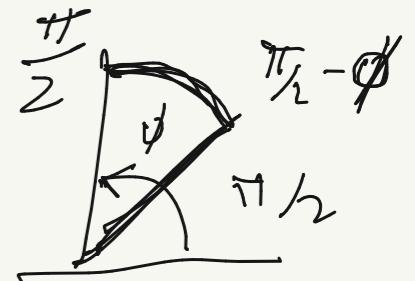


## Elliptic Functions / integrals:

- i) period of a simple pendulum beyond the small-angle approximation
- ii) circumference of an ellipse

A generalization of definition of circular functions (sines, cosines) to ellipses.

Standard notation:



$$\int_0^x \frac{dt}{\sqrt{1-k^2 t^2} \sqrt{1-t^2}} = F(\phi, k) = \sin^{-1} x$$

$$\int_0^x \frac{\sqrt{1-k^2 t^2}}{\sqrt{1-t^2}} dt = E(\phi, k)$$

where  $x = \sin \phi$  and  $0 \leq k \leq 1$

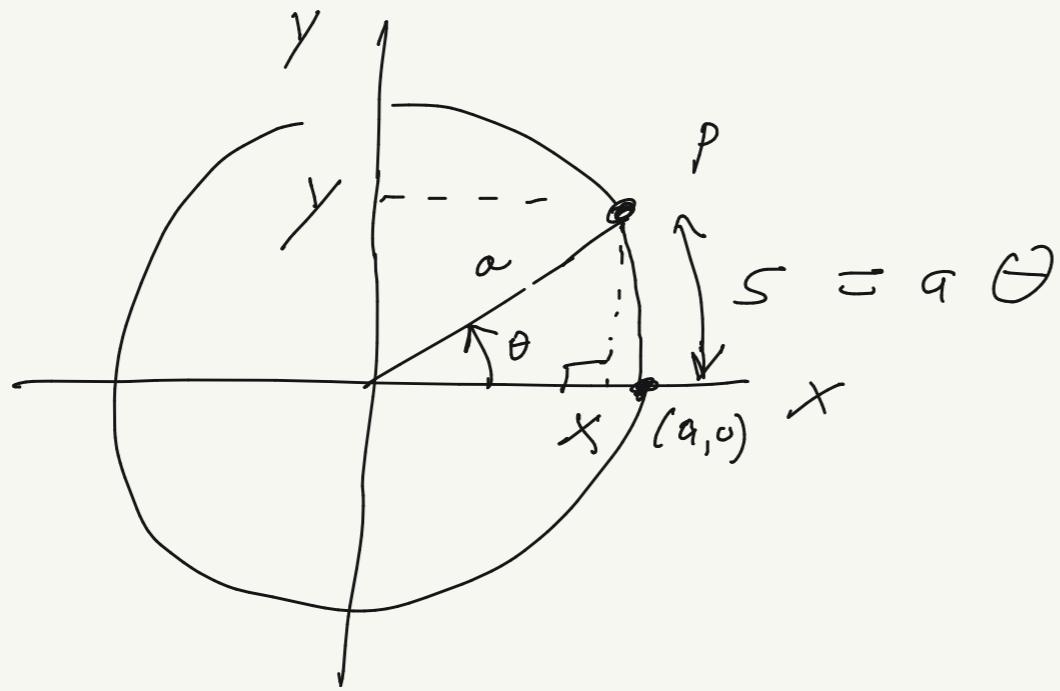
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If we change variables  $t \rightarrow \sin \theta$  in the integrals then

$$F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}$$

$$E(\phi, k) = \int_0^\phi \sqrt{1-k^2 \sin^2 \theta} d\theta$$

## Circular Functions:



$$\sin \theta = \frac{y}{a} \quad \cos \theta = \frac{x}{a}$$

where  $\theta = \frac{\text{arc length from } (a, 0) \text{ to } (x, y)}{a}$

$$= \frac{1}{a} \int_{(a, 0)}^{(x, y)} \sqrt{dx^2 + dy^2} \quad (= \int d\theta)$$

$$x^2 + y^2 = a^2 \rightarrow a^2 \cos^2 \theta + a^2 \sin^2 \theta = a^2$$

$$\rightarrow \boxed{\cos^2 \theta + \sin^2 \theta = 1}$$

## Derivatives:

$$\frac{d}{d\theta} \sin \theta = \frac{d}{d\theta} \left( \frac{y}{a} \right) = \frac{1}{a} \frac{dy}{d\theta} = \frac{dy}{\sqrt{dx^2 + dy^2}}$$

$$= \frac{1}{\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}}$$

$$= \frac{1}{\sqrt{\left(\frac{-y}{x}\right)^2 + 1}}$$

Now:  $x^2 + y^2 = a^2$   
 $\rightarrow 2x dx + 2y dy = 0$

$$\frac{dx}{dy} = -\frac{y}{x}$$

Then,

$$\frac{d}{d\theta} \sin \theta = \frac{1}{\sqrt{1 + \left(\frac{y}{x}\right)^2}} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} = \cos \theta$$

so  $\frac{d \sin \theta}{d\theta} = \cos \theta$

Similarly,  $\frac{d \cos \theta}{d\theta} = -\sin \theta$

Integrate :

$$\int \frac{d(\sin \theta)}{\cos \theta} = \int d\theta = \theta + \text{const}$$

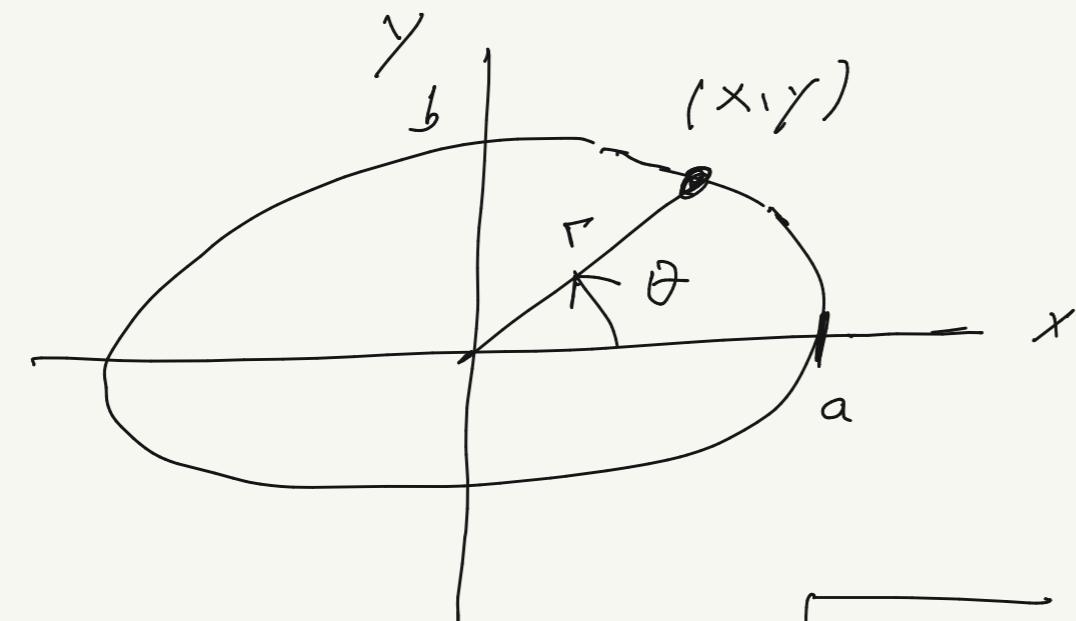
$$t = \sin \theta$$

$$\cos \theta = \sqrt{1-t^2}$$

$$dt = d(\sin \theta)$$

$$\rightarrow \boxed{\int \frac{dt}{1-t^2} = \theta + \text{const} = \sin^{-1} t + \text{const}}$$

