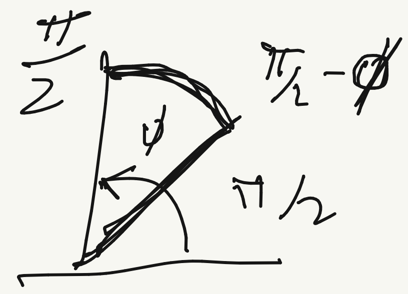
Elliptic functions / integrals:

i) period of a simple pendulum beyond the small-angle approximation

ii) circumference of an ellipse

A generalization of definition of circular functions (sines, cosines) to ellipses.



standard notation:

$$\int_0^x \frac{dt}{\sqrt{1-k^2t^2}\sqrt{1-t^2}} = F(\phi, k) = sn^{-1}x$$

$$\int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt = E(\phi, k)$$

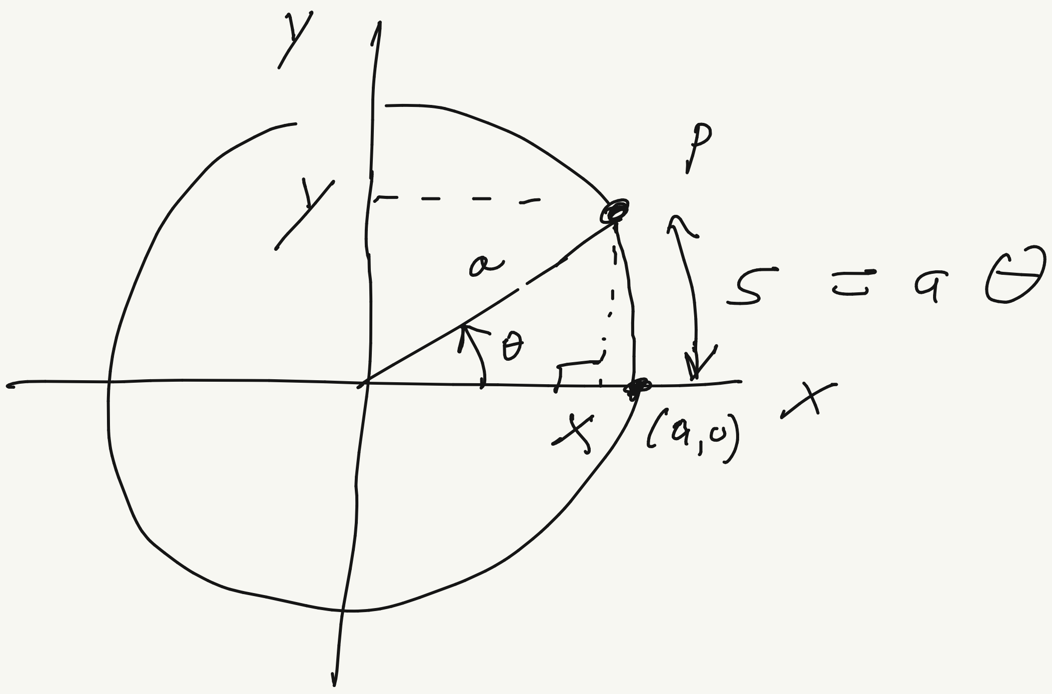
where $x \equiv \sin\phi$ and $0 \leq k \leq 1$

If we change variables $t \to \sin\theta$ in the integrals then

$$F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$E(\phi, k) = \int_0^\phi \sqrt{1-k^2\sin^2\theta}\, d\theta$$

Circular Functions:



$$\sin\theta \equiv \frac{y}{a}, \qquad \cos\theta \equiv \frac{x}{a}$$

where $\theta \equiv \dfrac{\text{arc length from } (a,0) \text{ to } (x,y)}{a}$

$$= \frac{1}{a}\int_{(a,0)}^{(x,y)} \sqrt{dx^2+dy^2} \qquad \left(= \int d\theta\right)$$

$$x^2+y^2 = a^2 \quad \rightarrow \quad a^2\cos^2\theta + a^2\sin^2\theta = a^2$$

$$\rightarrow \quad \boxed{\cos^2\theta + \sin^2\theta = 1}$$

Derivatives:

$$\frac{d}{d\theta}\sin\theta = \frac{d}{d\theta}\left(\frac{y}{a}\right) = \frac{1}{a}\frac{dy}{d\theta} = \frac{dy}{\sqrt{dx^2+dy^2}}$$

$$= \frac{1}{\sqrt{\left(\frac{dx}{dy}\right)^2+1}}$$

$$= \frac{1}{\sqrt{\left(\frac{-y}{x}\right)^2+1}}$$

Now: $x^2+y^2=a^2$

$$\rightarrow \quad 2x\,dx + 2y\,dy = 1$$

$$\frac{dx}{dy} = \frac{-y}{x}$$

Thus,

$$\frac{d}{d\theta}\sin\theta = \frac{1}{\sqrt{1+\left(\frac{-y}{x}\right)^2}} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{a} = \cos\theta$$

so $\frac{d\sin\theta}{d\theta} = \cos\theta$

Similarly, $\frac{d\cos\theta}{d\theta} = -\sin\theta$

Integrate:

$$\int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta = \theta + \text{const}$$

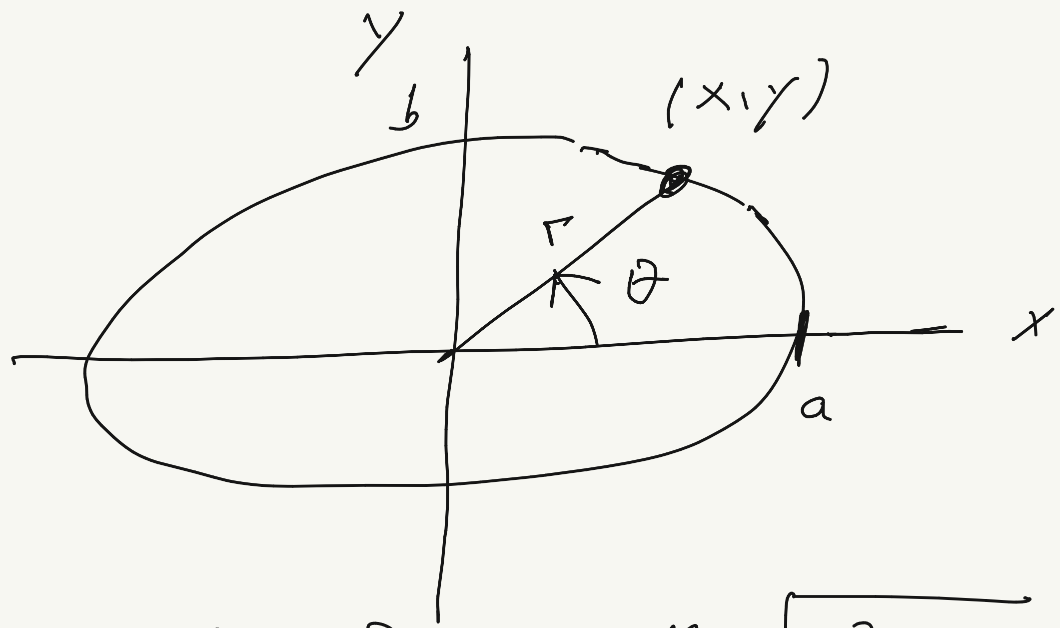
$t = \sin\theta$

$\cos\theta = \sqrt{1-t^2}$

$dt = d(\sin\theta)$

$$\rightarrow \boxed{\int \frac{dt}{1-t^2} = \theta + \text{const} = \sin^{-1} t + \text{const}}$$

Different parametrization of an ellipse:



$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

Let $a \geq b$

Eccentricity:

$$e^2 = 1-\left(\frac{b}{a}\right)^2$$

($e=0$ for a circle)

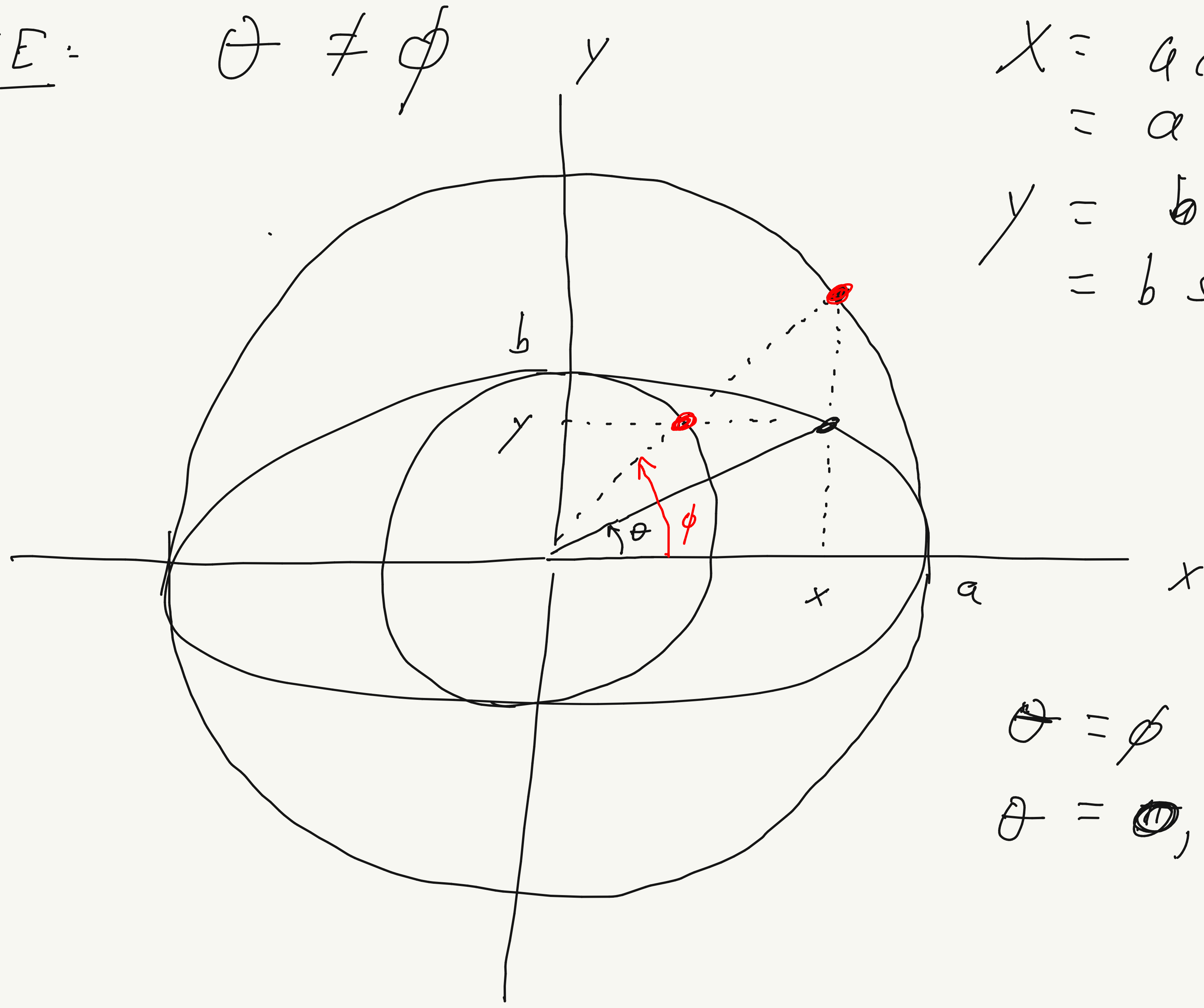
$$x = r\cos\theta, \quad y = r\sin\theta \qquad , \qquad r=\sqrt{x^2+y^2}$$

Another parameterization:

$$\left.\begin{aligned} x &= a\cos\phi \\ y &= b\sin\phi \end{aligned}\right\} \rightarrow \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

NOTE: $\theta \neq \phi$

$$\begin{aligned} x &= a\cos\phi \\ &= a\,\mathrm{cn}(u;k) \\ y &= b\sin\phi \\ &= b\,\mathrm{sn}(u;k) \end{aligned}$$



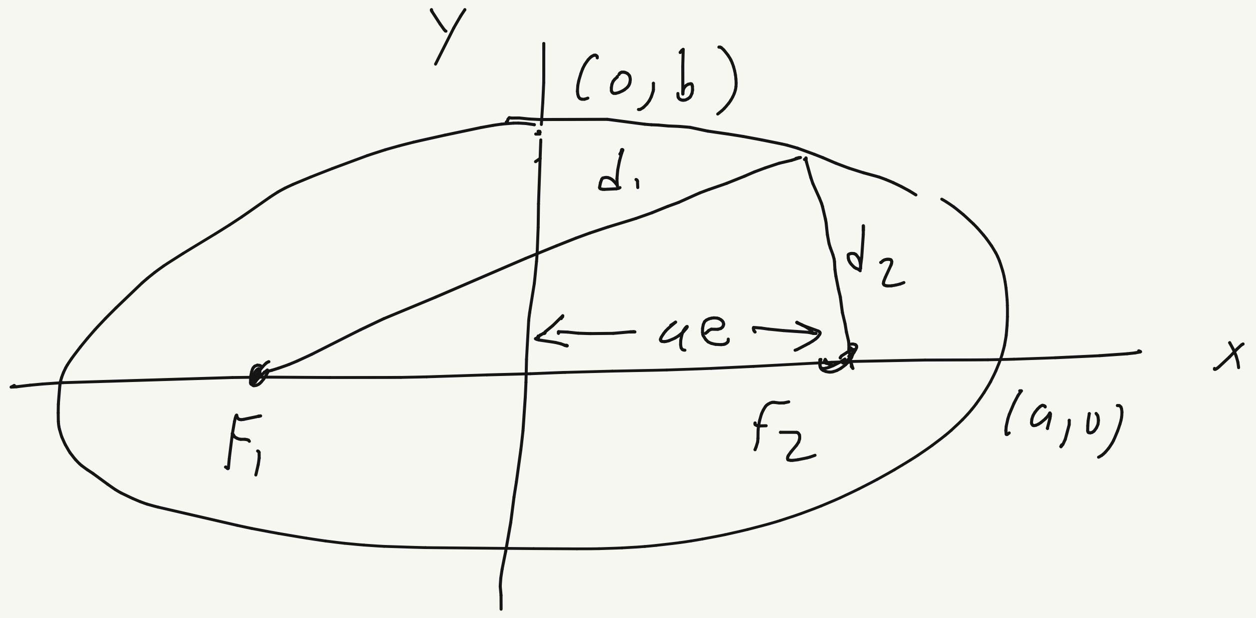
$\theta = \phi$ for

$\theta = 0, \frac{\pi}{2}, \ldots$

$$\tan\theta = \frac{y}{x} = \left(\frac{b}{a}\right)\tan\phi \rightarrow \theta = \arctan\left[\frac{b}{a}\tan\phi\right]$$

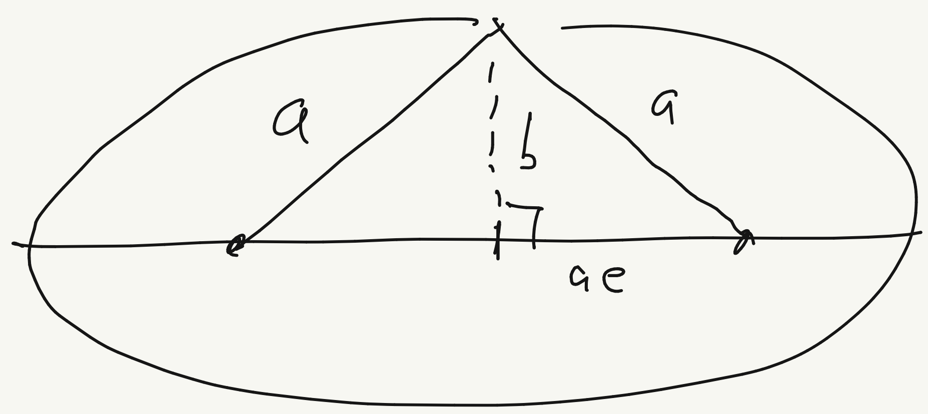
$$\phi = \arctan\left[\frac{a}{b}\tan\theta\right]$$

Elliptic functions:



$d_1 + d_2 = 2a$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

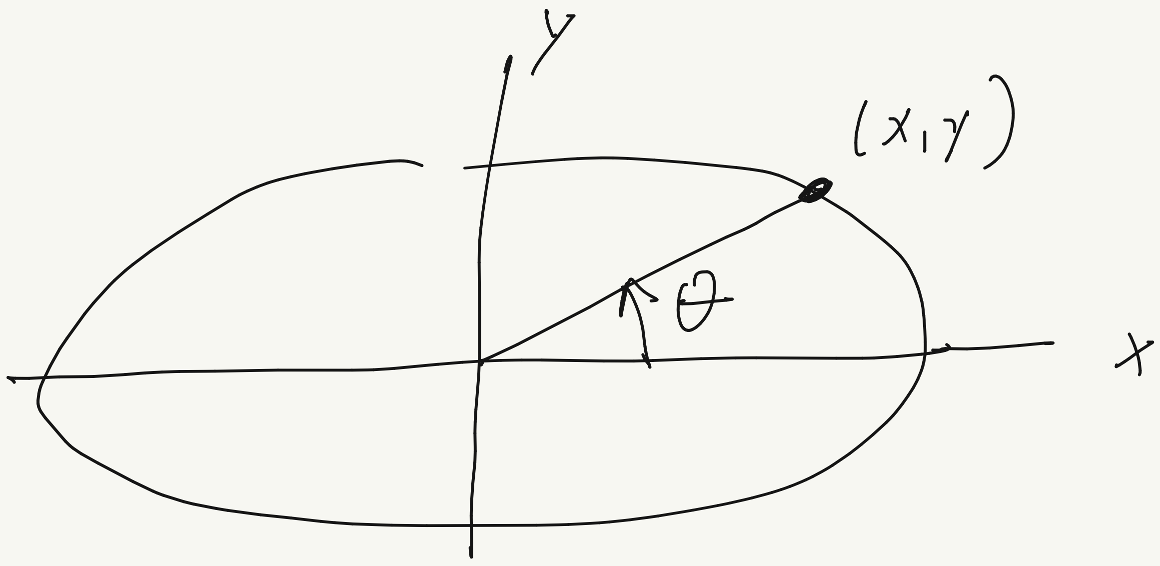


$a^2 = b^2 + a^2 e^2$

$a^2(1-e^2) = b^2$

$b = a\sqrt{1-e^2}$

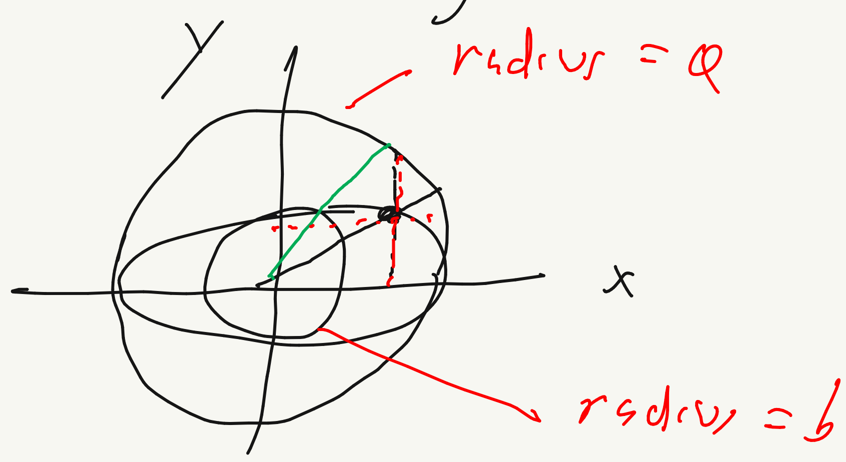
Also, $\left(\frac{b}{a}\right)^2 = 1 - e^2 \rightarrow \boxed{e = \sqrt{1 - \left(\frac{b}{a}\right)^2}}$



$x = r\cos\theta$

$y = r\sin\theta$

(but r changes)



$x = a\cos\phi$

$y = b\sin\phi$

$\phi \neq \theta$

Define: $\mathrm{cn}(u; k) \equiv x/a$

$\mathrm{sn}(u; k) \equiv y/b$

$\mathrm{dn}(u; k) \equiv r/a$

where $k = e$

$0 \leq k \leq 1$

~~and~~ where $u = \frac{1}{b}\int_0^\theta r\, d\theta$ — $u = \theta$ for circle

$u \neq$ arc length

since $ds = \sqrt{dr^2 + r^2 d\theta}$

NOTE: $b u = \int_0^{\theta} r\,d\theta \leq \int_0^{\theta} ds$

$\leq$ arc length from $(a,0)$ to (x,y)

Properties of sn, cn, dn:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \rightarrow \boxed{cn^2 u + sn^2 u = 1}$$

$$x^2 + y^2 = r^2 \rightarrow a^2 cn^2 u + b^2 sn^2 u = a^2 dn^2 u$$

$$a^2(1 - sn^2 u) + b^2 sn^2 u = a^2 dn^2 u$$

$$1 - sn^2 u + \frac{b^2}{a^2} sn^2 u = dn^2 u$$

$$1 - \left(1 - \frac{b^2}{a^2}\right) sn^2 u = dn^2 u$$

$$1 - k^2 sn^2 u = dn^2 u$$

Thus, $\boxed{dn^2 u + k^2 sn^2 u = 1}$

for circle $k = 0$, $dn\, u = 1$

Derivatives:

$$\frac{d}{du} sn\, u = \frac{d}{du}\left(\frac{y}{b}\right) = \frac{dy}{b\,du}$$

