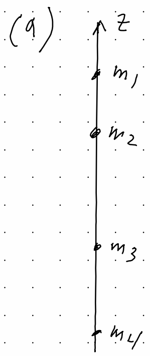
Sec 32, Prob 1:

(a)



$I_3 = 0$ since $x_a = y_a = 0$ for all masses

$$I_1 = \sum_a m_a (r_a^2 - \cancel{x_a^2}^{\,0}) = \sum_a m_a z_a^2$$

$$I_2 = \sum_a m_a (r_a^2 - \cancel{y_a^2}^{\,0}) = \sum_a m_a z_a^2$$

$$\rightarrow I_1 = I_2 \equiv I = \sum_a m_a z_a^2$$

(assuming COM at $z=0$)

If COM is not at $z=0$, but at z_{com}, then:

$$I = \sum_a m_a (z_a - z_{com})^2 , \quad z_{com} = \frac{1}{\mu}\sum_b m_b z_b$$

$$= \sum_a m_a (z_a^2 + z_{com}^2 - 2 z_{com} z_a)$$

$$= \sum_a m_a z_a^2 + \mu z_{com}^2 - 2 z_{com} \underbrace{\sum_a m_a z_a}_{\mu z_{com}}$$

$$= \sum_a m_a z_a^2 - \mu z_{com}^2$$

This last expression can be rewritten in terms of $l_{ab} \equiv |z_a - z_b|$ as follows:

$$I = \frac{1}{2}\sum_a m_a z_a^2 + \frac{1}{2}\sum_b m_b z_b^2 - \frac{1}{\mu}\Big(\sum_a m_a z_a\Big)\Big(\sum_b m_b z_b\Big)$$

$$= \frac{1}{2\mu}\sum_a\sum_b m_b m_a z_a^2 + \frac{1}{2\mu}\sum_a\sum_b m_a m_b z_b^2 - \frac{1}{\mu}\sum_a\sum_b m_a m_b z_a z_b$$

Thus

$$I = \frac{1}{2\mu}\sum_a\sum_b m_a m_b (z_a^2 + z_b^2 - 2 z_a z_b)$$

$$= \frac{1}{2\mu}\sum_a\sum_b m_a m_b (z_a - z_b)^2$$

$$= \frac{1}{2\mu}\sum_a\sum_b m_a m_b l_{ab}^2$$

NOTE: For just two masses:

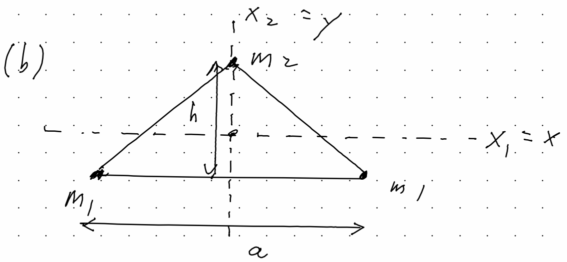
$$I = \frac{1}{2\mu}(m_1 m_2 l^2 + m_2 m_1 l^2)$$

$$= \frac{m_1 m_2}{\mu} l^2$$

$$= m l^2 \quad \text{where} \quad m \equiv \frac{m_1 m_2}{m_1 + m_2}$$

$$l = |z_1 - z_2|$$

(b)



Assume COM at $(x_1, x_2) = (x, y) = (0,0)$

Then $2m_1 y_1 + m_2 y_2 = 0$

where $y_2 - y_1 = h$

thus, $2m_1 y_1 + m_2(h + y_1) = 0$

$$(2m_1 + m_2) y_1 + m_2 h = 0$$

$$y_1 = \frac{-m_2 h}{\mu}, \quad \mu = 2m_1 + m_2 = \text{total mass}$$

and
$$y_2 = y_1 + h = \frac{-m_2 h}{\mu} + h = \frac{(\mu - m_2) h}{\mu} = \frac{2m_1 h}{\mu}$$

All masses have $z_a = 0$

Thus, $I_3 = \sum_a m_a (r_a^2 - z_a^2) = \sum_a m_a (x_a^2 + y_a^2)$

$$I_1 = \sum_a m_a (r_a^2 - x_a^2) = \sum_a m_a y_a^2$$

$$I_2 = \sum_a m_a (r_a^2 - y_a^2) = \sum_a m_a x_a^2$$

Thus, $I_3 = I_1 + I_2$

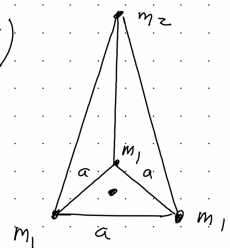
so need to calculate I_1, I_2.

