Adapting Escher's Rules for "Regular Division of the Plane" to Create *TesselMania!*®

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M.C. Escher's fascination with "regular division of the plane" is well documented both by his artistic works and numerous texts and articles. In his own words,

> *A plane, which should be considered limitless on all sides, can be filled with or divided into similar geometric figures that border on each other on all sides with leaving any "empty spaces." This can be carried on to infinity according to a limited number of systems.* ([1, p. 156]; also see [3, p. 15])

Mathematicians now call a "regular division of the plane" a tessellation (or tiling) and making such tessellations has become a wonderful activity for school children and teachers connecting mathematics and art. Jill Britton and Dale Seymour in their book *Introductions to Tessellations* [2] have introduced thousands of students and teachers to Escher's work and the joy of creating a tessellation. Unfortunately, due to time limitations in the classroom and the difficulty of creating tessellations by hand, students are able to experiment with just a few rudimentary tile *types*. A tile type is described by the specific geometric rules of how it is constructed and repeated to cover the plane without any gaps or overlaps.

The computer program, *TesselMania!*, was designed to let students easily create tessellations, therefore enabling them to explore many different tile types. The program was inspired by Escher's pioneering work on "regular division." In describing his study, he wrote:

> *At first I had no idea at all of the possibility of systematically building up my figures. I did now know any of the 'ground rules' and tried, almost without knowing what I was doing, to fit together congruent shapes that I attempted to give the form of animals. Gradually, designing new motifs became easier as a result of my study of the literature on the subject, as far as this was possible for some untrained in mathematics, and especially as a result of my putting forward my own layman's theory . . . It remains an extremely absorbing activity, a real mania to which I have become addicted. . .* [1, p. 164]

In creating the computer programs *TesselMania!* and *TesselMania! Deluxe* I experienced much of this same mania.

Escher's investigations of the geometry behind regular division is well documented by Doris Schattschneider in her excellent book: *M.C. Escher Visions of Symmetry* [7]. She chronicles Escher's initial introduction to the problem of regular division through his many years of investigations and discoveries.

He recorded his "layman's theory" in a notebook that was completed in 1942. This notebook was not published and relatively unknown until the first Escher Congress in 1986 when Schattschneider made it the subject of her talk and then wrote an article for the proceedings [3, pp. 82–96]. She expanded on it further in her book.

Escher's Initial Discoveries

Escher realized that when he started with a single tile (motif) there were only certain types of geometric moves, or motions, to apply that would not change the size or shape of the tile. These moves are known as isometries and math students learn that there are exactly four types: translation, rotation, reflection, and glide-reflection. In his lectures on regular division Escher would explain:

> *Anyone who wishes to achieve symmetry on a flat surface must take account of three fundamental principles of crystallography: repeated shifting (translation); turning about axes (rotation) and gliding mirror image (reflexion).* [4, p. 8]

Note that Escher states three fundamental principles, not four! He is clearly talking about translation, rotation, and glide reflection. But what about reflection? Escher certainly knew about reflection, but for creating tiles it is not a useful operation. If a side of a tile was used as a reflection line then that side is forced to be a straight line – not very helpful if you dream of creating tiles in the shape of lizards, fishes, or other creatures! For a further explanation, see [7, p. 33]. Interestingly, he did create some tiles with bilateral reflection symmetry (see [7, p. 118] for an example).

Escher initially did what every good mathematician does (although he claimed he was not a mathematician!) – he placed restrictions on his problem. When he set about to classify his tiles and tilings he initially limited himself to asymmetric tiles whose congruent copies produced tilings in which each tile was surrounded in the same way (hence the name "regular division of the plane"). An asymmetric tile is one that by itself has no symmetry. Moreover, he also required that in his tilings, no two tiles that shared a common border were the same color, and in achieving this, a minimum number of colors should be used. In Escher's view, tiling and coloring were inseparable problems. He actually did research in a field, eventually called color symmetry, that would not be fully explored by crystallographers until years later [7, p. 39]. He didn't really set out to classify tiles and tilings, but rather he set out to explore all the ways he could generate tiles.

The Idea for a Computer Program

The idea for a educational computer program to create Escher-like tessellations originated with Craig Solomonson and Shari Zehm at MECC (Minnesota

Educational Computing Corporation). They had noticed the popularity of sessions on tessellations at math teacher conferences. They attended several sessions on creating "Escher-like" tessellations in the classroom, including one by Jill Britton, and they were hooked. These workshops were teaching some of the geometric rules that Escher had discovered.