Now: $b\,du = r\,d\theta$

$$\rightarrow \frac{d}{du} sn\, u = \frac{dy}{r\,d\theta}$$

$$x = r\cos\theta \quad \rightarrow \quad dx = dr\cos\theta - r\sin\theta\, d\theta$$
$$y = r\sin\theta \qquad\qquad dy = dr\sin\theta + r\cos\theta\, d\theta$$

$$\rightarrow \quad -\sin\theta\, dx = -\sin\theta\cos\theta\, dr + r\sin^2\theta\, d\theta$$
$$+\cos\theta\, dy = \cos\theta\sin\theta\, dr + r\cos^2\theta\, d\theta$$

add: $\cos\theta\, dy - \sin\theta\, dx = r\, d\theta$

$$\rightarrow \quad \frac{x}{r}\, dy - \frac{y}{r}\, dx = r\, d\theta$$

thus,

$$\frac{d\,\mathrm{sn}\,u}{du} = \frac{dy}{r\,d\theta} = \frac{\cancel{dy}}{\frac{x}{r}\, dy - \frac{y}{r}\, d\cancel{x}}$$

$$= \frac{r}{x - y\frac{dx}{dy}}$$

Also: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \rightarrow \quad \frac{2x\,dx}{a^2} + \frac{2y\,dy}{b^2} = 0$

$$\frac{dx}{dy} = -\frac{y}{x}\left(\frac{a}{b}\right)^2$$

$$\rightarrow \quad \frac{d\,\mathrm{sn}\,u}{du} = \frac{r}{x + \frac{y^2}{x}\left(\frac{a}{b}\right)^2} = \frac{r\,x}{x^2 + y^2\left(\frac{a^2}{b^2}\right)}$$

$$= \frac{r}{a}\,\frac{x}{a}\,\frac{1}{\underbrace{\left(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2\right)}_{1}} = \boxed{\mathrm{cn}\,u \cdot \mathrm{dn}\,u}$$

Then using $\quad sn^2 u + cn^2 u = 1$

we have $\quad \cancel{2}\, sn\, u \dfrac{d}{du} sn\, u + \cancel{2}\, cn\, u \dfrac{d}{du} cn\, u = 0$

$$\rightarrow \quad \frac{d\, cn\, u}{du} = -\frac{sn\, u}{cn\, u} \frac{d}{du} sn\, u$$

$$= -\frac{sn\, u}{\cancel{cn\, u}} \cancel{cn\, u} \cdot dn\, u$$

$$= \boxed{-sn\, u \cdot dn\, u}$$

And using $\quad dn^2 u + k^2 sn^2 u = 1$:

$$\cancel{2}\, dn\, u \frac{d(dn\, u)}{du} + \cancel{2}\, k^2 sn\, u \frac{d}{du}(sn\, u) = 0$$

$$\rightarrow \quad \frac{d}{du}(dn\, u) = -k^2 \frac{sn\, u}{dn\, u} \frac{d(sn\, u)}{du}$$

$$= -k^2 \frac{sn\, u}{\cancel{dn\, u}} cn\, u \cdot \cancel{dn\, u}$$

$$= \boxed{-k^2 sn\, u\, cn\, u}$$

Summary:

$$\frac{d}{du} sn\, u = cn\, u \cdot dn\, u$$

$$\frac{d}{du} cn\, u = -sn\, u \cdot dn\, u$$

$$\frac{d}{du} dn\, u = -k^2 sn\, u\, cn\, u$$

Integrate:

$$\frac{d\,\mathrm{sn}\,u}{du} = \mathrm{cn}\,u \cdot \mathrm{dn}\,u$$

$$\int \frac{d(\mathrm{sn}\,u)}{\mathrm{cn}\,u \cdot \mathrm{dn}\,u} = \int du = u + \text{const}$$

Let $t = \mathrm{sn}\,u \to \mathrm{cn}\,u = \sqrt{1-t^2}$, $\mathrm{dn}\,u = \sqrt{1-k^2\mathrm{sn}^2 u}$

$$\to \int \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} = u + \text{const}$$

But since $t = \mathrm{sn}\,u$:

$$\boxed{\int \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} = \mathrm{sn}^{-1}(t;k) + \text{const}}$$

In handbooks:

$$\boxed{\int_0^{\sin\phi} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} = F(\phi,k)}$$

Incomplete Elliptic integral of the 1st Kind

$$\downarrow$$

$$\boxed{\int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\phi,k)}$$

where $t = \sin\theta$.

NOTE: $\boxed{\sin\phi = \mathrm{sn}\,u}$ ☆

Complete elliptic integral of 1st kind:

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}}$$

Connection to simple pendulum:

$$P = \frac{4}{\omega_0} K\left(k \equiv \sin\left(\frac{\phi_0}{2}\right)\right)$$

$$\omega_0 t = sn^{-1}\left(x \equiv \frac{\sin(\phi/2)}{\sin\left(\frac{\phi_0}{2}\right)},\ k \equiv \sin\left(\frac{\phi_0}{2}\right)\right) + \text{const}$$

$\sqrt{\frac{g}{l}}$

$-K(k)$
for $\phi = \phi_0$
when $t = 0$

$$\phi(t) = 2 \arcsin\left[k\ sn\left(\omega_0\left(t + \frac{P}{4}\right);\ k\right)\right]$$

Elliptic integral of 2nd kind (circumference of ellipse w.r.t. y-axis)

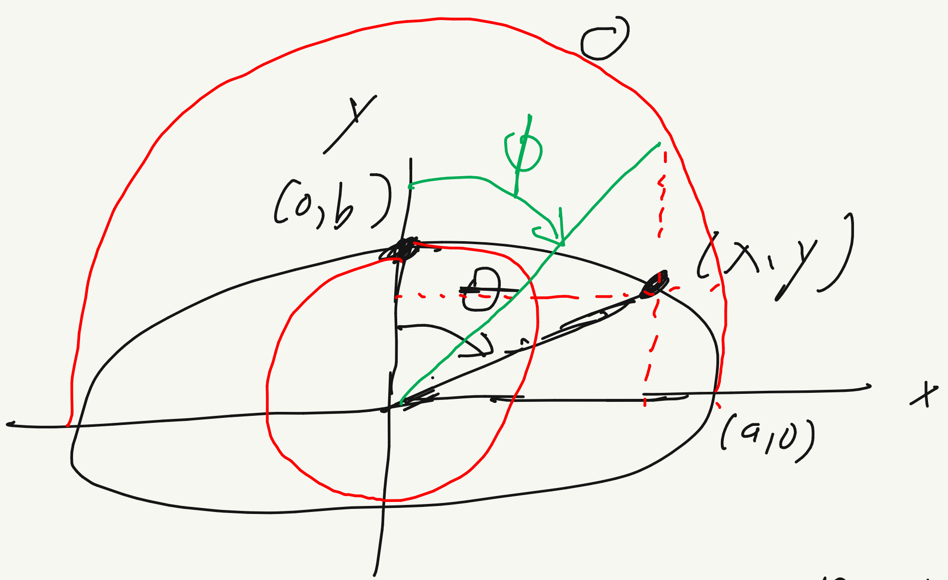
CW

$$E(\phi, k) \equiv \int_0^{\sin\phi} dt \, \frac{\sqrt{1-k^2 t^2}}{\sqrt{1-t^2}}$$

Rewrite: $t = \sin\bar{\phi}$

$dt = \cos\bar{\phi}\, d\bar{\phi} = \sqrt{1-t^2}\, d\bar{\phi}$

$$E(\phi, k) \equiv \int_0^{\phi} d\bar{\phi} \sqrt{1-k^2\sin^2\bar{\phi}} \quad \text{(scipy definition)}$$



$$k = e = \sqrt{1-\left(\frac{b}{a}\right)^2}$$

$x = r\sin\theta = a\sin\varphi = a\sin\phi$

$y = r\cos\theta = b\cos\varphi = b\cos\phi$

} sin, cos interchanged from previous page

$$ds = \sqrt{dx^2+dy^2}$$
$$= \sqrt{a^2\cos^2\phi + b^2\sin^2\phi}\; d\phi$$
$$= \sqrt{a^2(1-\sin^2\phi) + b^2\sin^2\phi}\; d\phi$$
$$= a\sqrt{1-\left(1-\left(\frac{b}{a}\right)^2\right)\sin^2\phi}\; d\phi$$
$$= a\sqrt{1-k^2\sin^2\phi}\; d\phi$$

$(\phi \neq \theta)$

$\tan\theta = \frac{a}{b}\tan\phi$

$\theta = \arctan\left(\frac{a}{b}\tan\phi\right)$

$\phi = \arctan\left(\frac{b}{a}\tan\theta\right)$

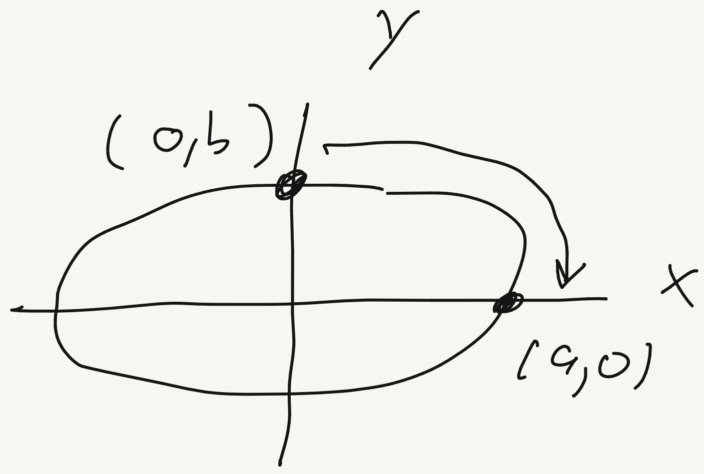
$$\boxed{S_{(0,b)\to(x,y)} = a\int_0^{\phi}\sqrt{1-k^2\sin^2\bar{\phi}}\; d\bar{\phi}} \equiv E(\phi, k)$$

Complete elliptic integral of 2nd kind:

$$E(k) \equiv E\left(\frac{\pi}{2}, k\right)$$

$$= \int_0^{\frac{\pi}{2}} d\phi \sqrt{1 - k^2 \sin^2\phi}$$

$$= \int_0^1 dt \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}$$



Circumference:

$$C = 4a \int_0^1 dt \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}$$

1st order correction (for $k \ll 1$; nearly circular)

$$C \cong 4a \int_0^1 \frac{dt}{\sqrt{1-t^2}} \left(1 - \frac{1}{2}k^2t^2\right)$$

$$= 4a \left[\underbrace{\int_0^1 \frac{dt}{\sqrt{1-t^2}}}_{\sin^{-1}(1) = \pi/2} - \frac{k^2}{2} \int_0^1 \frac{dt\ t^2}{\sqrt{1-t^2}} \right]$$

$t = \sin x$

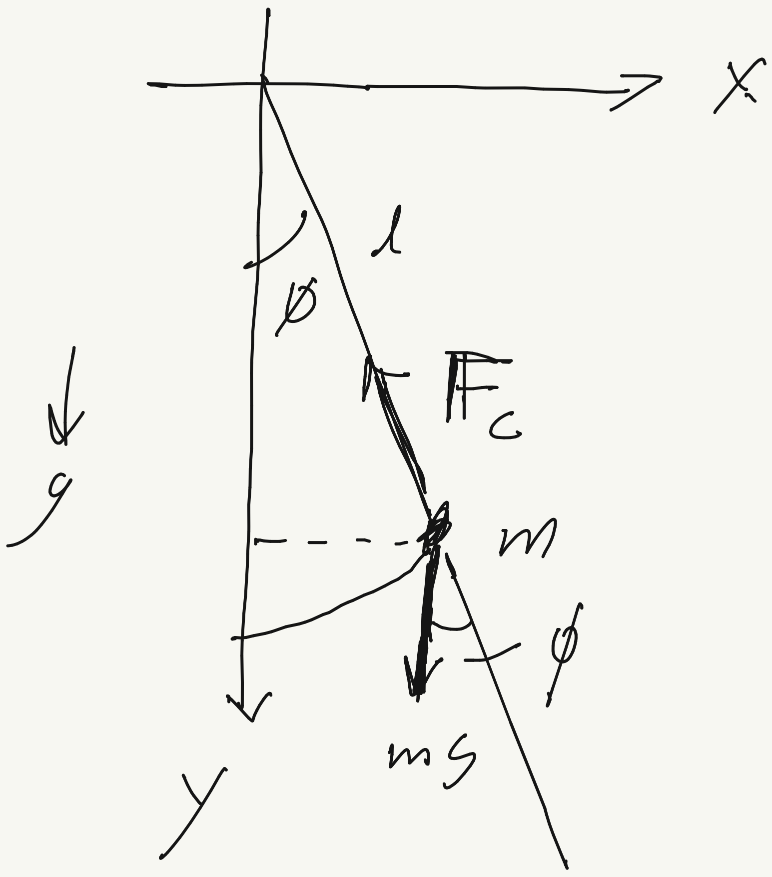
$$= 4a \left[\frac{\pi}{2} - \frac{k^2}{2} \int_0^{\pi/2} \frac{\cancel{\cos x}\, dx\ \sin^2 x}{\cancel{\sqrt{1-\sin^2 x}}} \right]$$

$$= 4a \left[\frac{\pi}{2} - \frac{k^2}{2} \frac{1}{2} \int_0^{\pi/2} (1 - \cancel{\cos 2x}^{\,0})\, dx \right]$$

$$= 4a \left[\frac{\pi}{2} - \frac{k^2}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right]$$

$$= 2\pi a \left(1 - k^2/4\right)$$

Simple pendulum (Freshman physics analysis):



$$mg\sin\phi = -ma_T = -ml\ddot{\phi}$$

$$\rightarrow \ddot{\phi} = -\frac{g}{l}\sin\phi$$

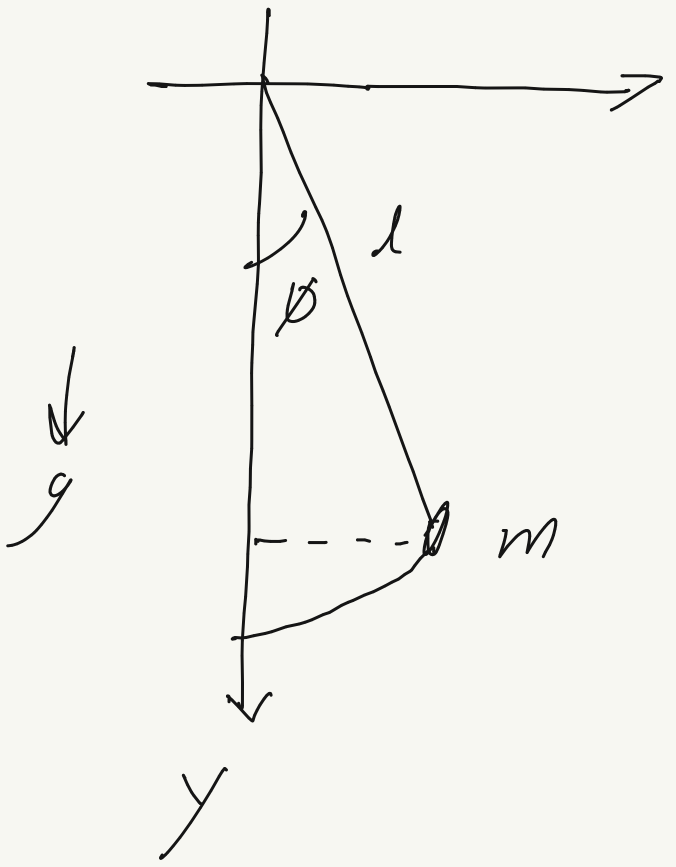
constraint force:

$$F_c - mg\cos\phi = ma_c = m\omega^2 l = ml\dot{\phi}^2$$

Thus, $F_c = mg\cos\phi + ml\dot{\phi}^2$

(non-zero at turning points as well as in vertical position – i.e., $\theta = 0$)

Period of a simple pendulum:



$U = -mgy$

where $y = l\cos\phi$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2} m l^2 \dot{\phi}^2$$

$$L = T - U$$
$$= \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$$\frac{d}{dt}\left(m l^2 \dot{\phi}\right) = -mgl\sin\phi$$

$$m l^2 \ddot{\phi} = -mgl\sin\phi$$

$$\rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi}$$ (same as before)

Small-angle approx: $\sin\phi \simeq \phi$

$$\Rightarrow \dot{\phi} \simeq -\frac{g}{l}\phi = -\omega^2\phi$$

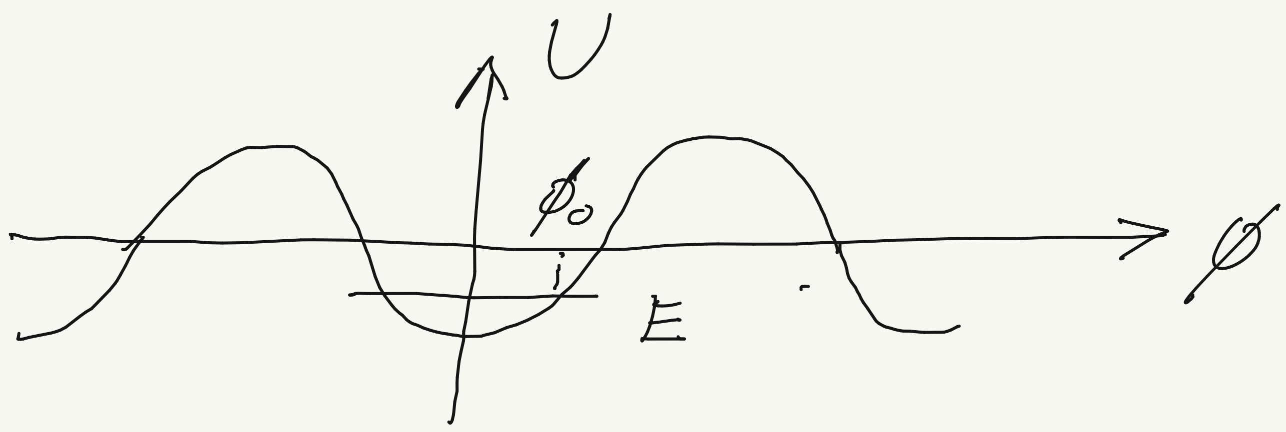
sol'n: $\phi(t) = \underset{\text{complex}}{A} e^{i\omega t}$, $\omega \equiv \sqrt{\frac{g}{l}}$, $P = \frac{2\pi}{\omega}$

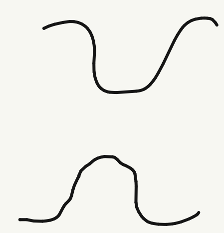
Beyond small-angle approx:

Cons. of energy:

$$E = T + U$$
$$= \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$$

$E = \text{const} = -mgl\cos\phi_0$ at turning points





Thus,

$$-mgl\cos\phi_0 = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$$

$$\frac{1}{2} m l^2 \dot{\phi}^2 = +mgl\underbrace{(\cos\phi - \cos\phi_0)}_{\geq 0 \text{ since } \phi \leq \phi_0}$$

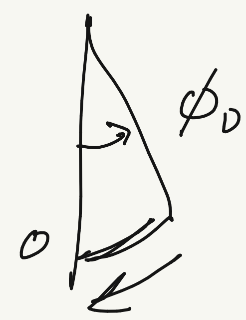
$$\rightarrow \quad \dot{\phi} = \pm\sqrt{2\frac{g}{l}(\cos\phi - \cos\phi_0)}$$

$$\int dt = \pm \int \frac{d\phi}{\sqrt{2\frac{g}{l}}\sqrt{\cos\phi - \cos\phi_0}}$$

$$\rightarrow \quad \boxed{t = \frac{\pm 1}{\sqrt{2}\,\omega_0} \int \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}} + \text{const}}, \quad \omega_0 \equiv \sqrt{\frac{g}{l}}$$

Period:

$$P = \frac{4}{\sqrt{2}\,\omega_0}\int_0^{\phi_0}\frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$



$\frac{1}{4}$ th of a complete cycle

Now: $\cos\phi = \cos\left(2\,\frac{\phi}{2}\right)$

$$= \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\rightarrow \quad \cos\phi - \cos\phi_0 = -2\sin^2\left(\frac{\phi}{2}\right) + 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$= 2\sin^2\left(\frac{\phi_0}{2}\right)\left[1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)}\right]$$

Let: $x \equiv \dfrac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}$ $\rightarrow$ $dx = \dfrac{1}{\sin\left(\frac{\phi_0}{2}\right)}\,\dfrac{1}{2}\cos\left(\frac{\phi}{2}\right)d\phi$

define $k \equiv \sin\left(\frac{\phi_0}{2}\right)$

$$= \frac{1}{2\sin\left(\frac{\phi_0}{2}\right)}\sqrt{1 - \sin^2\left(\frac{\phi}{2}\right)}\;d\phi$$

$$= \frac{1}{2k}\sqrt{1 - k^2x^2}\;d\phi$$

Thus,

$$P = \frac{4}{\sqrt{2}\,\omega_0}\int_0^1 \frac{dx\;\cancel{2k}}{\sqrt{1-k^2x^2}}\,\frac{1}{\cancel{\sqrt{2}\,k}\sqrt{1-x^2}}$$

$$= \boxed{\frac{4}{\omega_0}\int_0^1\frac{dx}{\sqrt{1-k^2x^2}\sqrt{1-x^2}}} = \frac{4}{\omega_0}K(k)$$

complete elliptic integral of 1st kind

Leading-order correction to period:

$$P = \frac{4}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1-k^2x^2}\sqrt{1-x^2}}$$

suppose $\phi_0 \ll 1 \rightarrow k = \sin\left(\frac{\phi_0}{2}\right) \approx \frac{\phi_0}{2} \ll 1$

