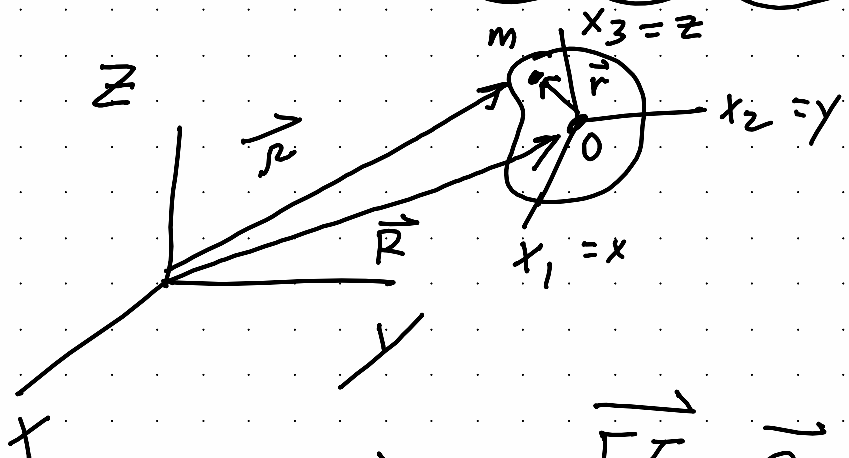
Lec #23: Tuesday Nov 10th

- Quiz #5: Thursday
- Midterm #2: Next Thursday 11/19 (scattering, small oscillations, some rigid body)
- Today: Rigid body motion (Sec 31–36, 38, 39)
  - 38: static equilibrium
  - 39: non-inertial



O: at COM (usually)

x_1, x_2, x_3: fixed in RB

$$\vec{V} = \frac{d\vec{R}}{dt}, \qquad \vec{\Omega} = \frac{d\vec{\phi}}{dt} \quad \Big\} \ \dot{q}_i$$

$(\vec{R}, \vec{\phi})$: 6 DOF $= q_i$

$$\vec{v} = \frac{d\vec{\kappa}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}$$

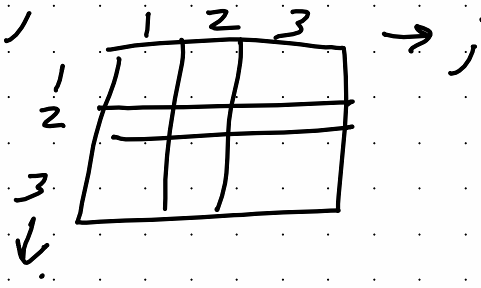
$\vec{\Omega}$: angular velocity vector

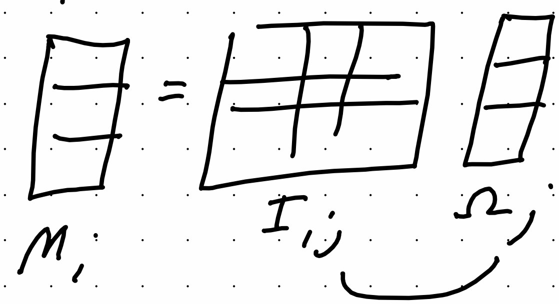
$$T = \frac{1}{2} m v^2 \qquad , \qquad \text{rotational quantities} \quad T_{rot} = \frac{1}{2} I \Omega^2$$

$$\vec{p} = m \vec{v} \qquad \vec{M} = I \vec{\Omega} \rightarrow M_i = \sum_j I_{ij} \Omega_j$$

I : moment of inertia

$\longrightarrow$ I_{ij} : inertia tensor





KE:

$$\vec{v}_a = \vec{V} + \vec{\Omega} \times \vec{r}_a$$

$$T = \frac{1}{2} \sum_a m_a |\vec{v}_a|^2$$

$$= \frac{1}{2} \sum_a m_a |\vec{V} + \vec{\Omega} \times \vec{r}_a|^2$$

$$|\vec{A} + \vec{B}|^2 = A^2 + B^2 + 2\vec{A}\cdot\vec{B}$$

$$= \frac{1}{2} \sum_a m_a \left(|\vec{V}|^2 + |\vec{\Omega} \times \vec{r}_a|^2 + 2\vec{V}\cdot(\vec{\Omega} \times \vec{r}_a) \right)$$

$$1 = \frac{1}{2} \sum_a m_a |\vec{V}|^2 = \boxed{\frac{1}{2} \mu V^2}$$

μ: total mass

$$3 = \sum_a m_a \vec{V}\cdot(\vec{\Omega} \times \vec{r}_a)$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$= \left(\sum_a m_a \vec{r}_a\right)\cdot(\vec{V}\times\vec{\Omega})$$

$$= \underbrace{\mu \vec{R}_{com}}_{=0 \text{ for}} \cdot (\vec{V}\times\vec{\Omega})$$

$\boxed{\text{O at com}}$

(2) $= \frac{1}{2} \sum_a m_a \left| \vec{\Omega} \times \vec{r_a} \right|^2$

$$= \frac{1}{2} \sum_a m_a \left(\vec{\Omega} \times \vec{r_a} \right) \cdot \left(\vec{\Omega} \times \vec{r_a} \right)$$

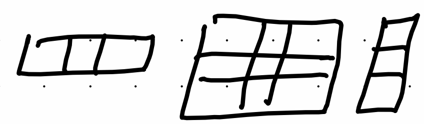
$$= \frac{1}{2} \sum_a m_a \; \vec{\Omega} \cdot \left(\vec{r_a} \times \left(\vec{\Omega} \times \vec{r_a} \right) \right)$$

$$= \frac{1}{2} \sum_a m_a \; \vec{\Omega} \cdot \left(\vec{\Omega}\, r_a^2 - \vec{r_a} \left(\vec{r_a} \cdot \vec{\Omega} \right) \right)$$

$$= \frac{1}{2} \sum_a m_a \left(\Omega^2 r_a^2 - \left(\vec{\Omega} \cdot \vec{r_a} \right)^2 \right)$$

$$= \frac{1}{2} \sum_a m_a \left(\sum_{i,j} \Omega_i \Omega_j \delta_{ij} r_a^2 - \sum_{i,j} \Omega_i r_{ai} \Omega_j r_{aj} \right)$$

$$= \frac{1}{2} \sum_{i,j} \left(\sum_a m_a \left(r_a^2 \delta_{ij} - r_{ai} r_{aj} \right) \right) \Omega_i \Omega_j$$

$= \boxed{\frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j}$ $\Big(= \frac{1}{2}$  $\Big)$

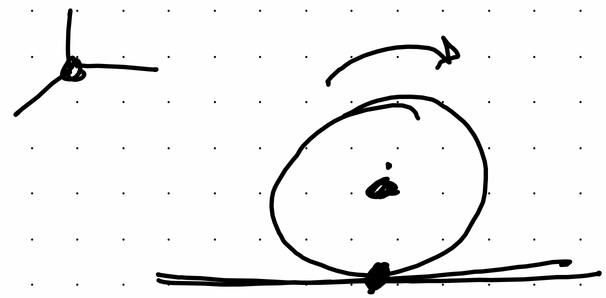
$$\vec{A} \times \left(\vec{B} \times \vec{C} \right) = \vec{B} \left(\vec{A} \cdot \vec{C} \right) - \vec{C} \left(\vec{A} \cdot \vec{B} \right)$$

$$\boxed{T = \underbrace{\frac{1}{2}\mu V^2}_{\text{trans}} + \underbrace{\frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\Omega_j}_{\text{rotational}}}$$