Craig and Shari quickly realized the advantage a computer program could bring to the process of creating tessellations. Craig then suggested the idea to me and used a computer paint program to illustrate how to create a few simple tiles. It didn't take long to figure out why Escher had found creating tessellations such an interesting problem and I, too, was caught up in the mania of trying figure out the geometric rules with the additional task of writing a computer program that applied the rules. Fortunately, Escher had already figured out the theory; I just had to be able to understand it and apply it in designing a computer program.

A Simple Quadrilateral System

The prototype for *TesselMania!* started with simple tile systems from Britton and Seymour's *Introduction to Tessellations*. By analyzing Escher's symmetry drawings, they managed to rediscover the 'ground rules' behind some of Escher's drawings. They describe geometric systems for creating Escher-like tessellations using their own graphic notation. As Escher discovered, the key to creating a tessellation is to start with a polygon that tessellates, and then alter the edges in such a way that the tessellating property is preserved. An **edge** of a tile connects two vertices. A **vertex** of a tile is defined as any point on the tile where more than two tiles in the tessellation meet. In Fig. 1 each tile has four vertices and four edges.

The first tile type implemented in *TesselMania!* was based on a parallelogram and used only translations. The initial parallelogram tiles the plane, forming the underlying grid of the tessellation. To transform the tessellation at the left in Fig. 1 to that on the right, two adjacent edges are altered and translated to opposite edges: the top edge is altered then translated to the bottom edge and the left edge is changed and translated to the right. The original vertex points do not change position and, in fact, form an underlying grid of parallelograms.

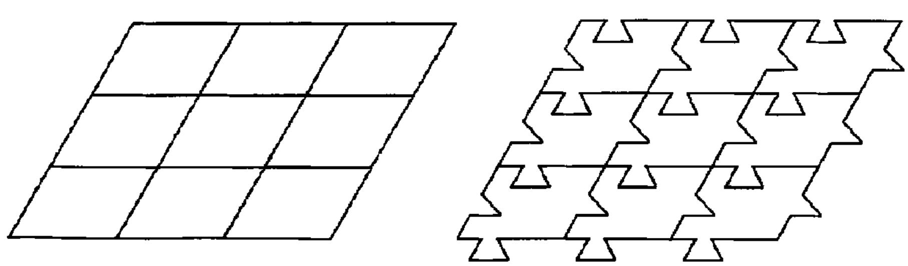

Fig. 1. Tessellation based on Translation

This strict application of the geometric rules guarantees that as a tile is copied and moved to the spot below it, the top edge of the copy will match exactly with the bottom edge of the original, or conversely, if the tile is moved to the position above it the bottom edge of the copy will match the top edge of the original. The same is true for moving the tile to the left or right.

Creating Escher-Like Tessellations

The tile-making process taught in many tessellation workshops for teachers uses an index card to create a tessellating polygon, usually starting with a square or a rectangle. A piece is carefully cut off one edge and taped to a corresponding edge according to the geometric rules for the tile type. This technique is discussed in more detail in the article by Jill Britton (page 305).

TesselMania! set out to mimic this process of transforming a polygon into an interesting tessellating shape, but adapted it to take advantage of the power of the computer. Students start by selecting a tile type (the geometric rules for the tile). Instead of cutting and taping a cardboard tile, they shape a virtual tile on the computer screen by using a tack tool that allows them to bump the sides of the initial polygon. As they introduce bumps on one side of the tile, corresponding bumps are automatically introduced on the related side. Unlike the index card method, students can re-shape their tile's edges at any time. As a further advantage, students see corresponding sides of the tile being formed simultaneously. Once they have completed the outline of their tile, computer paint tools can be used to add features to the interior of the tile. By clicking the "tessellate" button they see their tile repeated to cover the screen, complete with all the interior details.

The following illustration gives more details of this procedure. Here the tile type selected is a parallelogram with parallel sides related by translation (the same type of tile used in Fig. 1). Three tools are provided to manipulate the border of the tile: an arrow, a tack, and a scissors. The arrow is used to drag existing points on the tile, the tack is used to introduce and drag points on the border of the tile, and the scissors is used delete existing points. The tile outline is kept track of by the computer as a series of points that can be manipulated at any time. Figure 2 illustrates how the tile border tools are used to produce a tessellating tile. At every step in the construction the tile retains the ability to tessellate.

The last two stages show features added to the interior of the tile using the paint tools. The tools include a stamp tool with many pre-designed features like eyes, ears, noses, mouths, and hats to allow students to quickly add new features to their designs and to help the unfortunate student, who like me, cannot draw very well.

