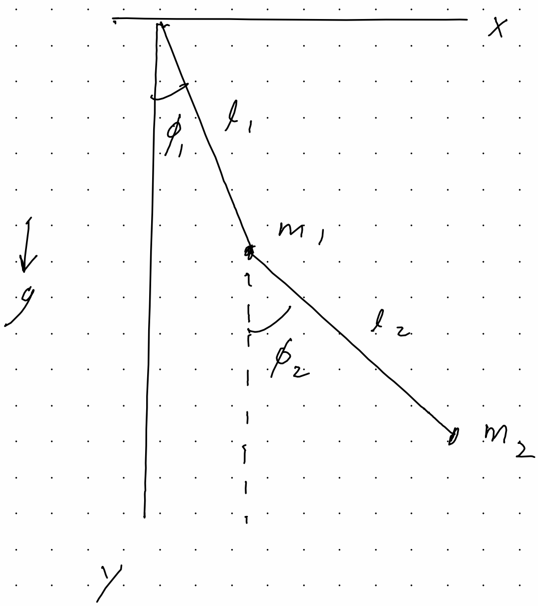
sec5, prob1



$x_1 = l_1 \sin\phi_1$
$y_1 = l_1 \cos\phi_1$
$x_2 = x_1 + l_2 \sin\phi_2 = l_1 \sin\phi_1 + l_2 \sin\phi_2$
$y_2 = y_1 + l_2 \cos\phi_2 = l_1 \cos\phi_1 + l_2 \cos\phi_2$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_1 g l_1 \cos\phi_1 - m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$
$$= -(m_1 + m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$\dot{x}_1 = l_1 \cos\phi_1 \dot{\phi}_1$ $\quad$ $\dot{y}_1 = -l_1 \sin\phi_1 \dot{\phi}_1$
$\dot{x}_1^2 = l_1^2 \cos^2\phi_1 \dot{\phi}_1^2$ $\quad$ $\dot{y}_1^2 = l_1^2 \sin^2\phi_1 \dot{\phi}_1^2$

Thus, $\dot{x}_1^2 + \dot{y}_1^2 = l_1^2(\sin^2\phi_1 + \cos^2\phi_1)\dot{\phi}_1^2$

$$= l_1^2\dot{\phi}_1^2$$

$$\dot{x}_2 = l_1\cos\phi_1\dot{\phi}_1 + l_2\cos\phi_2\dot{\phi}_2$$

$$\rightarrow \dot{x}_2^2 = l_1^2\cos^2\phi_1\dot{\phi}_1^2 + l_2^2\cos^2\phi_2\dot{\phi}_2^2 + 2l_1l_2\cos\phi_1\cos\phi_2\dot{\phi}_1\dot{\phi}_2$$

$$\dot{y}_2 = -l_1\sin\phi_1\dot{\phi}_1 - l_2\sin\phi_2\dot{\phi}_2$$

$$\rightarrow \dot{y}_2^2 = l_1^2\sin^2\phi_1\dot{\phi}_1^2 + l_2^2\sin^2\phi_2\dot{\phi}_2^2 + 2l_1l_2\sin\phi_1\sin\phi_2\dot{\phi}_1\dot{\phi}_2$$

Thus,

$$\dot{x}_2^2 + \dot{y}_2^2 = l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2l_1l_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2)\dot{\phi}_1\dot{\phi}_2$$

$$= l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$

so $T = \frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2)$

$$= \frac{1}{2}m_1l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2$$

$$+ m_2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$

$$= \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2 + m_2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$

