sec 16, prob 2

From (16.7) we have

$$p(\theta_0)d\theta_0 = \frac{1}{2}\sin\theta_0\, d\theta_0$$
$$= -\frac{1}{2}\, d(\cos\theta_0)$$

where θ_0 is the angle of one of the emitted particles in the com frame.

- We would like to find $p(\theta)d\theta$, where θ is the angle of one of the emitted particle in the lab frame.

- Since $p(\theta)d\theta = p(\theta_0)d\theta_0$

we just need to find θ_0 as a function of θ.

This is given by (16.6) which we first derive.

Proof: Given $\tan\theta = \dfrac{v_0\sin\theta_0}{v_0\cos\theta_0 + V}$

we have

$$\tan\theta(v_0\cos\theta_0 + V) = v_0\sqrt{1-\cos^2\theta_0}$$

Square both sides

$$\tan^2\theta(v_0^2\cos^2\theta_0 + V^2 + 2v_0V\cos\theta_0) = v_0^2(1-\cos^2\theta_0)$$

$$\underbrace{(1+\tan^2\theta)}_{\sec^2\theta}v_0^2\cos^2\theta_0 + 2v_0V\tan^2\theta\cos\theta_0 + (V^2\tan^2\theta - v_0^2) = 0$$

quadratic equation for $\cos\theta_0$

$$\rightarrow \cos\theta_0 = \frac{-2v_0V\tan^2\theta \pm\sqrt{4v_0^2V^2\tan^4\theta - 4v_0^2\sec^2\theta(V^2\tan^2\theta - v_0^2)}}{2v_0^2\sec^2\theta}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \frac{1}{\sec^2\theta}\sqrt{\left(\frac{V}{v_0}\right)^2\tan^4\theta - \sec^2\theta\left(\left(\frac{V}{v_0}\right)^2\tan^2\theta - 1\right)}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \frac{1}{\sec\theta}\sqrt{\left(\frac{V}{v_0}\right)^2\left(\frac{\sin^4\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}\right) + 1}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{\left(\frac{V}{v_0}\right)^2\frac{\sin^2\theta(\sin^2\theta - 1)}{\cos^2\theta} + 1}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{1 - \left(\frac{V}{v_0}\right)^2\sin^2\theta}$$

(for $v_0 > V$ take $+\sqrt{\ }$, for $v_0 < V$ take both signs.)

Now differentiate both sides:

$$d(\cos\theta_0) = -\frac{V}{v_0}2\sin\theta\cos\theta\, d\theta \mp \sin\theta\, d\theta\sqrt{\ }$$
$$\pm\cos\theta\left(\frac{1}{\cancel{2}}\right)\frac{1}{\sqrt{\ }}(-\cancel{2})\left(\frac{V}{v_0}\right)^2\sin\theta\cos\theta\, d\theta$$

$$= \sin\theta\, d\theta\left[-2\frac{V}{v_0}\cos\theta \mp\sqrt{\ } \mp\left(\frac{V}{v_0}\right)^2\cos^2\theta\frac{1}{\sqrt{\ }}\right]$$

Now:

$$\sqrt{\;} + \left(\frac{V}{v_0}\right)^2 \cos^2\theta \, \frac{1}{\sqrt{\;}}$$

$$= \frac{1-\left(\frac{V}{v_0}\right)^2 \sin^2\theta + \left(\frac{V}{v_0}\right)^2 \cos^2\theta}{\sqrt{\;}}$$

$$= \frac{1+\left(\frac{V}{v_0}\right)^2 (\cos^2\theta - \sin^2\theta)}{\sqrt{\;}}$$

$$= \frac{1+\left(\frac{V}{v_0}\right)^2 \cos 2\theta}{\sqrt{\;}}$$

Thus,

$$d(\cos\theta) = \underbrace{-\sin\theta\, d\theta}_{d(\cos\theta)} \left[2\frac{V}{v_0}\cos\theta \pm \frac{\left(1+\left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1-\left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

So:

$$p(\theta)\, d\theta = \frac{1}{2}\sin\theta\, d\theta \left[2\frac{V}{v_0}\cos\theta \pm \frac{\left(1+\left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1-\left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

Now: for $v_0 > V$, take $+$ sign $(\theta \in [0, \pi])$

for $v_0 < V$, as θ_0 increases from 0 to π

θ increases from $0 \mapsto \theta_{max}$

θ decreases from $\theta_{max} \mapsto 0$

Thus, for $v_0 < V$ need to take the difference of the $+$ and $-$ expressions:

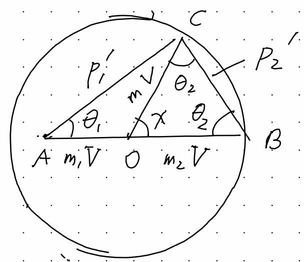
$$p(\theta)\, d\theta = \frac{1}{2}\sin\theta\, d\theta \left[2\frac{V}{v_0}\cos\theta + \frac{\left(1+\left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1-\left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

$$- \frac{1}{2}\sin\theta\, d\theta \left[2\frac{V}{v_0}\cos\theta - \frac{\left(1+\left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1-\left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

$$= \frac{\sin\theta\, d\theta \left(1+\left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1-\left(\frac{V}{v_0}\right)^2 \sin^2\theta}}$$

which is valid for $0 \le \theta \le \theta_{max} = \sin^{-1}\left(\frac{v_0}{V}\right)$

Sec 17, Prob 1:



$$\chi + 2\theta_2 = \pi$$
$$\chi = \pi - 2\theta_2$$

From the above figure:

$$(m_2 v_2')^2 = 2(mv)^2 - 2(mv)^2 \cos\chi$$
$$= 2m^2v^2\left(1-\cos(\pi-2\theta_2)\right)$$
$$= 2m^2v^2\left[1-\left(\underbrace{\cos(\pi)}_{-1}\cos(2\theta_2)+\cancelto{0}{\sin(\pi)}\sin(2\theta_2)\right)\right]$$
$$= 2m^2v^2\left(1+\cos(2\theta_2)\right)$$

$$\rightarrow \quad v_2' = \sqrt{2}\left(\frac{m}{m_2}\right)v\sqrt{1+(\cos^2\theta_2-\sin^2\theta_2)}$$
$$= \sqrt{2}\left(\frac{m_1}{m_1+m_2}\right)v\sqrt{2\cos^2\theta_2}$$
$$= 2\left(\frac{m_1}{m_1+m_2}\right)v\cos\theta_2$$

Thus, $\left(\frac{v_2'}{v}\right) = \left(\frac{2m_1}{m_1+m_2}\right)\cos\theta_2$

Also,

$$(mv)^2 = (m_1V)^2 + (m_1v_1')^2 - 2m_1^2 v_1' V\cos\theta_1$$
$$\rightarrow \quad (m_1v_1')^2 - 2m_1V\cos\theta_1(m_1v_1') + m_1^2V^2 - m^2v^2 = 0$$

Now: $V = \frac{m_1v_1 + m_2\cancelto{0}{v_2}}{m_1+m_2} = \frac{m_1v}{m_1+m_2}$

Thus,

$$(m_1v_1')^2 - 2\left(\frac{m_1^2 v}{m_1+m_2}\right)\cos\theta_1\, m_1v_1' + \frac{m_1^2 m_1^2 v^2}{(m_1+m_2)^2} - \frac{m_1^2m_2^2}{(m_1+m_2)^2}v^2 = 0$$

$$\rightarrow \quad (v_1')^2 - 2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1\, v_1' + \frac{m_1^2-m_2^2}{(m_1+m_2)^2}v^2 = 0$$

$$(v_1')^2 - 2\left(\frac{m_1v}{m_1+m_2}\right)\cos\theta_1\, v_1' + \left(\frac{m_1-m_2}{m_1+m_2}\right)v^2 = 0$$

$$\left(\frac{v_1'}{v}\right)^2 - 2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1\left(\frac{v_1'}{v}\right) + \left(\frac{m_1-m_2}{m_1+m_2}\right) = 0$$