$\left(\frac{1}{2} I \Omega^2\right)$ Freshman physics

for COM at Origin
OF RB Frame

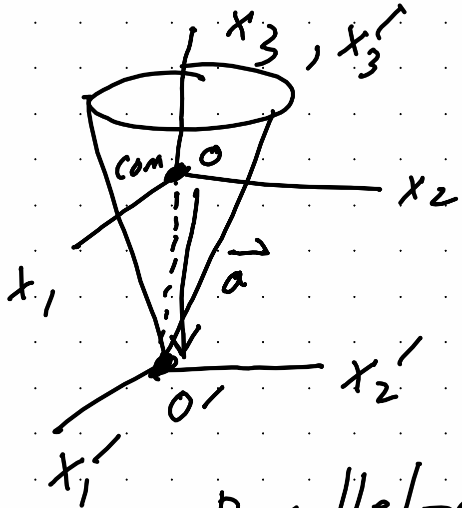
$\vec{M}$: wrt COM of body



$$\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a$$
$$= \sum_a m_a\, \vec{r}_a \times \vec{v}_a$$
$$= \sum_a m_a\, \vec{r}_a \times \left(\vec{V} + \vec{\Omega} \times \vec{r}_a\right)$$
$$= \sum_a m_a\, \vec{r}_a \times \left(\vec{\Omega} \times \vec{r}_a\right)$$
$$= \sum_a m_a \left(\vec{\Omega}\, r_a^2 - \vec{r}_a\,(\vec{r}_a \cdot \vec{\Omega})\right)$$

$$\boxed{M_i = \sum_j I_{ij}\,\Omega_j}$$

$\vec{M} = I\vec{\Omega}$ (Fresh. physics)



parallel-axis theorem

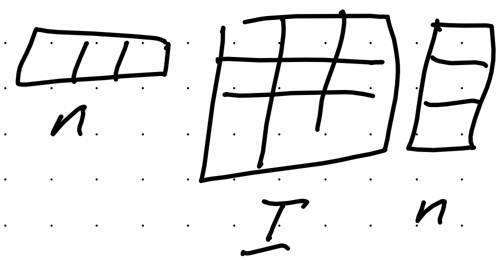
$$\underbrace{I'_{ij}}_{\text{wrt } O'} \;,\quad \underbrace{I_{ij}}_{\text{wrt } O \text{ (com)}}$$

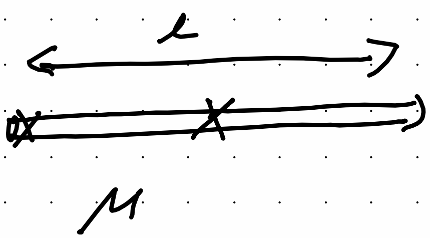
$$I'_{ij} = I_{ij} + \mu(a^2\delta_{ij} - a_i a_j)$$

$\vec{a}$: vector from O to O'

$$\underbrace{I(\hat{n})}_{\text{moment of inertia}} = \sum_{i,j} I_{ij} \underbrace{n_i n_j}$$

$\hat{n}$: axis of rotation





$$\boxed{\begin{aligned} I_{com} &= \tfrac{1}{12}\mu l^2 \\ I_{end} &= \tfrac{1}{3}\mu l^2 \end{aligned}}$$

$$\boxed{\underbrace{I_{ij}}_{\text{3x3 real symmetric}} = \sum_a m_a \left(r_a^2 \delta_{ij} - r_{ai} r_{aj} \right) = \int \underbrace{\rho\, dV}_{dm} \left(r^2 \delta_{ij} - r_i r_j \right)}$$

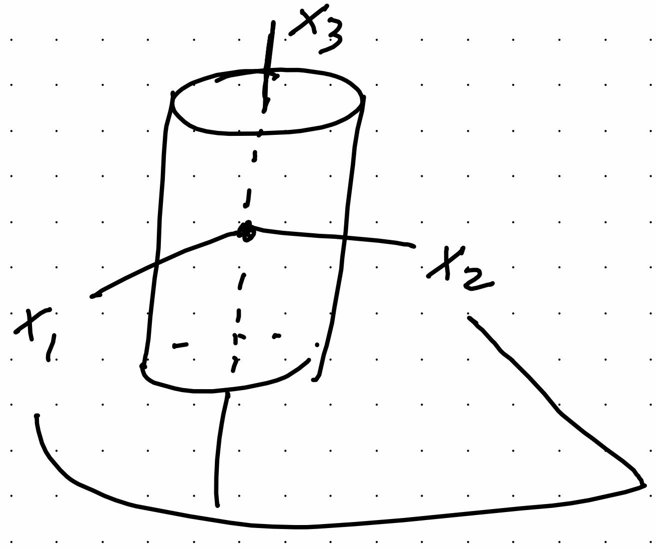
can always be diagonalized

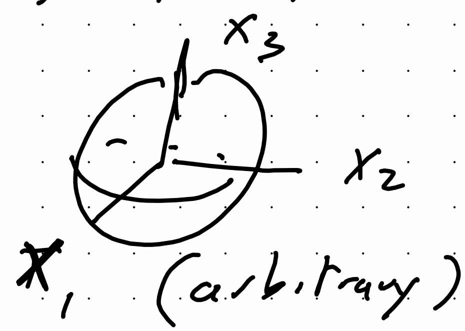
$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$I_{ij} = I_i \delta_{ij}$$

$$T_{rot} = \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j = \frac{1}{2} \left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 \right)$$

<u>principle axes:</u> (x_1, x_2, x_3)

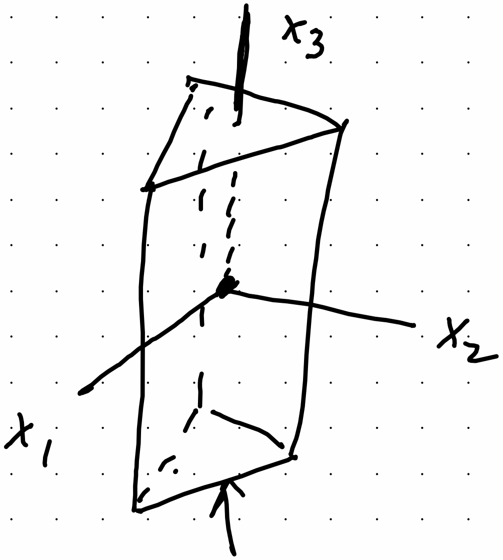




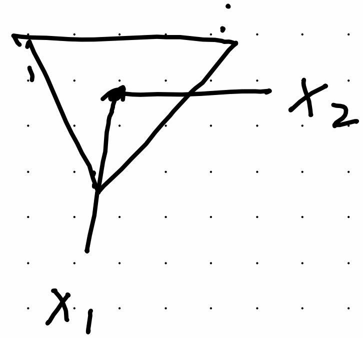
$$M_i = \sum_j I_{ij} \Omega_j$$

$$M_1 = I_1 \Omega_1, \; M_2 = I_2 \Omega_2$$

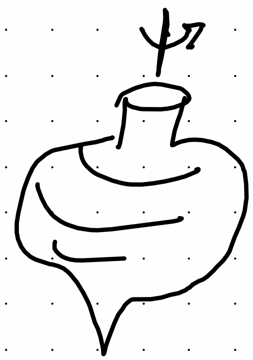
$$M_3 = I_3 \Omega_3$$

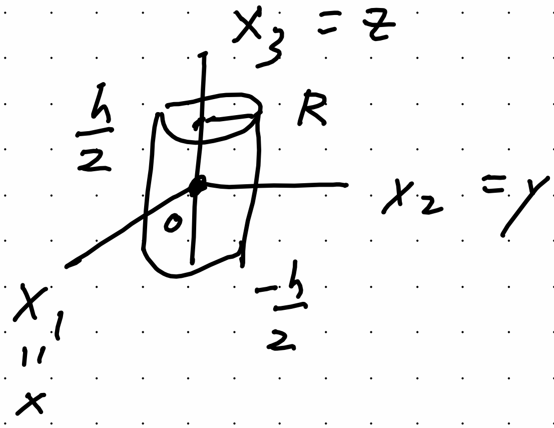


equilateral triang



$$\underbrace{I_1 = I_2 \neq I_3}_{\text{symmetrical top}}$$





total mass M

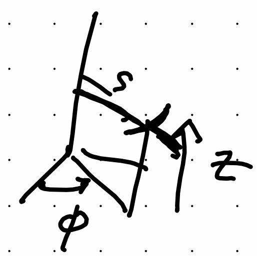
$$\rho = \frac{M}{\text{volume}} = \frac{M}{\pi R^2 h}$$

$$I_3 = I_{33} = \int \rho \, dV \left(r^2 \underbrace{\delta_{33}}_{1} - \underbrace{r_3 r_3}_{z^2} \right)$$

$$= \int \rho \, dV \left(r^2 - z^2 \right)$$

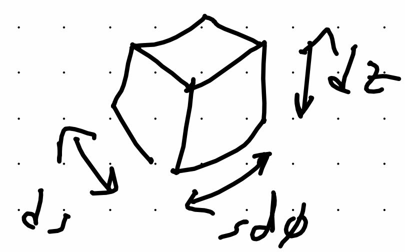
$$= \int \rho \, dV \left(x^2 + y^2 \right) = \int \rho \, dV s^2$$

<u>cylindrical:</u> s, ϕ, z $\quad s^2 = x^2 + y^2$



$$dV = ds \; s\,d\phi \; dz$$

$$= s\,ds\,d\phi\,dz$$



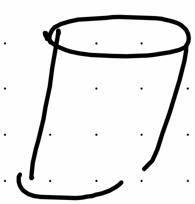
$$I_3 = \int \rho \, dV \, s^2$$

$$= \frac{M}{\pi R^2 h} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-h/2}^{h/2} dz}_{h} \underbrace{\int_0^{R} s^3 \, ds}_{\frac{R^4}{4}}$$

