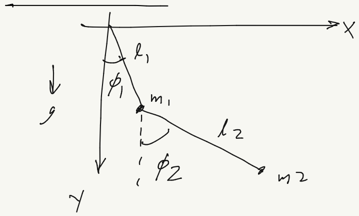
Sec 5, Prob 1:



$$x_1 = l_1 \sin\phi_1$$
$$y_1 = l_1 \cos\phi_1$$

$$x_2 = x_1 + l_2 \sin\phi_2$$
$$y_2 = y_1 + l_2 \cos\phi_2$$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_1 g l_1 \cos\phi_1 - m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$
$$= -(m_1+m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\dot{x}_1 = l_1 \dot{\phi}_1 \cos\phi_1 \rightarrow \dot{x}_1^2 = l_1^2 \dot{\phi}_1^2 \cos^2\phi_1$$
$$\dot{y}_1 = -l_1 \dot{\phi}_1 \sin\phi_1 \rightarrow \dot{y}_1^2 = l_1^2 \dot{\phi}_1^2 \sin^2\phi_1$$
$$\dot{x}_1^2 + \dot{y}_1^2 = l_1^2 \dot{\phi}_1^2$$

$$\dot{x}_2 = l_1 \dot{\phi}_1 \cos\phi_1 + l_2 \dot{\phi}_2 \cos\phi_2$$
$$\dot{y}_2 = -l_1 \dot{\phi}_1 \sin\phi_1 - l_2 \dot{\phi}_2 \sin\phi_2$$

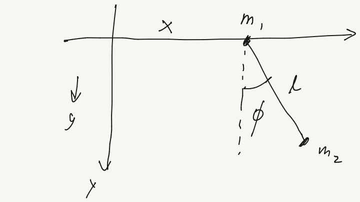
$$\rightarrow \dot{x}_2^2 = l_1^2 \dot{\phi}_1^2 \cos^2\phi_1 + l_2^2 \dot{\phi}_2^2 \cos^2\phi_2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos\phi_1 \cos\phi_2$$
$$\dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 \sin^2\phi_1 + l_2^2 \dot{\phi}_2^2 \sin^2\phi_2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin\phi_1 \sin\phi_2$$

so $$\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2)$$
$$= l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

Thus,
$$T = \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$
$$= \frac{1}{2} (m_1+m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

$$\rightarrow L = T - U$$
$$= \frac{1}{2} (m_1+m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$
$$+ (m_1+m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2$$

Sec 5, Prob 2:



$$(x_1, y_1) = (x, 0)$$
$$(x_2, y_2) = (x + l \sin\phi,\ l \cos\phi)$$

$$(\dot{x}_1, \dot{y}_1) = (\dot{x}, 0)$$
$$(\dot{x}_2, \dot{y}_2) = (\dot{x} + l\dot{\phi} \cos\phi,\ -l\dot{\phi} \sin\phi)$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$
$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}^2 + l^2 \dot{\phi}^2 \cos^2\phi + 2 l \dot{x} \dot{\phi} \cos\phi$$
$$+ l^2 \dot{\phi}^2 \sin^2\phi)$$

$$= \frac{1}{2} (m_1+m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \dot{x} \dot{\phi} \cos\phi$$

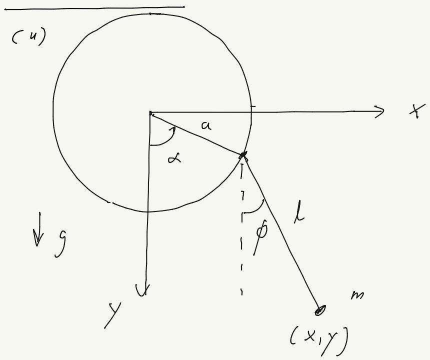
$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_2 g l \cos\phi$$

$$L = T - U$$
$$= \frac{1}{2} (m_1+m_2) \dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \dot{x} \dot{\phi} \cos\phi$$
$$+ m_2 g l \cos\phi$$

$$= \frac{1}{2} (m_1+m_2) \dot{x}^2 + \frac{1}{2} m_2 (l^2 \dot{\phi}^2 + l \dot{x} \dot{\phi} \cos\phi) + m_2 g l \cos\phi$$

Sec 5, Prob 3:

(a)



$$\alpha = \gamma t$$

$$x = a\sin\alpha + l\sin\phi$$

$$y = a\cos\alpha + l\cos\phi$$

$$U = -mgy$$

$$= \underbrace{-mga\cos\alpha}_{\text{prescribed function of time (ignore)}} - mgl\cos\phi$$

$$= -mgl\cos\phi$$

$$\dot{x} = a\gamma\cos\alpha + l\dot{\phi}\cos\phi$$

$$\dot{y} = -a\gamma\sin\alpha - l\dot{\phi}\sin\phi$$

$$\dot{x}^2 = a^2\gamma^2\cos^2\alpha + l^2\dot{\phi}^2\cos^2\phi + 2al\gamma\dot{\phi}\cos\alpha\cos\phi$$

$$\dot{y}^2 = a^2\gamma^2\sin^2\alpha + l^2\dot{\phi}^2\sin^2\phi + 2al\gamma\dot{\phi}\sin\alpha\sin\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left[a^2\gamma^2 + l^2\dot{\phi}^2 + 2al\gamma\dot{\phi}\cos(\alpha-\phi)\right]$$

$$= \underbrace{\frac{1}{2}m a^2\gamma^2}_{\text{prescribed function of time (ignore)}} + \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\cos(\gamma t-\phi)$$

$$= \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\cos(\gamma t-\phi)$$

$$L = T - U$$

$$= \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\cos(\gamma t-\phi) + mgl\cos\phi$$

Now:

$$\gamma\dot{\phi}\cos(\gamma t-\phi) = \frac{d}{dt}\underbrace{\left[-\gamma\sin(\gamma t-\phi)\right]}_{\text{can ignore since total time derivative.}} + \gamma^2\cos(\gamma t-\phi)$$

Thus,

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos(\gamma t-\phi) + mgl\cos\phi$$

NOTE:

EOM should be the same for both Lagrangians:

(1st): $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi}$

$$\frac{d}{dt}\left(ml^2\dot{\phi} + mal\gamma\cos(\gamma t-\phi)\right) = mal\gamma\dot{\phi}\sin(\gamma t-\phi) - mgl\sin\phi$$

$$ml^2\ddot{\phi} - mal\gamma^2\sin(\gamma t-\phi) + \cancel{mal\gamma\dot{\phi}\sin(\gamma t-\phi)} = \cancel{mal\gamma\dot{\phi}\sin(\gamma t-\phi)} - mgl\sin\phi$$

$$\rightarrow \ddot{\phi} = \frac{a}{l}\gamma^2\sin(\gamma t-\phi) - \frac{g}{l}\sin\phi$$

agree

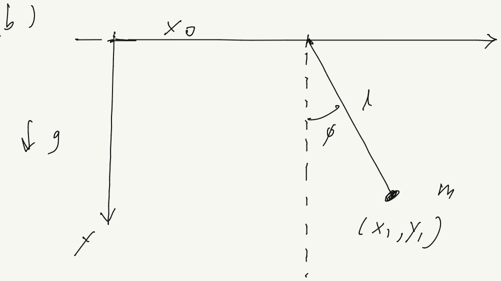
(2nd): $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi}$

$$\frac{d}{dt}\left(ml^2\dot{\phi}\right) = +mal\gamma^2\sin(\gamma t-\phi) - mgl\sin\phi$$

$$ml^2\ddot{\phi} = mal\gamma^2\sin(\gamma t-\phi) - mgl\sin\phi$$

$$\rightarrow \ddot{\phi} = \frac{a}{l}\gamma^2\sin(\gamma t-\phi) - \frac{g}{l}\sin\phi$$

(b)



$x_0 = a\cos\gamma t$

$\dot{x}_0 = -a\gamma\sin\gamma t$

$$x = x_0 + l\sin\phi$$

$$y = l\cos\phi$$

$$U = -mgy = -mgl\cos\phi$$

$$\dot{x} = \dot{x}_0 + l\dot{\phi}\cos\phi$$

$$= -a\gamma\sin\gamma t + l\dot{\phi}\cos\phi$$

$$\dot{y} = -l\dot{\phi}\sin\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left(a^2\gamma^2\sin^2\gamma t + l^2\dot{\phi}^2\cos^2\phi - 2al\gamma\dot{\phi}\sin\gamma t\cos\phi + l^2\dot{\phi}^2\sin^2\phi\right)$$

$$= \underbrace{\frac{1}{2}ma^2\gamma^2\sin^2\gamma t}_{\text{prescribed function of time (ignore)}} + \frac{1}{2}ml^2\dot{\phi}^2 - \underbrace{mal\gamma\dot{\phi}\sin\gamma t\cos\phi}$$

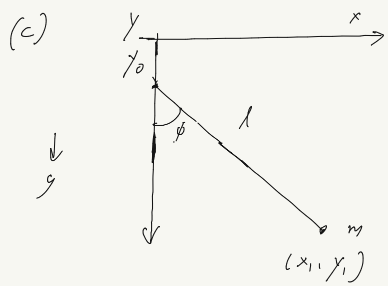
$$= \frac{d}{dt}\left(\gamma\sin\gamma t\sin\phi\right) - \gamma^2\cos\gamma t\sin\phi$$

ignore

$$= \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\sin\phi$$

$$\rightarrow L = T - U = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\sin\phi + mgl\cos\phi$$

(c)



$y = a\cos\gamma t$

$x = l\sin\phi\,,\quad y = y_0 + l\cos\phi$

$\dot{x} = l\dot{\phi}\cos\phi$

$\dot{y} = \dot{y}_0 + l\dot{\phi}\sin\phi$

$= -a\gamma\sin\gamma t - l\dot{\phi}\sin\phi$

$\dot{x}^2 = l^2\dot{\phi}^2\cos^2\phi$

$\dot{y}^2 = a^2\gamma^2\sin^2\gamma t + l^2\dot{\phi}^2\sin^2\phi + 2al\gamma\dot{\phi}\sin\gamma t\sin\phi$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$= \frac{1}{2}m(a^2\gamma^2\sin^2\gamma t + l^2\dot{\phi}^2 + 2al\gamma\dot{\phi}\sin\gamma t\sin\phi)$

$= \underbrace{\frac{1}{2}ma^2\gamma^2\sin^2\gamma t}_{\text{prescribed fcn of } t \text{ (ignore)}} + \frac{1}{2}ml^2\dot{\phi}^2 + \underbrace{mal\gamma\dot{\phi}\sin\gamma t\sin\phi}$

$= \frac{d}{dt}(-\gamma\sin\gamma t\cos\phi) + \gamma^2\cos\gamma t\cos\phi$

total time derivative (ignore)

