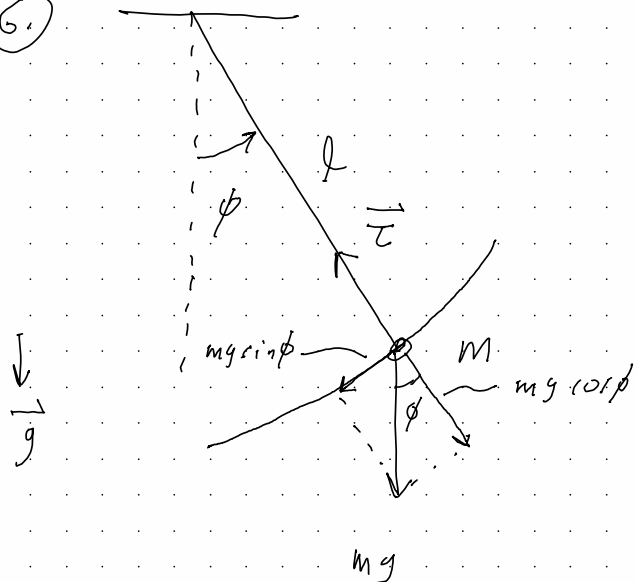


(6.)



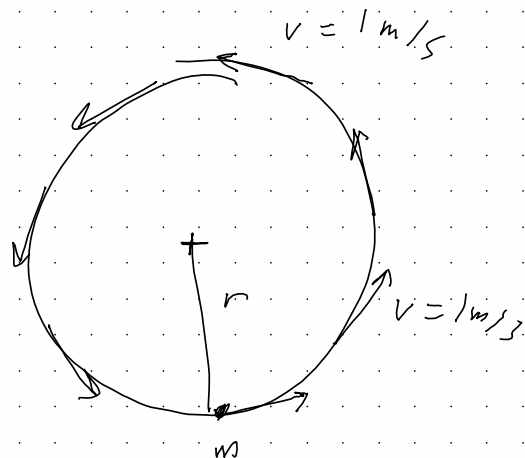
~~$T = mg \cos \phi$~~

$$\begin{aligned} T - mg \cos \phi &= F_{\text{centrifugal}} \\ &= m a_{\text{centrifugal}} \\ &= m l \dot{\phi}^2 \end{aligned}$$

Centrifugal force

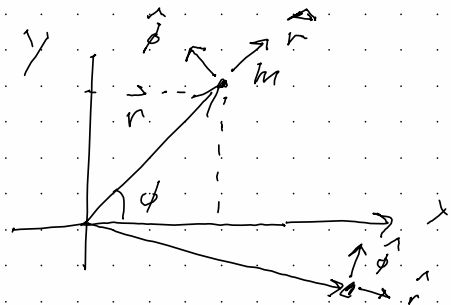
Centrifugal force

$$m \vec{a} = \vec{F}_{\text{applied}} + \vec{F}_{\text{fictitious}}$$



$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$a_{\text{centrifugal}} = \frac{v^2}{r} = \omega^2 r$$



2-d motion:

2-d motion:

$$\vec{r} = x \hat{x} + y \hat{y}$$

$$\vec{v} = \dot{\vec{r}} = \dot{\vec{r}}(t)$$

$$\vec{r} = r \hat{r}$$

$$\vec{v} = \dot{\vec{r}} = \dot{r} \hat{r} + r \dot{\hat{r}}$$

$$= \dot{r} \hat{r} + r \dot{\phi} \hat{\phi}$$

$$\hat{r} = \cos\phi \hat{x} + \sin\phi \hat{y}$$

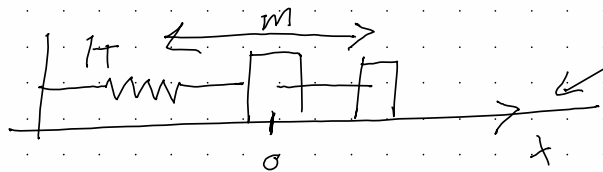
$$\hat{\phi} = -\sin\phi \hat{x} + \cos\phi \hat{y}$$

$$\vec{a} = \ddot{\vec{r}} = \ddot{r} \hat{r} + 2\dot{r}\dot{\phi} \hat{\phi} + r\ddot{\phi} \hat{\phi} - r\dot{\phi}^2 \hat{r}$$

~~kinetic~~ kinetic energy for a single mass m
in Cartesian, sph. polar, and polar coordinates.

