sec 16, prob 2

From (16.7) we have

$$p(\theta_0)\,d\theta_0 = \tfrac{1}{2}\sin\theta_0\, d\theta_0 = -\tfrac{1}{2}\, d(\cos\theta_0)$$

where θ_0 is the angle of one of the emitted particles in the com frame.

= We would like to find $p(\theta)d\theta$, where θ is the angle of one of the emitted particle in the lab frame.

= Since $p(\theta)d\theta = p(\theta_0)d\theta_0$

we just need to find θ_0 as a function of θ.

This is given by (16.6) which we first derive.

Proof: Given $\tan\theta = \dfrac{v_0 \sin\theta_0}{v_0\cos\theta_0 + V}$

we have

$$\tan\theta\,(v_0\cos\theta_0 + V) = v_0\sqrt{1-\cos^2\theta_0}$$

Square both sides

$$\tan^2\theta\,(v_0^2\cos^2\theta_0 + V^2 + 2v_0 V\cos\theta_0) = v_0^2(1-\cos^2\theta_0)$$

$$\underbrace{(1+\tan^2\theta)}_{\sec^2\theta} v_0^2\cos^2\theta_0 + 2v_0 V\tan^2\theta\cos\theta_0 + (V^2\tan^2\theta - v_0^2) = 0$$

quadratic equation for $\cos\theta_0$

$$\rightarrow \cos\theta_0 = \frac{-2v_0 V\tan^2\theta \pm \sqrt{4v_0^2V^2\tan^4\theta - 4v_0^2\sec^2\theta\,(V^2\tan^2\theta - v_0^2)}}{2v_0^2\sec^2\theta}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \frac{1}{\sec^2\theta}\sqrt{\left(\frac{V}{v_0}\right)^2\tan^4\theta - \sec^2\theta\left(\left(\frac{V}{v_0}\right)^2\tan^2\theta - 1\right)}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \frac{1}{\sec\theta}\sqrt{\left(\frac{V}{v_0}\right)^2\left(\frac{\sin^4\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}\right) + 1}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{\left(\frac{V}{v_0}\right)^2\frac{\sin^2\theta(\sin^2\theta - 1)}{\cos^2\theta} + 1}$$

$$= -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{1 - \left(\frac{V}{v_0}\right)^2\sin^2\theta}$$

(for $v_0 > V$ take $+\sqrt{}$, for $v_0 < V$ take both signs.)

Now differentiate both sides:

$$d(\cos\theta_0) = -\frac{V}{v_0}\,2\sin\theta\cos\theta\,d\theta \mp \sin\theta\,d\theta\sqrt{}$$

$$\pm\cos\theta\left(\frac{1}{\cancel{2}}\right)\frac{1}{\sqrt{}}(-\cancel{2})\left(\frac{V}{v_0}\right)^2\sin\theta\cos\theta\,d\theta$$

$$= \sin\theta\,d\theta\left[-2\frac{V}{v_0}\cos\theta \mp \sqrt{} \mp \left(\frac{V}{v_0}\right)^2\cos^2\theta\,\frac{1}{\sqrt{}}\right]$$

Now:

$$\sqrt{} + \left(\frac{V}{v_0}\right)^2 \cos^2\theta \frac{1}{\sqrt{}}$$

$$= \frac{1-\left(\frac{V}{v_0}\right)^2 \sin^2\theta + \left(\frac{V}{v_0}\right)^2 \cos^2\theta}{\sqrt{}}$$

$$= \frac{1 + \left(\frac{V}{v_0}\right)^2 \left(\cos^2\theta - \sin^2\theta\right)}{\sqrt{}}$$

$$= \frac{1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta}{\sqrt{}}$$

Thus,

$$d(\cos\theta) = \underbrace{-\sin\theta\, d\theta}_{d(\cos\theta)} \left[2\frac{V}{v_0}\cos\theta \pm \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1-\left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

So:

$$p(\theta)\, d\theta = \frac{1}{2}\sin\theta\, d\theta \left[2\frac{V}{v_0}\cos\theta \pm \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1-\left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

Now: for $v_0 > V$, take $+$ sign $(\theta \in [0, \pi])$

for $v_0 < V$, as θ_0 increases from 0 to π

θ increases from $0 \mapsto \theta_{max}$

θ decreases from $\theta_{max} \mapsto 0$

Thus, for $v_0 < V$ need to take the difference of the $+$ and $-$ expressions:

