

Notes: Thurs 8/27

- 1) Elliptic Functions      ↪ go beyond small  
2) Simple pendulum      angle approx

### Elliptic Functions:

$$\int \frac{dx}{\sqrt{1-x^2}} = ? = \sin^{-1}(x) + \text{const}$$

"                   $\arcsin(x)$

$$\rightarrow \sin^2 \theta + \cos^2 \theta = 1$$

subst:  $x = \sin \theta \rightarrow \sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta$   
 $dx = \cos \theta d\theta$

$$\int \frac{\cos \theta d\theta}{\cos \theta} = \int d\theta = \theta = \sin^{-1}(x) \quad \begin{matrix} x = \sin \theta \\ \theta = \sin^{-1}(x) \end{matrix}$$

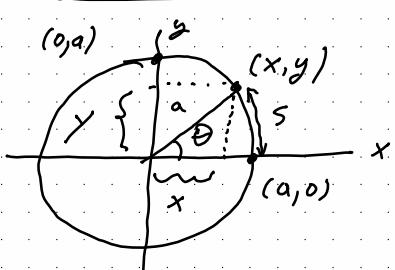
$$\frac{d}{d\theta} \sin \theta = \cos \theta \quad \lim_{h \rightarrow 0} \left( \frac{\sin(\theta+h) - \sin \theta}{h} \right)$$

$$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \dots$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

### Circular Functions..

$$x^2 + y^2 = a^2, \quad a = \text{radius}$$



$$\text{Def: } \sin \theta = \frac{y}{a}, \quad \cos \theta = \frac{x}{a}$$

$s$ : arc length from  $(0,0)$  to  $(x,y)$

$$s = a\theta \quad | \quad \theta = \frac{s}{a}$$

$$= \frac{1}{a} \int ds$$

$$= \frac{1}{a} \int \sqrt{dx^2 + dy^2}$$

$$= \int d\theta$$

$$\sqrt{dx^2 + dy^2} = ds$$

$(x+dx, y+dy)$

$dy$

$(x, y)$

$dx$

Given:  $x^2 + y^2 = a^2$

Follows: (i)  $a^2 \cos^2 \theta + a^2 \sin^2 \theta = a^2 \rightarrow [\cos^2 \theta + \sin^2 \theta = 1]$

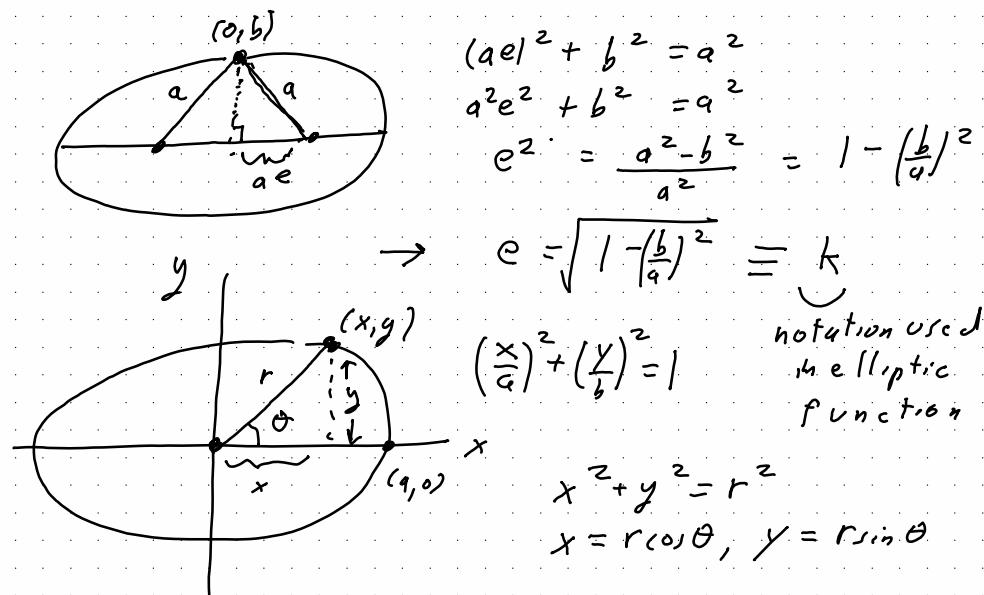
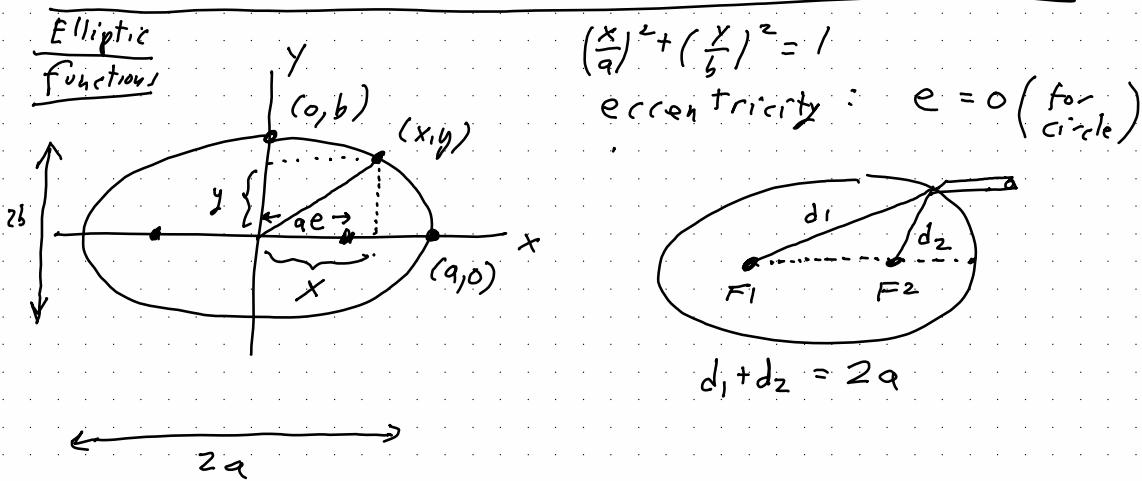
$$(ii) \left( \frac{d}{d\theta} \sin \theta \right) = \frac{1}{a} \frac{dy}{d\theta} = \frac{1}{a} \frac{dy}{\sqrt{dx^2 + dy^2}} = \frac{1}{\sqrt{(x)^2 + 1}}$$

$$2x dx + 2y dy = 0 \rightarrow dx = -\frac{y}{x} dy \quad | \quad \frac{1}{\sqrt{x^2 + y^2}} = \frac{x}{a} = \boxed{\cos \theta}$$

$$\frac{d(\sin \theta)}{d\theta} = \cos \theta \rightarrow \int \frac{d(\sin \theta)}{\cos \theta} = \int d\theta$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \theta = \sin^{-1}(x)$$

$x = \sin \theta$   
 $\cos \theta = \sqrt{1 - \sin^2 \theta}$   
 $= \sqrt{1 - x^2}$



Define:  $\operatorname{cn}(u; k) = \frac{x}{a}, \quad \operatorname{sn}(u; k) = \frac{y}{b}$   
 $\operatorname{dn}(u; k) = \frac{r}{a} \quad (=1 \text{ for circle})$

where  $u = \frac{1}{b} \int_0^\theta r d\theta$        $y = r \sin \theta$        $ds = \sqrt{dx^2 + dy^2}$   
 $(= \theta \text{ for a circle})$        $= \sqrt{dr^2 + r^2 d\theta^2}$

$$\text{Given: } \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad , \quad x^2 + y^2 = r^2 \quad dn(u; k) = \frac{r}{a}$$

$$\begin{aligned} \text{Follows: } & (i) \quad cn^2(u; k) + sn^2(u; k) = 1 \\ & (ii) \quad dn^2(u; k) + k^2 sn^2(u; k) = 1 \end{aligned} \quad \boxed{u = \int_0^\theta \int r d\theta}$$

$$(iii) \quad \frac{d}{du} sn(u; k) = cn(u; k) dn(u; k) \quad \left( \begin{array}{l} \text{Analogous to} \\ \frac{ds}{d\theta} = \cos \theta \end{array} \right)$$

$$\frac{d}{du} cn(u; k) = -sn(u; k) dn(u; k)$$

$$\frac{d}{du} dn(u; k) = -k^2 sn(u; k) cn(u; k)$$

$$\rightarrow \text{Integrate: } \frac{d}{du} sn(u; k) = cn(u; k) dn(u; k)$$

$$\int \frac{d}{du} sn(u; k) du = \int du = u$$

$$\int \frac{dx}{\sqrt{1-x^2} \sqrt{1-k^2 x^2}} = u + \text{const} = \sin^{-1}(x; k) + \text{const}$$

$$x \equiv sn(u; k)$$

$$\frac{d}{d\theta} \sin \theta = \cos \theta$$

$$\frac{ds}{d\theta} = d\theta$$

$$x = \sin \theta$$

$$\int \frac{dx}{\sqrt{1-x^2}} = d\theta$$

$$= \theta$$

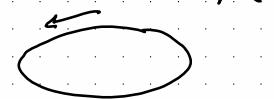
$$= \sin^{-1} x$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-k^2 x^2}} \equiv K(k) \rightarrow \begin{aligned} & \text{related to} \\ & \text{period of a pendulum} \\ & \text{going beyond} \\ & \text{small-angle} \\ & \text{approximation} \end{aligned}$$

(complete elliptic integral of 1st kind)

$$\int_0^1 \frac{\sqrt{1-k^2 x^2} dx}{\sqrt{1-x^2}} \equiv E(k) \rightarrow \begin{aligned} & \text{circumference} \\ & \text{around an ellip. sec} \end{aligned}$$

(complete elliptic integral of 2nd kind)



$$\text{circle: } C = 2\pi a$$

Notes: Tuesday 9/1

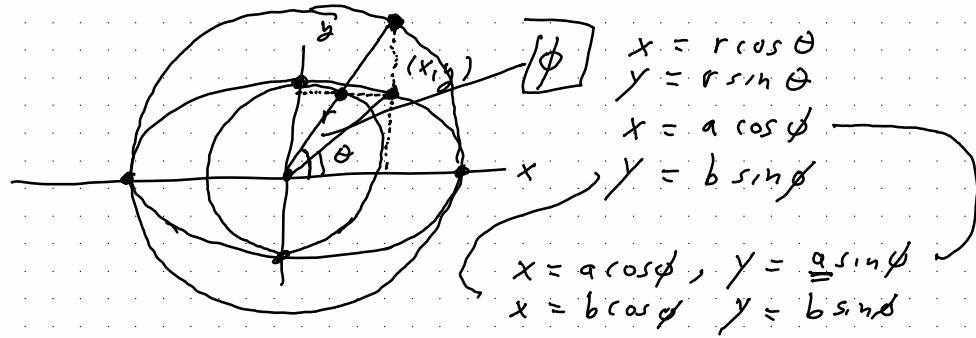
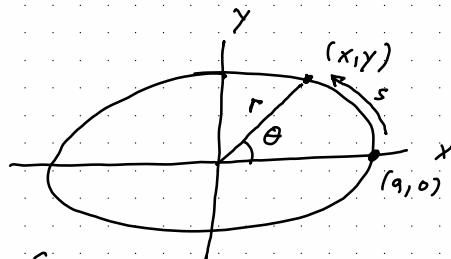
1) Review of elliptic functions,

2) Simple pendulum

$$U = \frac{1}{b} \int_{(a,0)}^{(x,y)} r d\theta$$

$$bu = \int_0^\theta r d\theta \leq \int_0^\theta ds = s$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\theta^2} \geq r d\theta$$



Simple pendulum:

(i) "Freshman physics"

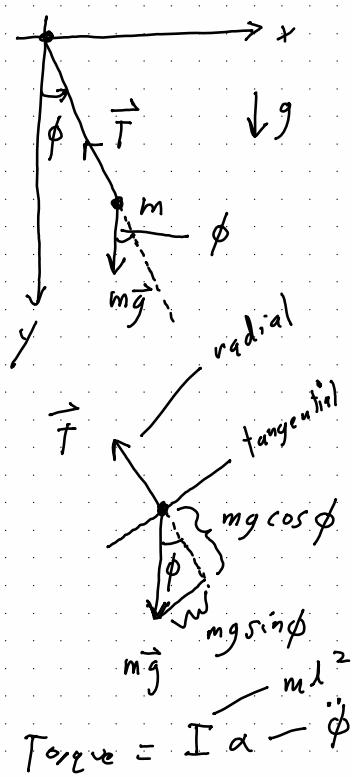
forces, free-body diagram

→ EOM, tension

tangential:

$$-mg \sin \phi = ma_{\text{tangential}}$$

$$-mg \sin \phi = m \ddot{\phi}$$



$\phi$ : angular displacement [rad]

$\dot{\phi}$ : angular velocity [rad/sec]

$\ddot{\phi}$ : angular accel [rad/sec<sup>2</sup>]

$$\boxed{\ddot{\phi} = -\frac{g}{l} \sin \phi} \quad (\text{EOM})$$

$$\text{radial: } T - mg \cos \phi = m a_{\text{centripetal}}$$

$$T - mg \cos \phi = m \dot{\phi}^2 l$$

$$\boxed{T = mg \cos \phi + m \dot{\phi}^2 l}$$

(ii) Small angle approximation:

$$\sin \phi \approx \phi \leftarrow \phi \ll 1 = 57 \text{ degrees}$$

$$\ddot{\phi} = -\frac{g}{l} \sin \phi \approx -\frac{g}{l} \phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l} \phi}$$

$$\phi(t) = \boxed{a} \cos(\omega t) + \boxed{b} \sin(\omega t)$$

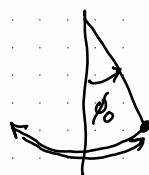
small angle approx.

where  $\omega = \sqrt{\frac{g}{l}}$

determined by initial conditions

I.Cs: If  $\phi(0) = \phi_0$  (at rest)

$$\text{then } \boxed{\phi(t) = \phi_0 \cos(\omega t)}$$



$$\text{Period: } P = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}}$$

independent of  $\phi_0$  !!

(iii) Lagrangian approach  $T \equiv \text{Kinetic Energy}$

$$L = T - U$$

$U \equiv \text{Potential Energy}$

$$T = \frac{1}{2} m l^2 \dot{\phi}^2 \quad (= \frac{1}{2} m (x^2 + y^2))$$

$$y = l \cos \phi$$

$$x = l \sin \phi$$

$$U = -mg l \cos \phi + \text{const}$$

$$U = mg l (1 - \cos \phi)$$

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mg l \cos \phi$$

$$S = \int_{t_1}^{t_2} dt L(\phi, \dot{\phi}, t)$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \quad \text{Lagrange's equation}$$

$$\frac{d}{dt} (m l^2 \dot{\phi}) = -mg l \sin \phi$$

$$m l^2 \ddot{\phi} = -mg l \sin \phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l} \sin \phi} \quad (\text{Eom})$$

$$(iiv) \text{ solving } \ddot{\phi} = -\frac{g}{l} \sin \phi \quad (2^{\text{nd}} \text{ order non-linear})$$

ODE ↑  
hard!!

$$\begin{aligned} E &= \text{const} \\ &= T + U \\ &= \frac{1}{2} m l^2 \dot{\phi}^2 - m g l \cos \phi \end{aligned}$$

$$\begin{aligned} E &= 0 - m g l \cos \phi_0 && \text{release from rest} \\ &= -m g l \cos \phi_0 && \text{from } \phi = \phi_0 \end{aligned}$$

$$-m g l \cos \phi_0 = \frac{1}{2} m l^2 \dot{\phi}^2 - m g l \cos \phi$$

$$-m g l (\cos \phi_0 - \cos \phi) = \frac{1}{2} m l^2 \dot{\phi}^2$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{-2 \frac{g}{l} (\cos \phi_0 - \cos \phi)} \quad |\phi| \leq \phi_0$$

$\int \frac{d\phi}{\sqrt{-2 \frac{g}{l} (\cos \phi_0 - \cos \phi)}} = \int dt = t + \text{const}$

Separable  
1st order  
ODE

$$t + t_0 = \int \frac{d\phi}{\sqrt{-2 \frac{g}{l} (\cos \phi_0 - \cos \phi)}} \quad \left| \frac{1}{\sqrt{ax^2}}$$

Substitution:

$$\begin{aligned} \cos \phi &= 1 - 2 \sin^2 \left( \frac{\phi}{2} \right) & \cos \phi &= \cos \left( 2 \left( \frac{\phi}{2} \right) \right) \\ \cos \phi_0 &= 1 - 2 \sin^2 \left( \frac{\phi_0}{2} \right) & & = \cos^2 \left( \frac{\phi}{2} \right) - \sin^2 \left( \frac{\phi}{2} \right) \\ & & & = 1 - 2 \sin^2 \left( \frac{\phi}{2} \right) \end{aligned}$$

$$\Rightarrow \cos \phi_0 - \cos \phi = -2 \left( \sin^2 \left( \frac{\phi}{2} \right) - \sin^2 \left( \frac{\phi_0}{2} \right) \right)$$

$$\begin{aligned} t + t_0 &= \int \frac{d\phi}{2 \sqrt{\frac{g}{l}} \sqrt{\sin^2 \left( \frac{\phi_0}{2} \right) - \sin^2 \left( \frac{\phi}{2} \right)}} \quad |\phi| \leq \phi_0 \\ &= \frac{1}{2 \sqrt{\frac{g}{l}}} \int \frac{d\phi}{\sin \left( \frac{\phi_0}{2} \right) \sqrt{1 - \frac{\sin^2 \left( \frac{\phi}{2} \right)}{\sin^2 \left( \frac{\phi_0}{2} \right)}}} \end{aligned}$$

let  $x = \sin \left( \frac{\phi}{2} \right)$   
 $\frac{1}{\sin \left( \frac{\phi_0}{2} \right)}$

$$x = \frac{\sin(\frac{\phi}{2})}{\sin(\frac{\phi_0}{2})} \rightarrow dx = \frac{1}{\sin(\frac{\phi_0}{2})} \frac{1}{2} \cos(\frac{\phi}{2}) d\phi$$

$\sqrt{1-x^2}$  ←  
↑  
denominator

Find this out

$$\begin{aligned} \textcircled{1} \quad \phi(t) &= \int \frac{dx}{\sqrt{1-\sin^2(\frac{\phi}{2})}} \\ \textcircled{2} \quad \text{Period} &= ? ? \\ \textcircled{3} \quad \text{Redo the analysis using Lagrange multipliers for find tension in strings} & \quad \text{---} \end{aligned}$$

$$t + t_0 = \int \frac{dx}{\sqrt{1-\sin^2(\frac{\phi}{2})}} \quad \begin{array}{l} \text{integrated} \\ \text{for } \sin^{-1}(x; k) \end{array} \quad \boxed{H = \sin(\frac{\phi_0}{2})}$$

Lec #4 : Thurs 9/3

$$\phi(t) = 2 \sin^{-1} \left[ H \sin \left( \omega_0 (t + \frac{P}{4}) \right); H \right] \star$$

$$H = \sin \left( \frac{\phi_0}{2} \right), \quad \omega_0 = \sqrt{\frac{g}{L}}$$

$$P = 4 \sqrt{\frac{L}{g}} \quad K(k) = 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1-x^2} \sqrt{1-H^2 x^2}}$$

small angle  
approx

$$P_0 = 2\pi \sqrt{\frac{L}{g}}$$

$$\frac{\sin(\frac{\phi}{2})}{\sin(\frac{\phi_0}{2})} = x = \sin \left[ \sqrt{\frac{g}{L}} (t + t_0); H \right]$$

$$\sqrt{\frac{L}{g}} \int \frac{dx}{\sqrt{1-x^2} \sqrt{1-H^2 x^2}} = \sqrt{\frac{L}{g}} \sin^{-1}(x; H) = t + t_0$$

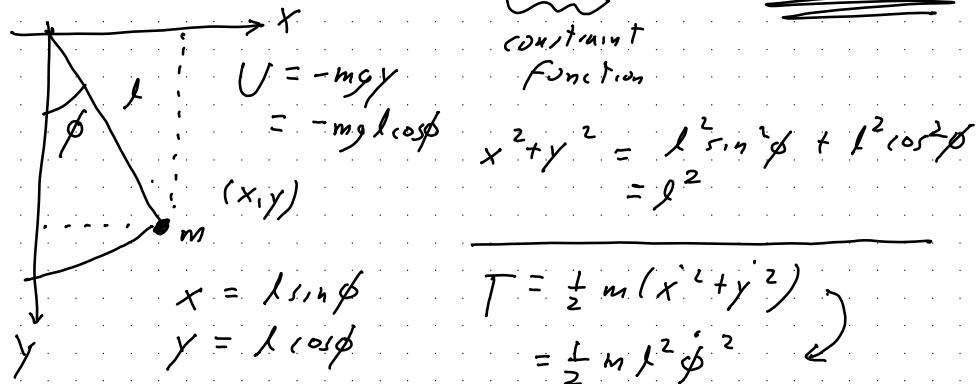
$$\sin^{-1}(x; H) = \sqrt{\frac{g}{L}} (t + t_0)$$

$$P = 4\sqrt{\frac{l}{g}} \text{kr}(k) = 2\pi \sqrt{\frac{l}{g}} \left( 1 + \frac{1}{16} \phi_0^2 + \dots \right)$$

Problem Landau II.1

Lagrange multiplier:

$$\varphi(x, y) = x^2 + y^2 - l^2 = 0$$



$$L = T - U + \lambda \varphi$$

$\uparrow \lambda, *$        $\square$  Lagrange multiplier

$L(x, \dot{x}, y, \dot{y}, t)$      $q = (x, y)$      $\dot{q} = (\dot{x}, \dot{y})$      $\lambda(t)$   
 $L(r, \dot{r}, \phi, \dot{\phi}, t)$      $q = (r, \phi)$      $\dot{q} = (r, \dot{\phi})$      $r(t)$   
 $\phi(t)$

$L(\phi, \dot{\phi}, t)$      $\varphi(x, y) = x^2 + y^2 - l^2 = 0$      $\varphi(r, \phi) = r - l = 0$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + mg r \cos\phi + \lambda (r - l)$$

$$r: \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} \rightarrow m \ddot{r} = mr \dot{\phi}^2 + mg \cos\phi + \lambda$$

$$\phi: \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \rightarrow \frac{d}{dt} (mr^2 \dot{\phi}) = -mg r \sin\phi$$

$$\lambda: r - l = 0 \quad \square \quad 2mr \ddot{r} \dot{\phi} + mr^2 \ddot{\phi} = -mg r \sin\phi$$

$$r - l = 0 \rightarrow r = l$$

$$\dot{r} = 0$$

$$\ddot{r} = 0$$

$$\rightarrow \ddot{\phi} = -\frac{g}{l} \sin \phi$$

$$m l \dot{\phi}^2 + mg \cos \phi + \lambda = 0$$

$$\lambda = -(mg \cos \phi + m l \dot{\phi}^2)$$

$$\lambda = -T$$

T

$$L = T - U + \lambda \phi$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt} (m \dot{x}) = -\frac{\partial U}{\partial x} + \lambda \frac{\partial \phi}{\partial x}$$

$$U = U(x, t)$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\phi(x, y, t)$$

$$U(x, y, t)$$

$$\frac{d}{dt} (m \dot{y}) = -\frac{\partial U}{\partial y} + \lambda \frac{\partial \phi}{\partial y}$$

$$\frac{d \vec{p}}{dt} = \vec{F}_{\text{net}}$$

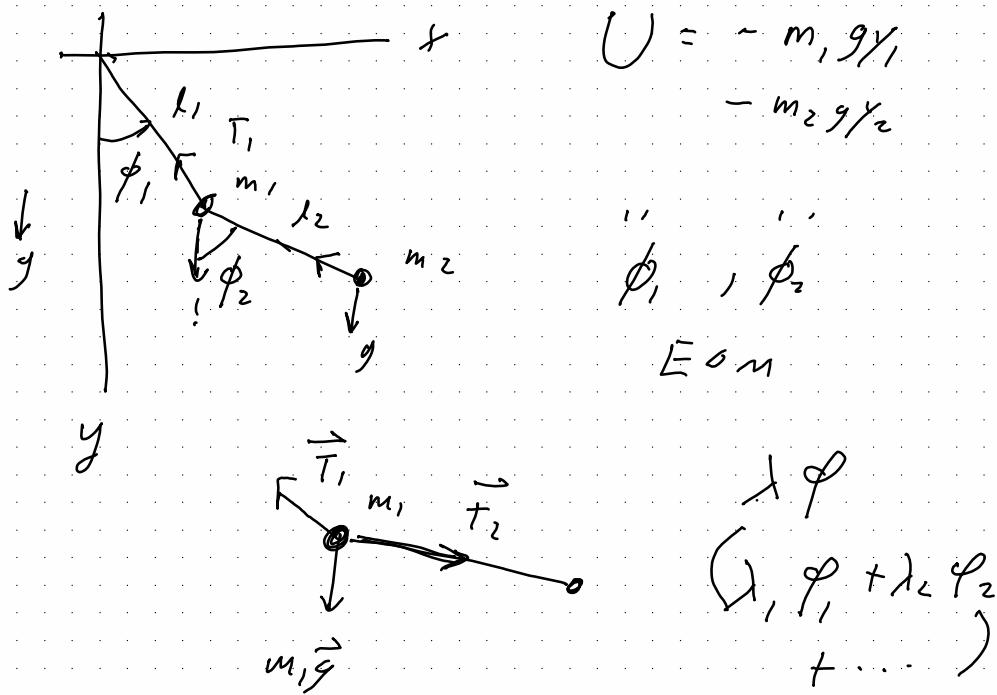
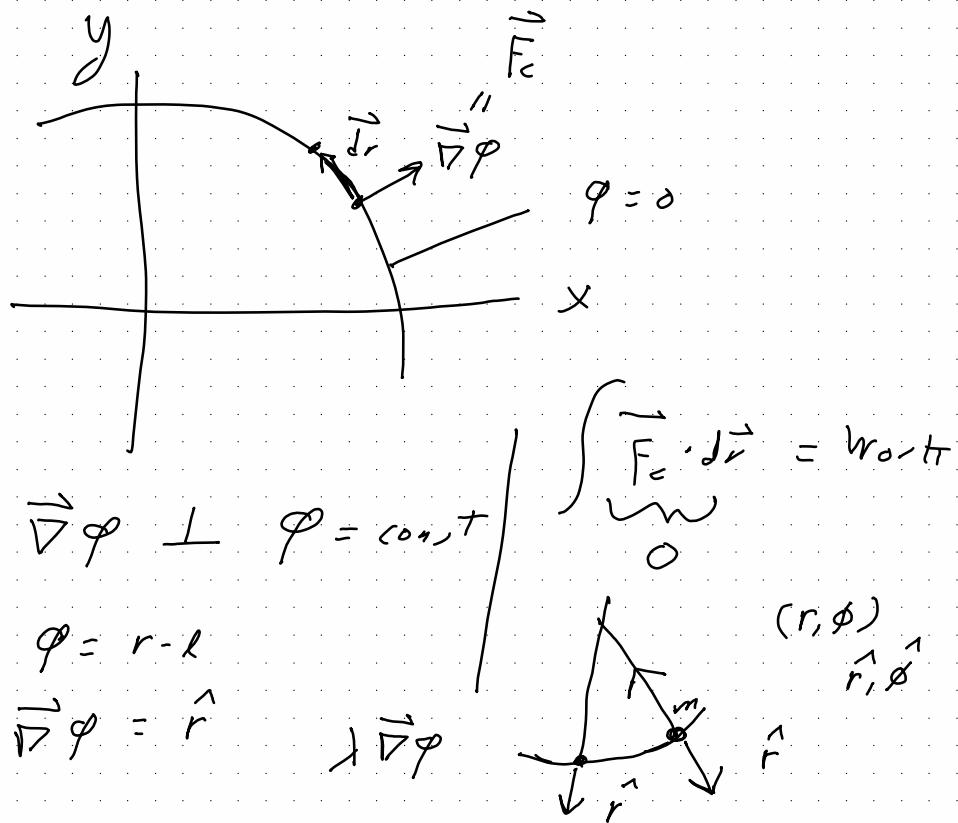
applied force      force

$$\frac{d p_x}{dt} = F_x + \lambda \frac{\partial \phi}{\partial x}$$

$$\frac{d p_y}{dt} = F_y + \lambda \frac{\partial \phi}{\partial y}$$

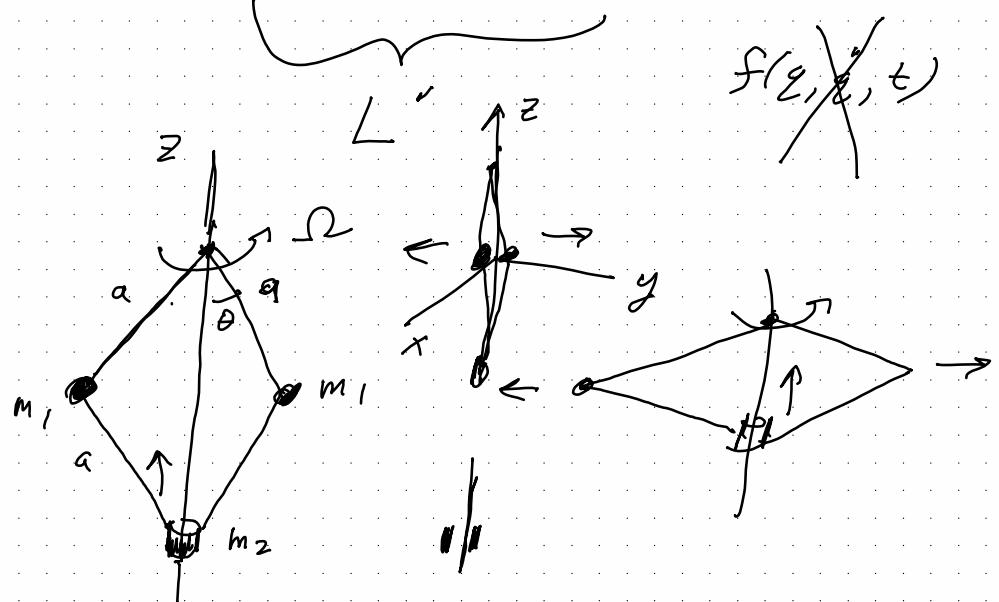
$$\frac{d \vec{p}}{dt} = \vec{F} + \lambda \vec{\nabla} \phi$$

$$\left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) = \vec{\nabla} \phi$$

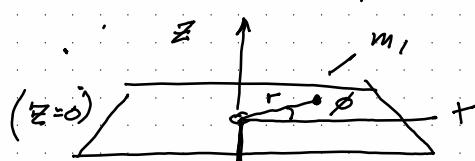


$$L \rightarrow EOM$$

$$L \rightarrow L + \frac{d}{dt} (f(\varphi, t))$$



Lec #5: Tuesday 9/18



$$r - z = l = \text{length of string}$$

$$L = T - U$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$v_2^2 = \dot{z}^2, \quad v_1^2 = \dot{r}^2 + r^2 \dot{\phi}^2$$

$$(= \dot{x}^2 + \dot{y}^2, \quad x = r \cos \phi, \quad y = r \sin \phi)$$

$$\boxed{T = \frac{1}{2} m_1 (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2}$$

$$= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2$$

$$U = m_2 g z = m_2 g (r - l) = m_2 g r - m_2 g l$$

$$U = m_2 g r$$

$$L = T - U = \boxed{\frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r}$$

$\Rightarrow$  constant

$$\frac{d}{dt} \left( \frac{\partial L}{\partial r} \right) = \frac{\partial L}{\partial r}, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \dot{\phi}} \Rightarrow 2^{\text{nd}} \text{ order EOMs}$$

No explicit  $t$  dependence:

$$E = \dot{r} \frac{\partial L}{\partial \dot{r}} + \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - L = T + U$$

$$E = \sum_i g_i \frac{\partial L}{\partial \dot{q}_i} - L$$

$$= p_i$$

No explicit  $\phi$  dependence:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} = M_z$$

$$M_z = m_1 r^2 \dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{m_1 r^2}$$