## Different parameterization of an ellipse



$$x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Let  $a \geq b$

Eccentricity:

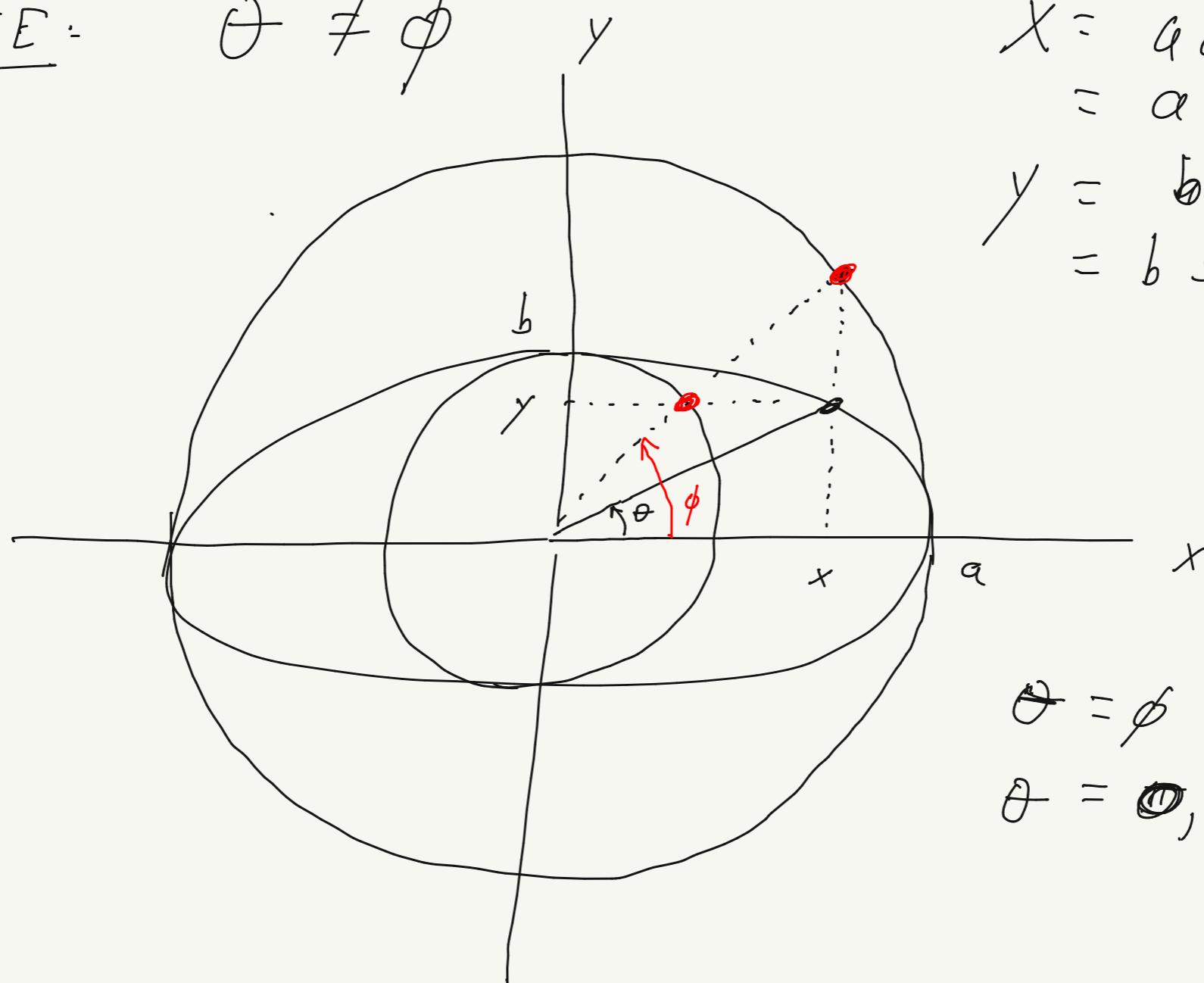
$$e^2 = 1 - \left(\frac{b}{a}\right)^2$$

( $e=0$  for a circle)

Another parameterization:

$$\begin{cases} x = a \cos \phi \\ y = b \sin \phi \end{cases} \rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Note:  $\theta \neq \phi$



$$\begin{aligned} x &= a \cos \phi \\ &= a \cos(u; \tau) \end{aligned}$$

$$\begin{aligned} y &= b \sin \phi \\ &= b \sin(u; \tau) \end{aligned}$$

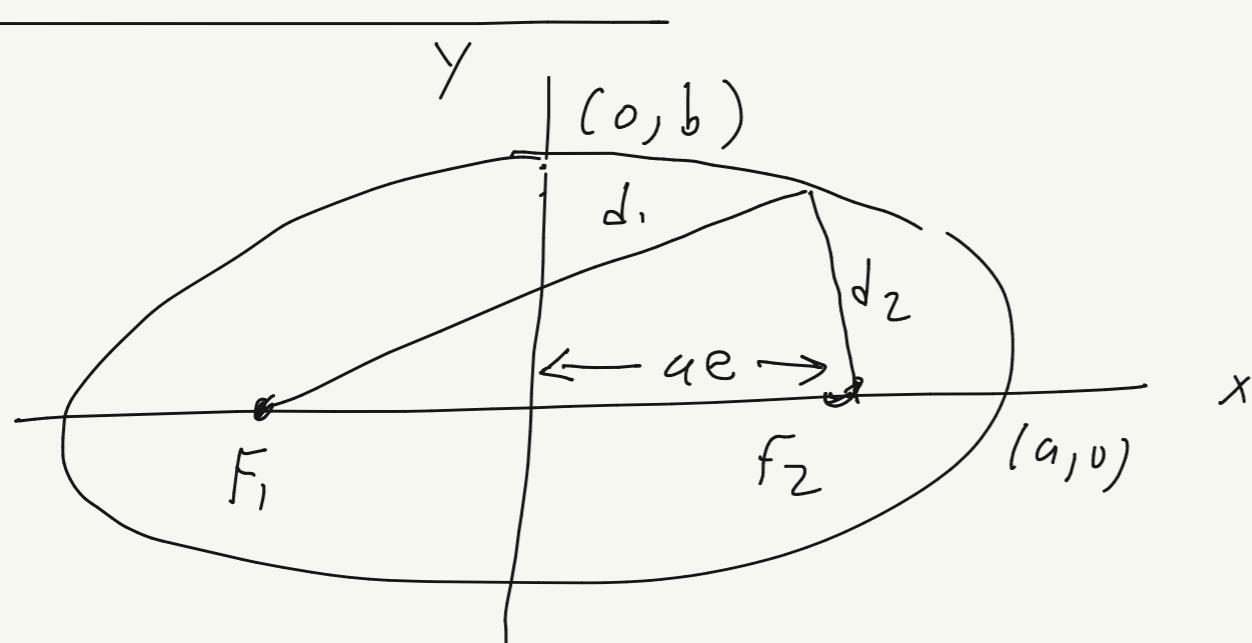
$\theta = \phi$  for

$$\theta = 0, \frac{\pi}{2}, \dots$$

$$\tan \theta = \frac{y}{x} = \left(\frac{b}{a}\right) \tan \phi \rightarrow \theta = \arctan \left[ \frac{b}{a} \tan \phi \right]$$

