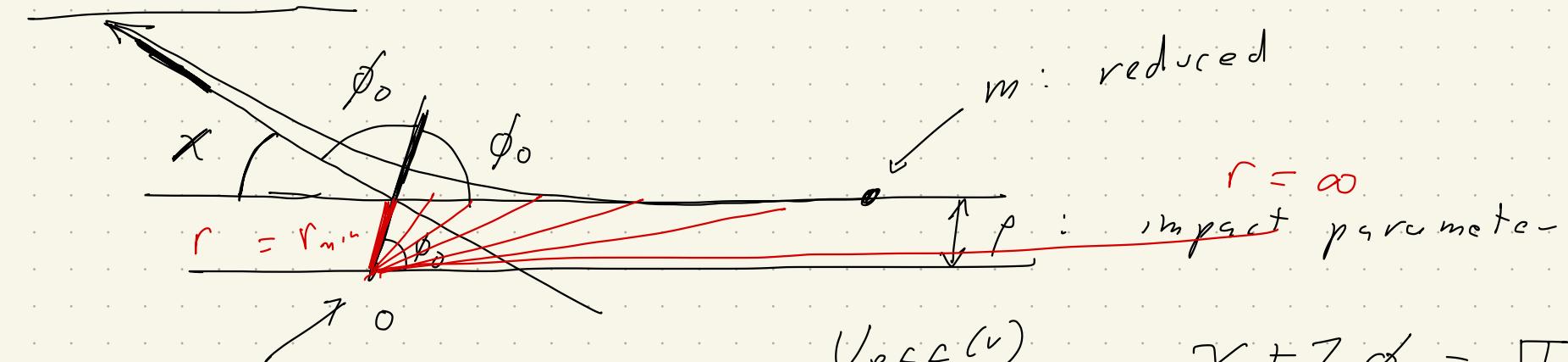


Lecture #16

Thurs 10/14



(CM
of original
system)

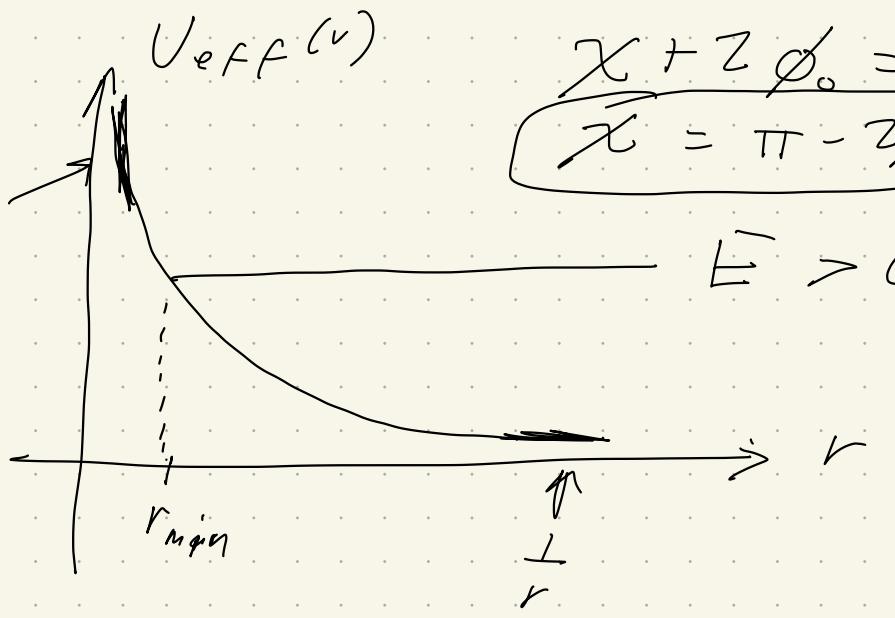
$$U(r) = \frac{\alpha}{r}, \quad \alpha > 0$$

(repulsive)

$$U_{\text{eff}}(r) = U(r) + \frac{M_z^2}{2mr^2}$$

$$= \frac{\alpha}{r} + \frac{M_z^2}{2mr^2}$$

↑ ↑



$$E = U_{\text{eff}}(r_{\min})$$

↑
turning point

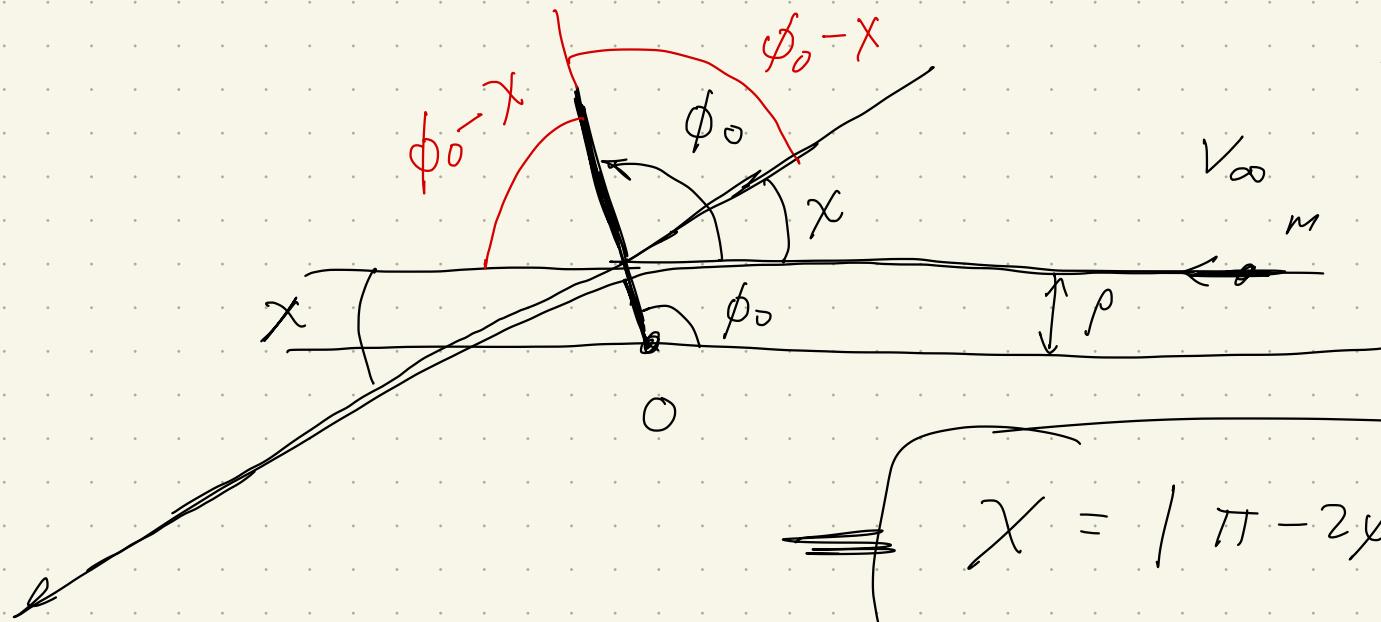
m : reduced

$r = \infty$

P : impact parameter

$$\begin{aligned} x + z\phi_0 &= \pi \\ x &= \pi - z\phi_0 \end{aligned}$$

$$E > 0$$



$$2(\phi_0 - x) + x = \pi$$

$$2\phi_0 - x = \pi$$

$$x = 2\phi_0 - \pi$$

$$x = |\pi - 2\phi_0|$$

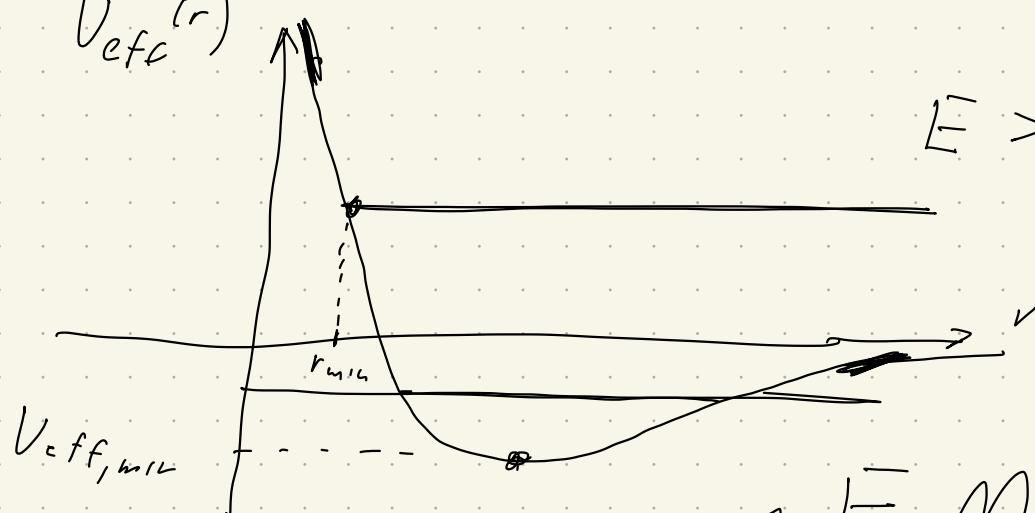
for both
repulsive &
attractive

$$U(r) = -\frac{\alpha}{r}$$

$$U_{eff}(r) = -\frac{\alpha}{r} + \frac{M_t^2}{2mr^2}$$



$$U_{eff}(r)$$



E, M

p, v_∞

$$E = \frac{1}{2}mv_\infty^2$$

$$M = mp v_\infty$$

Sect 14:

$$t = \pm \int \frac{dr}{\sqrt{\dots}} + \text{const}$$

$\Rightarrow \phi = \pm \int \frac{M dr/r^2}{\sqrt{2m(E - V(r)) - \frac{M^2}{r^2}}} + \text{const}$

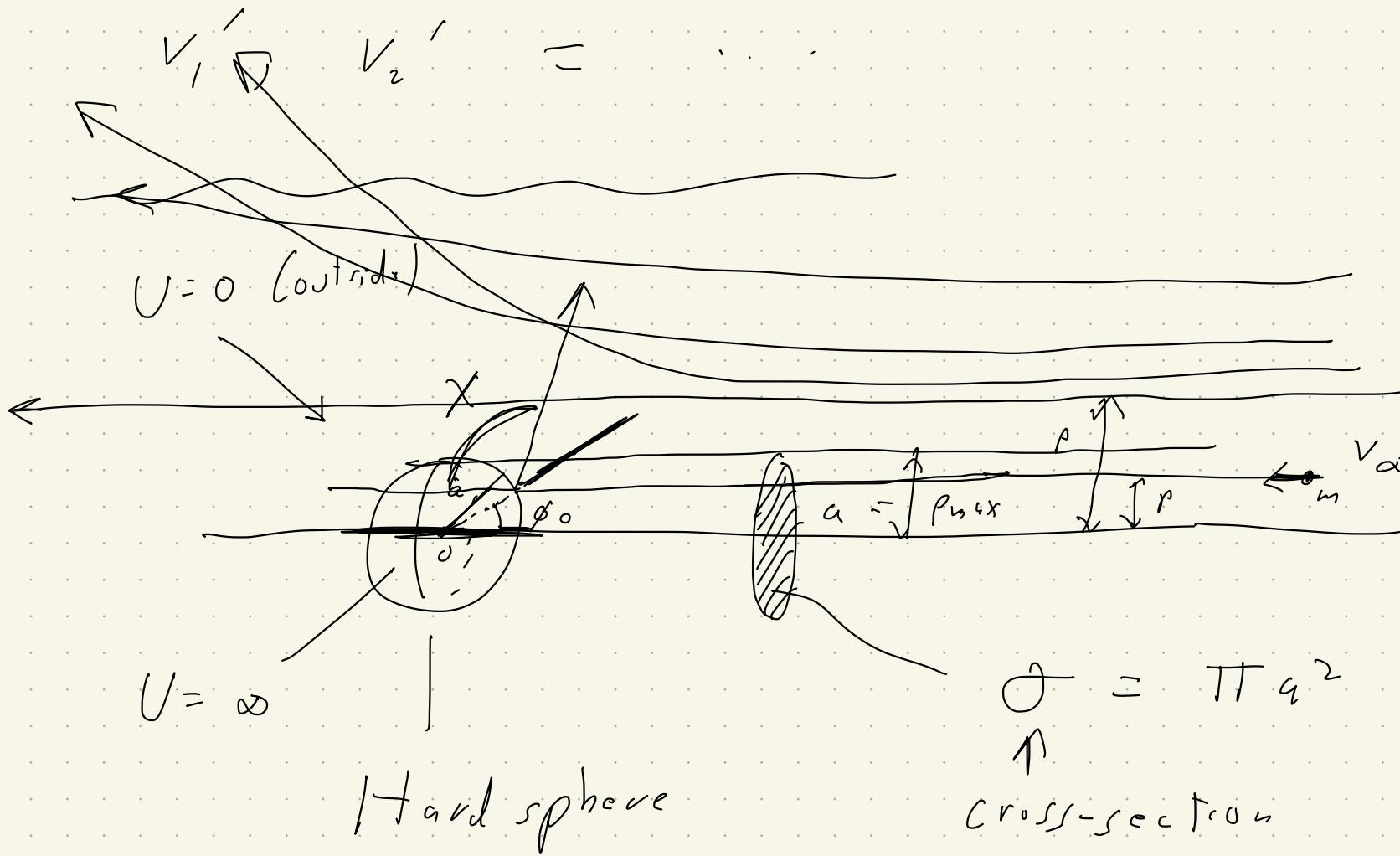
$$\begin{aligned} &= \int_{r_{min}}^{\infty} \frac{M dr/r^2}{\sqrt{2m(E - V(r)) - \frac{M^2}{r^2}}} \quad E = U_{eff}(r_{max}) \\ &= \int_{r_{min}}^{\infty} \frac{m\rho v_\infty dr/r^2}{\sqrt{2m\left(\frac{1}{2}mv_\infty^2 - V(r)\right) - \frac{m^2\rho^2 v_\infty^2}{r^2}}} \\ &= \int_{r_{min}}^{\infty} \frac{\rho dr/r^2}{\sqrt{1 - \frac{V(r)}{\frac{1}{2}mv_\infty^2} - \frac{\rho^2}{r^2}}} \end{aligned}$$

$M = m\rho v_\infty$
 $E = \frac{1}{2}mv_\infty^2$

$$\theta_2 = \frac{1}{2}(\pi - x)$$

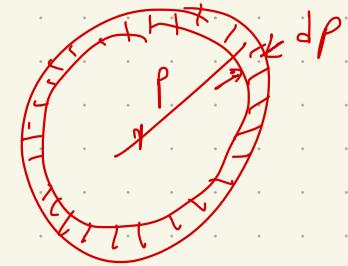
$$x = |\pi - 2\phi_0|$$

$$f_{q1}\theta_1 = \frac{m_2 \sin x}{m_1 + m_2 \cos x}$$



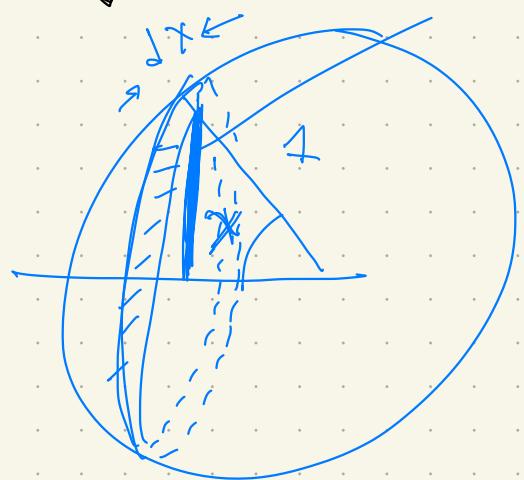
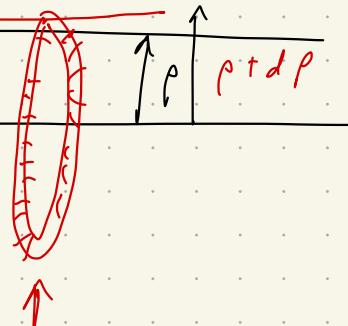
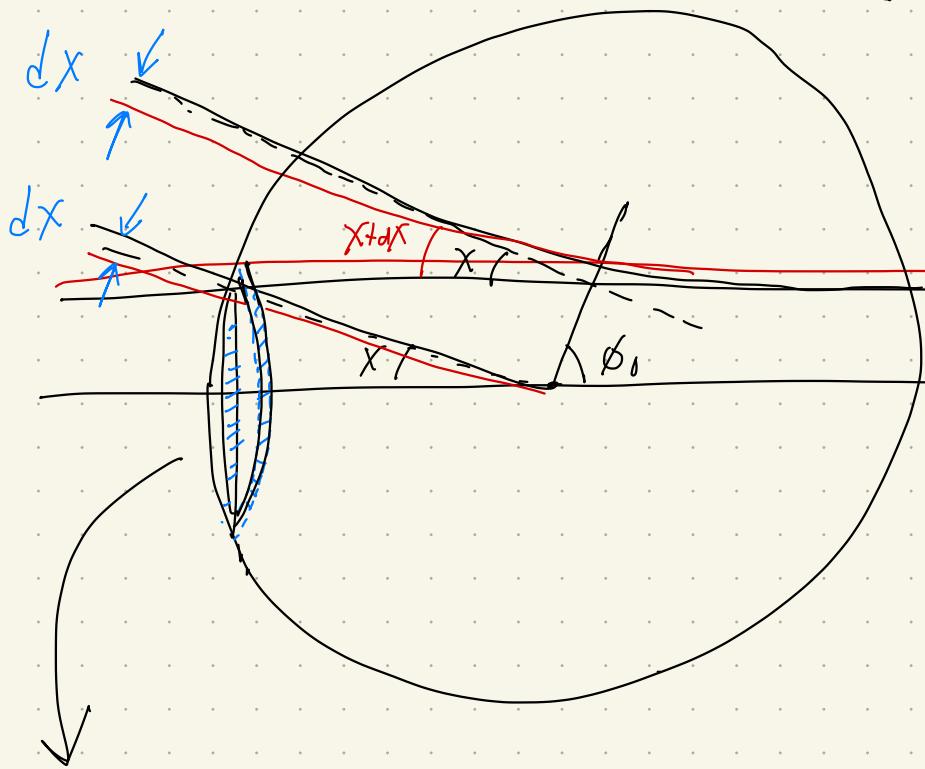
$$d\sigma = 2\pi \rho(x) \left| \frac{dp}{dx} \right| dx$$

$$d\sigma = 2\pi \rho dp$$



Differential cross sect.

