Notes: Thurs 8/27

1) Elliptic functions
2) Simple pendulum

go beyond small angle approx

Elliptic Functions:

$$\int \frac{dx}{\sqrt{1-x^2}} = ? = \underset{\text{arcsin}(x)}{\overset{\Vert}{\sin^{-1}(x)}} + \text{const}$$

$\rightarrow \sin^2\theta + \cos^2\theta = 1$

subst: $x = \sin\theta \rightarrow \sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$dx = \cos\theta \, d\theta$

$$\int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \int d\theta = \theta = \sin^{-1}(x)$$

$x = \sin\theta$
$\theta = \sin^{-1}(x)$

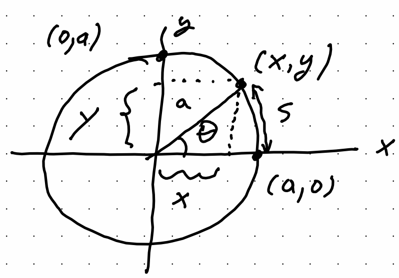
$$\frac{d}{d\theta}\sin\theta = \cos\theta \qquad \lim_{h\to 0}\left(\frac{\sin(\theta+h) - \sin\theta}{h}\right)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\sin(A+B) = \sin A\cos B + \cos A \sin B$$

circular functions:

$x^2 + y^2 = a^2$, a = radius



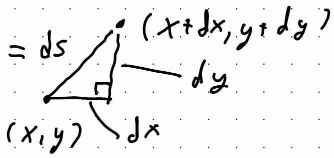
Def: $\sin\theta = \frac{y}{a}$, $\cos\theta = \frac{x}{a}$

s: arclength from $(a,0)$ to (x,y)

$s = a\theta$

$$\theta = \frac{s}{a} = \frac{1}{a}\int ds = \frac{1}{a}\int\sqrt{dx^2+dy^2} = \int d\theta$$

$\sqrt{dx^2+dy^2} = ds$



Given: $x^2+y^2 = a^2$

Follows: (i) $\cancel{a^2}\cos^2\theta + \cancel{a^2}\sin^2\theta = \cancel{a^2} \rightarrow \boxed{\cos^2\theta + \sin^2\theta = 1}$

(ii) $\boxed{\frac{d}{d\theta}\sin\theta} = \frac{1}{a}\frac{dy}{d\theta} = \frac{1}{\cancel{a}}\frac{dy}{\frac{1}{\cancel{a}}\sqrt{dx^2+dy^2}} = \frac{1}{\sqrt{\left(\frac{y}{x}\right)^2+1}}$

$\cancel{2}x\,dx + \cancel{2}y\,dy = 0 \rightarrow dx = -\frac{y}{x}dy$

$$= \frac{1}{\sqrt{\frac{x^2+y^2}{x^2}}} = \frac{x}{a} = \boxed{\cos\theta}$$

$$\frac{d(\sin\theta)}{d\theta} = \cos\theta \rightarrow \int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta$$

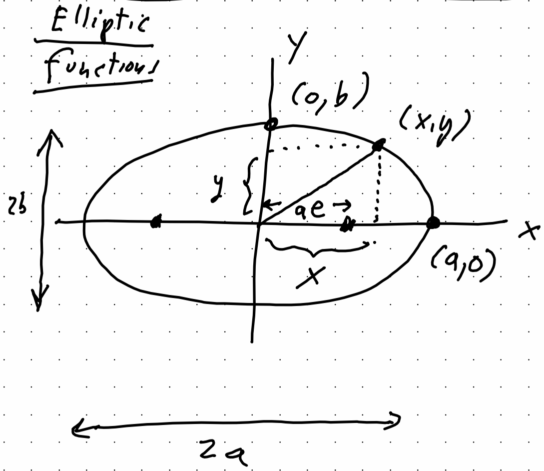
$x = \sin\theta$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$= \sqrt{1-x^2}$

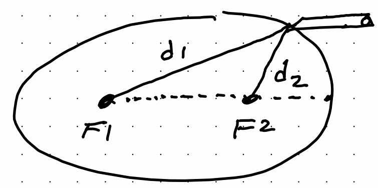
$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \theta = \sin^{-1}(x)}$$

Elliptic functions

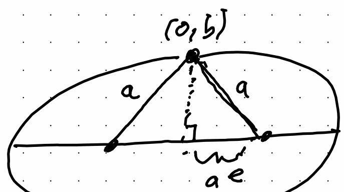


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

eccentricity : $e = 0$ (for circle)



$d_1 + d_2 = 2a$



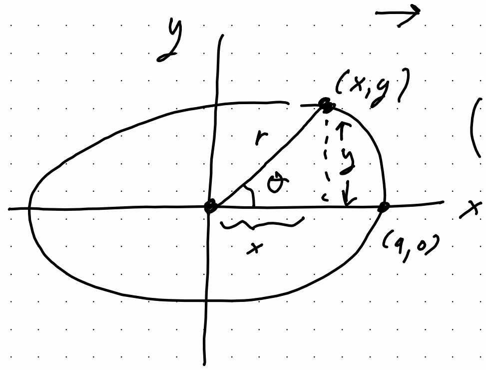
$(ae)^2 + b^2 = a^2$

$a^2e^2 + b^2 = a^2$

$e^2 = \frac{a^2-b^2}{a^2} = 1-\left(\frac{b}{a}\right)^2$

$\rightarrow e = \sqrt{1-\left(\frac{b}{a}\right)^2} \equiv k$

notation used in elliptic function



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$x^2 + y^2 = r^2$

$x = r\cos\theta, \ y = r\sin\theta$

Define: $cn(u;k) \equiv \frac{x}{a}$, $sn(u;k) \equiv \frac{y}{b}$

$dn(u;k) \equiv \frac{r}{a}$ — ($=1$ for a circle)

where $u = \frac{1}{b}\int_0^\theta r\, d\theta$ — $y = r\sin\theta$

($=\theta$ for a circle)

$ds = \sqrt{dx^2+dy^2}$

$= \sqrt{dr^2 + r^2 d\theta^2}$

Given: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, $x^2+y^2=r^2$ $\qquad dn(u;k) = \frac{r}{a}$

Follows: (i) $cn^2(u;k) + sn^2(u;k) = 1$

(ii) $dn^2(u;k) + k^2 sn^2(u;k) = 1$

$$u = \frac{1}{b}\int_0^\theta r\,d\theta$$

(iii) $\frac{d}{du} sn(u;k) = cn(u;k)\, dn(u;k)$

$\frac{d}{du} cn(u;k) = -sn(u;k)\, dn(u;k)$

$\frac{d}{du} dn(u;k) = -k^2 sn(u;k)\, cn(u;k)$

Analogous to

$\frac{d\sin\theta}{d\theta} = \cos\theta$

$\frac{d\sin\theta}{\cos\theta} = d\theta$

$x = \sin\theta$

$\int \frac{dx}{\sqrt{1-x^2}} = \int d\theta = \theta = \sin^{-1}x$

Integrate: $\frac{d\, sn(u;k)}{du} = cn(u;k)\, dn(u;k)$

$$\int \frac{d\, sn(u;k)}{cn(u;k)\, dn(u;k)} = \int du = u$$

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}} = u + \text{const} = sn^{-1}(x;k) + \text{const} \qquad x \equiv sn(u;k)}$$

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + \text{const}}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}} \equiv K(k) \longrightarrow$$

related to Period of a pendulum going beyond small-angle approximation

(complete elliptic integral of 1st kind)

$$\int_0^1 \frac{\sqrt{1-k^2x^2}\;dx}{\sqrt{1-x^2}} \equiv E(k) \longrightarrow$$

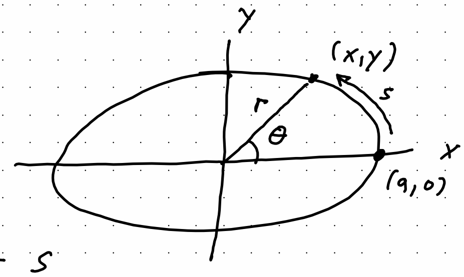
circumference around an ellipse

(complete elliptic integral of 2nd kind

circle: $C = 2\pi a$

Notes: Tuesday 9/1

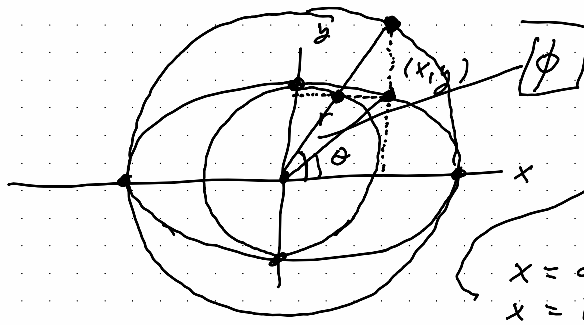
1) Review of elliptic functions
2) Simple pendulum



$$u = \frac{1}{b}\int_{(a,0)}^{(x,y)} r\,d\theta$$

$$bu = \int_0^{\theta} r\,d\theta \le \int_0^{\theta} ds = s$$

$$ds = \sqrt{dx^2+dy^2} = \sqrt{dr^2+r^2d\theta^2} \ge r\,d\theta$$



$\boxed{\phi}$

$x = r\cos\theta$

$y = r\sin\theta$

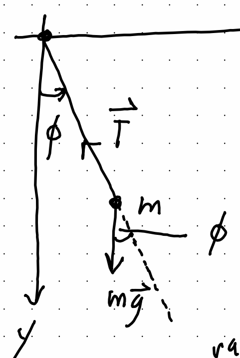
$x = a\cos\phi$

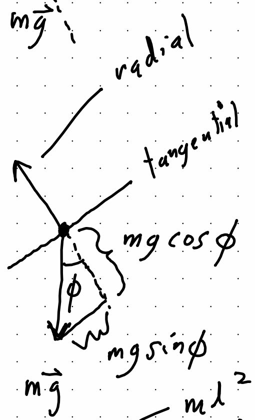
$y = b\sin\phi$

$x = a\cos\phi$, $y = \underline{\underline{a}}\sin\phi$

$x = b\cos\phi$ $y = b\sin\phi$

simple pendulum:





Torque $= I\alpha$ — ml^2 — $\ddot{\phi}$

(i) "Freshman physics"

forces, free-body diagram

→ EOM, tension

tangential:

$$-mg\sin\phi = m a_{tangential}$$

$$-\cancel{m}g\sin\phi = \cancel{m}l\ddot{\phi}$$

ϕ : angular displacement [rad]

$\dot{\phi}$: angular velocity [rad/sec]

$\ddot{\phi}$: angular accel [rad/sec^2]

$$\boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi} \quad \text{(EOM)}$$

radial: $T - mg\cos\phi = m a_{centripetal}$

$$T - mg\cos\phi = m\dot{\phi}^2 l$$

$T = mg\cos\phi + m\dot{\phi}^2 l$

(ii) Small angle approximation:

$$\sin\phi \approx \phi \leftarrow \phi \ll \underset{\text{radian}}{1} = 57 \text{ degrees}$$

$$\ddot{\phi} = -\frac{g}{l}\sin\phi \approx -\frac{g}{l}\phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\phi}$$

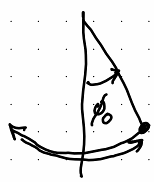
small angle approx.

$$\phi(t) = \boxed{a}\cos(\omega t) + \boxed{b}\sin(\omega t)$$

where $\omega \equiv \sqrt{\frac{g}{l}}$

determined by initial conditions

ICs: If $\phi(0) = \phi_0$ (at rest)



then $\boxed{\phi(t) = \phi_0 \cos(\omega t)}$

Period: $P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$

independent of ϕ_0 !!

(iii) Lagrangian approach

$L = T - U$

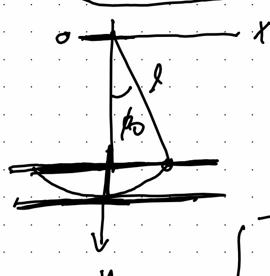
$T \equiv$ Kinetic energy

$U \equiv$ Potential energy

$$T = \frac{1}{2} m l^2 \dot{\phi}^2 \quad \left(= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)\right)$$

$y = l\cos\phi$

$x = l\sin\phi$



$$U = -mgl\cos\phi + \text{const} \leftarrow U = -mgy$$

$$U = mgl(1 - \cos\phi)$$

$$\boxed{L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi}$$

action

$$S = \int_{t_1}^{t_2} dt\, L(\phi, \dot{\phi}, t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \quad \text{Lagrange's equation}$$

$$\frac{d}{dt}\left(ml^2\dot{\phi}\right) = -mgl\sin\phi$$

$$ml^2\ddot{\phi} = -mgl\sin\phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi} \quad \text{(EOM)}$$

(iv) solving $\ddot{\phi} = -\frac{g}{l}\sin\phi$ $\left(2^{nd} \text{ order non-linear ODE}\right)$ hard!!

$$E = \text{const} = T + U = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$$

$$E = 0 - mgl\cos\phi_0 = -mgl\cos\phi_0$$

release from rest from $\phi = \phi_0$

$$-mgl\cos\phi_0 = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$$

$$-\cancel{m}gl(\cos\phi_0 - \cos\phi) = \frac{1}{2}\cancel{m} l^2 \dot{\phi}^2$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{-2\frac{g}{l}(\cos\phi_0 - \cos\phi)}$$

$|\phi| \leq \phi_0$

separable 1st order ODE

$$\int \frac{d\phi}{\sqrt{-\frac{2g}{l}(\cos\phi_0 - \cos\phi)}} = \int dt = t + \text{const}$$

$$t + t_0 = \int \frac{d\phi}{\sqrt{-2\frac{g}{l}(\cos\phi_0 - \cos\phi)}}$$

$$\int \frac{1}{\sqrt{a + bx^2}}$$

substitution:

$$\cos\phi = 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\cos\phi_0 = 1 - 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$\cos\phi = \cos\left(2\left(\frac{\phi}{2}\right)\right) = \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right) = 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\rightarrow \cos\phi_0 - \cos\phi = -2\left(\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)\right)$$

$$t + t_0 = \int \frac{d\phi}{2\sqrt{\frac{g}{l}}\sqrt{\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)}}$$

$$= \frac{1}{2\sqrt{\frac{g}{l}}} \int \frac{d\phi}{\sin\left(\frac{\phi_0}{2}\right)\sqrt{1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)}}}$$

$|\phi| \leq \phi_0$

let $x = \frac{\sin(\phi/2)}{\sin\left(\frac{\phi_0}{2}\right)}$

$$\boxed{x = \frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}} \rightarrow dx = \frac{1}{\sin\left(\frac{\phi_0}{2}\right)} \frac{1}{2} \cos\left(\frac{\phi}{2}\right) d\phi$$

$$d\phi = \frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\cos\left(\frac{\phi}{2}\right)} = \frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1 - \sin^2\left(\frac{\phi}{2}\right)}}$$

$$= \boxed{\frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1 - \underbrace{\sin^2\left(\frac{\phi_0}{2}\right)}_{k^2} x^2}}}$$

$\sqrt{1-x^2}$

denominator

Find this out

① $\phi(t) =$

② Period = ??

③ Redo the analysis using Lagrange multiplier for find tension in string

$t + t_0 = \int$ — integrand for $sn^{-1}(x; k)$

$k = \sin\left(\frac{\phi_0}{2}\right)$

Lec #4 : Thurs 9/3

$$\phi(t) = 2 \sin^{-1}\left[k \, sn\left(\omega_0\left(t + \frac{P}{4}\right); k\right)\right] ☆$$

$$k = \sin\left(\frac{\phi_0}{2}\right), \quad \omega_0 = \sqrt{\frac{g}{l}}$$

$$P = 4\sqrt{\frac{l}{g}} K(k) = 4\sqrt{\frac{l}{g}} \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

small angle approx

$$P_0 = 2\pi\sqrt{\frac{l}{g}}$$

$$\frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)} = x = sn\left[\sqrt{\frac{g}{l}}(t+t_0); k\right]$$

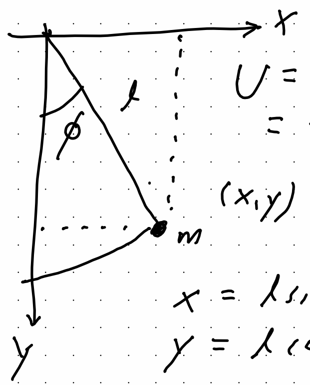
$$\sqrt{\frac{l}{g}} \int \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} = \sqrt{\frac{l}{g}} \, sn^{-1}(x; k) = t + t_0$$

$$sn^{-1}(x; k) = \sqrt{\frac{g}{l}}(t+t_0)$$

$$P = 4\sqrt{\frac{l}{g}}\,K(k) = 2\pi\sqrt{\frac{l}{g}}\underbrace{\left(1+\frac{1}{16}\phi_0^2+\dots\right)}_{\text{Problem Landau 11,1}}$$

Lagrange multiplier:

$$\underbrace{\varphi(x,y)}_{\text{constraint function}} = x^2+y^2-l^2=0$$



$$U = -mgy = -mgl\cos\phi$$

$$x^2+y^2 = l^2\sin^2\phi + l^2\cos^2\phi = l^2$$

$$x = l\sin\phi$$
$$y = l\cos\phi$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}ml^2\dot{\phi}^2$$

$$L = T - U + \lambda\varphi$$

Lagrange multiplier

$\lambda, \cancel{\dot{\lambda}}$

$L(x,\dot{x},y,\dot{y},t) \quad q=(x,y) \quad \dot{q}=(\dot{x},\dot{y})$

$L(r,\dot{r},\phi,\dot{\phi},t) \quad q=(r,\phi) \quad \dot{q}=(\dot{r},\dot{\phi})$

$\lambda(t)$
$r(t)$
$\phi(t)$

$L(\phi,\dot{\phi},t) \quad \varphi(x,y)=x^2+y^2-l^2=0 \quad \boxed{\varphi(r,\phi) = r-l=0}$

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + mg\underbrace{r\cos\phi}_{y} + \lambda(r-l)}$$

r: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \rightarrow \boxed{\cancel{m\ddot{r}} = mr\dot{\phi}^2 + mg\cos\phi + \lambda}$

ϕ: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \rightarrow \frac{d}{dt}(mr^2\dot{\phi}) = -mgr\sin\phi$

λ: $r-l=0$ ✓ $\quad \cancel{2mr\dot{r}\dot{\phi}} + mr^2\ddot{\phi} = -mgr\sin\phi$

$$r - l = 0 \quad \longrightarrow \quad r = l$$
$$\dot{r} = 0$$
$$\ddot{r} = 0$$

$$\rightarrow \quad \ddot{\phi} = -\frac{g}{l}\sin\phi$$

$$m l \dot{\phi}^2 + m g \cos\phi + \lambda = 0$$

$$\boxed{\lambda = -\underbrace{(m g \cos\phi + m l \dot{\phi}^2)}_{T}} \qquad \lambda = -T$$

$$L = T - U + \lambda \varphi$$

$U = U(x,t)$

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

$\varphi(x, y, t)$

$U(x, y, t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}(m\dot{x}) = -\frac{\partial U}{\partial x} + \lambda \frac{\partial \varphi}{\partial x}$$

$$\frac{d}{dt}(m\dot{y}) = -\frac{\partial U}{\partial y} + \lambda \frac{\partial \varphi}{\partial y}$$

$$\frac{d\vec{p}}{dt} = \vec{F}_{net}$$

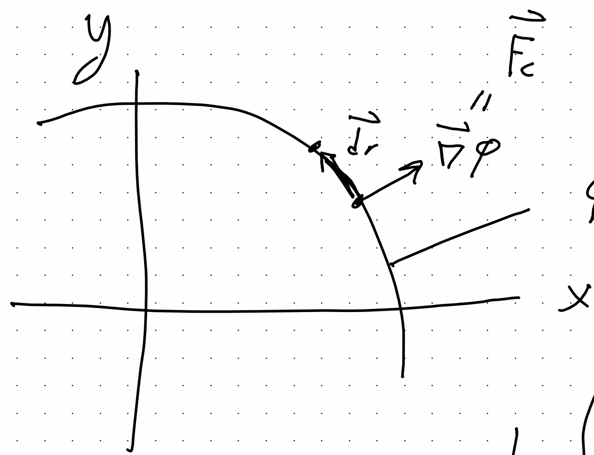
applied force   force constraint

$$\boxed{\frac{d\vec{p}}{dt} = \vec{F} + \lambda \vec{\nabla}\varphi}$$

$$\frac{dp_x}{dt} = F_x + \lambda \frac{\partial \varphi}{\partial x}$$

$$\frac{dp_y}{dt} = F_y + \lambda \frac{\partial \varphi}{\partial y}$$

$$\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) = \vec{\nabla}\varphi$$



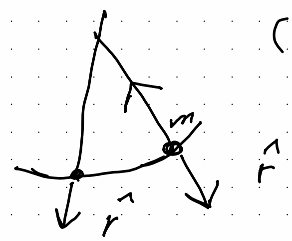
$\int \underbrace{\vec{F}_c \cdot d\vec{r}}_{0} = \text{Work}$

$\vec{\nabla}\varphi \perp \varphi = \text{const}$

$\varphi = r - l$

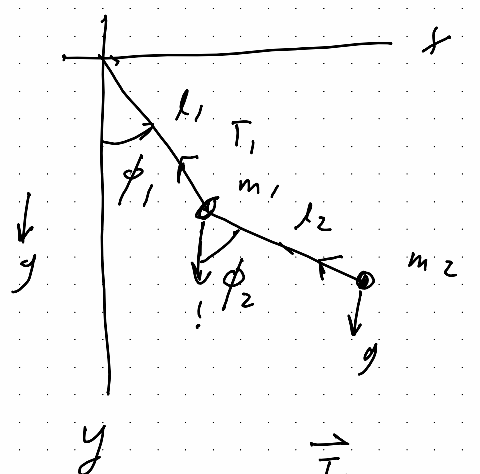
$\vec{\nabla}\varphi = \hat{r}$

$\lambda \vec{\nabla}\varphi$



(r, ϕ)

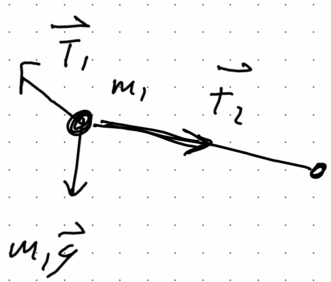
$\hat{r}, \hat{\phi}$



$U = -m_1 g y_1$
$\quad - m_2 g y_2$

$\ddot{\phi}_1, \ddot{\phi}_2$

EOM



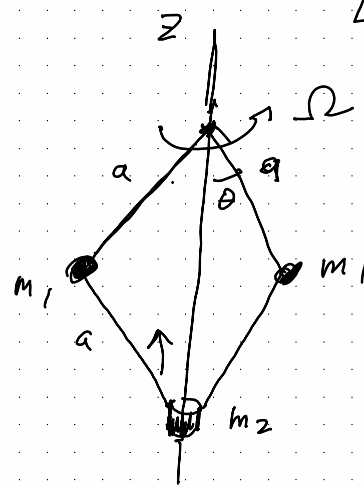
$\lambda\varphi$

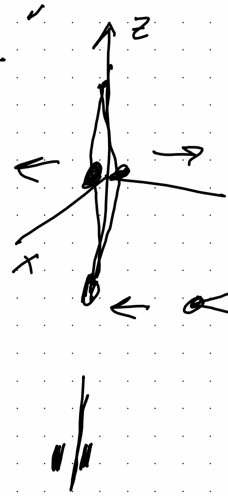
$(\lambda_1 \varphi_1 + \lambda_2 \varphi_2$
$+ \cdots)$

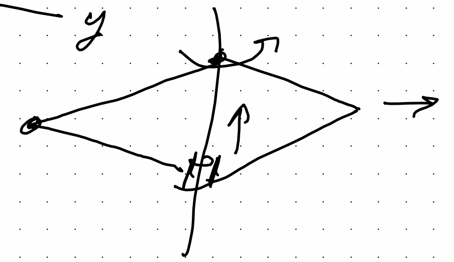
$$L \longrightarrow \text{EOM}$$

$$L \longrightarrow \underbrace{L + \frac{d}{dt}\left(f(q,t)\right)}_{L'}$$

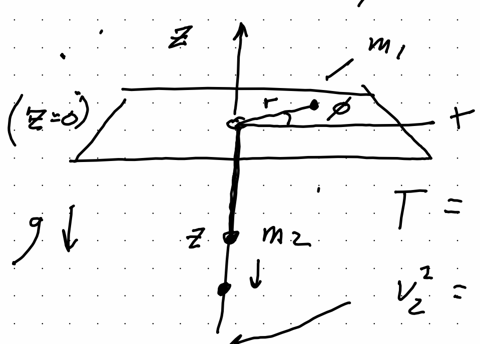
$$f(q, \cancel{\dot{q}}, t)$$







Lec #5: Tuesday 9/8



$v_2^2 = \dot{r}^2$

$\boxed{r - z = l}$ = length of string

$$L = T - U$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$v_2^2 = \dot{z}^2, \quad v_1^2 = \dot{r}^2 + r^2 \dot{\phi}^2$$

$$\left(= \dot{x}^2 + \dot{y}^2, \quad \begin{array}{l} x = r\cos\phi \\ y = r\sin\phi \end{array}\right)$$

$$\boxed{\begin{aligned} T &= \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2 \\ &= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 \end{aligned}}$$

$$U = m_2 g z = m_2 g (r - l) = m_2 g r - \underbrace{m_2 g l}_{\text{constant}}$$

$$\boxed{U = m_2 g r}$$

$$L = T - U = \boxed{\frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = \frac{\partial L}{\partial r}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi} \Rightarrow \text{2nd order EOMs}$$

No explicit t dependence:

$$E = \dot r \frac{\partial L}{\partial \dot r} + \dot\phi\frac{\partial L}{\partial\dot\phi} - L = \boxed{T+U}$$

$$E = \sum_i \dot q_i \underbrace{\frac{\partial L}{\partial \dot q_i}}_{=p_i} - L$$

No explicit ϕ dependence:

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = \text{const} = M_z$$

$$\boxed{M_z = m_1 r^2\dot\phi} \rightarrow \boxed{\dot\phi = \frac{M_z}{m_1 r^2}}$$

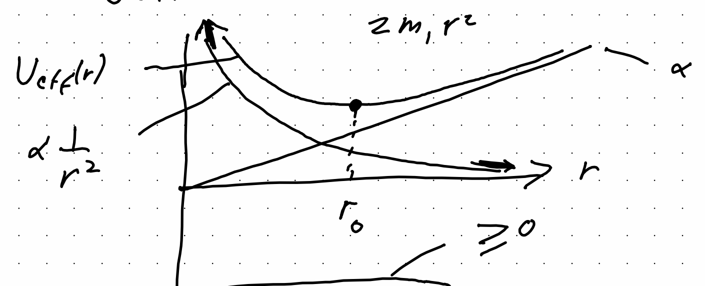
$\vec M$: angular momentum (L&L notation)

$$E = \frac{1}{2}m\dot x^2 + U(x)$$

$$\boxed{E = \frac{1}{2}(m_1+m_2)\dot r^2 + \frac{1}{2}m_1 r^2\dot\phi^2 + m_2 g r}$$

$$\boxed{E = \frac{1}{2}(m_1+m_2)\dot r^2 + \left(\frac{M_z^2}{2m_1 r^2} + m_2 g r\right)} \leftarrow \boxed{U_{eff}(r) = U(r) + \frac{M_z^2}{2m_1 r^2}}$$

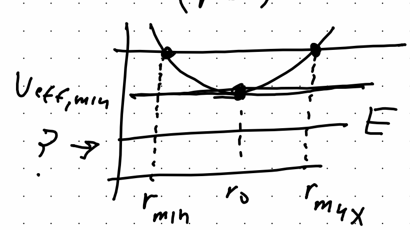
$$U_{eff}(r) = \frac{M_z^2}{2m_1 r^2} + m_2 g r \qquad U(r) = m_2 g r$$



$$E = \overbrace{\boxed{\frac{1}{2}(m_1+m_2)\dot r^2}}^{\geq 0} + U_{eff}(r)$$

E const

r_{min}, r_{max} : "turning points" ($\dot r = 0$)



i) $E = U_{eff,min} = U_{eff}(r_0)$

unif circular motion: $r = r_0$, $\dot\phi = \frac{M_z}{m_1 r_0^2}$

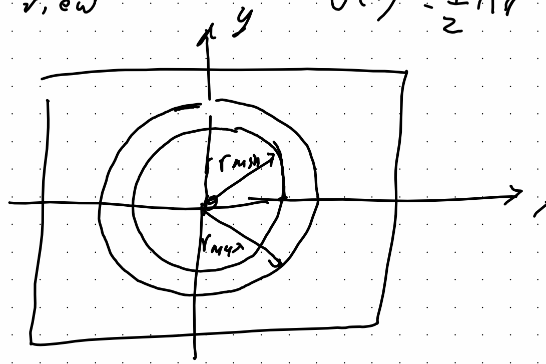
$E < U_{eff,min}$ (not allowed)

ii) $E > U_{eff,min}$

$E = U_{eff}(r_{min}) = U_{eff}(r_{max})$

$E \geq U_{eff,min}$

top view



$U(r) = \frac{1}{2}kr^2$ — closed bound

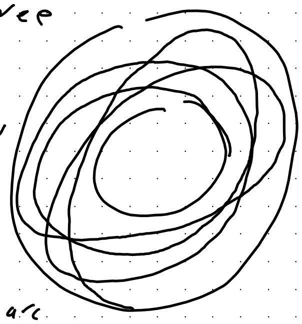
$U(r) = -\frac{Gm_1m_2}{r}$ orbits

$\underbrace{\qquad\qquad}_{\text{Newtonian gravity}}$

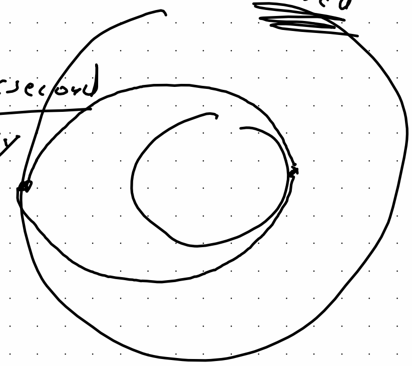
bound orbits $\equiv$ ellipse
are closed

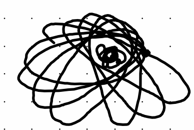
1 degree
"
60 mins of arc

1 min of arc
"
60 sec of arc



$\frac{43 \text{ arcsecond}}{\text{century}}$





perihelion precession of Mercury
└ closest approach to sun

$\underline{r_0}$: $\left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = 0$ (minimum)

$$0 = \frac{d}{dr}\left(\frac{M_z^2}{2m_1r^2} + m_2 g r\right)\Bigg|_{r=r_0}$$

$$= -\frac{M_z^2}{m_1 r_0^3} + m_2 g$$

$$\boxed{M_z^2 = m_1 m_2 g r_0^3}$$

tells you the value of M_z needed to have a specific r_0 value.