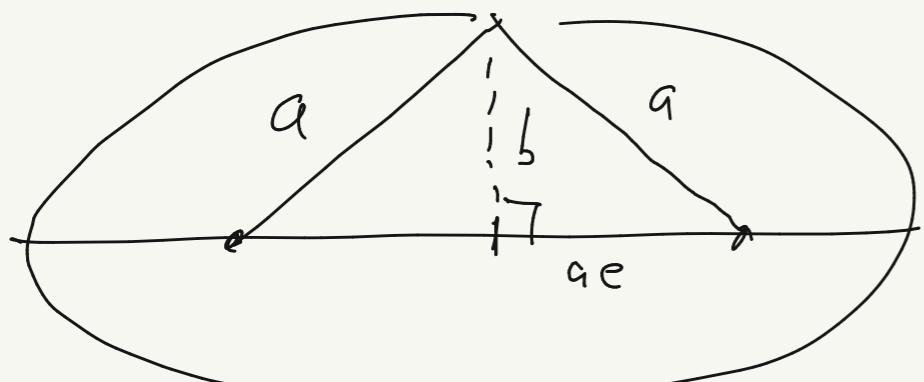
$$\phi = \arctan \left[ \frac{a}{b} \tan \theta \right]$$

Elliptic Functions :



$$d_1 + d_2 = 2a$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

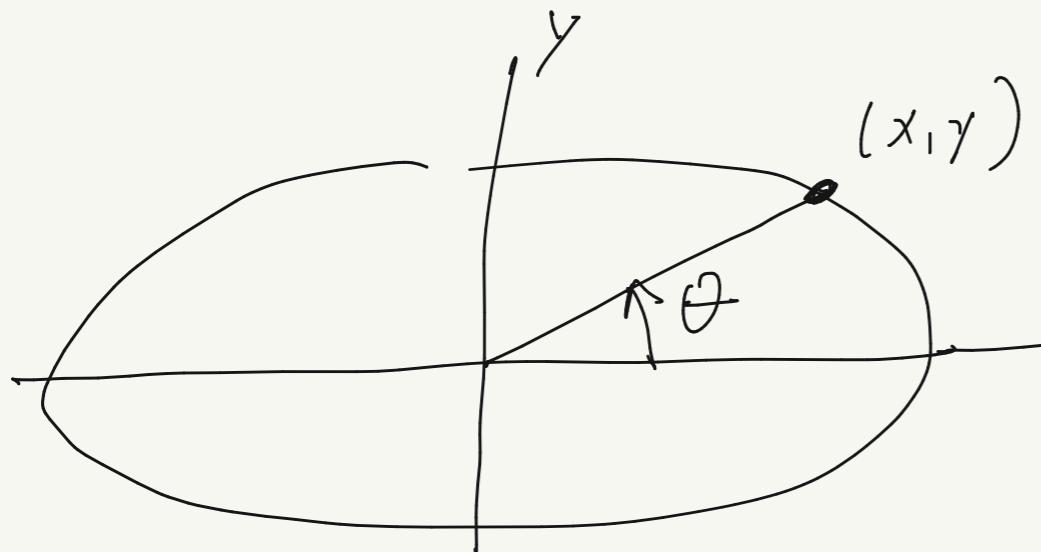


$$a^2 = b^2 + c^2$$

$$a^2(1-e^2) = b^2$$

$$b = a \sqrt{1-e^2}$$

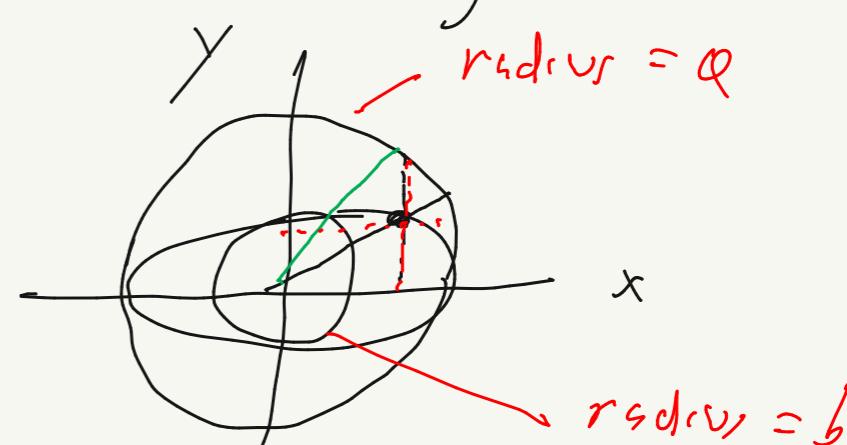
$$At (x, y), \quad \left(\frac{b}{a}\right)^2 = 1-e^2 \quad \rightarrow \quad e = \sqrt{1-\left(\frac{b}{a}\right)^2}$$



$$x = r \cos \theta$$

$$y = r \sin \theta$$

(but  $r$  changes)



$$x = a \cos \phi \quad \phi \neq \theta$$

$$y = b \sin \phi$$

Define:  $\operatorname{cn}(u; k) \equiv x/a$  where  $k = e$   
 $\operatorname{sn}(u; k) \equiv y/b$   $0 \leq k \leq 1$

$$\operatorname{dn}(u; k) \equiv r/a$$

~~and~~

where  $u = \int_b^{\theta} \sqrt{r^2 + r'^2} d\theta$

$u = \theta$  for circle

$u \neq \text{arc length}$

since  $ds = \sqrt{dr^2 + r'^2 d\theta}$

NOTE:  $b_u = \int_0^\theta r d\theta \leq \int_0^\theta ds \leq \text{arc length from } (a,0) \text{ to } (x,y)$

Properties of sn, cn, dn:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \rightarrow [cn^2 u + sn^2 u = 1]$$

$$x^2 + y^2 = r^2 \rightarrow a^2 cn^2 u + b^2 sn^2 u = a^2 dn^2 u$$

$$a^2(1 - sn^2 u) + b^2 sn^2 u = a^2 dn^2 u$$

$$1 - sn^2 u + \frac{b^2}{a^2} sn^2 u = dn^2 u$$

$$1 - \left(1 - \frac{b^2}{a^2}\right) sn^2 u = dn^2 u$$

$$1 - H^2 sn^2 u = dn^2 u$$

Thus,  $[dn^2 u + H^2 sn^2 u = 1]$

for circle  $H=0$ ,  $dn u = 1$

Derivatives:

$$\frac{d}{du} sn u = \frac{d}{du} \left( \frac{y}{b} \right) = \frac{dy}{b du}$$

Now:  $b du = r d\theta$

$$\rightarrow \frac{d}{du} sn u = \frac{dy}{r d\theta}$$

$$\begin{aligned}x &= r \cos \theta & dx &= dr \cos \theta - r \sin \theta d\theta \\y &= r \sin \theta & dy &= dr \sin \theta + r \cos \theta d\theta\end{aligned}$$

$$\begin{aligned}\rightarrow -\sin \theta dx &= -\sin \theta \cos \theta dr + r \sin^2 \theta d\theta \\+\cos \theta dy &= \cos \theta \sin \theta dr + r \cos^2 \theta d\theta\end{aligned}$$


---

add:  $\cos \theta dy - \sin \theta dx = r d\theta$

$$\rightarrow \frac{x}{r} dy - \frac{y}{r} dx = r d\theta$$

thus,

$$\begin{aligned}\frac{d \operatorname{cnu}}{dy} &= \frac{dy}{r d\theta} = \frac{\cancel{dy}}{\cancel{x} dy - \cancel{y} dx} \\&= \frac{r}{x - y \frac{dx}{dy}}\end{aligned}$$