$\cancel{4\pi}$

$$= \frac{M}{\cancel{\pi} \cancel{R^2} \cancel{h}} \cdot 2 \cancel{\pi} \cancel{h} \, \frac{R^{\cancel{4} \to 2}}{4}$$

$$= \boxed{\frac{1}{2} M R^2}$$



$$I_1 = I_2 \equiv I$$

$$I_1 = \int \rho \, dV \, (r^2 - x^2)$$

$$+ \; I_2 = \int \rho \, dV \, (r^2 - y^2)$$

$$2I = \int \rho \, dV \left(2r^2 - x^2 - y^2\right)$$

$$\left(\begin{array}{l} r^2 = s^2 + z^2 \\ x^2 + y^2 = s^2 \end{array}\right)$$

$$2I = \int \rho \, dV \left(s^2 + 2z^2\right)$$

$$I = \frac{1}{2} \underbrace{\int \rho \, dV \, s^2}_{I_3} + \int \rho \, dV \, z^2$$

$$= \frac{1}{2} I_3 + \int \rho \, dV \, z^2$$

easier to evaluate

$$\int \rho \, dV \, z^2 = \frac{M}{\pi R^2 h} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-h/2}^{h/2} dz \cdot z^2}_{\left.\frac{z^3}{3}\right|_{-h/2}^{h/2}} \underbrace{\int_0^R s\,ds}_{\frac{R^2}{2}}$$

$$\left.\frac{z^3}{3}\right|_{-h/2}^{h/2} = \frac{2}{3}\left(\frac{h}{2}\right)^3 = \frac{h^3}{12}$$

$$= \frac{M}{\cancel{\pi} \cancel{R^2} h} \cdot \cancel{2}\cancel{\pi} \cdot \frac{h^3}{12} \cdot \frac{\cancel{R^2}}{\cancel{2}}$$

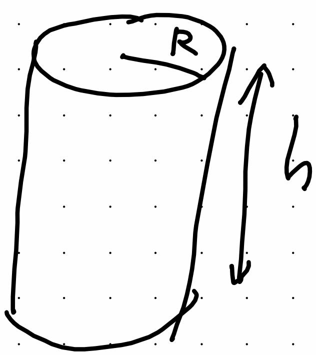
$$= \boxed{\frac{M h^2}{12}}$$

$$I = \frac{1}{2}\left(\frac{1}{2}\mu R^2\right) + \frac{\mu h^2}{12}$$

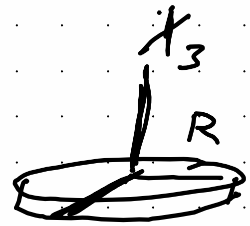
$$= \frac{1}{4}\mu R^2 + \frac{1}{12}\mu h^2$$

$$= \frac{1}{4}\mu\left(R^2 + \frac{1}{3}h^2\right) = I_1, I_2$$

$$I_3 = \frac{1}{2}\mu R^2$$



<u>Limiting Cases</u>:

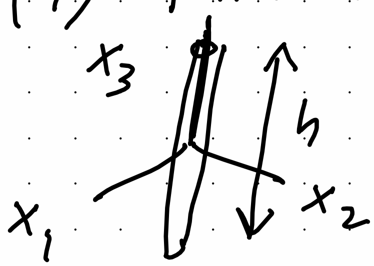


(i) Disk ($h \to 0$)

$I_3 = \frac{1}{2}\mu R^2$

$I_1, I_2 = \frac{1}{4}\mu R^2$

(i) thin rod ($R \to 0$)



$I_3 = 0$

$I_1 = I_2 = \frac{1}{12}\mu h^2$

$$L = T - U$$

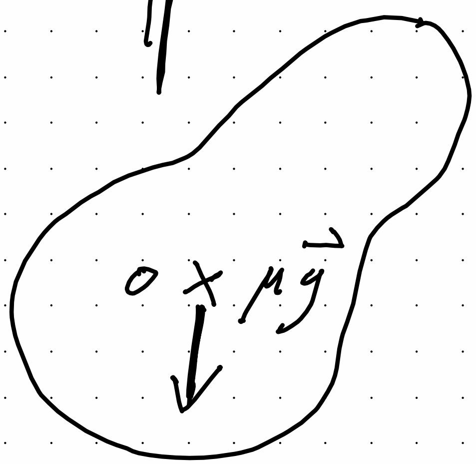
$$= \frac{1}{2}\mu V^2 + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\Omega_j - U$$

$$\vec{V} = \dot{\vec{R}} \qquad \Bigg| \qquad L(\vec{R}, \vec{\phi}, \dot{\vec{R}}, \dot{\vec{\phi}})$$

$$\vec{\Omega} = \dot{\vec{\phi}}$$

$\vec{g}$ $\downarrow$

uniform Field

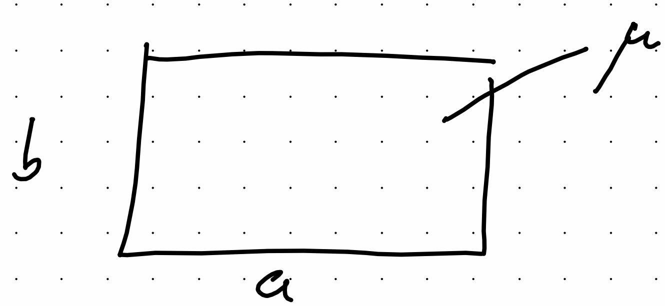


Lecture #24 : Thursday 11/12

- Quiz #5 (today)
- Midterm #2 (next Thursday) (scattering, small oscillations, RB motion)
- Today's topics:
  (1) RB EOMs
  (2) Euler's equations
  (3) Euler angles

Q5: Calculate the principal moments of inertia for a 2-d rectangle with sidelengths a, b.

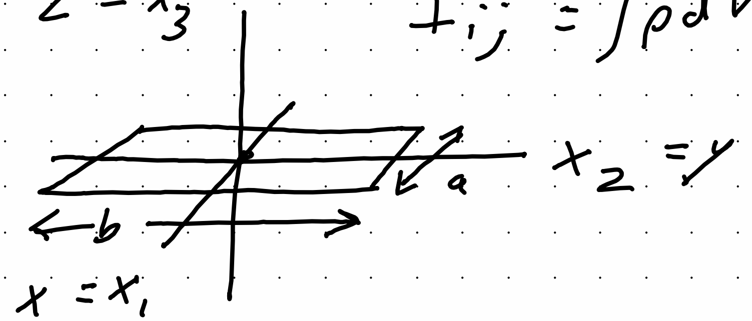
uniform

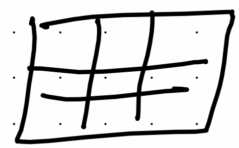


name_q5.pdf

$z = x_3$ $\qquad I_{ij} = \int \rho\, dV\, (r^2 \delta_{ij} - r_i r_j)$

$r^2 = x^2 + y^2 + z^2$





$dm = \rho\, dV$

$= \sigma\, dx\, dy$

$= \frac{M}{ab}\, dx\, dy$

$I_1 = \int \rho\, dV\, (r^2 - x^2)$

$\parallel$

I_{11}

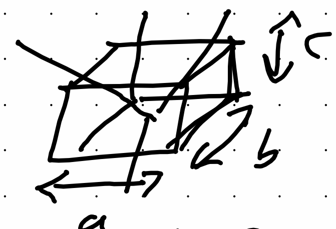
$= \int \rho\, dV\, y^2$

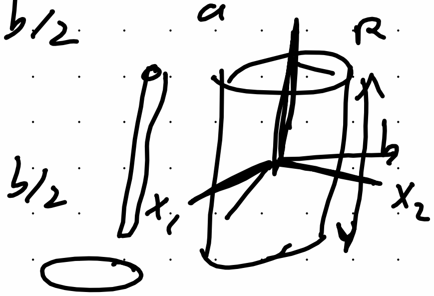
$= \frac{M}{ab} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\, y^2$

