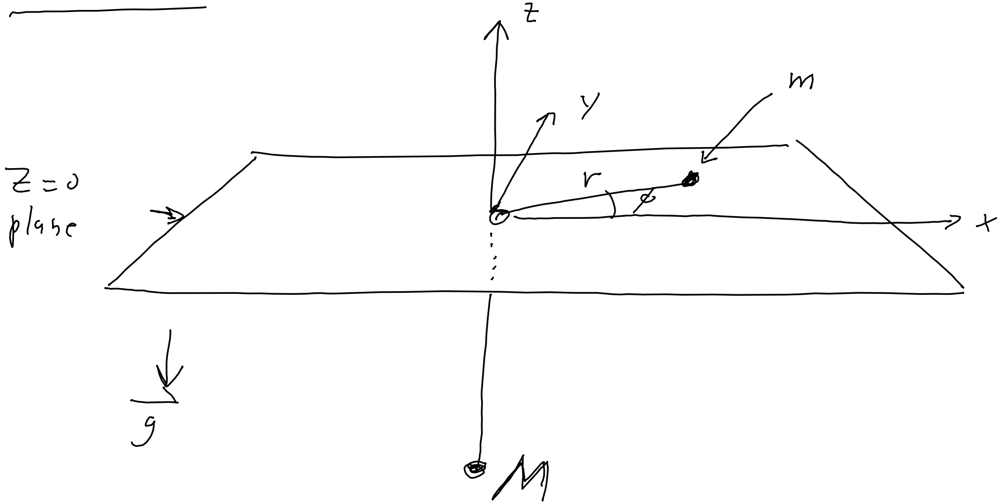
Example:



l = length of string

$z = r - l$

$l = r - z$

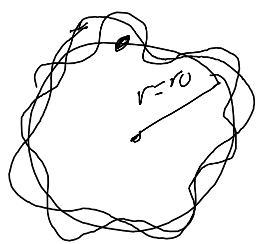
a) write down Lagrangian: $L(r, \phi, \dot{r}, \dot{\phi})$

b) determined conserved quantities

c) determine the effective potential $U_{eff}(r)$

d) qualitatively determine the allowed motions

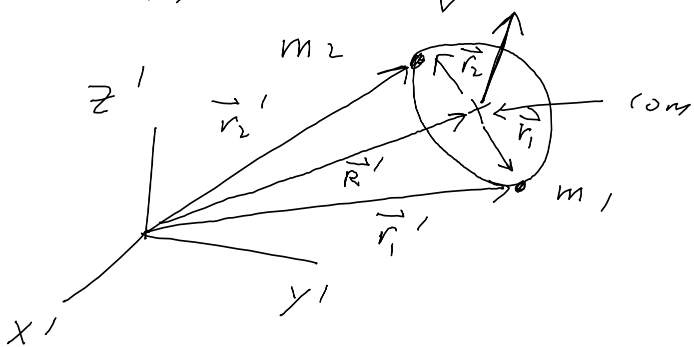
Top view:



Two body problem:

m_1, m_2 $\quad \vec{V},\ U(|\vec{r}_1 - \vec{r}_2|)$

$E, \vec{P}, \vec{M}$
$= \text{const}, = \text{const}, = \text{const}$



$$\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2 = \mu \vec{V}$$

$\mu = m_1 + m_2$

COM frame (X, Y, Z)

$\vec{V} = 0,\quad \vec{P} = 0,$

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = \mu \vec{R} = 0$$

$\vec{r}_1 = \vec{R} + \vec{r}_1$
$\vec{r}_2 = \vec{R} + \vec{r}_2$

$\vec{r}_1, \vec{r}_2$: position vector w-t COM frame

$$\vec{r} = \vec{r_1} - \vec{r_2} \qquad \text{relative separation vector}$$

$$0 = m_1 \vec{r_1} + m_2 \vec{r_2}$$

$$\boxed{\vec{r_1} = \frac{m_2 \vec{r}}{(m_1 + m_2)} \, , \qquad \vec{r_2} = -\frac{m_1 \vec{r}}{(m_1 + m_2)}}$$

$$T = \frac{1}{2}\left(m_1 |\dot{\vec{r_1}}|^2 + m_2 |\dot{\vec{r_2}}|^2 \right)$$

$$= \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2} \right) |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} m |\dot{\vec{r}}|^2$$

$$U = U(|\vec{r_1} - \vec{r_2}|) = U(|\vec{r_1} - \vec{r_2}|) = U(|\vec{r}|) = U(r)$$

$$\boxed{L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(r)}$$

two body problem

$\downarrow$

one body

choose z-axis of COM frame to point along $\vec{M}$

$$\vec{M} = \vec{r} \times \vec{p}$$

$\hat{z}$, $\vec{M} = M\hat{z}$

$\vec{r}$ lies in the x-y plane

$x = r\cos\phi$

$y = r\sin\phi$

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)}$$

$$M \equiv p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \text{const} \qquad \frac{\partial L}{\partial \phi} = 0$$