For a given M_z, this tells you what r_0 equals.

Energy equation:

$$E = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{M_z^2}{2m_1 r^2} + m_2 g r$$

$$\boxed{\dot{\phi} = \frac{M_z}{m_1 r^2}} \leftarrow \phi \text{ equation}$$

$$\frac{1}{2}(m_1+m_2)\dot{r}^2 = E - \frac{M_z^2}{2m_1 r^2} - m_2 g r$$

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}$$

$$\int \frac{dr}{\sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}} = \int dt = t + \text{const}$$

$$t(r) \longleftrightarrow r(t)$$

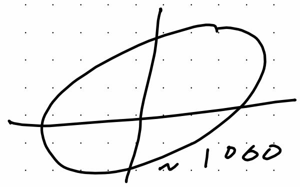
orbital equation: $\quad r = r(\phi) \longleftrightarrow \phi = \phi(r)$

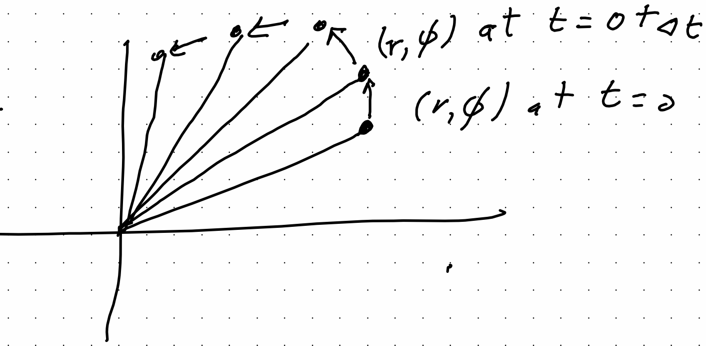
$$\frac{dr}{dt} = \dot{r} = \sqrt{\left(\frac{2}{m_1+m_2}\right)\left[E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right]}$$

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M_z}{m_1 r^2}$$

$$\rightarrow \frac{dr}{d\phi} = \frac{m_1 r^2}{M_z}\sqrt{\left(\frac{2}{m_1+m_2}\right)\left[\quad\right]}$$

$$\int \frac{dr}{\frac{m_1 r^2}{M_z}\sqrt{(\ )[\quad]}} = \int d\phi = \phi + \text{const}$$





Know: r, ϕ at some time t

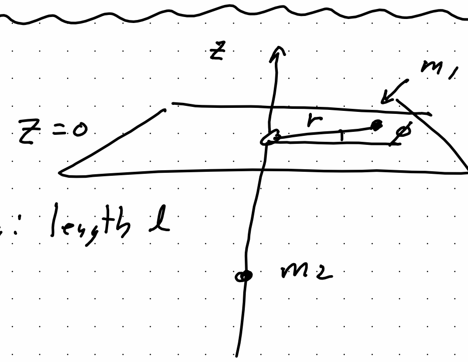
given: Δt need to know Δr and $\Delta\phi$

$$r(t+\Delta t) = r(t) + \Delta r(t) + \cdots$$
$$\phi(t+\Delta t) = \phi(t) + \Delta\phi(t) + \cdots$$

ignore if Δt is suff. small

Lecture #6 : Thursday 10 Sep

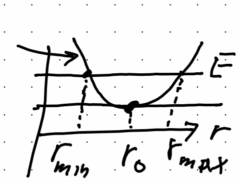
1) Secs 6–10 (today), Sec 40 (next Tuesday)
2) Finish up example from last time
3) Conservation of E, $\vec{P}$, $\vec{M}$
4) Mechanical similarity
5) Quiz: last 20 minutes (1:30pm)



string: length l

E, M_z = const

U_{eff}



$$\frac{d\phi}{dt} = \frac{M_z}{m_1 r^2}$$

$$\frac{dr}{dt} = \sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E - \underbrace{m_2 g r - \frac{M_z^2}{2m_1 r^2}}\right)} = \sqrt{0}$$

$$\frac{d\phi}{dt} = \frac{M_z}{m_1 r^2} \rightarrow \Delta\phi = \Delta t \, \frac{M_z}{m_1 r^2}$$

$$\frac{dr}{dt} = \sqrt{} \rightarrow \Delta r = \Delta t \sqrt{}$$

$r(0) = r_{min}$
$\phi(0) = 0$

$\phi(\Delta t) = \phi(0) + \Delta\phi$
$r(\Delta t) = r(0) + \Delta r$

$\phi(2\cdot\Delta t) = \phi(\Delta t) + \Delta\phi$
$r(2\cdot\Delta t) = r(\Delta t) + \Delta r$

$\vdots$

$\phi(t_i) = \phi(t_{i-1}) + \Delta\phi$
$r(t_i) = r(t_{i-1}) + \Delta r$

Cons. of $E, \vec{P}, \vec{M}$:

All of E, $\vec{P}$, $\vec{M}$ conserved for a closed system

no external forces
$U = U(\vec{r}_1 - \vec{r}_2, \vec{r}_2 - \vec{r}_3, \cdots)$
relative position vectors.

Even in the presence of external forces, you can still have cons. of E and some components of $\vec{P}$ and $\vec{M}$.

(i) $U = mgy$ $\quad \vec{F} = -m\vec{g}$
If U does not depend explicitly on time t, then E is conserved.

$$E \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

constant external field

(ii) e.g., $\downarrow \vec{F}_g = m\vec{g}$

$\longleftrightarrow$ x, y

$p_x = \text{const}$
$p_y = \text{const}$

If U is unchanged by a translation in some direction $\hat{t}$ then $\vec{P}\cdot\hat{t} = \text{const}$

$\vec{v} = \text{const}$

(iii) $\vec{M}$ depends on choice of origin

(a) uniform gravitational field $\vec{F} = m\vec{g}$ $\uparrow z$

$U = m_1 g z_1 + m_2 g z_2 + \cdots$

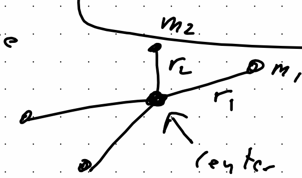
If U is unchanged by a rotation about a particular axis $\hat{n}$ then $\boxed{\vec{M}\cdot\hat{n} = \text{const}}$ (e.g., $\hat{n} = \hat{z}$, $M_z = \text{const}$) provided the origin is located on the axis.

(b) central force

$\boxed{\vec{M} = \text{const}}$

$U = U(r)$

$\vec{F} = -\vec{\nabla} U$
$= -\dfrac{dU}{dr}\hat{r}$



Mechanical similarity :

$L \rightarrow L' = c \cdot L$ same equations of motion

suppose we rescale position vectors $\vec{r}_a \rightarrow \alpha\vec{r}_a$

$U'(\vec{r}_i, \ldots) \equiv U(\alpha\vec{r}_1, \alpha\vec{r}_2, \cdots) = \alpha^k U(\vec{r}_1, \vec{r}_2, \cdots)$

potential is homogenous of degree k w.r.t. position vector

Example: i) $U = mgy$, $k = 1$
ii) $U = \frac{1}{2}kx^2$, $k = 2$
iii) $U = -\dfrac{Gm_1m_2}{r}$, $k = -1$

$U' = mg\alpha y$
$= \alpha mgy$
$= \alpha U$

$L' = T' - U' = T' - \alpha^k U = \text{const} \cdot L$
$= \alpha^k T - \alpha^k U$
$= \alpha^k (T - U) = \alpha^k L$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \frac{l^2}{t^2}$$

length

time

$$l \to l' = \alpha l$$
$$t \to t' = \beta t$$

$$\frac{l'}{l} = \alpha \qquad \frac{t'}{t} = \beta$$

$U = mgy$, $k = 1$

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{\frac{1}{2}}$$

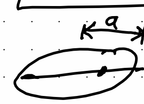
$U = \frac{1}{2} k x^2$, $k = 2$

$t'/t = \text{const}$

$$T' = \frac{1}{2} m v'^2 = \frac{1}{2} m \frac{l'^2}{t'^2}$$
$$= \frac{1}{2} m \frac{\alpha^2 l^2}{\beta^2 t^2}$$
$$= \frac{\alpha^2}{\beta^2} T$$

$$\boxed{\frac{\alpha^2}{\beta^2} = \alpha^k} \to \boxed{\beta = \alpha^{1-k/2}} \to \boxed{\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-k/2}}$$

$$\beta^2 = \alpha^{2-k}$$



$$\boxed{\frac{p^2}{a^3} = \text{const}}$$

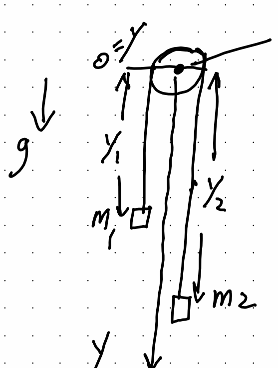
$U = -\frac{G m_1 m_2}{r}$

$k = -1$

$$\left(\frac{t'}{t}\right) = \left(\frac{l'}{l}\right)^{3/2}$$

$$p^2 = \text{Dist}^3$$

QUIZ #1: Atwood's machine



massless
frictionless
pulley

i) L ?

ii) EOM

iii) solve EOM

$(m_1 > m_2)$

string: length l
(massless
inextensible
...)

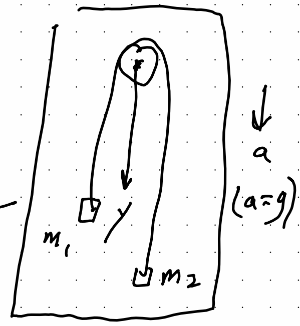
$$y_1 + y_2 = l \to y_2 = l - y_1 \to \dot{y}_2 = -\dot{y}_1$$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_1 g y_1 - m_2 g (l - y_1)$$
$$= -m_1 g y_1 \underbrace{- m_2 g l}_{\text{ignore}} + m_2 g y_1$$
$$= \boxed{-(m_1 - m_2) g y_1}$$

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \underbrace{\dot{y}_2^2}_{= \dot{y}_1^2}$$
$$= \boxed{\frac{1}{2}(m_1 + m_2)\dot{y}_1^2}$$

Lecture #7: Tuesday 9/15

1) Go over quiz #1
2) Modified Atwood problem
3) Finish mechanical similarity (sec 10)
4) Hamilton's equations (sec 40)

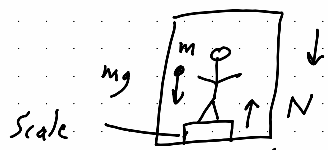


$$L = T - U$$
$$= \frac{1}{2}(m_1+m_2)\dot{y}_1^2 + (m_1-m_2)g y_1$$

EOMs: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) = \frac{\partial L}{\partial y_1} \rightarrow (m_1+m_2)\ddot{y}_1 = (m_1-m_2)g$

$$\boxed{\ddot{y}_1 = \frac{(m_1-m_2)g}{(m_1+m_2)}}$$

$$y_1(t) = y_0 + v_0 t + \frac{1}{2}\frac{(m_1-m_2)}{(m_1+m_2)} g t^2$$



$\downarrow a \quad ma = mg - N$

$N = m(g-a)$

apparent or effective weight

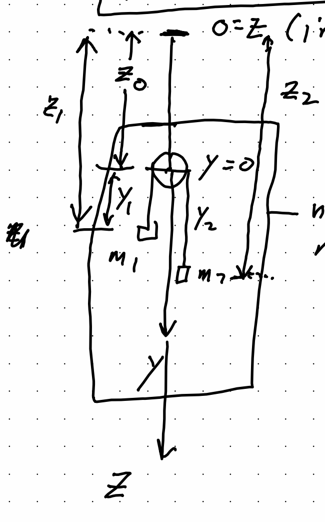
$$\boxed{\ddot{y}_1 = \frac{(m_1-m_2)}{m_1+m_2}(g-a)}$$

$\vec{F} = m\vec{a}$ (valid in an inertial ref frame)

$\vec{F} + \vec{F}_{\text{fictitious}} = m\vec{a}$ —— w.r.t. to a non-inertial ref. frame

sec 39 (L&L) $\qquad L' = L + f(t)$

$\boxed{L = T - U}$ (valid in an inertial ref frame)



$$T = \frac{1}{2}m_1\dot{z}_1^2 + \frac{1}{2}m_2\dot{z}_2^2$$
$$U = -m_1 g z_1 - m_2 g z_2$$

$z_1 = z_0 + y_1 \quad | \quad y_2 = l - y_1$

$z_2 = z_0 + y_2$

$$U = -m_1 g(z_0 + y_1) - m_2 g(z_0 + \underbrace{y_2}_{l - y_1})$$
$$= \underbrace{-m_1 g z_0 \quad - m_1 g y_1 \quad - m_2 g z_0}_{\text{prescribed function of time} = \text{ignore}} \quad \underbrace{-m_2 g l}_{\text{const}} + m_2 g y_1$$
$$= \boxed{-(m_1-m_2)g y_1}$$

Do this at home:

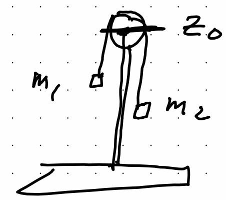
$$T = \frac{1}{2}(m_1+m_2)\dot{y}_1^2 + (m_1-m_2)\dot{z}_0\dot{y}_1$$

$$L' = L + \frac{d}{dt}(f(q,t)) \longrightarrow \text{same EOM}$$

$$(m_1-m_2)\dot{z}_0\dot{y}_1 = \underbrace{\frac{d}{dt}\Big[\underbrace{(m_1-m_2)\dot{z}_0 y_1}_{f(y_1,t)}\Big]}_{\text{ignore}} - (m_1-m_2)\ddot{z}_0 y_1 \qquad \equiv a$$

$$\boxed{T = \frac{1}{2}(m_1+m_2)\dot{y}_1^2 - (m_1-m_2)\,a\,y_1}$$

$z_0(t)$: given, not to be solved for



Hamilton's equations: $\qquad L(q,\dot{q},t)$

Hamiltonian: $\quad E = \underbrace{\sum \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L}_{E(q,\dot{q},t)}$

$H = H(q,p)$

not here if $L = L(q,\dot{q})$

$$H = \left(\sum_i p_i\dot{q}_i - L\right)\Bigg|_{\dot{q}=\dot{q}(q,p)}$$

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

Example:

$L = \frac{1}{2}m\dot{x}^2 - U(x)$ (single particle, 1-d, const external field)

$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \longrightarrow \dot{x} = \frac{p}{m}$

$$H = (p\dot{x} - L)\Big|_{\dot{x}=p/m}$$

$$= \left(p\dot{x} - \frac{1}{2}m\dot{x}^2 + U(x)\right)\Big|_{\dot{x}=p/m} = \frac{p^2}{m} - \frac{1}{2}m\left(\frac{p}{m}\right)^2 + U(x)$$

$$H = \frac{1}{2}\frac{p^2}{m} + U(x)$$

EOMs: (Hamilton's equations) ((39,6) Prob 2 Sec 40)

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, \dots, s \quad \text{w/ } s = \#\text{ of DOF}$$

Lagrange's equation:

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}}_{s\text{-equations, } 2^{nd} \text{ order ODE for } q_i} \qquad i = 1, \dots, s$$

2s-equations, 1^{st} order in $\dot{q}_i, \dot{p}_i$

$\dot{x} = p/m$

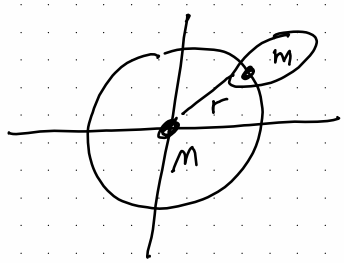
$p = m\dot{x}$

$\dot{p} = -\frac{\partial U}{\partial x}$

$$\boxed{m\ddot{x} = -\frac{\partial U}{\partial x}}$$

$$H = \frac{p^2}{2m} + U(x) \rightarrow \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}\,, \quad \dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$$

$$L = \frac{1}{2} m \dot{x}^2 - U(x) \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow m\ddot{x} = -\frac{\partial U}{\partial x}$$



$$U = -\frac{GMm}{r}$$

Problem: $U' = cU$

~~masses equal~~

$r' = r\,, \; m' = m$

$$\frac{GM\cancel{m}}{r^2} = \frac{\cancel{m} v^2}{r}$$

$$\boxed{v = \sqrt{\frac{GM}{r}}}$$

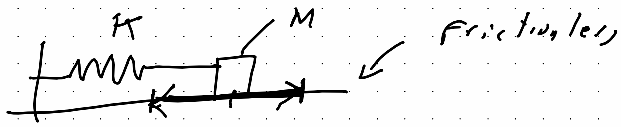
Suppose $M \rightarrow cM = M'$

$$U' = cU$$

$$\frac{2\pi r}{T'} = v' = \sqrt{\frac{cGM}{r}}$$

$$\frac{1}{T'} \propto \sqrt{c}$$

$$\sqrt{\frac{E}{E'}} = \sqrt{\frac{U'}{U}}$$



$U = \frac{1}{2} k x^2$

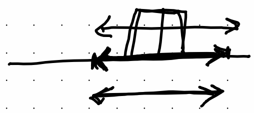
$T = \frac{1}{2} m \dot{x}^2$ $\quad L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$ $\quad \omega = \sqrt{\frac{k}{m}}$

$m\ddot{x} = -kx$

$\ddot{x} = -\frac{k}{m} x \rightarrow x(t) = a \cos \omega t + b \sin \omega t$

$U' = cU$ (for example $k' = ck$)

$T' = cU$



$\ddot{x} = -\frac{k'}{m} x$, $\omega' = \sqrt{\frac{k'}{m}}$

$T = \frac{1}{2} m \dot{x}^2$

$T' = cT$

$\frac{2\pi}{P'} = \sqrt{\frac{k'}{m}}$

$\frac{P}{P'} = \sqrt{\frac{U'}{U}}$

$L = T - U$

$L' = cL = cT - cU$

$L'' = T - cU \neq cL$

L, L' same EOMs

L, L'' different EOMs

$\frac{1}{t'^2} = c$

$T = \frac{1}{2} m \dot{x}^2$, $U = \frac{1}{2} k x^2$

$L = T - U \rightarrow m\ddot{x} = -kx$

$L' = cL \rightarrow \cancel{c} m\ddot{x} = -\cancel{c} k x$

$m\ddot{x} = -kx$

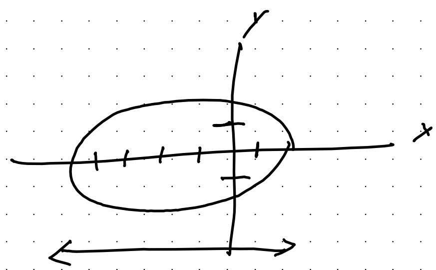
$\ddot{x} = -\frac{k}{m} x$

$L'' \rightarrow m\ddot{x} = -ckx$

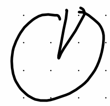
$c x'' = \ddot{x} = -c\left(\frac{k}{m}\right) x$

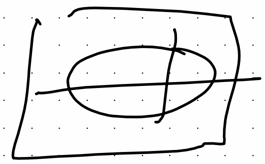
differ EOMs

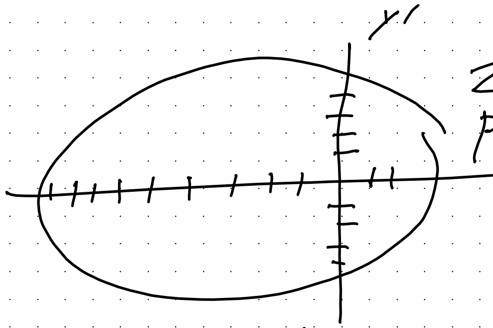
$x = a \cos(\omega' t) + b \sin(\omega' t)$



a, P



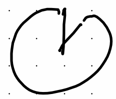




2

$P' = ?$

$$\frac{P^2}{a^3} = \text{const}$$



t'

a', P'

$U' = cU$

$P' = ?\, P$

$m' = m$

$l' = l$

"mechanical similarity"

$L' = \text{const}\ L$

$T' - U' = (\text{const})(T - U)$

$T' - cU = c\,(T - U) \quad cU = U'$

$$\frac{P'}{P} = \sqrt{\frac{U}{U'}}$$

2A

$\sqrt{\frac{1}{2}}$

k

$k \to k' = ck$

Periods

$g \to 2g$

1m $P' ?\ P$

$\frac{1}{\sqrt{2}}$

$T = \frac{1}{2} m \dot{x}^2$

$\cancel{T' = T}$

$U' = cU$

$\cancel{L' = cL}$

$L' = cL$

Lec #8: Thur Sep 17th

Today — 1-d motion (Sec 11)

Next two weeks — central force (Sec 13-15)

☆ Midterm 1 — Tues Oct 6th $\qquad T = \frac{1}{2} a(q)\dot{q}^2$

$$T = \frac{1}{2}\sum_{j,k} a_{jk}(q)\,\dot{q}_j\,\dot{q}_k \qquad \text{[single particle]}$$

$\uparrow$?? $\qquad j = 1, 2, 3$

$U(q_1, \dots, q_s, t)$

$$T = \frac{1}{2} m|\vec{v}|^2$$

$$= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad \text{cartesian } (x, y, z)$$

$$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) \qquad \text{sph. polar } (r, \theta, \phi)$$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

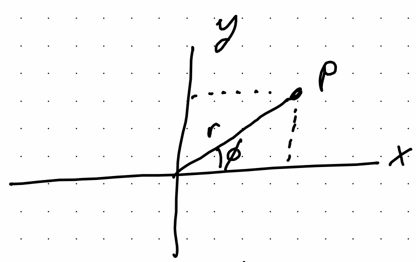
$q_1 = r$

$q_2 = \theta$

$q_3 = \phi$

$$a_{ij} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

$a_{11} = m$, $a_{22} = mr^2$, $a_{33} = mr^2\sin^2\theta$, $a_{12} = 0$, $a_{13} = 0$

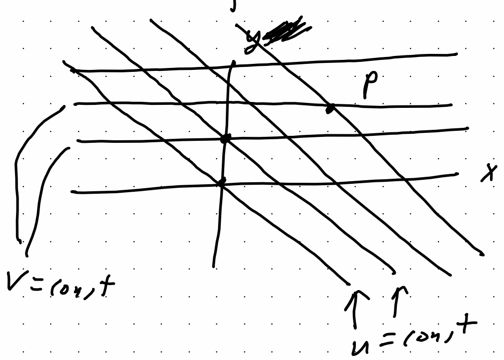


(x, y)

(r, ϕ)

$\dot{x} = \dot{u} - \dot{v}$

$\dot{y} = \dot{v}$



$u = x + y \quad | \quad x = u - v$

$v = y \quad | \quad y = v$

$u = \text{const}$?

$x + y = \text{const}$

$$\boxed{y = \underbrace{\text{const}}_{b}\ \underbrace{-x}_{m}}$$

$T = \frac{1}{2} m(\dot{u}^2 + \dot{v}^2)$??

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$ ✓

$= \frac{1}{2} m(\dot{u}^2 + \dot{v}^2 - 2\dot{u}\dot{v} + \dot{v}^2)$

$= \frac{1}{2} m(\dot{u}^2 + 2\dot{v}^2 - 2\dot{u}\dot{v})$

$$a_{ij} = m\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

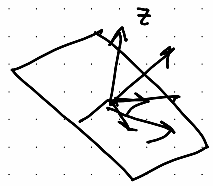
$$T = \frac{1}{2} m |\vec{v}|^2$$

$$\vec{v} = \sum_{i=1}^{3} v_i \hat{e}_i$$

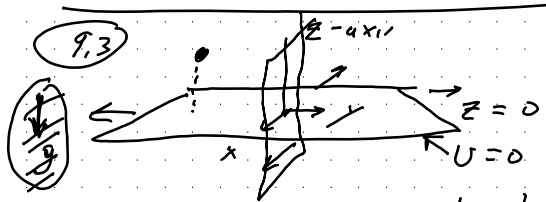
$$|\vec{v}|^2 = \vec{v}\cdot\vec{v}$$
$$= \left(\sum_i v_i \hat{e}_i\right)\cdot\left(\sum_j v_j \hat{e}_j\right)$$
$$= \sum_{i,j} v_i v_j \underbrace{\hat{e}_i \cdot \hat{e}_j}_{\text{might not}} = \delta_{ij} \text{ (orthogonal)}$$

$\hat{e}_1 \cdot \hat{e}_1 = 1$

$\hat{e}_i \cdot \hat{e}_j \neq 0$



9.3

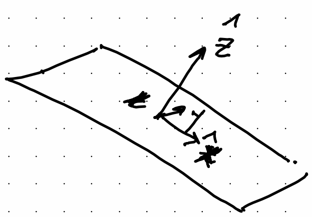


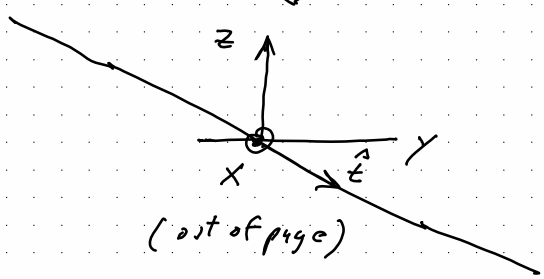
(a) $U(\cancel{x}, \cancel{y}, z)$

$mgz = U(z)$, $\vec{F} = -m\vec{g}$

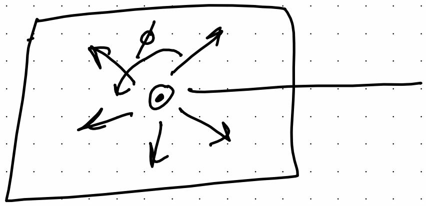
$\delta L = 0$ if you displace in $\hat{x}, \hat{y}$ → P_x, P_y = const, M_z = const

$$\vec{F} = -\vec{\nabla} U = -\frac{dU}{dz}\hat{z}$$



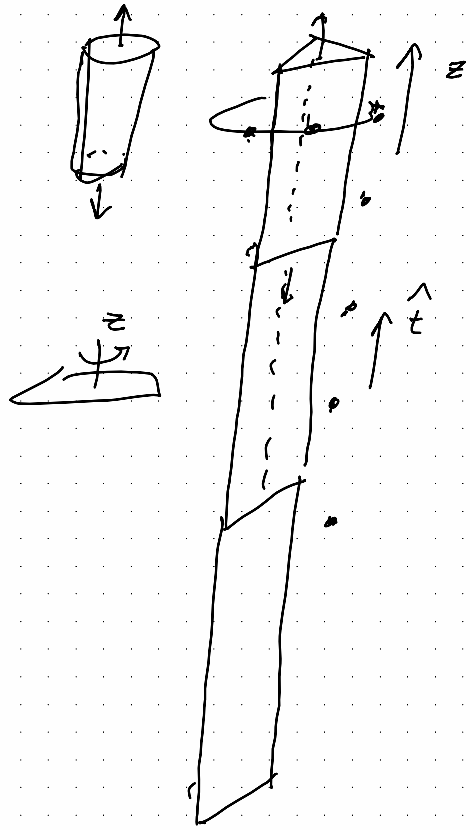


$$\vec{P} \cdot \hat{t} = \text{constant}$$



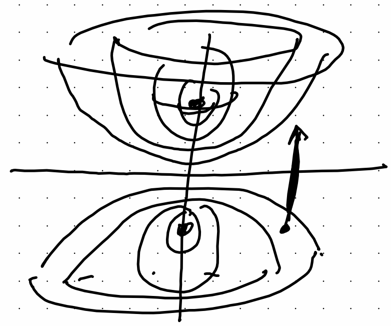
$\hat{z}$ out

$$\vec{M} \cdot \hat{z} = \text{const}$$



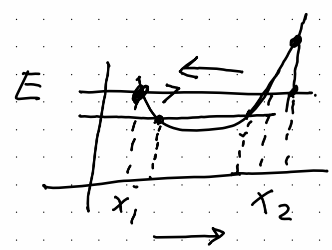
$U(x, y, \cancel{z})$

$\vec{P} \cdot \hat{t} = \text{const} = P_z$



$L = \frac{1}{2} m \dot{x}^2 - U(x)$

$E = \underbrace{\frac{1}{2} m \dot{x}^2}_{\text{positive}} + U(x) = \text{const}$