Thus,
$$\frac{1}{\sqrt{1-k^2x^2}} \approx 1 + \frac{1}{2}k^2x^2$$
$$= 1 + \frac{1}{2}\left(\frac{\phi_0^2}{4}\right)x^2$$
$$= 1 + \frac{1}{8}\phi_0^2 x^2$$

$$\rightarrow P \approx \frac{4}{\omega_0}\int_0^1 \frac{dx}{\sqrt{1-x^2}}\left[1 + \frac{1}{8}\phi_0^2 x^2\right]$$
$$= \frac{4}{\omega_0}\int_0^1 \frac{dx}{\sqrt{1-x^2}} + \frac{\phi_0^2}{2\omega_0}\int_0^1 \frac{dx\, x^2}{\sqrt{1-x^2}}$$
$$= \frac{4}{\omega_0}\sin^{-1}(1) + \frac{\phi_0^2}{2\omega_0}\int_0^{\pi/2} \frac{\cancel{\cos\theta}\, d\theta\, \sin^2\theta}{\cancel{\sqrt{1-\sin^2\theta}}}$$
$$= \frac{4}{\omega_0}\cdot\frac{\pi}{2} + \frac{\phi_0^2}{2\omega_0}\int_0^{\pi/2} d\theta\, \frac{1}{2}(1-\cos 2\theta)$$
$$= \frac{2\pi}{\omega_0} + \frac{\phi_0^2}{4\omega_0}\left(\frac{\pi}{2} - \frac{1}{2}\cancelto{0}{\sin 2\theta}\Big|_0^{\pi/2}\right)$$
$$= \boxed{\frac{2\pi}{\omega_0}\left(1 + \frac{\phi_0^2}{16}\right)}$$

General time dependence of oscillation:

$$\int_0^t dt = \frac{-1}{2w_0} \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$ where $\phi(0) = \phi_0$ and $\frac{d\phi}{dt} < 0$

$$\rightarrow t = -\frac{1}{w_0} \int_1^x \frac{dx}{\sqrt{1-k^2x^2}\sqrt{1-x^2}}$$

$$= -\frac{1}{w_0}\left[\int_1^0 + \int_0^x\right] \frac{dx}{\sqrt{1-k^2x^2}\sqrt{1-x^2}}$$

$$= +\frac{1}{w_0}\int_0^1 \frac{dx}{\sqrt{1-k^2x^2}\sqrt{1-x^2}} - \frac{1}{w_0}\int_0^x \frac{dx}{\sqrt{1-k^2x^2}\sqrt{1-x^2}}$$

$$= \frac{1}{w_0}\left[K(k) - sn^{-1}(x;k)\right]$$

Thus, $sn^{-1}(x;k) = K(k) - w_0 t$

$$\rightarrow \quad x = sn(\underbrace{K(k)}_{\frac{w_0 P}{4}} - w_0 t;\ k) = cn(w_0 t; k)$$

similar to $\cos\theta = \sin(\frac{\pi}{2} - \theta)$

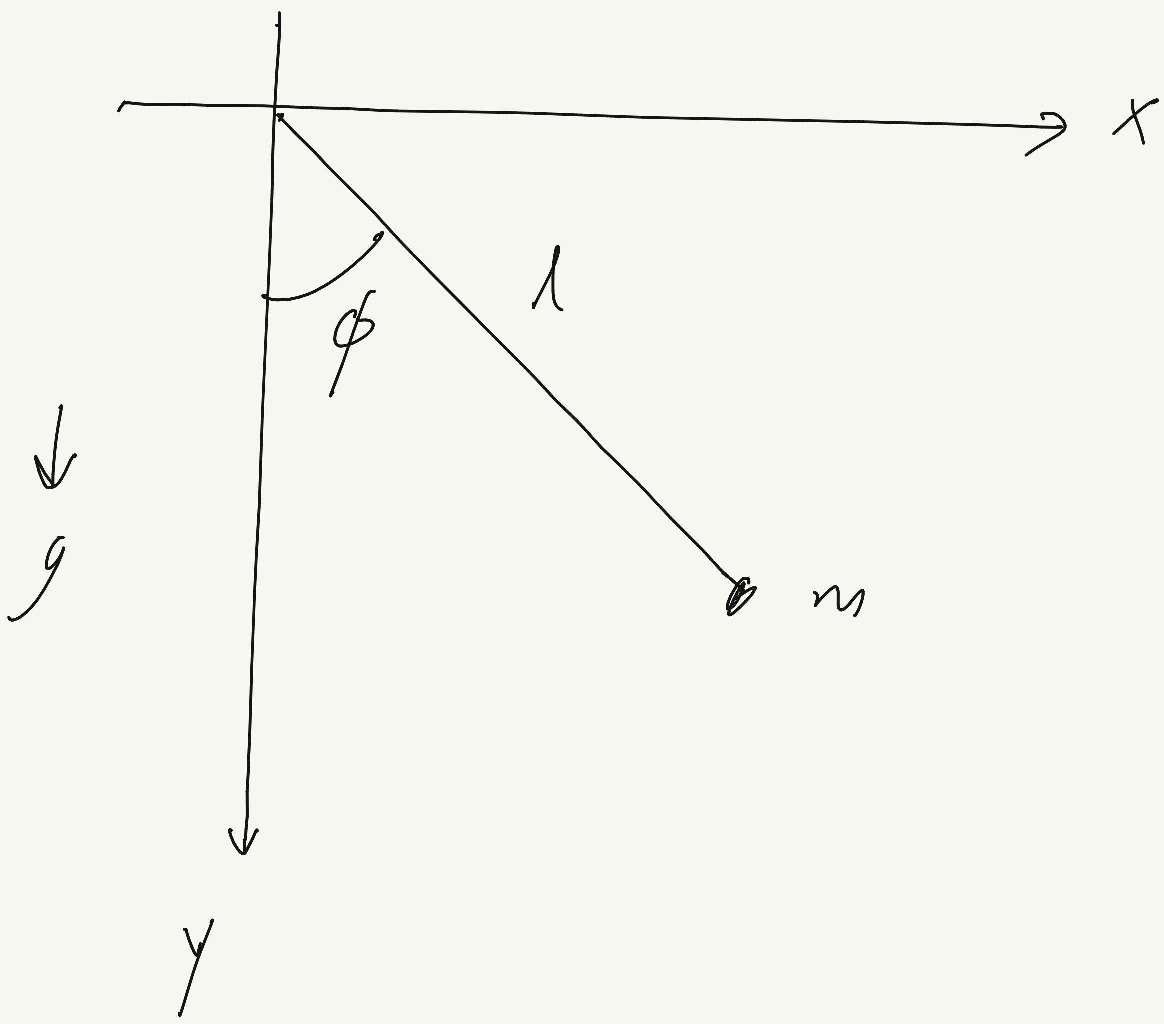
But $x = \sin(\phi/2)/\sin(\phi_0/2) = \frac{1}{k}\sin(\phi/2)$

$$\rightarrow \quad \sin\left(\frac{\phi}{2}\right) = k\, cn(w_0 t; k)$$

$$\boxed{\phi(t) = 2 \arcsin\left[k\ cn(w_0 t; k)\right]}$$

where $k = \sin\left(\frac{\phi_0}{2}\right)$

Lagrange multipliers:



constraint:

$$l = r = \sqrt{x^2+y^2}$$

$$U = -mgy$$
$$= -mgr\cos\phi$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$
$$= \frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2)$$

$$\varphi(r,\phi) \equiv l - r = 0$$

$$L = T - U + \lambda\varphi$$
$$= \frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2) + mgr\cos\phi + \lambda(l-r)$$

EOMs:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \qquad (2)$$

$$\varphi = 0 \qquad (3)$$

$$\frac{d}{dt}(m\dot{r}) = mr\dot{\phi}^2 + mg\cos\phi - \lambda$$

$$\boxed{\ddot{r} = r\dot{\phi}^2 + g\cos\phi - \frac{\lambda}{m}} \qquad (1)$$

$$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = -mgr\sin\phi$$

$$2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} = -mgr\sin\phi$$

$$\boxed{\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = -\frac{g}{r}\sin\phi} \qquad (2)$$

$$\boxed{l - r = 0} \qquad (3)$$

Differentiate constraint twice w.r.t time:

$\rightarrow \dot{r} = 0 \,,\quad \ddot{r} = 0$

Substitute for $\ddot{r}$ using (1):

$$0 = \ddot{r} = r\dot{\phi}^2 + g\cos\phi - \frac{\lambda}{m}$$

$$\rightarrow \lambda = mr\dot{\phi}^2 + mg\cos\phi$$

$$= ml\dot{\phi}^2 + mg\cos\phi$$

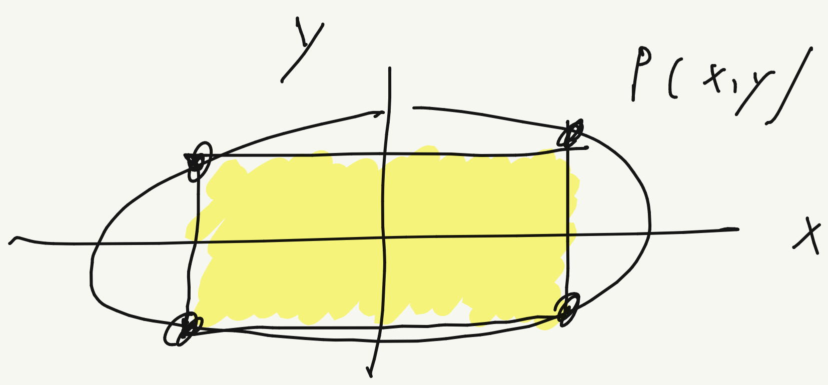
<u>Compare with freshman physics calculation</u>

$$F_c = mg\cos\phi + ml\dot{\phi}^2 \qquad (\text{in } -\hat{r} \text{ direction})$$

Thus,

$$\vec{F}_c = \lambda\vec{\nabla}\varphi \qquad (\text{since } \varphi = l - r \rightarrow \vec{\nabla}\varphi = -\hat{r})$$

Another example:



Suppose you want to maximize the area of a rectangle subject to the constraint that its corners lie on an ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ (with a, b given).

Two methods:

Area $= 4x \cdot y$

Constraint: $\varphi(x,y) = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$

1. Define:

$$F(x,y,\lambda) = 4x \cdot y - \lambda \varphi(x,y)$$
$$= 4xy - \lambda\left(1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right)$$

require: $\frac{\partial F}{\partial x} = 0 \quad (1)$

$\frac{\partial F}{\partial y} = 0 \quad (2)$

$\frac{\partial F}{\partial \lambda} = 0 \quad (3)$

(1) $0 = 4y + \frac{2\lambda x}{a^2}$

(2) $0 = 4x + \frac{2\lambda y}{b^2}$

(3) $0 = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$

3 equations, 3 unknowns (x, y, λ)

$$0 = \frac{4y^2}{b^2} + \frac{2\lambda xy}{a^2b^2}$$

$$0 = \frac{4x^2}{a^2} + \frac{2\lambda yx}{b^2a^2}$$

<u>subtract:</u> $0 = 4\left[\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right]$

$$\rightarrow \quad \frac{x}{a} = \pm\frac{y}{b}$$

<u>substitute into (3):</u> $0 = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^2$

$$2\left(\frac{x}{a}\right)^2 = 1$$

$$\frac{x}{a} = \frac{1}{\sqrt{2}}$$

$$\boxed{x = \frac{a}{\sqrt{2}}} \rightarrow \boxed{y = \frac{b}{\sqrt{2}}}$$

② <u>Reduced space method</u>

$F(x) = 4xy \;\Big|\; y = b\sqrt{1-\left(\frac{x}{a}\right)^2}$

$$= 4xb\sqrt{1-\left(\frac{x}{a}\right)^2}$$

Solve $F'(x) = 0$

$$0 = F'(x)$$

$$= 4b\sqrt{1-\left(\frac{x}{a}\right)^2} + 4xb\,\frac{1}{\cancel{2}}\,\frac{1}{\sqrt{\quad}}\left(-\frac{\cancel{2}x}{a^2}\right)$$

$$= 4b\left(\sqrt{1-\left(\frac{x}{a}\right)^2} - \left(\frac{x}{a}\right)^2\frac{1}{\sqrt{\quad}}\right)$$

$$= \frac{4b}{\sqrt{\quad}}\left[1-\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^2\right]$$

$$= \frac{4b}{\sqrt{\quad}}\left[1 - 2\left(\frac{x}{a}\right)^2\right]$$

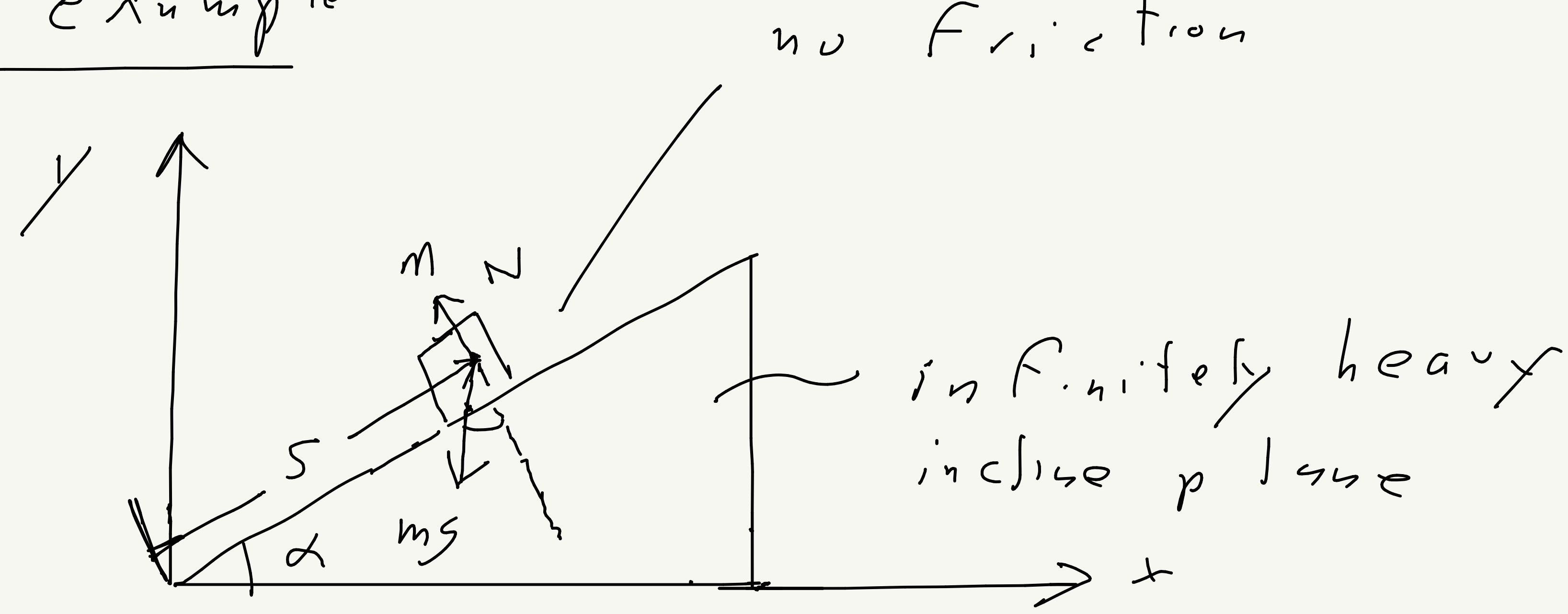
$$\to \left(\frac{x}{a}\right)^2 = \frac{1}{2} \to \boxed{x = \frac{a}{\sqrt{2}}}$$

$$\to \boxed{y} = b\sqrt{1-\left(\frac{x}{a}\right)^2} \quad \Bigg| \ \frac{x}{a} = \frac{1}{\sqrt{2}}$$

$$\boxed{= \frac{b}{\sqrt{2}}}$$

(same result as Lagrange multiplier method)

Another example



$$X = s\cos\alpha \qquad m\ddot{s} = -mg\sin\alpha$$
$$Y = s\sin\alpha \qquad \boxed{\ddot{s} = -g\sin\alpha} \quad \text{EOM}$$

$$N - mg\cos\alpha = 0$$

$$\rightarrow \boxed{N = mg\cos\alpha} \quad \text{constraint force (normal force)}$$

Using Lagrange multipliers:

(x, y) degrees of freedom

Constraint: $\dfrac{y}{x} = \tan\alpha \rightarrow \varphi \equiv y - x\tan\alpha = 0$

Lagrangian:

$$L = T - U$$
$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

Enforce constraint

$$L' = L + \lambda\varphi(x, y)$$
$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(y - x\tan\alpha)$$

EOMs:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} \qquad (2)$$

$$\varphi(x,y) = 0 \qquad (3)$$

(1) $m\ddot{x} = -\lambda \tan\alpha$

$\rightarrow \ddot{x} = -\frac{\lambda}{m}\tan\alpha$

(2) $m\ddot{y} = -mg + \lambda$

$\rightarrow \ddot{y} = -g + \frac{\lambda}{m}$

(3) $y - x\tan\alpha = 0$

$y = x\tan\alpha$

Differentiate contraint twice wrt t:

$$\ddot{y} = \ddot{x}\tan\alpha$$

Substitute from (1) and (2):

$$-g + \frac{\lambda}{m} = -\frac{\lambda}{m}\tan\alpha \cdot \tan\alpha$$

$$\frac{\lambda}{m}\left(1 + \tan^2\alpha\right) = g$$

$s^2 + c^2 = 1$
$1 + \tan^2 = \sec^2\alpha$

$$\lambda = \frac{mg}{\sec^2\alpha} = \boxed{mg\cos^2\alpha}$$

Constraint force:

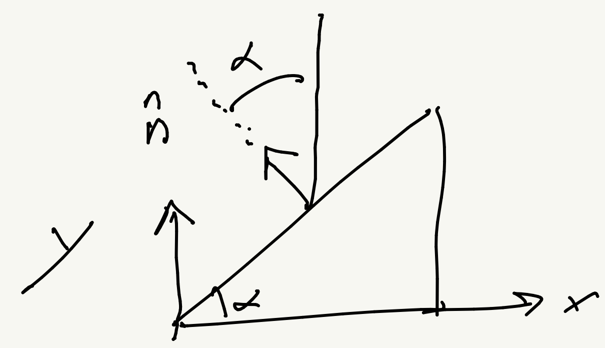
$$\vec{F}_c = \lambda \vec{\nabla} \phi$$

$$= mg\cos^2\alpha \left(\hat{y} - \hat{x} \tan\alpha \right)$$

$$= mg\cos\alpha \left(\cos\alpha\, \hat{y} - \sin\alpha\, \hat{x} \right)$$

$$= mg\cos\alpha\, \hat{n}$$

$$= N\hat{n}$$

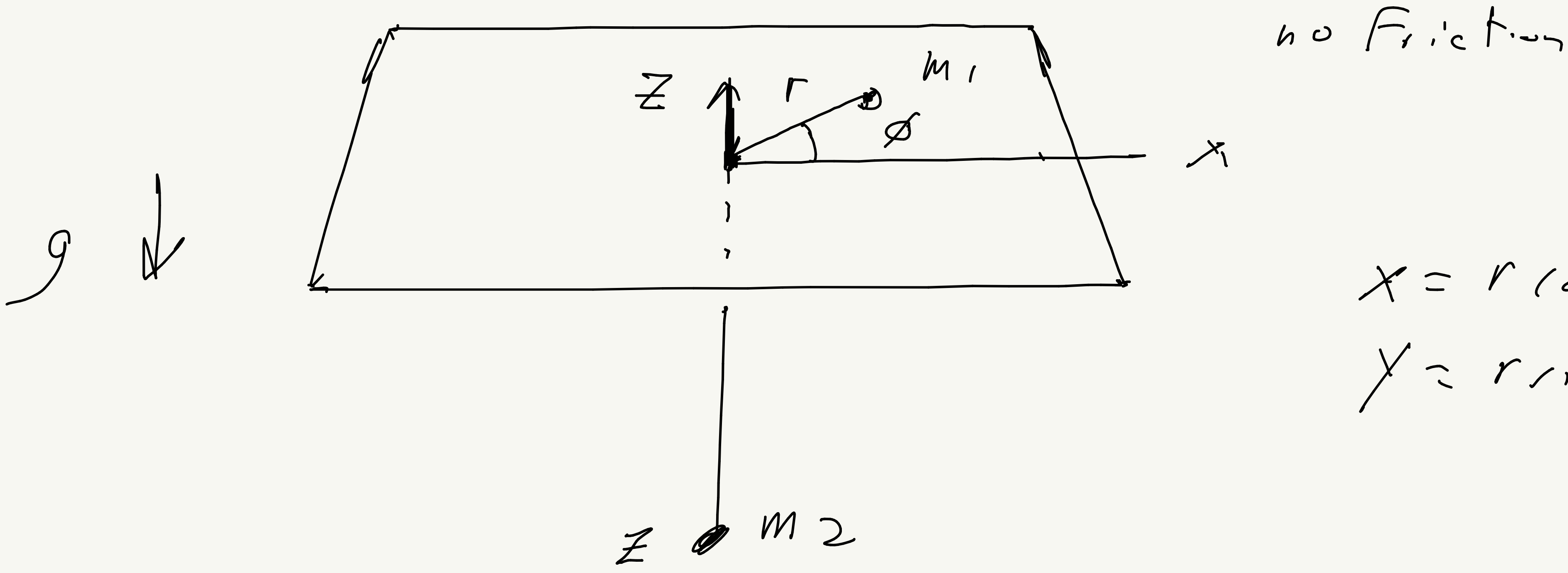


where $\hat{n} = -\sin\alpha\, \hat{x} + \cos\alpha\, \hat{y}$

is unit vector $\perp$ to surface of

incline plane

Conservation of momentum example



$x = r\cos\phi$
$y = r\sin\phi$

Constraint: $r - z = l$

$$T = \frac{1}{2} m_1 (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_2 \dot{z}^2$$

$$= \frac{1}{2} m_1 (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2$$

$$U = m_2 g z = m_2 g (r - l) = m_2 g r + \underbrace{\text{const}}_{\text{ignore}}$$

$$L = T - U$$

$$= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r$$

No t-dependence:

$$E = T + U = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 + m_2 g r$$

$$= \text{const}$$

No ϕ-dependence

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} = \text{const} \equiv M_z$$