All:  $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \Rightarrow \frac{2x dx}{a^2} + \frac{2y dy}{b^2} = 0$

$$\frac{dx}{dy} = -\frac{y}{x} \left(\frac{a}{b}\right)^2$$

$$\rightarrow \frac{d \operatorname{cnu}}{dy} = \frac{r}{x + \frac{y^2}{x} \left(\frac{a}{b}\right)^2} = \frac{r x}{x^2 + y^2 \left(\frac{a^2}{b^2}\right)}$$

$$= \frac{r}{a} \frac{x}{a} \underbrace{\frac{1}{\left(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2\right)}}_1 = \boxed{\operatorname{cnu} \cdot \operatorname{dn} u}$$

Then using

$$\sin^2 u + \cos^2 u = 1$$

we have  $\cancel{\frac{d}{du} \sin u} \frac{d}{du} \sin u + \cancel{\frac{d}{du} \cos u} \frac{d}{du} \cos u = 0$

$$\begin{aligned}\rightarrow \frac{d \cos u}{du} &= -\frac{\sin u}{\cos u} \frac{d}{du} \sin u \\ &= -\frac{\sin u}{\cos u} \cos u \cdot du \\ &= \boxed{-\sin u \cdot du}\end{aligned}$$

And using  $\sin^2 u + \pi^2 \sin^2 u = 1$ :

$$\cancel{\frac{d}{du} \sin u} \frac{d}{du} (\sin u) + \cancel{\pi^2 \sin u} \frac{d}{du} (\sin u) = 0$$

$$\begin{aligned}\rightarrow \frac{d}{du} (\sin u) &= -\pi^2 \frac{\sin u}{\cos u} \frac{d}{du} (\cos u) \\ &= -\pi^2 \frac{\sin u}{\cos u} \cos u \cdot du \\ &= \boxed{-\pi^2 \sin u \cos u}\end{aligned}$$

Summary:

$$\frac{d}{du} \sin u = \cos u \cdot du$$

$$\frac{d}{du} \cos u = -\sin u \cdot du$$

$$\frac{d}{du} \sin u = -\pi^2 \sin u \cos u$$

## Integration:

$$\frac{d \sin u}{du} = \csc u \cdot du$$

$$\int \frac{d(\sin u)}{\csc u \cdot du} = \int du = u + \text{const}$$

$$\text{Let } t = \sin u \Rightarrow \csc u = \sqrt{1-t^2}, \quad du = \sqrt{1-H^2 \sin^2 u}$$

$$\int \frac{dt}{\sqrt{1-t^2} \sqrt{1-H^2 t^2}} = u + \text{const}$$

But since  $t = \sin u$ :

$$\boxed{\int \frac{dt}{\sqrt{1-t^2} \sqrt{1-H^2 t^2}} = \sin^{-1}(t; H) + \text{const}}$$

In hand book:

$$\boxed{\int_0^{\sin \phi} \frac{dt}{\sqrt{1-t^2} \sqrt{1-H^2 t^2}} = F(\phi, H)}$$

Incomplete  
R.H.p.t.c  
integral of  
the 1st  
kind

$$\boxed{\int_0^{\phi} \frac{d\theta}{\sqrt{1-H^2 \sin^2 \theta}} = F(\phi, H)}$$

where  $t = \sin \theta$ ,

NOTE:  $\boxed{\sin \phi = \sin u} \star$

Complete elliptic integral of 1<sup>st</sup> kind:

$$K(\tau) = \int_0^1 \frac{dt}{\sqrt{1-t^2} \sqrt{1-\tau^2 t^2}}$$

~~~~~

Connection to simple pendulum:

$$\varphi = \frac{\omega_0}{t} K\left(\tau = \sin\left(\frac{\phi_0}{2}\right)\right)$$

$$\omega_0 t = \operatorname{sn}^{-1} \left( x \equiv \frac{\sin\left(\frac{\phi_0}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}, \tau \equiv \sin\left(\frac{\phi_0}{2}\right) \right) + \text{const}$$

↑  
 $\frac{g}{l}$       -  $\tau = E/H$ )

for  $\phi = \phi_0$   
when  $t = 0$

→  $\phi(t) = 2 \operatorname{arc sin} \left[ \tau \operatorname{sn} \left( \omega_0 \left( t + \frac{P}{q} \right); \tau \right) \right]$

Elliptic integral of 2<sup>nd</sup> kind (circumference of ellipse w.r.t y-axis)

$$E(\phi, \kappa) = \int_0^{\phi} dt \frac{\sqrt{1 - \kappa^2 t^2}}{\sqrt{1 - t^2}}$$

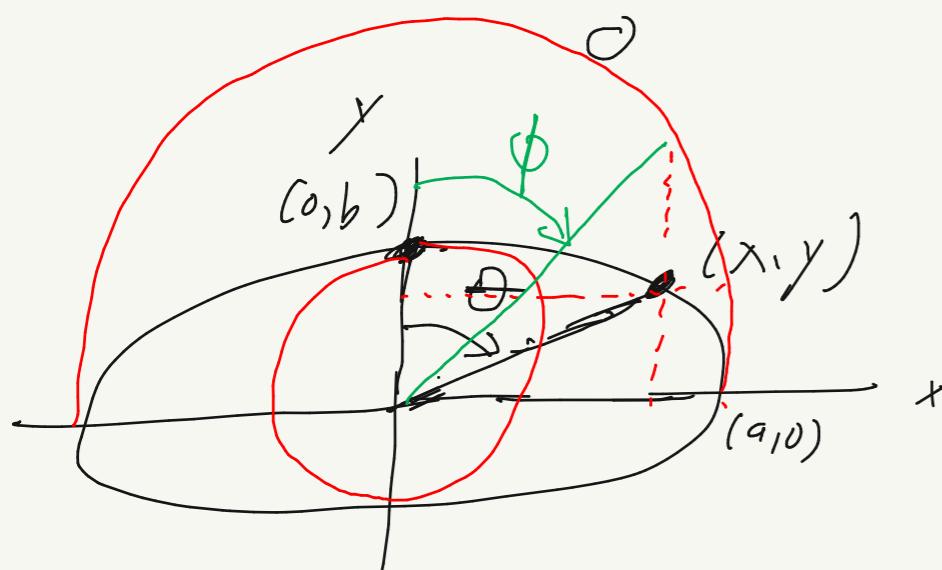
$\curvearrowright$   
cw

Rew., to:

$$t = \sin \bar{\phi}$$

$$dt = \cos \bar{\phi} d\bar{\phi} = \sqrt{1 - t^2} d\bar{\phi}$$

$$E(\phi, \kappa) = \int_0^{\phi} d\bar{\phi} \sqrt{1 - \kappa^2 \sin^2 \bar{\phi}} \quad (\text{scipy definition})$$



$$\kappa = e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

$$\begin{aligned} x &= r \sin \theta &= a \sin \psi &= a \sin \phi \\ y &= r \cos \theta &= b \cos \psi &= b \cos \phi \end{aligned} \quad \left. \begin{array}{l} \text{sin, cos} \\ \text{interchanged} \\ \text{from} \\ \text{previous} \\ \text{page} \end{array} \right\}$$

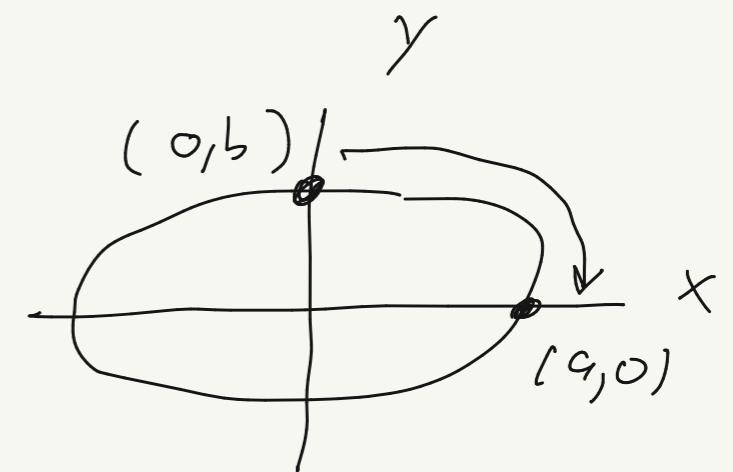
$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} d\phi \\ &= \sqrt{a^2 (1 - \sin^2 \phi) + b^2 \sin^2 \phi} d\phi \\ &= a \sqrt{1 - \left(1 - \left(\frac{b}{a}\right)^2\right) \sin^2 \phi} d\phi \\ &= a \sqrt{1 - \kappa^2 \sin^2 \phi} d\phi \end{aligned}$$

$$\begin{cases} (\phi \neq \theta) \\ \tan \theta = \frac{a}{b} \tan \phi \\ \theta = \arctan \left( \frac{a}{b} \tan \phi \right) \\ \phi = \arctan \left( \frac{b}{a} \tan \theta \right) \end{cases}$$

$$S_{(0,b) \rightarrow (x,y)} = a \int_0^{\phi} \sqrt{1 - \kappa^2 \sin^2 \bar{\phi}} d\bar{\phi} \quad \boxed{=} E(\phi, \kappa)$$

Complete elliptic integral of 2<sup>nd</sup> kind :

$$\begin{aligned} E(k) &= E\left(\frac{\pi}{2}, k\right) \\ &= \int_0^{\frac{\pi}{2}} d\phi \sqrt{1 - k^2 \sin^2 \phi} \\ &= \int_0^1 dt \frac{\sqrt{1 - k^2 t^2}}{\sqrt{1 - t^2}} \end{aligned}$$



Circumference:

$$C = 4a \int_0^1 dt \frac{\sqrt{1 - k^2 t^2}}{\sqrt{1 - t^2}}$$

1<sup>st</sup> order correction (for  $k \ll 1$ ; nearly circular)

$$C \approx 4a \int_0^1 \frac{dt}{\sqrt{1 - t^2}} \left( 1 - \frac{1}{2} k^2 t^2 \right)$$

$$\approx 4a \left[ \underbrace{\int_0^1 \frac{dt}{\sqrt{1 - t^2}}}_{\text{circumference of unit circle}} - \frac{k^2}{2} \int_0^1 \frac{dt}{\sqrt{1 - t^2}} t^2 \right]$$

$t = \sin x$

$$\sin^{-1}(1) = \frac{\pi}{2}$$

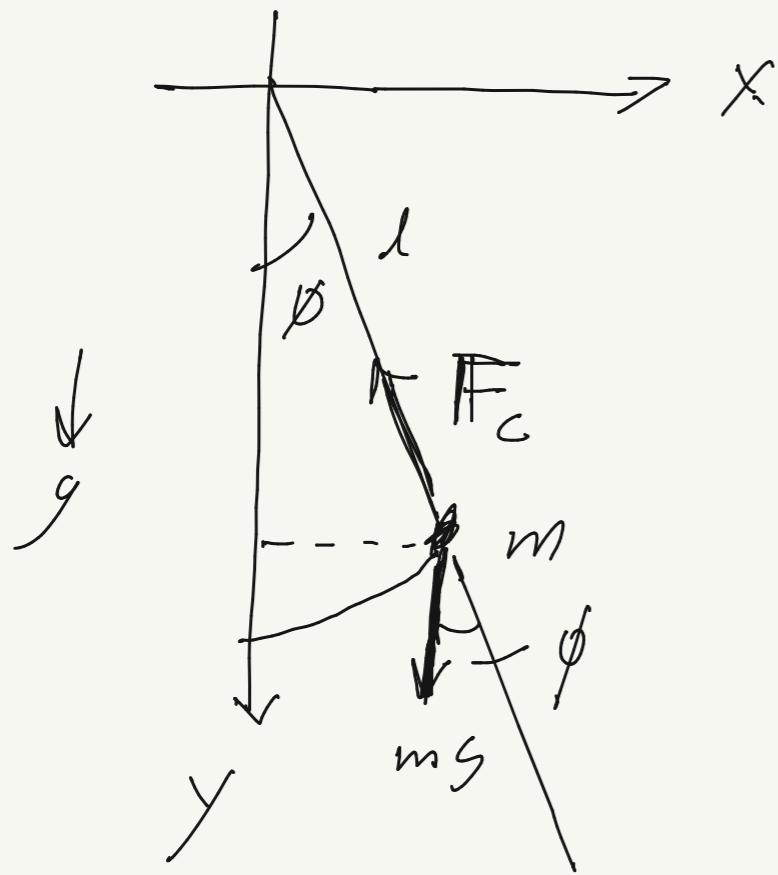
$$\approx 4a \left[ \frac{\pi}{2} - \frac{k^2}{2} \int_0^{\pi/2} \frac{\cos dx \sin^2 x}{\sqrt{1 - \sin^2 x}} \right]$$

$$\approx 4a \left[ \frac{\pi}{2} - \frac{k^2}{2} \int_0^{\pi/2} (1 - \cos^2 x)^{-1/2} dx \right]$$

$$= 4a \left[ \frac{\pi}{2} - \frac{k^2}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right]$$

$$= 2\pi a \left( 1 - \frac{k^2}{4} \right)$$

# Simple pendulum (freshman physics analysis) :



$$mg \sin \phi = -m\alpha_T \\ = -m l \ddot{\phi}$$