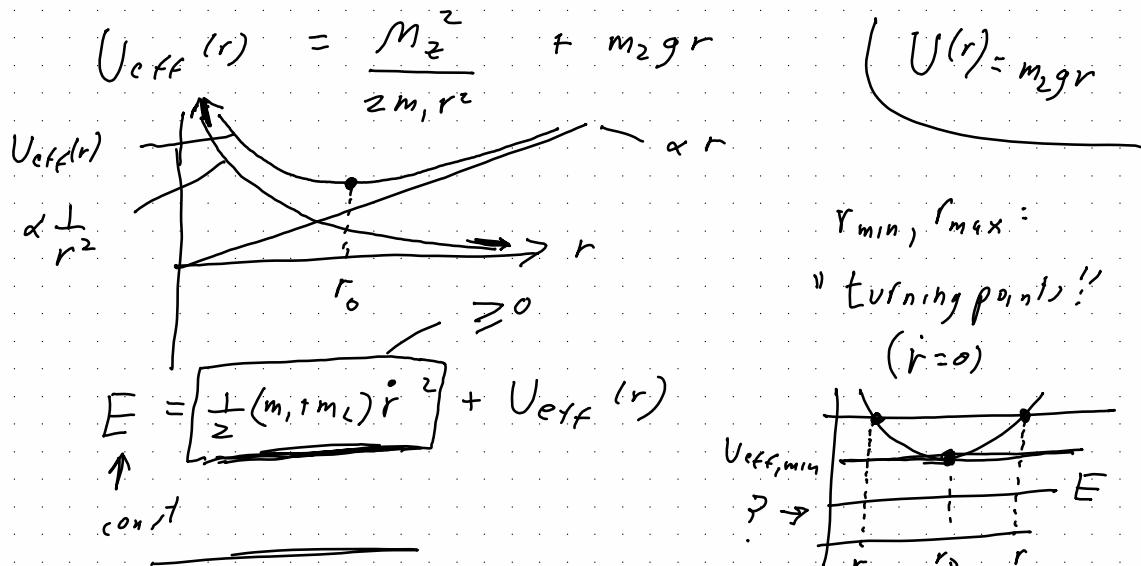
$M_z$ : angular momentum  
(L&L notation)

$$E = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 + m_2 g r$$

$$E = \frac{1}{2} m \dot{x}^2 + U(x)$$

$$E = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \left( \frac{M_z^2}{2 m_1 r^2} + m_2 g r \right)$$

$$U_{\text{eff}}(r) = U(r) + \frac{M_z^2}{2 m_1 r^2}$$



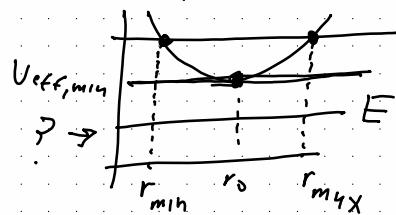
$$i) E = U_{\text{eff}, \min} = U_{\text{eff}}(r_0)$$

unit circular motion:  $r = r_0, \dot{\phi} = \frac{M_z}{m_1 r_0^2}$

$$ii) E > U_{\text{eff}, \min}$$

$$E = U_{\text{eff}}(r_{\min}) = U_{\text{eff}}(r_{\max})$$

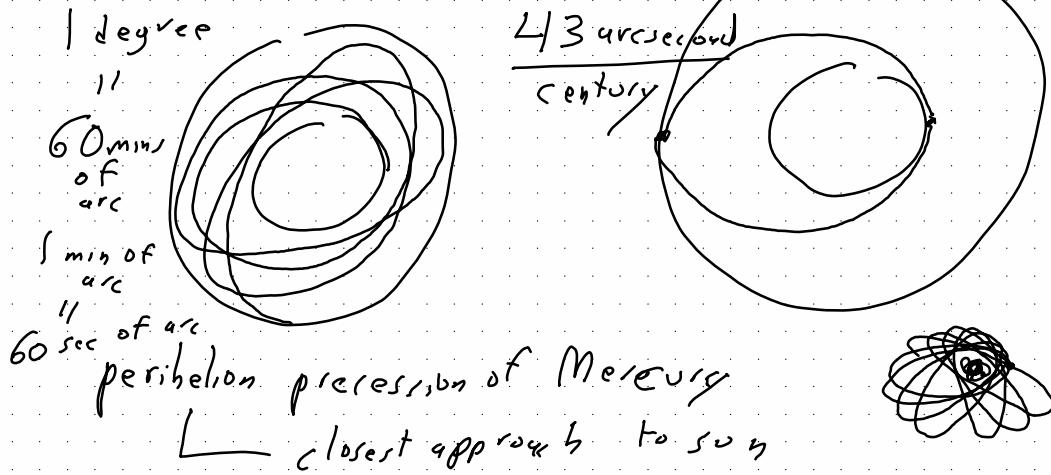
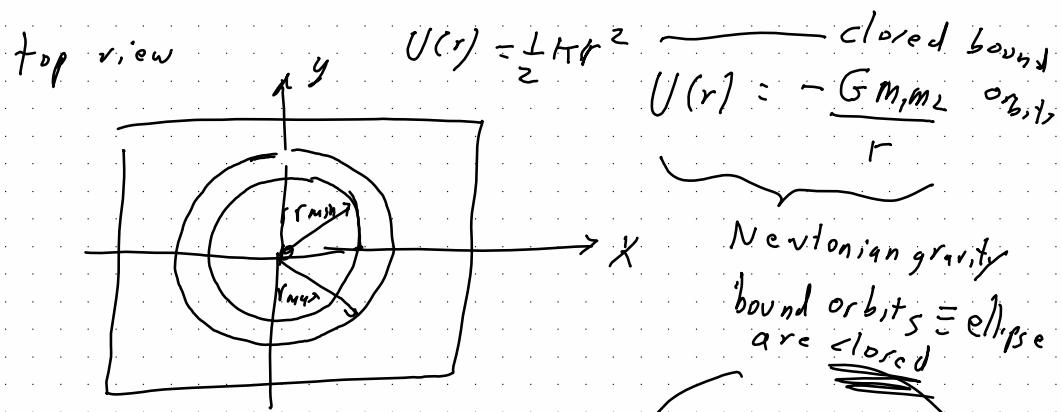
$r_{\min}, r_{\max}$ :  
"turning points"  
( $r=0$ )



$$E < U_{\text{eff}, \min}$$

(not allowed)

$$E \geq U_{\text{eff}, \min}$$



$$r_0 : \left. \frac{dU_{eff}}{dr} \right|_{r=r_0} = 0 \quad (\text{minimum})$$

$$0 = \left. \frac{d}{dr} \left( \frac{M_z^2}{2m_1 r^2} + m_2 gr \right) \right|_{r=r_0}$$

$$= -\frac{M_z^2}{m_1 r_0^3} + m_2 g$$

$$M_z^2 = m_1 m_2 g r_0^3$$

tells you the value of  $M_z$  needed to have a specific  $r_0$  value.

For a given  $M_z$ , this tells you what  $r_0$  equals.

### Energy equation:

$$E = \frac{1}{2} (m_1 + m_2) r^2 + \frac{M_2^2}{2m_1 r^2} + m_2 g r$$

$$\dot{\phi} = \frac{M_2}{m_1 r^2}$$

$\leftarrow \phi \text{ equation}$

$$\frac{1}{2} (m_1 + m_2) r^2 = E - \frac{M_2^2}{2m_1 r^2} - m_2 g r$$

$$r = \frac{dr}{dt} = \pm \sqrt{\left(\frac{2}{m_1 + m_2}\right) \left(E - \frac{M_2^2}{2m_1 r^2} - m_2 g r\right)}$$

$$\int \frac{dr}{\sqrt{\left(\frac{2}{m_1 + m_2}\right) \left(E - \frac{M_2^2}{2m_1 r^2} - m_2 g r\right)}} = \int dt = t + \text{const}$$

$t(r) \leftrightarrow r(t)$

### Orbital equation:

$$r = r(\phi) \Leftrightarrow \phi = \phi(r)$$

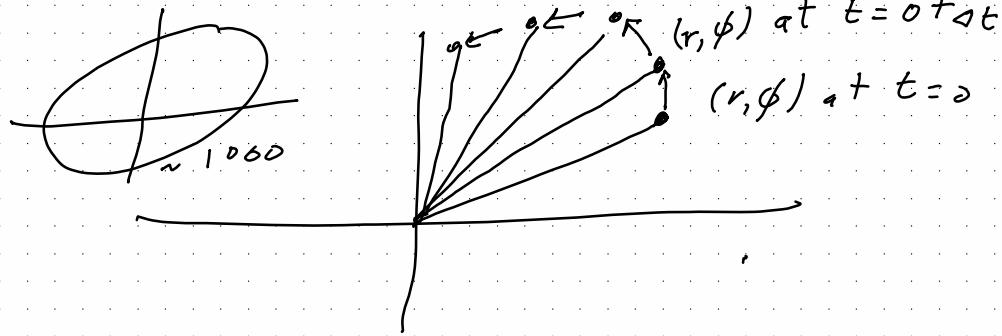
$$\frac{dr}{dt} = \dot{r} = \sqrt{\left(\frac{2}{m_1 + m_2}\right) \left[E - \frac{M_2^2}{2m_1 r^2} - m_2 g r\right]}$$

$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{M_2}{m_1 r^2}$$

with

$$\rightarrow \frac{dr}{d\phi} = \frac{m_1 r^2}{M_2} \sqrt{\left(\frac{2}{m_1 + m_2}\right) \left[\dots\right]}$$

$$\int \frac{dr}{\frac{m_1 r^2}{M_2} \sqrt{\left(\frac{2}{m_1 + m_2}\right) \left[\dots\right]}} = \int d\phi = \phi + \text{const}$$



Given:  $r, \phi$  at some time  $t$

Given:  $\Delta t$  need to know  $\Delta r$  and  $\Delta \phi$

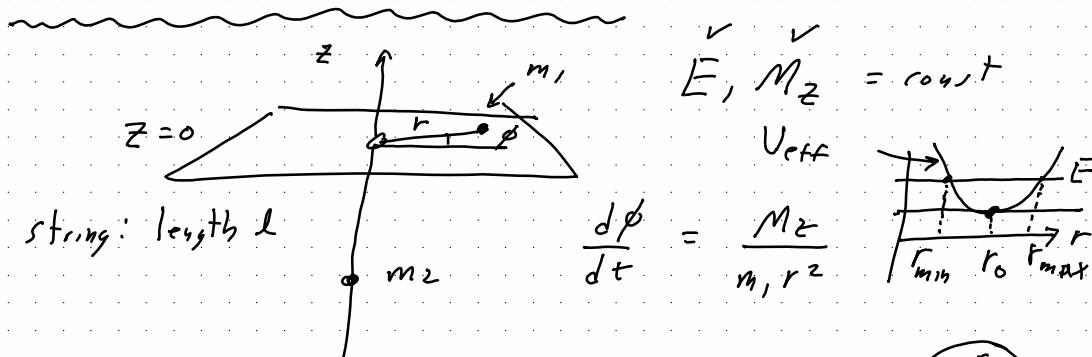
$$r(t+\Delta t) = r(t) + \Delta r(t) + \dots$$

$$\phi(t+\Delta t) = \phi(t) + \Delta \phi(t) + \dots$$

Ignore it if  $\Delta t$  is suff. small

### Lecture #6 : Thursday 10 Sep

- 1) Secs 6-10 (today), Sec 40 (next Tuesday)
- 2) Finish up example from last time
- 3) Conservation of  $E, \vec{P}, \vec{m}$
- 4) Mechanical similarity
- 5) Quiz: last 20 minutes (1:30 pm)



$$\frac{dr}{dt} = \sqrt{\left(\frac{2}{m_1 + m_2}\right) \left(E - m_2 gr - \frac{M_z^2}{2m_1 r^2}\right)} = \sqrt{\Theta}$$

$$\frac{d\phi}{dt} = \frac{M_2}{m_1 r^2} \rightarrow \Delta\phi = \Delta t \frac{M_2}{m_1 r^2}$$

$$\frac{dr}{dt} = \sqrt{\phi} \rightarrow dr = \Delta t \sqrt{\phi}$$

$$r(0) = r_{min}$$

$$\phi(0) = 0$$

$$\phi(\Delta t) = \phi(0) + \Delta\phi$$

$$r(\Delta t) = r(0) + \Delta r$$

$$\phi(2\Delta t) = \phi(\Delta t) + \Delta\phi$$

$$r(2\Delta t) = r(\Delta t) + \Delta r$$

:

$$\phi(t_i) = \phi(t_{i-1}) + \Delta\phi$$

$$r(t_i) = r(t_{i-1}) + \Delta r$$

Cons. of  $E, \vec{P}, \vec{M}$ :

All of  $E, \vec{P}, \vec{M}$  conserved for a closed system

$$\begin{aligned} &\text{no external forces} \\ &U = U(\vec{r}_1 - \vec{r}_2, \vec{r}_2 - \vec{r}_3, \dots) \end{aligned}$$

relative position vectors.

Even in the presence of external forces, you can still have cons. of  $E$  and some components of  $\vec{P}$  and  $\vec{M}$ .

$$(i) \quad U = mgx \quad \vec{F} = -mg\hat{j}$$

If  $U$  does not depend explicitly on time  $t$ , then  $E$  is conserved.

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

constant  
external  
field

(ii) e.g.,  $\downarrow \vec{F}_g = m\vec{g}$

$\xrightarrow{x, y}$

$P_x = \text{const}$

$P_y = \text{const}$

If  $U$  is unchanged by a translation in some direction  $\hat{E}$  then  $\vec{P} \cdot \hat{E} = \text{const}$

$$\vec{v} = \text{const}$$

(iii)  $\vec{M}$  depends on choice of origin

(a) uniform gravitational field  $\vec{F} = m\vec{g}$   $\uparrow z$   
 $U = m_1 g z_1 + m_2 g z_2 + \dots$

If  $U$  is unchanged by a rotation about a particular axis  $\hat{n}$  then  $\boxed{\vec{M} \cdot \hat{n} = \text{const}}$  (e.g.,  $\hat{n} = \hat{z}$ ,  $M_z = \text{const}$ )

(b) central force  $U = U(r)$

$\boxed{\vec{M} = \text{const}}$

$\vec{F} = -\nabla U = -\frac{dU}{dr} \hat{r}$

provided the origin is located on the axis.

### Mechanical similarity:

$$L \rightarrow L' = c \cdot L \quad \text{same equations of motion}$$

suppose we rescale position vectors  $\vec{r}_a \rightarrow \alpha \vec{r}_a$

$$U'(\vec{r}_1, \dots) = U(\alpha \vec{r}_1, \alpha \vec{r}_2, \dots) = \alpha^K U(\vec{r}_1, \vec{r}_2, \dots)$$

potential is homogeneous of degree  $K$  wrt position vector

Example: i)  $U = mg y$ ,  $K = 1$

ii)  $U = \frac{1}{2} k x^2$ ,  $K = 2$

iii)  $U = -\frac{G m_1 m_2}{r}$ ,  $K = -1$

$$\begin{aligned} U' &= mg \alpha y \\ &= \alpha mg y \\ &= \alpha U \end{aligned}$$

$$\begin{aligned} L' &= T' - U' = T' - \alpha^K U = \text{const} \cdot L \\ &= \alpha^K T - \alpha^K U \\ &= \alpha^K (T - U) = (\alpha^K) L \end{aligned}$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \frac{l^2}{t^2} \quad \begin{matrix} \text{length} \\ \text{time} \end{matrix}$$

$$\begin{matrix} l \rightarrow l' = \alpha l \\ t \rightarrow t' = \beta t \end{matrix}$$

$$\begin{cases} \frac{l'}{l} = \alpha & U = mgx, k=1 \\ \frac{t'}{t} = \beta & \frac{t'}{t} = (\frac{l'}{l})^{\frac{1}{2}} \\ & U = \frac{1}{2} kx^2, k=2 \\ & t'/t = \text{const} \end{cases}$$

$$T' = \frac{1}{2} m v'^2 = \frac{1}{2} m \frac{l'^2}{t'^2}$$

$$= \frac{1}{2} m \frac{\alpha^2 l^2}{\beta^2 t^2}$$

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-k/2}$$

$$= \frac{\alpha^2}{\beta^2} T$$

$$U = -\frac{GMm_2}{r}, k=-1$$

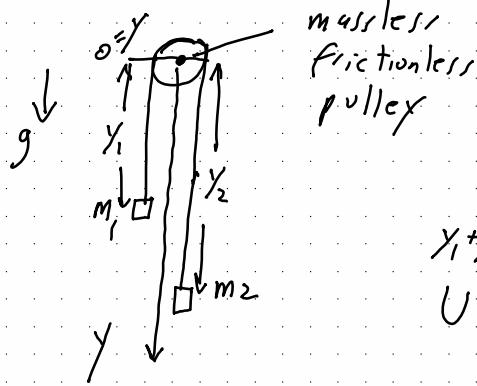
$$\frac{\alpha^2}{\beta^2} = \alpha^{k/2}$$

$$\beta = \alpha^{1-k/2} \quad \begin{matrix} \uparrow \\ \left(\frac{t'}{t}\right) = \left(\frac{l'}{l}\right)^{3/2} \end{matrix}$$

$$\beta^2 = \alpha^{2-k}$$

$$P^2 = D_1 s^3$$

### QUIZ #1 : Atwood's machine



i)  $L$  ?

ii) EoM

iii) solve EoM

$$U = -m_1 gy_1 - m_2 gy_2$$

$$= -m_1 gy_1 - m_2 g(l - y_1)$$

$$= -m_1 gy_1 - m_2 g l + m_2 g y_1$$

ignore

$$= [-(m_1 - m_2)g y_1] \quad \parallel \quad \ddot{y}_1 =$$

string: length l

(massless, inextensible)

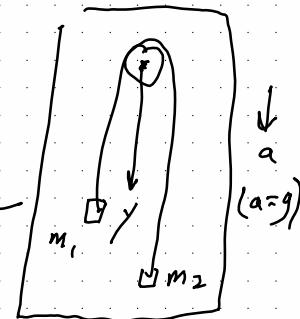
...)

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2$$

$$= \boxed{\frac{1}{2} (m_1 + m_2) \dot{y}_1^2}$$

## Lecture #7: Tuesday 9/15

- 1) Go over quiz #1
- 2) Modified Atwood problem
- 3) Finish mechanical similarity (sec 10)
- 4) Hamilton's equations (sec 40)



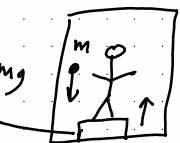
$$L = T - U$$

$$= \frac{1}{2} (m_1 + m_2) \dot{y}_1^2 + (m_1 - m_2) g y_1$$

EOMs:  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}_1} \right) = \frac{\partial L}{\partial y_1} \rightarrow (m_1 + m_2) \ddot{y}_1 = (m_1 - m_2) g$

$$\ddot{y}_1 = \frac{(m_1 - m_2) g}{(m_1 + m_2)}$$

$$y_1(t) = y_0 + v_0 t + \frac{1}{2} \frac{(m_1 - m_2) g}{(m_1 + m_2)} t^2$$

Scale: 

$$ma = mg - N$$

$$N = m(g-a)$$

$$\ddot{y}_1 = \frac{(m_1 - m_2) (g-a)}{m_1 + m_2}$$

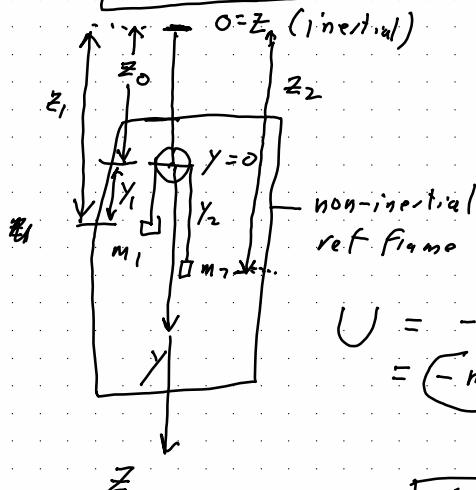
$$\vec{F} = \vec{m a} \quad (\text{valid in an inertial ref. frame})$$

$$\vec{F} + \vec{F}_{\text{fiction}} = \vec{m a} \quad \text{w.r.t. to a non-inertial ref. frame}$$

sec 39 (L & L)

$$L' = L + F(t)$$

$$L = T - U \quad (\text{valid in an inertial ref. frame})$$



$$T = \frac{1}{2} m_1 \dot{z}_1^2 + \frac{1}{2} m_2 \dot{z}_2^2$$

$$U = -m_1 g z_1 - m_2 g z_2$$

$$z_1 = z_0 + y_1 \quad | \quad y_2 = l - y_1$$

$$z_2 = z_0 + y_2 \quad | \quad l - y_1$$

$$U = -m_1 g (z_0 + y_1) - m_2 g (z_0 + y_2)$$

$$= (-m_1 g z_0) - m_1 g y_1 - m_2 g z_0 - m_2 g l + m_2 g y_1$$

$\downarrow$  const  
prescribed function of time = ignore

$$= [-m_1 - m_2] g y_1$$

Do this at home:

$$T = \frac{1}{2} (m_1 + m_2) \dot{y}_1^2 + (m_1 - m_2) \dot{z}_0 \dot{y}_1$$

$$L' = L + \frac{d}{dt}(f(y_1, t)) \rightarrow \text{same EoM}$$

$$(m_1 - m_2) \dot{z}_0 \dot{y}_1 = \frac{d}{dt} \left[ \underbrace{(m_1 - m_2)}_{f(y_1, t)} \dot{z}_0 \dot{y}_1 \right] - (m_1 - m_2) \ddot{z}_0 \dot{y}_1$$

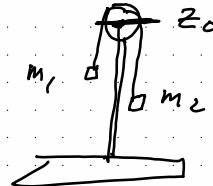
~~$\equiv 0$~~

ignore

$$T = \frac{1}{2} (m_1 + m_2) \dot{y}_1^2 - (m_1 - m_2) \dot{z}_0 \dot{y}_1$$

$z_0(t)$ : given, not to be solved for

$\equiv$



Hamilton's equations:

$L(y_i, \dot{y}_i, t)$

Hamiltonian:  $E = \sum \frac{\partial L}{\partial \dot{y}_i} \dot{y}_i - L$

$H = H(y_i, p_i)$   $\longleftrightarrow$   $E(y_i, \dot{y}_i, t)$  not here  
if  $L = L(y_i, \dot{y}_i)$

$H = \left( \sum p_i \dot{y}_i - L \right) |_{\dot{y}_i = \dot{y}(y_i, p_i)}$

$$p_i = \frac{\partial L}{\partial \dot{y}_i}$$

Example:

$L = \frac{1}{2} m \dot{x}^2 - U(x)$  (single particle, 1-d, const external field)

$p = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \rightarrow \dot{x} = \frac{p}{m}$

$H = (p \dot{x} - L) |_{\dot{x} = p/m}, \neq$

$$= \left( p \dot{x} - \frac{1}{2} m \dot{x}^2 + U(x) \right) |_{\dot{x} = p/m} = \frac{p^2}{m} - \frac{1}{2} m \left( \frac{p}{m} \right)^2 + U(x)$$

$H = \frac{1}{2} \frac{p^2}{m} + U(x)$

EOMs: (Hamilton's equations)

(39,6) Prob 2  
Sec 210

$$\ddot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad i=1, \dots, s$$

# of DOF

Lagrange's equation:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} \quad i=1, \dots, s$$

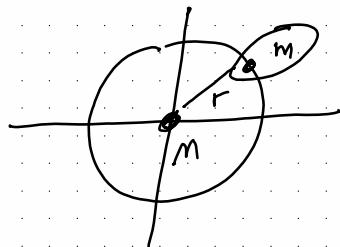
s-equations, 2nd order ODE for  $\ddot{q}_i$

$$\begin{aligned} \dot{x} &= p_m \\ p &= m\dot{x} \\ \dot{p} &= -\frac{\partial U}{\partial x} \\ m\ddot{x} &= -\frac{\partial U}{\partial x} \end{aligned}$$

2s-equations, 1st order in  $\dot{q}_i, \dot{p}_i$

$$H = \frac{p^2}{2m} + U(x) \rightarrow \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \rightarrow \dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$$

$$L = \frac{1}{2}m\dot{x}^2 - U(x) \rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \rightarrow m\ddot{x} = -\frac{\partial U}{\partial x}$$



$$U = -\frac{GM_m}{r}$$

$$\text{problem: } U' = cU$$

~~$m'/r' = cM/m$~~ 

$$r' = r, m' = m$$

$$\frac{GM_m}{r^2} = \frac{m v^2}{r}$$

$$v = \sqrt{\frac{GM}{r}}$$

Suppose  $M \rightarrow cM = m'$

$$U' = cU$$

$$\frac{2\pi r}{T'} = V' = \sqrt{\frac{cGM}{r}}$$

$$\frac{1}{T'} \propto \sqrt{c}$$

$$\sqrt{\frac{E}{E'}} = \sqrt{\frac{U'}{U}}$$

$V = \frac{1}{2} Kx^2$   
 $T = \frac{1}{2} m \dot{x}^2$   
 $L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} Kx^2$   
 $m \ddot{x} = -Kx$   
 $\ddot{x} = -\frac{K}{m} x$

friction, loss  
 $w = \sqrt{\frac{K}{m}}$   
 $x(t) = a \cos \omega t + b \sin \omega t$

---

$V' = cV$  (for example  $K' = cK$ )  
 $T' = cT$

$\ddot{x} = -\frac{K'}{m} x$ ,  $\omega' = \sqrt{\frac{K'}{m}}$   
 $T = \frac{1}{2} m \dot{x}^2$   
 $T' = cT$   
 $\frac{2\pi}{\omega'} = \sqrt{\frac{K'}{m}}$   
 $\frac{P}{P'} = \sqrt{\frac{c}{c}}$

---

$L = T - V$   
 $L' = cL = cT - cV$   
 $L'' = T - cV \neq cL$

$L, L'$   
 same E<sub>om</sub>,  
 $L, L''$   
 different E<sub>om</sub>,

---

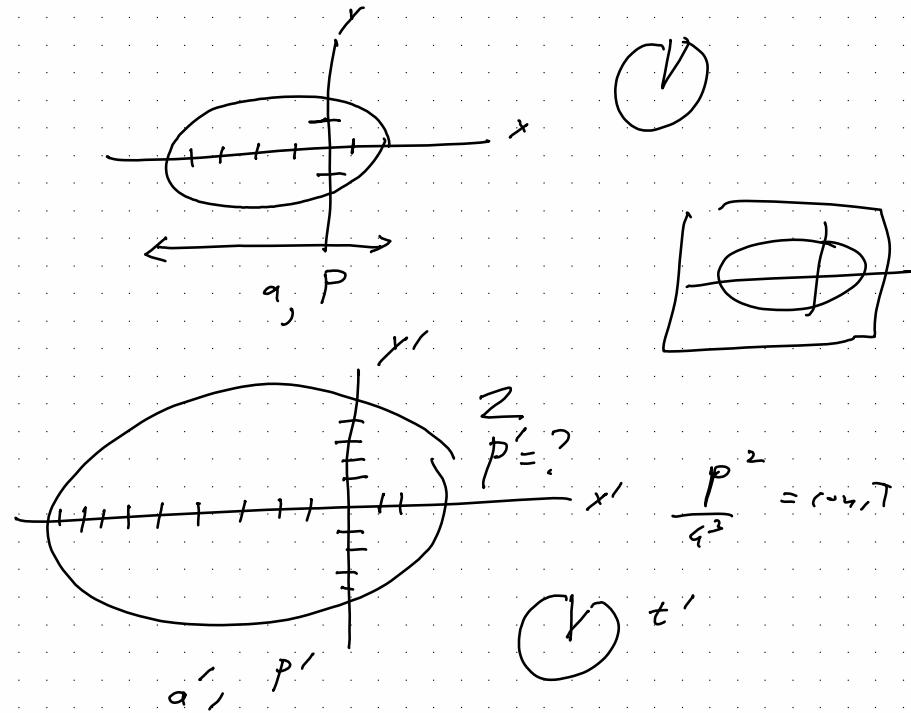
$T = \frac{1}{2} m \dot{x}^2$ ,  $V = \frac{1}{2} Kx^2$ ,  $\frac{1}{\omega'^2} = c$

$L = T - V \rightarrow m \ddot{x} = -Kx$   
 $L' = cL \rightarrow m \ddot{x} = -cKx$   
 $L'' \rightarrow m \ddot{x} = -cKx$

$c \ddot{x} = \ddot{x} = -c \left( \frac{K}{m} \right) x$   
 $x = a \cos(\omega' t) + b \sin(\omega' t)$

different E<sub>om's</sub>

---



$U' = cU$

$P' = ? P$

$m' = m$

$\lambda' = \lambda$

"mechanical similarity"

$L' = \cancel{c} \lambda + L$

$T' - U' = (c\lambda + t)(T - U)$

$T' - cU = c | (T - u) cU = U'$

$\frac{P'}{P} = \sqrt{\frac{U}{U'}}$

$T \rightarrow T' = ck$   
Periods

$g \rightarrow 2g$

$I_m \frac{P'}{P} = \frac{1}{\sqrt{2}}$

$L' = cL$

$U' = cU$

$T' = T$

$\cancel{T' = ck}$

Lec #8: Thur Sep 17<sup>th</sup>

Today - 1-d motion (Sec 11)

Next two weeks - central force (Sec 13-15)

\* Midterm 1 - Tues Oct 6<sup>th</sup>

$$\underline{T = \frac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)}$$

$$T = \frac{1}{2} \sum_{j=1}^3 q_j \ddot{q}_j \quad [ \text{single particle}]$$

$$T = \frac{1}{2} m \vec{v}^2 \quad U(q_1, \dots, q_s, t)$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad \text{cartesian } (x, y, z)$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \quad \text{sph. polar } (r, \theta, \phi)$$

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

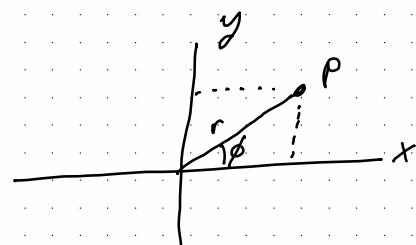
$$z = r \cos \theta$$

$$\begin{array}{l|l} q_1 = r & a_{ij} = m \\ q_2 = \theta & \\ q_3 = \phi & \end{array}$$

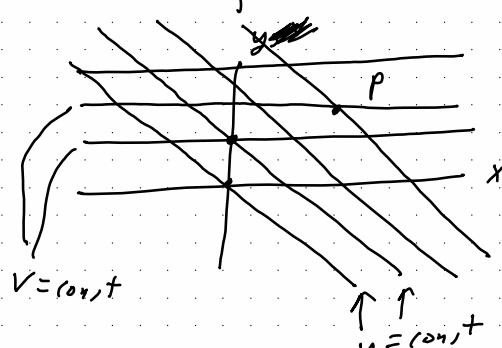
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$r^2 \sin^2 \theta$$

$$\underline{a_{11} = m}, \quad q_{22} = \frac{mr^2}{r^2 \sin^2 \theta}, \quad q_{33} = \frac{m r^2 \sin^2 \theta}{r^2 \sin^2 \theta}, \quad a_{13} = 0$$



$$(x, y) \quad (r, \phi) \quad \begin{array}{l} \dot{x} = \dot{u} - \dot{v} \\ \dot{y} = \dot{v} \end{array}$$



$$\begin{array}{l|l} u = x + y & x = u - v \\ v = y & y = v \end{array}$$

$$u = \text{const} ?$$

$$x + y = \text{const}$$

$$y = \text{const} - x$$

$$T = \frac{1}{2} m (\dot{u}^2 + \dot{v}^2) ??$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \checkmark$$

$$= \frac{1}{2} m (\dot{u}^2 + \dot{v}^2 - 2\dot{u}\dot{v} + \dot{v}^2)$$

$$= \frac{1}{2} m (\dot{y}^2 + 2\dot{v}^2 - 2\dot{u}\dot{v})$$

$$a_{ij} = m \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$T = \frac{1}{2} m |\vec{v}|^2$$

$$\vec{v} = \sum_{i=1}^3 v_i \hat{e}_i$$

$$\hat{e}_i \cdot \hat{e}_i = 1$$

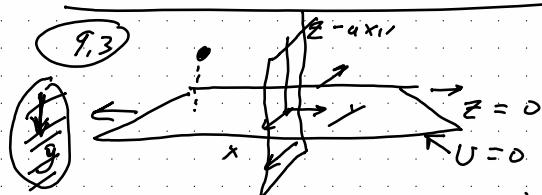
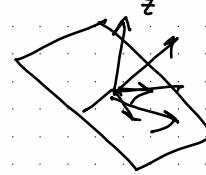
$$\hat{e}_i \cdot \hat{e}_j \neq 0$$

$$|\vec{v}|^2 = \vec{v} \cdot \vec{v}$$

$$= (\sum_i v_i \hat{e}_i) \cdot (\sum_j v_j \hat{e}_j)$$

$$= \sum_i v_i v_j \hat{e}_i \cdot \hat{e}_j$$

*might not*  $= \delta_{ij}$  (orthogonal)