Figure 3 illustrates part of the plane (the computer screen) tessellated with the fish. Coloring is a key part of Escher's tessellations; he used color to make it easier to recognize individual tiles. In a tiling such as in Fig. 3, *TesselMania!*

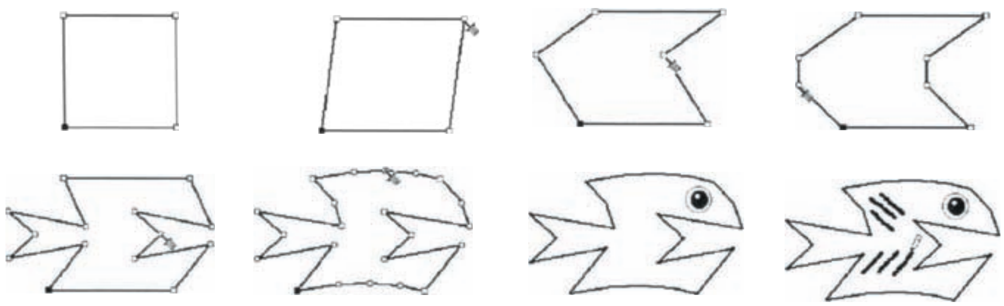

Fig. 2. Example of creating a tile using *TesselMania!*

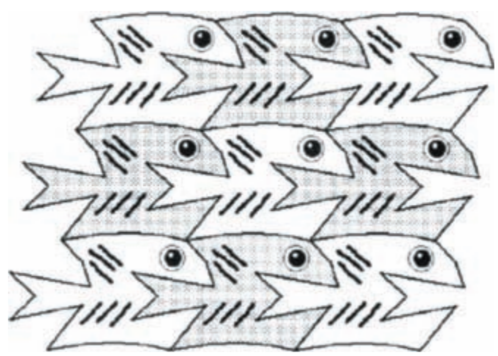

Fig. 3. A portion of the tessellated computer screen

places colors in contrasting pairs. Thus if a student paints inside the master tile, the surrounding tiles are automatically painted a contrasting color. This scheme is extended to triples of colors for tessellations that require a minimum of three colors for recognizability of tiles.

Tile Types and TesselMania!

The prototype for *TesselMania!* was created using the information about tile types from [2]. When I discovered *Visions of Symmetry*, I started to adopt Escher's numbering for his quadrilateral systems (Part I of his 1941–42 notebook) for some of the tile types. The remaining tile types were contained in his transition systems (Part II of the notebook) and his triangle systems (Part IV of the notebook). These two additional systems were harder to understand and I did not relish trying to explain the systems to school children. Eventually I switched to a system invented by the mathematician Heinrich Heesch. Heesch's classification system is very similar to Escher's system but mathematically more elegant and complete. Heesch proved there are 28 different ways (using combinations of translations, rotations or glide-reflections) to tile the plane in a regular way with an asymmetric tile. Each of the 28 ways determines a tile "type," since the geometric motions that fit copies snugly next to each other naturally determine certain relationships among the edges of a tile. Escher's work shows that he, working independently of Heesch, discovered 27 of the 28 tile types.

Escher's Quadrilateral Systems (Part I of his notebook)

Part I of Escher's notebook describes his two-colored quadrilateral systems. These are systems that begin with an underlying grid of some type of parallelogram. He identified ten systems using a Roman numeral to indicate the system and further specified the type of quadrilateral by using a letter (A, B, C, D, or E). Each tile system is classified by the motions needed to move the original tile to the tiles adjacent to it. In a quadrilateral system, for each tile, there are eight surrounding tiles, four that share an edge with the center tile and four that share just a vertex. This property assures that the tiling can be colored using a minimum of two colors with no two tiles that share an edge having the same color. In fact, the coloring is just that of a checkerboard (see [7], p. 59).

Escher identified the relationships of tiles that surrounded a given tile by using transversal and diagonal directions. In the fish tessellation of Fig. 3, the central tile translates in both transversal directions (up-down and left-right) and translates in both diagonal directions; this is the signature of his quadrilateral system I. In some quadrilateral systems, it is possible for a tile to be related to an adjacent tile by a glide-reflection or by a rotation of 180° (2-fold rotation) or 90° (4-fold rotation). The centers (axes) about which a tile rotates to an adjacent tile must be located at midpoints of the sides of the parallelogram or at its vertices. In his notebook, Escher published a summary table for his quadrilateral systems; it is reproduced in [7, p. 61].

Escher's classification scheme is a *local* classification system in which he specified the moves needed to generate a tiling from a single tile. (This is in contrast to the "global" view of crystallographers, who classify using symmetry groups. They look at the *entire* pattern rather than only a patch surrounding one tile, and collect (in the symmetry group) all symmetries of the entire pattern.) An advantage of Escher's local system is that it also tells how to construct the tile. Essentially, he determined the geometric relationships among the edges of a tile and therefore determined how edges could be modified yet keep the tile's ability to fill the plane.