!o



$$d\sigma = \frac{\rho(x)}{\sin x} \left| \frac{dp}{dx} \right| d\Omega$$

$$d\Omega = 2\pi \sin x dx$$

solid angle

$$\frac{d\sigma}{d\Omega} = \frac{2\pi \rho dp}{2\pi \sin x dx} = \frac{\rho(x)}{\sin x} \left| \frac{dp}{dx} \right|$$

$$\frac{d\sigma}{d\Omega} = \frac{p}{\sin x} \left| \frac{dp}{dx} \right|$$

Com frame

$$\frac{d\sigma_1}{d\Omega_1} = \frac{p}{\sin \theta_1} \left| \frac{dp}{d\theta_1} \right|$$

Lab frame
 θ_1

$$\frac{d\sigma_2}{d\Omega_2} = \frac{p}{\sin \theta_2} \left| \frac{dp}{d\theta_2} \right|$$

Lab frame
 θ_2

$$\begin{aligned} \frac{d\sigma_1}{d\Omega_1} &= \frac{\sin x}{\sin \theta_1} \left| \frac{dx}{d\theta_1} \right| \frac{d\sigma}{d\Omega} \\ &= \left| \frac{d(\cos x)}{d(\cos \theta_1)} \right| \left(\frac{d\sigma}{d\Omega} \right) \end{aligned}$$

calculated
for the
Com

Lecture #17: Tues Oct 19th

Next week: start small oscillations

Today / Thurs: Q & A (collisions and scattering)

Lecture #18: Thur Oct 21st

Next week: small oscillations

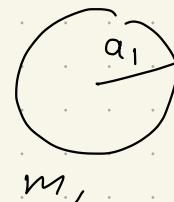
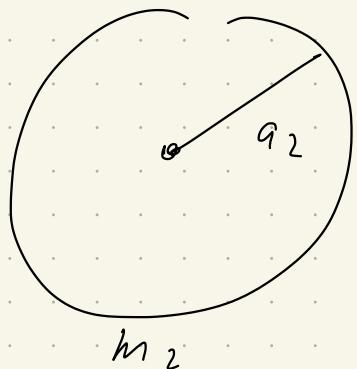
Today: Q & A, Quiz #3

(last 20 minutes)

Q3 - name.pdf

Quiz #3:

Two hard spheres with masses m_1, m_2 and radii a_1, a_2 :



$$\rho(X) = (a_1 + a_2) \cos\left(\frac{X}{2}\right)$$

a) Find $\rho = \rho(X)$ where $X = \text{scat. angle w.r.t. com}$

b) What value of ρ will give $\theta_2 = 60^\circ$?

$$2\theta_2 + X = \pi \rightarrow X = \pi - 2\theta_2 \\ = 60^\circ$$

$$\rho = (a_1 + a_2) \cos(30^\circ) \\ = \frac{\sqrt{3}}{2} (a_1 + a_2)$$

Lec #19:

10/26

Small oscillations:

Sec 21

, 22, 23

more
than
1d

Free oscillations
in 1-d

Forced oscillations
in 1d

Damping : Sec 25 (not covered)

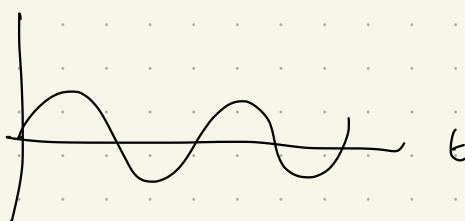
SHM, (Simple harmonic motion)

i) sinusoidal

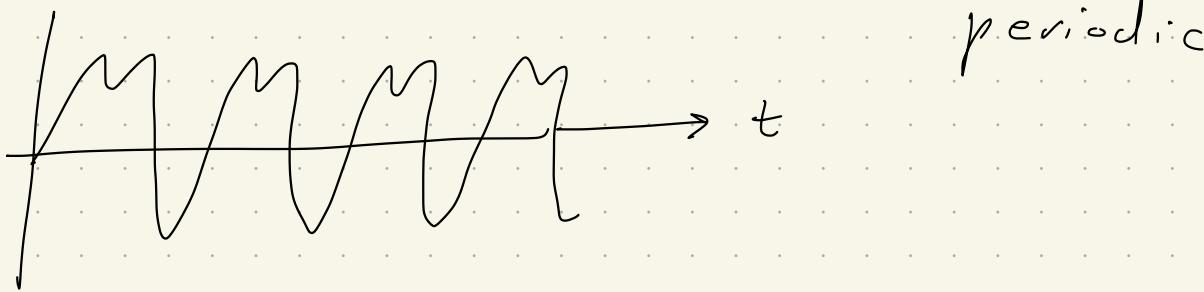
$$\omega = \frac{2\pi}{T}$$

$$x(t) = a \cos(\omega t + \alpha)$$

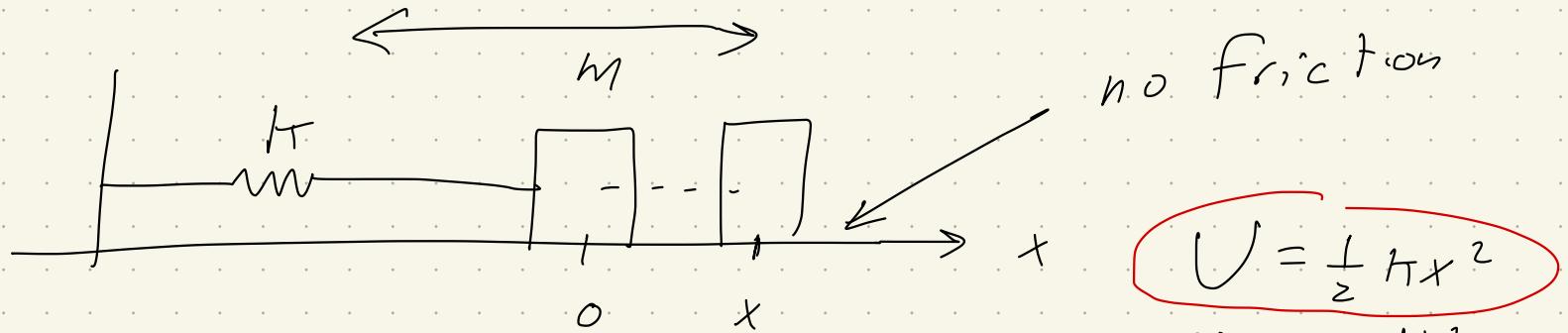
sinus



amplitude | phase
 |
 angular
 freq



Example:



$$F = -kx \quad \begin{matrix} \text{linear} \\ \text{restoring} \end{matrix}$$

$$F = m\ddot{x} = -kx \rightarrow \ddot{x} = -\frac{k}{m}x$$

$$\begin{aligned} F &= -\frac{dU}{dx} \\ &= -kx \end{aligned}$$

sol'n: $x = C_1 \cos \omega t + C_2 \sin \omega t$

$$\omega = \sqrt{\frac{k}{m}}$$

Alternatives:

$$x = a \cos(\omega t + \alpha)$$

$$x = \operatorname{Re}[A e^{i\omega t}], A = a e^{i\alpha}$$

$$T = \frac{1}{2} m \dot{x}^2$$

$$U = \frac{1}{2} k x^2$$

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \end{aligned}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \rightarrow m \ddot{x} = -kx$$

$$U(q) = q^3$$

$$q_0 = 0$$

More generally:

$$\left(\frac{d^2 U}{d q^2} \Big|_{q_0} = 0 \right)$$

pt. of inflection
(saddle)

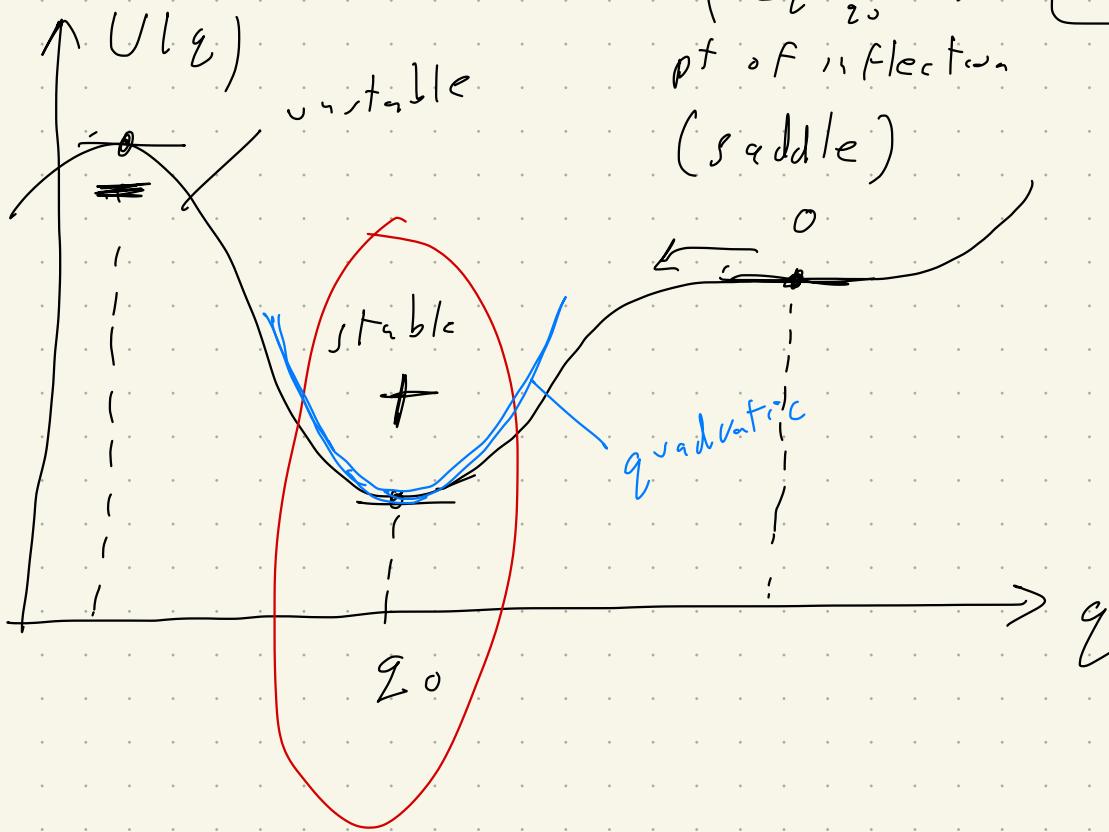
$$L = \frac{1}{2} a(q) \dot{q}^2 - U(q)$$

Equilibrium: ($q = q_0$)

$$0 = F = -\frac{dU}{dq} \Big|_{q_0}$$

stable:

$$\frac{d^2 U}{d q^2} \Big|_{q_0} \geq 0$$



$$U = c_{\text{const}}$$

Expansion: (about q_0)

$$U(q) = U(q_0) + \underbrace{\frac{dU}{dq} \Big|_{q_0} (q - q_0)}_{\text{const}} + \frac{1}{2} \left(\frac{d^2 U}{dq^2} \Big|_{q_0} \right) (q - q_0)^2$$

(ignore $\sim L$)

$K > 0$

$$+ \frac{1}{3!} \left(\frac{d^3 U}{dq^3} \Big|_{q_0} \right) (q - q_0)^3 + \dots$$

ignore for $q - q_0$ small

$$U(q) \approx \frac{1}{2} K x^2$$

$$\begin{aligned} T &= \frac{1}{2} q(q) \dot{q}^2 \\ &= \frac{1}{2} q(q_0) \dot{x}^2 \\ &\approx \frac{1}{2} m \dot{x}^2 \end{aligned}$$

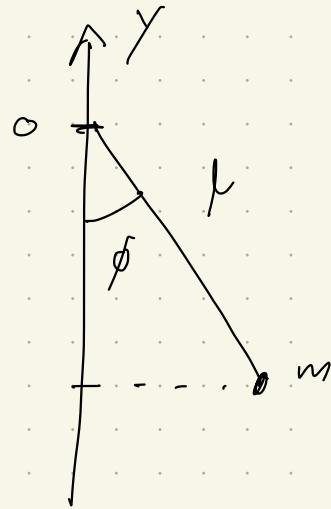
$$x \equiv q - q_0$$

$$K = \left. \frac{d^2 U}{dq^2} \right|_{q_0}$$

$$L \approx \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2$$

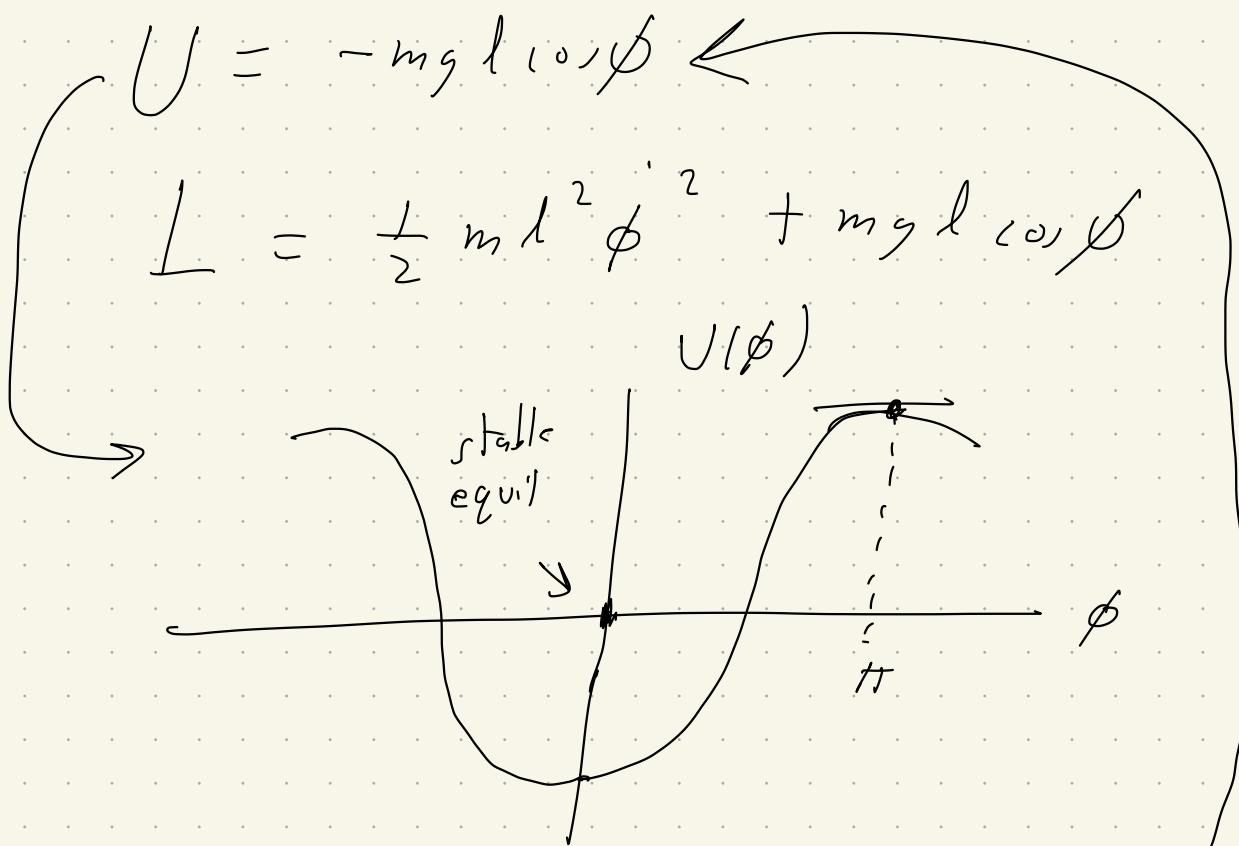
Simple pendulum:

$$T = \frac{1}{2} m l^2 \dot{\phi}^2$$

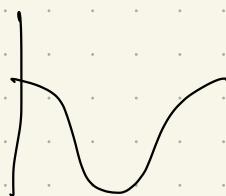


$$U = mgx$$

$$= -mgl \cos \phi$$



$$\phi_0 = 0 \text{ (equilibrium)}$$



$$\cos \phi = 1 - \frac{1}{2} \phi^2 + \dots$$

$$\cos \phi = \underbrace{\cos 0}_1 + \frac{d(\cos \phi)}{d\phi} \Big|_{\phi=0} \cdot \phi + \frac{1}{2} \frac{d^2(\cos \phi)}{d\phi^2} \Big|_{\phi=0} \phi^2$$

$\downarrow \sin \phi$

$$= 1 - \frac{1}{2} \phi^2 + \dots$$

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl \left(1 - \frac{1}{2} \phi^2 \right)$$

$$= \frac{1}{2} \cancel{m l^2} \dot{\phi}^2 - \frac{1}{2} \cancel{mgl} \phi^2 + mgl$$

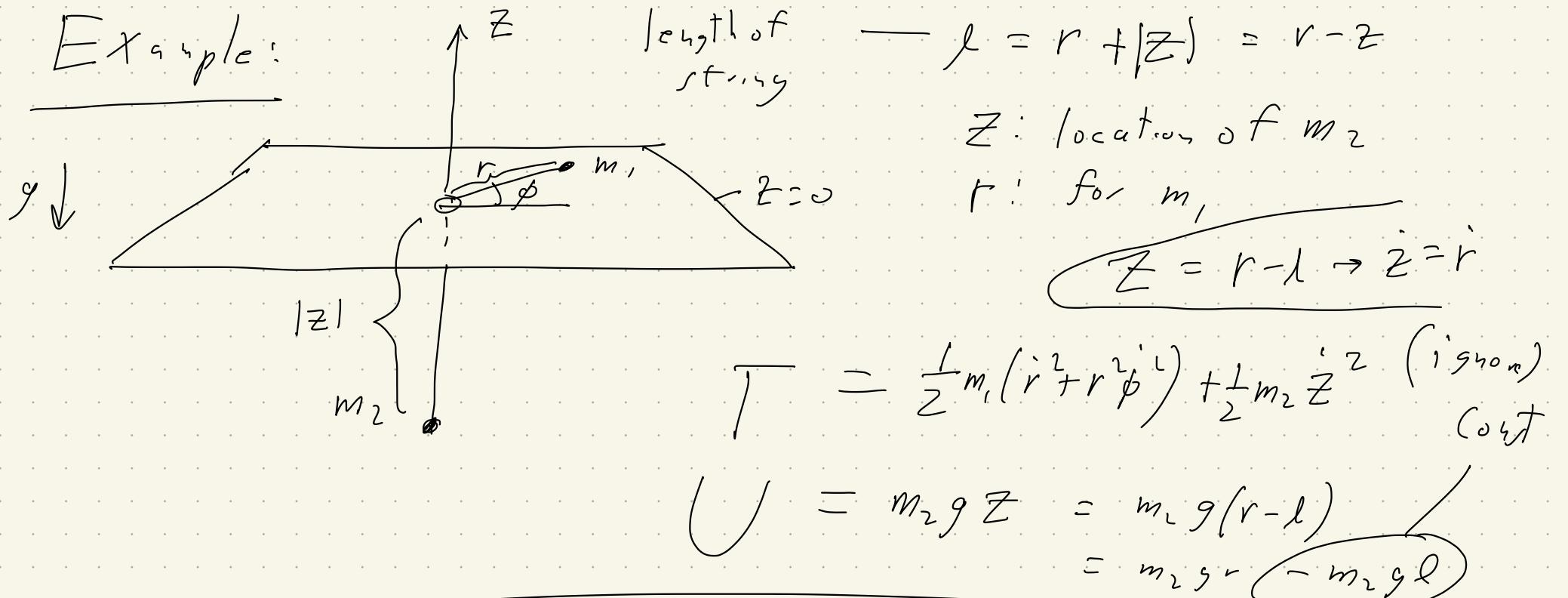
M H $l g u v e$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} \cancel{m l^2} \dot{\phi}^2$$