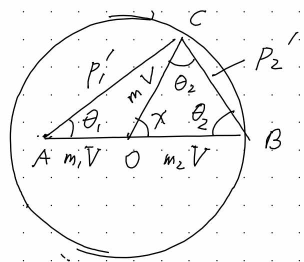
$$p(\theta)\, d\theta = \frac{1}{2}\sin\theta\, d\theta \left[2\frac{V}{v_0}\cos\theta + \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1-\left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

$$- \frac{1}{2}\sin\theta\, d\theta \left[2\frac{V}{v_0}\cos\theta - \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1-\left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

$$= \frac{\sin\theta\, d\theta \left(1 + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{1-\left(\frac{V}{v_0}\right)^2 \sin^2\theta}}$$

which is valid for $0 \le \theta \le \theta_{max} = \sin^{-1}\left(\frac{v_0}{V}\right)$

Sec 17, Prob 1:



$$\chi + 2\theta_2 = \pi$$

$$\chi = \pi - 2\theta_2$$

From the above figure:

$$(m_2 v_2')^2 = 2(mv)^2 - 2(mv)^2 \cos\chi$$

$$= 2m^2 v^2 \left(1 - \cos(\pi - 2\theta_2)\right)$$

$$= 2m^2 v^2 \left[1 - \left(\underbrace{\cos(\pi)}_{=-1}\cos(2\theta_2) + \cancelto{0}{\sin(\pi)}\sin(2\theta_2)\right)\right]$$

$$= 2m^2 v^2 \left(1 + \cos(2\theta_2)\right)$$

$$\rightarrow \quad v_2' = \sqrt{2}\left(\frac{m}{m_2}\right) v \sqrt{1 + (\cos^2\theta_2 - \sin^2\theta_2)}$$

$$= \sqrt{2}\left(\frac{m_1}{m_1+m_2}\right) v \sqrt{2\cos^2\theta_2}$$

$$= 2\left(\frac{m_1}{m_1+m_2}\right) V \cos\theta_2$$

Thus, $\left(\frac{v_2'}{V}\right) = \left(\frac{2m_1}{m_1+m_2}\right)\cos\theta_2$

Also,

$$(mv)^2 = (m_1 V)^2 + (m_1 v_1')^2 - 2 m_1^2 v_1' V \cos\theta_1$$

$$\rightarrow \quad (m_1 v_1')^2 - 2 m_1 V \cos\theta_1 (m_1 v_1') + m_1^2 V^2 - m^2 V^2 = 0$$

Now: $V = \frac{m_1 v_1 + \cancelto{0}{m_2 v_2}}{m_1 + m_2} = \frac{m_1 v}{m_1 + m_2}$

Thus,

$$(m_1 v_1')^2 - 2\left(\frac{m_1^2 v}{m_1+m_2}\right)\cos\theta_1 \, m_1 v_1' + \frac{m_1^2 m_1^2 v^2}{(m_1+m_2)^2} - \frac{m_1^2 m_2^2}{(m_1+m_2)^2} v^2 = 0$$

$$\rightarrow \quad (v_1')^2 - 2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \, v_1' + \frac{m_1^2 - m_2^2}{(m_1+m_2)^2} v^2 = 0$$

$$(v_1')^2 - 2\left(\frac{m_1 V}{m_1+m_2}\right)\cos\theta_1 \, v_1' + \left(\frac{m_1 - m_2}{m_1+m_2}\right) V^2 = 0$$

$$\left(\frac{v_1'}{V}\right)^2 - 2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \left(\frac{v_1'}{V}\right) + \left(\frac{m_1 - m_2}{m_1+m_2}\right) = 0$$

Quadratic equation

$$\frac{v_1'}{v} = \frac{2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \sqrt{4\left(\frac{m_1}{m_1+m_2}\right)^2\cos^2\theta_1 - 4\left(\frac{m_1-m_2}{m_1+m_2}\right)}}{2}$$

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_1^2\cos^2\theta_1 - (m_1-m_2)(m_1+m_2)}$$