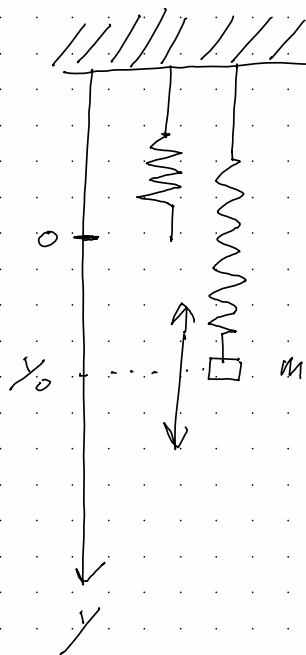
$$\begin{aligned} T &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \frac{1}{2} m (\quad) \quad \leftarrow (r, \theta, \phi) \text{ sph. polar} \\ &= \frac{1}{2} m (\quad) \quad \leftarrow (\rho, \phi, z) \text{ cylindrical} \end{aligned}$$

$$x = r \sin \theta \cos \phi, \text{ etc}$$



Frictionless

$$U = \frac{1}{2} k x^2$$



$$L = T - U$$

$$T = \frac{1}{2} m \dot{y}^2$$

$$U = \underbrace{\frac{1}{2} k y^2}_{U_s} - \underbrace{mgy}_{U_g}$$

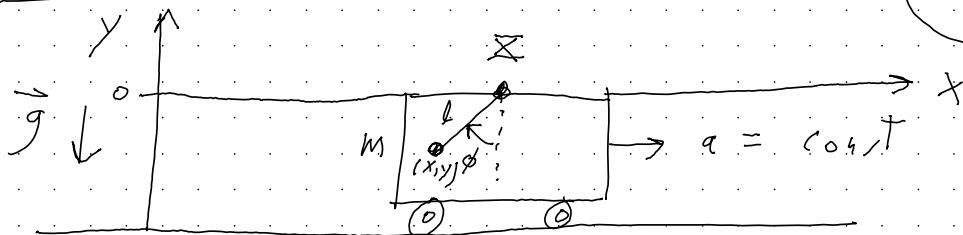
$$E = T + U$$

$$= \frac{1}{2} m \dot{y}^2 + \underbrace{\frac{1}{2} k y^2 - mgy}_{U_{\text{eff}}(y)}$$

$U_{\text{eff}}(y)$

plot this

9/11: Quiz #1



$$\vec{X} = \vec{X}_0 + \vec{V}_0 t + \frac{1}{2} a t^2$$

$$\dot{\vec{X}} = \vec{v}$$

$$\ddot{\vec{X}} = \vec{a}$$

$$L = \dot{P} = T - U$$

$$x = X - l \sin \phi$$

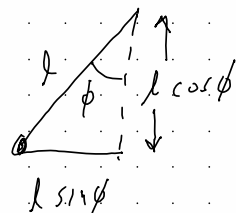
$$y = -l \cos \phi$$

$$U = mgy = -mgl \cos \phi$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{X} - l \dot{\phi} \cos \phi$$

$$\dot{y} = l \dot{\phi} \sin \phi$$



$$\vec{r} = x \hat{x} + y \hat{y}$$

$$\vec{g} = -g \hat{y}$$

$$\vec{F} = m \vec{g}$$

$$U = -m \vec{g} \cdot \vec{r}$$

$$\vec{F} = -\vec{\nabla} U$$

$$= -\frac{\partial U}{\partial r}$$

$$U = -m \vec{g} \cdot \vec{r} = mgy$$

$$\begin{aligned}\dot{x}^2 &= \left(\dot{X} - l \dot{\phi} \cos \phi \right)^2 \\ &= \dot{X}^2 + l^2 \dot{\phi}^2 \cos^2 \phi - 2l \dot{X} \dot{\phi} \cos \phi\end{aligned}$$