$\simeq \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\cos\phi$

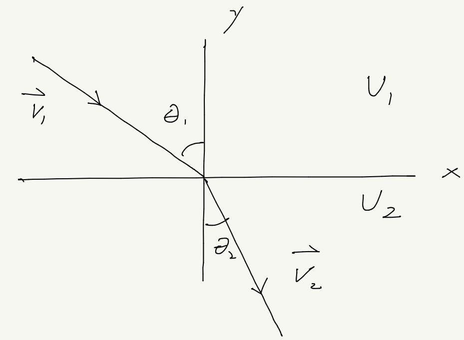
$U = -mgy = -mg(y + l\cos\phi)$

$= \underbrace{-mga\cos\gamma t}_{\text{prescribed fcn of } t \text{ (ignore)}} - mgl\cos\phi$

Thus,

$L = T - U = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\cos\phi + mgl\cos\phi$

Sec 7, Prob 1:



- Energy is conserved
- Component of linear momentum in x-direction is also conserved

i) $E = \frac{1}{2}mv_1^2 + U_1 = \frac{1}{2}mv_2^2 + U_2$

$\frac{1}{2}mv_2^2 = \frac{1}{2}mv_1^2 + (U_1 - U_2)$

$v_2 = \sqrt{v_1^2 + \frac{2}{m}(U_1 - U_2)}$

$\frac{v_2}{v_1} = \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2}mv_1^2}}$

ii) $p_x = \cancel{m}v_1\sin\theta_1 = \cancel{m}v_2\sin\theta_2$

$\rightarrow \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1}$

$= \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2}mv_1^2}}$

Sec 8, Prob 1:

$$S[q] = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t)$$

Let inertial frame K' move with velocity $\vec{V}$ wrt inertial frame K.

$$\sum_a m_a \vec{v}_a{}' = \mu \vec{R}'$$

Then:

$$\vec{v}_a = \vec{v}_a{}' + \vec{V}$$
$$\vec{r}_a = \vec{r}_a{}' + \vec{V} \cdot t$$

Thus,

$$L = \frac{1}{2} \sum_a m_a |\vec{v}_a|^2 - U(\vec{r}_1, \vec{r}_2, \dots, t)$$
$$= \frac{1}{2} \sum_a m_a |\vec{v}_a{}' + \vec{V}|^2 - U$$
$$= \frac{1}{2} \sum_a m_a |\vec{v}_a{}'|^2 + \frac{1}{2} \sum_a m_a V^2 + \left(\sum_a m_a \vec{v}_a{}'\right) \cdot \vec{V} - U$$
$$= L' + \vec{p}\,' \cdot \vec{V} + \frac{1}{2} \mu V^2$$

$$\rightarrow S = \int_{t_1}^{t_2} dt \left(L' + \vec{p}\,' \cdot \vec{V} + \frac{1}{2} \mu V^2\right)$$
$$= S' + \vec{V} \cdot \sum_a m_a \vec{r}_a{}' \Big|_{t_1}^{t_2} + \frac{1}{2} \mu V^2 \Big|_{t_1}^{t_2}$$
$$= S' + \mu \vec{V} \cdot \left(\vec{R}'(t_2) - \vec{R}'(t_1)\right) + \frac{1}{2} \mu V^2 (t_2 - t_1)$$

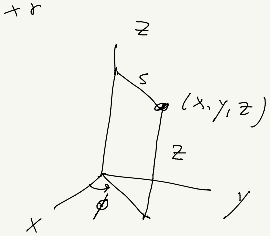
where $\vec{R}'$ is com of system wrt frame K'

Sec 9, Prob 1:

cylindrical coords (s, ϕ, z):

$$x = s\cos\phi, \quad y = s\sin\phi, \quad z = z$$

$$\vec{M} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v} = m \vec{r} \times \dot{\vec{r}}$$



Thus,

$$M_x = m(y\dot{z} - z\dot{y})$$
$$M_y = m(z\dot{x} - x\dot{z})$$
$$M_z = m(x\dot{y} - y\dot{x})$$

$$\dot{x} = \dot{s}\cos\phi - s\dot{\phi}\sin\phi$$
$$\dot{y} = \dot{s}\sin\phi + s\dot{\phi}\cos\phi$$
$$\dot{z} = \dot{z}$$

$$\rightarrow M_x = m\left[s\sin\phi\, \dot{z} - z(\dot{s}\sin\phi + s\dot{\phi}\cos\phi)\right]$$
$$= m\left[\sin\phi(s\dot{z} - z\dot{s}) - z s\dot{\phi}\cos\phi\right]$$

$$M_y = m\left[z(\dot{s}\cos\phi - s\dot{\phi}\sin\phi) - s\cos\phi\, \dot{z}\right]$$
$$= m\left[-\cos\phi(s\dot{z} - z\dot{s}) - z s\dot{\phi}\sin\phi\right]$$

$$M_z = m\left[s\cos\phi(\cancel{\dot{s}\sin\phi} + s\dot{\phi}\cos\phi) - s\sin\phi(\cancel{\dot{s}\cos\phi} - s\dot{\phi}\sin\phi)\right]$$
$$= m s^2 \dot{\phi}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$= m^2 \Big[\sin^2\phi\,(s\dot z - z\dot s)^2 + z^2 s^2 \dot\phi^2 \cos^2\phi$$

$$\cancel{- 2 z s\dot\phi \cos\phi \sin\phi\,(s\dot z - z\dot s)}$$

$$+ \cos^2\phi\,(s\dot z - z\dot s)^2 + z^2 s^2 \dot\phi^2 \sin^2\phi$$

$$\cancel{+ 2 z s\dot\phi \cos\phi \sin\phi\,(s\dot z - z\dot s)}$$

$$+ s^4 \dot\phi^2 \Big]$$

$$= m^2 \left[(s\dot z - z\dot s)^2 + z^2 s^2 \dot\phi^2 + s^4 \dot\phi^2 \right]$$

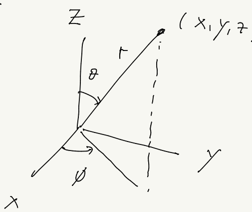
$$= m^2 \left[(s\dot z - z\dot s)^2 + s^2 (z^2 + s^2) \dot\phi^2 \right]$$

<u>Sec 9, Prob 2</u>:

spherical polar coords (r, θ, ϕ):

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$\vec M = \vec r \times \vec p = m\vec r \times \vec v = m\vec r \times \dot{\vec r}$$



Thus,

$$M_x = m(y\dot z - z\dot y)$$
$$M_y = m(z\dot x - x\dot z)$$
$$M_z = m(x\dot y - y\dot x)$$

$$\dot x = \dot r\sin\theta\cos\phi + r\dot\theta\cos\theta\cos\phi - r\dot\phi\sin\theta\sin\phi$$

$$\dot y = \dot r\sin\theta\sin\phi + r\dot\theta\cos\theta\sin\phi + r\dot\phi\sin\theta\cos\phi$$

$$\dot z = \dot r\cos\theta - r\dot\theta\sin\theta$$

$$\rightarrow M_x = m\Big[r\sin\theta\sin\phi\,(\cancel{\dot r\cos\theta} - r\dot\theta\sin\theta)$$
$$- r\cos\theta\,(\cancel{\dot r\sin\theta\sin\phi} + r\dot\theta\cos\theta\sin\phi + r\dot\phi\sin\theta\cos\phi)\Big]$$

$$= m\left[-r^2\dot\theta\sin\phi\,(\sin^2\theta + \cos^2\theta) - r^2\dot\phi\sin\theta\cos\theta\cos\phi \right]$$

$$= m\left[-r^2\dot\theta\sin\phi - r^2\dot\phi\sin\theta\cos\theta\cos\phi \right]$$

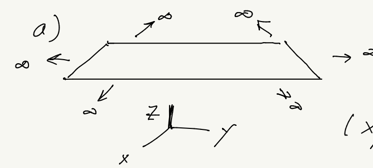
$$M_y = m\Big[r\cos\theta\,(\cancel{\dot r\sin\theta\cos\phi} + r\dot\theta\cos\theta\cos\phi - r\dot\phi\sin\theta\sin\phi)$$
$$- r\sin\theta\cos\phi\,(\cancel{\dot r\cos\theta} - r\dot\theta\sin\theta)\Big]$$

$$= m\left[r^2\dot\theta\cos\phi - r^2\dot\phi\sin\theta\cos\theta\sin\phi \right]$$

$$M_z = m\Big[r\sin\theta\cos\phi\big(\cancel{\dot r\sin\theta\sin\phi} + \cancel{r\dot\theta\cos\theta\sin\phi} + r\dot\phi\sin\theta\cos\phi\big)$$
$$- r\sin\theta\sin\phi\big(\cancel{\dot r\sin\theta\cos\phi} + \cancel{r\dot\theta\cos\theta\cos\phi} - r\dot\phi\sin\theta\sin\phi\big)\Big]$$
$$= m\left[r^2\dot\phi\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right)\right]$$
$$= m\, r^2\dot\phi\sin^2\theta$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$
$$= m^2\Big[\, r^4\dot\theta^2\sin^2\phi + r^4\dot\phi^2\sin^2\theta\cos^2\theta\cos^2\phi$$
$$\cancel{+2r^4\dot\theta\dot\phi\sin\theta\cos\theta\sin\phi\cos\phi}$$
$$+ r^4\dot\theta^2\cos^2\phi + r^4\dot\phi^2\sin^2\theta\cos^2\theta\sin^2\phi$$
$$\cancel{-2r^4\dot\theta\dot\phi\sin\theta\cos\theta\sin\phi\cos\phi}$$
$$+ r^4\dot\phi^2\sin^4\theta\,\Big]$$
$$= m^2r^4\left[\dot\theta^2 + \dot\phi^2\left(\sin^2\theta\cos^2\theta + \sin^4\theta\right)\right]$$
$$= m\, r^4\left[\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right]$$

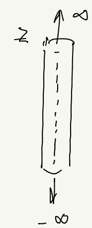
Sec 9, Prob 3:

a) 

P_x, P_y = const

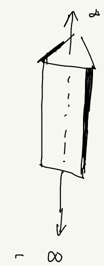
M_z = const

(xy-plane // to homog. plane)

b) 

P_z , M_z = const

(z-axis is axis of cylinder)

c) 

infinite homog. cylinder

P_z = const

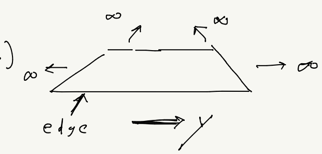
(z-axis is // to edges of prism

d) 

two points

M_z = const

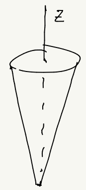
(z-axis passes through the two points)

e) 

∞ homog. $\frac{1}{2}$ plane

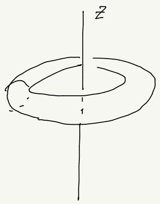
P_y = const

(y axis is // to edge of $\frac{1}{2}$ plane)

f) 

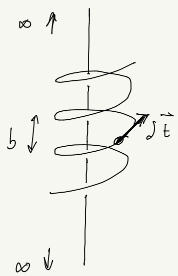
$M_z = \text{const}$

(z-axis is axis of cone)

g) 