Quadratic equation

$$\frac{v_1'}{v} = \frac{2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \sqrt{4\left(\frac{m_1}{m_1+m_2}\right)^2\cos^2\theta_1 - 4\left(\frac{m_1-m_2}{m_1+m_2}\right)}}{2}$$

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_1^2\cos^2\theta_1 - (m_1-m_2)(m_1+m_2)}$$

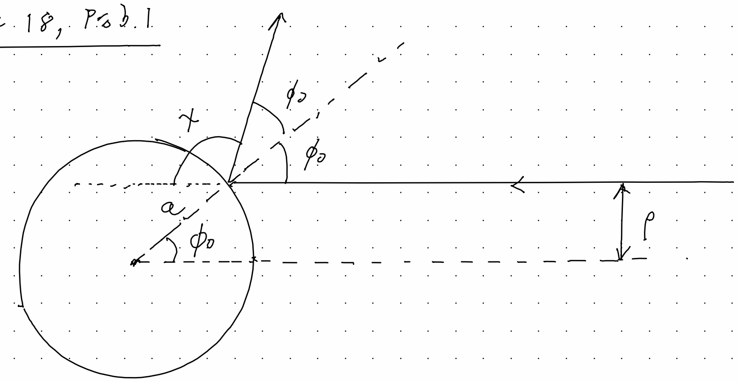
$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_1^2(\cos^2\theta_1 - 1) + m_2^2}$$

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_2^2 - m_1^2\sin^2\theta_1}$$

The + sign holds for $m_1 < m_2$

+/- signs hold for $m_1 > m_2$

Sec. 18, Prob. 1



$$\chi + 2\phi_0 = \pi \quad \rightarrow \quad \phi_0 = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\rho = a\sin\phi_0$$

$$= a\sin\left(\frac{\pi}{2} - \frac{\chi}{2}\right)$$

$$= a\left(\underset{\overset{\shortparallel}{1}}{\sin\left(\frac{\pi}{2}\right)}\cos\left(\frac{\chi}{2}\right) - \cancelto{0}{\cos\left(\frac{\pi}{2}\right)}\sin\left(\frac{\chi}{2}\right)\right)$$

$$= a\cos\left(\frac{\chi}{2}\right)$$

$$d\sigma = 2\pi\rho\, d\rho$$

$$= 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi$$

$$= \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$

where $d\Omega$ = solid angle

$$= \underset{\substack{\uparrow \\ \text{integral over} \\ d\phi}}{2}\pi\sin\chi\, d\chi$$

$$\rho = a\cos\left(\frac{\chi}{2}\right)$$

$$d\rho = -a\,\frac{1}{2}\sin\left(\frac{\chi}{2}\right)d\chi$$

$$= -\frac{a}{2}\sin\left(\frac{\chi}{2}\right)d\chi$$

thus,

$$d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$

$$= \frac{a\cos(\chi/2)}{\sin\chi}\,\frac{a}{2}\sin\left(\frac{\chi}{2}\right)d\Omega$$

$$= \frac{a^2}{2}\,\frac{\sin(\chi/2)\cos(\chi/2)}{\sin\chi}\,d\Omega$$

$$= \boxed{\frac{a^2}{4}\,d\Omega} \qquad \left(\text{since } \sin\chi = 2\sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)\right)$$

total cross section

$$\sigma = \int d\sigma = \frac{a^2}{4}\int d\Omega = \frac{a^2}{4}\,4\pi = \boxed{\pi a^2}$$

Now calculate differential cross section in lab frame for both m_1 and m_2

Use the result that

$$d\sigma_1 = \frac{\rho(\theta_1)}{\sin\theta_1}\left|\frac{d\rho}{d\theta_1}\right| d\Omega_1 = \rho\left|\frac{d\rho}{d(\cos\theta_1)}\right| d\Omega_1$$

Compare to:

$$d\sigma = \rho\left|\frac{d\rho}{d(\cos\chi)}\right| d\Omega$$

$$\rightarrow \frac{d\sigma_1}{d\Omega_1} = \rho\left|\frac{d\rho}{d(\cos\theta_1)}\right|$$

$$= \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|\frac{d\sigma}{d\Omega}$$

So we need to evaluate:

$$\frac{d(\cos\chi)}{d(\cos\theta_1)} \quad \text{and} \quad \frac{d(\cos\chi)}{d(\cos\theta_2)}$$

start with θ_2: (17.4)

$$\theta_2 = \frac{1}{2}(\pi - \chi) \rightarrow \boxed{\chi = \pi - 2\theta_2}$$

$$\rightarrow \cos\chi = \cos(\pi - 2\theta_2)$$

$$= \cos\pi\cos(2\theta_2) + \cancel{\sin\pi}^{\,0}\sin(2\theta_2)$$

$$= -\cos(2\theta_2)$$

$$= -(\cos^2\theta_2 - \sin^2\theta)$$

$$= -(2\cos^2\theta_2 - 1)$$

$$= -2\cos^2\theta_2 + 1$$

Thus, $\boxed{d(\cos\chi) = -4\cos\theta_2\, d(\cos\theta_2)}$ ✓

Thus,

$$\frac{d\sigma_2}{d\Omega_2} = \frac{d\sigma}{d\Omega}\left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right|$$

$$= \frac{1}{4}a^2 \cdot |4\cos\theta_2|$$

$$= a^2|\cos\theta_2|$$

so $\boxed{d\sigma_2 = a^2|\cos\theta_2|\,d\Omega_2}$

Now consider θ_1:

From (17.4):

$$\tan\theta_1 = \frac{m_2\sin\chi}{m_1 + m_2\cos\chi}$$

Compare with

$$\tan\theta = \frac{v_0\sin\theta_0}{V + v_0\cos\theta_0} \qquad (16.5)$$

make identifications: $\theta\to\theta_1$, $v_0\to m_2$, $V\to m_1$

then we can write down from (16.6).

$$\cos\chi = -\frac{m_1}{m_2}\sin^2\theta_1 \pm \cos\theta_1\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}$$

[see also sec 16, Prob. 2 where we derived this for θ and θ_0]

We also worked out the derivative:

$$d(\cos\theta_0) = d(\cos\theta)\left[2\frac{V}{v_0}\cos\theta \pm \frac{1+\left(\frac{V}{v_0}\right)^2\cos 2\theta}{\sqrt{1-\left(\frac{V}{v_0}\right)^2\sin^2\theta}}\right]$$

So we can similarly write dow

$$d(\cos\chi) = d(\cos\theta_1)\left[2\frac{m_1}{m_2}\cos\theta_1 \pm \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\right]$$

For $m_1 < m_2$: take + sign

For $m_1 > m_2$: As χ increases from 0 to π, θ_1 increases from 0 to θ_{max}, then θ_1 decreases from θ_{max} to 0. In that case

$$d(\cos\chi) = d(\cos\theta_1)[\cancel{\oplus}+\oplus] - d(\cos\theta_1)[\cancel{\oplus}-\ominus]$$

$$= 2\,d(\cos\theta_1)\,\frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

Use: (for both)

$$d\sigma_1 = \left(\frac{d\sigma}{d\Omega}\right)\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| d\Omega_1$$

$$= \frac{1}{4}a^2\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| d\Omega_1$$

Thus,

For $m_1 < m_2$:

$$d\sigma_1 = \frac{1}{4}a^2\left[2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\right]d\Omega_1$$

For $m_1 > m_2$:

$$d\sigma_1 = \frac{1}{4}a^2 \cdot 2\ \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\ d\Omega_1$$

$$= \frac{a^2}{2}\ \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\ d\Omega_1$$

Sec. 18, Prob. 2:

Hard sphere scattering again.
Calculate $d\sigma$ in terms of $d\epsilon$ where
ϵ = energy lost by scattered particle.