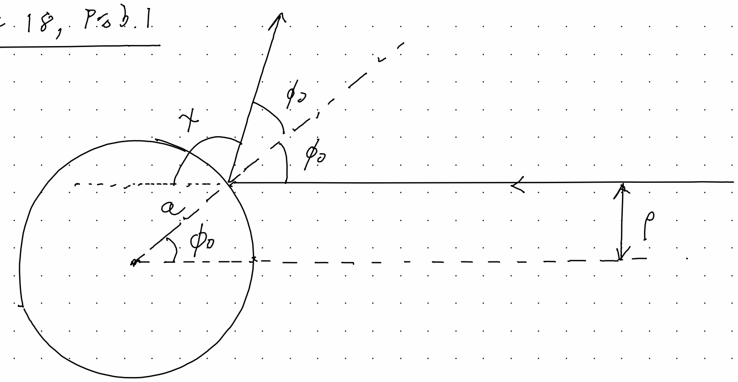
$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_1^2(\cos^2\theta_1-1) + m_2^2}$$

$$= \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{m_2^2 - m_1^2\sin^2\theta_1}$$

The + sign holds for $m_1 < m_2$

+/- signs hold for $m_1 > m_2$

Sec. 18, Prob. 1



$$\chi + 2\phi_0 = \pi \quad \rightarrow \quad \phi_0 = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\rho = a\sin\phi_0$$

$$= a\sin\left(\frac{\pi}{2} - \frac{\chi}{2}\right)$$

$$= a\left(\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}\cos\left(\frac{\chi}{2}\right) - \cancel{\cos\left(\frac{\pi}{2}\right)}^{0}\sin\left(\frac{\chi}{2}\right)\right)$$

$$= a\cos\left(\frac{\chi}{2}\right)$$

$$d\sigma = 2\pi\rho\, d\rho$$

$$= 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi$$

$$= \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$

where $d\Omega$ = solid angle

$$= \underbrace{2\pi}_{\text{integral over } d\phi}\sin\chi\, d\chi$$

$$\rho = a\cos\left(\frac{\chi}{2}\right)$$

$$d\rho = -a\,\frac{1}{2}\sin\left(\frac{\chi}{2}\right)d\chi$$

$$= -\frac{a}{2}\sin\left(\frac{\chi}{2}\right)d\chi$$

thus,

$$d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$

$$= \frac{a\cos(\chi/2)}{\sin\chi}\;\frac{a}{2}\sin\left(\frac{\chi}{2}\right)d\Omega$$

$$= \frac{a^2}{2}\,\frac{\sin(\chi/2)\cos(\chi/2)}{\sin\chi}\,d\Omega$$

$$= \boxed{\frac{a^2}{4}\,d\Omega} \qquad \left(\text{since } \sin\chi = 2\sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)\right)$$

total cross section

$$\sigma = \int d\sigma = \frac{a^2}{4}\int d\Omega = \frac{a^2}{4}\,4\pi = \boxed{\pi a^2}$$

Now calculate differential cross section in lab frame for both m_1 and m_2

Use the result that

$$d\sigma_1 = \frac{\rho(\theta_1)}{\sin\theta_1}\left|\frac{d\rho}{d\theta_1}\right| d\Omega_1 = \rho\left|\frac{d\rho}{d(\cos\theta_1)}\right| d\Omega_1$$

Compare to:

$$d\sigma = \rho\left|\frac{d\rho}{d(\cos\chi)}\right| d\Omega$$

$$\rightarrow \frac{d\sigma_1}{d\Omega_1} = \rho\left|\frac{d\rho}{d(\cos\theta_1)}\right|$$

$$= \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|\frac{d\sigma}{d\Omega}$$

So we need to evaluate:

$$\frac{d(\cos\chi)}{d(\cos\theta_1)} \quad \text{and} \quad \frac{d(\cos\chi)}{d(\cos\theta_2)}$$

start with θ_2: (17.4)

$$\theta_2 = \frac{1}{2}(\pi-\chi) \rightarrow \boxed{\chi = \pi - 2\theta_2}$$

$$\rightarrow \cos\chi = \cos(\pi-2\theta_2)$$

$$= \cos\pi\cos(2\theta_2) + \cancel{\sin\pi}^{\,0}\sin(2\theta_2)$$

$$= -\cos(2\theta_2)$$

$$= -(\cos^2\theta_2 - \sin^2\theta)$$

$$= -(2\cos^2\theta_2 - 1)$$

$$= -2\cos^2\theta_2 + 1$$

Thus, $\boxed{d(\cos\chi) = -4\cos\theta_2\, d(\cos\theta_2)}$

Thus,

$$\frac{d\sigma_2}{d\Omega_2} = \frac{d\sigma}{d\Omega}\left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right|$$

$$= \frac{1}{4}a^2 \cdot |4\cos\theta_2|$$

$$= a^2|\cos\theta_2|$$

so $\boxed{d\sigma_2 = a^2|\cos\theta_2|\, d\Omega_2}$

Now consider θ_1:

From (17.4):

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}$$

Compare with

$$\tan\theta = \frac{v_0\sin\theta_0}{V + v_0\cos\theta_0} \qquad (16.5)$$

Make identifications: $\theta \to \theta_1$, $v_0 \to m_2$, $V \to m_1$

Then we can write down from (16.6):

$$\cos\chi = -\frac{m_1}{m_2}\sin^2\theta_1 \pm \cos\theta_1\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}$$

[See also sec 16, Prob. 2 where we derived this for θ and θ_0]

We also worked out the derivative:

$$d(\cos\theta_0) = d(\cos\theta)\left[2\frac{V}{v_0}\cos\theta \pm \frac{1 + \left(\frac{V}{v_0}\right)^2\cos 2\theta}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2\sin^2\theta}}\right]$$

So we can similarly write dow

$$d(\cos\chi) = d(\cos\theta_1)\left[2\frac{m_1}{m_2}\cos\theta_1 \pm \frac{1 + \left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\right]$$

For $m_1 < m_2$: take + sign

For $m_1 > m_2$: As χ increases from 0 to π, θ_1 increases from 0 to θ_{max}, then θ_1 decreases from θ_{max} to 0. In that case

$$d(\cos\chi) = d(\cos\theta_1)[\cancel{\oslash} + \oslash] - d(\cos\theta_1)[\cancel{\oslash} - \oslash]$$

$$= 2\, d(\cos\theta_1)\frac{1 + \left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

Use: (for both)

$$d\sigma_1 = \left(\frac{d\sigma}{d\Omega}\right)\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| d\Omega_1$$

$$= \frac{1}{4}a^2\left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| d\Omega_1$$

Thus,

For $m_1 < m_2$:

$$d\sigma_1 = \frac{1}{4}a^2\left[2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\right]d\Omega_1$$

For $m_1 > m_2$:

$$d\sigma_1 = \frac{1}{4}a^2\cdot 2\,\frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\,d\Omega_1$$

$$= \frac{a^2}{2}\,\frac{1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\,d\Omega_1$$

Sec. 18, Prob 2:

Hard sphere scattering again.
Calculate $d\sigma$ in terms of $d\epsilon$ where
ϵ = energy lost by scattered particle.

Now: ϵ = energy lost by scattered particle
= energy gained by m_2
$= \frac{1}{2}m_2(v_2')^2$

From Fig. 16.), we have (law of cosines):

$$(m_2v_2')^2 = (mv)^2 + (mv)^2 - 2(mv)^2\cos\chi$$
$$= 2(mv)^2[1-\cos\chi]$$
$$= 2(mv)^2\,2\sin^2\left(\frac{\chi}{2}\right)$$

So $\quad m_2v_2' = 2\,mv\,\sin\left(\frac{\chi}{2}\right)$

$$\rightarrow \quad v_2' = \left(\frac{m}{m_2}\right)v\,\sin\left(\frac{\chi}{2}\right)$$
$$= \left(\frac{m_1}{m_1+m_2}\right)v\,\sin\left(\frac{\chi}{2}\right) \qquad (17.5)$$

NOTE: $\quad d\sigma = \frac{1}{4}a^2\,d\Omega$
$$= \frac{1}{4}a^2\,2\pi\sin\chi\,d\chi$$
$$= \frac{\pi a^2}{2}\,|d(\cos\chi)|$$

So we would like to relate $d\epsilon$ and $d(\cos\chi)$.