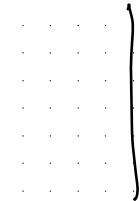
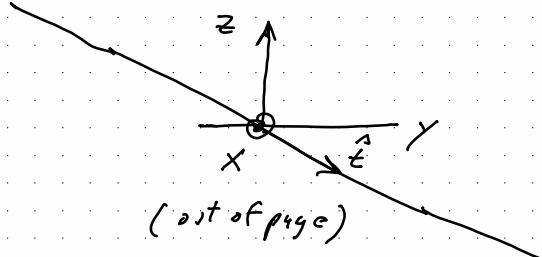
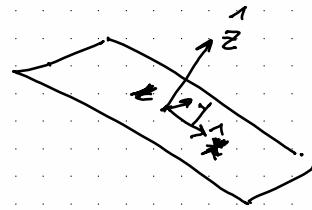
$$(a) U(x, y, z)$$

$$m g z = U(z), \vec{F} = -m \vec{g}$$

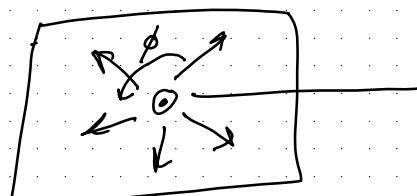
$\delta L = 0$  if you displace in  $\hat{x}, \hat{y}$   $\rightarrow P_x, P_y$

$$\vec{F} = -\nabla U = -\frac{dU}{dz} \hat{z}$$

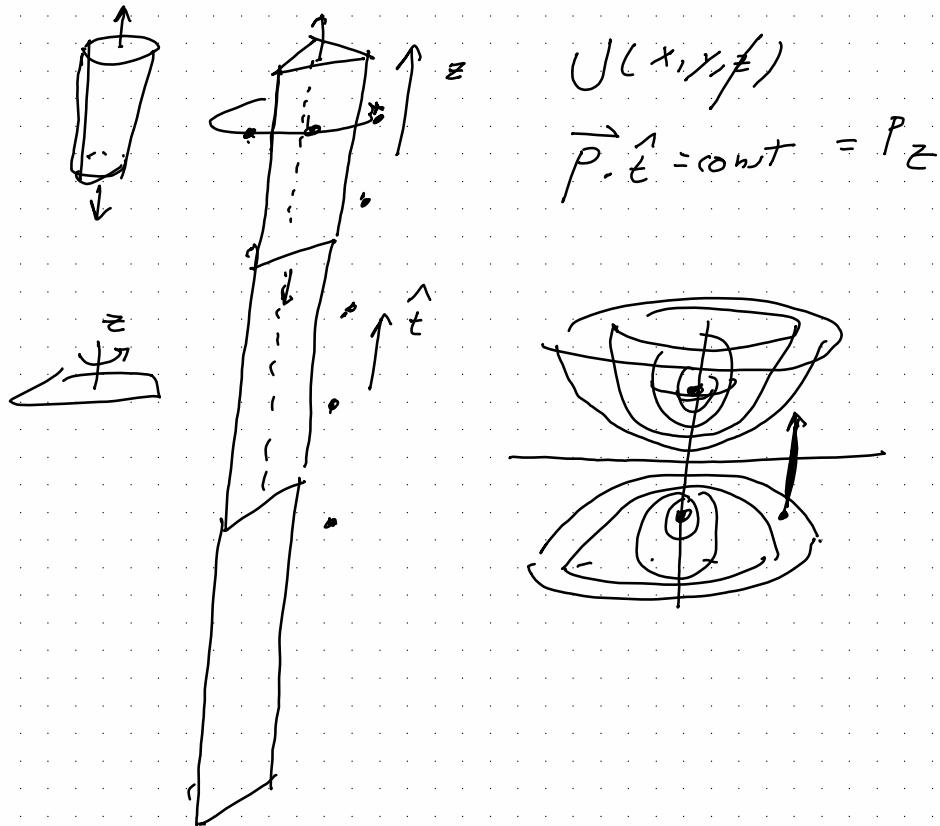
$M_z = \text{const}$



$$\vec{P} \cdot \hat{t} = \text{const}$$

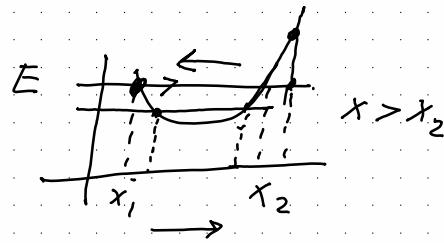


$$\vec{M} \cdot \hat{z}^{\text{out}} = \text{const}$$



$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$\textcircled{E} = \underbrace{\frac{1}{2} m \dot{x}^2}_{\text{positive}} + U(x) = \text{const}$$



$$\frac{1}{2} m \dot{x}^2 = E - U(x)$$

$$\frac{dx}{dt} = \dot{x} = \pm \sqrt{\frac{2}{m}(E - U(x))} \quad \text{— Separable 1st order}$$

$$\int \frac{dx}{\pm \sqrt{\frac{2}{m}(E - U(x))}} = \int dt$$

$$t = \int \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}} + \text{const}$$

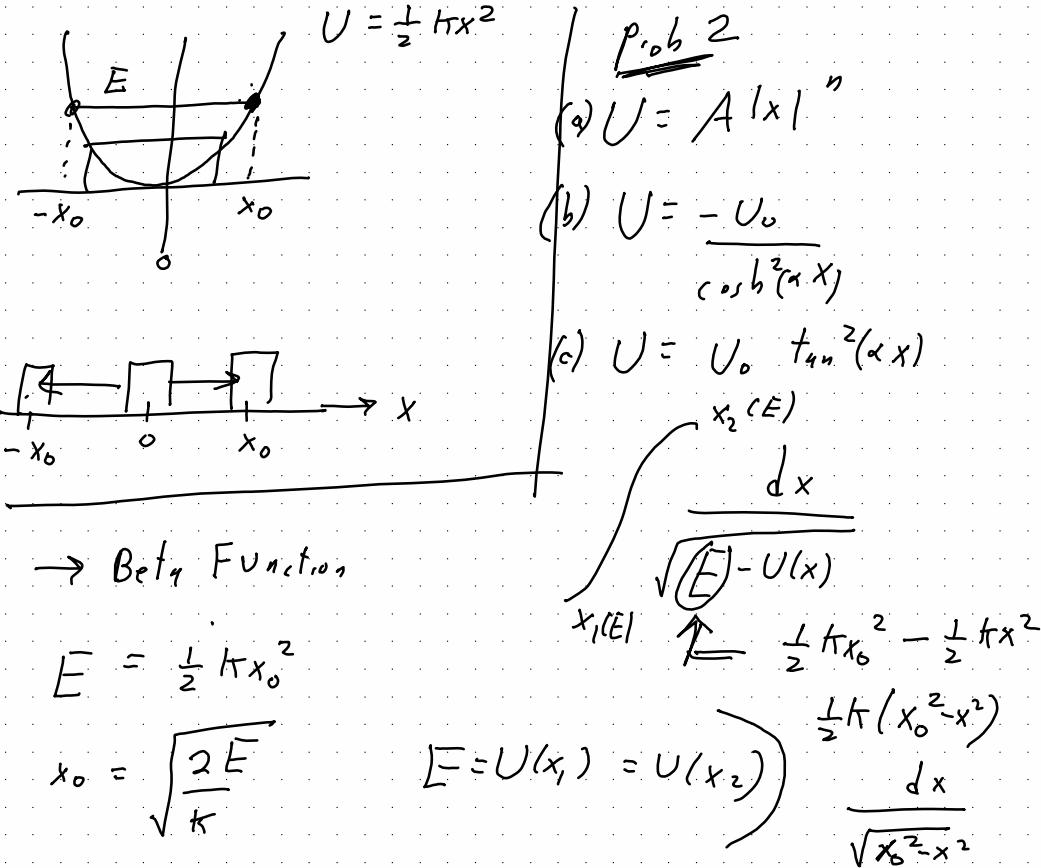
$$\boxed{\begin{aligned} ODE: & \frac{dx}{dt} \\ T(E) &= 2 \int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}} \end{aligned}}$$

$$U(x) = \frac{1}{2} k x^2$$

$$U(\phi) = mg \cos \phi$$

$$\phi = -\frac{g}{f} \sin \phi$$

$$\Delta x = \pm \Delta t \sqrt{\frac{2}{m}(E - U(x))}$$



Lecture #9: Tues 9/22      40%      60%

- 1) Midterm #1 : Tues Oct 6<sup>th</sup> (short answer; long problems)
- 2) Next 4 lectures (central force problem)      Sec 13-15

General Formalism:

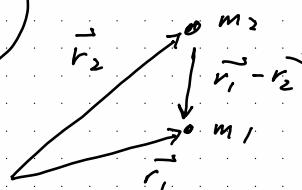
Two interacting particles (no external forces)

$m_1, m_2$

$$U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$$

$$L = T - U$$

$$= \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 - U(|\vec{r}_1 - \vec{r}_2|)$$



i) L unchanged by a translation  $\vec{r}_1 \rightarrow \vec{r}_1 + \delta \vec{x}$ ,  $\vec{r}_2 \rightarrow \vec{r}_2 + \delta \vec{x}$

$$\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2 = \text{const}$$

com moves with const velocity

choose ref frame such that COM at origin.  
inertia!

$$\frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \vec{R}_{\text{COM}} = 0$$

Define:  $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$

~~Defn:~~  $\vec{r}_2 = \vec{r}_1 - \vec{r}$

$$m_1 \vec{r}_1 + m_2 (\vec{r}_1 - \vec{r}) = 0$$

$$(m_1 + m_2) \vec{r}_1 = m_2 \vec{r} = 0$$

$$\vec{r}_1 = \left( \frac{m_2}{m_1 + m_2} \right) \vec{r}$$

$$U(|\vec{r}_1 - \vec{r}_2|)$$

$$U = U(r), r = |\vec{r}|$$

$$\vec{r}_2 = \left( \frac{m_2}{m_1 + m_2} \right) \vec{r} - \vec{r}$$

$$\vec{r}_2 = - \left( \frac{m_1}{m_1 + m_2} \right) \vec{r}$$

$$T = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2$$

$$T = \frac{1}{2} m_1 \left( \frac{m_2}{m_1 + m_2} \right)^2 |\dot{\vec{r}}|^2 + \frac{1}{2} m_2 \left( \frac{-m_1}{m_1 + m_2} \right)^2 |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)^2} |\dot{\vec{r}}|^2 \cancel{(m_2 + m_1)}$$

$$= \frac{1}{2} \left( \frac{m_1 m_2}{m_1 + m_2} \right) |\dot{\vec{r}}|^2$$

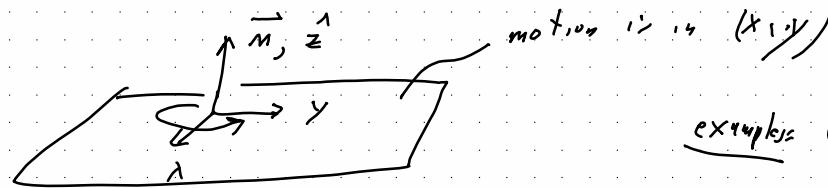
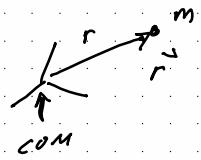
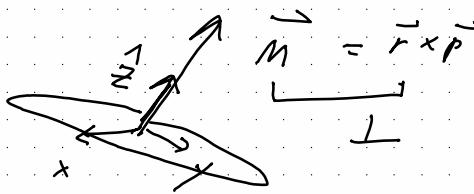
reduced mass:  $m$

$$= \frac{1}{2} m |\dot{\vec{r}}|^2$$

"effective"

one body Lagrangian

ii)  $L$  unchanged under  $\pi$  rotations  
 $\rightarrow \vec{M} = \text{const} \quad (\vec{M} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v})$



Newtonian gravity  
example:  $U(r) = -\frac{Gm_1m_2}{r}$

$$\boxed{L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r)}$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r)$$

$$U(r) = \frac{1}{2} k r^2$$

Sphere oscillator

i) No explicit t-dependence

$$\boxed{E = T + U = \text{const}}$$

$E, M$  constants  
of the motion

ii) No  $\phi$  dependence

$$M_z = p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \text{const} = mr^2 \dot{\phi} = M$$

$$\left\{ \begin{array}{l} E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + U(r), \\ M = m r^2 \dot{\phi} \end{array} \right. \rightarrow \boxed{\dot{\phi} = \frac{M}{mr^2}}$$

$$\left. \begin{array}{l} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r}, \\ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \end{array} \right.$$

$$\Rightarrow E = \frac{1}{2} m \dot{r}^2 + \underbrace{\left( \frac{M^2}{2mr^2} + U(r) \right)}_{U_{\text{eff}}(r)}$$

$$\frac{dr}{dt} = \dot{r} = \sqrt{\frac{2}{m} \left( E - U(r) - \frac{M^2}{2mr^2} \right)}$$

$$= \sqrt{\frac{2}{m} (E - U(r)) - \frac{M^2}{m^2 r^2}}$$

$$\cancel{\text{dt}} \int dt = \int \frac{dr}{\sqrt{\frac{2}{m} (E - U(r)) - \frac{M^2}{m^2 r^2}}} + \text{const}$$

$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{M}{mr^2} = \sqrt{\dots}$$

$$+ \int \frac{\frac{dr M}{mr^2}}{\sqrt{\dots}} = \int d\phi = \phi$$

$$\boxed{\phi = \int \frac{M dr/r^2}{\sqrt{2m(E-U(r)) - M^2/r^2}} + \text{const}}$$

$$\phi = \phi(r) \Leftrightarrow r = r(\phi)$$

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

$$\begin{aligned} r &= r(\phi) \\ r &= r(t) \\ \phi &= \phi(t) \end{aligned} \quad \left. \begin{aligned} r &= r(\phi) \\ r &= r(t) \\ \phi &= \phi(t) \end{aligned} \right\} \vec{r}(t)$$

Example:  $U(r) = \frac{1}{2}kr^2$  (space oscillator)  $(r, \phi)$

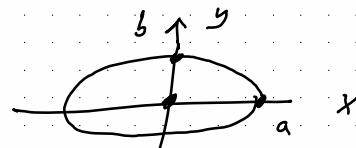
$$\begin{aligned} \underline{\underline{L}} &= \frac{1}{2}m(r^2 + r^2\dot{\phi}^2) - \frac{1}{2}kr^2 \quad (x, y) \\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2) \\ &= \left( \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \right) + \left( \frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2 \right) \end{aligned}$$

$$\begin{aligned} x(t) &= a \cos(\omega t + \phi), \quad \omega = \sqrt{\frac{k}{m}} \\ y(t) &= b \sin(\omega t + \phi), \quad \cancel{\omega} \end{aligned}$$

→ closed orbit

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

ellipse with center at origin

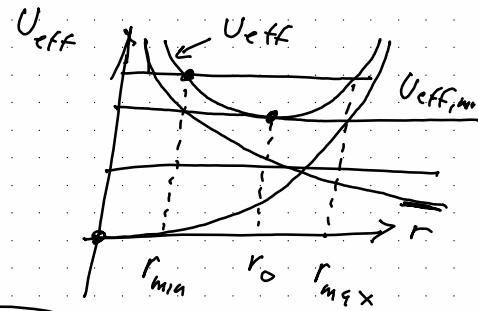


$U = -\frac{mk}{r}$ ,  $U = \frac{1}{2}kr^2$  are only two potentials that have closed bound orbits.

Hypotesis:  $U = \frac{1}{2}kr^2$

$$U_{\text{eff}} = U(r) + \frac{M^2}{2mr^2}$$

$$= \frac{1}{2}kr^2 + \frac{M^2}{2mr^2}$$



$$\phi = \int \frac{M dr/r^2}{\sqrt{2m(E - \frac{1}{2}kr^2) - \frac{M^2}{r^2}}} + \text{const}$$

bound orbits

Substitution:

$$u = \frac{1}{r}, \quad du = -\frac{dr}{r^2}$$

$$v = u^2$$

$$dv = 2u du$$

$$\phi = - \int \frac{M du}{\sqrt{2m(E - \frac{1}{2}u^2) - M^2u^2}} = - \int \frac{Mu du}{\sqrt{2m(Eu^2 - \frac{1}{2}) - M^2u^2}} + \text{const}$$

+ const

+ const

$$\phi = -\frac{1}{2} \int \frac{dv}{\sqrt{2m(Ev - \frac{1}{2}) - M^2v^2}} + \text{const}$$

Complete the square:  $-M^2v^2 + 2mv - \frac{1}{2}$

$$= -M^2 \left( v^2 - \frac{2mv}{M^2} + \frac{\frac{1}{2}}{M^2} \right)$$

$$= -M^2 \left( \left( v - \frac{m}{M^2} \right)^2 - \frac{m^2E^2}{M^4} + \frac{\frac{1}{2}}{M^2} \right)$$

$$= -M^2 \left( (v-A)^2 - B^2 \right)$$

$$= M^2 (B^2 - (v-A)^2)$$

$$A = \frac{mE}{M^2}, \quad B^2 = A^2 - \frac{m^2}{M^2}$$

$$\phi = -\frac{1}{2} \int \frac{dv}{\sqrt{B^2 - (v-A)^2}} + \text{const}$$

3rd substitution:

$$v-A = B \sin \theta$$

$$dv = B \cos \theta d\theta$$

$$\phi = -\frac{1}{2} \int \frac{B \cos \theta d\theta}{\sqrt{B^2 - B^2 \sin^2 \theta}} + \text{const}$$

$$= -\frac{1}{2} \theta + \text{const}$$

$$= -\frac{1}{2} \sin^{-1} \left( \frac{v-A}{B} \right) + \text{const}$$

$$= -\frac{1}{2} \sin^{-1} \left( \frac{\frac{1}{r^2} - A}{B} \right) + \text{const}$$

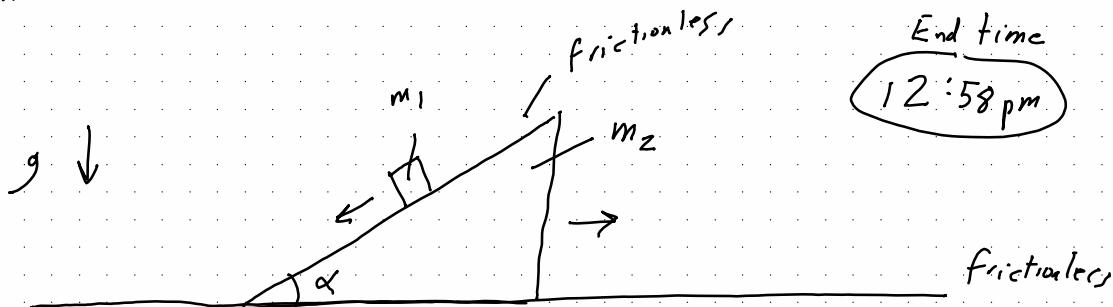
$v-A = \pm B$ 
 $\frac{1}{r^2} - A = \pm B$ 
 $r_{\max} = \frac{1}{\sqrt{A \pm B}}$ 
 $\frac{1}{r_{\min}^2} = A \pm B$ 
 $\frac{1}{r_{\min}^2} = A + B$ 
 $4 = \frac{1}{r}$ 
 $v = \frac{1}{r^2}$

$\int \frac{dx}{\sqrt{1-x^2}}$ 
 $x = \sin \theta$ 
 $dx = \cos \theta d\theta$ 
 $\int \frac{\cos \theta d\theta}{\sqrt{1-\sin^2 \theta}}$ 
 $= \theta$ 
 $= \sin^{-1}(x)$ 
 $V = 4^2$ 
 $4 = \frac{1}{r}$ 
 $v = \frac{1}{r^2}$

Lecture #10: 9/24

- Finite spring oscillator
- Quiz #2
- Next week: Q & A

lastname-q2.pdf



i) generalized coord.)

ii)  $L = T - U$

iii) EOM's

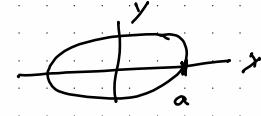
iv) special limiting cases:  $\alpha = 0, \alpha = \pi/2$   
 $m_1 \gg m_2, m_1 \ll m_2$

$$\phi = -\frac{1}{2} \sin^{-1} \left( \frac{\frac{1}{r^2} - A}{B} \right) + \underline{\text{const}} \quad E, M$$

choose const so that  $\phi=0 \Leftrightarrow r=r_{max}$

$$\frac{1}{r_{max}^2} = A - B$$

$$\frac{1}{r_{max}^2} - A = \frac{-B}{B} = -1$$



$$0 = -\frac{1}{2} \underbrace{\sin^{-1}(-1)}_{-\frac{\pi}{2}} + \text{const} +$$

$$= \frac{\pi}{4} + \text{const}$$

$$\boxed{\text{const} = -\frac{\pi}{4}}$$

$$\phi = -\frac{1}{2} \sin^{-1} \left( \frac{\frac{1}{r^2} - A}{B} \right) - \frac{\pi}{4}$$

$$\begin{aligned} -2(\phi + \frac{\pi}{4}) &= \sin^{-1} \left( \frac{\frac{1}{r^2} - A}{B} \right) \\ -\sin(2\phi + \frac{\pi}{2}) &= \left( \frac{1}{r^2} - A \right) / B \end{aligned}$$

$$-\sin(2\phi + \frac{\pi}{2}) = \left( \frac{1}{r^2} - A \right) / B$$

$$-B \cos(2\phi) = \frac{1}{r^2} - A$$

$$\boxed{\frac{1}{r^2} = A - B \cos(2\phi)}$$

$$\cos(2\phi) = \cos^2 \phi - \sin^2 \phi$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\Rightarrow \cos(2\phi) = \left( \frac{x}{r} \right)^2 - \left( \frac{y}{r} \right)^2$$

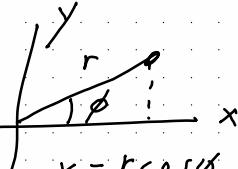
$$\left[ \frac{1}{r^2} = A - B \left( \left( \frac{x}{r} \right)^2 - \left( \frac{y}{r} \right)^2 \right) \right] \quad x^2 r^2 - \frac{(x)^2}{a^2} - \frac{(y)^2}{b^2} = 1$$

$$1 = Ar^2 - B(x^2 - y^2)$$

$$= A(x^2 + y^2) - B(x^2 - y^2) = (A-B)x^2 + (A+B)y^2 = 1$$

$$\boxed{A = mE/m^2, B^2 = A^2 - mk/m^2}$$

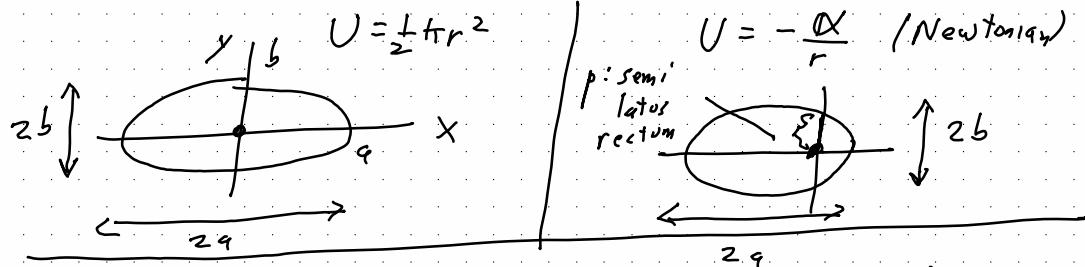
$$\begin{aligned} \sin(2\phi + \frac{\pi}{2}) &= \sin(2\phi) \cos(\frac{\pi}{2}) \\ &+ \cos(2\phi) \sin(\frac{\pi}{2}) \\ &= \cos(2\phi) = 0 \end{aligned}$$



$$x = r \cos \phi$$

$$y = r \sin \phi$$

$$\frac{1}{a^2} = A - B, \quad \frac{1}{b^2} = A + B$$



$$E = \frac{1}{2} m r^2 + \frac{M^2}{2mr^2} + \frac{1}{2} k r^2 \quad \begin{cases} (r=a) \\ (r=b) \end{cases}$$

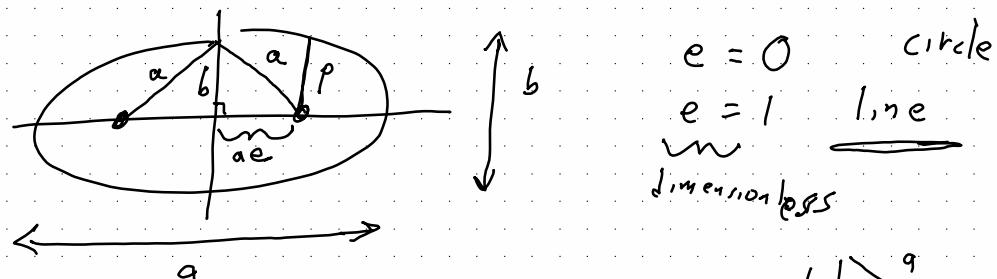
$\approx 0, t$   
 $r=a, r=b$

$$\left. \begin{aligned} E &= \frac{M^2}{2ma^2} + \frac{1}{2} k a^2 \\ E &= \frac{M^2}{2mb^2} + \frac{1}{2} k b^2 \end{aligned} \right\} \rightarrow \begin{aligned} E_a^2 &= \frac{M^2}{2m} + \frac{1}{2} k a^2 \\ E_b^2 &= \frac{M^2}{2m} + \frac{1}{2} k b^2 \end{aligned}$$

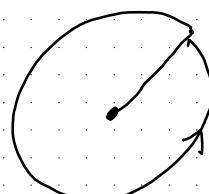
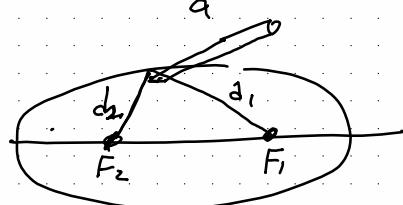
subtract  $E(a^2 - b^2) = \frac{1}{2} k (a^4 - b^4)$

$$(a^2 - b^2)(a^2 + b^2) \rightarrow E = \frac{1}{2} k (a^2 + b^2)$$

$$M^2 = m k a^2 b^2$$



$e=0$  circle  
 $e=1$  line  
dimensionless



$$a^2 = b^2 + a^2 e^2$$

$$e = \sqrt{\frac{a^2 - b^2}{a^2}}$$

$$d_1 + d_2 = 2a$$

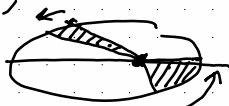
Kepler's Law: (For  $U = -k/r$ )

$$b = a \sqrt{1-e^2}$$

i) elliptical orbits with sun at one focus,  $\Delta t$

ii) equal areas in equal times

$$\frac{P^2}{a^3} = \text{const.}$$



$$U = \frac{1}{2} k r^2$$

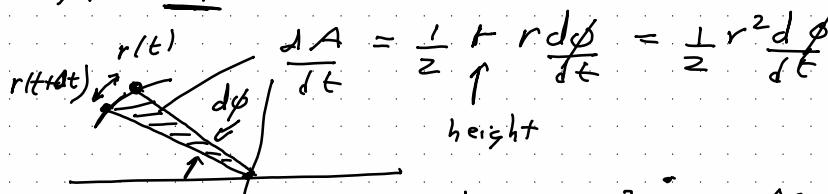
"Kepler's laws": (i) elliptical orbits with "Sun" at center  
 (ii) equal areas in equal times

(iii)

true for any

central potential

$$M = m r^2 \dot{\phi} = \text{const}$$



$$\frac{dA}{dt} = \text{const}$$

$$\frac{dA}{dt} = \frac{1}{2} r \frac{dr}{dt} \frac{d\phi}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt}$$

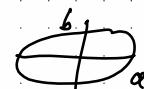
height

$$\frac{dA}{dt} = \int \frac{M}{zm} dt$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{M}{zm} = \text{const}$$

$$A = \frac{M}{zm} P$$

$$A = \pi r^2 \quad \text{circle}$$



$$\pi r^2 = \frac{M}{zm} P$$

$$P = \frac{2\pi m}{M} \pi r^2 b \quad , \quad M^2 = m \pi r^2 b^2$$

$$M = \sqrt{m \pi r^2 b}$$

$$= \frac{2\pi m \pi r^2 b}{\sqrt{m \pi r^2 b}}$$

$$= \frac{2\pi}{\sqrt{m/b}}$$

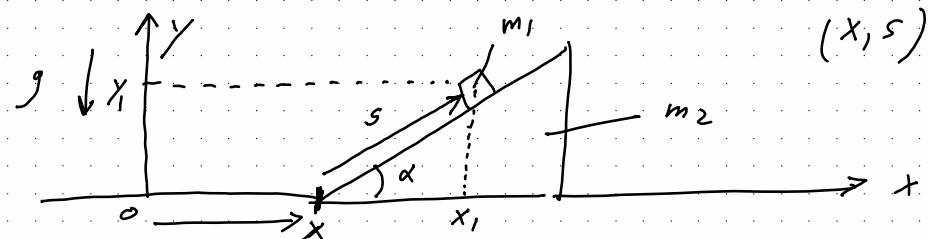
$$= \frac{2\pi}{\omega} = \text{const} \quad (\text{for any } a, b)$$

$$P = \text{const}$$

indep of  $a, b$  !!