Now: ϵ = energy lost by scattered particle
= energy gained by m_2
$= \frac{1}{2}m_2(v_2')^2$

From Fig. 16., we have (law of cosines):

$$(m_2v_2')^2 = (mv)^2 + (mv)^2 - 2(mv)^2\cos\chi$$
$$= 2(mv)^2[1-\cos\chi]$$
$$= 2(mv)^2\ 2\sin^2\left(\frac{\chi}{2}\right)$$

so $m_2v_2' = 2\,mv\sin\left(\frac{\chi}{2}\right)$

$\rightarrow\quad v_2' = \left(\frac{m}{m_2}\right)v\sin\left(\frac{\chi}{2}\right)$

$= \left(\frac{m_1}{m_1+m_2}\right)v\sin\left(\frac{\chi}{2}\right) \qquad (17.5)$

NOTE: $d\sigma = \frac{1}{4}a^2\,d\Omega$

$= \frac{1}{4}a^2\ 2\pi\sin\chi\ d\chi$

$= \frac{\pi a^2}{2}\left|d(\cos\chi)\right|$

So we would like to related $d\epsilon$ and $d(\cos\chi)$.

Now: $\epsilon = \frac{1}{2} m_2 (v_2')^2$

$$= \frac{1}{2} m_2 \frac{4 m_1^2 v^2}{(m_1+m_2)^2} \sin^2\left(\frac{\chi}{2}\right)$$

$$= 2 \frac{m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 \sin^2\left(\frac{\chi}{2}\right) \quad (\text{since } v = v_\infty)$$

$$\equiv \epsilon_{max} \sin^2\left(\frac{\chi}{2}\right)$$

where $\epsilon_{max} \equiv \frac{2 m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2$

$$= 4\left(\frac{m_1}{m_1+m_2}\right)\frac{m v_\infty^2}{2}$$

$$= 4\left(\frac{m_1}{m_1+m_2}\right) E$$

Thus, $d\epsilon = \epsilon_{max}\, 2 \sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)\frac{d\chi}{2}$

$$= \frac{1}{2}\epsilon_{max} \sin\chi\, d\chi$$

$$= \frac{1}{2}\epsilon_{max} |d(\cos\chi)|$$

So $d\sigma = \frac{\pi a^2}{2} |d(\cos\chi)|$

$$= \frac{\pi a^2}{2}\frac{2}{\epsilon_{max}} d\epsilon = \boxed{\frac{\pi a^2}{\epsilon_{max}} d\epsilon}$$

which is a uniform distribution wrt ϵ.

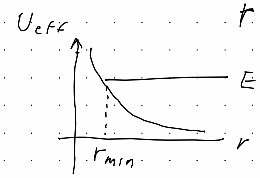
Sec 18, Prob. 4

Effective cross section to "fall" to center of $U(r) = -\alpha/r^2$ $(\alpha > 0)$

$$U_{eff}(r) = U(r) + \frac{M^2}{2mr^2}$$

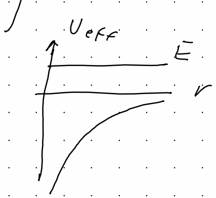
$$= -\frac{\alpha}{r^2} + \frac{M^2}{2mr^2}$$

$$= \frac{1}{r^2}\left(\frac{M^2}{2m} - \alpha\right)$$



$\frac{M^2}{2m} - \alpha > 0$

(don't fall to center since $r_{min} > 0$)



$\frac{M^2}{2m} - \alpha < 0$

Fall to center $(r=0)$

For a given $E = \frac{1}{2}mv_\infty^2 > 0$ need

$$\frac{M^2}{2m} - \alpha < 0$$

$$\frac{M^2}{2m} < \alpha$$

$$\rightarrow \quad M_{max} = \sqrt{2m\alpha}$$

Cross section: $\sigma = \pi \rho_{max}^2$, $M = \rho m v_\infty$

Thus,

$$\sigma = \pi \rho_{max}^2 \quad , \qquad \rho_{max} = \frac{M_{max}}{m v_\infty}$$

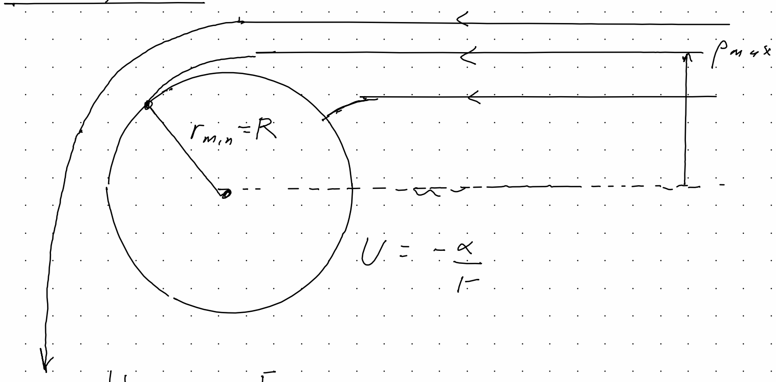
$$= \pi \frac{M_{max}^2}{m^2 v_\infty^2}$$

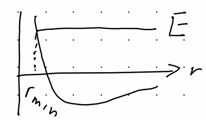
$$= \pi \frac{2 m \alpha}{m^2 v_\infty^2}$$

$$= \frac{\pi \alpha}{\frac{1}{2} m v_\infty^2}$$

$$= \boxed{\frac{\pi \alpha}{E}}$$

Sec 18, Prob 6





Turning point at $r = R$

$$0 = E - U_{eff}(R)$$

$$= E - U(R) - \frac{M_{max}^2}{2mR^2}$$

$$= E + \frac{\alpha}{R} - \frac{M_{max}^2}{2mR^2}$$

$$\rightarrow \frac{M_{max}^2}{2mR^2} = E + \frac{\alpha}{R}$$

$$M_{max} = \rho_{max} m v_\infty \quad , \qquad E = \frac{1}{2} m v_\infty^2$$

Thus,

$$\sigma = \pi \rho_{max}^2$$

$$= \pi \frac{M_{max}^2}{m^2 v_\infty^2}$$

$$= \pi \frac{1}{m^2 v_\infty^2} \quad 2mR^2\left(E + \frac{\alpha}{R}\right)$$

$$= \pi R^2 \underbrace{\left(\frac{2}{m v_\infty^2}\right)}_{\frac{1}{E}}\left(E + \frac{\alpha}{R}\right)$$

$$= \boxed{\pi R^2 \left(1 + \frac{\alpha}{ER}\right)}$$

where $E = \frac{1}{2} m v_\infty^2 = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right) v_\infty^2$

and $\alpha \equiv G m_1 m_2$

Sec 19, Prob. 1:

$$U = \frac{\alpha}{r^2} \quad , \quad \alpha > 0$$

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M\, dr / r^2}{\sqrt{2m[E - U(r)] - M^2/r^2}}$$

substitute: $E = \frac{1}{2} m v_\infty^2$

$M = \rho m v_\infty$

$$\rightarrow \phi_0 = \int_{r_{min}}^{\infty} \frac{\rho m v_\infty\, dr / r^2}{\sqrt{2m\left[\frac{1}{2} m v_\infty^2 - U(r)\right] - \rho^2 m^2 v_\infty^2 / r^2}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr / r^2}{\sqrt{1 - 2U(r)/m v_\infty^2 - \rho^2/r^2}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr / r^2}{\sqrt{1 - \rho^2/r^2 - 2U(r)/m v_\infty^2}}$$

substitute: $U(r) = \frac{\alpha}{r^2}$

$$\sqrt{} = \sqrt{1 - \rho^2/r^2 - \left(\frac{2\alpha}{m v_\infty^2}\right)\frac{1}{r^2}}$$

$$= \sqrt{1 - \left(\rho^2 + \frac{2\alpha}{m v_\infty^2}\right)\frac{1}{r^2}} = \sqrt{1 - \frac{A^2}{r^2}}$$

$$A^2 = \rho^2 + \frac{2\alpha}{m v_\infty^2}$$

Thus,

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1-\frac{A^2}{r^2}}}$$

$$= \int_{A}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1-\frac{A^2}{r^2}}}$$

