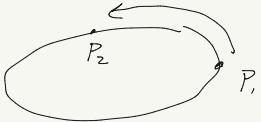


Lecture #1: Aug 24<sup>th</sup>

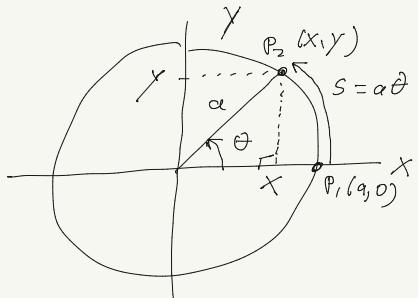
## Elliptic Functions / integrals :

i) circumference of an ellipse

ii) period of a simple pendulum beyond the small-angle approximation



Circular functions: sines, cosines,



$$ds^2 = dx^2 + dy^2$$

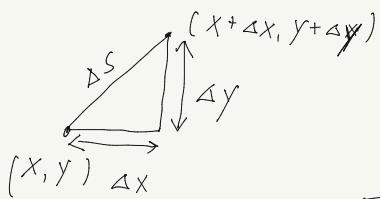
$a = \text{radius}$

$$x^2 + y^2 = a^2$$

$$\theta = \frac{s}{a}$$

$$= \frac{1}{a} \int \sqrt{dx^2 + dy^2}$$

$$= \int d\theta$$



$$x = a \cos \theta$$

$$y = a \sin \theta$$

$$\begin{aligned} \sin \theta &= y/a \\ \cos \theta &= x/a \end{aligned} \quad \left. \begin{aligned} &\text{defn. of } \sin, \cos, \\ &\text{defn. of } \theta \end{aligned} \right\}$$

$$\boxed{x^2 + y^2 = a^2} \rightarrow \boxed{\frac{d(\cos^2 \theta + \sin^2 \theta)}{d\theta} = 0}$$

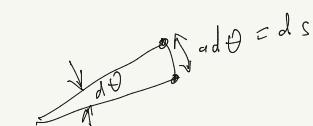
$$\boxed{\frac{d}{d\theta} \sin \theta = \cos \theta}$$

$$\text{Proof: } \frac{d}{d\theta} \sin \theta = \frac{d}{d\theta} \left( \frac{y}{a} \right)$$

$$= \frac{1}{a} \frac{dy}{d\theta}$$

$$= \frac{dy}{\sqrt{dx^2 + dy^2}}$$

$$= \frac{1}{\sqrt{dx^2 + dy^2}} = \sqrt{\left( \frac{dx}{dy} \right)^2 + 1}$$



$$d\theta = \sqrt{dx^2 + dy^2}$$

$$x^2 + y^2 = a^2 \Rightarrow 2x dx + 2y dy = 0$$

$$\frac{dx}{dy} = -\frac{y}{x}$$

$$\frac{d}{d\theta} \sin\theta = \frac{1}{\sqrt{(-\frac{y}{x})^2 + 1}} = \frac{1}{\sqrt{\frac{x^2}{x^2} + 1}} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{a} = \cos\theta$$

$$\boxed{\frac{d}{d\theta} \sin\theta = \cos\theta}$$

Similarly  $\boxed{\frac{d \cos\theta}{d\theta} = -\sin\theta}$

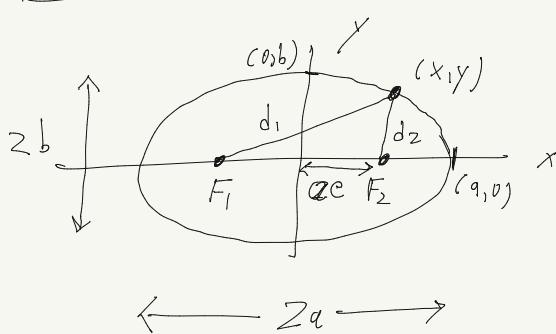
$$\int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta = \theta + \text{const}$$

$$t = \sin\theta$$

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - t^2}$$

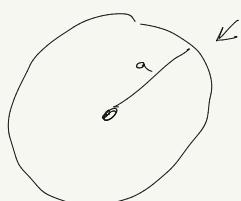
$$\left| \begin{array}{l} \int \frac{dt}{\sqrt{1-t^2}} = \theta + \text{const} \\ = \sin^{-1}t + \text{const} \end{array} \right.$$

Lec #2: Avg 26<sup>th</sup>



$$d_1 + d_2 = 2a \quad x^2 + y^2 = a^2$$

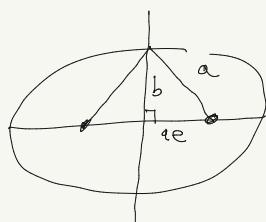
$$e \neq \frac{b}{a}$$



$$e \neq 1 - \frac{b}{a}$$

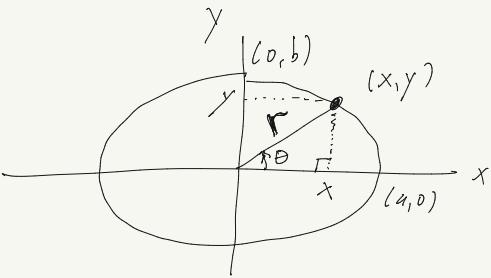
$$e^2 = \frac{b^2}{1-a^2}$$

$$\boxed{c = \sqrt{1 - \left(\frac{b}{a}\right)^2}}$$



Pf:

$$\begin{aligned} a^2 &= b^2 + (ae)^2 \\ a^2/(1-e^2) &= b^2 \\ 1-e^2 &= \left(\frac{b}{a}\right)^2 \\ c &= \sqrt{1 - \left(\frac{b}{a}\right)^2} \end{aligned}$$



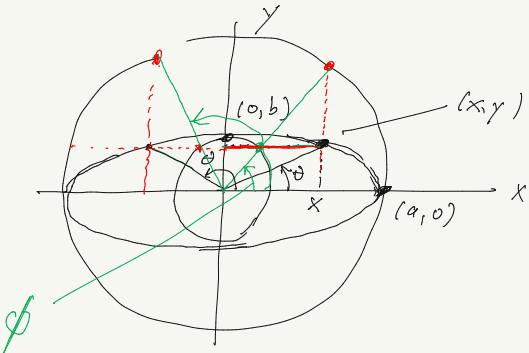
$$x = r \cos \theta$$

$$y = r \sin \theta$$

↑  
r changes

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\begin{cases} x = a \cos \phi \\ y = b \sin \phi \end{cases}$$



$$0 < e < 1$$

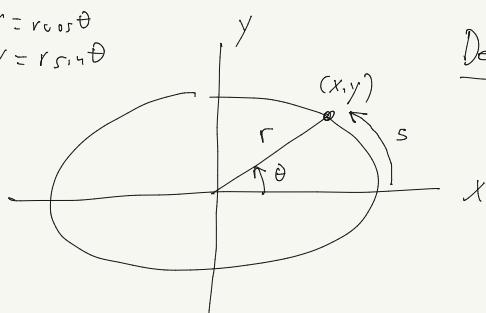
$e = 0$  circle

$e = 1$  parabola

$e > 1$  hyperbola

$$X = r \cos \theta$$

$$Y = r \sin \theta$$



Def'n:

$$\begin{cases} \operatorname{cn}(u; k) = x/a \\ \operatorname{sn}(u; k) = y/b \\ \operatorname{dn}(u; k) = r/a \end{cases}$$

$$u \equiv \frac{1}{b} \int_0^\theta r d\theta$$

↑  
not  $\theta$ , not arc length

$k = e$ ,  $0 < k < 1$   
Modulus

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{dr^2 + r^2 d\theta^2} \end{aligned}$$

$$\oint u = \int_0^\theta r d\theta < s$$

$$\begin{aligned} x &= r \cos \theta, y = r \sin \theta \\ dx &= dr \cos \theta - r \sin \theta d\theta \\ dy &= dr \sin \theta + r \cos \theta d\theta \end{aligned}$$

$\operatorname{cn}(u)$	periodic
$\operatorname{sn}(u)$	
$k = \sin\left(\frac{\phi_0}{2}\right)$	

Property:

$$\boxed{C_n^2 u + S_n^2 u = 1} \quad \leftarrow \quad \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1$$

$$\leftarrow x^2 + y^2 = r^2$$

$$\begin{aligned} d_n^2 u &= C_n^2 u + \left( \frac{b}{a} \right)^2 S_n^2 u \\ &= 1 - S_n^2 u + \left( \frac{b}{a} \right)^2 S_n^2 u \\ &= 1 - S_n^2 u \left( 1 - \left( \frac{b}{a} \right)^2 \right) \\ &= 1 - H^2 S_n^2 u \end{aligned}$$

$$\boxed{d_n^2 u + H^2 S_n^2 u = 1}$$

$$\begin{aligned} \frac{d}{du} S_n u &= \frac{1}{b} \frac{dy}{du} \\ &= \frac{dy}{r d\theta} \end{aligned}$$