Now: $\epsilon = \frac{1}{2} m_2 (v_2')^2$

$$= \frac{1}{2} m_2 \frac{4 m_1^2 v^2}{(m_1+m_2)^2} \sin^2\left(\frac{\chi}{2}\right)$$

$$= \frac{2 m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 \sin^2\left(\frac{\chi}{2}\right) \quad (\text{since } v = v_\infty)$$

$$\equiv \epsilon_{max} \sin^2\left(\frac{\chi}{2}\right)$$

where $\epsilon_{max} \equiv \frac{2 m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2$

$$= 4\left(\frac{m_1}{m_1+m_2}\right) \frac{m v_\infty^2}{2}$$

$$= 4\left(\frac{m_1}{m_1+m_2}\right) E$$

Thus, $d\epsilon = \epsilon_{max}\, 2 \sin\left(\frac{\chi}{2}\right) \cos\left(\frac{\chi}{2}\right) \frac{d\chi}{2}$

$$= \frac{1}{2} \epsilon_{max} \sin\chi \, d\chi$$

$$= \frac{1}{2} \epsilon_{max} |d(\cos\chi)|$$

So $d\sigma = \frac{\pi a^2}{2} |d(\cos\chi)|$

$$= \frac{\pi a^2}{2} \frac{2}{\epsilon_{max}} d\epsilon = \boxed{\frac{\pi a^2}{\epsilon_{max}} d\epsilon}$$

which is a uniform distribution wrt ϵ.

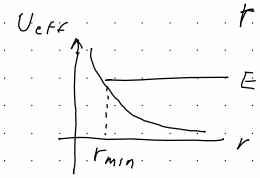
Sec 18, Prob. 4

Effective cross section to "fall" to center of
$U(r) = -\alpha/r^2 \quad (\alpha > 0)$

$$U_{eff}(r) = U(r) + \frac{M^2}{2mr^2}$$

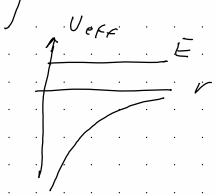
$$= -\frac{\alpha}{r^2} + \frac{M^2}{2mr^2}$$

$$= \frac{1}{r^2}\left(\frac{M^2}{2m} - \alpha\right)$$



$\frac{M^2}{2m} - \alpha > 0$

(don't fall to center since $r_{min} > 0$)



$\frac{M^2}{2m} - \alpha < 0$

Fall to center ($r=0$)

For a given $E = \frac{1}{2} m v_\infty^2 > 0$ need

$$\frac{M^2}{2m} - \alpha < 0$$

$$\frac{M^2}{2m} < \alpha$$

$$\rightarrow \quad M_{max} = \sqrt{2m\alpha}$$

Cross section: $\sigma = \pi \rho_{max}^2$, $M = \rho m v_\infty$

Thus,

$$\sigma = \pi \rho_{max}^2 \quad , \qquad \rho_{max} = \frac{M_{max}}{m V_\infty}$$

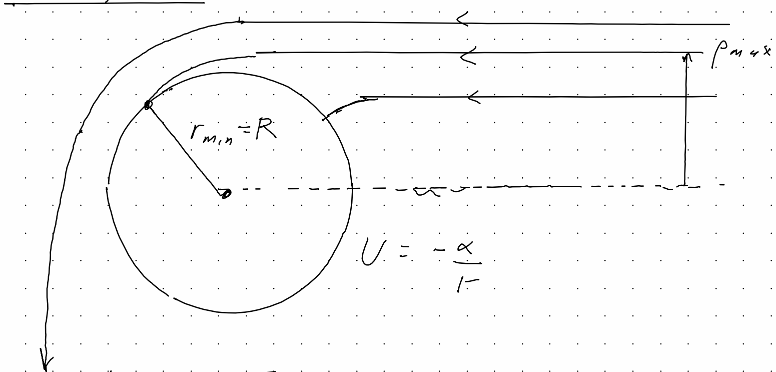
$$= \pi \frac{M_{max}^2}{m^2 V_\infty^2}$$

