<u>Notes:</u> Thurs 8/27

1) Elliptic functions
2) Simple pendulum

go beyond small angle approx

<u>Elliptic Functions:</u>

$$\int \frac{dx}{\sqrt{1-x^2}} = \, ? = \sin^{-1}(x) + \text{const}$$

$$\sin^{-1}(x) = \arcsin(x)$$

$\rightarrow \sin^2\theta + \cos^2\theta = 1$

<u>subst:</u> $x = \sin\theta \rightarrow \sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$dx = \cos\theta \, d\theta$

$$\int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \int d\theta = \theta = \sin^{-1}(x)$$

$x = \sin\theta$

$\theta = \sin^{-1}(x)$

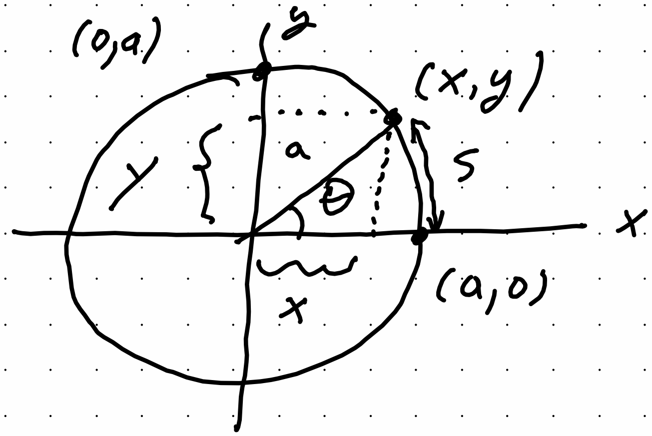
$$\frac{d}{d\theta}\sin\theta = \cos\theta \qquad \lim_{h\to 0}\left(\frac{\sin(\theta+h)-\sin\theta}{h}\right)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\sin(A+B) = \sin A\cos B + \cos A \sin B$$

circular functions..

$x^2+y^2=a^2$, a = radius



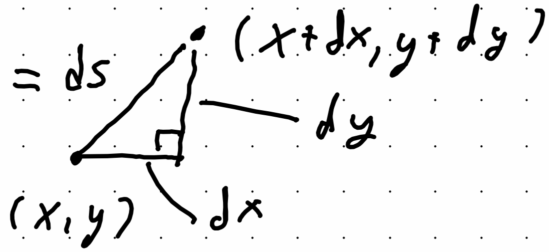
$\underline{\underline{\text{Def:}}}$ $\sin\theta = \dfrac{y}{a}$, $\cos\theta = \dfrac{x}{a}$

s : arclength from $(a,0)$ to (x,y)

$s = a\theta$

$$\theta = \frac{s}{a} = \frac{1}{a}\int ds = \frac{1}{a}\int\sqrt{dx^2+dy^2} = \int d\theta$$

$\sqrt{dx^2+dy^2} = ds$



Given: $x^2+y^2=a^2$

Follows: (i) $\cancel{a^2}\cos^2\theta + \cancel{a^2}\sin^2\theta = \cancel{a^2} \rightarrow \boxed{\cos^2\theta+\sin^2\theta=1}$

(ii) $\boxed{\dfrac{d}{d\theta}\sin\theta} = \dfrac{1}{a}\dfrac{dy}{d\theta} = \dfrac{\frac{1}{\cancel{a}}\,dy}{\frac{1}{\cancel{a}}\sqrt{dx^2+dy^2}} = \dfrac{1}{\sqrt{\left(\frac{dx}{dy}\right)^2+1}}$

