

Notes: Thurs 8/27

- 1) Elliptic Functions \hookrightarrow go beyond small angle approx
- 2) Simple pendulum

Elliptic Functions:

$$\int \frac{dx}{\sqrt{1-x^2}} = ? = \sin^{-1}(x) + \text{const}$$

"
 $\arcsin(x)$

$$\rightarrow \sin^2 \theta + \cos^2 \theta = 1$$

subst: $x = \sin \theta \rightarrow \sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta$

$$dx = \cos \theta d\theta$$

$$\int \frac{\cos \theta d\theta}{\cos \theta} = \int d\theta = \theta = \sin^{-1}(x) \quad \begin{matrix} x = \sin \theta \\ \theta = \sin^{-1}(x) \end{matrix}$$

$$\frac{d}{d\theta} \sin \theta = \cos \theta$$

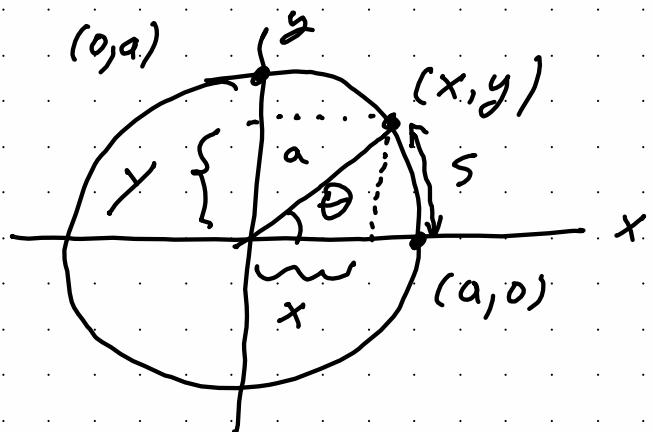
$$\lim_{h \rightarrow 0} \left(\frac{\sin(\theta+h) - \sin \theta}{h} \right)$$

$$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \dots$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

circular functions..

$$x^2 + y^2 = a^2, \quad a = \text{radius}$$



$$\underline{\text{Def:}} \quad \sin \theta = \frac{y}{a}, \quad \cos \theta = \frac{x}{a}$$

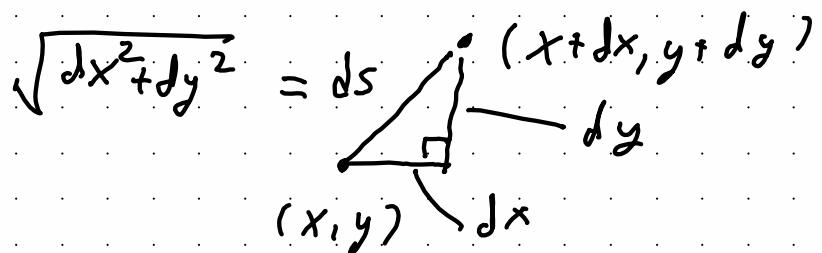
s : arc length from $(a, 0)$ to (x, y)

$$s = a\theta \quad | \quad \theta = \frac{s}{a}$$

$$= \frac{1}{a} \int ds$$

$$= \frac{1}{a} \int \sqrt{dx^2 + dy^2}$$

$$= \int d\theta$$



$$\underline{\text{Given:}} \quad x^2 + y^2 = a^2$$

$$\underline{\text{Follows:}} \quad (i) \quad a^2 \cos^2 \theta + a^2 \sin^2 \theta = a^2 \rightarrow \boxed{\cos^2 \theta + \sin^2 \theta = 1}$$

$$(ii) \quad \boxed{\frac{d \sin \theta}{d \theta}} = \frac{1}{a} \frac{dy}{d \theta} = \frac{1}{a} \frac{dy}{\sqrt{dx^2 + dy^2}} = \frac{1}{\sqrt{x^2 + y^2}}$$

$$2x dx + 2y dy = 0 \rightarrow dx = -\frac{y}{x} dy \quad | \quad \Rightarrow \frac{1}{\sqrt{x^2 + y^2}} = \frac{1}{\sqrt{x^2}} = \frac{x}{a} = \boxed{\cos \theta}$$

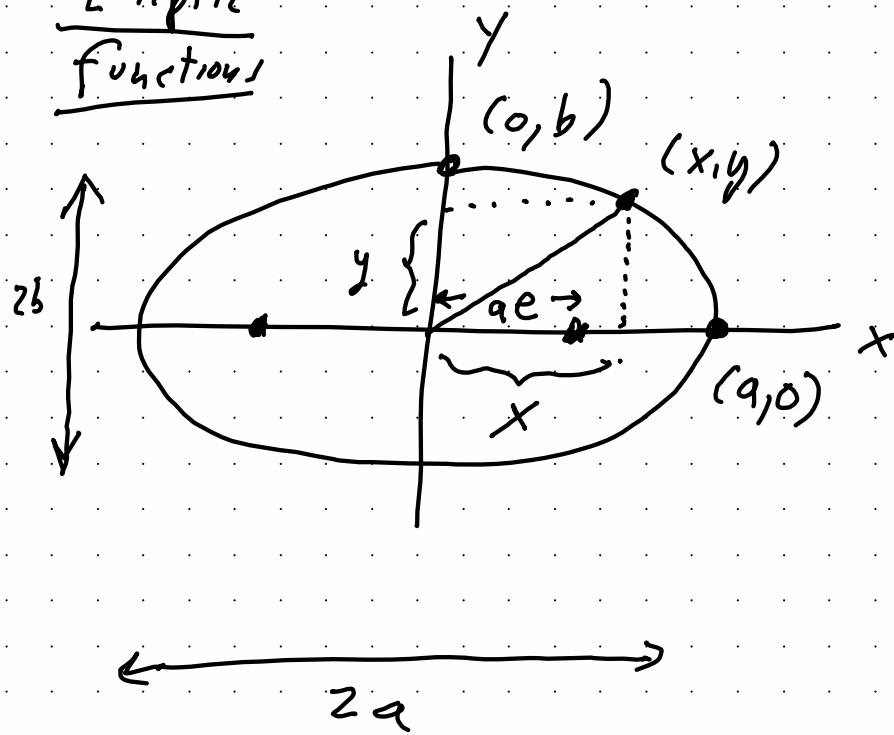
$$\frac{d(\sin \theta)}{d\theta} = \cos \theta \rightarrow \int \frac{d(\sin \theta)}{\cos \theta} = \int d\theta$$

$$x = \sin \theta$$

$$\begin{aligned}\cos \theta &= \sqrt{1 - \sin^2 \theta} \\ &= \sqrt{1 - x^2}\end{aligned}$$

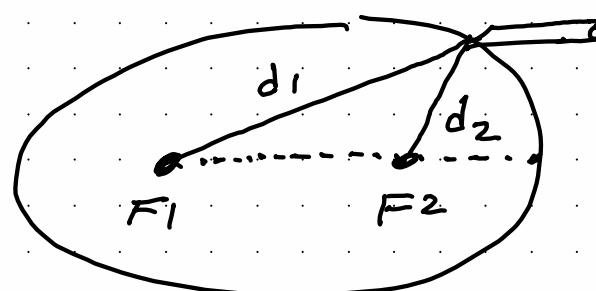
$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \theta = \sin^{-1}(x)}$$

Elliptic functions

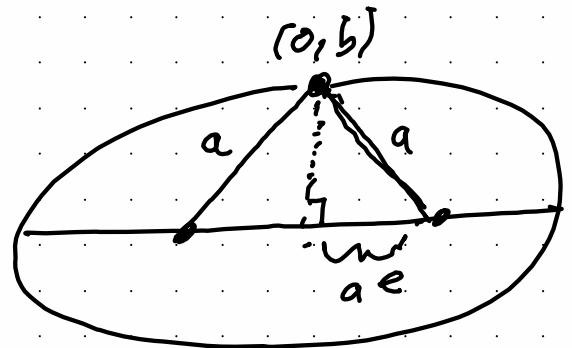


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

eccentricity : $e = 0$ (for circle)



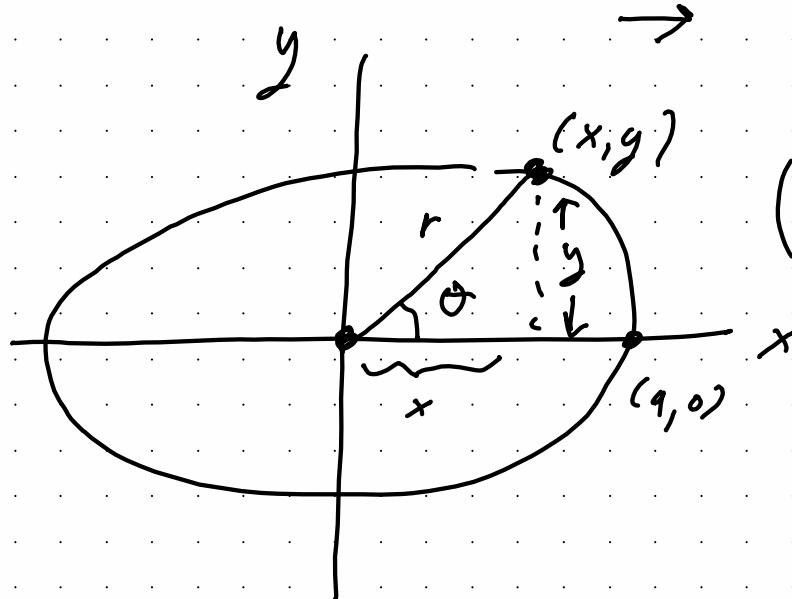
$$d_1 + d_2 = 2a$$



$$(ae)^2 + b^2 = a^2$$

$$a^2e^2 + b^2 = a^2$$

$$e^2 = \frac{a^2 - b^2}{a^2} = 1 - \left(\frac{b}{a}\right)^2$$



$$\rightarrow e = \sqrt{1 - \left(\frac{b}{a}\right)^2} = k$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

notation used
in elliptic
function

$$x^2 + y^2 = r^2$$

$$x = r\cos\theta, y = r\sin\theta$$

Define: $\operatorname{cn}(u; \kappa) \equiv \frac{x}{a}$, $\operatorname{sn}(u; \kappa) \equiv \frac{y}{b}$

$$\operatorname{dn}(u; \kappa) \equiv \frac{r}{a} \quad (=1 \text{ for a circle})$$

where $u = \frac{1}{b} \int_0^\theta r d\theta$ $y = r\sin\theta$ $ds = \sqrt{dx^2 + dy^2}$
 $(= \theta \text{ for a circle})$ $= \sqrt{dr^2 + r^2 d\theta^2}$

$$\text{Given: } \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad , \quad x^2 + y^2 = r^2 \quad dn(u; k) = \frac{r}{a}$$

$$\begin{aligned} \text{Follows: (i)} \quad & cn^2(u; k) + sn^2(u; k) = 1 \\ \text{(ii)} \quad & dn^2(u; k) + k^2 sn^2(u; k) = 1 \end{aligned} \quad u = \int_0^\theta r d\theta$$

$$(iii) \quad \frac{d}{du} sn(u; k) = cn(u; k) dn(u; k)$$

$$\frac{d}{du} cn(u; k) = -sn(u; k) dn(u; k)$$

$$\frac{d}{du} dn(u; k) = -k^2 sn(u; k) cn(u; k)$$

Integrate: $\frac{d sn(u; k)}{dn(u; k)} = cn(u; k) dn(u; k)$

$$\int \frac{d sn(u; k)}{cn(u; k) dn(u; k)} = \int du = u$$

$$\int \frac{dx}{\sqrt{1-x^2}} = u + \text{const} = \sin^{-1}(x; k) + \cos, t$$

$$x \equiv sn(u; k)$$

Analogous to
 $\frac{ds \sin \theta}{d\theta} = \cos \theta$

$$\frac{ds \sin \theta}{\cos \theta} = d\theta$$

$$x = \sin \theta$$

$$\int \frac{dx}{\sqrt{1-x^2}} = d\theta = \theta = \sin^{-1} x$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + \cos, t$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-k^2 x^2}} \equiv K(k) \rightarrow$$

(Complete elliptic integral of 1st kind)

related to
Period of a pendulum
going beyond
small-angle
approximation

$$\int_0^1 \frac{\sqrt{1-k^2 x^2} dx}{\sqrt{1-x^2}} \equiv E(k) \rightarrow$$

(Complete elliptic integral of 2nd kind)

circumference
around an ellip.



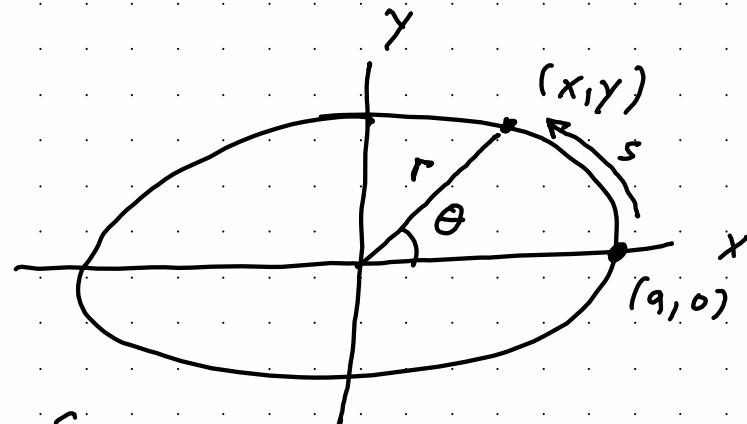
circle: $C = 2\pi a$

Notes: Tuesday 9/1

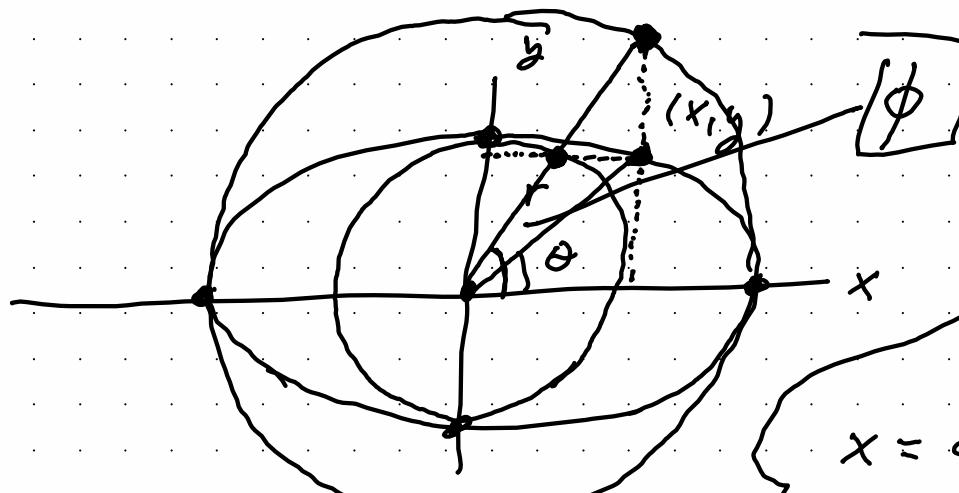
- 1) Review of elliptic functions
- 2) Simple pendulum

$$u = \frac{1}{b} \int_{(a,0)}^{(x,y)} r d\theta$$

$$bu = \int_0^\theta r d\theta \leq \int_0^\theta ds = s$$



$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\theta^2} \geq r d\theta$$



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$x = a \cos \phi$$

$$y = b \sin \phi$$

$$x = a \cos \phi, y = a \sin \phi$$

$$x = b \cos \phi, y = b \sin \phi$$

Simple pendulum:

(i) "Freshman physics"

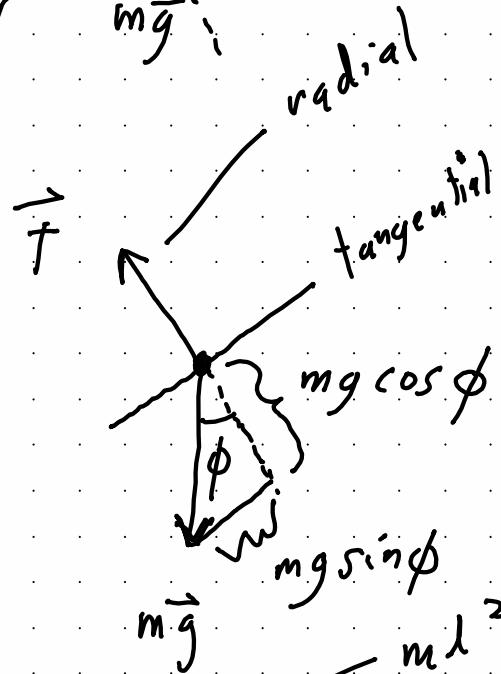
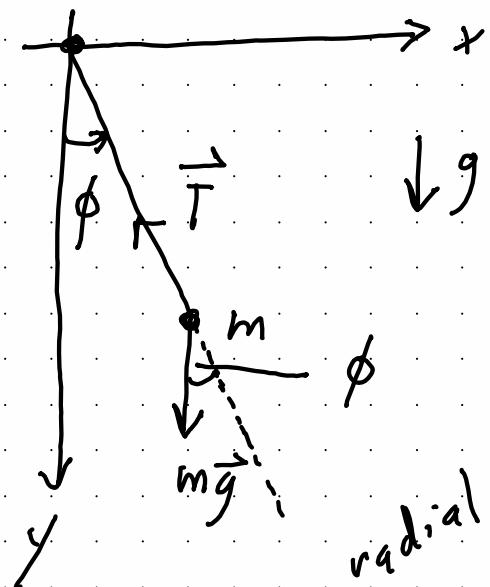
forces, free-body diagrams

→ EoM, tension

tangential:

$$-mg \sin \phi = m a_{\text{tangential}}$$

$$-g \sin \phi = m \ddot{\phi}$$



$$\text{Torque} = I \alpha - \dot{\phi}$$

ϕ : angular displacement [rad]

$\dot{\phi}$: angular velocity [rad/sec]

$\ddot{\phi}$: angular accel [rad/sec²]

$$\boxed{\ddot{\phi} = -\frac{g}{l} \sin \phi} \quad (\text{EoM})$$

radial: $T - mg \cos \phi = m a_{\text{centripetal}}$

$$T - mg \cos \phi = m \dot{\phi}^2 l$$

$$\boxed{T = mg \cos \phi + m \dot{\phi}^2 l}$$

(ii) Small angle approximation:

$$\sin \phi \approx \phi \leftarrow \phi \ll 1 = 57 \text{ degrees}$$

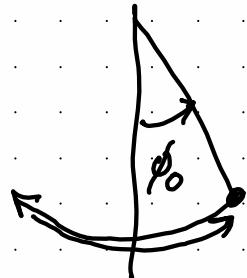
$$\ddot{\phi} = -\frac{g}{l} \sin \phi \approx -\frac{g}{l} \phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l} \phi}$$

$$\phi(t) = \boxed{a} \cos(\omega t) + \boxed{b} \sin(\omega t)$$

where $\omega = \sqrt{\frac{g}{l}}$ determined by initial conditions

I.Cs: If $\phi(0) = \phi_0$ (at rest)

then $\boxed{\phi(t) = \phi_0 \cos(\omega t)}$



Period: $P = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}}$

independent of ϕ_0 !!

(iii) Lagrangian approach $T \equiv$ Kinetic Energy

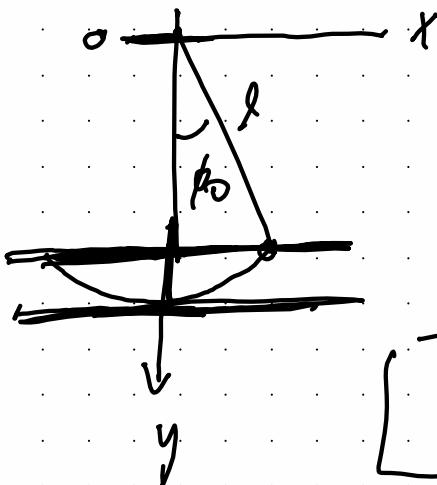
$$L = T - U$$

$U \equiv$ Potential energy

$$T = \frac{1}{2} m l^2 \dot{\phi}^2 \quad (= \frac{1}{2} m (x^2 + y^2))$$

$$y = l \cos \phi$$

$$x = l \sin \phi$$



$$U = -mg l \cos \phi + \text{const}$$

$$U = -mgy \quad \text{action}$$

$$U = mgy l (1 - \cos \phi)$$

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mg l \cos \phi$$

$$S = \int_{t_1}^{t_2} dt L(\phi, \dot{\phi}, t)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \quad \text{Lagrange's equation}$$

$$\frac{d}{dt} (ml^2 \dot{\phi}) = -mg l \sin \phi$$

$$ml^2 \ddot{\phi} = -mg l \sin \phi$$

$$\ddot{\phi} = -\frac{g}{l} \sin \phi \quad (\text{Lam})$$

(iv) solving $\ddot{\phi} = -\frac{g}{l} \sin \phi$ (2^{nd} order non-linear)

$$E = \text{const}$$

$$= T + U$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 - m g l \cos \phi$$

ODE ↑

hard!!

$$E = 0 - m g l \cos \phi_0$$

release from rest

$$= -m g l \cos \phi_0$$

from $\phi = \phi_0$

$$-m g l \cos \phi_0 = \frac{1}{2} m l^2 \dot{\phi}^2 - m g l \cos \phi$$

$$-m g l (\cos \phi_0 - \cos \phi) = \frac{1}{2} m l^2 \dot{\phi}^2$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{-2 \frac{g}{l} (\cos \phi_0 - \cos \phi)}$$

$|\phi| \leq \phi_0$

$$\int \frac{d\phi}{\sqrt{-2 \frac{g}{l} (\cos \phi_0 - \cos \phi)}} = \int dt = t + \text{const}$$

Separable
1st order
ODE

$$t + t_0 = \int \frac{d\phi}{\sqrt{-2 \frac{g}{l} (\cos \phi_0 - \cos \phi)}}$$

Substitution:

$$\cos \phi = 1 - 2 \sin^2 \left(\frac{\phi}{2} \right)$$

$$\cos \phi_0 = 1 - 2 \sin^2 \left(\frac{\phi_0}{2} \right)$$

$$\cos \phi = \cos \left(2 \left(\frac{\phi}{2} \right) \right)$$

$$= \cos^2 \left(\frac{\phi}{2} \right) - \sin^2 \left(\frac{\phi}{2} \right)$$

$$= 1 - 2 \sin^2 \left(\frac{\phi}{2} \right)$$

$$\Rightarrow \cos \phi_0 - \cos \phi = -2 \left(\sin^2 \left(\frac{\phi_0}{2} \right) - \sin^2 \left(\frac{\phi}{2} \right) \right)$$

$$t + t_0 = \int \frac{d\phi}{2 \sqrt{\frac{g}{l}} \sqrt{\sin^2 \left(\frac{\phi_0}{2} \right) - \sin^2 \left(\frac{\phi}{2} \right)}}$$

$$|\phi| \leq \phi_0$$

$$= \frac{1}{2 \sqrt{\frac{g}{l}}} \int \frac{d\phi}{\sin \left(\frac{\phi_0}{2} \right) \sqrt{1 - \frac{\sin^2 \left(\frac{\phi}{2} \right)}{\sin^2 \left(\frac{\phi_0}{2} \right)}}}$$

let $x = \sin \left(\frac{\phi}{2} \right)$

$$\sin \left(\frac{\phi_0}{2} \right)$$

$$x = \frac{\sin(\frac{\phi}{2})}{\sin(\frac{\phi_0}{2})}$$

$$\sqrt{1-x^2}$$

denominator

Find this out

① $\phi(t) =$

② Period = ??