$M_z = \text{const}$

(z-axis is axis of torus)

h) 

a = radius of helix

b = height between neighboring coils of helix

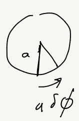
$h \equiv b/a$ (pitch of helix)

Lagrangian invariant w.r.t translations around the helix:

$$\delta\vec{t} = a\,\delta\phi\,\hat{\phi} + \frac{b\,\delta\phi}{2\pi}\,\hat{z}$$

$$= a\,\delta\phi\left[\hat{\phi} + \frac{b/a}{2\pi}\,\hat{z}\right]$$

$$= a\,\delta\phi\left[\hat{\phi} + \frac{h}{2\pi}\,\hat{z}\right]$$



$$\delta L = \frac{\partial L}{\partial\phi}\,\delta\phi + \frac{\partial L}{\partial z}\,\delta z$$

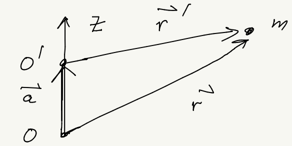
$$= \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right)\delta\phi + \frac{d}{dt}\left(\frac{\partial L}{\partial\dot z}\right)\frac{h}{2\pi}\,\delta\phi$$

$$= \delta\phi\,\frac{d}{dt}\Big[\underbrace{p_\phi}_{M_z} + p_z\frac{h}{2\pi}\Big] \rightarrow \boxed{M_z + \frac{p_z h}{2\pi} = \text{const}}$$

NOTE: M_z is independent of location of origin on z-axis

$$\vec{M} = \vec{r}\times\vec{p} = m\vec{r}\times\dot{\vec{r}}$$

Change origin by shifting along $\vec{a} = a\hat{z}$



$$\vec{r} = \vec{r}\,' + \vec{a}$$

$$\vec{M} = m\vec{r}\times\dot{\vec{r}}$$

$$= m(\vec{r}\,'+\vec{a})\times\frac{d}{dt}(\vec{r}\,'+\vec{a})$$

$$= m\vec{r}\,'\times\dot{\vec{r}}\,' + m\vec{a}\times\dot{\vec{r}}\,'$$

$$= \vec{M}' + \vec{a}\times\vec{p}\,' \quad \text{(for arbitrary } \vec{a}\text{)}$$

Thus,

$$M_z = \vec{M}\cdot\hat{z}$$

$$= (\vec{M}' + \vec{a}\times\vec{p}\,')\cdot\hat{z}$$

$$= M'_z + a(\hat{z}\times\vec{p}\,')\cdot\hat{z}$$

$$= M'_z + a\,\cancelto{0}{(\hat{z}\times\hat{z})}\cdot\vec{p}\,'$$

$$= M'_z$$

Sec 10, Prob 1:

same path, different masses, same potential energies

$\rightarrow$ $x' = x$, $m' \neq m$, $U' = U$, $t' \neq t$

$$L = T - U = \frac{1}{2} m \dot{x}^2 - U$$

$$L' = \frac{1}{2} m' \left(\frac{dx}{dt'}\right)^2 - U$$

$$= \frac{1}{2} m' \left(\frac{t}{t'}\right)^2 \dot{x}^2 - U$$

Thus, $L' = L$ $\rightarrow$ $m' \left(\frac{t}{t'}\right)^2 = m$

$$\left(\frac{t'}{t}\right)^2 = \frac{m'}{m}$$

$$\rightarrow \quad \frac{t'}{t} = \sqrt{\frac{m'}{m}}$$

Sec 10, Prob 2:

same path, same mass, potential energies differing by a constant factor ($U' = c\,U$)

$\rightarrow$ $x = x'$, $m = m'$, $t' \neq t$

$$L = T - U$$
$$= \frac{1}{2} m \dot{x}^2 - U$$

$$L' = \frac{1}{2} m \left(\frac{dx}{dt'}\right)^2 - U'$$

$$= \frac{1}{2} m \left(\frac{t}{t'}\right)^2 \dot{x}^2 - c\,U$$

Thus, need $\left(\frac{t}{t'}\right)^2 = c$ to get same EOM,

$$\rightarrow \quad \frac{t'}{t} = \sqrt{\frac{1}{c}}$$
$$= \sqrt{\frac{U}{U'}}$$

Sec 40, Prob 1

Hamiltonian for a single particle

$$L = T - U$$
$$= \frac{1}{2}m|\dot{\vec{r}}|^2 - U(\vec{r},t)$$

Cartesian:

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - U$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = p_x/m$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \rightarrow \dot{y} = p_y/m$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$\rightarrow \quad H = \left(\sum_i p_i \dot{q}_i - L\right)_{\dot{q}_i = \dot{q}_i(q,p)}$$

$$= \left(p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + U(x,y,z,t)\right)\Big|_{\dot{x}=p_x/m,\,etc}$$

$$= \frac{1}{2m}\left(p_x^2+p_y^2+p_z^2\right) + U(x,y,z,t)$$

cylindrical: (s,ϕ,z)

$$L = \frac{1}{2}m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2) - U(s,\phi,z,t)$$

$$p_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s} \rightarrow \dot{s} = p_s/m$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ms^2\dot{\phi} \rightarrow \dot{\phi} = p_\phi/ms^2$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$\rightarrow H = \left(p_s\dot{s} + p_\phi\dot{\phi} + p_z\dot{z} - L\right)_{\dot{s}=p_s/m,\,etc}$$

$$= \frac{p_s^2}{m} + \frac{p_\phi^2}{ms^2} + \frac{p_z^2}{m}$$
$$- \frac{1}{2}m\left[\left(\frac{p_s}{m}\right)^2 + s^2\left(\frac{p_\phi}{ms^2}\right)^2 + \left(\frac{p_z}{m}\right)^2\right] + U(s,\phi,z,t)$$

$$= \frac{1}{2m}\left(p_s^2 + \frac{p_\phi^2}{s^2} + p_z^2\right) + U(s,\phi,z,t)$$

spherical polar: (r,θ,ϕ)

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - U(r,\theta,\phi,t)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \longrightarrow \dot{r} = p_r/m$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \rightarrow \dot{\theta} = p_\theta/mr^2$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi} \rightarrow \dot{\phi} = p_\phi/mr^2\sin^2\theta$$

$$\rightarrow H = \left(p_r \dot r + p_\theta \dot\theta + p_\phi \dot\phi - L\right)\Big|_{\dot r = p_r/m,\ etc.}$$

$$= p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{mr^2}\right) + p_\phi\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)$$

$$- \frac{1}{2} m\left(\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{mr^2}\right)^2 + r^2\sin^2\theta\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)^2\right)$$

$$+ U(r,\theta,\phi,t)$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r,\theta,\phi,t)$$

Sec 40, Prob 2:

For a uniformly rotating ref. frame:

$$L = \frac{1}{2} m v^2 + m\vec v\cdot(\vec\Omega\times\vec r) + \frac{1}{2} m|\vec\Omega\times\vec r|^2 - U$$

Hamiltonian

$$H = \left(\sum_i p_i \dot q_i - L\right)\Big|_{\dot q = \dot q(q,p)}$$

where

$$\vec p \equiv \frac{\partial L}{\partial \vec v}$$

$$= m\vec v + m(\vec\Omega\times\vec r)$$

$$= m[\vec v + \vec\Omega\times\vec r] \text{ ——— velocity wrt inertial frame}$$

$$\rightarrow \vec v = \frac{\vec p}{m} - \vec\Omega\times\vec r$$

thus,

$$H = (\vec p\cdot\vec v - L)\Big|_{\vec v = \vec p/m - \vec\Omega\times\vec r}$$

$$= \vec p\cdot\left(\frac{\vec p}{m} - \vec\Omega\times\vec r\right) - \frac{1}{2} m\left|\frac{\vec p}{m} - \vec\Omega\times\vec r\right|^2$$

$$- m\left(\frac{\vec p}{m} - \vec\Omega\times\vec r\right)\cdot(\vec\Omega\times\vec r) - \frac{1}{2} m|\vec\Omega\times\vec r|^2 + U$$

$$= \frac{p^2}{m} - \cancel{\vec p\cdot(\vec\Omega\times\vec r)} - \frac{1}{2} m\left(\frac{p^2}{m^2} + \cancel{|\vec\Omega\times\vec r|^2} - \cancel{\frac{2}{m}\vec p\cdot(\vec\Omega\times\vec r)}\right)$$

$$- \vec p\cdot(\vec\Omega\times\vec r) + \cancel{m|\vec\Omega\times\vec r|^2} - \cancel{\frac{1}{2} m|\vec\Omega\times\vec r|^2} + U$$

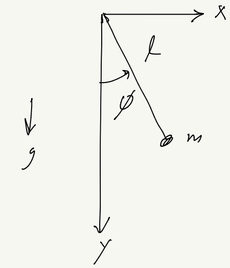
$$H = \frac{p^2}{2m} - \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + U$$

$$= \frac{p^2}{2m} - \vec{\Omega}\cdot(\vec{r}\times\vec{p}) + U$$

$$= \frac{p^2}{2m} - \vec{\Omega}\cdot\vec{M} + U$$

Sec 11, Prob 1:

Simple pendulum:



Cons. of energy:

$$E = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$$

$$= \text{const.}$$

Assume pendulum bob released from rest from $\phi = \phi_0$.

$$\rightarrow E = -mgl\cos\phi_0$$

Thus, $-mgl\cos\phi_0 = \frac{1}{2} ml^2\dot{\phi}^2 - mgl\cos\phi$

$$\frac{d\phi}{dt} \equiv \dot{\phi} = \pm\sqrt{\frac{2g}{l}(\cos\phi - \cos\phi_0)}$$

$$= \pm\sqrt{2}\,\omega_0\sqrt{\cos\phi - \cos\phi_0}$$

where $\omega_0 \equiv \sqrt{\frac{g}{l}}$ (angular freq in small-angle approximation)

Separable differential equation:

$$\sqrt{2}\,\omega_0\int dt = \pm\int\frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$

$\frac{1}{4}$ period for $\phi = \phi_0 \rightarrow \phi = 0$

$$\sqrt{2}\,\omega_0\frac{P}{4} = \int_0^{\phi_0}\frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$

$$\rightarrow \quad P = \frac{1}{\omega_0}\frac{4}{\sqrt{2}}\int_0^{\phi_0}\frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$

substitute: $\cos\phi = \cos^2(\phi/2) - \sin^2(\phi/2)$

$$= 1 - 2\sin^2(\phi/2)$$

$$\cos\phi_0 = 1 - 2\sin^2(\phi_0/2)$$

$$\rightarrow \sqrt{\ } = \sqrt{2}\sqrt{\sin^2(\phi_0/2) - \sin^2(\phi/2)}$$

$$= \sqrt{2}\,\sin(\phi_0/2)\sqrt{1 - \frac{\sin^2(\phi/2)}{\sin^2(\phi_0/2)}}$$

$$= \sqrt{2}\,\sin\left(\frac{\phi_0}{2}\right)\sqrt{1 - x^2}$$

where $x \equiv \dfrac{\sin(\phi/2)}{\sin(\phi_0/2)}$

NOTE: $\phi = 0,\ \phi_0 \rightarrow x = 0,\ 1$

$$dx = \frac{1}{k}\frac{1}{2}\cos\left(\frac{\phi}{2}\right)d\phi$$

$$= \frac{1}{2k}\sqrt{1 - \sin^2(\phi/2)}\,d\phi$$

$$= \frac{1}{2k}\sqrt{1 - k^2x^2}\,d\phi, \quad k \equiv \sin(\phi_0/2)$$

Thus, $d\phi = \dfrac{2k\,dx}{\sqrt{1 - k^2x^2}}$

$$\rightarrow \quad P = \frac{1}{\omega_0}\frac{4}{\cancel{\sqrt{2}}}\int_0^1 \frac{\cancel{2k}\,dx}{\sqrt{1 - k^2x^2}}\ \frac{1}{\cancel{\sqrt{2}}\ \cancel{k}\sqrt{1 - x^2}}$$

$$= \frac{4}{\omega_0}\int_0^1 \frac{dx}{\sqrt{1 - x^2}\sqrt{1 - k^2x^2}}$$

$$= \frac{4}{\omega_0}K(k)$$

complete elliptic integral of the 1st kind.