→

$$\omega = \sqrt{\frac{H}{m}} = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}$$

Example:



$$T = \frac{1}{2}m_1(r^2 + r^2\dot{\phi}^2) + \frac{1}{2}m_2\dot{z}^2 \quad (\text{ision})$$

(cont)

$$U = m_2 g z = m_2 g(r - l)$$

$$= m_2 g r - m_2 g l$$

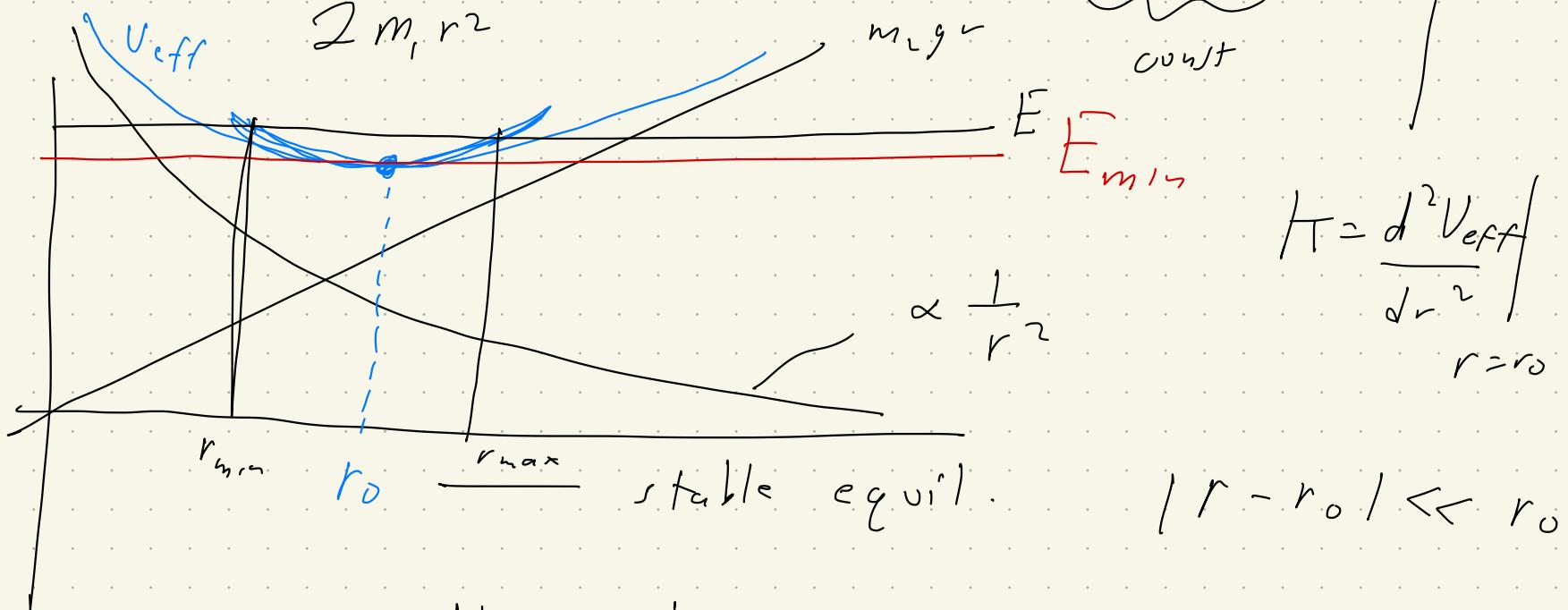
$$L = \frac{1}{2}(m_1 + m_2)r^2 + \frac{1}{2}m_1r^2\dot{\phi}^2 - m_2gr$$

$$\text{i)} \quad M_z = \frac{\partial L}{\partial \dot{\phi}} = m_1r^2\dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{m_1r^2}$$

$$\text{ii)} \quad E = T + U = \frac{1}{2}(m_1 + m_2)r^2 + \underbrace{\frac{1}{2}m_1r^2\dot{\phi}^2}_{\frac{M_z^2}{2m_1r^2}} + m_2gr$$

$$E = \frac{1}{2} \cancel{(m_1 + m_2)} r^2 + U_{\text{eff}}(r)$$

$$U_{\text{eff}}(r) = \frac{M_z^2}{2m_1 r^2} + m_2 g r = U(r_0) + \frac{1}{2} k(r - r_0)^2$$



$$r_0 = ?$$

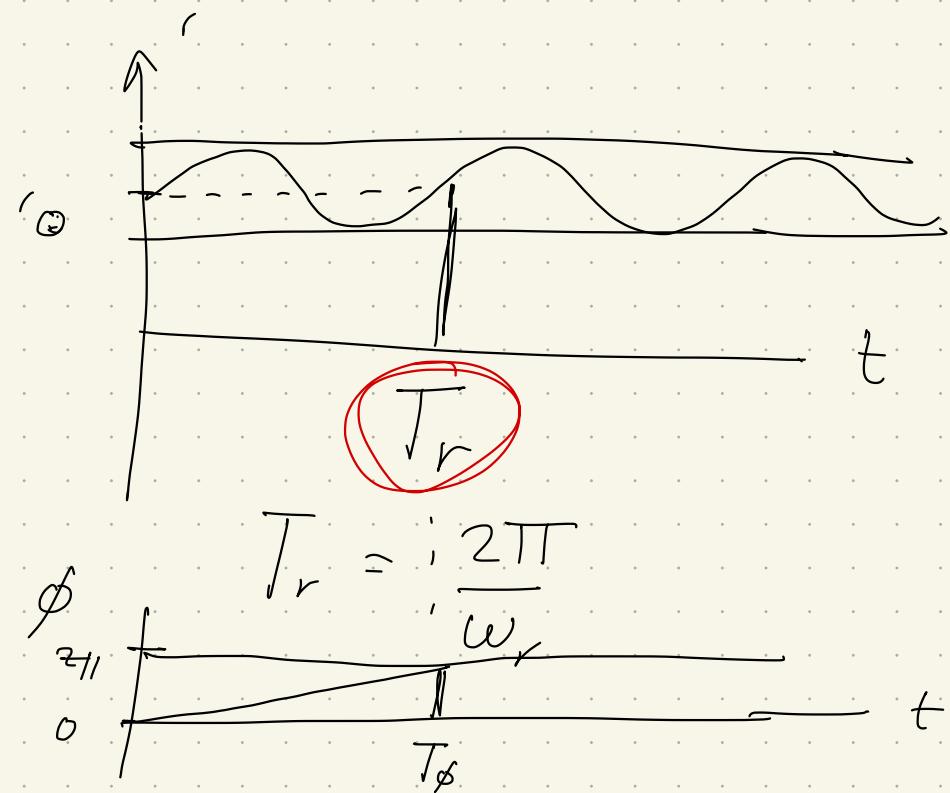
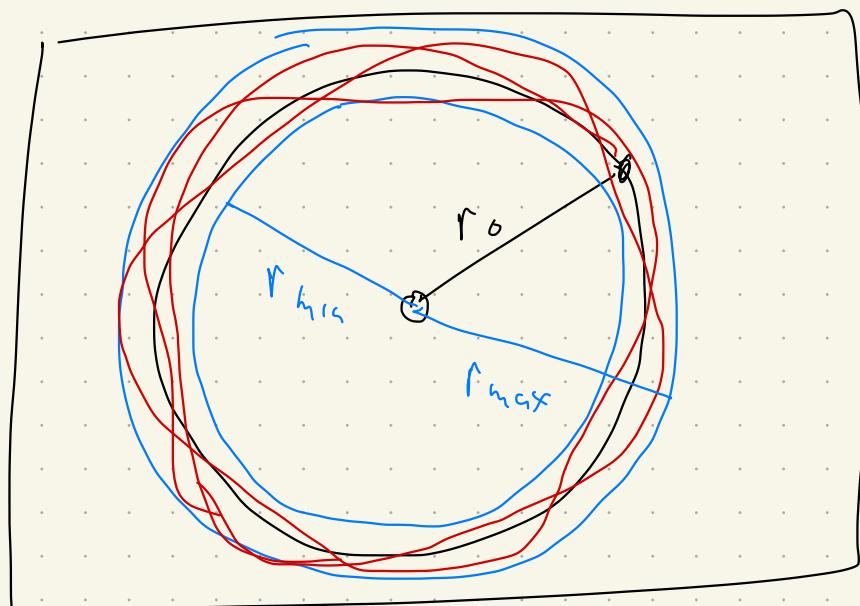
$$0 = \left. \frac{d U_{\text{eff}}}{dr} \right|_{r=r_0}$$

$$= - \frac{M_z^2}{m_1 r_0^3} + m_2 g$$



$$\boxed{M_z^2 = m_1 m_2 g r_0^3}$$

$$\begin{aligned}
 \frac{\int_0^r U_{\text{eff}}}{\int_{r_0}^r} &= \frac{3M_2^2}{m_1 r_0^4} \\
 &= \frac{3m_1 m_2 g r_0^3}{m_1 r_0^4} \\
 &= \frac{3m_2 g}{r_0} = k
 \end{aligned}$$



$$T_\phi = \frac{2\pi}{\omega_\phi} = \frac{2\pi}{\dot{\phi}/r_0}$$

$$\dot{\phi} = \frac{M_z}{m_1 r^2}$$

$$\omega_\phi = \dot{\phi} \Big|_{r=r_0} = \frac{M_z}{m_1 r_0^2} = \frac{\sqrt{m_1 m_2 g r_0^3}}{m_1 r_0^2}$$

$$\omega_\phi = \sqrt{\frac{m_2 g}{m_1 r_0}}$$

$$\omega_r = \sqrt{\frac{3 m_2 g}{(m_1 + m_2) r_0}}$$

Compare to

$$f \quad \frac{3}{m_1 + m_2} = \frac{1}{m_1} \quad \text{then} \quad \omega_r = \omega_{\phi}$$

$$\frac{3m_1}{m_1 + m_2} = m_1 + m_2$$

$$\boxed{2m_1 = m_2}$$



Lec #20: 10/28

Forced oscillations:

$$m\ddot{x} = -kx + F(t)$$

$$\ddot{x} + \frac{k}{m}x = \frac{F(t)}{m}$$

$$\boxed{\ddot{x} + \omega^2 x = \frac{F(t)}{m}}, \quad \omega = \sqrt{\frac{k}{m}}$$

general sol'n:

$$x(t) = x_h(t) + x_p(t)$$

↑
homogeneous
($F(t)=0$)

particular

(any solution for $F(t)$)

$$\boxed{x_h(t) = a \cos(\omega t + \alpha)}$$

a, α : two constants
(initial conditions)

Suppose: $F(t) = f \cos(\gamma t + \beta)$

$$x_p'' + \omega^2 x_p = \frac{f}{m} \cos(\gamma t + \beta)$$

Guess !

$$x_p(t) = b \cos(\gamma t + \beta)$$

$$\rightarrow -b\gamma^2 \cancel{\cos(\gamma t + \beta)} + \omega^2 b \cancel{\cos(\gamma t + \beta)} = \frac{f}{m} \cancel{\cos(\gamma t + \beta)}$$

$$b(\omega^2 - \gamma^2) = \frac{f}{m}$$

$$\rightarrow b = \frac{f}{m(\omega^2 - \gamma^2)}$$

$$x_p(t) = \frac{f}{m(\omega^2 - \gamma^2)} \cos(\gamma t + \beta)$$

$$+ 10 \cos(\omega t)$$

$$x(t) = a \cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)} \cos(\gamma t + \beta)$$

$$x(t) = a \cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)} [\cos(\gamma t + \beta) - \cos(\omega t + \beta)]$$



L'Hopital's

$$= \frac{\frac{d}{d\gamma} (num)}{\frac{d}{d\gamma} (den)} \Big|_{\gamma \rightarrow \omega}$$

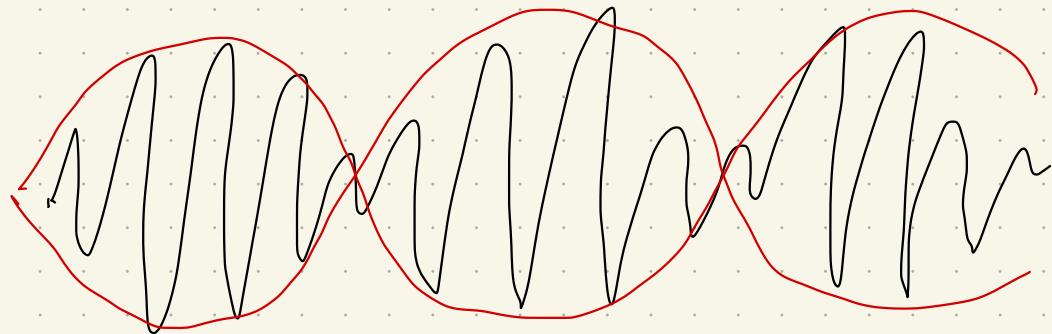
$$= \frac{+ft \sin(\omega t + \beta)}{+2m\omega}$$

at resonance

$$\downarrow (\gamma = \omega)$$

$$= \frac{ft}{2m\omega} \sin(\omega t + \beta)$$

$$x(t) = a \cos(\omega t + \alpha) + \frac{ft}{2m\omega} \sin(\omega t + \beta)$$



$$\frac{\omega + \gamma}{2}$$

$$|\omega - \gamma| = \omega_{\text{beat}}$$

General: for arbitrary $F(t)$

$$F(t) = \operatorname{Re} \int_{-\infty}^{\infty} d\gamma \tilde{F}(\gamma) e^{i\gamma t}$$

$\underbrace{\hspace{10em}}$

Fourier transform

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}$$

$$y' + P(x)y = Q(x)$$

Math methods

$$Y = Y(x)$$

Let:

$$\xi = \dot{x} + i\omega x$$

complex

$$\begin{aligned}\xi &= \dot{x} + i\omega x \\ &= \ddot{x} + i\omega (\dot{x} - i\omega x) \\ &= \ddot{x} + i\omega \dot{x} + \omega^2 x\end{aligned}$$

$$\rightarrow \xi' - i\omega \xi = \ddot{x} + \omega^2 x = \frac{F(t)}{m}$$

Homog: $\xi_h' - i\omega \xi_h = 0 \rightarrow \xi_h(t) = A e^{i\omega t}$

Guess: $\xi_p(t) = A(t) e^{i\omega t}$

replace
Complex constant

$$\boxed{\ddot{\xi}_p - i\omega \dot{\xi}_p = \frac{F(t)}{m}}, \quad \xi_p = A(t) e^{i\omega t}$$

$$\rightarrow \dot{A} e^{i\omega t} + \cancel{iAw e^{i\omega t}} - \cancel{i\omega A e^{i\omega t}} = \frac{F(t)}{m}$$

$$\dot{A} = \frac{e^{-i\omega t} F(t)}{m}$$

$$A(t) = \int dt \frac{F(t)}{m} e^{-i\omega t} + \text{const}$$

$$\boxed{\xi(t) = e^{i\omega t} \left[\int_0^t d\bar{t} \frac{F(\bar{t})}{m} e^{-i\omega \bar{t}} + \xi_0 \right]}$$

$$\boxed{\xi = x + i\omega x \rightarrow x(t) = \frac{1}{\omega} \text{Im}(\xi(t))}$$

Complex
constant
(I.C.)

$$y' + P(x)y = Q(x) \quad f(x)dx = g(y)dy$$

$$\frac{dy}{dx} + P(x)y - Q(x) = 0$$

$$\boxed{1 dy + (P(x)y - Q(x))dx = 0} \neq dU$$

$$dU(x,y) = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

$$\frac{\partial^2 U}{\partial y \partial x}$$

$$=$$

$$\frac{\partial^2 U}{\partial x \partial y}$$

M. Boas
~~Mathematica~~

$$\mu(x) [dy + (P(x)y - Q(x))dx] = dU$$

$$\frac{d\mu}{dx} = \frac{\partial}{\partial y} (P(x)y - Q(x)) \mu(x)$$

$$\frac{d\mu}{dx} = P(x) \mu(x)$$

$$\int \frac{d\mu}{\mu} = \int P(x) dx$$

$$\ln \mu = \int P(x) dx$$
$$\mu(x) = e^{\int p(x) dx}$$

Lec #21: Nov 2nd

Today: Sec 23 Free oscillations in 2 or more dimensions.