$$\dot{y}^2 = l^2 \dot{\phi}^2 \sin^2 \phi$$

$$\rightarrow \dot{x}^2 + \dot{y}^2 = \dot{X}^2 + l^2 \dot{\phi}^2 - 2l \dot{X} \dot{\phi} \cos \phi$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m \dot{X}^2 + \frac{1}{2} m l^2 \dot{\phi}^2 - m l \dot{X} \dot{\phi} \cos \phi$$

$$U = -mgl \cos \phi$$

$$L = \frac{1}{2} m \dot{X}^2 + \frac{1}{2} m l^2 \dot{\phi}^2 - m l \dot{X} \dot{\phi} \cos \phi + mgl \cos \phi$$

$$L = L(\phi, \dot{\phi}, t) \rightarrow L' = L + \frac{d}{dt} f(\phi, t)$$

$$L = \underbrace{\frac{1}{2} m \dot{x}^2}_{\text{prescribed function of time} \rightarrow \text{ignore}} + \frac{1}{2} m l^2 \dot{\phi}^2 - m l \dot{x} \dot{\phi} \cos \phi + m g l \cos \phi$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi}$$

$$\dot{x} \dot{\phi} \cos \phi = \frac{d}{dt} \left(\dot{x} \sin \phi \right) = \ddot{x} \sin \phi + \dot{x} \dot{\phi} \cos \phi$$

$$- m l \dot{x} \dot{\phi} \cos \phi = \frac{d}{dt} (-m l \dot{x} \sin \phi) + m l \ddot{x} \sin \phi$$

ignore this in Lagrangian

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + m l a \sin \phi + m g l \cos \phi$$

i) show that both Lagrangians give the same EOMs

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \iff \frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{\phi}} \right) = \frac{\partial L'}{\partial \phi}$$

ii) $L = \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \sin \phi + m g l \cos \phi$
 does not depend explicitly on time

$$\rightarrow E = h = H = \underbrace{\dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - L}_{\text{more generally}}$$

$$= \sum_i \underbrace{\dot{q}_i}_{w} \frac{\partial L}{\partial \dot{q}_i} - L \quad (\text{more generally})$$

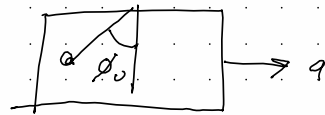
$p_i =$ momentum conjugate to q_i

$i \rightarrow$ general \rightarrow

$\neq T + U \leftarrow$ total mechanical energy

iii) Equil solution: $\dot{\phi} = 0$

$$\tan \phi_0 = \frac{a}{g}$$



Use EOM, from Lagrangian to show this

$$E = \frac{1}{2} m l^2 \dot{\phi}^2 - m l a \sin \phi - m g l \cos \phi$$

$$= \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - L$$

$$E = \frac{1}{2} m l^2 \dot{\phi}^2 + U_{\text{eff}}(\phi)$$

$$U_{\text{eff}}(\phi) = -m l (a \sin \phi + g \cos \phi)$$

★ graph it



$$0 = \left. \frac{d U_{\text{eff}}}{d \phi} \right|_{\phi = \phi_0}$$

$$V_{\text{eff}}(\phi) = V_{\text{eff}}(\phi_0) + \cancel{\left. \frac{dV_{\text{eff}}}{d\phi} \right|_{\phi_0}} (\phi - \phi_0) + \frac{1}{2} \left. \frac{d^2 V_{\text{eff}}}{d\phi^2} \right|_{\phi_0} (\phi - \phi_0)^2 + \dots$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^3 + \dots$$

$$E = T + V_{\text{eff}}(\phi)$$

$$= T + V_{\text{eff}}(\phi_0) + \frac{1}{2} \mathcal{K} (\phi - \phi_0)^2$$

$$= \frac{1}{2} m \dot{\phi}^2 + \underbrace{V_{\text{eff}}(\phi_0)}_{\text{const}} + \frac{1}{2} \mathcal{K} \underbrace{(\phi - \phi_0)^2}$$