$x > x_2$

$\frac{1}{2} m \dot{x}^2 = E - U(x)$

$\frac{dx}{dt} = \dot{x} = \pm \sqrt{\frac{2}{m}(E - U(x))}$ — separable 1st order ODE

$\int \frac{dx}{\pm\sqrt{\frac{2}{m}(E-U(x))}} = \int dt$

$\boxed{t = \int \frac{dx}{\sqrt{\frac{2}{m}(E-U(x))}} + \text{const}}$

$\boxed{T(E) = 2 \int_{x_1 = x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{\frac{2}{m}(E-U(x))}}}$

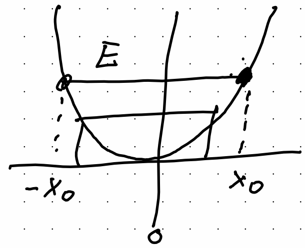
$\Delta x = \pm \Delta t \sqrt{\frac{2}{m}(E - U(x))}$

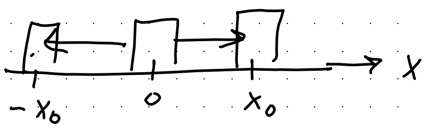
$U(x) = \frac{1}{2} k x^2$

$U(\phi) = mg \cos\phi$

$\ddot{\phi} = \frac{-g}{l} \sin\phi$

$U = \frac{1}{2} k x^2$





Prob 2

(a) $U = A|x|^n$

(b) $U = \dfrac{-U_0}{\cosh^2(\alpha x)}$

(c) $U = U_0 \tan^2(\alpha x)$

$$\int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E - U(x)}}$$

$E \leftarrow \frac{1}{2} k x_0^2 - \frac{1}{2} k x^2$

$\frac{1}{2} k (x_0^2 - x^2)$

$$\frac{dx}{\sqrt{x_0^2 - x^2}}$$

→ Beta Function

$E = \frac{1}{2} k x_0^2$

$x_0 = \sqrt{\dfrac{2E}{k}}$

$E = U(x_1) = U(x_2)$

Lecture #9: Tues 9/22

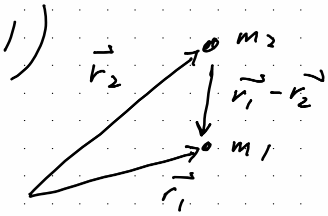
1) Midterm #1 : Tues Oct 6th (short answer; long problems) 40% 60%
2) Next 4 lecture (central force problem)
Sec 13 - 15

General Formalism:

Two interacting particles (no external forces)

m_1, m_2

$U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$



$$L = T - U$$
$$= \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 - U(|\vec{r}_1 - \vec{r}_2|)$$

i) L unchanged by a translation $\vec{r}_1 \to \vec{r}_1 + \delta\vec{x}$, $\vec{r}_2 \to \vec{r}_2 + \delta\vec{x}$

$\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2 = \text{const}$

COM moves with const velocity

choose inertial ref frame suchthat COM at origin.
" COM Frame

$$\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = \vec{R}_{com} = 0$$

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0$$

$U(|\vec{r}_1-\vec{r}_2|)$

Define: $\boxed{\vec{r} \equiv \vec{r}_1 - \vec{r}_2}$

$U = U(r), \quad r = |\vec{r}|$

$$\vec{r}_2 = \vec{r}_1 - \vec{r}$$

$$m_1\vec{r}_1 + m_2(\vec{r}_1 - \vec{r}) = 0$$

$$(m_1+m_2)\vec{r}_1 - m_2\vec{r} = 0$$

$$\boxed{\vec{r}_1 = \left(\frac{m_2}{m_1+m_2}\right)\vec{r}}$$

$$\vec{r}_2 = \left(\frac{m_2}{m_1+m_2}\right)\vec{r} - \vec{r}$$

$$\boxed{\vec{r}_2 = -\left(\frac{m_1}{m_1+m_2}\right)\vec{r}}$$

$$T = \frac{1}{2}m_1|\dot{\vec{r}}_1|^2 + \frac{1}{2}m_2|\dot{\vec{r}}_2|^2$$

$$T = \frac{1}{2}m_1\left(\frac{m_2}{m_1+m_2}\right)^2|\dot{\vec{r}}|^2 + \frac{1}{2}m_2\left(\frac{-m_1}{m_1+m_2}\right)^2|\dot{\vec{r}}|^2$$

$$= \frac{1}{2}\frac{m_1 m_2}{(m_1+m_2)^{\cancel{2}}}|\dot{\vec{r}}|^2 \cancel{(m_2+m_1)}$$

$$= \frac{1}{2}\underbrace{\left(\frac{m_1 m_2}{m_1+m_2}\right)}_{\text{reduced mass}: \ m}|\dot{\vec{r}}|^2$$

$$= \frac{1}{2}m|\dot{\vec{r}}|^2$$

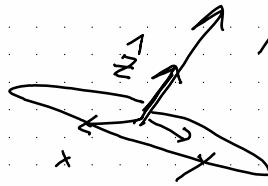
central potential

$$\boxed{L = \frac{1}{2}m|\dot{\vec{r}}|^2 - U(r)}$$

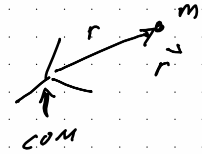
"effective" one body Lagrangian

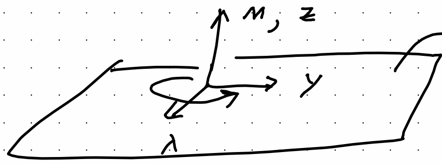
ii) L unchanged under a rotation

$\rightarrow$ $\underline{\vec{M} = \text{const}}$ $\quad (\vec{M} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v})$



$\vec{M} = \underbrace{\vec{r} \times \vec{p}}_{\perp}$





motion is in (x, y)

examples: $U(r) = \frac{-Gm_1m_2}{r}$ Newtonian gravity

$U(r) = \frac{1}{2}kr^2$ space oscillator

$$\boxed{L = \frac{1}{2}m|\dot{\vec{r}}|^2 - U(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)}$$

i) No explicit t-dependence

$\boxed{E = T + U = \text{const}}$

E, M constants of the motion

ii) No ϕ dependence

$M_z = p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \text{const} = \boxed{mr^2\dot{\phi} \equiv M}$

$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + U(r)$,

$M = mr^2\dot{\phi} \longrightarrow \boxed{\dot{\phi} = \frac{M}{mr^2}}$

$\rightarrow E = \frac{1}{2}m\dot{r}^2 + \underbrace{\left(\frac{M^2}{2mr^2} + U(r)\right)}_{U_{eff}(r)}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} = 0$

$$\frac{dr}{dt} = \dot{r} = \sqrt{\frac{2}{m}\left(E - U(r) - \frac{M^2}{2mr^2}\right)}$$

$$= \sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2r^2}}$$

$$\int dt = \boxed{t = \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2r^2}}} + \text{const}}$$

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M}{mr^2} = \sqrt{}$$

$$\text{const} + \int \frac{\frac{dr\, M}{mr^2}}{\sqrt{}} = \int d\phi = \phi$$

$$\boxed{\phi = \int \frac{M\, dr/r^2}{\sqrt{2m(E-U(r)) - M^2/r^2}} + \text{const}}$$

$$\phi = \phi(r) \longleftrightarrow r = r(\phi)$$

$$m = \frac{m_1 m_2}{m_1 + m_2} \qquad \begin{aligned} r &= r(\phi) \\ r &= r(t) \\ \phi &= \phi(t) \end{aligned} \Big\} \ \vec{r}(t)$$

<u>Example:</u> $U(r) = \frac{1}{2}kr^2$ (space oscillator) (r,ϕ)

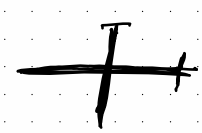
<u>Easy:</u> $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}kr^2$ (x,y)

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2+y^2)$$

$$= \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) + \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2\right)$$

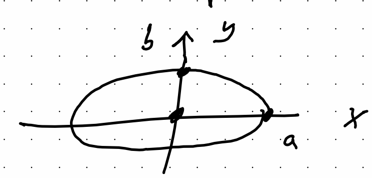
$$x(t) = a\cos(\omega t + \cancel{\alpha}^{\,0}), \quad \omega = \sqrt{\frac{k}{m}}$$

$$y(t) = b\sin(\omega t + \cancel{\beta}^{\,0}),$$



$\rightarrow$ closed orbit

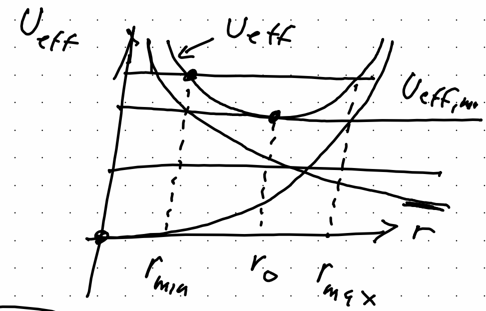
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



ellipse with center at origin

$U = -\frac{\alpha}{r}$, $U = \frac{1}{2}kr^2$ are <u>only</u> two potentials that have closed <u>bound orbits</u>

Harder: $U = \frac{1}{2} k r^2$



$$U_{eff} = U(r) + \frac{M^2}{2mr^2}$$

$$= \frac{1}{2} k r^2 + \frac{M^2}{2mr^2}$$

$$\phi = \int \frac{M\,dr/r^2}{\sqrt{2m\left(E - \frac{1}{2} k r^2\right) - \frac{M^2}{r^2}}} + \text{const}$$

bound orbits

substitutions

$$u = \frac{1}{r}, \quad du = \frac{-dr}{r^2}$$

$$v = u^2$$
$$dv = 2u\,du$$

$$\phi = -\int \frac{M\,du}{\sqrt{2m\left(E - \frac{1}{2}\frac{k}{u^2}\right) - M^2 u^2}} + \text{const} = -\int \frac{M\,u\,du}{\sqrt{2m\left(Eu^2 - \frac{k}{2}\right) - M^2 u^4}} + \text{const}$$

$$\phi = -\frac{1}{2}\int \frac{dv\,M}{\sqrt{2m\left(Ev - \frac{k}{2}\right) - M^2 v^2}} + \text{const}$$

Complete the square:

$$-M^2 v^2 + 2mEv - mk$$

$$= -M^2\left(v^2 - \frac{2mEv}{M^2} + \frac{mk}{M^2}\right)$$

$$= -M^2\left(\left(v - \frac{mE}{M^2}\right)^2 - \frac{m^2E^2}{M^4} + \frac{mk}{M^2}\right)$$

$$= -M^2\left((v-A)^2 - B^2\right)$$

$$= M^2\left(B^2 - (v-A)^2\right)$$

$$A = \frac{mE}{M^2}, \quad B^2 = A^2 - \frac{mk}{M^2}$$

$$\phi = -\frac{1}{2}\int \frac{dv}{\sqrt{B^2-(v-A)^2}} + \text{const}$$

$$\int \frac{dx}{\sqrt{1-x^2}}$$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$\int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \theta = \sin^{-1}(x)$$

3rd substitution:

$$v - A = B\sin\theta$$
$$dv = B\cos\theta\, d\theta$$

$v - A = \pm B$

$\frac{1}{r^2} - A = \pm B$

r_{max}

$\frac{1}{r^2} = A \pm B$

$$\phi = -\frac{1}{2}\int \frac{\cancel{B\cos\theta}\, d\theta}{\cancel{\sqrt{B^2 - B^2\sin^2\theta}}} + \text{const}$$

$$= -\frac{1}{2}\theta + \text{const}$$

$$= -\frac{1}{2}\sin^{-1}\left(\frac{v-A}{B}\right) + \text{const}$$

$$= -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right) + \text{const}$$

$\frac{1}{r_{max}^2} = A - B$

$\frac{1}{r_{min}^2} = A + B$

$v = u^2$

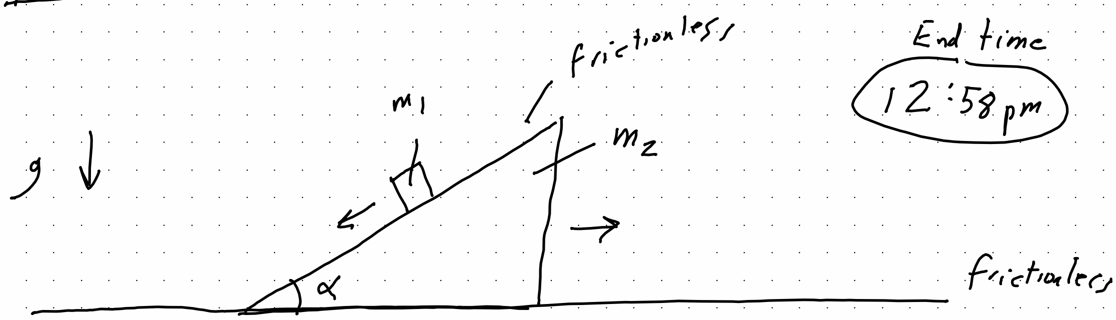
$u = \frac{1}{r}$

$v = \frac{1}{r^2}$

Lecture #10: 9/24

- Finish space oscillator
- Quiz #2
- Next week: Q&A

lastname_q2.pdf



End time

12:58 pm

i) generalized coords

ii) $L = T - U$

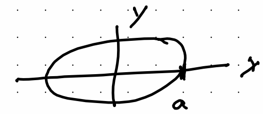
iii) EOMs

iv) special limiting cases: $\alpha = 0$, $\alpha = \pi/2$

$m_1 \gg m_2$, $m_1 \ll m_2$

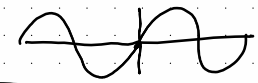
$$\phi = -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right) + \underline{\underline{\text{const}}} \qquad E, M$$

choose const so that $\phi = 0 \longleftrightarrow r = r_{max}$



$$\frac{1}{r_{max}^2} = A - B$$

$$\frac{\frac{1}{r_{max}^2} - A}{+B} = \frac{-B}{B} = -1$$



$$0 = -\frac{1}{2}\underbrace{\sin^{-1}(-1)}_{-\frac{\pi}{2}} + \text{const}$$

$$= \frac{\pi}{4} + \text{const}$$

$$\boxed{\text{const} = -\frac{\pi}{4}}$$

$$\phi = -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right) - \frac{\pi}{4}$$

$$-2\left(\phi + \frac{\pi}{4}\right) = \sin^{-1}\left(\frac{\frac{1}{r^2}-A}{B}\right)$$

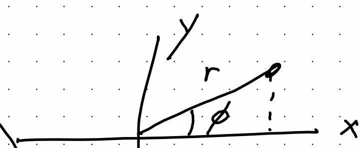
$$-\sin\left(2\phi + \frac{\pi}{2}\right) = \left(\frac{1}{r^2} - A\right)/B$$

$$-\sin\left(2\phi + \frac{\pi}{2}\right) = \left(\frac{1}{r^2} - A\right)/B$$

$$-B\cos(2\phi) = \frac{1}{r^2} - A$$

$$\boxed{\frac{1}{r^2} = A - B\cos(2\phi)}$$

$$\sin\left(2\phi + \frac{\pi}{2}\right) = \sin(2\phi)\underbrace{\cos\left(\frac{\pi}{2}\right)}_{\to 0} + \cos(2\phi)\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=0} = \cos(2\phi)$$



$x = r\cos\phi$

$y = r\sin\phi$

$$\cos(2\phi) = \cos^2\phi - \sin^2\phi$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(2\phi) = \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2$$

$$\left[\frac{1}{r^2} = A - B\left(\left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2\right)\right] \times r^2 \qquad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

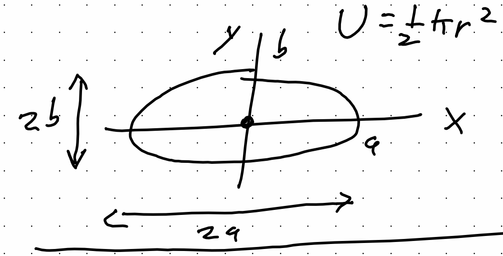
$$\frac{1}{a^2} = A - B, \quad \frac{1}{b^2} = A + B$$

$$1 = Ar^2 - B(x^2 - y^2)$$

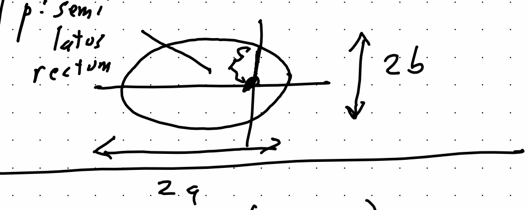
$$= A(x^2+y^2) - B(x^2-y^2) = (A-B)x^2 + (A+B)y^2 = 1$$

$$A = mE/M^2, \quad B^2 = A^2 - mk/M^2$$

$U = \frac{1}{2} k r^2$



$U = -\frac{\alpha}{r}$ (Newtonian)



$$E = \underbrace{\frac{1}{2} m \dot{r}^2}_{\substack{=0 \text{ at} \\ r=a,\ r=b}} + \frac{M^2}{2mr^2} + \frac{1}{2} k r^2 \qquad \begin{pmatrix} r=a \\ r=b \end{pmatrix}$$

$$\boxed{\begin{aligned} E &= \frac{M^2}{2ma^2} + \frac{1}{2} k a^2 \\ E &= \frac{M^2}{2mb^2} + \frac{1}{2} k b^2 \end{aligned}}$$

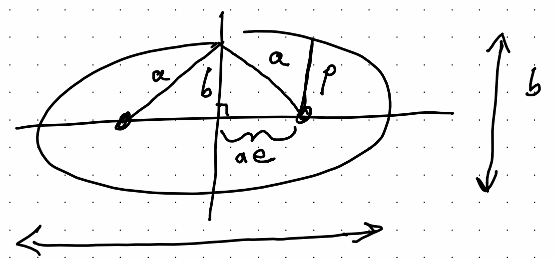
$$\rightarrow E a^2 = \frac{M^2}{2m} + \frac{1}{2} k a^4$$

$$\rightarrow E b^2 = \frac{M^2}{2m} + \frac{1}{2} k b^2$$

subtract $E\cancel{(a^2-b^2)} = \frac{1}{2} k \underbrace{(a^4 - b^4)}_{\cancel{(a^2-b^2)}(a^2+b^2)}$

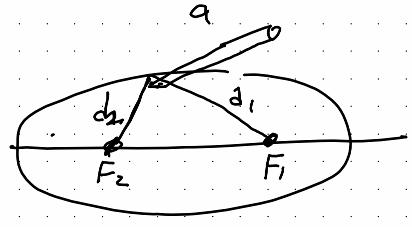
$$\rightarrow \boxed{E = \frac{1}{2} k (a^2 + b^2)}$$

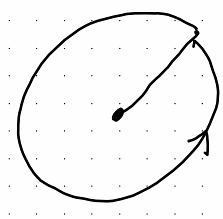
$$\rightarrow \boxed{M^2 = m k a^2 b^2}$$

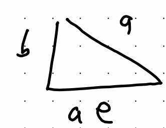


$e = 0$ circle

$\underbrace{e = 1}_{\text{dimensionless}}$ line







$a^2 = b^2 + a^2 e^2$

$e = \sqrt{\frac{a^2 - b^2}{a^2}}$

$d_1 + d_2 = 2a$

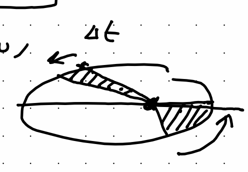
$$\boxed{b = a\sqrt{1-e^2}}$$

<u>Kepler's laws</u> (for $U = -\alpha/r$)

i) elliptical orbits with sun at one focus

ii) equal areas in equal times

iii) $\frac{P^2}{a^3} = \text{const}$



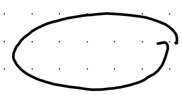
$U = \frac{1}{2} k r^2$

"Kepler's laws": (i) elliptical orbits with "Sun" at center of the ellipse

(ii) equal areas in equal times

(iii)

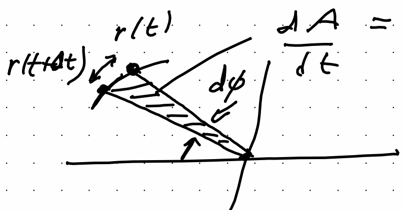
true for any central potential



$M = m r^2 \dot{\phi} \equiv \text{const}$

$$\frac{dA}{dt} = \frac{M}{2m}$$

$$\int dA = \int \frac{M}{2m}\, dt$$



$$\frac{dA}{dt} = \frac{1}{2}\, r\, r\frac{d\phi}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt}$$

height

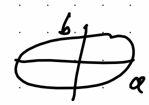
$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{M}{2m} = \text{const}$$

$$\frac{dA}{dt} = \text{const}$$

$$A = \frac{M}{2m} P$$

$A = \pi r^2$ circle

$= \pi a b$



$$\pi a b = \frac{M}{2m} P$$

$$P = \frac{2m}{M} \pi a b \,, \qquad M^2 = m k a^2 b^2$$

$$M = \sqrt{mk}\, a b$$

$$= \frac{2m\, \pi \cancel{ab}}{\sqrt{mk}\, \cancel{ab}}$$

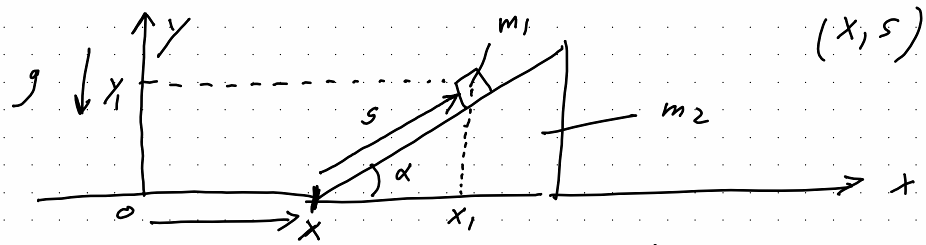
$$= \frac{2\pi}{\sqrt{k/m}}$$

$$= \frac{2\pi}{\omega} = \text{const} \quad (\text{for any } a, b)$$

$P = \text{const}$ indep of a, b !!

Lec #11: Tuesday 9/29

- Midterm #1: Next Tuesday Oct 6^{th}
- Trial midterm on BlackBoard (do it before Thursday)
- Today: a) QUIZ #2
  - b) Finish $U = \frac{1}{2}kr^2$
  - c) Q & A



$$L = T - U \qquad T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$x_1 = X + s\cos\alpha \rightarrow \dot{x}_1 = \dot{X} + \dot{s}\cos\alpha$$

$$y_1 = s\sin\alpha \rightarrow \dot{y}_1 = \dot{s}\sin\alpha$$

$$x_2 = X \rightarrow \dot{x}_2 = \dot{X}$$

$$y_2 = 0 \rightarrow \dot{y}_2 = 0$$

$$\dot{x}_1^2 = (\dot{X} + \dot{s}\cos\alpha)^2 = \dot{X}^2 + \underline{\dot{s}^2\cos^2\alpha} + 2\dot{X}\dot{s}\cos\alpha$$

$$\dot{y}_1^2 = \underline{\dot{s}^2\sin^2\alpha}$$

$$T = \frac{1}{2} m_1 (\dot{X}^2 + \dot{s}^2 + 2\dot{X}\dot{s}\cos\alpha) + \frac{1}{2} m_2 \dot{X}^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{X}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1 \dot{X}\dot{s}\cos\alpha$$

$$U = m_1 g y_1 = m_1 g s \sin\alpha$$

$$\boxed{L = \frac{1}{2}(m_1 + m_2)\dot{X}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1 \dot{X}\dot{s}\cos\alpha - m_1 g s \sin\alpha}$$

EOMs: X: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = \cancel{\frac{\partial L}{\partial X}} \rightarrow 0$ $\quad E = T + U$

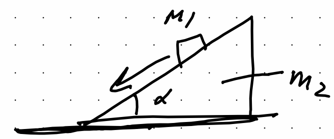
$$\boxed{P_X = (m_1 + m_2)\dot{X} + m_1 \dot{s}\cos\alpha = \text{const}}$$

$$\rightarrow \boxed{(m_1 + m_2)\ddot{X} + m_1 \ddot{s}\cos\alpha = 0}$$

s:

$$\frac{d}{dt}(\cancel{m_1}\dot{s} + \cancel{m_1}\dot{X}\cos\alpha) = -\cancel{m_1} g \sin\alpha \qquad \boxed{\ddot{s} + \ddot{X}\cos\alpha = -g\sin\alpha}$$

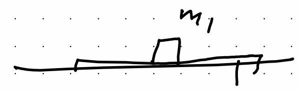
$$\ddot{x} = -\left(\frac{m_1}{m_1+m_2}\right)\ddot{s}\cos\alpha$$



$$\ddot{s} = -g\sin\alpha\left(\frac{m_1+m_2}{m_1\sin^2\alpha+m_2}\right)$$

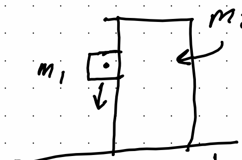
---

i) $m_2 >> m_1$: $\ddot{s} \approx -g\sin\alpha$

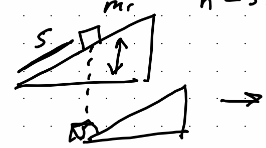


ii) $\alpha = 0$ $\ddot{s}=0$, $\ddot{x}=0$

iii) $\alpha = \pi/2$ $\ddot{s}=-g$, $\ddot{x}=0$



iv) $m_1 >> m$: $\begin{cases} \ddot{s} \approx -\dfrac{g}{\sin\alpha} \\ \ddot{x} \approx +\dfrac{g}{\tan\alpha} \end{cases}$



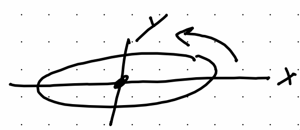
$h = s\sin\alpha$

$$\ddot{h} = -g = \ddot{s}\sin\alpha \rightarrow \ddot{s} = \frac{-g}{\sin\alpha}$$

Space Oscillator: $U = \frac{1}{2}kr^2$ $U = \frac{-\alpha}{r}$

"Kepler's" law:

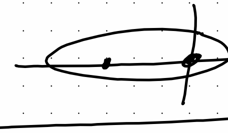
(i) ellipses with origin at center of ellipse



(ii) equal areas in equal times — applies to all central potentials

(iii) $P = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}}$ indep. of a, b

$\frac{P^2}{a^3} = \text{const}$ ordinary Kepler



$$t = -\int \frac{dr}{\sqrt{\frac{2}{m}(E-U(r)) - \frac{M^2}{m^2r^2}}} + \text{const}$$

angular moment

reduced mass

$\frac{1}{2}kr^2$

$E = \frac{1}{2}k(a^2+b^2)$

want t to increase as r decreases.