$= \frac{M}{\cancel{a}b}\, \cancel{a}\, \frac{y^3}{3}\Big|_{-b/2}^{b/2}$

$= \frac{M}{b}\, \frac{2}{3} \left(\frac{b}{2}\right)^3$

$= \frac{M}{b}\, \frac{2}{3}\, \frac{b^3}{8} = \boxed{\frac{1}{12} M b^2}$





$\boxed{I_2 = \frac{1}{12} M a^2}$

$I_3 = \int \sigma\, dx\, dy\, (x^2 + y^2)$

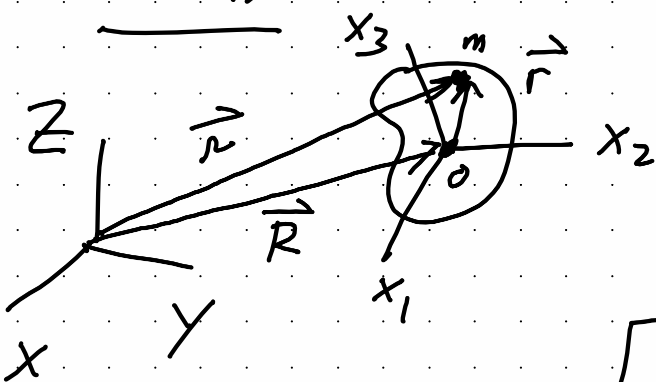
$= I_1 + I_2$

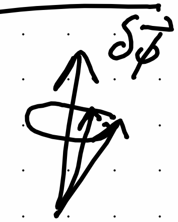
$\boxed{I_3 = \frac{1}{12} M (a^2 + b^2)}$

EOMs: Final result:

$$\frac{d\vec{P}}{dt} = \vec{F} = \sum \vec{f}$$

$$\frac{d\vec{M}}{dt} = \vec{K} = \sum \vec{r}\times\vec{f}$$





$$\vec{\mathfrak{r}} = \vec{R} + \vec{r}\,, \quad \boxed{\delta\vec{\mathfrak{r}} = \delta\vec{R} + \delta\vec{\phi}\times\vec{r}}$$

$\delta\Omega_k$

$$L = T - U$$

$$= \frac{1}{2}\mu|\vec{V}|^2 + \frac{1}{2}\sum_{i,j} I_{ij}\Omega_i\Omega_j - U(\vec{\mathfrak{r}})$$

$$\delta L = \mu\vec{V}\cdot\delta\vec{V} + \sum_{i,j} I_{ij}\Omega_j\delta\Omega_i \boxed{- \sum\frac{\partial U}{\partial\vec{\mathfrak{r}}}\cdot\delta\vec{\mathfrak{r}}}$$

$$- \sum\frac{\partial U}{\partial\vec{\mathfrak{r}}}\cdot\left(\delta\vec{R} + \delta\vec{\phi}\times\vec{r}\right)$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial\vec{V}}\right) = \frac{\partial L}{\partial\vec{R}}}\,, \quad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial\vec{\Omega}}\right) = \frac{\partial L}{\partial\vec{\phi}}}$$

$$\frac{1}{2}\sum_{i,j} I_{ij}\, \underline{\delta\Omega_i} \cdot \Omega_j + \frac{1}{2}\sum_{i,j} I_{ij}\, \Omega_i \cdot \delta\Omega_j$$

$$= \quad \text{''} \quad + \frac{1}{2}\sum_{j,i} I_{ji}\, \Omega_j \cdot \underline{\delta\Omega_i}$$

$$= \quad \text{''} \quad + \frac{1}{2}\sum_{\underbrace{j,i}_{\text{swap}}} I_{ij}\, \Omega_j \cdot \delta\Omega_i$$

$$= \frac{1}{2}\sum_{i,j} I_{ij}\, \Omega_j \cdot \delta\Omega_i + \frac{1}{2}\sum_{i,j} I_{ij}\, \Omega_j \cdot \delta\Omega_i$$

$$= \boxed{\sum_{i,j} I_{ij}\, \Omega_j \cdot \delta\Omega_i}$$

$$\delta L = m\vec{V}\cdot \underline{\delta \vec{V}} + \sum_{ij} I_{ij}\Omega_j \delta\Omega_i \qquad \vec{A}\cdot(\vec{B}\times\vec{C}) \,/\, \vec{B}\cdot(\vec{C}\times\vec{A})$$

$$- \left(\sum \frac{\partial U}{\partial \vec{r}}\right)\cdot \delta\vec{R} \;-\; \underbrace{\sum \frac{\partial U}{\partial \vec{r}}\cdot(\delta\vec{\phi}\times\vec{r})}$$

$$= -\sum \delta\vec{\phi}\cdot\left(\vec{r}\times\frac{\partial U}{\partial \vec{r}}\right)$$

$$= -\,\delta\vec{\phi}\cdot\left(\sum \vec{r}\times\frac{\partial U}{\partial \vec{r}}\right)$$

$$\sum_i M_i\,\delta\Omega_i = \vec{M}\cdot\delta\vec{\Omega}$$

$$dF = \_\,dx + \_\,dy$$

$$\boxed{\delta L = \vec{P}\cdot\delta\vec{V} + \vec{M}\cdot\delta\vec{\Omega} + \left(\sum\vec{f}\right)\cdot\delta\vec{R} + \delta\vec{\phi}\cdot\left(\sum\vec{r}\times\vec{f}\right)}$$

$$\vec{P} = \frac{\partial L}{\partial \vec{V}},\quad \vec{M} = \frac{\partial L}{\partial \vec{\Omega}},\quad \sum\vec{f} = \frac{\partial L}{\partial \vec{R}},\quad \sum\vec{r}\times\vec{f} = \frac{\partial L}{\partial\vec{\phi}}$$

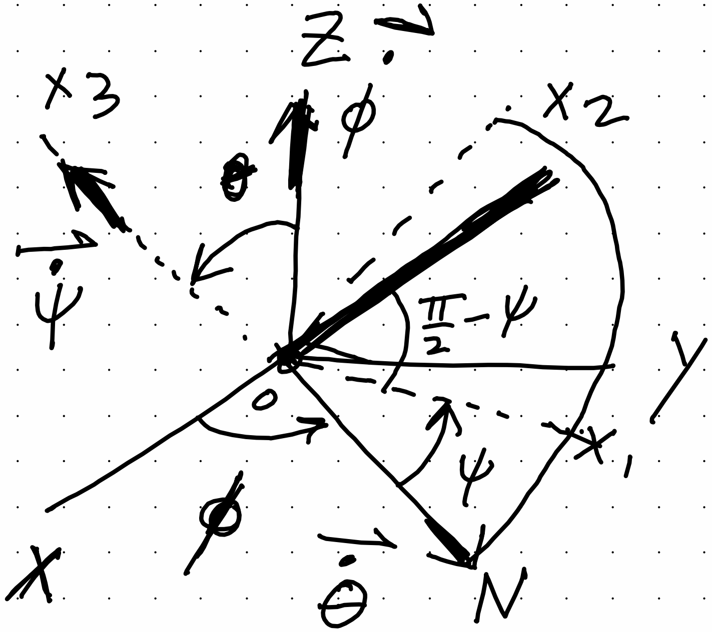
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{V}}\right) = \frac{\partial L}{\partial \vec{R}} \rightarrow \boxed{\frac{d\vec{P}}{dt} = \sum\vec{f} \equiv \vec{F}} \quad\Big|\quad \frac{d}{dt}\left(\frac{\partial L}{\partial\vec{\Omega}}\right) = \frac{\partial L}{\partial\vec{\phi}} \rightarrow \boxed{\frac{d\vec{M}}{dt} = \sum\vec{r}\times\vec{f} = \vec{K}}$$

Euler's equations / Euler angles ☆

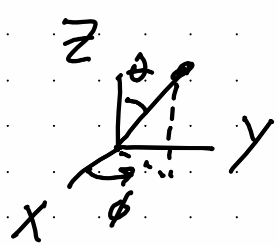
$\vec{R}, \vec{\phi}$

Ω_i : ith component of $\vec{\Omega}$

L wrt $\hat{x}_i$



(line of node)



$$\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta} + \vec{\dot\psi}$$

$$\vec{\dot\psi} = \dot\psi\, \hat{x}_3$$

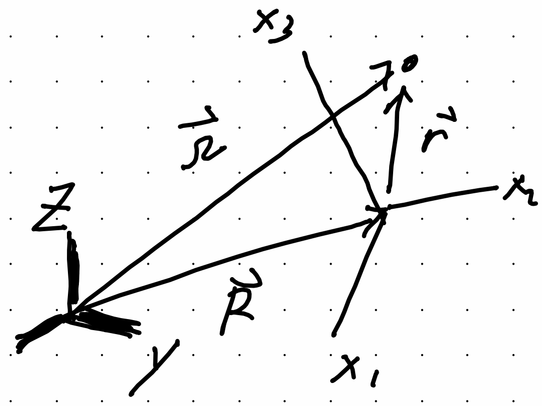
$$\vec{\dot\theta} = \dot\theta \cos\psi\, \hat{x}_1 - \dot\theta \sin\psi\, \hat{x}_2$$

$$\vec{\dot\phi} = \dot\phi \cos\theta\, \hat{x}_3 + \dot\phi \sin\theta \left(\underbrace{\sin\psi}_{\cos(\frac{\pi}{2}-\psi)} \hat{x}_1 + \underbrace{\cos\psi}_{\sin(\frac{\pi}{2}-\psi)} \hat{x}_2 \right)$$