Let: $u = \frac{1}{r} \rightarrow du = -\frac{1}{r^2}dr$

$$\frac{A^2}{r^2} = A^2 u^2$$

$$\phi_0 = -\int_{\frac{1}{A}}^{0} \frac{\rho\, du}{\sqrt{1-A^2u^2}}$$

$$= \int_{0}^{\frac{1}{A}} \frac{\rho\, du}{\sqrt{1-A^2u^2}}$$

Let: $\sin\theta = Au$

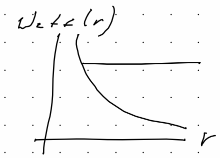
$\cos\theta\, d\theta = A\, du$

$u = 0, \frac{1}{A} \rightarrow \theta = 0, \pi/2$

$\sqrt{1-A^2u^2} = \sqrt{1-\sin^2\theta} = \cos\theta$

$$\rightarrow \phi_0 = \int_0^{\pi/2} \frac{\rho\, \cancel{\cos\theta}\, d\theta/A}{\cancel{\cos\theta}} = \frac{\rho}{A}\frac{\pi}{2}$$

$$\boxed{\phi_0 = \frac{\pi}{2}\frac{\rho}{\sqrt{\rho^2 + \frac{2\alpha}{m v_\infty^2}}}}$$



Repulsive scattering: $\chi + 2\phi_0 = \pi$

$\chi = \pi - 2\phi_0$

$$\chi = \pi - \pi\frac{\rho}{\sqrt{\rho^2 + \frac{2\alpha}{m v_\infty^2}}} = \pi\left[1 - \frac{1}{\sqrt{1+\frac{2\alpha}{\rho^2 m v_\infty^2}}}\right]$$

$$\left(\frac{\pi\, \rho}{\sqrt{\quad}}\right)^2 = (\pi - \chi)^2$$

$$\frac{\pi^2\rho^2}{\rho^2 + \frac{2\alpha}{m v_\infty^2}} = (\pi-\chi)^2$$

$$\pi^2\rho^2 = (\pi-\chi)^2\rho^2 + (\pi-\chi)^2\frac{2\alpha}{m v_\infty^2}$$

$$(\pi^2 - (\pi-\chi)^2)\rho^2 = (\pi-\chi)^2\frac{2\alpha}{m v_\infty^2}$$

$$(\cancel{\pi^2} - \cancel{\pi^2} - \chi^2 + 2\pi\chi)\rho^2 = (\pi-\chi)^2\frac{2\alpha}{m v_\infty^2}$$

$$\rho^2 = \frac{(\pi-\chi)^2}{2\pi\chi - \chi^2}\frac{2\alpha}{m v_\infty^2}$$

$$\boxed{\rho = \frac{(\pi-\chi)}{\sqrt{2\pi\chi-\chi^2}}\sqrt{\frac{2\alpha}{m v_\infty^2}}}$$

Differential cross-section:

$$d\sigma = 2\pi \rho \, d\rho$$
$$= 2\pi \, \rho(\chi) \left|\frac{d\rho}{d\chi}\right| d\chi$$
$$= \frac{\rho(\chi)}{\sin\chi} \left|\frac{d\rho}{d\chi}\right| d\Omega \quad , \quad d\Omega = 2\pi \sin\chi \, d\chi$$

Thus,

$$\frac{d\rho}{d\chi} = \sqrt{\frac{2\alpha}{m v_\infty^2}} \; \frac{-\sqrt{2\pi\chi-\chi^2} - \frac{1}{2\sqrt{\ }}(2\pi-2\chi)(\pi-\chi)}{2\pi\chi-\chi^2}$$

$$= -\sqrt{\frac{2\alpha}{m v_\infty^2}} \; \frac{\sqrt{2\pi\chi-\chi^2} + \frac{(\pi-\chi)^2}{\sqrt{\ }}}{2\pi\chi-\chi^2}$$

$$= -\sqrt{\frac{2\alpha}{m v_\infty^2}} \; \frac{2\pi\chi-\chi^2 + (\pi-\chi)^2}{(2\pi\chi-\chi^2)^{3/2}}$$

$$= -\sqrt{\frac{2\alpha}{m v_\infty^2}} \; \frac{\cancel{2\pi\chi} - \cancel{\chi^2} + \pi^2 + \cancel{\chi^2} - \cancel{2\pi\chi}}{(2\pi\chi-\chi^2)^{3/2}}$$

$$= -\sqrt{\frac{2\alpha}{m v_\infty^2}} \; \frac{\pi^2}{(2\pi\chi-\chi^2)^{3/2}}$$

So,

$$d\sigma = \frac{(\pi-\chi)}{\sqrt{2\pi\chi-\chi^2}} \sqrt{\frac{2\alpha}{m v_\infty^2}} \; \frac{1}{\sin\chi} \sqrt{\frac{2\alpha}{m\alpha}} \; \frac{\pi^2}{(2\pi\chi-\chi^2)^{3/2}} \, d\Omega$$

$$= \boxed{\left(\frac{2\alpha}{m v_\infty^2}\right) \frac{d\Omega}{\sin\chi} \; \frac{\pi^2(\pi-\chi)}{(2\pi\chi-\chi^2)^{3/2}}}$$

Sec 20, Prob. 1 Small-angle scattering

Start with (18.4):

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho \, dr/r^2}{\sqrt{1-\rho^2/r^2 - 2U/m v_\infty^2}}$$

Assume U is weak so that $2U/m v_\infty^2 \ll 1$

$$\frac{1}{\sqrt{\ }} = \frac{1}{\sqrt{(1-\rho^2/r^2)\left(1 - \frac{2U/m v_\infty^2}{(1-\rho^2/r^2)}\right)}}$$

$$\approx \frac{1}{\sqrt{1-\rho^2/r^2}} \left(1 + \frac{1}{\cancel{2}} \frac{\cancel{2}U/m v_\infty^2}{1-\rho^2/r^2}\right)$$

$$= \frac{1}{\sqrt{1-\rho^2/r^2}} + \frac{U/m v_\infty^2}{(1-\rho^2/r^2)^{3/2}}$$

Can replace r_{min} limit by ρ:

$$\int_{\rho}^{\infty} \frac{\rho \, dr/r^2}{\sqrt{1-\rho^2/r^2}} = -\int_{\frac{1}{\rho}}^{0} \frac{\rho \, du}{\sqrt{1-\rho^2 u^2}}$$

let: $u = \frac{1}{r}$

$du = -\frac{1}{r^2} dr$

$$= \int_0^{\frac{1}{\rho}} \frac{\rho \, du}{\sqrt{1-\rho^2 u^2}}$$

let $\rho u = \sin\theta$

$\rho \, du = \cos\theta \, d\theta$

$u = \frac{1}{\rho} \to \theta = \pi/2$

$$= \boxed{\frac{\pi}{2}}$$

Then,

$$\phi_0 \simeq \frac{\pi}{2} + \frac{1}{m v_\infty^2} \int_\rho^\infty \frac{\rho\, dr/r^2\, U(r)}{(1 - \rho^2/r^2)^{3/2}}$$

$$= \frac{\pi}{2} + \frac{1}{m v_\infty^2} \frac{\partial}{\partial \rho}\left[\int_\rho^\infty \frac{U(r)\, dr}{\sqrt{1-\rho^2/r^2}} \right]$$

