= \begin{vmatrix} \omega_+^2 m & 0 \\ 0 & \omega_-^2 m \end{vmatrix} \quad \rightarrow \quad \mathcal{E}_+ = \omega_+^2 m \\ \mathcal{E}_- = \omega_-^2 m$$

$$\text{Thus, } L = \frac{1}{2} m (\dot{\theta}_+^2 + \dot{\theta}_-^2) - \frac{1}{2} (m \omega_+^2 \dot{\theta}_+^2 + m \omega_-^2 \dot{\theta}_-^2) \\ = \left(\frac{1}{2} m \dot{\theta}_+^2 - \frac{1}{2} m \omega_+^2 \dot{\theta}_+^2 \right) + \left(\frac{1}{2} m \dot{\theta}_-^2 - \frac{1}{2} m \omega_-^2 \dot{\theta}_-^2 \right)$$

Quiz #4:

Calculate freq of small oscillations around
the circular orbit



α fixed

(Sph. polar coords)

$$\text{Solution } T = r_0$$

- Lagrangian?
- Determine r_0
(relationship between r_0 and M_z)
- Determine ω_r

~~q4~~

q4 - name.pdf

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