$$\rightarrow \ddot{\phi} = -\frac{g}{l} \sin \phi$$

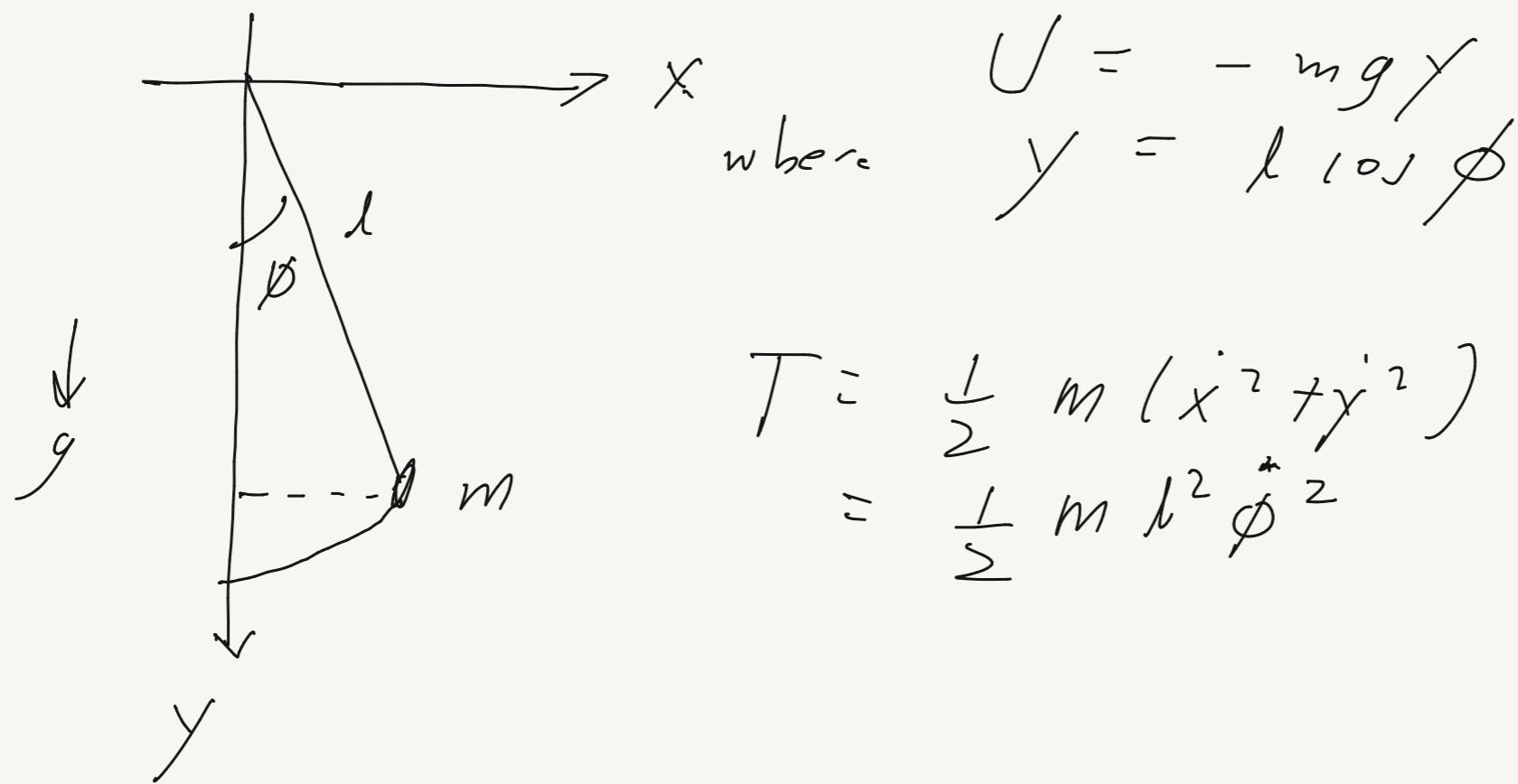
constraint force :

$$F_c - mg \cos \phi = m a_c \\ = m \omega^2 l \\ = m l \dot{\phi}^2$$

$$\text{Thus, } F_c = mg \cos \phi + m l \dot{\phi}^2$$

(non-zero at turning points as well as in vertical position - i.e.,  $\theta = 0$ )

## Period of a simple pendulum:



$$T = \frac{1}{2} m (x^2 + y^2) \\ = \frac{1}{2} m l^2 \dot{\phi}^2$$

$$L = T - U \\ = \frac{1}{2} m l^2 \dot{\phi}^2 + mg l \cos \phi$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi}$$

$$\frac{d}{dt} (m l^2 \dot{\phi}) = -m g l \sin \phi$$

$$m l^2 \ddot{\phi} = -m g l \sin \phi$$

$$\rightarrow \boxed{\ddot{\phi} = -\frac{g}{l} \sin \phi} \quad (\text{same as before})$$

Small-angle approx:  $\sin \phi \approx \phi$

$$\Rightarrow \ddot{\phi} \approx -\frac{g}{l} \phi = -\omega^2 \phi$$

Sol:  $\phi(t) = Ae^{i\omega t}, \quad \omega = \sqrt{\frac{g}{l}}, \quad P = \frac{2\pi}{\omega}$

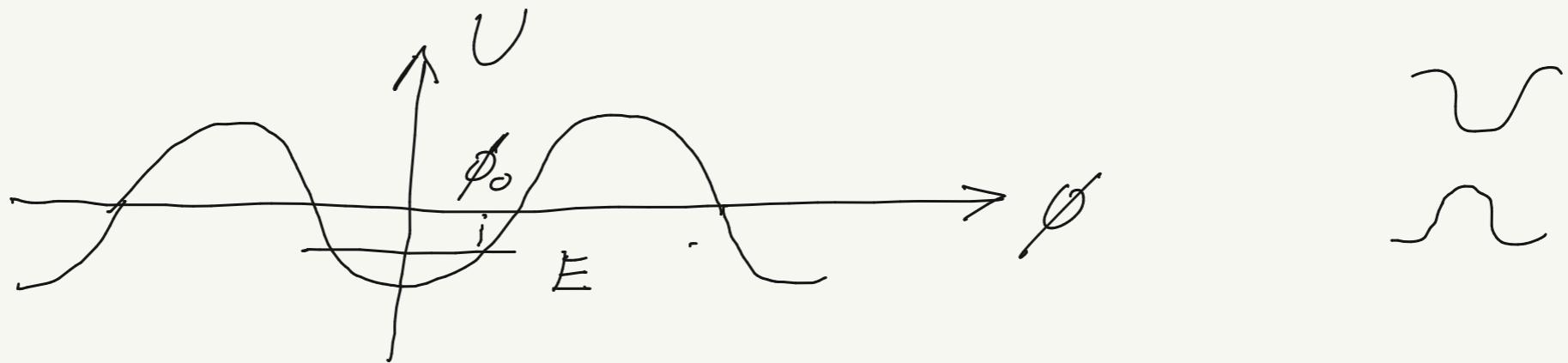
Complex

Beyond small-angle approx:

Cons. of Energy:

$$\begin{aligned} E &= T + U \\ &= \frac{1}{2} m l^2 \dot{\phi}^2 - m g l \cos \phi \end{aligned}$$

$$E = \text{const} = -m g l \cos \phi_0 \quad \text{at turning points}$$



Thus,

$$-m g l \cos \phi_0 = \frac{1}{2} m l^2 \dot{\phi}^2 - m g l \cos \phi$$

$$\begin{aligned} \frac{1}{2} m l^2 \dot{\phi}^2 &= +m g l (\cos \phi - \cos \phi_0) \\ &\geq 0 \quad \text{since } \phi \leq \phi_0 \end{aligned}$$

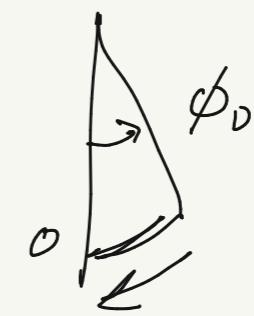
$$\rightarrow \dot{\phi} = \pm \sqrt{2 \frac{g}{l} (\cos \phi - \cos \phi_0)}$$

$$\int dt = \pm \int \frac{d\phi}{\sqrt{2 \frac{g}{l} \sqrt{\cos \phi - \cos \phi_0}}}$$

$$\rightarrow \left[ t = \frac{1}{\sqrt{2} \omega_0} \int \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_0}} + \text{const} \right] \quad \omega_0 = \sqrt{\frac{g}{l}}$$

Period:

$$\frac{\text{Period}}{P} = \frac{4}{\sqrt{2} \omega_0} \int_0^{\phi_0} \frac{d\phi}{\sqrt{(\omega_0)\phi - (\omega_0)\phi_0}}$$



$$\begin{aligned}
 \text{Now: } \cos \phi &= \cos\left(2 \frac{\phi}{2}\right) \\
 &= \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right) \\
 &= 1 - 2 \sin^2\left(\frac{\phi}{2}\right)
 \end{aligned}$$

$$\rightarrow (0/\phi - 0/\phi_0) = -2 \sin^2\left(\frac{\phi}{2}\right) + 2 \sin^2\left(\frac{\phi_0}{2}\right)$$

$$= 2 \sin^2\left(\frac{\phi_0}{2}\right) \left[ 1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)} \right]$$

$$\begin{aligned}
 \text{Let: } x &= \frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)} \quad \rightarrow \quad dx = \frac{1}{\sin\left(\frac{\phi_0}{2}\right)} \frac{1}{2} \cos\left(\frac{\phi}{2}\right) d\phi \\
 &\qquad \nearrow \\
 \text{definer } H &= \sin\left(\frac{\phi_0}{2}\right)
 \end{aligned}$$

Thurs,

$$P = \frac{H}{\sqrt{2} w_0} \int_0^R \frac{dx}{\sqrt{1 - H^2 x^2}}$$

$$= \left[ \frac{4}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1-t^2x^2} \sqrt{1-x^2}} \right] = \frac{4}{\omega_0} E(t)$$

complete elliptic integral  
of the first kind

Leading-order correction to period:

$$P = \frac{4}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1-H^2x^2}} \sqrt{1-x^2}$$

$$\text{Suppose } \phi_0 \ll 1 \rightarrow H \approx \sin\left(\frac{\phi_0}{2}\right) \approx \frac{\phi_0}{2} \ll 1$$