Thurs: Rigid body motion

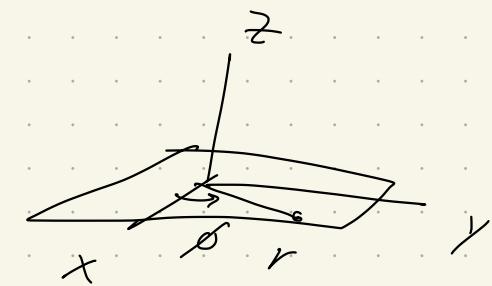
~~Exam~~ Exam 2: Nov 18th → Nov 23rd

Quiz #4: Today (at end of class)

Example: space oscillator

$$U = \frac{1}{2} kr^2 = \frac{1}{2} k(x^2 + y^2)$$

$$\begin{aligned} T &= \frac{1}{2} m(r^2 + r^2\dot{\phi}^2) \\ &= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) \end{aligned}$$



$$L = T - U$$

$$= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k(x^2 + y^2)$$

$$= \underbrace{\left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \right)}_{w_x = \sqrt{\frac{k}{m}}} + \underbrace{\left(\frac{1}{2} m \dot{y}^2 - \frac{1}{2} k y^2 \right)}_{w_y = \sqrt{\frac{k}{m}}}$$

$$w_x = \sqrt{\frac{k}{m}}$$

$$w_y = \sqrt{\frac{k}{m}}$$

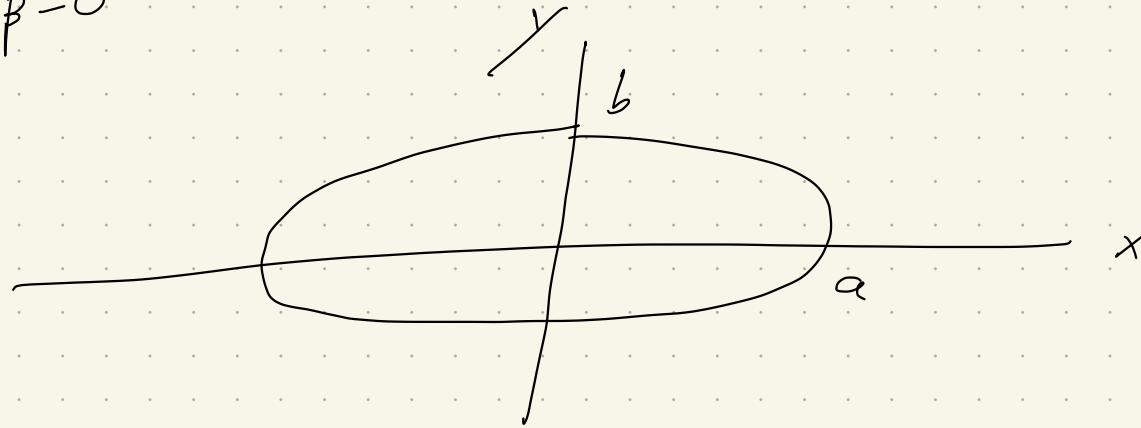
$$\omega = \sqrt{\frac{k}{m}}$$

$$x = a \cos(\omega t + \alpha)$$

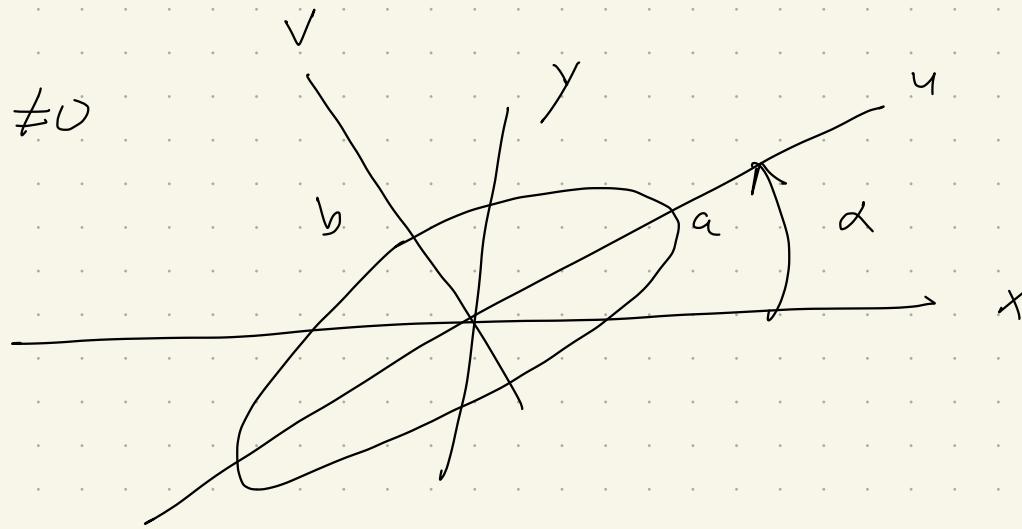
$$y = b \sin(\omega t + \beta)$$

general solutions

$$\alpha = \beta = 0$$

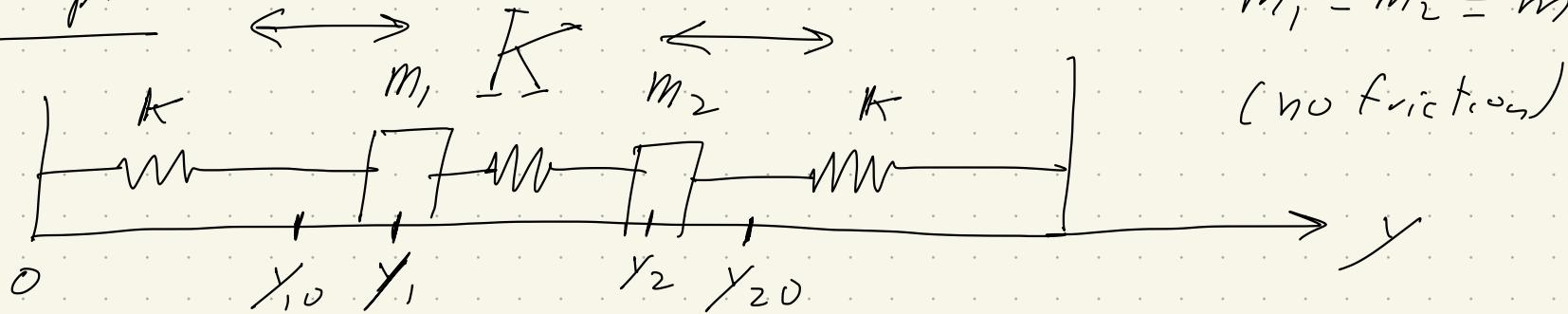


$$\alpha = \beta \neq 0$$



$$\alpha \neq \beta \longrightarrow \text{ellipse}$$

Example:



$$m_1 = m_2 = m$$

(no friction)

$$L = T - U$$

$$x_1 = y_1 - y_{10}$$

$$x_2 = y_2 - y_{20}$$

$$T = \frac{1}{2} m \dot{y}_1^2 + \frac{1}{2} m \dot{y}_2^2 = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$U = \frac{1}{2} \kappa x_1^2 + \frac{1}{2} \kappa x_2^2 + \frac{1}{2} K (x_2 - x_1)^2$$

$$= \frac{1}{2} \kappa x_1^2 + \frac{1}{2} \kappa x_2^2 + \frac{1}{2} K (x_2^2 + x_1^2 - 2x_1 x_2)$$

$$= \frac{1}{2} (\kappa + K) x_1^2 + \frac{1}{2} (\kappa + K) x_2^2 - \cancel{\frac{1}{2} K x_1 x_2} - \cancel{\frac{1}{2} K x_1 x_2} - \cancel{\frac{1}{2} K x_2 x_1}$$

$$T = \frac{1}{2} \sum_{i=1}^n m_i \kappa \dot{x}_i^2$$

$$U = \frac{1}{2} \sum_{i,j} K_{ij} x_i x_j$$

$$m_i \kappa = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$K_{ij} = \begin{pmatrix} K & -K \\ -K & K+2K \end{pmatrix}$$

$$L = \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k - \frac{1}{2} \sum_{i,k} t_{ijk} \dot{x}_i \dot{x}_k$$

geht auf)

$$\cancel{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_j} \right)} = \frac{\partial L}{\partial x_j} \quad j = 1, 2, \dots$$

$$\cancel{\frac{d}{dt} \left(\sum_k m_{jk} \dot{x}_k \right)} = - \sum_k t_{jk} \dot{x}_k$$

$$\cancel{\frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k} = \frac{1}{2} (m_{11} \dot{x}_1^2 + m_{12} \dot{x}_1 \dot{x}_2 + m_{13} \dot{x}_1 \dot{x}_3 + m_{21} \dot{x}_2 \dot{x}_1 + m_{22} \dot{x}_2^2 + m_{23} \dot{x}_2 \dot{x}_3 + m_{31} \dot{x}_3 \dot{x}_1 + m_{32} \dot{x}_3 \dot{x}_2 + m_{33} \dot{x}_3^2)$$

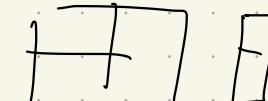
$$\begin{aligned} \frac{\partial L}{\partial \dot{x}_2} &= \frac{1}{2} (m_{12} \dot{x}_1 + m_{21} \dot{x}_1 + 2m_{22} \dot{x}_2 + m_{23} \dot{x}_3 + m_{32} \dot{x}_3) \\ &= \frac{1}{2} (m_{21} \dot{x}_1 + 2m_{22} \dot{x}_2 + 2m_{23} \dot{x}_3) \\ &= \sum_k m_{2k} \dot{x}_k \end{aligned}$$

$$\sum_{\pi} m_{j\pi} x_{\pi} = - \sum_{\pi} \tau_{j\pi} x_{\pi}$$

Guess: $x_{\pi} = A_{\pi} e^{i\omega t}$

$$\rightarrow \dot{x}_{\pi} = -\omega^2 A_{\pi} e^{i\omega t}$$

$$\sum_{\pi} -\omega^2 m_{j\pi} A_{\pi} e^{i\omega t} = - \sum_{\pi} \tau_{j\pi} A_{\pi} e^{i\omega t}$$



$$\sum_{\pi} (\tau_{j\pi} - \omega^2 m_{j\pi}) A_{\pi} = 0 \quad \text{0 vector}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

↑

TF invertible

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{M} \cdot \vec{v} = \lambda \vec{v}$$

$$(\underline{M} - \lambda \underline{1}) \cdot \vec{v} = 0$$

$$\det = 0$$

$$0 = \det(K_j H - \omega^2 m_j H)$$

characteristic equations

polynomial $(\omega^2)^N$
equation \rightarrow

Normal mode

freqs,

characteristic freqs,

eigen freqs

ω^2

label
the

N different
eigen
freqs

$$0 = \det \left(\begin{bmatrix} K+K & -K \\ -K & K+K \end{bmatrix} - \omega^2 \begin{bmatrix} m_0 & 0 \\ 0 & m_0 \end{bmatrix} \right)$$

$$= \det \begin{vmatrix} K+K-\omega^2 m_0 & -K \\ -K & K+K-\omega^2 m_0 \end{vmatrix}$$

$$= ((K+K)-\omega^2 m_0)^2 - K^2$$

$$((k + K) - \omega^2 m)^2 = K^2$$

$$(k + K) - \omega^2 m = \pm K$$

$$\rightarrow \omega^2 = \frac{(k + K) \pm K}{m}$$

$$\boxed{\omega_+^2 = \frac{k + 2K}{m}, \quad \omega_-^2 = \frac{K}{m}}$$

eigen
freqs

solve for eigenvectors:



$$\omega_+^2 = \frac{k + 2K}{m}$$

$$V_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

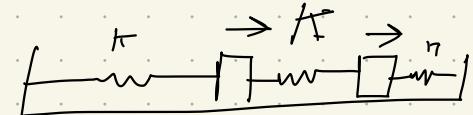
$$\omega_-^2 = \frac{4}{m}$$

$$V_- = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

normal mode
oscillation

$$\Delta_{\pi_\alpha} = \frac{1}{2} \begin{bmatrix} V_+ & V_- \end{bmatrix} \begin{bmatrix} K \\ m \end{bmatrix}$$

Π matrix of eigenvectors



$$x_H = \operatorname{Re} \left[\sum_{\alpha} \Delta_{\pi_\alpha} c_\alpha e^{i\omega_\alpha t} \right]$$

Complex const
determined by
 $\pm C's$

$$= \sum_{\alpha} \Delta_{\pi_\alpha} \theta_\alpha$$

$$\theta_\alpha = \operatorname{Re} [c_\alpha e^{i\omega_\alpha t}]$$

normal
coords

$$L = T - U$$

$$= \frac{1}{2} \sum_{i, \pi} m_{i, \pi} \dot{x}_i \cdot \dot{x}_{\pi} - \frac{1}{2} \sum_{i, \pi} K_{i, \pi} x_i \cdot x_{\pi}$$

$$= \frac{1}{2} \dot{x}^T M \dot{x} - \frac{1}{2} x^T K x$$

$$= \frac{1}{2} \theta^T (\Delta^T_m \Delta) \theta - \frac{1}{2} \theta^T (\Delta^T K \Delta) \theta$$

$$= \frac{1}{2} \sum_{\alpha} M_{\alpha} \dot{\theta}_{\alpha}^2 - \frac{1}{2} \sum_{\alpha} J_{\alpha} \theta_{\alpha}^2$$

★ ★

where $M = \text{diagonal matrix} = \Delta^T_m \Delta$
 $J = \dots \quad \quad \quad = \Delta^T_K \Delta$

Eigenvectors
diagonalize
both $m_{i, \pi}$ and
 $K_{i, \pi}$ with

CHECK:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\tilde{J}_{\alpha} = M_{\alpha} \omega_{\alpha}^2$$

$$= \frac{m}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{m}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rightarrow \quad M_+ = m, M_- = \omega$$

$$\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} K+K & -K \\ -K & K+K \end{vmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} K+2K & K \\ -K-2K & K \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 2K+4K & 0 \\ 0 & 2K \end{vmatrix}$$

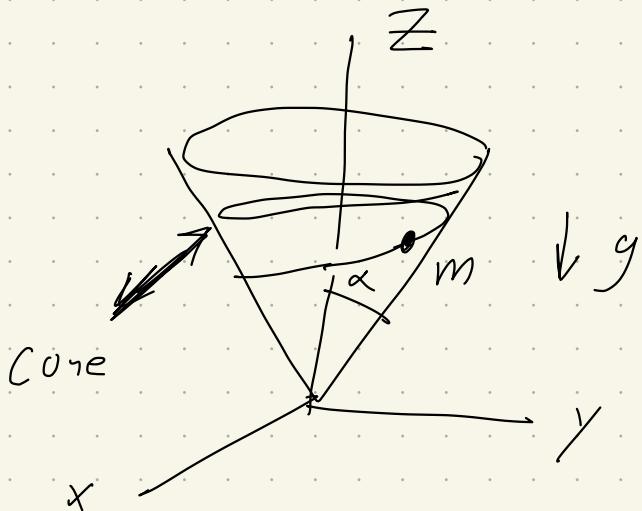
$$= \begin{vmatrix} K+2K & 0 \\ 0 & K \end{vmatrix}$$

$$= \begin{vmatrix} \omega_+^2 m & 0 \\ 0 & \omega_-^2 m \end{vmatrix} \quad \rightarrow \quad \mathcal{E}_+ = \omega_+^2 m \\ \mathcal{E}_- = \omega_-^2 m$$

$$\text{Thus, } L = \frac{1}{2} m (\dot{\theta}_+^2 + \dot{\theta}_-^2) - \frac{1}{2} (m \omega_+^2 \dot{\theta}_+^2 + m \omega_-^2 \dot{\theta}_-^2) \\ = \left(\frac{1}{2} m \dot{\theta}_+^2 - \frac{1}{2} m \omega_+^2 \dot{\theta}_+^2 \right) + \left(\frac{1}{2} m \dot{\theta}_-^2 - \frac{1}{2} m \omega_-^2 \dot{\theta}_-^2 \right)$$

Quiz #4:

Calculate freq of small oscillations around
the circular orbit



α fixed

(Sph. polar coords)

$$\text{Solution } T = r_0$$

a) Lagrangian?

b) Determine r_0

(relationship between r_0 and M_z)

c) Determine ω_r

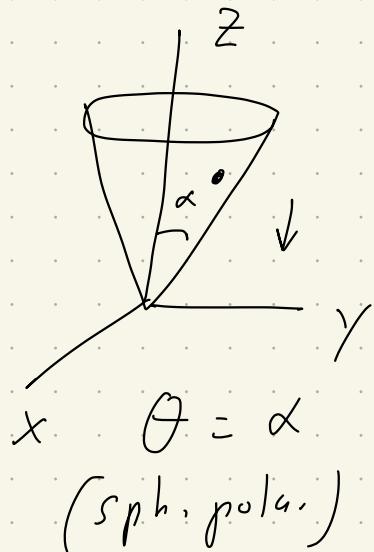
~~q4~~

q4 - name.pdf

joseph.d.romano@tu.edu

Lec #22: Thurs Nov 4th

- Next 5 lectures, rigid body motion 31-36, 38
- EXAM 2 - Tues 11/23 (not Thur 11/18)



$$T = \frac{1}{2} m(r^2 + r^2 \sin^2 \alpha \dot{\phi}^2)$$

$$U = mgz = mgr \cos \alpha$$

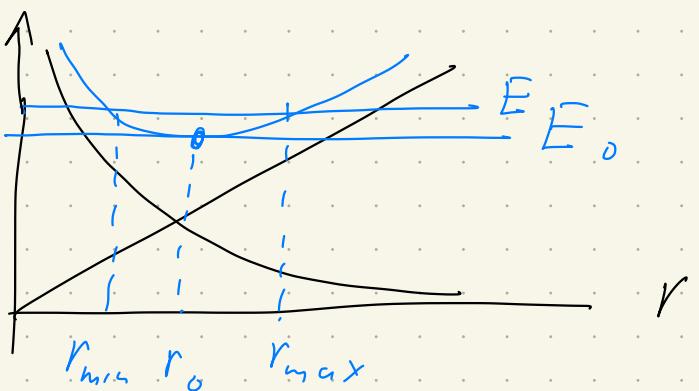
$$L = T - U$$

$$= \frac{1}{2} m(r^2 + r^2 \sin^2 \alpha \dot{\phi}^2) - mgr \cos \alpha$$

$$M_z = mr^2 \sin^2 \alpha \dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{mr^2 \sin^2 \alpha}$$

$$E = T + U = \sum_i \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right)$$

$$= \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{M_z}{2mr^2 \sin^2 \alpha}}_{U_{eff}(r)} + mgr \cos \alpha$$



$$\omega_r = \sqrt{\frac{k}{m}}$$

$$k = \frac{d^2 V_{\text{eff}}}{dr^2}$$

$$r = r_0$$

$$\omega = \left. \frac{d V_{\text{eff}}}{dr} \right|_{r=r_0}$$

$$= \frac{3mg \cos \alpha}{r_0}$$

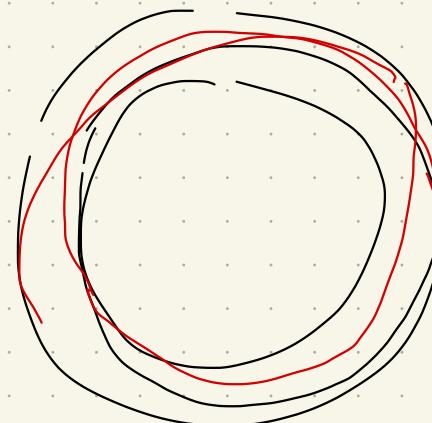
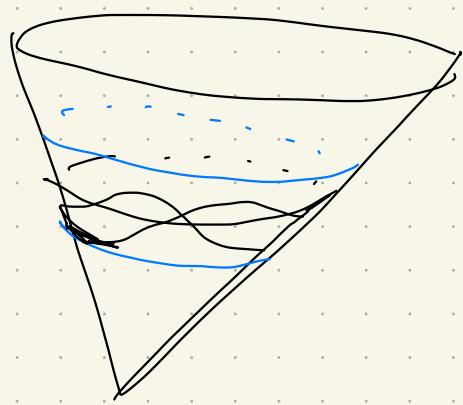
$$M_Z^2 = m^2 g r_0^3 \sin^2 \alpha \cos \alpha$$

$$\rightarrow \omega_r = \sqrt{\frac{k}{m}} = \sqrt{\frac{3g \cos \alpha}{r_0}}$$

$$\omega_\phi = \left. \dot{\phi} \right|_{r=r_0} = \sqrt{\frac{g \cos \alpha}{r_0 \sin^2 \alpha}}$$

don't agree

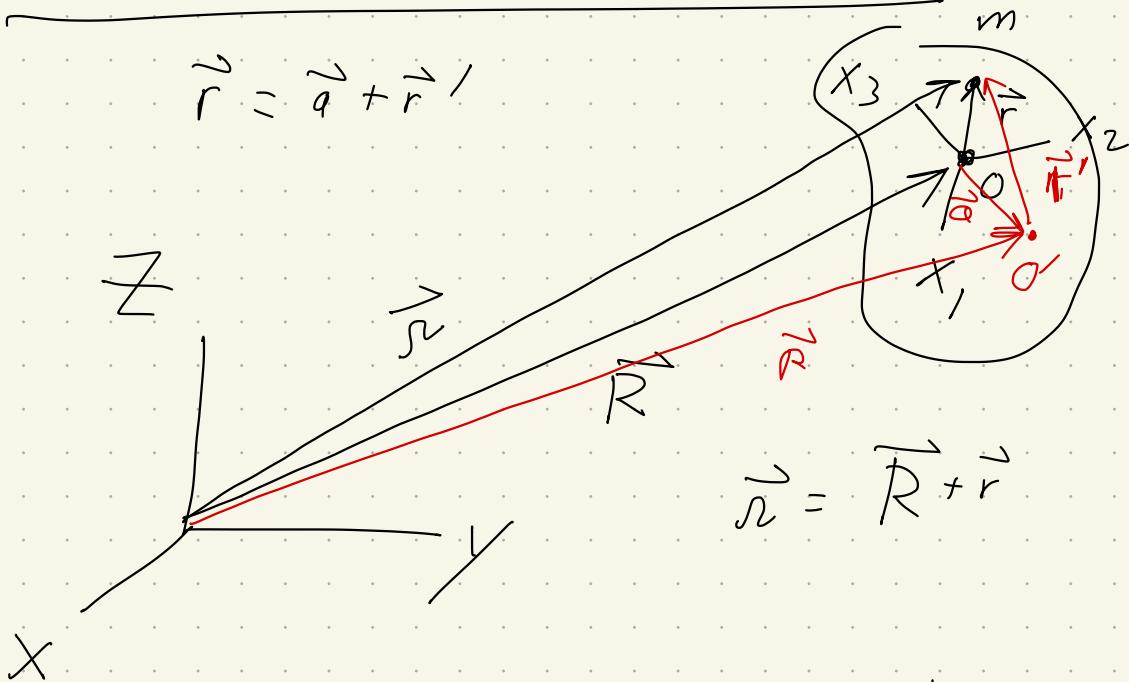
(not closed)