Heesch's System

Before exploring more of Escher's systems it will be useful to know how Heesch's system works. In fact I will use Heesch's notation to explain Escher's transition and triangle systems. I had implemented about eleven of Escher's tile types in the software when Doris Schattschneider suggested I look at Heesch's system. She had mentioned Heesch in her book and included his table of 28 types of asymmetric tiles that can fill the plane in a regular manner without using reflections [7, p. 326 Table 2].

Heinrich Heesch was a German mathematician who in 1932 investigated and classified the possible asymmetric shapes that could tile the plane in a regular manner, not allowing reflections[1]. Unfortunately for Escher, this work was not published until 1963. Apparently, in 1963 they briefly corresponded [7, p. 44].

Heesch's table proved to be a computer scientist's dream, at least for a computer scientist who dreamed of creating a tessellations program. Not only does it classify all possible ways to create the tiles, but the notation system contains the algorithm that describes how each tile is made! Heesch's system is very similar to Escher's in that it uses local information about how the tile is constructed to classify the tile. He used a letter code, with subscripts when needed, to denote the geometric properties of the tile construction. For each edge of the tile there is a letter (and possibly a subscript) that indicates how it is constructed in relation to another edge in the tile. The number of letters equals the number of edges. He used the letter **T** to denote translation, **G** to denote glide-reflection, and **C** for rotation. For the letter **T** a subscript is never needed since it is clear that a translated side is matched with the opposite, parallel side of the tile. In the case of G, subscripts were used, if necessary, to denote corresponding sides. For rotation a subscript was used to indicate the amount of rotation: the letter **C** without a subscript represents 180° rotation (2-fold), $\mathbf{C}_3$ represents 120° rotation (3-fold), $\mathbf{C}_4$ represents 90° (4-fold) rotation, and $\mathbf{C}_6$ represents 60° rotation (6-fold). Since the tilings produced with these tiles are periodic two-dimensional patterns, no other angle possibilities exist.

Figure 4 gives a stepwise explanation of the Heesch type for the tile in Fig. 2. In Fig. 4a the top and bottom edges are related by translation so they are each labeled with **T**. Similarly, the right and the left edges are related by translation (Fig. 4b). All the labels are shown in Fig. 4c and the Heesch type is found by proceeding (counterclockwise) around the tile, recording the letters: **TTTT**.

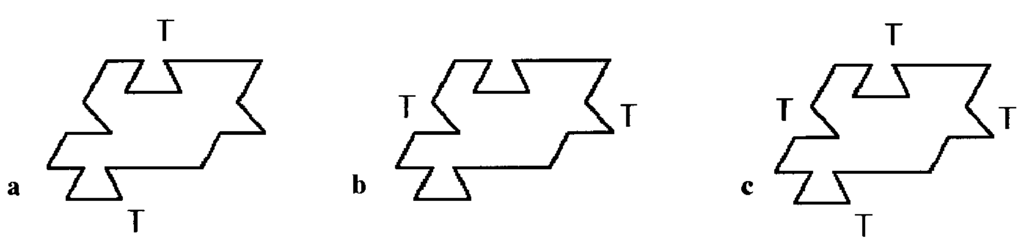

Fig. 4. Explanation of Heesch Type TTTT

The next example illustrates the use of subscripts to identify a tile type that uses a glide-reflection. It also involves reverse-engineering a tessellation, which

[1] A complete classification of isohedral tiles without the asymmetric restriction was done by Branko Grübaum and G. C. Shephard in their book *Tilings and Patterns* [6].