$$u = \left( \frac{1}{b} \right) \int_0^\theta (r d\theta)$$

$$du = \frac{r d\theta}{b} \rightarrow b du = r d\theta$$

~~~~~

$$x = r \cos \theta, y = r \sin \theta$$

$$\begin{aligned} dx &= dr \cos \theta - r \sin \theta d\theta & \rightarrow \sin \theta dx = \sin \theta \cos \theta dr - r \sin^2 \theta d\theta \\ dy &= dr \sin \theta + r \cos \theta d\theta & \rightarrow -\cos \theta dy = -\cos \theta \sin \theta dr - r \cos^2 \theta d\theta \end{aligned}$$

$$\text{add } \sin \theta dx - \cos \theta dy = -r d\theta$$

$$\frac{y}{r} dx - \frac{x}{r} dy = -r d\theta$$

$$\rightarrow \boxed{r d\theta = \frac{-y}{r} dx + \frac{x}{r} dy}$$

$$\frac{d}{du} S_n u = \frac{dy}{r dx + x dy} = \frac{r}{-y dx + x dy}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \frac{\partial}{\partial x} \frac{x}{a^2} + \frac{\partial}{\partial y} \frac{y}{b^2} = 0$$

$$\frac{\partial x}{\partial y} = -\frac{y}{x} \frac{a^2}{b^2}$$

$$\begin{aligned} \frac{d \sin u}{du} &= \frac{r}{y \left(\frac{-y}{x}\right) \frac{a^2}{b^2} + x} = \frac{r}{y^2 \left(\frac{a}{b}\right)^2 + x^2} \\ &= \frac{r}{a^2} \frac{x}{\left(\frac{y}{b}\right)^2 + \left(\frac{x}{a}\right)^2} \\ &= \frac{du \cdot \cos u}{1} \\ \boxed{\frac{d}{du} \sin u = \cos u \cdot du} \end{aligned}$$

$$\begin{cases} \frac{d}{du} \boxed{\cos u} = -\sin u \cdot du \\ \frac{d}{du} \cos u = -H^2 \sin u \cdot \cos u \end{cases}$$

$$\boxed{\frac{d}{du} \sin u = \cos u \cdot du}$$

Integrate!

$$\sin^2 u + \cos^2 u = 1$$

$$du^2 + H^2 \sin^2 u = 1$$

$$\int \frac{d(\cos \theta)}{\cos \theta} = \int d\theta = \theta$$

$$t = \sin^{-1} \theta \quad \theta = \sin^{-1} t$$

$$\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t$$

$$\int \frac{d(\sin u)}{\cos u \cdot du} = \int du = u + \text{const}$$

$$t = \sin u \quad \boxed{\int \frac{dt}{\sqrt{1-t^2}} \frac{1}{\sqrt{1-H^2 t^2}} = \sin^{-1}(t; H) + \text{const}}$$

$$F(\phi, k) = \int_0^{\sin \phi} \frac{dt}{\sqrt{1-t^2} \sqrt{1-k^2 t^2}}$$

incomplete elliptic integral of 1st kind (angular dependence, period of a simple pendulum)

$$E(\phi, k) = \int_0^{\sin \phi} \frac{dt}{\sqrt{1-t^2} \sqrt{1-k^2 t^2}}$$

incomplete elliptic integral of 2nd kind (arc length along ellipse)

$$\phi = \frac{\pi}{2}$$

$$\int_0^1 \frac{dt}{\sqrt{1-t^2} \sqrt{1-k^2 t^2}} = K(k)$$

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} \sqrt{1-k^2 t^2} = E(k)$$

$\begin{cases} \sin u = \sin \phi \\ \frac{y}{l} = \sin \phi \\ \cos u = \cos \phi \end{cases}$

$\begin{cases} \frac{x}{a} = \cos \phi \end{cases}$

complete elliptic integrals of 1st and 2nd kind

Lec #3: Aug 31<sup>st</sup>

Simple pendulum: ("Freshman physics")

Free-body diagram:

$a_c = \frac{v^2}{r} = \omega^2 r$

$a_t = \alpha r$

$\alpha = \ddot{\phi}, \omega = \dot{\phi}$

Free-body diagram of a simple pendulum bob of mass  $m$  at angle  $\phi$  from the vertical. The forces shown are tension  $T$ , weight  $mg$ , and normal force  $N$ . The components of weight are  $mg \sin \phi$  perpendicular to the string and  $mg \cos \phi$  tangent to the string. The components of tension are  $T \cos \phi$  perpendicular to the string and  $T \sin \phi$  tangent to the string.

1)  $T - mg \cos \phi = m \ddot{\phi} l$

$$T = m \dot{\phi}^2 l + mg \cos \phi$$

2)  $m g \sin \phi = -m \alpha_{\text{tangential}}$

$$\ddot{\phi} = -\frac{g}{l} \sin \phi$$

EOM

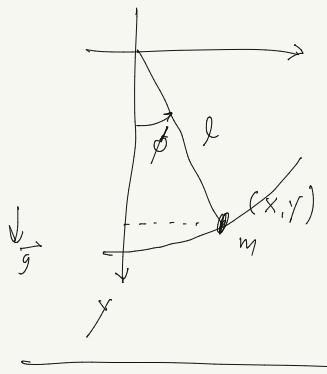
$$\ddot{\phi} = -\frac{g}{l} \sin \phi \quad (\text{hard to solve; 2nd order nonlinear ODE})$$

Small-angle approx:  $\phi \ll 1 \text{ rad} \approx 57^\circ$  ( $\pi \text{ radians} = 180^\circ$ )

$$\Rightarrow \ddot{\phi} = -\frac{g}{l} \phi \quad \omega_0 = \sqrt{\frac{g}{l}}$$

$$\begin{aligned}\phi(t) &= A e^{-i \sqrt{\frac{g}{l}} t} \\ &= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \\ &= a \cos(\omega_0 t + \alpha)\end{aligned}$$

Lagrangian formalism:



$$\begin{aligned}(x, y) &\quad x = r \cos \phi \\ (r, \phi) &\quad y = r \sin \phi \\ T &= \frac{1}{2} m(r^2 \dot{\phi}^2 + r^2 \dot{\phi}^2) \end{aligned}$$

$$L = T - U$$

$$\begin{aligned}U &= -m g y \\ &= -m g l \cos \phi \\ T &= \frac{1}{2} m (x^2 + y^2) \\ &= \frac{1}{2} m l^2 \dot{\phi}^2 \\ L &= \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \cos \phi \quad \boxed{\text{Euler}} \end{aligned}$$

Lagrange's equation

$$\begin{aligned}\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) &= \frac{\partial L}{\partial \phi} \\ \frac{d}{dt} (m l^2 \dot{\phi}) &= -m g l \sin \phi \\ m l^2 \ddot{\phi} &= -m g l \sin \phi \\ \ddot{\phi} &= -\frac{g}{l} \sin \phi\end{aligned}$$

$L$  does not depend explicitly on time  $t \Rightarrow E$  is conserved.

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$$

$$= T + U$$

Example:

$$\boxed{E} = \frac{\partial L}{\partial \dot{\phi}} \phi - L$$

$$= ml^2 \dot{\phi} \dot{\phi} - \left( \frac{1}{2} ml^2 \dot{\phi}^2 + mgl \cos \phi \right)$$

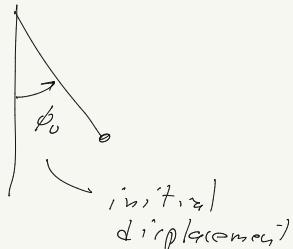
$$= \frac{1}{2} ml^2 \dot{\phi}^2 - mgl \cos \phi$$

$$= T + U$$

$$E = -mgl \cos \phi_0$$

$$-mgl \cos \phi_0 = \frac{1}{2} l^2 \dot{\phi}^2 - mgl \cos \phi$$

$$\dot{\phi}^2 = \frac{2}{l^2} (mgl \cos \phi - mgl \cos \phi_0)$$



$$\dot{\phi}^2 = 2 \frac{g}{l} (\cos \phi - \cos \phi_0)$$

$$\frac{d\phi}{dt} = \dot{\phi} = \pm \sqrt{2} \frac{g}{l} \sqrt{\cos \phi - \cos \phi_0}$$

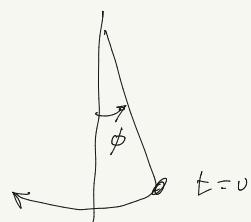
$$\int_{\phi_0}^{\phi} dt = \int_{\phi_0}^{\phi} \frac{-d\phi}{\sqrt{2} \omega_0 \sqrt{\cos \phi - \cos \phi_0}}$$

$$\omega_0 t + \text{const.} = \pm \int \frac{d\phi}{\sqrt{2} \sqrt{\cos \phi - \cos \phi_0}}$$

$L \& L II. 1$

$$\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t + \text{const.}$$

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} =$$



$$\cos \phi = \cos \left( 2 \cdot \frac{\phi}{2} \right) = \cos^2 \left( \frac{\phi}{2} \right) - \sin^2 \left( \frac{\phi}{2} \right) = 1 - 2 \sin^2 \left( \frac{\phi}{2} \right)$$

$$\cos \phi_0 = 1 - 2 \sin^2 \left( \frac{\phi_0}{2} \right)$$

$$\Rightarrow \sqrt{\cos \phi - \cos \phi_0} = \sqrt{2} \sqrt{\sin^2 \left( \frac{\phi_0}{2} \right) - \sin^2 \left( \frac{\phi}{2} \right)} = \sqrt{2} \left| \sin \left( \frac{\phi_0}{2} \right) \right| \sqrt{1 - \frac{\sin^2 \left( \frac{\phi_0}{2} \right)}{\sin^2 \left( \frac{\phi}{2} \right)}}$$

$$\begin{aligned}
 k &= |\sin\left(\frac{\phi_0}{2}\right)| \quad (0 < \kappa < 1) \\
 x &= \frac{\sin\left(\frac{\phi}{2}\right)}{\kappa} = \frac{\sin\left(\frac{\phi}{2}\right)}{\kappa} \rightarrow dx = \frac{1}{\pi} \frac{1}{2} \cos\left(\frac{\phi}{2}\right) d\phi \\
 \sqrt{2 \sqrt{\cos\phi - \cos\phi_0}} &= 2\kappa \sqrt{1-x^2} \quad \left. \begin{array}{l} \downarrow \\ d\phi = \frac{2\kappa dx}{\sqrt{1-\kappa^2 x^2}} \end{array} \right\} \\
 \end{aligned}$$

$$\begin{aligned}
 \omega_0 t + \text{const} &= \pm \int \frac{2\kappa dx}{\sqrt{1-\kappa^2 x^2}} \xrightarrow{-2\kappa \sqrt{1-x^2}} \\
 &\stackrel{*}{=} \int_0^x \frac{dx}{\sqrt{1-x^2} \sqrt{1-\kappa^2 x^2}} \\
 &= \sin^{-1}(x; \kappa) \quad \left. \begin{array}{l} x = \sin\left(\frac{\phi}{2}\right) \\ \kappa = |\sin\left(\frac{\phi_0}{2}\right)| \end{array} \right\}
 \end{aligned}$$

$$t = 0 \iff \phi = \phi_0$$

$$\text{const} = \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-\kappa^2 x^2}} \underset{\sim}{=} K(\kappa)$$

complete elliptic integral of the first kind

$$\omega_0 t + K(\kappa) \underset{\sim}{=} \sin^{-1}\left(\frac{\sin\left(\frac{\phi_0}{2}\right)}{\kappa}; \kappa\right)$$

$$\sin(\omega_0 t + K(\kappa); \kappa) = \frac{1}{\kappa} \sin\left(\frac{\phi}{2}\right)$$

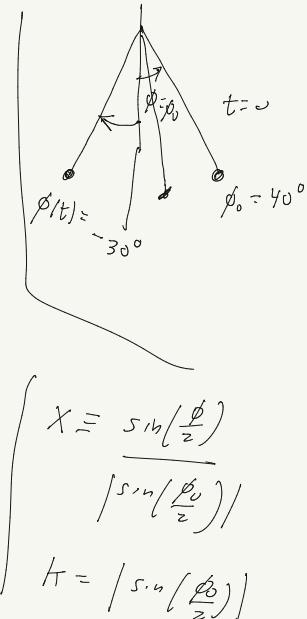
$$\rightarrow \boxed{\phi(t) = 2 \arcsin\left(\kappa \sin(\omega_0 t + K(\kappa); \kappa)\right)}$$

$$P = \frac{4}{\omega_0} K(\kappa) \rightarrow \omega_0 \frac{P}{4} \approx K(\kappa)$$

Lec #4: 2 Sep 2021

$$\omega_0 \int_0^t dt = - \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\Theta}}$$

$$\begin{aligned}\omega_0 t &= - \left[ \int_{\phi_0}^0 + \int_0^{\phi} \right] \frac{d\phi}{\sqrt{\Theta}} \\ &= + \int_0^{\phi_0} \frac{d\phi}{\sqrt{\Theta}} - \int_0^{\phi} \frac{d\phi}{\sqrt{\Theta}} \\ &= \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-\tilde{H}^2 x^2}} - \int_0^x \frac{dx}{\sqrt{1-x^2} \sqrt{1-\tilde{H}^2 x^2}} \\ &= \tilde{K}(\tilde{H}) - \sin^{-1}(x; \tilde{H})\end{aligned}$$



$$\begin{aligned}\sin^{-1}(x; \tilde{H}) &= \tilde{K}(\tilde{H}) - \omega_0 t \\ \frac{\sin(\frac{\phi}{2})}{\tilde{H}} &= x = \sin(\tilde{K}(\tilde{H}) - \omega_0 t; \tilde{H}) = \sin(\omega_0 t; \tilde{H}) \quad \omega_0 = \sqrt{\frac{g}{x}} \\ \rightarrow \boxed{\phi(t) = 2 \arcsin(\tilde{H} \sin(\tilde{K}(\tilde{H}) - \omega_0 t; \tilde{H}))}\end{aligned}$$

$$\begin{aligned}\frac{1}{\sqrt{1-\tilde{H}^2 x^2}} &\approx 1 + \frac{1}{2} \tilde{H}^2 x^2 \\ (1+\epsilon)^p &\approx 1 + p\epsilon\end{aligned}$$

$$\boxed{\omega_0 \frac{P}{4} = \tilde{K}(\tilde{H})} \rightarrow \boxed{P = \frac{4}{\omega_0} \tilde{K}(\tilde{H})}$$