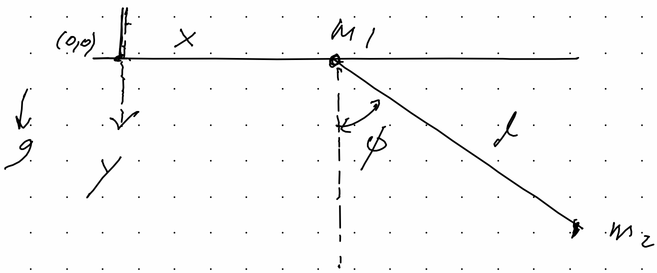
$$U = -(m_1+m_2)gl_1\cos\phi_1 - m_2gl_2\cos\phi_2$$

$$L = T - U$$

$$= \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2 + m_2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$

$$+ (m_1+m_2)gl_1\cos\phi_1 + m_2gl_2\cos\phi_2$$

Sec5 Prob. 2



Generalised coords: x, ϕ

$$(x_1, y_1) = (x, 0)$$
$$(x_2, y_2) = (x + l\sin\phi, \, l\cos\phi)$$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_2 g l \cos\phi$$
$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

Now: $\dot{x}_1^2 + \dot{y}_1^2 = \dot{x}^2$

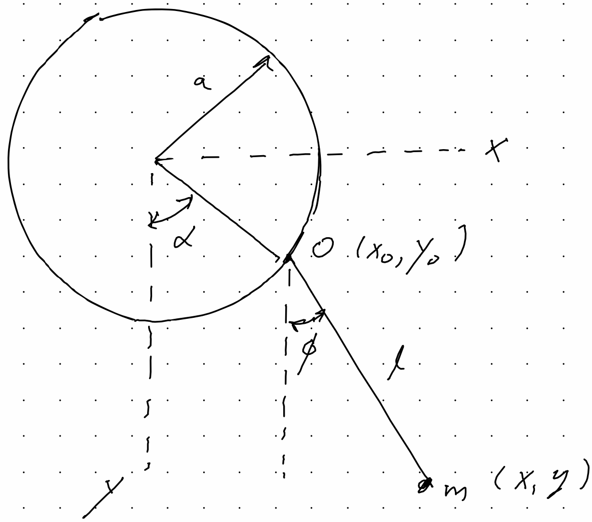
$$\dot{x}_2^2 + \dot{y}_2^2 = (\dot{x} + l\cos\phi\,\dot{\phi})^2 + (-l\sin\phi\,\dot{\phi})^2$$
$$= \dot{x}^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi} + l^2\sin^2\phi\,\dot{\phi}$$
$$= \dot{x}^2 + l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi}$$

$$\to T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi})$$
$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2 l\cos\phi\,\dot{x}\dot{\phi}$$

$$L = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2 l\cos\phi\,\dot{x}\dot{\phi} + m_2 g l\cos\phi$$

Sec 5, Prob 3

(a)



point of support O moves along circle:

$$x_0 = a\sin\alpha \quad , \quad y_0 = a\cos\alpha$$

where $\alpha = \gamma t$, $\gamma = \text{const}$

Pendulum bob:

(x, y) : $x = x_0 + l\sin\phi$

$y = y_0 + l\cos\phi$

$$U = -mgy = \underbrace{-mgy_0}_{} - mgl\cos\phi$$

specified function of time
[can ignore in L]

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{x}_0 + l\cos\phi\ \dot{\phi}$$

$$\dot{x}^2 = \dot{x}_0^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\ \dot{x}_0\dot{\phi}$$

$$\dot{y} = \dot{y}_0 - l \sin\phi\, \dot{\phi}$$

$$\dot{y}^2 = \dot{y}_0^2 + l^2 \sin^2\phi\, \dot{\phi}^2 - 2 l \sin\phi\, \dot{y}_0 \dot{\phi}$$

Thus,

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m \left(\dot{x}_0^2 + l^2 \cos^2\phi\, \dot{\phi}^2 + 2 l \cos\phi\, \dot{x}_0 \dot{\phi} + \dot{y}_0^2 + l^2 \sin^2\phi\, \dot{\phi}^2 - 2 l \sin\phi\, \dot{y}_0 \dot{\phi} \right)$$

$$= \frac{1}{2} m \left(\dot{x}_0^2 + \dot{y}_0^2 \right) + \frac{1}{2} m l^2 \dot{\phi}^2 + m l \dot{\phi} \left(\dot{x}_0 \cos\phi - \dot{y}_0 \sin\phi \right)$$