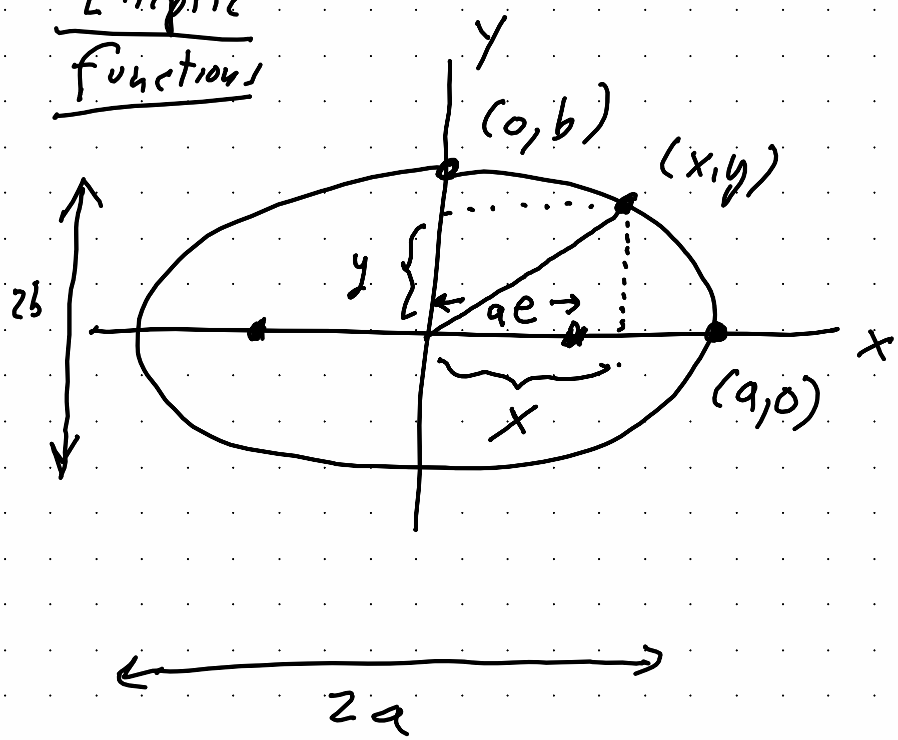
$\cancel{2}x\,dx + \cancel{2}y\,dy = 0 \rightarrow dx = -\dfrac{y}{x}dy$ $\quad\Big|\quad = \dfrac{1}{\sqrt{\dfrac{x^2+y^2}{x^2}}} = \dfrac{x}{a} = \boxed{\cos\theta}$

$$\frac{d(\sin\theta)}{d\theta} = \cos\theta \longrightarrow \int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta$$

$$x = \sin\theta$$
$$\cos\theta = \sqrt{1-\sin^2\theta}$$
$$= \sqrt{1-x^2}$$

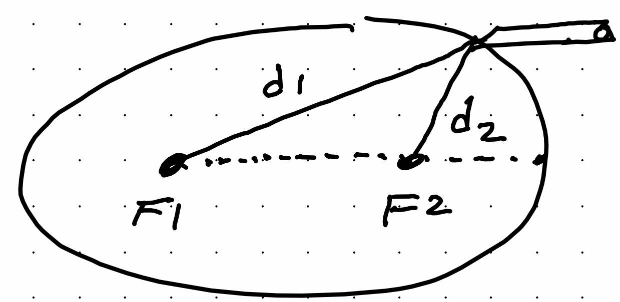
$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \theta = \sin^{-1}(x)}$$

Elliptic functions

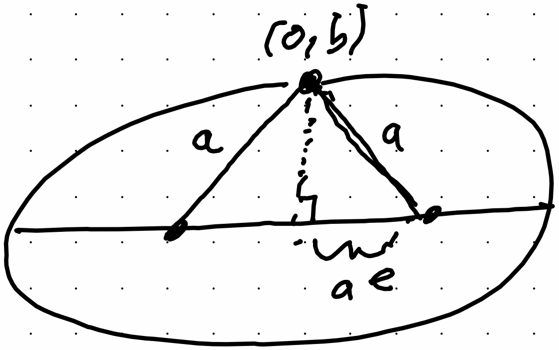


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

eccentricity : $e = 0$ (for circle)