$$I_1 = \sum_a m_a y_a^2 = 2m_1 y_1^2 + m_2 y_2^2 = 2m_1 \frac{m_2^2 h^2}{\mu^2} + m_2 \frac{4m_1^2 h^2}{\mu^2} = \frac{2m_1 m_2 h^2}{\mu^2}(m_2 + 2m_1) = \boxed{\frac{2m_1 m_2}{\mu} h^2}$$

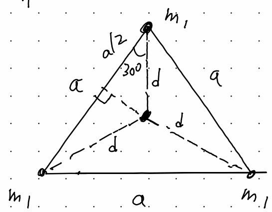
$$I_2 = \sum_a m_a x_a^2 = m_1 \left(\frac{a}{2}\right)^2 + m_1 \left(\frac{-a}{2}\right)^2 = \boxed{\frac{m_1 a^2}{2}}$$

$$I_3 = I_1 + I_2 = \boxed{\frac{2m_1 m_2}{\mu} h^2 + \frac{m_1 a^2}{2}}$$

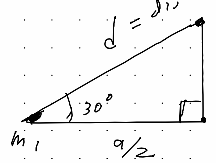
(c)



tetrahedron, height h
base: equilateral Δ with side length a



base



$$\cos 30^\circ = \frac{a}{2d} = \frac{\sqrt{3}}{2}$$

$$\rightarrow \quad d = \frac{a}{\sqrt{3}}$$

COM lies on axis of symmetry (z-axis)
Assume COM has $z=0$
Then $0 = m_2 z_2 + 3m_1 z_1$, $\quad z_2 - z_1 = h$

$$= m_2(z_1 + h) + 3m_1 z_1$$
$$= (3m_1 + m_2) z_1 + m_2 h$$

$$\rightarrow \quad z_1 = \frac{-m_2 h}{3m_1 + m_2} = -\frac{m_2 h}{\mu}$$

$$z_2 = h + z_1$$
$$= h - \frac{m_2 h}{\mu}$$
$$= \frac{3m_1 h}{\mu}$$

Since a tetrahedron has 3-fold rotational symmetry, the x_1 principal axes can be chosen arbitrarily in the plane $\perp$ to the symmetry axis ($x_3 \equiv z$). [x_2 is $\perp$ to x_1, x_3]

Thus, $I_1 = I_2 \equiv I$

$$I_3 = \sum_a m_a (r_a^2 - z_a^2)$$
$$= \sum_a m_a s_a^2 \qquad \text{where } s^2 = r^2 - z^2$$

$$\left.\begin{array}{l} I_1 = \sum_a m_a (r_a^2 - x_a^2) \\ I_2 = \sum_a m_a (r_a^2 - y_a^2) \end{array}\right\} \text{equal } (I_1 = I_2 \equiv I)$$

$$2I = I_1 + I_2$$
$$= \sum_a m_a (2r_a^2 - x_a^2 - y_a^2)$$
$$= \sum_a m_a (2(s_a^2 + z_a^2) - s_a^2)$$
$$= \sum_a m_a s_a^2 + 2\sum_a m_a z_a^2$$
$$= I_3 + 2\sum_a m_a z_a^2$$

Thus,

$$I = \frac{1}{2} I_3 + \sum_a m_a z_a^2$$

Now

$$I_3 = \sum_a m_a s_a^2$$
$$= 3 m_1 d^2$$
$$= \cancel{3} m_1 \frac{a^2}{\cancel{3}} = \boxed{m_1 a^2}$$

Also,

$$\sum_a m_a z_a^2 = 3m_1 z_1^2 + m_2 z_2^2$$

$$= 3m_1\left(-\frac{m_2 h}{\mu}\right)^2 + m_2\left(\frac{3m_1 h}{\mu}\right)^2$$

$$= \frac{3m_1 m_2^2 h^2}{\mu^2} + 9\frac{m_1^2 m_2 h^2}{\mu}$$

$$= \frac{3m_1 m_2 h^2}{\mu^2}\underbrace{(m_2 + 3m_1)}_{\mu}$$

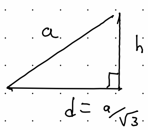
$$= \frac{3m_1 m_2 h^2}{\mu}$$

Thus,

$$I = \frac{1}{2}I_3 + \sum_a m_a z_a^2$$

$$= \boxed{\frac{1}{2}m_1 a^2 + \frac{3m_1 m_2 h^2}{\mu}}$$

Regular tetrahedron: $m_1 = m_2$



$h^2 + \frac{a^2}{3} = a^2 \rightarrow h = \sqrt{\frac{2}{3}}\,a$

$\mu = 4m_1$

$I_3 = m_1 a^2$

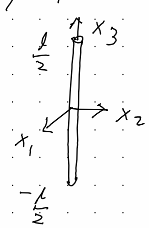
$I = \frac{1}{2}m_1 a^2 + \frac{\cancel{3}\cancel{m_1} m_1}{4\cancel{m_1}}\left(\frac{2}{\cancel{3}}\right)a^2$