Now:

$$\int_\rho^\infty \frac{U(r)\,dr}{\sqrt{1-\rho^2/r^2}} = uv \Big|_\rho^\infty - \int_\rho^\infty v\, du$$

where $u = U(r)$

$$dv = \frac{dr}{\sqrt{1-\rho^2/r^2}} = \frac{r\,dr}{\sqrt{r^2-\rho^2}} \qquad x = r^2-\rho^2, \quad dx = 2r\,dr$$

$$= \frac{dx/2}{\sqrt{x}}$$

$$\rightarrow v = \frac{1}{2}\int \frac{dx}{\sqrt{x}} = \sqrt{x} + \text{const}$$

$$= \sqrt{r^2-\rho^2} + \text{const}$$

So:

$$\int_\rho^\infty \frac{U(r)\,dr}{\sqrt{1-\rho^2/r^2}} = \cancel{U(r)\sqrt{r^2-\rho^2}\Big|_\rho^\infty} - \int_\rho^\infty \left(\frac{dU}{dr}\right) dr \sqrt{r^2-\rho^2}$$

assuming $U(r) \to 0$ faster than $\frac{1}{r}$ as $r \to \infty$

So

$$\phi_0 = \frac{\pi}{2} + \frac{1}{m v_\infty^2}\frac{\partial}{\partial\rho}\left[-\int_\rho^\infty \frac{dU}{dr}\, dr \sqrt{r^2-\rho^2}\right]$$

$$= \frac{\pi}{2} + \frac{1}{m v_\infty^2}(-)\int_\rho^\infty \frac{dU}{dr}\, dr\, \frac{1}{\cancel{2}\sqrt{r^2-\rho^2}}(-\cancel{2}\rho)$$

$$= \frac{\pi}{2} + \frac{\rho}{m v_\infty^2}\int_\rho^\infty dr\, \frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

Scattering angle χ:

$$2\phi_0 + \chi = \pi$$

$$\chi = \pi - 2\phi_0$$

$$\rightarrow \chi = \cancel{\pi} - 2\left(\cancel{\frac{\pi}{2}} + \frac{\rho}{m v_\infty^2}\int_\rho^\infty dr\, \frac{dU/dr}{\sqrt{r^2-\rho^2}}\right)$$

$$= -\frac{2\rho}{m v_\infty^2}\int_\rho^\infty dr\, \frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

In terms of θ_1:

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi} \rightarrow \theta_1 \simeq \frac{m_2 \chi}{m_1+m_2}$$

Thus,

$$\theta_1 \simeq \left(\frac{m_2}{m_1+m_2}\right)\chi$$

$$\simeq \left(\frac{m_2}{m_1+m_2}\right)\left(\frac{-2\rho}{m v_\infty^2}\right)\int_\rho^\infty dr \frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

$$= \frac{-2\rho}{m_1 v_\infty^2}\int_\rho^\infty dr \frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

which is Eq. (20,3)

Sec 21, Prob 1

$x = a\cos(\omega t+\alpha)$

Express a, α in terms of x_0, v_0

$x_0 = a\cos\alpha$

$v = \dot{x} = -a\omega\sin(\omega t+\alpha)$

$\rightarrow v_0 = -a\omega\sin\alpha$

Thus, $\dfrac{v_0}{x_0} = -\omega\tan\alpha$

$\rightarrow \boxed{\tan\alpha = \dfrac{-v_0}{\omega x_0}}$

Also: $x_0 = a\cos\alpha$

$\dfrac{v_0}{\omega} = -a\sin\alpha$

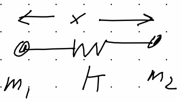
$\rightarrow x_0^2 + \left(\dfrac{v_0}{\omega}\right)^2 = a^2\cos^2\alpha + a^2\sin^2\alpha$

$= a^2$

so $\boxed{a = \sqrt{x_0^2 + \left(\dfrac{v_0}{\omega}\right)^2}}$

Sec 21, Prob 2:

Diatomic molecules, different isotopes m_1, m_2 and m_1', m_2'



2-body $\rightarrow$ 1-body w.r.t COM

$$T = \frac{1}{2} m \dot{x}^2, \quad m = \frac{m_1 m_2}{m_1 + m_2}$$

$$U = \frac{1}{2} k x^2$$

Thus, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}$

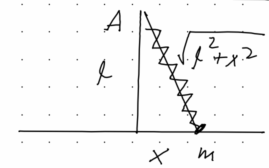
Similarly, for masses m_1', m_2' with $k' = k$:

$$\omega' = \sqrt{\frac{k}{m'}} = \sqrt{\frac{k(m_1' + m_2')}{m_1' m_2'}}$$

$$\rightarrow \frac{\omega'}{\omega} = \sqrt{\frac{\cancel{k}(m_1' + m_2')}{m_1' m_2'}} \sqrt{\frac{m_1 m_2}{\cancel{k}(m_1 + m_2)}}$$

$$= \sqrt{\left(\frac{m_1' + m_2'}{m_1 + m_2}\right) \frac{m_1 m_2}{m_1' m_2'}}$$

Sec 21, Prob 3:



$$U \approx F \delta\ell$$

$$\delta\ell = \sqrt{\ell^2 + x^2} - \ell$$

$$= \ell\left(\sqrt{1 + \left(\frac{x}{\ell}\right)^2} - 1\right)$$

$$= \ell\left(\cancel{1} + \frac{1}{2}\left(\frac{x}{\ell}\right)^2 + \dots - \cancel{1}\right)$$

$$\approx \frac{1}{2} \frac{x^2}{\ell}$$

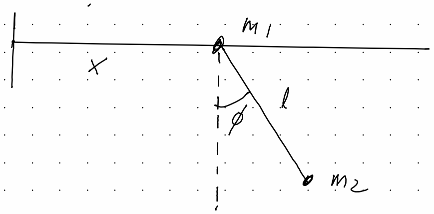
Thus, $U \approx \frac{1}{2}\left(\frac{F}{\ell}\right) x^2 = \frac{1}{2} k x^2$ for $k \equiv \frac{F}{\ell}$

Also, $T = \frac{1}{2} m \dot{x}^2$

$$\rightarrow \omega = \sqrt{\frac{F}{\ell m}}$$

Sec 21, Prob 5.

Figure 2 from Sec 5:



Recall from Sec 14, Prob 3:

$$E = \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left(1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right) - m_2 g l \cos\phi$$

Now $\phi = 0$ corresponds to stable equilibrium.

Small oscillations: $\phi \ll 1$

Since $\dot{\phi}^2$ is small can set $\phi = 0$ in $\left(1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right)$

$$\rightarrow 1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi \rightarrow 1 - \frac{m_2}{m_1+m_2} = \frac{m_1}{m_1+m_2}$$

Then

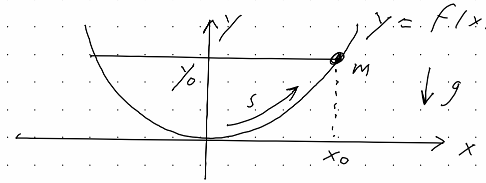
$$E = \frac{1}{2} m l^2 \dot{\phi}^2 - m_2 g l \left(1 - \frac{1}{2}\phi^2\right)$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + \frac{1}{2} m_2 g l \phi^2 - \underbrace{m_2 g l}_{\text{const}}$$

Thus,

$$\omega = \sqrt{\frac{m_2 g l}{m l^2}} = \sqrt{\left(\frac{m_1+m_2}{m_1}\right)\frac{g}{l}} \quad \left(\rightarrow \sqrt{\frac{g}{l}} \text{ for } m_1 \gg m_2\right)$$

Sec 21, Prob 6.



Find $y = f(x)$ so that period $P(y_0)$ is indep. of y_0.