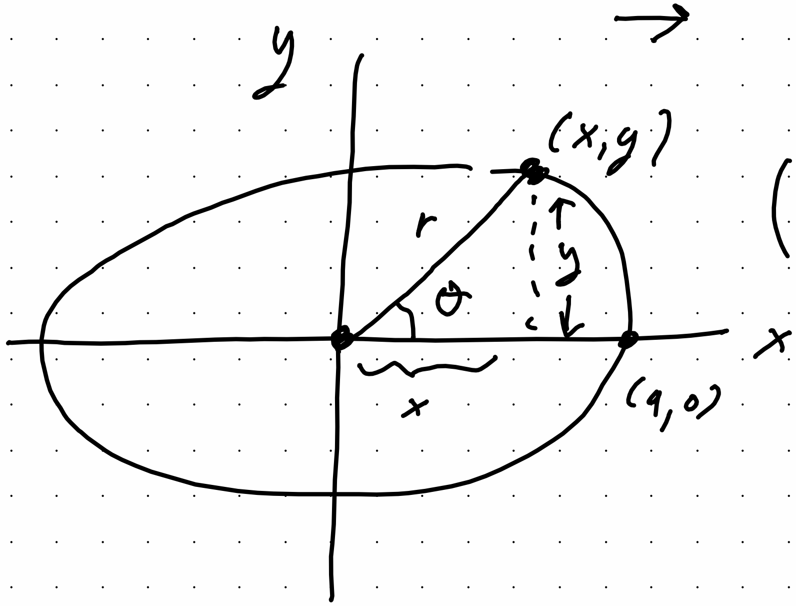
$$d_1 + d_2 = 2a$$



$$(ae)^2 + b^2 = a^2$$
$$a^2e^2 + b^2 = a^2$$
$$e^2 = \frac{a^2 - b^2}{a^2} = 1 - \left(\frac{b}{a}\right)^2$$

$$\rightarrow \quad e = \sqrt{1 - \left(\frac{b}{a}\right)^2} \equiv k$$

notation used in elliptic function



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$x^2 + y^2 = r^2$$
$$x = r\cos\theta, \quad y = r\sin\theta$$

<u>Define:</u> $\mathrm{cn}(u;k) \equiv \frac{x}{a}$, $\quad \mathrm{sn}(u;k) \equiv \frac{y}{b}$

$\mathrm{dn}(u;k) \equiv \frac{r}{a}$ — (=1 for a circle)

where $u = \frac{1}{b}\int_0^\theta r\,d\theta$ — $y = r\sin\theta$

(= θ for a circle)

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\theta^2}$$

Given: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, $x^2 + y^2 = r^2$

$dn(u;k) = \frac{r}{a}$

Follow: (i) $cn^2(u;k) + sn^2(u;k) = 1$

(ii) $dn^2(u;k) + k^2 sn^2(u;k) = 1$

$u = \frac{1}{b}\int_0^\theta r\,d\theta$

(iii) $\frac{d}{du} sn(u;k) = cn(u;k)\,dn(u;k)$

$\frac{d}{du} cn(u;k) = -sn(u;k)\,dn(u;k)$

$\frac{d}{du} dn(u;k) = -k^2 sn(u;k)\,cn(u;k)$

Integrate: $\frac{d\,sn(u;k)}{du} = cn(u;k)\,dn(u;k)$

$$\int \frac{d\,sn(u;k)}{cn(u;k)\,dn(u;k)} = \int du = u$$

Analogous to

$\frac{d\sin\theta}{d\theta} = \cos\theta$

$\frac{d\sin\theta}{\cos\theta} = d\theta$

$x = \sin\theta$

$\int \frac{dx}{\sqrt{1-x^2}} = \int d\theta = \theta = \sin^{-1} x$

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}} = u + \text{const} = sn^{-1}(x;k) + \text{const} \qquad x \equiv sn(u;k)}$$

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + \text{const}}$$

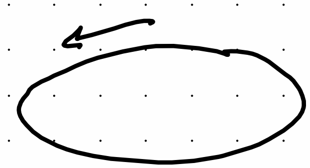
$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \equiv K(k) \longrightarrow$$

related to Period of a pendulum going beyond small-angle approximation

(complete elliptic integral of 1^{st} kind)

$$\int_0^1 \frac{\sqrt{1-k^2x^2}\; dx}{\sqrt{1-x^2}} \equiv E(k) \longrightarrow$$

circumference around an ellipse



circle: $C = 2\pi a$

(complete elliptic integral of 2^{nd} kind)