$M^2 = mka^2b^2$

$$t = -\int \frac{dr}{\sqrt{\frac{\cancel{2}}{m}\left(\frac{1}{\cancel{2}}k(a^2+b^2) - \frac{1}{\cancel{2}}kr^2\right) - \frac{\cancel{m}ka^2b^2}{m^{\cancel{2}}r^2}}}$$

(r, ϕ)
(x, y)

$$= -\frac{1}{\sqrt{\frac{k}{m}}}\int \frac{dr}{\sqrt{(a^2+b^2) - r^2 - \frac{a^2b^2}{r^2}}}$$

$r^2 = x^2+y^2$

$$= -\frac{1}{\sqrt{\frac{k}{m}}}\int \frac{r\,dr}{\sqrt{r^2(a^2+b^2) - r^4 - a^2b^2}}$$

$$= - \frac{1}{\omega}\int \frac{r\,dr}{\sqrt{-\underbrace{(r^4 - r^2(a^2+b^2) + a^2b^2)}_{= (r^2-a^2)(r^2-b^2)}}}$$

$b < r < a$

$\sqrt{a^2(1-\sin^2\theta)} = a\cos\theta$

$\int \frac{dx}{\sqrt{a^2 - x^2}}$

$x = a\sin\theta$

substitution

$$\boxed{r^2 = a^2\cos^2\xi + b^2\sin^2\xi}$$

$\boxed{x = a\cos\xi}$ $\boxed{y = b\sin\xi}$

$\rightarrow \quad r^2 = a^2\cos^2\xi + b^2\sin^2\xi$

Differentiate:

$$2r\,dr = -2a^2\cos\xi\sin\xi + 2b^2\sin\xi\cos\xi$$

$$= -2\sin\xi\cos\xi(a^2-b^2)$$

$$r\,dr = -\sin\xi\cos\xi(a^2-b^2)$$

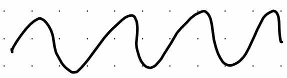
$$\sqrt{-(r^2-a^2)(r^2-b^2)} = \sqrt{-(\underbrace{a^2\cos^2\xi + b^2\sin^2\xi - a^2}_{-a^2\sin^2\xi})(\underbrace{a^2\cos^2\xi + b^2\sin^2\xi - b^2}_{-b^2\cos^2\xi})}$$

$$= \sqrt{\underbrace{-(-)}_{+}(a^2-b^2)\sin^2\xi\,(a^2-b^2)\cos^2\xi}$$

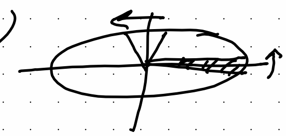
$$= \boxed{(a^2-b^2)\sin\xi\cos\xi}$$

$$t = \frac{1}{\omega}\int d\xi + \underbrace{\text{const}}_{=0} \;\rightarrow\; \boxed{\omega t = \xi} + \text{const}$$

$$\boxed{\begin{aligned} x &= a\cos(\omega t) \\ y &= b\sin(\omega t) \end{aligned}} \;\rightarrow\; \begin{aligned} r^2 &= x^2 + y^2 \\ &= a^2\cos^2(\omega t) + b^2\sin^2(\omega t) \end{aligned}$$



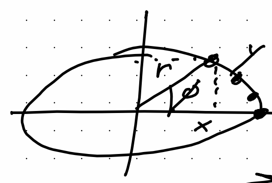
$$\underbrace{\boxed{r = \sqrt{a^2\cos^2(\omega t) + b^2\sin^2(\omega t)}}}_{r(t)}$$



$\phi = \phi(t)$

$(x, y) \longleftrightarrow (r, \phi)$

$$\boxed{\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}}$$



$r\cos\phi = a\cos(\omega t)$

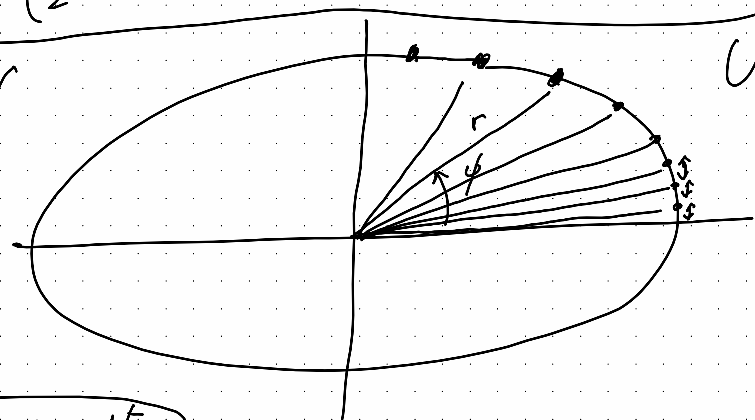
$r\sin\phi = b\sin(\omega t)$

Divide 2nd by 1st: $\tan\phi = \frac{b}{a}\tan(\omega t)$

$$\rightarrow \boxed{\phi = \tan^{-1}\left(\frac{b}{a}\tan(\omega t)\right)}$$

$\rightarrow \phi = \omega t$ if $b = a$

$$L = \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) + \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2\right)$$



$U = -\frac{\alpha}{r}$

$\cancel{+1/2}$

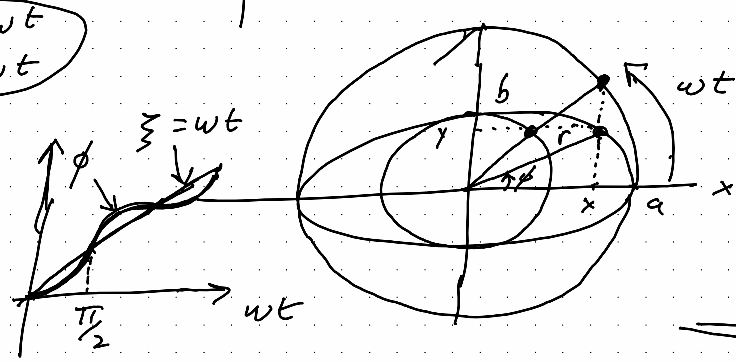
$\boxed{+1/2}$

$x = a\cos\omega t$

$y = b\sin\omega t$

$x = r\cos\phi$

$y = r\sin\phi$



Lec #12:

- Midterm Exam 1 : Tuesday Oct 6th
- Do trial exam by end of the day today
- Today : Q&A

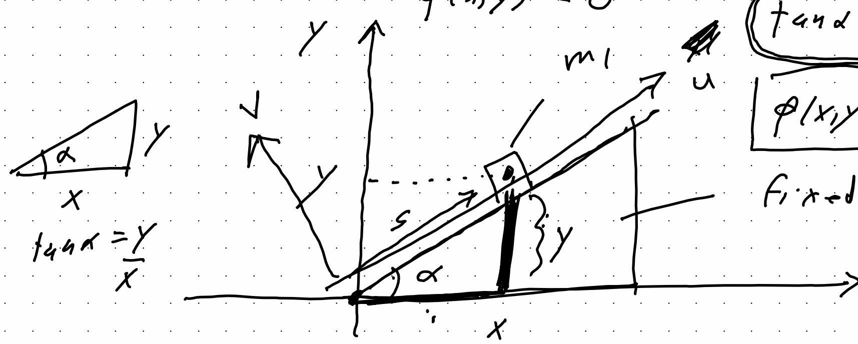
$$x^2+y^2-r_0^2=0$$

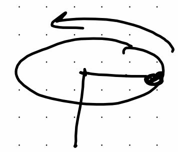
$$r-r_0=0$$

$$\varphi(x,y)=0$$

$$\tan\alpha = \frac{y}{x}$$

$$\boxed{\varphi(x,y) = y - x\tan\alpha}$$





$$T = \frac{1}{2}m_1(\dot{x}^2+\dot{y}^2) \quad , \quad U = m_1 g y \qquad (x,y)$$

$$L = T - U + \lambda\,(y - x\tan\alpha) \qquad (r,\phi)$$

EOMs:

$$x : \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow \boxed{m_1\ddot{x} = -\lambda\tan\alpha}$$

$$y : \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} \rightarrow \boxed{m_1\ddot{y} = -mg + \lambda}$$

constraint:

$$\boxed{y - x\tan\alpha = 0}$$

$$\boxed{\ddot{y} - \ddot{x}\tan\alpha = 0}$$

$$(m_1\ddot{x} = -\lambda\tan\alpha)\tan\alpha$$

$$+\, m_1\ddot{x}\tan\alpha = -mg + \lambda$$

$$0 = -\lambda\tan^2\alpha - \lambda + m_1 g$$

$$= -\lambda(1+\tan^2\alpha) + mg$$

$$\sin^2 + \cos^2 = 1$$

$$\tan^2 + 1 = \sec^2$$

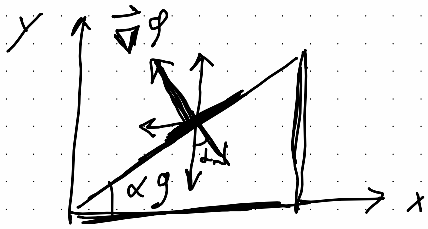
$$\boxed{L = \frac{1}{2}m_1(\dot{x}^2+\dot{y}^2) - mgy + \lambda(y - x\tan\alpha)}$$

$$\lambda = \frac{mg}{\sec^2\alpha} \propto mg\cos\alpha$$

$$\varphi(x,y) = y - x\tan\alpha$$

$$\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{x} + \frac{\partial\varphi}{\partial y}\hat{y}$$

$$= -\tan\alpha\,\hat{x} + \hat{y} = -\frac{\sin\alpha}{\cos\alpha}\hat{x} + \hat{y}$$



$$= \frac{-\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}}{\cos\alpha}$$

$$\boxed{\begin{array}{l}\vec{N} = mg\cos\alpha \\ \perp \text{ to incline}\end{array}}$$

$$\vec{\nabla}\varphi \propto -\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}$$

$$y = x\tan\alpha$$

$$dy = \tan\alpha\, dx \rightarrow \boxed{\cos\alpha\, dy = \sin\alpha\, dx}$$

$$\vec{\nabla}\varphi\cdot d\vec{r} = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = -\cancel{\tan\alpha\, dx} + 1\,\cancel{\tan\alpha}\, dx$$
$$= 0$$

$$(u,v) \qquad \varphi(u,v) = v = 0$$

$$L = T - U + \lambda v$$
$$v = 0$$

$$\boxed{L = T - U + \lambda\varphi}$$

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - U(x)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}(m\dot{x}) = -\frac{\partial U}{\partial x}$$

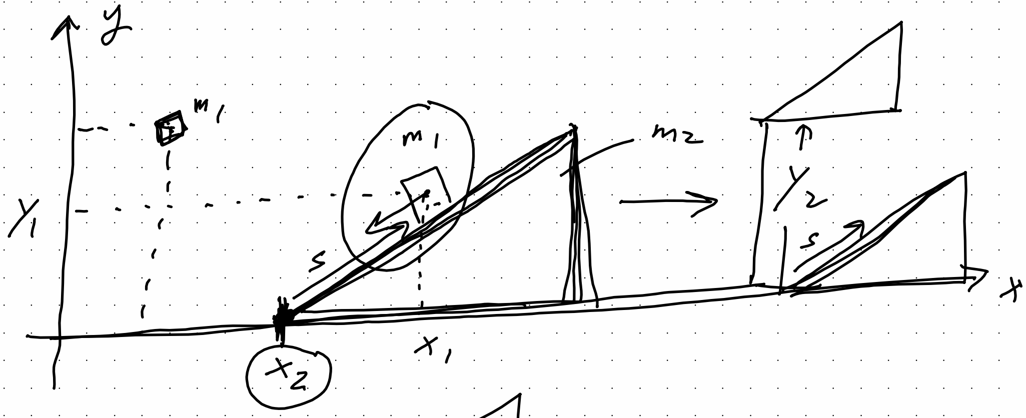
$$\boxed{\frac{dp}{dt} = -\frac{\partial U}{\partial x} = F}$$

$$\frac{dp}{dt} = F$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = -\frac{\partial L}{\partial \vec{r}}$$

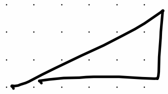
$$\frac{d\vec{p}}{dt} = -\frac{\partial U}{\partial \vec{r}} + \lambda\frac{\partial\varphi}{\partial\vec{r}}$$

$$\frac{d\vec{p}}{dt} = -\vec{\nabla}U + \vec{\nabla}\varphi$$
$$= \underbrace{\vec{F}_{\text{applied}} + \vec{F}_{\text{constraint}}}$$



(x_1, y_1, x_2, y_2)

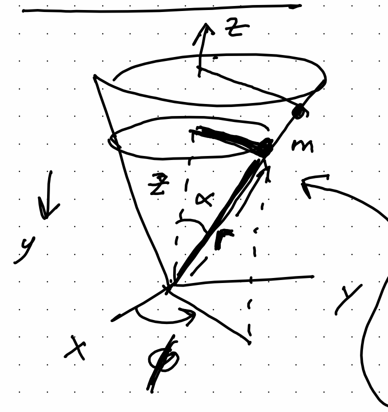
m_1

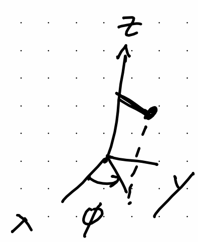


$y_2 = 0$

$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \cancel{\dot{y}_2^2})$ $\to 0$

Sec 14, Prob 2:





~~cylindrical~~

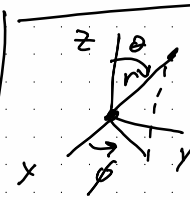
(r, ϕ, z)

$r^2 = x^2 + y^2$

alt. notation

$s^2 = x^2 + y^2$

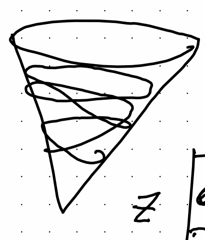
$\rho^2 = x^2 + y^2$

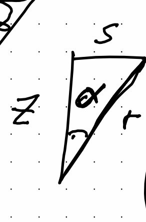


sph. polar

(r, ϕ, θ)

$r^2 = x^2 + y^2 + z^2$





$\boxed{\tan\alpha = \frac{s}{z}}$

$\theta = \alpha$

(ϕ, s) (ϕ, z)

$$T = \frac{1}{2}m\left(\dot r^2 + \cancel{r^2\dot\theta^2}_{\to 0} + r^2\underbrace{\sin^2\theta}_{\sin^2\alpha}\,\dot\phi^2\right)$$

sph. polar

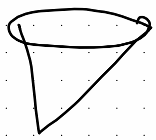
$\theta = \alpha$

$$T = \frac{1}{2}m\left(\underbrace{\dot s^2 + s^2\dot\phi^2}_{\text{plane polar coord}} + \dot z^2\right)$$

cyl. coords

s, ϕ, z

$\tan\alpha = \dfrac{s}{z}$



$$z = \frac{s}{\tan\alpha} \qquad s = z\tan\alpha$$

$$\dot z = \frac{\dot s}{\tan\alpha} \qquad \dot s = \dot z\tan\alpha$$

$$L = T - U + \lambda\varphi$$

$$\varphi = s - z\tan\alpha$$

$$f(x,y) = e^{-\frac{1}{2}(x^2+y^2)}$$

No constraint: $\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases}$

$g(x,y,\lambda)$

$$\equiv f(x,y) + \lambda\varphi(x,y)$$

$$\frac{\partial g}{\partial x} = 0$$

$$\frac{\partial g}{\partial y} = 0, \quad \frac{\partial g}{\partial \lambda} = 0$$

---

suppose constraint $y = \frac{1}{2}x^2 + 1$

$$F(x) = f(x,y)\Big|_{y = \frac{1}{2}x^2+1}$$

$$\boxed{\frac{dF}{dx} = 0} \qquad \underbrace{\boxed{\varphi = y - \tfrac{1}{2}x^2 - 1}}_{=0}$$

$$L = T - U + \lambda\varphi$$

x, y, λ

COM:

$m_1, \; m_2, \; m_3, \quad \vec{r}_1, \; \vec{r}_2, \; \vec{r}_3$

$$\vec{R}_{com} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3}{m_1 + m_2 + m_3}$$

$$0 = \frac{\sum_a m_a \vec{r}_a}{\sum_b m_b}$$

$\vec{P} = M_{tot} \dot{\vec{R}}_{com}$

$= const$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_a}\right) = 0$$

$$\vec{P} = \sum_a \frac{\partial L}{\partial \dot{\vec{r}}_a} = const$$

~~Pcom~~ closed system: $\vec{P} = const$

$$L = \frac{1}{2} \sum_a m_a |\dot{\vec{r}}_a|^2 - U\left(|\vec{r}_1 - \vec{r}_2|, |\vec{r}_1 - \vec{r}_3|, \text{etc.} \right)$$

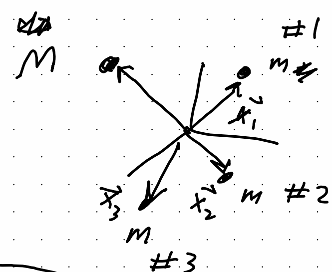
$\vec{r}_a \to \vec{r}_a + \delta \vec{x}$

$L \to L$

$$\frac{\partial L}{\partial \vec{r}} = 0$$

$M, \; \underbrace{m, \ldots, m}_{n}$

$\vec{X} \quad \vec{x}_1, \ldots, \vec{x}_n$



$$M \vec{X} + m \sum_{a=1}^{n} \vec{x}_a = 0$$

$$\boxed{\begin{aligned} \vec{r}_1 &= \vec{x}_1 - \vec{X} \\ \vec{r}_2 &= \vec{x}_2 - \vec{X} \\ &\vdots \\ \vec{r}_n &= \vec{x}_n - \vec{X} \end{aligned}}$$

$\vec{r}_1, \vec{r}_2$

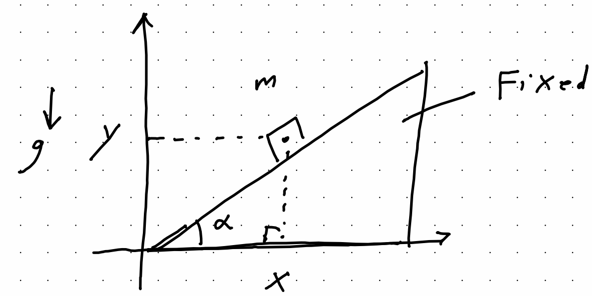
$\vec{r} = \vec{r}_1 - \vec{r}_2$

1 hr + 20 min

80 min

120 mins

2:35 pm



$\tan\alpha = \dfrac{y}{x}$

so $y = x\tan\alpha$

$\underbrace{\varphi(x,y) = y - x\tan\alpha = 0}_{\text{constraint}}$

$$L = T - U + \lambda\varphi$$
$$= \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy + \lambda(y - x\tan\alpha)$$

EOMs:

x: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) = \dfrac{\partial L}{\partial x} \rightarrow m\ddot{x} = -\lambda\tan\alpha \qquad (1)$

y: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{y}}\right) = \dfrac{\partial L}{\partial y} \rightarrow m\ddot{y} = -mg + \lambda \qquad (2)$

λ: $y - x\tan\alpha = 0 \rightarrow y = x\tan\alpha \qquad (3)$

Differentiate constraint twice:

$$\ddot{y} = \ddot{x}\tan\alpha$$

substitute for $\ddot{y}$, $\ddot{x}$ using (1) and (2)

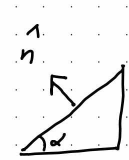
$$-g + \frac{\lambda}{m} = -\frac{\lambda}{m}\tan\alpha \cdot \tan\alpha$$

$$\frac{\lambda}{m}\big(\underbrace{1+\tan^2\alpha}_{\sec^2\alpha}\big) = g$$

$$\boxed{\lambda = \frac{mg}{\sec^2\alpha}}$$

constraint Force:

$$\vec{F}_c = \lambda\vec{\nabla}\varphi \qquad \text{where} \quad \varphi = y - x\tan\alpha$$
$$= \frac{mg}{\sec^2\alpha}\left(-\tan\alpha\,\hat{x} + \hat{y}\right)$$
$$= \frac{mg}{\sec^2\alpha}\left(-\frac{\sin\alpha}{\cos\alpha}\hat{x} + \hat{y}\right)$$
$$= mg\cos\alpha\left(-\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}\right)$$
$$= \boxed{mg\cos\alpha\;\hat{n}}$$



(where $\hat{n} = -\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}$ is $\perp$ to incline)

Return to EOMs:

$$\ddot{x} = -\frac{\lambda}{m}\tan\alpha$$

$$\ddot{y} = -g + \frac{\lambda}{m}$$

where $\lambda = \dfrac{mg}{\sec^2\alpha} = mg\cos^2\alpha$

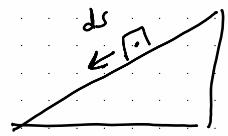
$$\rightarrow \quad \ddot{x} = -g\cos^2\alpha\tan\alpha$$

$$= -g\sin\alpha\cos\alpha$$

$$\ddot{y} = -g + g\cos^2\alpha$$

$$= -g(1-\cos^2\alpha)$$

$$= -g\sin^2\alpha$$



$ds = \sqrt{dx^2 + dy^2}$

Acceleration down the incline:

$$\ddot{s} = -\sqrt{\ddot{x}^2 + \ddot{y}^2} = -g\sin\alpha\sqrt{\cos^2\alpha + \sin^2\alpha} = \boxed{-g\sin\alpha}$$

standard result

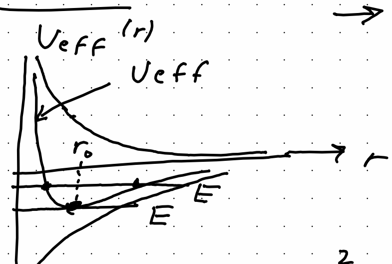
Lecture #14: Thurs Oct 8th

- midterm 1: Avg $\approx 14/20$
- midterm 2: Nov 19th (before Thanksgiving)
- oral final

Collisions (2) & Scattering (3) (Sec 16 - 20)

Scattering: closed system, two bodies

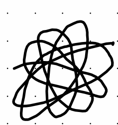
$\rightarrow$ central force problem

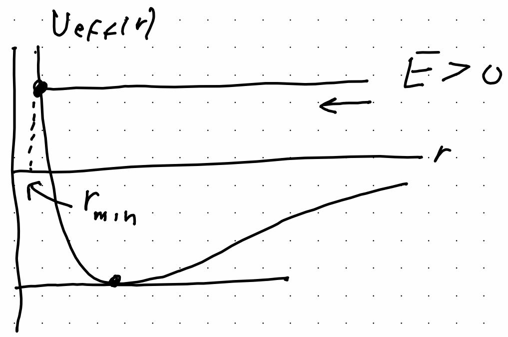


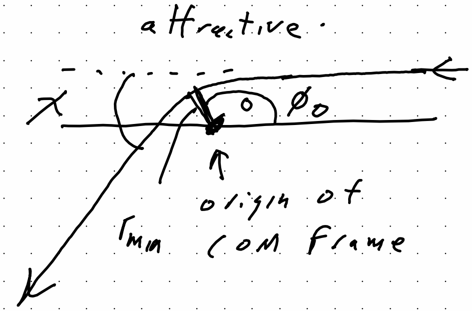
$$U_{eff} = -\alpha/r + \frac{M_z^2}{2mr^2}$$

$E = U_{eff,min} \rightarrow r = r_0$ (circular)

$U_{eff_{min}} < E < 0 \rightarrow \underbrace{\text{bound orbit}}_{\text{ellipse for } -\alpha/r}$

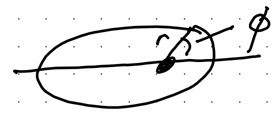


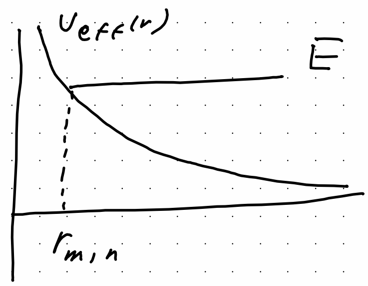


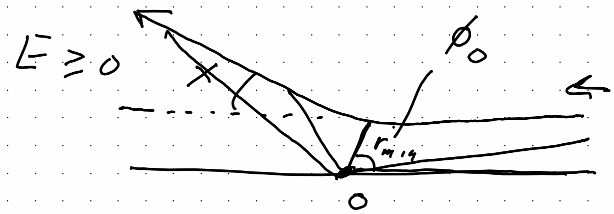


$U(r) = +\alpha/r$

$U_{eff}(r) = \alpha/r + M_z^2/2mr^2$







$$\int_0^{\phi_0} d\phi = \int_{r_{min}}^{\infty} \frac{M_z\, dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M_z^2}{r^2}}}$$ (14.7) L&L

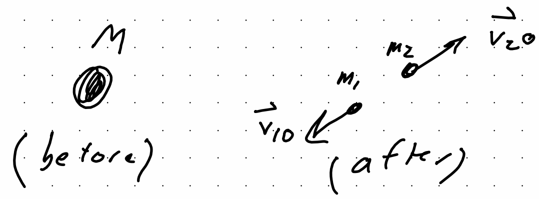
$E, M_z, U(r)$

$$\boxed{\phi_0 = \int_{r_{min}}^{\infty} \frac{M_z\, dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M_z^2}{r^2}}}}$$

Fig 18 in L & L

Collisions: (Sec 16, 17)

- Elastic collisions of two particles
- Spontaneous disintegration of a single particle



Analyze in COM frame (to begin with)

$\vec{p}_{tot,0} = \vec{0}$

$\llcorner$ COM

cons of momentum

$\vec{p}_0$: momentum of mass m_1 after disintegration

$-\vec{p}_0$: " m_2

cons. of energy

$$\underbrace{E_i}_{\substack{\text{internal}\\\text{energy of}\\\text{mass } M = m_1+m_2}} = \underbrace{E_{1i}}_{\substack{\text{internal}\\\text{energy}\\\text{of } m_1}} + \underbrace{T_{10}}_{\substack{\text{KE}\\\text{in}\\\text{COM}\\\text{frame}}} + E_{2i} + T_{20}$$

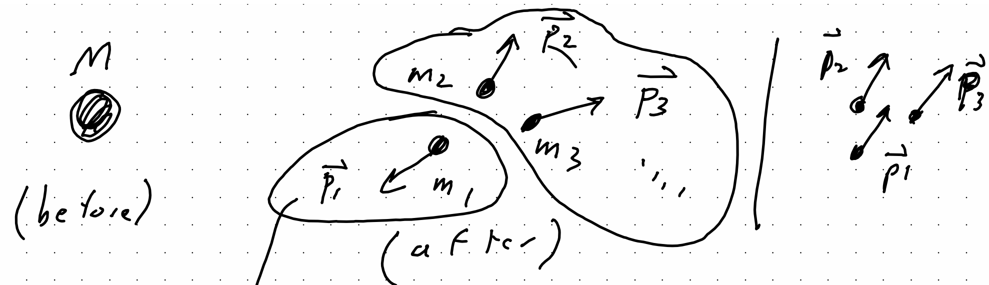
$$T_{10} = \frac{1}{2} m_1 |\vec{v}_{10}|^2 = \frac{|\vec{p}_0|^2}{2m_1}$$

$$E_i = E_{1i} + \frac{|\vec{p}_0|^2}{2m_1} + E_{2i} + \frac{|\vec{p}_0|^2}{2m_2}$$

$$\underbrace{E_i - E_{1i} - E_{2i}}_{\substack{\epsilon = \text{disintegration}\\ \geq 0 \quad \text{energy}}} = \frac{p_0^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

$$= \frac{p_0^2}{2}\left(\frac{m_1+m_2}{m_1 m_2}\right) = \frac{p_0^2}{2m}$$

$$p_0 = \sqrt{2m\epsilon} \rightarrow v_{10} = p_0/m_1, \quad v_{20} = p_0/m_2$$



$\vec{p}_{tot} = 0$

$0 = \vec{p}_{tot} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots$

$\vec{p}_1 = \vec{p}_0$

$T_{10} = \dfrac{p_0^2}{2m_1}$

Q: what condition is there on the velocities of $m_2, m_3, \ldots$ such that KE of m_1 is largest?