Announcements

- Midterm II is this thursday
- Today:
  - i) Euler angles
  - ii) Euler's equation for RB motion
  - iii) Free rotation with $\vec{\Omega} = \text{const}$
  - iv) " of a symmetric top $(I_1 = I_2)$
  - v) Heavy symmetrical top with one point fixed [prob 35.1]

$$\vec{\Omega} = (\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)\,\hat{x}_1 + (\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)\,\hat{x}_2 + (\dot\psi + \dot\phi \cos\theta)\,\hat{x}_3$$



$$\boxed{\begin{aligned} \Omega_1 &= \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi \\ \Omega_2 &= \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi \\ \Omega_3 &= \dot\psi + \dot\phi \cos\theta \end{aligned}}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}$$

$\frac{d'\vec{r}}{dt}$ wrt RB

Euler's equations:

$$\frac{d\vec{P}}{dt} = \sum \vec{f} = \vec{F}$$

$$\frac{d\vec{M}}{dt} = \sum \vec{r} \times \vec{f} = \vec{K}$$

(wrt RB axes) $\vec{A}$: any vector

$$\boxed{\underbrace{\frac{d\vec{A}}{dt}}_{\text{wrt inertial frame}} = \underbrace{\frac{d'\vec{A}}{dt}}_{\text{wrt rotating frame}} + \underbrace{\vec{\Omega} \times \vec{A}}_{\text{angular velocity of the rotating frame}}}$$

$$\left(\frac{d'\vec{A}}{dt}\right)_i = \frac{dA_i}{dt} \equiv \dot{A}_i$$

cartesian components wrt rotating frame

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}$$

$$= \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$$

$$\cancel{= \ddot{r}\hat{r}}$$

$$\vec{A} = \sum_i \underline{\underline{A_i}}\ \underline{\underline{\hat{x}_i}}$$

$$\frac{d\vec{A}}{dt} = \underbrace{\sum_i \left(\frac{dA_i}{dt}\right)\hat{x}_i}_{\frac{d'\vec{A}}{dt}} + \underbrace{\sum_i A_i \frac{d\hat{x}_i}{dt}}_{\vec{\Omega}\times\vec{A}}$$

$$\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\Omega}\times\vec{A}, \qquad \left(\frac{d'\vec{A}}{dt}\right)_i = \dot{A}_i$$

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d'\vec{P}}{dt} + \vec{\Omega}\times\vec{P}$$

$$\vec{P} = \mu\vec{V}$$

$$\rightarrow \boxed{F_1} = \dot{P}_1 + (\vec{\Omega}\times\vec{P})_1$$

$$= \dot{P}_1 + \Omega_2 P_3 - \Omega_3 P_2$$

$$\boxed{= \mu(\dot{V}_1 + \Omega_2 V_3 - \Omega_3 V_2)}$$

similar equations for F_2, F_3

$$\boxed{\vec{K} = \frac{d\vec{M}}{dt}} = \frac{d'\vec{M}}{dt} + \vec{\Omega}\times\vec{M}$$

$$M_i = I_i \Omega_i$$

$$K_x = \dot{M}_x$$

$$\boxed{K_1} = \dot{M}_1 + \Omega_2 M_3 - \Omega_3 M_2$$

$$= I_1\dot{\Omega}_1 + \Omega_2 I_3 \Omega_3 - \Omega_3 I_2 \Omega_2$$

$$\boxed{= I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_2)}$$

*(similar equations for K_2, K_3)

Free rotation: $K_i = 0$, $F_i = 0$

$$0 = I_1\dot\Omega_1 + \Omega_2\Omega_3(I_3 - I_2),\ \cancel{\text{etc.}}$$
$$0 = I_2\dot\Omega_2 + \Omega_3\Omega_1(I_1 - I_3)$$
$$0 = I_3\dot\Omega_3 + \Omega_1\Omega_2(I_2 - I_1)$$

Free rotation with $\vec\Omega = \text{const}$:

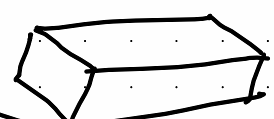
$$\frac{d\vec\Omega}{dt} = 0$$
$$\frac{d'\vec\Omega}{dt} = 0$$
$$\boxed{\frac{d\vec\Omega}{dt} = \frac{d'\vec\Omega}{dt}} + \cancel{\vec\Omega\times\vec\Omega}$$
$$\frac{d\vec M}{dt} \neq \frac{d'\vec M}{dt}$$

$$0 = \Omega_2\Omega_3(I_3 - I_2)$$
$$0 = \Omega_3\Omega_1(I_1 - I_3)$$
$$0 = \Omega_1\Omega_2(I_2 - I_1)$$

$$\boxed{\Omega_1 = \text{const},\ \Omega_2 = 0,\ \Omega_3 = 0}, \qquad I_1 < I_2 < I_3$$

← stable

unstable



$$\boxed{\Omega_2 = \text{const},\ \Omega_1 = 0,\ \Omega_3 = 0}$$

$$\boxed{\Omega_3 = \text{const},\ \Omega_1 = 0,\ \Omega_2 = 0}$$

$\Omega_1 = \text{const}, \quad \Omega_2 = 0, \quad \Omega_3 = 0$: exact

$$\boxed{\begin{aligned} \Omega_1 &= \text{const} + \epsilon_1 \\ \Omega_2 &= \epsilon_2 \\ \Omega_3 &= \epsilon_3 \end{aligned}}$$

$\epsilon_{1,2,3}$: small time dependent perturbations

Keep 0^{th} and 1^{st} order terms.
Ignore 2^{nd} order, e.g. $\epsilon_2 \epsilon_3$

$$0 = I_1 \frac{d}{dt}(\underset{\to 0}{\text{const}} + \epsilon_1) + \underbrace{\epsilon_2 \epsilon_3 (I_3 - I_2)}_{2^{nd} \text{ order} \to \text{ignore}}$$

$$\approx I \dot{\epsilon}_1$$

$$\to \epsilon_1 = \text{const} \to \Omega_1 = \text{const}$$

$$0 = I_2 \dot{\epsilon}_2 + \epsilon_3 \Omega_1 (I_1 - I_3)$$

$$0 = I_3 \dot{\epsilon}_3 + \Omega_1 \epsilon_2 (I_2 - I_1)$$

coupled 1^{st} order diff. equations

Differentiate

$$0 = I_2 \ddot{\epsilon}_2 + \dot{\epsilon}_3 \Omega_1 (I_1 - I_3)$$

$$= I_2 \ddot{\epsilon}_2 - \frac{\Omega_1 \epsilon_2 (I_2 - I_1)}{I_3} \Omega_1 (I_1 - I_3)$$

$I_1 < I_2 < I_3$

$$0 = I_2 \ddot{\epsilon}_2 + \dot{\epsilon}_3 \Omega_1 (I_1 - I_3)$$

$$= I_2 \ddot{\epsilon}_2 - \frac{\Omega_1 \epsilon_2 (I_2 - I_1)}{I_3} \Omega_1 (I_1 - I_3)$$

$$= \ddot{\epsilon}_2 + \underbrace{\frac{\Omega_1^2 (I_2 - I_1)(I_3 - I_1)}{I_2 I_3}}_{\omega^2} \epsilon_2$$

$$\boxed{\begin{matrix}(I_3 - I_2)\\(I_1 - I_2)\end{matrix}}$$

$$= \ddot{\epsilon}_2 + \omega^2 \epsilon_2$$

$$\ddot{\epsilon}_2 = -\omega^2 \epsilon_2 \longrightarrow \text{SHM}$$

$$\epsilon_2 = A \cos \omega t + B \sin \omega t$$

$$= C \cos(\omega t + \alpha)$$

Similarly,

$$\ddot{\epsilon}_3 = -\omega^2 \epsilon_3 \longrightarrow \epsilon_3 = D \cos(\omega t + \beta)$$

(ϵ_2, ϵ_3 are bound by their initial deviations away from 0)