③ Redo the analysis using Lagrange multiplier for find tension in strings

$$t + t_0 = \int \text{---} \quad \begin{matrix} \text{integrated} \\ \text{for } \sin^{-1}(x_j/T) \end{matrix}$$

$$T = \sin\left(\frac{\phi_0}{2}\right)$$

$$dx = \frac{1}{\sin(\frac{\phi_0}{2})} \cdot \frac{1}{2} \cos\left(\frac{\phi}{2}\right) d\phi$$

$$d\phi = \frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\cos\left(\frac{\phi}{2}\right)}$$

$$= \frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1 - \sin^2\left(\frac{\phi}{2}\right)}}$$

$$= \frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1 - \sin^2\left(\frac{\phi_0}{2}\right) x^2}}$$

$$\sqrt{1 - \sin^2\left(\frac{\phi_0}{2}\right) x^2}$$

$$\hookrightarrow T^2$$

Lec #4 : Thurs 9/3

$$\phi(t) = 2 \sin^{-1} \left[k \operatorname{sn} \left(\omega_0 \left(t + \frac{P}{4} \right); k \right) \right] \star$$

$$k \equiv \sin \left(\frac{\phi_0}{2} \right), \quad \omega_0 = \sqrt{\frac{g}{L}}$$

$$P = 4\sqrt{\frac{L}{g}} \quad K(k) = 4\sqrt{\frac{L}{g}} \int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt{1-k^2 x^2}}$$

small angle
approx

$$P_0 = 2\pi \sqrt{\frac{L}{g}}$$

$$\frac{\sin \left(\frac{\phi}{2} \right)}{\sin \left(\frac{\phi_0}{2} \right)} = x = \operatorname{sn} \left[\sqrt{\frac{g}{L}} (t + t_0); k \right]$$

$$\sqrt{\frac{L}{g}} \int \frac{dx}{\sqrt{1-x^2} \sqrt{1-k^2 x^2}} = \sqrt{\frac{L}{g}} \operatorname{sn}^{-1}(x; k) = t + t_0$$

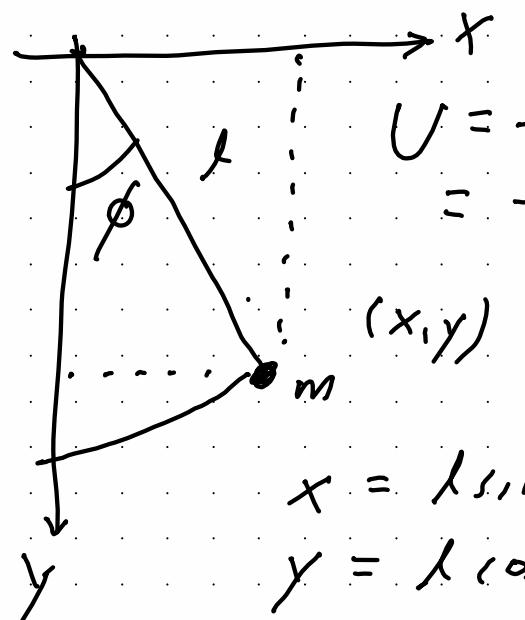
$$\operatorname{sn}^{-1}(x; k) = \sqrt{\frac{g}{L}} (t + t_0)$$

$$P = 4\sqrt{\frac{L}{g}} T(H) = 2\pi \sqrt{\frac{L}{g}} \left(1 + \frac{1}{16} \phi_0^2 + \dots \right)$$

Problem Landau II, 1

Lagrange multiplier:

$$\varphi(x, y) = x^2 + y^2 - l^2 = 0$$



$$\begin{aligned} U &= -mg y \\ &= -mgl \cos \phi \end{aligned}$$

$$\begin{aligned} x^2 + y^2 &= l^2 \sin^2 \phi + l^2 \cos^2 \phi \\ &= l^2 \end{aligned}$$

$$\begin{aligned} x &= l \sin \phi \\ y &= l \cos \phi \end{aligned}$$

$$\begin{aligned} T &= \frac{1}{2} m (x'^2 + y'^2) \\ &= \frac{1}{2} m l^2 \dot{\phi}^2 \end{aligned}$$

$$L = T - U + \lambda \phi$$

\uparrow
 $L(x, \dot{x}, y, \dot{y}, t)$

$$L(x, \dot{x}, y, \dot{y}, t)$$

$$L(r, \dot{r}, \phi, \dot{\phi}, t)$$

\square Lagrange multiplier

$$q = (x, y) \quad \dot{q} = (\dot{x}, \dot{y})$$

$$q = (r, \phi) \quad \dot{q} = (r\dot{\phi}, \dot{\phi})$$

$\lambda(t)$
 $r(t)$
 $\phi(t)$

$$L(\phi, \dot{\phi}, t) \quad \phi(x, y) = x^2 + y^2 - l^2 = 0$$

$$\phi(r, \phi) = r - l = 0$$

$$L = \frac{1}{2} m (\ddot{r}^2 + r^2 \dot{\phi}^2) + mg \underbrace{r \cos \phi}_{y} + \lambda (r - l)$$

$$r: \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} \rightarrow \cancel{m \ddot{r}} = mr\dot{\phi}^2 + mg \cos \phi + \lambda$$

$$\phi: \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \rightarrow \cancel{\frac{d}{dt} (mr^2 \dot{\phi})} = -mg r \sin \phi$$

$$\lambda: \cancel{r - l = 0} \quad \checkmark \quad \cancel{2mr\ddot{\phi} + mr^2 \ddot{\phi}} = -mg r \sin \phi$$

$$r - l = 0 \rightarrow r = l$$

$$\dot{r} = 0$$

$$\ddot{r} = 0$$

$$\rightarrow \ddot{\phi} = -\frac{g}{l} \sin \phi$$

$$ml\dot{\phi}^2 + mg \cos \phi + \lambda = 0$$

$$\boxed{\lambda = -(mg \cos \phi + ml\dot{\phi}^2)}$$

$$\lambda = -T$$

T

$$L = T - U + \lambda \phi$$

$$U = U(x, t)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\frac{d}{dt} (m \dot{x}) = - \frac{\partial U}{\partial x} + \lambda \frac{\partial \phi}{\partial x}$$

$$\phi(x, y, t)$$

$$U(x, y, t)$$

$$\frac{d}{dt} (m \dot{y}) = - \frac{\partial U}{\partial y} + \lambda \frac{\partial \phi}{\partial y}$$

$$\frac{d \vec{P}}{dt} = \vec{F}_{\text{net}}$$

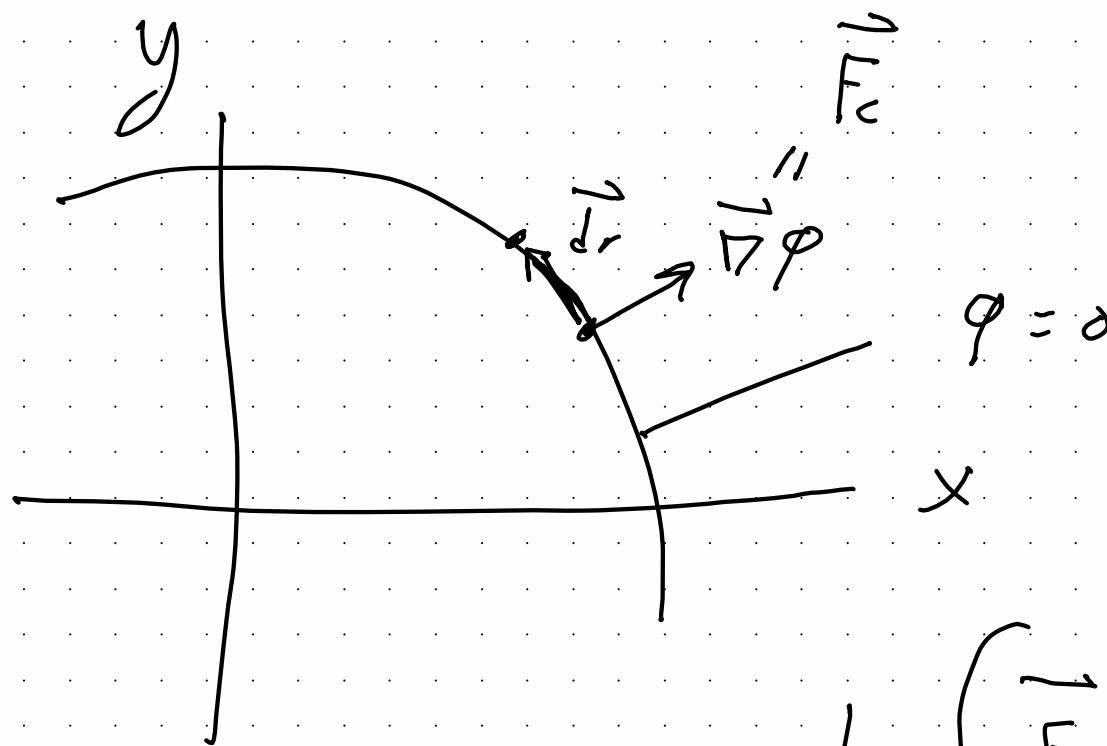
applied force force

$$\frac{d p_x}{dt} = F_x + \lambda \frac{\partial \phi}{\partial x}$$

$$\frac{d \vec{P}}{dt} = \vec{F} + \lambda \vec{\nabla} \phi$$

$$\frac{d p_y}{dt} = F_y + \lambda \frac{\partial \phi}{\partial y}$$

$$\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) = \vec{\nabla} \phi$$



$$\nabla \varphi \perp \varphi = \text{const} +$$

$$\int \vec{F}_c \cdot d\vec{r} = \text{Wort}$$

$$\varphi = r - \ell$$

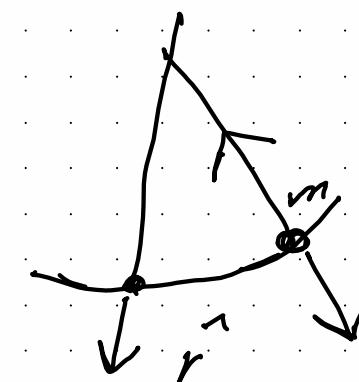
$$\nabla \varphi = \hat{r}$$

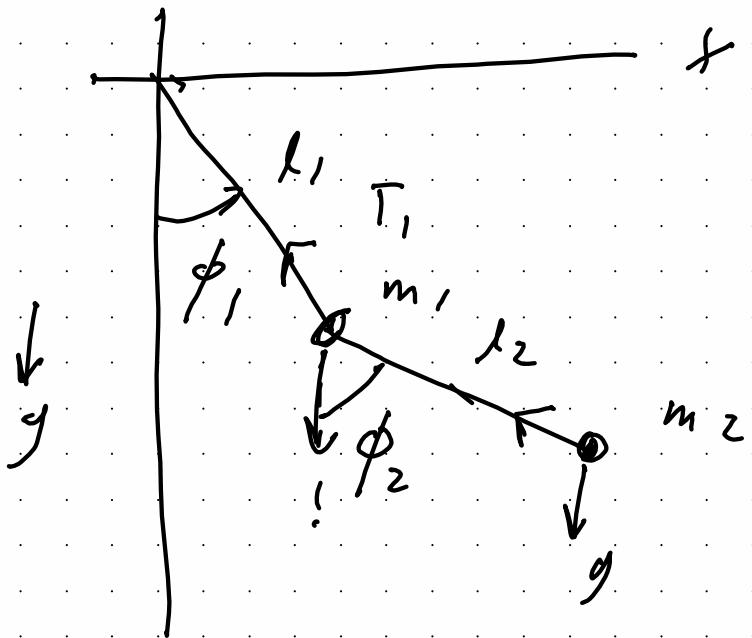
$$\nabla \varphi$$

$$(r, \varphi)$$

$$\hat{r}, \hat{\varphi}$$

$$\hat{r}$$

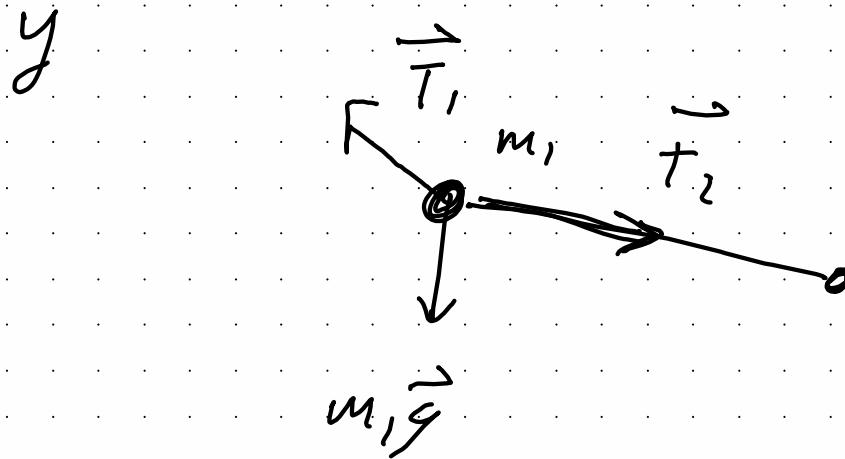




$$U = -m_1 g y_1 - m_2 g y_2$$

ϕ_1, ϕ_2

E_{om}

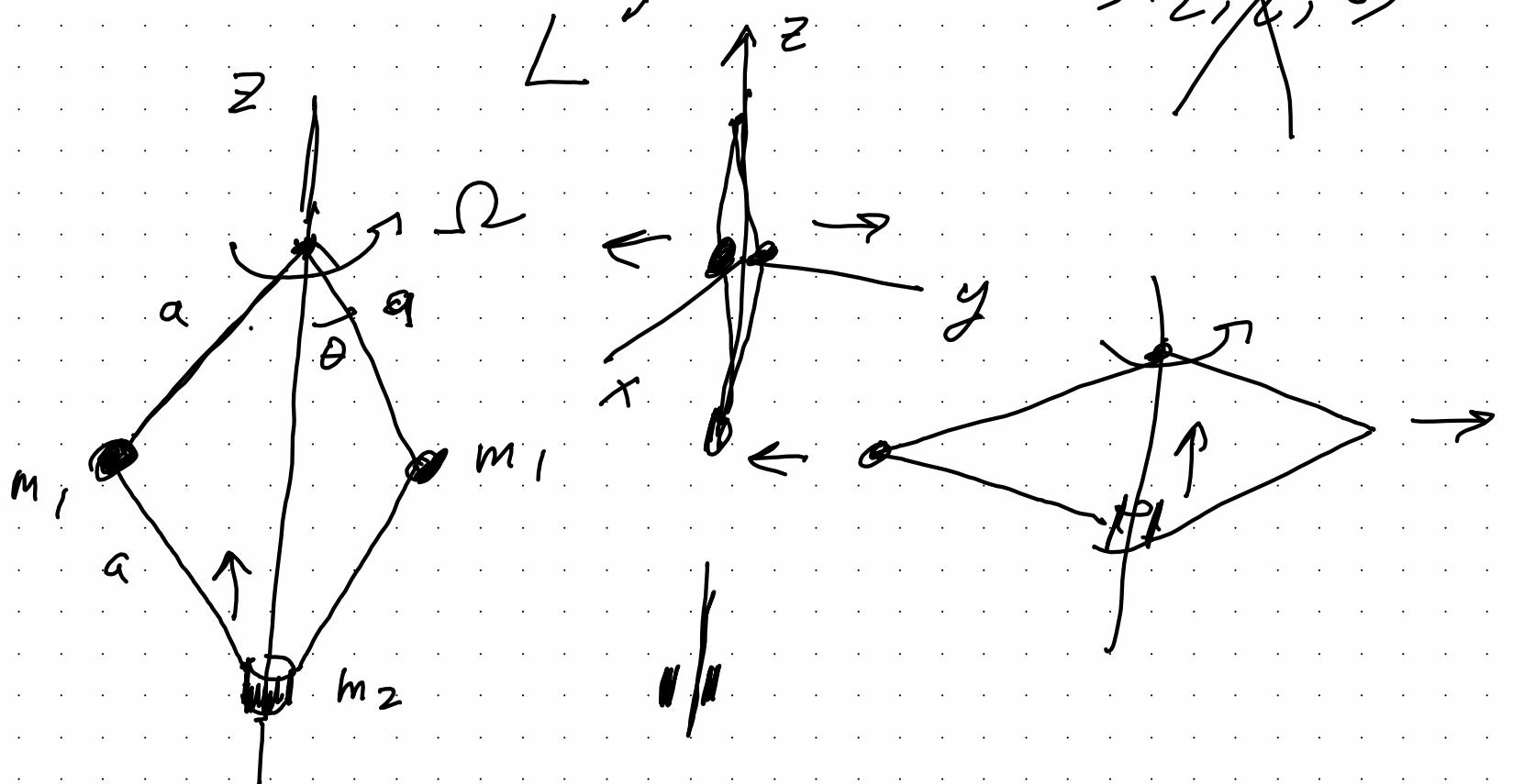


$$\begin{aligned} & T \phi \\ & (\lambda_1 \phi_1 + \lambda_2 \phi_2 + \dots) \end{aligned}$$

L  EOM  L

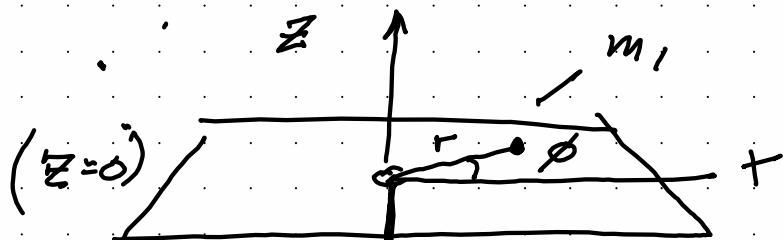
$$L \rightarrow L + \frac{d}{dt} (f(q, t))$$

$$\cancel{f(q, \dot{q}, t)}$$

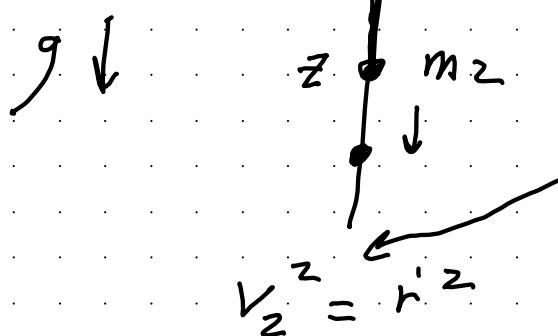


Lec #5: Tuesday 9/18

$$r - z = \ell = \text{length of string}$$



$$L = T - U$$



$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$v_2^2 = \dot{z}^2, \quad v_1^2 = \dot{r}^2 + r^2 \dot{\phi}^2 \\ (= \dot{x}^2 + \dot{y}^2, \quad x = r \cos \phi, \quad y = r \sin \phi)$$

$$\boxed{T = \frac{1}{2} m_1 (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2 \\ = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2}$$

$$U = m_2 g z = m_2 g (r - \ell) = m_2 g r - m_2 g \ell$$

$$\boxed{U = m_2 g r}$$

$$L = T - U = \boxed{\frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r}$$

constant

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r}, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \Rightarrow \text{2nd order EOMs}$$

No explicit t dependence:

$$E = \dot{r} \frac{\partial L}{\partial \dot{r}} + \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - L = \boxed{T + U}$$

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

\downarrow

$$= p_i$$

No explicit ϕ dependence:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \text{const} = M_z$$

$$M_z = m_1 r^2 \dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{m_1 r^2}$$

M : angular momentum
(L&L notation)

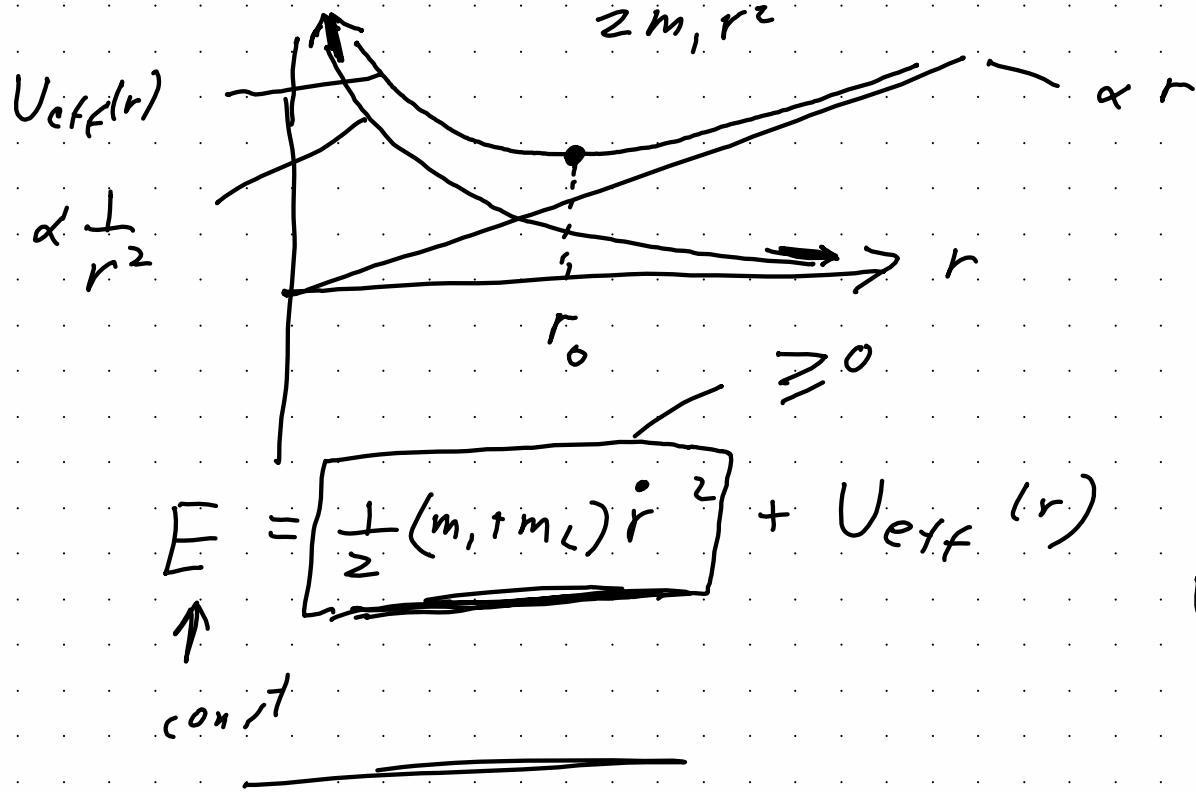
$$E = \frac{1}{2} m \dot{r}^2 + U(r)$$

$$E = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 + m_2 g r$$

$$E = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \left(\frac{M_z^2}{2 m_1 r^2} + m_2 g r \right)$$

$$U_{\text{eff}}(r) = U(r) + \frac{M_z^2}{2 m_1 r^2}$$

$$U_{eff}(r) = \frac{M_2^2}{2m_1 r^2} + m_2 g r$$



i) $E = U_{eff, min} = U_{eff}(r_0)$

unif circular motion: $r = r_0$, $\dot{\phi} = \frac{M_2}{m_1 r_0^2}$

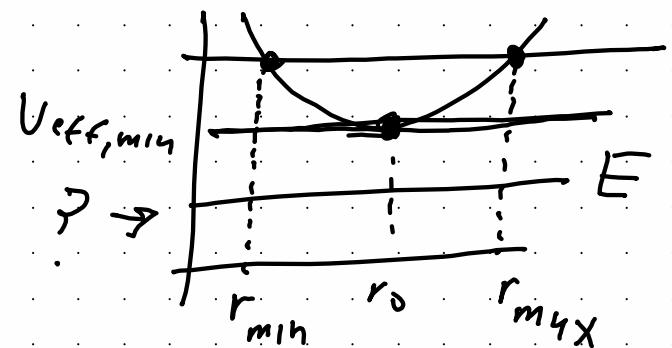
ii) $E > U_{eff, min}$

$$E = U_{eff}(r_{min}) = U_{eff}(r_{max})$$

$$U(r) = m_2 g r$$

r_{min}, r_{max} :

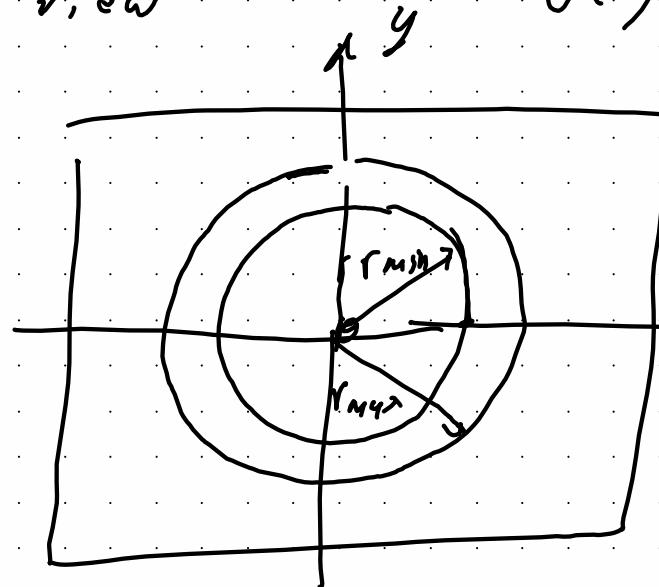
"turning points":
($r=0$)



$E < U_{eff, min}$
(not allowed)

$$E \geq U_{eff, min}$$

top view



$$U(r) = \frac{1}{2} k r^2$$

closed bound

$$U(r) = -\frac{G m_1 m_2}{r}$$

r

Newtonian gravity
bound orbits = ellipse
are closed

1 degree

11

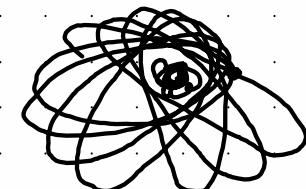
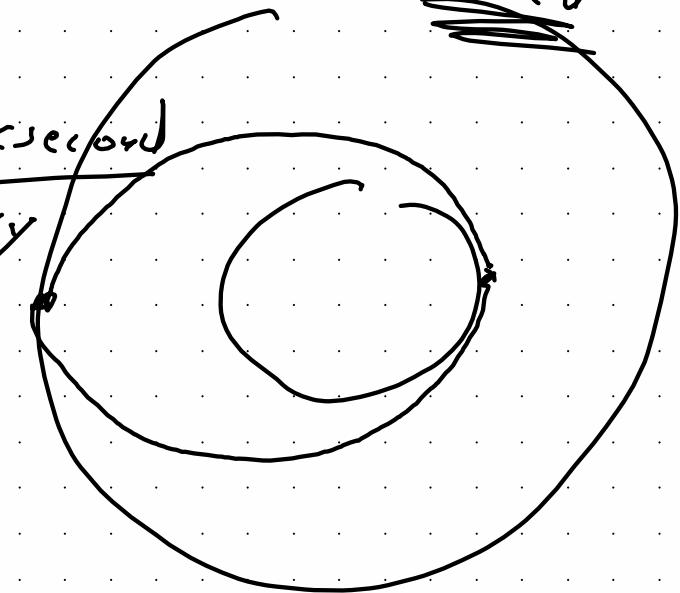
60 mins
of arc

1 min of
arc

11
60 sec of arc

perihelion precession of Mercury

closest approach to sun



$$\underline{r_0} : \left. \frac{dU_{\text{eff}}}{dr} \right|_{r=r_0} = 0 \quad (\text{minimum})$$

$$0 = \left. \frac{d}{dr} \left(\frac{M_z^2}{2m_1 r^2} + m_2 gr \right) \right|_{r=r_0}$$

$$= -\frac{M_z^2}{m_1 r_0^3} + m_2 g$$

$$M_z^2 = m_1 m_2 g r_0^3$$

tells you the value of M_z needed to have a specific r_0 value.