Consider COM of $m_2, m_3, \ldots$

E_i' : internal energy of $m_2, m_3, \ldots$

$\vec{p}_2 + \vec{p}_3 + \cdots = -\vec{p}_0$

$$KE = \frac{p_0^2}{2\underbrace{(M-m_1)}_{(m_2+m_3+\cdots)}}$$

Cons. of energy:

$$\underbrace{E_i}_{\text{int. energy of } M} = \underbrace{E_{1i} + \frac{p_0^2}{2m_1}}_{\text{int. + KE of } m_1} \underbrace{+ E_i' + \frac{p_0^2}{2(M-m_1)}}_{\text{int + KE of } m_2, m_3, \ldots}$$

$$E_i - E_{1i} - E_i' = \frac{p_0^2}{2}\left(\frac{1}{m_1} + \frac{1}{M-m_1}\right)$$

$$E_i - E_{1i} - E_i' = \frac{p_0^2}{2m_1}\frac{M}{(M-m_1)}$$

$E_i' \neq E_{2i} + E_{3i} + \ldots$

$$\boxed{T_{10} = \frac{p_0^2}{2m_1} = \left(\frac{M-m_1}{M}\right)(E_i - E_{1i} - E_i')}$$

T_{10} ~~is~~ maximum when E_i' is minimum.

$$E_i' + \frac{p_0^2}{2(M-m_1)} = \underbrace{E_{2i} + \frac{p_2^2}{2m_2} + E_{3i} + \frac{p_3^2}{2m_3} + \cdots}$$

int energy + KE of

masses $m_2, m_3, \ldots$

$\vec{p}_2 = m_2\vec{v}_2$, etc.

$$= E_{2i} + \tfrac{1}{2}m_2|\vec{v}_2|^2 + E_{3i} + \tfrac{1}{2}m_3|\vec{v}_3|^2 + \cdots$$

$$E_i' = (E_{2i} + E_{3i} + \cdots) + \left(\tfrac{1}{2}m_2|\vec{v}_2|^2 + \tfrac{1}{2}m_3|\vec{v}_3|^2 + \cdots\right) - \frac{|\vec{p}_2 + \vec{p}_3 + \cdots|^2}{2(m_2+m_3+\cdots)}$$

minimized

$= |\vec{v}_0|^2$

$$= (E_{2i} + E_{3i} + \cdots) + \left(\tfrac{1}{2}m_2|\vec{v}_2|^2 + \tfrac{1}{2}m_3|\vec{v}_3|^2 + \cdots\right) - \frac{1}{2}\frac{|m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots|^2}{m_2+m_3+\cdots}$$

$$|\vec{v}_2 + \vec{v}_3|^2 = |\vec{v}_2|^2 + |\vec{v}_3|^2 + 2\vec{v}_2\cdot\vec{v}_3$$

IF $\vec{v}_2 = \vec{v}_3 = \cdots \equiv \vec{v}_0$

then: $\tfrac{1}{2}m_2|\vec{v}_2|^2 + \tfrac{1}{2}m_3|\vec{v}_3|^2 + \cdots$

$$= \tfrac{1}{2}(m_2+m_3+\cdots)|\vec{v}_0|^2$$

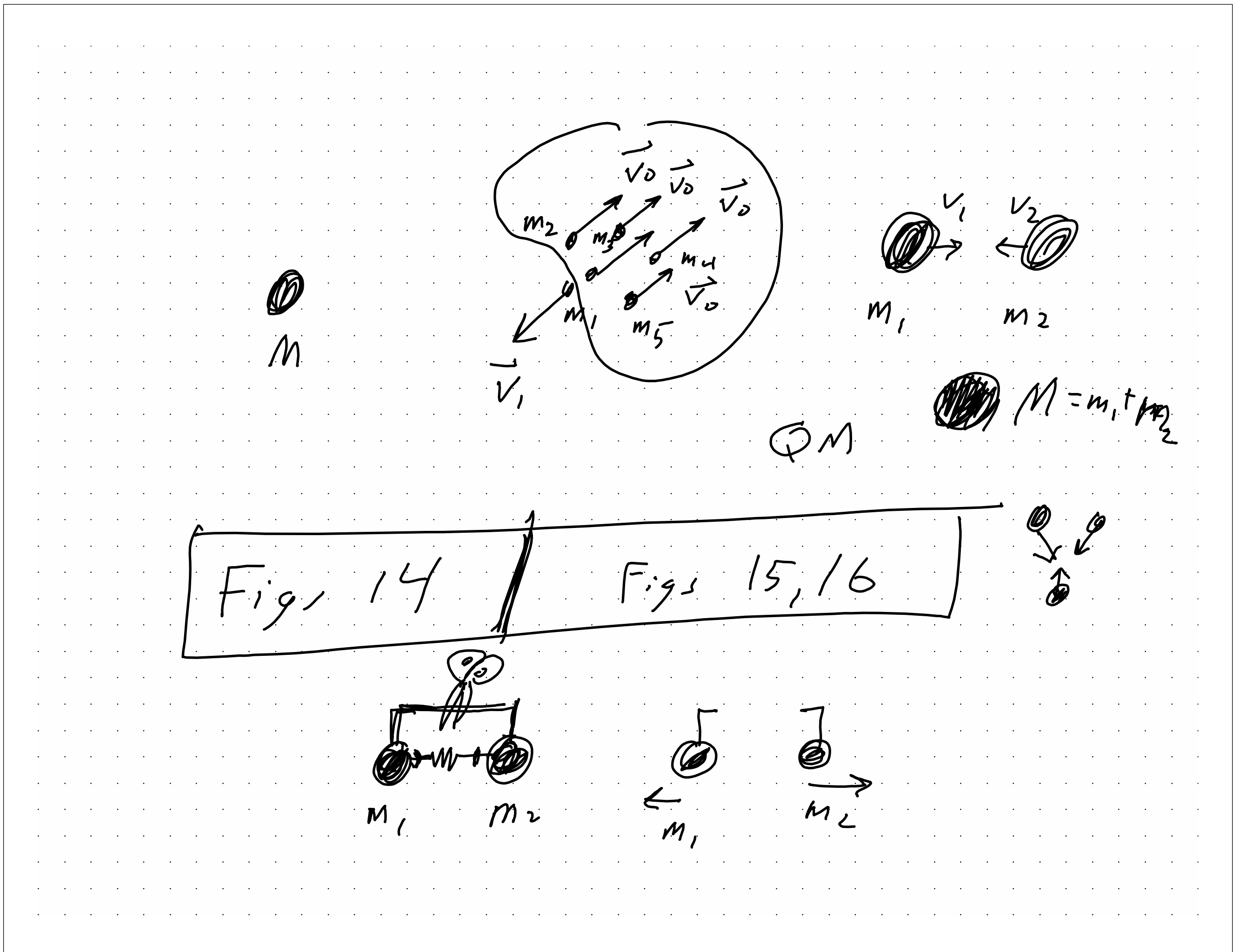
and $$\frac{1}{2}\frac{(m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots)^2}{m_2+m_3+\cdots}$$

$$= \frac{1}{2}|\vec{v}_0|^2 \frac{(m_2+m_3+\cdots)^{\cancel{2}}}{\cancel{m_2+m_3+\cdots}}$$

$$= \tfrac{1}{2}|\vec{v}_0|^2(m_2+m_3+\cdots)$$

$=$

$$T_{10,max} = \left(\frac{M-m_1}{M}\right)\epsilon$$



Lec #15: Tues 10/13

- solutions to midterm #1 posted
- next two weeks : Sec 16 - 20
  Collisions and scattering

Figures 14, 15, 16

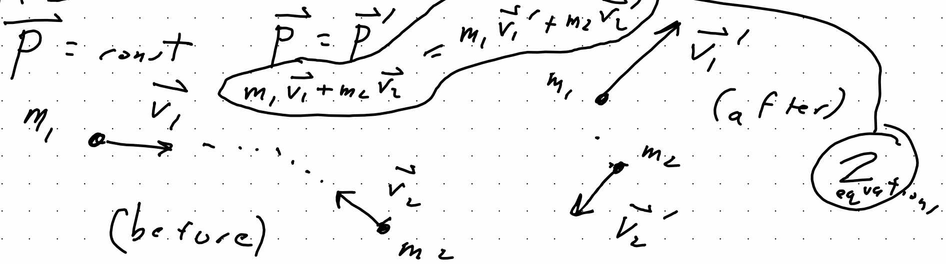
Elastic collision of two particles

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1(v_1')^2 + \frac{1}{2}m_2(v_2')^2$$ (1)

closed system: E, $\vec{P}$, $\vec{M}$ are conserved

✓ i) KE = const [ignore internal energies]

✓ ii) $\vec{P}$ = const $\quad \vec{P} = \vec{P}' = m_1\vec{v}_1' + m_2\vec{v}_2'$, $\quad m_1\vec{v}_1 + m_2\vec{v}_2$ (2) equations



After: $\vec{v}_1{}'$, $\vec{v}_2{}'$: $\boxed{\begin{array}{l}4\ \text{DOF}\\ 3\ \text{equations}\end{array}}$

Elastic collision:

$\vec{v} \equiv \vec{v}_1 - \vec{v}_2$ $= \vec{v}_{10} - \vec{v}_{20}$ relative velocity vector

$\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ $= \vec{r}_{10} - \vec{r}_{20}$ relative position vector

$\vec{v} = \dot{\vec{r}}$

$\vec{v}_1 = \vec{V} + \vec{v}_{10}$

$\vec{v}_2 = \vec{V} + \vec{v}_{20}$

$\vec{r}_1 = \vec{R} + \vec{r}_{10}$

$\vec{r}_2 = \vec{R} + \vec{r}_{20}$

$\vec{v}_1$: velocity of particle 1 wrt lab frame before collision

$\vec{v}_{10}$: " wrt COM frame before collision

$\vec{v}_1{}'$, $\vec{v}_{10}{}'$, $\vec{v}_2{}'$, $\vec{v}_{20}{}'$: velocities after collision

---

$$T_0 = \frac{1}{2} m_1 |\dot{\vec{r}}_{10}|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_{20}|^2$$

$$\vec{v}_{10} = \dot{\vec{r}}_{10} = \left(\frac{m_2}{m_1+m_2}\right)\dot{\vec{r}} \quad \dot{\vec{r}} = \vec{v} \qquad \vec{r} \equiv \vec{r}_{10} - \vec{r}_{20}$$

$$\vec{v}_{20} = \dot{\vec{r}}_{20} = -\left(\frac{m_1}{m_1+m_2}\right)\dot{\vec{r}}$$

$$T_0 = \frac{1}{2} m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 |\dot{\vec{r}}|^2 + \frac{1}{2} m_2 \left(\frac{-m_1}{m_1+m_2}\right)^2 |\dot{\vec{r}}|^2$$

$$= \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)|\dot{\vec{r}}|^2$$

$$= \frac{1}{2} m |\dot{\vec{r}}|^2 = \frac{1}{2} m |\vec{v}|^2$$

$T_0 = T_0{}'$

$\cancel{\tfrac{1}{2}}\cancel{m}|\vec{v}|^2 = \cancel{\tfrac{1}{2}}\cancel{m}|\vec{v}'|^2$

$\boxed{v = v'}$

~~relative~~ magnitude of relative velocity vector = const

Before collision

$$\underline{\vec{V}_{10}} = \left(\frac{m_2}{m_1+m_2}\right)\underset{\vec{V}_1}{\overset{\shortparallel}{\vec{V}}}, \qquad \underline{\vec{V}_{20}} = \left(\frac{-m_1}{m_1+m_2}\right)\underset{\vec{V}_1}{\overset{\shortparallel}{\vec{V}}}$$

$$\vec{V} = \underline{\vec{V}_1} - \underline{\cancel{\vec{V}_2}}$$

$$\vec{V}_1 = \vec{V}$$
$$\underbrace{\vec{V}_2 = 0}_{\text{lab frame}}$$

After collision:

$$\vec{V}_{10}' = \underbrace{\left(\frac{m_2}{m_1+m_2}\right) V}_{\text{magnitude of } \vec{V}} \underbrace{\hat{n}_0}_{\text{unit vector pointing in some direction (angle} = \chi \text{ wrt COM frame)}}$$

$$\vec{V}_{20}' = \left(\frac{-m_1}{m_1+m_2}\right) V\hat{n}_0$$

$$\boxed{\vec{V}_1' = \vec{V}_{10}' + \vec{V} = \left(\frac{m_2}{m_1+m_2}\right) V\hat{n}_0 + \frac{m_1\vec{V}_1 + m_2\vec{V}_2}{m_1+m_2}}$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \rightarrow \vec{V} = \frac{m_1\vec{V}_1 + m_2\vec{V}_2}{m_1+m_2}$$

$$\vec{V}_1' = \vec{V}_{10}' + \vec{V} = \boxed{\left(\frac{m_2}{m_1+m_2}\right) V\hat{n}_0 + \frac{m_1\vec{V}_1 + m_2\vec{V}_2}{m_1+m_2} = \vec{V}_1'}$$

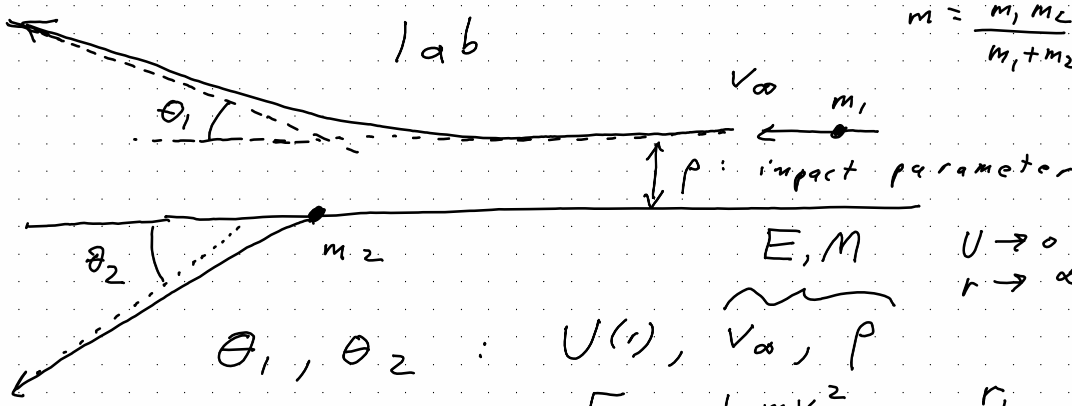
$$\vec{V}_2' = \vec{V}_{20}' + \vec{V} = \boxed{\left(\frac{-m_1}{m_1+m_2}\right) V\hat{n}_0 + \frac{m_1\vec{V}_1 + m_2\vec{V}_2}{m_1+m_2} = \vec{V}_2'}$$

$$\boxed{\vec{p}_1' = m_1\vec{v}_1' = mV\hat{n}_0 + m_1\underbrace{\left(\frac{\vec{p}_1 + \vec{p}_2}{m_1+m_2}\right)}_{\vec{V}}}$$

$$\boxed{\vec{p}_2' = m_2\vec{v}_2' = -mV\hat{n}_0 + m_2\left(\frac{\vec{p}_1 + \vec{p}_2}{m_1+m_2}\right)}$$

Lec #16: Thurs 10/15

- Quiz #3 next week
- General remarks about scattering
- Example: Prob 18.1 (Hard-sphere)



$m = \dfrac{m_1 m_2}{m_1 + m_2}$

E, M

$U \to 0$
$r \to \infty$

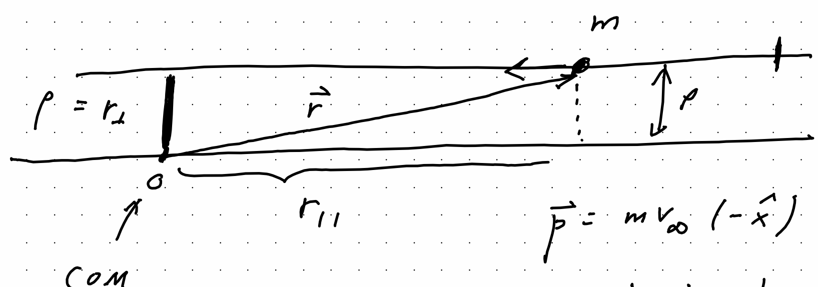
θ_1, θ_2 : $U(r)$, v_∞, ρ

χ : scattering angle w/t COM frame

$E = \frac{1}{2} m v_\infty^2$

$r_\perp$

$M = \rho m v_\infty$ | $|\vec{r} \times \vec{p}| = M$

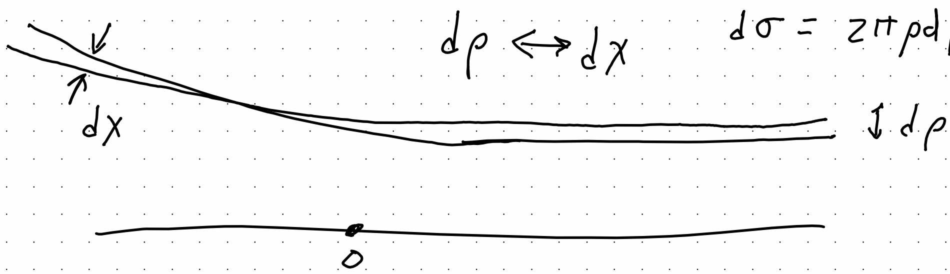


$\vec{p} = m v_\infty (-\hat{x})$

$$\boxed{\begin{aligned} E &= \tfrac{1}{2} m v_\infty \\ M &= \rho m v_\infty \end{aligned}}$$

$$\begin{aligned} M &= |\vec{r} \times \vec{p}| \\ &= r_\perp m v_\infty \\ &= \rho m v_\infty \end{aligned}$$

$d\rho \leftrightarrow d\chi$ $\qquad d\sigma = 2\pi \rho \, d\rho$



$$d\sigma = 2\pi\, \rho(\chi) \left|\frac{d\rho}{d\chi}\right| d\chi$$

$$d\sigma = \frac{\rho(\chi)}{\sin\chi} \left|\frac{d\rho}{d\chi}\right| d\Omega = \rho \left|\frac{d\rho}{d(\cos\chi)}\right| d\Omega$$

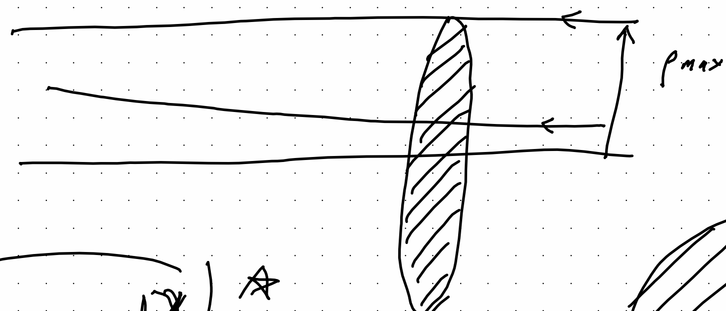
$$\boxed{\frac{d\sigma}{d\Omega} = \rho\left|\frac{d\rho}{d(\cos\chi)}\right|}$$ in COM frame

$$\boxed{\frac{d\sigma_1}{d\Omega_1}} = \rho\left|\frac{d\rho}{d(\cos\theta_1)}\right|$$ in lab frame θ_1

$$= \frac{d\sigma}{d\Omega}\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|$$

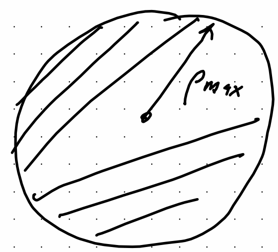
$$\frac{d\sigma_2}{d\Omega_2} = \frac{d\sigma}{d\Omega}\left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right|$$

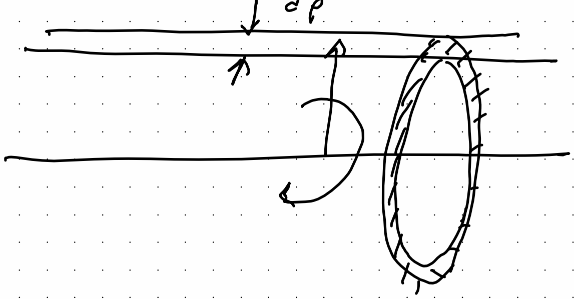
wrt lab frame (θ_1, θ_2)



$$\boxed{d\rho \leftrightarrow d\chi}$$

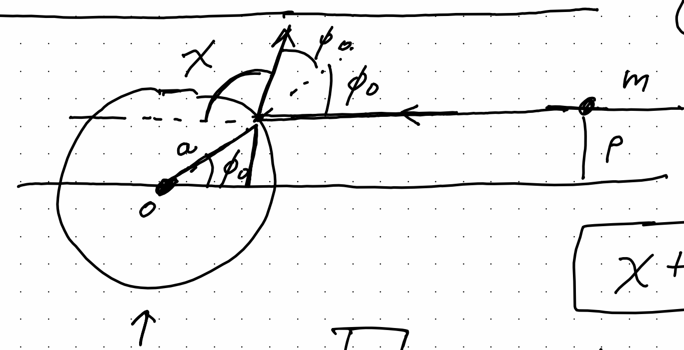
perspective





$$\sigma = \pi \rho_{max}^2$$

Hard sphere: Prob 18.1 $U(r) = \begin{cases} \infty & r<a \\ 0 & r>a \end{cases}$



radius a

$$\boxed{\chi \longleftrightarrow \rho}$$

$$\boxed{\chi + 2\phi_0 = \pi}$$

$$\phi_0 = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\boxed{\rho = a \sin\phi_0 = a\sin\left(\frac{\pi}{2}-\frac{\chi}{2}\right) = a\cos\left(\frac{\chi}{2}\right)}$$

$$\frac{d\rho}{d\chi} = -\frac{a}{2}\sin\left(\frac{\chi}{2}\right)$$

$$d\sigma = 2\pi\rho\, d\rho = 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi$$

$$d\sigma = \cancel{2}\pi\; a\cos\left(\frac{\chi}{2}\right)\frac{a}{\cancel{2}}\sin\left(\frac{\chi}{2}\right)d\chi$$

$$= \pi a^2 \sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)d\chi$$

$$= \boxed{\frac{\pi a^2}{2}\sin\chi\, d\chi}$$

$$d\Omega = 2\pi\sin\chi\, d\chi \quad\rightarrow\quad \sin\chi\, d\chi = \frac{1}{2\pi}d\Omega$$

$$d\sigma = \frac{\cancel{\pi} a^2}{2}\cdot\frac{1}{2\cancel{\pi}}\,d\Omega$$

$$= \boxed{\frac{1}{4}a^2\, d\Omega}$$ —— uniform distribution

---

Total cross-section:

$$\sigma_{tot} \equiv \sigma = \int_{\text{unit sphere}} d\sigma = \frac{1}{4}a^2\underbrace{\int_{\text{sphere}} d\Omega}_{=4\pi} = \boxed{\pi a^2}$$

$$\frac{d\sigma_1}{d\Omega_1} = \frac{d\sigma}{d\Omega}\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|$$

$$\frac{1}{4}a^2$$

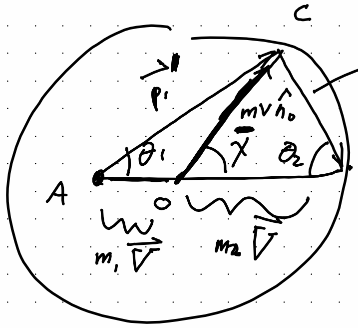
$$\frac{d\sigma_2}{d\Omega_2} = \frac{d\sigma}{d\Omega}\left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right|$$

$$\boxed{\begin{aligned}\chi &= \pi - 2\theta_2 \\ \cos\chi &= -\frac{m_1}{m_2}\sin^2\theta_1 \pm \cos\theta_1\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}\end{aligned}}$$

$$\sin\theta_1\, d\theta_1 = -d(\cos\theta_1)$$

Lecture #17 Tue, 10/20

- Quiz #3 — this week
- Q&A: $\vec{p}_1{}' + \vec{p}_2{}' = \vec{P}_{tot}{}'$
  $= \vec{P}_{tot}$



$\vec{p}_1 + \cancel{\vec{p}_2}^{\to 0}$
$= \vec{p}_1$

$m_1 < m_2,\ v_2 = 0$

$$m_1\vec{V} + m_2\vec{V} = (m_1+m_2)\vec{V} = \vec{P}_{tot}$$

$$\vec{V} = \frac{\vec{p}_1 + \cancel{\vec{p}_2}^{\to 0}}{m_1+m_2}$$

com velocity

$$= \frac{\vec{p}_1}{m_1+m_2} = \frac{m_1\vec{v}_1}{m_1+m_2}$$

$$m_1\vec{V} = m_1\left(\frac{m_1\vec{v}_1}{m_1+m_2}\right) = \left(\frac{m_1^2}{m_1+m_2}\right)\vec{v}_1$$

$$\vec{v}_2 = 0 \to \boxed{\vec{p}_2 = 0}$$

$$m_1 \vec{r_1} + m_2 \vec{r_2} = 0$$

$$\vec{r} = \vec{r_1} - \vec{r_2}$$

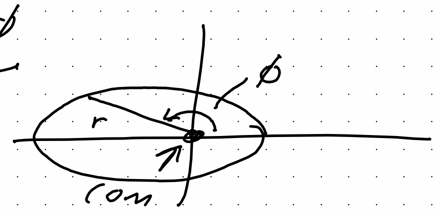
$$T = \frac{1}{2} m_1 |\dot{\vec{r_1}}|^2 + \frac{1}{2} m_2 |\dot{\vec{r_2}}|^2$$

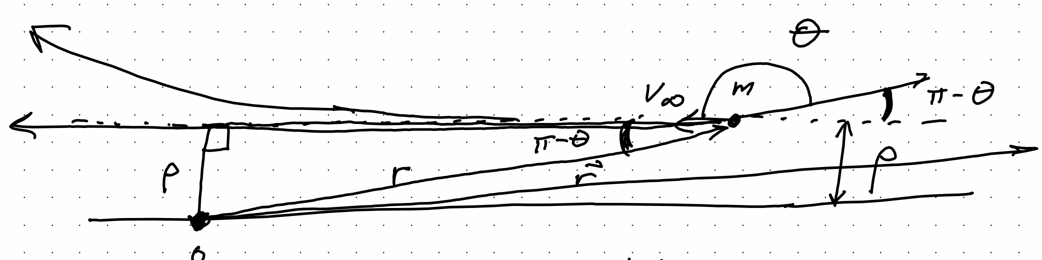
$$= \frac{1}{2} m |\dot{\vec{r}}|^2 \qquad m = \frac{m_1 m_2}{m_1 + m_2}$$

$\underbrace{r \text{ vs } \phi}$

$$\vec{r_1} = \frac{m_2}{m_1 + m_2} \vec{r}$$

$$\vec{r_2} = \frac{-m_1}{m_1 + m_2} \vec{r}$$





$$U = 0$$

$$M = \rho m v_\infty$$

$$\vec{M} = \vec{r} \times \vec{p} \qquad \vec{p} = m \vec{v_\infty}$$

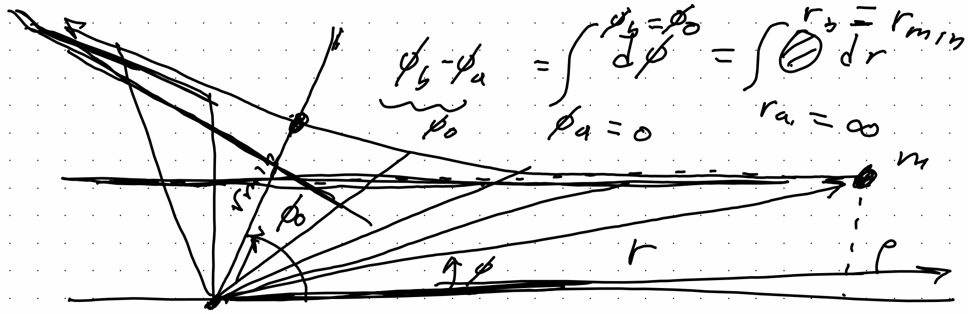
$$= m \vec{r} \times \vec{v_\infty}$$

$$M = |\vec{M}| = m r v_\infty \sin\theta$$

$$= m v_\infty \underbrace{r \sin(\pi - \theta)}_{\rho}$$

$$= \rho m v_\infty$$

$$\sin(\pi - \theta)$$
$$= \sin\pi \cos\theta - \cos\pi \sin\theta$$
$$= \sin\theta$$



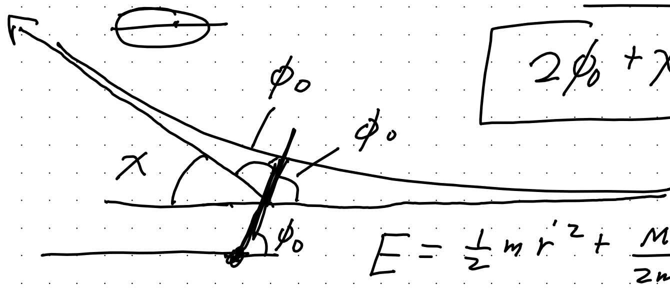
$$\underbrace{\phi_b-\phi_a}_{\phi_0} = \int_{\phi_a=0}^{\phi_b=\phi_0} d\phi = \int_{r_a=\infty}^{r_b=r_{min}} \oslash \, dr$$

$$U(r) = \frac{\alpha}{r}$$

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M\,dr/r^2}{\pm\sqrt{\underbrace{2m(E-U) - M^2/r^2}_{>0}}}$$