Lec #11: Tuesday 9/29

- Midterm #1: Next Tuesday Oct 6<sup>th</sup>
- Final midterms on Blackboard (do it before Thursday)
- Today: a) Quiz #2  
b) Finish  $U = \frac{1}{2} k r^2$   
c) Q & A



$$L = T - U \quad T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{s}^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{0}^2)$$

$$x_1 = x + s \cos \alpha \rightarrow \dot{x}_1 = \dot{x} + \dot{s} \cos \alpha$$

$$y_1 = s \sin \alpha \rightarrow \dot{y}_1 = \dot{s} \sin \alpha$$

$$x_2 = x \rightarrow \dot{x}_2 = \dot{x}$$

$$y_2 = 0 \rightarrow \dot{y}_2 = 0$$

$$\dot{x}_1^2 = (\dot{x} + \dot{s} \cos \alpha)^2 = \dot{x}^2 + \underline{\dot{s}^2 \cos^2 \alpha} + 2\dot{x}\dot{s} \cos \alpha$$

$$\dot{y}_1^2 = \underline{\dot{s}^2 \sin^2 \alpha}$$

$$T = \frac{1}{2} m_1 (\dot{x}^2 + \dot{s}^2 + 2\dot{x}\dot{s} \cos \alpha) + \frac{1}{2} m_2 \dot{x}^2$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1 \dot{x} \dot{s} \cos \alpha$$

$$U = m_1 g y_1 = m_1 g s \sin \alpha$$

$$L = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1 \dot{x} \dot{s} \cos \alpha - m_1 g s \sin \alpha$$

$$\text{EOMs! } x: \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \cancel{\frac{\partial L}{\partial x}} \rightarrow 0 \quad E = T + U$$

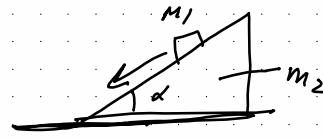
$$P_x = (m_1 + m_2) \dot{x} + m_1 \dot{s} \cos \alpha = \text{const}$$

$$s: \rightarrow [(m_1 + m_2) \ddot{x} + m_1 \ddot{s} \cos \alpha = 0]$$

$$\frac{d}{dt} (m_1 \dot{s} + m_1 \dot{x} \cos \alpha) = -m_1 g s \sin \alpha \quad \boxed{\ddot{s} + \ddot{x} \cos \alpha = -g \sin \alpha}$$

$$\ddot{x} = -\left(\frac{m_1}{m_1 + m_2}\right) \ddot{s} \cos \alpha$$

$$\ddot{s} = -g \sin \alpha \left( \frac{m_1 + m_2}{m_1 \sin^2 \alpha + m_2} \right)$$



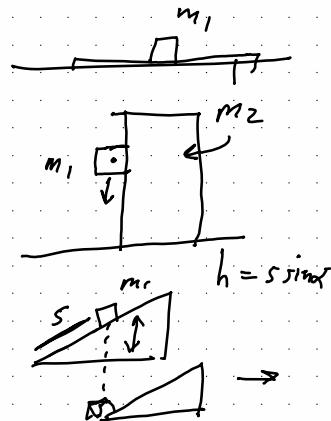
i)  $m_2 \gg m_1$ :  $\ddot{s} \approx -g \sin \alpha$

ii)  $\alpha = 0$   $\ddot{s} = 0, \ddot{x} = 0$

iii)  $\alpha = \pi/2$   $\ddot{s} = -g, \ddot{x} = 0$

iv)  $m_1 \gg m$ :  $\begin{cases} \ddot{s} \approx -g \\ \ddot{x} \approx +\frac{g}{\tan \alpha} \end{cases}$

$$h = -g = \ddot{s} \sin \alpha \rightarrow \ddot{s} = -\frac{g}{\sin \alpha}$$



Space oscillator:  $V = \frac{1}{2} k r^2$

$$V = \frac{-\alpha}{r}$$

"Kepler's" law:

(i) ellipses with origin at center of ellipse



(ii) equal areas in equal times — applies to all

$$P = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}} \text{ ind. of } a, b$$

central potential

$$\frac{p^2}{q^3} = \text{const} + \text{Ordinary Kepler}$$



$$t = -\int \frac{dr}{\sqrt{\frac{2}{m}(E - V(r)) - \frac{M^2}{m^2 r^2}}} + \text{const}$$

reduced mass,  $\frac{1}{2} k r^2$ ,  $E = \frac{1}{2} k(a^2 + b^2)$ ,  $M^2 = m k a^2 b^2$

want t to increase as r decreases,

$$\begin{aligned}
 t &= - \int \frac{dr}{\sqrt{\frac{1}{m}(\frac{1}{2}\hbar^2(a^2+b^2) - \frac{1}{2}\hbar^2r^2) - \frac{m\hbar^2a^2b^2}{m^2r^2}}} \\
 &= - \frac{1}{\sqrt{\frac{\hbar^2}{m}}} \int \frac{dr}{\sqrt{(a^2+b^2) - r^2 - \frac{a^2b^2}{r^2}}} \quad (r, \phi) \\
 &= - \frac{1}{\sqrt{\frac{\hbar^2}{m}}} \int \frac{r dr}{\sqrt{r^2(a^2+b^2) - r^4 - a^2b^2}} \quad (x, y) \\
 &= - \frac{1}{\sqrt{\frac{\hbar^2}{m}}} \int \frac{r dr}{\sqrt{- (r^4 - r^2(a^2+b^2) + a^2b^2)}} \\
 &\quad \boxed{\sqrt{a^2(1-\sin^2\theta)} = a_{10,\theta}} \quad \boxed{r^2 = x^2+y^2} \\
 &\quad \boxed{\frac{dx}{\sqrt{a^2-x^2}}} \quad \boxed{a^2 = a^2\sin^2\theta} \\
 &\quad \boxed{x = a\sin\theta} \quad \boxed{\text{substitution}} \quad \boxed{b < r < a} \\
 &\quad \boxed{r^2 = a^2\cos^2\xi + b^2\sin^2\xi}
 \end{aligned}$$

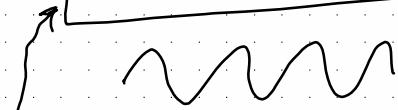
$$\begin{aligned}
 X &= a\cos\xi \\
 Y &= b\sin\xi \\
 \rightarrow r^2 &= a^2\cos^2\xi + b^2\sin^2\xi \\
 \text{Differentiates} \quad 2rdr &= -2a^2\cos\xi\sin\xi + 2b^2\sin\xi\cos\xi \\
 &= -2\sin\xi\cos\xi(a^2-b^2)
 \end{aligned}$$

$$r dr = -\sin\xi\cos\xi(a^2-b^2)$$

$$\begin{aligned}
 \sqrt{-(r^2-a^2)/(r^2-b^2)} &= \sqrt{-(a^2\cos^2\xi + b^2\sin^2\xi - a^2)(a^2\cos^2\xi + b^2\sin^2\xi - b^2)} \\
 &\quad - a^2\sin^2\xi \quad - b^2\cos^2\xi \\
 &= \sqrt{(-1/a^2-b^2)\sin^2\xi (a^2-b^2)\cos^2\xi} \\
 &= \boxed{((a^2-b^2)\sin\xi\cos\xi)}
 \end{aligned}$$

$$t = \frac{1}{\omega} \int d\xi + \text{const} \rightarrow \boxed{\omega t = \xi} + \frac{\cos \xi}{\omega} = 0$$

$$\begin{aligned} x &= a \cos(\omega t) \\ y &= b \sin(\omega t) \end{aligned}$$



$$\phi = \phi(t)$$

$$(x, y) \leftrightarrow (r, \phi)$$

$$\begin{aligned} x &= r \cos \phi \\ y &= r \sin \phi \end{aligned}$$

$$r \cos \phi = a \cos(\omega t)$$

$$r \sin \phi = b \sin(\omega t)$$

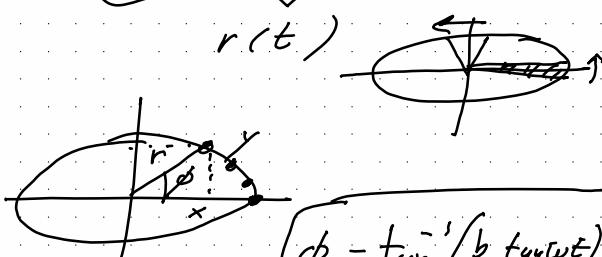
$$\text{Divide 2nd by 1st: } \tan \phi = \frac{b}{a} \tan(\omega t) \rightarrow \phi = \omega t \text{ if } b = a$$

$$r^2 = x^2 + y^2$$

$$= a^2 \cos^2(\omega t) + b^2 \sin^2(\omega t)$$

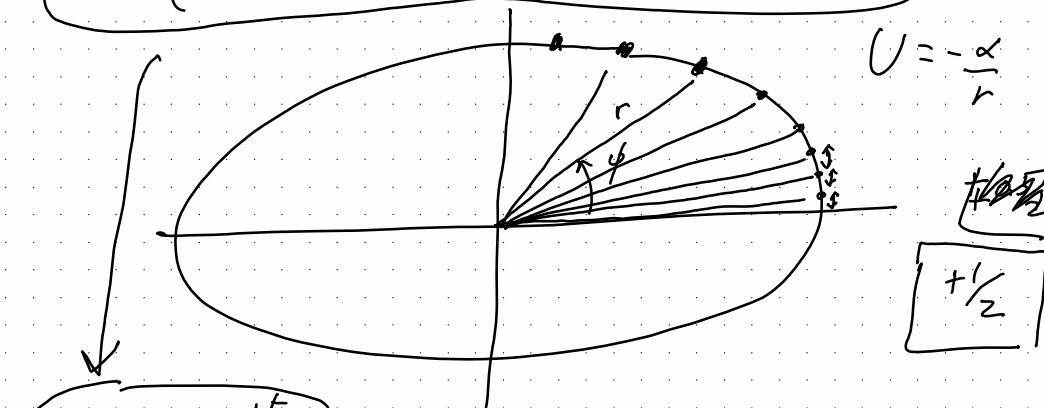
$$r = \sqrt{a^2 \cos^2(\omega t) + b^2 \sin^2(\omega t)}$$

$$r(t)$$



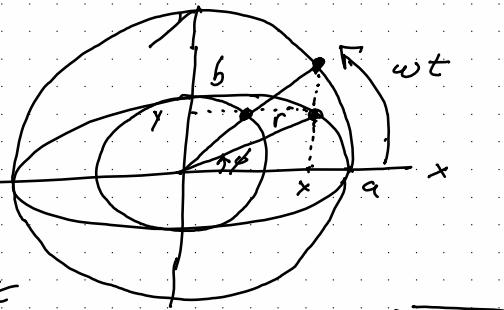
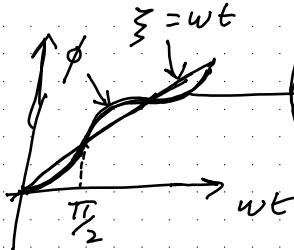
$$\phi = \tan^{-1} \left( \frac{b}{a} \tan(\omega t) \right)$$

$$L = \left( \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2 \right) + \left( \frac{1}{2} m \dot{y}^2 - \frac{1}{2} K y^2 \right)$$



$$\begin{aligned} x &= a \cos \omega t \\ y &= b \sin \omega t \end{aligned}$$

$$\begin{aligned} x &= r \cos \phi \\ y &= r \sin \phi \end{aligned}$$



## Lec #12:

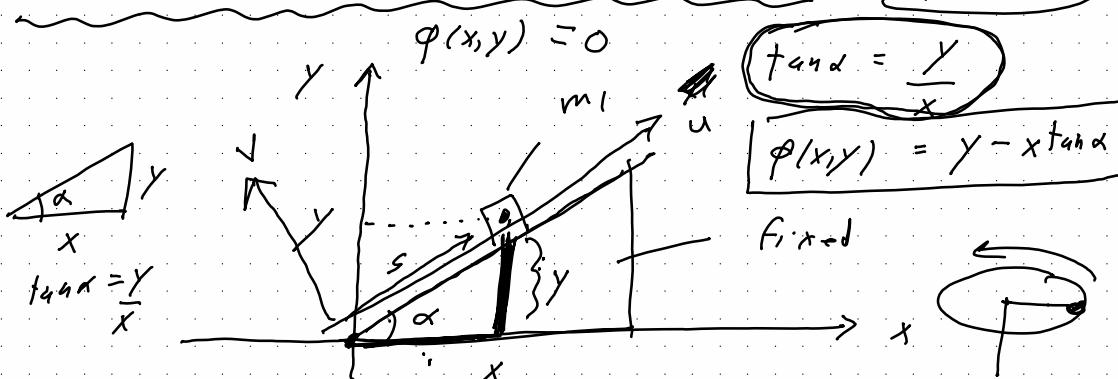
— Midterm Exam 1 : Tuesday Oct 6 + 4

— Do trial exam by end of the day today

— Today : Q&A

$$x^2 + y^2 - r_0^2 = 0$$

$$r - r_0 = 0$$



$$T = \frac{1}{2} m_1 (x'^2 + y'^2), \quad U = m_1 g y \quad (x, y)$$

$$L = T - U + \lambda (y - x \tan \alpha) \quad \checkmark \quad (r, \phi)$$

## Eqms:

$$x : \frac{d}{dt} \left( \frac{\partial L}{\partial x} \right) = \frac{\partial L}{\partial x} \rightarrow \boxed{m_1 \ddot{x} = -\lambda \tan \alpha} \quad \checkmark$$

$$y : \frac{d}{dt} \left( \frac{\partial L}{\partial y} \right) = \frac{\partial L}{\partial y} \rightarrow \boxed{m_1 \ddot{y} = -m_1 g + \lambda} \quad \checkmark$$

constraint:

$$\begin{cases} y - x \tan \alpha = 0 \\ y' - x' \tan \alpha = 0 \end{cases}$$

$$(m_1 \ddot{x} = -\lambda \tan \alpha) \tan \alpha + m_1 \ddot{x} \tan \alpha = -m_1 g + \lambda$$

$$\begin{aligned} 0 &= -\lambda \tan^2 \alpha - \lambda \\ &\quad + m_1 g \\ &= -\lambda (1 + \tan^2 \alpha) + m_1 g \end{aligned}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

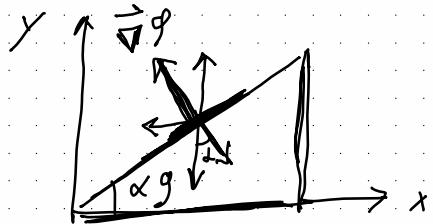
$$L = \frac{1}{2} m_1 (x'^2 + y'^2) - m_1 g y + \lambda (y - x \tan \alpha)$$

$$\lambda = \frac{m_1 g}{\sec^2 \alpha} \propto m_1 g \cos^2 \alpha$$

$$\varphi(x, y) = y - x \tan \alpha$$

$$\vec{\nabla} \varphi = \frac{\partial \varphi}{\partial x} \hat{x} + \frac{\partial \varphi}{\partial y} \hat{y}$$

$$= -\tan \alpha \hat{x} + \hat{y} = -\frac{\sin \alpha \hat{x} + \cos \alpha \hat{y}}{\cos \alpha}$$



$$= -\frac{\sin \alpha \hat{x} + \cos \alpha \hat{y}}{\cos \alpha}$$

$$\boxed{N = mg \cos \alpha}$$

$\perp$  to incline

$$\vec{\nabla} \varphi \propto -\sin \alpha \hat{x} + \cos \alpha \hat{y}$$

$$y = x \tan \alpha$$

$$dy = \tan \alpha dx \rightarrow \boxed{\cos \alpha dy = \sin \alpha dx}$$

$$\vec{\nabla} \varphi \cdot d\vec{r} = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy = -\tan \alpha dx + \frac{1}{\cos \alpha} dx = 0$$

$$(u, v) \quad \varphi(u, v) = v = 0$$

$$L = T - U + \lambda v$$

$$v = 0$$

$$\boxed{L = T - U + \lambda \varphi}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt} (m \dot{x}) = -\frac{\partial U}{\partial x}$$

$$\boxed{\frac{dp}{dt} = -\frac{\partial U}{\partial x} = F}$$

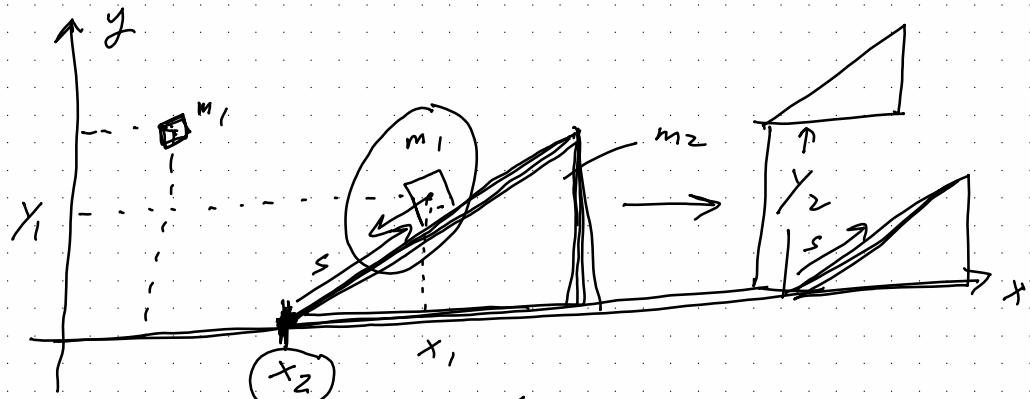
$$\frac{dp}{dt} = F$$

$$\begin{aligned} L(x, \dot{x}) \\ = \frac{1}{2} m \dot{x}^2 - U(x) \end{aligned}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = -\frac{\partial L}{\partial r}$$

$$\frac{d\vec{p}}{dt} = -\frac{\partial U}{\partial r} + \lambda \frac{\partial \varphi}{\partial r}$$

$$\begin{aligned} \frac{d\vec{p}}{dt} &= -\vec{\nabla} U + \vec{\nabla} \varphi \\ &= \vec{F} + \underbrace{\vec{F}_{\text{constraint}}}_{\text{applied}} \end{aligned}$$



$(x_1, y_1, x_2, y_2)$

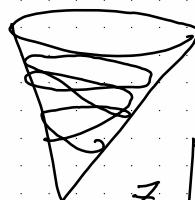
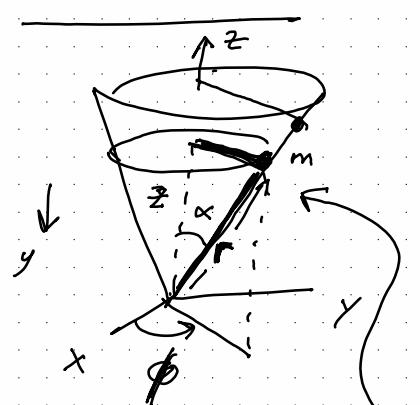
$m_1$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$y_2 = 0$$

$\downarrow_0$

Sec 14, Prob 2:



cylindrical  
 $(r, \phi, z)$

a/t.  
notation

$$\begin{cases} r^2 = x^2 + y^2 \\ z^2 = x^2 + y^2 \\ \rho^2 = x^2 + y^2 \end{cases}$$

sph. polar  
 $(r, \phi, \theta)$

$$r^2 = x^2 + y^2 + z^2$$

$$\theta = \alpha$$

$$\tan \alpha = \frac{z}{r}$$

$(\phi, S)$

$(\phi, z)$

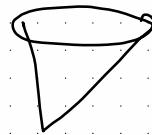
$$T = \frac{1}{2} m \left( r^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) \quad \text{sp. pol., } \theta = \alpha$$

---


$$T = \frac{1}{2} m \left( s^2 + s^2 \dot{\phi}^2 + \dot{z}^2 \right) \quad \text{cyl. coords, } s, \phi, z$$

plane polar  
coord

$$\tan \alpha = \frac{s}{z}$$



$$L = T - U + \lambda \varphi$$

$$\varphi = s - z \tan \alpha$$

$$z = \frac{s}{\tan \alpha} \quad \begin{cases} s = z \tan \alpha \\ \dot{s} = \dot{z} \tan \alpha \end{cases}$$

$$f(x, y) = e^{-\frac{1}{2}(x^2+y^2)}$$

No constraint:  $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$

$$g(x, y, \lambda) \equiv f(x, y) + \lambda \varphi(x, y)$$

$$\frac{\partial g}{\partial x} = 0, \quad \frac{\partial g}{\partial y} = 0, \quad \frac{\partial g}{\partial \lambda} = 0$$

---

Suppose constraint  $y = \frac{1}{2}x^2 + 1$

$$F(x) = f(x, y) \Big|_{y = \frac{1}{2}x^2 + 1}$$

$$L = T - U + \lambda \varphi$$

$$\boxed{\frac{dF}{dx} = 0}$$

$$\boxed{\varphi = y - \frac{1}{2}x^2 - 1 = 0}$$

$x, y, \lambda$

COM:

$$m_1, m_2, m_3, \vec{r}_1, \vec{r}_2, \vec{r}_3 \quad \vec{P} = M_{\text{tot}} \dot{\vec{R}}_{\text{com}}$$

$$\vec{R}_{\text{com}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3}{m_1 + m_2 + m_3} = \text{const}$$

$$0 = \sum_a m_a \vec{r}_a \leq m_b$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \vec{r}_a} \right) = 0 \quad \vec{P} = \sum_a \frac{\partial L}{\partial \vec{r}_a} = \vec{e}_{0, i}$$

~~closed system~~:  $\vec{P} = \text{const}$

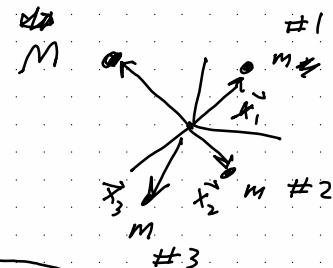
$$L = \frac{1}{2} \sum_a m_a |\vec{r}_a|^2 - U(\vec{r}_1 - \vec{r}_2, \vec{r}_2 - \vec{r}_3, \dots)$$

$$\vec{r}_a \rightarrow \vec{r}_a + \delta \vec{x}$$

$$\frac{\partial L}{\partial \vec{r}} = 0$$

$$M, \underbrace{m_1, \dots, m_n}_n$$

$$\vec{X}, \vec{x}_1, \dots, \vec{x}_n$$



$$M \vec{X} + m \sum_{a=1}^n \vec{x}_a = 0$$

$$\begin{aligned} \vec{r}_1 &= \vec{x}_1 - \vec{X} \\ \vec{r}_2 &= \vec{x}_2 - \vec{X} \\ &\vdots \\ \vec{r}_n &= \vec{x}_n - \vec{X} \end{aligned}$$

1 h + 20 ms

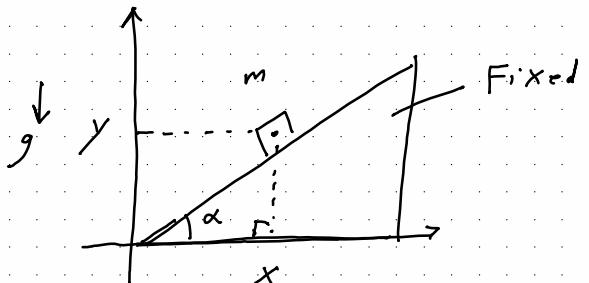
80 ms

120 ms

$$\vec{r}_1, \vec{r}_2$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

2 : 35 ps



$$L = T - U + \lambda \varphi$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mg y + \lambda(y - x \tan \alpha)$$

$$\tan \alpha = \frac{y}{x}$$

$$so \quad y = x \tan \alpha$$

$$\varphi(x, y) = y - x \tan \alpha = 0$$

constraint

Eoms:

$$x: \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \rightarrow m \ddot{x} = -\lambda \tan \alpha \quad (1)$$

$$y: \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) = \frac{\partial L}{\partial y} \rightarrow m \ddot{y} = -mg + \lambda \quad (2)$$

$$\lambda: y - x \tan \alpha = 0 \rightarrow y = x \tan \alpha \quad (3)$$

Differentiate constraint twice:

$$\ddot{y} = \ddot{x} \tan \alpha$$

Substitute for  $\ddot{y}$ ,  $\ddot{x}$  using (1) and (2)

$$-g + \frac{\lambda}{m} = -\frac{\lambda}{m} \tan \alpha \cdot \tan \alpha$$

$$\frac{\lambda}{m} \left( 1 + \tan^2 \alpha \right) = g$$

$$\boxed{\lambda = \frac{mg}{\sec^2 \alpha}}$$

constraint Force:

$$\vec{F}_c = \lambda \vec{\nabla} \varphi \quad \text{where} \quad \varphi = y - x \tan \alpha$$

$$= \frac{mg}{\sec^2 \alpha} \left( -\tan \alpha \hat{x} + \hat{y} \right)$$

$$= \frac{mg}{\sec^2 \alpha} \left( -\frac{\sin \alpha}{\cos \alpha} \hat{x} + \hat{y} \right)$$

$$= mg \cos \alpha \left( -\sin \alpha \hat{x} + \cos \alpha \hat{y} \right)$$

$$= \boxed{mg \cos \alpha \hat{n}}$$

(where  $\hat{n} = -\sin \alpha \hat{x} + \cos \alpha \hat{y}$   
is  $\perp$  to incline)



Return to EOMs:

$$\ddot{x} = -\frac{\lambda}{m} \tan \alpha$$

$$\ddot{y} = -g + \frac{\lambda}{m}$$

$$\text{where } \lambda = \frac{mg}{\sec^2 \alpha} = mg \cos^2 \alpha$$

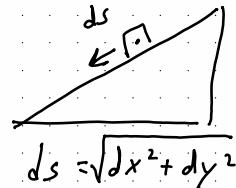
$$\rightarrow \ddot{x} = -g \cos^2 \alpha \tan \alpha$$

$$= -g \sin \alpha \cos \alpha$$

$$\ddot{y} = -g + g \cos^2 \alpha$$

$$= -g (1 - \cos^2 \alpha)$$

$$= -g \sin^2 \alpha$$



Acceleration down the incline:

$$\ddot{s} = -\sqrt{\ddot{x}^2 + \ddot{y}^2} = -g \sin \alpha \sqrt{\cos^2 \alpha + \sin^2 \alpha} = \boxed{-g \sin \alpha}$$

*standard result*

Lecture #14: Thurs Oct 8<sup>th</sup>

- mid term 1: Avg  $\approx 14/20$

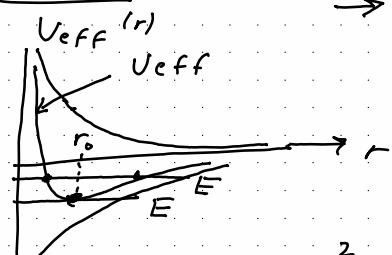
- mid term 2: Nov 19<sup>th</sup> (before Thanksgiving)

- oral final

(Collisions & Scattering (Sec 16-20))

Scattering: closed system, two bodies

$\rightarrow$  central force problem

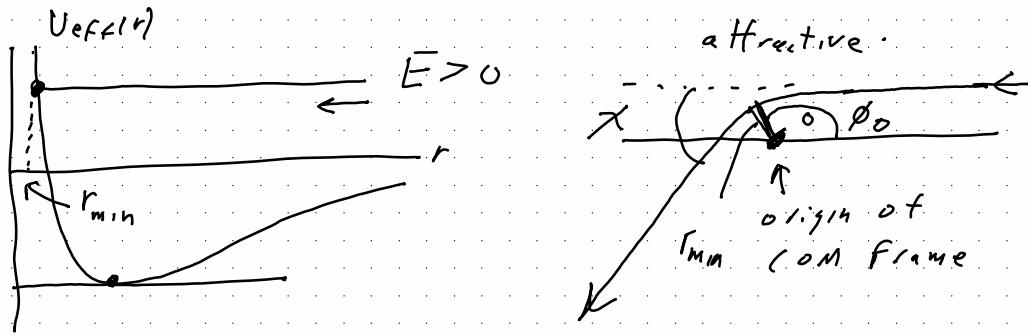


$$E = U_{\text{eff}, \min} \rightarrow r = r_0 \quad (\text{circular})$$

$U_{\text{eff}, \min} < E < 0 \rightarrow$  bound orbit

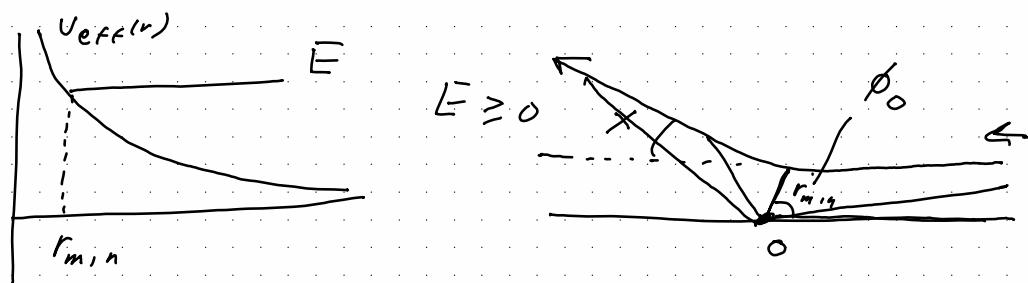
$$U_{\text{eff}} = -\frac{\alpha}{r} + \frac{M_2^2}{2mr^2}$$

~~ellipses~~  
for  $-\alpha/r$



$$U(r) = +\alpha/r$$

$$U_{\text{eff}}(r) = \alpha/r + \frac{M_e^2}{2mr^2}$$



$$\int_0^{\phi_0} d\phi = \int_{r_{\min}}^{\infty} \frac{M_e dr / r^2}{\sqrt{2m(E - U(r)) - \frac{M_e^2}{r^2}}} \quad (14.7) \text{ L\&L}$$

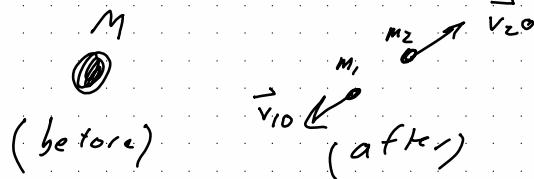
$$\boxed{\phi_0 = \int_{r_{\min}}^{\infty} \frac{M_e dr / r^2}{\sqrt{2m(E - U(r)) - \frac{M_e^2}{r^2}}}} \quad | \quad E, M_e, U(r)$$

Fig 18 in L\&L

## Collisions: (Sec 16, 17)

- Elastic collisions of two particles

✗ Spontaneous disintegration of a single particle



Analyze in COM Frame (to begin w/t)

$$\vec{p}_{\text{tot}, 0} = \vec{0}$$

L com

cons. of momentum

$\vec{p}_0$ : momentum of mass  $m_1$ , after disintegration  
 $-\vec{p}_0$ :  $m_1$   $m_2$

cons. of energy

$$E_i = E_{i,c} + T_{1,0} + E_{2,c} + T_{2,0}$$

internal energy of mass  $M$  in com frame

$$E_{i,c} = \frac{1}{2} m_1 |\vec{v}_{1,0}|^2 + \frac{1}{2} m_2 |\vec{v}_{2,0}|^2$$

$$E_i = E_{i,c} + \frac{|\vec{p}_0|^2}{2m_1} + E_{2,i} + \frac{|\vec{p}_0|^2}{2m_2}$$

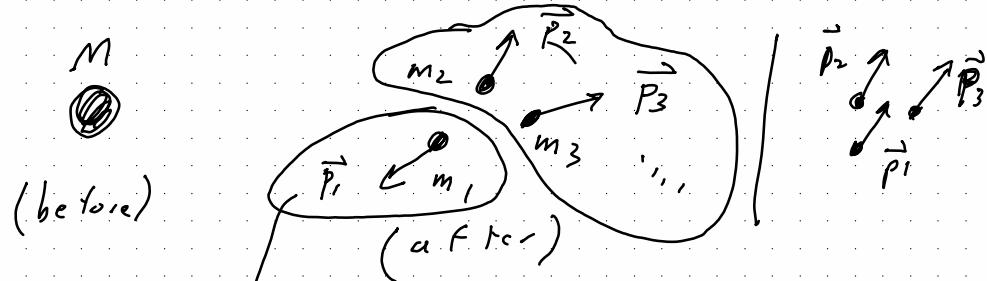
$$E_i - E_{i,c} - E_{2,i} = \frac{p_0^2}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right)$$

$E =$  disintegration energy

$$\geq 0 \quad = \frac{p_0^2}{2} \left( \frac{m_1 + m_2}{m_1 m_2} \right) = \frac{p_0^2}{2m}$$

$$\vec{p}_0 = \sqrt{2mE}$$

$$\rightarrow \vec{v}_{1,0} = \vec{p}_0/m_1, \quad \vec{v}_{2,0} = \vec{p}_0/m_2$$



$$\vec{p}_{\text{tot}} = 0$$

$$0 = \vec{p}_{\text{tot}} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \dots$$

$$\vec{p}_1 = \vec{p}_0$$

Q: what condition is there on the velocities of  $m_2, m_3, \dots$  such that KE of  $m_1$  is largest?

$$T_{1,0} = \frac{p_0^2}{2m}$$

Consider COM of  $m_2, m_3, \dots$

$$\vec{p}_2 + \vec{p}_3 + \dots = -\vec{p}_0$$

$E_i'$ : internal energy of  $m_2, m_3, \dots$

$$KE = \frac{p_0^2}{2(M-m_1)}$$

$\underbrace{\phantom{p_0^2 / 2}}$   
 $(m_2 + m_3 + \dots)$

cons. of energy:

$$E_i = E_{i,i} + \frac{p_0^2}{2m_1}$$

$\underbrace{\phantom{p_0^2 / 2}}$   
int. energy of M

$$+ E_i' + \frac{p_0^2}{2(M-m_1)}$$

$\underbrace{\phantom{p_0^2 / 2}}$   
int + KE of  $m_2, m_3, \dots$

$$E_i - E_{i,i} - E_i' = \frac{p_0^2}{2} \left( \frac{1}{m_1} + \frac{1}{M-m_1} \right)$$

$$E_i - E_{i,i} - E_i' = \frac{p_0^2}{2} \frac{M}{m_1(M-m_1)}$$

$E_{2,i} + E_{3,i} + \dots$

X

$$T_{1,0} = \frac{p_0^2}{2m_1} = \left( \frac{M-m_1}{M} \right) (E_i - E_{i,i} - E_i')$$