Expand $K(k)$ keeping 1st non-zero correction:

$$k = \sin\left(\frac{\phi_0}{2}\right) \approx \frac{\phi_0}{2}$$

$$K(k) = \int_0^1 \frac{dx}{\sqrt{1 - x^2}\sqrt{1 - k^2x^2}}$$

$$\cong \int_0^1 \frac{dx}{\sqrt{1 - x^2}}\left(1 + \frac{1}{2}k^2x^2\right)$$

$$= \int_0^1 \frac{dx}{\sqrt{1 - x^2}} + \frac{1}{2}k^2\int_0^1 \frac{dx\ x^2}{\sqrt{1 - x^2}}$$

$$= \sin^{-1}(1) + \frac{1}{2}k^2\int_0^1 \frac{dx\ x^2}{\sqrt{1 - x^2}}$$

Now. $\sin^{-1}(1) = \pi/2$

$$\int_0^1 dx\,\frac{x^2}{\sqrt{1-x^2}} = \int_0^{\pi/2} \frac{\cancel{\cos\theta}\, d\theta\, \sin^2\theta}{\cancel{\sqrt{1-\sin^2\theta}}}$$

$x = \sin\theta$
$dx = \cos\theta\, d\theta$

$$= \int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}(1-\cos 2\theta)$$

$$\left(\text{using } \cos 2\theta = 1 - 2\sin^2\theta \rightarrow \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{2}\cancelto{0}{\sin 2\theta}\Big|_0^{\pi/2}\right)$$

$$= \frac{\pi}{4}$$

Thus,

$$K(k) \approx \frac{\pi}{2} + \frac{1}{2}k^2\frac{\pi}{4}$$

$$= \frac{\pi}{2}\left(1 + \frac{1}{4}k^2\right) \qquad k \approx \frac{\phi_0}{2}$$

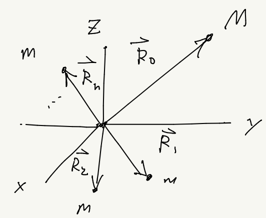
$$= \frac{\pi}{2}\left(1 + \frac{1}{16}\phi_0^2\right)$$

$$\rightarrow P = \frac{4}{\omega_0}K(k) \cong \frac{2\pi}{\omega_0}\left(1 + \frac{1}{16}\phi_0^2\right)$$

Sec 13, Prob 1:

$\vec{R}_0$: position vector for mass M

$\vec{R}_i$, $i = 1, 2, \ldots, n$: position vectors for n masses all with mass m



COM frame : $\vec{0} = M\vec{R}_0 + m\sum_i \vec{R}_i$

Relative position vectors:

$$\vec{r}_i \equiv \vec{R}_i - \vec{R}_0$$

Thus,
$$\vec{0} = M\vec{R}_0 + m\sum_i\left(\vec{R}_0 + \vec{r}_i\right)$$

$$= \underbrace{(M + nm)}_{\text{total mass } \mu}\vec{R}_0 + m\sum_i \vec{r}_i$$

$$\rightarrow \vec{R}_0 = -\frac{m}{\mu}\sum_i \vec{r}_i$$

Potential energy:

$$U = U\left(|\vec{R}_1 - \vec{R}_2|, \ldots, |\vec{R}_1 - \vec{R}_0|, \ldots, |\vec{R}_n - \vec{R}_0|\right)$$

$$= \underbrace{U\left(|\vec{r}_1 - \vec{r}_2|, \ldots, |\vec{r}_1|, \ldots, |\vec{r}_n|\right)}_{\text{depends only on the relative position vectors } \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n}$$

Kinetic energy:

$$T = \frac{1}{2} M |\dot{\vec{R}}_0|^2 + \frac{1}{2} m \sum_i |\dot{\vec{R}}_i|^2$$

Now: $\dot{\vec{R}}_i = \dot{\vec{r}}_i + \dot{\vec{R}}_0$

$$\to |\dot{\vec{R}}_i|^2 = |\dot{\vec{r}}_i|^2 + |\dot{\vec{R}}_0|^2 + 2\, \dot{\vec{r}}_i \cdot \dot{\vec{R}}_0$$

and $\dot{\vec{R}}_0 = -\frac{m}{\mu} \sum_i \dot{\vec{r}}_i$

$$\to |\dot{\vec{R}}_0|^2 = \frac{m^2}{\mu^2} \left|\sum_i \dot{\vec{r}}_i\right|^2$$

thus,

$$T = \frac{1}{2} \frac{M m^2}{\mu^2} \left|\sum_i \dot{\vec{r}}_i\right|^2 + \frac{1}{2} m \sum_i |\dot{\vec{r}}_i|^2 + \frac{1}{2} m n |\dot{\vec{R}}_0|^2 + m \sum_i \dot{\vec{r}}_i \cdot \dot{\vec{R}}_0$$

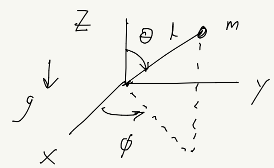
$$= \frac{1}{2} m \sum_i |\dot{\vec{r}}_i|^2 + \frac{1}{2} \frac{M m^2}{\mu^2} \left|\sum_i \dot{\vec{r}}_i\right|^2 + \frac{1}{2} m n \frac{m^2}{\mu^2} \left|\sum_i \dot{\vec{r}}_i\right|^2 - \frac{m^2}{\mu} \left|\sum_i \dot{\vec{r}}_i\right|^2$$

$$= \frac{1}{2} m \sum_i |\dot{\vec{r}}_i|^2 + \frac{1}{2} \frac{m^2}{\mu^2} \left|\sum_i \dot{\vec{r}}_i\right|^2 (\underbrace{M + mn}_{\mu} - 2\mu)$$

$$= \frac{1}{2} m \sum_i |\dot{\vec{r}}_i|^2 - \frac{1}{2} \frac{m^2}{\mu} \left|\sum_i \dot{\vec{r}}_i\right|^2$$

$$\to \boxed{L = \frac{1}{2} m \sum_i |\dot{\vec{r}}_i|^2 - \frac{1}{2} \frac{m^2}{\mu} \left|\sum_i \dot{\vec{r}}_i\right|^2 - U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n)}$$

Sec 14, Prob 1



$$x = l \sin\theta \cos\phi$$
$$y = l \sin\theta \sin\phi$$
$$z = l \cos\theta$$

$$U = mgz = mgl\cos\theta$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$= \frac{1}{2} m (l^2 \dot\theta^2 + l^2 \sin^2\theta\, \dot\phi^2)$$

$$L = T - U$$
$$= \frac{1}{2} m l^2 (\dot\theta^2 + \sin^2\theta\, \dot\phi^2) - mgl\cos\theta$$

No explicit t dependence or ϕ dependence

$$\to E \equiv T + U = \text{const}, \quad M_z \equiv p_\phi \equiv \frac{\partial L}{\partial \dot\phi} = \text{const}$$

$$E = \frac{1}{2} m l^2 (\dot\theta^2 + \sin^2\theta\, \dot\phi^2) + mgl\cos\theta$$

$$M_z = m l^2 \sin^2\theta\, \dot\phi \to \dot\phi = \frac{M_z}{m l^2 \sin^2\theta}$$

Thus,

$$E = \frac{1}{2} m l^2 \dot\theta^2 + \frac{1}{2} m l^2 \sin^2\theta \left(\frac{M_z^2}{m^2 l^4 \sin^4\theta}\right) + mgl\cos\theta$$

$$= \frac{1}{2} m l^2 \dot\theta^2 + \underbrace{\frac{M_z^2}{2 m l^2 \sin^2\theta} + mgl\cos\theta}_{\text{effective potential} \equiv V_{eff}(\theta)}$$

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + U_{eff}(\theta)$$

$$\rightarrow \dot{\theta} = \pm\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}$$

so $$\frac{d\theta}{\pm\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}} = dt$$

$$\rightarrow \boxed{t = \int \frac{d\theta}{\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}} + \text{const}}$$

$$\longrightarrow t = t(\theta)$$

Trajectory :

To find $\theta = \theta(\phi)$ write

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d\theta}{d\phi}\dot{\phi} = \frac{d\theta}{d\phi}\frac{M_z}{ml^2\sin^2\theta}$$

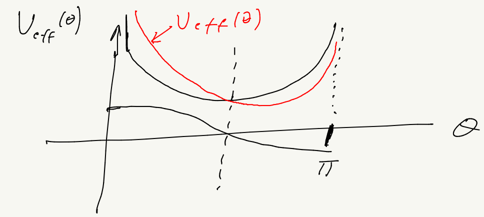
so $$\frac{d\theta}{d\phi}\frac{M_z}{ml^2\sin^2\theta} = \pm\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}$$

$$\frac{d\theta/\sin^2\theta}{\pm\frac{ml^2}{M_z}\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}} = d\phi$$

$$\rightarrow \boxed{\phi = \int \frac{d\theta/\sin^2\theta}{\sqrt{\frac{2ml^2}{M_z^2}\left(E - U_{eff}(\theta)\right)}} + \text{const}}$$