$$\begin{aligned}\tilde{K}(\tilde{H}) &= \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-\tilde{H}^2 x^2}} \\ \underline{\underline{\tilde{H}=0}}: \quad \tilde{K}(0) &= \int_0^1 \frac{dx}{\sqrt{1-x^2}} \approx \sin^{-1} 1 = \frac{\pi}{2} \\ P &= \frac{4}{\omega_0} \frac{\pi}{2} = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{g}{x}}\end{aligned}$$

$$0 < \tilde{H} < 1 \quad \tilde{K}(\tilde{H}) = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \left( 1 + \frac{1}{2} \tilde{H}^2 x^2 \right) = \frac{\pi}{2} + \int_0^1 \frac{dx}{\sqrt{1-x^2}} \frac{1}{2} \tilde{H}^2 x^2$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

Pf:  $\sin\left(\frac{\pi}{2} - \theta\right) = \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} \cos \theta - \cos\left(\frac{\pi}{2}\right) \sin \theta$

$$= \cos \theta$$

$$\sin(-\theta) = -\sin \theta, \quad \cos(-\theta) = \cos \theta$$

$\sin(\pi - x; \pi) = \sin(x)$

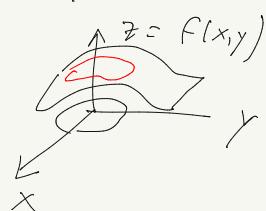
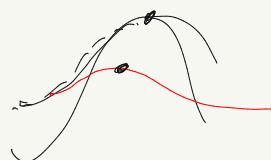
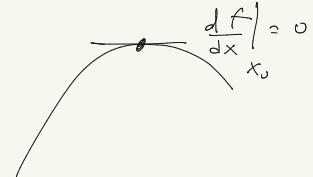
Lagrange multipliers

$$f(x) : \max \text{ or } \min ?$$

$$f(x, y) :$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$



$f(x, y)$ : Find extreme value subject to constraint  $\varphi(x, y) = 0$

1) "reduced space method":

$$\text{solve constraint } \varphi(x, y) = 0 \rightarrow y = g(x)$$

$$F(x) = f(x, y)$$

2) "method of Lagrange multipliers"

$$F(x, y, \lambda) = f(x, y) + \lambda \varphi(x, y)$$

$$\text{e.g., } x^2 + y^2 = 1$$

$$2x dx + 2y dy = 0$$

$$dx = \frac{-y}{x} dy$$

$$\begin{cases} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial \lambda} = 0 \end{cases}$$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0$$

$$F(x, y, \lambda) = f(x, y) + \lambda \varphi(x, y)$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0$$

$$\frac{\partial F}{\partial \lambda} = \varphi(x, y) = 0$$

$$\varphi(r, \phi) = 0 = r - \ell$$

$$L'(r, \dot{\phi}, r, \dot{\phi}, t) + \lambda(r - \ell) = L'$$

$$\frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{q}} \right) = \frac{\partial L'}{\partial q}$$

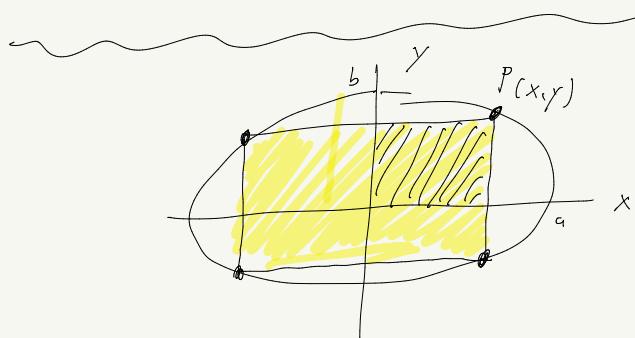
$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q} + \lambda \frac{\partial \varphi}{\partial q}$$

$$\frac{dP}{dt} = -\frac{\partial U}{\partial q} + \lambda \frac{\partial \varphi}{\partial q} = F_{pp} + F_{constraint}$$

Lec #5: 9/7

1) Lagrange multiplier

2) Example: 2-d oscillating orbit



Maximize the area of a rectangle whose corners lie on the ellipse

$$\rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$A(x, y) = 4xy$$

$$\varphi(x, y) = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 0$$

$$F(x, y, \lambda) = 4xy + \lambda \varphi(x, y)$$

$$\boxed{\begin{aligned} \frac{\partial F}{\partial x} &= 4y + \lambda \left(-\frac{2x}{a^2}\right) = 0 \\ \frac{\partial F}{\partial y} &= 4x + \lambda \left(-\frac{2y}{b^2}\right) = 0 \\ \frac{\partial F}{\partial \lambda} &= 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 0 \end{aligned}}$$

$$\begin{aligned}
 \frac{\partial F}{\partial x} &= 4x + \lambda \left( -\frac{2x}{a^2} \right) = 0 & \rightarrow 4y^2 a^2 - 2\lambda xy &= 0 \quad \leftarrow \\
 \frac{\partial F}{\partial y} &= 4y + \lambda \left( -\frac{2y}{b^2} \right) = 0 & \rightarrow 4x^2 b^2 - 2\lambda yx &= 0 \\
 \frac{\partial F}{\partial \lambda} &= 1 - \left( \frac{x}{a} \right)^2 - \left( \frac{y}{b} \right)^2 = 0 \quad \leftarrow & \text{subtract} \\
 && 4y^2 a^2 - 4x^2 b^2 &= 0 \\
 && x^2 a^2 = x^2 b^2 & \\
 &\Rightarrow \boxed{\frac{x}{b} = \pm \frac{y}{a}} & = & \\
 0 &= 1 - \left( \frac{x}{a} \right)^2 - \left( \frac{y}{a} \right)^2 & & \\
 &= 1 - 2 \left( \frac{x}{a} \right)^2 & & \\
 \frac{x}{a} &= \frac{1}{\sqrt{2}} \quad \rightarrow \boxed{x = \frac{a}{\sqrt{2}}} & , \boxed{y = \frac{b}{\sqrt{2}}} & \rightarrow A_{max} = 4xy \\
 && & = 4 \frac{a}{\sqrt{2}} \frac{b}{\sqrt{2}} \\
 && & = \boxed{2ab}
 \end{aligned}$$

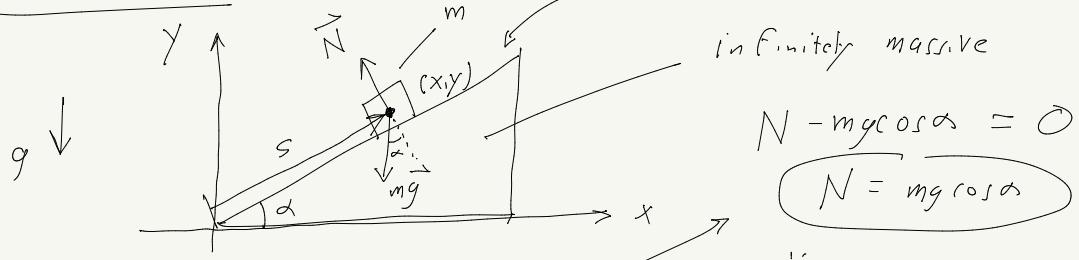
- Reduced space method

$$\begin{aligned}
 F(x) &= 4xy \quad | \quad \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1 \\
 &\quad | \quad y = b \sqrt{1 - \left( \frac{x}{a} \right)^2} \quad | \quad y = b \sqrt{1 - \left( \frac{x}{a} \right)^2} \\
 &= 4b \times \sqrt{1 - \left( \frac{x}{a} \right)^2}
 \end{aligned}$$

Maximize:

$$\begin{aligned}
 0 = F'(x) &= 4b \sqrt{1 - \left( \frac{x}{a} \right)^2} + 4b \times \left( \frac{1}{a} \right) \frac{1}{\sqrt{1 - \left( \frac{x}{a} \right)^2}} \left( -\frac{2x}{a^2} \right) \\
 &= 4b \left( \sqrt{1 - \left( \frac{x}{a} \right)^2} - \left( \frac{x}{a} \right)^2 \frac{1}{\sqrt{1 - \left( \frac{x}{a} \right)^2}} \right) \\
 &= \frac{4b}{\sqrt{1 - \left( \frac{x}{a} \right)^2}} \left( 1 - \left( \frac{x}{a} \right)^2 - \left( \frac{x}{a} \right)^2 \right) \\
 &= \frac{4b}{\sqrt{1 - \left( \frac{x}{a} \right)^2}} \left( 1 - 2 \left( \frac{x}{a} \right)^2 \right) \quad \rightarrow \quad \boxed{\left( \frac{x}{a} \right)^2 = \frac{1}{2}} \\
 &\quad | \quad x = \pm \frac{a}{\sqrt{2}}
 \end{aligned}$$

Mechanics problem:



$$N - mg \cos \alpha = 0$$
$$N = mg \cos \alpha$$

Freshman physics analysis:

$$m \ddot{s} = -mg \sin \alpha$$
$$\ddot{s} = -g \sin \alpha$$

Lagrangian analysis:

$$x = s \cos \alpha, y = s \sin \alpha$$
$$\frac{y}{x} = \tan \alpha$$

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$
$$y = x \tan \alpha$$

$$U = mg y$$

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mg y$$

$$\varphi(x, y) = y - x \tan \alpha = 0$$

$$L' = L + \lambda \varphi$$
$$F = A + \lambda \varphi$$

$$= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mg y + \lambda(y - x \tan \alpha)$$

$$\frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{x}} \right) = \frac{\partial L'}{\partial x} \rightarrow \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} \right)$$

$$\frac{d p_x}{dt} = F_x + \lambda \frac{\partial \varphi}{\partial x}$$
$$\frac{d p_y}{dt} = F_y + \lambda \frac{\partial \varphi}{\partial y}$$
$$\frac{d \vec{p}}{dt} = \vec{F} + \lambda \vec{\nabla} \varphi$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) = \frac{\partial L}{\partial y} + \lambda \frac{\partial \varphi}{\partial y}$$

$$\varphi(x, y) = y - x \tan \alpha = 0$$

$$\frac{d}{dt}(m \dot{x}) = 0 + \lambda(-\tan \alpha) \rightarrow m \ddot{x} = -\lambda \tan \alpha$$

$$\ddot{x} = -\frac{\lambda}{m} \tan \alpha$$

$$\frac{d}{dt}(m \dot{y}) = -mg + \lambda \rightarrow m \ddot{y} = -g + \frac{\lambda}{m}$$

$$y - x \tan \alpha = 0$$

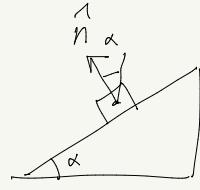
$$\ddot{y} - \dot{x} \tan \alpha = 0 \rightarrow \ddot{y} = \dot{x} \tan \alpha$$

$$\left( -g + \frac{\lambda}{m} \right) = -\frac{\lambda}{m} \tan \alpha \cdot \tan \alpha$$

$$\phi = y - x \tan \alpha$$

$$\frac{\lambda}{m} (1 + \tan^2 \alpha) = g$$

$\sec^2 \alpha = \frac{1}{\cos^2 \alpha}$



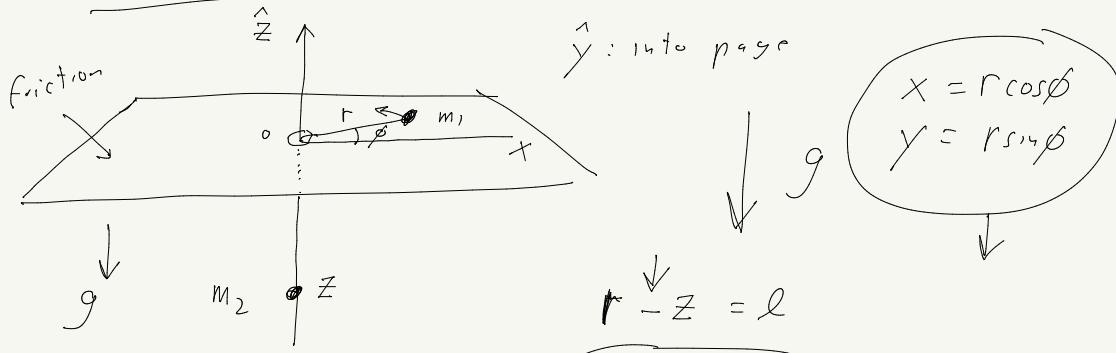
$$\frac{\lambda}{m} = g \cos^2 \alpha$$

$\boxed{\lambda = mg \cos^2 \alpha}$

$$\begin{aligned} \vec{F}_c &= \lambda \vec{\nabla} \phi \\ &= \lambda (\hat{y} - \hat{x} \tan \alpha) \end{aligned} \quad \text{unit} \quad \left| \begin{aligned} \vec{F}_c &= \lambda \left( \hat{y} - \hat{x} \frac{\sin \alpha}{\cos \alpha} \right) \\ &= \frac{\lambda}{\cos \alpha} \left( \cos \hat{y} - \hat{x} \sin \alpha \right) \\ &= \frac{\lambda}{\cos \alpha} \hat{n} = \boxed{mg \cos \alpha \hat{n}} \end{aligned} \right.$$

(2.) 2-d Example:

string length  $\ell$



$$\begin{aligned} T &= \frac{1}{2} m_1 (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_2 \dot{z}^2 \\ &= \frac{1}{2} m_1 (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2 \end{aligned}$$

$$= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2$$

$$U = m_2 g z = m_2 g (r - \ell) = m_2 g r \quad \text{ignore } \ell \text{ (const)}$$

$$\boxed{L = T - U}$$

$$= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r$$

$$(r, \phi) \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \cancel{\left( \frac{\partial L}{\partial \phi} \right)} = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = \text{const} = \boxed{m_1 r^2 \dot{\phi} = M_z}$$

No explicit time-dependence:

$$E = \underbrace{\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i}_{\text{const}} - L = \overbrace{T + U}$$

$$E = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 + m_2 g r \quad \left. \right\} \text{const}$$

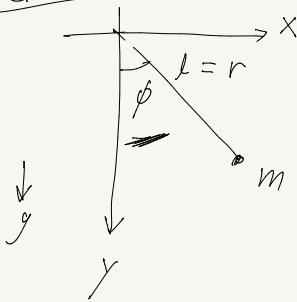
$$M_z = m_1 r^2 \dot{\phi} \rightarrow \boxed{\dot{\phi} = \frac{M_z}{m_1 r^2}}$$

$$\begin{aligned} \rightarrow E &= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \left( \frac{M_z}{m_1 r^2} \right)^2 + m_2 g r \\ &= \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \underbrace{\frac{M_z^2}{2 m_1 r^2}}_{U_{eff}(r)} + m_2 g r \end{aligned}$$

Lec #6:

Sept. 9<sup>th</sup>

Quiz #1



Calculate tension in the string  
using the method of  
Lagrange multipliers.

joseph.d.romano@ttu.edu

constraint:  $\varphi = \ell - r = 0$        $U = -mgy$   
 $\underline{=}$        $= -mgyr\cos\varphi$

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}m(r^2 + r^2\dot{\varphi}^2) + mgyr\cos\varphi \end{aligned}$$

$\boxed{\text{Engg. Nirajay}} \quad \boxed{\text{Muhammad}}$