$\checkmark \quad \checkmark \quad \underbrace{\phantom{p_0^2 / 2}}$

$T_{1,0}$  ~~is~~ maximum when  $E_i'$  is minimum

$$E_i' + \frac{p_0^2}{2(M-m_1)} = E_{2i} + \frac{p_2^2}{2m_2} + E_{3i} + \frac{p_3^2}{2m_3} + \dots$$

int energy + T.E. of

$$\vec{p}_2 = m_2 \vec{v}_2, e + c.$$

$$= E_{2i} + \frac{1}{2} m_2 |\vec{v}_2|^2 + E_{3i} + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots$$

$$E_i' = (E_{2i} + E_{3i} + \dots) + \left( \frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots \right)$$

$$\text{W} \quad - \quad \frac{|\vec{p}_2 + \vec{p}_3 + \dots|^2}{2(m_2 + m_3 + \dots)} = |\vec{v}_0|^2$$

$$= (E_{2i} + E_{3i} + \dots) + \left( \frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots \right)$$

$$- \frac{1}{2} \frac{|\vec{m}_2 \vec{v}_2 + \vec{m}_3 \vec{v}_3 + \dots|^2}{m_2 + m_3 + \dots}$$

$$\begin{aligned} & |\vec{v}_2 + \vec{v}_3|^2 \\ & = |\vec{v}_2|^2 + |\vec{v}_3|^2 \\ & + 2 \vec{v}_2 \cdot \vec{v}_3 \end{aligned}$$

$$\text{If } \vec{v}_2 = \vec{v}_3 = \dots \equiv \vec{v}_0$$

$$\text{then: } \frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots$$

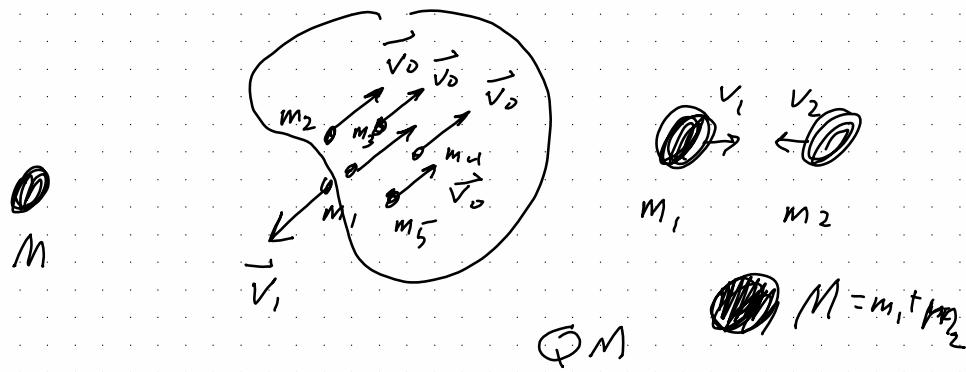
$$= \left( \frac{1}{2} (m_2 + m_3 + \dots) \right) |\vec{v}_0|^2$$

$$\text{and } \frac{1}{2} \frac{(m_2 \vec{v}_2 + m_3 \vec{v}_3 + \dots)^2}{m_2 + m_3 + \dots}$$

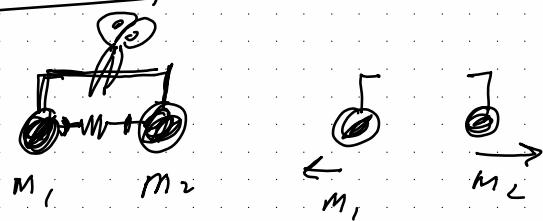
$$= \frac{1}{2} |\vec{v}_0|^2 \frac{(m_2 + m_3 + \dots)^2}{m_2 + m_3 + \dots}$$

$$= \frac{1}{2} |\vec{v}_0|^2 (m_2 + m_3 + \dots)$$

$$T_{10, \max} = \left( \frac{M-m_1}{M} \right) E$$



Fig, 14      Figs 15, 16



Lec #15: Tues 10/13

— solutions to midterm #1 posted

— next two weeks: Sec 16 - 20

Collisions and scattering

Figures 14, 15, 16

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1(v'_1)^2 + \frac{1}{2}m_2(v'_2)^2$$

Elastic collision of two particles

closed system:  $E$ ,  $\vec{P}$ ,  $\vec{M}$  are conserved

✓ i)  $H\bar{E} = \text{const}$  [ ignore internal energies ]

✓ ii)  $\vec{P} = \text{const}$

After:  $\vec{v}_1'$ ,  $\vec{v}_2'$ :

4 DOF  
3 equations

Elastic collision:

$$\vec{V} = \vec{v}_1 - \vec{v}_2 = \vec{v}_{10} - \vec{v}_{20}$$

relative velocity vector

$$\vec{r} = \vec{r}_1 - \vec{r}_2 = \vec{r}_{10} - \vec{r}_{20}$$

relative position vector

$$\vec{V} = \dot{\vec{r}}$$

$$\vec{v}_1 = \vec{V} + \vec{v}_{10}$$

$$\vec{v}_2 = \vec{V} + \vec{v}_{20}$$

$$\vec{r}_1 = \vec{R} + \vec{r}_{10}$$

$$\vec{r}_2 = \vec{R} + \vec{r}_{20}$$

$\vec{v}_1$ : velocity of particle 1 wrt lab frame  
before collision

$\vec{v}_{10}$ : " wrt COM Frame  
before collision

$\vec{v}_1'$ ,  $\vec{v}_{10}'$ ,  $\vec{v}_2'$ ,  $\vec{v}_{20}'$ : velocities after collision

$$T_0 = \frac{1}{2} m_1 |\vec{r}_{10}|^2 + \frac{1}{2} m_2 |\vec{r}_{20}|^2$$

$$\vec{v}_{10} = \vec{r}_{10} = \left( \frac{m_2}{m_1 + m_2} \right) \vec{r} = \vec{V} \quad \vec{r} = \vec{r}_{10} - \vec{r}_{20}$$

$$\vec{v}_{20} = \vec{r}_{20} = - \left( \frac{m_1}{m_1 + m_2} \right) \vec{r}$$

$$T_0 = \frac{1}{2} m_1 \left( \frac{m_2}{m_1 + m_2} \right)^2 |\vec{r}|^2$$

$$+ \frac{1}{2} m_2 \left( \frac{-m_1}{m_1 + m_2} \right)^2 |\vec{r}|^2$$

$$= \frac{1}{2} \left( \frac{m_1 m_2}{m_1 + m_2} \right) |\vec{r}|^2$$

$$= \frac{1}{2} m |\vec{r}|^2 = \frac{1}{2} m |\vec{V}|^2$$

$$T_0 = T_0'$$

$$|\vec{r}| = |\vec{r}'|$$

$$= \frac{1}{2} m' |\vec{V}'|^2$$

$$V = V'$$

~~constant~~  
magnitude of relative velocity vector = const

Before collision

$$\overrightarrow{v}_{1,0} = \left( \frac{m_2}{m_1 + m_2} \right) \overrightarrow{v}, \quad \overrightarrow{v}_{2,0} = \left( \frac{-m_1}{m_1 + m_2} \right) \overrightarrow{v}$$

$$\overrightarrow{V} = \overrightarrow{v}_1 - \overrightarrow{v}_2$$

$\overrightarrow{v}_1' = \overrightarrow{v}$   
 $\overrightarrow{v}_2' = 0$

After coll., nos:

$$\overrightarrow{v}'_1 = \left( \frac{m_2}{m_1 + m_2} \right) v \hat{n}_o$$

magnitude  
of  $\overrightarrow{v}'$

$$\overrightarrow{v}'_{2,0} = \left( \frac{-m_1}{m_1 + m_2} \right) v \hat{n}_o$$

(angle =  $\chi$  wrt  
COM  
frame)

$$\boxed{\overrightarrow{v}'_1 = \overrightarrow{v}'_{1,0} + \overrightarrow{V} = \left( \frac{m_2}{m_1 + m_2} \right) v \hat{n}_o + \frac{m_1 \overrightarrow{v}_1 + m_2 \overrightarrow{v}_2}{m_1 + m_2}}$$

$$\overrightarrow{R} = \frac{m_1 \overrightarrow{r}_1 + m_2 \overrightarrow{r}_2}{m_1 + m_2} \rightarrow \overrightarrow{V} = \frac{m_1 \overrightarrow{v}_1 + m_2 \overrightarrow{v}_2}{m_1 + m_2}$$

$$\overrightarrow{v}'_1 = \overrightarrow{v}'_{1,0} + \overrightarrow{V} = \left( \frac{m_2}{m_1 + m_2} \right) v \hat{n}_o + \frac{m_1 \overrightarrow{v}_1 + m_2 \overrightarrow{v}_2}{m_1 + m_2} = \overrightarrow{v}'_1$$

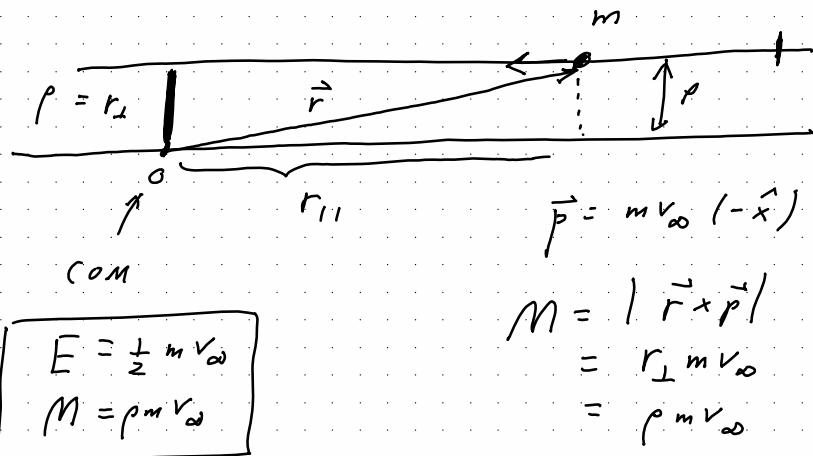
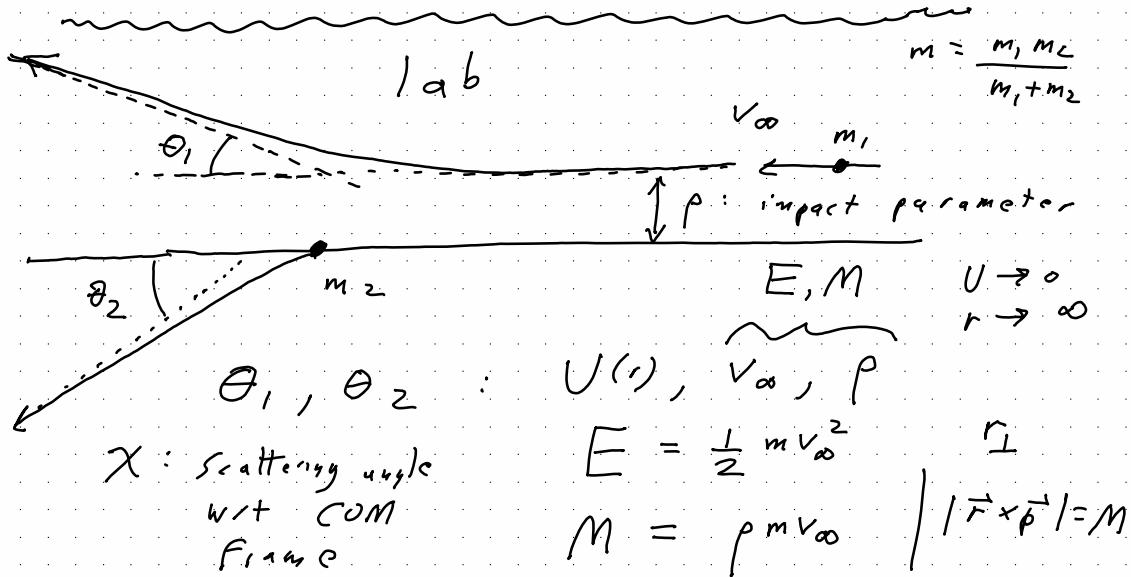
$$\overrightarrow{v}'_2 = \overrightarrow{v}'_{2,0} + \overrightarrow{V} = \left( \frac{-m_1}{m_1 + m_2} \right) v \hat{n}_o + \frac{m_1 \overrightarrow{v}_1 + m_2 \overrightarrow{v}_2}{m_1 + m_2} = \overrightarrow{v}'_2$$

$$\boxed{\overrightarrow{p}_1' = m_1 \overrightarrow{v}_1' = m v \hat{n}_o + m_1 \left( \frac{\overrightarrow{p}_1 + \overrightarrow{p}_2}{m_1 + m_2} \right)}$$

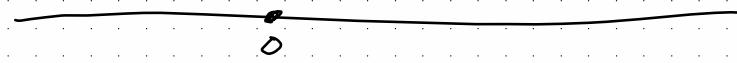
$$\boxed{\overrightarrow{p}_2' = m_2 \overrightarrow{v}_2' = -m v \hat{n}_o + m_2 \left( \frac{\overrightarrow{p}_1 + \overrightarrow{p}_2}{m_1 + m_2} \right)}$$

Lec #16: Thurs 10/15

- Quiz #3 next week
- General remarks about scattering
- Example: Prob 18.1 (Hard-sphere)



$$d\rho \leftrightarrow dx \quad d\sigma = 2\pi p d\rho$$



$$d\sigma = 2\pi \rho(x) \left| \frac{d\rho}{dx} \right| dx$$

$$d\sigma = \frac{\rho(x)}{\sin x} \left| \frac{d\rho}{dx} \right| d\Omega = \rho \left| \frac{d\rho}{d(\cos x)} \right| d\Omega$$

$$\boxed{\frac{d\sigma}{d\Omega} = \rho \left| \frac{d\rho}{d(\cos x)} \right|}$$

in COM frame

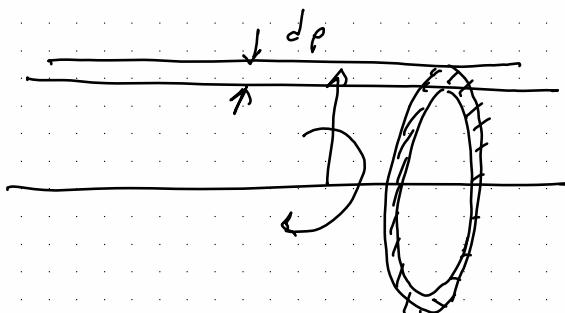
$$\boxed{\frac{d\sigma_1}{d\Omega_1} = \rho \left| \frac{d\rho}{d(\cos \theta_1)} \right|} \quad \text{in lab frame } \theta_1,$$

$$= \frac{d\sigma}{d\Omega} \left| \frac{d(\cos x)}{d(\cos \theta_1)} \right|$$

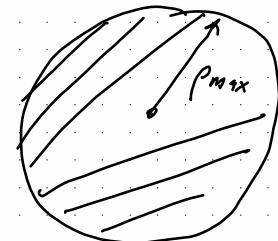
$$\boxed{\frac{d\sigma_2}{d\Omega_2} = \frac{d\sigma}{d\Omega} \left| \frac{d(\cos x)}{d(\cos \theta_2)} \right|}$$

wrt  
lab frame  
( $\theta_1, \theta_2$ )

$$d\rho \leftrightarrow d\chi$$

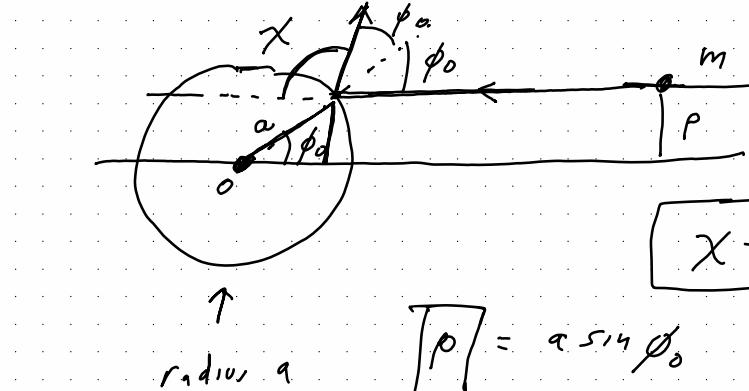


perspective



$$\sigma = \pi \rho_{max}^2$$

Hard sphere: prob 18.1  $U(r) = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases}$



$$x \leftrightarrow \rho$$

$$\begin{aligned} \rho &= a \sin \phi_a \\ &= a \sin\left(\frac{\pi}{2} - \frac{x}{a}\right) \\ &= a \cos\left(\frac{x}{a}\right) \end{aligned}$$

$$x + 2\phi_0 = \pi$$

$$\phi_0 = \frac{\pi}{2} - \frac{x}{a}$$

$$d\sigma = 2\pi \rho d\rho = 2\pi \rho(x) \left| \frac{d\rho}{dx} \right| dx$$

$$d\sigma = \cancel{2\pi} a \cos\left(\frac{x}{a}\right) \frac{a}{a} \sin\left(\frac{x}{a}\right) dx$$

$$= \pi a^2 \sin\left(\frac{x}{a}\right) \cos\left(\frac{x}{a}\right) dx$$

$$= \boxed{\frac{\pi a^2}{2} \sin x dx}$$

$$d\Omega = 2\pi \sin x dx \rightarrow \sin x dx = \frac{1}{2\pi} d\Omega$$

$$d\sigma = \frac{\pi a^2}{2} \cdot \frac{1}{2\pi} d\Omega$$

$$= \boxed{\frac{1}{4} a^2 d\Omega} \quad \text{--- uniform distribution}$$

Total cross-section:

$$\sigma_{tot} = \sigma = \int_{unit\ sphere} d\sigma = \frac{1}{4} a^2 \int_{sphere} d\Omega = \boxed{\pi a^2} \quad \text{--- } 4\pi$$

$$\frac{d\sigma_1}{d\Omega_1} = \frac{d\sigma}{d\Omega} \left| \frac{d(\cos X)}{d(\cot\theta_1)} \right|$$

$$\frac{d\sigma_2}{d\Omega_2} = \frac{d\sigma}{d\Omega} \left| \frac{d(\cot X)}{d(\cot\theta_2)} \right|$$

$$\boxed{\begin{aligned} X &= \pi - 2\theta_2 \\ \cos X &= -\frac{m_1}{m_2} \sin^2 \theta_1 \pm \cos \theta_1 \sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2 \theta_1} \end{aligned}}$$

$$\sin \theta_1 d\theta_1 = -d(\cot\theta_1)$$

Lecture #17      Tues, 10/20

- Q & A #3 - this week
- Q & A :  $\vec{p}_1 + \vec{p}_2' = \vec{P}_{tot}'$

$$\vec{V} = \frac{\vec{p}_1' + \vec{p}_2'}{m_1 + m_2}$$

$$c = \frac{\vec{P}_{tot}'}{m_1 + m_2}$$

$$v_{CM} = \frac{\vec{p}_1'}{m_1 + m_2}$$

$$v_{CM} = \frac{m_1 \vec{v}_1}{m_1 + m_2}$$

$$m_1 \vec{V} = m_1 \left( \frac{m_1 \vec{v}_1}{m_1 + m_2} \right)$$

$$\vec{v}_2 = 0 \rightarrow \boxed{\vec{p}_2' = 0} = \left( \frac{m_1^2}{m_1 + m_2} \right) \vec{r}_1$$

$m_1 < m_2, v_2 = 0$

$m_1$

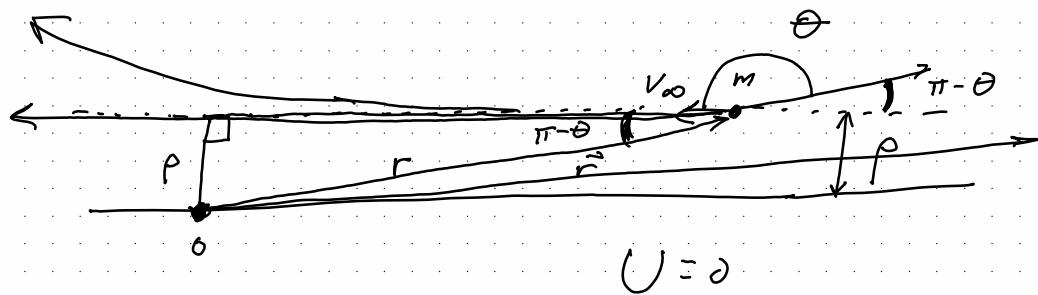
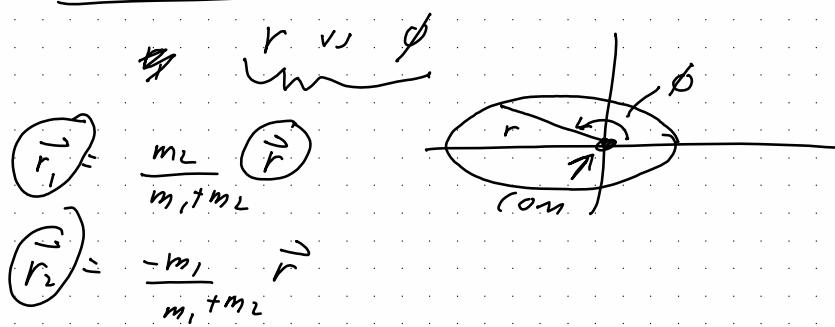
$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$T = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2$$

$$= \frac{1}{2} m |\dot{\vec{r}}|^2$$

$$m = \frac{m_1 m_2}{m_1 + m_2}$$



$$J = \partial$$

$$M = \rho m v_\infty$$

$$\vec{M} = \vec{r} \times \vec{p}$$

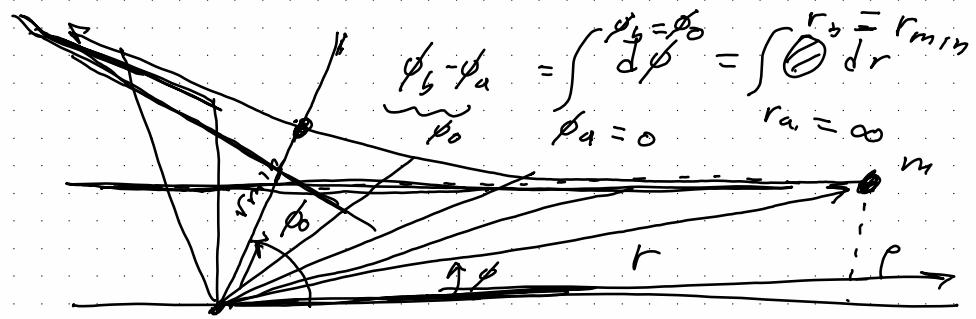
$$= m \vec{r} \times \vec{v}_\infty$$

$$M = |\vec{M}| = m r v_\infty \sin \theta$$

$$= m v_\infty r \sin(\pi - \theta)$$

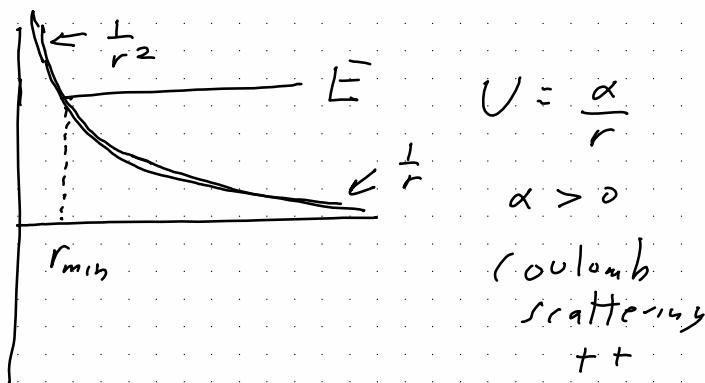
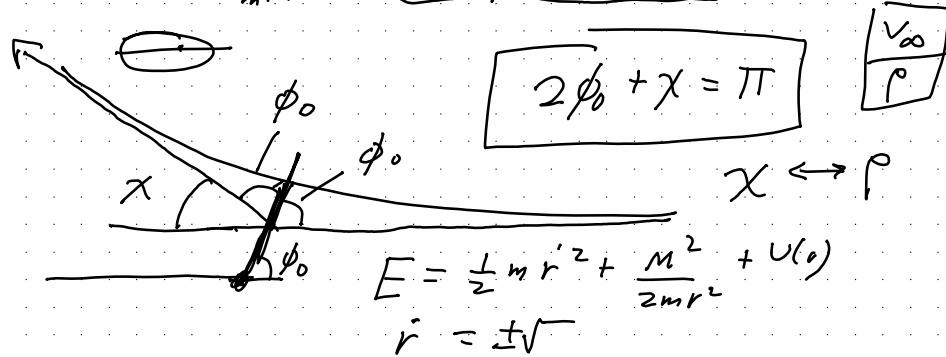
$$= \rho m v_\infty r$$

$$\begin{aligned} & \sin(\pi - \theta) \\ &= \sin 90^\circ \\ &= \cos \theta \\ &= \sin \theta \end{aligned}$$



$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M dr / r^2}{\sqrt{2m(E-U) - M^2/r^2}} \quad U(r) = \frac{\alpha}{r}$$

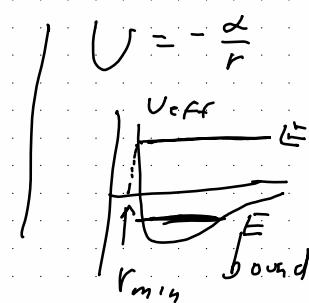
$M = pmv_\infty$   
 $E = \frac{1}{2}mv_\infty^2$



Coulomb scattering  
++  
--

$$U_{eff} = U(r) + \frac{M^2}{2mr^2}$$

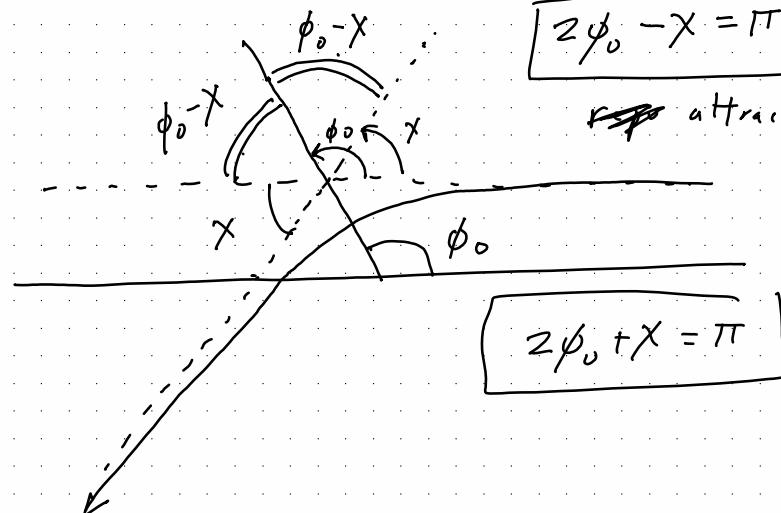
$$E = U_{eff}(r_{min})$$



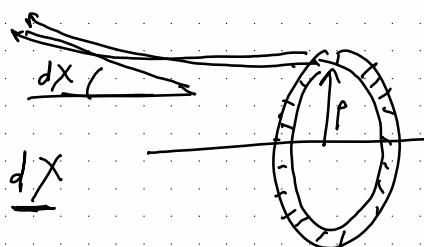
$$2(\phi_0 - x) + x = \pi$$

$$2\phi_0 - x = \pi$$

~~attractive~~



$$\begin{aligned} d\sigma &= 2\pi \rho d\rho \\ &= 2\pi \rho(x) \left| \frac{d\rho}{dx} \right| dx \end{aligned}$$



$$= \frac{\rho(x)}{\sin x} \left| \frac{d\rho}{dx} \right| d\Omega$$

~~d\rho > 0~~

~~dx < 0~~

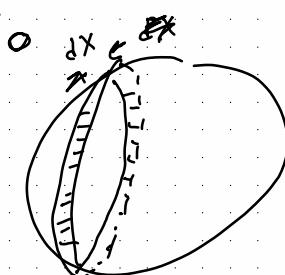
$$d\Omega = 2\pi \sin x dx$$

w.r.t Com Frame  
To go to lab frame

$$x \rightarrow \theta_1, \theta_2$$

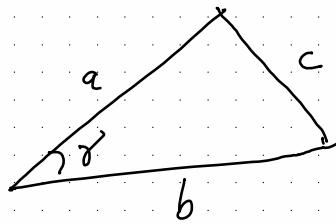
$$d\sigma_1, d\sigma_2$$

$$d\theta_1, d\theta_2, d\Omega_1, d\Omega_2$$



Lecture #18: Thurs 10/22

$$c^2 = a^2 + b^2 - 2ab \cos \gamma \quad \underline{\underline{}}$$



$$\cos^2 x + \sin^2 x = 1$$

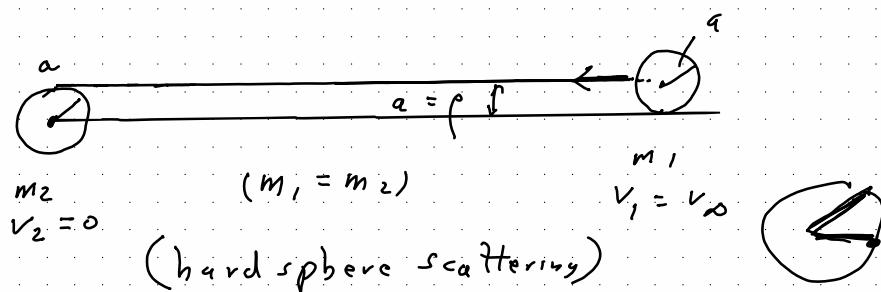
$$\sin(A \pm B) = \sin A \cos B \\ \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \\ \mp \sin A \sin B$$

QUIZ #3:

Lab Frame

name-q3.pdf



1) calculate  $\chi$  (scattering angle of the reduced mass wrt COM frame)

2) calculate  $\theta_1, \theta_2$  (scattering angles wrt Lab Frame)

$$\tan \theta_1 = \frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi}, \quad \theta_2 = \frac{1}{2}(\pi - \chi)$$

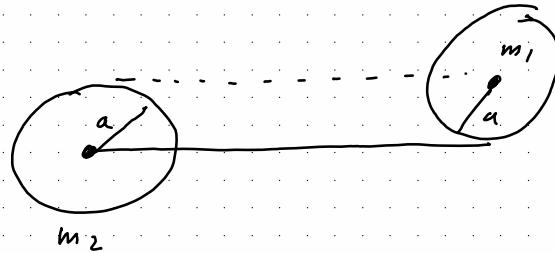
3) How do  $\theta_1, \theta_2$  change if masses  $m_1$  and  $m_2$  change? ( $m_1 \gg m_2$ ;  $m_1 \ll m_2$ )

elast  
scattering

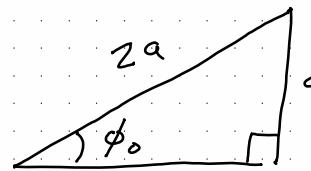
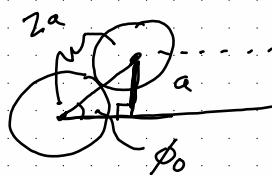
$m_2$

at rest

$$U(r) = \begin{cases} \infty & r < 2a \\ 0 & r > 2a \end{cases}$$

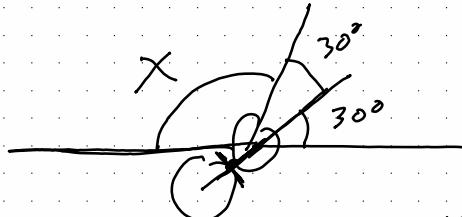


$$\vec{r} = \vec{r}_1 - \vec{r}_2$$



$$\sin \phi_0 = \frac{a}{2a} = \frac{1}{2}$$

$$\boxed{\phi_0 = \pi/6 = 30^\circ}$$



$$\boxed{x = 120^\circ} = \frac{2\pi}{3}$$

independent of  $m_1$  and  $m_2$

$$\frac{2\pi}{3} = 120^\circ$$

$$\boxed{\theta_2 = 120^\circ}$$

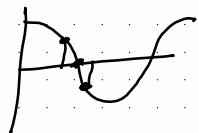
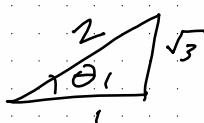
$$2) \quad \theta_2 = \frac{1}{2}(\pi - x) = \frac{1}{2}(180^\circ - 120^\circ) = \boxed{30^\circ}$$