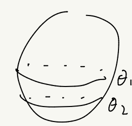
$$U_{eff}(\theta) = \frac{M_z^2}{2ml^2\sin^2\theta} + mgl\cos\theta$$

$$= \frac{M_z^2}{2ml^2(1-\cos^2\theta)} + mgl\cos\theta$$



Turning points : $\theta = \theta_1, \theta_2$ for which

$$E - U_{eff}(\theta) = 0$$



$$E = U_{eff}(\theta)$$

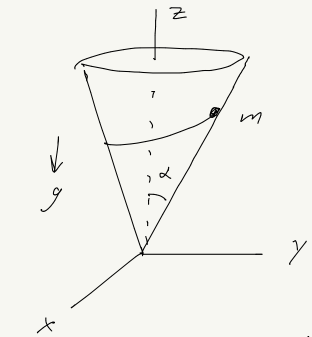
$$= \frac{M_z^2}{2ml^2(1-\cos^2\theta)} + mgl\cos\theta$$

$$2ml^2E(1-\cos^2\theta) = M_z^2 + 2m^2gl^3\cos\theta(1-\cos^2\theta)$$

$$2ml^2E - 2ml^2E\cos^2\theta = M_z^2 + 2m^2gl^3\cos\theta - 2m^2gl^3\cos^3\theta$$

$$\rightarrow \underbrace{\left(2ml^2E - M_z^2\right) - 2m^2gl^3\cos\theta - 2ml^2E\cos^2\theta + 2m^2gl^3\cos^3\theta = 0}_{\text{Cubic equation for } \cos\theta}$$

Sec 14, Prob 2



spherical coords: (r, θ, ϕ)

$$\theta = \alpha$$

$$\rightarrow \quad x = r \sin\alpha \cos\phi$$
$$y = r \sin\alpha \sin\phi$$
$$z = r \cos\alpha$$

$$T = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$= \tfrac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2)$$
$$= \tfrac{1}{2} m (\dot{r}^2 + r^2 \sin^2\alpha\, \dot{\phi}^2)$$

$$U = mgz = mgr\cos\alpha$$

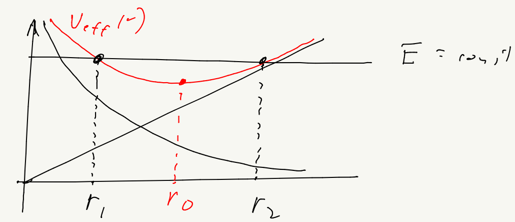
$$L = T - U$$

No explicit t, ϕ dependence $\rightarrow$

$$E = T + U = \text{const}$$
$$M_z \equiv p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\alpha\, \dot{\phi} = \text{const}$$
$$\rightarrow \quad \dot{\phi} = \frac{M_z}{m r^2 \sin^2\alpha}$$

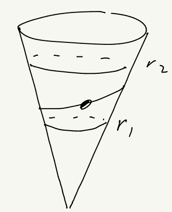
$$E = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m r^2 \sin^2\alpha \left(\frac{M_z}{m r^2 \sin^2\alpha}\right)^2 + mgr\cos\alpha$$
$$= \tfrac{1}{2} m \dot{r}^2 + \frac{M_z^2}{2 m r^2 \sin^2\alpha} + mgr\cos\alpha$$
$$= \tfrac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = \frac{M_z^2}{2 m r^2 \sin^2\alpha} + mgr\cos\alpha$$



Turning points: $r = r_1, r_2$ when $E = U_{eff}(r)$

$$E = \frac{M_z^2}{2 m r^2 \sin^2\alpha} + mgr\cos\alpha$$



$$2mE r^2 \sin^2\alpha = M_z^2 + 2m^2 g r^3 \sin^2\alpha \cos\alpha$$

$$\underbrace{0 = M_z - 2mEr^2\sin^2\alpha - 2m^2 g r^3 \sin^2\alpha\cos\alpha}_{\text{cubic equation for } r}$$

$$r_1 \le r \le r_2$$

Integrals for $t = t(r)$, $\phi = \phi(r)$:

$$E = \tfrac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

$$\pm\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)} = \frac{dr}{dt}$$

$$\rightarrow \quad \boxed{t = \pm \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}} + \text{const}}$$

Orbit equation:

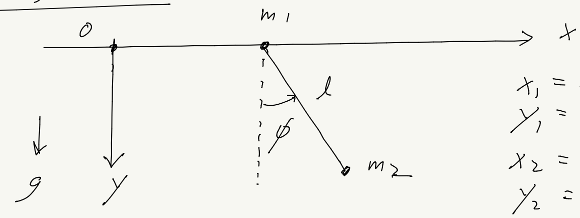
$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M_z}{mr^2\sin^2\alpha}$$

so
$$\frac{dr}{d\phi}\frac{M_z}{mr^2\sin^2\alpha} = \pm\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}$$

$$\frac{dr/r^2}{\frac{m\sin^2\alpha}{M_z}\sqrt{\cdots}} = \pm\, d\phi$$

$$\rightarrow \boxed{\phi = \frac{\pm M_z}{\sin^2\alpha}\int\frac{dr/r^2}{\sqrt{2m\left(E - U_{eff}(r)\right)}} + \text{const}}$$

Sec 14, Prob 3:



$x_1 = x$
$y_1 = 0$
$x_2 = x + l\sin\phi$
$y_2 = l\cos\phi$

$$U = -mgy_2$$
$$= -mgl\cos\phi$$

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$
$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left[(\dot{x} + l\dot{\phi}\cos\phi)^2 + l^2\dot{\phi}^2\sin^2\phi\right]$$
$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left[\dot{x}^2 + 2l\dot{x}\dot{\phi}\cos\phi + l^2\dot{\phi}^2\cos^2\phi + l^2\dot{\phi}^2\sin^2\phi\right]$$
$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2 l\dot{x}\dot{\phi}\cos\phi$$

$$L = T - U$$
$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2 l\dot{x}\dot{\phi}\cos\phi + mgl\cos\phi$$

No explicit t, x dependence $\rightarrow$

$$E = T + U = \text{const}$$
$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2 l\dot{\phi}\cos\phi = \text{const}$$

Since $p_x = \text{const}$, we can work in the x-com frame where $x_{com} = 0$ and $p_x = 0$

In this frame:

$$0 = x_{com}$$
$$= \frac{m_1 x + m_2 (x + l \sin\phi)}{m_1 + m_2}$$
$$= \frac{(m_1 + m_2) x + m_2 l \sin\phi}{m_1 + m_2}$$
$$= x + \left(\frac{m_2}{m_1+m_2}\right) l \sin\phi$$

$$\rightarrow \boxed{x = -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi}$$

Thus, $\dot{x} = -\left(\frac{m_2}{m_1+m_2}\right) l \dot{\phi} \cos\phi$

$$\rightarrow E = \frac{1}{2}(m_1+m_2)\frac{m_2^2}{(m_1+m_2)^2} l^2 \dot{\phi}^2 \cos^2\phi + \frac{1}{2} m_2 l^2 \dot{\phi}^2$$
$$- \frac{m_2^2 l^2}{m_1+m_2} \dot{\phi}^2 \cos^2\phi - m_2 g l \cos\phi$$
$$= \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left(1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right) - m_2 g l \cos\phi$$
$$= \frac{1}{2}\left(\frac{m_2}{m_1+m_2}\right) l^2 \dot{\phi}^2 \left(m_1 + m_2 \sin^2\phi\right) - m_2 g l \cos\phi$$

Solve for $\dot{\phi}$:

$$E + m_2 g l \cos\phi = \frac{1}{2}\left(\frac{m_2}{m_1+m_2}\right) l^2 \dot{\phi}^2 (m_1 + m_2 \sin^2\phi)$$

$$\pm\sqrt{\frac{\frac{2}{l^2}\left(\frac{m_1+m_2}{m_2}\right)\left(E + m_2 g l \cos\phi\right)}{m_1 + m_2 \sin^2\phi}} = \frac{d\phi}{dt}$$

$$\rightarrow \boxed{t = \pm\sqrt{\frac{l^2}{2}\left(\frac{m_2}{m_1+m_2}\right)} \int d\phi \sqrt{\frac{m_1 + m_2 \sin^2\phi}{E + m_2 g l \cos\phi}} + \text{const}}$$

NOTE: In the x-com frame $x = -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi$

$$x_2 = x + l \sin\phi$$
$$= -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi + l \sin\phi$$
$$= \left(\frac{m_1}{m_1+m_2}\right) l \sin\phi$$
$$\equiv b \sin\phi$$

$$y_2 = l \cos\phi \equiv a \cos\phi$$

Thus,

$$\left(\frac{x_2}{b}\right)^2 + \left(\frac{y_2}{a}\right)^2 = \sin^2\phi + \cos^2\phi = 1$$

So m_2 traces out an ellipse in the x-com frame.

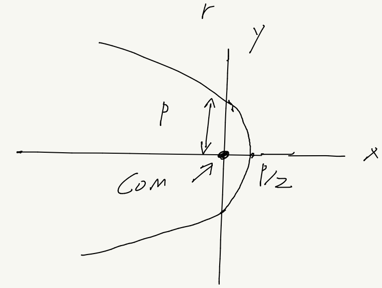
Sec 15, Prob 1:

$E = 0$ (parabola), $e = 1$

$U = -\alpha/r$

$$\left.\begin{aligned}\frac{p}{r} &= 1 + e\cos\phi \\ &= 1 + \cos\phi\end{aligned}\right\}$$

$\phi = 0 \rightarrow \frac{p}{r} = 2 \rightarrow r_{min} = \frac{p}{2}$



Time dependence:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{2}{m}(\cancel{E}^{\,0} + \frac{\alpha}{r}) - \frac{M^2}{m^2r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{M^2}{m^2r^2}}} + \text{const}$$

Recall: $p = \frac{M^2}{m\alpha} \rightarrow M^2 = m\alpha p$

$$\rightarrow t = \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{m\alpha p}{M^2 r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{\alpha}{m}}\sqrt{\frac{2}{r} - \frac{p}{r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha p}} \int \frac{r\,dr}{\sqrt{\frac{2r}{p} - 1}} + \text{const}$$

Let: $\frac{2r}{p} - 1 = \xi^2$ $\quad \left(\xi = 0 \rightarrow \frac{2r}{p} - 1 = 0 \rightarrow r = p/2 \right)$

Thus, $\frac{\cancel{2}\,dr}{p} = \cancel{2}\,\xi\,d\xi$

$$dr = p\,\xi\,d\xi$$

Also: $\frac{2r}{p} = 1 + \xi^2 \rightarrow r = \frac{p}{2}(1+\xi^2)$

$$\rightarrow t = \sqrt{\frac{m}{\alpha p}} \int \frac{\frac{p}{2}(1+\xi^2)\, p\,\cancel{\xi}\,d\xi}{\cancel{\sqrt{\xi^2}}} + \text{const}$$

$$= \frac{p^2}{2}\sqrt{\frac{m}{\alpha p}} \int (1+\xi^2)\,d\xi + \text{const}$$

so $t = \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}}\left(\xi + \frac{\xi^3}{3}\right) + \text{const}$

choose const $= 0$ so that $t = 0 \longleftrightarrow \xi = 0$

Thus,
$$\boxed{\begin{aligned} r &= \frac{p}{2}(1+\xi^2) \\ t &= \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}}\left(\xi + \frac{1}{3}\xi^3\right) \end{aligned}}$$

Recall: $\frac{p}{r} = 1 + \cos\phi$

so $1 + \cos\phi = \frac{2}{1+\xi^2}$

$$\boxed{\begin{aligned} \cos\phi &= \frac{2}{1+\xi^2} - 1 \\ &= \frac{1-\xi^2}{1+\xi^2} \end{aligned}}$$