For a given M_z , this tells you what r_0 equals.

Energy equation:

$$E = \frac{1}{2} (m_1 + m_2) r^2 + \frac{M_z^2}{2m_1 r^2} + m_2 gr$$

$$\boxed{\dot{\phi} = \frac{M_z}{m_1 r^2}} \quad \leftarrow \quad \phi \text{ equation}$$

$$\frac{1}{2} (m_1 + m_2) r^2 = E - \frac{M_z^2}{2m_1 r^2} - m_2 gr$$

$$\dot{r} = \frac{dr}{dt} = \pm \sqrt{\left(\frac{2}{m_1 + m_2} \right) \left(E - \frac{M_z^2}{2m_1 r^2} - m_2 gr \right)}$$

$$\int \frac{dr}{\sqrt{\left(\frac{2}{m_1 + m_2} \right) \left(E - \frac{M_z^2}{2m_1 r^2} - m_2 gr \right)}} = \int dt = t + \text{const}$$

$r(t) \Leftrightarrow r(t)$

orbital equations:

$$r = r(\phi) \iff \phi = \phi(r)$$

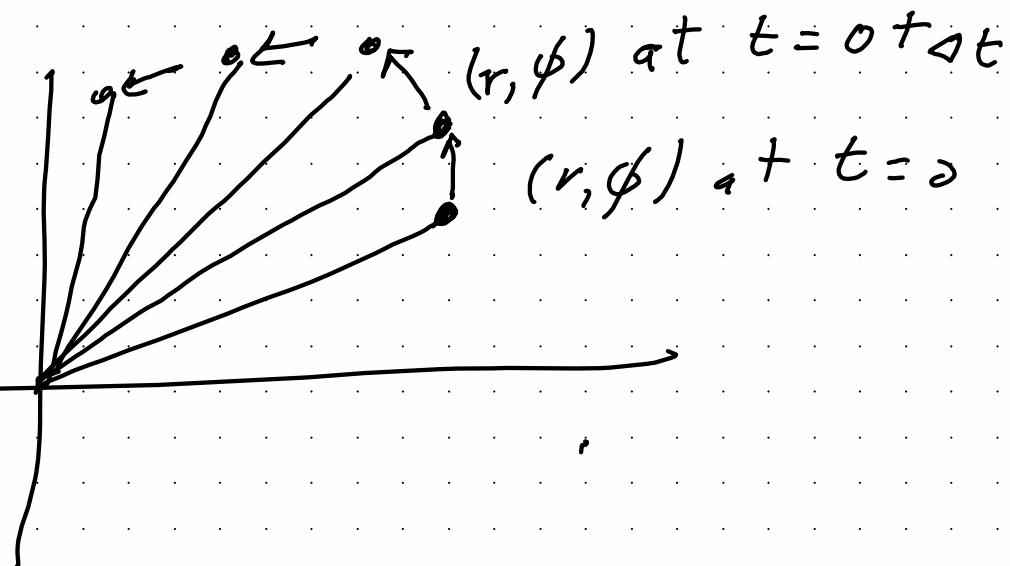
$$\frac{dr}{dt} = \dot{r} = \sqrt{\frac{2}{m_1 + m_2}} \left[E - \frac{M_Z^2}{2m_1 r^2} - m_2 g r \right]$$

$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{M_Z}{m_1 t^2}$$

with

$$\rightarrow \frac{dr}{d\phi} = \frac{m_1 r^2}{M_Z} \sqrt{\frac{2}{m_1 + m_2}} \left[\quad \right]$$

$$\int \frac{dr}{\frac{m_1 r^2}{M_Z} \sqrt{\left[\quad \right]}} = \int d\phi = \phi + \cos t$$



Show, r, ϕ at some time t

Given: Δt need to know Δr and $\Delta \phi$

$$r(t+\Delta t) = r(t) + \Delta r(t) + \dots$$

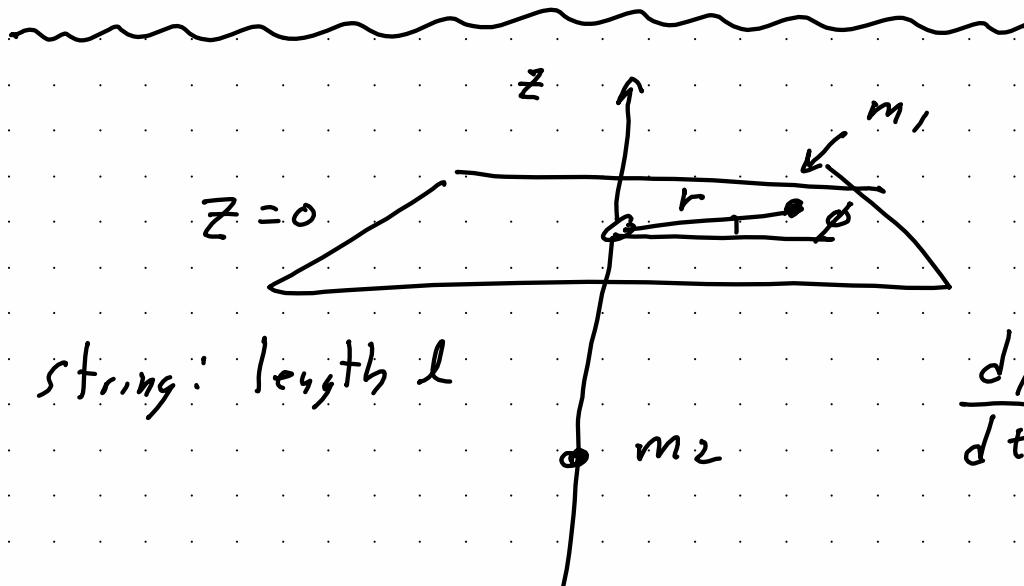
$$\phi(t+\Delta t) = \phi(t) + \Delta \phi(t) + \dots$$

(W)

ignore it if Δt
is suff. small

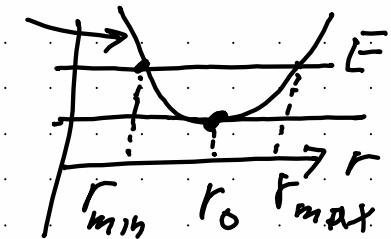
Lecture #6 : Thursday 10 Sep

- 1) Secs 6-10 (today), sec 40 (next Tuesday)
- 2) Finish up example from last time
- 3) Conservation of E , \vec{P} , \vec{m}
- 4) Mechanical similarity
- 5) Quiz: last 20 minutes (1:30 pm)



$$E, M_Z = \text{const} \quad v \quad v$$

v_{eff}



$$\frac{d\phi}{dt} = \frac{M_Z}{m_1 r^2}$$

$$\frac{dr}{dt} = \sqrt{\left(\frac{2}{m_1 + m_2}\right) \left(E - m_2 gr - \frac{M_Z^2}{2m_1 r^2} \right)} = \sqrt{\Theta}$$



$$\frac{d\phi}{dt} = \frac{M_2}{m_1 r^2} \rightarrow \Delta\phi = \Delta t \frac{M_2}{m_1 r^2}$$

$$\frac{dr}{dt} = \sqrt{\textcircled{2}} \rightarrow \Delta r = \Delta t \sqrt{\textcircled{3}}$$

$$r(0) = r_{min}$$

$$\phi(0) = 0$$

$$\phi(\Delta t) = \phi(0) + \Delta\phi$$

$$r(\Delta t) = r(0) + \Delta r$$

$$\phi(2\Delta t) = \phi(\Delta t) + \Delta\phi$$

$$r(2\Delta t) = r(\Delta t) + \Delta r$$

⋮

$$\phi(t_i) = \phi(t_{i-1}) + \Delta\phi$$

$$r(t_i) = r(t_{i-1}) + \Delta r$$

Cons. of E , \vec{P} , \vec{M} :

All of E , \vec{P} , \vec{M} conserved for a closed system

no external forces

$$U = U(\vec{r}_1 - \vec{r}_2, \vec{r}_2 - \vec{r}_3, \dots)$$

relative position vectors

Even in the presence of external forces, you can still have cons. of E and some components of \vec{P} and \vec{M} .

(i) $U = mgx \quad \vec{F} = -m\vec{g}$

If U does not depend explicitly on time t , then E is conserved.

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

constant
external
field

(ii) e.g. $\downarrow \vec{F}_g = m\vec{g}$

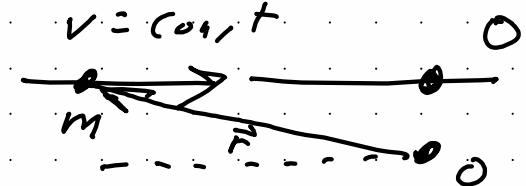
$$\xleftarrow{\hspace{1cm}} \quad x, y \quad \xrightarrow{\hspace{1cm}}$$

$$P_x = \text{const}$$

$$P_y = \text{const}$$

If U is unchanged by a translation in some direction \hat{E} then $\vec{P} \cdot \hat{E} = \text{const}$

$$\vec{v} = \text{const}$$



(iii) \vec{M} depends on choice of origin

(a) uniform gravitational field



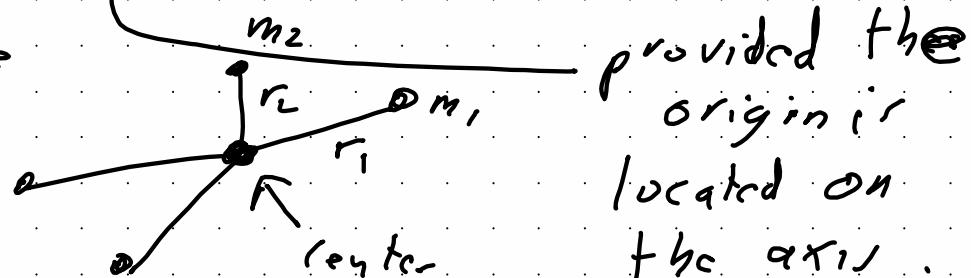
If U is unchanged by a rotation about a particular axis \hat{n}

then $\boxed{\vec{M} \cdot \hat{n} = \text{const}}$ (e.g., $\hat{n} = \hat{z}$, $M_z = \text{const}$)

(b) central force

$$\boxed{M = \text{const}} \quad U = U(r)$$

$$\begin{aligned} \vec{F} &= -\nabla U \\ &= -\frac{dU}{dr} \hat{r} \end{aligned}$$



provided the origin is located on the axis.

Mechanical similarity :

$$L \rightarrow L' = c \cdot L$$

same equations of motion

suppose we rescale position vectors $\vec{r}_a \rightarrow \alpha \vec{r}_a$

$$U'(\vec{r}_1, \dots) = U(\alpha \vec{r}_1, \alpha \vec{r}_2, \dots) = \alpha^K U(\vec{r}_1, \vec{r}_2, \dots)$$

potential is homogeneous of degree K
w.r.t position vector

Example: i) $U = mgx$, $K = 1$

ii) $U = \frac{1}{2}kx^2$, $K = 2$

iii) $U = -\frac{Gm_1 m_2}{r}$, $K = -1$

$$\begin{aligned} U' &= mg \alpha x \\ &= \alpha mgx \\ &= \alpha U \end{aligned}$$

$$\begin{aligned} L' &= T' - U' = T' - \alpha^K U = \text{const.} \cdot L \\ &= \alpha^{K+1} T - \alpha^K U \\ &= \alpha^{K+1} (T - U) = \alpha^{K+1} L \end{aligned}$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \frac{l^2}{t^2} \quad \begin{array}{l} \text{length} \\ \text{time} \end{array}$$

$$\begin{aligned} l &\rightarrow l' = \alpha l \\ t &\rightarrow t' = \beta t \end{aligned}$$

$$T' = \frac{1}{2} m v'^2 = \frac{1}{2} m \frac{l'^2}{t'^2}$$

$$= \frac{1}{2} m \frac{\alpha^2 l^2}{\beta^2 t^2}$$

$$= \frac{\alpha^2}{\beta^2} T$$

$$\boxed{\frac{\alpha^2}{\beta^2} = \alpha^{4t}}$$

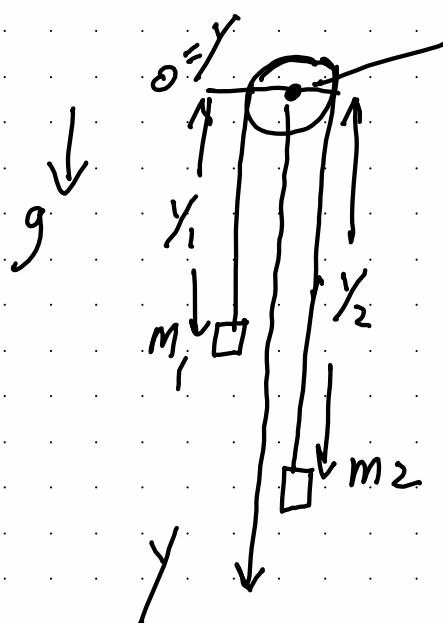
$$\beta^2 = \alpha^{2-4t}$$

$$\begin{cases} \frac{l'}{l} = \alpha \\ \frac{t'}{t} = \beta \end{cases} \quad \boxed{\begin{array}{l} U = mgy, k=1 \\ \frac{t'}{t} = \left(\frac{l'}{l}\right)^{\frac{1}{2}} \\ U = \frac{1}{2} kx^2, k=2 \\ \frac{t'}{t} = \text{const} \end{array}}$$

$$\boxed{\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-t/2}}$$

$$\begin{array}{c} \uparrow \\ \boxed{\beta = \alpha^{1-t/2}} \\ \uparrow \\ \boxed{\frac{P^2}{GJ} = \text{const}} \end{array} \quad \boxed{\begin{array}{l} U = -\frac{GM_1 M_2}{r} \\ kT = -1 \\ \left(\frac{t'}{t}\right) = \left(\frac{l'}{l}\right)^{3/2} \\ P^2 = \text{Dist}^3 \end{array}}$$

QUIZ #1 : Atwood's machine



$$(m_1 > m_2)$$

string: length l

(mass less
inextensible
...)

massless
frictionless
pulley

$$i) \quad L ?$$

$$ii) \quad EoM$$

$$iii) \quad \text{solve EoM}$$

$$y_1 + y_2 = l \rightarrow y_2 = l - y_1 \rightarrow \dot{y}_2 = -\dot{y}_1$$

$$U = -m_1 g y_1 - m_2 g y_2$$

$$= -m_1 g y_1 - m_2 g (l - y_1)$$

$$= -m_1 g y_1 - \underbrace{m_2 g l}_{\text{ignore}} + m_2 g y_1$$

$$= \boxed{-(m_1 - m_2) g y_1} \quad \parallel \dot{y}_1^2$$

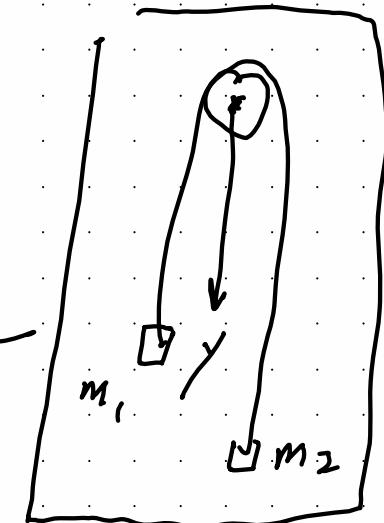
$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2$$

$$= \boxed{\frac{1}{2} (m_1 + m_2) \dot{y}_1^2}$$

Lecture #7: Tuesday 9/15

- 1) Go over quiz #1
- 2) Modified Atwood problem
- 3) Finish mechanical similarity (sec 10)
- 4) Hamilton's equations (sec 40)

$$\downarrow a \quad (a=g)$$



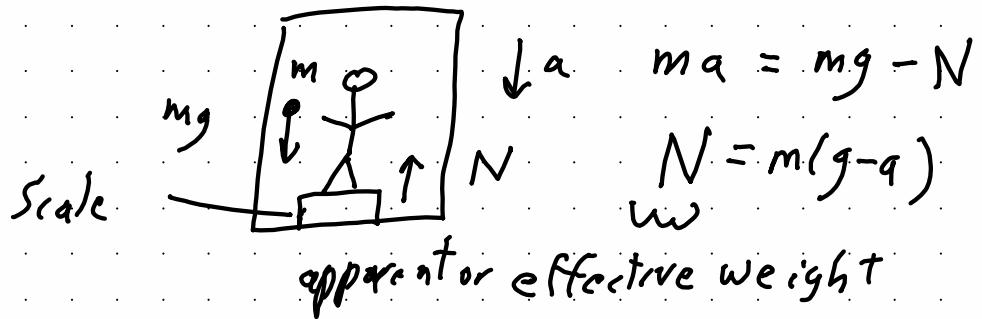
$$L = T - U$$

$$= \frac{1}{2} (m_1 + m_2) \dot{y}_1^2 + (m_1 - m_2) g y_1$$

EOMs: $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}_1} \right) = \frac{\partial L}{\partial y_1} \rightarrow \frac{(m_1 + m_2) \ddot{y}_1}{m_1 - m_2} = g$

$$\boxed{\ddot{y}_1 = \frac{(m_1 - m_2) g}{(m_1 + m_2)}}$$

$$y_1(t) = y_0 + v_0 t + \frac{1}{2} \frac{(m_1 - m_2) g t^2}{(m_1 + m_2)}$$



$$\boxed{\ddot{y}_1 = \frac{(m_1 - m_2) (g-a)}{m_1 + m_2}}$$

$$\vec{F} = \vec{m}\vec{a} \quad (\text{valid in an inertial ref frame})$$

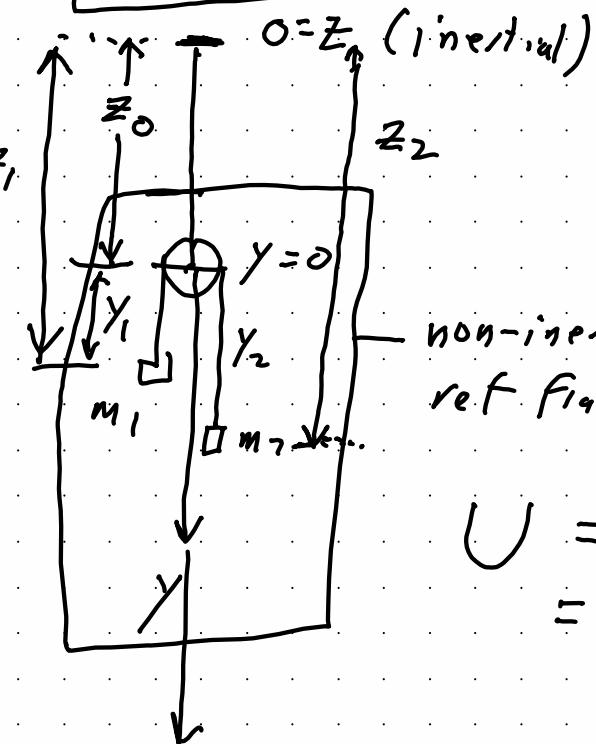
$$\vec{F} + \vec{F}_{\text{fictional}} = \vec{m}\vec{a} \quad \text{w.r.t. to a non-inertial ref. frame}$$

sec 3q (L&L)

$$L' = L + F(t)$$

$$L = T - U$$

(valid in an inertial ref frame)



non-inertial
ref frame

$$T = \frac{1}{2}m_1 \dot{z}_1^2 + \frac{1}{2}m_2 \dot{z}_2^2$$

$$U = -m_1 g z_1 - m_2 g z_2$$

$$z_1 = z_0 + y_1 \quad | \quad y_2 = l - y_1$$

$$z_2 = z_0 + y_2$$

$$l - y_1$$

$$U = -m_1 g (z_0 + y_1) - m_2 g (z_0 + y_2)$$

$$= \boxed{-m_1 g z_0} - m_1 g y_1 \boxed{-m_2 g z_0} \boxed{-m_2 g l + m_2 g y_1}$$

"const"

prescribed function of time = ignore

$$= \boxed{-(m_1 + m_2)g y_1}$$

Do this at home:

$$T = \frac{1}{2} (m_1 + m_2) \dot{y}_1^2 + (m_1 - m_2) z_0 \dot{y}_1$$

$$L' = L + \frac{d}{dt} (f(q, t)) \rightarrow \text{same EoM}$$

$$(m_1 - m_2) z_0 \dot{y}_1 = \frac{d}{dt} [(m_1 - m_2) z_0 \dot{y}_1] - (m_1 - m_2) z_0'' y_1$$

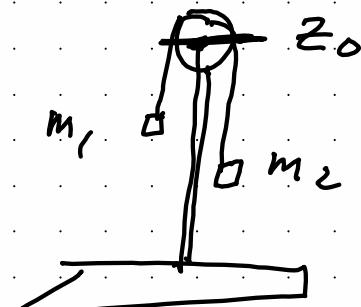
$f(y_1, t)$

$\equiv \alpha$

ignore

$$T = \frac{1}{2} (m_1 + m_2) \dot{y}_1^2 - (m_1 - m_2) \alpha y_1$$

$z_0(t)$: given, not to be solved for



Hamilton's equations:

$$L(q_i, \dot{q}_i, t)$$

Hamiltonian: $E = \sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$

$$H = H(q, p)$$

$$E(q, \dot{q}, t)$$

not here

if $L = L(q, \dot{q})$

$$H = \left(\sum p_i \dot{q}_i - L \right) \Big|_{\dot{q} = \dot{q}(q, p)}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Example:

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

(single particle, 1-d,
const external field)

$$p = \frac{\partial L}{\partial \dot{x}} = m \dot{x}$$



$$\dot{x} = \frac{p}{m}$$

$$H = (p \dot{x} - L) \Big|_{\dot{x} = p/m}$$

~~t~~

$$= \left(p \dot{x} - \frac{1}{2} m \dot{x}^2 + U(x) \right) \Big|_{\dot{x} = p/m} = \frac{p^2}{m} - \frac{1}{2} m \left(\frac{p}{m} \right)^2 + U(x)$$

$$H = \frac{1}{2} \frac{p^2}{m} + U(x)$$

EOMs: (Hamilton's equations) (39,6) Prob 2
Sec 40

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad i=1, \dots, s \quad \# \text{ of DOF}$$