$= m_1 a^2 \; (= I_1 = I_2)$

so $I_1 = I_2 = I_3 = m_1 a^2$

Sec 32, Prob 2:

(a) Thin rod of length ℓ:



$I_3 = \boxed{0}$

and $I_1 = I_2 \equiv I$

$$I = \int \rho\, dV\,(r^2 - x^2)$$

$$= \int \rho\, dV\, z^2$$

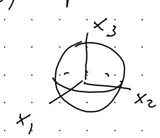
$$= \int_{-\ell/2}^{\ell/2} dz\left(\frac{\mu}{\ell}\right) z^2$$

$$= \frac{\mu}{\ell}\frac{z^3}{3}\Big|_{-\ell/2}^{\ell/2}$$

$$= \frac{\mu}{\ell}\,\frac{2}{3}\,\frac{\ell^3}{8}$$

$$= \boxed{\frac{1}{12}\mu \ell^2}$$

(b) Sphere of radius R:



$I_1 = I_2 = I_3 = I$

$$I = \frac{1}{3}(I_1 + I_2 + I_3)$$

$$= \frac{1}{3}\left[\int \rho\, dV\,(r^2 - x^2) + \int \rho\, dV\,(r^2 - y^2) + \int \rho\, dV\,(r^2 - z^2)\right]$$

$$= \frac{1}{3}\int \rho\, dV\,[3r^2 - x^2 - y^2 - z^2]$$

$$= \frac{2}{3}\int \rho\, dV\, r^2$$

$$I = \frac{2}{3}\int \rho\, dV\, r^2 \qquad \rho = \frac{M}{\frac{4}{3}\pi R^3}$$

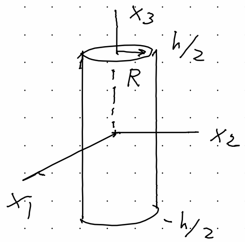
$$= \frac{2}{\cancel{3}} \frac{M}{\frac{4}{\cancel{3}}\pi R^3} \int_0^R r^4 dr \underbrace{\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi}_{4\pi}$$

$$= \frac{M}{2\pi R^3} \cdot 4\pi \int_0^R r^4 dr$$

$$= \frac{2M}{R^3} \frac{R^5}{5}$$

$$= \boxed{\frac{2}{5} M R^2}$$

(c)



$$\rho = \frac{M}{\pi R^2 \cdot h}$$

$$dV = ds\, s\, d\phi\, dz$$

where $s^2 = x^2 + y^2$

$$I_1 = I_2 \equiv I$$

$$2I = I_1 + I_2$$

$$= \int \rho\, dV (r^2 - x^2) + \int \rho\, dV (r^2 - y^2)$$

$$= \int \rho\, dV (2r^2 - s^2)$$

$$= \int \rho\, dV (2(s^2 + z^2) - s^2)$$

$$= \int \rho\, dV\, s^2 + 2\int \rho\, dV\, z^2$$

$$\rightarrow I = \frac{1}{2} I_3 + \int \rho\, dV \cdot z^2$$

$$I_3 = \int \rho\, dV\, s^2$$

$$= \frac{M}{\pi R^2 h} \int_0^R s^3 ds \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz$$

$$= \frac{M}{\cancel{\pi} R^2 \cancel{h}} \frac{R^4}{4} \cdot 2\cancel{\pi} \cdot \cancel{h}$$

$$= \boxed{\frac{1}{2} M R^2}$$

$$I = \frac{1}{2} I_3 + \int \rho\, dV\, z^2$$

$$\int \rho\, dV\, z^2 = \frac{M}{\pi R^2 h} \int_0^R s\, ds \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} z^2 dz$$

$$= \frac{M}{\cancel{\pi} \cancel{R^2} h} \frac{\cancel{R^2}}{\cancel{2}} \cdot \cancel{2}\cancel{\pi} \cdot \frac{z^3}{3}\Big|_{-h/2}^{h/2}$$

$$= \frac{M}{h} \frac{2}{3} \frac{h^3}{8}$$

$$= \frac{1}{12} M h^2$$

Thus,

$$I = \frac{1}{2}\left(\frac{1}{2} M R^2\right) + \frac{1}{12} M h^2$$

$$= \frac{1}{4} M R^2 + \frac{1}{12} M h^2$$

$$= \boxed{\frac{1}{4} M \left(R^2 + \frac{1}{3} h^2\right)}$$

NOTE: special limiting case:

(i) Thin rod $(R \to 0)$

$$I_3 = 0$$

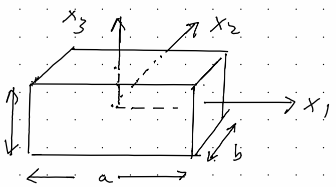
$$I_1 = I_2 = \frac{1}{12} M h^2$$

(ii) Thin disk $(h \to 0)$

$$I_3 = \frac{1}{2} M R^2$$

$$I_1 = I_2 = \frac{1}{4} M R^2$$

(d)



$$\rho = \frac{M}{abc}$$

$$dV = dx\,dy\,dz$$

$$I_1 = \int \rho\, dV\,(r^2 - x^2)$$

$$= \int \rho\, dV\,(y^2 + z^2)$$

$$= \frac{M}{abc} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz\,(y^2 + z^2)$$

$$= \frac{M}{\cancel{a}bc}\, \cancel{a} \int_{-b/2}^{b/2} dy \left(y^2 z + \frac{z^3}{3}\right)\Big|_{-c/2}^{c/2}$$

$$= \frac{M}{bc} \int_{-b/2}^{b/2} dy \left(c y^2 + \frac{2}{3}\frac{c^3}{8}\right)$$

$$= \frac{M}{bc}\left[c \frac{y^3}{3}\Big|_{-b/2}^{b/2} + \frac{1}{12} c^3 y \Big|_{-b/2}^{b/2} \right]$$

$$I_1 = \frac{M}{bc}\left[c\,\frac{2}{3}\frac{b^3}{8} + \frac{1}{12} b c^3 \right]$$

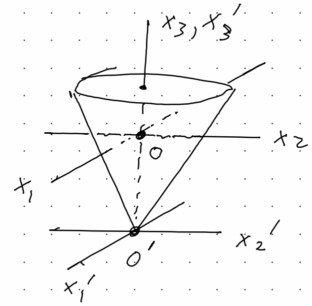
$$= \frac{M}{12bc}\left[c b^3 + b c^3 \right]$$

$$= \frac{M}{12}\left(b^2 + c^2\right)$$

$$I_2 = \frac{M}{12}\left(c^2 + a^2\right)$$

$$I_3 = \frac{M}{12}\left(a^2 + b^2\right)$$

(e)



First calculate I'_{ij} (w.r.t. x_1', x_2', x_3').

Then calculate I_{ij} via

$$I_{ij} = I'_{ij} - M(a^2 \delta_{ij} - a_i a_j)$$

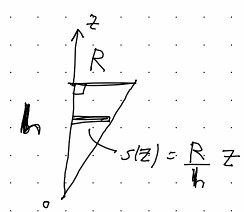
where $\vec{a} = (0, 0, -d)$

d: height of com above O'

Volume of cone:

$$V = \int dV$$

$$= \int_0^h dz \int_0^{2\pi} d\phi \int_0^{\frac{R}{h}z} s\, ds$$



$$= \cancel{2}\pi \int_0^h dz\, \frac{s^2}{\cancel{2}}\Big|_0^{Rz/h}$$

$$V = \pi \int_0^h dz \; \frac{R^2 z^2}{h^2}$$

$$= \frac{\pi R^2}{h^2} \; \frac{z^3}{3}\Big|_0^h$$

$$= \frac{\pi R^2}{h^2} \; \frac{h^3}{3}$$

$$= \boxed{\frac{1}{3}\pi R^2 h}$$

Thus, $\rho = \frac{M}{V} = \boxed{\frac{M}{\frac{1}{3}\pi R^2 h}}$ (mass density)

