Appendix A: Elliptic Functions

This appendix is added to make the paper completely self-contained and to show that everything one needs to know about elliptic functions is relatively easy to develop. In particular, this development is trigonometric and makes no explicit use of complex analysis. The Jacobian elliptic functions are like trigonometric functions except that they are defined on the ellipse

$$\frac{x^2}{a^2} + y^2 = 1 \tag{A1}$$

rather than on the unit circle. The shape of the ellipse is controlled by the eccentricity

$$k = \sqrt{1 - 1/a^2}. \tag{A2}$$

For a given k, the sn and cn functions are defined by analogy to sine and cosine, namely

$$\mathrm{sn}(u,k) = y \qquad \text{and} \qquad \mathrm{cn}(u,k) = x/a. \tag{A3}$$

In this context, k is called the modulus, and the argument u of the elliptic functions, namely

$$u = \int_0^\theta r\, d\theta, \tag{A4}$$

is the integral along the ellipse from the x intercept $(a, 0)$ to the point (x, y). Note that u is neither the arc length nor the area subtended. In terms of the polar angle θ, the upper limit is such that $\sin\theta = y/r$. This is not to be confused with Jacobi's amplitude $\phi = \arcsin y = \mathrm{Am}(u,k)$ such that $\sin(\mathrm{Am}(u,k)) = \mathrm{sn}(u,k)$. The latter equation will serve to define the elliptic amplitude function $\mathrm{Am}(u,a)$.

Because the radius is not constant for an ellipse, there is a third elementary elliptic function, not corresponding to any trigonometric function, in addition to $\mathrm{sn}(u,k)$ and $\mathrm{cn}(u,k)$, namely

$$\mathrm{dn}(u,k) = \frac{r}{a}. \tag{A5}$$

From the ellipse equation, we have

$$\mathrm{cn}(u,k)^2 + \mathrm{sn}(u,k)^2 = 1. \tag{A6}$$

From this result and the Pythagorean relation $x^2 + y^2 = r^2$, we have

$$\mathrm{dn}(u,k)^2 + k^2 \mathrm{sn}(u,k)^2 = 1. \tag{A7}$$

In the normal trigonometry based on a circle, the latter two identities reduce to one.

It is relatively easy to show from the definitions that

$$\frac{d}{du}\mathrm{sn}(u,k) \;=\; \mathrm{cn}(u,k)\mathrm{dn}(u,k), \tag{A8}$$

and because all the calculus properties of the Jacobi functions follow from this, we devote a few lines to showing it here.

The first step is to express $du = r\,d\theta$ in Cartesian form. Starting from $x = r\,\cos\theta$ and $y = r\,\sin\theta$, we find

$$x\,dy - y\,dx \;=\; r^2\,d\theta, \tag{A9}$$

so

$$du \;=\; \frac{x\,dy - y\,dx}{r}. \tag{A10}$$

We use the ellipse equation to eliminate x and dx, namely

$$x\,dx + a^2 y\,dy \;=\; 0, \qquad dx \;=\; -a^2\frac{y}{x}\,dy. \tag{A11}$$

Substituting this and multiplying by x in Eq. (A10) gives

$$\frac{x^2 dy - xy\,dx}{r} \;=\; x\,du. \tag{A12}$$

Now we use the ellipse equation again in the form $x^2 + a^2y^2 \;=\; a^2$ to obtain

$$\frac{(x^2 + a^2y^2)dy}{r} \;=\; \frac{a^2}{r}\,dy \;=\; x\,du, \tag{A13}$$

which gives

$$\frac{dy}{du} \;=\; \frac{x}{a}\,\frac{r}{a} \qquad \text{or} \qquad \frac{d}{du}\,\mathrm{sn}(u,k) \;=\; \mathrm{cn}(u,k)\,\mathrm{dn}(u,k), \tag{A14}$$

which is the desired result. By differentiating the two algebraic identities, Eq. (A6) and Eq. (A7), shown above and substituting Eq. (A14), one arrives at

$$\frac{d}{du}\mathrm{cn}(u,k) \;=\; -\,\mathrm{sn}(u,k)\mathrm{dn}(u,k) \qquad \text{and} \qquad \frac{d}{du}\mathrm{dn}(u,k) \;=\; -k^2\,\mathrm{sn}(u,k)\mathrm{cn}(u,k). \tag{A15}$$

These three differential equations match the Euler equations almost perfectly.

Elliptic functions are related quite closely to the elliptic integrals of Legendre. From the separable differential equation

$$\frac{d}{du}\mathrm{sn}(u,k) \;=\; \mathrm{cn}(u,k)\,\mathrm{dn}(u,k) \;=\; \sqrt{1-\mathrm{sn}(u,k)^2}\sqrt{1-k^2\mathrm{sn}(u,k)^2}, \tag{A16}$$

we have that

$$u \;=\; \int_0^{\mathrm{sn}(u,k)} \frac{dt}{\sqrt{(1-t^2)\,(1-k^2\,t^2)}}. \tag{A17}$$

This shows the Elliptic integral of the First kind commonly denoted as the F function to be the inverse of the sn function. However, the argument of F has been taken historically to be the Jacobi amplitude angle ϕ such that $\sin\phi = \mathrm{sn}(u,k)$ rather than $\mathrm{sn}(u,k)$, so

$$F(\phi,k) = \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)\,(1-k^2\,t^2)}}. \tag{A18}$$

Thus written, the first elliptic integral function $F(\phi,k)$ of Legendre is related to the inverse sn function by

$$\mathrm{sn}^{-1}(y,k) = F(\sin^{-1} y, k) \tag{A19}$$

over the standard domain of $\sin^{-1}$.

There are also the second and third types of Legendre elliptic integrals. These are integrals of simple elliptic function expressions. The second elliptic integral is the integral of Jacobi's dn squared, namely

$$E(\phi,k) = \int_0^{\phi} \sqrt{1-k^2\sin^2\phi}\,d\phi \;=\; \int_0^{u} (\mathrm{dn}(u,k))^2\,du, \tag{A20}$$

where in the second integral we made the elliptic substitution $\sin\phi = \mathrm{sn}(u,k)$.

Legendre's third elliptic integral contains an extra parameter n, which can be any real number, namely

$$\Pi(n;\phi,k) \;=\; \int_0^{\phi} \frac{1}{1-n\,\sin^2\phi}\frac{d\phi}{\sqrt{1-k^2\,\sin^2\phi}}, \tag{A21}$$

and on making the same elliptic substitution we obtain

$$\Pi(n;\phi,k) \;=\; \int_0^{u} \frac{du}{1-n\,\mathrm{sn}^2(u,k)}. \tag{A22}$$

These three elliptic integral functions are standard and their properties are tabulated on the internet, and in most textbooks regarding Elliptic functions.