Rigid body motion

$$\begin{aligned} \vec{x}_i &= (x_1, x_2, x_3) \\ &= (x, y, z) \end{aligned}$$

$$\vec{r} = \vec{q} + \vec{r}'$$

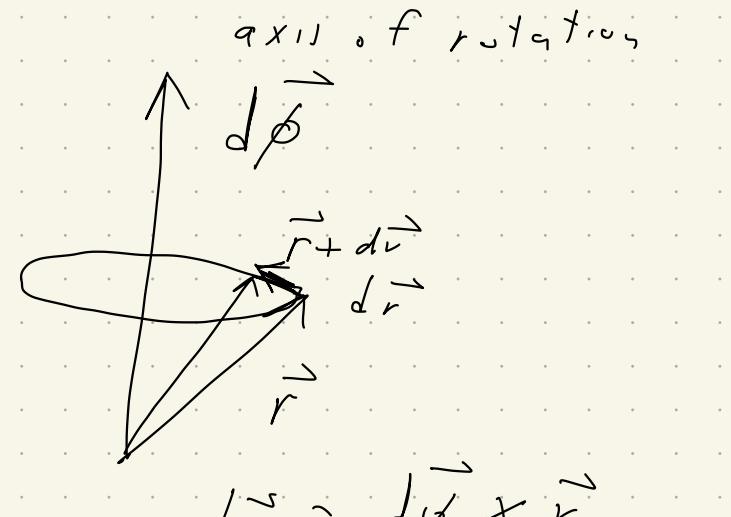


$$\vec{r} = \vec{R} + \vec{r}'$$

$$\begin{aligned} \vec{r} &= \frac{d\vec{r}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}'}{dt} \\ &= \vec{V} + \vec{\omega} \times \vec{r} \end{aligned}$$

fixed
inertial
frame

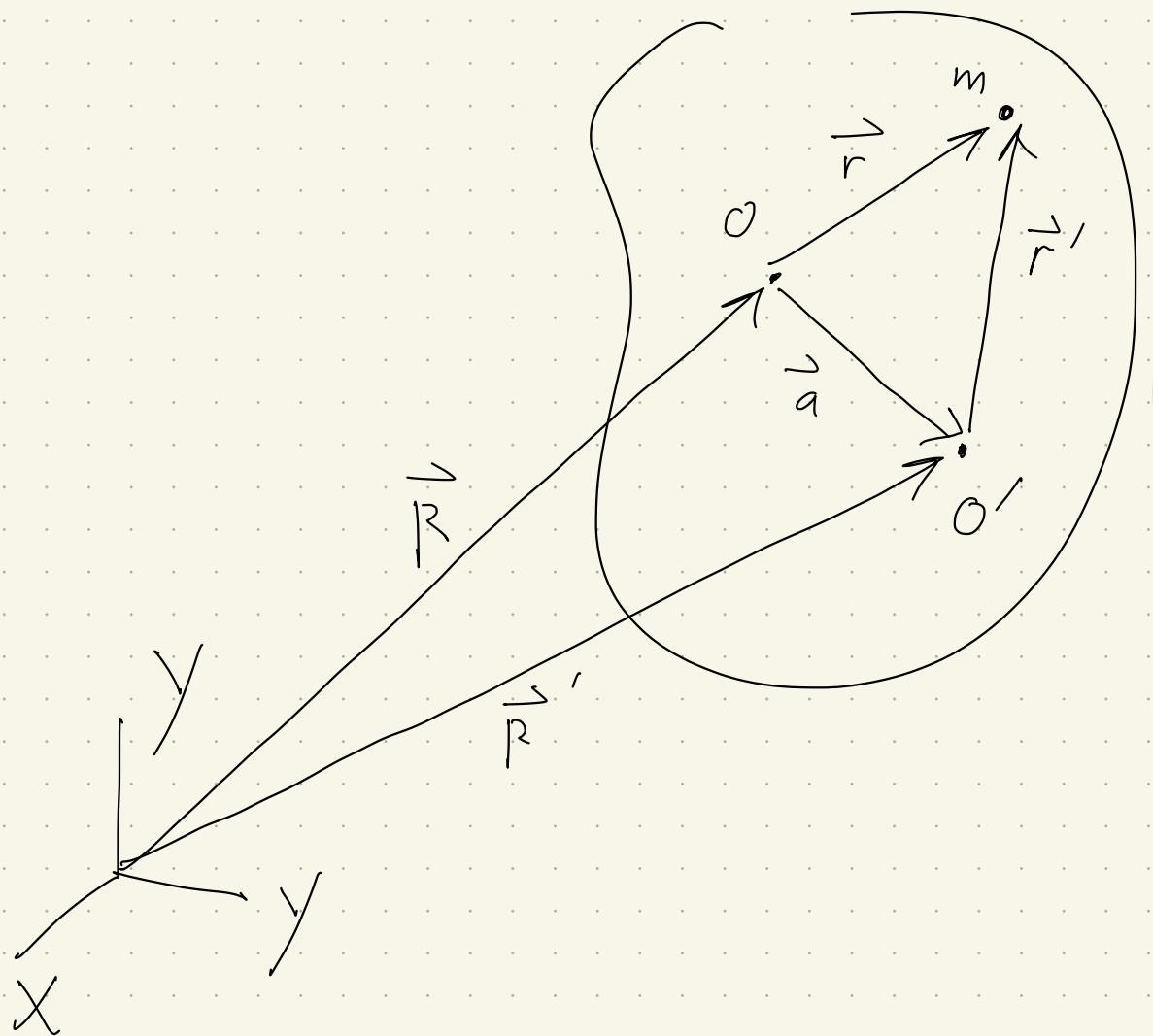
$$|d\vec{\phi}| = d\phi$$



most convenient if O is at com of RB.

$$\frac{d\vec{r}}{dt} = \frac{d\vec{\phi}}{dt} \times \vec{r} = \vec{\omega} \times \vec{r}$$

$$\vec{r} = \vec{a} + \vec{r}' \quad \rightarrow \quad \vec{v} = \vec{V} + \vec{\omega} \times \vec{r}$$



$$\vec{v}' = \vec{V}' + \vec{\omega}' \times \vec{r}'$$

$$\vec{v}' = \vec{V} + \vec{\omega} \times (\vec{a} + \vec{r}')$$

$$= \vec{V} + \vec{\omega} \times \vec{a} + \vec{\omega} \times \vec{r}'$$

$$\vec{V}' = \vec{V} + \vec{\omega} \times \vec{a}$$

$$\vec{\omega}' = \vec{\omega}$$

2-d rotational motion:

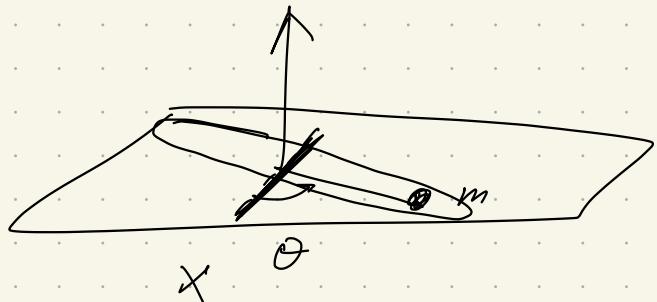
$$\vec{F} = \frac{d\vec{P}}{dt}$$

$$\vec{P} = \mu \vec{V}$$

$$\theta, \frac{d\theta}{dt} = \omega$$

$$T_{rot} = \frac{1}{2} I \omega^2$$

axis of rotation (fixed)



Moment of inertia

3-d

$$\underline{L} = I \omega$$

\underline{I}_{ij} : inertia tensor

$$M_i = \sum_k I_{ik} \omega_k$$

(3x3 symmetric matrix)

$$\overline{\underline{M}} = \underline{\underline{I}} \overline{\underline{\omega}}$$

$$\overline{\underline{N}} = \frac{d \overline{\underline{M}}}{dt}$$

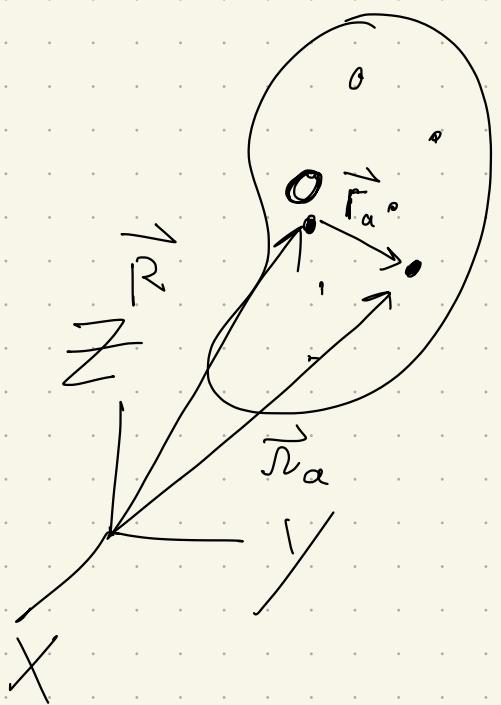
TE

m_a : label the mass point
in the rigid body

$$T = \sum_a \frac{1}{2} m_a |\vec{v}_a|^2$$

$$\vec{v}_a = \vec{V} + \vec{\omega} \times \vec{r}_a$$

$$\begin{aligned}
 |\vec{v}_a|^2 &= (\vec{V} + \vec{\omega} \times \vec{r}_a) \cdot (\vec{V} + \vec{\omega} \times \vec{r}_a) \\
 &= \cancel{|\vec{V}|^2} + 2 \vec{V} \cdot (\vec{\omega} \times \vec{r}_a) \\
 &\quad + \cancel{(\vec{\omega} \times \vec{r}_a) \cdot (\vec{\omega} \times \vec{r}_a)}
 \end{aligned}$$



1st term

$$\sum_a \frac{1}{2} m_a |\vec{V}|^2 = \frac{1}{2} \mu |\vec{V}|^2$$

$$\mu = \sum_a m_a = \text{total mass}$$

$$\begin{aligned}
 2^{\text{nd}} \text{ term} &= \sum_a \oint_R \chi m_a \vec{V} \cdot (\vec{r} \times \vec{r}_a) \\
 &= \left(\sum_a m_a \vec{r}_a \right) \bullet (\vec{V} \times \vec{r}) = O \left(\begin{array}{l} \text{if} \\ \text{origin} \\ \text{is at} \\ \text{com} \end{array} \right)
 \end{aligned}$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$\begin{aligned}
 3^{\text{rd}} \text{ term} &= (\vec{r} \times \vec{r}_a) \cdot (\vec{r} \times \vec{r}_a) \\
 \vec{A} \times (\vec{B} \times \vec{C}) &= \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B}) \\
 &= \vec{Q} \cdot (\vec{r}_a \times (\vec{r} \times \vec{r}_a)) \\
 &= \vec{Q} \cdot \left(\vec{Q} |\vec{r}_a|^2 - \vec{r}_a (\vec{r}_a \cdot \vec{Q}) \right) \\
 &= \vec{Q} \cdot \vec{Q} r_a^2 - (\vec{Q} \cdot \vec{r}_a)(\vec{Q} \cdot \vec{r}_a)
 \end{aligned}$$

$$T = \frac{1}{2} M |\vec{V}|^2 + \sum_a \frac{1}{2} m_a \left(\underbrace{\vec{r}_a \cdot \vec{r}_a}_{\approx r_a^2} - (\vec{r}_a \cdot \vec{r}_c)(\vec{r}_a \cdot \vec{r}_c) \right)$$

$\sum_i r_i^2 = \sum_{i,j} r_i \cdot r_j \cdot \delta_{ij}$

$\sum_i r_i \cdot x_{ai} \cdot \sum_j r_j \cdot x_{aj} \approx$

$$= \frac{1}{2} M |\vec{V}|^2 + \sum_a \frac{1}{2} m_a \sum_{i,j} (r_a^2 \delta_{ij} - x_{ai} x_{aj}) \underline{r_i r_j}$$

$$= \frac{1}{2} M |\vec{V}|^2 + \frac{1}{2} \sum_{i,j} \left(\sum_a m_a (r_a^2 \delta_{ij} - x_{ai} x_{aj}) \right) \underline{r_i r_j}$$

$$= \boxed{\frac{1}{2} M |\vec{V}|^2 + \frac{1}{2} \sum_{i,j} I_{ij} \underline{r_i r_j}}$$

I_{ij}

Lec #23:

Nov 9th

Quiz #5: next Tuesday

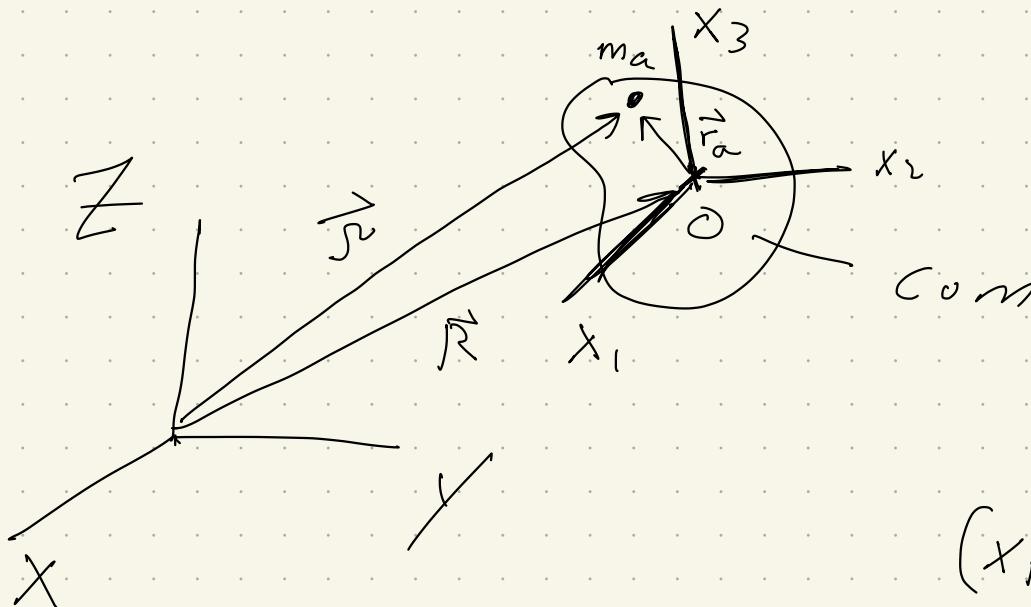
Exam 2: Tuesday 11/23

Toddy: Inertia tensor

Thurs: Equations of motion

Last time:

$$T = \frac{1}{2} \mu \vec{V}^2 + \frac{1}{2} I_{ij} \Omega_i \Omega_j$$



$$\underbrace{\sum_{i=1}^3 \sum_{j=1}^3 I_{ij} \Omega_i \Omega_j}_{\text{"drop"}}$$

$$(x_1, x_2, x_3) = (x, y, z)$$

$$I_{ij} = \sum_a m_a (r_a^2 \delta_{ij} - x_{ai} x_{aj})$$

$$r_a^2 = \sum_i x_{ai}^2$$

$$dm = \rho dV$$

↑

mass
density

$$I_{ij} = \int \rho dV (r^2 \delta_{ij} - x_i x_j)$$

Angular momentum: (wrt COM)

$$\begin{aligned} \vec{M} &= \sum_a \vec{r}_a \times \vec{p}_a & \vec{p}_a &= m_a \vec{v}_a \\ &= \sum_a m_a \vec{r}_a \times \vec{v}_a \\ &= \sum_a m_a \vec{r}_a \times (\cancel{\vec{\omega}^0} + \vec{\Omega} \times \vec{r}_a) \\ &= \sum_a m_a \vec{r}_a \times (\vec{\Omega} \times \vec{r}_a) \end{aligned}$$

$$\vec{M} = \sum_a m_a (\vec{\Omega} \cdot \vec{r}_a^2 - \vec{r}_a (\vec{\Omega} \cdot \vec{r}_a))$$

$$M_i = \sum_a m_a (\Omega_i \cdot \vec{r}_a^2 - x_{ai} (\Omega_i \cdot \vec{x}_{aj}))$$

$$\delta_{ij} \cdot \Omega_j$$

$$= \sum_a m_a (\delta_{ij} \cdot \vec{r}_a^2 - x_{ai} \cdot x_{aj}) \Omega_j$$

$$= I_{ij} \cdot \Omega_j$$

$$\boxed{M = I \cdot \Omega}$$

$$M_i = I_{ij} \cdot \Omega_j, \quad T = \frac{1}{2} \mu V^2 + \frac{1}{2} I_{ij} \cdot \Omega_i \cdot \Omega_j$$

↑
COM Frame

Properties of I_{ij} :

Inertia tensor
(3×3 matrix)

1) I_{ij} real, symmetric ($I_{ij} = I_{ji}$)

Can always diagonalize:

(x_1, x_2, x_3) for such

axes are called

"principle axes"

$$I_{ij} = \begin{vmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{vmatrix}$$

$$= \delta_{ij} I_i \quad (\text{no summation})$$

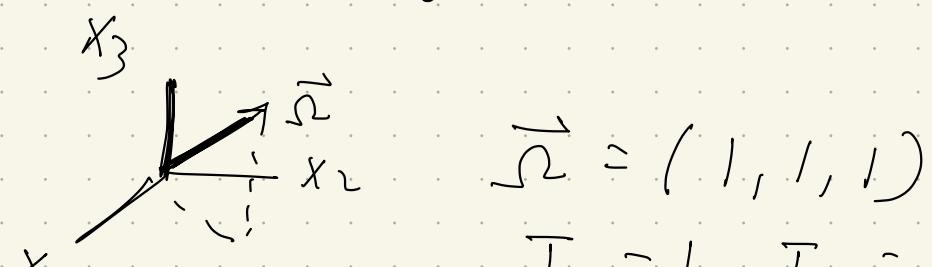
$$T = \frac{1}{2} \mu V^2 + \frac{1}{2} I_i \omega_i^2$$

$$M_i = I_i \omega_i$$

$$(M_1 = I_1 \omega_1)$$

$$M_2 = I_2 \omega_2$$

$$M_3 = I_3 \omega_3$$

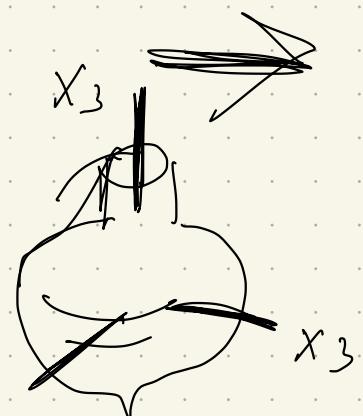
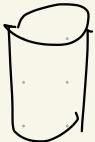


$$\vec{\omega} = (1, 1, 1)$$

$$I_1 = 1, I_2 = 1, I_3 = 2$$

$$\vec{m} = (1, 1, 2)$$

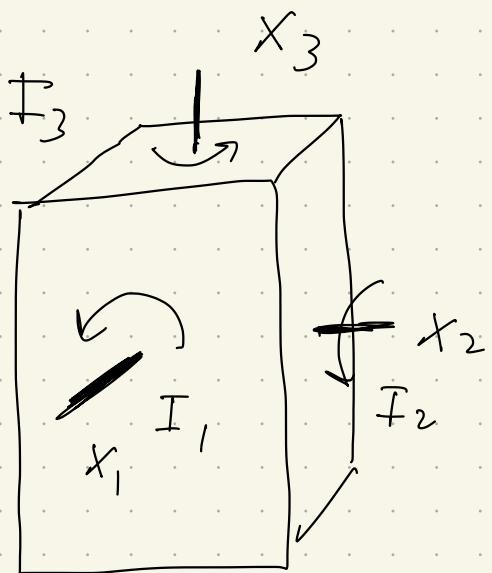
$I_1 \neq I_2 \neq I_3$ (in general)



x_1 \hat{n} :

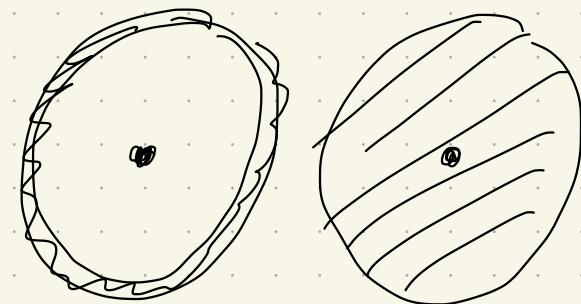
$$I(\hat{n}) = I_{ij} n_i n_j$$

moment of inertia
about \hat{n}

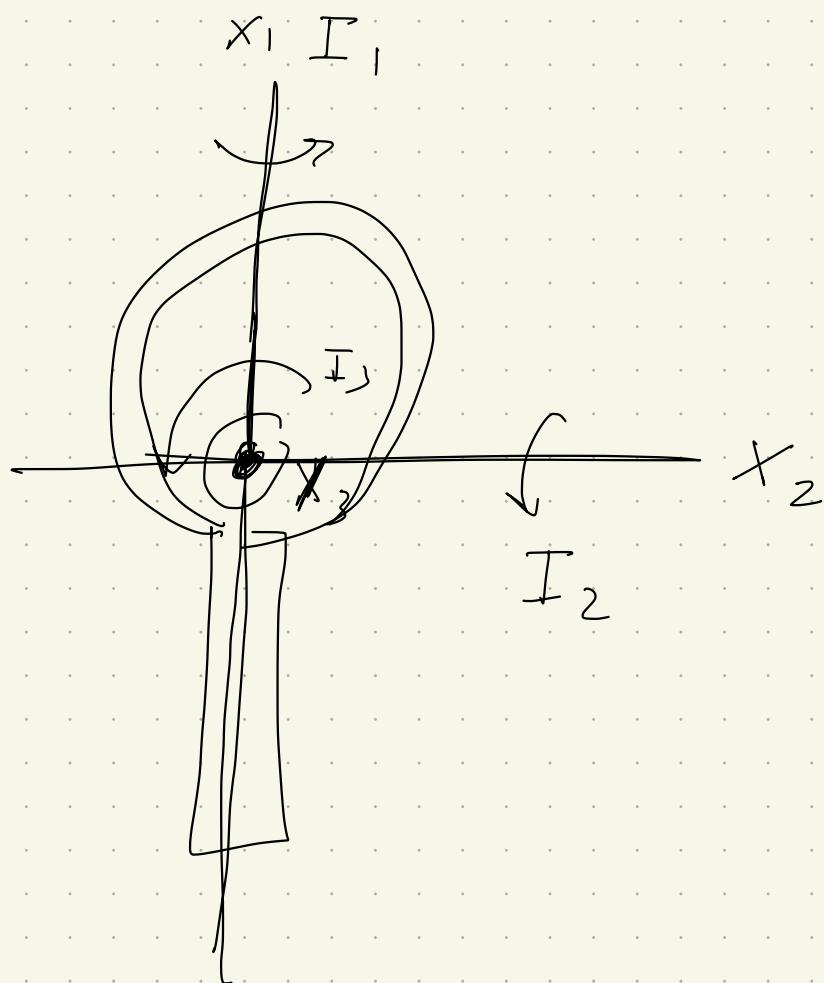


$$I_1 > I_2 > I_3$$

Moment of
inertia
about axis x_1, x_2, x_3

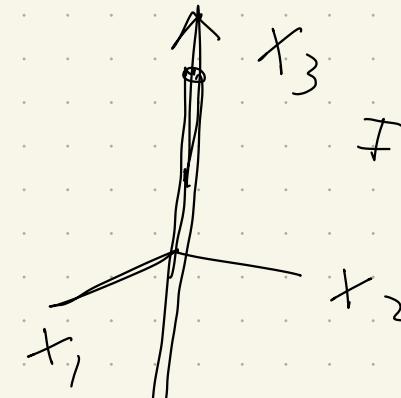


$$I_1 < I_2 < I_3$$



$I_3 = 0$ "rotato-"

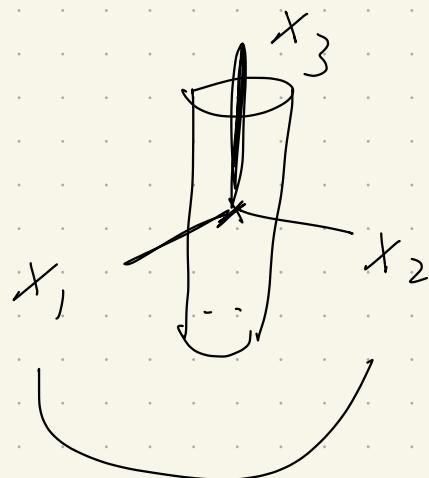
$I_1 = I_2 = I_3$ "sphere"
"cube"



$$F_3 = 0$$

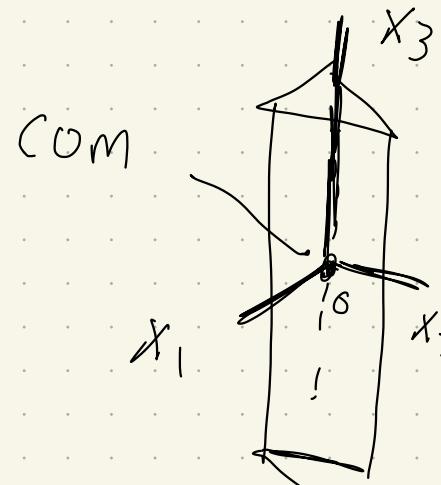
$$I_1 = I_2 \neq 0$$

$I_1 = I_2$, "symmetric top"

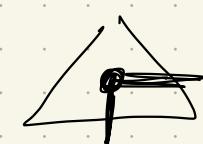


$$I_1 \neq I_2 \neq I_3$$

"asymmetric top"

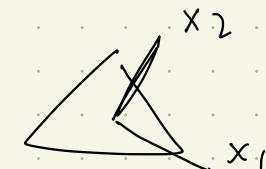


(top view)



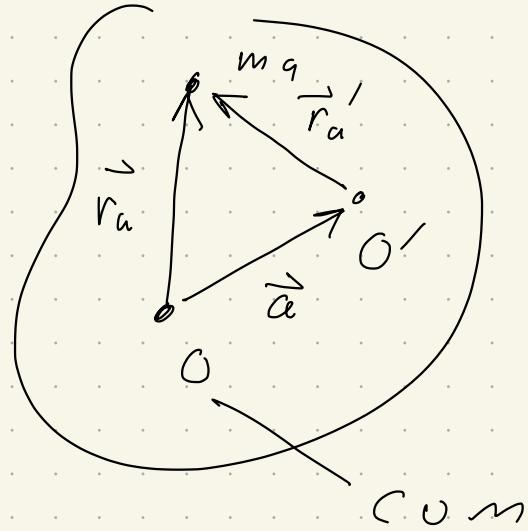
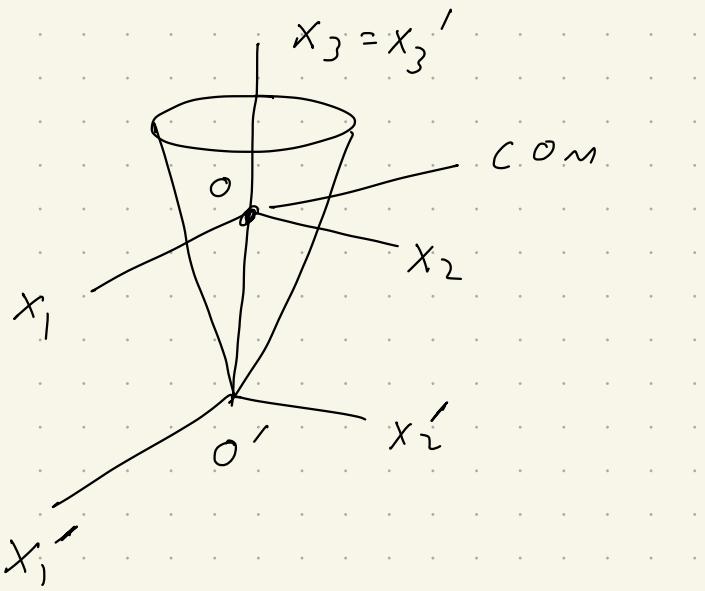
arbitrarily chosen
in plane \perp to x_3

$$I_1 = I_2$$



equivalent
triangle

Shift the origins:



$$\vec{r}_a = \vec{a} + \vec{r}'_a$$

$$\vec{r}'_a = \vec{r}_a - \vec{a}$$

$$I'_{ij} = \sum_a m_a \left(S_{ij} (\vec{r}'_a)^2 - \underbrace{\vec{x}'_{ai} \cdot \vec{x}'_{aj}}_{|r'_a|^2 + |\vec{a}|^2 + 2\vec{r}'_a \cdot \vec{a}} \right)$$

$$(x_{ai} - a_i)(x_{aj} - a_j) = \underbrace{x_{ai} \cdot x_{aj}}_{-a_i \cdot x_{aj} - a_j \cdot x_{ai}} + a_i \cdot a_j$$

$$F_{ij}' = \sum_a m_a (s_{ij} \cdot |\vec{r}_a|^2 - x_{ai} \cdot x_{aj})$$

I_{ij}

$$+ \sum_a m_a (s_{ij} \cdot |\vec{a}|^2 - q_i \cdot q_j)$$

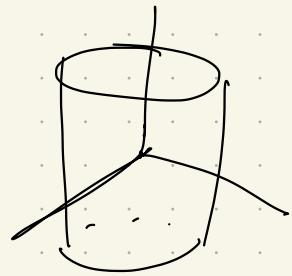
μ

$$+ 2 \left(\sum_a m_a \vec{r}_a \right) \cdot \vec{a} - \left(\sum_a m_a x_{aj} \right) q_i$$

$$- \left(\sum_a m_a x_{ai} \right) q_j$$

$M \vec{R}_{COM}$

$$I_{ij}' = I_{ij} + \mu (s_{ij} \cdot a^2 - q_i \cdot q_j)$$

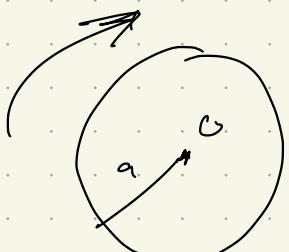


$$I_1, I_2, I_3$$

Unit cylinder : M, R, b

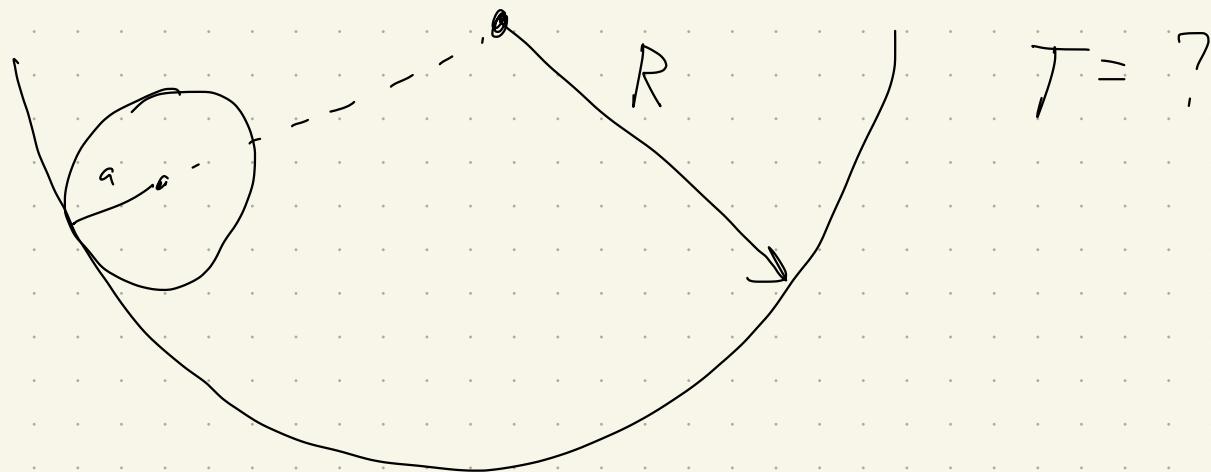
Sec 32, Prob 2c

rolling w. bout



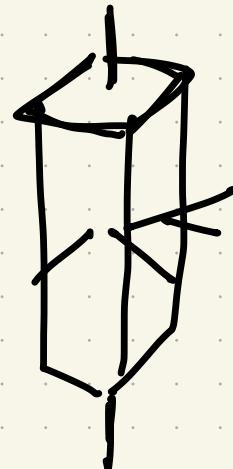
slipping

$$T = ?$$



Lecture #24: Thurs Nov 11th

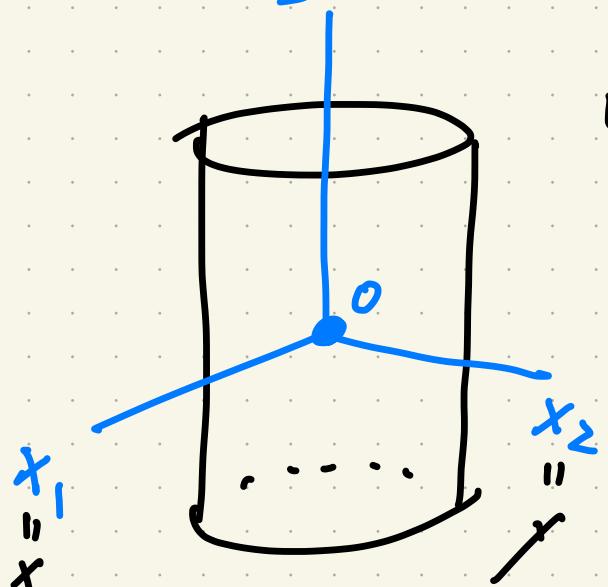
Quiz # 5 - Tuesday Nov 16th
Exam # 2 - Tuesday Nov 23rd



Inertia tensor:

$$I_{ij} = \int \rho dV (r^2 \delta_{ij} - x_i x_j)$$

$$z = x_3$$



Unit cylinder

mass: μ

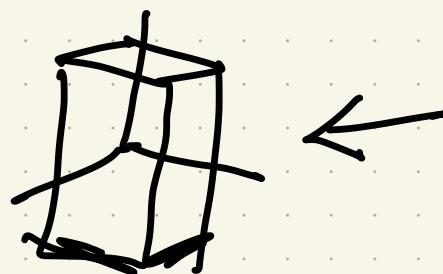
radius: R

height: h

$$\rho = \frac{\mu}{\pi R^2 \cdot h}$$

$$I_3 =$$

$$I = I_1 = I_2$$



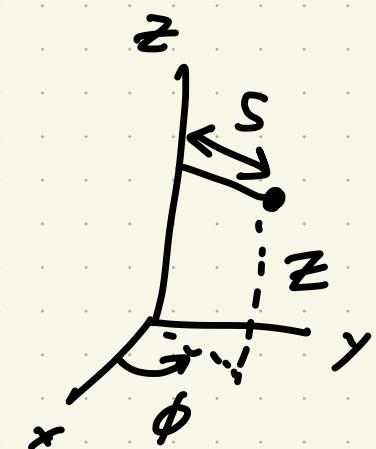
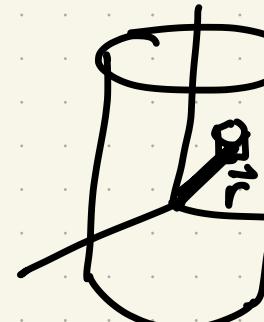
O : Com of the cylinder.