$$\begin{aligned} (1) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) &= \frac{\partial L}{\partial r} + \lambda \frac{\partial \varphi}{\partial r} & \varphi = 0 \\ (2) \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}} \right) &= \frac{\partial L}{\partial \varphi} + \lambda \frac{\partial \varphi}{\partial \varphi} & \rightarrow r = \ell \quad (3) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}(mr) &= mr\dot{\varphi}^2 + mg\cos\varphi - \lambda \\ mr'' &= mr\dot{\varphi}^2 + mg\cos\varphi - \lambda \end{aligned} \quad (1) \quad ml^2\ddot{\varphi} = -mg\sin\varphi$$

$$\begin{aligned} \frac{d}{dt}(mr^2\dot{\varphi}) &= -mgyr\sin\varphi \\ 2mr\dot{r}\dot{\varphi} + mr^2\ddot{\varphi} &= -mgyr\sin\varphi \end{aligned} \quad (2) \quad \dot{\varphi} = \frac{-g\sin\varphi}{l}$$

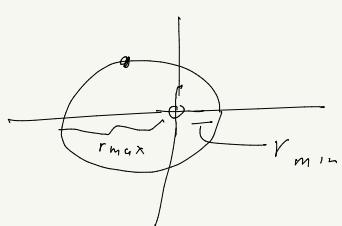
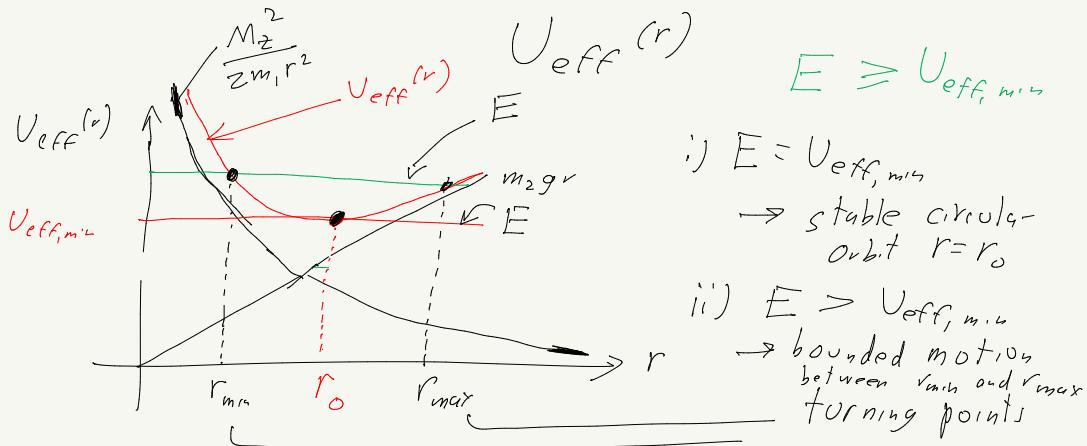
$$\boxed{r = \ell} \rightarrow \boxed{\dot{r} = 0}, \boxed{\ddot{r} = 0} \quad (3)$$

$$\begin{aligned} 0 &= ml\dot{\varphi}^2 + mg\cos\varphi - \lambda \\ \lambda &= ml\dot{\varphi}^2 + mg\cos\varphi \quad , \quad \boxed{\nabla\varphi = -\vec{r}} \\ \lambda - mg\cos\varphi &= ml\dot{\varphi}^2 \\ \boxed{\vec{F}_c = -(ml\dot{\varphi}^2 + mg\cos\varphi)\vec{r}} \end{aligned}$$

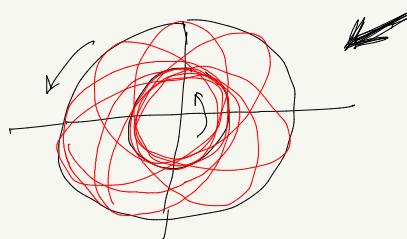
Revision: oscillating orbit example

$$E = \frac{1}{2}(m_1 + m_2) \dot{r}^2 + \frac{1}{2}m_1 r^2 \left( \frac{M_z}{m_1 r^2} \right)^2 + m_2 g r$$

$$= \frac{1}{2}(m_1 + m_2) \dot{r}^2 + \underbrace{\frac{M_z^2}{2m_1 r^2}}_{+ m_2 g r}$$



Closed bound orbit



bound orbit  
that's not closed

$r_0$ : minimum of the effective potential /

$$\circ = \frac{dU_{\text{eff}}}{dr} \Big|_{r=r_0} = \left( m_2 g - \frac{M_z^2}{m_1 r^3} \right) \Big|_{r=r_0}$$

$$\circ = m_2 g - \frac{M_z^2}{m_1 r_0^3}$$

$$\boxed{M_z^2 = m_1 m_2 g r_0^3}$$

$$E = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + U_{\text{eff}}(r),$$

$$E - U_{\text{eff}}(r) = \frac{1}{2} (m_1 + m_2) \dot{r}^2$$

$$\boxed{U_{\text{eff}}(r) = m_2 gr + \frac{M_z^2}{2m_1 r^2}}$$

$$\frac{dr}{dt} = \dot{r} = \pm \sqrt{\left(\frac{2}{m_1 + m_2}\right) (E - U_{\text{eff}}(r))}$$

$$\int dt = \pm \int \frac{dr}{\sqrt{\left(\frac{2}{m_1 + m_2}\right) (E - U_{\text{eff}}(r))}} \rightarrow t = t(r)$$

$$M_z = m_1 r^2 \dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{m_1 r^2}$$

$$\frac{dr}{dt} = \pm \sqrt{\textcircled{1}} \quad , \quad \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{M_z}{m_1 r^2} \frac{dr}{d\phi}$$

$$\frac{M_z}{m_1 r^2} \frac{dr}{d\phi} = \pm \sqrt{\textcircled{1}}$$

Orbit equations

$$\int \frac{M_z dr}{m_1 r^2 \sqrt{\textcircled{1}}} = \int \pm d\phi \rightarrow \phi = \phi(r)$$

$$r = r(\phi)$$

$$\boxed{\frac{dr}{dt} = \pm \sqrt{\textcircled{1}}}$$

$$\boxed{\frac{d\phi}{dt} = \frac{M_z}{m_1 r^2}}$$

$$\boxed{\Delta r = \pm \sqrt{\textcircled{1}} \Delta t}$$

$$\boxed{\Delta \phi = \frac{M_z}{m_1 r^2} \Delta t}$$

$$\textcircled{1} = \frac{2}{m_1 + m_2} (E - U_{\text{eff}}(r))$$

~~start~~ choose  $E, M_z, m_1, m_2$ , etc.

start system off at some values of  $r$  and  $\phi$  at  $t=0$

stop 1:  ~~$r_1, \phi_1$~~   
stop 2:  $r_2 = r_1 + \Delta r$ ,  $\phi_2 = \phi_1 + \Delta \phi$   
 (repeat)

Lec #7:

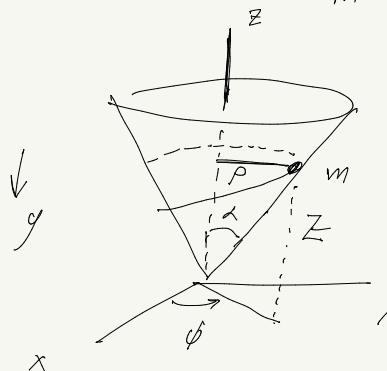
Sep 14<sup>th</sup>

Lec #8:

Sep 16<sup>th</sup>

Q #1:

Determine constraint force vary  
method of Lagrange multiplier



$$\phi(r, \theta, z) = r - z \tan \alpha = 0$$

cylindrical coord.  $(r, \theta, z)$

$$z \begin{cases} r \\ \theta \\ \alpha \end{cases} \quad \boxed{r \tan \alpha = \frac{p}{z}} \rightarrow p = z \tan \alpha$$

$$\begin{aligned} T &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) \leftarrow \\ &= \frac{1}{2} m (\dot{z}^2 \tan^2 \alpha + \dot{z}^2 + \dot{z}^2) \\ &= \frac{1}{2} m (\dot{z}^2 (1 + \tan^2 \alpha) + \dot{z}^2 + \dot{z}^2) \end{aligned}$$

$$U = mgz$$

$$\begin{aligned} T &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \\ \boxed{\theta = \alpha} &= \text{const} \\ T &= \frac{1}{2} m (\dot{r}^2 + r^2 \sin^2 \alpha \dot{\phi}^2) \\ U &= mgz = mg r \cos \alpha \\ \phi(r, \theta, \phi) &= \alpha - \theta = \omega \text{ const} \\ L' &= \left( \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - mg r \cos \alpha \right) \\ &\quad + \lambda (\alpha - \theta) \end{aligned}$$

$(r, \theta, \phi), \quad \phi, \quad r = l$

$$\begin{aligned}
 \frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{r}} \right) &= \frac{\partial L'}{\partial r} \\
 \frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{\theta}} \right) &= \frac{\partial L'}{\partial \theta} \\
 \frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{\phi}} \right) &= \frac{\partial L'}{\partial \phi} \\
 \varphi = \alpha - \dot{\theta} &= 0 \rightarrow \boxed{\dot{\theta} = \alpha} \\
 \end{aligned}$$

$L' = L_w + \lambda \varphi$   
 $T - U$   
 $\frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r}$   
 $+ \lambda \frac{\partial \varphi}{\partial r}$   
 etc.

Differentiate  $\dot{\theta} = 0$  constant w.r.t. time  
 $\dot{\varphi} = 0, \ddot{\varphi} = 0$   
 $\dot{\theta} = 0$   
 $\ddot{\theta} = 0$

solve for  $\lambda$   
 (algebraic)  
 $\vec{F}_c = \lambda \vec{\nabla} \varphi = -\lambda \hat{\varphi}$

Hamiltonian:  
 $L = T - U$   
 $H = \left( \sum_i p_i \dot{q}_i - L \right)$   
 $w$   
 $\vec{E}(\vec{z}, \dot{\vec{z}})$   
 $\dot{q}_i = \dot{q}^i(\vec{z}, p)$   
 $H(\vec{z}, p, t)$   
 $\# DOF$

$q \equiv q_i$  generalized coord,  
 $p \equiv p_i$  generalized momenta  
 $i = 1, \dots, s$   
 $i = 1, \dots, s$

$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$  (def'n) =  $f(\vec{z}, \dot{\vec{z}})$

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

Energy conserved? Yes, because  $\frac{\partial L}{\partial t} = 0$

$$E = \frac{1}{2} m \dot{x}^2 + U(x)$$

Momentum conjugate to  $x$ ?  $p_x, p_y, p_z$

$$p = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \rightarrow \dot{x} = \frac{p}{m}$$

$$\begin{aligned} H &= (p \dot{x} - L)_{\dot{x}} = p/m \\ &= (p \dot{x} - (\frac{1}{2} m \dot{x}^2 - U(x))) \\ &= p \frac{p}{m} - \frac{1}{2} m \left( \frac{p}{m} \right)^2 + U(x) \\ &= \frac{p^2}{2m} + U(x) \end{aligned}$$

Hamilton's equations:

$$\begin{cases} \dot{p}_i = -\frac{\partial H}{\partial q_i} & i = 1, \dots, 3 \\ \dot{q}_i = \frac{\partial H}{\partial p_i} & i = 1, \dots, 3 \end{cases}$$

(2s) 1<sup>st</sup> order ordinary differential equations,  
(coupled)

Euler-Lagrange

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$$

$i = 1, 2, \dots, 3$

$$L = L(q, \dot{q})$$

is 2<sup>nd</sup> order ordinary  
diff equations  
(coupled)

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$H = \frac{p^2}{2m} + U(x)$$

EL equations:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

$$m \ddot{x} = -\frac{\partial U}{\partial x} \quad (m\ddot{x} = F)$$

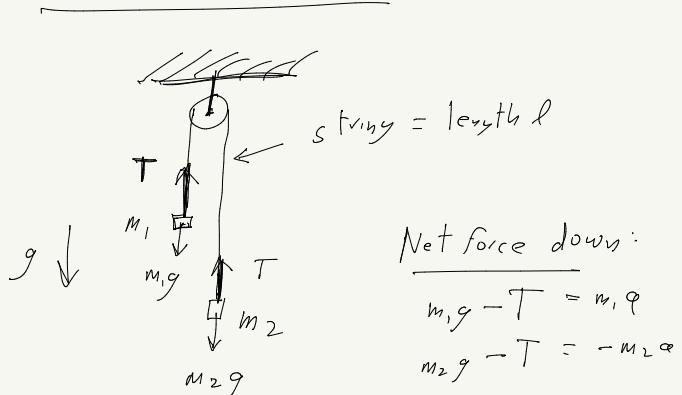
Hamilton's equations:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$$

$$m \ddot{x} = -\frac{\partial U}{\partial x}$$

A two-mass spring:



$$a = \frac{g(m_1 - m_2)}{m_1 + m_2}$$

(down)

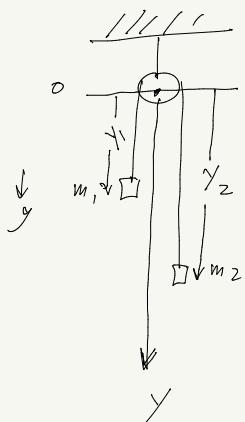
Net force down:

$$m_1 g - T = m_1 a$$

$$m_2 g - T = -m_2 a$$

subtract  $(m_1 - m_2)g = (m_1 + m_2)a$

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2}$$



$$L = T - U$$

$$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 \quad \Rightarrow \quad y_2 = l - y_1$$

$$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_1^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{y}_1^2$$

$$U = -m_1g y_1 - m_2g y_2$$

$$= -m_1g y_1 - m_2g(l-y_1)$$

$$= -(m_1+m_2)g y_1 - \underbrace{m_2 g l}_{\text{const ignore}}$$

$$\dot{y}_1 + \dot{y}_2 = l$$

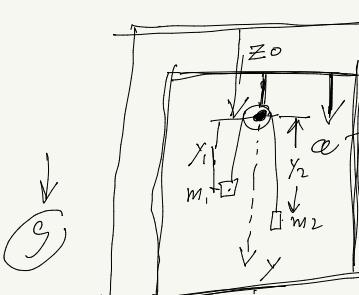
$$y_2 = l - y_1$$

$$L = \frac{1}{2}(m_1+m_2)\dot{y}_1^2 + (m_1-m_2)g y_1$$
~~$$L = \frac{1}{2}(m_1+m_2)\dot{y}_1^2 + (m_1-m_2)g y_1$$~~

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}_1} \right) = \frac{\partial L}{\partial y_1} \rightarrow (m_1+m_2)\ddot{y}_1 = (m_1-m_2)g$$

$$\ddot{y}_1 = \frac{(m_1-m_2)g}{m_1+m_2}$$

Analyze Atwood's machine in an accelerating reference frame.



$$z=0 \text{ (inertial)}$$

$$Z_0 = \frac{1}{2}a t^2$$

$$y_1 = ?$$

$$x \quad y \quad z$$

$$L = T - U$$

is valid only wrt an inertial reference frame

$$\vec{F} = m\vec{a} \quad (\text{only wrt inertial frame})$$



Lec #9: Sep 21<sup>st</sup>

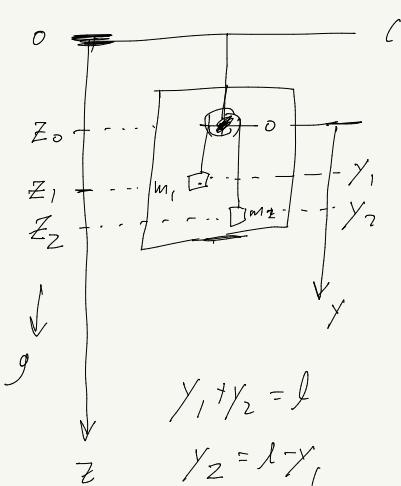
$$\vec{F} = m\vec{a} \quad (\text{valid only in an inertial frame})$$

$$\vec{F} + \underbrace{\vec{F}_{\text{ fictitious}}}_{\text{Coriolis force } \leftarrow \text{ + centrifugal force } \leftarrow \text{ + linear acceleration } \text{ + angular acceleration}} = m\vec{a} \quad \text{wrt a non-inertial frame}$$

coriolis force  $\leftarrow$   
+ centrifugal force  $\leftarrow$   
+ linear acceleration  
+ angular acceleration

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} \quad \begin{matrix} \text{same form of EOMs} \\ \text{in inertial and non-inertial ref frames} \end{matrix}$$

$$L = T - V \quad \text{valid only in an inertial frame}$$



(inertial frame)

$$z_1 = z_0 + y_1$$

$$z_2 = z_0 + y_2$$