Cartesian: $x = r\cos\phi$

$$= \frac{p}{2}\cancel{(1+\xi^2)}\left(\frac{1-\xi^2}{\cancel{1+\xi^2}}\right)$$

$$= \frac{p}{2}(1-\xi^2)$$

$$y = r\sin\phi = \frac{p}{2}(1+\xi^2)\sqrt{1 - \left(\frac{1-\xi^2}{1+\xi^2}\right)^2}$$

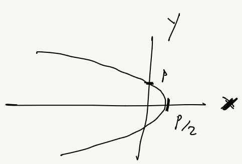
$$y = \frac{p}{2}\cancel{(1+\xi^2)}\,\frac{1}{\cancel{(1+\xi^2)}}\sqrt{(1+\xi^2)^2-(1-\xi^2)^2}$$

$$= \frac{p}{2}\sqrt{\cancel{1}+\cancel{\xi^4}+2\xi^2-(\cancel{1}+\cancel{\xi^4}-2\xi^2)}$$

$$= \frac{p}{2}\sqrt{4\xi^2}$$

$$= p\xi$$

Thus,
$$\boxed{\begin{aligned} x &= \frac{p}{2}(1-\xi^2) \\ y &= p\xi \end{aligned}}$$



NOTE:

$$x = \frac{p}{2} - \frac{p}{2}\xi^2$$

$$\boxed{= \frac{p}{2} - \frac{y^2}{2p}}$$

parabola

$\cancel{x} = y^2$

$\underline{y=0}$: $x = p/2$

$\underline{x=0}$: $y = \pm p$

Sec 15, Prob 3:

$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U)-M^2/r^2}} \qquad (14.10)$$

Consider a small perturbation $\delta U(r)$ to the potential energy:

$$\rightarrow \Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U-\delta U)-M^2/r^2}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U)-M^2/r^2-2m\delta U}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U)-M^2/r^2}\sqrt{1-\frac{2m\delta U}{2m(E-U)-M^2/r^2}}}$$

$$\approx 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U)-M^2/r^2}}\left(1+\frac{m\,\delta U}{2m(E-U)-M^2/r^2}\right)$$

$$= \underbrace{2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U)-M^2/r^2}}}_{\Delta\phi_0} + \underbrace{2\int_{r_{min}}^{r_{max}} \frac{Mm\,\delta U\,dr/r^2}{\left(2m(E-U)-M^2/r^2\right)^{3/2}}}_{\delta\phi}$$

Now: $\frac{\partial}{\partial M}\left(\frac{1}{\sqrt{2m(E-U)-M^2/r^2}}\right) = -\frac{1}{2}\frac{(-2M/r^2)}{(2m(E-U)-M^2/r^2)^{3/2}}$

$$= \frac{M/r^2}{(2m(E-U)-M^2/r^2)^{3/2}}$$

Thus,

$$\delta\phi = \frac{\partial}{\partial M}\left(\int_{r_{min}}^{r_{max}} \frac{2m\,\delta U\,dr}{\sqrt{2m(E-U)-M^2/r^2}}\right)$$

Consider the case $U(r) = -\alpha/r$

Then: $\Delta\phi_0 = 2\pi$ (since a bound orbit is an ellipse, which is closed)

Also: $\delta\phi = \frac{\partial}{\partial M}\left(\int_{r_{min}}^{r_{max}} \frac{2m\,\delta U\,dr}{\sqrt{2m(E+\alpha/r)-M^2/r^2}}\right)$

In the integral, we can use the solution for the unperturbed motion:

$$\frac{p}{r} = 1+e\cos\phi \rightarrow -\frac{p}{r^2}dr = -e\sin\phi\,d\phi$$

so $dr = \frac{e}{p}r^2\sin\phi\,d\phi$

$r = r_{min}, r_{max} \longleftrightarrow \phi = 0, \pi$

$\sqrt{} = \sqrt{2m(E+\frac{\alpha}{r}) - \frac{M^2}{r^2}}$

$$= \sqrt{2mE + \frac{2m\alpha(1+e\cos\phi)}{p} - M^2\frac{(1+e\cos\phi)^2}{p^2}}$$

$$= \frac{1}{p}\sqrt{2mEp^2 + 2m\alpha p(1+e\cos\phi) - M^2(1+e\cos\phi)^2}$$

Recall: $p = \frac{M^2}{m\alpha}$

$e = \sqrt{1+\frac{2EM^2}{m\alpha^2}}$

Thus, $2mEp^2 = 2mE\frac{M^4}{m^2\alpha^2} = \left(\frac{2EM^2}{m\alpha^2}\right)M^2$

$2m\alpha p = 2M^2$

$$\rightarrow \sqrt{} = \frac{m\alpha}{M^2}\sqrt{\left(\frac{2EM^2}{m\alpha^2}\right)M^2 + 2M^2(1+\cancel{e\cos\phi}) - M^2(1+e^2\cos^2\phi + \cancel{2e\cos\phi})}$$

$$= \frac{m\alpha}{M}\sqrt{\underbrace{\frac{2EM^2}{m\alpha^2}+1}_{e^2} - e^2\cos^2\phi}$$

$$= \frac{m\alpha e}{M}\sin\phi$$

Thus,

$$\delta\phi = \frac{\partial}{\partial M}\left(\int_0^{\pi} \frac{2\cancel{m}\,\delta U\, r^2 \frac{\cancel{e}}{p} \cancel{\sin\phi}\, d\phi}{\frac{\cancel{m}\alpha \cancel{e}}{M} \cancel{\sin\phi}} \right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2M}{\alpha p} \int_0^{\pi} r^2 \delta U \, d\phi \right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m}{M} \int_0^{\pi} r^2 \delta U \, d\phi \right) \qquad \left(\text{using } p = \frac{M^2}{m\alpha} \right)$$

where $\frac{p}{r} = 1 + e\cos\phi$

(a) $\delta U = \frac{\beta}{r^2}$

$$\rightarrow \delta\phi = \frac{\partial}{\partial M}\left(\frac{2m}{M} \int_0^{\pi} \cancel{r^2} \frac{\beta}{\cancel{r^2}} \, d\phi \right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m}{M} \beta \pi \right)$$

$$= -\frac{2\pi m \beta}{M^2} \quad \boxed{= -\frac{2\pi\beta}{\alpha p}}$$

(b) $\delta U = \frac{\gamma}{r^3}$

$$\rightarrow \delta\phi = \frac{\partial}{\partial M}\left(\frac{2m}{M} \int_0^{\pi} r^2 \frac{\gamma}{r^3} \, d\phi \right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m\gamma}{M} \int_0^{\pi} \frac{d\phi}{r} \right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m\gamma}{M} \frac{1}{p} \int_0^{\pi} (1 + e\cos\phi)\, d\phi \right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m\gamma}{Mp} \left[\pi + \underbrace{e\sin\phi \Big|_0^{\pi}}_{0} \right] \right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2\pi m\gamma}{Mp} \right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2\pi m^2 \gamma \alpha}{M^3} \right) \longleftarrow \text{using } p = \frac{M^2}{m\alpha}$$

$$= -\frac{6\pi m^2 \gamma \alpha}{M^4}$$

$$= -\frac{6\pi \cancel{m^2} \gamma \alpha}{p^2 \cancel{m^2} \alpha^2}$$

$$= \boxed{-\frac{6\pi\gamma}{p^2 \alpha}}$$

sec 16, prob 2

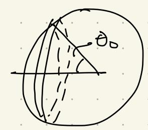
dN = Fraction of particles entering $d\Omega_0$

$$= \frac{d\Omega_0}{4\pi}$$

$$= \frac{2\pi \sin\theta_0 \, d\theta_0}{4\pi}$$

$$= \frac{1}{2} \sin\theta \, d\theta_0$$

$$= -\frac{1}{2} d(\cos\theta_0)$$



Now: (16.6)

$$\cos\theta_0 = -\frac{V}{v_0} \sin^2\theta \pm \cos\theta \sqrt{1 - \frac{V^2}{v_0^2} \sin^2\theta}$$

i) For $V < v_0$, take $+\sqrt{}$

$$d(\cos\theta_0) = -\frac{2V}{v_0} \sin\theta \cos\theta \, d\theta$$

$$- \sin\theta \, d\theta \sqrt{}$$

$$+ \frac{\cos\theta}{\sqrt{}} \frac{1}{\cancel{2}} \left(-\frac{V^2}{v_0^2}\right) \cancel{2} \sin\theta \cos\theta \, d\theta$$

$$= -\sin\theta \, d\theta \left\{ \frac{2V}{v_0} \cos\theta + \sqrt{} + \left(\frac{V}{v_0}\right)^2 \frac{\cos^2\theta}{\sqrt{}} \right\}$$

Thus,

$$dN = \frac{1}{2} \sin\theta \, d\theta \left[\frac{2V}{v_0} \cos\theta + \frac{\left(1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{}} \right\}$$

$$= \frac{1}{2} \sin\theta \, d\theta \left\{ \frac{2V}{v_0} \cos\theta + \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right\}$$

where $0 \le \theta \le \pi$

ii) For $V > v_0$, there are two solutions corresponding to the $+\sqrt{}$ and $-\sqrt{}$ in (16.6).

- For the $+\sqrt{}$ we have $d\theta/d\theta_0 > 0$
- For the $-\sqrt{}$ we have $d\theta/d\theta_0 < 0$

so we should subtract the two contributions

$$dN = dN_+ - dN_-$$

where dN_+ = above expression

$$= \frac{1}{2} \sin\theta \, d\theta \left\{ \frac{2V}{v_0} \cos\theta + \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right\}$$

and

$$dN_- = \frac{1}{2} \sin\theta \, d\theta \left\{ \frac{2V}{v_0} \cos\theta - \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right\}$$

Thus,

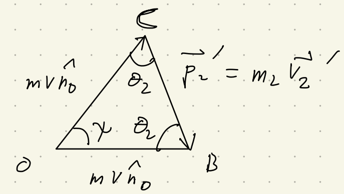
$$dN = \frac{\sin\theta \, d\theta \left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}}$$

where $0 \leq \theta \leq \theta_{max}$

Sec 17, Prob 1

Want to determine v_1', v_2' as functions of θ_1, θ_2 (as opposed to functions of χ)

From Fig 16, triangle OBC:



$$\chi + 2\theta_2 = \pi$$
$$\chi = \pi - 2\theta_2$$

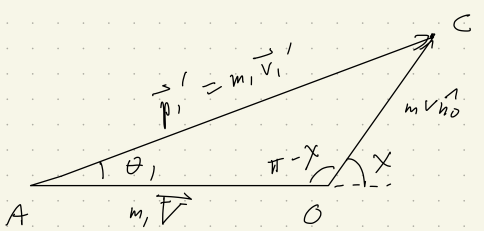
$$(m_2 v_2')^2 = 2(mv)^2 - 2(mv)^2 \cos\chi$$
$$= 2(mv)^2 (1 - \cos\chi)$$