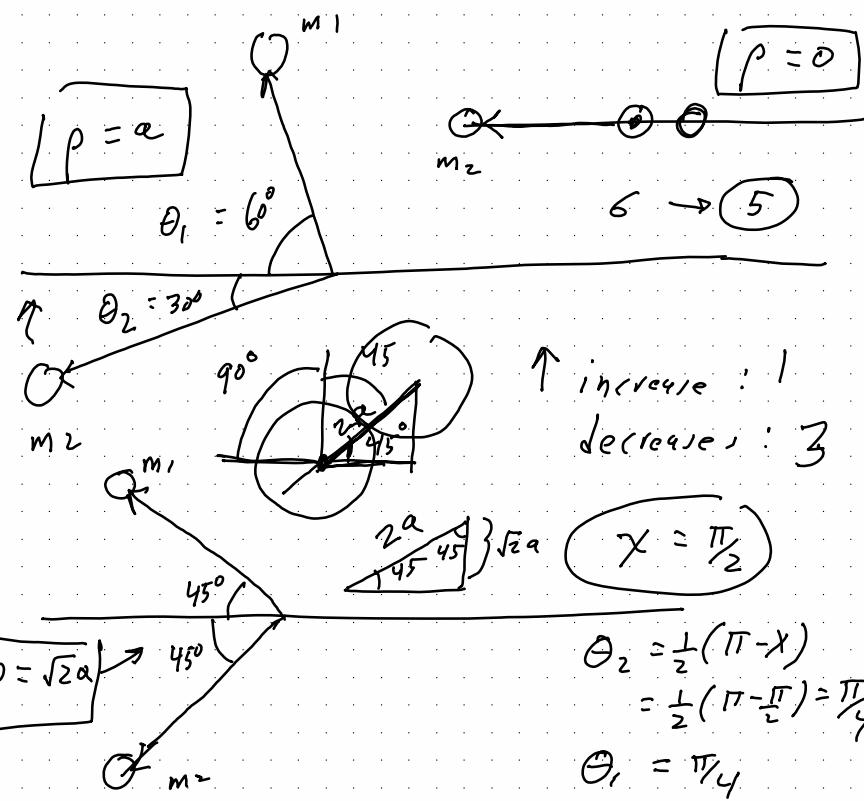
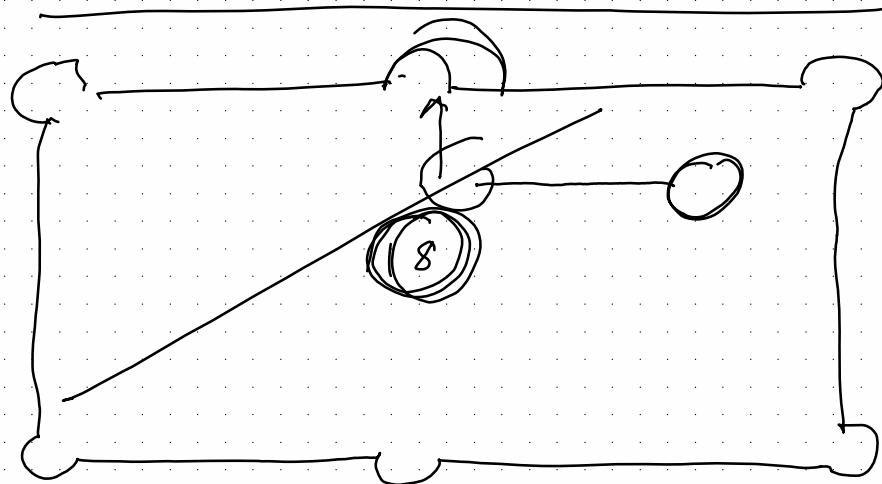
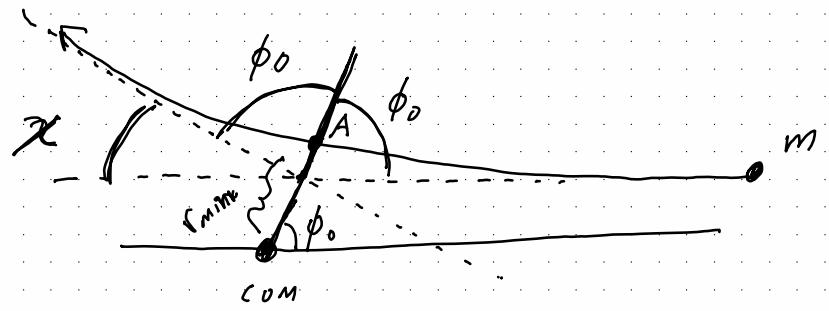
$$\tan \theta_1 = \frac{m_2 \sin(120^\circ)}{m_1 + m_2 \cos(120^\circ)} = \frac{\sin(120^\circ)}{1 + \cos(120^\circ)}$$

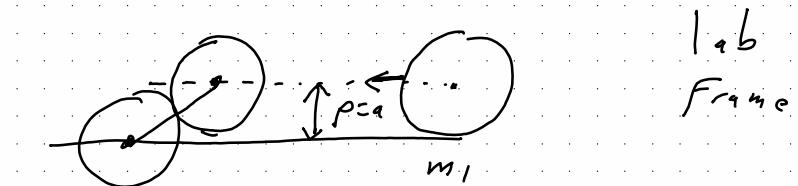
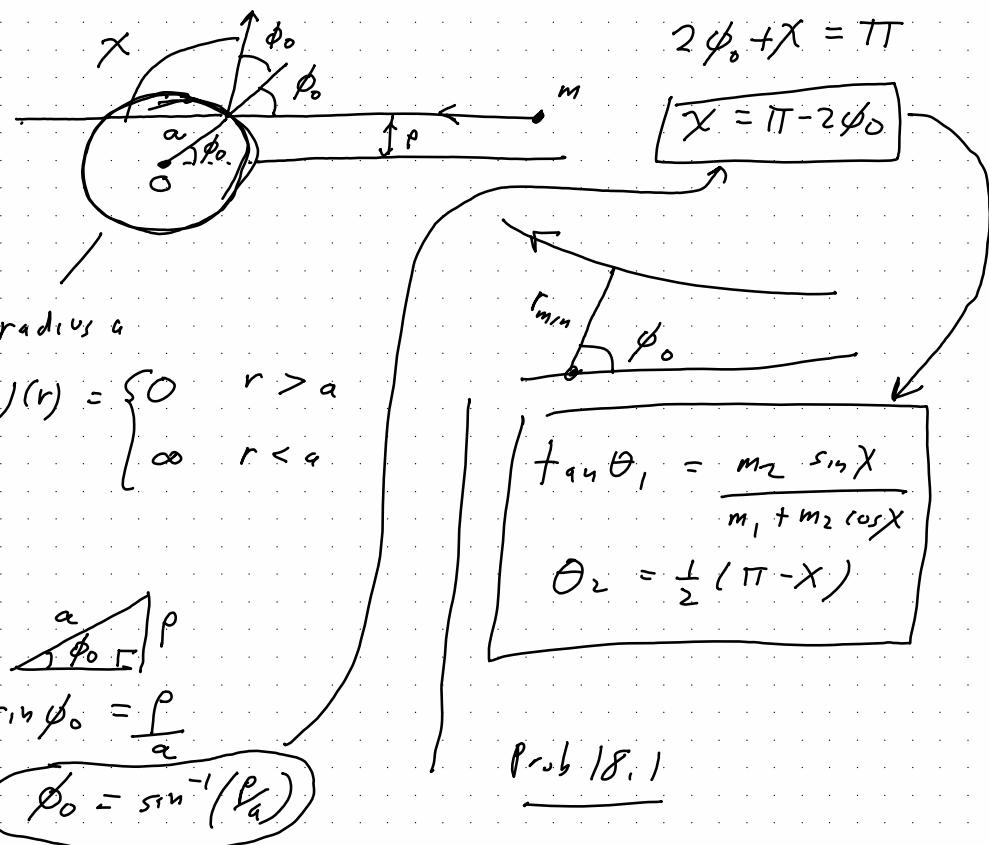
$$= \frac{\sqrt{3}/2}{1 - 1/2}$$

$$= \sqrt{3}$$

$$\boxed{\theta_1 = 60^\circ}$$

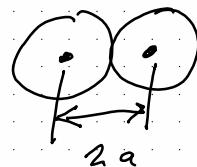






$$m_2 \quad (V_2 = v)$$

$$m_1 = m_2 \quad p = a$$



$$U(r) = \begin{cases} 0 & r > 2a \\ \infty & r < 2a \end{cases}$$

Diagram showing two spheres separated by a distance of  $2a$  with a central mass  $m$  at a distance  $p=a$  from the center of mass of the system. An angle  $\phi_0$  is shown between the line of centers and the line to  $m$ .

$$\sin \phi_0 = \frac{1}{2}$$

$$\phi_0 = 30^\circ$$

COM Frame

$$X = \pi - 2\phi = 180^\circ - 2 \cdot 30^\circ = 120^\circ$$

$$\begin{cases} \tan \theta_1 = \frac{m_2 \sin X}{m_1 + m_2 \cos X} \\ \theta_2 = \frac{1}{2}(\pi - X) \end{cases}$$

independent  
of  $m_1, m_2$



$$\theta_2 = \frac{1}{2}(180^\circ - 120^\circ) = 30^\circ \quad (\text{indep. of } m_1, m_2)$$

$$\tan \theta_1 = \frac{\sin(120^\circ)}{1 + \cos(120^\circ)} = \frac{\sqrt{3}/2}{1 - 1/2} = \sqrt{3}$$

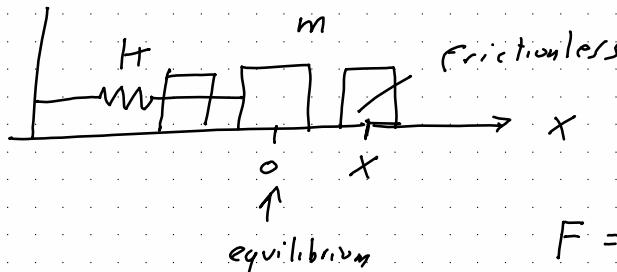
$$\theta_1 = 60^\circ$$

$$\begin{array}{l} m_1 \ll m_2 \rightarrow \tan \theta_1 \approx \tan X \rightarrow \theta_1 = 120^\circ \\ m_1 \gg m_2 \rightarrow \tan \theta_1 \approx \frac{m_2}{m_1} \sin X \rightarrow \theta_1 \approx 0^\circ \end{array}$$

Loc #19: Tuesday 10/27

- Small oscillations (next 3 classes)

Sec 21, 22, 23  
 (Free oscillation)  
 in 1-d      (Forced oscillation)  
 in 1-d      several dimensions



$$F = -kx$$

spring  
constant +

$$F = m \ddot{x}$$

$$-kx = m \ddot{x}$$

$$\rightarrow \ddot{x} = -\frac{k}{m}x \rightarrow x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

$\omega = \sqrt{\frac{k}{m}}$  (Angular freq.)

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

[ ] determined by ICs.

$$x(t) = a \cos(\omega t + \alpha)$$

| initial phase  
amplitude

$$x(t) = \operatorname{Re} [A e^{i\omega t}], A = a e^{i\alpha}$$

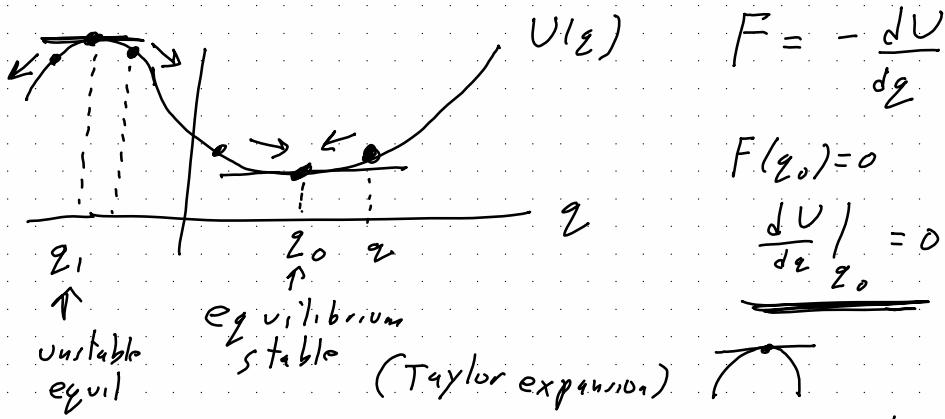
[ ] complex

$$\begin{aligned}\operatorname{Re} [a e^{i\alpha} e^{i\omega t}] &= \operatorname{Re} [a e^{i(\omega t + \alpha)}] \\ &= \operatorname{Re} [a (\cos(\omega t + \alpha) \\ &\quad + i \sin(\omega t + \alpha))] \\ &= a \cos(\omega t + \alpha)\end{aligned}$$

Small oscillation:  $f(x) = f(x_0) + f'(x_0)(x - x_0)$   
 $\quad \quad \quad + \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots$

1-d system:  $q$  generalized coord

$$L = \frac{1}{2} a(q) \dot{q}^2 - U(q)$$

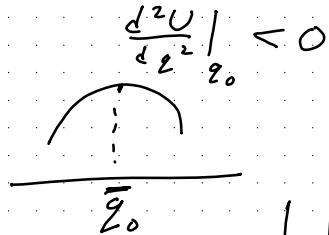
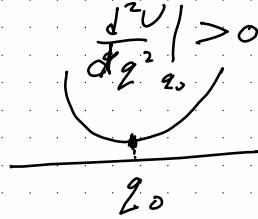


$$U(q) = U(q_0) + \underbrace{\frac{dU}{dq}}_0 \Big|_{q_0} (q - q_0) + \frac{1}{2} \underbrace{\frac{d^2U}{dq^2}}_{q_0} (q - q_0)^2 + \dots$$

Small displacement away from equilibrium

$$|q - q_0| \ll 1 \quad (\text{ignore } O(3))$$

$$U(q) \approx U(q_0) + \frac{1}{2} \left. \frac{d^2 U}{dq^2} \right|_{q_0} (q - q_0)^2$$



$$K = \left. \frac{d^2 U}{dq^2} \right|_{q_0}$$

$$U(q) \approx U(q_0) + \underbrace{\frac{1}{2}}_{\text{const}} K (q - q_0)^2$$

$$F = -\frac{dU}{dq}$$

$$= -K(q - q_0)$$

=

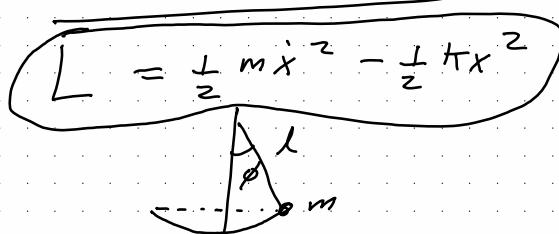
-Kx

$$U = \frac{1}{2} Kx^2$$

$$L = \frac{1}{2} a(q) \dot{q}^2 - U(q)$$

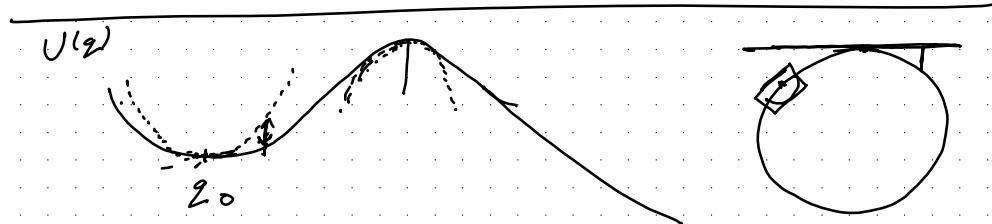
$$= \frac{1}{2} \left. a(q_0) \right|_{m''} \dot{x}^2 - U(q_0) - \frac{1}{2} Kx^2$$

$$x = q - q_0 \rightarrow \dot{x} = \dot{q}$$



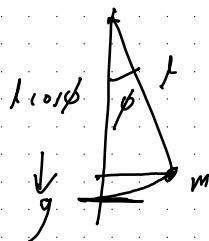
$$\rightarrow m \ddot{x} = -Kx$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$



Example:

Simple pendulum



$$L = \frac{1}{2} m l^2 \dot{\phi}^2 - m g l (1 - \cos \phi)$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \cos \phi$$

$$= T - U$$

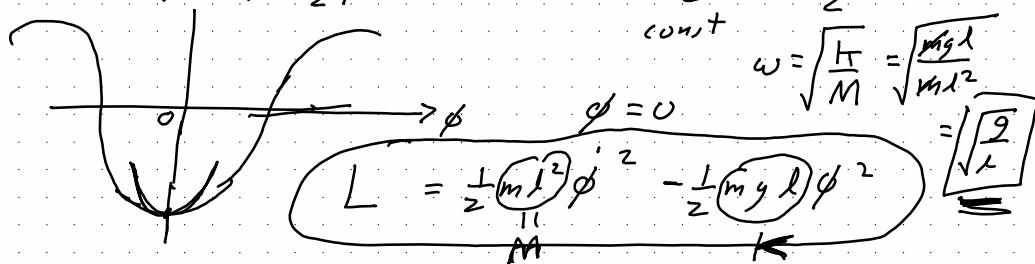
$$mgY$$

$$y = l = l \cos \phi$$

$$= l (1 - \cos \phi) = -m g l \left( \frac{1 - \cos \phi}{2} \right)$$

$$U(\phi) = -m g l \cos \phi$$

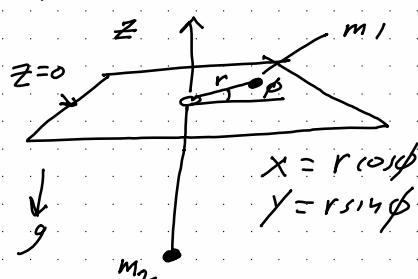
$$\cos \phi = 1 - \frac{1}{2} \phi^2 + \dots = -\underbrace{m g l}_{\text{const}} + \frac{+m g l \phi^2}{2}$$



$$\omega = \sqrt{\frac{T}{M}} = \sqrt{\frac{m g l}{M l^2}}$$

$$= \sqrt{\frac{g}{l}}$$

Example: 2-d problem



$$T = \frac{1}{2} m_1 (\dot{r}^2 + r^2 \dot{\phi}^2)$$

$$+ \frac{1}{2} m_2 \frac{\dot{z}^2}{r^2}$$

$$T = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2$$

string: length l

$$r + |z| = l$$

$$r - z = l$$

$$\boxed{z = r - l}$$

$$\rightarrow \dot{z} = \dot{r}$$

$$L = T - U$$

$$L = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r$$

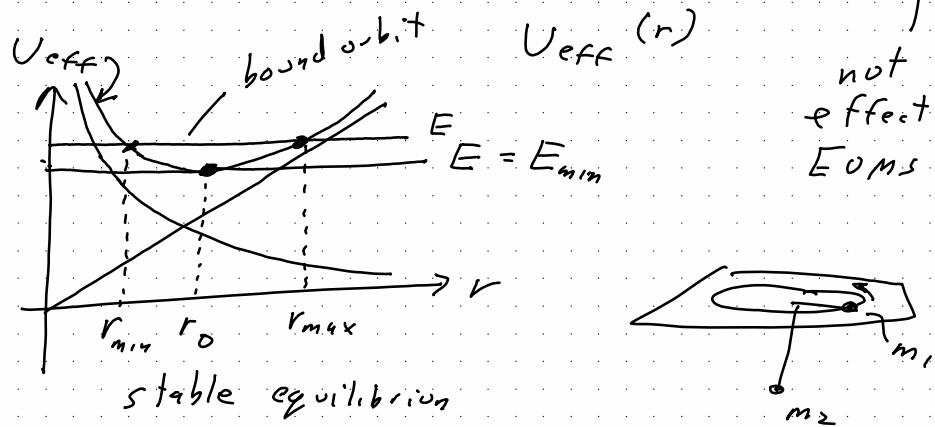
$$\dot{\phi} = \frac{M_z}{m_1 r^2}$$

No  $\phi$  dependence

$$\leftarrow M_z = \frac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} = \text{const.}$$

No explicit  $t$ ,  $m_1$  dependence  $\rightarrow E = \text{const}$

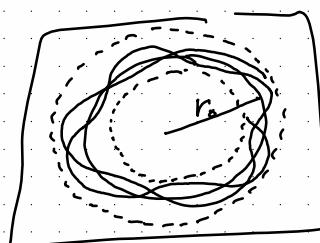
$$\begin{aligned} E &= T + U \\ &= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_1\dot{\phi}^2 + m_2 g r \\ &= \boxed{\frac{1}{2}(m_1 + m_2)\dot{r}^2} + \frac{M_z^2}{2m_1 r^2} + m_2 g r \quad (-m_2 g \theta) \end{aligned}$$



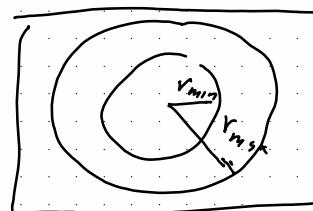
$$\begin{aligned} \Omega &= \left. \frac{dU_{\text{eff}}}{dr} \right|_{r=r_0}, \quad U_{\text{eff}}(r) \\ &= -\frac{M_z^2}{m_1 r_0^3} + m_2 g \end{aligned}$$

$$M_z^2 = m_1 m_2 g r_0^3$$

Small oscillations around  $r_0$ :  $\frac{|r-r_0|}{r_0} \ll 1$



top view



$$\frac{dU_{\text{eff}}}{dr} = -\frac{M_2^2}{m_1 r^3} + m_2 g$$

$$\rightarrow \frac{d^2 U_{\text{eff}}}{dr^2} \Big|_{r_0} = \frac{3 M_2^2}{m_1 r_0^4}, \quad M_2^2 = m_1 m_2 g r_0^3$$

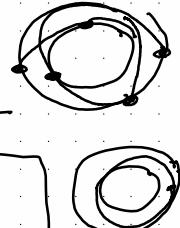
$$K = \frac{3 m_1 m_2 g r_0^3}{m_1 r_0^4} = \boxed{\frac{3 m_2 g}{r_0}}$$

$$E = \frac{1}{2} (m_1 + m_2) r^{-2} + \underbrace{U_{\text{eff}}(r_0)}_{\text{const}} + \frac{1}{2} K (r - r_0)^2$$

$$x = r - r_0$$

$$E = \frac{1}{2} (m_1 + m_2) x^{-2} + \frac{1}{2} K x^2 + \text{const}$$

$$\omega_r = \sqrt{\frac{K}{m_1 + m_2}} = \boxed{\sqrt{\frac{3 m_2 g}{(m_1 + m_2) r_0}}}$$



$$\omega_\phi = \dot{\phi} \Big|_{r_0}$$

$$M_2 = m_1 r^2 \dot{\phi}$$

$$= \frac{M_2}{m_1 r_0^2}$$

$$= \frac{\sqrt{m_1 m_2 g r_0^3}}{m_1 r_0^2}$$

$$= \frac{\sqrt{m_1 m_2 g r_0^3}}{\sqrt{m_1^2 r_0^4}}$$

$$= \boxed{\sqrt{\frac{m_2 g}{m_1 r_0}}}$$

$$\omega_r = \boxed{\sqrt{\frac{3 m_2 g}{(m_1 + m_2) r_0}}}$$

$$0 \rightarrow 2\pi$$

not equal is general

For  $\omega_r = \omega_\phi$

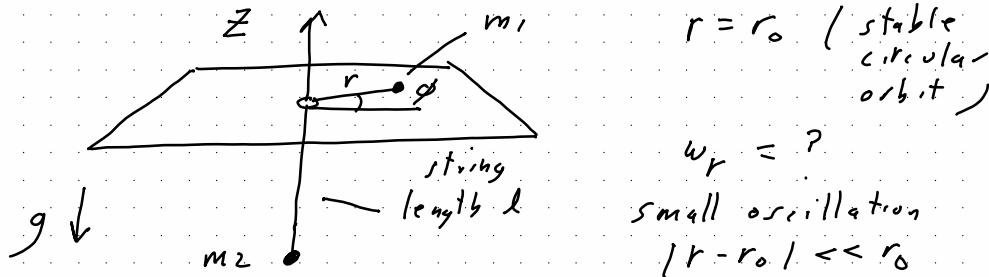
$$\frac{3}{m_1 + m_2} = \frac{1}{m_1}$$

$$\frac{3}{m_1 + m_2} = \frac{1}{3 m_1}$$

*closed  
bound  
Orbit*

## Lecture #20

- Quiz #4 - Tuesday
- Today: (i) Finish up example from last time  
(ii) Forced oscillations



$$\omega_r = \sqrt{\frac{3m_2 g}{(m_1 + m_2)r_0}}, \quad \omega_\phi = \sqrt{\frac{m_2 g}{m_1 r_0}} = \frac{\dot{\phi}}{r_0}$$

$$\omega_r = \omega_\phi \rightarrow \text{closed orbit} \quad \frac{3}{m_1 + m_2} = \frac{1}{m_1} \rightarrow \frac{3m_1}{m_1 + m_2} = m_1 \quad \boxed{m_1 = \frac{1}{2}m_2}$$

### Forced oscillations: (1-d)

$$m\ddot{x} = -kx + F(t)$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}, \quad \omega = \sqrt{\frac{k}{m}}$$

General sol'n

$$x(t) = \underbrace{x_h(t)}_{\text{general soln to } F(t)=0} + \underbrace{x_p(t)}_{\text{particular soln}}$$

$$a \cos(\omega t + \alpha)$$

$$\left. \begin{aligned} L &= \frac{1}{2} m \dot{x}^2 \\ &\quad - \frac{1}{2} k x^2 + x F(t) \end{aligned} \right\}$$

any solution to the equation with RHS =  $F(t)$

Example:  $F(t) = f \cos(\gamma t + \beta)$

$$x_p(t) = b \cos(\gamma t + \beta) = \frac{f}{m} \left( \frac{1}{\omega^2 - \gamma^2} \right)$$

$$-b\gamma^2 \cos(\gamma t + \beta) + \omega^2 b \cos(\gamma t + \beta) = \frac{f}{m} \cos(\gamma t + \beta)$$

$$b = \frac{f}{m} \left( \frac{1}{\omega^2 - \gamma^2} \right)$$

Resonance:  $\sigma = \omega$

$$x(t) = a \cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)} (\cos(\gamma t + \beta) - \cos(\omega t + \alpha))$$

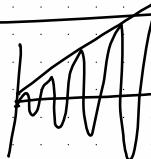
particular solution

$\frac{\partial}{\partial \gamma}$

L'Hopital's

$$\left. \frac{d}{d\gamma} (\text{num}) \right|_{\gamma=\omega} = \frac{tf t \sin(\omega t + \beta)}{+ 2m\omega}$$

$$x(t) = a \cos(\omega t + \alpha) + \frac{ft}{2m\omega} \sin(\omega t + \beta)$$



after some time  
this term no longer  
invalidates small  
osc. approx.

General  $F(t)$ :

$$F(t) = \int_{-\infty}^{\infty} d\gamma \tilde{F}(\gamma) e^{i\gamma t}$$

$\tilde{F}(\gamma)$  complex

Example

$$F(t) = f_2 \sin(2\omega t) + f_3 \sin(3\omega t) + f_4 \sin(4\omega t)$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m} \quad (2^{\text{nd}} \text{ order})$$

$$\begin{aligned} \xi &= \dot{x} + i\omega x \\ \dot{\xi} &= \ddot{x} + i\omega \dot{x} \end{aligned} \quad \rightarrow \boxed{\ddot{\xi} - i\omega \dot{\xi} = \frac{F(t)}{m}} \quad (1^{\text{st}} \text{ order})$$

$$y' + P(x)y = Q(x)$$

$$y' = \frac{dy}{dx}$$

Boas  
"Math method"

$$\dot{x} - i\omega x = \frac{F(t)}{m}$$

$$\boxed{dy + (P(x)y - Q(x))dx = 0} \quad \text{--- not exact!}$$

"Exact" differential:

$$\text{If exact } \textcircled{1} dy + (P(x)y - Q(x))dx = dU(x,y)$$

$$dU(x,y) = \left( \frac{\partial U}{\partial x} \right) dx + \left( \frac{\partial U}{\partial y} \right) dy$$

$$\rightarrow \frac{\partial^2 U}{\partial y \partial x} = \underline{\frac{\partial^2 U}{\partial x \partial y}} \quad \left| \begin{array}{l} \frac{\partial}{\partial x} = P(x) \\ 0 \neq P(x) \end{array} \right.$$

Claim:  $\boxed{= 0 \text{ (original equation)}}$

$$\mu(x) \left[ dy + (P(x)y - Q(x))dx \right] = dU$$

$$\underbrace{\mu}_{\substack{\text{integrating} \\ \text{Factor}}} \quad \text{RHS} = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

$$\mu(x) = \frac{\partial U}{\partial y}$$

$$\mu(x) (P(x)y - Q(x)) = \frac{\partial U}{\partial x}$$

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x} \rightarrow \mu'(x) = \mu(x) P(x)$$

$$\int \frac{\mu'}{\mu} = \int P(x) dx \quad \text{I}(x)$$

$$\ln \mu = \int P(x) dx \rightarrow \mu = e^{\int P(x) dx}$$

$\int P(x) dx$

$$\mu(x) = e^{I(x)}, \quad I(x) = \int dx P(x)$$

$$\frac{\partial U}{\partial y} = \mu(x)$$

$$U(x,y) = \mu(x)y + g(x) = C$$

$$\frac{\partial U}{\partial x} = \mu(x)(P(x)y - Q(x))$$

$$\cancel{\mu(x)y} + g'(x) = \mu(x) \cancel{(P(x)y - Q(x))}$$

$$g'(x) = -\mu(x)Q(x)$$

$$\rightarrow g(x) = - \int dx Q(x) \mu(x) = - \int dx Q(x) e^{I(x)}$$

$$y = \frac{1}{\mu(x)}(C - g(x)), \quad \mu(x) = e^{I(x)}$$

$$= e^{-I(x)} \left( C + \int dx Q(x) e^{I(x)} \right), \quad I(x) = \int dx P(x)$$

$$y' + P(x)y = Q(x)$$

$$x(t) = \text{Im}\left(\frac{\xi(t)}{\omega}\right)$$

$$\dot{\xi} - i\omega \xi = F(t)$$

$$y_H \rightarrow \xi(t)$$

$$\xi = \dot{x} + i\omega x$$

$$P(x) \rightarrow -i\omega$$

$$I(x) \rightarrow I(t) = -i\omega t$$

$$Q(x) \rightarrow F(t)/m$$

$$\xi(t) = e^{i\omega t} \left[ \xi_0 + \int_0^t d\bar{t} e^{-i\omega \bar{t}} F(\bar{t})/m \right]$$