<u>NOTE</u>: $\dot{x}_0^2 + \dot{y}_0^2 = a^2 \dot{\alpha}^2 = a^2 \gamma^2$

since this is a specified function of time, we can ignore it in the Lagrangian;

Thus, $$L = \frac{1}{2} m l^2 \dot{\phi}^2 + m l \dot{\phi} \left(\dot{x}_0 \cos\phi - \dot{y}_0 \sin\phi \right) + m g l \cos\phi$$

We can rewrite the second term:

$$x_0 = a \sin\alpha \rightarrow \dot{x}_0 = a \cos\alpha\, \dot{\alpha} \qquad (\dot{\alpha} \equiv \gamma)$$

$$y_0 = a \cos\alpha \rightarrow \dot{y}_0 = -a \sin\alpha\, \dot{\alpha}$$

Thus,

$$m l \dot{\phi} (\dot{x}_0 \cos\phi - \dot{y}_0 \sin\phi) = m l \dot{\phi} a \gamma (\cos\alpha \cos\phi + \sin\alpha \sin\phi)$$

$$= m l \dot{\phi} a \gamma \cos(\phi - \alpha)$$

$$= m l \dot{\phi} a \gamma \cos(\phi - \gamma t)$$

Now;
$$\frac{d}{dt}\left[ml a\gamma \sin(\phi-\gamma t)\right]$$

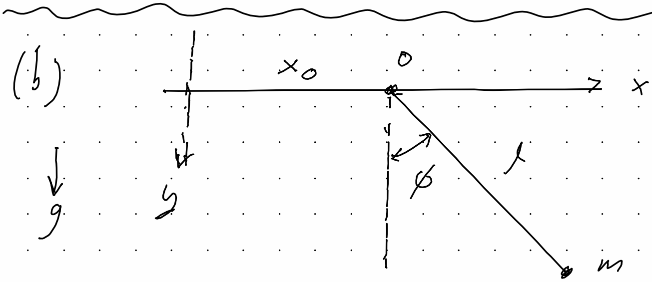
$$= mla\gamma\cos(\phi-\gamma t)(\dot\phi-\gamma)$$
$$= mla\dot\phi\gamma\cos(\phi-\gamma t) - mla\gamma^2\cos(\phi-\gamma t)$$

Thus,
$$mla\dot\phi\gamma\cos(\phi-\gamma t) = \frac{d}{dt}\left[mla\gamma\sin(\phi-\gamma t)\right] + mla\gamma^2\cos(\phi-\gamma t)$$

(and we can ignore the total time derivative in the Lagrangian)

$$\rightarrow L = \frac{1}{2}ml^2\dot\phi^2 + mgl\cos\phi + mla\gamma^2\cos(\phi-\gamma t)$$

(b)



point O moving according to $x_0 = a\cos\gamma t$

$$x = x_0 + l\sin\phi$$
$$y = l\cos\phi$$
$$U = -mgy = -mgl\cos\phi$$

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{x}_0 + l\cos\phi\,\dot{\phi} \qquad , \qquad x_0 = a\cos\gamma t$$

$$= -a\sin(\gamma t)\gamma + l\cos\phi\,\dot{\phi}$$

$$\rightarrow \dot{x}^2 = a^2\gamma^2\sin^2\gamma t + l^2\cos^2\phi\,\dot{\phi}^2$$

$$- 2al\gamma\dot{\phi}\sin(\gamma t)\cos\phi$$

$$\dot{y} = -l\sin\phi\,\dot{\phi}$$

$$\rightarrow \dot{y}^2 = l^2\sin^2\phi\,\dot{\phi}^2$$

Thus, $T = \frac{1}{2} m\left(a^2\gamma^2\sin^2\gamma t + l^2\cos^2\phi\,\dot{\phi}^2\right.$

$$\left. - 2al\gamma\dot{\phi}\sin(\gamma t)\cos\phi + l^2\sin^2\phi\,\dot{\phi}^2\right)$$

$$= \frac{1}{2} m l^2\dot{\phi}^2 + \frac{1}{2} m a^2\gamma^2\sin^2\gamma t$$

specified function of time (ignore)

$$- mal\gamma\dot{\phi}\sin(\gamma t)\cos\phi$$

$$L = \frac{1}{2} ml^2\dot{\phi}^2 - mal\gamma\dot{\phi}\sin(\gamma t)\cos\phi + mgl\cos\phi$$