For $\Omega_2 = \text{const}$ solution

perturbations

$$\ddot{\epsilon}_1 = + \omega^2 \epsilon_1$$

$$\ddot{\epsilon}_3 = + \omega^2 \epsilon_3$$

$$\rightarrow \quad \epsilon_1(t) = A \underbrace{e^{\omega t}}_{\text{grows exponentially}} + B \underbrace{e^{-i\omega t}}_{\text{damped exponentially}}$$

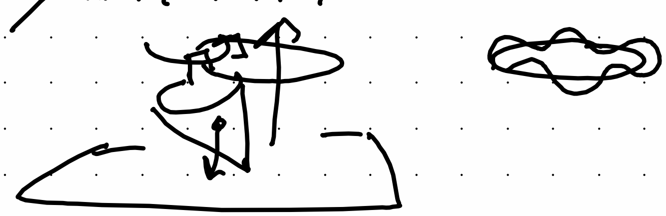
$\rightarrow$ instability

Lecture #27   Tues 11/24

- Midterm II — Avg $\approx$ 12/20 (solns posted)
- Quiz #6 — Next Tuesday (last day)
- Oral Final — google doc sign up sheet ??
  Sat 12/5 1:30pm – 4:00pm

- Today, next time:
  i) Free rotation of a symmetric top
  ii) motion in a non-inertial ref. frame

- on your own:
  (i) Sec 35, Prob 1: motion of a heavy symmetrical top
  (ii) static equil: (Sec 38)



Free rotation of a symmetric top:

no force, torque $\rightarrow$ $\vec{P} = \text{const}$, $\vec{M} = \text{const}$

Euler's equations:

$I_1 = I_2$

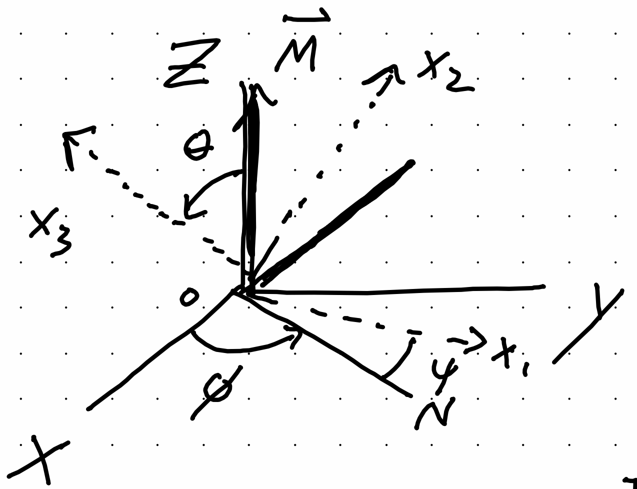
const

$$0 = I_1\dot{\Omega}_1 + \Omega_2\Omega_3\left(I_3 - \overset{I_1}{\overset{\shortparallel}{I_2}}\right) \qquad (1)$$

$$0 = I_2\dot{\Omega}_2 + \Omega_3\Omega_1(I_1 - I_3) \qquad (2)$$

$$0 = I_3\dot{\Omega}_3 + \Omega_1\Omega_2\,\cancel{(I_2 - I_1)} \rightarrow 0 \qquad (3)$$

$$\dot{\Omega}_3 = 0 \rightarrow \boxed{\Omega_3 = \text{const}}$$



$$\Omega_1 = \dot{\phi}\sin\theta\sin\psi\ \cancel{+\dot{\theta}\cos\psi} \rightarrow 0$$

$$\Omega_2 = \dot{\phi}\sin\theta\cos\psi\ \cancel{-\dot{\theta}\sin\psi} \rightarrow 0$$

$$\Omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$$

$$I_1\Omega_1 = M_1 = M\sin\theta\sin\psi \qquad (4)$$

$$I_1\Omega_2 = M_2 = M\sin\theta\cos\psi \qquad (5)$$

$$I_3\Omega_3 = M_3 = M\cos\theta \qquad (6)$$

$\Omega_3 = \text{const} \rightarrow \boxed{\theta = \text{const}}$, $\cos\theta = \dfrac{I_3 \Omega_3}{M}$

$$0 = I_1 \dot{\Omega}_1 + \Omega_2 \Omega_3 (I_3 - I_1)$$

$$0 = \dot{\Omega}_1 + \Omega_2 \left(\frac{\Omega_3 (I_3 - I_1)}{I_1} \right) \leftarrow \boxed{\omega \equiv \frac{\Omega_3 (I_3 - I_1)}{I_1}}$$

$$0 = I_1 \dot{\Omega}_2 + \Omega_1 \Omega_3 (I_1 - I_3)$$

$M_1 = I_1 \Omega_1$, $M_2 = I_1 \Omega_2$

$$0 = \dot{\Omega}_2 - \Omega_1 \left(\frac{\Omega_3 (I_3 - I_1)}{I_1} \right)$$

$\sqrt{\Omega_1^2 + \Omega_2^2} = a$

$$0 = \dot{\Omega}_1 + \Omega_2 \omega$$

$$0 = \dot{\Omega}_2 - \Omega_1 \omega$$

$$0 = \ddot{\Omega}_1 + \dot{\Omega}_2 \omega$$

$$= \ddot{\Omega}_1 + \Omega_1 \omega^2$$

$\boxed{\Omega_2 = a \sin(\omega t + \alpha)}$

$\uparrow$

$\ddot{\Omega}_1 = -\omega^2 \Omega_1 \rightarrow \boxed{\Omega_1 = a \cos(\omega t + \alpha)}$

$$\xi = \Omega_1 + i \Omega_2$$

$$\dot{\xi} = \dot{\Omega}_1 + i \dot{\Omega}_2$$

$\rightarrow \dot{\xi} = i \omega \xi \rightarrow \xi(t) = A e^{i \omega t}$

A: complex

$$I_1 \dot\phi \cancel{\sin\theta \sin\psi} = M \cancel{\sin\theta \sin\psi} \qquad \boxed{\theta = \text{const}}$$

$$\boxed{\dot\phi = \frac{M}{I_1} = \text{const}} \rightarrow \boxed{\phi(t) = \phi_0 + \frac{M}{I_1} t}$$

$$I_3 \left(\dot\psi + \cos\theta \dot\phi \right) = M \cos\theta$$

$$\boxed{-\omega =} \dot\psi = \frac{M \cos\theta}{I_3} - \cos\theta \dot\phi$$

$$= \frac{M\cos\theta}{I_3} - \cos\theta \frac{M}{I_1}$$

$$= M \cos\theta \left(\frac{1}{I_3} - \frac{1}{I_1} \right) = \text{const}$$

$$\rightarrow \boxed{\psi(t) = \psi_0 + M\cos\theta \left(\frac{1}{I_3} - \frac{1}{I_1} \right)}$$

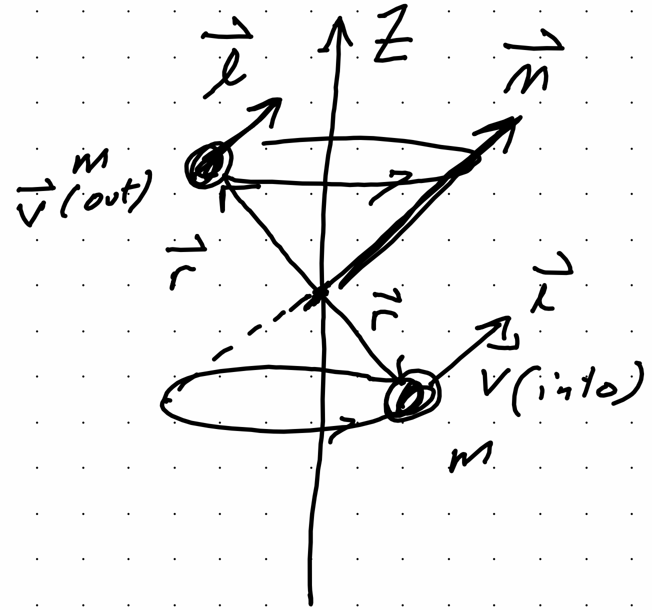
$$\omega = \frac{\Omega_3 (I_3 - I_1)}{I_1}$$

$$\left. \begin{aligned} M_3 &= I_3 \Omega_3 \\ M \cos\theta &= M_3 \end{aligned} \right\}$$

$$\Omega_3 = \frac{M\cos\theta}{I_3}$$

$$\omega = \frac{M\cos\theta (I_3 - I_1)}{I_1 I_3} \qquad \boxed{-\dot\psi} \; //$$

$$= M\cos\theta \left(\frac{1}{I_1} - \frac{1}{I_3} \right)$$



$\vec{\Omega} = \text{const}$? Yes $\vec{\Omega} = \Omega\hat{Z}$

$\vec{M} = \text{const}$? No $\vec{M}$ precesses around $\hat{Z}$ axis

$\vec{l} = \vec{r} \times \vec{p}$

$= m\vec{r} \times \vec{v}$

---

wobble vs spin freq

$\text{wobble} = \dot{\phi}$ — precession freq of 3-axis around $\vec{M}$ (or $\hat{Z}$)