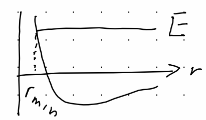
$$= \pi \frac{2 m \alpha}{m^2 V_\infty^2}$$

$$= \frac{\pi \alpha}{\frac{1}{2} m V_\infty^2}$$

$$= \boxed{\frac{\pi \alpha}{E}}$$

Sec 18, Prob 6





Turning point at $r = R$

$$0 = E - U_{eff}(R)$$

$$= E - U(R) - \frac{M_{max}^2}{2 m R^2}$$

$$= E + \frac{\alpha}{R} - \frac{M_{max}^2}{2 m R^2}$$

$$\rightarrow \frac{M_{max}^2}{2 m R^2} = E + \frac{\alpha}{R}$$

$$M_{max} = \rho_{max} m V_\infty \quad , \qquad E = \frac{1}{2} m V_\infty^2$$

Thus,

$$\sigma = \pi \rho_{max}^2$$

$$= \pi \frac{M_{max}^2}{m^2 v_\infty^2}$$

$$= \pi \frac{1}{m^2 v_\infty^2} \quad 2mR^2\left(E + \frac{\alpha}{R}\right)$$

$$= \pi R^2 \underbrace{\left(\frac{2}{m v_\infty^2}\right)}_{\frac{1}{E}}\left(E + \frac{\alpha}{R}\right)$$

$$= \boxed{\pi R^2 \left(1 + \frac{\alpha}{ER}\right)}$$

where $E = \frac{1}{2} m v_\infty^2 = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right) v_\infty^2$

and $\alpha \equiv G m_1 m_2$

Sec 19, Prob. 1:

$$U = \frac{\alpha}{r^2} \quad , \quad \alpha > 0$$

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M\, dr/r^2}{\sqrt{2m[E - U(r)] - M^2/r^2}}$$

Substitute: $E = \frac{1}{2} m v_\infty^2$

$M = \rho m v_\infty$

$$\rightarrow \phi_0 = \int_{r_{min}}^{\infty} \frac{\rho m v_\infty\, dr/r^2}{\sqrt{2m\left[\frac{1}{2} m v_\infty^2 - U(r)\right] - \rho^2 m^2 v_\infty^2/r^2}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - 2U(r)/m v_\infty^2 - \rho^2/r^2}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \rho^2/r^2 - 2\, U(r)/m v_\infty^2}}$$

Substitute: $U(r) = \frac{\alpha}{r^2}$

$$\sqrt{} = \sqrt{1 - \rho^2/r^2 - \left(\frac{2\alpha}{m v_\infty^2}\right)\frac{1}{r^2}}$$

$$= \sqrt{1 - \left(\rho^2 + \frac{2\alpha}{m v_\infty^2}\right)\frac{1}{r^2}} = \sqrt{1 - \frac{A^2}{r^2}}$$

$$A^2 = \rho^2 + \frac{2\alpha}{m v_\infty^2}$$

Thus,

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \frac{A^2}{r^2}}}$$

$$= \int_{A}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \frac{A^2}{r^2}}}$$

Let: $u = \frac{1}{r} \rightarrow du = -\frac{1}{r^2} dr$

$$\frac{A^2}{r^2} = A^2 u^2$$

$$\phi_0 = -\int_{\frac{1}{A}}^{0} \frac{\rho\, du}{\sqrt{1 - A^2 u^2}}$$

$$= \int_{0}^{\frac{1}{A}} \frac{\rho\, du}{\sqrt{1 - A^2 u^2}}$$

Let: $\sin\theta = Au$

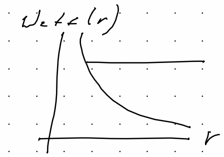
$\cos\theta\, d\theta = A\, du$

$u = 0, \frac{1}{A} \rightarrow \theta = 0, \pi/2$

$\sqrt{1 - A^2u^2} = \sqrt{1 - \sin^2\theta} = \cos\theta$

$$\rightarrow \phi_0 = \int_0^{\pi/2} \frac{\rho\, \cancel{\cos\theta}\, d\theta / A}{\cancel{\cos\theta}} = \frac{\rho}{A}\frac{\pi}{2}$$

$$\boxed{\phi_0 = \frac{\pi}{2} \frac{\rho}{\sqrt{\rho^2 + \frac{2\alpha}{m v_\infty^2}}}}$$