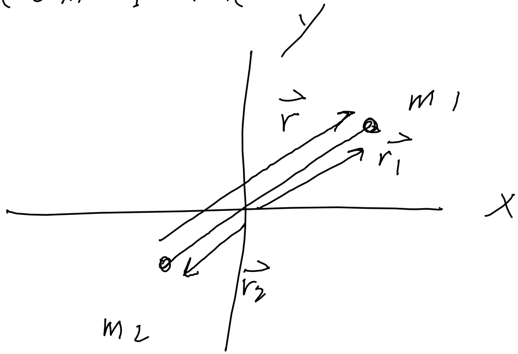
$$M = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + U(r) = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L$$

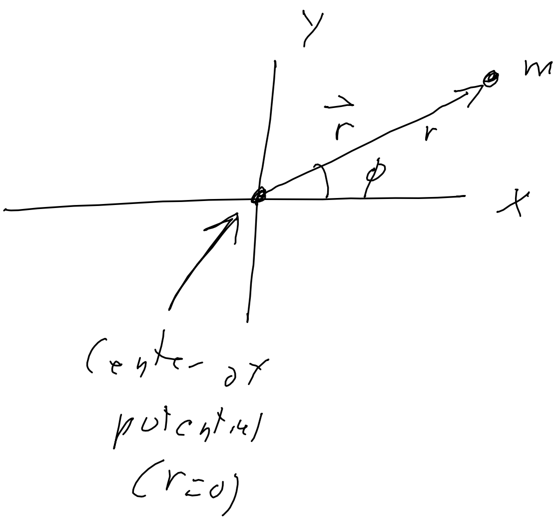
$$= \frac{1}{2}m\dot{r}^2 + \frac{M^2}{2mr^2} + U(r)$$

$$= \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$$

COM frame



y
x
$\vec{r}$
m_1
$\vec{r}_1$
$\vec{r}_2$
m_2



y
x
m
$\vec{r}$
r
ϕ
Center of potential (r=0)

$$E = \frac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

$$\dot{\phi} = \frac{M}{mr^2}$$

$$\pm\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)} = \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M}{mr^2}$$

$$= \pm\sqrt{}$$

$$dt = \frac{\pm\, dr}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}}$$

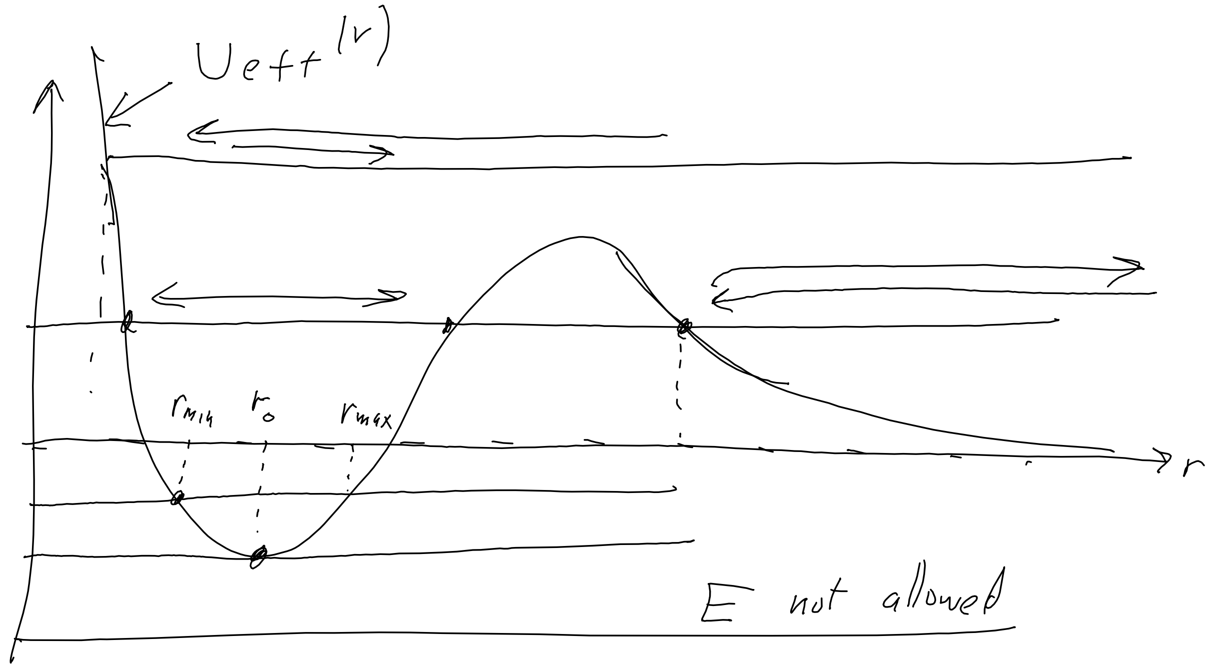
$$\boxed{t = \pm \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}} + \text{const}}$$

$$d\phi = \pm \frac{dr\, M/mr^2}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}}$$

$$u = \frac{1}{r}$$

$$du = -\frac{1}{r^2}\, dr$$

$$\boxed{\phi = \pm \int \frac{M\, dr/r^2}{m\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}} + \text{const}}$$



$$E = \frac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

$$= \frac{1}{2} m \dot{r}^2 + \left(\frac{M^2}{2mr^2} + U(r) \right)$$