$$z_0 = \frac{1}{2} \alpha t^2$$

unif acc@lent

$$T = \frac{1}{2} m_1 \dot{z}_1^2 + \frac{1}{2} m_2 \dot{z}_2^2$$

$$= \frac{1}{2} m_1 (\dot{y}_1^2 + \alpha^2 t^2 + 2\alpha t \dot{y}_1) + \frac{1}{2} m_2 (\dot{y}_2^2 + \alpha^2 t^2 + 2\alpha t \dot{y}_2)$$

$$= \left( \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 \right) + \frac{1}{2} (m_1 + m_2) \alpha^2 t^2 + \alpha t (m_1 \dot{y}_1 + m_2 \dot{y}_2)$$

$$= \underbrace{\frac{d}{dt} (\alpha t (m_1 \dot{y}_1 + m_2 \dot{y}_2))}_{\text{ignore}} - \alpha (m_1 \dot{y}_1 + m_2 \dot{y}_2)$$

(ignore)  
prescr. fncs  
of trns

$$L = T - V$$

wrt inertial frame

$$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 - \alpha(m_1y_1 + m_2y_2)$$

$$y_2 = \ell y_1 \quad \rightarrow \quad \dot{y}_2 = \dot{\ell} y_1$$

$$\boxed{T = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - \alpha(m_1y_1 + m_2\ell - m_2y_1)} \\ = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - \alpha(m_1 - m_2)y_1 \quad \underbrace{- \alpha m_2 \ell}_{\text{ignore}}$$

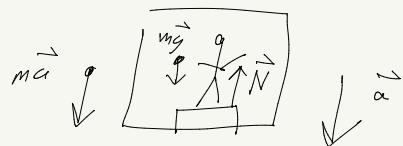
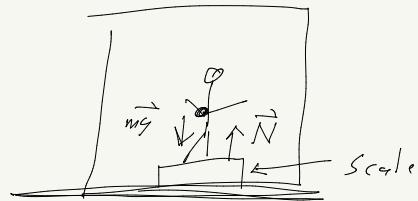
$$\boxed{U = -m_1gZ_1 - m_2gZ_2} \\ = -m_1g(Z_0 + y_1) - m_2g(Z_0 + y_2) \\ = -m_1gy_1 - m_2gy_2 \quad \underbrace{-(m_1 + m_2)gZ_0}_{\text{ignore}} \\ = -(m_1 - m_2)gy_1 \quad \underbrace{-m_2g\ell}_{\text{ignore}} \quad \underbrace{+ (m_1 - m_2)gZ_0}_{\text{ignore}}$$

$$\boxed{H = T - U = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - \alpha(m_1 - m_2)y_1 + (m_1 - m_2)gY_1} \\ = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 + (m_1 - m_2)(g - \alpha)y_1$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}_1} \right) = \frac{\partial L}{\partial y_1}$$

$$(m_1 + m_2)\ddot{y}_1 = (m_1 - m_2)(g - \alpha)$$

$$\rightarrow \boxed{\ddot{y}_1 = \left( \frac{m_1 - m_2}{m_1 + m_2} \right)(g - \alpha)}$$



$$m\vec{a} = \vec{F} \\ = \vec{mg} + \vec{N}$$

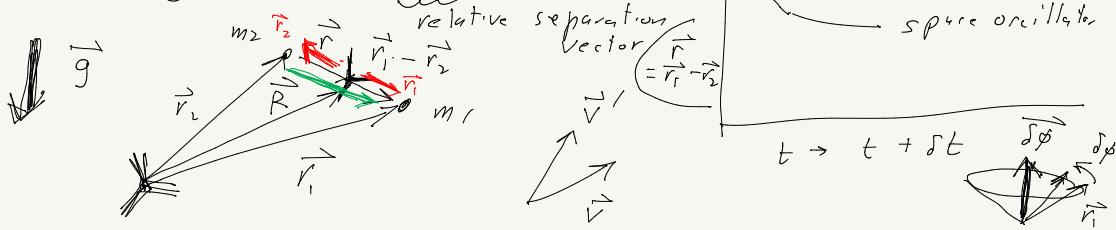
$$\vec{N} = -m(\vec{g} - \vec{a})$$

$$\vec{N} = 0 \quad \text{if } \vec{a} = \vec{g}$$

## Central force motion:

Two masses, closed system, interact via a central potential

$$U = U(|\vec{r}_1 - \vec{r}_2|)$$

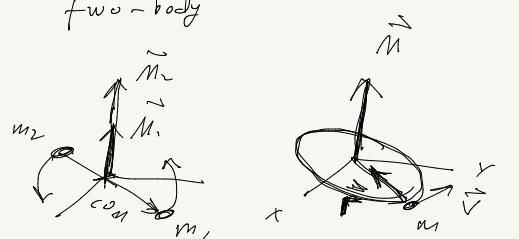


$$L = T - U$$

$$= \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 - U(|\vec{r}_1 - \vec{r}_2|)$$

- (1) no explicit time dependence  $\Rightarrow E = T + U = \text{const.}$   $\left( \begin{array}{l} \vec{r}_1 + \delta x \\ \vec{r}_2 + \delta x \end{array} \right)$
- (2) translational invariance  $\vec{r}_q \rightarrow \vec{r}_q + \delta x : \vec{p} = \sum_m m_q \vec{v}_q = \text{const.}$   $\left( \begin{array}{l} \vec{r}_1 + \delta x \\ \vec{r}_2 + \delta x \end{array} \right)$
- (3) rotational invariance:  $\left( \begin{array}{l} \vec{r}_q \rightarrow \vec{r}_q + \vec{\delta \phi} \times \vec{r}_q \\ \vec{v}_q \rightarrow \vec{v}_q + \vec{\delta \phi} \times \vec{v}_q \end{array} \right) \right| \begin{array}{l} L + \delta L \\ L' \end{array}$

$$\begin{aligned} \vec{M} &= (\vec{v}_1 \times \vec{p}_1) + (\vec{v}_2 \times \vec{p}_2) \quad \leftarrow \\ O \cdot \vec{M} &= \vec{v}_1 \times \vec{p}_1 + \vec{r}_1 \times \dot{\vec{p}}_1 + \vec{v}_2 \times \vec{p}_2 + \vec{r}_2 \times \dot{\vec{p}}_2 \\ &\quad \vec{p}_1 = m_1 \vec{v}_1 \\ &= \vec{r}_1 \times \dot{\vec{p}}_1 + \vec{r}_2 \times \dot{\vec{p}}_2 \quad \text{two-body} \\ \vec{M} &= \vec{M}_1 + \vec{M}_2 \\ \vec{P} &= \vec{p}_1 + \vec{p}_2 \\ &= m_1 \vec{v}_1 + m_2 \vec{v}_2 \\ \dot{\vec{P}} &= 0 \end{aligned}$$



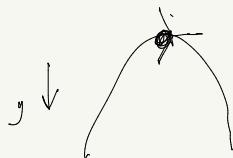
$$\begin{aligned} \vec{M} &= m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2 \\ O \cdot \vec{M} &= m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2 \\ O \cdot \vec{M} &= m_1 \vec{r}_1 \times \vec{v}_1 \end{aligned}$$

$$O \cdot \vec{M} = m_1 \vec{r}_1 \times \vec{v}_1$$

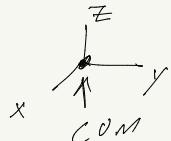
$$\vec{P} = \text{const}$$

com moves with const velocity

$$\begin{aligned}\vec{P} &= m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 \\ &= \frac{d}{dt} \left( \underbrace{m_1 \vec{r}_1 + m_2 \vec{r}_2}_{m_1 + m_2} \right) \cdot (m_1 + m_2) \\ &= (m_1 + m_2) \boxed{\frac{d\vec{R}}{dT}}\end{aligned}$$



We can go to the COM Frame.



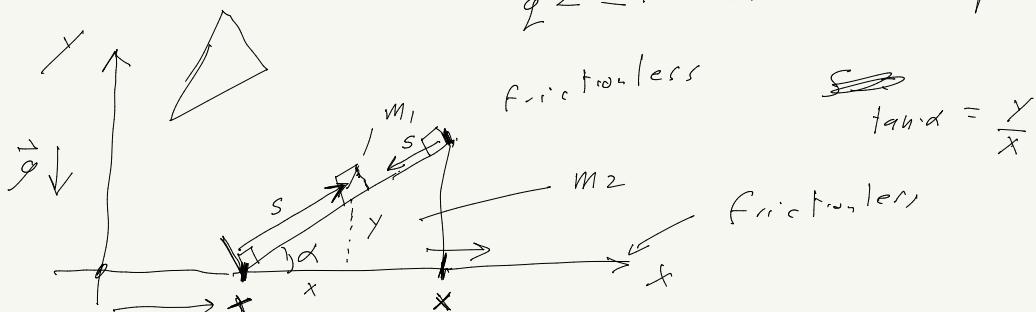
$$\boxed{\begin{array}{l} m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 \\ \vec{r} = \vec{r}_1 - \vec{r}_2 \end{array}}$$

$$\begin{aligned}\vec{P} &= 0 \\ \frac{d\vec{R}}{dT} &= 0 \\ \vec{R} &= \text{const} \\ &= \vec{\odot} \\ \text{Oris}'s\end{aligned}$$

QVIZ #2:

joseph.dimonaco@tutu.edu

qz - Firstname - lastname.pdf



1) Write down Lagrangian in terms of generalized coord

2) what quantities (if any) are conserved?  
why?

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\begin{cases} x_2 = x \\ y_2 = 0 \\ x_1 = x + s \cos \alpha \\ y_1 = s \sin \alpha \end{cases}$$

Lec #10: (Th 9/23)

Midterm 1: Tues Oct 5<sup>th</sup>

QZ:  $\dot{x}_1 = \dot{x} + \dot{s} \cos \alpha$        $x_1 = x + s \cos \alpha$   
 $\dot{y}_1 = \dot{s} \sin \alpha$   
 $\dot{x}_2 = \dot{x}$   
 $\dot{y}_2 = 0$

$$T = \frac{1}{2} m_1 \left( \dot{x}^2 + \dot{s}^2 \cos^2 \alpha + 2 \dot{x} \dot{s} \cos \alpha \right) + \frac{1}{2} m_2 \dot{x}^2$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1 \dot{x} \dot{s} \cos \alpha$$

$$U = m_1 g y_1 = m_1 g s \sin \alpha$$

$$L = T - U, \quad E = T + U \text{ conserved (no explicit time dependence)}$$

$$\frac{\partial L}{\partial x} = 0 \rightarrow \frac{\partial L}{\partial \dot{x}} = P_x \text{ conserved (no } x \text{ dependence)}$$

Con:  $m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 \quad (1)$   
 $\vec{r}_1 - \vec{r}_2 = \vec{r} \quad (2)$

$\vec{r} = \vec{r}_1 - \vec{r}_2$

for rotational motion  
com

$\vec{r}_2 = \vec{r}_1 - \vec{r}$

$m_1 \vec{r}_1 + m_2 (\vec{r}_1 - \vec{r}) = 0$

$(m_1 + m_2) \vec{r}_1 - m_2 \vec{r} = 0$

$\vec{r}_1 = \left( \frac{m_2}{m_1 + m_2} \right) \vec{r}$

$\vec{r}_2 = \left( \frac{m_1}{m_1 + m_2} \right) \vec{r}$

+ by  
com  
frame

$$\dot{\vec{r}}_1 = \left( \frac{m_2}{m_1 + m_2} \right) \dot{\vec{r}}, \quad \dot{\vec{r}}_2 = - \left( \frac{m_1}{m_1 + m_2} \right) \dot{\vec{r}}$$

$T = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2$

$= \frac{1}{2} m |\dot{\vec{r}}|^2$

$\Rightarrow T = \frac{1}{2} m \left( \frac{m_1 + m_2}{m_1 m_2} \right)^2 \dot{\vec{r}}^2$

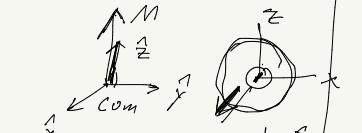
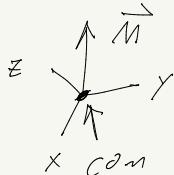
$m \equiv \frac{m_1 m_2}{m_1 + m_2}$

reduced mass of system

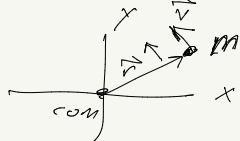
$$U = U(|\vec{r}_1 - \vec{r}_2|) = U(|\vec{r}|) = U(r)$$

$$\boxed{L = T - U = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(r)} \quad (\text{effective one-body problem})$$

i) const. of angular momentum



$\vec{F}_1$  must lie in the  $xy$  plane



$$\begin{aligned} \vec{M} &= \text{const} \\ &= m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2 \\ &= m \vec{r} \times \vec{v} \\ &= m \vec{r} \times \dot{\vec{r}} \end{aligned}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\begin{aligned} L &= \cancel{\frac{1}{2} m |\dot{\vec{r}}|^2} - U(r) \\ &= \cancel{\frac{1}{2} m (r^2 \dot{\phi}^2 + r^2 \dot{\theta}^2)} - U(r) \\ &\Rightarrow \cancel{r^2 \dot{\theta}^2} \end{aligned} \quad \begin{array}{l} \text{using plane} \\ \text{polar coordinates} \\ (\vec{r}, \phi) \end{array}$$

$$\frac{\partial L}{\partial \dot{\phi}} = 0 \rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{const} = mr^2 \dot{\phi} \quad \boxed{M_z = mr^2 \dot{\phi}} = \text{const}$$

$$\cancel{M = M_z}$$

not  
total  
mass

magnitude  
of angular vector  $\vec{M}$ ,  
momentum

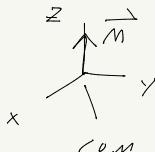
$$\frac{d \vec{M}}{dt} = 0$$

i) cons. of linear momentum  $\rightarrow$  COM Frame

$$L = \frac{1}{2} m |\vec{r}|^2 - U(r) \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

ii) cons. of angular momentum  $\rightarrow$  choose  $\vec{z}$ -axis to point along  $\vec{M}$



$$\vec{M} = \vec{r} \times \vec{p} = m \vec{r} \times \dot{\vec{r}} \rightarrow \text{motion } (\vec{r}, \dot{\vec{r}}) \text{ lies in the } x-y \text{ plane}$$

using plane-polar coord  $(r, \phi)$ :

$$L = \frac{1}{2} m (r^2 + r^2 \dot{\phi}^2) - U(r)$$

$$\frac{\partial L}{\partial \phi} = 0 \rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{const} = \boxed{mr^2 \dot{\phi} = M_z}$$

$$p_\phi = M_z = M$$

$$\boxed{\dot{\phi} = \frac{M_z}{mr^2}}$$

iii) cons. of energy  $E = T + U = \text{const}$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 + U(r) = \frac{1}{2} m \dot{r}^2 + \left( \frac{M_z^2}{2mr^2} + U(r) \right)$$

$\uparrow$   $U_{\text{eff}}(r)$

$$\boxed{E = \frac{1}{2} m \dot{r}^2 + U_{\text{eff}}(r)} \quad \text{where } U_{\text{eff}}(r) = U(r) + \frac{M_z^2}{2mr^2}$$

$$E - U_{\text{eff}}(r) = \frac{1}{2} m \dot{r}^2 \quad \pm \sqrt{\frac{2}{m} (E - U_{\text{eff}}(r))} = \dot{r} = \frac{dr}{dt} \quad \leftarrow \frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt}$$

$$\int dt = \pm \int \frac{dr}{\sqrt{\frac{2}{m} (E - U_{\text{eff}}(r))}} \quad \text{next page}$$

$$\boxed{\pm \sqrt{\frac{2}{m} (E - U_{\text{eff}}(r))} = \frac{dr}{d\phi} \frac{M_z}{mr^2}}$$