$$\sum_{i,j} I_{ij} \cdot n_i \cdot n_j = I(\hat{n})$$

$$I_3 = \sum_{i,j} I_{ij} \cdot (x_3)_i \cdot (x_3)_j = I_{33}$$

$$I_1 = I_{11}$$

$$I_2 = I_{22}$$



$$I_3 = \int \rho dV (r^2 \delta_{33} - x_3^2)$$

$$= \int \rho dV (r^2 - z^2)$$

$$= \int \rho dV (x^2 + y^2)$$

sp^{l.} polar
radius

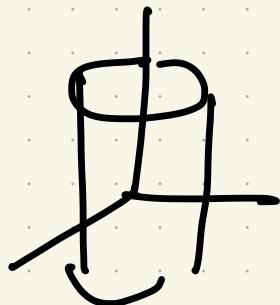
$$dV = s ds d\phi dz$$

$$I_3 = \rho \int_0^R \int_0^{2\pi} \int_{-h/2}^{h/2} s ds d\phi dz s^2$$

$s=0 \quad \phi=0 \quad z=-\frac{h}{2}$

$$= \frac{M}{\pi R^2 h} 2\pi h$$

$$= \boxed{\frac{1}{2} M R^2}$$



$$\int_{s=0}^R s^3 ds$$

$$\left. \frac{s^4}{4} \right|_0^R$$

$$\frac{R^4}{4}$$

$$I = I_1 = \rho \int dV (\underbrace{r^2 - x_1^2}_{y^2 + z^2})$$

$$= \rho \int dV (y^2 + z^2)$$

$$I = I_2 = \rho \int dV (x^2 + z^2)$$

(s, ϕ, z)

$$y = s \sin \phi$$

$$x = s \cos \phi$$

$$2I = I_1 + I_2$$

$$= \rho \int dV (x^2 + y^2 + z^2)$$

$$= \rho \int dV s^2 + 2\rho \int dV z^2$$

$$\rightarrow \boxed{I = \frac{1}{2} I_3 + \rho \int dV z^2} \quad \leftarrow$$

$$I_1 = I_2 = \frac{1}{2} I_3 + \rho \int dV z^2$$

$$= \frac{1}{4} M R^2 + \frac{M}{\pi R^2 h} \int_{s=0}^R \int_{z=-\frac{h}{2}}^{\frac{h}{2}} \int_{\phi=0}^{2\pi} s ds dz d\phi z^2$$

$$= \frac{1}{4} M R^2 + \frac{M}{\pi R^2 h} 2\pi \left(\frac{z^3}{3} \right) \Big|_{-\frac{h}{2}}^{\frac{h}{2}} \frac{s^2}{2} \Big|_0^R$$

$$+ \frac{M}{\cancel{\pi R^2 h}} \cancel{2\pi} \cancel{\left(\frac{2}{3} \left(\frac{h}{2} \right)^3 \right)} \cancel{\frac{R^2}{2}}$$

$$\frac{1}{12} M h^2$$

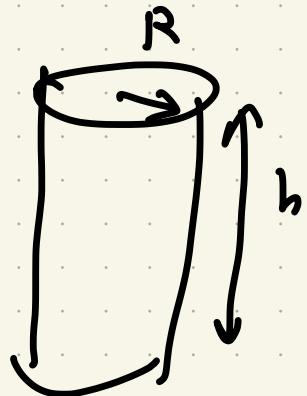
$$\downarrow I_1 = I_2$$

$$= \frac{1}{4} M R^2 + \frac{1}{12} M h^2$$

$$= \boxed{\frac{1}{4} M \left(R^2 + \frac{h^2}{3} \right)}$$

$$I_1 = I_2 = \frac{1}{4} M \left(R^2 + \frac{1}{3} h^2 \right)$$

$$I_3 = \frac{1}{2} M R^2$$



i) thin disk ($h \rightarrow 0$)



$$\begin{cases} I_3 = \frac{1}{2} M R^2 \\ I_1 = I_2 = \frac{1}{4} M R^2 \end{cases}$$

ii) (R → 0)



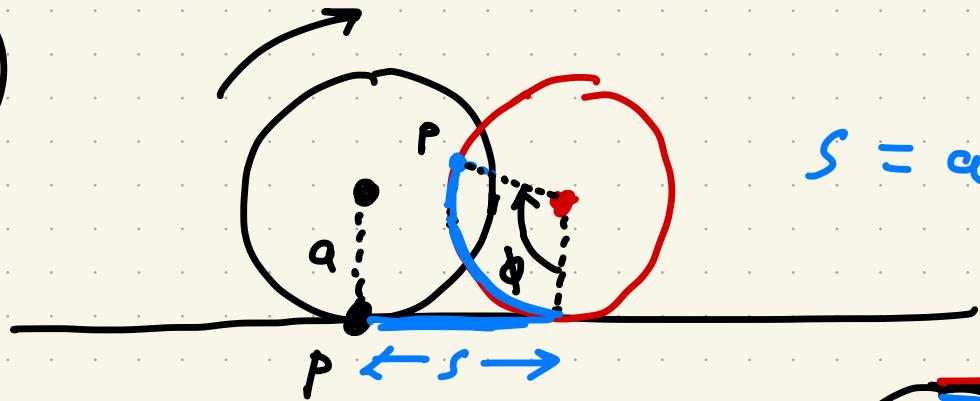
thin rod

$$\begin{cases} I_3 = 0 \\ I_1 = I_2 = \frac{1}{12} M h^2 \end{cases}$$

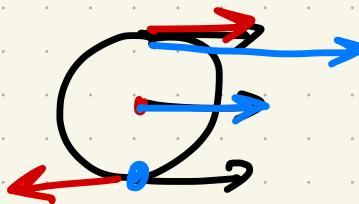
Calculate: KE of a rolling cylinder (rolling w/o JST, slipping)
 $(M, a, h = \text{height} = h)$

$$\begin{aligned} T &= \frac{1}{2} M |\vec{V}|^2 + \frac{1}{2} I_{\text{tot}} \omega^2 \\ &= \frac{1}{2} M |\vec{V}|^2 + \frac{1}{2} (I_1 n_1^2 \\ &\quad + I_2 n_2^2 \\ &\quad + I_3 n_3^2) \end{aligned}$$

i)



$$s = a\phi$$



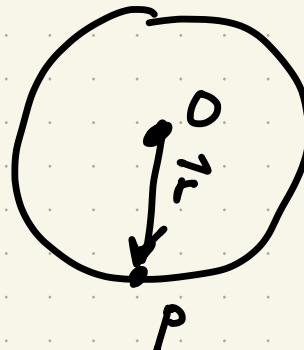
$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$$

$$V = \dot{s} = a\dot{\phi}$$

$$\Omega = \frac{V}{a} = \frac{a\dot{\phi}}{a} = \dot{\phi}$$

$$\Gamma = \frac{1}{2} M a^2 \dot{\phi}^2 + \frac{1}{2} \left(\frac{1}{2} M a^2 \right) \dot{\phi}^2$$

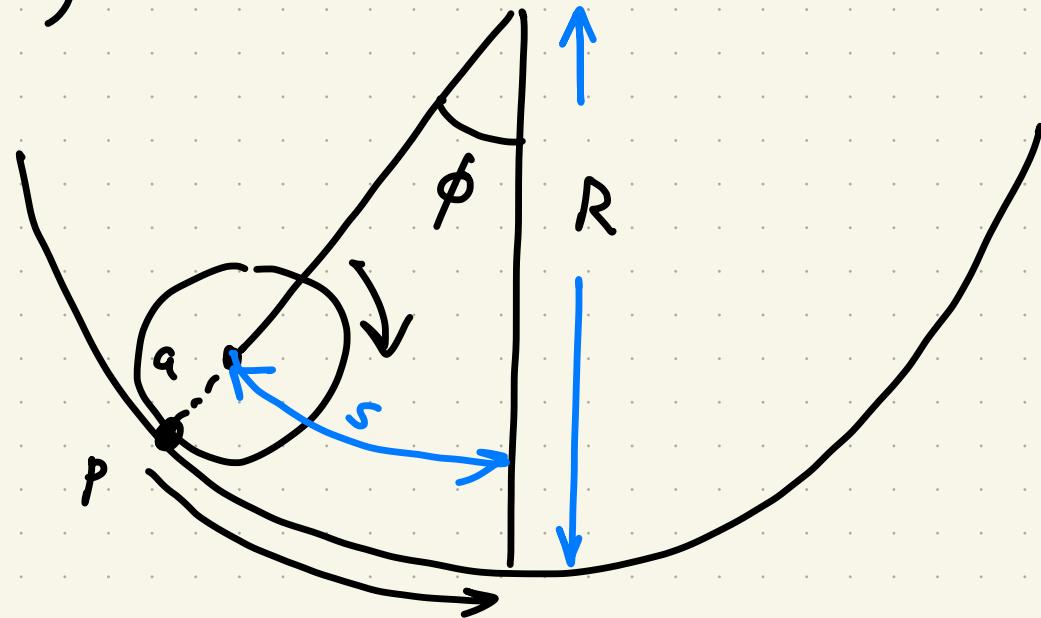
$$= \boxed{\frac{3}{4} M a^2 \dot{\phi}^2}$$



$$\sigma = \vec{V} + \vec{\Omega} \times (-a\hat{n})$$

$$\boxed{V = \Omega a} \quad \text{into plane}$$

ii)



$$s = (R-a)\dot{\phi}$$

$$\begin{aligned} V &= \dot{s} \\ &= (R-a)\dot{\phi} \end{aligned}$$

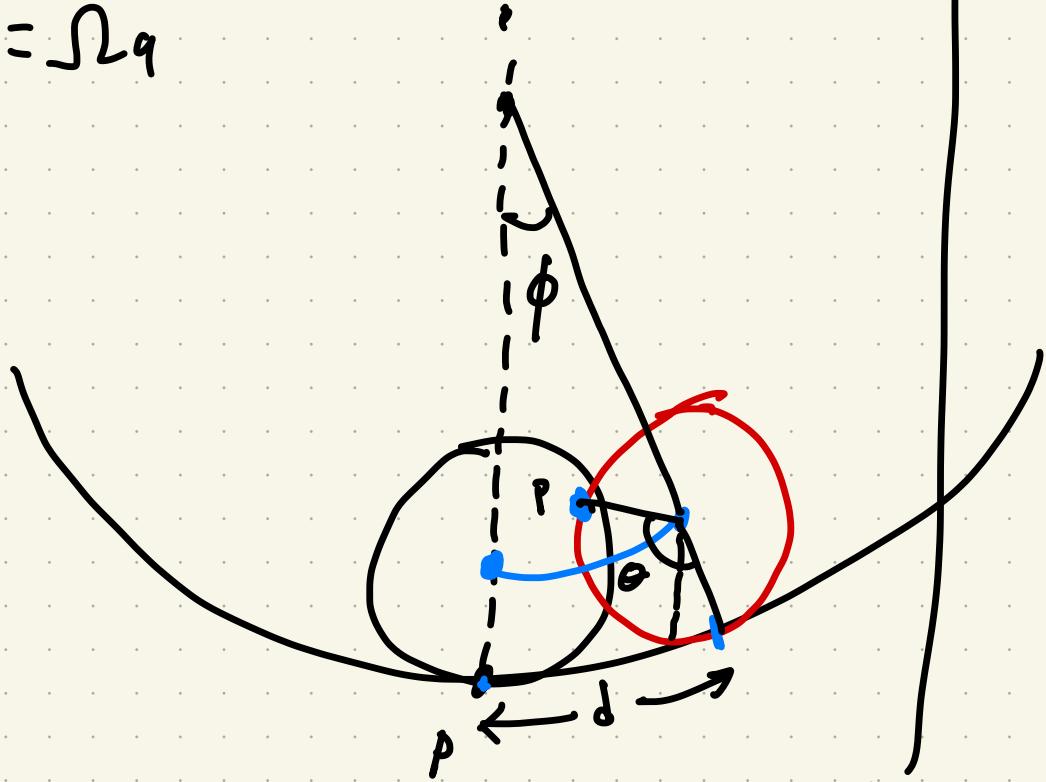
$$V = \Omega a$$

$$V = \Omega a$$

$$\Omega = \left(\frac{R-a}{a} \right) \dot{\phi}$$

$$\rightarrow T = \frac{3}{4} M (R-a)^2 \dot{\phi}^2$$

$$V = \Omega a$$



$$\begin{aligned}
 \dot{\alpha} - \dot{\phi} &= \dot{\theta} - \dot{\phi} \\
 \dot{\alpha} &= \dot{\theta} - \dot{\phi} \\
 &= \frac{R}{a} \dot{\phi} - \dot{\phi} \\
 &= \left(\frac{R}{a} - 1 \right) \dot{\phi} \\
 &= \left(\frac{R-a}{a} \right) \dot{\phi} \\
 &= \Omega
 \end{aligned}$$

$$a\dot{\theta} = d\dot{\phi} \rightarrow \dot{\theta} = \frac{R}{a}\dot{\phi}$$

$$s = (R-a)\dot{\phi}$$

$$V = s = (R-a)\dot{\phi} = \Omega a \rightarrow \boxed{\Omega = \left(\frac{R-a}{a} \right) \dot{\phi}}$$

Lecture #25: Nov 16th

EXAM 2 - Next tuesday 11/23

(Zoom review session: 7pm - 8pm, Sunday)

Today: - Quiz #5

- RB motion

Thursday - RB motion / statics ??
(Sec 38)

Tues: 11/23 EXAM 2

Thurs: Thanksgiving

Tues: 11/30: non-inertial ref frames

Sat 12/4: 1:30 - 4:00pm { Sat
Sun
Mon }

EOM_s:

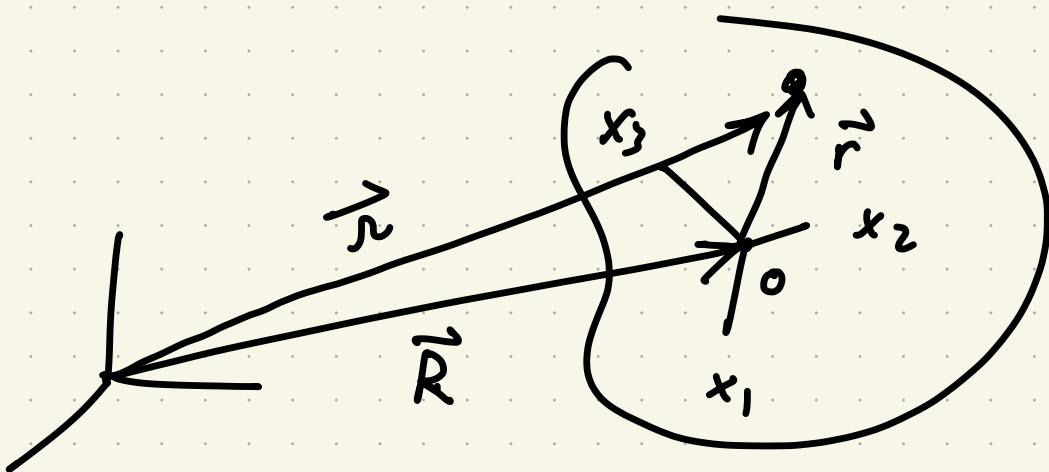
$$\frac{d \vec{M}}{dt} = \vec{F} = \sum_f r_f \vec{f}_f$$

$$\frac{d \vec{P}}{dt} = \vec{F} = \sum_f \vec{f}_f$$

$\uparrow \sum_a r_a \vec{f}_a$

6 DOF's: $(\vec{R}, \vec{\phi}) = q_i$

$$(\vec{V}, \vec{\omega}) = \dot{q}_i$$



inertial

$$\begin{aligned}\vec{v} &= \frac{d \vec{r}}{dt} \\ &= \frac{d}{dt} (\vec{r}^l + \vec{r}) \\ &= \vec{V} + \vec{\omega} \times \vec{r}^l\end{aligned}$$

Derive:

$$\frac{d\vec{P}}{dt} = \frac{d}{dt} (\sum \vec{p}) = \sum \frac{d\vec{p}}{dt} = \sum \vec{f}$$

$$\begin{aligned}\frac{d\vec{M}}{dt} &= \frac{d}{dt} (\sum \vec{r} \times \vec{p}) = \sum \left(\cancel{\vec{r} \times \dot{\vec{p}}}^0 + \vec{r} \times \cancel{\dot{\vec{p}}}^{\cancel{m\vec{v}}} \right) \\ &= \sum \vec{r} \times \vec{f}\end{aligned}$$

$$m\vec{v}$$

$$m \frac{d\vec{r}}{dt}$$

$$m \frac{d\vec{r}}{dt}$$

$$m \vec{r}$$

$$\vec{r} = \vec{\Omega} \times \vec{r}$$

Alternative derivation:

$$L = \underbrace{\frac{1}{2} \mu |\vec{V}|^2 + \frac{1}{2} \sum_{i,j} I_{ij} \cdot \vec{r}_i \cdot \vec{r}_j}_{\tau} - U(\vec{r})$$

$\sum_{i,j}$ implied
"are"

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \vec{V}} \right) = \frac{\partial L}{\partial \vec{R}} \quad \longleftrightarrow \quad \frac{d \vec{P}}{dt} = \sum \vec{f}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \vec{r}_i} \right) = \frac{\partial L}{\partial \vec{v}_i} \quad \longleftrightarrow \quad \frac{d \vec{M}_i}{dt} = \sum \vec{r} \times \vec{f}$$

$$\begin{aligned} \frac{\partial L}{\partial \vec{V}} &= \mu \vec{V} = \vec{P} \\ \frac{\partial L}{\partial \vec{r}_i} &= \vec{M}_i \end{aligned} \quad \left| \quad \begin{aligned} \frac{\partial L}{\partial r_i} &= \sum_j I_{ij} \cdot \vec{r}_j \\ &= M_i \end{aligned} \right.$$

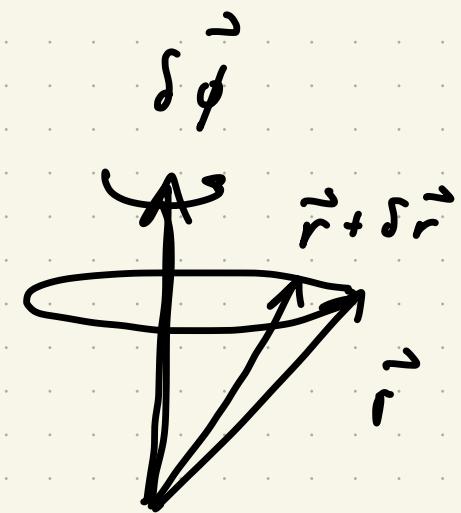
$$\frac{1}{2} I_{ij} \cdot \nabla_i \cdot \nabla_j = \frac{1}{2} \left(I_1 \nabla_1^2 + I_2 \nabla_2^2 + I_3 \nabla_3^2 \right)$$

$$\frac{\partial L}{\partial \dot{r}_3} = I_3 \ddot{r}_3 = M_3$$

$$\delta U(\vec{r}) = \sum_a \frac{\partial U}{\partial \vec{r}_a} \cdot \delta \vec{r}_a$$

$$\vec{r} = \vec{R} + \vec{r}$$

$$\begin{aligned} \delta \vec{r} &= \delta \vec{R} + \delta \vec{r} \\ &= \delta \vec{R} + \delta \vec{\phi} \times \vec{r} \end{aligned}$$



$$\delta U = \sum_a \underbrace{\frac{\partial U}{\partial \vec{r}_a}}_{f_a} \cdot (\delta \vec{R} + \delta \vec{\phi} \times \vec{r})$$

$$\delta U = - \left(\sum \vec{f} \right) \cdot \delta \vec{R} - \underbrace{\sum \vec{f} \cdot (\delta \vec{\phi} \times \vec{r})}_{\delta \vec{\phi} \cdot \sum (\vec{r} \times \vec{f})}$$

$$\boxed{\frac{\partial U}{\partial \vec{R}}} = - \sum \vec{f} \\ = - \vec{F}$$

$$\boxed{\frac{\partial U}{\partial \vec{\phi}}} = - \sum \vec{r} \times \vec{f} \\ = - \vec{F}$$

$$\frac{\partial L}{\partial \vec{R}} = - \frac{\partial U}{\partial \vec{R}} = \vec{F} \quad , \quad \frac{\partial L}{\partial \vec{\phi}} = - \frac{\partial U}{\partial \vec{\phi}} = \vec{F}$$

$$U(x, y) \quad dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

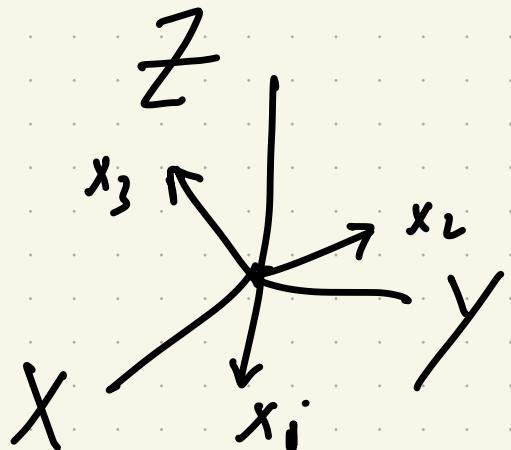
$$\delta U = \frac{\partial U}{\partial \vec{R}} \cdot \delta \vec{R} + \frac{\partial U}{\partial \vec{\phi}} \cdot \delta \vec{\phi}$$