2nd term:

$$-\frac{d}{dt}\left[mal\gamma\sin(\gamma t)\sin\phi\right]$$

$$= -mal\gamma^2\cos(\gamma t)\sin\phi - mal\gamma\dot{\phi}\sin(\gamma t)\cos\phi$$

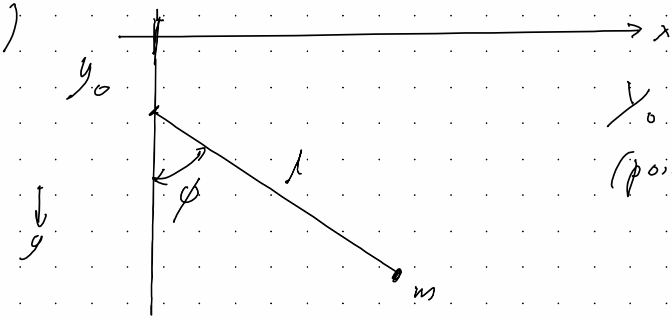
so

$$-mal\gamma\dot{\phi}\sin(\gamma t)\cos\phi = \underbrace{-\frac{d}{dt}[\ ]}_{\text{ignore}} + mal\gamma^2\cos(\gamma t)\sin\phi$$

Thus, ignoring total time derivatives,

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi + mal\gamma^2 \cos(\gamma t)\sin\phi$$

(c)



$y_0 = a\cos\gamma t$

(point of support)

$$x = l\sin\phi$$
$$y = y_0 + l\cos\phi$$
$$= a\cos\gamma t + l\cos\phi$$

$$U = -mgy$$
$$= \underbrace{-mga\cos\gamma t}_{} - mgl\cos\phi$$

specified function of time [can ignore]

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = l\cos\phi\,\dot{\phi}$$
$$\dot{x}^2 = l^2\cos^2\phi\,\dot{\phi}^2$$
$$\dot{y} = -a\gamma\sin(\gamma t) - l\sin\phi\,\dot{\phi}$$
$$\dot{y}^2 = \underbrace{a^2\gamma^2\sin^2(\gamma t)}_{} + l^2\sin^2\phi\,\dot{\phi}^2 + 2al\gamma\sin(\gamma t)\sin\phi\,\dot{\phi}$$

specified function of time [can ignore]

Thus, ignoring this function of time

$$T = \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \dot{\phi} \sin(\gamma t) \sin\phi$$

$$\rightarrow L = T - U$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \gamma \dot{\phi} \sin(\gamma t) \sin\phi + m g l \cos\phi$$

<u>Rewrite 2nd term:</u>

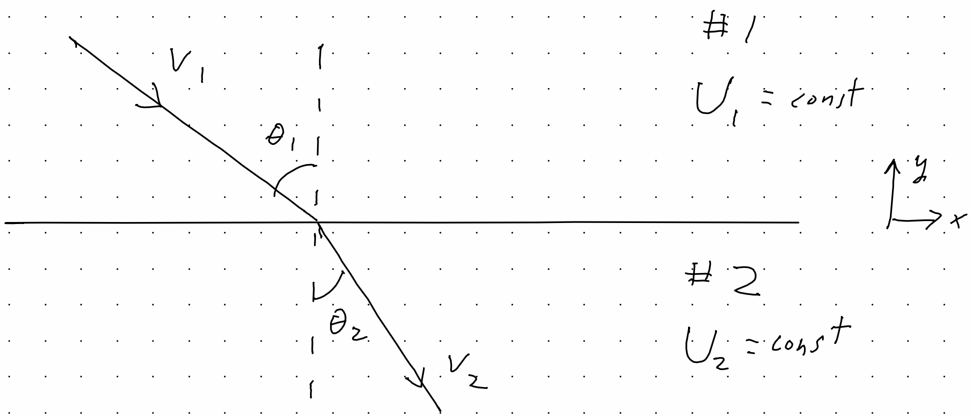
$$-\frac{d}{dt}\left[m a l \gamma \sin(\gamma t) \cos\phi\right] = -m a l \gamma^2 \cos(\gamma t) \cos\phi + m a l \gamma \sin(\gamma t) \sin\phi \, \dot{\phi}$$