Lagrange's equation:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} \quad i=1, \dots, s$$

s -equations, 2nd order ODE for q_i

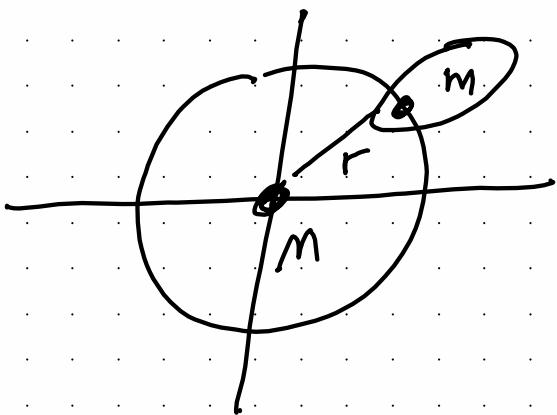
→ 2s-equations, 1st order in q_i, \dot{p}_i

$$H = \frac{p^2}{2m} + U(x) \rightarrow \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$$

$$L = \frac{1}{2} m \dot{x}^2 - U(x) \rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \rightarrow m \ddot{x} = -\frac{\partial U}{\partial x}$$

$$\begin{aligned} x &= p/m \\ p &= m\dot{x} \\ \dot{p} &= -\frac{\partial U}{\partial x} \end{aligned}$$

$$m\ddot{x} = -\frac{\partial U}{\partial x}$$



$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$v = \sqrt{\frac{GM}{r}}$

$$U = -\frac{GMm}{r}$$

Problem: $U' = cU$

~~masses = constant~~

$$r' = r, m' = m$$

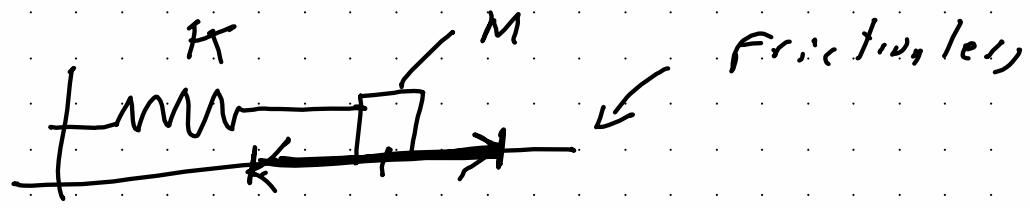
Suppose $M \rightarrow cM = M'$

$$U' = cU$$

$$\frac{2\pi r}{T'} = v' = \sqrt{\frac{cGM}{r}}$$

$$\frac{1}{T'} \propto \sqrt{c}$$

$$\sqrt{\frac{E}{E'}} = \sqrt{\frac{U'}{U}}$$



$$U = \frac{1}{2} Kx^2$$

$$T = \frac{1}{2} m \dot{x}^2$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} Kx^2$$

$$\omega = \sqrt{\frac{K}{m}}$$

$$m \ddot{x} = -Kx$$

$$\ddot{x} = -\frac{K}{m}x$$

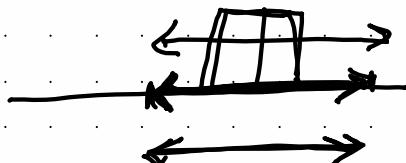
$$\rightarrow x(t) = a \cos \omega t + b \sin \omega t$$

$$U' = cU \quad (\text{for example } T' = ct)$$

$$T' = cT$$

$$\ddot{x} = -\frac{K'}{m}x$$

$$\omega' = \sqrt{\frac{K'}{m}}$$



$$T = \frac{1}{2} m \dot{x}^2$$

$$T' = cT$$

$$\frac{2\pi}{P'} = \sqrt{\frac{K'}{m}}$$

$$\frac{P}{P'} = \sqrt{\frac{U'}{U}}$$

$$L = T - U$$

L, L'

same E_{0m}

$$L' = cL \quad = cT - cU$$

L, L''

$$L'' = T - cU \neq cL$$

different E_{0m}

$$T = \frac{1}{2}mx^2, \quad U = \frac{1}{2}\kappa x^2$$

$$\frac{1}{\kappa} = c$$

$$L = T - U \rightarrow mx'' = -\kappa x$$

$$L' = cL \rightarrow \cancel{mx''} = -\cancel{c}\kappa x$$

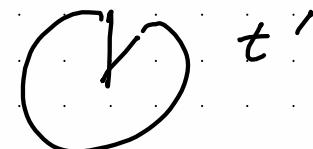
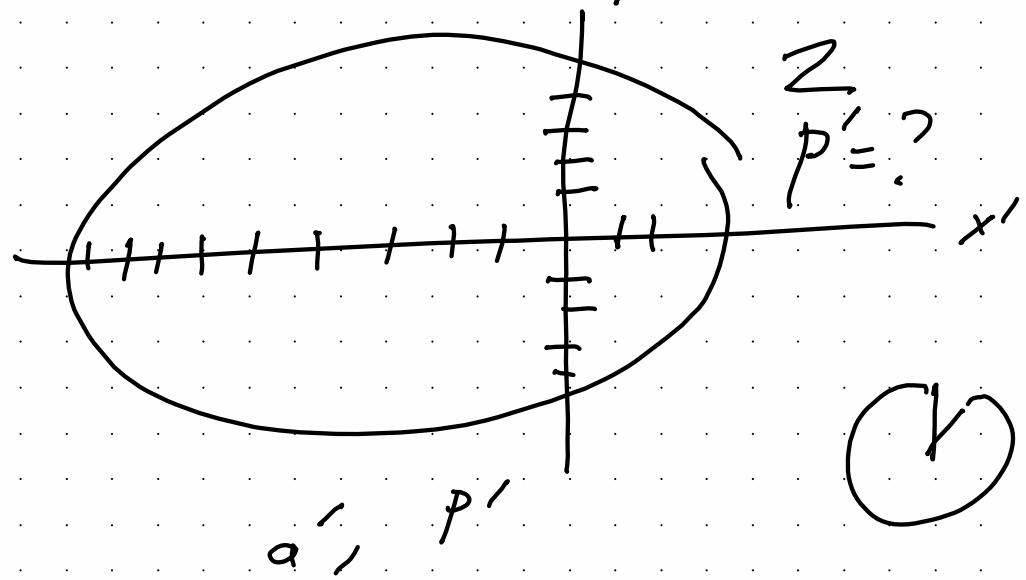
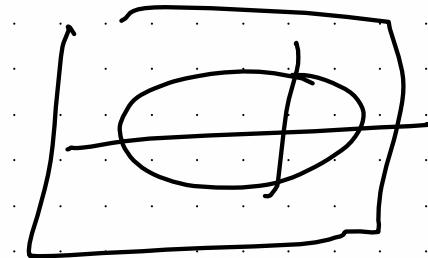
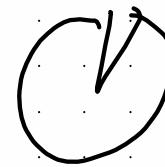
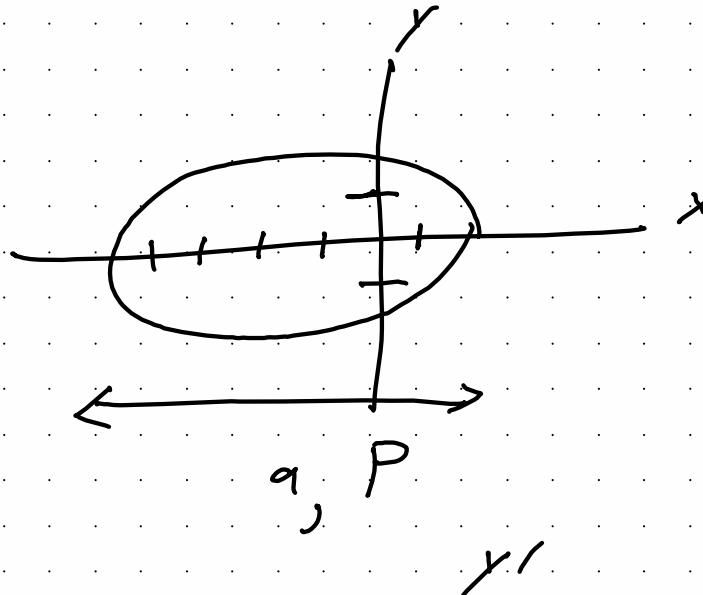
$$mx'' = -\kappa x$$

$$L'' \rightarrow mx'' = -c\kappa x$$

$$cx'' = \cancel{x''} = -c\left(\frac{\kappa}{m}\right)x$$

different
 $E_{0m's}$

$$x = a \cos(\omega't) + b \sin(\omega't)$$



$$\stackrel{z}{P'} = ?$$

$$\frac{P^2}{a^2} = \cos z$$

$$U' = cU$$

$$\cancel{P'} = ? \cancel{P}$$

$$m' = m$$

$$l' = l$$

$$L' = \cancel{(c\omega_0 + L)}$$

$$T' - U' = (c\omega_0 + L) - U'$$

$$T' - cU = c \cancel{(T-U)} \cancel{+ cU} = U'$$

$\cancel{+}$
 $\cancel{-}$

$$\frac{P'}{P} = \sqrt{\frac{U}{U'}}$$

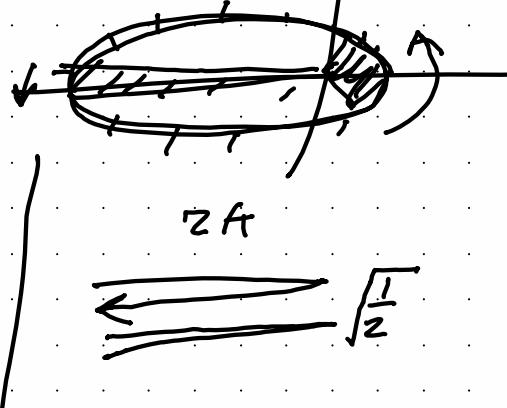
"mechanical similarity"

$$H \rightarrow H' = ck$$

Periods

$$g \rightarrow 2g$$

$$1m \left[P' ? P \right] \frac{1}{\sqrt{2}}$$



$$T = \frac{1}{2} m \dot{x}^2$$

$$T' = T$$

$$U' = cU$$

$$\cancel{T' \neq cT}$$

$$L' = cL$$



Lec #8: Thu Sep 17th

Today — 1-d motion (Sec 11)

Next two weeks — central force (Sec 13-15)

* Midterm 1 — Tues Oct 6th

$$\underline{T = \frac{1}{2} m(\dot{q})^2}$$

$$T = \frac{1}{2} \sum_{j=1}^N q_j \dot{q}_j \quad [\text{single particle}]$$

$$\bar{T} = \frac{1}{2} m \vec{v}^2$$

$$U(q_1, \dots, q_N, t)$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad \text{cartesian } (x, y, z)$$

$$= \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \quad \begin{matrix} \text{sp. polar} \\ (r, \theta, \phi) \end{matrix}$$

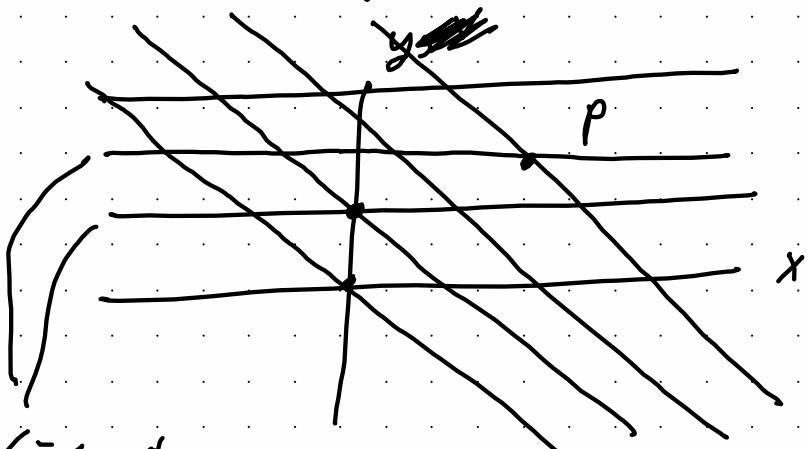
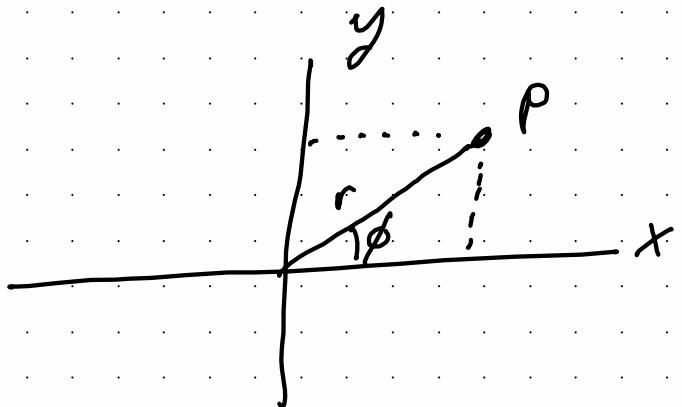
$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\left[\begin{array}{l} q_1 = r \\ q_2 = \theta \\ q_3 = \phi \end{array} \right] \quad a_{ij} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{r^2 \sin^2 \theta}$$

$$\underline{\underline{a_{11} = m}}, \quad \underline{q_{22} = mr^2}, \quad \underline{q_{33} = mr^2 \sin^2 \theta}, \quad \underline{q_{12} = 0}, \quad \underline{q_{13} = 0}$$



$$V = \text{const}, t$$

$$\begin{matrix} \uparrow & \uparrow \\ u & = \text{const} \end{matrix}$$

$$T = \frac{1}{2} m (\dot{u}^2 + \dot{v}^2) ??$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \checkmark$$

$$= \frac{1}{2} m (\dot{u}^2 + \dot{v}^2 - 2\dot{u}\dot{v} + \dot{v}^2)$$

$$= \frac{1}{2} m (\dot{u}^2 + 2\dot{v}^2 - 2\dot{u}\dot{v})$$

$$(x, y)$$

$$(r, \phi)$$

$$\begin{matrix} \dot{x} = \ddot{u} - \ddot{v} \\ \dot{y} = \ddot{v} \end{matrix}$$

$$u = x + y$$

$$\cancel{u = x - y}$$

$$v = y$$

$$x = u - v$$

$$y = v$$

$$u = \text{const} ?$$

$$x + y = \text{const}$$

$$y = \text{const} - x$$

$$\begin{matrix} b \\ m \end{matrix}$$

$$a_{ij} = m \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$T = \frac{1}{2} m |\vec{v}|^2$$

$$\vec{v} = \sum_{i=1}^3 v_i \hat{e}_i$$

$$|\vec{v}|^2 = \vec{v} \cdot \vec{v}$$

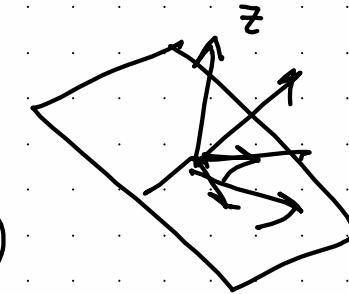
$$= (\sum_i v_i \hat{e}_i) \cdot (\sum_j v_j \hat{e}_j)$$

$$= \sum_{i,j} v_i v_j \hat{e}_i \cdot \hat{e}_j$$

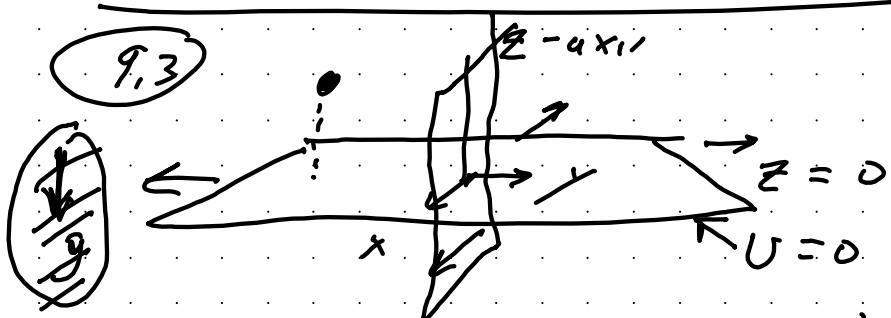
$\underbrace{m_i g + n_i t}_{\text{might not}} = \delta_{ij} \quad (\text{orthogonal})$

$$\hat{e}_i \cdot \hat{e}_i = 1$$

$$\hat{e}_i \cdot \hat{e}_j \neq 0$$



9.3



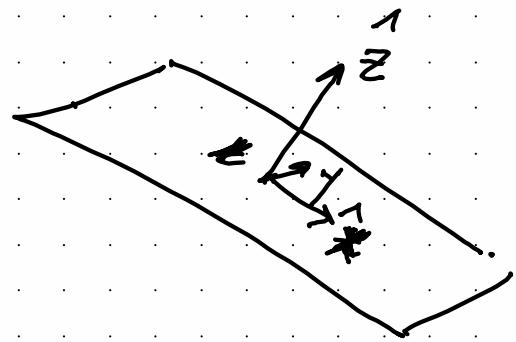
$$(a) U(x, y, z)$$

$$mgz = U(z), \vec{F} = -m\vec{g}$$

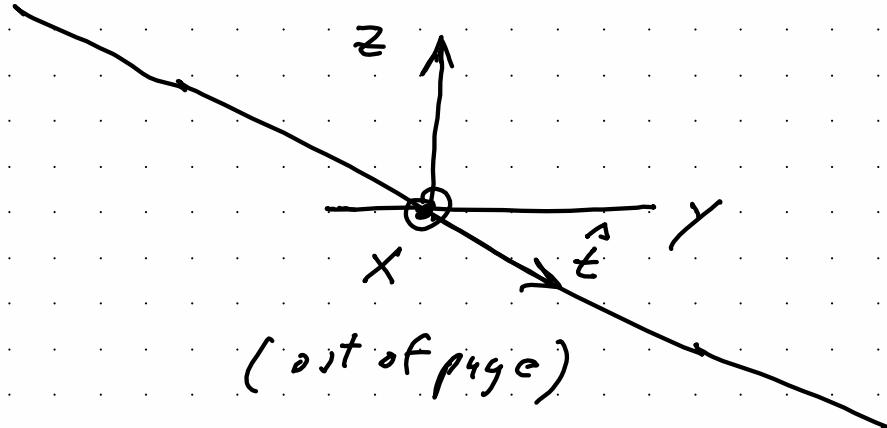
$\delta L = 0$; if you displace in \hat{x}, \hat{y} $\rightarrow [P_x, P_y] = \text{const}$

$$\vec{F} = -\vec{\nabla}U = -\frac{dU}{dz} \hat{z}$$

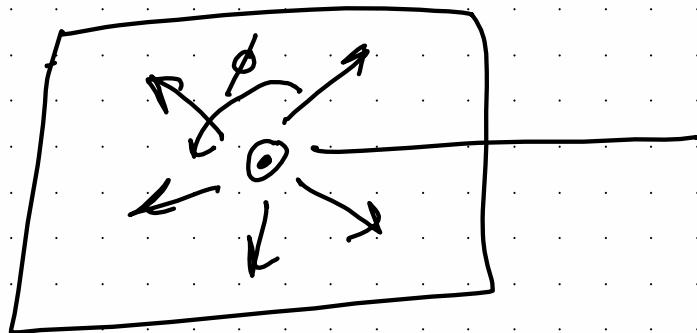
$$M_z = \text{const}$$



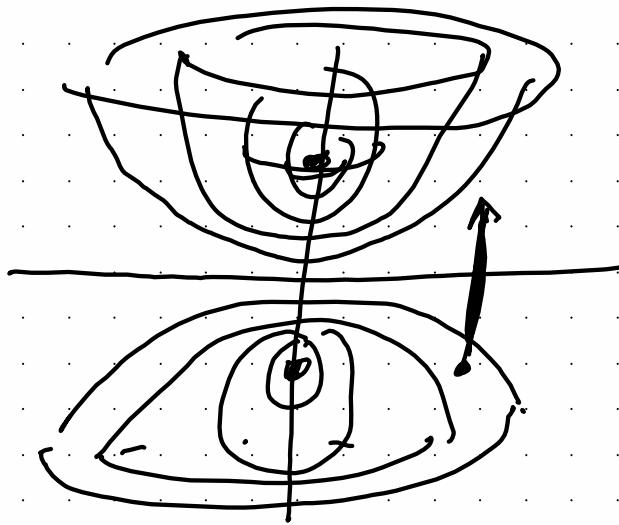
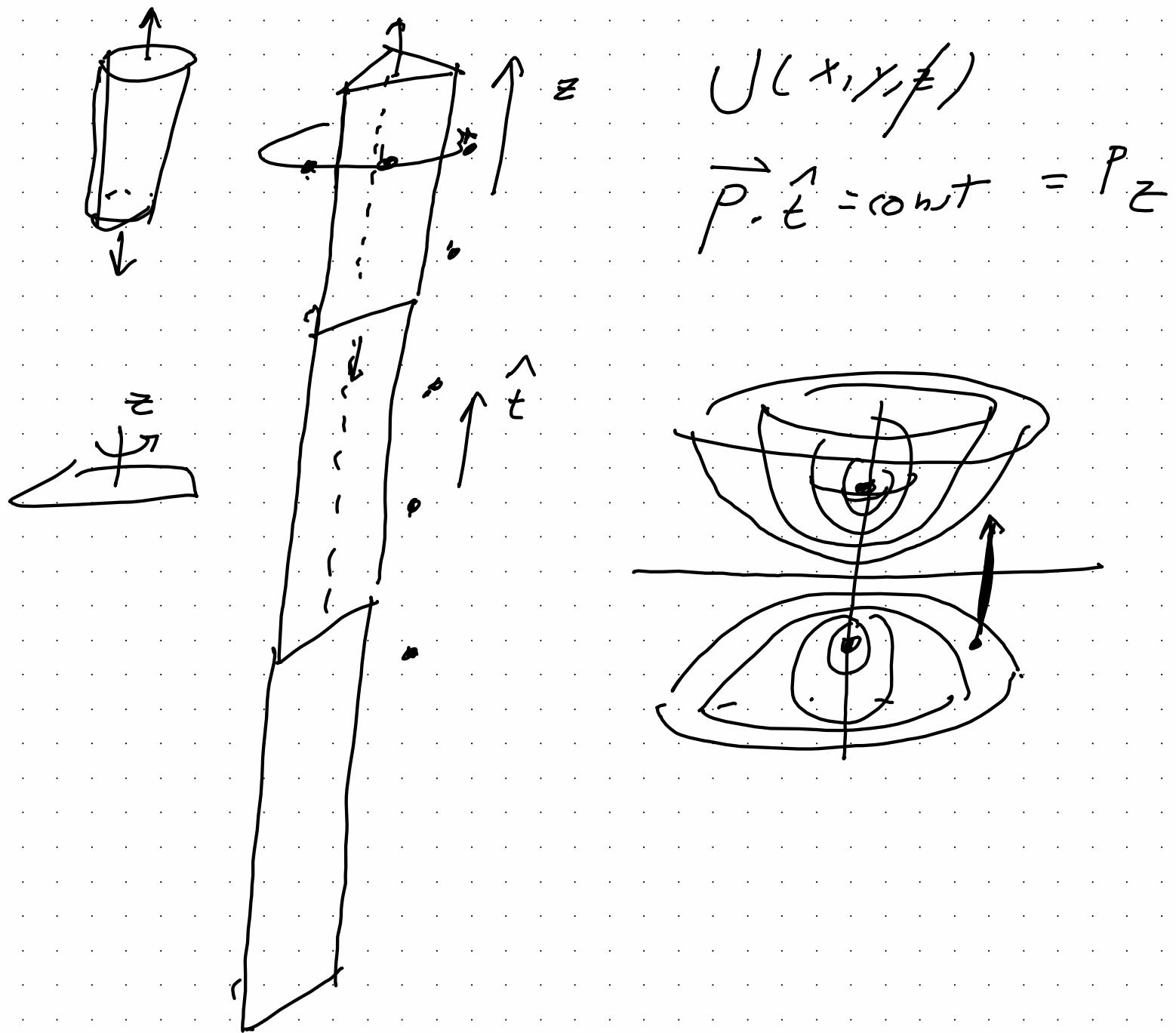
}



$$\vec{P} \cdot \hat{t} = \text{constant}$$



$$\vec{M} \cdot \hat{z} = \text{constant}$$



$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$(E) = \frac{1}{2} m \dot{x}^2 + U(x) = \text{const}$$

positive

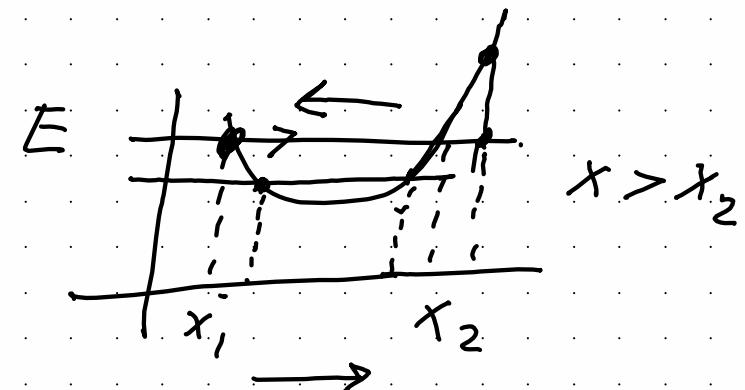
$$\frac{1}{2} m \dot{x}^2 = E - U(x)$$

$$\frac{dx}{dt} = \dot{x} = \pm \sqrt{\frac{2}{m}(E - U(x))} \quad \text{— separable, 1st order}$$

$$\int \pm \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}} = \int dt$$

$t = \int \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}} + \text{const}$

$$\Delta x = \pm \Delta t \sqrt{\frac{2}{m}(E - U(x))}$$



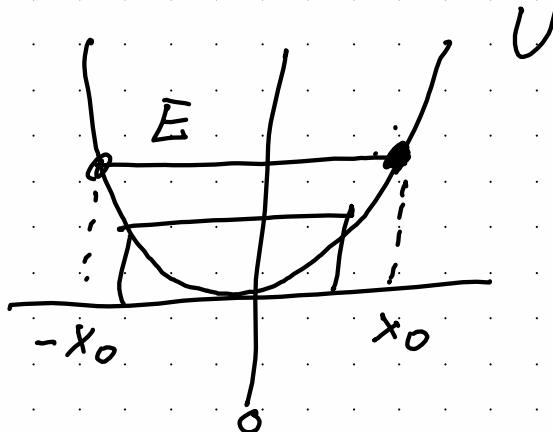
O D E $\int x_2(E)$

$$T(E) = 2 \int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}}$$

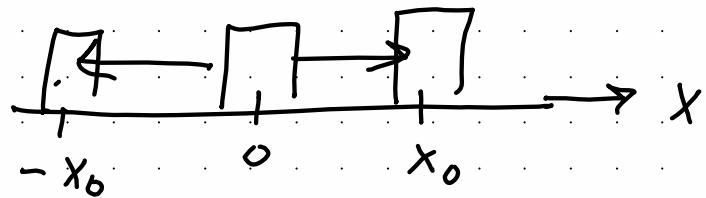
$$U(x) = \frac{1}{2} k x^2$$

$$U(\phi) = mg \cos \phi$$

$$\phi = \frac{g}{l} \sin \phi$$



$$U = \frac{1}{2} kx^2$$



→ Beta Functions

$$E = \frac{1}{2} kx_0^2$$

$$x_0 = \sqrt{\frac{2E}{k}}$$

$$E = U(x_1) = U(x_2)$$

Prob 2

$$(a) U = A|x|^n$$

$$(b) U = -\frac{U_0}{\cosh^2(\alpha x)}$$

$$(c) U = U_0 + \tan^2(\alpha x)$$

$$\frac{dx}{\sqrt{E - U(x)}}$$

$$x_1(E) \uparrow \frac{1}{2} kx_0^2 - \frac{1}{2} kx^2$$

$$\frac{\frac{1}{2} k(x_0^2 - x^2)}{\sqrt{x_0^2 - x^2}}$$

Lecture #9: Tues 9/22

40%

60%

- 1) Midterm #1: Tues Oct 6th (short answer; long problems)
- 2) Next 4 lecture (central force problem)
Sec 13 - 15