$\text{spin freq} = \Omega_3$ — component of angular along 3-axis

$$\dot{\phi} = \frac{M}{I_1}$$

$$\Omega_3 = \frac{M\cos\theta}{I_3}$$

$$\boxed{\frac{\Omega_3}{\dot{\phi}} = \frac{\cancel{M}\cos\theta}{I_3}\,\frac{I_1}{\cancel{M}} = \frac{I_1\cos\theta}{I_3}}$$

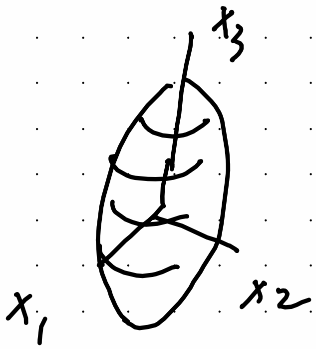
$$\frac{\Omega_3}{\dot{\phi}} = \frac{I_1}{I_3}\cos\theta \longrightarrow \frac{I_1}{I_3} \qquad (\theta \to 0)$$

$\sim$

$\frac{\text{spin}}{\text{wobble}}$

$I_1 = I_3$ sphere

$$\boxed{\frac{\Omega_3}{\dot{\phi}} = 1}$$



football

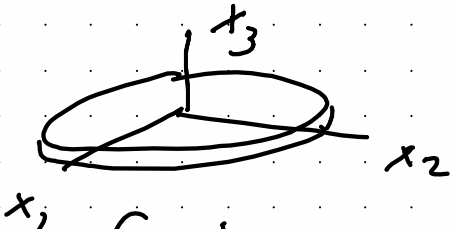
$I_1 > I_3$

football

$$\boxed{\frac{\Omega_3}{\dot{\phi}} > 1}$$

frisbee

$$\boxed{\frac{\Omega_3}{\dot{\phi}} = \frac{1}{2} < 1}$$



frisbee

$$\underset{\frac{1}{4}MR^2}{\underset{\uparrow}{I_1}} < \underset{\frac{1}{2}MR^2}{\underset{\uparrow}{I_3}}$$

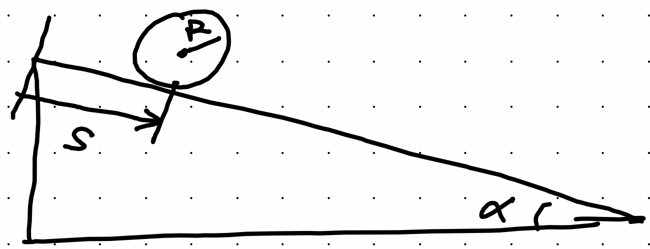
$I_1 = \frac{1}{2} I_3$

(uniform disk)

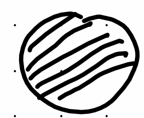
Lecture #28:

name = q6.pdf

— course evaluations (please submit by EOB today)
— sign-up for oral final exam date/time
— Quiz #6



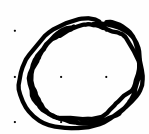
uniform disk $I_3 = \frac{1}{2} M R^2$



a) calculate KE of the uniform disk as it rolls without slipping down the incline

$$T = \text{something}(M, \dot{s}, R)$$

b) repeat for a hoop



$$I_3 = M R^2$$

$$T = \text{something}$$

which object will reach the bottom first? (or do both reach the bottom at the same time)

$$T = \frac{1}{2}\mu V^2 + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\Omega_j$$

$$= \frac{1}{2}\mu V^2 + \frac{1}{2}\sum_{i} I_i\,\Omega_i^2$$

$$= \frac{1}{2}\mu V^2 + \frac{1}{2} I_3\,\Omega_3^2$$

$$V = \dot{s}\,, \quad \Omega_3 \overset{?}{=} \dot{s} \longrightarrow \Omega_3 = \frac{\dot{s}}{R}$$

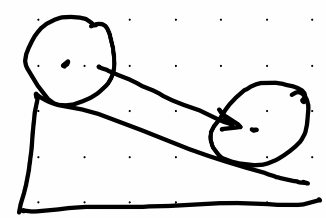
$$T = \frac{1}{2}\mu \dot{s}^2 + \frac{1}{2} I_3 \frac{\dot{s}^2}{R^2}$$

$$= \frac{1}{2}\mu \dot{s}^2 + \frac{1}{2}\left(\frac{1}{2}\mu \cancel{R^2}\right)\frac{\dot{s}^2}{\cancel{R^2}}$$

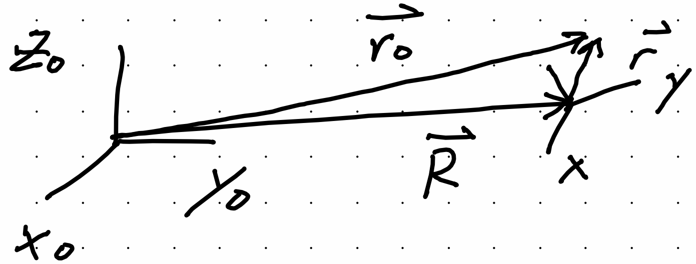
$$= \boxed{\frac{3}{4}\mu \dot{s}^2} \quad \text{(unif disk)} \longrightarrow \dot{s} = \sqrt{\frac{4}{3}\frac{T}{\mu}}$$

$$\boxed{T = \mu \dot{s}^2} \quad \text{(hoop)}$$

$$\longrightarrow \dot{s} = \sqrt{\frac{T}{\mu}}$$



Motion in a non-inertial rot frame (Sec 39)



(non-inertial)

$$\vec{r}_0 = \vec{R} + \vec{r}$$

(inertial)

$$m\vec{a}_0 = \vec{F} \quad \text{(2nd law in inertial frame)}$$

$$m\vec{a} = \vec{F} \; ??$$

$$\boxed{\left.\frac{d\vec{A}}{dt}\right|_0 = \underbrace{\frac{d\vec{A}}{dt}}_{\text{wrt rotating frame}} + \vec{\Omega}\times\vec{A}} \longleftarrow \text{use this}$$

$$\vec{a}_0 = \left.\frac{d\vec{v}_0}{dt}\right|_0, \quad \vec{v}_0 = \left.\frac{d\vec{r}_0}{dt}\right|_0$$

$$\vec{v}_0 = \left.\frac{d}{dt}(\vec{R}+\vec{r})\right|_0 = \left.\frac{d\vec{R}}{dt}\right|_0 + \left.\frac{d\vec{r}}{dt}\right|_0 = \boxed{\vec{V} + \overbrace{\vec{v}}^{\frac{d\vec{r}}{dt}} + \vec{\Omega}\times\vec{r}}$$

$$\vec{v_0} = \vec{V} + \vec{v} + \vec{\Omega}\times\vec{r}$$

$$\vec{a_0} = \frac{d\vec{v_0}}{dt}\Big|_0$$

$$= \frac{d\vec{V}}{dt}\Big|_0 + \frac{d\vec{v}}{dt}\Big|_0 + \frac{d\vec{\Omega}}{dt}\Big|_0\times\vec{r} + \vec{\Omega}\times\frac{d\vec{r}}{dt}\Big|_0$$

$$= \underbrace{\vec{W}}_{\text{accel of origin of non-inertial frame}} + \Big(\underbrace{\frac{d\vec{v}}{dt}}_{=\,\vec{a}} + \vec{\Omega}\times\vec{v}\Big) + \Big(\underbrace{\frac{d\vec{\Omega}}{dt}}_{=\,\dot{\vec{\Omega}}} + \cancelto{0}{\vec{\Omega}\times\vec{\Omega}}\Big)\times\vec{r}$$

$$+\ \vec{\Omega}\times\Big(\underbrace{\frac{d\vec{r}}{dt}}_{=\,\vec{v}} + \vec{\Omega}\times\vec{r}\Big)$$

$$\boxed{\vec{a_0} = \vec{a} + \underbrace{\vec{W}}_{\text{trans accel.}} + \underbrace{\dot{\vec{\Omega}}\times\vec{r}}_{\text{rot. accel.}} + \underbrace{2\vec{\Omega}\times\vec{v}}_{\text{coriolis}} + \underbrace{\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}_{\text{centrifugal}}}$$

$$m\vec{a_0} = \vec{F}$$

$$m(\vec{a} + \dots) = \vec{F}$$

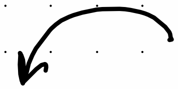
$$\rightarrow \boxed{m\vec{a} = \vec{F} - m\vec{W} - m\dot{\vec{\Omega}}\times\vec{r} - 2m\vec{\Omega}\times\vec{v} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$$

$$\frac{d\vec{p}}{dt} = \sum f = \vec{F}$$

$$\boxed{\frac{d'\vec{p}}{dt} + \vec{\Omega}\times\vec{p} = \vec{F}} \longrightarrow \boxed{\frac{d'\vec{M}}{dt} + \vec{\Omega}\times\vec{M} = \vec{K}}$$