so $m a l \gamma \dot{\phi} \sin(\gamma t) \sin\phi = -\frac{d}{dt}[\ ] + m a l \gamma^2 \cos(\gamma t) \cos\phi$

Thus, ignoring total time derivative

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + m g l \cos\phi + m a l \gamma^2 \cos(\gamma t) \cos\phi$$

Sec 7, Prob 1



- Energy conserved, since no time dependence.
- Also momentum in x-direction (// to interface) is conserved, since no x-dependence of the potential

$$U(x,y) = \begin{cases} U_1 & y \geq 0 \\ U_2 & y < 0 \end{cases}$$

v_1 : given

$$E = \frac{1}{2} m v_1^2 + U_1 = \frac{1}{2} m v_2^2 + U_2$$

$$\rightarrow \frac{1}{2} m v_2^2 = \frac{1}{2} m v_1^2 + (U_1 - U_2)$$

$$v_2^2 = v_1^2 + \frac{2(U_1 - U_2)}{m}$$

so, $$v_2 = v_1 \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

The angles θ_1, θ_2 are related by

$$p_{1x} = p_{2x}$$

$$\cancel{m} v_1 \sin\theta_1 = \cancel{m} v_2 \sin\theta_2$$

thus, $$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

Sec 8, Prob 1

Transformation of action $S = \int L\,dt$

K, K': two inertial frames

K' moves with velocity $\vec{V}$ wrt K

Assume that K, K' coincide at $t=0$ so $\vec{r}_a = \vec{r}_a{}'$ wrt these two frames

Now: $\vec{v}_a = \vec{V} + \vec{v}_a{}'$

$$L = T - U$$

$$= \sum_a \frac{1}{2} m_a |\vec{v}_a|^2 - U(\vec{r}_1, \vec{r}_2, \dots, t)$$

$$|\vec{v}_a|^2 = |\vec{V} + \vec{v}_a{}'|^2$$

$$= |\vec{V}|^2 + |\vec{v}_a{}'|^2 + 2\vec{V}\cdot\vec{v}_a{}'$$

so

$$L = \sum_a \frac{1}{2} m_a \left(|\vec{V}|^2 + |\vec{v}_a{}'|^2 + 2\vec{V}\cdot\vec{v}_a{}'\right) - U$$

$$= \frac{1}{2}\mu V^2 + T' + \vec{V}\cdot\sum_a m_a \vec{v}_a{}' - U$$

$$= T' - U + \frac{1}{2}\mu V^2 + \vec{P}'\cdot\vec{V}$$

$$= L' + \frac{1}{2}\mu V^2 + \vec{P}'\cdot\vec{V}$$

where $\vec{P}'$ = total momentum wrt K'

$\mu = \sum_a m_a$ = total mass

$$S = \int_{t_1}^{t_2} L\,dt$$

$$= \int_{t_1}^{t_2} \left(L' + \tfrac{1}{2}\mu V^2 + \vec{p}\,' \cdot \vec{V} \right) dt$$

$$= S' + \underbrace{\tfrac{1}{2}\mu V^2 (t_2 - t_1)}_{\text{doesn't change EOMs}} + \vec{V} \cdot \int_{t_1}^{t_2} \vec{p}\,' \, dt$$