Need: $E = \frac{1}{2} m v^2 + m g y$

$$= \frac{1}{2} m \left(\frac{ds}{dt}\right)^2 + \frac{1}{2} k s^2$$

where s = arc length along the curve

$$= \int_0^s ds$$

$$= \int_0^s \sqrt{dx^2 + dy^2}$$

$$= \int_0^x dx \sqrt{1 + y'^2}$$

so $m g y = \frac{1}{2} k s^2$

$$y = \frac{1}{2g} \frac{k}{m} s^2 \equiv A^2 s^2$$

$$\rightarrow \sqrt{y} = A s, \quad A = \frac{1}{\sqrt{2g}} \sqrt{\frac{k}{m}}$$

$$\frac{1}{2\sqrt{y}} \frac{dy}{dx} = A \frac{ds}{dx} = A \sqrt{1 + y'^2}$$

Thus, $\frac{1}{2}\frac{y'}{\sqrt{y}} = A\sqrt{1+y'^2}$

$$y'^2 = 4A^2 y(1+y'^2)$$

$$y'^2(1-4A^2y) = 4A^2y$$

$$\frac{dy}{dx} = y' = \sqrt{\frac{4A^2y}{1-4A^2y}}$$

so $\boxed{x = \int_0^y dy\sqrt{\frac{1-4A^2y}{4A^2y}}}$

make substitution

$$y = \frac{1}{8A^2}(1-\cos\theta)$$

$$dy = \frac{1}{8A^2}\sin\theta\, d\theta = \frac{1}{8A^2}\sqrt{1-\cos^2\theta}\, d\theta$$

$$\rightarrow \frac{1-4A^2y}{4A^2y} = \frac{1-\frac{1}{2}(1-\cos\theta)}{\frac{1}{2}(1-\cos\theta)} = \frac{\cancel{\frac{1}{2}}(1+\cos\theta)}{\cancel{\frac{1}{2}}(1-\cos\theta)}$$

$$= \frac{1+\cos\theta}{1-\cos\theta}$$

Thus,

$$x = \int \frac{1}{8A^2}\sqrt{1-\cos^2\theta}\, d\theta\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$

$$= \frac{1}{8A^2}\int d\theta\sqrt{\frac{\cancel{(1-\cos\theta)}(1+\cos\theta)(1+\cos\theta)}{\cancel{1-\cos\theta}}}$$

$$x = \frac{1}{8A^2}\int d\theta\,(1+\cos\theta)$$

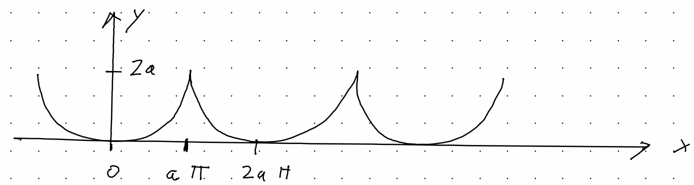
$$= \frac{1}{8A^2}(\theta + \sin\theta)$$

thus,

$$x = \frac{1}{8A^2}(\theta+\sin\theta) = a(\theta+\sin\theta)$$

$$y = \frac{1}{8A^2}(1-\cos\theta) = a(1-\cos\theta)$$

which are parametric equations for a cycloid



NOTE: $8A^2 = 8\frac{1}{2g}\frac{k}{m} = \frac{4}{g}\frac{k}{m} = \frac{4}{g}\omega^2$

Sec 22, Prob 1:

(a) $F = F_0 = \text{const}$

$x = \dot{x} = 0$ at $t=0 \rightarrow \xi_0 = 0$

$$\rightarrow \quad \xi(t) = e^{i\omega t}\left[\cancel{\xi_0}^{\,0} + \int_0^t d\bar{t}\, \frac{F(\bar{t})}{m} e^{-i\omega \bar{t}}\right]$$

$$= e^{i\omega t} \frac{F_0}{m} \int_0^t d\bar{t}\, e^{-i\omega \bar{t}}$$

$$= e^{i\omega t}\left(\frac{F_0}{-im\omega}\right) e^{-i\omega \bar{t}}\Big|_0^t$$

$$= e^{i\omega t}\left(\frac{F_0}{-im\omega}\right)\left(e^{-i\omega t} - 1\right)$$

$$= \frac{iF_0}{m\omega}\left(1 - e^{i\omega t}\right)$$

$$= \frac{iF_0}{m\omega}\left(1 - \cos\omega t - i\sin\omega t\right)$$

Now: $x(t) = \dfrac{\mathrm{Im}(\xi(t))}{\omega}$, $\dot{x}(t) = \mathrm{Re}(\xi(t))$

Thus,

$$\boxed{x(t) = \frac{F_0}{m\omega^2}\left(1 - \cos\omega t\right)}$$

(b) $F(t) = at$

$$\xi(t) = e^{i\omega t} \frac{a}{m} \int_0^t d\bar{t}\, \bar{t}\, e^{-i\omega \bar{t}}$$

Let $u = \bar{t} \rightarrow du = d\bar{t}$

$dv = e^{-i\omega \bar{t}} d\bar{t} \rightarrow v = \dfrac{1}{-i\omega} e^{-i\omega \bar{t}}$

Thus,

$$\xi(t) = e^{i\omega t} \frac{a}{m}\left[\frac{\bar{t}}{-i\omega} e^{-i\omega \bar{t}}\Big|_0^t + \frac{1}{i\omega}\int_0^t d\bar{t}\, e^{-i\omega \bar{t}}\right]$$

$$= e^{i\omega t} \frac{a}{m}\left[\frac{t}{-i\omega} e^{-i\omega t} + \frac{1}{\omega^2}\left(e^{-i\omega t} - 1\right)\right]$$

$$= \frac{a}{m}\left[\left(\frac{it}{\omega} + \frac{1}{\omega^2}\right) - \frac{1}{\omega^2} e^{i\omega t}\right]$$

$$= \frac{a}{m}\left[\left(\frac{it}{\omega} + \frac{1}{\omega^2}\right) - \frac{1}{\omega^2}\left(\cos\omega t + i\sin\omega t\right)\right]$$

$$= \frac{a}{m}\left[\frac{1}{\omega^2}\left(1 - \cos\omega t\right) + i\left(\frac{t}{\omega} - \frac{\sin\omega t}{\omega^2}\right)\right]$$

$$\rightarrow \quad x(t) = \frac{\mathrm{Im}(\xi(t))}{\omega}$$

$$= \boxed{\frac{a}{m\omega^3}\left(\omega t - \sin\omega t\right)}$$

(c) $F(t) = F_0 \exp(-\alpha t)$

$$\xi(t) = e^{i\omega t}\frac{F_0}{m}\int_0^t d\bar{t}\; e^{-\alpha \bar{t}}\, e^{-i\omega \bar{t}}$$

$$= \frac{F_0}{m} e^{i\omega t}\frac{1}{-(i\omega+\alpha)}\, e^{-(i\omega+\alpha)\bar{t}}\Big|_0^t$$

$$= \frac{F_0}{m} e^{i\omega t}\frac{1}{-(i\omega+\alpha)}\left(e^{-(i\omega+\alpha)t}-1\right)$$

NOTE: $\frac{1}{-(i\omega+\alpha)}\cdot\left(\frac{i\omega-\alpha}{i\omega-\alpha}\right) = \frac{i\omega-\alpha}{\alpha^2+\omega^2}$

$$\rightarrow \xi(t) = \frac{F_0}{m}\left(\frac{i\omega-\alpha}{\alpha^2+\omega^2}\right)\left[e^{-\alpha t} - e^{i\omega t}\right]$$

$$= \frac{F_0}{m}\left(\frac{i\omega-\alpha}{\alpha^2+\omega^2}\right)\left[e^{-\alpha t} - \cos\omega t - i\sin\omega t\right]$$

$$= \frac{F_0}{m(\alpha^2+\omega^2)}\left[i\left(\omega e^{-\alpha t} - \omega\cos\omega t + \alpha\sin\omega t\right) - \alpha e^{-\alpha t} + \alpha\cos\omega t + \omega\sin\omega t\right]$$