$$I_3' = \int \rho \, dV \, (r^2 - z^2)$$

$$= \int \rho \, dV \; s^2$$

$$= \rho \int_0^h dz \int_0^{2\pi} d\phi \int_0^{Rz/h} s^3 \, ds$$

$$= \frac{\rho}{4} \cdot 2\pi \int_0^h dz \; s^4 \Big|_0^{\frac{Rz}{h}}$$

$$= \frac{\rho \pi}{2} \int_0^h dz \left(\frac{R^4}{h^4}\right) z^4$$

$$= \frac{\rho \pi R^4}{2 h^4} \; \frac{h^5}{5}$$

$$= \frac{\rho \pi}{10} R^4 h$$

Thus,

$$I_3' = \frac{M}{\frac{1}{3}\cancel{\pi} R^2 \cancel{h}} \; \frac{\cancel{\pi}}{10} R^4 \cancel{h}$$

$$= \boxed{\frac{3}{10} M R^2}$$

Similar to the cylinder, we have $I_1' = I_2' \equiv I'$ where

$$I' = \frac{1}{2} I_3' + \int \rho \, dV \; z^2$$

Now:

$$\int \rho \, dV z^2 = \rho \int_0^h dz \; z^2 \int_0^{2\pi} d\phi \int_0^{Rz/h} s \, ds$$

$$= \cancel{2}\pi\rho \int_0^h dz \; z^2 \; \frac{s^2}{\cancel{2}}\Big|_0^{Rz/h}$$

$$= \pi\rho \frac{R^2}{h^2} \int_0^h dz \; z^4$$

$$= \pi\rho \frac{R^2}{h^2} \cdot \frac{h^5}{5}$$

$$= \frac{\pi}{5} \rho R^2 h^3$$

$$= \frac{\cancel{\pi}}{5}\left(\frac{M}{\frac{1}{3}\cancel{\pi}\cancel{R^2} h}\right) \cancel{R^2} h^3$$

$$= \frac{3}{5} M h^2$$

So

$$I' = \frac{1}{2}\left(\frac{3}{10}\mu R^2\right) + \frac{3}{5}\mu h^2$$

$$= \boxed{\frac{3}{5}\mu\left(\frac{R^2}{4} + h^2\right)} = I_1' = I_2'$$

Need to find location of COM:

$$d = \frac{1}{\mu}\int \rho \, dV \, z$$

$$= \frac{1}{\mu}\rho \int_0^h z\,dz \int_0^{2\pi} d\phi \int_0^{Rz/h} s\,ds$$

$$= \frac{1}{\mu}\rho \cdot \cancel{2}\pi \int_0^h z\,dz \; \frac{1}{\cancel{2}}\left(\frac{R}{h}\right)^2 z^2$$

$$= \frac{\pi\rho}{\mu}\frac{R^2}{h^2}\frac{z^4}{4}\Big|_0^h$$

$$= \frac{\pi\rho}{4\mu} R^2 h^2$$

$$= \frac{\cancel{\pi}}{4\cancel{\mu}} \cdot \frac{\cancel{\mu}}{\frac{1}{3}\cancel{\pi}\cancel{R^2}h} \cancel{R^2} h^2$$

$$= \boxed{\frac{3}{4}h}$$

Thus,

$$I_{ij} = I_{ij}' - \mu(a^2\delta_{ij} - a_i a_j)$$

where $\vec{a} = (0, 0, -\frac{3}{4}h) \rightarrow a^2 = \frac{9}{16}h^2$

$$\rightarrow I_1 = I_1' - \mu a^2$$

$$= \frac{3}{5}\mu\left(\frac{R^2}{4} + h^2\right) - \mu\frac{9}{16}h^2$$

$$= \frac{3}{20}\mu R^2 + \mu h^2 \left(\underbrace{\frac{3}{5} - \frac{9}{16}}_{\frac{48-45}{80} = \frac{3}{80}}\right)$$

$$= \boxed{\frac{3}{20}\mu\left(R^2 + \frac{h^2}{4}\right)}$$

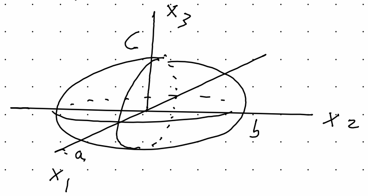
Also, $I_2 = I_1$

Finally, $I_3 = I_3' - \mu(\cancel{a^2 - a^2}) \rightarrow 0$

$$= I_3'$$

$$= \boxed{\frac{3}{10}\mu R^2}$$

(f) Ellipsoid with semi-axes a, b, c



$(a, b, c) \leftrightarrow (x_1, x_2, x_3)$

Define rescaled coordinates:

$$(u, v, w) \equiv \left(\frac{x_1}{a}, \frac{x_2}{b}, \frac{x_3}{c}\right)$$

So that boundary of ellipsoid

$$1 = \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 + \left(\frac{x_3}{c}\right)^2 = \underbrace{u^2 + v^2 + w^2}_{\text{unit 2-sphere}}$$