Lecture #21: Tues 11/3

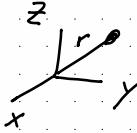
Q4: on Thursday

Thursday - rigid body motion / non-inertial  
reference frames

Sec 23: Free oscillations  
in 2-d or higher

Prob 3, Sec 23:  $U = \frac{1}{2} \nabla r^2$

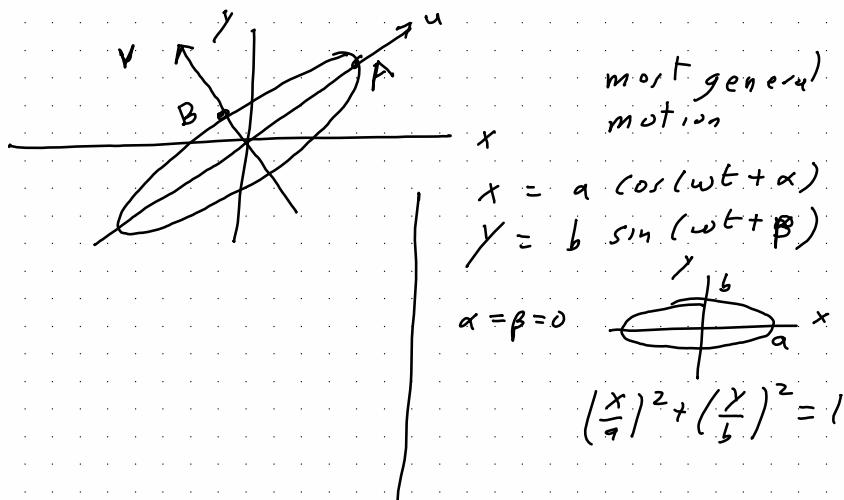
motion is in 2-d plane  
( $x, y$ )



spare oscillator

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} \nabla (x^2 + y^2) \\ &= \left( \frac{1}{2} m \dot{x}^2 - \frac{1}{2} \nabla x^2 \right) + \left( \frac{1}{2} m \dot{y}^2 - \frac{1}{2} \nabla y^2 \right) \end{aligned}$$

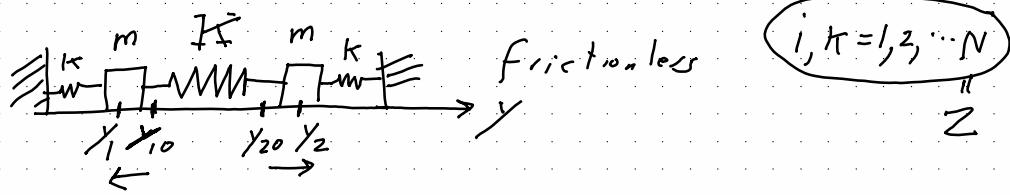
2 independent oscillations in the  $x, y$ -directions,  
Ang freq:  $\omega_x = \sqrt{\frac{k}{m}}, \omega_y = \sqrt{\frac{k}{m}} \rightarrow \omega_x = \omega_y$



Normal coordinates:

N independent oscillations with  
Frequency  $\omega_1, \omega_2, \dots, \omega_N$

Example:



$$x_1 = y_1 - y_{10} \quad \left. \begin{array}{l} \text{small deviations away from} \\ \text{equilibrium} \end{array} \right\}$$

$$L = T - U$$

$$\boxed{T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2} \quad \left. \begin{array}{l} (y_2 - y_1) - (y_{20} - y_{10}) \\ = (y_2 - y_{20}) - (y_1 - y_{10}) \\ = x_2 - x_1 \end{array} \right\}$$

$$\boxed{U = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} K (x_2 - x_1)^2} \quad \left. \begin{array}{l} \frac{1}{2} \sum_{i=1}^2 m_{ii} \dot{x}_i \dot{x}_i \\ = \frac{1}{2} (k + K) x_1^2 + \frac{1}{2} (k + K) x_2^2 \\ - K x_1 x_2 \end{array} \right\} \quad \begin{array}{l} (x_1^2 + x_2^2 - 2 x_1 x_2) \\ = \frac{1}{2} \sum_{i=1}^2 k_{ii} x_i x_i \end{array}$$

$$\begin{aligned} T &= \frac{1}{2} \sum_{i=1}^2 m_{ii} \dot{x}_i \dot{x}_i = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 \\ &= \frac{1}{2} [m_{11} \dot{x}_1^2 + m_{12} \dot{x}_1 \dot{x}_2 + m_{21} \dot{x}_2 \dot{x}_1 + m_{22} \dot{x}_2^2] \\ &= \frac{1}{2} [m_{11} \dot{x}_1^2 + m_{22} \dot{x}_2^2 + 2m_{12} \dot{x}_1 \dot{x}_2] \end{aligned}$$

$$\rightarrow m_{11} = m, \quad m_{22} = m, \quad m_{12} = \theta = m_{21}$$

$$\boxed{m_{ii} = \begin{array}{|c|c|} \hline m & 0 \\ \hline 0 & m \\ \hline \end{array}}$$

$$U = \frac{1}{2} \sum_{i=1}^2 k_{ii} x_i x_i = \frac{1}{2} (k + K) x_1^2 + \frac{1}{2} (k + K) x_2^2 - 2 K x_1 x_2$$

$$\boxed{k_{ii} = \begin{array}{|c|c|} \hline k + K & -K \\ \hline -K & k + K \\ \hline \end{array}}$$

$$L = T - U$$

$$= \frac{1}{2} \sum_{j,k} m_{jk} \ddot{x}_j \dot{x}_k - \frac{1}{2} \sum_{j,k} K_{jk} x_j \ddot{x}_k \quad \left| \begin{array}{l} \delta_{jk} \\ \delta_{kj} \end{array} \right.$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) = \frac{\partial L}{\partial x_i} \quad (i=1, 2, \dots, N)$$

$$\boxed{\sum_k m_{ik} \ddot{x}_k = - \sum_k K_{ik} x_k}$$

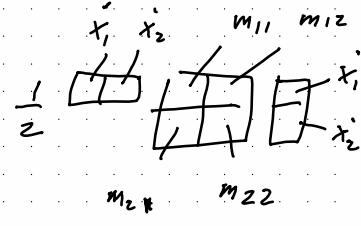
wave

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\uparrow \quad \uparrow$   
 $m \quad x$

$$\frac{\partial}{\partial x_1} (K_{11} x_1^2 + K_{22} x_2^2 + 2K_{12} x_1 x_2) \left( \frac{1}{2} \right)$$

$$= - (K_{11} x_1 + K_{22} x_2) = - \sum_k K_{ik} x_k$$



$$\frac{1}{2} \sum_{j,k} m_{jk} \ddot{x}_j \dot{x}_k$$

$$\frac{1}{2} \sum_{j,k} m_{jk} \ddot{x}_j \dot{x}_k$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_1} \right) = \frac{\partial L}{\partial x_1}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_2} \right) = \frac{\partial L}{\partial x_2}$$

}

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) = \frac{\partial L}{\partial x_i} \quad i=1, 2$$

$$\sum_k m_{ik} \ddot{x}_k = - \sum_k K_{ik} x_k$$

$$\sum_k (m_{ik} \ddot{x}_k + K_{ik} x_k) = 0$$

Trial solution:

$$x_k = A_k e^{i\omega t}$$

$$\ddot{x}_k = -\omega^2 A_k e^{i\omega t}$$

0 vector

$$\sum_k (-m_{ik} \omega^2 + K_{ik}) A_k e^{i\omega t} = 0$$

matrix

$$\boxed{\boxed{0}} = \boxed{0}$$

$$M^{-1}(M \underline{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix})$$

$$\underline{v} = M^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$K_{tt} - \omega^2 m_{tt}$  must be a singular matrix

$$\det(K_{tt} - \omega^2 m_{tt}) = 0 \quad \text{characteristic equation for angular freq}$$

Eigenvalues  $\rightarrow \omega_1, \omega_2, \dots, \omega_N$   
(normal mode freqs)

Eigenvector  $\rightarrow \underline{v}_1, \underline{v}_2, \dots, \underline{v}_N$   
(normal mode oscillation)

$$\det \left( \begin{array}{cc} K + EI & -EI \\ -EI & K + EI \end{array} - \omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \right) = 0$$

$$\det \begin{pmatrix} (K + EI) - \omega^2 m & -EI \\ -EI & (K + EI) - \omega^2 m \end{pmatrix} = 0$$

$$\omega^4 + \omega^2 + 1 = 0$$

$\rightarrow$  quadratic equation for  $\omega^2 (= \lambda)$

$$\lambda^2 + \lambda + 1 = 0$$

solve quadratic equation:  $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\boxed{\begin{aligned} \omega_+^2 &= \frac{K + 2EI}{m} \\ \omega_-^2 &= \frac{K}{m} \end{aligned}}$$

$$\omega_+^2:$$

$$\begin{array}{|c|c|} \hline k + I - m\omega_+^2 & -II \\ \hline -II & k + I - m\omega_+^2 \\ \hline \end{array}$$

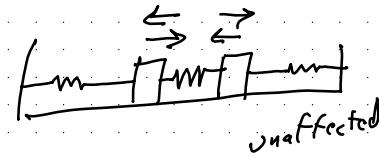
w

$$\begin{array}{|c|c|} \hline A_1 & 0 \\ \hline A_2 & 0 \\ \hline \end{array}$$

$$A_2 = -A_1$$

$V_+$  : eigenvector  
assoc. with  $\omega_+$

Normal mode oscillations



$$\omega_-^2:$$

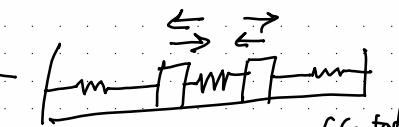
$$V_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

General solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \text{Re} \left( C_+ V_+ e^{i\omega_+ t} + C_- V_- e^{-i\omega_- t} \right)$$

complex constant determined by initial

conditions



Lec 22: Nov 5<sup>th</sup>

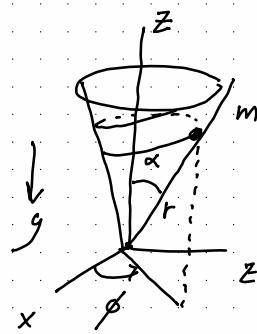
- Quiz #4 today (Quiz #5 - next Thursday  
Quiz #6 - last day of class)

- Rigid body motion: Sec 31-36

Non-inertial ref frames: Sec 38, 39

Q4:

name - q4.pdf



a) Lagrangian write down, t.e.

b) Find  $r_0$  for a stable circular orbit for fixed angular momentum  $M_z$ .

c) Find freq of small radial oscillations about  $r_0$ :

$$z = r \cos \alpha, \quad T = \frac{1}{2} m (r^2 + r^2 \sin^2 \alpha \dot{\phi}^2 + \cancel{r^2 \dot{\theta}^2})$$

$$E = \frac{1}{2} mr^2 + U_{\text{eff}}(r) \quad M_z^2 = m^2 r_0^3 \cos^2 \alpha$$

$$\frac{M_z^2}{2mr^2 \sin^2 \alpha} + mgr \cos \alpha$$

$$U_{\text{eff}} = M_z^2 / (2mr^2 \sin^2 \alpha)$$

~~Ans~~

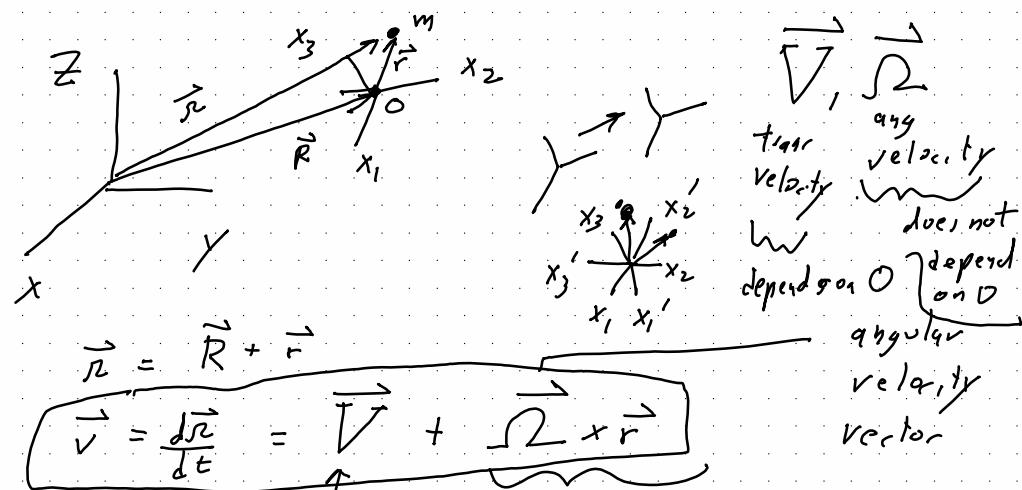
$$T = \frac{d^2 U_{\text{eff}}}{dr^2} \Big|_{r_0} = \frac{3mg \cos \alpha}{r_0}$$

$$\omega_r = \sqrt{\frac{T}{m}} = \sqrt{\frac{3g \cos \alpha}{r_0}}$$

$$\omega_\phi = \sqrt{\frac{g \cos \alpha}{r_0 \sin^2 \alpha}} \neq \omega_r$$

$$x = g - z_0$$

$$\begin{aligned} \dot{\phi} &= \frac{M_z}{mr^2 \sin^2 \alpha} \\ &= \frac{M_z}{mr_0^2 \sin^2 \alpha} \\ &= \frac{\sqrt{m^2 g r_0^3 \cos^2 \sin^2 \alpha}}{\sqrt{m^2 r_0^4 \sin^4 \alpha}} \end{aligned}$$

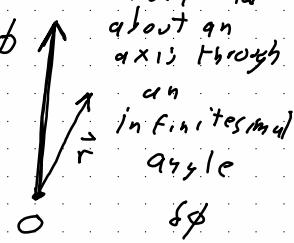
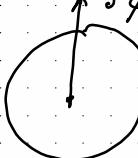


translational  
velocity of  
O

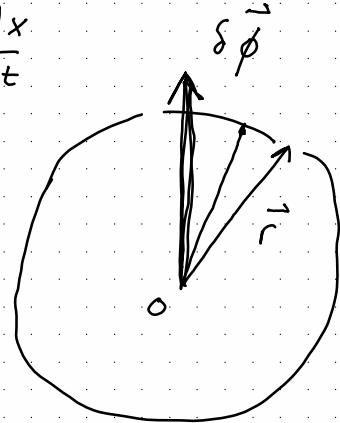
rotational  
velocity

infinitesimal  
rotational  
about an  
axis through  
an  
infinitesimal  
angle

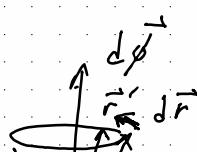
$$\vec{R} \rightarrow \vec{R} + d\vec{R}, \quad \frac{d\vec{R}}{dt} = \vec{V} + \vec{\omega} \times \vec{r}$$



$$\vec{v} = \frac{d\vec{x}}{dt}$$



$$\frac{d\vec{r}}{dt}$$



$$\boxed{\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\omega} \times \vec{r}}$$

$$\vec{V} = \frac{d\vec{R}}{dt} \quad \text{3 DOF associated with origin}$$

$\vec{dR}$  of body frame

$$O (\phi, \theta, \psi)$$

$$\vec{R} \quad \vec{\phi}: \text{3 DOF assoc w/ orientation}$$

$$\frac{d\vec{r}}{dt} = \vec{\phi} \times \vec{r}$$

$$\frac{d\vec{r}}{dt} = \left[ \begin{array}{c} \vec{\phi} \\ \frac{d\vec{r}}{dt} \end{array} \right]_{\text{body frame wrt inertial frame}}$$

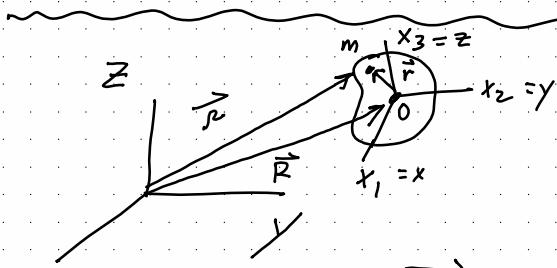
Lec #23: Tuesday Nov 10<sup>th</sup>

- Quiz #5: Thursday

- Midterm #2: Next Thursday 11/19 (scattering, small oscillations, some rigid body)

- Today: Rigid body motion  
(Sec 31-36, 38, 39)

non-inertial  
static equilibrium



$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$= \vec{V} + \vec{\omega} \times \vec{r}$$

$\vec{\omega}$ : angular velocity vector

O: at COM (usually)

$x_1, x_2, x_3$ : fixed in RB

$$\left. \begin{aligned} \vec{V} &= \frac{d\vec{R}}{dt} \\ \vec{\omega} &= \frac{d\vec{\phi}}{dt} \end{aligned} \right\} \text{2i}$$

$$(\vec{R}, \vec{\phi}) : 6 \text{ DOF} = 2i$$

$$T = \frac{1}{2} m v^2$$

$$\vec{p} = m \vec{v}$$

rotational quantities

$$T_{\text{rot}} = \frac{1}{2} I \omega^2$$

$$\vec{M} = I \vec{\omega} \rightarrow M_i = \sum I_{ij} \omega_j$$

I: moment of inertia

→  $I_{ij}$ : inertia tensor

$$\begin{matrix} & 1 & 2 & 3 \\ 1 & \square & & \\ 2 & & \square & \\ 3 & & & \square \\ \downarrow & & & \\ & & & j \end{matrix}$$

$$\begin{matrix} M_i & = & \begin{matrix} \square & & \square \\ & \square & \\ & & \square \end{matrix} & \begin{matrix} \square \\ I_{ij} \\ \omega_j \end{matrix} \end{matrix}$$

KE:

$$T = \frac{1}{2} \sum_a m_a |\vec{v}_a|^2$$

$$= \frac{1}{2} \sum_a m_a \left( |\vec{V} + \vec{\omega} \times \vec{r}_a|^2 \right)$$

$$= \frac{1}{2} \sum_a m_a \left( |\vec{V}|^2 + |\vec{\omega} \times \vec{r}_a|^2 + 2 \vec{V} \cdot (\vec{\omega} \times \vec{r}_a) \right)$$

$$\vec{v}_a = \vec{V} + \vec{\omega} \times \vec{r}_a$$

$$\left| |\vec{A} + \vec{B}|^2 \right| = A^2 + B^2 + 2 \vec{A} \cdot \vec{B}$$

$$\textcircled{1} = \frac{1}{2} \sum_a m_a |\vec{V}|^2 = \boxed{\frac{1}{2} \mu V^2}$$

total mass

$$\textcircled{3} = \sum_a m_a \vec{V} \cdot (\vec{\omega} \times \vec{r}_a)$$

$$\vec{A} \cdot (\vec{B} \times \vec{C})$$

$$= \left( \sum_a m_a \vec{r}_a \right) \cdot (\vec{V} \times \vec{\omega})$$

$$= \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$= \mu \vec{R}_{\text{com}} \cdot (\vec{V} \times \vec{\omega})$$

$$= \vec{C} \cdot (\vec{A} \times \vec{B})$$

= 0 for

$\vec{O}$  at com

$$\begin{aligned}
 \textcircled{2} &= \frac{1}{2} \sum_a m_a (\vec{\omega} \times \vec{r}_a)^2 \\
 &= \frac{1}{2} \sum_a m_a (\vec{\omega} \times \vec{r}_a) \cdot (\vec{\omega} \times \vec{r}_a) \\
 &= \frac{1}{2} \sum_a m_a \vec{\omega} \cdot (\vec{r}_a \times (\vec{\omega} \times \vec{r}_a)) \\
 &= \frac{1}{2} \sum_a m_a \vec{\omega} \cdot (\vec{\omega} r_a^2 - \vec{r}_a (\vec{r}_a \cdot \vec{\omega})) \\
 &= \frac{1}{2} \sum_a m_a (\vec{\omega}^2 r_a^2 - (\vec{\omega} \cdot \vec{r}_a)^2) \\
 &= \frac{1}{2} \sum_a m_a \left( \sum_{i,j} \Omega_i \Omega_j \delta_{ij} r_a^2 - \sum_{i,j} \Omega_i r_{ai} \Omega_j r_{aj} \right) \\
 &= \frac{1}{2} \sum_{i,j} \left( \sum_a m_a (r_a^2 \delta_{ij} - r_{ai} r_{aj}) \right) \Omega_i \Omega_j \\
 &= \boxed{\frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j} \quad (= \frac{1}{2} \boxed{\text{trans}} \boxed{\text{rotational}} \boxed{\text{kinetic energy}})
 \end{aligned}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\boxed{T = \underbrace{\frac{1}{2} \mu V^2}_{\text{trans}} + \underbrace{\frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j}_{\text{rotational}}} + \textcircled{2} \quad \left( \frac{1}{2} I \vec{\omega}^2 \right) \quad \text{Frischmeyer}$$

for COM at origin  
of RB Frame

$\vec{M}$ : wrt COM of body

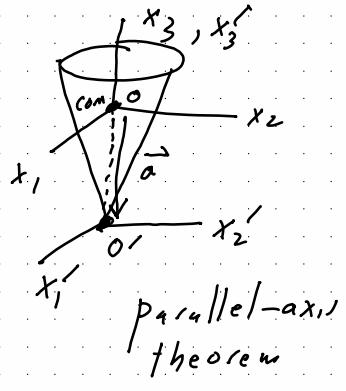


$$\begin{aligned}
 \vec{M} &= \sum_a \vec{r}_a \times \vec{p}_a \\
 &= \sum_a m_a \vec{r}_a \times \vec{v}_a \\
 &= \sum_a m_a \vec{r}_a \times (\vec{V} + \vec{\omega} \times \vec{r}_a) \\
 &= \sum_a m_a \vec{r}_a \times (\vec{\omega} \times \vec{r}_a)
 \end{aligned}$$

$$= \sum_a m_a (\vec{\omega} r_a^2 - \vec{r}_a (\vec{r}_a \cdot \vec{\omega}))$$

$$M_i = \sum_j I_{ij} \Omega_j$$

$$\vec{M} = I \vec{\omega} \quad (\text{Frisch. phys.})$$



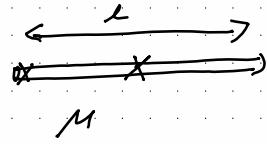
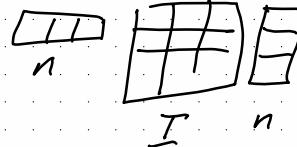
$$I_{ij}' \text{ wrt } O' \quad I_{ij} \text{ wrt } O \text{ (com)}$$

$$I_{ij}' = I_{ij} + m(a^2 \delta_{ij} - a_i a_j)$$

$\vec{a}$  : Vector from  $O$  to  $O'$

$$I(\hat{n}) = \sum_{i,j} I_{ij} n_i n_j$$

~~moment of inertia~~  
 $\hat{n}$ : axis of rotation



$$\boxed{I_{\text{com}} = \frac{1}{2} M l^2}$$

$$I_{\text{end}} = \frac{1}{3} M l^2$$

$$I_{ij} = \sum_a m_a (r_a^2 \delta_{ij} - r_{ai} r_{aj})$$

~~3x3 real symmetric~~

$$= \int_V \rho dV (r^2 \delta_{ij} - r_i r_j)$$

can always be diagonalized

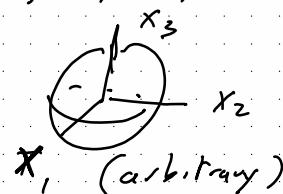
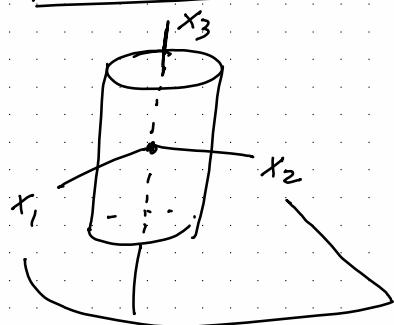
$$\begin{matrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{matrix}$$

$$I_{ij} = I_{ij} \delta_{ij}$$

$$T_{\text{rot}} = \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j$$

Principal axes:  $(x_1, x_2, x_3)$

$$= \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$$

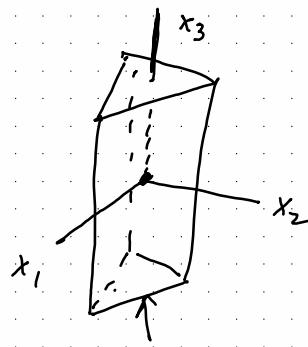


arb. trary

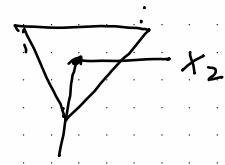
x<sub>1</sub> (arbitrary)

$$M_i = \sum_j I_{ij} \Omega_j$$

$$M_1 = I_1 \Omega_1, M_2 = I_2 \Omega_2, M_3 = I_3 \Omega_3$$



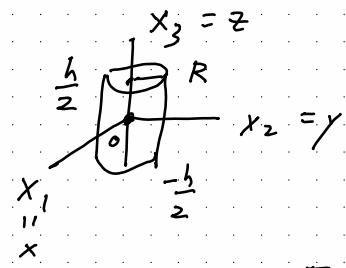
equilateral  
triangle



$I_1 = I_2 \neq I_3$



symmetrical  
top

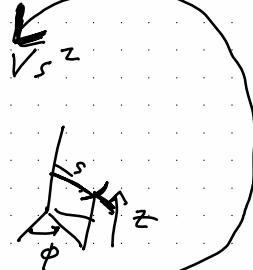


total mass  $\mu$

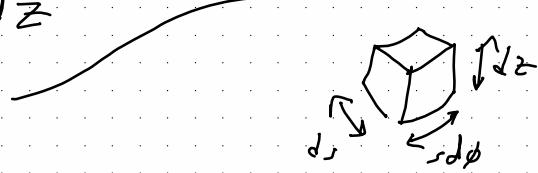
$$\rho = \frac{\mu}{\text{volume}} = \frac{\mu}{\pi R^2 h}$$

$$\begin{aligned} I_3 &= I_{33} = \int \rho dV \left( r^2 s_{33} - \frac{r_3^2 r_3}{z^2} \right) \\ &\doteq \int \rho dV (r^2 - z^2) \\ &= \int \rho dV (x^2 + y^2) = \int \rho dV s^2 \end{aligned}$$

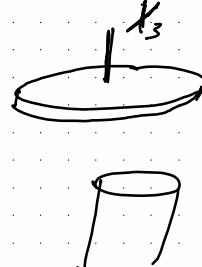
cylindrical:  $s, \phi, z$        $s^2 = x^2 + y^2$



$$\begin{aligned} dV &= ds s d\phi dz \\ &= s ds d\phi dz \end{aligned}$$



$$\begin{aligned}
 I_3 &= \int \rho dV s^2 \\
 &= \frac{M}{\pi R^2 h} \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz \int s^3 ds \\
 &\quad \text{with } s = \sqrt{r^2 + z^2} \\
 &= \frac{M}{\pi R^2 h} \cdot 2\pi h \frac{R^4}{4} \\
 &= \boxed{\frac{1}{2} M R^2}
 \end{aligned}$$



$$\begin{aligned}
 I_1 &= I_2 \equiv I \\
 I_1 &= \int \rho dV (r^2 - x^2) \\
 + I_2 &= \int \rho dV (r^2 - y^2)
 \end{aligned}$$

$$\begin{aligned}
 2I &= \int \rho dV (2r^2 - x^2 - y^2) \\
 &\quad \left( r^2 = s^2 + z^2 \right) \\
 &\quad x^2 + y^2 = s^2
 \end{aligned}$$

$$\begin{aligned}
 2I &= \int \rho dV (s^2 + 2z^2) \\
 \boxed{I} &= \frac{1}{2} \underbrace{\int \rho dV s^2}_{I_3} + \int \rho dV z^2 \\
 &= \frac{1}{2} I_3 + \int \rho dV z^2
 \end{aligned}$$

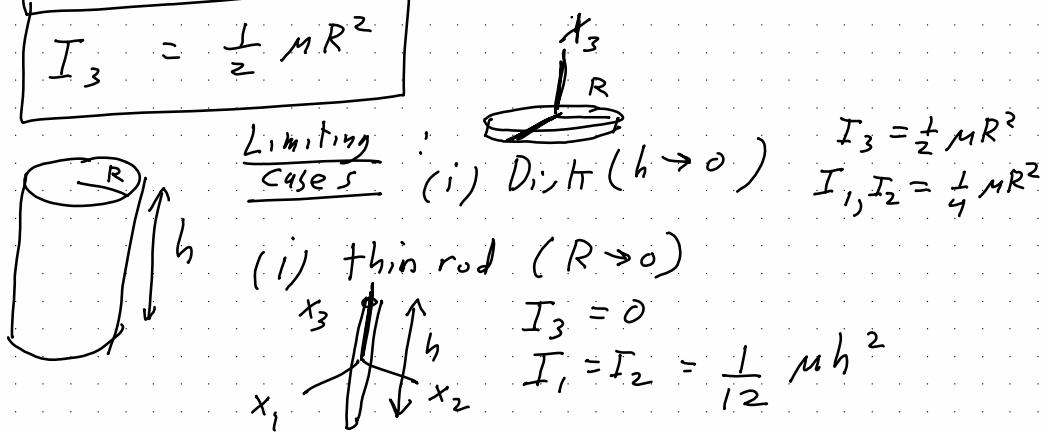
evalute

$$\int \rho dV z^2 = \frac{M}{\pi R^2 h} \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz \cdot z^2 \int_0^R s ds$$

$= \frac{z^4}{3} \Big|_{-h/2}^{h/2}$   
 $= \frac{h^4}{12}$   
 $= \frac{M}{\pi R^2 h} \cdot \frac{1}{2} \pi R^2 \cdot \frac{h^3}{12}$   
 $= \boxed{\frac{M h^2}{12}}$

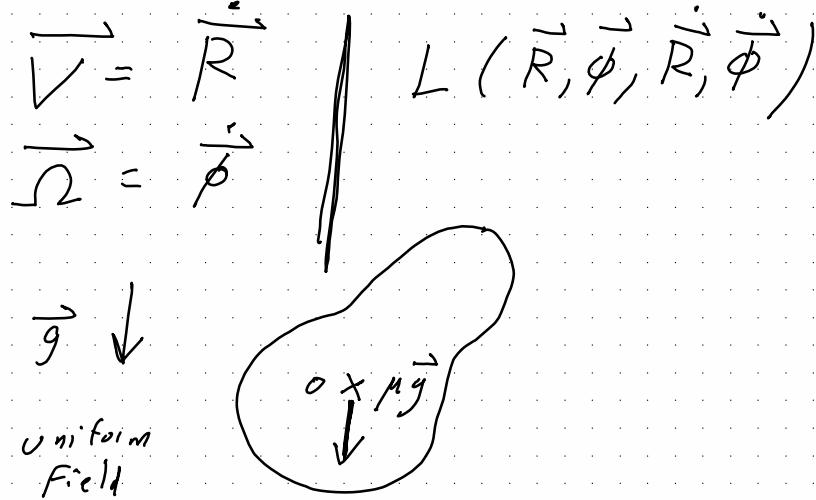
$$\begin{aligned}
 I &= \frac{1}{2} \left( \frac{1}{2} M R^2 \right) + \frac{M h^2}{12} \\
 &= \frac{1}{4} M R^2 + \frac{1}{12} M h^2 \\
 &= \boxed{\frac{1}{4} M \left( R^2 + \frac{1}{3} h^2 \right)} = I_1, I_2
 \end{aligned}$$

$$I_3 = \frac{1}{2} M R^2$$



$$L = T - U$$

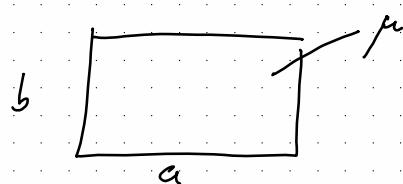
$$= \frac{1}{2} m \vec{V}^2 + \frac{1}{2} \leq I_{ij} \Omega_j \Omega_j - U$$



Lecture #24 : Thursday 11/12

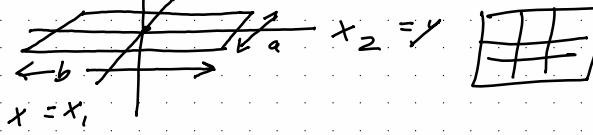
- Quiz #5 (+today)
- Midterm #2 (next Thursday) (scattering, small oscillations, RB motion)
- Today's topics :
  - (1) RB EOMs
  - (2) Euler's equations
  - (3) Euler angles

Q5: Calculate the principal moments of inertia for a 2-d rectangle with side lengths  $a, b$ . uniform