Repulsive scattering: $\chi + 2\phi_0 = \pi$

$$\chi = \pi - 2\phi_0$$

$$\chi = \pi - \pi \frac{\rho}{\sqrt{\rho^2 + \frac{2\alpha}{m v_\infty^2}}} = \pi\left[1 - \frac{1}{\sqrt{1 + \frac{2\alpha}{\rho^2 m v_\infty^2}}}\right]$$

$$\left(\frac{\pi\, \rho}{\sqrt{}}\right)^2 = (\pi - \chi)^2$$

$$\frac{\pi^2 \rho^2}{\rho^2 + \frac{2\alpha}{m v_\infty^2}} = (\pi - \chi)^2$$

$$\pi^2 \rho^2 = (\pi - \chi)^2 \rho^2 + (\pi - \chi)^2 \frac{2\alpha}{m v_\infty^2}$$

$$\left(\pi^2 - (\pi - \chi)^2\right)\rho^2 = (\pi - \chi)^2 \frac{2\alpha}{m v_\infty^2}$$

$$\left(\cancel{\pi^2} - \cancel{\pi^2} - \chi^2 + 2\pi\chi\right)\rho^2 = (\pi - \chi)^2 \frac{2\alpha}{m v_\infty^2}$$

$$\rho^2 = \frac{(\pi - \chi)^2}{2\pi\chi - \chi^2} \frac{2\alpha}{m v_\infty^2}$$

$$\boxed{\rho = \frac{(\pi - \chi)}{\sqrt{2\pi\chi - \chi^2}} \sqrt{\frac{2\alpha}{m v_\infty^2}}}$$

Differential cross-section:

$$d\sigma = 2\pi\rho\, d\rho$$
$$= 2\pi\, \rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi$$
$$= \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega \quad , \quad d\Omega = 2\pi \sin\chi\, d\chi$$

Thus,

$$\frac{d\rho}{d\chi} = \sqrt{\frac{2\alpha}{m v_\infty^2}}\; \frac{-\sqrt{2\pi\chi-\chi^2} - \frac{1}{2\sqrt{\ }}(2\pi-2\chi)(\pi-\chi)}{2\pi\chi-\chi^2}$$

$$= -\sqrt{\frac{2\alpha}{m v_\infty^2}}\; \frac{\sqrt{2\pi\chi-\chi^2} + \frac{(\pi-\chi)^2}{\sqrt{\ }}}{2\pi\chi-\chi^2}$$

$$= -\sqrt{\frac{2\alpha}{m v_\infty^2}}\; \frac{2\pi\chi-\chi^2 + (\pi-\chi)^2}{(2\pi\chi-\chi^2)^{3/2}}$$

$$= -\sqrt{\frac{2\alpha}{m v_\infty^2}}\; \frac{\cancel{2\pi\chi}-\cancel{\chi^2}+\pi^2+\cancel{\chi^2}-\cancel{2\pi\chi}}{(2\pi\chi-\chi^2)^{3/2}}$$

$$= -\sqrt{\frac{2\alpha}{m v_\infty^2}}\; \frac{\pi^2}{(2\pi\chi-\chi^2)^{3/2}}$$

So,

$$d\sigma = \frac{(\pi-\chi)}{\sqrt{2\pi\chi-\chi^2}}\sqrt{\frac{2\alpha}{m v_\infty^2}}\;\frac{1}{\sin\chi}\sqrt{\frac{2\alpha}{m\alpha}}\;\frac{\pi^2}{(2\pi\chi-\chi^2)^{3/2}}\, d\Omega$$

$$= \boxed{\left(\frac{2\alpha}{m v_\infty^2}\right)\frac{d\Omega}{\sin\chi}\;\frac{\pi^2(\pi-\chi)}{(2\pi\chi-\chi^2)^{3/2}}}$$

Sec 20, Prob. 1 Small-angle scattering

Start with (18.4):

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1-\rho^2/r^2 - 2U/m v_\infty^2}}$$

Assume U is weak so that $2U/m v_\infty^2 \ll 1$

$$\frac{1}{\sqrt{\ }} = \frac{1}{\sqrt{(1-\rho^2/r^2)\left(1 - \frac{2U/m v_\infty^2}{(1-\rho^2/r^2)}\right)}}$$

$$\approx \frac{1}{\sqrt{1-\rho^2/r^2}}\left(1 + \frac{1}{\cancel{2}}\frac{\cancel{2}U/m v_\infty^2}{1-\rho^2/r^2}\right)$$

$$= \frac{1}{\sqrt{1-\rho^2/r^2}} + \frac{U/m v_\infty^2}{(1-\rho^2/r^2)^{3/2}}$$