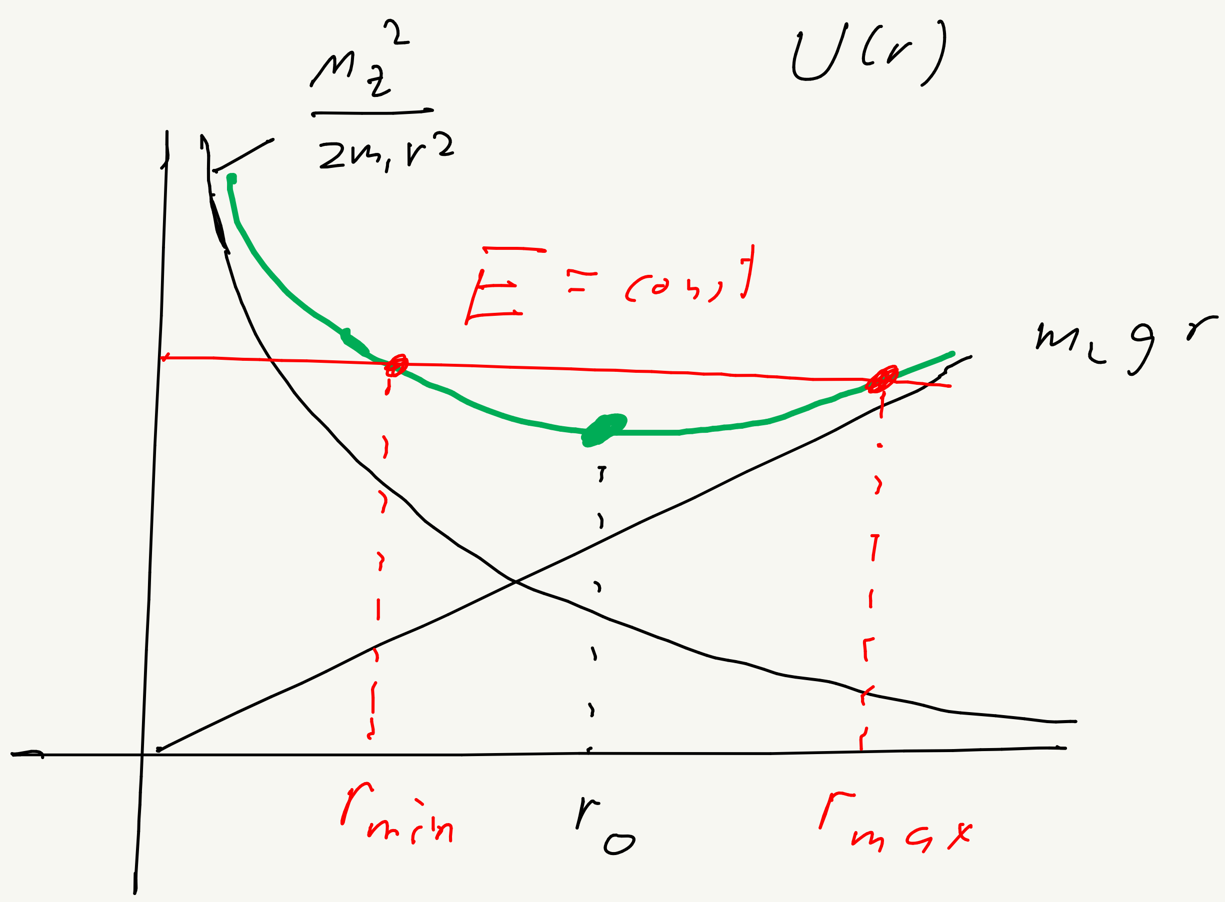
Then, $\dot{\phi} = \dfrac{M_z}{m_1 r^2}$

$$E = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \left(\frac{M_z^2}{m_1^2 r^4}\right) + m_2 g r$$

$$= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{M_z^2}{2m_1 r^2} + m_2 g r$$

$$= \frac{1}{2}(m_1+m_2)\dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = \frac{M_z^2}{2m_1 r^2} + \underbrace{m_2 g r}_{U(r)}$$



r_{min}, r_{max} : turning points

$E = U_{eff}(r_0) \rightarrow$ stable circular orbit $(r = r_0)$

$E \geq U_{eff}(r_0)$

$$0 = \frac{dU_{eff}}{dr}\Big|_{r=r_0}$$

$$= \left(-\frac{M_z^2}{m_1 r^3} + m_2 g\right)\Big|_{r=r_0}$$

$$= \frac{-M_z^2}{m_1 r_0^3} + m_2 g$$

$$M_z^2 = m_1 m_2 g r_0^3$$

so $\boxed{M_z = \sqrt{m_1 m_2 g r_0^3}}$

angular momentum ~~necessary~~ to give ~~you~~ a circular orbit at $r = r_0$

$$U_{eff,min} = U_{eff}(r_0)$$

$$= \left(\frac{M_z^2}{2 m_1 r_0^2} + m_2 g r_0\right)\Big|_{M_z = \sqrt{m_1 m_2 g r_0^3}}$$

$$= \frac{\cancel{m_1} m_2 g r_0^{\cancel{3}}}{2 \cancel{m_1} \cancel{r_0^2}} + m_2 g r_0$$

$$= \frac{3}{2} m_2 g r_0$$

EOM's:

$$E = \frac{1}{2}(m_1+m_2)\dot{r}^2 + U_{eff}(r)$$

$$= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{M_z^2}{2m_1 r^2} + m_2 g r$$

$$\rightarrow \frac{2}{(m_1+m_2)}\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right) = \dot{r}^2$$

$$\frac{dr}{dt} = \pm\sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}$$

$$\boxed{\int \pm\, dt = \int \frac{dr}{\sqrt{\frac{2}{m_1+m_2}\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}}}$$

Integrate to get $t = t(r)$

Orbit equation:

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M_z}{m_1 r^2}$$

Thus,
$$\frac{dr}{d\phi}\frac{M_z}{m_1 r^2} = \pm\sqrt{\quad}$$

$$\frac{dr}{r^2} = \pm\frac{m_1}{M_z}\sqrt{\quad}\, d\phi$$

$$\boxed{\int \pm\, d\phi = \int \frac{dr/r^2}{\frac{m_1}{M_z}\sqrt{\frac{2}{m_1+m_2}\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}}}$$

Can try to evaluate these integrals analytically making substitutions of the type

$$u = \frac{1}{r} \rightarrow du = -\frac{1}{r^2}\, dr \quad , \text{etc.}$$

<u>Numerically</u>!

$$dr = \pm\sqrt{\circledast}\, dt, \qquad \circledast = \frac{2}{m_1+m_2}\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)$$

$$d\phi = \frac{M_z}{m_1 r^2}\, dt$$

So start the system at time $t = t_0$ with some values of r and ϕ for given E, M_z

<u>$t = t_0$</u>: $\quad r(t_0)\ ,\ \phi(t_0)$

<u>$t = t_0 + \Delta t$</u>: $\quad r(t_0) + \Delta r\ ,\ \phi(t_0) + \Delta\phi$

where $\Delta r = \pm\sqrt{\circledast}\,\Delta t\ ,\quad \Delta\phi = \dfrac{M_z}{m_1 r(t_0)}\,\Delta t$

$\vdots$

(repeat)

Virial theorem:

Assume potential energy is a homogeneous function of degree k:

$$U(\alpha \vec{r}_1, \alpha \vec{r}_2, \dots) = \alpha^k \, U(\vec{r}_1, \vec{r}_2, \dots)$$

Then:

$$\sum_a \frac{\partial U}{\partial \vec{r}_a} \cdot \vec{r}_a = k U$$

NOTE:

$$\sum_a \frac{\partial T}{\partial \vec{v}_a} \cdot \vec{v}_a = 2T = \sum_a \frac{\partial L}{\partial \vec{v}_a} \cdot \vec{v}_a$$

$$= \sum_a \vec{p}_a \cdot \vec{v}_a$$

$U = mgy \rightarrow k = 1$

$U = -\dfrac{G m_1 m_2}{r} \rightarrow k = -1$

$U = \frac{1}{2} k x^2 \rightarrow k = 2$

Thus,

$$k U = \sum_a \frac{\partial U}{\partial \vec{r}_a} \cdot \vec{r}_a$$

$$= -\sum_a \frac{\partial L}{\partial \vec{r}_a} \cdot \vec{r}_a$$

$$= -\sum_a \frac{d}{dt}\Bigg(\underbrace{\frac{\partial L}{\partial \vec{v}_a}}_{\vec{p}_a}\Bigg) \cdot \vec{r}_a$$

$$= -\frac{d}{dt}\left(\sum_a \vec{p}_a \cdot \vec{r}_a\right) + \sum_a \vec{p}_a \cdot \vec{v}_a$$

Now:

$$\sum_a \vec{p}_a \cdot \vec{v}_a = \sum_a m_a |\vec{v}_a|^2 = 2T$$

Thus, $$\boxed{k U = -\frac{d}{dt}\left(\sum_a \vec{p}_a \cdot \vec{r}_a\right) + 2T}$$

Also assume that the motion is bounded, so

$\sum_a \vec{p}_a \cdot \vec{r}_a$ = finite for all t.

Then if we take the time average:

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} dt\, f(t) = \bar{f}$$

we find:

$$k \bar{U} = \cancel{-\lim_{\tau \to \infty} \frac{1}{\tau}\left(\underbrace{\sum_a \vec{p}_a \cdot \vec{r}_a \Big|_{t=\tau}}_{\text{finite}} - \underbrace{\sum_a \vec{p}_a \cdot \vec{r}_a \Big|_{t=0}}_{\text{finite}} \right)} \to 0$$

$$+ 2\bar{T}$$

$$= 2\bar{T}$$

so $\boxed{k\bar{U} = 2\bar{T}}$

Hamiltonian mechanics:

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \quad \left(= \text{const if } L \text{ does not depend explicitly on } t \right)$$

$$= E(q, \dot{q}, t)$$

Proof: suppose $\frac{\partial L}{\partial t} = 0$

$$\rightarrow \quad \frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right)$$

$$= \sum_i \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right]$$

$$= \sum_i \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right]$$

$$= \frac{d}{dt} \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right)$$

Thus, $$\frac{d}{dt} \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right) = 0$$

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \quad = \text{const}$$

Hamiltonian: $H = \left(\sum p_i \dot{q}_i - L \right) \Big|_{\dot{q} = \dot{q}(q, p)}$

Hamiltonian is a function of (q_i, p_i, t)

EOMs:

$$dL = \sum_i \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right) + \frac{\partial L}{\partial t}$$

$$= \sum_i \left(\dot{p}_i \, dq_i + p_i \, d\dot{q}_i \right) + \frac{\partial L}{\partial t}$$

$$= \sum_i \left(\dot{p}_i \, dq_i + d(p_i \dot{q}_i) - dp_i \, \dot{q}_i \right) + \frac{\partial L}{\partial t}$$

$$d\left(L - \sum_i p_i \dot{q}_i\right) = \sum_i \left(\dot{p}_i \, dq_i - \dot{q}_i \, dp_i \right) + \frac{\partial L}{\partial t}$$

$$d\Big(\underbrace{\sum_i p_i \dot{q}_i - L}_{H}\Big) = \sum_i \left(-\dot{p}_i \, dq_i + \dot{q}_i \, dp_i \right) - \frac{\partial L}{\partial t}$$

$$\rightarrow \boxed{\underbrace{\frac{\partial H}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial p_i} = \dot{q}_i}_{\text{Hamilton's equations } 2s \text{ } 1^{st} \text{ order equation}}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

Conservation of energy

Example:

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = p/m$$

$$\rightarrow H = \left(p\dot{x} - L \right)\Big|_{\dot{x} = p/m}$$

$$= p^2/m - \left(\frac{1}{2} m \left(\frac{p}{m}\right)^2 + U(x) \right)$$

$$= \frac{p^2}{2m} + U(x)$$

$$\left.\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} = p/m \\ \dot{p} &= -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x} \end{aligned}\right\}$$

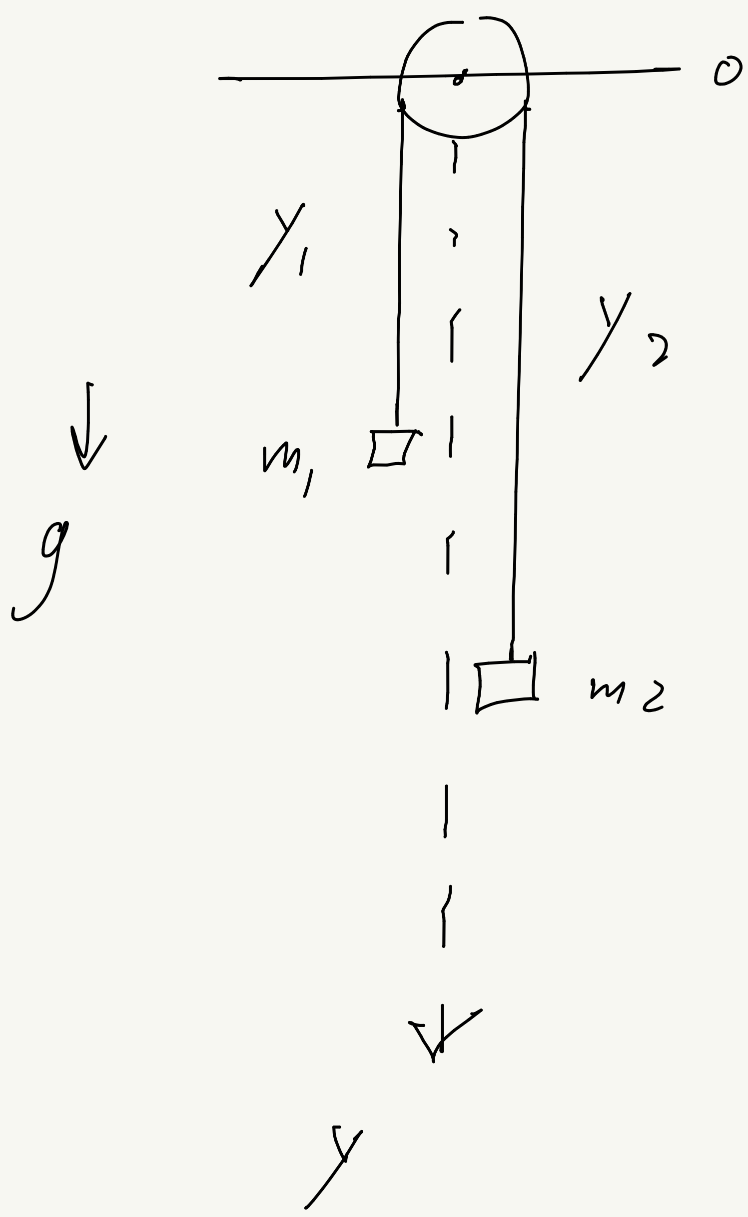
combine:

$$p = m\dot{x}$$

$$\rightarrow \dot{p} = m\ddot{x}$$

$$\rightarrow m\ddot{x} = -\frac{\partial U}{\partial x}$$

Atwood machine



$$y_1 + y_2 = l \longrightarrow y_2 = l - y_1$$

$$U = -m_1 g y_1 - m_2 g y_2$$

$$= -m_1 g y_1 - m_2 g (l - y_1)$$

$$= -(m_1 - m_2) g y_1 \underbrace{- m_2 g l}_{\text{const, ignore}}$$

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2$$

$$= \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_1^2$$

$$= \frac{1}{2} (m_1 + m_2) \dot{y}_1^2$$

$$L = T - U$$

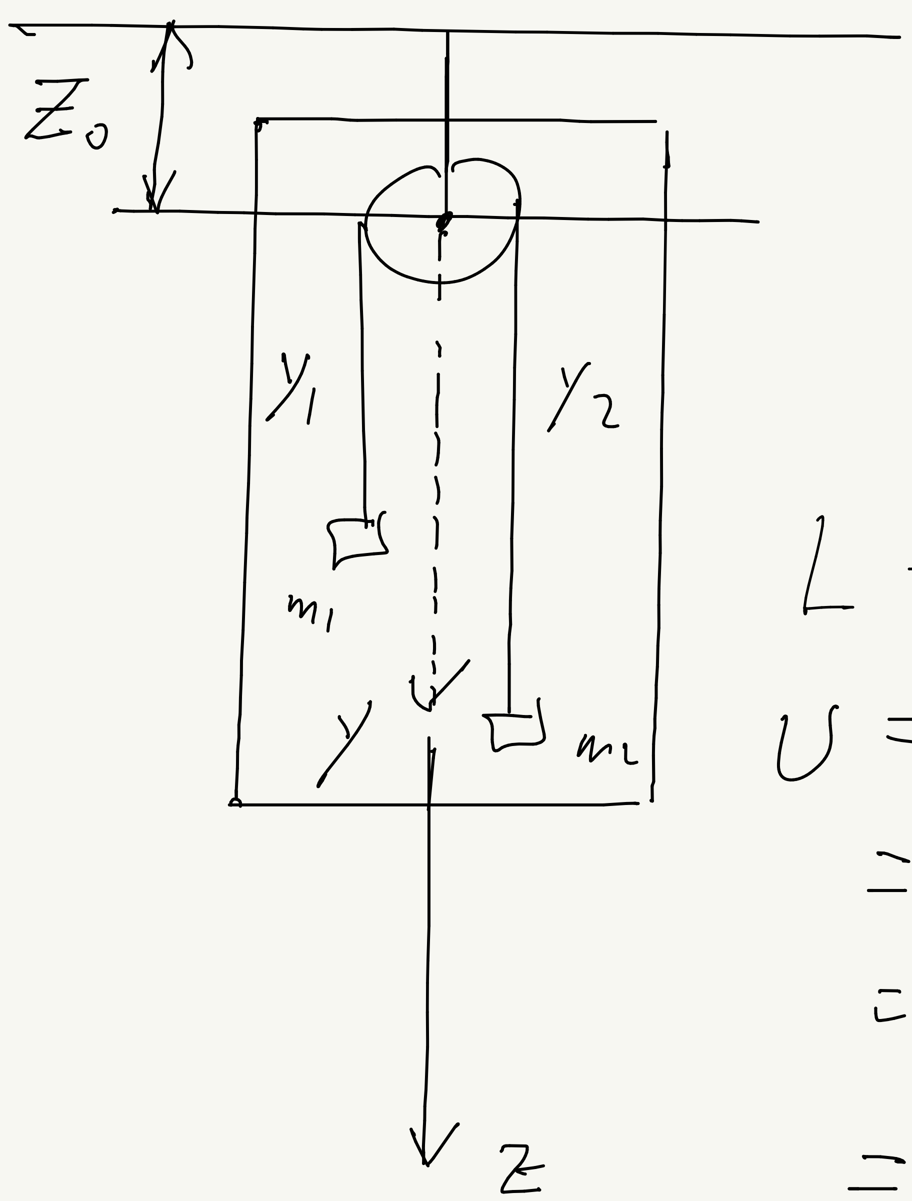
$$= \frac{1}{2} (m_1 + m_2) \dot{y}_1^2 + (m_1 - m_2) g y_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) = \frac{\partial L}{\partial y_1}$$

$$(m_1 + m_2) \ddot{y}_1 = (m_1 - m_2) g$$

$$\boxed{\ddot{y}_1 = \frac{(m_1 - m_2)}{(m_1 + m_2)} g}$$

What happens if the Atwood machine is in an accelerating reference frame?



$$Z_0 = \frac{1}{2} a t^2$$

$$Z_1 = Z_0 + y_1$$

$$Z_2 = Z_0 + y_2$$

$L = T - U$ (w.r.t. inertial frame)

$$U = -m_1 g Z_1 - m_2 g Z_2$$

$$= -m_1 g y_1 - m_2 g y_2 \underbrace{-(m_1+m_2) g Z_0}_{\text{prescribed fcn of time (ignore)}}$$

$$= -(m_1 - m_2) g y_1 \underbrace{- m_2 g l}_{\text{const (ignore)}}$$

$$= -(m_1 - m_2) g y_1$$

$y_1 + y_2 = l$

$$T = \frac{1}{2}\left(m_1 \dot{Z}_1^2 + m_2 \dot{Z}_2^2\right)$$

$$= \frac{1}{2} m_1 \left(a^2 t^2 + \dot{y}_1^2 + 2at\dot{y}_1\right)$$

$$+ \frac{1}{2} m_2 \left(a^2 t^2 + \dot{y}_2^2 + 2at\dot{y}_2\right)$$

$$\dot{Z}_1 = \dot{Z}_0 + \dot{y}_1 = at + \dot{y}_1$$

$$\dot{Z}_2 = \dot{Z}_0 + \dot{y}_2 = at + \dot{y}_2$$

$$= \underbrace{\frac{1}{2}(m_1+m_2) a^2 t^2}_{\text{prescribed function of } t \text{ (ignore)}} + \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 + at\left(m_1 \dot{y}_1 + m_2 \dot{y}_2\right)$$

Now: $a t\,(m_1 \dot{y}_1 + m_2 \dot{y}_2) = \frac{d}{dt}\left[a t (m_1 y_1 + m_2 y_2)\right] - a(m_1 y_1 + m_2 y_2)$

thus,

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m \dot{y}_2^2 - a(m_1 y_1 + m_2 y_2) + \underbrace{\text{total time derivative}}_{\text{(ignore)}}$$

$$= \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - a(m_1 y_1 + m_2(l - y_1))$$

$$= \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - a(m_1 - m_2) y_1 \underbrace{- a m_2 l}_{\text{ignore}}$$