$M = \rho m v_\infty$

$E = \frac{1}{2} m v_\infty^2$



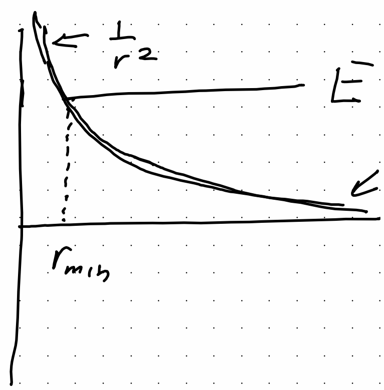
$$\boxed{2\phi_0 + \chi = \pi}$$

$$\boxed{\frac{v_\infty}{\rho}}$$

$\chi \leftrightarrow \rho$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} + U(r)$$

$$\dot{r} = \pm\sqrt{\ }$$



$U = \frac{\alpha}{r}$

$\alpha > 0$

coulomb scattering

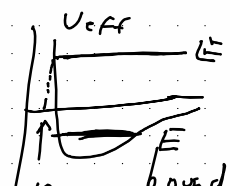
$+\,+$

$-\,-$

$$U_{eff} = U(r) + \frac{M^2}{2mr^2}$$

$$E = U_{eff}(r_{min})$$

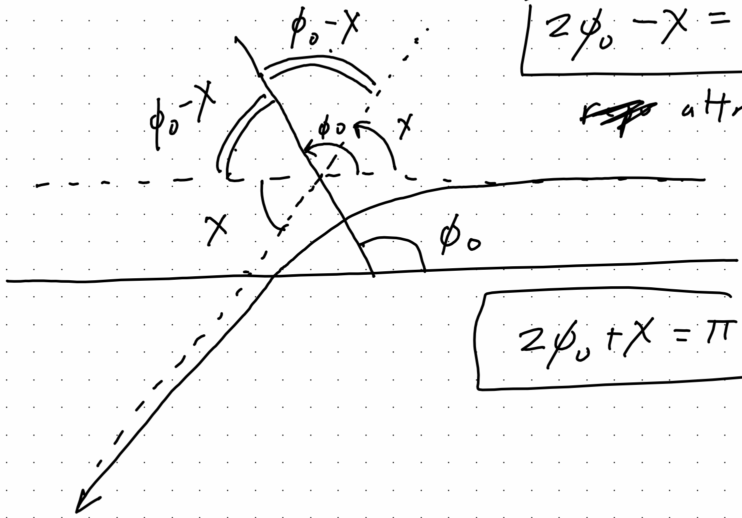
$U = -\frac{\alpha}{r}$



$2(\phi_0-\chi)+\chi=\pi$

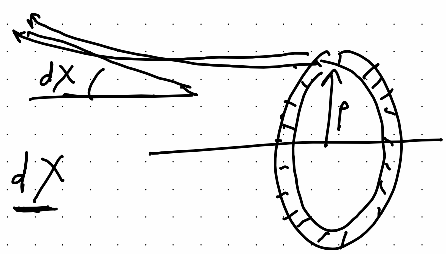
$\boxed{2\phi_0-\chi=\pi}$

~~rep~~ attractive



$\boxed{2\phi_0+\chi=\pi}$

$d\sigma = 2\pi\rho\, d\rho$



$= 2\pi\,\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi$

$= \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$

$d\rho>0$

$d\chi<0$

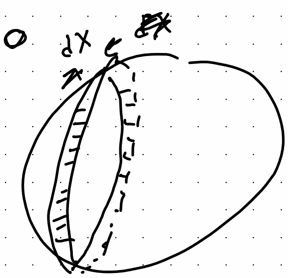
wrt COM frame

$d\Omega = 2\pi\sin\chi\, d\chi$

to go to lab frame

$\chi \rightarrow \theta_1$

or θ_2

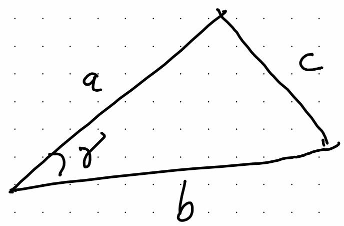


$d\sigma_1$, $d\sigma_2$

$d\theta_1$, $d\theta_2$, $d\Omega_1$, $d\Omega_2$

Lecture #18: Thurs 10/22

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$



$$\cos^2 x + \sin^2 x = 1$$

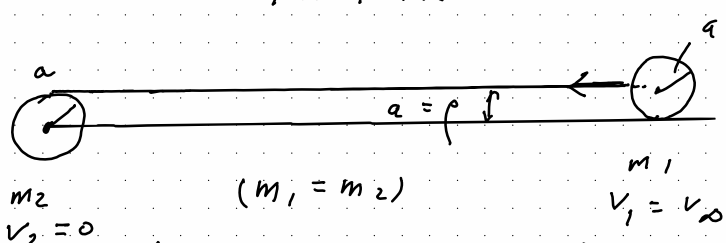
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

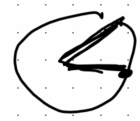
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

QUIZ #3:

name_q3.pdf

lab Frame





(hard sphere scattering)

1) calculate χ (scattering angle of the reduced mass w/t COM Frame)

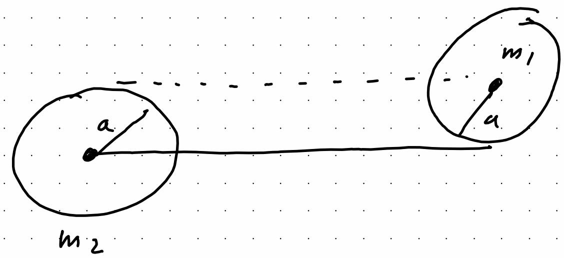
2) calculate θ_1, θ_2 (scattering angles w/t lab Frame)

$$\boxed{\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}}, \quad \boxed{\theta_2 = \frac{1}{2}(\pi - \chi)}$$

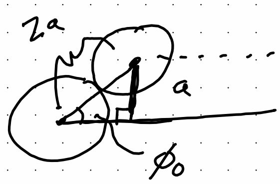
elastic scattering m_2 initially at rest

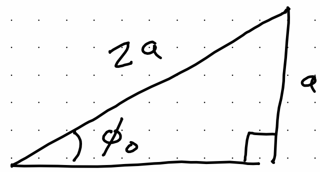
3) How do θ_1, θ_2 change if masses m_1 and m_2 change? ($m_1 >> m_2$; $m_1 << m_2$)

$$U(r) = \begin{cases} \infty & r < 2a \\ 0 & r > 2a \end{cases}$$



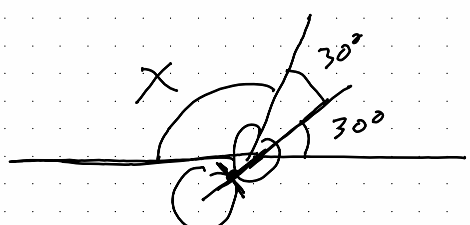
$$\vec{r} = \vec{r}_1 - \vec{r}_2$$





$$\sin\phi_0 = \frac{a}{2a} = \frac{1}{2}$$

$$\boxed{\phi_0 = \pi/6 = 30^\circ}$$



$$\boxed{\chi = 120^\circ} = \frac{2\pi}{3}$$

independent of m_1 and m_2

$$\frac{2\pi}{3} = 120^\circ$$

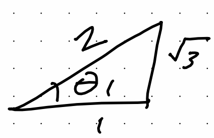
2) $\theta_2 = \frac{1}{2}(\pi - \chi) = \frac{1}{2}(180^\circ - 120^\circ) = \boxed{30^\circ}$ $\boxed{\theta_2}$

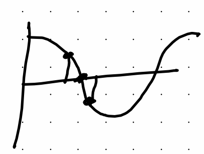
$$\tan\theta_1 = \frac{\cancel{m_2}\sin(120^\circ)}{\cancel{m_1} + \cancel{m_2}\cos(120^\circ)} = \frac{\sin(120^\circ)}{1 + \cos(120^\circ)}$$

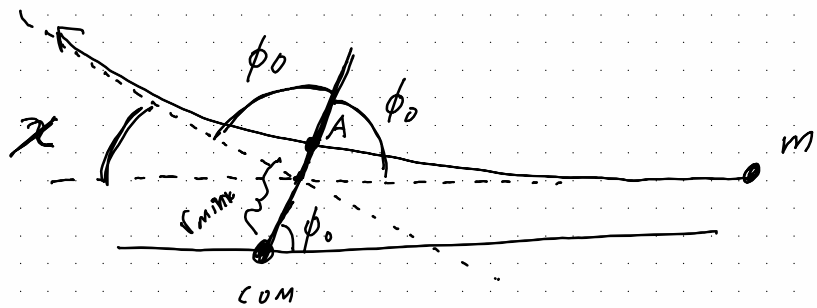
$$= \frac{\sqrt{3}/2}{1 - 1/2}$$

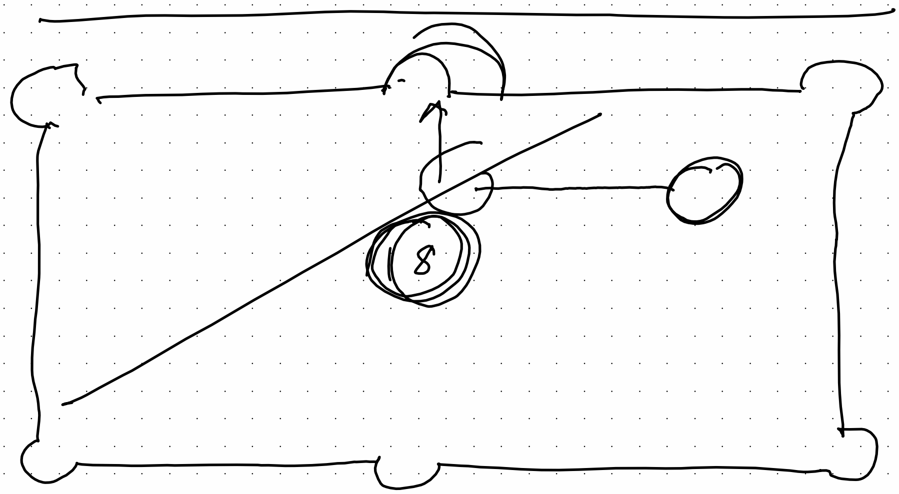
$$= \sqrt{3}$$

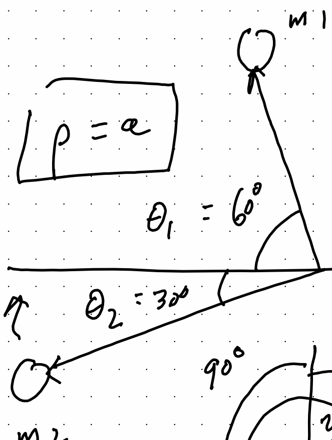
$$\boxed{\theta_1 = 60^\circ}$$

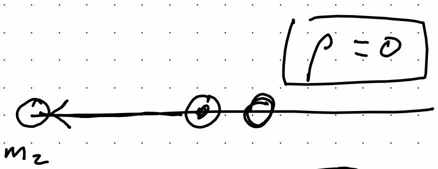




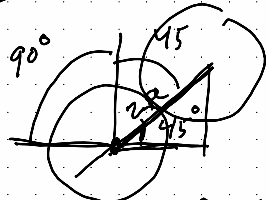




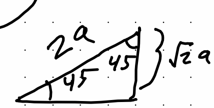




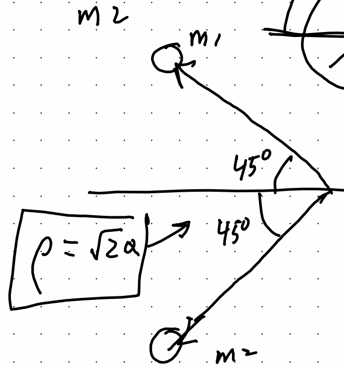
6 → 5



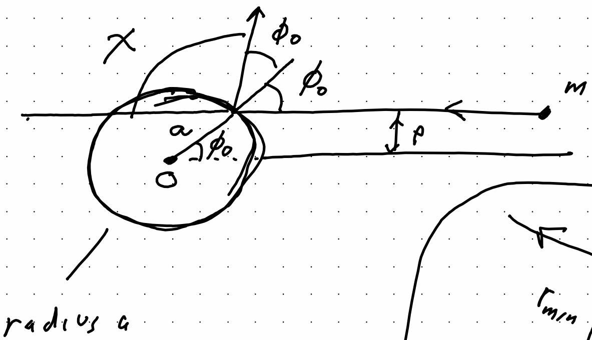
↑ increase : 1
decrease : 3



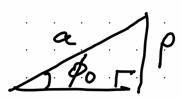
$\chi = \pi/2$



$\theta_2 = \frac{1}{2}(\pi - \chi)$
$= \frac{1}{2}(\pi - \frac{\pi}{2}) = \pi/4$
$\theta_1 = \pi/4$



$$U(r) = \begin{cases} 0 & r > a \\ \infty & r < a \end{cases}$$

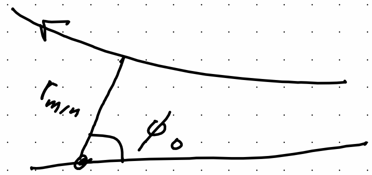


$$\sin\phi_0 = \frac{\rho}{a}$$

$$\phi_0 = \sin^{-1}(\rho/a)$$

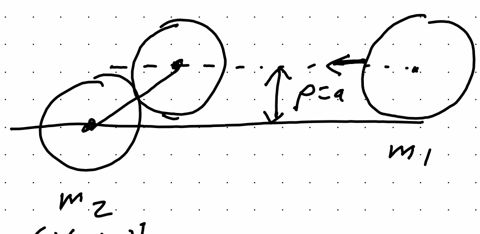
$$2\phi_0 + \chi = \pi$$

$$\boxed{\chi = \pi - 2\phi_0}$$

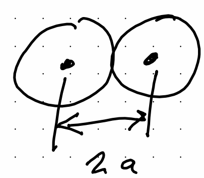


$$\boxed{\begin{aligned} \tan\theta_1 &= \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi} \\ \theta_2 &= \frac{1}{2}(\pi - \chi) \end{aligned}}$$

Prob 18.1

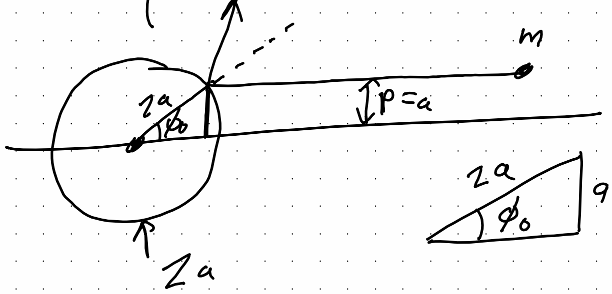


lab Frame



$m_1 = m_2$ $\rho = a$

$$U(r) = \begin{cases} 0 & r > 2a \\ \infty & r < 2a \end{cases}$$



COM Fram

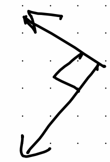
$$\sin\phi_0 = \frac{1}{2}$$

$$\phi_0 = 30°$$

$$\chi = \pi - 2\phi_0 \quad = 180^\circ - 2 \cdot 30^\circ = \boxed{120^\circ}$$

independent of m_1, m_2

$$\boxed{\begin{aligned} \tan\theta_1 &= \frac{m_2 \sin\chi}{m_1 + m_2 \cos\chi} \\ \theta_2 &= \tfrac{1}{2}(\pi - \chi) \end{aligned}}$$



$$\theta_2 = \tfrac{1}{2}(180^\circ - 120^\circ) = \boxed{30^\circ} \quad (\text{indep. of } m_1, m_2)$$

$$\tan\theta_1 = \frac{\sin(120^\circ)}{1 + \cos(120^\circ)} = \frac{\sqrt{3}/2}{1 - 1/2} = \sqrt{3}$$

$$\boxed{\theta_1 = 60^\circ}$$

$m_1 \ll m_2 \rightarrow \tan\theta_1 \approx \tan\chi \longrightarrow \boxed{\theta_1 = 120^\circ}$

$m_1 \gg m_2 \rightarrow \tan\theta_1 \approx \frac{m_2}{m_1}\sin\chi \longrightarrow \boxed{\theta_1 \approx 0^\circ}$

Lec #19: Tuesday 10/27

– Small oscillation (next 3 classes)

sec 21, 22, 23

(Free oscillation in 1-d) (Forced oscillation in 1-d) several dimensions

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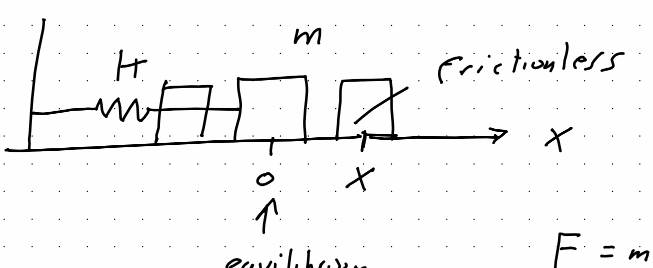


$F = -kx$

spring constant

$$F = m\ddot{x}$$
$$-kx = m\ddot{x}$$

$$\rightarrow \ddot{x} = -\frac{k}{m}x \longrightarrow x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

$\omega = \sqrt{\frac{k}{m}}$ (angular freq)

$$x(t) = \underbrace{c_1 \cos \omega t + c_2 \sin \omega t}$$ determined by ICs.

$$x(t) = a \cos(\omega t + \alpha)$$

a: amplitude; α: initial phase

$$x(t) = \mathrm{Re}\left[A e^{i\omega t} \right], \quad A = a e^{i\alpha}$$

A: complex

$$\mathrm{Re}\left[a e^{i\alpha} e^{i\omega t} \right] = \mathrm{Re}\left[a e^{i(\omega t + \alpha)} \right]$$

$$= \mathrm{Re}\left[a \left(\cos(\omega t + \alpha) + i \sin(\omega t + \alpha) \right) \right]$$

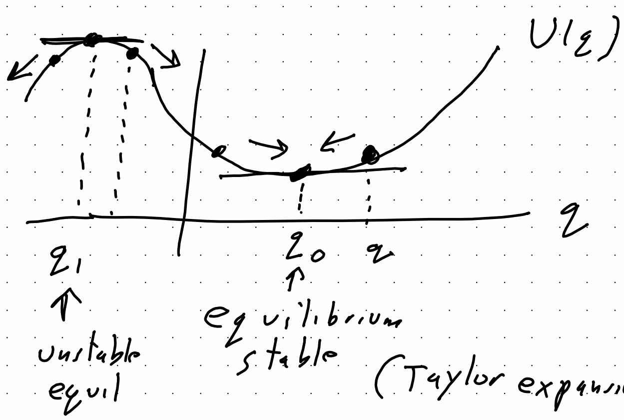
$$= a \cos(\omega t + \alpha)$$

Small oscillation:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \ldots$$

1-d system: q generalized coord

$$L = \frac{1}{2} a(q) \dot{q}^2 - U(q)$$

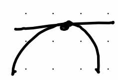


$$F = -\frac{dU}{dq}$$

$$F(q_0) = 0$$

$$\left. \frac{dU}{dq} \right|_{q_0} = 0$$

(Taylor expansion)



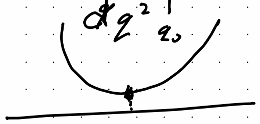
$$U(q) = \underline{U(q_0)} + \underbrace{\cancel{\left. \frac{dU}{dq} \right|_{q_0}}}_{0} (q - q_0) + \frac{1}{2} \left. \frac{d^2U}{dq^2} \right|_{q_0} (q - q_0)^2 + \ldots$$

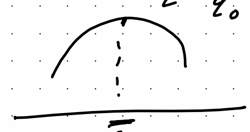
Small displacement away from equilibrium

$$\frac{|q-q_0|}{q_0} \ll 1 \quad (\text{ignore } O(3))$$

$$\boxed{U(q) \cong U(q_0) + \frac{1}{2}\left.\frac{d^2U}{dq^2}\right|_{q_0}(q-q_0)^2}$$

$$\left.\frac{d^2U}{dq^2}\right|_{q_0} > 0 \qquad\qquad \left.\frac{d^2U}{dq^2}\right|_{q_0} < 0$$





$$k = \left.\frac{d^2U}{dq^2}\right|_{q_0}$$

$$U(q) \cong \underbrace{U(q_0)}_{\text{const}} + \frac{1}{2}k\underbrace{(q-q_0)^2}_{x^2}$$

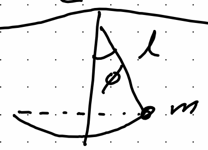
$$F = -\frac{dU}{dq}$$
$$= -k(q-q_0)$$
$$= -kx$$
$$U = \frac{1}{2}kx^2$$

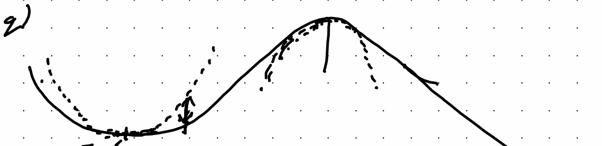
$$L = \frac{1}{2}a(q)\dot{q}^2 - U(q)$$
$$= \frac{1}{2}\underbrace{a(q_0)}_{m}\dot{x}^2 - \underbrace{U(q_0)}_{\text{const}} - \frac{1}{2}kx^2$$

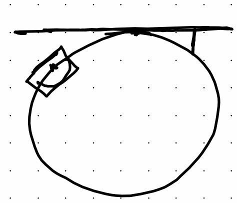
$$x = q - q_0 \rightarrow \dot{x} = \dot{q}$$

$$\boxed{L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2} \xrightarrow[\text{EOM}]{} m\ddot{x} = -kx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

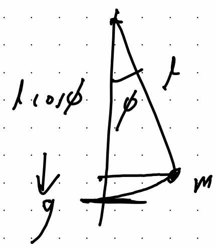






Example:

simple pendulum



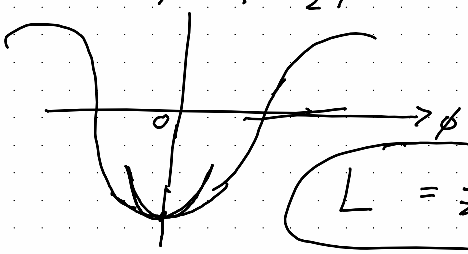
$$L = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl(1-\cos\phi)$$
$$= \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi$$
$$= T - U$$

$\uparrow y$

mgy

$$y = l - l\cos\phi$$
$$= l(1-\cos\phi)$$

$$U(\phi) = -mgl\cos\phi \Leftarrow -mgl\left(1-\frac{1}{2}\phi^2\right)$$
$$\cos\phi = 1 - \frac{1}{2}\phi^2 + \dots \qquad = \underbrace{-mgl}_{\text{const}} + \frac{1}{2} mgl\phi^2$$

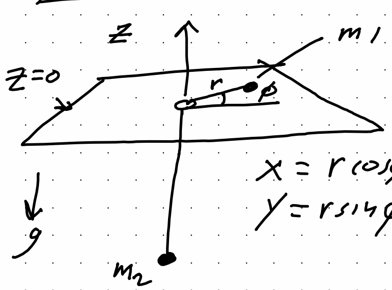


$$\omega = \sqrt{\frac{K}{M}} = \sqrt{\frac{\cancel{m}gl}{\cancel{m}l^2}} = \boxed{\sqrt{\frac{g}{l}}}$$

$\phi = 0$

$$\boxed{L = \frac{1}{2}\underbrace{ml^2}_{M}\dot{\phi}^2 - \frac{1}{2}\underbrace{mgl}_{K}\phi^2}$$

Example: 2-d problem



$$T = \frac{1}{2} m_1 (\overbrace{\dot{r}^2 + r^2\dot{\phi}^2}^{\dot{x}^2+\dot{y}^2}) + \frac{1}{2} m_2 \underbrace{\dot{z}^2}_{\dot{r}^2}$$

$$x = r\cos\phi$$
$$y = r\sin\phi$$

$$\boxed{T = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2}$$

string: length l

$$r + |z| = l$$
$$r - z = l$$
$$z = r - l$$
$$\rightarrow \dot{z} = \dot{r}$$

$$\boxed{U = m_2 g z} = m_2 g(r - \underset{\text{const}}{\underset{\uparrow}{l}})$$

$$L = T - U$$
$$\boxed{L = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r}$$

$\uparrow z \quad \downarrow \vec{g}$

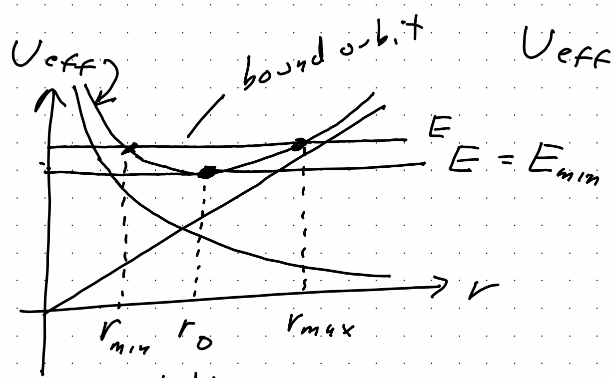
No ϕ dependence

$$\dot{\phi} = \frac{M_z}{m_1 r^2} \leftarrow M_z = \frac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} = \text{const}$$

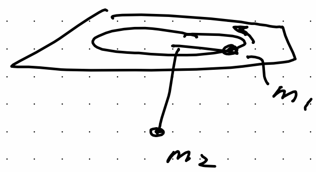
No explicit timedependence $\rightarrow$ $E = \text{const}$

$$E = T + U$$
$$= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\phi}^2 + m_2 g r$$
$$= \boxed{\frac{1}{2}(m_1+m_2)\dot{r}^2} + \underbrace{\frac{M_z^2}{2m_1 r^2} + m_2 g r}_{U_{eff}(r)} \quad - m_2 g \ell$$

not effect EOMs



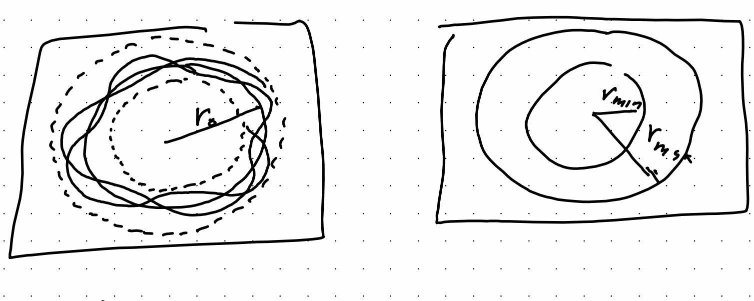
stable equilibrium



$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0}, \quad U_{eff}(r) = \frac{M_z^2}{2m_1 r^2} + m_2 g r$$
$$= \frac{-M_z^2}{m_1 r_0^3} + m_2 g$$

$$\boxed{M_z^2 = m_1 m_2 g r_0^3}$$

Small oscillations around r_0: $\quad \frac{|r-r_0|}{r_0} \ll 1$



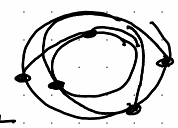
top view

$$\frac{dU_{eff}}{dr} = \frac{-M_z^2}{m_1 r^3} + m_2 g$$

$$\rightarrow \left.\frac{d^2 U_{eff}}{dr^2}\right|_{r_0} = \frac{3M_z^2}{m_1 r_0^4}, \qquad M_z^2 = m_1 m_2 g r_0^3$$

$$K = \frac{3\cancel{m_1} m_2 g r_0^3}{\cancel{m_1} r_0^4} = \boxed{\frac{3 m_2 g}{r_0}}$$

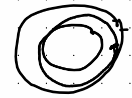
$$E = \frac{1}{2}\underline{(m_1+m_2)}\dot{r}^2 + \underbrace{U_{eff}(r_0)}_{\text{const}} + \frac{1}{2}K(r-r_0)^2$$



$$x = r - r_0$$

$$E = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}Kx^2 + \text{const}$$

$$\omega_r = \sqrt{\frac{K}{m_1+m_2}} = \boxed{\sqrt{\frac{3 m_2 g}{(m_1+m_2) r_0}}}$$



$$\omega_\phi \equiv \dot{\phi}\Big|_{r_0} \qquad M_z = m_1 r^2 \dot{\phi}$$

$$= \frac{M_z}{m_1 r_0^2}$$

$$= \frac{\sqrt{m_1 m_2 g r_0^3}}{m_1 r_0^2} = \frac{\sqrt{m_1 m_2 g r_0^3}}{\sqrt{m_1^2 r_0^4}}$$

$$= \boxed{\sqrt{\frac{m_2 g}{m_1 r_0}}}$$

$0 \rightarrow 2\pi$

not equal in general

$$\boxed{\omega_r = \sqrt{\frac{3 m_2 g}{(m_1+m_2) r_0}}}$$

For $\omega_r = \omega_\phi$

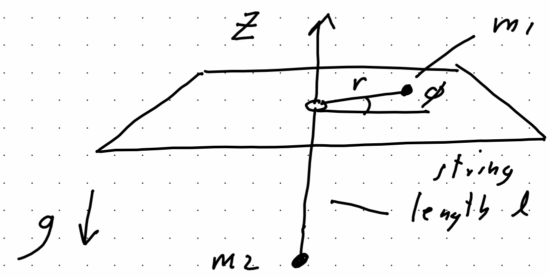
$$\frac{3}{m_1+m_2} = \frac{1}{m_1}$$

$$m_1 + m_2 = 3m_1$$

<u>closed bound orbits</u>

Lecture #20

- Quiz #4 – Tuesday
- Today: (i) Finish up example from last time
  (ii) Forced oscillations



$r = r_0$ (stable circular orbit)

$\omega_r = ?$

small oscillation $|r - r_0| << r_0$

$$\omega_r = \sqrt{\frac{3 m_2 g}{(m_1 + m_2) r_0}}, \quad \omega_\phi = \sqrt{\frac{m_2 g}{m_1 r_0}} = \dot\phi\Big|_{r_0}$$

$\omega_r = \omega_\phi \rightarrow$ closed orbit $\frac{3}{m_1 + m_2} = \frac{1}{m_1} \rightarrow 3m_1 = m_1 + m_2$

$m_1 = \frac{1}{2} m_2$

Forced oscillations: (1-d)

$$m\ddot{x} = -kx + F(t)$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}, \quad \omega \equiv \sqrt{\frac{k}{m}}$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + x F(t)$$

General sol'n:

$$x(t) = x_h(t) + x_p(t)$$

$x_h(t)$: general soln to $F(t) = 0$, $\underbrace{\qquad}_{a \cos(\omega t + \alpha)}$

$x_p(t)$: particular soln — any solution to the equation with RHS = $F(t)$

Example: $F(t) = f \cos(\gamma t + \beta)$

$$x_p(t) = b \cos(\gamma t + \beta) = \frac{f}{m}\left(\frac{1}{\omega^2 - \gamma^2}\right)$$

$$-b\gamma^2 \cancel{\cos(\gamma t + \beta)} + \omega^2 b \cancel{\cos(\gamma t + \beta)} = \frac{f}{m} \cancel{\cos(\gamma t + \beta)}$$

$$b = \frac{f}{m}\left(\frac{1}{\omega^2 - \gamma^2}\right)$$

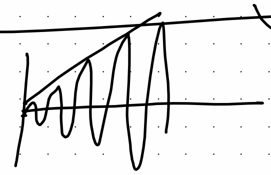
Resonance: $\gamma = \omega$

$$x(t) = a\cos(\omega t + \alpha) + \underbrace{\frac{f}{\underset{0}{m(\omega^2-\gamma^2)}}\left(\overset{0}{\cos(\gamma t+\beta) - \cos(\omega t+\beta)}\right)}_{\text{particular solution}}$$

$\frac{0}{0}$ L'Hopital's

$$\frac{\left.\frac{d}{d\gamma}(\text{num})\right|_{\gamma=\omega}}{\left.\frac{d}{d\gamma}(\text{den})\right|_{\gamma=\omega}} = \frac{+f\,t\sin(\omega t+\beta)}{+2m\omega}$$

$$\boxed{x(t) = a\cos(\omega t+\alpha) + \underbrace{\frac{f\,t}{2m\omega}\sin(\omega t+\beta)}}$$



after some time t, this term no longer $\longrightarrow$ invalidates small osc. approx.