$$\boxed{t = \pm \int \frac{dr}{\sqrt{\frac{2}{m} (E - U(r)) - \frac{M_z^2}{mr^2}}} + \text{const}} \quad (14.6) \quad \begin{array}{l} L \& L \\ \rightarrow t = t(r) \\ r = r(t) \end{array}$$

$$\frac{d\phi}{dt} = \frac{M_z}{mr^2} \rightarrow \int d\phi = \int \frac{M_z}{mr^2(t)} dt$$

$$\rightarrow \phi = \int \frac{M_z}{mr^2(t)} dt + \text{const}$$

$$\int d\phi = \int \frac{\pm M_2 dr/r^2}{\sqrt{\frac{2}{m}(E - V(r)) - \frac{M_2^2}{r^2}}} \quad (14.7)$$

L&L

$$\boxed{\phi = \pm M_2 \int \frac{dr/r^2}{\sqrt{\frac{2m(E - V(r)) - M_2^2}{r^2}}} + \text{const}} \quad \uparrow$$

$$\phi = \phi(r) \leftrightarrow r = r(\phi)$$

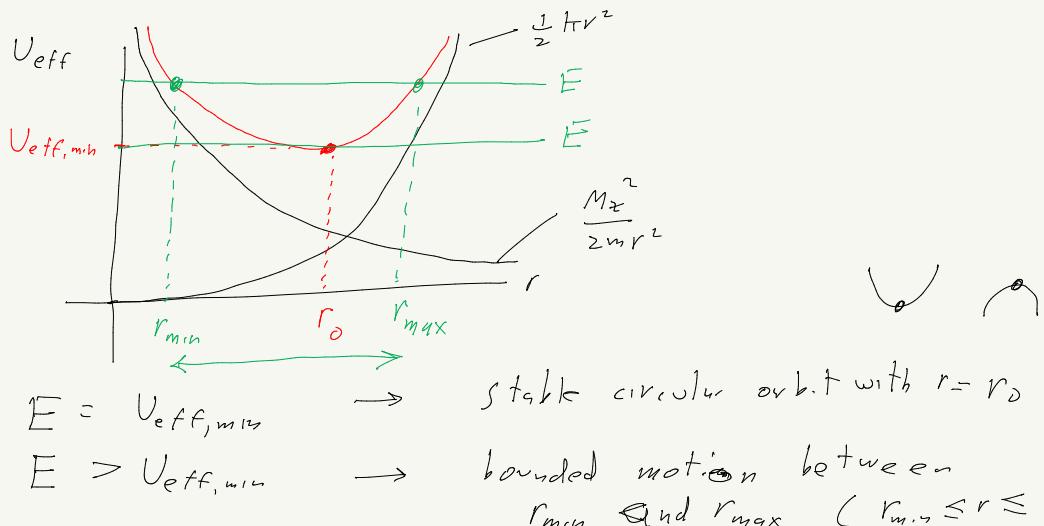
\* \*  $V(r) = \frac{1}{2} \pi r^2$  (spiral oscillator) —

 $V(r) = -\frac{\alpha}{r} = -\frac{G m_1 m_2}{r}$  (Newtonian gravity)

$$U_{\text{eff}}(r) = V(r) + \frac{M_2^2}{2mr^2}$$

$$\frac{dU_{\text{eff}}}{dr} \Big|_{r_0} = 0$$

$$= \frac{1}{2} \pi r^2 + \frac{M_2^2}{2mr^2}$$



$$\phi = M_2 \int \frac{dr/r^2}{\sqrt{2m(E - V(r)) - \frac{M_2^2}{r^2}}} + \text{const}$$

$$= M_2 \int \frac{dr/r^2}{\sqrt{\frac{2m(E - \frac{1}{2}mr^2) - M_2^2}{r^2}}} + \text{const}$$

$$= M_2 \int \frac{(dr/r^2)}{\sqrt{\frac{2mE - mr^2 - M_2^2/r^2}{r^2}}} + \text{const}$$

$$\sqrt{ } = \sqrt{a + bx + cx^2}$$

e.g.,

$$\begin{aligned} \sin^{-1}x &= \theta = \int d\theta \\ &\quad \leftarrow \qquad \qquad \qquad \begin{aligned} \int \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1}x + \text{const} \\ x &= \sin\theta \end{aligned} \rightarrow \begin{aligned} dx &= (\cos\theta)d\theta \\ \sqrt{ } &= \sqrt{1-\sin^2\theta} \\ &= \cos\theta \end{aligned} \end{aligned}$$

$$\boxed{u = \frac{1}{r}} \rightarrow du = -\frac{1}{r^2} dr \rightarrow \frac{dr}{r^2} = -du$$

$$\begin{aligned} \sqrt{ } &= \sqrt{2mE - \frac{mr^2}{u^2} - M_2^2 u^2} \\ &= \frac{1}{u} \sqrt{2mEu^2 - mr^2 - M_2^2 u^4} \end{aligned}$$

$$\phi = -M_2 \int \frac{u du}{\sqrt{2mEu^2 - mr^2 - M_2^2 u^4}} + \text{const}$$

$$\boxed{v = u^2} \rightarrow dv = 2u du \rightarrow u du = \frac{dv}{2}$$

$$\sqrt{ } = \sqrt{2mEv - mr^2 - M_2^2 v^2}$$

$$\boxed{\phi = -\frac{M_2}{2} \int \frac{dv}{\sqrt{2mEv - mr^2 - M_2^2 v^2}}} + \text{const}$$

complete the square:

$$\begin{aligned}
 \overbrace{-M_z^2 v^2 + 2mE v - m\hbar}^{\uparrow} &= -M_z^2 \left( v^2 - \frac{2mE}{M_z^2} v + \frac{m\hbar^2}{M_z^2} \right) \\
 &= -M_z^2 \left( \left( v - \frac{mE}{M_z^2} \right)^2 - \frac{m^2 E^2}{M_z^4} + \frac{m\hbar^2}{M_z^2} \right) \\
 &\quad \uparrow \qquad \uparrow \\
 &= -M_z^2 \left( (v-A)^2 - B^2 \right) \\
 &= M_z^2 (B^2 - (v-A)^2) \\
 A &= \frac{mE}{M_z^2} \\
 B^2 &= A^2 - \frac{m\hbar^2}{M_z^2}
 \end{aligned}$$

$$E \geq v_{\text{eff, min}}$$

$$\phi = -\frac{M_z}{2} \int \frac{dv}{\sqrt{M_z^2 (B^2 - (v-A)^2)}} + \text{const}$$

$$\begin{aligned}
 \int \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x \\
 &= \text{const}
 \end{aligned}$$

$$\text{Let: } v-A = B \sin \theta \quad B^2 - (v-A)^2 = B^2 - B^2 \sin^2 \theta$$

$$\begin{aligned}
 dv &= B \cos \theta d\theta \\
 &\uparrow \\
 &= B \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \boxed{\phi} &= -\frac{1}{2} \int d\theta + \text{const} \\
 &= -\frac{1}{2} \theta + \text{const} \\
 &= -\frac{1}{2} \sin^{-1} \left( \frac{v-A}{B} \right) + \text{const} \\
 &= -\frac{1}{2} \sin^{-1} \left( \frac{v-A}{B} \right) + \underline{\underline{\text{const}}}
 \end{aligned}$$

$$v-A \approx B \sin \theta$$

$$\sin \theta = \frac{v-A}{B}$$

$$\theta = \sin^{-1} \left( \frac{v-A}{B} \right)$$

$$\begin{aligned}
 v &= u^2 = v_r^2 \\
 u &= \frac{r}{v}
 \end{aligned}$$

Choose const so that  $\phi = 0 \iff r = r_{\max}$

$$\begin{aligned} O &= -\frac{1}{2} \sin^{-1} \left( \frac{\frac{1}{r^2} - A}{B} \right) + \text{const} \\ &\quad \text{at turning point } (r = r_{\max} \text{ or } r = r_{\min}) \\ B^2 &= (V - A)^2 = 0 \rightarrow V - A = \pm B \\ \Rightarrow O &= -\frac{1}{2} \sin^{-1} \left( \frac{-B}{B} \right) + \text{const} \\ &= -\frac{1}{2} \sin^{-1}(-1) + \text{const} \\ &= -\frac{1}{2} \left( -\frac{\pi}{2} \right) + \text{const} \\ &= \frac{\pi}{4} + \text{const} \\ \text{const} &= -\frac{\pi}{4} \end{aligned}$$

$$\begin{cases} \frac{1}{r_{\min}^2}, \frac{1}{r_{\max}^2} = A \pm B \\ \frac{1}{r_{\max}^2} = A - B \\ \frac{1}{r_{\min}^2} = A + B \end{cases}$$

$$A = \frac{m E}{M_e^2}$$

$$B^2 = A^2 - \frac{m k}{M_e^2}$$

$$\phi = -\frac{1}{2} \sin^{-1} \left( \frac{\frac{1}{r^2} - A}{B} \right) - \frac{\pi}{4}$$

$$\phi + \frac{\pi}{4} = -\frac{1}{2} \sin^{-1} \left( \frac{\frac{1}{r^2} - A}{B} \right)$$

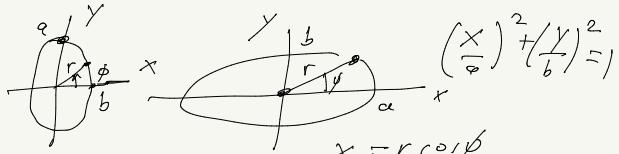
$$-\frac{1}{2}(2(\phi + \frac{\pi}{4})) = \sin^{-1} \left( \frac{\frac{1}{r^2} - A}{B} \right)$$

$$-\sin(2(\phi + \frac{\pi}{4})) = \frac{\frac{1}{r^2} - A}{B}$$

$\sin(-x) \approx -\sin x$

$$\begin{aligned} LHS &= -\sin(2\phi + \frac{\pi}{2}) \\ &= -\left( \sin(2\phi) \cos(\frac{\pi}{2}) + \cos(2\phi) \sin(\frac{\pi}{2}) \right) \\ &= -\cos 2\phi \end{aligned}$$

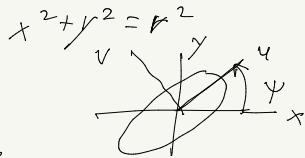
$$-10r^2\phi = \frac{1}{r^2} - A$$



$$-Br^2\cos^2\phi = \frac{1}{r^2} - A$$

$$\boxed{-\frac{1}{r^2} = A - Br^2\cos^2\phi}$$

elliptic with  
center of ellipse  
at origin



$$\begin{aligned} I &= Ar^2 - Br^2\cos^2\phi \\ &\quad \cos^2\phi \rightarrow \sin^2\phi \\ &= Ar^2 - Br^2(\cos^2\phi - \sin^2\phi) \\ &= A(x^2 + y^2) - B(x^2 - y^2) \\ &= (A-B)x^2 + (A+B)y^2 \\ &= \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \end{aligned}$$

$$\begin{aligned} \frac{1}{a^2} &= A - B \\ \frac{1}{b^2} &= A + B \\ a &= \sqrt{AB}, \quad b = \sqrt{A+B} \end{aligned}$$

Lec #11: Tues 9/28

- Midterm #1, Tues Oct 5<sup>th</sup> (In class,  
MOM 269)  
8 short answer - 40%  
2 longer problems - 60%

*notify me if  
you are not  
feeling well*

- Blackboard Zoom recordings

i) deriving  $t = t(r)$ ,  $\phi = \phi(r)$  integrals

ii) solving for  $\phi = \phi(r)$  for  $U = \frac{1}{2}kr^2$

- Today

i) clear up constant calculations

ii) complete space oscillator potential problem

- Thursday: Q&A

$$\frac{1}{a^2} = A - B \quad , \quad \frac{1}{b^2} = A + B$$

$$a^2 + b^2 = \frac{1}{A-B} + \frac{1}{A+B}$$

$$= \frac{A+B+A-B}{A^2-B^2}$$

$$= \frac{2A}{A^2-B^2} = \frac{2mE/M_2^2}{1/k/M_2^2} = \frac{2E}{\hbar}$$

$$\boxed{E = \frac{1}{2}\hbar(a^2+b^2)} \quad \leftarrow$$

$$E = \frac{1}{2}\hbar x_0^2$$

$\uparrow$   
max displacement  
with zero  
initial velocity

$$\frac{m\hbar}{M_2^2} = A^2 - B^2 = (A-B)(A+B)$$

$$= \frac{1}{a^2} \frac{1}{b^2}$$

$$\frac{m\hbar}{M_2^2} = \frac{1}{a^2 b^2}$$

$$\boxed{M_2 = \sqrt{m\hbar} ab}$$

$$U = -\frac{\alpha}{r} = -\frac{G m_1 m_2}{r}$$

(1.)

$E = -\frac{\alpha}{2\alpha}$

planets orbits the Sun in an elliptical orbit with Sun at focal point

(2) Equal areas in equal times



(3).  $\frac{P^2}{a^3} \propto \text{area}$   $\text{area eq.}$

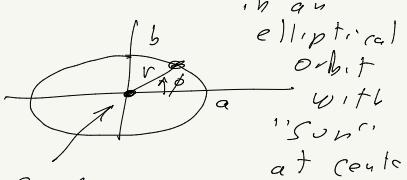
$$\frac{P^2}{a^3} = \text{const}$$

$$\frac{P_1^2}{a_1^3} = \frac{P_2^2}{a_2^3}$$

period

$$U = \frac{1}{2}\hbar r^2$$

(1.) "Planet" moves around "Sun" in an elliptical orbit with "Sun" at centre



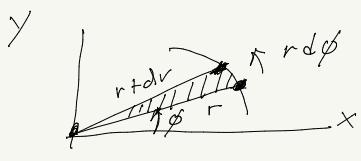
(2.) Equal areas in equal time



$$(3.) P = \frac{2\pi}{\omega}, \omega = \sqrt{\frac{k}{m}}$$

size of ellipse  $\propto$  semi-major axis

$a, b$   $a, e$   $a, p$



$$\begin{aligned} \dot{A}(t) &= \left( \frac{dA}{dt} \right) = \frac{1}{2} r^2 \frac{d\phi}{dt} \\ &= \frac{1}{2} r^2 \frac{M_z}{mr^2} \\ &= \frac{1}{2} \frac{M_z}{m} \\ &= \text{const.} \end{aligned}$$

$$\begin{aligned} dA &= \text{const.} \cdot dt \\ \hline \frac{dA}{dt} &= \frac{1}{2} \frac{M_z}{m} \end{aligned}$$

$$\begin{aligned} dA &= \frac{1}{2} \text{base} \cdot \text{height} \\ &\approx \frac{1}{2} r d\phi \cdot (r + dr) \\ &= \frac{1}{2} r^2 d\phi \quad \left( + \frac{1}{2} r d\phi \cdot dr \right) \\ M_z &= mr^2 \dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{mr^2} \end{aligned}$$

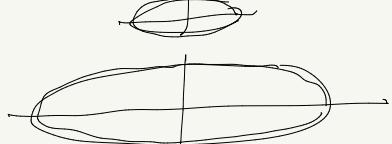
(Equal areas in equal times if ignore)

Consequence of const. of angular momentum (hold for all central potentials)

2nd order small

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{2} \frac{M_z}{m} \\ \int dA &= \int \frac{1}{2} \frac{M_z}{m} dt \\ A &= \frac{1}{2} \frac{M_z}{m} P \\ \pi &= \frac{1}{2} \frac{\sqrt{mH} ab}{m} P \\ \pi &= \frac{1}{2} \sqrt{\frac{H}{m}} P \end{aligned}$$

$$\boxed{\frac{2\pi}{\omega} = P}$$

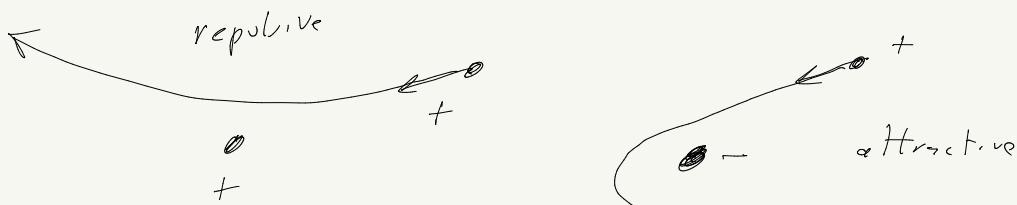


$$M_z = \sqrt{mH} ab$$



$$\sqrt{\frac{H}{m}} = \omega$$

$$\begin{aligned} m_2 &\rightarrow \cup(\vec{r}_1, \vec{r}_2) \\ m_1 &\rightarrow \cup(\vec{r}_1, -\vec{r}_2) \\ &\text{Consequence of angular momentum} \end{aligned}$$

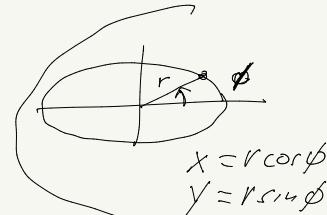


Solving for  $r = r(t)$ ,  $\phi(t)$ ,  $x(t)$ ,  $y(t)$ : (For  $V = \frac{1}{2} k r^2$ )

$$t = - \int \frac{dr}{\sqrt{\frac{2}{m}(E - \frac{1}{2}kr^2) - \frac{M_e^2}{m^2r^2}}} + \text{const}$$

$$\begin{aligned} E &= \frac{1}{2} k(a^2 + b^2) \\ M_H &= \sqrt{m k} ab \end{aligned} \quad \begin{aligned} \therefore \theta &= \frac{2}{m} \left( \frac{1}{2} k(a^2 + b^2) - \frac{1}{2} kr^2 \right) - \frac{M_H^2 a^2 b^2}{m^2 r^2} \\ &= \frac{k}{m} \left( a^2 + b^2 - r^2 - \frac{a^2 b^2}{r^2} \right) \end{aligned}$$