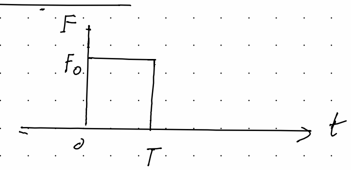
Thus,

$$x(t) = \mathrm{Im}(\xi(t))/\omega$$

$$= \frac{F_0}{m\omega(\alpha^2+\omega^2)}\left(\omega e^{-\alpha t} - \omega\cos\omega t + \alpha\sin\omega t\right)$$

$$= \boxed{\frac{F_0}{m(\alpha^2+\omega^2)}\left(e^{-\alpha t} - \cos\omega t + \frac{\alpha}{\omega}\sin\omega t\right)}$$

Sec 22, Prob 3:



Initial condition

$\xi_0 = 0$ (since $x_0 = \dot{x}_0 = 0$)

For $t > T$:

$$\xi(t) = e^{i\omega t}\int_0^T d\bar{t}\,\frac{F_0}{m}\, e^{-i\omega\bar{t}}$$

$$= e^{i\omega t}\frac{F_0}{m}\left(\frac{1}{-i\omega}\right)\left(e^{-i\omega T}-1\right)$$

$$= \frac{iF_0}{m\omega}\left(e^{i\omega(t-T)} - e^{i\omega t}\right)$$

$$= \frac{iF_0}{m\omega}\left[\cos(\omega(t-T)) + i\sin(\omega(t-T)) - \cos\omega t - i\sin\omega t\right]$$

$$= \frac{iF_0}{m\omega}\left(\cos(\omega(t-T)) - \cos(\omega t)\right) - \frac{F_0}{m\omega}\left(\sin(\omega t - T) - \sin(\omega t)\right)$$

$$\rightarrow x(t) = \frac{\mathrm{Im}(\xi(t))}{\omega}$$

$$= \frac{F_0}{m\omega^2}\left(\cos(\omega(t-T)) - \cos(\omega t)\right)$$

We can also write for $t > T$:

$$x(t) = c_1 \cos(\omega(t-T)) + c_2 \sin(\omega(t-T))$$

$$\rightarrow \quad \dot{x}(t) = -c_1 \omega \sin(\omega(t-T)) + c_2 \omega \cos(\omega(t-T))$$

Thus, $x(T) = c_1$, $\quad \dot{x}(T) = c_2 \omega$

Match with:

$$x(t) = \frac{F_0}{m\omega^2}\left(\cos(\omega(t-T)) - \cos(\omega t)\right)$$

$$\dot{x}(t) = \frac{F_0}{m\omega}\left(-\sin(\omega(t-T)) + \sin(\omega t)\right)$$

at $t = T$,

$$\rightarrow \quad c_1 = \frac{F_0}{m\omega^2}\left(1 - \cos(\omega T)\right)$$

$$c_2 \omega = \frac{F_0}{m\omega} \sin(\omega T)$$

Thus,

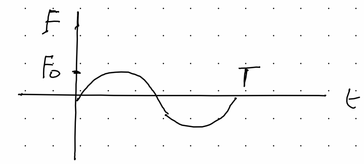
$$a \equiv \sqrt{c_1^2 + c_2^2}$$

$$= \frac{F_0}{m\omega^2}\sqrt{1 + \cos^2(\omega T) - 2\cos(\omega T) + \sin^2(\omega T)}$$

$$= \frac{F_0}{m\omega^2}\sqrt{2(1 - \cos(\omega T))}$$

$$= \boxed{\frac{2F_0}{m\omega^2}\sin\left(\frac{\omega T}{2}\right)} \quad \text{using } 1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$$

Sec. 22, Prob 5:



$$T = \frac{2\pi}{\omega}$$

$F(t) = F_0 \sin\omega t$, $\quad \xi_0 = 0$

For $t > T$:

$$\xi(t) = e^{i\omega t}\frac{F_0}{m}\int_0^T d\bar{t}\, \sin\omega\bar{t}\, e^{-i\omega\bar{t}}$$

$$= \frac{F_0}{m} e^{i\omega t}\frac{1}{2i}\int_0^T d\bar{t}\left(e^{i\omega\bar{t}} - e^{-i\omega\bar{t}}\right)e^{-i\omega\bar{t}}$$

$$= \frac{F_0}{2mi} e^{i\omega t}\int_0^T d\bar{t}\left(1 - e^{-i2\omega\bar{t}}\right)$$

$$= \frac{F_0}{2mi} e^{i\omega t}\left(T + \cancelto{0}{\frac{1}{i2\omega}\left(\underbrace{e^{-i2\omega T}}_{1} - 1\right)}\right)$$

$$= \frac{F_0 T}{2mi} e^{i\omega t}$$

Thus, $x(T) = \dfrac{\mathrm{Im}(\xi(T))}{\omega} = -\dfrac{F_0 T}{2m\omega}$ $\quad\Big|\quad$ $\dot{x}(T) = \mathrm{Re}(\xi(T)) = 0$

We can also write for $t > T$:

$$x(t) = c_1 \cos(\omega(t-T)) + c_2 \sin(\omega(t-T))$$

$$\rightarrow \quad \dot{x}(t) = -c_1 \omega \sin(\omega(t-T)) + c_2 \omega \cos(\omega(t-T))$$

Thus,

$$x(T) = c_1 \quad , \quad \dot{x}(T) = c_2\omega$$

match with

$$x(T) = \frac{-F_0 T}{2m\omega} \quad , \quad \dot{x}(T) = 0$$

$$\rightarrow \quad c_1 = \frac{-F_0 T}{2m\omega} \quad , \quad c_2 = 0$$

so $a = \sqrt{c_1^2 + c_2^2}$

$$= \frac{F_0 T}{2m\omega}$$

$$= \frac{F_0 \cancel{2}\pi/\omega}{\cancel{2} m\omega}$$

$$= \boxed{\frac{F_0 \pi}{m\omega^2}}$$

Sec 23, Prob 1:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}\omega_0^2(x^2+y^2) + \alpha xy$$

$$= \frac{1}{2}\sum_{i,k} m_{ik}\dot{x}_i\dot{x}_k - \frac{1}{2}\sum_{i,k} k_{ik} x_i x_k$$

where $m_{ik} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$$\sum_{i,k} k_{ik} x_i x_k = \omega_0^2(x^2+y^2) - 2\alpha xy$$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so $k_{ik} = \begin{bmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{bmatrix}$

EOMs:

$$\frac{d}{dt}\sum_k m_{ik}\dot{x}_k = -\sum_k k_{ik} x_k$$

$$\sum_k m_{ik}\ddot{x}_k = -\sum_k k_{ik} x_k$$

$$\sum_k (m_{ik}\ddot{x}_k + k_{ik} x_k) = 0$$

Trial solution: $x_k = A_k e^{i\omega t}$

$$\rightarrow \sum_k (-\omega^2 m_{ik} + k_{ik}) A_k e^{i\omega t} = 0$$

Thus,

$$\sum_k (K_{ik} - \omega^2 m_{ik}) A_k = 0$$

$\rightarrow \quad \det(K_{ik} - \omega^2 m_{ik}) = 0$

$$0 = \det \begin{bmatrix} \omega_0^2 - \omega^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega^2 \end{bmatrix}$$

$$= (\omega_0^2 - \omega^2)^2 - \alpha^2$$

$$= \omega^4 - 2\omega^2\omega_0^2 + (\omega_0^4 - \alpha^2)$$

Quadratic equation:

$$\omega_\pm^2 = \frac{2\omega_0^2 \pm \sqrt{\cancel{4\omega_0^4} - 4(\cancel{\omega_0^4} - \alpha^2)}}{2}$$

$$= \frac{2\omega_0^2 \pm 2\alpha}{2}$$

$$= \boxed{\omega_0^2 \pm \alpha} \quad \text{(normal mode freqs)}$$

Normal mode vectors:

$\omega_+^2 = \omega_0^2 + \alpha$:

$$\sum_k (K_{ik} - \omega_+^2 m_{ik}) A_k = 0$$

$$\begin{bmatrix} \omega_0^2 - \omega_+^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega_+^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\alpha & -\alpha \\ -\alpha & -\alpha \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $-\alpha(A_1 + A_2) = 0$

$$A_2 = -A_1$$

so eigenvector is $\boxed{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = V_+}$

$\omega_0^2 = \omega_-^2$:

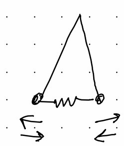
$$\begin{bmatrix} \omega_0^2 - \omega_-^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega_-^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $A_2 = A_1$

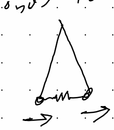
$\rightarrow \quad \boxed{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = V_-}$

V_+ corresponds to out-of-phase motion:



$$V_+ e^{i\omega_+ t} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\sqrt{\omega_0^2 + \alpha}\, t}$$

V_- corresponds to in-phase motion:



$$V_- e^{i\omega_- t} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\sqrt{\omega_0^2 - \alpha}\, t}$$

NOTE:

For weak coupling, $\alpha \ll \omega_0^2$, we have

$$\omega_\pm^2 = \omega_0^2 \pm \alpha$$
$$= \omega_0^2\left(1 \pm \frac{\alpha}{\omega_0^2}\right)$$

$$\rightarrow \omega_\pm = \omega_0\sqrt{1 \pm \frac{\alpha}{\omega_0^2}}$$

$$\simeq \omega_0\left(1 \pm \frac{1}{2}\frac{\alpha}{\omega_0^2}\right)$$ nearly equal freqs

Beat freq:

$$\omega_+ - \omega_- \simeq \omega_0\left(\cancel{1} + \frac{1}{2}\frac{\alpha}{\omega_0^2}\right) - \omega_0\left(\cancel{1} - \frac{1}{2}\frac{\alpha}{\omega_0^2}\right)$$
$$= \boxed{\frac{\alpha}{\omega_0}}$$

Sec 23, Prob 3:

$U = \frac{1}{2}kr^2$, space oscillator

central potential $\rightarrow$ motion in a plane (x,y) or (r,ϕ)

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{1}{2}k(x^2+y^2)$$
$$= \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) + \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2\right)$$

EOMs:

$m\ddot{x} = -kx \rightarrow x = a\cos(\omega t+\alpha)$

$m\ddot{y} = -ky \rightarrow y = b\cos(\omega t+\beta)$

where $\omega \equiv \sqrt{\frac{k}{m}}$

Write $\omega t+\beta = \omega t+\alpha+(\beta-\alpha)$
$= \phi+\delta$

where $\phi \equiv \omega t+\alpha$, $\delta \equiv \beta-\alpha$

Then $x = a\cos\phi \longrightarrow \boxed{\cos\phi = \frac{x}{a}}$

$y = b\cos(\phi+\delta)$
$= b(\cos\phi\cos\delta - \sin\phi\sin\delta)$

$$\rightarrow \frac{y}{b} = \cos\phi\cos\delta - \sin\phi\sin\delta$$
$$= \left(\frac{x}{a}\right)\cos\delta - \sin\phi\sin\delta$$

so $$\boxed{\sin\phi = \frac{1}{\sin\delta}\left[\left(\frac{x}{a}\right)\cos\delta - \frac{y}{b}\right]}$$

Square and add two boxed expressions

$$\cos^2\phi = \left(\frac{x}{a}\right)^2$$

$$+ \quad \sin^2\phi = \frac{1}{\sin^2\delta}\left[\left(\frac{x}{a}\right)^2\cos^2\delta + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab}\cos\delta\right]$$

$$1 = \left(\frac{x}{a}\right)^2\left(1 + \frac{\cos^2\delta}{\sin^2\delta}\right) + \left(\frac{y}{b}\right)^2\frac{1}{\sin^2\delta} - \frac{2xy}{ab}\frac{\cos\delta}{\sin^2\delta}$$

$$\rightarrow \sin^2\delta = \left(\frac{x}{a}\right)^2\underbrace{(\sin^2\delta + \cos^2\delta)}_{=1} + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab}\cos\delta$$

$$= \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab}\cos\delta$$

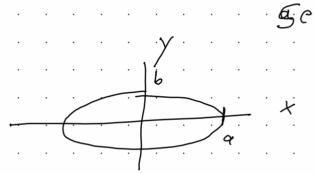
Thus, $\boxed{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab}\cos\delta = \sin^2\delta}$

suppose $\delta = \frac{\pi}{2}$:

Then $\cos\delta = 0$, $\sin\delta = 1$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

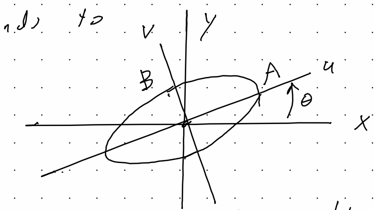
(ellipse centered at origin with semi-major axis = a, semi-minor axis = b)



In general

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab}\cos\delta = \sin^2\delta$$

corresponds to



Find angle θ such that the above expression maps to

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$$

Now: $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$\rightarrow \sin^2\delta = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \frac{2xy}{ab}\cos\delta$$

$$= \frac{1}{a^2}(\cos\theta\, u - \sin\theta\, v)^2 + \frac{1}{b^2}(\sin\theta\, u + \cos\theta\, v)^2$$

$$- \frac{2(\cos\theta\, u - \sin\theta\, v)(\sin\theta\, u + \cos\theta\, v)}{ab}\cos\delta$$

$$= \frac{1}{a^2}\left(u^2\cos^2\theta + v^2\sin^2\theta - 2uv\sin\theta\cos\theta\right)$$
$$+ \frac{1}{b^2}\left(u^2\sin^2\theta + v^2\cos^2\theta + 2uv\sin\theta\cos\theta\right)$$
$$- \frac{2}{ab}\cos\delta\left(u^2\sin\theta\cos\theta - v^2\sin\theta\cos\theta + uv\cos 2\theta\right)$$

$$= u^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} - \frac{2\sin\theta\cos\theta\cos\delta}{ab}\right)$$
$$+ v^2\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} + \frac{2\sin\theta\cos\theta\cos\delta}{ab}\right)$$
$$+ 2uv\left(-\frac{\sin\theta\cos\theta}{a^2} + \frac{\sin\theta\cos\theta}{b^2} - \frac{\cos 2\theta\cos\delta}{ab}\right)$$

We can make the factor multiplying uv equal to zero by choosing θ appropriately:

$$-\sin\theta\cos\theta\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{\cos 2\theta\cos\delta}{ab}$$

$$\frac{1}{2}\sin 2\theta\left(\frac{a^2-b^2}{a^2b^2}\right) = \frac{\cos 2\theta\cos\delta}{ab}$$

$$\boxed{\tan 2\theta = \left(\frac{2ab}{a^2-b^2}\right)\cos\delta}$$

NOTE: if $a=b$ then $0 = \cos 2\theta$

$$\rightarrow \theta = \frac{\pi}{4}$$

So choosing θ as above, we have

$$\sin^2\delta = u^2 \oslash + v^2 \oslash$$

$$1 = u^2\frac{\oslash}{\sin^2\delta} + v^2\frac{\oslash}{\sin^2\delta}$$

$$= \left(\frac{u}{A}\right)^2 + \left(\frac{v}{B}\right)^2$$

where
$$A^2 = \frac{\sin^2\delta}{\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} - \frac{2\sin\theta\cos\theta\cos\delta}{ab}\right)}$$

$$B^2 = \frac{\sin^2\delta}{\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} + \frac{2\sin\theta\cos\theta\cos\delta}{ab}\right)}$$