General Formalism:

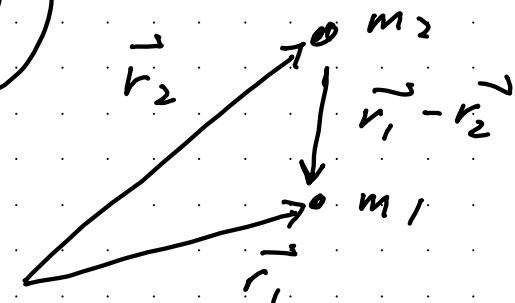
Two interacting particles (no external forces)

m_1, m_2

$$U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$$

$$L = T - U$$

$$\begin{aligned} &= \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 \\ &\quad - U(|\vec{r}_1 - \vec{r}_2|) \end{aligned}$$



i) L unchanged by a translation

$$\begin{aligned} \vec{r}_1 &\rightarrow \vec{r}_1 + \delta \vec{x} \\ \vec{r}_2 &\rightarrow \vec{r}_2 + \delta \vec{x} \end{aligned}$$

$$\vec{p} = m_1 \vec{v}_1 + m_2 \vec{v}_2 = \text{const}$$

com moves with const velocity

choose ref frame such that COM at orig'.

(
inertial)

III
COM Frame

$$\frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \vec{R}_{\text{COM}} = 0$$

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

Define: $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$

$$U(|\vec{r}_1 - \vec{r}_2|)$$

$$U = U(r), \quad r = |\vec{r}|$$

~~***~~

$$\vec{r}_2 = \vec{r}_1 - \vec{r}$$

$$m_1 \vec{r}_1 + m_2 (\vec{r}_1 - \vec{r}) = 0$$

$$(m_1 + m_2) \vec{r}_1 = m_2 \vec{r} = 0$$

$$\vec{r}_1 = \left(\frac{m_2}{m_1 + m_2} \right) \vec{r}$$

$$T = \frac{1}{2} m_1 |\vec{r}_1|^2 + \frac{1}{2} m_2 |\vec{r}_2|^2$$

$$\vec{r}_2 = \left(\frac{m_2}{m_1 + m_2} \right) \vec{r} - \vec{r}$$

$$\vec{r}_2 = - \left(\frac{m_1}{m_1 + m_2} \right) \vec{r}$$

$$T = \frac{1}{2} m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 |\dot{\vec{r}}|^2 + \frac{1}{2} m_2 \left(\frac{-m_1}{m_1 + m_2} \right)^2 |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)^2} |\dot{\vec{r}}|^2 \cancel{(m_2 + m_1)}$$

$$= \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) |\dot{\vec{r}}|^2$$

reduced mass : m

$$= \frac{1}{2} m |\dot{\vec{r}}|^2$$

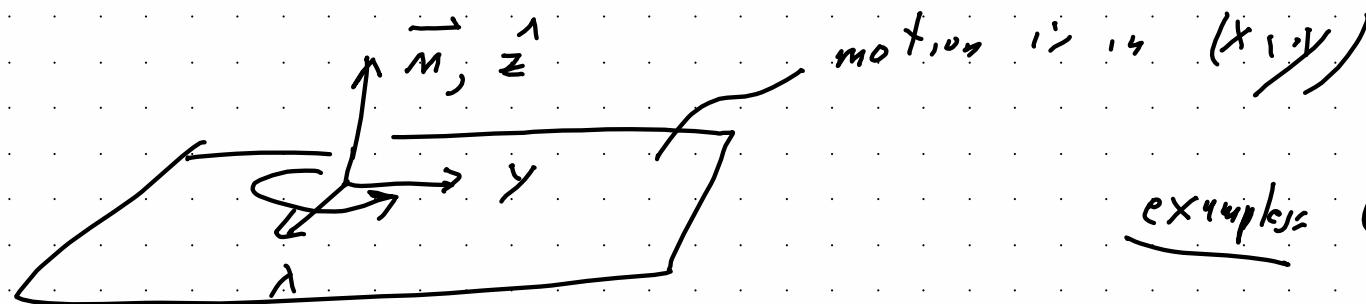
central potential

$$\boxed{L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(r)}$$

"effective"
one body Lagrangian

ii) L unchanged under τ rotations

$$\rightarrow \overrightarrow{M} = \text{const} \quad (\overrightarrow{M} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v})$$



motion is (x, y)

Newtonian gravity
example: $U(r) = -\frac{Gm_1m_2}{r}$

$$L = \frac{1}{2} m |\vec{r}|^2 - U(r)$$

$$U(r) = \frac{1}{2} k r^2$$

$$= \frac{1}{2} m (r^2 + r^2 \dot{\phi}^2) - U(r)$$

Space oscillator

i) No explicit t -dependence

$$E = T + V = \text{const}$$

E, M constants
of the motion

ii) No ϕ dependence

$$M_z = p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \text{const} = mr^2 \ddot{\phi} = M$$

$$mr^2 \ddot{\phi} = M$$

$$\left. \begin{array}{l} E = \frac{1}{2}m(r^2 + r^2\dot{\phi}^2) + U(r), \\ M = mr^2\dot{\phi} \end{array} \right\} \rightarrow \boxed{\dot{\phi} = \frac{M}{mr^2}}$$

$$\left. \begin{array}{l} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L_{\phi}}{\partial \phi} \end{array} \right\} \rightarrow E = \frac{1}{2}mr^2 + \left(\frac{M^2}{2mr^2} + U(r) \right)$$

$\underbrace{\hspace{10em}}_{U_{\text{eff}}(r)}$

$$\begin{aligned} \frac{dr}{dt} &= \dot{r} = \sqrt{\frac{2}{m} \left(E - U(r) - \frac{M^2}{2mr^2} \right)} \\ &= \sqrt{\frac{2}{m} (E - U(r)) - \frac{M^2}{m^2 r^2}} \end{aligned}$$

~~dt~~

$$dt = \sqrt{\frac{dr}{\frac{2}{m} (E - U(r)) - \frac{M^2}{m^2 r^2}}} + \text{const}$$

$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{M}{mr^2} = \sqrt{\quad}$$

$\int \frac{\frac{dr}{d\phi} M}{mr^2} dt + \text{const}$ = $\int d\phi = \phi$

$$\phi = \int \frac{M dr/r^2}{\sqrt{2m(E - U(r)) - M^2/r^2}} + \text{const}$$

$$\phi = \phi(r) \iff r = r(\phi)$$

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

$$m_1 + m_2$$

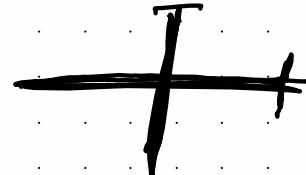
$$\begin{aligned} r &= r(\phi) \\ r &= r(t) \\ \phi &= \phi(t) \end{aligned} \quad \left. \begin{array}{l} r(t) \\ \phi(t) \end{array} \right\} \vec{r}(t)$$

Example: $U(r) = \frac{1}{2}kr^2$ (space oscillator) (r, ϕ)

$$\begin{aligned}\text{Egy: } L &= \frac{1}{2}m(r'^2 + r^2\dot{\phi}^2) - \frac{1}{2}kr^2 \quad (x, y) \\ &= \frac{1}{2}m(x'^2 + y'^2) - \frac{1}{2}k(x^2 + y^2) \\ &= \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) + \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2\right)\end{aligned}$$

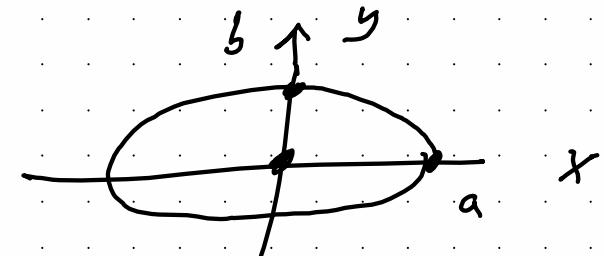
$$x(t) = a \cos(\omega t + \alpha), \quad \omega = \sqrt{\frac{k}{m}}$$

$$y(t) = b \sin(\omega t + \beta), \quad \underline{\alpha}$$



→ closed orbit

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



ellipse with center at origin

$U = \frac{-\alpha}{r}$, $U = \frac{1}{2}kr^2$ are only two potentials that have closed bound orbits.

Harter: $U = \frac{1}{2} k r^2$

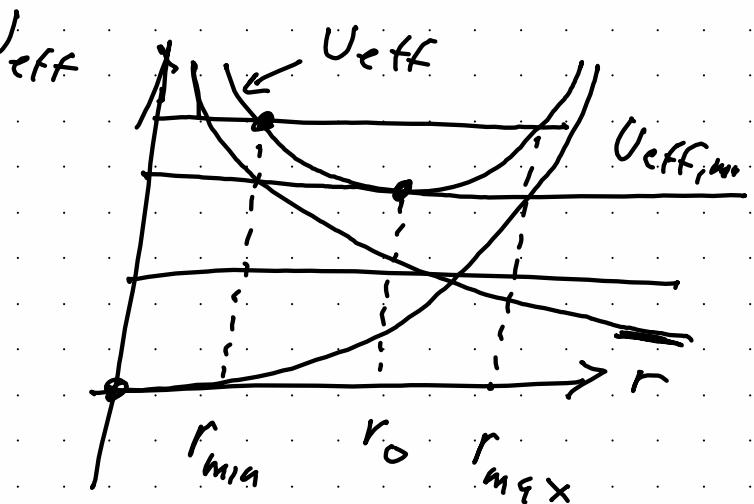
$$U_{\text{eff}} = U(r) + \frac{M^2}{2mr^2}$$

$$= \frac{1}{2} k r^2 + \frac{M^2}{2mr^2}$$

$$\phi = \int \frac{M dr/r^2}{\sqrt{2m(E - \frac{1}{2}kr^2) - \frac{M^2}{r^2}}} + \text{const}$$

substitutes

$$u = \frac{1}{r}, du = -\frac{dr}{r^2}$$



bound orbits

$$v = u^2$$

$$dv = 2u du$$

$$\phi = - \int \frac{M du}{\sqrt{2m(E - \frac{1}{2}\frac{k}{u^2}) - M^2 u^2}} = - \int \frac{Mu du}{\sqrt{2m(Eu^2 - \frac{k}{2}) - M^2 u^2}} + \text{const}$$

+ const

$$\phi = -\frac{1}{2} \int \frac{dv M}{\sqrt{2m(Ev - \frac{k}{2}) - M^2 v^2}} + \text{const}$$

Complete the square: $-M^2 v^2 + 2mEv - m k$

$$= -M^2 \left(v^2 - \frac{2mEv}{M^2} + \frac{m k}{M^2} \right)$$

$$= -M^2 \left(\left(v - \frac{mE}{M^2} \right)^2 - \frac{m^2 E^2}{M^4} + \frac{m k}{M^2} \right)$$

$$= -M^2 \left((v-A)^2 - B^2 \right)$$

$$= M^2 (B^2 - (v-A)^2)$$

$$A = \frac{mE}{M^2}, \quad B^2 = A^2 - \frac{m k}{M^2}$$

$$\phi = -\frac{1}{2} \int \frac{dv}{\sqrt{B^2 - (v-A)^2}} + C_{\text{const}}$$

3rd substitution:

$$v-A = B \sin \theta$$

$$dv = B \cos \theta d\theta$$

$$v-A = \pm B$$

$$\frac{1}{r^2} - A = \pm B$$

$$r_{\max}$$

$$\frac{1}{r^2} = A \pm B$$

$$\int \frac{dx}{\sqrt{1-x^2}}$$

$$x = \sin \theta$$

$$dx = \cos \theta d\theta$$

$$\int \frac{\cos \theta d\theta}{\sqrt{1-\sin^2 \theta}}$$

$$= \theta$$

$$= \sin^{-1}(x)$$

$$\phi = -\frac{1}{2} \int \frac{B \cos \theta d\theta}{\sqrt{B^2 - B^2 \sin^2 \theta}} + C_{\text{const}}$$

$$= -\frac{1}{2} \theta + C_{\text{const}}$$

$$= -\frac{1}{2} \sin^{-1} \left(\frac{v-A}{B} \right) + C_{\text{const}}$$

$$= -\frac{1}{2} \sin^{-1} \left(\frac{\frac{1}{r^2} - A}{B} \right) + C_{\text{const}}$$

$$\frac{1}{r_{\max}^2} = A - B$$

$$\frac{1}{r_{\min}^2} = A + B \quad v = 4^2$$

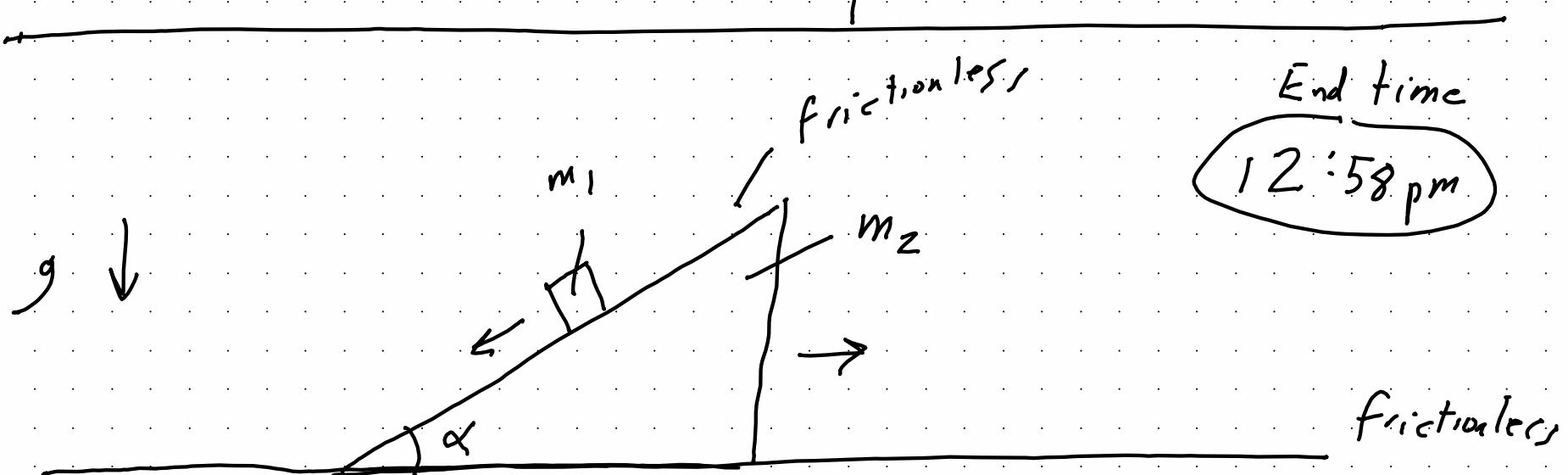
$$4 = \frac{1}{r}$$

$$v = \frac{1}{r^2}$$

Lecture #10: 9/24

- Finsler spring oscillator
- Quiz #2
- Next week: Q&A

lastname - 9.2.pdf



End time
12:58 pm

frictionless

i) generalized coord)

ii) $L = T - V$

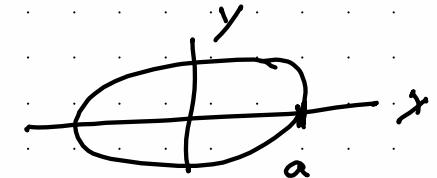
iii) EOMs

iv) special limiting cases: $\omega = 0$, $\alpha = \pi/2$
 $m_1 \gg m_2$, $m_1 \ll m_2$

$$\phi = -\frac{1}{2} \sin^{-1} \left(\frac{\frac{1}{r^2} - A}{B} \right) + \underline{\text{const}} \quad E, M$$

choose const so that $\phi=0 \leftrightarrow r=r_{\max}$

$$\frac{1}{r_{\max}^2} = A - B$$



$$\frac{\frac{1}{r_{\max}^2} - A}{+B} = \frac{-B}{B} = -1$$

$$\begin{aligned} 0 &= -\frac{1}{2} \underbrace{\sin^{-1}(-1)}_{-\frac{\pi}{2}} + \text{const} \\ &= \frac{\pi}{4} + \text{const} \end{aligned}$$

$$\boxed{\text{const} = -\frac{\pi}{4}}$$

$$\phi = -\frac{1}{2} \sin^{-1} \left(\frac{\frac{1}{r^2} - A}{B} \right) - \frac{\pi}{4}$$

$$\begin{aligned} -2(\phi + \frac{\pi}{4}) &= \sin^{-1} \left(\frac{\frac{1}{r^2} - A}{B} \right) \\ -\sin \left(2\phi + \frac{\pi}{2} \right) &= \left(\frac{\frac{1}{r^2} - A}{B} \right) \end{aligned}$$

$$-\sin\left(2\phi + \frac{\pi}{2}\right) = \left(\frac{1}{r^2} - A\right)/B$$

$$-B \cos(2\phi) = \frac{1}{r^2} - A$$

$$\boxed{\frac{1}{r^2} = A - B \cos(2\phi)}$$

$$\cos(2\phi) = \cos^2\phi - \sin^2\phi$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(2\phi) = \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2$$

$$\left[\frac{1}{r^2} = A - B \left(\left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 \right) \right]$$

$$1 = Ar^2 - B(x^2 - y^2)$$

$$= A(x^2 + y^2) - B(x^2 - y^2) =$$

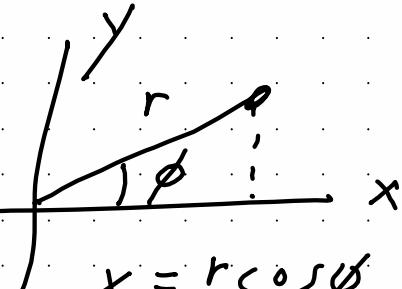
$$A = mE/m^2, B^2 = A^2 - mk/m^2$$

$$\sin\left(2\phi + \frac{\pi}{2}\right)$$

$$= \sin(2\phi) \cos\left(\frac{\pi}{2}\right)$$

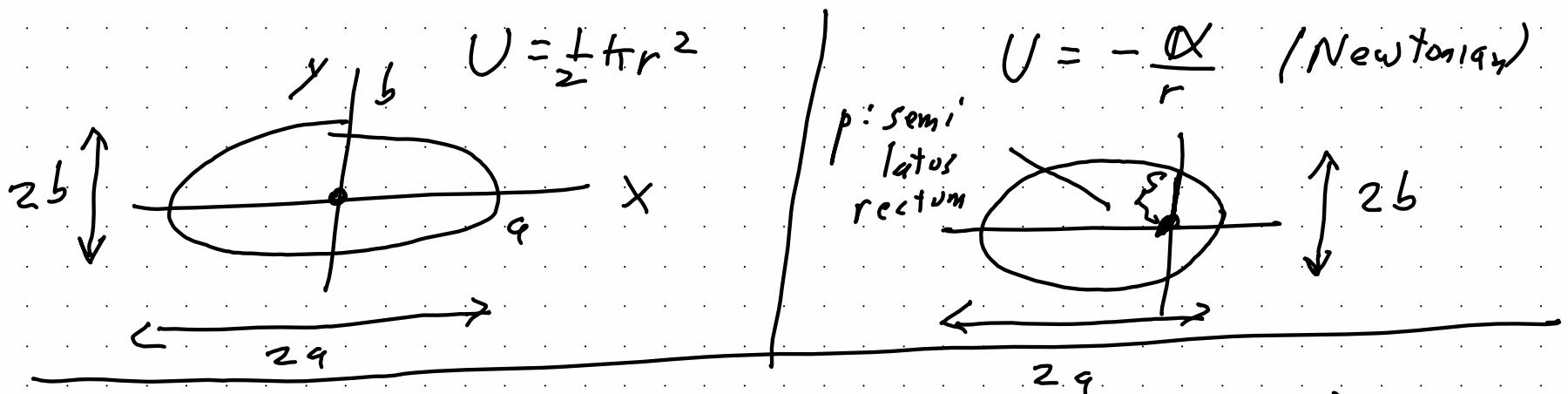
$$+ \cos(2\phi) \sin\left(\frac{\pi}{2}\right)$$

$$= \cos(2\phi) = 0$$



$$\frac{1}{r^2} = A - B \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} \right) \quad \boxed{\frac{1}{a^2} = A - B, \frac{1}{b^2} = A + B}$$

$$(A-B)x^2 + (A+B)y^2 = 1$$



$$E = \underbrace{\frac{1}{2} m r^2}_{=0, +} + \frac{M^2}{2mr^2} + \frac{1}{2} k r^2 \quad \begin{pmatrix} r=a \\ r=b \end{pmatrix}$$

$$\boxed{E = \frac{M^2}{2ma^2} + \frac{1}{2} k a^2}$$

$$\boxed{E = \frac{M^2}{2mb^2} + \frac{1}{2} k b^2}$$

$$\rightarrow E_a^2 = \frac{M^2}{2m} + \frac{1}{2} k a^2$$

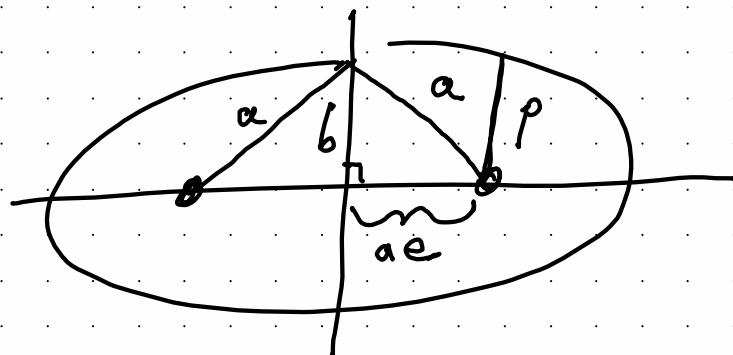
$$\rightarrow E_b^2 = \frac{M^2}{2m} + \frac{1}{2} k b^2$$

$$\rightarrow E = \frac{1}{2} k (a^2 + b^2)$$

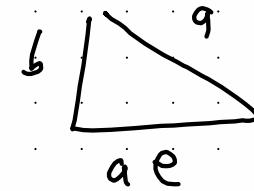
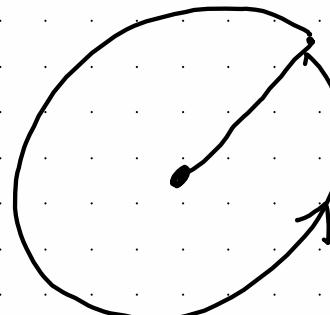
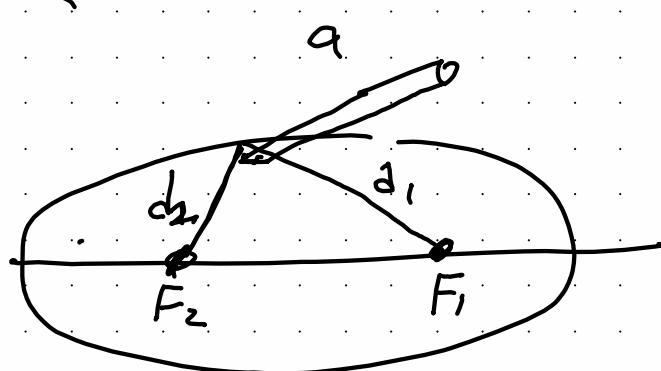
$$\rightarrow M^2 = m k a^2 b^2$$

$$\text{subtract } E(a^2 + b^2) = \frac{1}{2} k (a^4 - b^4)$$

$$(a^2 + b^2)(a^2 - b^2) \rightarrow$$



$e = 0$ circle
 $e = 1$ line
 dimensionless



$$a^2 = b^2 + a^2 e^2$$

$$e = \sqrt{\frac{a^2 - b^2}{a^2}}$$

$$d_1 + d_2 = 2a$$

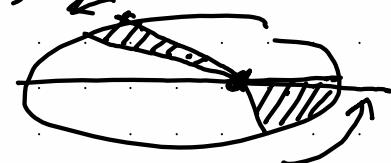
Kepler's Law: (For $V = -\alpha/r$)

$$b = a \sqrt{1-e^2}$$

i) elliptical orbits with sun at one focus

ii) equal areas in equal times

$$\frac{P^2}{a^3} = \text{const}$$



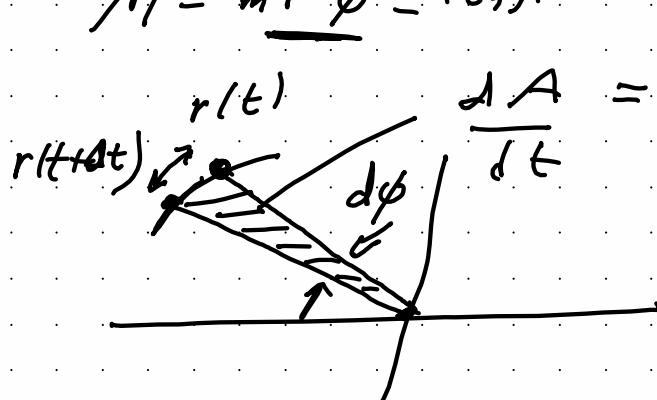
$$U = \frac{1}{2} k r^2$$

"Kepler's laws": (i) elliptical orbits with "Sun" at center
of the ellipse
(ii) equal areas in equal times