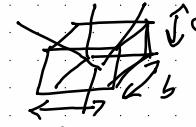
name-q5.pdf

$$z = x_3 \quad I_{ij} = \int \rho dV (r^2 s_{ij} - r_i r_j)$$



$$r^2 = x_1^2 + x_2^2 + z^2$$

$$\begin{aligned} I_{11} &= \int \rho dV (r^2 - x_1^2) \\ I_{11}'' &= \int \rho dV y^2 \\ I_{11} &= \frac{M}{ab} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy y^2 \\ &= \frac{M}{ab} \alpha \frac{y^3}{3} \Big|_{-b/2}^{b/2} \\ &= \frac{M}{b} \frac{2}{3} \left(\frac{b}{2}\right)^3 \\ &= \frac{M}{b} \frac{2}{3} \frac{b^3}{8} \\ &= \frac{1}{12} M b^2 \end{aligned}$$



$$\begin{aligned} dm &= \rho dV \\ &= \sigma dx dy \\ &= \frac{M}{ab} dx dy \end{aligned}$$

$$I_2 = \frac{1}{12} M a^2$$

$$I_3 = \int \sigma dx dy (x^2 + y^2)$$

$$= I_1 + I_2$$

$$I_3 = \frac{1}{12} M (a^2 + b^2)$$

EOMs: Final result:

$\vec{r} = \vec{R} + \vec{r}$ ,  $\delta \vec{r} = \delta \vec{R} + \delta \phi \times \vec{r}$

$L = T - U$

$\delta L = \mu \vec{V} \cdot \delta \vec{V} + \sum_{ij} I_{ij} \delta r_i \delta r_j - \sum_i \frac{\partial U}{\partial \vec{r}} \cdot \delta \vec{r}$

$\frac{d}{dt} \vec{P} = \vec{F} = \sum f$

$\frac{d}{dt} \vec{M} = \vec{K} = \sum \vec{r} \times \vec{f}$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \vec{V}} \right) = \frac{\partial L}{\partial \vec{r}}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \vec{r}} \right) = \frac{\partial L}{\partial \vec{p}}$$

$$\begin{aligned}
 & \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j + \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \delta\Omega_j \\
 &= \quad \text{II} \quad + \frac{1}{2} \sum_{j,i} I_{ji} \Omega_j \delta\Omega_i \\
 &= \quad \text{II} \quad + \frac{1}{2} \sum_{j,i} I_{ji} \Omega_j \delta\Omega_i \\
 &= \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \delta\Omega_j + \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \delta\Omega_i \\
 &= \boxed{\sum_{i,j} I_{ij} \Omega_i \delta\Omega_i}
 \end{aligned}$$

$$\begin{aligned}
 \delta L &= \underbrace{\left( M \nabla \cdot \vec{\delta V} \right)}_{\leq M_i \delta r_i} + \sum_{i,j} I_{ij} \delta r_i \delta r_j + \frac{\vec{A} \cdot (\vec{B} \times \vec{C})}{B \cdot (\vec{C} \times \vec{A})} \\
 &\quad - \left( \sum \frac{\partial U}{\partial r_i} \right) \cdot \delta \vec{r} - \sum \frac{\partial U}{\partial \vec{r}} \cdot (\vec{\delta p} \times \vec{r}) \\
 &= - \sum \delta \vec{p} \cdot \left( \vec{r} \times \frac{\partial U}{\partial \vec{r}} \right) \\
 &= - \sum \vec{\delta p} \cdot \left( \sum \vec{r} \times \frac{\partial U}{\partial \vec{r}} \right) \\
 dF &= \bigcirc dx + \bigcirc dy
 \end{aligned}$$

$$\delta L = \overrightarrow{P} \cdot \delta \overrightarrow{V} + \overrightarrow{M} \cdot \delta \overrightarrow{\omega} + (\sum \vec{F}) \cdot \delta \vec{R} + \delta \vec{\phi} \cdot (\sum \vec{r} \times \vec{F})$$

$$\vec{P} = \frac{\partial L}{\partial \dot{\vec{r}}}, \quad \vec{M} = \frac{\partial L}{\partial \dot{\vec{\phi}}}, \quad \sum \vec{f} = \frac{\partial L}{\partial \vec{R}}, \quad \sum \vec{r} \times \vec{f} = \frac{\partial L}{\partial \vec{\phi}}$$

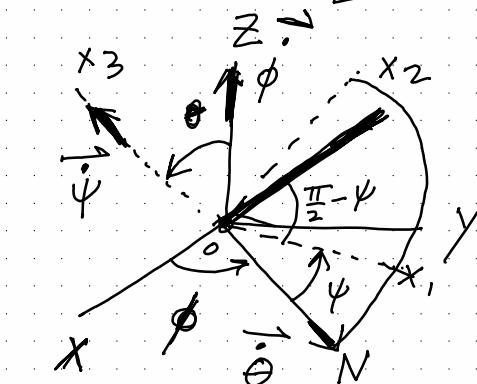
$$\frac{d}{dt} \left( \frac{\partial L}{\partial \vec{F}} \right) = \frac{\partial L}{\partial \vec{P}} \rightarrow \boxed{\frac{d \vec{P}}{d t} = \sum \vec{F} = \vec{F}} \quad \left| \frac{d}{dt} \left( \frac{\partial L}{\partial \vec{\Omega}} \right) = \frac{\partial L}{\partial \vec{M}} \rightarrow \frac{d \vec{M}}{d t} = \sum \vec{r} \times \vec{F} = \vec{F} \right.$$

## Euler's equations / Euler angles

$$\vec{R}, \vec{\phi}$$

$\Omega_i$  :  $i$ th component of  $\vec{\Omega}$

L wrt  $\hat{x}_i$



$$\vec{\Omega} = \dot{\phi} \hat{x}_3 + \dot{\theta} \hat{x}_1 + \dot{\psi} \hat{x}_2$$

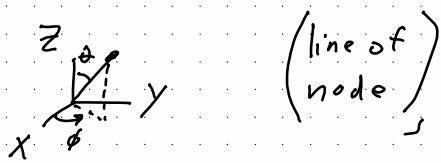
$$\dot{\psi} = \dot{\psi} \hat{x}_3$$

$$\dot{\theta} = \dot{\theta} \cos \psi \hat{x}_1 - \dot{\theta} \sin \psi \hat{x}_2$$

$$\dot{\phi} = \dot{\phi} \cos \theta \hat{x}_3$$

$$+ \dot{\phi} \sin \theta \left( \underline{\sin \psi \hat{x}_1} + \underline{\cos \psi \hat{x}_2} \right)$$

$$\cos(\frac{\pi}{2} - \psi) \quad \sin(\frac{\pi}{2} - \psi)$$



### Announcements

- Midterm II is this Thursday

- Today:

i) Euler angles

ii) Euler's equation for RB motion

iii) Free rotation with  $\vec{\Omega} = \text{const}$

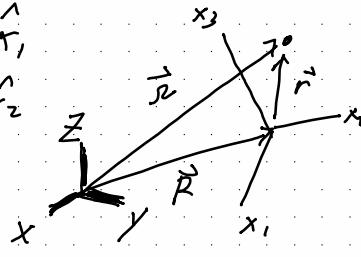
iv) II of a symmetric top ( $I_1 = I_2$ )

v) Heavy symmetrical top with one point fixed [prob 35.1]



$$\vec{\Omega} = (\dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi) \hat{x}_1 \\ + (\dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi) \hat{x}_2 \\ + (\dot{\psi} + \dot{\phi} \cos \theta) \hat{x}_3$$

$$\begin{aligned}\Omega_1 &= \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \\ \Omega_2 &= \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \\ \Omega_3 &= \dot{\psi} + \dot{\phi} \cos \theta\end{aligned}$$



$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}$$

~~+  $\frac{d' \vec{r}}{dt}$~~  wrt RB

Euler's equations: (wrt RB axes)  $\vec{A}$ : any vector

$$\frac{d\vec{P}}{dt} = \sum \vec{F} = \vec{F}$$

$$\frac{d\vec{M}}{dt} = \sum \vec{r}_x \vec{F} = \vec{K}$$

$$\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\Omega} \times \vec{A}$$

wrt inertial frame      wrt rotating frame      angular velocity of the rotating frame

$$\left( \frac{d'\vec{A}}{dt} \right)_i = \frac{dA_i}{dt} = \dot{A}_i$$

wrt cartesian components  
wrt rotating frame

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} \\ = \ddot{x} \hat{x} + \ddot{y} \hat{y} \\ + \ddot{z} \hat{z}$$

~~=  $\ddot{x} \hat{x} + \ddot{y} \hat{y} + \ddot{z} \hat{z}$~~

$$\vec{A} = \sum_i A_i \hat{x}_i$$

$$\frac{d\vec{A}}{dt} = \sum_i \left( \frac{dA_i}{dt} \right) \hat{x}_i + \sum_i A_i \frac{dx_i}{dt}$$

$\frac{d'\vec{A}}{dt}$        $\vec{\Omega} \times \vec{A}$

$$\frac{d\vec{A}}{dt} = \frac{d' \vec{A}}{dt} + \vec{\omega} \times \vec{A} \quad , \quad \left( \frac{d' \vec{A}}{dt} \right)_i = \dot{A}_i$$

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d' \vec{P}}{dt} + \vec{\omega} \times \vec{P}$$

$$\rightarrow \boxed{\vec{F}_1 = \dot{\vec{P}}_1 + (\vec{\omega} \times \vec{P})_1, \quad \vec{P} = \mu \vec{V}}$$

$$= \dot{\vec{P}}_1 + \omega_2 P_3 - \omega_3 P_2$$

$$= \mu (V_1 + \omega_2 V_3 - \omega_3 V_2)$$

similar equations  
for  $F_2, F_3$

$$\vec{K} = \frac{d\vec{M}}{dt} = \frac{d' \vec{M}}{dt} + \vec{\omega} \times \vec{M}$$

$$\boxed{\vec{K}_1 = \dot{\vec{M}}_1 + \omega_2 M_3 - \omega_3 M_2, \quad M_i = I_i \omega_i}$$

$$= I_1 \dot{\omega}_1 + \omega_2 I_3 \omega_3 - \omega_3 I_2 \omega_2$$

$$= I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) \quad K_x = \dot{M}_x$$

\*(similar  
equations)  
For  $K_2, K_3$ )

Free rotation:  $\vec{H}_i = 0, \vec{F}_i = 0$

$$\vec{O} = I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2), \quad \cancel{\text{---}}$$

$$\vec{O} = I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3)$$

$$\vec{O} = I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1)$$

Free rotation with  $\vec{\omega} = \text{const}$ :

$$\vec{O} = \omega_2 \omega_3 (I_3 - I_2)$$

$$\vec{O} = \omega_3 \omega_1 (I_1 - I_3)$$

$$\vec{O} = \omega_1 \omega_2 (I_2 - I_1)$$

$$\boxed{\omega_1 = \text{const}, \quad I_1 < \textcircled{I}_2 < I_3}$$

$$\omega_2 = 0 \quad \leftarrow \text{stable}$$

$$\omega_3 = 0 \quad \downarrow \text{unstable}$$

$$\omega_2 = \text{const}, \omega_1 = 0, \omega_3 = 0$$

$$\frac{d\vec{\omega}}{dt} = \vec{0}$$

$$\frac{d' \vec{\omega}}{dt} = \vec{0}$$

$$\frac{d\vec{\omega}}{dt} = \frac{d' \vec{\omega}}{dt} + \vec{\omega} \times \vec{\omega}$$

$$\frac{d\vec{M}}{dt} \neq \frac{d' \vec{M}}{dt}$$



$$\omega_3 = \text{const}, \omega_1 = 0, \omega_2 = 0$$

$\Omega_1 = \text{const}$ ,  $\Omega_2 = 0$ ,  $\Omega_3 = 0$  : exact

$$\boxed{\begin{aligned}\Omega_1 &= \text{const} + \epsilon_1 \\ \Omega_2 &= \epsilon_2 \\ \Omega_3 &= \epsilon_3\end{aligned}}$$

$\epsilon_{1,2,3}$  : small time dependent perturbations

Keep 0<sup>th</sup> and 1<sup>st</sup> order terms.

Ignore 2<sup>nd</sup> order, e.g.  $\epsilon_1 \epsilon_3$

$$O = I_1 \frac{d}{dt} (\underline{\text{ign} + \epsilon_1}) + \underbrace{\epsilon_2 \epsilon_3 (I_3 - I_2)}_{\text{2nd order} \rightarrow \text{ignore}}$$

$$\approx I \dot{\epsilon}_1$$

$$\rightarrow \epsilon_1 = \text{const} \rightarrow \boxed{\Omega_1 = \text{const}}$$

$$O = I_2 \dot{\epsilon}_2 + \epsilon_3 \Omega_1 (I_1 - I_3) \quad \} \quad \text{coupled}$$

$$O = I_3 \dot{\epsilon}_3 + \Omega_1 \epsilon_2 (I_2 - I_1) \quad \} \quad \text{1st order diff. equations}$$

Differentiate ..

$$\begin{aligned}O &= I_2 \ddot{\epsilon}_2 + \dot{\epsilon}_3 \Omega_1 (I_1 - I_3) \\ &= I_2 \ddot{\epsilon}_2 - \frac{\Omega_1 \epsilon_2 (I_2 - I_1) \Omega_1 (I_1 - I_3)}{I_3}\end{aligned}$$

$$I_1 < I_2 < I_3$$

$$\begin{aligned}O &= I_2 \ddot{\epsilon}_2 + \dot{\epsilon}_3 \Omega_1 (I_1 - I_3) \\ &= I_2 \ddot{\epsilon}_2 - \frac{\Omega_1 \epsilon_2 (I_2 - I_1) \Omega_1 (I_1 - I_3)}{I_3}\end{aligned}$$

$$= \ddot{\epsilon}_2 + \frac{\Omega_1^2 (I_2 - I_1) (I_3 - I_1)}{I_2 I_3} \epsilon_2$$

$$\boxed{\begin{aligned}(I_3 - I_2) \\ (I_1 - I_2)\end{aligned}}$$

$$= \ddot{\epsilon}_2 + \omega^2 \epsilon_2$$

$$\ddot{\epsilon}_2 = -\omega^2 \epsilon_2 \rightarrow \text{SITM} \quad \epsilon_2 = A \cos \omega t + B \sin \omega t$$

Similarly,

aaaa

$$= C \cos(\omega t + \phi)$$

$$\ddot{\epsilon}_3 = -\omega^2 \epsilon_3 \rightarrow \epsilon_3 = D \cos(\omega t + \rho)$$

( $\epsilon_2, \epsilon_3$  are bound by their initial deviations away from 0)

For  $\Omega_2 = \text{const}$ , the solution

perturbations,

$$\epsilon_1 = +w^2 \epsilon_1$$

$$\epsilon_3'' = + w^2 \epsilon_3$$

Lecture #27      Tues 11/24

- Midterm II - Avg  $\approx$  12/20 (sols posted)
  - Quiz #6 - Next Tuesday (1st day)
  - Oral Final - google doc sign up sheet ??  
Sat 12/15 1:30pm - 4:00pm
  - Today, next time:
    - i) Free rotation of a symmetric top
    - ii) motion in a non-inertial ref. frame
  - on your own:
    - (i) Sec 35, Prob 1: motion of a heavy, symmetrical top
    - (ii) static equil: 
    - (Sec 38)

Free rotation of a symmetric top:

no force, torque  $\rightarrow \vec{P} = \text{const}, \vec{M} = \text{const}$

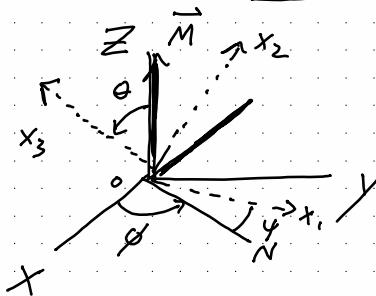
Euler's equations:

$$\dot{\Omega}_1 = I_1 \dot{\Omega}_1 + (I_2 \Omega_3 - I_3 \Omega_2) \quad (1)$$

$$\dot{\Omega}_2 = I_2 \dot{\Omega}_2 + I_3 \Omega_1 (I_1 - I_3) \quad (2)$$

$$\dot{\Omega}_3 = I_3 \dot{\Omega}_3 + \Omega_1 \Omega_2 (I_2 - I_1) \quad (3)$$

$$\Omega_3 = 0 \rightarrow \boxed{\Omega_3 = \text{const}}$$



$$\Omega_1 = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi$$

$$\Omega_2 = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi$$

$$\Omega_3 = \dot{\psi} + \dot{\phi} \cos \theta$$

$$I_1 \Omega_1 = M_1 = M \sin \theta \sin \psi \quad (4)$$

$$I_1 \Omega_2 = M_2 = M \sin \theta \cos \psi \quad (5)$$

$$I_3 \Omega_3 = M_3 = M \cos \theta \quad (6)$$

$$\Omega_3 = \text{const} \rightarrow \boxed{\theta = \text{const}}, \cos \theta = \frac{I_3 \Omega_3}{M}$$

$$\dot{\Omega}_1 = I_1 \dot{\Omega}_1 + \Omega_2 \Omega_3 (I_3 - I_1)$$

$$\dot{\Omega}_1 = \dot{\Omega}_1 + \Omega_2 \left( \frac{\Omega_3 (I_3 - I_1)}{I_1} \right) \leftarrow \boxed{\omega \equiv \frac{\Omega_3 (I_3 - I_1)}{I_1}}$$

$$\dot{\Omega}_2 = I_2 \dot{\Omega}_2 + \Omega_1 \Omega_3 (I_1 - I_3) \quad M_1 = I_1 \Omega_1, M_2 = I_2 \Omega_2$$

$$\dot{\Omega}_2 = \dot{\Omega}_2 - \Omega_1 \left( \frac{\Omega_3 (I_3 - I_1)}{I_1} \right) \sqrt{\Omega_1^2 + \Omega_3^2} = a$$

$$\dot{\Omega}_1 = \dot{\Omega}_1 + \Omega_2 \omega$$

$$\dot{\Omega}_2 = \dot{\Omega}_2 - \Omega_1 \omega$$

$$\dot{\Omega}_1 = \ddot{\Omega}_1 + \Omega_2 \omega \quad \boxed{\Omega_2 = a \sin(\omega t + \alpha)}$$

$$\dot{\Omega}_1 = \ddot{\Omega}_1 + \Omega_2 \omega^2 \quad \uparrow$$

$$\ddot{\Omega}_1 = -\omega^2 \Omega_1 \rightarrow \boxed{\Omega_1 = a \cos(\omega t + \alpha)}$$

$$\xi = \Omega_1 + i \Omega_2$$

$$\dot{\xi} = \dot{\Omega}_1 + i \dot{\Omega}_2$$

$$\dot{\xi} = i \omega \xi \rightarrow \xi(t) = A e^{i \omega t}$$

complex

$$I_1 \dot{\phi} \sin \theta \cos \theta = M \cos \theta \sin \theta \quad \boxed{\theta = \text{const}}$$

$$\boxed{\dot{\phi} = \frac{M}{I_1} = \text{const}} \rightarrow \boxed{\phi(t) = \phi_0 + \frac{M t}{I_1}}$$

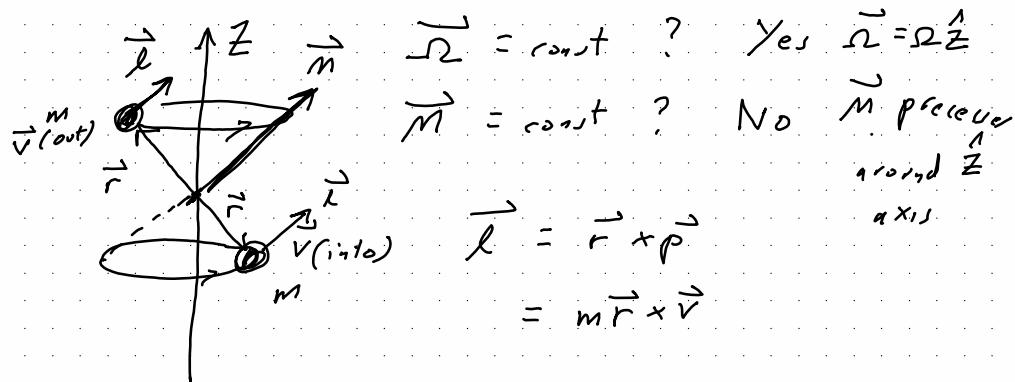
$$I_3 (\dot{\psi} + \cos \theta \dot{\phi}) = M_{\text{cos}} \theta$$

$$\begin{aligned} -\omega &= \dot{\psi} = \frac{M_{\text{cos}} \theta}{I_3} - \cos \theta \dot{\phi} \\ &= \frac{M_{\text{cos}} \theta}{I_3} - \cos \theta \frac{M}{I_1} \\ &= M_{\text{cos}} \theta \left( \frac{1}{I_3} - \frac{1}{I_1} \right) \end{aligned}$$

$\omega = \sqrt{I_3 (I_3 - I_1)}$   
 $I_3 = I_3 \Omega_3$   
 $M_{\text{cos}} \theta = M_3$   
 $\Omega_3 = M_{\text{cos}} \theta$

$$\rightarrow \boxed{\psi(t) = \psi_0 + M_{\text{cos}} \theta \left( \frac{1}{I_3} - \frac{1}{I_1} \right) t}$$

$\omega = \frac{M_{\text{cos}} \theta (I_3 - I_1)}{I_1 I_3} \quad \boxed{\ddot{\psi}}$   
 $= M_{\text{cos}} \theta \left( \frac{1}{I_1} - \frac{1}{I_3} \right)$



wobble  $\leftrightarrow$  spin freq  $\dot{\phi} = \frac{M}{I_1}$

$\dot{\phi}$   $\Omega_3$   $\Omega_3 = \frac{M \cos \theta}{I_3}$

precession freq of angular along 3-axis

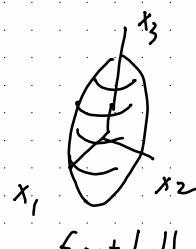
3-axis  
around M  
(or  $\hat{z}$ )

$$\boxed{\frac{\Omega_3}{\dot{\phi}} = \frac{M \cos \theta}{I_3} \frac{I_1}{M} = \frac{I_1 \cos \theta}{I_3}}$$

$$\frac{I_3}{\dot{\phi}} = \frac{I_1 \cos \theta}{I_3} \rightarrow \frac{I_1}{I_3} (\theta \geq 0)$$

w

spin  
wobble



$$I_1 > I_3$$

$$I_1 = I_3$$

sphere

$$\frac{I_3}{\dot{\phi}} = 1$$

football

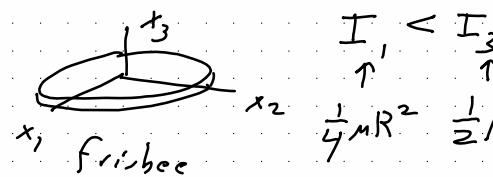
$$\frac{I_3}{\dot{\phi}} > 1$$

frisbee

$$\frac{I_3}{\dot{\phi}} = \frac{1}{2} < 1$$

$$I_1 < I_3$$

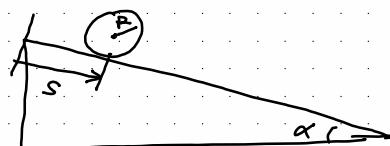
$\frac{1}{4}MR^2 \quad \frac{1}{2}MR^2 \quad (\text{uniform dist})$



Lecture #28:

name = g6.pdf

- course evaluations (please submit by EOD today)
- sign-up for oral final exam date/time
- Quiz #6



$$\text{uniform dist} \quad I_3 = \frac{1}{2}MR^2$$



- a) calculate KE of the uniform disk as it rolls without slipping down the incline

$$T = \text{something } (M, s, R)$$

- b) repeat for a hoop

$$I_3 = MR^2$$

$$T = \text{something}$$



which object will reach the bottom first?  
(or do both reach the bottom at the same time)

$$T = \frac{1}{2} \mu V^2 + \frac{1}{2} \sum_i I_i \omega_i^2$$

$$= \frac{1}{2} M V^2 + \frac{1}{2} \sum_i I_i \omega_i^2$$

$$= \frac{1}{2} M V^2 + \frac{1}{2} I_3 \omega_3^2$$

$$V = \dot{s}, \quad \omega_3 = \dot{s} \rightarrow s \omega_3 = \frac{\dot{s}}{R}$$

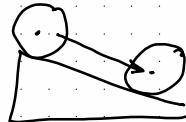
$$T = \frac{1}{2} \mu \dot{s}^2 + \frac{1}{2} I_3 \frac{\dot{s}^2}{R^2}$$

$$= \frac{1}{2} M \dot{s}^2 + \frac{1}{2} \left( \frac{1}{2} M R^2 \right) \frac{\dot{s}^2}{R^2}$$

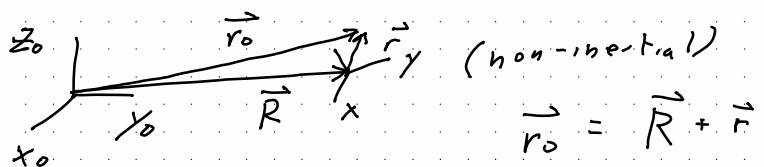
$$= \boxed{\frac{3}{4} M \dot{s}^2} \quad (\text{Unif Jis, k}) \rightarrow \dot{s} = \sqrt{\frac{4}{3} \frac{T}{M}}$$

$$\boxed{T = \mu \dot{s}^2} \quad (\text{loop})$$

$$\rightarrow \dot{s} = \sqrt{\frac{T}{M}}$$



Motion in a non-inertial ref frame (Sec 3g)



(inertial)

$$m \ddot{a}_0 = \vec{F} \quad (\text{2nd law in inertial Frame})$$

$$m \ddot{a} = \vec{F} \quad ??$$

$$\boxed{\frac{d \vec{A}}{dt} \Big|_0 = \frac{d \vec{A}}{dt} + \vec{\omega} \times \vec{A}} \quad \leftarrow \text{use this}$$

wrt rotating frame

$$\vec{a}_0 = \frac{d \vec{v}_0}{dt} \Big|_0, \quad \vec{v}_0 = \frac{d \vec{r}_0}{dt} \Big|_0$$

$$\vec{v}_0 = \frac{d}{dt} (\vec{R} + \vec{r}) \Big|_0 = \frac{d \vec{R}}{dt} \Big|_0 + \frac{d \vec{r}}{dt} \Big|_0 = \boxed{\vec{V} + \vec{v} + \vec{\omega} \times \vec{r}}$$

$$\vec{v}_D = \vec{V} + \vec{w} + \vec{q}$$

$$\vec{a}_0 = \frac{d\vec{v}_0}{dt} |_0$$

$$= \frac{d\vec{V}}{dt} \Big|_o + \frac{d\vec{v}}{dt} \Big|_o + \frac{d\vec{r}}{dt} \Big|_o \times \vec{r} + \vec{r} \times \frac{d\vec{r}}{dt} \Big|_o$$

$$= \overrightarrow{W} + \left( \frac{d\vec{v}}{dt} + \vec{\omega} \times \vec{v} \right) + \left( \frac{d\vec{\omega}}{dt} + \vec{\omega} \times \vec{\omega} \right) \times \vec{r}$$

all  
of orig

of non-inertial frame

$$+ \vec{r} \times \left( \frac{d\vec{r}}{dt} + \vec{v} \times \vec{r} \right)$$

$$\vec{q}_o = \vec{a} + \vec{W} + \vec{\Omega} \times \vec{r} + 2\vec{\Omega} \times \vec{v} + \vec{\omega}(\vec{v} \times \vec{r})$$

$$m \vec{a}_0 = \vec{F} \quad \text{transl. accel.} \quad | \quad \text{rot. accel.} \quad | \quad \text{Coriolis} \quad | \quad \text{Centrifugal} \quad |$$

$$m(\vec{a} + \dots) = \vec{F} - m\vec{V} - m\vec{\Omega} \times \vec{r} - 2m\vec{\Omega} \times \vec{v} - m\vec{\Omega} \times (\vec{\omega} \times \vec{r})$$

$$\rightarrow m\vec{a} = \vec{F} - m\vec{V} - m\vec{\Omega} \times \vec{r}$$

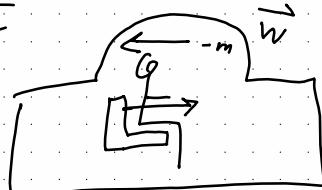
$$\frac{dp}{dt} = \nabla f = \vec{F}$$

$$\frac{d}{dt} \vec{P} + \vec{\Omega} \times \vec{P} = \vec{F}$$

$$\frac{d\vec{M}}{dt} + \vec{\omega} \times \vec{M} = \vec{K}$$

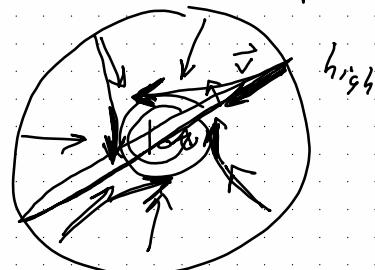
wrt  
rigid body Frame

$$\frac{d\vec{P}}{dt} = \vec{F} - \vec{\Sigma} \times \vec{P}$$



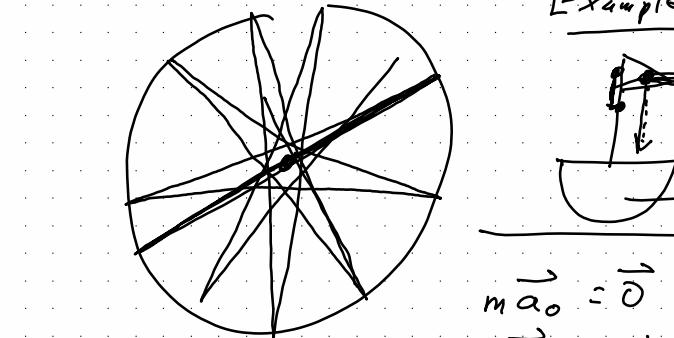
 cyclonic

100

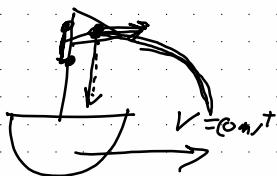


Saw

out  
of  
page



Example: merry-go-round



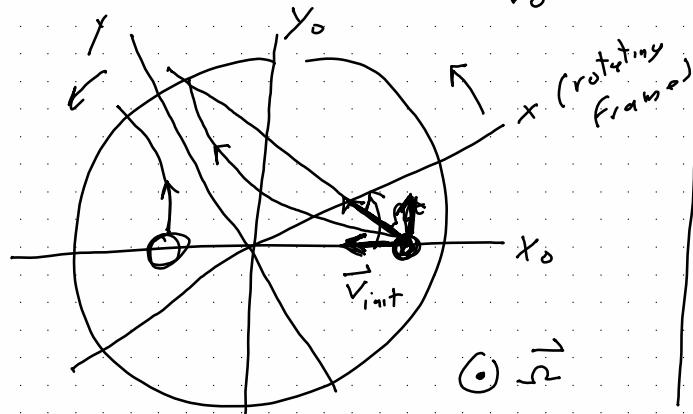
$$\vec{m}\vec{\alpha}_0 = \vec{0}$$

$$\vec{V}_0 = \text{const}$$

solve 2nd law  
in merry-go-round  
frame:

$$m \frac{d\vec{V}}{dt} = -2m\vec{\Omega} \times \vec{V}$$

$$-m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$



$$\vec{r} = \hat{x}\hat{x} + \hat{y}\hat{y}$$

$$\vec{V} = \dot{\hat{x}}\hat{x} + \dot{\hat{y}}\hat{y}$$

$$\frac{d\vec{V}}{dt} = \ddot{\hat{x}}\hat{x} + \ddot{\hat{y}}\hat{y}$$

$$\vec{\Omega} = \hat{z}$$

$$\begin{aligned}\ddot{x} &= 2\Omega\dot{y} + \Omega^2 x \\ \ddot{y} &= -2\Omega\dot{x} + \Omega^2 y\end{aligned}$$

$$\xi = x + iy$$

$$\dot{\xi} = \dot{x} + i\dot{y}$$

$$\ddot{\xi} = \ddot{x} + i\ddot{y}$$

$$\ddot{\xi} + i2\Omega\dot{\xi} - \Omega^2 \xi = 0$$

$$\text{Guess: } \xi = e^{i\lambda t}$$

$$\lambda = -\Omega \quad (\text{double root})$$

$$\xi = (A + Bt) e^{-i\Omega t}$$

complex const

determined by initial  
condition,

$$\xi(t) = x(t) + iy(t)$$

$$\begin{bmatrix} x \\ y \end{bmatrix}^l = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