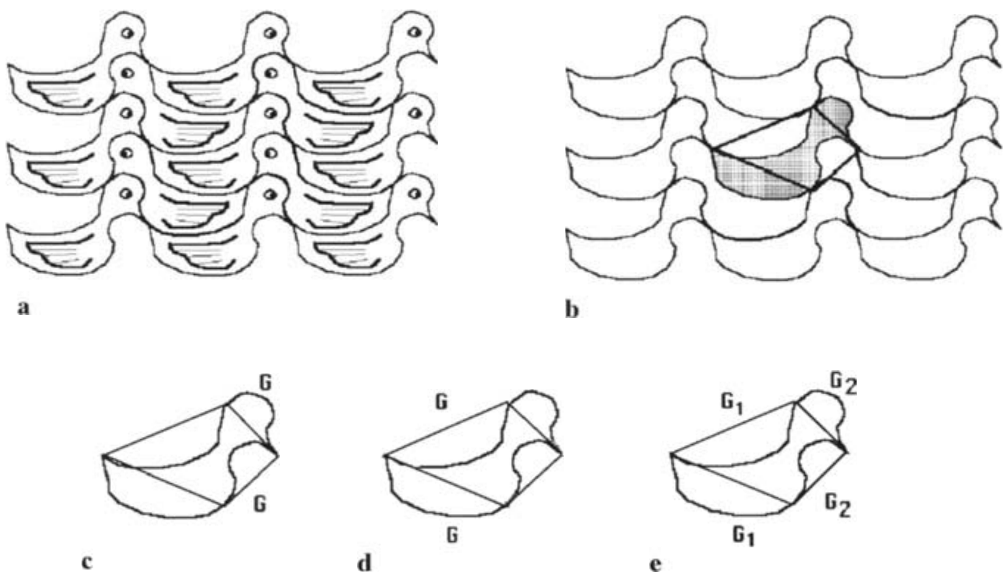

Fig. 5. Finding the Heesch type of a tile

is an activity students seem to enjoy, especially if the tessellation was created by Escher. Figure 5a shows a tessellation of ducks. The first step is to pick one duck and find its vertices by going around the boundary of the duck and finding points where more than two tiles meet. In this case there are four vertices of the tile, which also means there are also four edges. In Fig. 5b a single duck is highlighted and the four vertices have been connected by line segments to show the initial polygon for this tile.

The next task is to figure out the relationships between the four edges of a single tile. In Fig. 5c the top right edge is related to the bottom right edge by a glide-reflection along a vertical axis. The two edges each are labeled with the letter G. The top left edge and bottom left edge are also related by a glide-reflection along a different vertical axis of the tiling so in Fig. 5d these two edges are also labeled with the letter G. But now there is a dilemma: it is possible to have glide reflections relate adjacent sides (as is the case here) or opposite sides. Heesch used subscripts (as shown in Fig. 5e) to resolve the possible ambiguity. The Heesch type for this particular tile is $G_1G_1G_2G_2$. *TesselMania!* has a feature to animate the creation of any tile to make it easier for the student to visualize the geometric relationships between pairs of edges.

For each of the 28 tile types, Heesch's notation system generates a descriptive label that contains the geometric information necessary to understand the tile type. If I were to use Escher's labeling system, I would need to include a separate explanation for each tile type. There is a fair amount of work involved in understanding Escher's three separate systems for the types of tiles I wished to implement. So I decided to drop Escher's numbering system in the software and replace it with Heesch's labels.

The relationship between Escher's ten quadrilateral systems and Heesch's system is shown in Table 1. Even though the *TesselMania!* software uses

Table 1. Comparison of Escher's ten quadrilateral systems and Heesch's system.

Escher's Quadrilateral System	Heesch System
I	**TTTT**
II	**TCTC**
III	**CCCC** (III^A) **CCC** (III^C)
IV	$G_1G_1G_2G_2$
V	**TGTG**
VI	**CCGG** **CGG**
VII	**CGCG**
VIII	$G_1G_2G_1G_2$
IX	$C_4C_4C_4C_4$
X	CC_4C_4

Heesch's system it will be useful to examine Escher's transition and triangle systems and correlate them to Heesch types in order to count the total number of Heesch tile types Escher discovered. Table 1 accounts for 12 Heesch types.

Escher's Transition Systems (Part II of his notebook)

In the second part of his notebook, Escher explained how his quadrilateral systems that require only two colors can be modified to require three colors for

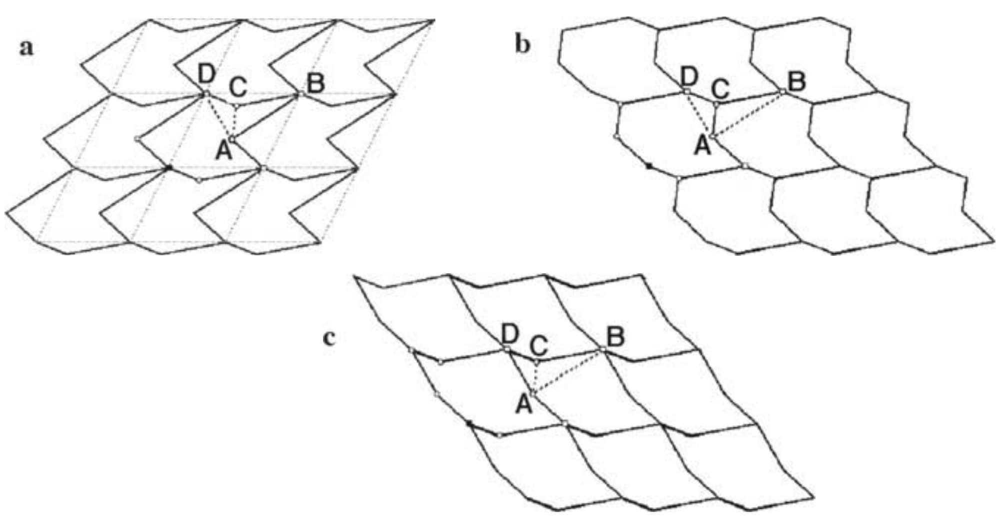

Fig. 6. Escher's Transition system I^A–I^A

Table 2. Comparison of Escher's Transition systems and Heesch's system.