(iii)

true for any
central potential

$M = mr^2\dot{\phi} = \text{const}$



$$\frac{dA}{dt} = \text{const}$$



$$\frac{dA}{dt} = \frac{M}{zm}$$

$$\int dA = \int \frac{M}{zm} dt$$

$$\frac{dA}{dt} = \frac{1}{2} r \frac{dr}{dt} \phi + r \frac{d\phi}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt}$$

height

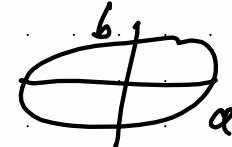
$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{M}{zm} = \text{const}$$

$$A = \frac{M}{2m} P$$

$$A = \pi r^2 \quad \text{circle}$$

$$= \pi ab$$

$$\pi ab = \frac{M}{2m} P$$



$$P = \frac{2m}{M} \pi ab$$

$$M^2 = m \pi a^2 b^2$$

$$M = \sqrt{m \pi a b}$$

$$= \frac{2m \pi ab}{\sqrt{m \pi a b}}$$

$$= \frac{2\pi}{\sqrt{K/m}}$$

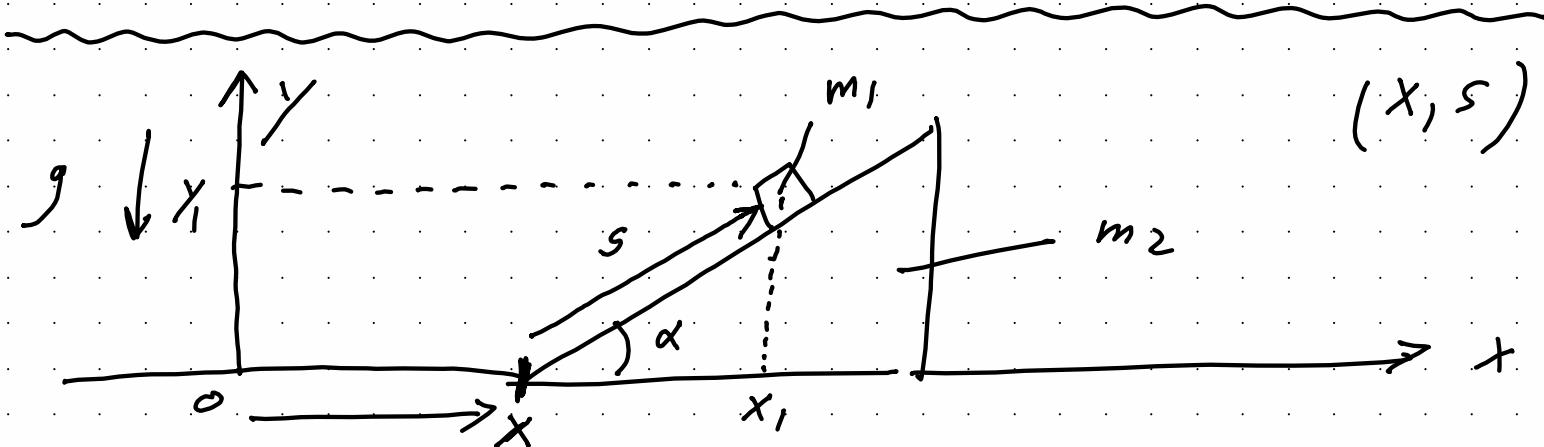
$$= \frac{2\pi}{\omega} = \text{const} \quad (\text{for any } a, b)$$

$$P = \text{const}$$

indep of a, b !!

Lec #11: Tuesday 9/29

- Midterm #1: Next Tuesday Oct 6th
- Trial midterm on Blackboard (do it before Thursday)
- Today:
 - a) Quiz #2
 - b) F. nsh $U = \frac{1}{2} k r^2$
 - c) Q & A



$$L = T - U \quad T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$x_1 = x + s \cos \alpha \rightarrow \dot{x}_1 = \dot{x} + s \cos \alpha$$

$$y_1 = s \sin \alpha \rightarrow \dot{y}_1 = s \sin \alpha$$

$$x_2 = x \rightarrow \dot{x}_2 = \dot{x}$$

$$y_2 = 0 \rightarrow \dot{y}_2 = 0$$

$$\dot{x}_1^2 = (\dot{x} + s \cos \alpha)^2 = \dot{x}^2 + \underline{s^2 \cos^2 \alpha} + 2\dot{x}s \cos \alpha$$

$$\dot{x}_1^2 = \underline{s^2 \sin^2 \alpha}$$

$$T = \frac{1}{2} m_1 (\dot{x}^2 + \dot{s}^2 + 2\dot{x}s \cos \alpha) + \frac{1}{2} m_2 \dot{x}^2$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1 \dot{x} s \cos \alpha$$

$$U = m_1 g y_1 = m_1 g s \sin \alpha$$

$$L = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1 \dot{x} s \cos \alpha - m_1 g s \sin \alpha$$

Euler's: x : $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \cancel{\frac{\partial L}{\partial x}} \rightarrow 0 \quad | \quad E = T + U$

$$P_x = (m_1 + m_2) \dot{x} + m_1 s \cos \alpha = \text{const}$$

$$\rightarrow \boxed{(m_1 + m_2) \ddot{x} + m_1 \ddot{s} \cos \alpha = 0}$$

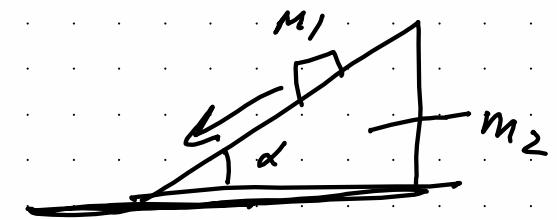
s :

$$\frac{d}{dt} (m_1 \dot{s} + m_1 \dot{x} \cos \alpha) = -m_1 g \sin \alpha$$

$$\dot{s} + \dot{x} \cos \alpha = -g \sin \alpha$$

$$\ddot{x} = - \left(\frac{m_1}{m_1 + m_2} \right) \ddot{s} \cos \alpha$$

$$\ddot{s} = -g \sin \alpha \left(\frac{m_1 + m_2}{m_1 \sin^2 \alpha + m_2} \right)$$



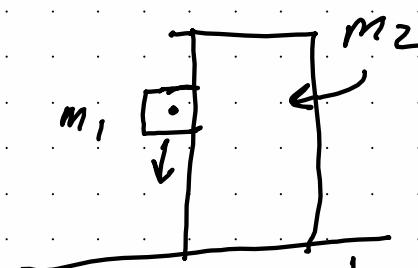
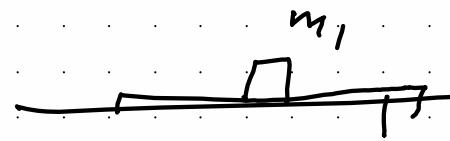
i) $m_2 \gg m_1$: $\ddot{s} \approx -g \sin \alpha$

ii) $\alpha = 0$ $\ddot{s} = 0, \ddot{x} = 0$

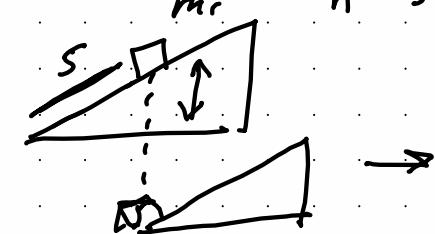
iii) $\alpha = \frac{\pi}{2}$ $\ddot{s} = -g, \ddot{x} = 0$

iv) $m_1 \gg m$: $\left\{ \begin{array}{l} \ddot{s} \approx -\frac{g}{\sin \alpha} \\ \ddot{x} \approx +\frac{g}{\tan \alpha} \end{array} \right.$

$h = -g = \ddot{s} \sin \alpha \rightarrow \ddot{s} = -\frac{g}{\sin \alpha}$



$$h = s \sin \alpha$$

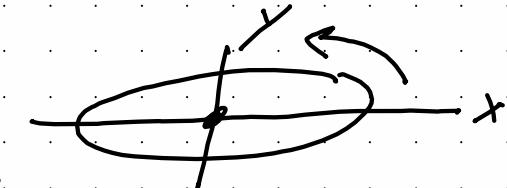


Space oscillator: $U = \frac{1}{2}kr^2$

$$U = -\frac{\alpha}{r}$$

"Kepler's" law:

(i) ellipses with origin at center of ellipse

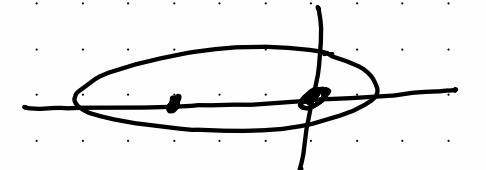


(ii) equal areas in equal times — approach to all central potentials

(iii) $P = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}}$ indep. of a, b

Central potential

$$\frac{P^2}{a^2} = \text{const} + \text{ordinary Kepler}$$



$$t = \int \frac{dr}{\sqrt{\frac{2}{m}(E-U(r)) - \frac{M^2}{m^2r^2}}} + \text{const}$$

reduced mass,

want t to increase as r decrease,

$$\frac{1}{2}kr^2 \quad E = \frac{1}{2}k(a^2+b^2) \\ M^2 = m^2a^2b^2$$

$$t = - \int \frac{dr}{\sqrt{\frac{1}{m} \left(\frac{1}{2} k(a^2 + b^2) - \frac{1}{2} kr^2 \right) - \frac{m + a^2 b^2}{m^2 r^2}}}$$

(t, ϕ)
 (x, y)

$$= - \frac{1}{\sqrt{\frac{k}{m}}} \int \frac{dr}{\sqrt{(a^2 + b^2) - r^2 - \frac{a^2 b^2}{r^2}}}$$

$$\begin{aligned} r^2 &= \\ x^2 + y^2 &\end{aligned}$$

$$= - \frac{1}{\sqrt{\frac{k}{m}}} \int \frac{r dr}{\sqrt{r^2(a^2 + b^2) - r^4 - a^2 b^2}}$$

$$= - \frac{1}{\sqrt{w}} \int \frac{r dr}{\sqrt{- (r^4 - r^2(a^2 + b^2) + a^2 b^2)}}$$

$$\boxed{\sqrt{a^2(1 - \sin^2 \theta)} = a \cos \theta}$$

$$\begin{aligned} dx & \\ \sqrt{a^2 - x^2} m & \\ x = a \sin \theta & \end{aligned}$$

$$\text{subst. } t = \theta \quad = (r^2 - a^2)(r^2 - b^2)$$

$$\boxed{r^2 = a^2 \cos^2 \xi + b^2 \sin^2 \xi}$$

$$b < r < a$$

$$x = a \cos \xi$$

$$y = b \sin \xi$$

$$\rightarrow r^2 = a^2 \cos^2 \xi + b^2 \sin^2 \xi$$

Differentiates

$$\begin{aligned} 2rdr &= -2a^2 \cos \xi \sin \xi + 2b^2 \sin \xi \cos \xi \\ &= -2 \sin \xi \cos \xi (a^2 - b^2) \end{aligned}$$

$$r dr = -\sin \xi \cos \xi (a^2 - b^2)$$

$$\sqrt{-(r^2 - a^2)(r^2 - b^2)} = \sqrt{- (a^2 \cos^2 \xi + b^2 \sin^2 \xi - a^2)(a^2 \cos^2 \xi + b^2 \sin^2 \xi - b^2)}$$
$$-\underline{a^2 \sin^2 \xi} \quad \quad \quad -\underline{b^2 \cos^2 \xi}$$

$$= \sqrt{- (-1/(a^2 - b^2)) \sin^2 \xi (a^2 - b^2) \cos^2 \xi}$$

$$= \boxed{(a^2 - b^2) \sin \xi \cos \xi}$$

$$t = \frac{1}{\omega} \int d\xi + \text{const} \rightarrow \boxed{\omega t = \xi} + \frac{\text{const}}{\omega} = 0$$

$$\begin{cases} x = a \cos(\omega t) \\ y = b \sin(\omega t) \end{cases}$$

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= a^2 \cos^2(\omega t) + b^2 \sin^2(\omega t) \end{aligned}$$

$$r = \sqrt{a^2 \cos^2(\omega t) + b^2 \sin^2(\omega t)}$$

$$\phi = \phi(t)$$

$$(x, y) \leftrightarrow (r, \phi)$$

$$\begin{cases} x = r \cos \phi \\ y = r \sin \phi \end{cases}$$

$$r \cos \phi = a \cos(\omega t)$$

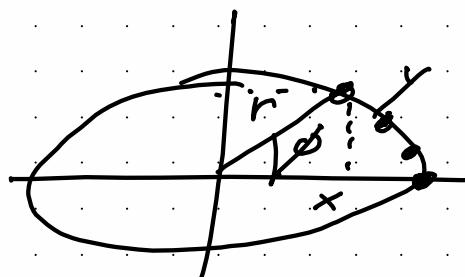
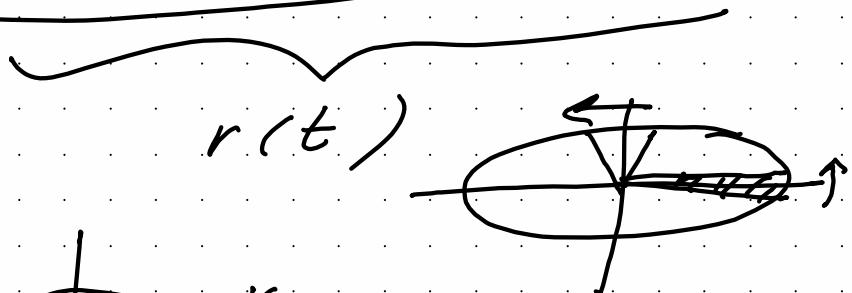
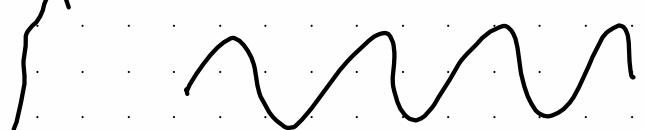
$$r \sin \phi = b \sin(\omega t)$$

Divide 2nd by 1st:

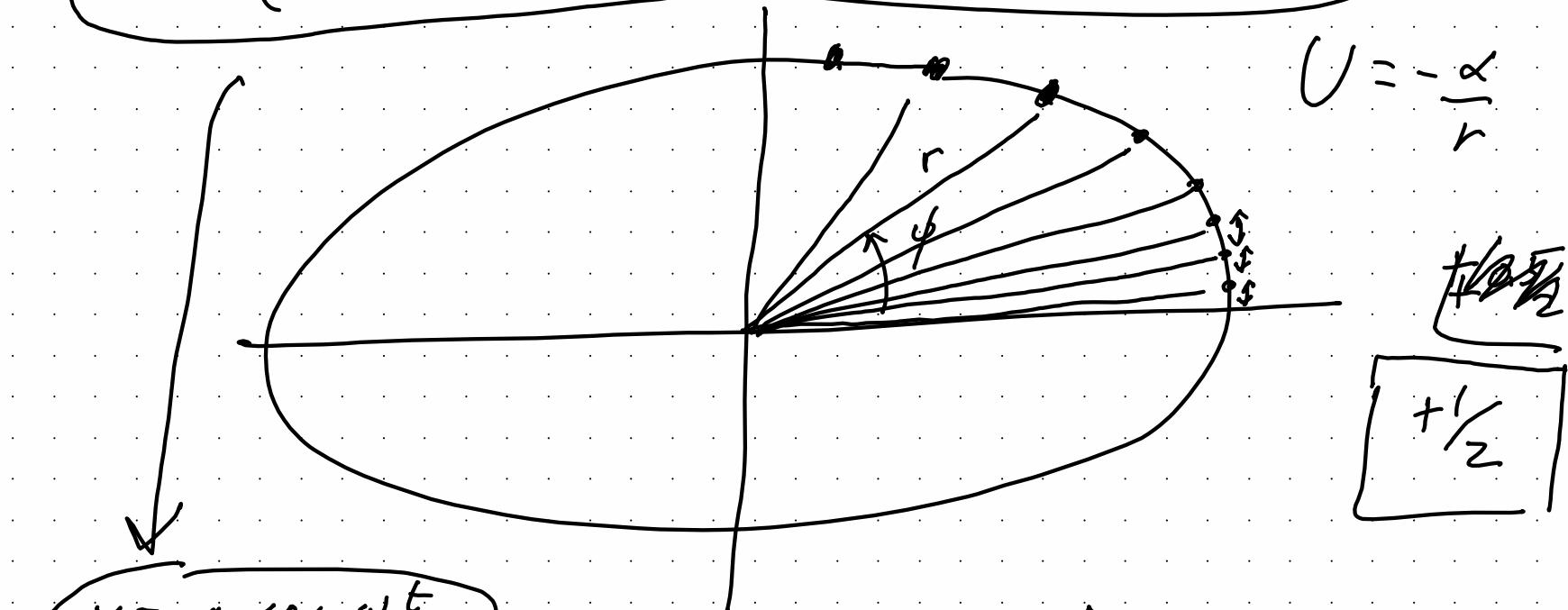
$$\tan \phi = \frac{b}{a} \tan(\omega t)$$

$$\phi = \tan^{-1} \left(\frac{b}{a} \tan(\omega t) \right)$$

$$\rightarrow \phi = \omega t \text{ if } b = a$$



$$L = \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2 \right) + \left(\frac{1}{2} m \dot{y}^2 - \frac{1}{2} K y^2 \right)$$



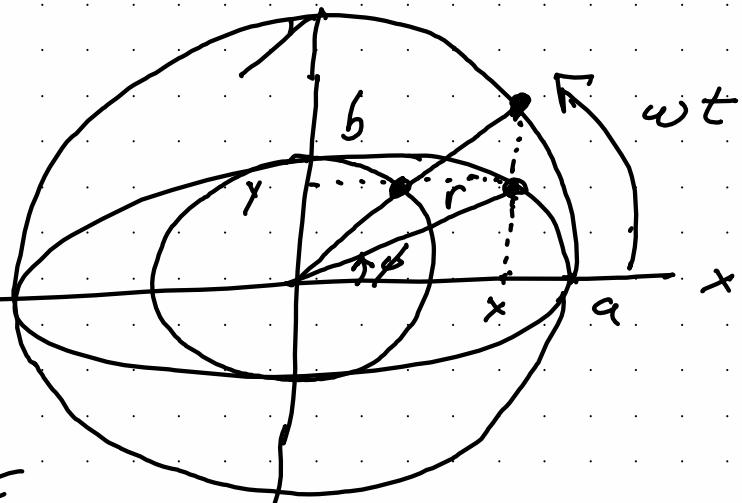
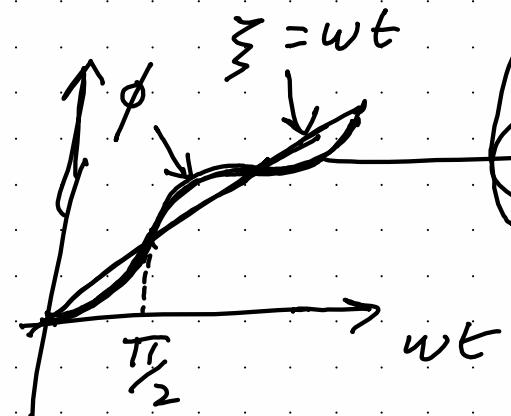
$$U = -\frac{\alpha}{r}$$

~~+1/2~~

$$+\frac{1}{2}$$

$$\begin{aligned} x &= a \cos \omega t \\ y &= b \sin \omega t \end{aligned}$$

$$\begin{aligned} x &= r \cos \phi \\ y &= r \sin \phi \end{aligned}$$

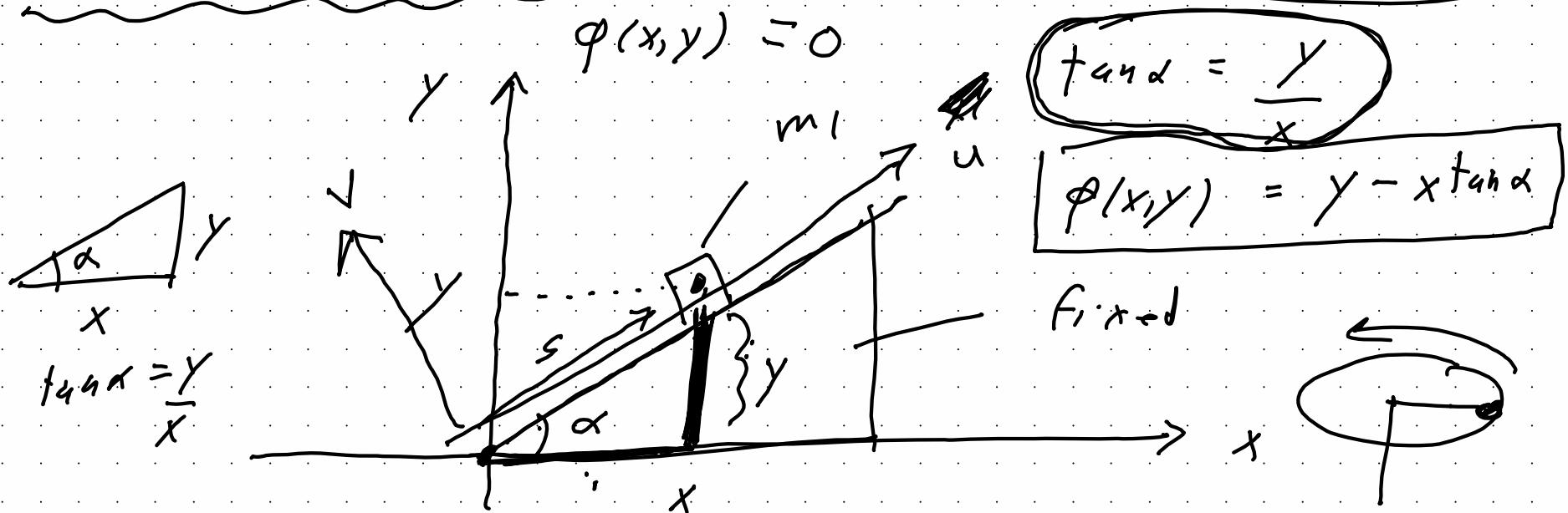


Lec #12:

- Midterm Exam 1 : Tuesday Oct 6 + 5
- Do trial exam by end of the day today
- Today : Q&A

$$x^2 + y^2 - r_0^2 = 0$$

$$r - r_0 = 0$$



$$T = \frac{1}{2} m_1 (x^2 + y^2), \quad U = m_1 g y \quad (x, y)$$

$$L = T - U + \underline{\underline{(y - x \tan \alpha)}} \quad \checkmark \quad (r, \phi)$$

L-oms:

$$x : \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \rightarrow m_1 \ddot{x} = -\lambda \tan \alpha$$

$$y : \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) = \frac{\partial L}{\partial y} \rightarrow m_1 \ddot{y} = -mg + \lambda$$

constraint:

$$\dot{y} - \dot{x} \tan \alpha = 0$$

$$\ddot{y} - \ddot{x} \tan \alpha = 0$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2$$

$$(m_1 \ddot{x} = -\lambda \tan \alpha) \text{ times}$$

$$+ m_1 \dot{x} \tan \alpha = -mg + \lambda$$

$$0 = -\lambda \tan^2 \alpha - \lambda$$

$$+ m_1 g$$

$$= -\lambda (1 + \tan^2 \alpha) + mg$$

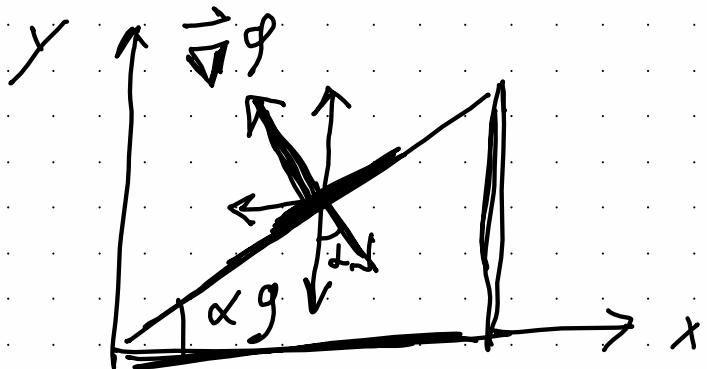
$$L = \frac{1}{2} m_1 (\dot{x}^2 + \dot{y}^2) - mg y + \lambda (\dot{y} - \dot{x} \tan \alpha)$$

$$\lambda = \frac{mg}{\sec^2 \alpha} \propto mg \cos \alpha$$

$$\varphi(x, y) = y - x \tan \alpha$$

$$\vec{\nabla} \varphi = \frac{\partial \varphi}{\partial x} \hat{x} + \frac{\partial \varphi}{\partial y} \hat{y}$$

$$= -\tan \alpha \hat{x} + \hat{y} = \frac{-\sin \alpha \hat{x} + \cos \alpha \hat{y}}{\cos \alpha}$$



$$= \frac{-\sin \alpha \hat{x} + \cos \alpha \hat{y}}{\cos \alpha}$$

$\vec{N} = mg \cos \alpha$

↓ to incline

$$\vec{\nabla} \varphi \propto -\sin \alpha \hat{x} + \cos \alpha \hat{y}$$

$$y = x \tan \alpha$$

$$dy = \tan \alpha dx \rightarrow \boxed{\cos \alpha dy = \sin \alpha dx}$$

$$\vec{\nabla} \varphi \cdot d\vec{r} = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy = -\tan \alpha dx + \frac{1}{\cos \alpha} dx = 0$$

$$(u, v)$$

$$\varphi(u, v) = v = 0$$

$$L = T - U + \lambda v$$

$$v=0$$

$$L = T - U + \lambda \varphi$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt} (m \dot{x}) = - \frac{\partial U}{\partial x}$$

$$\boxed{\frac{d \vec{p}}{dt} = - \frac{\partial U}{\partial x} = F}$$

$$\frac{d \vec{p}}{dt} = F$$

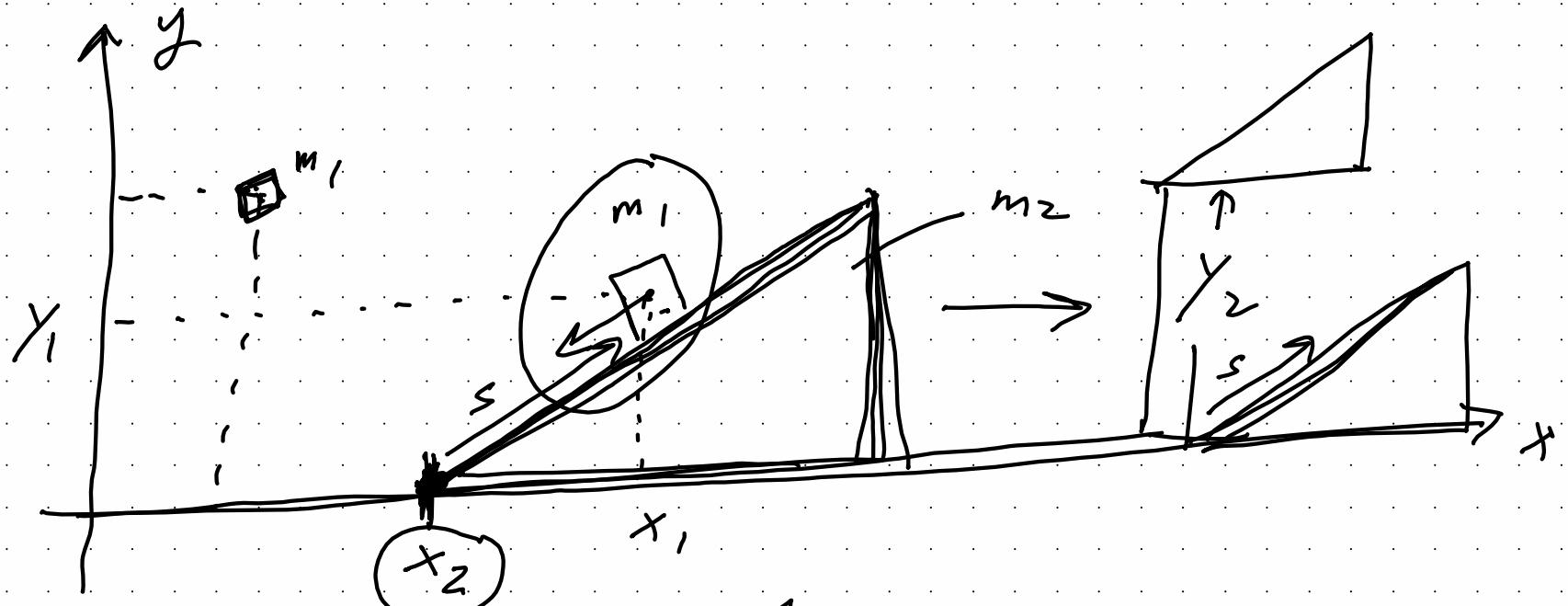
$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = - \frac{\partial L}{\partial r}$$

$$\frac{d \vec{p}}{dt} = - \frac{\partial U}{\partial r} + \lambda \frac{\partial \varphi}{\partial r}$$

$$\frac{d \vec{p}}{dt} = - \vec{\nabla} U + \vec{\nabla} \varphi$$

applied



(x_1, y_1, x_2, y_2)

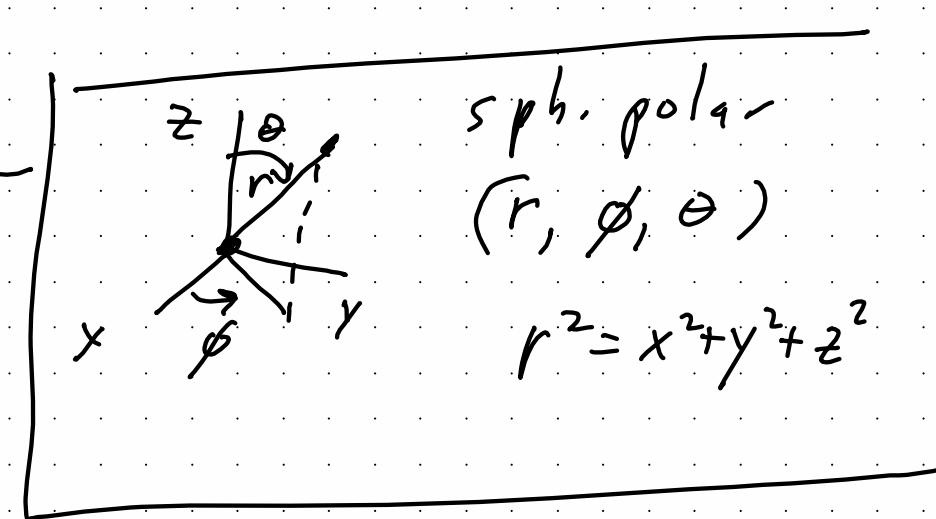
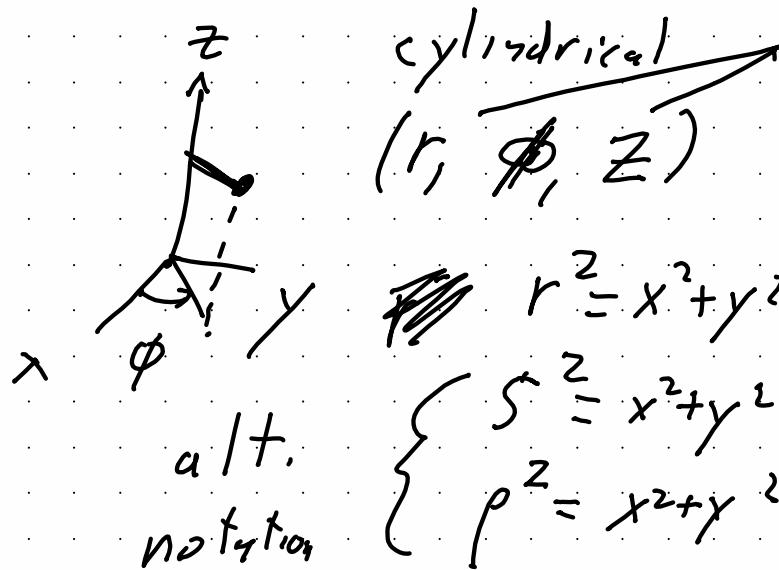
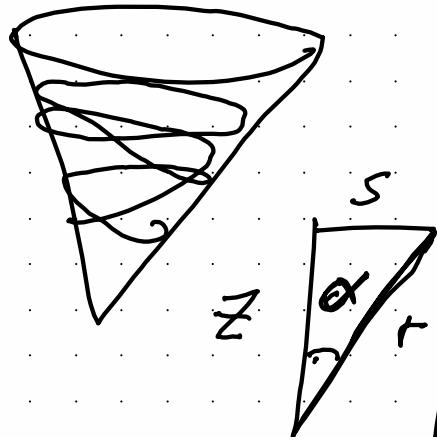
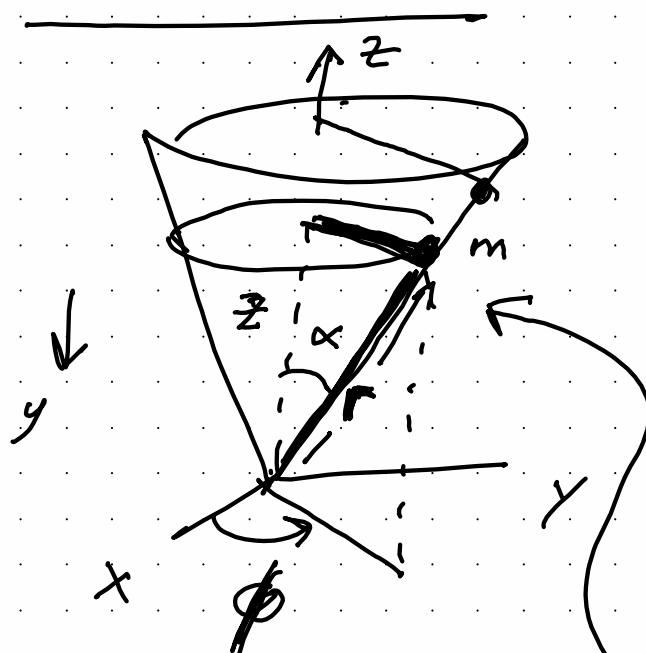
\approx

m_1

$y_2 = 0$

$$\underline{T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)}$$

Sec 14, Prob 2:



$$\tan \alpha = \frac{s}{z}$$

$$\theta = \alpha$$

$$\phi, s$$

$$\phi, z$$

$$T = \frac{1}{2} m \left(r^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right)$$

\downarrow

θ $\sin^2 \alpha$

sp^b. pol.,
 $\theta = \alpha$

$$T = \frac{1}{2} m \left(s^2 + s^2 \dot{\phi}^2 + \dot{z}^2 \right)$$

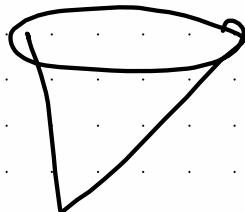
\curvearrowleft

cyl. coord,
 s, ϕ, z

~~theta~~

$\tan \alpha = \frac{s}{z}$

plane polar
coord



$$L = T - V + \lambda \varphi$$

$$\varphi = s - z \tan \alpha$$

$$z = \frac{s}{\tan \alpha}$$

$$\dot{z} = \frac{s}{\tan \alpha}$$

$$\begin{cases} s = z \tan \alpha \\ s = z \tan \alpha \end{cases}$$

$$f(x, y) = e^{-\frac{1}{2}(x^2+y^2)}$$

No constraint: $\left\{ \begin{array}{l} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{array} \right.$

$$\begin{aligned} g(x_1, y_1, \lambda) &= f(x_1, y_1) + \lambda \varphi(x_1, y_1) \\ &= \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial x} &= 0 \\ \frac{\partial g}{\partial y} &= 0, \quad \frac{\partial g}{\partial \lambda} = 0 \end{aligned}$$

Suppose constraint $y = \frac{1}{2}x^2 + 1$

$$F(x) = f(x, y) \Big| y = \frac{1}{2}x^2 + 1$$

$$L = T - U + \lambda \varphi$$

$$x_1, y_1, \lambda$$

$$\boxed{\frac{dF}{dx} = 0}$$

$$\boxed{\varphi = y - \frac{1}{2}x^2 - 1 = 0}$$

COM:

$$m_1, m_2, m_3, \vec{r}_1, \vec{r}_2, \vec{r}_3 \quad \left. \right\} \vec{P} = M_{\text{tot}} \vec{R}_{\text{com}}$$

$$\vec{R}_{\text{com}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3}{m_1 + m_2 + m_3} = \text{const}$$

$$0 = \sum_a m_a \vec{r}_a$$

$\sum_b m_b$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}_a} \right) = 0$$

$$\vec{P} = \sum_a \frac{\partial L}{\partial \dot{r}_a} = \text{const}$$

~~closed system~~: $\vec{P} = \text{const}$

$$L = \frac{1}{2} \sum_a m_a |\vec{r}_a|^2 - U(\vec{r}_1 - \vec{r}_2), (\vec{r}_2 - \vec{r}_3), \dots$$

$$\vec{r}_a \rightarrow \vec{r}_a + \delta \vec{x}$$

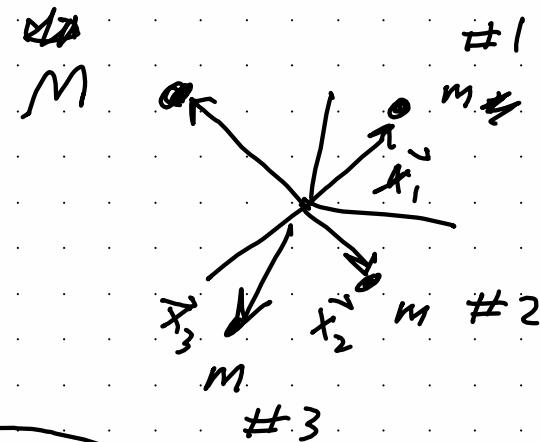
$$L \rightarrow L$$

$$\frac{\partial L}{\partial \vec{r}} = 0$$

$$M, \underbrace{m_1, \dots, m_n}_n$$

$$\overrightarrow{X} \quad x_1, \dots, x_n$$

$$M \overrightarrow{X} + m \sum_{a=1}^n \overrightarrow{x_a} = 0$$



$$\overrightarrow{r_1}, \overrightarrow{r_2}$$

$$\overrightarrow{r} = \overrightarrow{r_1} - \overrightarrow{r_2}$$

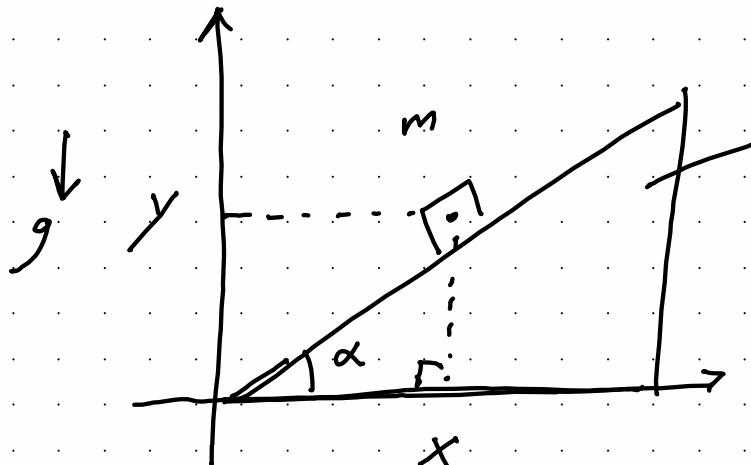
$$\begin{aligned} \cancel{\overrightarrow{r}} \\ \overrightarrow{r_2} &= \overrightarrow{x_1} - \overrightarrow{X} \\ \vdots \\ \overrightarrow{r_n} &= \overrightarrow{x_n} - \overrightarrow{X} \end{aligned}$$

$1h + 20mm$

$80mm$

$120mm$

$2:35pm$



$$L = T - U + \lambda \phi$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mg y + \lambda(y - x \tan \alpha)$$

Eoms:

$$x: \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \rightarrow m \ddot{x} = -\lambda \tan \alpha \quad (1)$$

$$y: \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) = \frac{\partial L}{\partial y} \rightarrow m \ddot{y} = -mg + \lambda \quad (2)$$

$$\lambda: y - x \tan \alpha = 0 \rightarrow y = x \tan \alpha \quad (3)$$

Differentiate constraint twice:

$$\ddot{y} = \ddot{x} \tan \alpha$$

Subst. to eq for \ddot{y} , \ddot{x} using (1) and (2)

$$\tan \alpha = \frac{y}{x}$$

$$\text{so } y = x \tan \alpha$$

$$\phi(x, y) = y - x \tan \alpha = 0$$

constraint

$$-g + \frac{1}{m} = -\frac{1}{m} \tan \alpha \cdot \tan \alpha$$

$$\frac{1}{m} (1 + \tan^2 \alpha) = g$$

$\tan^2 \alpha$

$$\lambda = \frac{mg}{\sec^2 \alpha}$$

constraint Force:

$$\overrightarrow{F}_c = \lambda \nabla \phi \quad \text{where} \quad \phi = y - x \tan \alpha$$

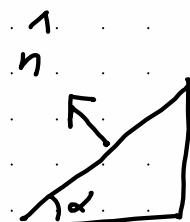
$$= \frac{mg}{\sec^2 \alpha} \left(-\tan \alpha \hat{x} + \hat{y} \right)$$

$$= \frac{mg}{\sec^2 \alpha} \left(-\frac{\sin \alpha}{\cos \alpha} \hat{x} + \hat{y} \right)$$

$$= mg \cos \alpha \left(-\sin \alpha \hat{x} + \cos \alpha \hat{y} \right)$$

$$= mg \cos \alpha \hat{n}$$

(where $\hat{n} = -\sin \alpha \hat{x} + \cos \alpha \hat{y}$
is \perp to incline)



Return to Eoms:

$$\ddot{x} = -\frac{\lambda}{m} \tan \alpha$$

$$\ddot{y} = -g + \frac{\lambda}{m}$$

where $\lambda = \frac{mg}{\sec^2 \alpha} = mg \cos^2 \alpha$

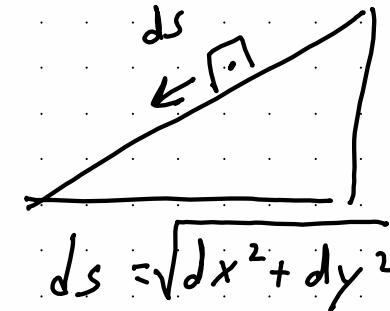
$$\rightarrow \ddot{x} = -g \cos^2 \alpha \tan \alpha$$

$$= -g \sin \alpha \cos \alpha$$

$$\ddot{y} = -g + g \cos^2 \alpha$$

$$= -g (1 - \cos^2 \alpha)$$

$$= -g \sin^2 \alpha$$



Acceleration down the incline:

$$\ddot{s} = -\sqrt{\ddot{x}^2 + \ddot{y}^2} = -g \sin \alpha \sqrt{\cos^2 \alpha + \sin^2 \alpha} = \boxed{-g \sin \alpha}$$

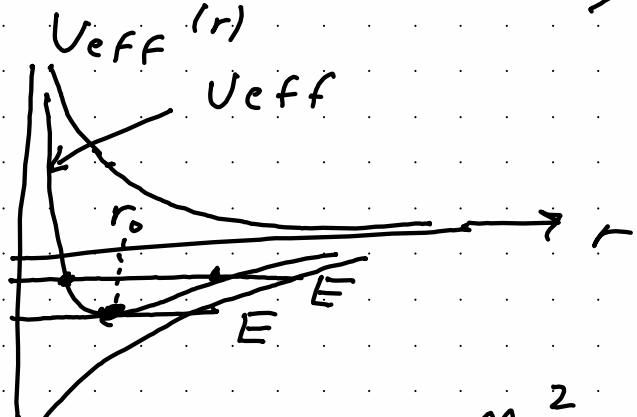
s standard result

Lecture #14: Thurs Oct 8th

- midterm 1: Avg $\approx 14/20$
- midterm 2: Nov 19th (before Thanksgiving)
- org final

Collisions & Scattering (Sec 16-20)
(2) (3)

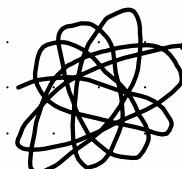
Scattering: closed system, two bodies,
 \rightarrow central force problem