Can replace r_{min} limit by ρ:

$$\int_{\rho}^{\infty}\frac{\rho\, dr/r^2}{\sqrt{1-\rho^2/r^2}} = -\int_{\frac{1}{\rho}}^{0}\frac{\rho\, du}{\sqrt{1-\rho^2 u^2}}$$

let: $u = \frac{1}{r}$

$du = -\frac{1}{r^2}dr$

$$= \int_0^{\frac{1}{\rho}}\frac{\rho\, du}{\sqrt{1-\rho^2 u^2}}$$

let $\rho u = \sin\theta$

$\rho\, du = \cos\theta\, d\theta$

$u = \frac{1}{\rho} \to \theta = \pi/2$

$$= \boxed{\frac{\pi}{2}}$$

Thus,

$$\phi_0 \simeq \frac{\pi}{2} + \frac{1}{m v_\infty^2} \int_\rho^\infty \frac{\rho\, dr / r^2\, U(r)}{(1-\rho^2/r^2)^{3/2}}$$

$$= \frac{\pi}{2} + \frac{1}{m v_\infty^2} \frac{\partial}{\partial \rho}\left[\int_\rho^\infty \frac{U(r)\, dr}{\sqrt{1-\rho^2/r^2}} \right]$$

Now:

$$\int_\rho^\infty \frac{U(r)\,dr}{\sqrt{1-\rho^2/r^2}} = uv \Big|_\rho^\infty - \int_\rho^\infty v\, du$$

where $u = U(r)$

$$dv = \frac{dr}{\sqrt{1-\rho^2/r^2}} = \frac{r\,dr}{\sqrt{r^2-\rho^2}} \qquad x = r^2-\rho^2, \quad dx = 2r\,dr$$

$$= \frac{dx/2}{\sqrt{x}}$$

$$\rightarrow v = \frac{1}{2}\int \frac{dx}{\sqrt{x}} = \sqrt{x} + \text{const}$$

$$= \sqrt{r^2-\rho^2} + \text{const}$$

So:

$$\int_\rho^\infty \frac{U(r)dr}{\sqrt{1-\rho^2/r^2}} = \cancel{U(r)\sqrt{r^2-\rho^2}\Big|_\rho^\infty} - \int_\rho^\infty \left(\frac{dU}{dr}\right) dr \sqrt{r^2-\rho^2}$$

assuming $U(r) \to 0$ faster than $\frac{1}{r}$ as $r \to \infty$

So

$$\phi_0 = \frac{\pi}{2} + \frac{1}{m v_\infty^2} \frac{\partial}{\partial\rho}\left[- \int_\rho^\infty \frac{dU}{dr} \cdot dr \sqrt{r^2-\rho^2} \right]$$

$$= \frac{\pi}{2} + \frac{1}{m v_\infty^2} (-) \int_\rho^\infty \frac{dU}{dr}\, dr \frac{1}{\cancel{2}\sqrt{r^2-\rho^2}} (-\cancel{2}\rho)$$

$$= \frac{\pi}{2} + \frac{\rho}{m v_\infty^2} \int_\rho^\infty dr \frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

Scattering angle χ:

$$2\phi_0 + \chi = \pi$$

$$\chi = \pi - 2\phi_0$$

$$\rightarrow \chi = \cancel{\pi} - 2\left(\cancel{\frac{\pi}{2}} + \frac{\rho}{m v_\infty^2} \int_\rho^\infty dr \frac{dU/dr}{\sqrt{r^2-\rho^2}} \right)$$

$$= -\frac{2\rho}{m v_\infty^2} \int_\rho^\infty dr \frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

In terms of θ_1:

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2 \cos\chi} \rightarrow \theta_1 \simeq \frac{m_2 \chi}{m_1 + m_2}$$

Thus,

$$\theta_1 \simeq \left(\frac{m_2}{m_1+m_2}\right)\chi$$

$$\simeq \left(\frac{m_2}{m_1+m_2}\right)\left(\frac{-2\rho}{m v_\infty^2}\right)\int_\rho^\infty dr\,\frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

$$= \frac{-2\rho}{m_1 v_\infty^2}\int_\rho^\infty dr\,\frac{dU/dr}{\sqrt{r^2-\rho^2}}$$

which is Eq. (20.3)