Escher's Transition System	Heesch System
I^A–I^A	***TTTTTT***
II^A–III^A	***TCCTCC*** ***TCTCC***
V^C–IV^B	$\boldsymbol{TG_1G_2TG_2G_1}$
VII^C–VI^B	***TCCTGG***
$VIII^C$–VI^C	$\boldsymbol{CG_1CG_2G_2G_1}$
$VIII^C$–VII^C	$\boldsymbol{CG_1G_2G_2G_1}$
IX^D–X^E	$\boldsymbol{CC_4C_4C_4C_4}$

recognizability of tiles; this formed the basis of his "transitional" systems. I will illustrate one transitional system here, Escher's label for it is I^A–I^A. Later I will use a similar procedure to investigate a Heesch tile type Escher missed. (See Escher's notebook along with Schattschneider's comments in [7] for a complete description of all his transitional systems).

Step 1. (Figure 6a) Start with quadrilateral system I^A. Mark vertices **B** and **D** of one edge; mark any point **C** on edge **BD** and mark any point **A** on the adjacent edge with endpoint **B** as shown. (Heesch System **TTTT**.)

Step 2. (Figure 6b) Remove the portion of the edge from **A** to **B** and connect **A** to **C**. Do this to every tile in the tessellation. This new tessellation, in which each tile has 6 edges, requires a minimum of 3 colors. Escher denoted this as transition system I^A–I^A. (Heesch System **TTTTTT**.)

Step 3. (Figure 6c) The transition process can be continued. Now remove the portion of the edge from **A** to **C** and connect **A** to **D**. Do this to every tile in the tessellation. This returns us to a (different) two-color quadrilateral system I^A. (Heesch System **TTTT**)

Table 2 shows that Escher's transition systems account for nine more Heesch types.

Escher's Triangle Systems (Part IV of his notebook)[2]

Tilings that had 3-fold and 6-fold rotations were classified by Escher as triangle systems since they were based on an underlying grid of equilateral triangles. Heesch's tablei [7] shows there are only six tile types that include a C_3 or C_6

[2] The astute reader will notice that I skipped from part II of Escher's notebook to part IV. Part III is about a technique of splitting a single tile into two motifs so as to produce tessellations with two distinct interlocked shapes.

Table 3. Escher's Triangle systems and Heesch's systems.

Escher System	Heesch System
Tr I A_3 type 1	$C_3C_3C_3C_3C_3C_3$
Tr I A_3 type 2	$C_3C_3C_3C_3$
Tr I B_2 type 1	CC_6C_6
Tr I B_3 type 1	$C_3C_3C_6C_6$
Tr I B_3 type 2	$CC_3C_3C_6C_6$

symbol. Escher's triangle systems account for five of these six (see Table 3). (For further explanation of Escher's triangle systems see [7] or [3, p. 92]).
Escher's notes indicate he knew about the sixth Heesch type, CC_3C_3, since he specifically mentioned that he was not considering systems where more than 6 motifs meet at one vertex [7, p. 79]. Heesch type CC_3C_3 is the only tile type that produces such tilings, and in fact, these tilings have vertices where 12 motifs meet.

Counting Escher Tile Types: A Surprise and Mystery

Tables 1, 2, and 3 show that 25 of the 28 Heesch types have associated Escher systems. Heesch type CC_3C_3 was explicitly omitted by Escher. That leaves only two types to account for: $TG_1G_1TG_2G_2$ and TCTGG. Escher's symmetry drawing 78 of unicorns (color plate 3) contains the marginal note: "New System?" He could not find this system among those he enumerated in his notebook. Surprise! This is Heesch type $TG_1G_1TG_2G_2$. So, working independently, Escher had discovered 27 of the possible 28 Heesch types! Amazing for a man who claimed to have been poor at mathematics.

So here is the mystery: What about the remaining Heesch Type, TCTGG? Was there something special about this type that it was not included in Escher's classification system? It requires a minimum of three colors and it has no 3-fold or 6-fold rotation centers so if it could arise using his techniques, it would have to be a transition system. I set out to see if I could discover such a transition system. Since *TesselMania!* had been written to create Escher-like tessellations, perhaps I could use it to create an Escher-like transition system! I did manage to find a transition system to generate TCTGG, but it took one little trick. Figure 7 illustrates the steps.