Now:

$$\vec{V} \cdot \int_{t_1}^{t_2} \vec{p}\,' \, dt = \vec{V} \cdot \int_{t_1}^{t_2} \sum_a m_a \vec{v}_a\,' \, dt$$

$$= \vec{V} \cdot \sum_a m_a \int_{t_1}^{t_2} \left(\frac{d\vec{r}_a}{dt} \right) dt$$

$$= \vec{V} \cdot \sum_a m_a \vec{r}_a \Big|_{t_1}^{t_2}$$

$$= \vec{V} \cdot \left(\mu \vec{R}(t_2) - \mu \vec{R}(t_1) \right)$$

$$= \mu \vec{V} \cdot \Big(\underbrace{\vec{R}(t_2) - \vec{R}(t_1)}_{\text{difference in COM positions}} \Big)$$

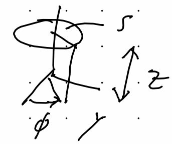
So:

$$S = S' + \tfrac{1}{2}\mu V^2 (t_2 - t_1) + \mu \vec{V} \cdot \left(\vec{R}(t_2) - \vec{R}(t_1) \right)$$

Sec 9, Prob 1

cylindrical coordinates (s, ϕ, z)

$s^2 = x^2 + y^2$



$x = s\cos\phi$

$y = s\sin\phi$

$z = z$

$\vec{M} = \vec{r} \times \vec{p} = m\vec{r} \times \dot{\vec{r}}$

Now: $M_x = m(y\dot{z} - z\dot{y})$

$M_y = m(z\dot{x} - x\dot{z})$

$M_z = m(x\dot{y} - y\dot{x})$

$M = \sqrt{M_x^2 + M_y^2 + M_z^2}$

$\dot{z} = \dot{z}$

$\dot{y} = \dot{s}\sin\phi + s\cos\phi\,\dot{\phi}$

$\dot{x} = \dot{s}\cos\phi - s\sin\phi\,\dot{\phi}$

Thus, $M_x = m\left(s\sin\phi\,\dot{z} - z(\dot{s}\sin\phi + s\cos\phi\,\dot{\phi})\right)$

$= m(s\sin\phi\,\dot{z} - z\sin\phi\,\dot{s} - zs\cos\phi\,\dot{\phi})$

$M_y = m\left(z(\dot{s}\cos\phi - s\sin\phi\,\dot{\phi}) - s\cos\phi\,\dot{z}\right)$

$= m(z\cos\phi\,\dot{s} - zs\sin\phi\,\dot{\phi} - s\cos\phi\,\dot{z})$

$M_z = m[s\cos\phi(\cancel{\dot{s}\sin\phi} + s\cos\phi\,\dot{\phi})$

$- s\sin\phi(\cancel{\dot{s}\cos\phi} - s\sin\phi\,\dot{\phi})]$

$= ms^2\dot{\phi}$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$= m^2 \Big\{ \left(\sin\phi \,(s\dot{z} - z\dot{s}) - z s \cos\phi \,\dot{\phi}\right)^2$$

$$+ \left(-\cos\phi \,(s\dot{z} - z\dot{s}) - z s \sin\phi \,\dot{\phi}\right)^2$$

$$+ \left(s^2 \dot{\phi}\right)^2 \Big\}$$

$$= m^2 \Big\{ \sin^2\phi \,(s\dot{z} - z\dot{s})^2 + z^2 s^2 \cos^2\phi \,\dot{\phi}^2$$

$$\cancel{- 2 z s \sin\phi \cos\phi \,\dot{\phi}\,(s\dot{z} - z\dot{s})}$$

$$+ \cos^2\phi \,(s\dot{z} - z\dot{s})^2 + z^2 s^2 \sin^2\phi \,\dot{\phi}^2$$

$$\cancel{+ 2 z s \sin\phi \cos\phi \,\dot{\phi}\,(s\dot{z} - z\dot{s})}$$

$$+ s^4 \dot{\phi}^2 \Big\}$$

$$= m^2 \left\{ (s\dot{z} - z\dot{s})^2 + z^2 s^2 \dot{\phi}^2 + s^4 \dot{\phi}^2 \right\}$$

$$= m^2 \left[(s\dot{z} - z\dot{s})^2 + s^2 \dot{\phi}^2 (z^2 + s^2) \right]$$

Sec 9, Prob 2

repeat for spherical polar coords.