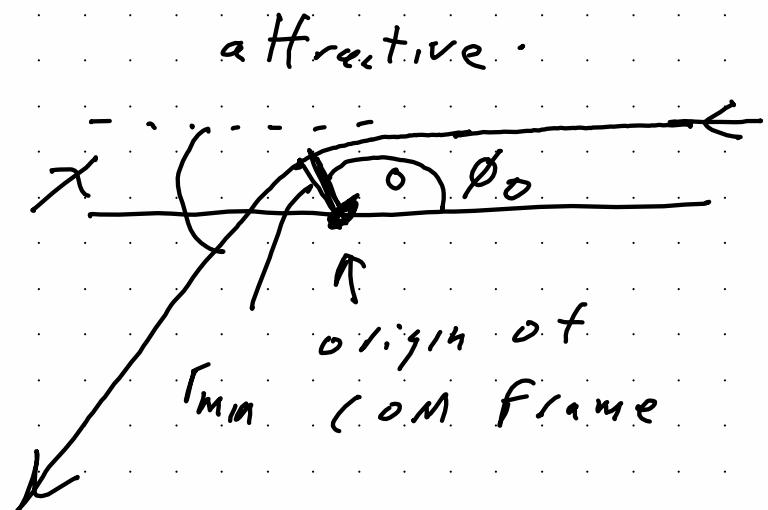
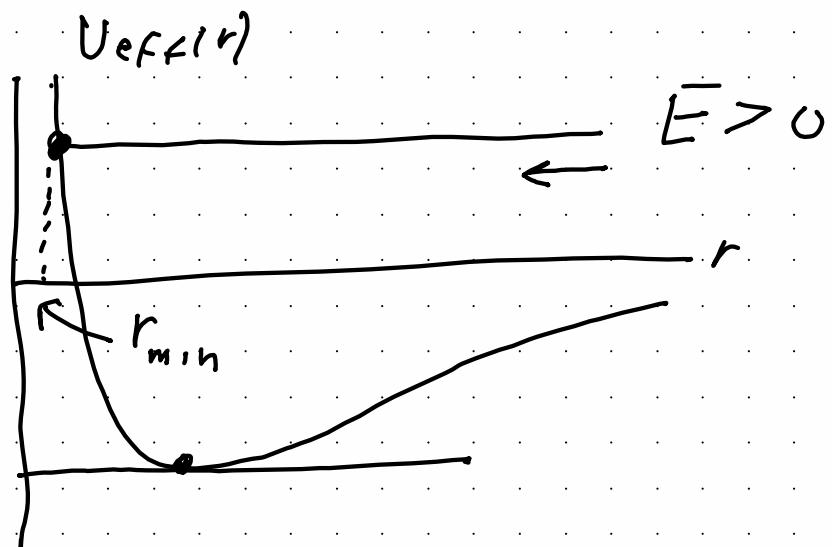
$$U_{\text{eff}} = -\frac{\alpha}{r} + \frac{M_2^2}{2mr^2}$$

$$E = U_{\text{eff}, \min} \rightarrow r = r_0 \quad (\text{circular})$$

$U_{\text{eff}, \min} < E < 0 \rightarrow$ bound orbit

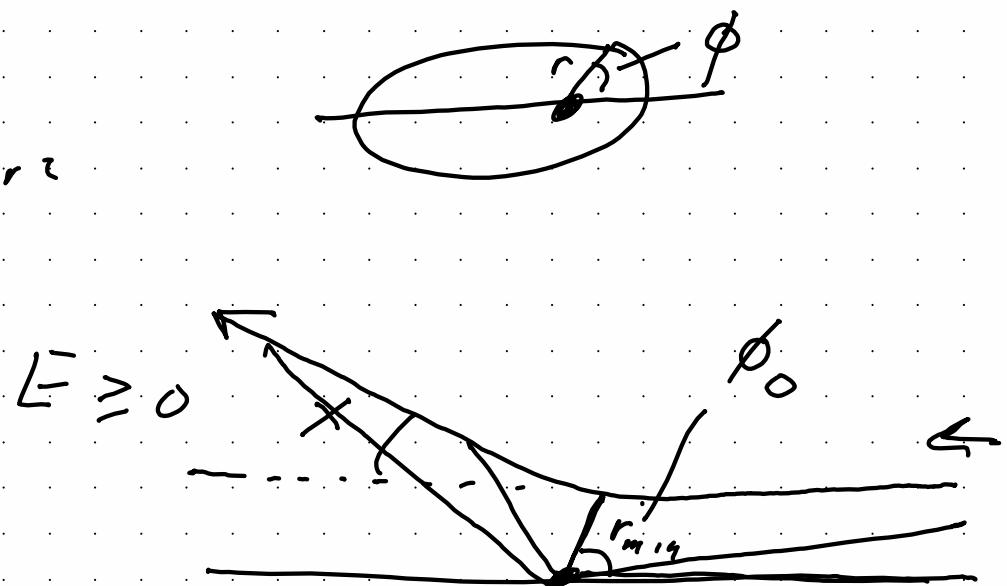
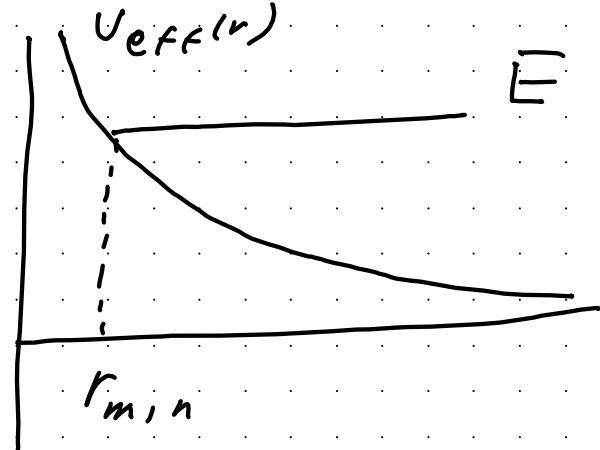


ellipse
for $-\alpha/r$



$$U(r) = +\alpha/r$$

$$U_{\text{eff}}(r) = \alpha/r + M_z^2/2mr^2$$



$$\int_0^{\phi_0} d\phi = \int_{r_{min}}^{\infty} \frac{M_e dr / r^2}{\sqrt{2m(E - U(r)) - \frac{M_e^2}{r^2}}} \quad (14.7) L \& L$$

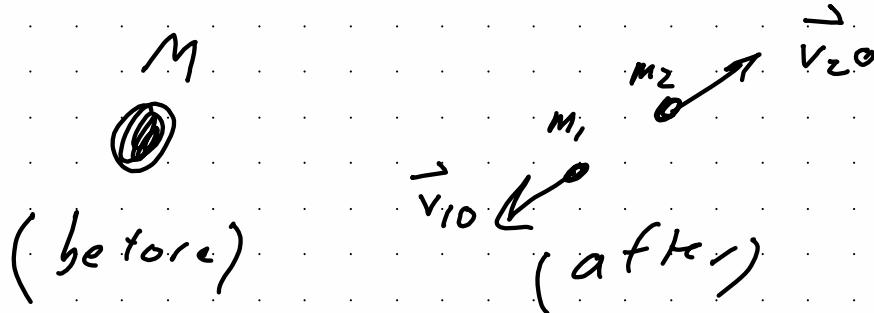
$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M_e dr / r^2}{\sqrt{2m(E - U(r)) - \frac{M_e^2}{r^2}}} \quad E, M_e, U(r)$$

Fig 18 in L & L

Collisions: (Sec 16, 17)

- Elastic collision of two particles

(*) Spontaneous disintegration of a single particle



Analyze in COM Frame (to begin with)

$$\vec{p}_{\text{tot}, 0} = \vec{0}$$

L_{COM}

cons. of momentum

$$\begin{aligned} \vec{p}_0 \\ - \vec{p}_0 \end{aligned}$$

: momentum of mass m_1 , after disintegration

11

m_2

cons. of energy

$$E_i = E_{1c} + T_{10} + E_{2c} + T_{20}$$

in
internal
energy of
mass $M = m_1 + m_2$

intergal KE in COM frame

$$= \frac{1}{2} m_1 \vec{v}_{10}^2$$
$$= \frac{|\vec{p}_0|^2}{2m_1}$$

$$E_i = E_{1c} + \frac{|\vec{p}_0|^2}{2m_1} + E_{2c} + \frac{|\vec{p}_0|^2}{2m_2}$$

$$\underbrace{E_i - E_{1c} - E_{2c}}_{\text{Energy}} = \frac{p_0^2}{2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right)$$

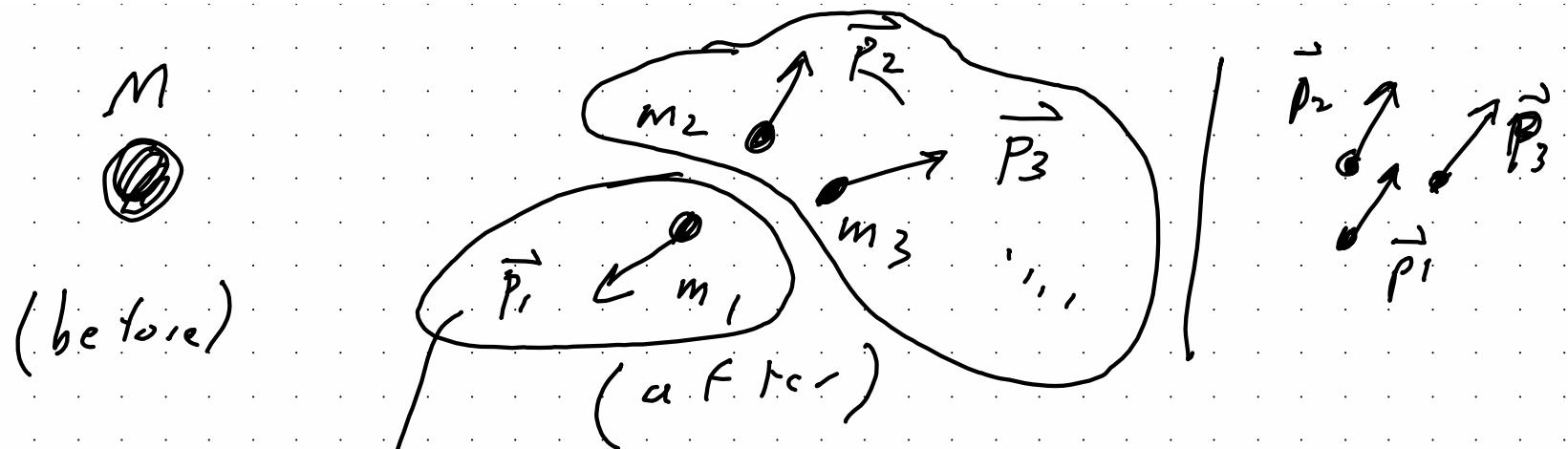
ϵ = d/integration

≥ 0 energy

$$= \frac{p_0^2}{2} \left(\frac{m_1 + m_2}{m_1 m_2} \right) = \frac{p_0^2}{2m}$$

$$\rho_0 = \sqrt{2m\epsilon}$$

$$\rightarrow v_{10} = p_0/m_1, v_{20} = p_0/m_2$$



$$\vec{p}_{\text{tot}} = 0$$

$$\vec{p}_1 = \vec{p}_0$$

$$T_{1,0} = \frac{p_0^2}{2m}$$

Q: what conditions are there on the velocities of m_2, m_3, \dots such that KE of m_1 is largest?

Consider COM of m_2, m_3, \dots

E_i^{int} : internal energy of m_2, m_3, \dots

$$\vec{p}_2 + \vec{p}_3 + \dots = -\vec{p}_0$$

$$KE = \frac{p_0^2}{2(M-m_1)}$$

$$(m_2 + m_3 + \dots)$$

Cons. of energy.

$$E_i = E_{i,i} + \frac{p_0^2}{2m_1}$$

ht. energy
of M

ht. + KE
of m_1

$$+ E_{i'} + \frac{p_0^2}{2(M-m_1)}$$

int + KE
of m_2, m_3, \dots

$$E_i - E_{i,i} - E_{i'} = \frac{p_0^2}{2} \left(\frac{1}{m_1} + \frac{1}{M-m_1} \right)$$

$$E_i - E_{i,i} - E_{i'} = \frac{p_0^2}{2} \frac{M}{m_1(M-m_1)}$$

$E_{2i} + E_{3i} + \dots$

X

$$T_{10} = \frac{p_0^2}{2m_1} = \left(\frac{M-m_1}{M} \right) (E_i - E_{i,i} - E_{i'})$$

T_{10} ~~is~~ maximum when $E_{i'}$ is minimum.

$$E_i' + \frac{p_0^2}{2(M-m_i)} = E_{2i} + \frac{p_2^2}{2m_2} + E_{3i} + \frac{p_3^2}{2m_3} + \dots$$

$\vec{p}_2 = m_2 \vec{v}_2, \text{ etc.}$

int energy + KE of
masses m_2, m_3, \dots

$$= E_{2i} + \frac{1}{2} m_2 |\vec{v}_2|^2 + E_{3i} + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots$$

$$E_i' = (E_{2i} + E_{3i} + \dots) + \left(\frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots \right)$$

W $\frac{|\vec{p}_2 + \vec{p}_3 + \dots|^2}{2(m_2 + m_3 + \dots)} = |\vec{v}_0|^2$

$$= (E_{2i} + E_{3i} + \dots) + \left(\frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots \right)$$

$\ominus \frac{1}{2} \frac{|\vec{m}_2 \vec{v}_2 + \vec{m}_3 \vec{v}_3 + \dots|^2}{M_2 + M_3 + \dots}$

$\begin{aligned} & |\vec{v}_2 + \vec{v}_3|^2 \\ &= |\vec{v}_2|^2 + |\vec{v}_3|^2 \\ &+ 2 \vec{v}_2 \cdot \vec{v}_3 \end{aligned}$

$$\text{If } \vec{v}_2 = \vec{v}_3 = \dots \equiv \vec{v}_0$$

then: $\frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \dots$

$$= \left(\frac{1}{2} (m_2 + m_3 + \dots) \right) |\vec{v}_0|^2$$

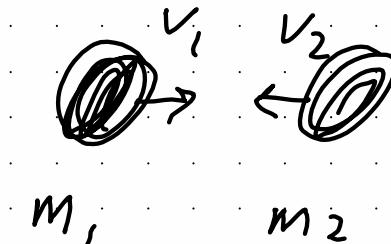
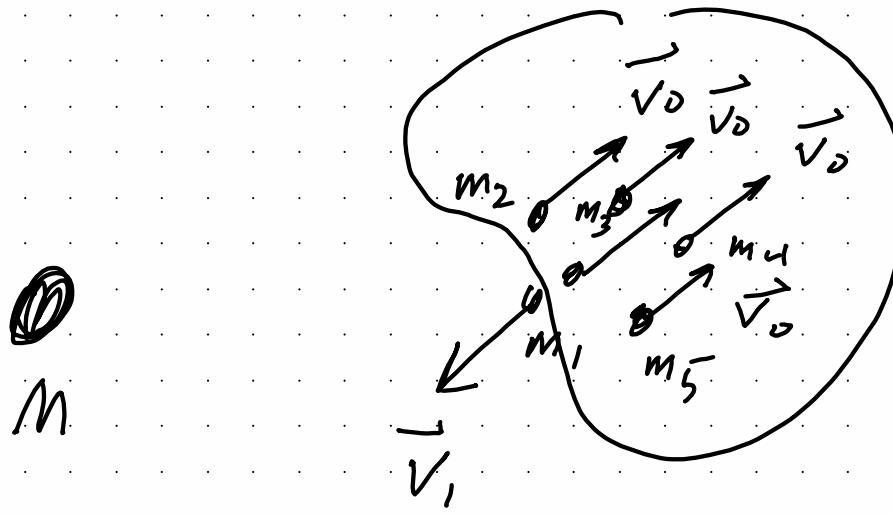
and

$$\frac{1}{2} \frac{(m_2 \vec{v}_2 + m_3 \vec{v}_3 + \dots)^2}{m_2 + m_3 + \dots}$$

$$= \frac{1}{2} |\vec{v}_0|^2 \frac{(m_2 + m_3 + \dots)^2}{\cancel{m_2 + m_3 + \dots}}$$

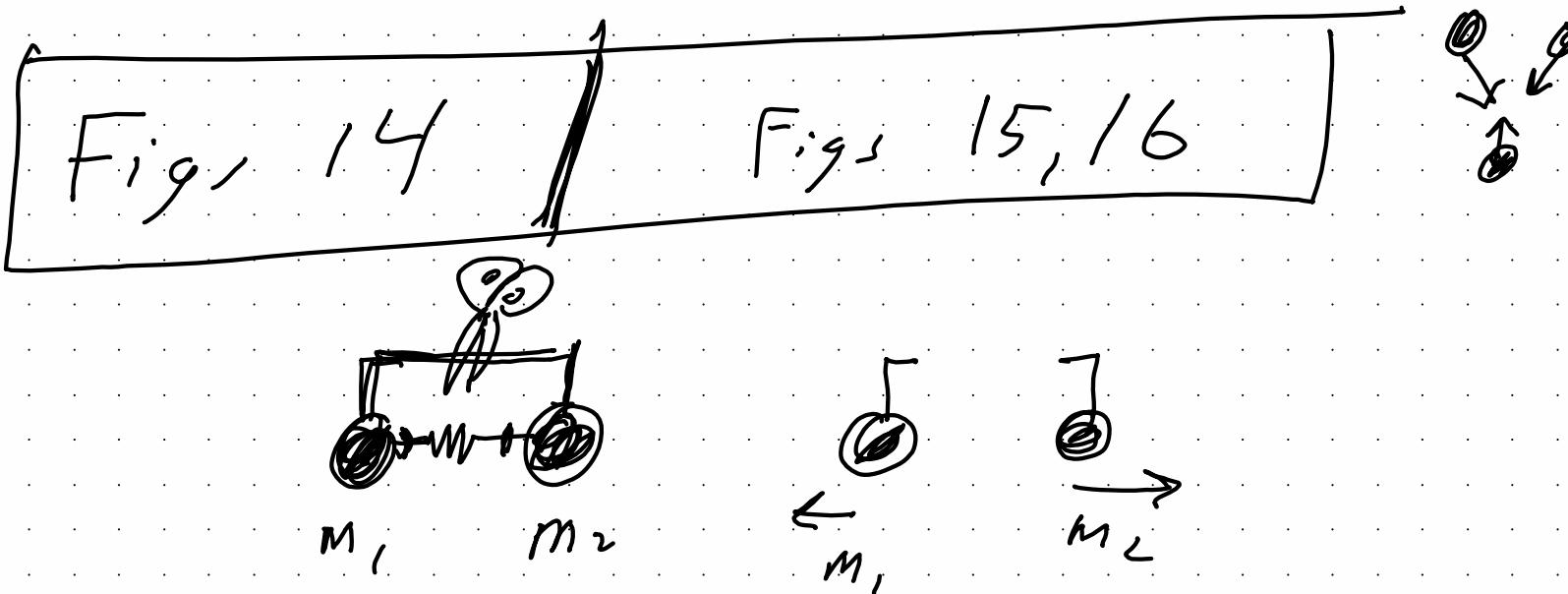
$$= \frac{1}{2} |\vec{v}_0|^2 (m_2 + m_3 + \dots)$$

$$T_{IO_1, \max} = \left(\frac{M-m_1}{M} \right) \in$$



QM

$$M = m_1 + m_2$$



Lec #15: Tues 10/13

— solutions to midterm #1 posted

— next two weeks: Sec 16-20

Collisions and scattering

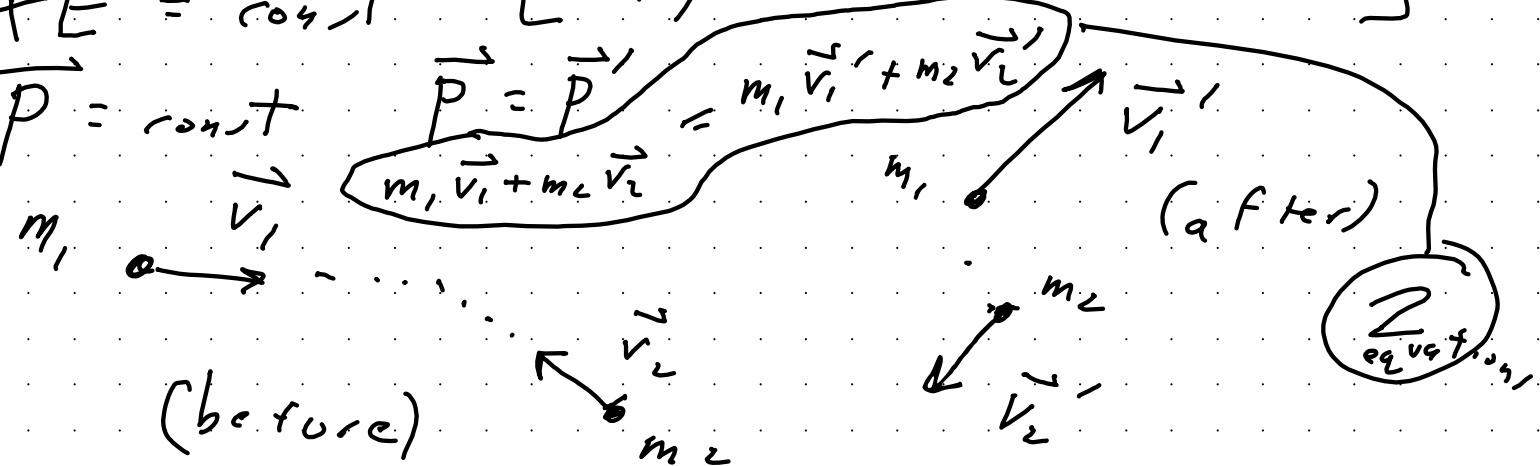
Figures 14, 15, 16

Elastic collision of two particles

Closed system: E , \vec{P} , \vec{M} are conserved

✓ i) $\cancel{KE} = \text{const}$ [ignore internal energies]

✓ ii) $\vec{P} = \text{const}$



After: \vec{v}_1' , \vec{v}_2' : 4 DOF
3 equations

Elastic collision:

$$\vec{V} \equiv \vec{v}_1 - \vec{v}_2 = \vec{v}_{10} - \vec{v}_{20} \quad \text{relative velocity vector}$$

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 = \vec{r}_{10} - \vec{r}_{20} \quad \text{relative position vector}$$

$$\vec{v}_1 = \vec{r}$$

$$\vec{v}_1 = \vec{V} + \vec{v}_{10}$$

$$\vec{v}_2 = \vec{V} + \vec{v}_{20}$$

$$\vec{r}_1 = \vec{R} + \vec{r}_{10}$$

$$\vec{r}_2 = \vec{R} + \vec{r}_{20}$$

\vec{v}_1 : velocity of particle 1 wrt lab frame
before collision

\vec{v}_{10} : " wrt COM Frame
before collision

\vec{v}_1' , \vec{v}_{10}' , \vec{v}_2' , \vec{v}_{20}' ; velocities after collision

$$T_0 = \frac{1}{2} m_1 |\dot{\vec{r}}_{10}|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_{20}|^2$$

$$\begin{aligned} \vec{v}_{10} &= \dot{\vec{r}}_{10} = \left(\frac{m_2}{m_1 + m_2} \right) \dot{\vec{r}} = \vec{V} & \vec{r} &= \vec{r}_{10} - \vec{r}_{20} \\ \vec{v}_{20} &= \dot{\vec{r}}_{20} = -\left(\frac{m_1}{m_1 + m_2} \right) \dot{\vec{r}} \end{aligned}$$

$$T_0 = \frac{1}{2} m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 |\dot{\vec{r}}|^2$$

$$+ \frac{1}{2} m_2 \left(\frac{-m_1}{m_1 + m_2} \right)^2 |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} m |\dot{\vec{r}}|^2 = \frac{1}{2} m |\vec{V}|^2$$

$$T_0 = T_0'$$

$$\frac{1}{2} m |\vec{V}|^2$$

$$= \frac{1}{2} m' |\vec{V}'|^2$$

$$\boxed{V = V'}$$

~~relative~~
magnitude
of relative
velocity
vector = const

Before collision

$$\underline{\underline{\overrightarrow{v}_{1,0}}} = \left(\frac{m_2}{m_1+m_2} \right) \underline{\underline{\overrightarrow{v}}}, \quad \underline{\underline{\overrightarrow{v}_{2,0}}} = \left(\frac{-m_1}{m_1+m_2} \right) \underline{\underline{\overrightarrow{v}}}$$

$$\underline{\underline{\overrightarrow{V}}} = \underline{\underline{\overrightarrow{v}_1}} - \underline{\underline{\overrightarrow{v}_2}}$$

After collision:

$$\overrightarrow{v}'_{1,0} = \left(\frac{m_2}{m_1+m_2} \right) v \hat{n}_0$$

magnitude
of \overrightarrow{V}

unit vector
pointing in
some direction
(angle = χ w.r.t
COM frame)

$$\overrightarrow{v}'_{2,0} = \left(\frac{-m_1}{m_1+m_2} \right) v \hat{n}_0$$

$$\boxed{\overrightarrow{v}'_1 = \overrightarrow{v}'_{1,0} + \overrightarrow{V} = \left(\frac{m_2}{m_1+m_2} \right) v \hat{n}_0 + \frac{m_1 \overrightarrow{v}_1 + m_2 \overrightarrow{v}_2}{m_1+m_2}}$$

$$\overrightarrow{R} = \frac{m_1 \overrightarrow{r}_1 + m_2 \overrightarrow{r}_2}{m_1+m_2} \rightarrow \overrightarrow{V} = \frac{m_1 \overrightarrow{v}_1 + m_2 \overrightarrow{v}_2}{m_1+m_2}$$

$$\vec{v}_1' = \vec{v}_{10} + \vec{V} = \left(\frac{m_2}{m_1+m_2} \right) \vec{v}_{n_0}^\wedge + \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1+m_2} = \vec{v}_1'$$

$$\vec{v}_2' = \vec{v}_{20} + \vec{V} = \left(\frac{-m_1}{m_1+m_2} \right) \vec{v}_{n_0}^\wedge + \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1+m_2} = \vec{v}_2'$$

$$\vec{p}_1' = m_1 \vec{v}_1' = m \vec{v}_{n_0}^\wedge + m_1 \left(\frac{\vec{p}_1 + \vec{p}_2}{m_1+m_2} \right)$$

$$\vec{p}_2' = m_2 \vec{v}_2' = -m \vec{v}_{n_0}^\wedge + m_2 \left(\frac{\vec{p}_1 + \vec{p}_2}{m_1+m_2} \right)$$