$$\begin{aligned} \sqrt{\theta} &= \sqrt{\frac{1}{r^2} \frac{k}{m} (r^2(a^2 + b^2) - r^4 - a^2 b^2)} \\ &= \frac{1}{r} \omega \sqrt{-(r^2 - a^2)(r^2 - b^2)} \end{aligned}$$



$$\rightarrow t = - \int \frac{dr}{\frac{\omega}{r} \sqrt{-(r^2 - a^2)(r^2 - b^2)}} + \text{const}$$

$$\omega t = - \int \frac{r dr}{\sqrt{-(r^2 - a^2)(r^2 - b^2)}} + \text{const}$$

Let:  $r^2 = a^2 \cos^2 \xi + b^2 \sin^2 \xi$

$\int r dr = \int a^2 \cos^2 \xi (-\sin \xi) d\xi + \int b^2 \sin^2 \cos \xi d\xi$

$\int r dr = -(a^2 - b^2) \sin \xi \cos \xi d\xi$

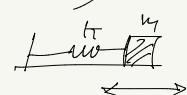
$\square$  in numerator

$$\begin{cases} r^2 = x^2 + y^2 \\ x = a \cos \xi \\ y = b \sin \xi \\ \text{parameter: } \xi \neq \phi \end{cases}$$

$$\begin{aligned}
 - (r^2 - a^2)(r^2 - b^2) &= (a^2 - b^2)^2 \sin^2 \xi \cos^2 \xi \\
 r^2 - a^2 &= a^2 (\cos^2 \xi - 1) + b^2 \sin^2 \xi \\
 &= -(a^2 - b^2) \sin^2 \xi \\
 r^2 - b^2 &= a^2 \cos^2 \xi + b^2 (\sin^2 \xi - 1) \\
 &= (a^2 - b^2) \cos^2 \xi
 \end{aligned}
 \quad \left. \begin{array}{l} r^2 = a^2 \cos^2 \xi + b^2 \sin^2 \xi \\ \hline \end{array} \right\}$$

$$\begin{aligned}
 \sqrt{r^2 - a^2} &= (a^2 - b^2) \sin \xi \cos \xi \\
 r dr = - (a^2 - b^2) \sin \xi \cos \xi d\xi &\quad \Rightarrow 0 \\
 \rightarrow \omega t = + \int d\xi + \text{const} &\quad \rightarrow \omega t = \xi + \text{const} \\
 &\quad \text{---} \\
 &\quad \text{---} \\
 &\quad \text{---}
 \end{aligned}$$

$$\boxed{r^2 = a^2 \cos^2(\omega t) + b^2 \sin^2(\omega t)} \quad \leftarrow r = r(t)$$



$$\boxed{\begin{aligned} x &= a \cos(\omega t) \\ y &= b \sin(\omega t) \end{aligned}} \quad \leftarrow \begin{aligned} x(t), \\ y(t) \end{aligned} \quad \omega = \sqrt{\frac{k}{m}} = \text{const} \quad v = \frac{1}{2} \pi r^2$$

$$\boxed{\begin{aligned} x &= r \cos \phi \\ y &= r \sin \phi \end{aligned}} \quad \div \quad \tan \phi = \frac{y}{x} = \frac{b}{a} + \tan(\omega t) \quad \boxed{\phi = \arctan \left( \frac{b}{a} + \tan(\omega t) \right)} \quad \leftarrow \phi = \phi(t)$$

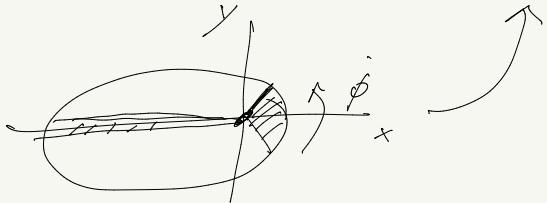
$$\begin{aligned} M_z &= m r^2 \dot{\phi} = \sqrt{m k} a b \\ r^2 \dot{\phi} &= \sqrt{\frac{k}{m}} a b \quad \rightarrow \boxed{\dot{\phi} = \frac{\omega a b}{r^2}} \quad \begin{array}{l} \text{const} \\ a = b = r \end{array} \end{aligned}$$

Angular velocity:  $(r, \dot{\phi})$



$$M_z = mr^2\dot{\phi}$$

$$\dot{\phi} = \frac{M_z}{mr^2}$$



Lecture #12

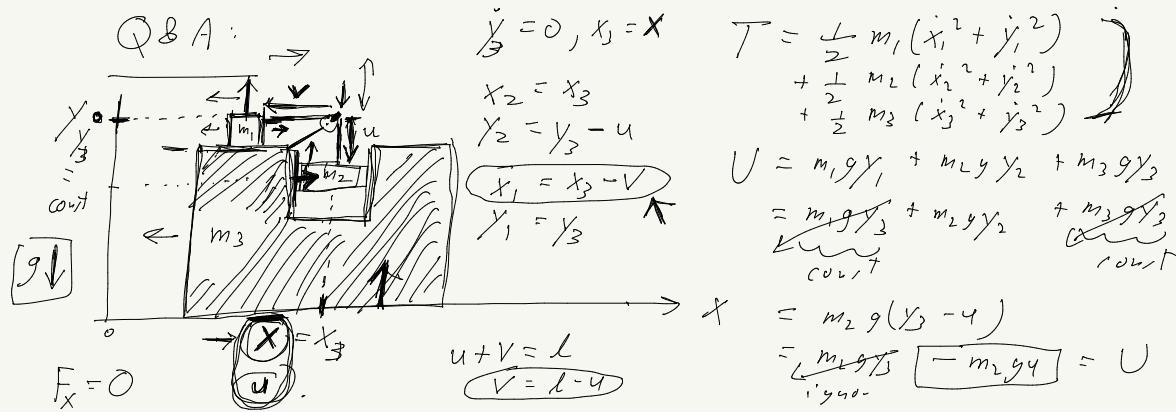
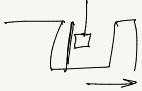
Thursday 9/30

EXAM 1: Tues Oct 5<sup>th</sup>, MCOM Z69

(★ If you are feeling sick, please contact me before the class to make arrangements for a remote exam.)

8 short answer questions  
2 longer problems

- 8 pts  
- 12 pts  
20 pts (total)



$$\begin{aligned}
 \rightarrow x_1 &= x_3 - v = x - (\ell - u) & \dot{x}_1 &= \dot{x} + \dot{u} & \leftarrow \\
 x_2 &= x_3 = x & \dot{x}_2 &= \dot{x} \\
 x_3 &= x & \dot{x}_3 &= \dot{x} \\
 y_1 &= y_3 = \text{const} & \dot{y}_1 &= 0 \\
 y_2 &= y_3 - u & \dot{y}_2 &= -\dot{u} \\
 y_3 &= \text{const} & \dot{y}_3 &= 0
 \end{aligned}$$

$$\begin{aligned}
 T &= \frac{1}{2} m_1 (\dot{x} + \dot{u})^2 + \frac{1}{2} m_2 (\dot{x} + \dot{u})^2 + \frac{1}{2} m_3 \dot{x}^2 \\
 &= \frac{1}{2} m_1 (\dot{x}^2 + \dot{u}^2 + 2\dot{x}\dot{u}) + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_3 \dot{x}^2 + \frac{1}{2} m_2 \dot{u}^2 \\
 &= \frac{1}{2} (m_1 + m_2 + m_3) \dot{x}^2 + \frac{1}{2} m_2 \dot{u}^2 + \boxed{m_1 \dot{x}\dot{u}} + \frac{1}{2} m_2 \dot{u}^2 \\
 &= \frac{1}{2} (m_1 + m_2 + m_3) \dot{x}^2 + \frac{1}{2} (m_1 + m_2) \dot{u}^2 + \boxed{m_1 \dot{x}\dot{u}}
 \end{aligned}$$

$\cup = -m_2 g \dot{u}$

$$\begin{aligned}
 L &= T - \cup & E &= T + \cup = \text{const} \\
 \text{No } x\text{-dependence} \rightarrow 0 &= p_x = \text{const} & \boxed{\dot{p}_x = (m_1 + m_2 + m_3) \dot{x} + m_1 \dot{u}} \\
 x_{\text{com}} &= 0
 \end{aligned}$$

$$(m_1 + m_2 + m_3) x_{\text{com}} = m_1 x_1 + m_2 x_2 + m_3 x_3 = 0$$

$x_1 = x - \ell + u$      $x_2 = x$      $x_3 = x$

$$\begin{aligned}
 &= (m_1 + m_2 + m_3) x - m_1 \ell + m_1 u \\
 &= 0
 \end{aligned}$$

$$(m_1 + m_2 + m_3) x = m_1 (\ell - u)$$

$z_i$  doesn't appear in Lagrangian

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}_i} \right) = \cancel{\frac{\partial L}{\partial z_i}} = 0 \quad p_i = \frac{\partial L}{\partial \dot{z}_i} = \text{const}$$

$\cancel{\frac{\partial L}{\partial z_i}}$ 
  
 const

Constraints:

- $y_3 = \text{const} \Rightarrow 0$  ✓
- $x_1 = \text{const}$  ✓
- $x_2 = x_3 \Rightarrow x$

$$x_1 = x_3 - (l - u) = x_3 - l + y_2 \rightarrow x_1 - x_3 + y_2 = -l$$

Generalized coordinates:

$$y_2 = x_3 - u \Rightarrow -u$$

$$x_3 = x$$

DOF constraints:

$$\frac{6 \text{ DOF}}{4 \text{ constraints}} = 2 \text{ DOF}$$

Final equations:

$$y_3 = 0, \quad y_1 = \text{const}, \quad x_2 - x_3 = 0$$

$$x_1 - x_3 + y_2 = -l$$

Radial forces:

$$T_{\text{radial}}: T - mg \cos \phi = m \ddot{\phi}_{\text{radial}}$$

Tangential forces:

$$T_{\text{tangential}}: mg \sin \phi = m \ddot{\phi}_{\text{tangential}}$$

$$-l \ddot{\phi} = \ddot{\phi}_{\text{tangential}}$$

$$l \ddot{\phi}^2 = \ddot{\phi}_{\text{radial}}$$

Don't do work:

$$L = T - U + \lambda \dot{\phi}$$

$$\dot{\phi} = \dot{T} - l \dot{\phi} = 0$$

$$T = \frac{1}{2} m (r^2 + l^2 \dot{\phi}^2)$$

$$U = -mgY$$

$$= -mg r \cos \phi$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

$$\ddot{\phi} = r - l = 0$$

$$T = mg \cos \phi + m l \dot{\phi}^2$$

$$mg \sin \phi = -m l \ddot{\phi}$$

$$\ddot{\phi} = -\frac{g}{l} \sin \phi$$

$$F_c = \lambda \nabla \phi$$

$$F_c = -T r$$

torque

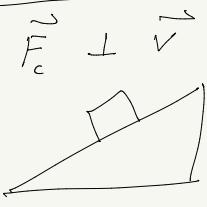
$$\vec{F}_c = - (m g \cos \phi + m l \dot{\phi}^2) \hat{r}$$

$$= - \vec{\nabla} U_{\text{const}}$$

$$\text{Work} = \int_{\text{path}} \vec{F}_c \cdot d\vec{l}$$

$$= \int_{\text{path}} \vec{F}_c \cdot \vec{v} dt$$

$$= 0$$



$$\frac{d\vec{l}}{dt} dt$$

$$\vec{v} dt$$



Lecture #14: Thu 10/7

Next two lectures — collisions  
Following 3 lectures — scattering

1.) Spontaneous disintegration of a single mass  $M = m_1 + m_2$  into two particles  $m_1$  and  $m_2$

closed system  $\rightarrow$  COM Frame (<sup>conservation of total linear momentum</sup>)

(after)

$$m_1 \xrightarrow{\text{after}} \vec{p}_1 \equiv \vec{p}_0$$

$$m_2 \xleftarrow{\text{after}} \vec{p}_2 = -\vec{p}_0$$

$$\vec{p}_1 + \vec{p}_2 = 0 \rightarrow \vec{p}_2 = -\vec{p}_1$$

(before)  $\vec{p} = 0$

Cons. of energy:

$$\Rightarrow E_i = E_{i,i} + T_{10} + E_{2,i} + T_{20}$$

$\uparrow$

internal energy of  $M$

$$= E_{i,i} + E_{2,i} + \frac{p_0^2}{2m_1} + \frac{p_0^2}{2m_2}$$

$$= E_{i,i} + E_{2,i} + \frac{p_0^2}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right)$$

$\curvearrowleft$

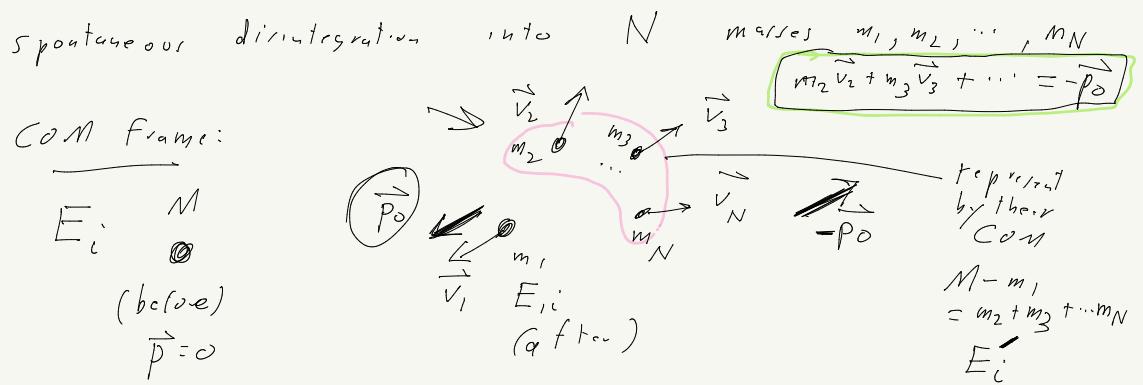
$$E_i - E_{ii} - E_{2,i} = \frac{p_0^2}{2m}$$

$\curvearrowleft$

$E : \text{disintegration energy}$

$$\frac{p_0^2}{2m} = \frac{1}{m_1 + m_2} = \frac{1}{m}$$

$$\rightarrow p_0 = \sqrt{2mE} \rightarrow \boxed{V_{10} = \frac{p_0}{m_1}, V_{20} = \frac{p_0}{m_2}}$$



Upper limit on potential of a single particle  $m_1$

cons. of energy

$$\Rightarrow E_i = \frac{\vec{p}_0^2}{2m_1} + E_{i,i} + \frac{\vec{p}_0^2}{2(M-m_1)} + E_i' = E_{i,i} + E_i' + \frac{\vec{p}_0^2}{2} \left( \frac{1}{m_1} + \frac{1}{M-m_1} \right) \xrightarrow{M \ll m_1 (M-m_1)}$$

$$E_i - E_{i,i} - E_i' = \frac{\vec{p}_0^2}{2m_1} \left( \frac{M}{M-m_1} \right)$$

$E_{i,i} + E_{i,i}' + \dots$   
 for maximum  $E_i'$  for  $m_1$

$$\frac{\vec{p}_0^2}{2m_1} = (E_i - E_{i,i} - E_i') \left( \frac{M-m_1}{M} \right)$$

$T_0$

maximum when  $E_i'$  is minimum

$$E_i' + \frac{\vec{p}_0^2}{2(M-m_1)} = (E_{i,i} + \frac{1}{2} m_2 |\vec{v}_2|^2 + E_{i,i}' + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots)$$

$$= E_{i,i} + E_{i,i}' + \dots + \frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots$$

$$-\vec{p}_0 = m_2 \vec{v}_2 + m_3 \vec{v}_3 + \dots \rightarrow \vec{p}_0^2 = |m_2 \vec{v}_2 + m_3 \vec{v}_3 + \dots|^2$$

$$E_i' = \underbrace{E_{2,i} + E_{3,i} + \dots}_{\text{Polarizable}} + \frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots$$

$\xrightarrow{\quad}$

$$\frac{1}{2(m_2 + m_3 + \dots)} |\vec{m}_2 \vec{v}_2 + \vec{m}_3 \vec{v}_3 + \dots|^2$$

$$|\vec{m}_2 \vec{v}_2 + \vec{m}_3 \vec{v}_3 + \dots|^2 = m_2^2 |\vec{v}_2|^2 + m_3^2 |\vec{v}_3|^2 + \dots + 2 m_2 m_3 \vec{v}_2 \cdot \vec{v}_3 + \dots$$

Suppose:  $\vec{v}_2 = \vec{v}_3 = \dots \equiv \vec{v}$

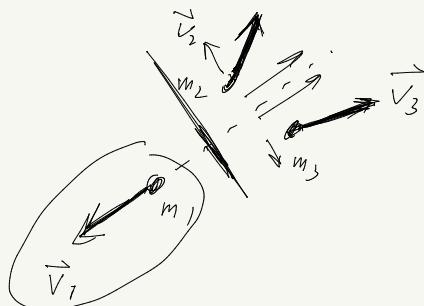
$$\frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots = \frac{1}{2} (m_2 + m_3 + \dots) |\vec{v}|^2$$

$$- \frac{1}{2(m_2 + m_3 + \dots)} (m_2 + m_3 + \dots)^2 |\vec{v}|^2 = -\frac{1}{2} (m_2 + m_3 + \dots) |\vec{v}|^2$$

$$\Rightarrow E_i' = E_{2,i} + E_{3,i} + \dots$$

$$M$$

$\vec{p} = 0$



$$\frac{p_0^2}{2m_1} = \bar{r}_{10}$$

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 = 0$$

$$\underbrace{p_0}_{-\vec{p}_0}$$



$$E_i' = E_{2,i} + E_{3,i} + \dots$$