$$\begin{aligned} \text{Then, } \frac{1}{\sqrt{1-H^2x^2}} &\approx 1 + \frac{1}{2} H^2 x^2 \\ &= 1 + \frac{1}{2} \left(\frac{\phi_0}{2}\right)^2 x^2 \\ &= 1 + \frac{1}{8} \phi_0^2 x^2 \end{aligned}$$

$$\begin{aligned} \rightarrow P &\approx \frac{4}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1-x^2}} \left[ 1 + \frac{1}{8} \phi_0^2 x^2 \right] \\ &= \frac{4}{\omega_0} \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \frac{\phi_0^2}{2\omega_0} \int_0^1 \frac{dx}{\sqrt{1-x^2}} x^2 \\ &= \frac{4}{\omega_0} \sin^{-1}(1) + \frac{\phi_0^2}{2\omega_0} \int_0^{\pi/2} \frac{\cos \theta d\theta \sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \\ &= \frac{4}{\omega_0} \cdot \frac{\pi}{2} + \frac{\phi_0^2}{2\omega_0} \int_0^{\pi/2} d\theta \frac{1}{2} (1 - \sin 2\theta) \\ &= \frac{2\pi}{\omega_0} + \frac{\phi_0^2}{4\omega_0} \left( \frac{\pi}{2} - \frac{1}{2} \sin 2\theta \Big|_0^{\pi/2} \right) \\ &= \boxed{\frac{2\pi}{\omega_0} \left( 1 + \frac{\phi_0^2}{16} \right)} \end{aligned}$$

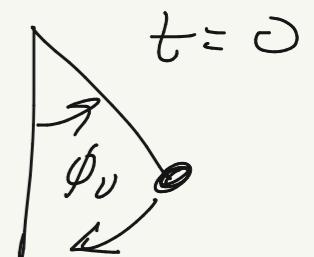
General time dependence of oscillation:

$$t = \frac{1}{\sqrt{2} \omega_0} \int \frac{d\phi}{\sqrt{\cos(\phi) - \cos(\phi_0)}} + \text{const} \quad ; \quad \omega_0 = \sqrt{\frac{g}{\ell}}$$

$$= \frac{1}{\omega_0} \int \frac{dx}{\sqrt{1-t^2 x^2} \sqrt{1-x^2}} + \text{const}$$

$$= \frac{1}{\omega_0} \left[ \operatorname{sn}^{-1} \left( x = \frac{\sin(\phi/2)}{\sin(\phi_0/2)} ; \; \bar{t} = \sin\left(\frac{\phi_0}{2}\right) \right) + \text{const} \right]$$

choose const so that  $t=0 \iff \phi=\phi_0$



$$\phi = \operatorname{sn}^{-1}(1, \bar{t}) + \text{const}$$

$$\begin{aligned} \rightarrow \text{const} &= -\operatorname{sn}^{-1}(1, \bar{t}) = -F\left(\frac{\pi}{2}, \bar{t}\right) = -\vec{F}_r/\bar{t} \\ &= -\frac{\omega_0}{4} P \end{aligned}$$

thus,

$$\omega_0 t = \operatorname{sn}^{-1}(x, \bar{t}) - \frac{\omega_0}{4} P$$

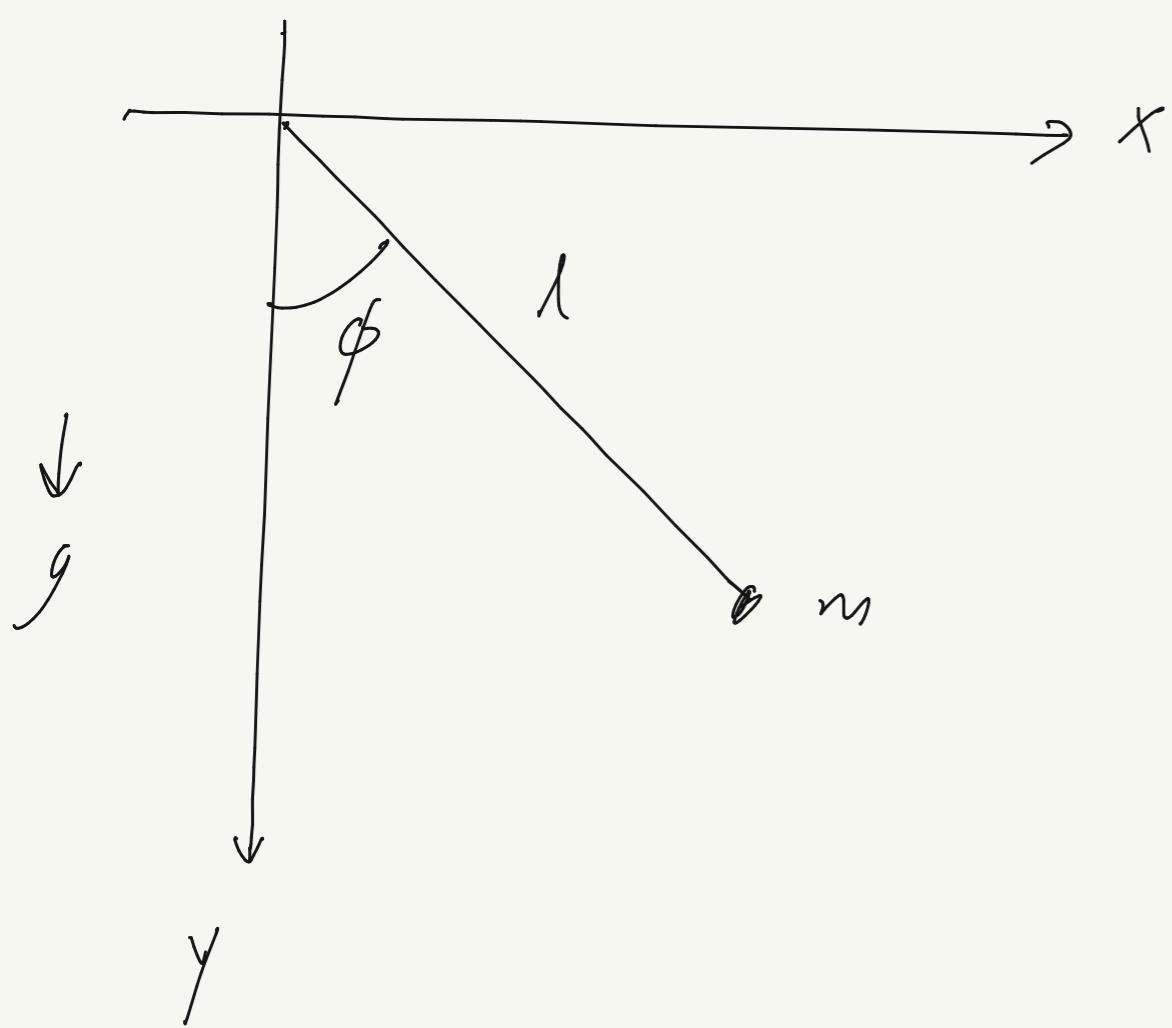
$$\omega_0 \left( t + \frac{P}{4} \right) = \operatorname{sn}^{-1}(x, \bar{t})$$

$$\operatorname{sn} \left[ \omega_0 \left( t + \frac{P}{4} \right); \bar{t} = \sin\left(\frac{\phi_0}{2}\right) \right] = x = \frac{\sin(\phi/2)}{\sin(\phi_0/2)}$$

$$\rightarrow \boxed{\phi(t) = 2 \arcsin \left[ \underbrace{\sin\left(\frac{\phi_0}{2}\right)}_K \operatorname{sn} \left( \omega_0 \left( t + \frac{P}{4} \right); \bar{t} = \sin\left(\frac{\phi_0}{2}\right) \right) \right]}$$

Lagrange multipliers:

constraint:



$$\ell = r = \sqrt{x^2 + y^2}$$

$$U = -mgy$$

$$= -mg r \cos\phi$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m (r^2 \dot{\phi}^2 + \dot{r}^2)$$

$$\varphi(r, \phi) = \ell - r = 0$$

$$L = T - U + \lambda \varphi$$

$$= \frac{1}{2} m (r^2 \dot{\phi}^2 + \dot{r}^2) + mg r \cos\phi + \lambda (\ell - r)$$

Euler-Lagrange:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} \quad (1)$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \quad (2)$$

$$\dot{\phi} = 0 \quad (3)$$

$$\frac{d}{dt} (mr) = m\dot{r} + r\dot{\phi}^2 + mg \cos\phi - \lambda$$

$\ddot{r} = r\dot{\phi}^2 + g \cos\phi - \frac{\lambda}{m}$

(1)

$$\frac{d}{dt} (mr^2\dot{\phi}) = -mgv \sin \phi$$

$$2mr\ddot{r}\dot{\phi} + mr^2\ddot{\phi} = -mg r \sin \phi$$

$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = -\frac{g}{r} \sin \phi$