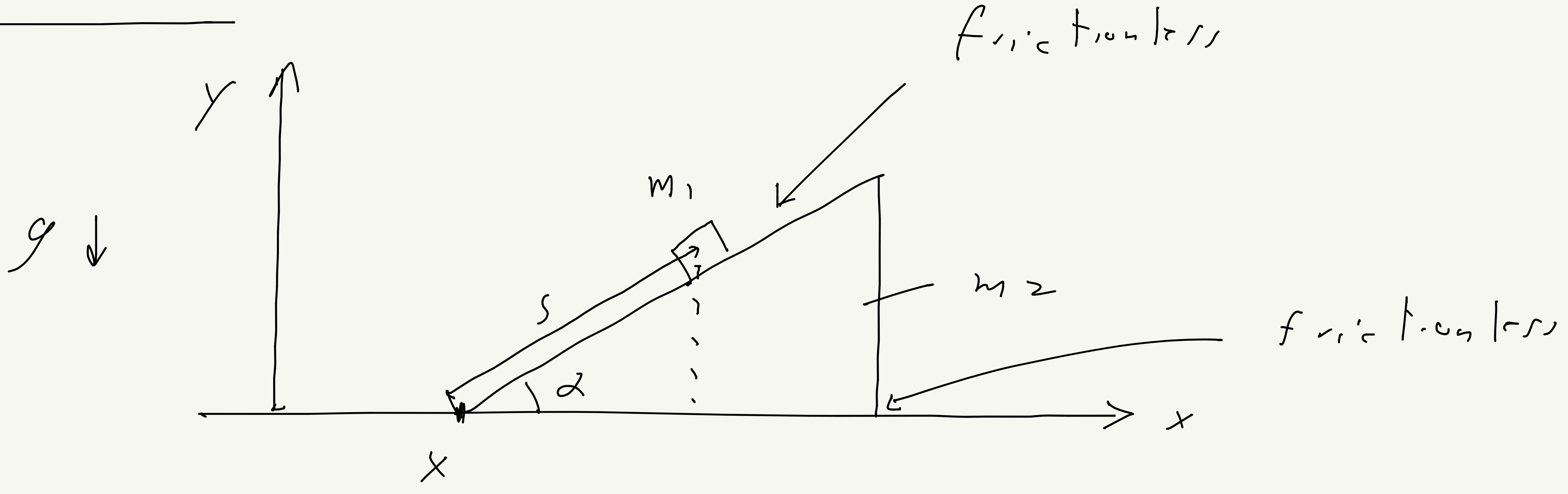
$$L = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - a(m_1 - m_2) y_1 + (m_1 - m_2) g y_1$$

$$= \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 + (m_1 - m_2)(g - a) y_1$$

$$\rightarrow \boxed{\ddot{y}_1 = \frac{(m_1 - m_2)(g - a)}{m_1 + m_2}}$$

So the acceleration of gravity g got modified by the acceleration of the non-inertial reference frame

Quiz #2:



$x_2 = x$
$y_2 = 0$

$U = m_1 g y$
$= m_1 g s \sin\alpha$

$x_1 = x + s\cos\alpha$
$y_1 = s\sin\alpha$

$\dot{x}_1 = \dot{x} + \dot{s}\cos\alpha$
$\dot{y}_1 = \dot{s}\sin\alpha$

$\dot{x}_2 = \dot{x}$
$\dot{y}_2 = 0$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_1 (\dot{x}^2 + \underline{\dot{s}^2 \cos^2\alpha} + 2\dot{x}\dot{s}\cos\alpha + \underline{\dot{s}^2 \sin^2\alpha})$$

$$+ \frac{1}{2} m_2 \dot{x}^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1 \dot{x}\dot{s}\cos\alpha$$

$$\boxed{\begin{aligned} L &= T - U \\ &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1 \dot{x}\dot{s}\cos\alpha - m_1 g s \sin\alpha \end{aligned}}$$

i) L does not depend explicitly on t

$\rightarrow \quad E = T + U = \text{const}$

ii) L does not depend on x

$$\rightarrow \quad p_x \equiv \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} + m_1 \dot{s} \cos\alpha$$

$$= \text{const}.$$

Thus, can choose a reference frame such that $p_x = 0$ and $x_{com} = 0$

$$x_{com} = m_1 x_1 + m_2 x_2$$

$$= m_1 (x + s\cos\alpha) + m_2 x$$

$$= (m_1 + m_2) x + m_1 s \cos\alpha$$

$$x_{com} = 0 \quad \rightarrow \quad \boxed{x = -\left(\frac{m_1}{m_1 + m_2}\right) s\cos\alpha}$$

Central force motion:

Reduction of 2-body problem

$$L = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 - U(|\vec{r}_1 - \vec{r}_2|)$$

i) No explicit time dependence $\rightarrow$ $E = T + U$ conserved

ii) Translationally invariant ($\vec{r}_a \rightarrow \vec{r}_a + \delta\vec{x}$) $\rightarrow$ total momentum $\vec{P} \equiv \sum_a m_a \vec{v}_a$ = constant

iii) Rotationally invariant ($\vec{r}_a \rightarrow \vec{r}_a + \delta\vec{\phi} \times \vec{r}_a$
$\vec{v}_a \rightarrow \vec{v}_a + \delta\vec{\phi} \times \vec{v}_a$)

$\rightarrow$ total angular momentum conserved $\vec{M} \equiv \sum_a \vec{r}_a \times \vec{p}_a$

Using $\vec{P}$ = const $\rightarrow$ can go to COM frame where

$$\sum_a m_a \vec{r}_a = 0 \iff m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

Define: (relative position vector)

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2$$

Then $U = U(r)$, $r = |\vec{r}| = |\vec{r}_1 - \vec{r}_2|$

and
$$0 = m_1 \vec{r}_1 + m_2 \vec{r}_2$$
$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$\rightarrow \vec{r}_2 = \vec{r}_1 - \vec{r}$ $\rightarrow$ $0 = m_1 \vec{r}_1 + m_2 (\vec{r}_1 - \vec{r})$
$= (m_1 + m_2)\vec{r}_1 - m_2 \vec{r}$

so $$\boxed{\vec{r}_1 = \frac{m_2 \vec{r}}{m_1 + m_2}}$$

$$\vec{r}_2 = \vec{r}_1 - \vec{r}$$
$$= \frac{m_2}{m_1+m_2}\vec{r} - \vec{r}$$
$$= \left(\frac{-m_1}{m_1+m_2}\right)\vec{r}$$

so $\boxed{\vec{r}_2 = \frac{-m_1}{m_1+m_2}\vec{r}}$

Thus,

$$T = \frac{1}{2}m_1|\dot{\vec{r}}_1|^2 + \frac{1}{2}m_2|\dot{\vec{r}}_2|^2$$
$$= \frac{1}{2}m_1\left(\frac{m_2}{m_1+m_2}\right)^2|\dot{\vec{r}}|^2 + \frac{1}{2}m_2\left(\frac{-m_1}{m_1+m_2}\right)^2|\dot{\vec{r}}|^2$$
$$= \frac{1}{2}\frac{m_1m_2}{(m_1+m_2)^{\cancel{2}}}\cancel{[m_1+m_2]}|\dot{\vec{r}}|^2$$
$$= \frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)|\dot{\vec{r}}|^2$$
$$= \frac{1}{2}m|\dot{\vec{r}}|^2$$

where $m \equiv \left(\frac{m_1m_2}{m_1+m_2}\right) \equiv$ reduced mass

$$\rightarrow \boxed{L = \frac{1}{2}m|\dot{\vec{r}}|^2 - U(r)}$$

Using $\vec{M}$ = const $\longrightarrow$ take $\vec{z}$ along $\vec{M}$.

Since

$$\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a$$

$$= m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2$$

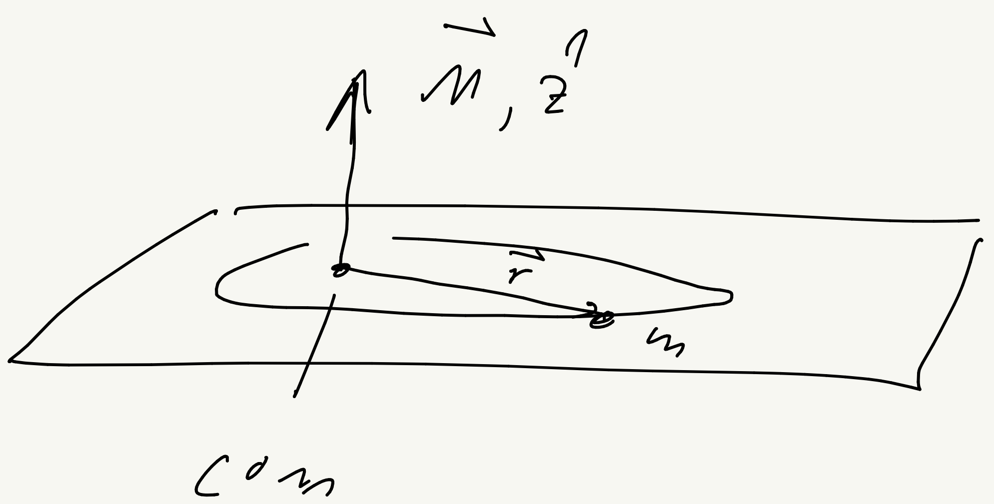
$$= m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 \vec{r} \times \dot{\vec{r}} + m_2 \left(\frac{-m_1}{m_1+m_2}\right)^2 \vec{r} \times \dot{\vec{r}}$$

$$= \frac{m_1 m_2}{(m_1+m_2)^{\cancel{2}}} \cancel{[m_2 + m_1]} \, \vec{r} \times \dot{\vec{r}}$$

$$= m \, \vec{r} \times \dot{\vec{r}}$$

we have $\vec{r}, \dot{\vec{r}} \perp$ to $\vec{M}$.

$\longrightarrow$ so motion is in xy plane



$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2\right) - U(r)$$

But, not only is direction of $\vec{M}$ constant, so is its magnitude! $M \equiv M_z$

$$\rightarrow \quad p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi} = \text{const} \quad \left(\text{since } \frac{\partial L}{\partial \phi} = 0\right)$$

$$\equiv M_z$$

Integral equations:

Cons. of energy

$$E = T + U = \text{const}$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + U(r)$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \left(\frac{M_z}{mr^2}\right)^2\right) + U(r)$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{M_z^2}{2mr^2} + U(r)$$

$$= \frac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = U(r) + \frac{M_z^2}{2mr^2}$$

Motion in 1-d subject to $U = U_{eff}(r)$

$$\frac{dr}{dt} = \dot{r} = \pm\sqrt{\frac{2}{m}(E - U_{eff}(r))}$$

$$dt = \pm \frac{dr}{\sqrt{\frac{2}{m}(E - U_{eff}(r))}}$$

(14.6)

$$\boxed{t = \pm \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M_z^2}{m^2 r^2}}} + \text{const}}$$

orbit equation: $\phi = \phi(r)$

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M_z}{mr^2}$$

Thus, $$\frac{dr}{d\phi}\frac{M_z}{mr^2} = \pm\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}$$

$$\rightarrow \quad d\phi = \pm \frac{M_z\,dr/r^2}{m\sqrt{\frac{2}{m}\left(E-U(r)\right) - \frac{M_z^2}{m^2r^2}}}$$

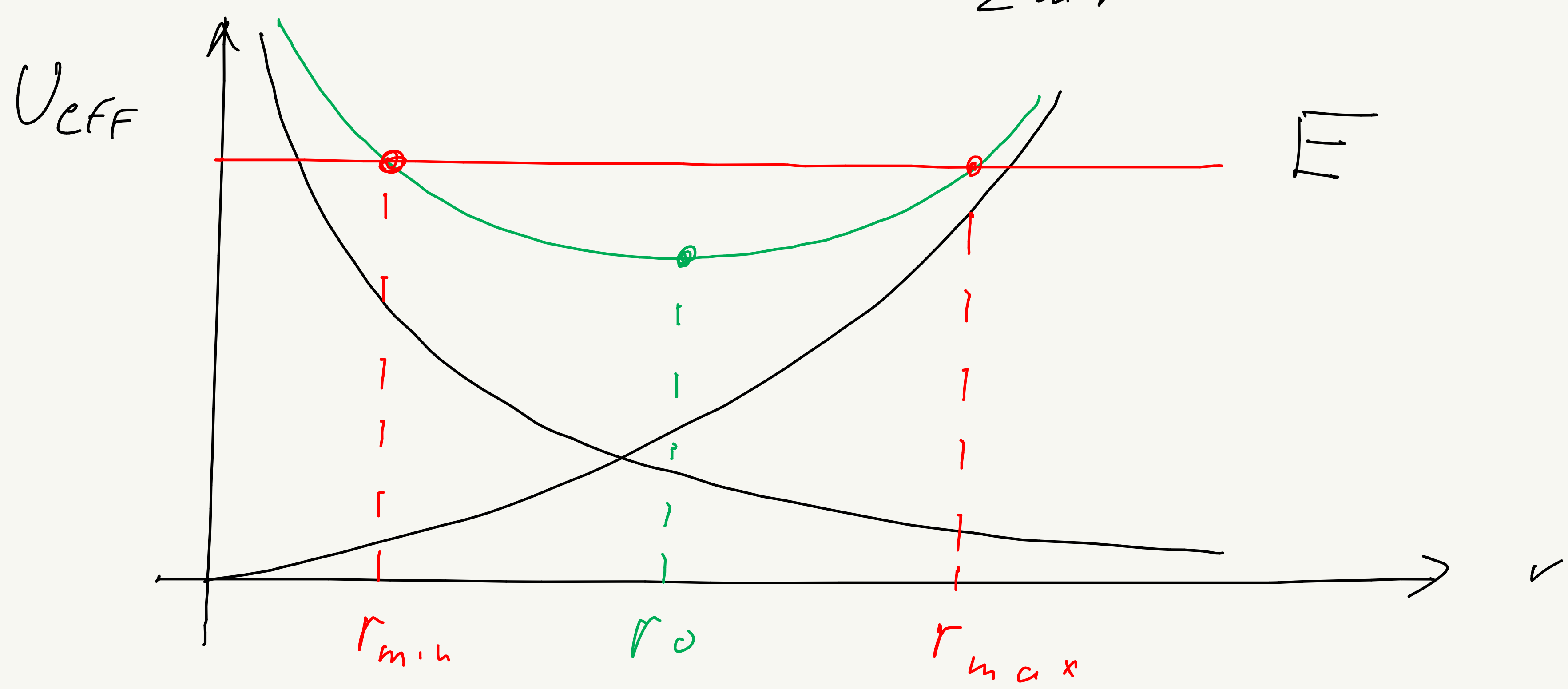
$$= \pm \frac{M_z\,dr/r^2}{\sqrt{2m\left(E-U(r)\right) - M_z^2/r^2}}$$

$$\boxed{\phi = \pm M_z \int \frac{dr/r^2}{\sqrt{2m\left(E-U(r)\right) - M_z^2/r^2}} + \text{const}} \qquad (14.7)$$

Example: space oscillator $U(r) = \frac{1}{2} k r^2$

$$U_{eff}(r) = U(r) + \frac{M_z^2}{2mr^2}$$

$$= \frac{1}{2} k r^2 + \frac{M_z^2}{2mr^2}$$



└ stable circular orbit

$$\phi = M_z \int \frac{dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M_z^2}{r^2}}} + \text{const}$$

$$= M_z \int \frac{dr/r^2}{\sqrt{2mE - mkr^2 - \frac{M_z^2}{r^2}}}$$

Let: $u = 1/r \rightarrow du = -\frac{1}{r^2} dr$

$$\phi = -M_z \int \frac{du}{\sqrt{2mE - \frac{mk}{u^2} - M_z^2 u^2}}$$

$$\phi = -M_z \int \frac{u\,du}{\sqrt{2mEu^2 - mk - M_z^2 u^4}} + \text{const}$$

<u>Let:</u> $v = u^2$, $dv = 2u\,du$

$$\phi = -\frac{M_z}{2} \int \frac{dv}{\sqrt{2mEv - mk - M_z^2 v^2}} + \text{const}$$

<u>complete the square:</u>

$$-M_z^2 v^2 + 2mEv - mk = -M_z^2\left(v^2 - \frac{2mE}{M_z^2} v + \frac{mk}{M_z^2}\right)$$

$$= -M_z^2\left[\left(v - \frac{mE}{M_z^2}\right)^2 - \frac{m^2E^2}{M_z^4} + \frac{mk}{M_z^2}\right]$$

$$= -M_z^2\left[(v-A)^2 - B^2\right]$$

$$A = \frac{mE}{M_z^2}, \quad B^2 = A^2 - \frac{mk}{M_z^2}$$

$$\rightarrow \phi = -\frac{1}{2}\int \frac{dv}{\sqrt{B^2 - (v-A)^2}} + \text{const}$$

Let $v - A = B \sin\theta$

$$\rightarrow \quad dv = B\cos\theta \, d\theta$$

$$\rightarrow \quad \sqrt{B^2 - (v-A)^2} = B\sqrt{1-\sin^2\theta} = B\cos\theta$$

So:

$$\phi = -\frac{1}{2}\int \frac{\cancel{B\cos\theta}\, d\theta}{\cancel{B\cos\theta}} + \text{const}$$

$$= -\frac{1}{2}\theta + \text{const}$$

$$= -\frac{1}{2}\sin^{-1}\left(\frac{v-A}{B}\right) + \text{const}$$

$$= -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right) + \text{const}$$

$v = u^2$

$u = \frac{1}{r}$

$\rightarrow v = \frac{1}{r^2}$

Choose const so that $\phi = 0 \longleftrightarrow r = r_{max}$.

Turning points $r = r_{min}$, r_{max} correspond to zeros of the $\sqrt{}$

$$0 = B^2 - (v-A)^2$$

$$= B^2 - \left(\frac{1}{r^2} - A\right)^2$$

$$\pm B = \frac{1}{r^2} - A$$

$$\rightarrow \quad \frac{1}{r^2} = A \pm B, \qquad \frac{1}{r_{max}^2} = A - B$$

Thus,

$$0 = -\frac{1}{2}\sin^{-1}\left(\frac{\cancel{A}-B-\cancel{A}}{B}\right) + \text{const}$$

$$= -\frac{1}{2}\sin^{-1}(-1) + \text{const}$$

$$= -\frac{1}{2}\left(\frac{-\pi}{2}\right) + \text{const}$$

$$\boxed{\text{Const} = -\frac{\pi}{4}}$$

Thus, $\phi = -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right) - \frac{\pi}{4}$

$$-2\left(\phi + \frac{\pi}{4}\right) = \sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right)$$

$$-\sin\left(2\phi + \frac{\pi}{2}\right) = \frac{\frac{1}{r^2}-A}{B}$$

$$-\left[\sin(2\phi)\cancelto{0}{\cos\frac{\pi}{2}} + \cos(2\phi)\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}\right] = \frac{\frac{1}{r^2}-A}{B}$$

$$-\cos(2\phi) = \frac{\frac{1}{r^2}-A}{B}$$

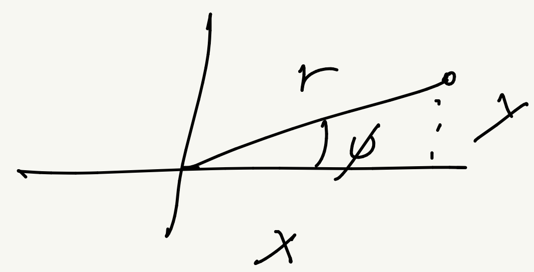
$$\boxed{A - B\cos(2\phi) = \frac{1}{r^2}}$$

actually an ellipse with origin at center

$$\frac{1}{r^2} = A - B\cos(2\phi), \qquad B^2 \equiv A^2 - \frac{mk}{M_z^2}$$

$$A \equiv \frac{mE}{M_z^2}$$

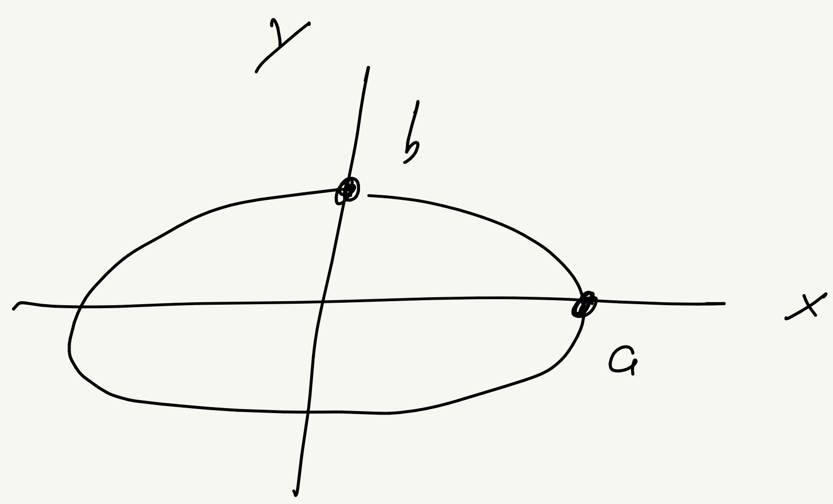
$$1 = Ar^2 - Br^2(\cos^2\phi - \sin^2\phi)$$



$$= A(x^2+y^2) - B(x^2-y^2)$$

$$= (A-B)x^2 + (A+B)y^2$$

$$= \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$



where $a^2 = \dfrac{1}{A-B} = r_{max}^2$

$b^2 = \dfrac{1}{A+B} = r_{min}^2$

Recall: at turning points $E = U_{eff}(r)$

$$E = \frac{1}{2}kr^2 + \frac{M_z^2}{2mr^2}$$

$$\rightarrow \quad E = \frac{1}{2}ka^2 + \frac{M_z^2}{2ma^2} \quad \longleftarrow \text{ multiply by } a^2$$

$$\rightarrow \quad E = \frac{1}{2}kb^2 + \frac{M_z^2}{2mb^2} \quad \longleftarrow \underbrace{\text{multiply by } b^2}_{\text{then subtract}}$$

$$E(a^2-b^2) = \frac{1}{2}k(a^4-b^4)$$

$$= \frac{1}{2}k(a^2-b^2)(a^2+b^2)$$

$$\rightarrow \quad E = \frac{1}{2}k(a^2+b^2)$$

$$\boxed{E = \frac{1}{2}ka^2 + \frac{1}{2}kb^2}$$

<u>NOTE</u>: $$a^2+b^2 = \frac{1}{A-B} + \frac{1}{A+B}$$

$$= \frac{A+B+A-B}{A^2-B^2}$$

$$= \frac{2A}{A^2-B^2}$$

$$= \frac{2\cancel{m}E/\cancel{M_z^2}}{\cancel{m}k/\cancel{M_z^2}}$$

$$= \frac{2E}{k}$$

$$\rightarrow \quad E = \frac{1}{2}k(a^2+b^2)$$

<u>Subtract:</u> $$0 = \frac{1}{2}k(a^2-b^2) + \frac{M_z^2}{2m}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

$$= \frac{1}{2}k(a^2-b^2) + \frac{M_z^2}{2ma^2b^2}(b^2-a^2)$$

$$= \frac{1}{2}(a^2-b^2)\left[k - \frac{M_z^2}{ma^2b^2}\right]$$

$$\rightarrow \boxed{M_z^2 = \kappa m a^2 b^2} \rightarrow \boxed{M_z = \sqrt{\kappa m}\, ab}$$