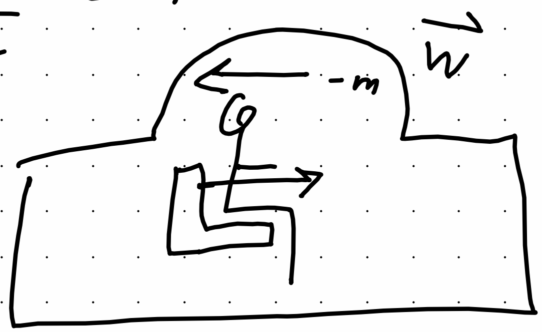
~~w~~
wrt
rigid body frame

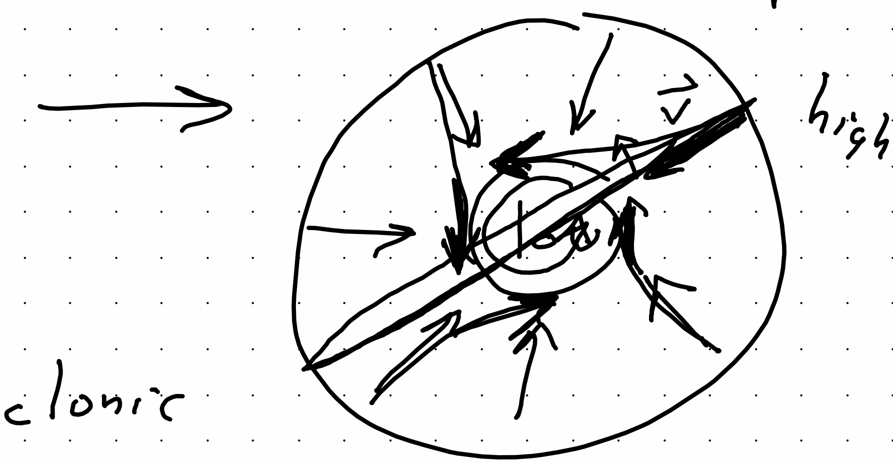
ccw

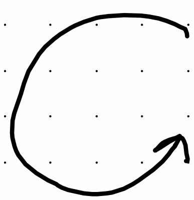


$\odot\,\vec{\Omega}$ out of page

$$\frac{d'\vec{p}}{dt} = \vec{F} - \vec{\Omega}\times\vec{p}$$

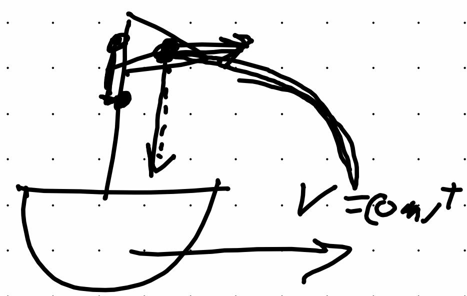


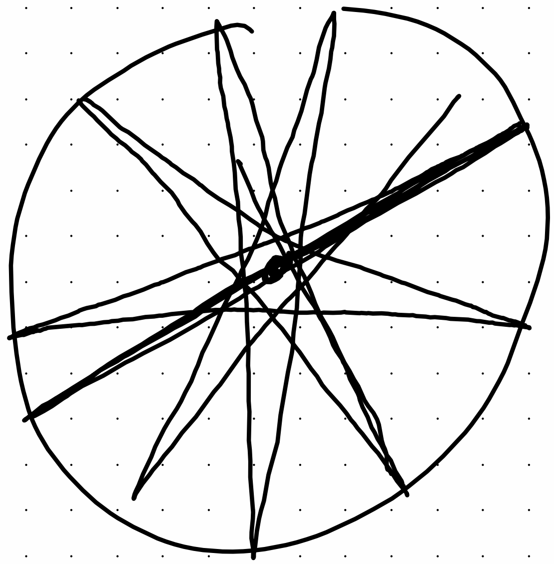


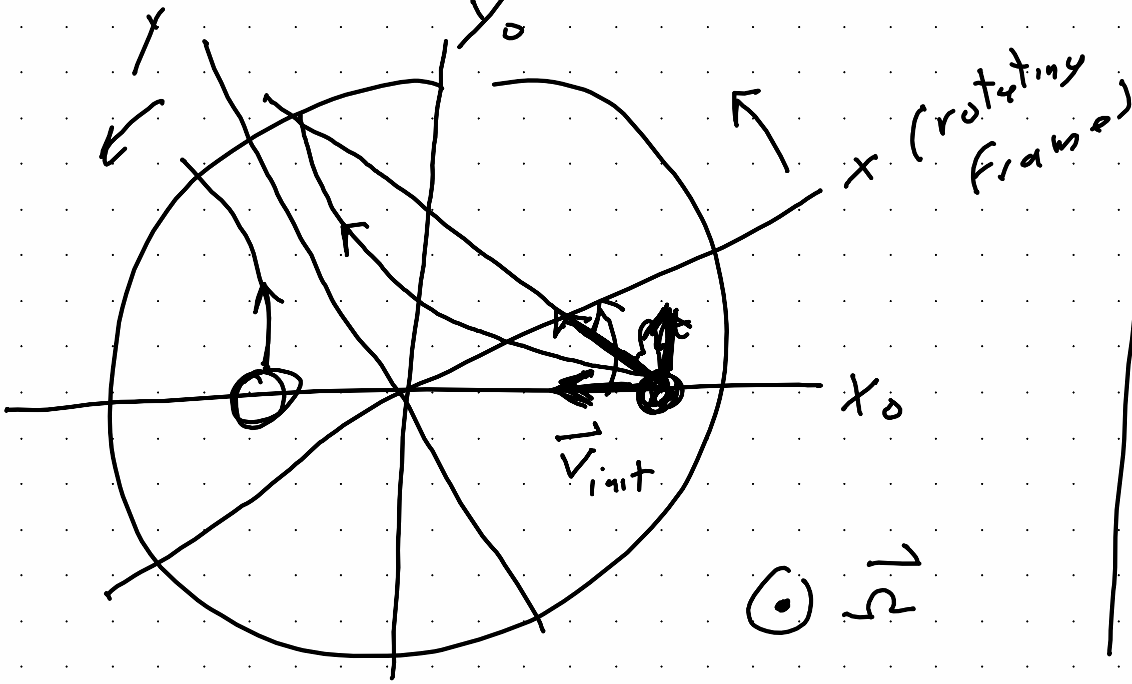


cyclonic

Example: merry-go-round







$m\vec{a}_0 = \vec{0}$

$\vec{V}_0 = \text{const}$

solve 2nd law in merry-go-round frame:

$$m\frac{d\vec{V}}{dt} = -2m\vec{\Omega}\times\vec{V} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\vec{r} = x\hat{x} + y\hat{y}$$

$$\vec{V} = \dot{x}\hat{x} + \dot{y}\hat{y}$$

$$\frac{d\vec{V}}{dt} = \ddot{x}\hat{x} + \ddot{y}\hat{y}$$

$$\vec{\Omega} = \Omega\hat{z}$$

$$\boxed{\begin{aligned} \ddot{x} &= 2\Omega\dot{y} + \Omega^2 x \\ \ddot{y} &= -2\Omega\dot{x} + \Omega^2 y \end{aligned}}$$

$$\left.\begin{aligned} \xi &= x + iy \\ \dot{\xi} &= \dot{x} + i\dot{y} \\ \ddot{\xi} &= \ddot{x} + i\ddot{y} \end{aligned}\right\} \quad \ddot{\xi} + i2\Omega\dot{\xi} - \Omega^2\xi = 0$$

<u>Guess:</u> $\xi = e^{i\lambda t}$

$\lambda = -\Omega$ (double root)

$$\boxed{\xi = (A + Bt)\, e^{-i\Omega t}}$$

complex const determined by initial conditions.

$\xi(t) = x(t) + iy(t)$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$