General $F(t)$:

$$F(t) = \int_{-\infty}^{\infty} d\gamma\, \underbrace{\tilde{F}(\gamma)}_{\text{complex}}\, e^{i\gamma t}$$

Examples

$$F(t) = f_2\sin(2\omega t) + f_3\sin(3\omega t) + f_4\sin(4\omega t)$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m} \quad (2^{nd}\text{ order})$$

$$\left.\begin{aligned} \xi &= \dot{x} + i\omega x \\ \dot{\xi} &= \ddot{x} + i\omega\dot{x} \end{aligned}\right\} \rightarrow \boxed{\dot{\xi} - i\omega\xi = \frac{F(t)}{m}} \quad (1^{st}\text{ order})$$

$$y' + P(x)\,y = Q(x)$$

by Boas "Math method"

$$y' = \frac{dy}{dx}$$

$$\dot{\xi} - i\omega\xi = \frac{F(t)}{m}$$

$$\boxed{dy + (P(x)y - Q(x))dx = 0}$$ — not exact!

"Exact" differential:

If exact $1\,dy + ((P(x)y - Q(x))dx = dU(x,y)$

$$dU(x,y) = \left(\frac{\partial U}{\partial x}\right)dx + \left(\frac{\partial U}{\partial y}\right)dy$$

$$\rightarrow \frac{\partial^2 U}{\partial y\,\partial x} = \frac{\partial^2 U}{\partial x\,\partial y}$$

$$\frac{\partial 1}{\partial x} \stackrel{?}{=} P(x)$$

$$0 \neq P(x)$$

Claim:

$$\mu(x)\Big[\overbrace{dy + (P(x)y - Q(x))dx}^{=0 \text{ (original equation)}}\Big] = dU$$

integrating Factor

$$RHS = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

$$\mu(x) = \frac{\partial U}{\partial y}$$

$$\mu(x)\,(P(x)y - Q(x)) = \frac{\partial U}{\partial x}$$

$$\frac{\partial^2 U}{\partial x\,\partial y} = \frac{\partial^2 U}{\partial y\,\partial x} \rightarrow \mu'(x) = \mu(x)\,P(x)$$

$$\int \frac{\mu'}{\mu} = \int P(x)\,dx$$

$$\ln\mu = \int P(x)\,dx \rightarrow \mu = e^{\int P(x)dx}$$

$$I(x) = \int P(x)dx$$

$$\mu(x) = e^{I(x)}, \quad I(x) = \int dx\, P(x)$$

$$\frac{\partial U}{\partial y} = \mu(x)$$

$$U(x,y) = \mu(x)\, y + g(x) = C$$

const

$$\frac{\partial U}{\partial x} = \mu(x)\left(P(x)y - Q(x)\right)$$

$$\cancel{\mu'(x) y} + g'(x) = \mu(x)\left(\cancel{P(x) y} - Q(x)\right)$$

$$g'(x) = -\mu(x)\, Q(x)$$

$$\rightarrow \quad g(x) = -\int dx\, Q(x)\mu(x) = -\int dx\, Q(x) e^{I(x)}$$

$$\boxed{y = \frac{1}{\mu(x)}\left(C - g(x)\right) = e^{-I(x)}\left(C + \int dx\, Q(x) e^{I(x)}\right)}, \quad \mu(x) = e^{I(x)}, \quad I(x) = \int dx\, P(x)$$

$$y' + P(x) y = Q(x)$$

$$x(t) = \mathrm{Im}\left(\frac{\xi(t)}{\omega}\right)$$

$$\dot{x}(t) = \mathrm{Re}(\xi(t))$$

$$\dot{\xi} - i\omega\xi = \frac{F(t)}{m}$$

$$\xi = \dot{x} + i\omega x$$

$$y(x) \rightarrow \xi(t)$$

$$P(x) \rightarrow -i\omega$$

$$I(x) \rightarrow I(t) = -i\omega t$$

$$Q(x) \rightarrow F(t)/m$$

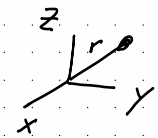
$$\boxed{\xi(t) = e^{i\omega t}\left[\xi_0 + \int_0^t d\bar{t}\, e^{-i\omega \bar{t}} F(\bar{t})/m\right]}$$

Lecture #21: Tues 11/3

Q4: on Thursday

Thursday — rigid body motion / non-inertial reference frames

Sec 23: Free oscillations in 2-d or higher



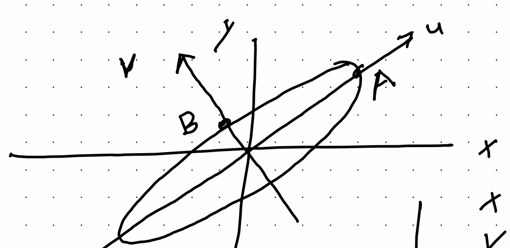
Prob 3, Sec 23: $U = \frac{1}{2} k r^2$ — space oscillator

motion is in 2-d plane (x, y)

$$L = T - U$$
$$= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k(x^2 + y^2)$$
$$= \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2\right) + \left(\frac{1}{2} m\dot{y}^2 - \frac{1}{2} ky^2\right)$$

2 independent oscillation in the x, y-directions

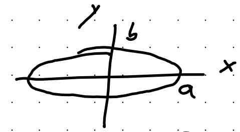
Ang freq: $\omega_x = \sqrt{\frac{k}{m}}$, $\omega_y = \sqrt{\frac{k}{m}}$ $\rightarrow \omega_x = \omega_y$



most general motion

$$x = a\cos(\omega t + \alpha)$$
$$y = b\sin(\omega t + \beta)$$

$\alpha = \beta = 0$

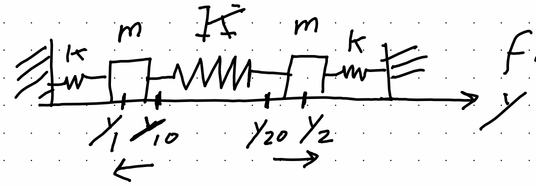


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Normal coordinates:

N independent oscillations with frequency $\omega_1, \omega_2, \ldots, \omega_N$

Example:



frictionless

$i, k = 1, 2, \cdots N$

$N = 2$

$$x_1 = y_1 - y_{10}$$
$$x_2 = y_2 - y_{20}$$

} small deviations away from equilibrium

$$(y_2 - y_1) - (y_{20} - y_{10}) = (y_2 - y_{20}) - (y_1 - y_{10}) = x_2 - x_1$$

$$L = T - U$$

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) = \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k$$

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} K (x_2 - x_1)^2$$

$$(x_1^2 + x_2^2 - 2 x_1 x_2)$$

$$= \frac{1}{2} (k + K) x_1^2 + \frac{1}{2} (k + K) x_2^2 - K x_1 x_2 = \frac{1}{2} \sum_{i,k} k_{ik} x_i x_k$$

$$T = \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$= \frac{1}{2} [m_{11} \dot{x}_1^2 + m_{12} \dot{x}_1 \dot{x}_2 + m_{21} \dot{x}_2 \dot{x}_1 + m_{22} \dot{x}_2^2]$$

$$= \frac{1}{2} [m_{11} \dot{x}_1^2 + m_{22} \dot{x}_2^2 + 2 m_{12} \dot{x}_1 \dot{x}_2]$$

$\rightarrow m_{11} = m, \quad m_{22} = m, \quad m_{12} = 0 = m_{21}$

$$m_{ik} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$U = \frac{1}{2} \sum_{i,k} k_{ik} x_i x_k = \frac{1}{2} (k + K) x_1^2 + \frac{1}{2} (k + K) x_2^2 - 2 K x_1 x_2$$

$$K_{ik} = \begin{bmatrix} k + K & -K \\ -K & k + K \end{bmatrix}$$

$$L = T - U$$

$$= \frac{1}{2}\sum_{j,k} m_{jk}\,\dot{x}_j\,\dot{x}_k - \frac{1}{2}\sum_{j,k} k_{jk}\,x_j\,x_k \qquad \Big|\ \delta_{ji},\ \delta_{ki}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \qquad (i = 1, 2, \cdots N)$$

$$\boxed{\sum_k m_{ik}\,\ddot{x}_k = -\sum_k k_{ik}\,x_k}$$

$$\underbrace{\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}}_{m}\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \begin{bmatrix} \ \\ \ \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} \dot{x}_1 & \dot{x}_2 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$\parallel$$

$$\frac{1}{2}\sum_{i,k} m_{ik}\,\dot{x}_i\,\dot{x}_k$$

$$\frac{1}{2}\sum_{j,l} m_{jl}\,\dot{x}_j\,\dot{x}_l$$

$$\frac{\partial}{\partial x_1}\left(k_{11}x_1^2 + k_{22}x_2^2 + 2k_{12}x_1x_2\right)\left(-\frac{1}{2}\right)$$

$$= -\left(k_{11}x_1 + k_{12}x_2\right) = -\sum_k k_{1k}\,x_k$$

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) &= \frac{\partial L}{\partial x_1} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) &= \frac{\partial L}{\partial x_2} \end{aligned}\right\} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \quad i = 1, 2$$

$$\sum_k m_{ik}\,\ddot{x}_k = -\sum_k k_{ik}\,x_k$$

$$\sum_k \left(m_{ik}\,\ddot{x}_k + k_{ik}\,x_k\right) = 0$$

Trial solution: $\quad x_k = A_k e^{i\omega t}$

$$\ddot{x}_k = -\omega^2 A_k e^{i\omega t}$$

$$\sum_k \Big(\underbrace{-m_{ik}\omega^2 + k_{ik}}_{\text{matrix}}\Big) A_k e^{i\omega t} = 0 \quad \text{0 vector}$$

$$\begin{bmatrix} \ & \ \\ \ & \ \end{bmatrix}\begin{bmatrix} \ \\ \ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$M^{-1}\left(M\,\underline{v} = \begin{bmatrix}0\\0\end{bmatrix}\right)$$

$$\underline{v} = M^{-1}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$K_{ik} - \omega^2 m_{ik}$ must be a singular matrix

$$\boxed{\det(K_{ik} - \omega^2 m_{ik}) = 0}$$ characteristic equation for angular freq

eigenvalues $\rightarrow \omega_1^2, \omega_2^2, \cdots, \omega_N^2$
(normal mode freqs)

eigenvector $\rightarrow \underline{v_1}, \underline{v_2}, \cdots, \underline{v_N}$
(normal mode oscillation)

$$\det\left(\begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} (k+K)-\omega^2 m & -K \\ -K & (k+K)-\omega^2 m \end{bmatrix} = 0$$

$$\bigcirc\,\omega^4 + \bigcirc\,\omega^2 + \bigcirc = 0$$

$\rightarrow$ quadratic equation for ω^2 $(\equiv \lambda)$

$$\bigcirc\,\lambda^2 + \bigcirc\,\lambda + \bigcirc = 0$$

solve quadratic equation: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\boxed{\begin{aligned} \omega_+^2 &= \frac{k+2K}{m} \\ \omega_-^2 &= \frac{k}{m} \end{aligned}}$$

$\underline{\omega_+^2}$:

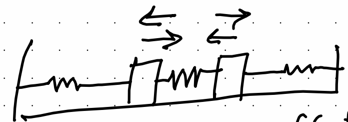
$$\begin{bmatrix} k+\bar{K}-m\omega_+^2 & -\bar{K} \\ -\bar{K} & k+\bar{K}-m\omega_+^2 \end{bmatrix} \underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_{V_+} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

V_+ : eigen vector assoc. with ω_+

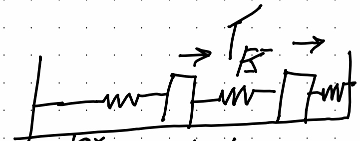
$A_2 = -A_1$

$$\boxed{V_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

Normal mode oscillation



unaffected



$\underline{\omega_-^2}$:

$$\boxed{V_- = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

complex constant determined by initial conditions

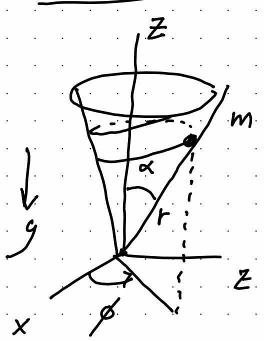
general solution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathrm{Re}\left(C_+ V_+ e^{i\omega_+ t} + C_- V_- e^{i\omega_- t} \right)$$

Lec 22: Nov 5^{th}

- Quiz #4 today (Quiz #5 - next thursday
  Quiz #6 - last day of class)
- Rigid body motion: Sec 31-36
  Non-inertial ref frames: Sec 38, 39

Q4: name_q4.pdf



a) Write down the Lagrangian
b) Find r_0 for a stable circular orbit for fixed angular momentum M_z.
c) Find angular freq of small radial oscillations about r_0.

$z = r\cos\alpha$, $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2 + \cancel{r^2\dot{\theta}^2}\right)$

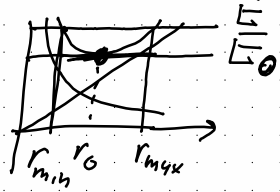
$$E = \frac{1}{2} m \dot{r}^2 + \underbrace{U_{eff}(r)}$$

$$M_z^2 = m^2 g r_0^3 \cos\alpha \sin^2\alpha$$

$$\frac{M_z^2}{2mr^2 \sin^2\alpha} + mgr\cos\alpha$$

$$M_z = mr^2 \sin^2\alpha\, \dot{\phi}$$

U_{eff}



$$\omega = \sqrt{\frac{k}{m}}$$

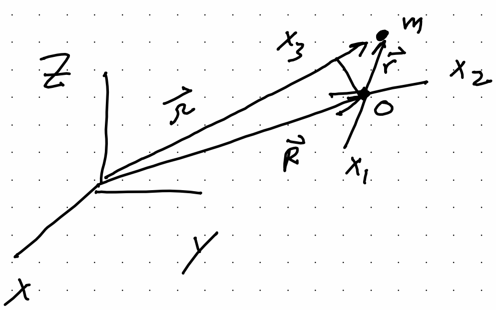
$$x = q - q_0$$

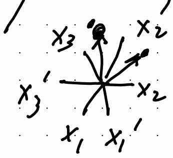
$$k = \left.\frac{d^2 U_{eff}}{dr^2}\right|_{r_0} = \frac{3mg\cos\alpha}{r_0}$$

$$\omega_r = \sqrt{\frac{k}{m}} = \boxed{\sqrt{\frac{3g\cos\alpha}{r_0}}}$$

$$\dot{\phi} = \frac{M_z}{mr^2\sin^2\alpha} = \frac{M_z}{mr_0^2\sin^2\alpha} = \frac{\sqrt{m^2 g r_0^3 \cos\alpha \sin^2\alpha}}{\sqrt{m^2 r_0^4 \sin^4\alpha}}$$

$$\omega_\phi = \sqrt{\frac{g}{r_0}\frac{\cos\alpha}{\sin^2\alpha}} \neq \omega_r$$





$\vec{V}$, $\vec{\Omega}$

linear velocity — depend on O

ang velocity — does not depend on O

angular velocity vector

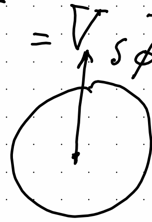
$$\vec{\rho} = \vec{R} + \vec{r}$$

$$\boxed{\vec{v} = \frac{d\vec{\rho}}{dt} = \vec{V} + \underbrace{\vec{\Omega} \times \vec{r}}}$$

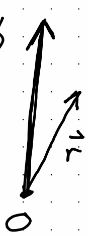
translational velocity of O

rotational velocity

$$\vec{R} \to \vec{R} + d\vec{R}, \quad \frac{d\vec{R}}{dt} = \vec{V}\,\delta\vec{\phi}$$



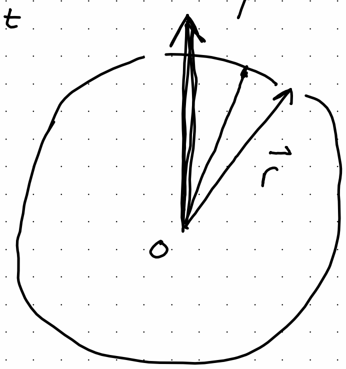
$\delta\vec{\phi}$



infinitesimal rotational about an axis through an infinitesimal angle $\delta\phi$

$\textcircled{v} = \dfrac{dx}{dt}$

$\delta\vec{\phi}$



$\vec{V} = \dfrac{d\vec{R}}{dt}$ — 3 DOF associated with origin of body frame

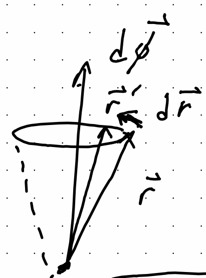
$d\vec{R}$ O

O

(ϕ, θ, ψ)

$d\vec{R}$ $\quad \vec{\phi}$: 3 DOF assoc with orientation of body frame wrt inertial frame

$\dfrac{\to 0}{\to 0}$



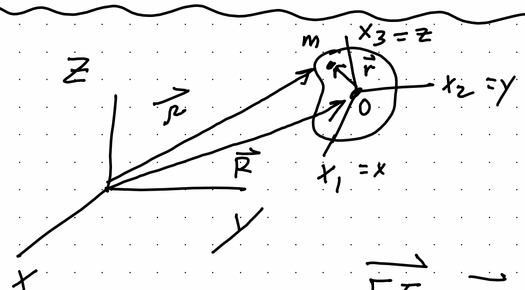
$d\vec{r} = d\vec{\phi} \times \vec{r}$

$\dfrac{d\vec{r}}{dt} = \underbrace{\dfrac{d\vec{\phi}}{dt}}_{\vec{\Omega}} \times \vec{r}$

$$\boxed{\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}}$$

Lec #23: Tuesday Nov 10th

- Quiz #5: Thursday
- Midterm #2: Next Thursday 11/19 (Scattering, small oscillations, some rigid body)
- Today: Rigid body motion (Sec 31–36, 38, 39)
  - 38: non-inertial
  - 31–36: static equilibrium



O: at COM (usually)

x_1, x_2, x_3: fixed in RB

$\left.\begin{aligned} \vec{V} &= \frac{d\vec{R}}{dt} \\ \vec{\Omega} &= \frac{d\vec{\phi}}{dt} \end{aligned}\right\} \dot{q}_i$

$\vec{v} = \dfrac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}$

$\vec{\Omega}$: angular velocity vector

$(\vec{R}, \vec{\phi})$: 6 DOF $= q_i$

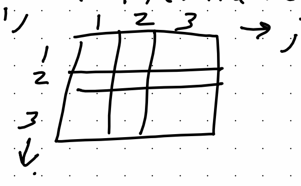
rotational quantities

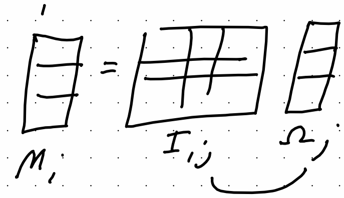
$T = \frac{1}{2} m v^2$, $T_{rot} = \frac{1}{2} I \Omega^2$

$\vec{p} = m \vec{v}$, $\vec{M} = I \vec{\Omega} \rightarrow M_i = \sum_j I_{ij} \Omega_j$

I : moment of inertia

$\longrightarrow I_{ij}$: inertia tensor





KE:

$T = \frac{1}{2} \sum_a m_a |\vec{v}_a|^2$, $\vec{v}_a = \vec{V} + \vec{\Omega} \times \vec{r}_a$

$= \frac{1}{2} \sum_a m_a |\vec{V} + \vec{\Omega} \times \vec{r}_a|^2$

$|\vec{A} + \vec{B}|^2 = A^2 + B^2 + 2\vec{A}\cdot\vec{B}$

$= \frac{1}{2} \sum_a m_a \left(|\vec{V}|^2 + |\vec{\Omega} \times \vec{r}_a|^2 + 2 \vec{V} \cdot (\vec{\Omega} \times \vec{r}_a) \right)$

① $= \frac{1}{2} \sum_a m_a |\vec{V}|^2 = \boxed{\frac{1}{2} \underbrace{\mu}_{\text{total mass}} V^2}$

③ $= \sum_a m_a \vec{V} \cdot (\vec{\Omega} \times \vec{r}_a)$

$= \left(\sum_a m_a \vec{r}_a \right) \cdot (\vec{V} \times \vec{\Omega})$

$= \underbrace{\mu \vec{R}_{com}}_{=0 \text{ for } \boxed{O \text{ at COM}}} \cdot (\vec{V} \times \vec{\Omega})$

$\vec{A} \cdot (\vec{B} \times \vec{C})$
$= \vec{B} \cdot (\vec{C} \times \vec{A})$
$= \vec{C} \cdot (\vec{A} \times \vec{B})$

② $= \frac{1}{2} \sum_a m_a |\vec{\Omega}\times\vec{r_a}|^2$

$= \frac{1}{2} \sum_a m_a \left(\vec{\Omega}\times\vec{r_a}\right)\cdot\left(\vec{\Omega}\times\vec{r_a}\right)$

$= \frac{1}{2} \sum_a m_a \ \vec{\Omega}\cdot\left(\vec{r_a}\times(\vec{\Omega}\times\vec{r_a})\right)$

$= \frac{1}{2} \sum_a m_a \ \vec{\Omega}\cdot\left(\vec{\Omega}\, r_a^2 - \vec{r_a}\,(\vec{r_a}\cdot\vec{\Omega})\right)$

$= \frac{1}{2} \sum_a m_a \left(\Omega^2 r_a^2 - (\vec{\Omega}\cdot\vec{r_a})^2\right)$

$= \frac{1}{2} \sum_a m_a \left(\sum_{i,j} \Omega_i \Omega_j \delta_{ij} r_a^2 - \sum_{i,j} \Omega_i r_{ai} \Omega_j r_{aj}\right)$

$= \frac{1}{2} \sum_{i,j} \left(\sum_a m_a \left(r_a^2 \delta_{ij} - r_{ai} r_{aj}\right)\right) \Omega_i \Omega_j$

$= \boxed{\frac{1}{2} \sum_{i,j} I_{ij}\, \Omega_i \Omega_j}$ $\left(= \frac{1}{2}\right)$

$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

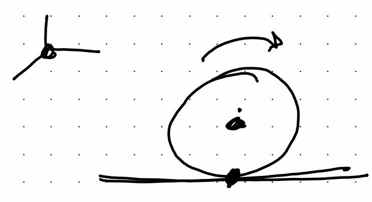
$$\boxed{T = \underbrace{\frac{1}{2}\mu V^2}_{\text{trans}} + \underbrace{\frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i \Omega_j}_{\text{rotational}}}$$

$+$ [illegible]

$\left(\frac{1}{2} I \Omega^2\right)$ Freshman physics

for COM at origin OF RB Frame

$\vec{M}$: wrt COM of body



$\vec{M} = \sum_a \vec{r_a}\times\vec{p_a}$

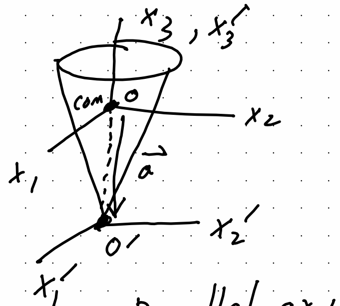
$= \sum_a m_a \vec{r_a}\times\vec{v_a}$

$= \sum_a m_a \vec{r_a}\times\left(\vec{V} + \vec{\Omega}\times\vec{r_a}\right)$

$= \sum_a m_a \vec{r_a}\times\left(\vec{\Omega}\times\vec{r_a}\right)$

$= \sum_a m_a \left(\vec{\Omega}\, r_a^2 - \vec{r_a}\,(\vec{r_a}\cdot\vec{\Omega})\right)$

$\boxed{M_i = \sum_j I_{ij}\,\Omega_j}$ $\vec{M} = I\vec{\Omega}$ (Fresh. physics)



I'_{ij} , I_{ij}

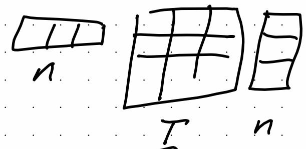
$\underbrace{I'_{ij}}_{\text{wrt } O'}$, $\underbrace{I_{ij}}_{\text{wrt } O \text{ (com)}}$

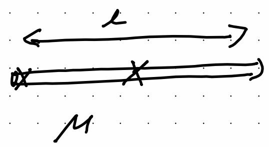
parallel-axis theorem

$$I'_{ij} = I_{ij} + \mu(a^2\delta_{ij} - a_i a_j)$$

$\vec{a}$: vector from O to O'

$$\underbrace{I(\hat{n})}_{\text{moment of inertia}} = \sum_{i,j} I_{ij} \underbrace{n_i n_j}_{\hat{n}:\ \text{axis of rotation}}$$





$$\boxed{\begin{aligned} I_{com} &= \frac{1}{12}\mu l^2 \\ I_{end} &= \frac{1}{3}\mu l^2 \end{aligned}}$$

$$\boxed{\underbrace{I_{ij}}_{3\times 3 \text{ real symmetric}} = \sum_a m_a (r_a^2\delta_{ij} - r_{ai} r_{aj}) = \int \underbrace{\rho\, dV}_{dm} (r^2\delta_{ij} - r_i r_j)}$$

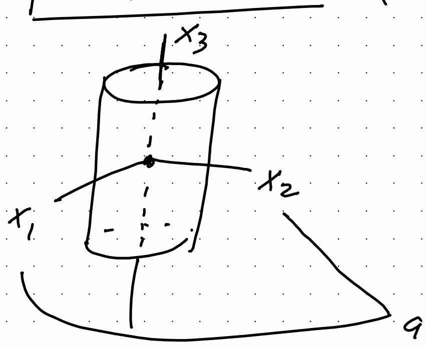
can always be diagonalized

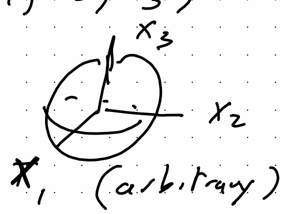
I_1	0	0
0	I_2	0
0	0	I_3

$$I_{ij} = I_i \delta_{ij}$$

$$T_{rot} = \frac{1}{2}\sum_{i,j} I_{ij}\Omega_i\Omega_j = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)$$

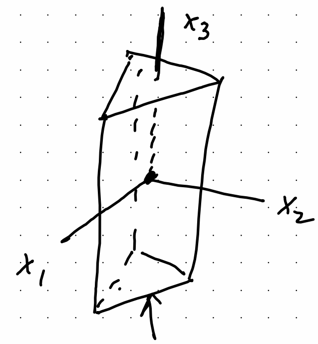
principle axes: (x_1, x_2, x_3)



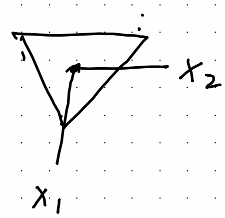


$$M_i = \sum_j I_{ij}\Omega_j$$

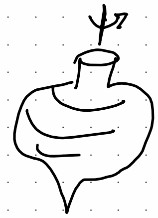
$$M_1 = I_1\Omega_1, \quad M_2 = I_2\Omega_2$$
$$M_3 = I_3\Omega_3$$