Euler's equations for RB motion:

$$\underbrace{\frac{d\vec{A}}{dt}}_{\text{w.r.t inertial frame}} = \underbrace{\frac{d'\vec{A}}{dt}}_{\text{w.r.t rotating frame}} + \vec{\omega} \times \vec{A}$$

↑
ang. velocity
of the
rotating frame.

$$\vec{A} = \sum_i A_i \hat{x}_i \quad \left(= A_x \hat{X} + A_y \hat{Y} + A_z \hat{Z} \right)$$



$$\frac{d\vec{A}}{dt} = \sum_i \frac{dA_i}{dt} \hat{x}_i + \sum_i A_i \cdot \frac{d\hat{x}_i}{dt}$$

$\frac{d'\vec{A}}{dt}$ $\vec{\omega} \times \vec{A}$

$$\left(\frac{d' \vec{A}}{dt} \right)_i = \frac{d A_i}{dt}$$

$$\frac{d \vec{A}}{dt} = \frac{d' \vec{A}}{dt} + \vec{\omega} \times \vec{A}$$

$$\vec{F} = \frac{d \vec{P}}{dt} = \frac{d' \vec{P}}{dt} + \vec{\omega} \times \vec{P}$$

$$P_i = m V_i$$

$$F_i = \left(\frac{d' \vec{P}}{dt} \right)_i + (\vec{\omega} \times \vec{P})_i$$

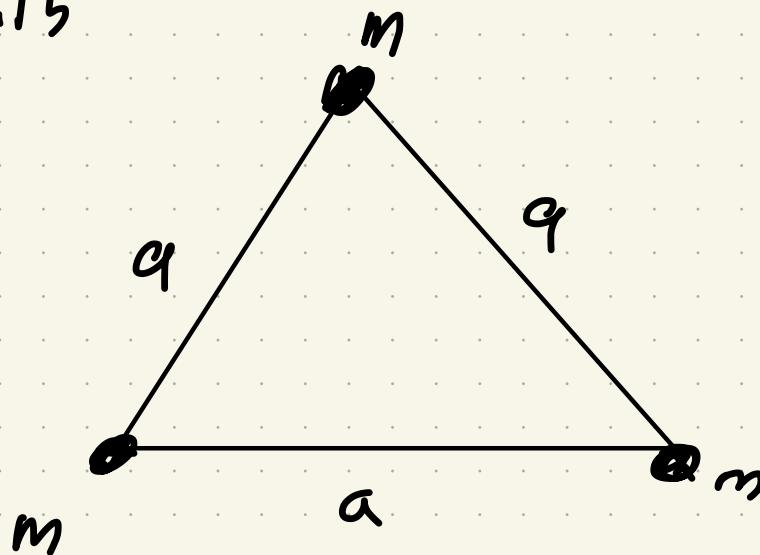
$$= \frac{d P_i}{dt} + \Omega_2 P_3 - \Omega_3 P_2$$

$$= m \left(\frac{d V_i}{dt} + \Omega_2 V_3 - \Omega_3 V_2 \right)$$

+ cyclic

q5 - Lastname :

Calculate the principal moments of inertia for a rigid body which consists of 3 equal masses at the corners of an equilateral triangle with side = a



Solution: Take points in xy plane. Since masses are equal, and an equilateral Δ has 120° rotational symmetry \rightarrow symmetric top ($\therefore I_1 = I_2 \leq I$)

Also, because points don't have a Z -component:

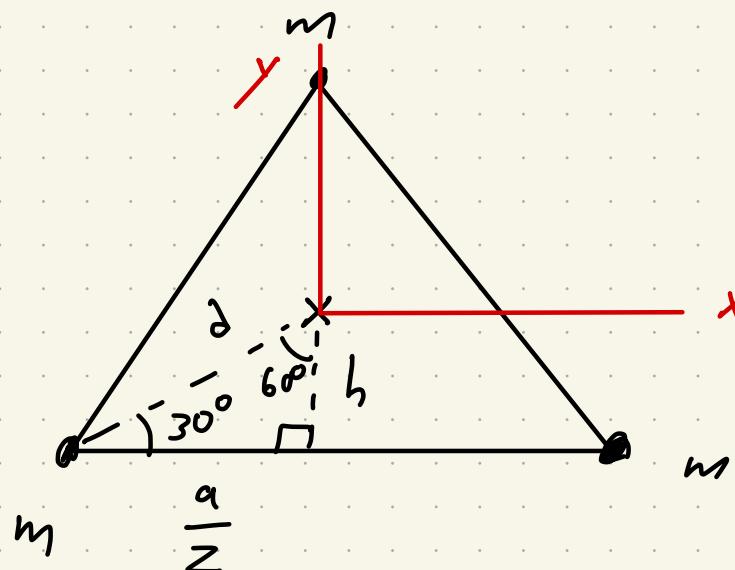
$$I_1 = \sum_a m_a (y_a^2 + z_a^2) = \sum_a m_a y_a^2$$

$$I_2 = \sum_a m_a (z_a^2 + x_a^2) = \sum_a m_a x_a^2$$

$$\rightarrow I_3 = \sum_a m_a (x_a^2 + y_a^2) = I_1 + I_2 = 2I$$

$$\text{Thus, } I = \sum_a I_3$$

To find I_3 , note that $x_a^2 + y_a^2 = \frac{1}{\text{distance of mass point from Z-axis}}$



$$\sin 30^\circ = \frac{h}{d} = \frac{1}{2}$$

$$\sin 60^\circ = \frac{a/2}{d} = \frac{\sqrt{3}}{2}$$

→

$$d = \frac{a}{\sqrt{3}}$$

$$h = \frac{d}{2} = \frac{a}{2\sqrt{3}}$$

Thus,

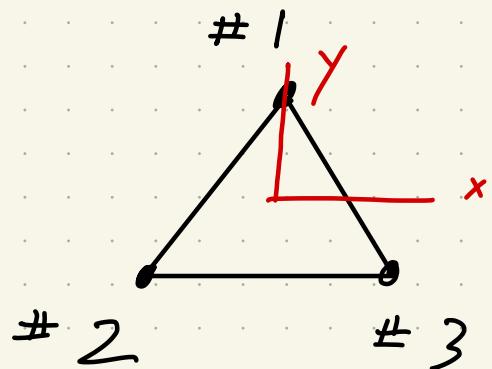
$$I_3 = \sum_a m_a (x_a^2 + y_a^2)$$

$$= \sum_a m_a d_a^2 = 3m \left(\frac{a}{\sqrt{3}}\right)^2 = \boxed{ma^2}$$

$$\rightarrow I_1 = I_2 = \frac{1}{2} I_3 = \boxed{\frac{1}{2} ma^2}$$

Alternative: (explicit calculation of I_1 , I_2)

Take (x_1, x_2) to be (x, y)



Then #1: $(x, y) = (0, \frac{a}{\sqrt{3}})$

#2: $(x, y) = (-\frac{a}{2}, -\frac{a}{2\sqrt{3}})$

#3: $(x, y) = (\frac{a}{2}, -\frac{a}{2\sqrt{3}})$

Then $I_1 = \sum_a m_a y_a^2 = m \left(\left(\frac{a}{\sqrt{3}}\right)^2 + \left(-\frac{a}{2\sqrt{3}}\right)^2 + \left(\frac{a}{2\sqrt{3}}\right)^2 \right)$

$$= m a^2 \left(\frac{1}{3} + \frac{1}{12} + \frac{1}{12} \right)$$

$$= \boxed{\frac{1}{2} m a^2}$$

$$I_2 = \sum_a m_a x_a^2 = m \left(0^2 + \left(\frac{-a}{2}\right)^2 + \left(\frac{a}{2}\right)^2 \right)$$

$$= \boxed{\frac{1}{2} m a^2}$$

$$\rightarrow I_3 = I_1 + I_2 = \boxed{m a^2}$$

Recall:

$$\frac{d\vec{A}}{dt} = \underbrace{\frac{d'\vec{A}}{dt}}_{\substack{\text{w.r.t} \\ \text{inertial} \\ \text{frame}}} + \underbrace{\vec{\omega} \times \vec{A}'}_{\substack{\text{w.r.t} \\ \text{rotating} \\ \text{frame}}}$$

L angular
velocity
of rotating
frame

$$\vec{A}' = \sum_i A'_i \hat{x}_i$$

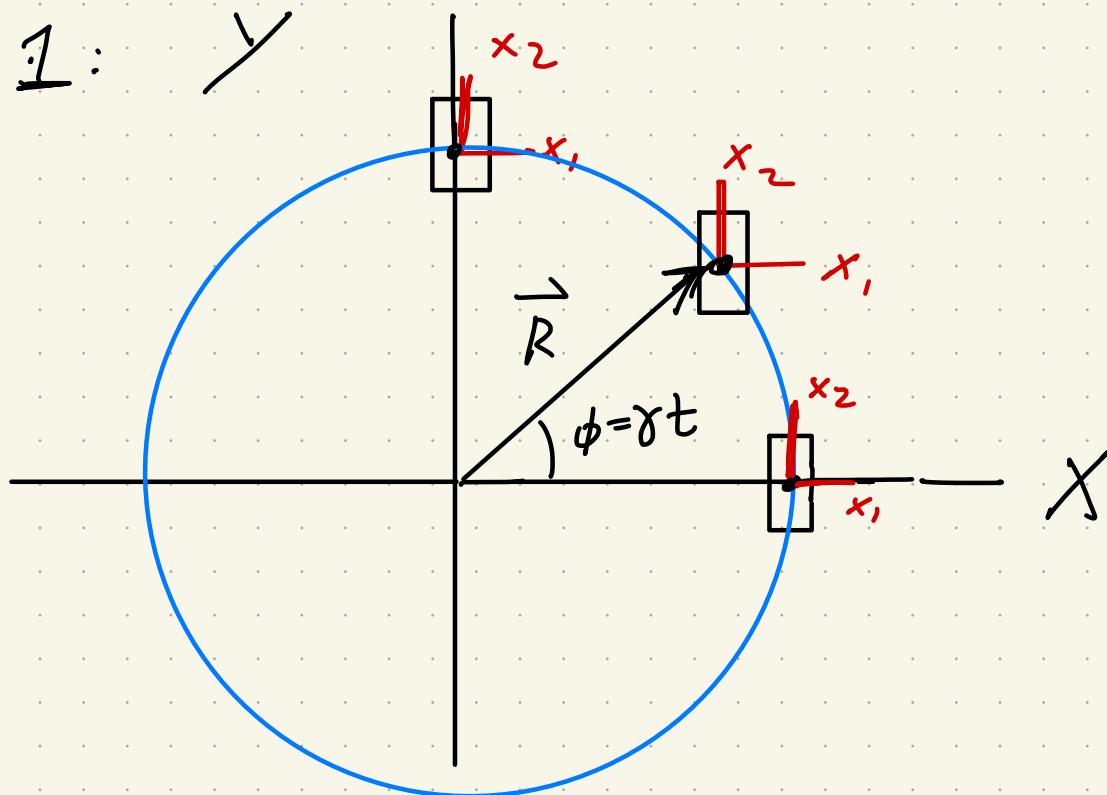
basis vectors (time-dependent)
for rotating frame

$$\frac{d\vec{A}}{dt} = \sum_i \underbrace{\frac{dA'_i}{dt} \hat{x}_i}_{\frac{d'\vec{A}}{dt}} + \sum_i A'_i \underbrace{\frac{d\hat{x}_i}{dt}}_{\vec{\omega} \times \vec{A}'}$$

NOTE: i) $\frac{d\vec{A}}{dt}$, $\frac{d'\vec{A}}{dt}$ involve the same vector

ii) Inertial frame and rotating frame do not have to have the same origin.

Example 1:



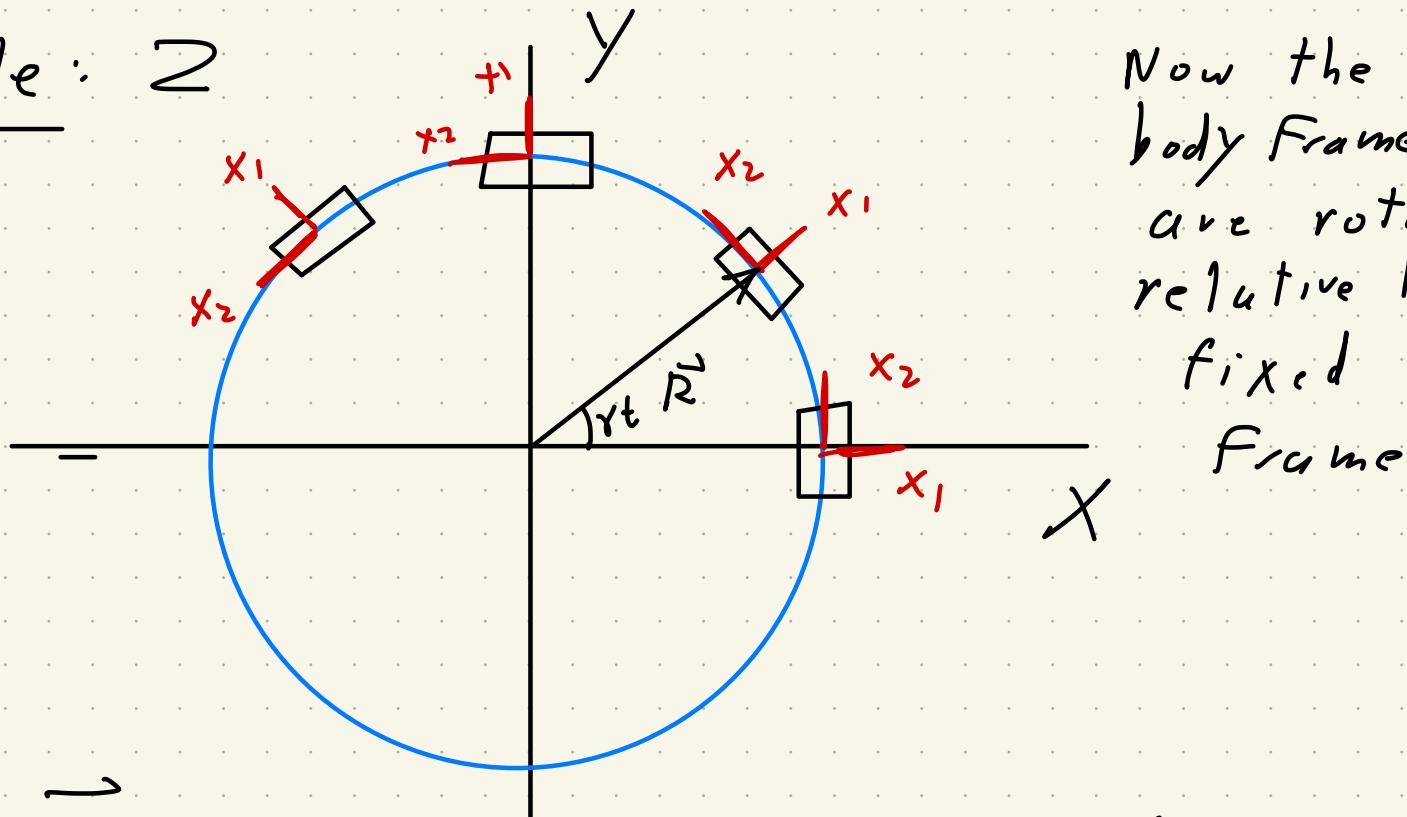
Com move in circle of radius R with unit angular velocity γ .

Thus, $\vec{R} = R \cos(\gamma t) \hat{X} + R \sin(\gamma t) \hat{Y}$

$$\vec{V} = \frac{d\vec{R}}{dt} = R\gamma (-\sin(\gamma t) \hat{X} + \cos(\gamma t) \hat{Y})$$

Since (\hat{x}_1, \hat{x}_2) point in same direction as (\hat{X}, \hat{Y})
then $\frac{d\vec{A}}{dt} = \frac{d\vec{A}'}{dt}$ for any \vec{A} , including \vec{V}

Example: 2



Now the rigid body frame axes are rotating relative to the fixed inertial frame

\vec{R} , \vec{V} same as for example 1.

$$\hat{x}_1 = \cos(\gamma t) \hat{X} + \sin(\gamma t) \hat{Y}$$

$$\hat{x}_2 = -\sin(\gamma t) \hat{X} + \cos(\gamma t) \hat{Y}$$

$$\frac{d\hat{x}_1}{dt} = -\gamma \sin(\gamma t) \hat{X} + \gamma \cos(\gamma t) \hat{Y} = \gamma \hat{x}_2$$

$$\frac{d\hat{x}_2}{dt} = -\gamma \cos(\gamma t) \hat{X} - \gamma \sin(\gamma t) \hat{Y} = -\gamma \hat{x}_1$$

$$\Gamma_{h_0}, \quad \dot{x}_i = A_i; \quad \frac{d\hat{x}_i}{dt} = \gamma A_1 \hat{x}_2 - \gamma A_2 \hat{x}_1 \\ = \vec{\Omega} \times \vec{A}$$

where $\vec{\Omega} = \gamma \hat{Z} = \gamma \hat{x}_3$

check:

$$(\vec{\Omega} \times \vec{A})_1 = \cancel{A_2}^o A_3 - \Omega_3 A_2 = -\gamma A_2$$

$$(\vec{\Omega} \times \vec{A})_2 = \Omega_3 A_1 - \cancel{A_1}^o A_3 = \gamma A_1$$

$$(\vec{\Omega} \times \vec{A})_3 = \cancel{\Omega_1}^o A_2 - \cancel{\Omega_2}^o A_1 = 0$$

so $\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\Omega} \times \vec{A}$ as claimed

NOTE: In rotating frame $\vec{R} = R \hat{x}_1, \quad \vec{V} = R \gamma \hat{x}_2$

$$\text{so } \frac{d'\vec{R}}{dt} = 0, \quad \frac{d'\vec{V}}{dt} = 0 \implies \vec{V} = \frac{d\vec{R}}{dt} = \vec{\Omega} + \vec{\Omega} \times \vec{R} \\ = -\gamma R_{\sin(\gamma t)} \hat{x}_1 + \gamma R_{\cos(\gamma t)} \hat{x}_2$$