Step 1. (Figure 7a) Start with a tile of Heesch type **CCGG** (Escher system VI). Mark **B**, the common vertex of the edges with 2-fold rotation centers, marked as **C** and **E**. Mark **D** the other vertex on the edge with points **B** and **C**. Mark any point **A** on the edge between **B** and **E**. Then mark **A'**, **B'**, **C'**, and **D'**, the images of **A**, **B**, **C**, and **D**, under a 180° rotation about **E**.

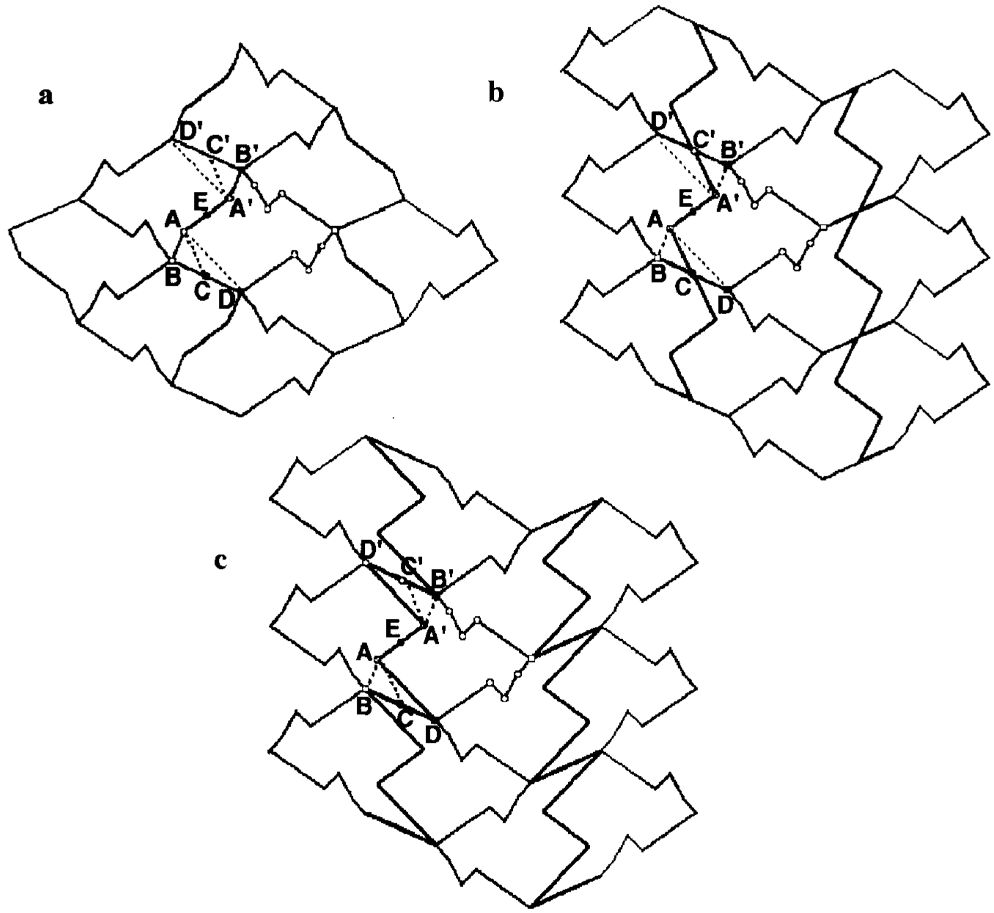

Fig. 7. An Escher-like transition system for Heesch Type TCTGG. Using Escher's notation, this transition system would be called VI-VI

Step 2. (Figure 7b) Remove the portion of the edge from **A** to **B** and connect **A** to **C**. To preserve the 2-fold rotation center **E**, remove the edge segment **A'B'** and connect **A'** to **C'**. Do this to every tile in the tessellation. Beginning with edge **B'C'**, travel counterclockwise around the new tile: **B'C'** gets Heesch label **T**, edge **C'C** is labeled **C**, edge **CD** is labeled **T**, and the two remaining (unaltered) edges are labeled **GG**, so the tile is Heesch type **TCTGG**. Vertex **B** of the original tile has been moved to point **C**, a center of 2-fold rotation.

Step 3. (Figure 7c) The process can be continued to return to a quadrilateral system. Remove the portion of the edge from **A** to **C** and connect **A** to **D**. Correspondingly, replace **A'C'** by **A'D'**. Do this to every tile. The resulting tiling is Escher's quadrilateral system VI (Heesch type **CCGG**).