<u>NOTE</u>:

$$\frac{M_z^2}{\kappa m} = a^2 b^2$$

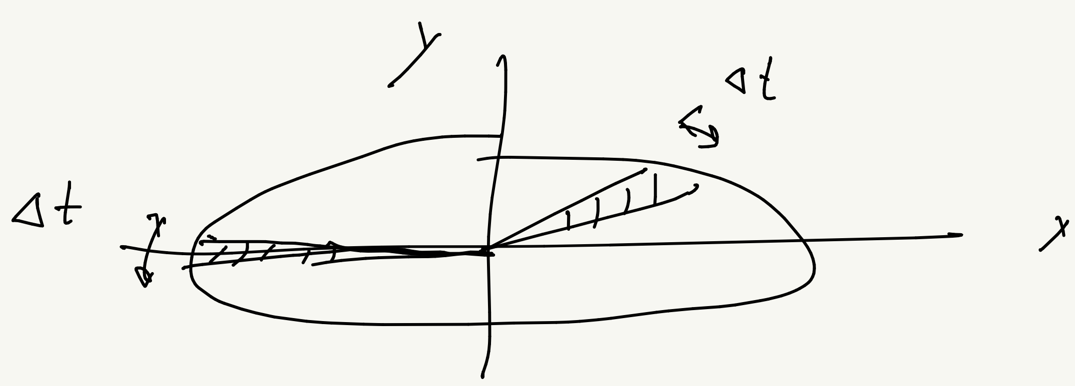
$$= \frac{1}{A-B} \cdot \frac{1}{A+B}$$

$$= \frac{1}{A^2 - B^2}$$

$$= \frac{1}{\frac{m\kappa}{M_z^2}}$$

$$= \frac{M_z^2}{m\kappa} \quad \checkmark$$

<u>All central potentials have equal areas in equal times</u>



$$dA = \frac{1}{2} r (r\, d\phi)$$

$$dA = \frac{1}{2} r^2 d\phi$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt} = \frac{1}{2} \frac{M_z}{m} = \text{const}$$

$$M_z = m r^2 \dot{\phi}$$

Integrate:

$$dA = \frac{1}{2}\frac{M_z}{m}\,dt$$

$$A = \frac{1}{2}\frac{M_z}{m}P$$

$$\pi ab = \frac{1}{2}\frac{M_z}{m}P$$

$$P = 2\pi\, ab\,\frac{m}{M_z}$$

$$= 2\pi\,\frac{M_z}{\sqrt{km}}\,\frac{m}{M_z}$$

$$= \frac{2\pi}{\sqrt{k/m}}$$

$$\equiv \frac{2\pi}{\omega}$$

Thus "Kepler's 3 law" for space oscillator potential:

1) orbits are ellipses with center of ellipse at origin of the potential

2) radius vector sweeps out equal areas in equal time

3) $P = \text{const} = \dfrac{2\pi}{\sqrt{k/m}}$ (versus $\dfrac{P^2}{a^3} = \text{const}$)
independent of size of ellipse.

Time dependence of orbit

$$t = -\int \frac{dr}{\sqrt{\frac{2}{m}(E-U(r)) - \frac{M_z^2}{m^2 r^2}}} + \text{const}$$

$$E = \frac{1}{2}k(a^2+b^2) \quad , \quad M_z^2 = mk a^2 b^2$$

$$U = \frac{1}{2}kr^2$$

$$\rightarrow \sqrt{} = \sqrt{\frac{\cancel{2}}{m}\left(\frac{1}{\cancel{2}}k(a^2+b^2) - \frac{1}{\cancel{2}}kr^2\right) - \frac{mka^2b^2}{m^2r^2}}$$

$$= \sqrt{\frac{k}{m}(a^2+b^2-r^2) - \frac{k}{m}\frac{a^2b^2}{r^2}}$$

$$= \sqrt{\frac{k}{m}}\sqrt{a^2+b^2-r^2 - \frac{a^2b^2}{r^2}}$$

$$= \sqrt{\frac{k}{m}}\left(\frac{1}{r}\right)\sqrt{-r^4 + r^2(a^2+b^2) - a^2b^2}$$

$$= \sqrt{\frac{k}{m}}\left(\frac{1}{r}\right)\sqrt{-\underbrace{(r^2-a^2)(r^2-b^2)}}$$

$$x = a\cos\xi$$
$$y = b\sin\xi$$

$$r^2 = x^2+y^2 = a^2\cos^2\xi + b^2\sin^2\xi$$

$$r^2 - a^2 = a^2(\cos^2\xi - 1) + b^2\sin^2\xi = (b^2-a^2)\sin^2\xi$$

$$r^2 - b^2 = a^2\cos^2\xi + b^2(\sin^2\xi - 1) = (a^2-b^2)\cos^2\xi$$

Thus,

$$\sqrt{} = \sqrt{\frac{k}{m}}\left(\frac{1}{r}\right)\sqrt{-(b^2-a^2)\sin^2\xi\,(a^2-b^2)\cos^2\xi}$$

$$= \sqrt{\frac{k}{m}}\left(\frac{1}{r}\right)(a^2-b^2)\sin\xi\cos\xi$$

$$\rightarrow\ t = -\int \frac{dr}{\sqrt{\frac{k}{m}}\left(\frac{1}{r}\right)(a^2-b^2)\sin\xi\cos\xi} + \text{const}$$

$$= -\frac{1}{\omega}\int \frac{r\,dr}{(a^2-b^2)\sin\xi\cos\xi} + \text{const}$$

<u>Now:</u> $\quad r^2 = a^2\cos^2\xi + b^2\sin^2\xi$

$$\rightarrow\ 2r\,dr = 2a^2\cos\xi(-\sin\xi)\,d\xi + 2b^2\sin\xi\cos\xi\,d\xi$$

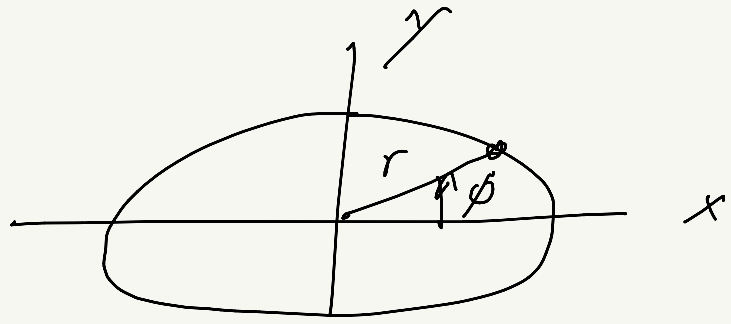
$$= -2(a^2-b^2)\sin\xi\cos\xi\,d\xi$$

$$\rightarrow\ r\,dr = -(a^2-b^2)\sin\xi\cos\xi\,d\xi$$

$$\rightarrow\ t = \frac{-1}{\omega}\int \frac{-\cancel{(a^2-b^2)}\,\cancel{\sin\xi\cos\xi}\,d\xi}{\cancel{(a^2-b^2)}\,\cancel{\sin\xi\cos\xi}} + \text{const}$$

$$= -\frac{\xi}{\omega} + \underbrace{\text{const}}_{=0 \text{ for } t=0 \ \Leftrightarrow\ \xi=0}$$

Thus, $\boxed{\xi = \omega t}$, $\boxed{\begin{array}{l} x = a\cos(\omega t) \\ y = b\sin(\omega t) \end{array}}$



$$x = a\cos(\omega t) = r\cos\phi$$

$$y = b\sin(\omega t) = r\sin\phi$$

$$r^2 = a^2\cos^2\xi + b^2\sin^2\xi$$

$$= a^2\cos^2(\omega t) + b^2\sin^2(\omega t)$$

$$\tan\phi = \frac{y}{x} = \frac{b}{a}\,\frac{\sin(\omega t)}{\cos(\omega t)} = \frac{b}{a}\tan(\omega t)$$

$$\boxed{\phi = \tan^{-1}\left[\frac{b}{a}\tan(\omega t)\right]}$$

Relation between $\dot{\phi}$ and ω:

$$M_z = m r^2 \dot{\phi}$$

$$M_z = \sqrt{mk}\,ab = \sqrt{\frac{k}{m}}\,mab = \omega mab$$

Thus, $\cancel{m} r^2 \dot{\phi} = \cancel{m}\omega ab$

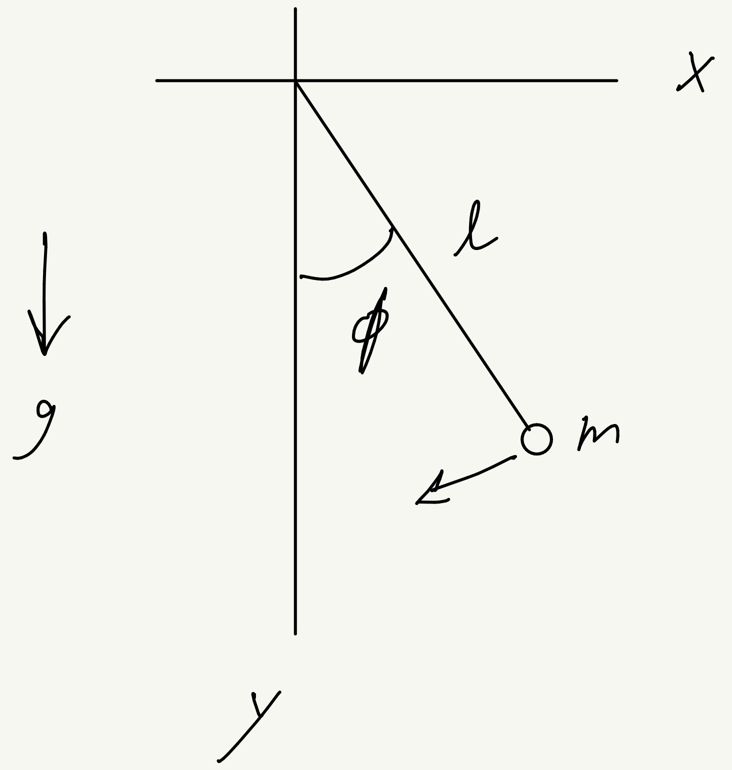
$$\boxed{\dot{\phi} = \frac{\omega ab}{r^2}}$$

Constraint forces: (simple pendulum)

$$E = T + U$$

$\Delta E = W_{nc}$ = work done by non-conservative forces, like friction

Constraint force does no (virtual) work



$$\varphi = r - l = 0$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$$

$$U = -mgy$$
$$= -mgr\cos\phi$$

$$L = T - U$$

1) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} + \lambda \frac{\partial \varphi}{\partial r}$

2) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} + \lambda \frac{\partial \varphi}{\partial \phi}$

3) $\varphi \equiv r - l = 0 \rightarrow r = l \rightarrow \dot{r} = 0, \ \ddot{r} = 0$

Thus, $m\ddot{r} = mr\dot{\phi}^2 + mg\cos\phi + \lambda$ (1)

$\frac{d}{dt}(mr^2\dot{\phi}) = -mgr\sin\phi$ (2)

$\rightarrow 2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} = -mgr\sin\phi$

Impose constraint: $r = l, \ \dot{r} = 0, \ \ddot{r} = 0$

$\rightarrow \lambda = -ml\dot{\phi}^2 - mg\cos\phi \ , \quad \ddot{\phi} = -\frac{g}{l}\sin\phi$

Constraint Force

$$\vec{F}_c = \lambda \vec{\nabla} \varphi$$

$$= \left(-m l \dot{\phi}^2 - m g \cos\phi\right) \hat{r}$$

virtual (and actual) displacement

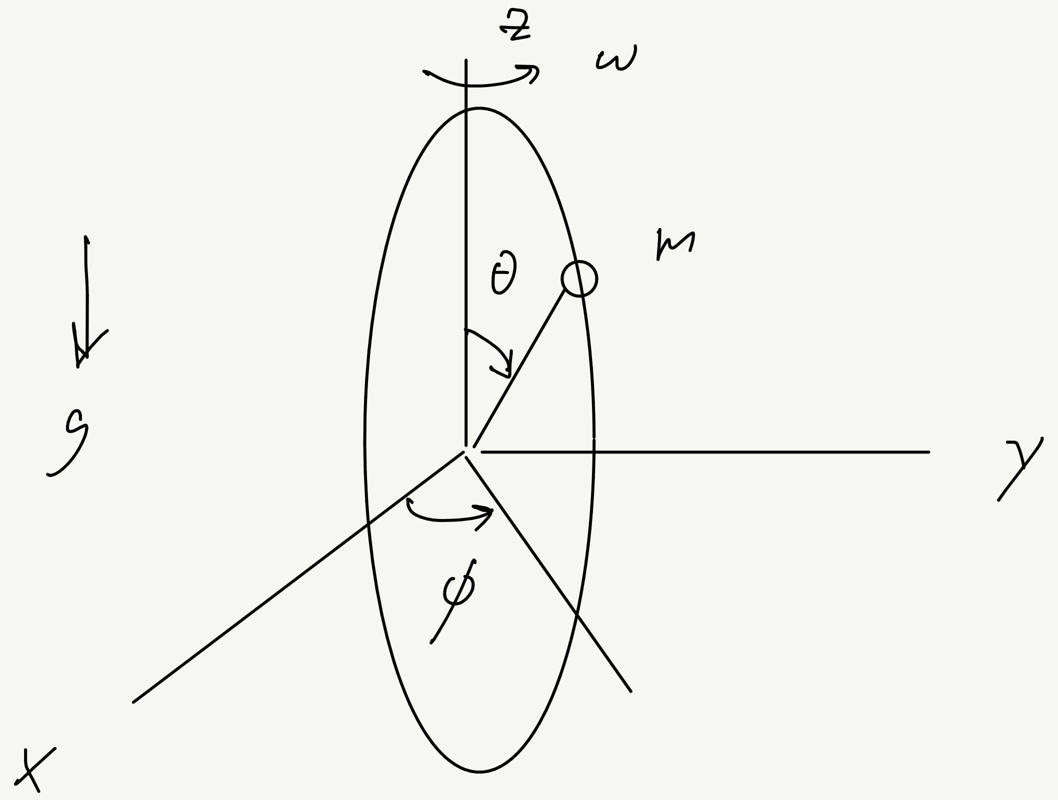
$$\delta \vec{r} = l \, \delta\phi \, \hat{\phi}$$

$$\rightarrow \quad \vec{F}_c \cdot \delta \vec{r} = \left(-m l \dot{\phi}^2 - m g \cos\phi\right) \hat{r} \cdot l \, \delta\phi \, \hat{\phi}$$

$$= 0$$

since $\hat{r} \perp \hat{\phi}$.

Rotating hoop:



$\phi = \omega t$ (specified) $\rightarrow \dot\phi = \omega$

$r = R \rightarrow \dot r = 0$

$$T = \frac{1}{2} m(\dot r^2 + r^2 \dot\theta^2 + r^2 \sin^2\theta \, \dot\phi^2)$$

$$= \frac{1}{2} m(R^2 \dot\theta^2 + R^2 \omega^2 \sin^2\theta)$$

$$U = mgz$$

$$= mgR\cos\theta$$

$$L = \frac{1}{2} m(R^2\dot\theta^2 + R^2\omega^2 \sin^2\theta) - mgR\cos\theta$$

no explicit t dependence $\rightarrow h = \frac{\partial L}{\partial \dot\theta}\dot\theta - L$

$= \text{const}$

$$\frac{\partial L}{\partial \dot\theta} = mR^2\dot\theta$$

$$\rightarrow \quad h = mR^2\dot\theta^2 - \frac{1}{2} mR^2\dot\theta^2 - \frac{1}{2} mR^2\omega^2\sin^2\theta + mgR\cos\theta$$

$$= \frac{1}{2} mR^2\dot\theta^2 - \frac{1}{2} m R^2\omega^2\sin^2\theta + mgR\cos\theta$$

Note: $h = \text{const}$ but $h \neq T + U \equiv E$

Determining constraint force:

$$\varphi_1 \equiv r - R = 0$$

$$\varphi_2 \equiv \phi - \omega t = 0$$

$$\vec{F}_c = \lambda_1 \vec{\nabla} \varphi_1 + \lambda_2 \vec{\nabla} \varphi_2$$

$$= \lambda_1 \hat{r} + \lambda_2 \frac{1}{r \sin\theta} \hat{\phi}$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2)$$

$$U = m g r \cos\theta$$

$$L = T - U$$

1) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} + \lambda_1 \frac{\partial \varphi_1}{\partial r} + \lambda_2 \frac{\partial \varphi_2}{\partial r}$$

2) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} + \lambda_1 \frac{\partial \varphi_1}{\partial \theta} + \lambda_2 \frac{\partial \varphi_2}{\partial \theta}$$

3) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} + \lambda_1 \frac{\partial \varphi_1}{\partial \phi} + \lambda_2 \frac{\partial \varphi_2}{\partial \phi}$$

4) $$r - R = 0 \rightarrow r = R \rightarrow \dot{r} = 0, \ \ddot{r} = 0$$

5) $$\phi - \omega t = 0 \rightarrow \phi = \omega t \rightarrow \dot{\phi} = \omega, \ \ddot{\phi} = 0$$

$$\frac{d}{dt}(m\dot{r}) = mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\phi}^2 - mg\cos\theta + \lambda_1$$

$$\rightarrow \boxed{\ddot{r} = r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2 - g\cos\theta + \frac{\lambda_1}{m}}$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = mr^2\sin\theta\cos\theta\,\dot{\phi}^2 + mg\sin\theta$$

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = mr^2\sin\theta\cos\theta\,\dot{\phi}^2 + mg\sin\theta$$

$$\boxed{2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r^2\sin\theta\cos\theta\,\dot{\phi}^2 + g\sin\theta}$$

$$\frac{d}{dt}(mr^2\sin^2\theta\,\dot{\phi}) = +\lambda_2$$

$$\boxed{\begin{aligned} &2mr\dot{r}\sin^2\theta\,\dot{\phi} + 2mr^2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi} \\ &\quad + mr^2\sin^2\theta\,\ddot{\phi} = \lambda_2 \end{aligned}}$$

U/c $\dot{r} = 0$, $\ddot{r} = 0$, $r = R$, $\phi = \omega t$, $\dot{\phi} = \omega$, $\ddot{\phi} = 0$

$$\rightarrow \quad 0 = R\dot{\theta}^2 + R\sin^2\theta\,\omega^2 - g\cos\theta + \frac{\lambda_1}{R}$$

$$0 + R^2\ddot{\theta} = R^2\omega^2\sin\theta\cos\theta + g\sin\theta$$

$$0 + 2mR^2\sin\theta\cos\theta\,\dot{\theta}\,\omega + 0 = \lambda_2$$

$$\rightarrow \boxed{\lambda_2 = 2mR^2\omega\sin\theta\cos\theta\,\dot{\theta}}$$

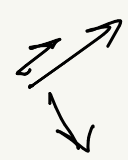
$$\boxed{\lambda_1 = -mR^2\dot{\theta}^2 - mR^2\sin^2\theta\,\omega^2 + mg\cos\theta}$$

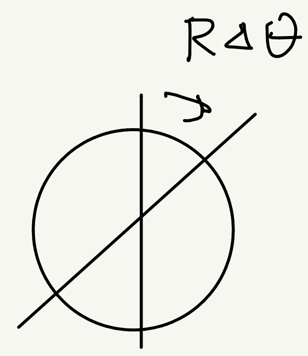
$$\vec{F}_c = \lambda_1 \hat{r} + \lambda_2 \frac{1}{R\sin\theta}\hat{\phi}$$

$$= \left(-mR^2\dot{\theta}^2 - mR^2\sin^2\theta\,\omega^2 + mg\cos\theta\right)\hat{r} + 2mR\omega\dot{\theta}\cos\theta\,\hat{\phi}$$

Virtual displacement: (constant time

$$\delta\vec{r} = R\,\delta\theta\,\hat{\theta}$$





$$\rightarrow \quad \vec{F}_c \cdot \delta\hat{r} = 0$$

Actual displacement:

$$\delta\vec{r} = \underbrace{\cancel{\delta r}}_{0}\,\hat{r} + R\,\delta\theta\,\hat{\theta} + R\sin\theta\,\delta\phi\,\hat{\phi}$$

$$= R\dot{\theta}\,\delta t\,\hat{\theta} + R\sin\theta\,\omega\,\delta t\,\hat{\phi}$$

$$= \delta t\left(\dot{\theta}\hat{\theta} + R\omega\sin\theta\,\hat{\phi}\right)$$

$$\vec{F}_c \cdot \delta\vec{r} = \delta t\, 2mR\omega\dot{\theta}\cos\theta\,\omega\sin\theta$$

$$= \delta t\, 2mR^2\omega^2\dot{\theta}\sin\theta\cos\theta$$

$$= \delta\left[mR^2\omega^2\sin^2\theta\right]$$

Thus,

$$\vec{F}_c = -\frac{\partial U_c}{\partial\vec{r}}, \quad U_c = -mR^2\omega^2\sin^2\theta$$

$$W_c = \Delta T + \Delta U = \Delta E$$

$$E = \frac{1}{2} m \left(R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2\theta \right) + mgR\cos\theta$$

$$\Delta E = \frac{1}{2} m \left(R^2 \left(\dot{\theta}_2^2 - \dot{\theta}_1^2\right) + R^2 \omega^2 \left(\sin^2\theta_2 - \sin^2\theta_1\right) \right)$$
$$+ mgR\left(\cos\theta_2 - \cos\theta_1\right)$$

$$W_c = mR^2\omega^2\left(\sin^2\theta_2 - \sin^2\theta_1\right)$$
$$= -\Delta U_c$$