Now: $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$
$\rightarrow 2\sin^2\theta = 1 - \cos 2\theta$

$$\rightarrow (m_2 v_2')^2 = 4(mv)^2 \sin^2\left(\frac{\chi}{2}\right)$$

$$\boxed{v_2' = 2\left(\frac{mv}{m_2}\right)\sin\left(\frac{\pi}{2} - \theta_2\right) = 2\left(\frac{m_1}{m_1+m_2}\right) v \cos\theta_2}$$

From Fig 16, $\triangle AOC$



$$\vec V = \frac{m_1\vec v_1 + m_2\cancel{\vec v_2}^{\,0}}{m_1+m_2} = \frac{m_1\vec v_1}{m_1+m_2} = \frac{m_1\vec v}{m_1+m_2}$$

Use law of cosines for OC:

$$OC^2 = AC^2 + AO^2 - 2\,AC\cdot AO\cos\theta_1$$

$$(mV)^2 = (m_1v_1')^2 + (m_1V)^2 - 2\,m_1v_1'm_1V\cos\theta_1$$

quadratic equation for v_1':

$$0 = (m_1v_1')^2 - 2\,m_1^2v_1'V\cos\theta_1 + (m_1V)^2 - (mV)^2$$

$$= m_1^2v_1'^2 - 2\frac{m_1^3}{m_1+m_2}v_1'v\cos\theta_1 + \left(\frac{m_1^4}{(m_1+m_2)^2} - \frac{m_1^2m_2^2}{(m_1+m_2)^2}\right)v^2$$

$$= m_1^2v^2\left[\left(\frac{v_1'}{v}\right)^2 - 2\left(\frac{m_1}{m_1+m_2}\right)\left(\frac{v_1'}{v}\right)\cos\theta_1 + \underbrace{\frac{m_1^2-m_2^2}{(m_1+m_2)^2}}_{=\left(\frac{m_1-m_2}{m_1+m_2}\right)}\right]$$

$$\rightarrow \quad \frac{v_1'}{v} = \frac{2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \sqrt{4\left(\frac{m_1}{m_1+m_2}\right)^2\cos^2\theta_1 - 4\left(\frac{m_1-m_2}{m_1+m_2}\right)}}{2}$$

$$\frac{v_1'}{v} = \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{\underbrace{m_1^2\cos^2\theta_1 - (m_1^2-m_2^2)}_{=\,m_1^2(\cos^2\theta_1-1)+m_2^2\ =\ m_2^2 - m_1^2\sin^2\theta_1}}$$

thus,

$$\boxed{\frac{v_1'}{v} = \frac{m_1}{m_1+m_2}\cos\theta_1 \pm \frac{1}{m_1+m_2}\sqrt{m_2^2 - m_1^2\sin^2\theta_1}}$$

For $m_1 > m_2$, the $\sqrt{\ }$ has both $\pm$ signs

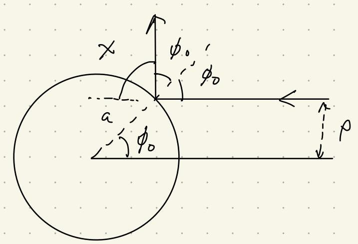
For $m_1 < m_2$, the $\sqrt{\ }$ should have the $+$ sign in order for

$$\frac{v_1'}{v} \xrightarrow[\theta_1\to 0]{} \left(\frac{m_1}{m_1+m_2}\right) + \frac{m_2}{m_1+m_2} = 1$$

Sec. 18, Probl:

Hard sphere

$$U = \begin{cases} 0 & r > a \\ \infty & r < a \end{cases}$$



$$\chi + 2\phi_0 = \pi \quad \rightarrow \quad \phi_0 = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\sin\phi_0 = \rho/a$$

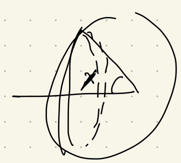
Thus, $\sin\left(\frac{\pi}{2} - \frac{\chi}{2}\right) = \frac{\rho}{a}$

$$\rightarrow \quad \cos\left(\frac{\chi}{2}\right) = \frac{\rho}{a}$$

Effective cross section:

$$d\sigma = 2\pi\rho\, d\rho$$

$$d\Omega = 2\pi \sin\chi\, d\chi$$



$$\rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{\rho\, d\rho}{\sin\chi\, d\chi}$$

$$= \frac{\rho(\chi)}{\sin\chi} \left|\frac{d\rho}{d\chi}\right|$$

$$\frac{d\sigma}{d\Omega} = \frac{a\cos\left(\frac{\chi}{2}\right)}{\sin\chi} \left| \frac{d}{d\chi}\left(a\cos\left(\frac{\chi}{2}\right)\right)\right|$$

$$= \frac{a^2 \cos\left(\frac{\chi}{2}\right) \frac{1}{2} \sin\left(\frac{\chi}{2}\right)}{\sin\chi}$$

$$= \frac{a^2}{2} \frac{\cancel{\cos\left(\frac{\chi}{2}\right)\sin\left(\frac{\chi}{2}\right)}}{2\cancel{\sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)}}$$

$$= \boxed{\frac{1}{4}a^2}$$

Total cross section

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega \quad , \quad d\Omega = 2\pi\sin\chi\, d\chi$$

$$= \frac{1}{4}a^2 \int_0^\pi 2\pi \sin\chi\, d\chi$$

$$= \frac{1}{2}\pi a^2 \left(-\cos\chi\right)\Big|_0^\pi \qquad \cos\pi = -1, \quad \cos 0 = 1$$

$$= \frac{1}{2}\pi a^2 (-1)(-1-1)$$

$$= \boxed{\pi a^2}$$

Calculate $d\sigma$ wrt θ_1 and θ_2

$$d\sigma = 2\pi\rho\, d\rho = 2\pi\rho \left|\frac{d\rho}{d\chi}\right| d\chi$$

$$d\Omega = 2\pi \sin\chi\, d\chi \quad \to \quad 2\pi\, d\chi = \frac{d\Omega}{\sin\chi}$$

$$\to \quad d\sigma = \frac{\rho}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$

Similarly $$d\sigma_1 = \frac{\rho}{\sin\theta_1}\left|\frac{d\rho}{d\theta_1}\right| d\Omega_1$$

$$d\sigma_2 = \frac{\rho}{\sin\theta_2}\left|\frac{d\rho}{d\theta_2}\right| d\Omega_2$$

$$\to \quad \frac{d\sigma_1}{d\Omega_1} = \frac{\sin\chi\, d\chi}{\sin\theta_1\, d\theta_1}\frac{d\sigma}{d\Omega}$$

$$= \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| \frac{d\sigma}{d\Omega}$$

and $$\frac{d\sigma_2}{d\Omega_2} = \left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right| \frac{d\sigma}{d\Omega}$$

where $\frac{d\sigma}{d\Omega} = \frac{1}{4}a^2$ (for hard sphere)

Now: $2\theta_2 + \chi = \pi$ (always)

$$\chi = \pi - 2\theta_2$$

$$\begin{aligned}
\to \cos\chi &= \cos(\pi - 2\theta_2) \\
&= -\cos(2\theta_2) \\
&= -\cos^2\theta_2 + \sin^2\theta_2 \\
&= 1 - 2\cos^2\theta_2
\end{aligned}$$

$$d(\cos\chi) = -4\cos\theta_2\, d(\cos\theta_2)$$

so $$\left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right| = 4|\cos\theta_2|$$

Thus, $$\boxed{\frac{d\sigma_2}{d\Omega_2} = \cancel{4}\cos\theta_2 \frac{1}{\cancel{4}}a^2 = a^2|\cos\theta_2|}$$

Relating θ_1 to χ:

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}$$

Want to find $\cos\chi$ in terms of θ_1

Compare to (16.5), (16.6):

$$\tan\theta = \frac{v_0 \sin\theta_0}{v_0\cos\theta_0 + V}$$

$$\cos\theta_0 = -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{1 - \frac{V^2}{v_0^2}\sin^2\theta}$$

Thus,

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_2\cos\chi + m_1}$$

$$\rightarrow \boxed{\cos\chi = -\left(\frac{m_1}{m_2}\right)\sin^2\theta_1 \pm \cos\theta_1\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

where $\pm$ sign for $m_1 > m_2$ and $+$ sign for $m_2 > m_1$

(i) <u>For $m_2 > m_1$:</u> (take $+\sqrt{\ }$)

$$d(\cos\chi) = -2\left(\frac{m_1}{m_2}\right)\sin\theta_1\cos\theta_1\,d\theta_1 + d(\cos\theta_1)\sqrt{\ }$$

$$+ \frac{\cos\theta_1}{\sqrt{\ }}\left(\frac{1}{\cancel{2}}\right)(-\cancel{2})\left(\frac{m_1}{m_2}\right)^2\sin\theta_1\cos\theta_1\,d\theta_1$$

<u>Now:</u> $\sin\theta_1\cos\theta_1\,d\theta_1 = -\cos\theta_1\,d(\cos\theta_1)$

$$d(\cos\chi) = d(\cos\theta_1)\left[2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \sqrt{\ } + \frac{\cos^2\theta_1}{\sqrt{\ }}\left(\frac{m_1}{m_2}\right)^2\right]$$

$$\frac{d(\cos\chi)}{d(\cos\theta_1)} = 2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1 - \left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1 + \cos^2\theta_1\left(\frac{m_1}{m_2}\right)^2}{\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

$$= 2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1 + \left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1}{\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

$$\rightarrow \frac{d\sigma_1}{d\Omega_1} = \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|\frac{d\sigma}{d\Omega} \quad \text{for } 0 \le \theta_1 \le \pi$$

$$= \frac{1}{4}a^2\left[2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1 + \left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1}{\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\right]$$

(ii) <u>For $m_1 > m_2$:</u> contribution from both $\pm$ signs

$+$ sign: $\frac{d\chi}{d\theta_1} > 0$

$-$ sign: $\frac{d\chi}{d\theta_1} < 0$

$\Big\}$ so need to subtract

$$\frac{d(\cos\chi)}{d(\cos\theta_1)} = 2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1 + \left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1}{\sqrt{\ }}$$

$$- \left(2\left(\frac{m_1}{m_2}\right)\cos\theta_1 - \frac{1 + \left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1}{\sqrt{\ }}\right)$$

$$\rightarrow \frac{d(\cos\chi)}{d(\cos\theta_1)} = 2\,\frac{\left(1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1\right)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