(2)

$\ell - r = 0$

(3)

Differentiate constraint twice w.r.t time:

$$\rightarrow \dot{r} = 0, \quad \ddot{r} = 0$$

Substitute for  $\ddot{r}$  using (1) :

$$0 = \ddot{\phi} = r\dot{\phi}^2 + g \cos \phi - \frac{\lambda}{m}$$

$$\begin{aligned} \rightarrow \lambda &= mr\dot{\phi}^2 + mg \cos \phi \\ &= ml\dot{\phi}^2 + mg \cos \phi \end{aligned}$$

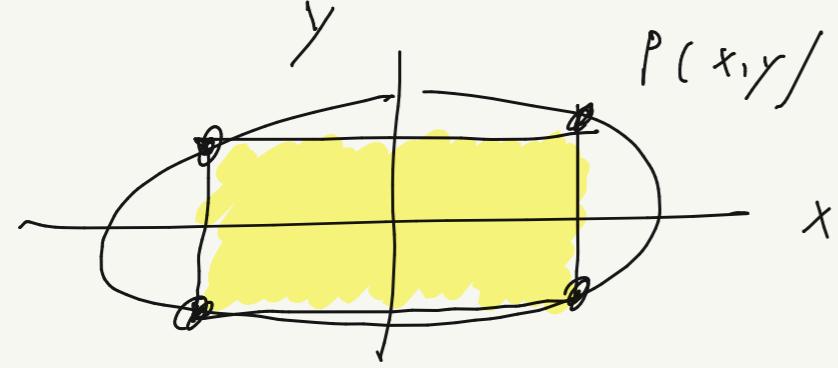
Compare with freshman physics calculation

$$F_c = mg \cos \phi + ml\dot{\phi}^2 \quad (\text{in } -\vec{r} \text{ direction})$$

Thus,

$$\overrightarrow{F}_c = \lambda \vec{\nabla} \phi \quad (\text{since } \phi = \ell - r \rightarrow \vec{\nabla} \phi = -\vec{r})$$

Another example:



Suppose you want to maximize the area of a rectangle subject to the constraint that its corners lie on an ellipse  $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$  (with  $a, b$  given).

Two methods:

$$\text{Area} = 4xy$$

$$\text{Constraint: } \varphi(x, y) = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$$

① Define:

$$\begin{aligned} F(x, y, \lambda) &= 4xy - \lambda \varphi(x, y) \\ &= 4xy - \lambda \left( 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 \right) \end{aligned}$$

require:  $\frac{\partial F}{\partial x} = 0 \quad (1)$

$$\frac{\partial F}{\partial y} = 0 \quad (2)$$

$$\frac{\partial F}{\partial \lambda} = 0 \quad (3)$$

$$(1) \quad 0 = 4y + \frac{2\lambda x}{a^2} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad \begin{array}{l} 3 \text{ equations,} \\ 3 \text{ unknowns,} \\ (x, y, \lambda) \end{array}$$

$$(2) \quad 0 = 4x + \frac{2\lambda y}{b^2}$$

$$(3) \quad 0 = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$$

$$O = \frac{4y^2}{b^2} + \frac{2\lambda xy}{a^2 b^2}$$

$$O = \frac{4x^2}{a^2} + \frac{2\lambda yx}{b^2 a^2}$$

Subtract:  $O = 4 \left[ \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 \right]$

$$\rightarrow \frac{x}{a} = \pm \frac{y}{b}$$

Substitute into (3) :-  $O = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$

$$2 \left(\frac{x}{a}\right)^2 = 1$$

$$\frac{x}{a} = \sqrt{\frac{1}{2}}$$

$$\boxed{x = \frac{a}{\sqrt{2}}} \rightarrow \boxed{y = \frac{b}{\sqrt{2}}}$$

(2.) Reduced space method

$$F(x) \equiv 4xy \quad |$$

$$y = b \sqrt{1 - \left(\frac{x}{a}\right)^2}$$

$$= 4x b \sqrt{1 - \left(\frac{x}{a}\right)^2}$$

Solve  $F'(x) = 0$

$$\begin{aligned}
 O &= F'(x) \\
 &= 4b \sqrt{1 - \left(\frac{x}{a}\right)^2} + 4x b \frac{1}{a^2} \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \left(-\frac{2x}{a^2}\right) \\
 &= 4b \left( \sqrt{1 - \left(\frac{x}{a}\right)^2} - \left(\frac{x}{a}\right)^2 \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \right) \\
 &= \frac{4b}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \left[ 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^2 \right] \\
 &= \frac{4b}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \left[ 1 - 2\left(\frac{x}{a}\right)^2 \right]
 \end{aligned}$$

$$\begin{aligned}
 \rightarrow \left(\frac{x}{a}\right)^2 &= \frac{1}{2} \rightarrow \boxed{x = \frac{a}{\sqrt{2}}} \\
 \rightarrow \boxed{y = b \sqrt{1 - \left(\frac{x}{a}\right)^2}} &\quad \Big| \quad \frac{x}{a} = \frac{1}{\sqrt{2}} \\
 &= \frac{b}{\sqrt{2}}
 \end{aligned}$$

(same result as Lagrange multiplier method)