Thus, $\boxed{0 = \Delta T + \Delta U + \Delta U_c = \Delta h}$

where $h = T + U + U_c$

$$= \frac{1}{2} m \left(R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2\theta \right)$$
$$+ mgR\cos\theta - mR^2\omega^2\sin^2\theta$$

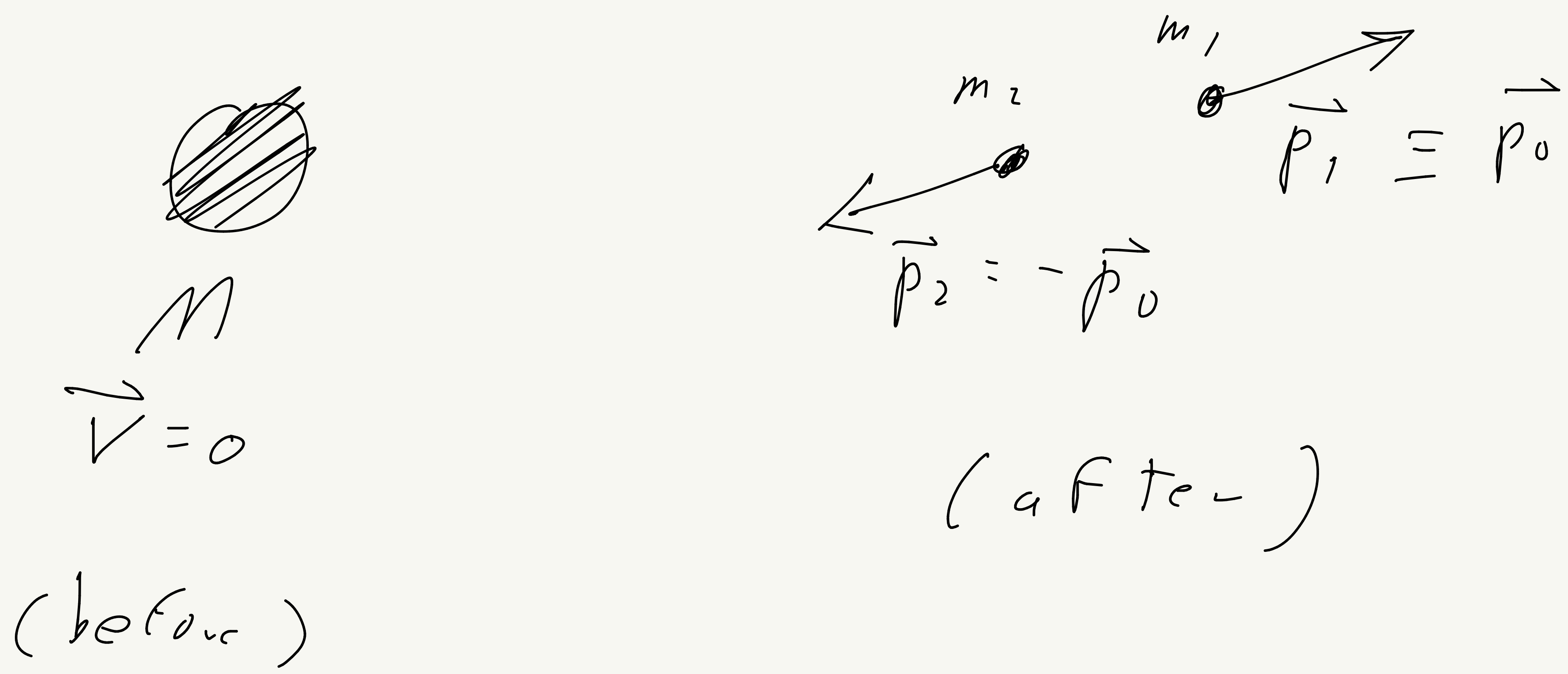
$$= \frac{1}{2} m \left(R^2 \dot{\theta}^2 - R^2 \omega^2 \sin^2\theta \right) + mgR\cos\theta$$

Collisions / Scattering

Start with spontaneous disintegration of a single particle of mass M into two particles with mass m_1, m_2.

Closed system: $\rightarrow$ energy, momentum, angular momentum conserved

$\rightarrow$ COM Frame



Cons. of energy.

$$E_i = E_{1i} + T_{10} + E_{2i} + T_{20}$$

$$= E_{1i} + E_{2i} + \frac{p_0^2}{2m_1} + \frac{p_0^2}{2m_2}$$

$$\rightarrow \underbrace{E_i - E_{1i} - E_{2i}}_{\epsilon} = \frac{p_0^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

$$\epsilon = \frac{p_0^2}{2m}, \quad m \equiv \frac{m_1 m_2}{m_1 + m_2}$$

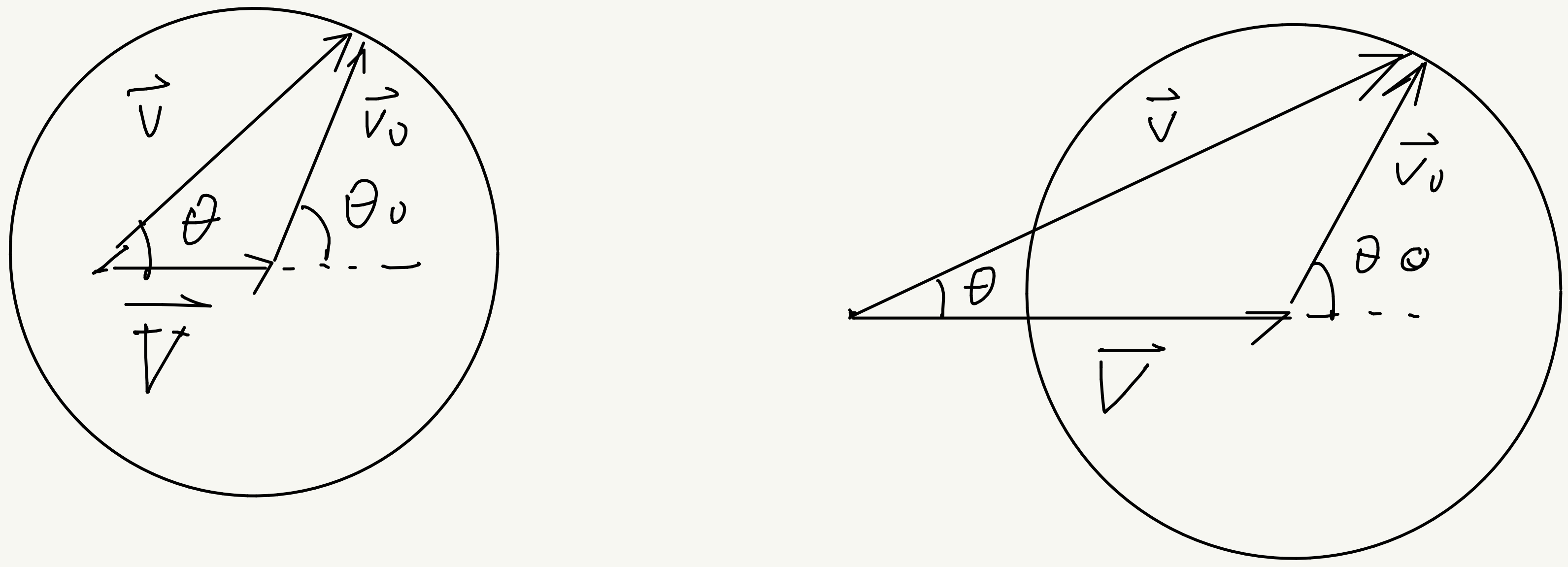
reduced mass

disintegration energy

So, $p_0 = \sqrt{2mE}$

$\rightarrow \quad v_1 = \dfrac{p_0}{m_1} \quad , \quad v_2 = \dfrac{p_0}{m_2}$

Lab frame:



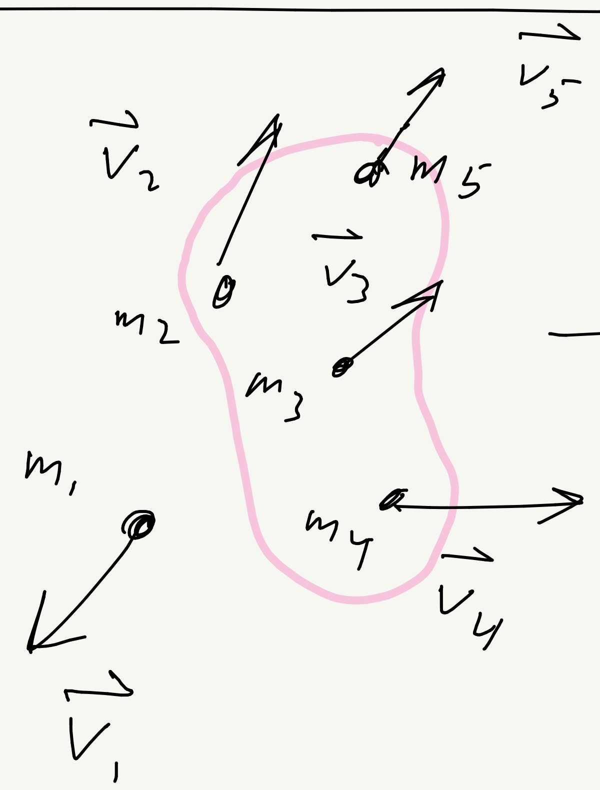
$\vec{v}$: w.r.t lab frame (for m_1 or m_2)
$\vec{v}_0$: w.r.t COM fram (for m_1 or m_2)
$\vec{V}$: velocity of COM (velocity of mass M w.r.t lab

$$\rightarrow \boxed{v_0^2 = v^2 + V^2 - 2vV\cos\theta}$$

$$v\sin\theta = v_0 \sin\theta_0$$
$$v\cos\theta = v_0\cos\theta_0 + V$$

$$\rightarrow \boxed{\tan\theta = \frac{v_0\sin\theta_0}{v_0\cos\theta_0 + V}}$$

Disintegration into more than two particles



Consider COM of the system of particles $m_2, m_3, \cdots, m_N$

COM Frame:

$$\vec{p}_0 = m_1 \vec{v}_1 , \qquad -\vec{p}_0 = m_2 \vec{v}_2 + m_3 \vec{v}_3 + \cdots + m_N \vec{v}_N$$

$$E_i = E_{1i} + \frac{p_0^2}{2m_1} + E_i' + \frac{p_0^2}{2(M-m_1)}$$

$$E_i - E_{1i} - E_i' = \frac{p_0^2}{2}\left(\frac{1}{m_1} + \frac{1}{M-m_1}\right)$$

$$= \frac{p_0^2}{2} \frac{M}{m_1(M-m_1)}$$

$$\rightarrow \quad \frac{p_0^2}{2m_1} = \left(\frac{M-m_1}{M}\right)\left(E_i - E_{1i} - E_i'\right)$$

This is maximized when E_i' is minimized

Now:

$$E_i' + \frac{p_0^2}{2(M-m_1)} = E_{2i} + \frac{1}{2} m_2 |\vec{v}_2|^2 + E_{3i} + \frac{1}{2} m_3 |\vec{v}_3|^2 + \cdots$$

Thus,

$$E_i' = E_{2i} + E_{3i} + \cdots + \frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \cdots - \frac{1}{2} \frac{|m_2 \vec{v}_2 + m_3 \vec{v}_3 + \cdots|^2}{(m_2 + m_3 + \cdots)}$$

Now:

$$\frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \cdots = T'$$

= KE of $m_2, m_3, \ldots$ wrt original com frame

$$\frac{1}{2} \frac{|m_2 \vec{v}_2 + m_3 \vec{v}_3 + \cdots|^2}{(m_2 + m_3 + \cdots)} = \frac{1}{2} \frac{|\vec{p}_0|^2}{(m_2 + m_3 + \cdots)} = T'_{com}$$

= KE of COM of $m_2, m_3, \ldots$ wrt original COM frame

Thus, $T' - T'_{com} = T_0'$

= KE of $m_2, m_3, \ldots$ wrt COM of $m_2, m_3, \ldots$

≥ 0

$$\rightarrow \boxed{E_i' = E_{2i} + E_{3i} + \cdots + T_0'}$$

$E_i' = \min$ when $T_0' = 0 \rightarrow m_2, m_3, \ldots$ move with the same velocity

Elastic collision of two particles:

i) Elastic: ignore internal energies

$\rightarrow E = \text{const} \Rightarrow T = \text{const}$

ii) Momentum and angular momentum also conserved

$\rightarrow$ motion in a plane (2-d)

4 DOFs: Final velocities of m_1, m_2 (magnitude, direction in 2-d)

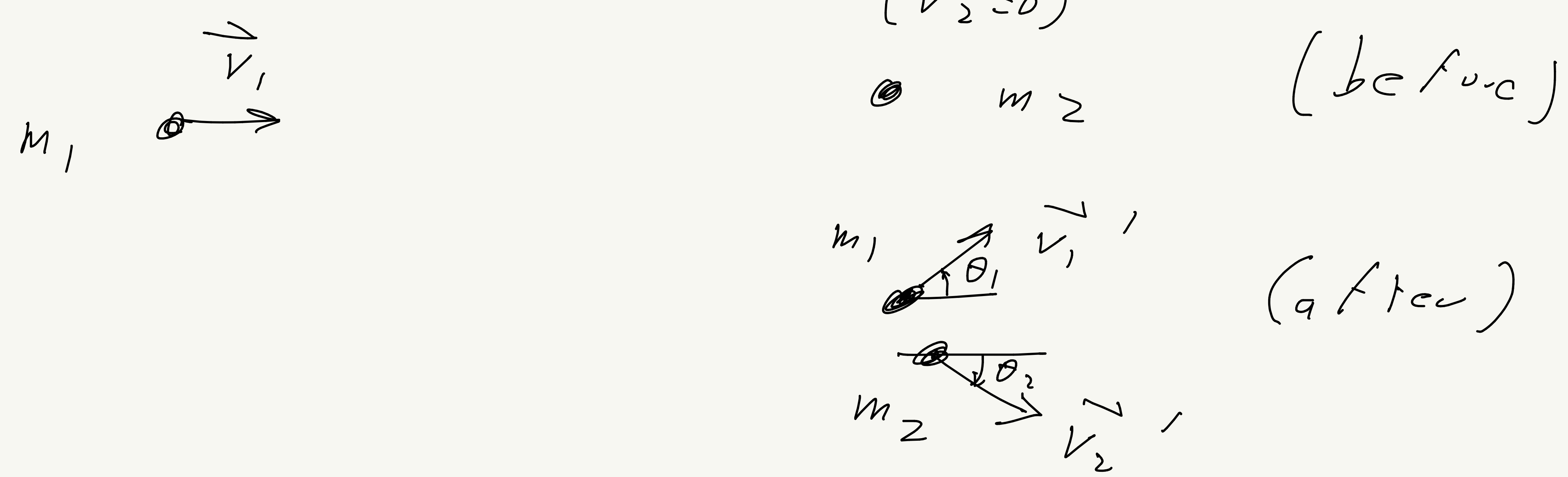
Constraints: $T = \text{const} \rightarrow 1$

$\vec{P} = \text{const} \rightarrow 2$ (in 2-d)

so 3 equations but 4 unknowns.

Example:

m_1, m_2 (initially at rest) ($\vec{v}_2 = 0$)



$$\vec{P} = m_1 v_1 \hat{x}$$
$$= m_1 \vec{v}_1{}' + m_2 \vec{v}_2{}'$$
$$= (m_1 v_1' \cos\theta_1 + m_2 v_2' \cos\theta_2)\hat{x} + (m_1 v_1' \sin\theta_1 - m_2 v_2' \sin\theta_2)\hat{y}$$

So

$$\boxed{\begin{aligned} m_1 v_1 &= m_1 v_1' \cos\theta_1 + m_2 v_2' \cos\theta_2 \\ 0 &= m_1 v_1' \sin\theta_1 - m_2 v_2' \sin\theta_2 \end{aligned}}$$

Also

$$\begin{aligned} T &= \tfrac{1}{2} m_1 |\vec{v}_1|^2 \\ &= \tfrac{1}{2} m_1 |\vec{v}_1{}'|^2 + \tfrac{1}{2} m_2 |\vec{v}_2{}'|^2 \end{aligned}$$

$$\rightarrow \boxed{m_1 v_1^2 = m_1 v_1'^2 + m_2 v_2'^2}$$

Numerical example:

$m_1 = 1\ \text{kg}$ $\qquad$ $v_1 = 1\ \text{m/s}$

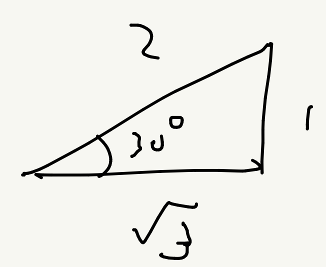
$m_2 = 2\ \text{kg}$ $\qquad$ $v_2 = 0$

$v_1' = ??$ $\qquad$ $\theta_1 = ??$

$v_2' = ??$ $\qquad$ $\theta_2 = 60^\circ$

Use above 3 equations:

$$1 = v_1' \cos\theta_1 + 2 v_2' \underbrace{\cos 60^\circ}_{\frac{1}{2}}$$



$$\rightarrow \boxed{1 = v_1' \cos\theta_1 + v_2'}$$

$$0 = v_1' \sin\theta_1 - 2 v_2' \underbrace{\sin 60^\circ}_{\frac{\sqrt{3}}{2}}$$

$$\boxed{0 = v_1' \sin\theta_1 - \sqrt{3}\, v_2'}$$

$$\boxed{1 - v_1'^2 = 2 v_2'^2}$$

Now: $v_1' \sin\theta_1 = \sqrt{3}\, v_2'$

$v_1' \cos\theta_1 = 1 - v_2'$

$\rightarrow \quad v_1'^2 \sin^2\theta_1 + v_1'^2 \cos^2\theta_1 = 3 v_2'^2 + 1 - 2v_2' + v_2'^2$

$v_1'^2 = 4 v_2'^2 - 2 v_2' + 1$

Also: $v_1'^2 = 1 - 2 v_2'^2$

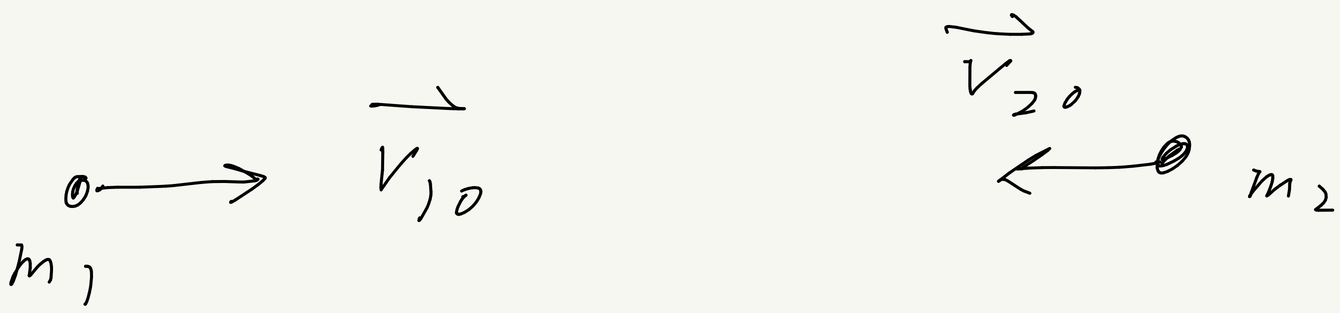
subtract: $0 = 6 v_2'^2 - 2 v_2'$

$= 2 v_2' (3 v_2' - 1) \quad \rightarrow \quad \boxed{v_2' = \frac{1}{3}\ \mathrm{m/s}}$

$\rightarrow \quad v_1'^2 = 1 - \frac{2}{9} = \frac{7}{9} \quad \rightarrow \quad \boxed{v_1' = \frac{\sqrt{7}}{3}\ \mathrm{m/s}}$

Finally: $\sin\theta_1 = \dfrac{\sqrt{3}\, v_2'}{v_1'} = \dfrac{\sqrt{3}\,\frac{1}{\cancel{3}}}{\frac{\sqrt{7}}{\cancel{3}}} = \sqrt{\frac{3}{7}} \rightarrow \boxed{\theta_1 = 41^\circ}$

Simpler to analyze in COM Frame:



$\vec{P}_0 = 0 = m_1 \vec{v}_{10} + m_2 \vec{v}_{20}$

$\vec{V} = \dfrac{m_1 \vec{v}_1 + m \vec{v}_2}{m_1 + m_2}$ (velocity of com w-r-t lab frame)

Relative position vector: $\vec{r} \equiv \vec{r}_{10} - \vec{r}_{20}$ $(= \vec{r}_1 - \vec{r}_2)$

" velocity " : $\vec{v} \equiv \vec{v}_{10} - \vec{v}_{20}$ $(= \vec{v}_1 - \vec{v}_2)$