$$x \equiv \phi - \phi_0$$

$$|x| \ll 1$$

const \rightarrow ignore

$$\dot{x} = \dot{\phi}$$

$$E =$$

$$\frac{1}{2} \underbrace{m}_{M} \dot{x}^2$$

$$+ \frac{1}{2} \mathcal{K} x^2$$

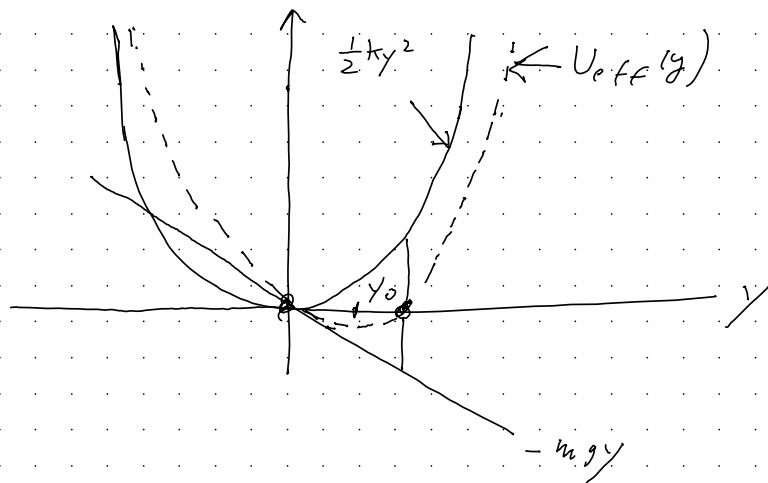
$$\omega = \sqrt{\frac{\mathcal{K}}{M}} = \sqrt{\frac{\mathcal{K}}{m \ell^2}}$$

Answer, to problems posed at the end of the last class

$$(1) \vec{a} = (\underbrace{\ddot{r} - r\dot{\phi}^2}_{\text{centrifugal acceleration}}) \hat{r} + (\underbrace{2\dot{r}\dot{\phi} + r\ddot{\phi}}_{\text{tangential acceleration}}) \hat{\phi}$$

$$(2) T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \\ = \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2)$$

$$(3) \text{ plot } U_{\text{eff}}(y) = \frac{1}{2} ky^2 - mgy$$



$$U_{\text{eff}}(y) = U_{\text{eff}}(y_0) + \frac{1}{2} \mathcal{E} (y - y_0)^2$$

where y_0 is the solution to

$$0 = \left. \frac{dU_{\text{eff}}}{dy} \right|_{y_0}$$

and \mathcal{E} is given by

$$\mathcal{E} = \left. \frac{d^2 U_{\text{eff}}}{dy^2} \right|_{y_0}$$

$$0 = \left. \frac{dU_{\text{eff}}}{dy} \right|_{y_0} = ky_0 - mg \rightarrow y_0 = \frac{mg}{k}$$

$$\mathcal{K} = \left. \frac{d^2 U_{\text{eff}}}{dy^2} \right|_{y_0} = k \quad (\text{so } \mathcal{K} \text{ is the same as } k \text{ for this problem})$$

$$U_{\text{eff}}(y_0) = \frac{1}{2}ky_0^2 - mgy_0 = \frac{1}{2}k\left(\frac{mg}{k}\right)^2 - mg\left(\frac{mg}{k}\right) = -\frac{1}{2}\frac{m^2g^2}{k}$$

$$\rightarrow \boxed{U_{\text{eff}}(y) = -\frac{1}{2}\frac{m^2g^2}{k} + \frac{1}{2}k\left(y - \frac{mg}{k}\right)^2}$$

you can also obtain the same expression for $U_{\text{eff}}(y)$ by completing the square:

$$\begin{aligned} U_{\text{eff}}(y) &= \frac{1}{2}ky^2 - mgy \\ &= \frac{1}{2}k\left(y^2 - \frac{2mg}{k}y\right) \\ &= \frac{1}{2}k\left[\left(y - \frac{mg}{k}\right)^2 - \frac{m^2g^2}{k^2}\right] \\ &= \frac{1}{2}k\left(y - \frac{mg}{k}\right)^2 - \frac{1}{2}\frac{m^2g^2}{k} \end{aligned}$$