It seems that Escher just missed this type. The little trick of moving a vertex along to a center of rotation would not have been much of a trick to Escher, who created many more amazing feats with regular division.

In some sense I feel it is unfair (uncharitable) to even point out the missing transition system. *I* know it is missing since I have Heesch's table and further-more I know exactly the tile type to look for. Escher wandered "... through the garden of the regular division of the plane all alone..." [1, p. 162] discovering different systems of regular division with no prior knowledge of the exact number of possibilities. To have found 27 of the 28 possible types through his explorations and investigations was an amazing accomplishment.

A Colorful Confession

I will conclude by giving an example of how I am continually surprised and always learning from Escher's pioneering work. The original *TesselMania!* implemented 15 Heesch types. In the later *TesselMania! Deluxe* the remaining 13 were added, which included Heesch type $CC_4C_4C_4C_4$. Each type is automatically colored with a minimum number of colors, which is either two or three. This meets Escher's criterion that tiles sharing a common edge must be of different colors. But there was a further coloring criterion that Escher implemented: the coloring of the tiles should be compatible with all the symmetries of the uncolored tiling. This condition is now called "perfect coloring" (see [7] for a further explanation of perfect coloring). Figure 8 shows the coloring scheme implemented in TesselMania! Deluxe for Heesch type $CC_4C_4C_4C_4$. Here letters indicate colors: **R** stands for red, **G** for green, and **B** for blue.

If a tiling is perfectly colored, every symmetry of the uncolored tiling either must preserve all colors or permute the colors in the colored tiling. The coloring in Fig. 8 is not perfect. To see this, note that a 90° rotation of the tiling about the marked center is a symmetry of the uncolored tiling, that is, it moves every

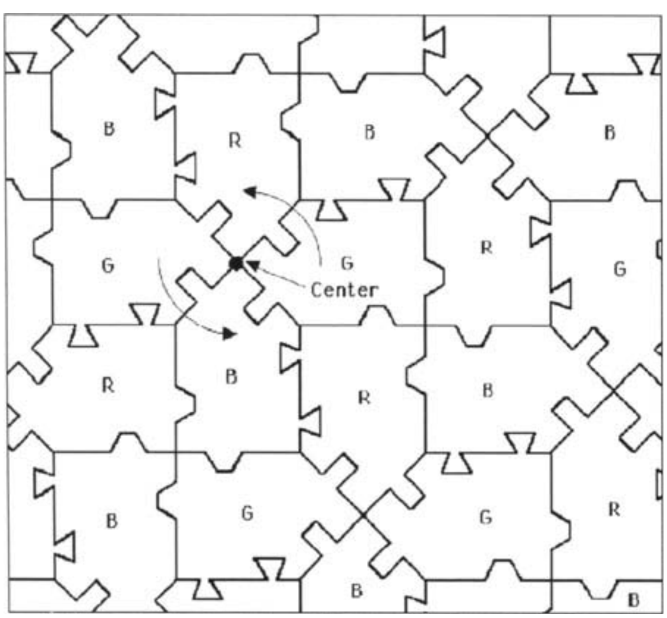

Fig. 8. Three-Colored $CC_4C_4C_4C_4$

tile exactly onto another tile. But look at the four tiles that surround this rotation center. The rotation moves one green tile to a blue tile and the other green tile to a red tile. If this tiling were perfectly colored, then this rotation would have to move every green tile to the same color. It is a nice exercise to try to perfectly color this tessellation using four colors (it can't be done using three colors – see Shephard's article [3, pp. 111–122].

Early on in his studies, Escher became aware of perfect coloring. His symmetry drawing 14 with 4-fold rotations uses the minimum three colors and has the same problem as that of Fig. 8. Made a short time afterward, his symmetry drawing number 20 of fish (color plate 2) could also have been done in three colors but he choose to use four colors and then perfectly colored it. I would like to think he would forgive me for having implemented this tile type in three colors!

Addendum

Following in footsteps of Escher and Heesch, my fascination with tessellations has continued. Since writing this article I have developed a completely new tiling program called *Tessellation Exploration* [Tom Snyder Productions, Cambridge, MA 2001]. This new program takes advantage of what I learned since creating *TesselMania!* (e.g. all tiles are perfectly colored!) It also takes advantage of the increased speed and power of the new microprocessors. A demonstration version has been included on the accompanying CD with a special set of slide show tiles that illustrates all 28 Heesch types.

In 1998 Bigalke and Wippermann [8] have extended Heesch's classification to 43 tile types by including edges with reflections. In *Tessellation Exploration* I implemented five of the new types in a modified way, the reflected edge is used to generate a larger tile with bilateral symmetry. Escher quickly discovered that an edge based on reflection can not be re-shaped!

Acknowledgment

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