Volume:

$$V = \int dx_1 \int dx_2 \int dx_3$$

$$= abc \int du \int dv \int dw$$

$$= abc \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{1} r^2 dr$$

$$= abc \; 2\pi \cdot 2 \cdot \frac{r^3}{3}\Big|_0^1$$

$$= \boxed{\frac{4}{3}\pi abc}$$

$$\longrightarrow \rho = \frac{M}{\frac{4}{3}\pi abc}$$

$$I_3 = \int \rho \, dV \, (r^2 - z^2)$$

$$= \int \rho \, dV \, (x^2 + y^2)$$

$$= \frac{M}{\frac{4}{3}\pi abc} \iiint dx \, dy \, dz \, (x^2 + y^2)$$

$$= \frac{M}{\frac{4}{3}\pi \cancel{abc}} \cancel{abc} \iiint du \, dv \, dw \, (a^2 u^2 + b^2 v^2)$$

$$= \frac{M}{\frac{4}{3}\pi} \int_0^1 r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \left(a^2 r^2 \sin^2\theta \cos^2\phi + b^2 r^2 \sin^2\theta \sin^2\phi\right)$$

<u>Now:</u>

$$\int_0^{\pi} \sin\theta \, d\theta \, \sin^2\theta = \int_{-1}^{1} d(\cos\theta)(1 - \cos^2\theta)$$

$$= \int_{-1}^{1} dx \, (1 - x^2)$$

$$= \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$= 2 \cdot \frac{2}{3} = \boxed{\frac{4}{3}}$$

$$\int_0^1 r^4 dr = \frac{r^5}{5}\Big|_0^1 = \boxed{\frac{1}{5}}$$

$$\int_0^{2\pi} d\phi \begin{Bmatrix} \sin^2\phi \\ \cos^2\phi \end{Bmatrix} = 2\pi \cdot \frac{1}{2} = \boxed{\pi}$$

Thus,

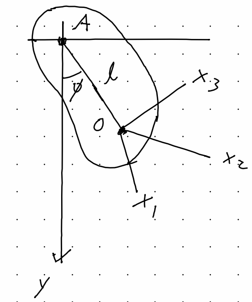
$$I_3 = \frac{M}{\cancel{\frac{4}{3}}\cancel{\pi}}\left(a^2 \cancel{\tfrac{4}{3}} \cdot \frac{1}{5} \cdot \cancel{\pi} + b^2 \cancel{\tfrac{4}{3}} \cdot \frac{1}{5} \cdot \cancel{\pi}\right)$$

$$= \boxed{\frac{M}{5}\left(a^2+b^2\right)}$$

cyclically permuting a, b, c $\rightarrow$

$$\boxed{\begin{aligned} I_1 &= \frac{M}{5}\left(b^2+c^2\right) \\ I_2 &= \frac{M}{5}\left(c^2+a^2\right) \end{aligned}}$$

Sec 32, Prob 3:



com at O
rotation axis at A, out of page

$$\vec{\Omega} = \dot{\phi}\,\hat{n}$$

$$U = Mgl\left(1-\cos\phi\right)$$

$$\approx \frac{1}{2} Mgl\phi^2 \quad \text{for } \phi \ll 1$$

$$L = T - U$$

$$T = \frac{1}{2} M V^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$\vec{\Omega} = \dot{\phi}\,\hat{n} \rightarrow \quad \begin{aligned} \Omega_1 &= \dot{\phi}\,\hat{n}\cdot\hat{x}_1 = \dot{\phi}\cos\alpha \\ \Omega_2 &= \dot{\phi}\,\hat{n}\cdot\hat{x}_2 = \dot{\phi}\cos\beta \\ \Omega_3 &= \dot{\phi}\,\hat{n}\cdot x_3 = \dot{\phi}\cos\gamma \end{aligned}$$

$$V = l\dot{\phi}$$

Thus,

$$T = \frac{1}{2} M l^2 \dot{\phi}^2 + \frac{1}{2}\dot{\phi}^2\left(I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)$$

so

$$L = \frac{1}{2}\dot{\phi}^2\left(Ml^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right) - \frac{1}{2}Mgl\phi^2$$

$$\rightarrow \omega = \sqrt{\frac{Mgl}{Ml^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma}}$$