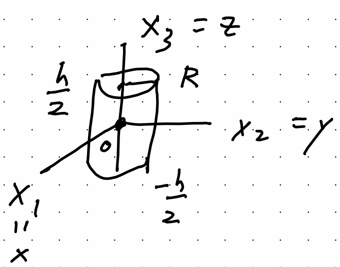


equilateral triang



$$\underbrace{I_1 = I_2 \neq I_3}_{\text{symmetrical top}}$$





total mass M

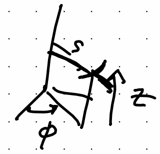
$$\rho = \frac{M}{\text{volume}} = \frac{M}{\pi R^2 h}$$

$$I_3 = I_{33} = \int \rho\, dV \left(r^2 \underbrace{\delta_{33}}_{1} - \underbrace{r_3 r_3}_{z^2} \right)$$

$$= \int \rho\, dV \left(r^2 - z^2 \right)$$

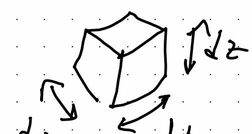
$$= \int \rho\, dV \left(x^2 + y^2 \right) = \int \rho\, dV s^2$$

cylindrical: s, ϕ, z $\quad s^2 = x^2 + y^2$



$$dV = ds\ s\,d\phi\ dz$$

$$= s\, ds\, d\phi\, dz$$

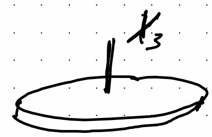


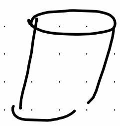
$$I_3 = \int \rho\, dV\, s^2$$

$$= \frac{M}{\pi R^2 h} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-h/2}^{h/2} dz}_{h} \underbrace{\int_0^{R} s^3\, ds}_{\frac{R^4}{4}}$$

$$= \frac{M}{\cancel{\pi} R^{\cancel{2}} \cancel{h}} \cdot 2\cancel{\pi}\, \cancel{h}\, \frac{R^{\cancel{4} \to 2}}{4}$$

$$= \boxed{\frac{1}{2} M R^2}$$





$$I_1 = I_2 \equiv I$$

$$I_1 = \int \rho\, dV (r^2 - x^2)$$

$$+\ I_2 = \int \rho\, dV (r^2 - y^2)$$

$$2I = \int \rho\, dV (2r^2 - x^2 - y^2)$$

$$\left(\begin{array}{l} r^2 = s^2 + z^2 \\ x^2 + y^2 = s^2 \end{array} \right)$$

$$2I = \int \rho\, dV (s^2 + 2z^2)$$

$$\boxed{I} = \frac{1}{2} \underbrace{\int \rho\, dV\, s^2}_{I_3} + \int \rho\, dV\, z^2$$

$$\boxed{= \frac{1}{2} I_3 + \int \rho\, dV\, z^2}$$

easier to evaluate

$$\int \rho\, dV\, z^2 = \frac{M}{\pi R^2 h} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-h/2}^{h/2} dz \cdot z^2}_{\frac{z^3}{3}\Big|_{-h/2}^{h/2}} \underbrace{\int_0^R s\, ds}_{\frac{R^2}{2}}$$

$$\frac{z^3}{3}\Big|_{-h/2}^{h/2} = \frac{2}{3}\left(\frac{h}{2}\right)^3 = \frac{h^3}{12}$$

$$= \frac{M}{\cancel{\pi} \cancel{R^2} h} \cdot \cancel{2}\cancel{\pi} \cdot \frac{h^3}{12} \cdot \frac{\cancel{R^2}}{\cancel{2}}$$

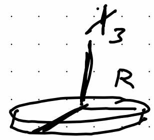
$$= \boxed{\frac{M h^2}{12}}$$

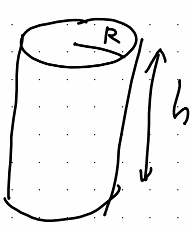
$$I = \frac{1}{2}\left(\frac{1}{2} M R^2\right) + \frac{M h^2}{12}$$

$$= \frac{1}{4} M R^2 + \frac{1}{12} M h^2$$

$$= \boxed{\frac{1}{4} M \left(R^2 + \frac{1}{3} h^2\right)} = I_1, I_2$$

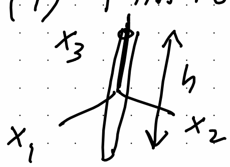
$$\boxed{I_3 = \frac{1}{2} M R^2}$$





Limiting cases: (i) Disk ($h \to 0$) $I_3 = \frac{1}{2} M R^2$, $I_1, I_2 = \frac{1}{4} M R^2$

(i) thin rod ($R \to 0$)



$I_3 = 0$

$I_1 = I_2 = \frac{1}{12} M h^2$

$$L = T - U$$

$$= \frac{1}{2} \mu V^2 + \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j - U$$

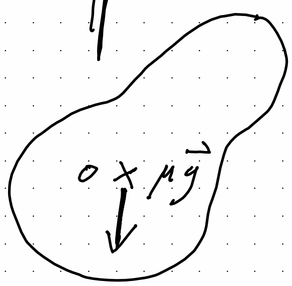
$$\vec{V} = \dot{\vec{R}}$$

$$\vec{\Omega} = \dot{\vec{\phi}}$$

$$L(\vec{R}, \vec{\phi}, \dot{\vec{R}}, \dot{\vec{\phi}})$$

$\vec{g} \downarrow$

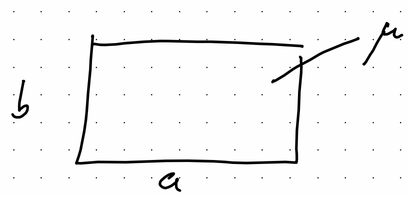
uniform field



Lecture #24: Thursday 11/12

- Quiz #5 (today)
- Midterm #2 (next Thursday) (scattering, small oscillations, RB motion)
- Today's topics:
  (1) RB EOMs
  (2) Euler's equations
  (3) Euler angles

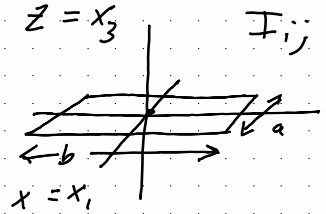
Q5: Calculate the principal moments of inertia for a uniform 2-d rectangle with sidelengths a, b.

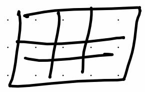


name_q5.pdf

$z = x_3$ $\qquad I_{ij} = \int \rho\, dV \left(r^2 \delta_{ij} - r_i r_j \right)$

$r^2 = x^2 + y^2 + z^2$





$$I_1 = \int \rho\, dV \left(r^2 - x^2 \right)$$

$I_1 \,\|\, I_{11}$

$$= \int \rho\, dV\, y^2$$

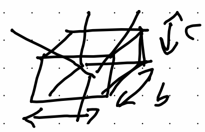
$$= \frac{\mu}{ab} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\, y^2$$

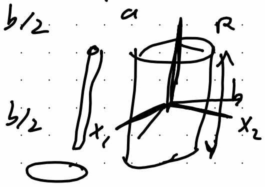
$$= \frac{\mu}{ab}\, a\, \frac{y^3}{3} \Big|_{-b/2}^{b/2}$$

$$= \frac{\mu}{b} \frac{2}{3} \left(\frac{b}{2} \right)^3$$

$$= \frac{\mu}{b} \frac{2}{3} \frac{b^3}{8} = \boxed{\frac{1}{12} \mu b^2}$$

$$dm = \rho\, dV = \sigma\, dx\, dy = \frac{\mu}{ab}\, dx\, dy$$





$$\boxed{I_2 = \frac{1}{12} \mu a^2}$$

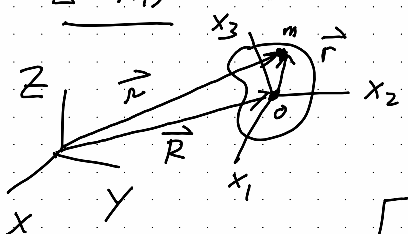
$$I_3 = \int \sigma\, dx\, dy \left(x^2 + y^2 \right) = I_1 + I_2$$

$$\boxed{I_3 = \frac{1}{12} \mu \left(a^2 + b^2 \right)}$$

EOMs: $\qquad$ Final result:

$$\frac{d\vec{P}}{dt} = \vec{F} = \sum \vec{f}$$

$$\frac{d\vec{M}}{dt} = \vec{K} = \sum \vec{r} \times \vec{f}$$



$$\vec{\rho} = \vec{R} + \vec{r}, \quad \boxed{\delta\vec{\rho} = \delta\vec{R} + \delta\vec{\phi} \times \vec{r}}$$

$\delta\Omega_k$

$$L = T - U$$

$$= \frac{1}{2} \mu |\vec{V}|^2 + \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j - U(\vec{\rho})$$

$$\delta L = \mu \vec{V} \cdot \delta\vec{V} + \sum_{i,j} I_{ij} \Omega_j \delta\Omega_i - \sum \frac{\partial U}{\partial \vec{\rho}} \cdot \delta\vec{\rho}$$

$$- \sum \frac{\partial U}{\partial \vec{\rho}} \cdot \left(\delta\vec{R} + \delta\vec{\phi} \times \vec{r} \right)$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{V}} \right) = \frac{\partial L}{\partial \vec{R}}}, \quad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{\Omega}} \right) = \frac{\partial L}{\partial \vec{\phi}}}$$

$$\frac{1}{2}\sum_{i,j} I_{ij}\,\underline{\delta\Omega_i}\,\Omega_j + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\delta\Omega_j$$

$$= \quad '' \quad + \frac{1}{2}\sum_{j,i} I_{ji}\,\Omega_j\,\underline{\delta\Omega_i}$$

$$= \quad '' \quad + \frac{1}{2}\underbrace{\sum_{j,i}}_{\text{swap}} I_{ij}\,\Omega_j\,\delta\Omega_i$$

$$= \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i + \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i$$

$$= \boxed{\sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i}$$

$$\delta L = m\vec{V}\cdot\underline{\delta\vec{V}} + \sum_{i,j} I_{ij}\,\Omega_j\,\delta\Omega_i$$

$\vec{A}\cdot(\vec{B}\times\vec{C})$
$\vec{B}\cdot(\vec{C}\times\vec{A})$

$$-\left(\sum \frac{\partial U}{\partial \vec{r}}\right)\cdot\delta\vec{R} - \underbrace{\sum \frac{\partial U}{\partial \vec{r}}\cdot\left(\delta\vec{\phi}\times\vec{r}\right)}_{}$$

$$= -\sum \delta\vec{\phi}\cdot\left(\vec{r}\times\frac{\partial U}{\partial \vec{r}}\right)$$

$$= -\,\delta\vec{\phi}\cdot\left(\sum \vec{r}\times\frac{\partial U}{\partial \vec{r}}\right)$$

$$\sum_i M_i\,\delta\Omega_i = \vec{M}\cdot\delta\vec{\Omega}$$

$$dF = \bigcirc\,dx + \bigcirc\,dy$$

$$\boxed{\delta L = \vec{P}\cdot\delta\vec{V} + \vec{M}\cdot\delta\vec{\Omega} + \left(\sum\vec{f}\right)\cdot\delta\vec{R} + \delta\vec{\phi}\cdot\left(\sum\vec{r}\times\vec{f}\right)}$$

$$\vec{P} = \frac{\partial L}{\partial \vec{V}},\quad \vec{M} = \frac{\partial L}{\partial \vec{\Omega}},\quad \sum\vec{f} = \frac{\partial L}{\partial \vec{R}},\quad \sum\vec{r}\times\vec{f} = \frac{\partial L}{\partial \vec{\phi}}$$

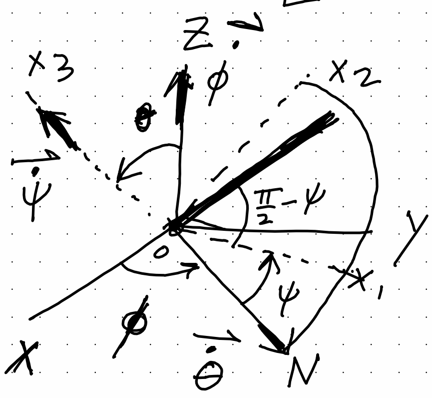
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{V}}\right) = \frac{\partial L}{\partial \vec{R}} \rightarrow \boxed{\frac{d\vec{P}}{dt} = \sum\vec{f} \equiv \vec{F}} \quad\Bigg|\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{\Omega}}\right) = \frac{\partial L}{\partial \vec{\phi}} \rightarrow \boxed{\frac{d\vec{M}}{dt} = \sum\vec{r}\times\vec{f} = \vec{K}}$$

Euler's equations / Euler angles ✩

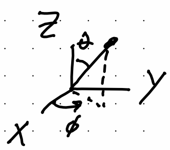
$\vec{R}, \vec{\phi}$

Ω_i : ith component of $\vec{\Omega}$

L wrt $\hat{x}_i$



(line of node)



$$\vec{\Omega} = \vec{\dot{\phi}} + \vec{\dot{\theta}} + \vec{\dot{\psi}}$$

$$\vec{\dot{\psi}} = \dot{\psi}\,\hat{x}_3$$

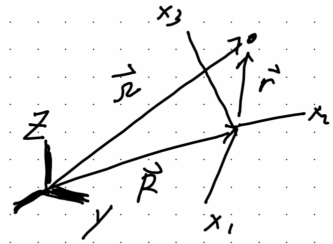
$$\vec{\dot{\theta}} = \dot{\theta}\cos\psi\,\hat{x}_1 - \dot{\theta}\sin\psi\,\hat{x}_2$$

$$\vec{\dot{\phi}} = \dot{\phi}\cos\theta\,\hat{x}_3 + \dot{\phi}\sin\theta\Big(\underbrace{\sin\psi}_{\cos(\frac{\pi}{2}-\psi)}\hat{x}_1 + \underbrace{\cos\psi}_{\sin(\frac{\pi}{2}-\psi)}\hat{x}_2\Big)$$

Announcements

- Midterm II is this thursday
- Today:
  - i) Euler angles
  - ii) Euler's equation for RB motion
  - iii) Free rotation with $\vec{\Omega} = $ const
  - iv) " of a symmetric top ($I_1 = I_2$)
  - v) Heavy symmetrical top with one point fixed [prob 35.1]

$$\vec{\Omega} = (\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)\,\hat{x}_1 + (\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)\,\hat{x}_2 + (\dot\psi + \dot\phi \cos\theta)\,\hat{x}_3$$



$$\boxed{\begin{aligned} \Omega_1 &= \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi \\ \Omega_2 &= \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi \\ \Omega_3 &= \dot\psi + \dot\phi \cos\theta \end{aligned}}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega}\times\vec{r} + \underbrace{\frac{d'\vec{r}}{dt}}_{\text{wrt RB}}$$

Euler's equations: (wrt RB axes) $\vec{A}$: any vector

$$\frac{d\vec{P}}{dt} = \sum \vec{f} \equiv \vec{F}$$

$$\frac{d\vec{M}}{dt} = \sum \vec{r}\times\vec{f} = \vec{K}$$

$$\boxed{\underbrace{\frac{d\vec{A}}{dt}}_{\text{wrt inertial frame}} = \underbrace{\frac{d'\vec{A}}{dt}}_{\text{wrt rotating frame}} + \underbrace{\vec{\Omega}}_{\text{angular velocity of the rotating frame}}\times\vec{A}}$$

$$\underbrace{\left(\frac{d'\vec{A}}{dt}\right)_i}_{\text{cartesian components wrt rotating frame}} = \frac{dA_i}{dt} \equiv \dot{A}_i$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}\,\hat{x} + \ddot{y}\,\hat{y} + \ddot{z}\,\hat{z}$$

$$\cancel{= \ddot{r}\,\hat{r}}$$

$$\vec{A} = \sum_i A_i\,\hat{x}_i$$

$$\frac{d\vec{A}}{dt} = \underbrace{\sum_i \left(\frac{dA_i}{dt}\right)\hat{x}_i}_{\frac{d'\vec{A}}{dt}} + \underbrace{\sum_i A_i \frac{d\hat{x}_i}{dt}}_{\vec{\Omega}\times\vec{A}}$$

$$\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\Omega}\times\vec{A}\ , \quad \left(\frac{d'\vec{A}}{dt}\right)_i = \dot{A}_i$$

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d'\vec{P}}{dt} + \vec{\Omega}\times\vec{P}$$

$$\vec{P} = \mu\vec{V}$$

$$\rightarrow \boxed{\begin{aligned} F_1 &= \dot{P}_1 + (\vec{\Omega}\times\vec{P})_1 \\ &= \dot{P}_1 + \Omega_2 P_3 - \Omega_3 P_2 \\ &= \mu(\dot{V}_1 + \Omega_2 V_3 - \Omega_3 V_2) \end{aligned}}$$

similar equations for F_2, F_3

$$\boxed{\vec{K} = \frac{d\vec{M}}{dt}} = \frac{d'\vec{M}}{dt} + \vec{\Omega}\times\vec{M}$$

$M_i = I_i\Omega_i$

$$K_x = \dot{M}_x$$

$$\boxed{\begin{aligned} K_1 &= \dot{M}_1 + \Omega_2 M_3 - \Omega_3 M_2 \\ &= I_1\dot{\Omega}_1 + \Omega_2 I_3 \Omega_3 - \Omega_3 I_2 \Omega_2 \\ &= I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_2) \end{aligned}}$$

* (similar equations for K_2, K_3)

Free rotation: $K_i = 0$, $F_i = 0$

$$0 = I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_2), \cancel{\text{etc.}}$$
$$0 = I_2\dot{\Omega}_2 + \Omega_3\Omega_1(I_1 - I_3)$$
$$0 = I_3\dot{\Omega}_3 + \Omega_1\Omega_2(I_2 - I_1)$$

Free rotation with $\vec{\Omega}$ = const:

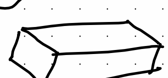
$$0 = \Omega_2\Omega_3(I_3 - I_2)$$
$$0 = \Omega_3\Omega_1(I_1 - I_3)$$
$$0 = \Omega_1\Omega_2(I_2 - I_1)$$

$$\frac{d\vec{\Omega}}{dt} = 0$$
$$\frac{d'\vec{\Omega}}{dt} = 0$$
$$\boxed{\frac{d\vec{\Omega}}{dt} = \frac{d'\vec{\Omega}}{dt}} + \cancel{\vec{\Omega}\times\vec{\Omega}}$$

$$\boxed{\frac{d\vec{M}}{dt} \neq \frac{d'\vec{M}}{dt}}$$

$\boxed{\begin{aligned}\Omega_1 &= \text{const},\\ \Omega_2 &= 0\\ \Omega_3 &= 0\end{aligned}}$ ← stable

$I_1 < I_2 < I_3$



unstable ↓

$\boxed{\Omega_2 = \text{const},\ \Omega_1 = 0,\ \Omega_3 = 0}$

$\boxed{\Omega_3 = \text{const},\ \Omega_1 = 0,\ \Omega_2 = 0}$

$\Omega_1 = \text{const}, \quad \Omega_2 = 0, \quad \Omega_3 = 0$: exact

$$\boxed{\begin{aligned} \Omega_1 &= \text{const} + \epsilon_1 \\ \Omega_2 &= \epsilon_2 \\ \Omega_3 &= \epsilon_3 \end{aligned}}$$

$\epsilon_{1,2,3}$: small time dependent perturbations

Keep 0^{th} and 1^{st} order terms.
Ignore 2^{nd} order, e.g. $\epsilon_2 \epsilon_3$

$$0 = I_1 \frac{d}{dt}(\text{const} + \epsilon_1) + \underbrace{\epsilon_2 \epsilon_3 (I_3 - I_2)}_{2^{nd}\text{ order} \to \text{ignore}}$$

$$\approx I \dot{\epsilon}_1$$

$\to \epsilon_1 = \text{const} \to \Omega_1 = \text{const}$

$$\left.\begin{aligned} 0 &= I_2 \dot{\epsilon}_2 + \epsilon_3 \Omega_1 (I_1 - I_3) \\ 0 &= I_3 \dot{\epsilon}_3 + \Omega_1 \epsilon_2 (I_2 - I_1) \end{aligned}\right\}$$ coupled 1st order diff. equations

Differentiate

$$0 = I_2 \ddot{\epsilon}_2 + \dot{\epsilon}_3 \Omega_1 (I_1 - I_3)$$
$$= I_2 \ddot{\epsilon}_2 - \frac{\Omega_1 \epsilon_2 (I_2 - I_1)}{I_3} \Omega_1 (I_1 - I_3)$$

$I_1 < I_2 < I_3$

$$0 = I_2 \ddot{\epsilon}_2 + \dot{\epsilon}_3 \Omega_1 (I_1 - I_3)$$
$$= I_2 \ddot{\epsilon}_2 - \frac{\Omega_1 \epsilon_2 (I_2 - I_1)}{I_3} \Omega_1 (I_1 - I_3)$$
$$= \ddot{\epsilon}_2 + \underbrace{\frac{\Omega_1^2 (I_2 - I_1)(I_3 - I_1)}{I_2 I_3}}_{\omega^2} \epsilon_2$$

$$\boxed{\begin{matrix} (I_3 - I_2) \\ (I_1 - I_2) \end{matrix}}$$

$$= \ddot{\epsilon}_2 + \omega^2 \epsilon_2$$

$\ddot{\epsilon}_2 = -\omega^2 \epsilon_2 \longrightarrow$ SHM

$\epsilon_2 = A \cos \omega t + B \sin \omega t$
$= C \cos(\omega t + \alpha)$

Similarly,

$\ddot{\epsilon}_3 = -\omega^2 \epsilon_3 \longrightarrow \epsilon_3 = D \cos(\omega t + \beta)$

(ϵ_2, ϵ_3 are bound by their initial deviations away from 0)

For $\Omega_2 = \text{const}$ solution

perturbations

$$\ddot{\epsilon}_1 = +\omega^2 \epsilon_1$$

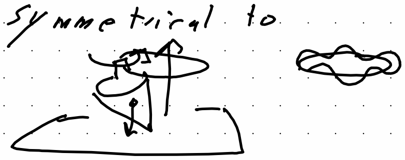
$$\ddot{\epsilon}_3 = +\omega^2 \epsilon_3$$

$$\rightarrow \quad \epsilon_1(t) = A \underbrace{e^{\omega t}}_{\text{grows exponentially}} + B \underbrace{e^{-\omega t}}_{\text{damped exponentially}}$$

$\rightarrow$ instability

Lecture #27 Tues 11/24

- Midterm II — Avg $\approx$ 12/20 (solns posted)
- Quiz #6 — Next Tuesday (last day)
- Oral Final — google doc sign up sheet ??
  Sat 12/5 1:30pm – 4:00pm
- Today, next time:
  i) Free rotation of a symmetric top
  ii) motion in a non-inertial ref. frame
- On your own:
  (i) Sec 35, Prob 1: motion of a heavy symmetrical top
  (ii) static equil: (Sec 38)



Free rotation of a symmetric top:

no force, torque → $\vec{P} = \text{const}$, $\vec{M} = \text{const}$

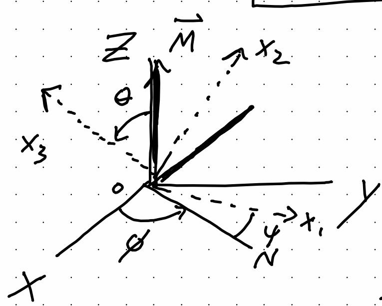
Euler's equations: $I_1 = I_2$

$$0 = I_1\dot{\Omega}_1 + \underbrace{\Omega_2\Omega_3}_{\text{const}}(I_3 - \overset{I_1}{\overset{\parallel}{I_2}}) \qquad (1)$$

$$0 = I_2\dot{\Omega}_2 + \Omega_3\Omega_1(I_1 - I_3) \qquad (2)$$

$$0 = I_3\dot{\Omega}_3 + \Omega_1\Omega_2\,\cancel{(I_2 - I_1)} \to 0 \qquad (3)$$

$\dot{\Omega}_3 = 0 \to \boxed{\Omega_3 = \text{const}}$



$$\Omega_1 = \dot{\phi}\sin\theta\sin\psi + \cancel{\dot{\theta}\cos\psi} \to 0$$

$$\Omega_2 = \dot{\phi}\sin\theta\cos\psi - \cancel{\dot{\theta}\sin\psi} \to 0$$

$$\Omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$$

$$I_1\Omega_1 = M_1 = M\sin\theta\sin\psi \qquad (4)$$

$$I_1\Omega_2 = M_2 = M\sin\theta\cos\psi \qquad (5)$$

$$I_3\Omega_3 = M_3 = M\cos\theta \qquad (6)$$

$\Omega_3 = \text{const} \to \boxed{\theta = \text{const}}$, $\cos\theta = \dfrac{I_3\Omega_3}{M}$

$$0 = I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_1)$$

$$0 = \dot{\Omega}_1 + \Omega_2\left(\frac{\Omega_3(I_3 - I_1)}{I_1}\right) \leftarrow \boxed{\omega \equiv \frac{\Omega_3(I_3 - I_1)}{I_1}}$$

$$0 = I_1\dot{\Omega}_2 + \Omega_1\Omega_3(I_1 - I_3) \qquad M_1 = I_1\Omega_1,\ M_2 = I_1\Omega_2$$

$$0 = \dot{\Omega}_2 - \Omega_1\left(\frac{\Omega_3(I_3 - I_1)}{I_1}\right) \qquad \sqrt{\Omega_1^2 + \Omega_2^2} = a$$

$$0 = \dot{\Omega}_1 + \Omega_2\omega$$

$$0 = \dot{\Omega}_2 - \Omega_1\omega$$

$$0 = \ddot{\Omega}_1 + \dot{\Omega}_2\omega = \ddot{\Omega}_1 + \Omega_1\omega^2$$

$$\ddot{\Omega}_1 = -\omega^2\Omega_1 \to \boxed{\Omega_1 = a\cos(\omega t + \alpha)}$$

$$\boxed{\Omega_2 = a\sin(\omega t + \alpha)}$$

$$\xi = \Omega_1 + i\Omega_2$$

$$\dot{\xi} = \dot{\Omega}_1 + i\dot{\Omega}_2$$

$$\to \dot{\xi} = i\omega\xi \to \xi(t) = \underset{\text{complex}}{A}\,e^{i\omega t}$$

$$I_1 \dot\phi \cancel{\sin\theta \sin\psi} = M \cancel{\sin\theta \sin\psi} \qquad \boxed{\theta = \text{const}}$$

$$\boxed{\dot\phi = \frac{M}{I_1} = \text{const}} \rightarrow \boxed{\phi(t) = \phi_0 + \frac{M}{I_1} t}$$

$$I_3 \left(\dot\psi + \cos\theta \dot\phi\right) = M \cos\theta$$

$$\boxed{-\omega =} \dot\psi = \frac{M\cos\theta}{I_3} - \cos\theta \dot\phi$$

$$= \frac{M\cos\theta}{I_3} - \cos\theta \frac{M}{I_1}$$

$$= \boxed{M\cos\theta \left(\frac{1}{I_3} - \frac{1}{I_1}\right)} = \text{const}$$

$$\rightarrow \boxed{\psi(t) = \psi_0 + M\cos\theta\left(\frac{1}{I_3} - \frac{1}{I_1}\right)}$$

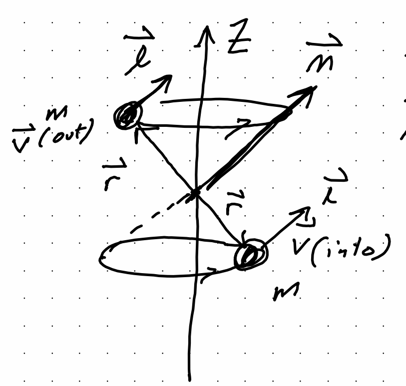
$$\omega = \frac{\Omega_3 (I_3 - I_1)}{I_1}$$

$$\left.\begin{aligned} M_3 &= I_3 \Omega_3 \\ M\cos\theta &= M_3 \end{aligned}\right\}$$

$$\Omega_3 = \frac{M\cos\theta}{I_3}$$

$$\omega = \frac{M\cos\theta (I_3 - I_1)}{I_1 I_3} \quad \boxed{-\dot\psi}$$

$$= M\cos\theta\left(\frac{1}{I_1} - \frac{1}{I_3}\right)$$



$\vec{\Omega} = \text{const}$? Yes $\vec{\Omega} = \Omega \hat{z}$

$\vec{M} = \text{const}$? No $\vec{M}$ precesses around $\hat{z}$ axis

$$\vec{l} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v}$$

---

wobble vs spin Freq

$$\underbrace{\dot\phi}_{\text{precession freq of 3-axis around } \vec{M} \text{ (or } \hat{z})} \qquad \underbrace{\Omega_3}_{\text{Component of angular along 3-axis}}$$

$$\dot\phi = \frac{M}{I_1}$$

$$\Omega_3 = \frac{M\cos\theta}{I_3}$$

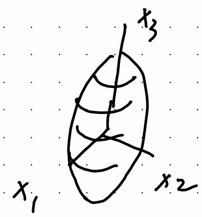
$$\boxed{\frac{\Omega_3}{\dot\phi} = \frac{\cancel{M}\cos\theta}{I_3} \frac{I_1}{\cancel{M}} = \frac{I_1 \cos\theta}{I_3}}$$

$$\frac{\Omega_3}{\dot{\phi}} = \frac{I_1}{I_3}\cos\theta \longrightarrow \frac{I_1}{I_3} \qquad (\theta \to 0)$$

spin / wobble

$I_1 = I_3$ sphere

$$\boxed{\frac{\Omega_3}{\dot{\phi}} = 1}$$

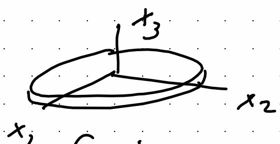


football

$I_1 > I_3$

football

$$\boxed{\frac{\Omega_3}{\dot{\phi}} > 1}$$



frisbee

$$\underset{\frac{1}{4}MR^2}{\underset{\uparrow}{I_1}} < \underset{\frac{1}{2}MR^2}{\underset{\uparrow}{I_3}}$$

$I_1 = \frac{1}{2} I_3$ (uniform disk)

frisbee

$$\boxed{\frac{\Omega_3}{\dot{\phi}} = \frac{1}{2} < 1}$$