$M_x = m(y\dot{z} - z\dot{y})$, cyclic

$M^2 = M_x^2 + M_y^2 + M_z^2$

Now: $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$\rightarrow \dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi}$

$\dot{y} = \dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi}$

$\dot{z} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$

Thus,

$M_x = m(y\dot{z} - z\dot{y})$

$= m\{ r\sin\theta\sin\phi(\cancel{\dot{r}\cos\theta} - r\sin\theta\,\dot{\theta})$

$- r\cos\theta(\cancel{\dot{r}\sin\theta\sin\phi} + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi})\}$

$= m\{ -r^2\sin^2\theta\sin\phi\,\dot{\theta} - r^2\cos^2\theta\sin\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\cos\phi\,\dot{\phi}\}$

$= m\{ -r^2\sin\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\cos\phi\,\dot{\phi}\}$

$= -mr^2[\sin\phi\,\dot{\theta} + \sin\theta\cos\theta\cos\phi\,\dot{\phi}]$

$$M_y = m(z\dot{x} - x\dot{z})$$

$$= m\{ r\cos\theta\,(\cancel{\dot{r}\sin\theta\cos\phi} + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi})$$

$$- r\sin\theta\cos\phi\,(\cancel{\dot{r}\cos\theta} - r\sin\theta\,\dot{\theta})\}$$

$$= m\{ r^2\cos^2\theta\cos\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\sin\phi\,\dot{\phi}$$

$$+ r^2\sin^2\theta\cos\phi\,\dot{\theta}\}$$

$$= m[\, r^2\cos\phi\,\dot{\theta} - r^2\sin\theta\cos\theta\sin\phi\,\dot{\phi}\,]$$

$$= mr^2[\cos\phi\,\dot{\theta} - \sin\theta\cos\theta\sin\phi\,\dot{\phi}]$$

$$M_z = m(x\dot{y} - y\dot{x})$$

$$= m\{ r\sin\theta\cos\phi\,(\cancel{\dot{r}\sin\theta\sin\phi} + \cancel{r\cos\theta\sin\phi\,\dot{\theta}} + r\sin\theta\cos\phi\,\dot{\phi})$$

$$- r\sin\theta\sin\phi\,(\cancel{\dot{r}\sin\theta\cos\phi} + \cancel{r\cos\theta\cos\phi\,\dot{\theta}} - r\sin\theta\sin\phi\,\dot{\phi})\}$$

$$= m[\, r^2\sin^2\theta\cos^2\phi\,\dot{\phi} + r^2\sin^2\theta\sin^2\phi\,\dot{\phi}\,]$$

$$= mr^2\sin^2\theta\,\dot{\phi}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$= m^2r^4[\sin\phi\,\dot{\theta} + \sin\theta\cos\theta\cos\phi\,\dot{\phi}]^2$$

$$+ m^2r^4[\cos\phi\,\dot{\theta} - \sin\theta\cos\theta\sin\phi\,\dot{\phi}]^2$$

$$+ m^2r^4\sin^4\theta\,\dot{\phi}^2$$

cross terms will cancel

$$M^2 = m^2 r^4 \left\{ \sin^2\phi\, \dot\theta^2 + \sin^2\theta \cos^2\theta \cos^2\phi\, \dot\phi^2 \right.$$

$$+ \cos^2\phi\, \dot\theta^2 + \sin^2\theta \cos^2\theta \sin^2\phi\, \dot\phi^2$$

$$\left. + \sin^4\theta\, \dot\phi^2 \right\}$$

$$= m^2 r^4 \left[\dot\theta^2 + \sin^2\theta \cos^2\theta\, \dot\phi^2 + \sin^4\theta\, \dot\phi^2 \right]$$

$$= m^2 r^4 \left[\dot\theta^2 + \sin^2\theta\, \dot\phi^2 \left(\cos^2\theta + \sin^2\theta \right) \right]$$

$$= m^2 r^4 \left[\dot\theta^2 + \sin^2\theta\, \dot\phi^2 \right]$$