Added discussion: (meaning of  $E_i'$ )

conservation of total energy:  $E_i = E_{1i} + \frac{1}{2} m_1 \vec{v}_1'^2 + E_{2i} + \frac{1}{2} m_2 \vec{v}_2'^2 + E_{3i} + \frac{1}{2} m_3 \vec{v}_3'^2 + \dots$

$\underbrace{\quad}_{\substack{\text{internal + HE} \\ \text{of mass } m_1}}$        $\underbrace{\quad}_{\substack{\text{same for} \\ m_2}}$        $\underbrace{\quad}_{\substack{\text{same for} \\ m_3}}$

Let:  $T' = \frac{1}{2} m_2 \vec{v}_2'^2 + \frac{1}{2} m_3 \vec{v}_3'^2 + \dots$  (= HE of  $m_2, m_3, \dots$ )  
 $= \frac{1}{2} m_2 |\vec{V}' + \vec{v}_2'|^2 + \frac{1}{2} m_3 |\vec{V}' + \vec{v}_3'|^2 + \dots$

where  $\vec{V}' = \text{velocity of COM of } m_2, m_3, \dots$

$\vec{v}_2' = \text{velocity of } m_2 \text{ with respect to COM of } m_2, m_3, \dots$

$\vec{v}_3' = \text{velocity of } m_3 \text{ with respect to COM of } m_2, m_3, \dots$

...

$T'_0 = \text{HE of } m_2, m_3, \dots$   
 w.r.t.  
 COM of  
 $m_2, m_3, \dots$

$\rightarrow T' = \frac{1}{2} m_2 (|\vec{V}'|^2 + \vec{v}_2'^2 + 2 \vec{V}' \cdot \vec{v}_2') + \dots$

$= \left( \frac{1}{2} (m_1 + m_2 + \dots) |\vec{V}'|^2 \right) + \left( \frac{1}{2} m_2 |\vec{v}_2'|^2 + \frac{1}{2} m_3 |\vec{v}_3'|^2 + \dots \right)$

$T'_{\text{com}} = \text{HE of}$   
 COM of  
 $m_2, m_3, \dots$

$+ \underbrace{(m_2 \vec{v}_2' + m_3 \vec{v}_3' + \dots)}_{=0 \text{ by definition of COM for } m_2, m_3, \dots} \cdot \vec{V}'$

so  $T' = T'_{\text{com}} + T'_0$

$\rightarrow E_i = E_{1i} + \frac{1}{2} m_1 \vec{v}_1'^2 + (E_{2i} + E_{3i} + \dots) + T'_{\text{com}} + T'_0$   
 $= E_{1i} + \frac{1}{2} m_1 \vec{v}_1'^2 + (E_{2i} + E_{3i} + \dots) + \frac{p_0^2}{2(m_1 + m_2 + \dots)} + T'_0$

compare to

$$E_i = \left( E_{1i} + \frac{1}{2} m_1 \vec{v}_1'^2 \right) + \left( E_{2i}' + \frac{p_0^2}{2(M - m_1)} \right)$$

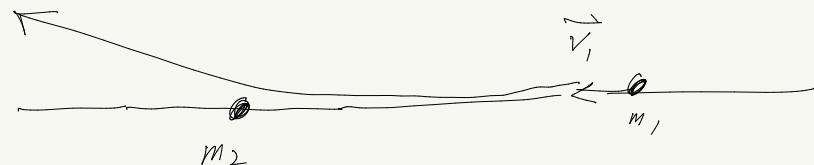
$\uparrow$   
 $m_2 + m_3 + \dots$

$\rightarrow E_i' = E_{2i} + E_{3i} + \dots + T'_0$   
 $= (\text{internal energy of } m_2, m_3, \dots) + (\text{HE of } m_2, m_3, \dots \text{ w.r.t. COM of } m_2, m_3, \dots)$

NOTE: when  $m_2, m_3, \dots$  all move with the same velocity, they are moving together with the COM of  $m_2, m_3, \dots$ . Then  $T'_0 = 0$  and  $E_i' = \text{internal energy of } m_2, m_3, \dots$

COM Frame       $v_c$        $I_{ab}$  Frame

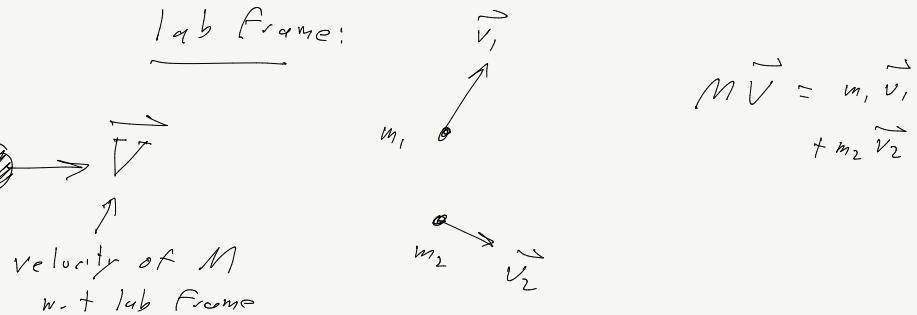
Example:



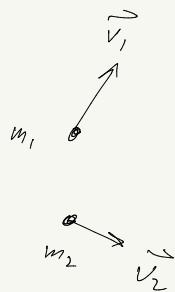
Lab Frame      ( $a + \text{rest}$ )

$$\vec{P}_{tot} = m_1 \vec{v}_1 + m_2 \vec{v}_2 \neq 0 = m_1 \vec{v}_1$$

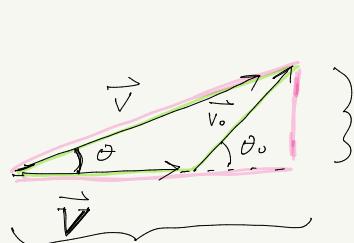
$I_{ab}$  Frame:



velocity of  $M$   
w.r.t. lab frame



$$M\vec{V} = m_1 \vec{v}_1 + m_2 \vec{v}_2$$



$\theta$ : w.r.t. lab frame  
 $\theta_0$ : w.r.t. COM frame

$\vec{v}_0$ : velocity of  $m_1$  (or  $m_2$ ) w.r.t. COM frame

$\vec{V}$ : velocity of  $M$  w.r.t.  $I_{ab}$  frame

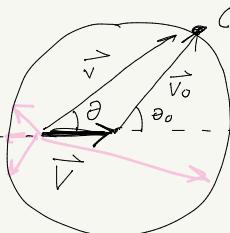
$\vec{v}$ : velocity of  $m_1$  (or  $m_2$ ) w.r.t. lab frame

$$v_0^2 = v^2 + V^2 - 2vV \cos \theta$$

$$v \sin \theta = v_0 \sin \theta_0$$

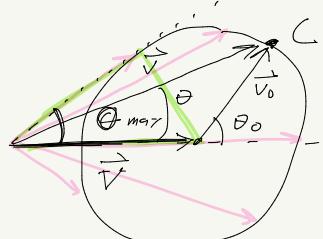
$$v \cos \theta = V + v_0 \cos \theta_0$$

$$\therefore \rightarrow \boxed{\tan \theta = \frac{v_0 \sin \theta_0}{V + v_0 \cos \theta_0}}$$



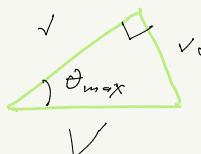
$$V < v_0$$

$\vec{v}$  can point in  
any direction



$$V > v_0$$

$\vec{v}$  can only  
point in the  
forward direction



$$\sin \theta_{\max} \approx \frac{v_0}{V}$$

Lec # 15:

$$\text{grade: } \max = 20$$

rescale by  $20/16$

(part (b) of long  
problem # 2)

Average :  $12/20$   
(with rescaling)

$$9/20$$

$$\begin{aligned} \text{Example: } \frac{14}{20} &\rightarrow 14 \times \frac{20}{16} = 14 \times \frac{10}{8} \\ &= \frac{140}{8} \\ &= 17. \dots \end{aligned}$$

Q viz # 3 — next week sometime

## Elastic collisions: (Sec 17)

→ motion w/  
in 2-d plane

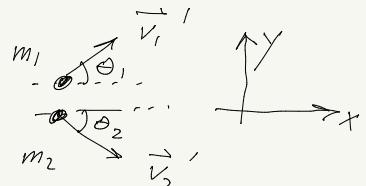
→ const. of linear momentum, angular momentum,  
total energy

Elastic collision → no change in internal energy  
→ Kinetic energy is conserved

Specify masses  $m_1, m_2$  and initial velocities  $\vec{v}_1, \vec{v}_2$

$$m_1 \quad \vec{v}_1 \quad m_2 \quad (\vec{v}_2 = 0)$$

(before)



(after)

(lab frame)

Unknowns: (4)  
 $v_1', v_2', \theta_1, \theta_2$

Equations:

$$\rightarrow T_{\text{before}} = T_{\text{after}} \quad (1 \text{ equation})$$

$$\vec{P}_{\text{before}} = \vec{P}_{\text{after}} \quad (2 \text{ equations})$$

3 equations

We normally specify  $\theta_2$  in order to solve  
for  $v_1', v_2', \theta_1$

Numerical:

$$m_1 = 1 \text{ kg}$$

$$m_2 = 2 \text{ kg}$$

$$v_1 = 1 \text{ m/s}$$

$$v_2 = 0$$

$$\theta_2 = 60^\circ$$

$$\left\{ \begin{array}{l} v_1' = ? \Rightarrow = \frac{1}{3} \text{ m/s} \\ v_2' = ? \Rightarrow = \frac{\sqrt{7}}{3} \text{ m/s} \\ \theta_1 = ? \Rightarrow \approx 41^\circ \end{array} \right.$$

$$\frac{1}{2} m_1 \|\vec{v}_1\|^2 = \frac{1}{2} m_1 (\vec{v}_1')^2 + \frac{1}{2} m_2 (\vec{v}_2')^2$$

$$m_1 v_i^2 = m_1 (v_1')^2 + m_2 (v_2')^2 \quad v_1', v_2', \theta_1$$

$$x: \boxed{m_1 v_i = m_1 v_1' \cos \theta_1 + m_2 v_2' \cos \theta_2}$$

$$y: \boxed{0 = m_1 v_1' \sin \theta_1 - m_2 v_2' \sin \theta_2}$$

Analyze in COM frame instead



$$m_1 \vec{v}_{1,0} + m_2 \vec{v}_{2,0} = \vec{0}$$

$\vec{p}_0 = \vec{0}$  (before and after the collision)

$$\vec{V} = \frac{\vec{m}_1 \vec{v}_1 + \vec{m}_2 \vec{v}_2}{m_1 + m_2} \quad (\text{velocity of COM in the lab frame})$$

$$\vec{v}_1 = \vec{V} + \vec{v}_{1,0}, \quad \vec{v}_2 = \vec{V} + \vec{v}_{2,0}$$

$$\vec{v}_1' = \vec{V} + \vec{v}_{1,0}', \quad \vec{v}_2' = \vec{V} + \vec{v}_{2,0}'$$

↑  
Lab frame      COM frame

$$\begin{aligned} \vec{r} &= \vec{r}_{1,0} - \vec{r}_{2,0} \\ &= \vec{r}_1 - \vec{r}_2 \\ \vec{v} &= \vec{v}_{1,0} - \vec{v}_{2,0} \\ &= \vec{v}_1 - \vec{v}_2 \end{aligned}$$

Relative vector

$\vec{R}'$

$$\boxed{T_0 = \frac{1}{2} m_1 (\vec{v}_{10})^2 + \frac{1}{2} m_2 (\vec{v}_{20})^2}$$

$$= \frac{1}{2} m (\vec{v})^2$$

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

$$\vec{r}_{10} = \frac{m_2}{m_1 + m_2} \vec{r}, \quad \vec{r}_{20} = \frac{-m_1}{m_1 + m_2} \vec{r}$$

$$\boxed{\vec{v}_{10} = \frac{m_2}{m_1 + m_2} \vec{v}}, \quad \boxed{\vec{v}_{20} = \frac{-m_1}{m_1 + m_2} \vec{v}}$$

$$T' = \frac{1}{2} m (\vec{v}')^2 = T_0 = \frac{1}{2} m (\vec{v})^2$$

[ ]

$|\vec{v}'| = |\vec{v}|$

$v' = v$   
 $\vec{v}'_{10} = \vec{v}_{10}$   
 $\vec{v}'_{20} = \vec{v}_{20}$

$$\vec{v}' = \cancel{\vec{v}} \hat{n}_0, \quad \hat{n}_0 : \text{unit vector & makes an angle}$$

$\chi$  w.r.t com velocity

$$\vec{v} = \vec{v}_{10} - \vec{v}_{20}$$

$$= \vec{v}_1 - \vec{v}_2$$

$\hat{n}_0$

$\vec{v}$

$\vec{v}$  frame

$w.r.t$  com velocity

$$\vec{v}' = \vec{V} + \vec{v}'_{10}$$

w.r.t com Frame

$$= \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} + \frac{m_2}{m_1 + m_2} v \hat{n}_0'$$

$$\vec{v}'_2 = \vec{V} + \vec{v}'_{20}$$

$$= \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} - \frac{m_1}{m_1 + m_2} v \hat{n}_0'$$

$$\vec{p}'_1 = m_1 \vec{v}'_1$$

$$\vec{p}'_2 = m_2 \vec{v}'_2$$

$$\boxed{\vec{p}'_1 = m_1 \vec{V} + m v \hat{n}_0}$$

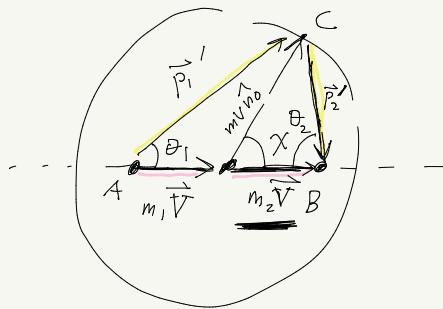
$$\boxed{\vec{p}'_2 = m_2 \vec{V} - m v \hat{n}_0}$$

$$\vec{p}'_1 + \vec{p}'_2 = (m_1 + m_2) \vec{V} = \vec{P}$$

$$= \vec{p}_1 + \vec{p}_2$$

$\vec{v}_1, \vec{v}_2$  are not b. to  $y$

$A, B : \overrightarrow{P}$



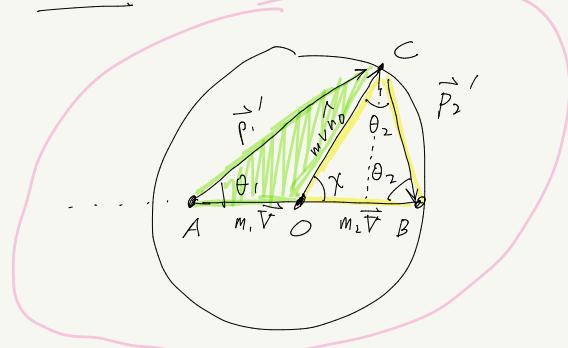
$\vec{v}_2 = 0$  : (standard lab frame)

$$\vec{v} = \vec{v}_1 - \vec{v}_2 = \vec{v}_1$$

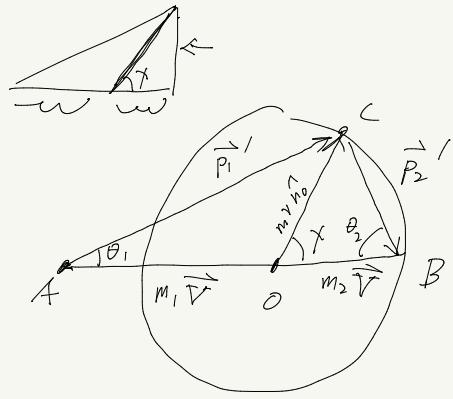
$$\vec{V} = \frac{\vec{m}_1 \vec{v}_1 + \vec{m}_2 \vec{v}_2}{\vec{m}_1 + \vec{m}_2} = \frac{\vec{m}_1 \vec{v}_1}{\vec{m}_1 + \vec{m}_2} = \frac{\vec{m}_1 \vec{V}}{\vec{m}_1 + \vec{m}_2}$$

$$m_2 \vec{V} = \frac{m_2 m_1 \vec{V}}{m_1 + m_2} = m \vec{V} \rightarrow B \text{ lies on the } C \text{ circle}$$

$\vec{v}_2 = 0$  :



$$m_1 < m_2$$



$$m_1 > m_2$$

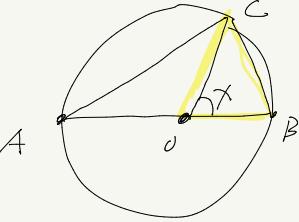
$$\chi + 2\theta_2 = \pi$$

$$\left[ \theta_2 = \frac{\pi}{2} - \frac{\chi}{2} \right] \Rightarrow$$

$$\tan \theta_1 = \frac{\text{opp}}{\text{adj}}$$

$$= \frac{m_1 V \sin \chi}{m_1 V + m_2 V \cos \chi}$$

$$\begin{aligned}
 & \left| \text{f}_\theta \theta_1 \right| = \frac{\left( \frac{m_1 m_2}{m_1 + m_2} \right) v \sin \chi}{\left( \frac{m_1 v}{m_1 + m_2} \right) + \left( \frac{m_1 m_2}{m_1 + m_2} \right) v \cos \chi} \\
 & = \frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi}
 \end{aligned}$$



$$\begin{aligned}
 m_2^2 |v'_2|^2 &= 2 m^2 v^2 - 2 m v^2 \cos \chi \\
 &= 2 m^2 v^2 (1 - \cos \chi) \\
 &= 2 \frac{m_1^2 m_2}{(m_1 + m_2)^2} v^2 = 2 \sin^2 \left( \frac{\chi}{2} \right)
 \end{aligned}$$

$$\left| v'_2 \right| = \left( \frac{2 m_1 v}{m_1 + m_2} \right) \sin \left( \frac{\chi}{2} \right)$$

$$\begin{aligned}
 \cos \chi &= \cos \left( 2 \frac{\chi}{2} \right) \\
 &= \cos^2 \left( \frac{\chi}{2} \right) - \sin^2 \left( \frac{\chi}{2} \right) \\
 &= 1 - 2 \sin^2 \left( \frac{\chi}{2} \right) \\
 1 - \cos \chi &= 2 \sin^2 \left( \frac{\chi}{2} \right)
 \end{aligned}$$

$$v'_1 = \frac{v}{m_1 + m_2} \sqrt{m_1^2 + m_2^2 + 2 m_1 m_2 \cos \chi}$$