Recall:

$$\vec{r}_{10} = \frac{m_2}{m_1+m_2}\vec{r}, \qquad \vec{r}_{20} = \frac{-m_1}{m_1+m_2}\vec{r}$$

$$\rightarrow \quad \vec{v}_{10} = \frac{m_2}{m_1+m_2}\vec{v}, \qquad \vec{v}_{20} = \frac{-m_1}{m_1+m_2}\vec{v}$$

$$T = \frac{1}{2} m_1 |\vec{v}_{10}|^2 + \frac{1}{2} m_2 |\vec{v}_{20}|^2$$

$$= \frac{1}{2} m |\vec{v}|^2, \qquad m \equiv \frac{m_1 m_2}{m_1+m_2}$$

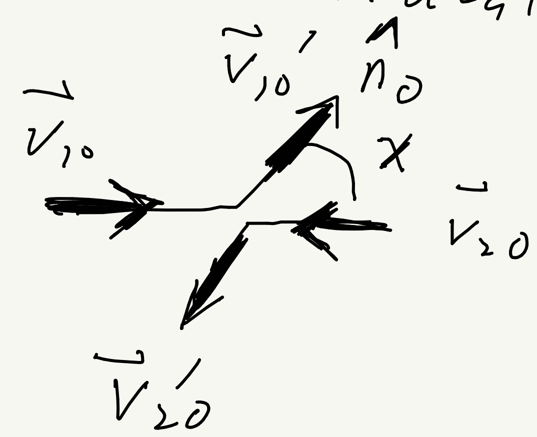
$$T' = \frac{1}{2} m |\vec{v}'|^2$$

Elastic: $T = T' \rightarrow v = v'$

$\rightarrow v_{10} = v_{10}'$

$\rightarrow v_{20} = v_{20}'$

Thus, relative velocity vector is unchanged in magnitude $\rightarrow$ magnitudes of individual velocities are unchanged.



$$\boxed{\begin{aligned} \vec{v}_{10}' &= \frac{m_2}{m_1+m_2}\hat{n}\, v \\ \vec{v}_{20}' &= \frac{-m_1}{m_1+m_2}\hat{n}\, v \end{aligned}}$$

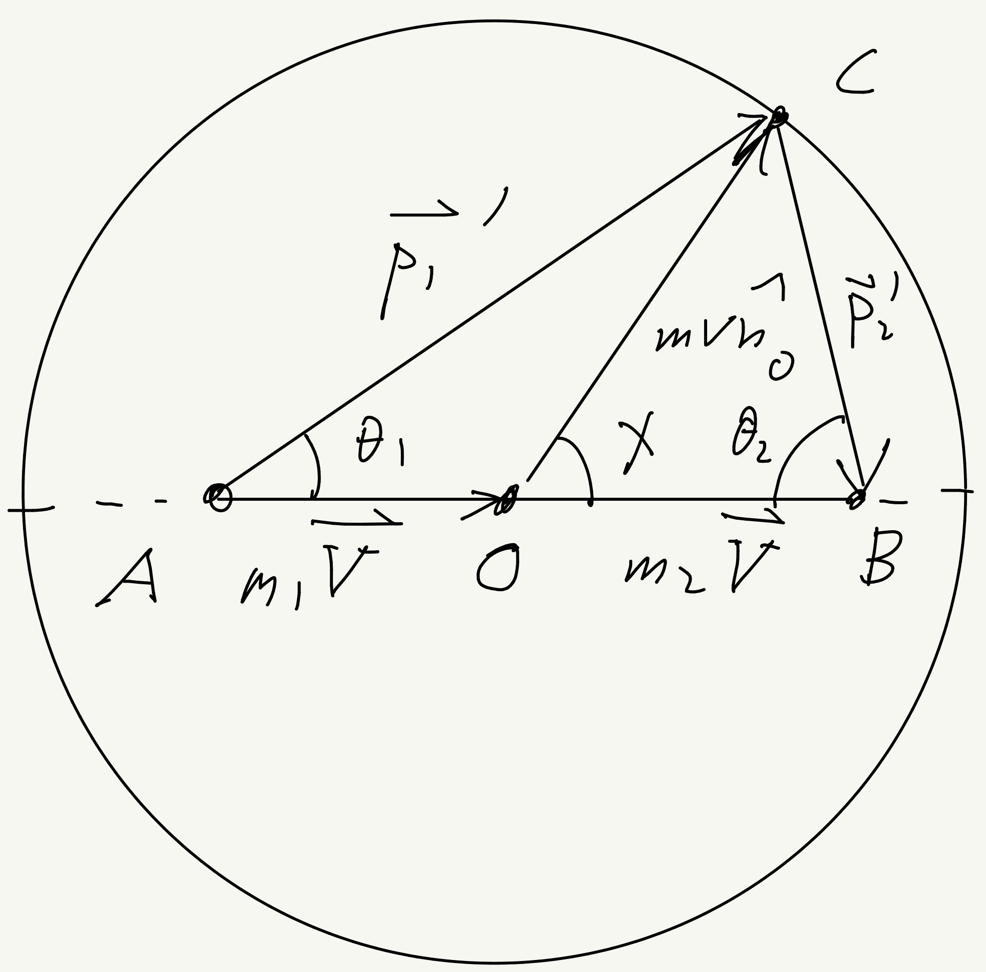
In lab frame

$$\vec{v}_1' = \vec{V} + \vec{v}_{10}'$$
$$\vec{v}_2' = \vec{V} + \vec{v}_{20}'$$

Thus,
$$\vec{v}_1' = \left(\frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}\right) + \left(\frac{m_2}{m_1+m_2}\right)\hat{n}_0 v$$
$$\vec{v}_2' = \left(\frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}\right) - \left(\frac{m_1}{m_1+m_2}\right)\hat{n}_0 v$$

Convert to momenta:

$$\vec{p}_1' = m_1\vec{v}_1' = m_1\vec{V} + m\hat{n}_0 v$$
$$\vec{p}_2' = m_2\vec{v}_2' = m_2\vec{V} - m\hat{n}_0 v$$



AB: $\vec{P} = (m_1+m_2)\vec{V}$

AO: $m_1 \vec{V}$

BO: $m_2 \vec{V}$

Suppose $v_2 = 0$

Then $\vec{OB} = m_2\left(\frac{m_1\vec{v}_1 + \cancel{m_2\vec{v}_2}^{\,0}}{m_1+m_2}\right)$

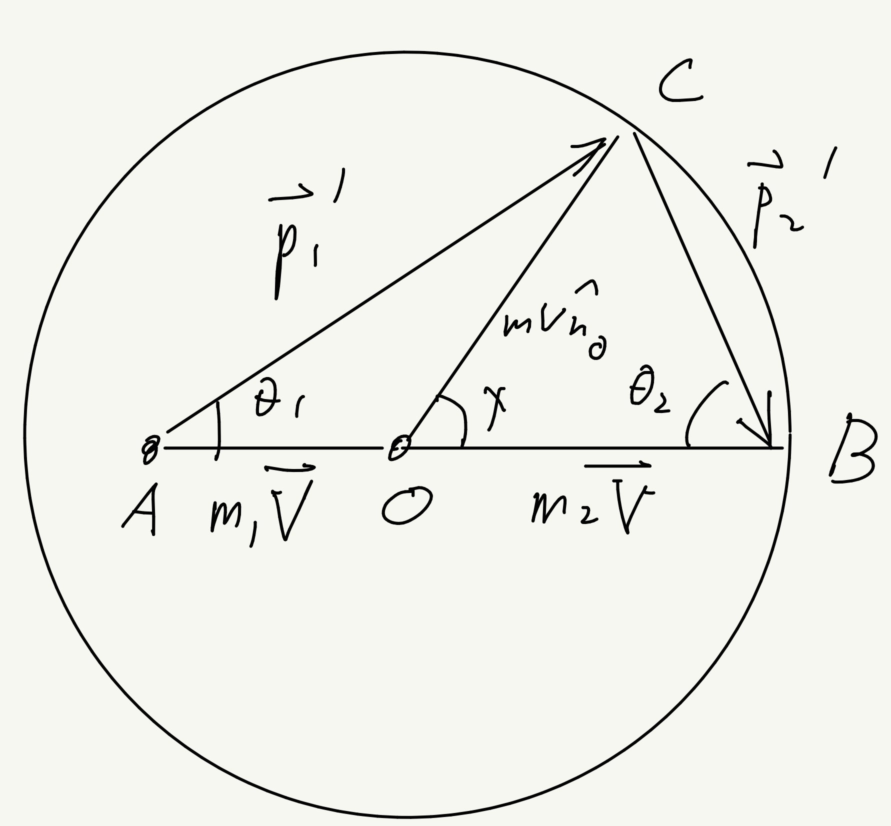
$= m\vec{v}_1$

$= m\vec{v}$

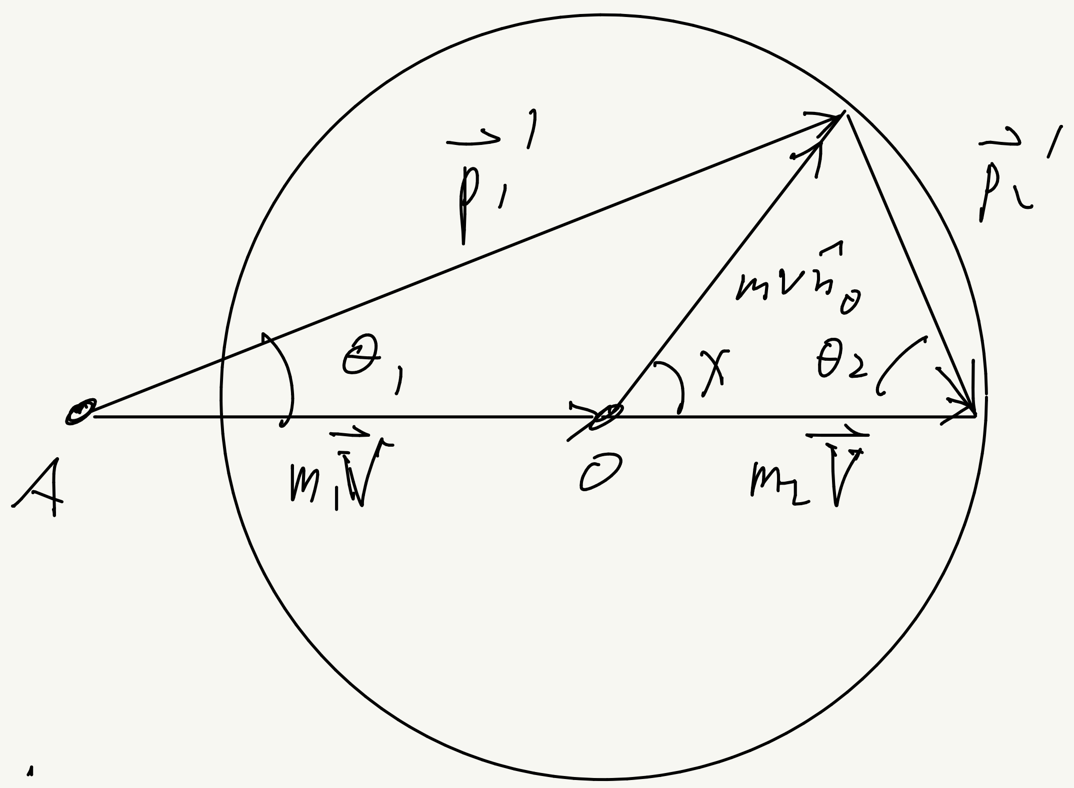
$|\vec{OB}| = |\vec{OC}|$

For $\vec{V}_2 = 0$:

$m_1 < m_2$



$m_1 > m_2$



Relate angles:

$$\chi + 2\theta_2 = \pi \longrightarrow \boxed{\theta_2 = \frac{\pi - \chi}{2}}$$

$$\tan\theta_1 = \frac{mV \sin\chi}{m_1 V + mV\cos\chi}$$

$$= \frac{\left(\frac{m_1 m_2}{\cancel{m_1+m_2}}\right)\cancel{V} \sin\chi}{m_1\left(\frac{\cancel{m_1}\cancel{V}}{\cancel{m_1+m_2}}\right) + \frac{\cancel{m_1} m_2 \cancel{V}}{\cancel{m_1+m_2}}\cos\chi}$$

$$= \boxed{\frac{m_2 \sin\chi}{m_1 + m_2\cos\chi} = \tan\theta_1} \qquad (17.4)$$

Suppose $m_1 = m_2$: (billiard balls)

$$\tan\theta_1 = \frac{\sin\chi}{1+\cos\chi}$$

$$= \frac{\cancel{2}\sin\left(\frac{\chi}{2}\right)\cancel{\cos\left(\frac{\chi}{2}\right)}}{\cancel{2}\cos^{\cancel{2}}\left(\frac{\chi}{2}\right)}$$

$$= \tan\left(\frac{\chi}{2}\right)$$

Now:

$$\cos(\chi) = \cos\left(2\,\frac{\chi}{2}\right)$$

$$= \cos^2\left(\frac{\chi}{2}\right) - \sin^2\left(\frac{\chi}{2}\right)$$

$$= 2\cos^2\left(\frac{\chi}{2}\right) - 1$$

$$1 + \cos\chi = 2\cos^2\left(\frac{\chi}{2}\right)$$

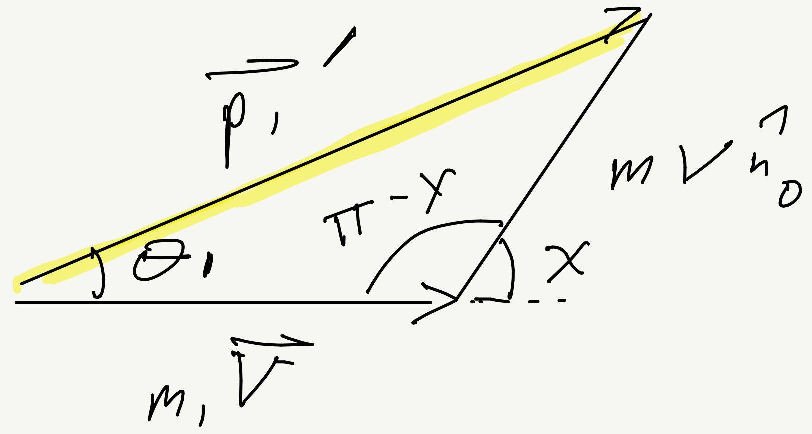
$$1 - \cos\chi = 2\sin^2\left(\frac{\chi}{2}\right)$$

Thus, $\theta_1 = \frac{\chi}{2}$

Also $\theta_2 = \frac{\pi}{2} - \frac{\chi}{2}$

$\rightarrow \quad \theta_1 + \theta_2 = \frac{\pi}{2}$

Relate velocities:



$\vec{V} = \vec{v}_1$ (since $\vec{v}_2 = 0$)

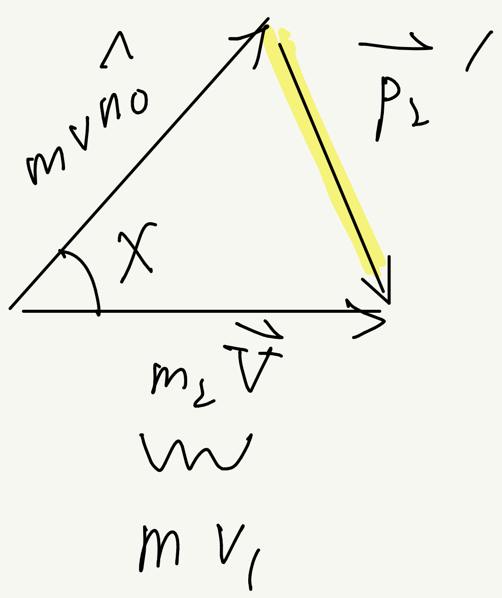
Thus, $$m_1^2 |v_1'|^2 = m_1^2 V^2 + m^2 v^2 - 2\, m_1 V\, m v \cos(\pi - \chi)$$

$$\Rightarrow \cancel{m_1^2}\, v_1'^2 = \cancel{m_1^2}\left(\frac{m_1 V}{m_1+m_2}\right)^2 + \left(\frac{\cancel{m_1} m_2}{m_1+m_2}\right)^2 v^2$$

$$+ 2\, m_1 \left(\frac{\cancel{m_1} V}{m_1+m_2}\right)\left(\frac{\cancel{m_1} m_2}{m_1+m_2}\right) v \cos\chi$$

$$\boxed{v_1' = \left(\frac{V}{m_1+m_2}\right)\sqrt{m_1^2 + m_2^2 + 2 m_1 m_2 \cos\chi}}$$

Also:



$$\cancel{m_2^2}\, v_2'^2 = 2 m^2 v^2 - 2 m^2 v^2 \cos\chi$$

$$= \frac{2\, m_1^2 \cancel{m_2^2}}{(m_1+m_2)^2}\, v^2 \underbrace{\left(1 - \cos\chi\right)}$$

$$\boxed{v_2' = \left(\frac{2\, m_1 V}{m_1+m_2}\right)\sin\left(\frac{\chi}{2}\right)} \qquad (17.5)$$

Resolve example problem:

$m_1 = 1\,kg, \quad m_2 = 2\,kg, \quad v_1 = 1\,m/s, \quad v_2 = 0, \quad \theta_2 = 60^\circ$

Find v_1', v_2', θ_1

$$\chi + 2\theta_2 = \pi \longrightarrow \chi = \pi - 2\left(\frac{\pi}{3}\right) = \frac{\pi}{3}$$

so $\boxed{\chi = 60^\circ}$

$v = v_1 = 1\,m/s$

$$v_2' = \left(\frac{2m_1 v}{m_1 + m_2}\right)\sin\left(\frac{\chi}{2}\right) = \frac{2}{3}\sin 30^\circ = \boxed{\frac{1}{3}}\ m/s$$

$$v_1' = \left(\frac{v}{m_1+m_2}\right)\sqrt{m_1^2 + m_2^2 + 2m_1m_2\cos\chi}$$

$$= \frac{1}{3}\sqrt{1 + 4 + 2\cdot\cancel{2}\ \underbrace{\cos 60^\circ}_{\cancel{\frac{1}{2}}}}$$

$$= \boxed{\frac{\sqrt{7}}{3}}\ m/s$$

Revisit original equations:

$$m_1 v_1 = m_1 v_1' \cos\theta_1 + m_2 v_2' \cos\theta_2 \qquad (1)$$

$$0 = m_1 v_1' \sin\theta_1 - m_2 v_2' \sin\theta_2 \qquad (2)$$

$$m_1 v_1^2 = m_1 v_1'^2 + m_2 v_2'^2 \qquad (3)$$

$$\rightarrow \quad m_1 v_1' \sin\theta_1 = m_2 v_2' \sin\theta_2$$

$$m_1 v_1' \cos\theta_1 = m_1 v_1 - m_2 v_2' \cos\theta_2$$

square and add:

$$m_1^2 v_1'^2 = m_2^2 v_2'^2 \sin^2\theta_2 + m_1^2 v_1^2 + m_2^2 v_2'^2 \cos^2\theta_2 - 2 m_1 m_2 v_1 v_2' \cos\theta_2$$

$$= m_1^2 v_1^2 + m_2^2 v_2'^2 - 2 m_1 m_2 v_1 v_2' \cos\theta_2$$

Also:

$$m_1^2 v_1'^2 = m_1^2 v_1^2 - m_1 m_2 v_2'^2$$

$$\rightarrow \quad 0 = (m_2^2 + m_1 m_2) v_2'^2 - 2 m_1 m_2 v_1 v_2' \cos\theta_2$$

$$= m_2 (m_1 + m_2) v_2'^2 - 2 m_1 m_2 v_1 v_2' \cos\theta_2$$

$$= v_2'^2 - 2\left(\frac{m_1}{m_1 + m_2}\right) v_2' v_1 \cos\theta_2$$

$$= v_2' \left[v_2' - 2\left(\frac{m_1}{m_1 + m_2}\right) v_1 \cos\theta_2 \right]$$

Thus, $$\boxed{v_2' = \frac{2 m_1 v_1}{m_1 + m_2} \cos\theta_2}$$

Using $m_1 v_1^2 = m_1 v_1'^2 + m_2 v_2'^2$

$$\rightarrow \quad v_1'^2 = v_1^2 - \frac{m_2}{m_1} v_2'^2$$

$$= v_1^2 - \frac{m_2}{\cancel{m_1}} \frac{4 m_1^{\cancel{2}} v_1^2}{(m_1+m_2)^2} \cos^2\theta_2$$

$$= \frac{v_1^2}{(m_1+m_2)^2} \left[(m_1+m_2)^2 - 4 m_1 m_2 \cos^2\theta_2 \right]$$

Thus, $$\boxed{v_1' = \left(\frac{v_1}{m_1+m_2}\right) \sqrt{m_1^2 + m_2^2 + 2 m_1 m_2 (1 - 2\cos^2\theta_2)}}$$

NOTE:

If we write $\theta_2 = \frac{\pi}{2} - \frac{\chi}{2}$, then

$$\cos\theta_2 = \cos\left(\frac{\pi}{2} - \frac{\chi}{2}\right) = \cancel{\underbrace{\cos\frac{\pi}{2}}_{0}} \cos\frac{\chi}{2} + \underbrace{\sin\frac{\pi}{2}}_{=1} \sin\frac{\chi}{2} = \sin\frac{\chi}{2}$$

$$1 - 2\cos^2\theta_2 = 1 - 2\sin^2\left(\frac{\chi}{2}\right) = \cos\chi$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

Expression for θ_1:

$$m_1 v_1' \sin\theta_1 = m_2 v_2' \sin\theta_2$$
$$m_1 v_1' \cos\theta_1 = m_1 v_1 - m_2 v_2' \cos\theta_2$$

Divide:

$$\tan\theta_1 = \frac{m_2 v_2' \sin\theta_2}{m_1 v_1 - m_2 v_2' \cos\theta_2}$$

$$= \frac{m_2 \left(\frac{2 \cancel{m_1 v_1}}{m_1 + m_2}\right) \cos\theta_2 \sin\theta_2}{\cancel{m_1 v_1} - m_2 \left(\frac{2 \cancel{m_1 v_1}}{m_1 + m_2}\right) \cos^2\theta_2}$$

$$= \boxed{\frac{m_2 \sin(2\theta_2)}{m_1 + m_2 (1 - 2\cos^2\theta_2)}}$$

For $\theta_2 = \frac{\pi}{2} - \frac{\chi}{2}$:

$$\sin(2\theta_2) = \sin(\pi - \chi) = \sin\pi \cos\chi - \cos\pi \sin\chi = \sin\chi$$

$$1 - 2\cos^2\theta_2 = \cos\chi$$