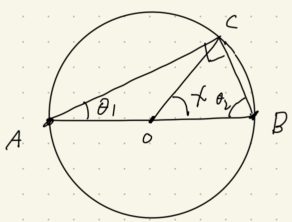
$$\boxed{\frac{d\sigma_1}{d\Omega_1} = \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| \frac{d\sigma}{d\Omega} \leftarrow \frac{1}{4}a^2}$$

$$\boxed{= \frac{1}{2}a^2\,\frac{\left(1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)\right)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}}$$

for $0 \le \theta_1 \le \theta_{max}$

(iii) For $m_1 = m_2$, $\theta_1 + \theta_2 = \pi/2$

$2\theta_2 + \chi = \pi \rightarrow \theta_2 = \frac{\pi}{2} - \frac{\chi}{2}$



$$\rightarrow \theta_1 = \frac{\pi}{2} - \theta_2 = \frac{\pi}{2} - \left(\frac{\pi}{2} - \frac{\chi}{2}\right) = \frac{\chi}{2}$$

Thus, $\frac{d(\cos\chi)}{d(\cos\theta_1)} = \frac{d(\cos(2\theta_1))}{d(\cos\theta_1)}$

But $\cos(2\theta_1) = 2\cos^2\theta_1 - 1$

$$\rightarrow \frac{d(\cos\chi)}{d(\cos\theta_1)} = \frac{4\cos\theta_1\,\cancel{d(\cos\theta_1)}}{\cancel{d\cos\theta_1}} = 4\cos\theta_1$$

Thus,

$$\boxed{\frac{d\sigma_1}{d\Omega_1} = \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|\frac{d\sigma}{d\Omega} = 4|\cos\theta_1|\,\frac{1}{4}a^2 = a^2|\cos\theta_1|}$$

Sec 18, Prob 2:

From problem 1, $d\sigma = \frac{1}{4} a^2 d\Omega$

$$= \frac{1}{4} a^2 \, 2\pi \sin\chi \, d\chi$$

$$= \frac{1}{2} \pi a^2 \sin\chi \, d\chi$$

Want to replace $\sin\chi \, d\chi$ by some function involving ϵ, the energy lost by the scattered particle.

ϵ = energy lost by scattered particle
= energy gained by scattering particle
$= \frac{1}{2} m_2 v_2'^2$

Now: $v_2' = \frac{2 m_1 v}{m_1 + m_2} \sin\left(\frac{\chi}{2}\right)$, $\quad v = v_1 - \cancel{v_2}^{\,0} = v_\infty$

$$= 2 \frac{m}{m_2} v_\infty \sin\left(\frac{\chi}{2}\right)$$

$$\rightarrow \epsilon = \frac{1}{2} m_2 \frac{4 m^2 v_\infty^2}{m_2^2} \sin^2\left(\frac{\chi}{2}\right)$$

$$= 2 \frac{m^2}{m_2} v_\infty^2 \sin^2\left(\frac{\chi}{2}\right) = 2 \frac{m_1^2 m_2 v_\infty^2}{(m_1+m_2)^2} \sin^2\left(\frac{\chi}{2}\right)$$

$\cos 2\theta = 1 - 2\sin^2\theta \longrightarrow \sin^2\frac{\chi}{2} = \frac{1-\cos\chi}{2}$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$\epsilon = \frac{m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 (1 - \cos\chi)$$

$$d\epsilon = \frac{m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 \sin\chi \, d\chi$$

Thus, $d\sigma = \frac{1}{2} \pi a^2 \sin\chi \, d\chi$

$$= \frac{1}{2} \pi a^2 \frac{(m_1+m_2)^2}{m_1^2 m_2 v_\infty^2} d\epsilon$$

NOTE: $\epsilon = \frac{m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 (1 - \cos\chi)$ $\therefore$

has $\epsilon_{max} = \frac{2 m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2$ when $\chi = \pi$

$$\rightarrow \boxed{d\sigma = \frac{\pi a^2}{\epsilon_{max}} d\epsilon}$$

(Uniform distribution in ϵ between 0 and ϵ_{max})

Sec 18, Prob 4:

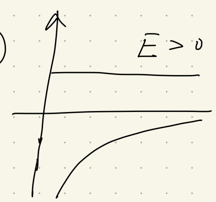
Effective cross section for particle to fall to center of potential $U = -\frac{\alpha}{r^2}$

$$U_{eff}(r) = \frac{M^2}{2mr^2} + U(r)$$

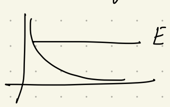
$$= \frac{M^2}{2mr^2} - \frac{\alpha}{r^2}$$

$$= \frac{1}{r^2}\left(\frac{M^2}{2m} - \alpha\right)$$

Need $\alpha \geq \frac{M^2}{2m}$ so that $\rightarrow$ $U_{eff}(r)$

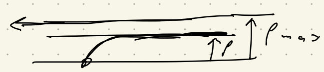


otherwise for $\alpha < \frac{M^2}{2m}$, the effective potential is repulsive and particle can't reach the center.



$$\alpha \geq \frac{M^2}{2m} = \frac{m^2\rho^2 V_\infty^2}{2m} = \rho^2\left(\frac{1}{2} m V_\infty^2\right)$$

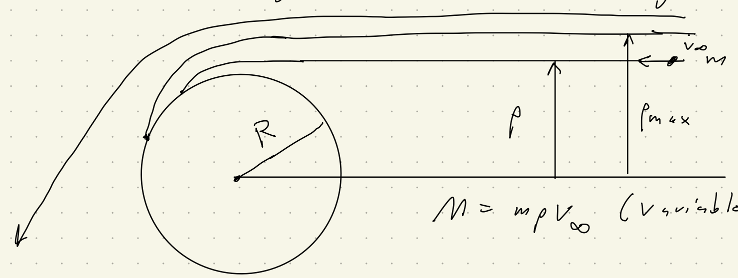
$$\rho \leq \sqrt{\frac{\alpha}{\frac{1}{2}mv_\infty^2}} = \rho_{max}$$



$$\boxed{\sigma = \pi\rho_{max}^2 = \frac{\pi\alpha}{\frac{1}{2}mv_\infty^2}}$$

Sec 18, Prob 6:

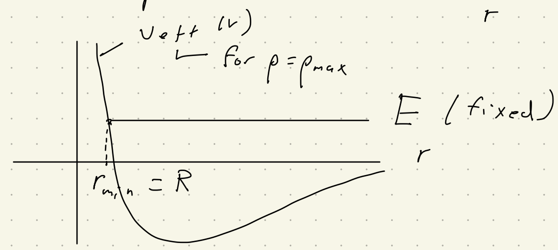
Effective cross section for particle of mass m_1 to strike a sphere of mass m_2 and radius R subject to Newtonian gravity



$E = \frac{1}{2}mv_\infty^2$ (given)

$M = m\rho V_\infty$ (variable)

outside sphere $U = \frac{-Gm_1m_2}{r} = -\frac{\alpha}{r}$



$$\frac{1}{2}mv_\infty^2 = U_{eff}(r_{min})$$

$$= U_{eff}(R)$$

$$= -\frac{\alpha}{R} + \frac{M^2}{2mR^2}$$

$$= -\frac{\alpha}{R} + \frac{m^2\rho_{max}^2 V_\infty^2}{2mR^2}$$

$$= -\frac{\alpha}{R} + \left(\frac{1}{2}mv_\infty^2\right)\frac{\rho_{max}^2}{R^2}$$

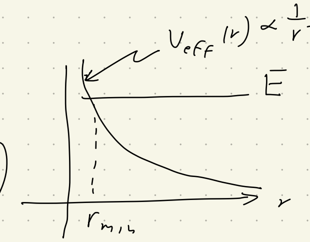
Thus, $E = -\frac{\alpha}{R} + E\,\frac{\rho_{max}^2}{R^2}$

$$\rightarrow \quad \frac{E + \alpha/R}{E} = \frac{\rho_{max}^2}{R^2}$$

$$\begin{aligned}
\rightarrow \quad \sigma &= \pi \rho_{max}^2 \\
&= \pi R^2 \left(1 + \frac{\alpha}{R} \cdot \frac{1}{E} \right) \\
&= \pi R^2 \left(1 + \frac{G \cancel{m_1 m_2}}{R} \, \frac{1}{\frac{1}{2} \frac{\cancel{m_1 m_2}}{(m_1+m_2)} v_\infty^2} \right) \\
&= \pi R^2 \left(1 + \frac{2G(m_1+m_2)}{R v_\infty^2} \right)
\end{aligned}$$

Sec 19, Prob 1: $U = \frac{\alpha}{r^2}$, $\alpha > 0$

$$\begin{aligned}
U_{eff}(r) &= U(r) + \frac{M^2}{2mr^2} \\
&= \frac{\alpha}{r^2} + \frac{\rho^2 m^2 v_\infty^2}{2mr^2} \\
&= \frac{1}{r^2}\left(\alpha + \frac{1}{2} m v_\infty^2 \cdot \rho^2 \right)
\end{aligned}$$



$$\begin{aligned}
\phi_0 &= \int_{r_{min}}^{\infty} \frac{\rho \, dr/r^2}{\sqrt{1 - \rho^2/r^2 - 2U/mv_\infty^2}} \\
&= \int_{r_{min}}^{\infty} \frac{\rho \, dr/r^2}{\sqrt{1 - \left(\rho^2 + \frac{2\alpha}{mv_\infty^2}\right)\frac{1}{r^2}}} \\
&= \int_{r_{min}}^{\infty} \frac{\rho \, dr/r^2}{\sqrt{1 - \beta^2/r^2}}
\end{aligned}
\qquad \Bigg| \quad \begin{aligned} \beta^2 &\equiv \rho^2 + \frac{2\alpha}{mv_\infty^2} \\ &= \rho^2 + \frac{\alpha}{E} \end{aligned}$$

Let: $u = \frac{1}{r} \rightarrow du = -\frac{1}{r^2}\,dr$

$$\phi_0 = \int_0^{\frac{1}{r_{min}}} \frac{\rho \, du}{\sqrt{1 - \beta^2 u^2}}, \qquad 1 - \frac{\beta^2}{r_{min}^2} = 0 \rightarrow \boxed{\beta = r_{min}}$$

$$\phi_0 = \int_0^{\frac{1}{\beta}} \frac{\rho\, du}{\sqrt{1-\beta^2 u^2}}$$

Let: $\beta u = \sin\theta \rightarrow du = \frac{1}{\beta}\cos\theta\, d\theta$

$$\sqrt{1-\beta^2u^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$

$u = 0 \rightarrow \theta = 0$

$u = \frac{1}{\beta} \rightarrow \sin\theta = 1 \rightarrow \theta = \pi/2$

$$\rightarrow \phi_0 = \int_0^{\pi/2} \frac{\rho}{\beta} \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}}$$

$$= \frac{\pi}{2}\frac{\rho}{\beta}$$

$$= \frac{\pi}{2}\frac{\rho}{\sqrt{\rho^2 + \frac{\alpha}{E}}}$$

Square both sides

$$\left(\frac{2\phi_0}{\pi}\right)^2 = \frac{\rho^2}{\rho^2 + \frac{\alpha}{E}}$$

$$\left(\rho^2 + \frac{\alpha}{E}\right)\left(\frac{2\phi_0}{\pi}\right)^2 = \rho^2$$

$$\frac{\alpha}{E}\left(\frac{2\phi_0}{\pi}\right)^2 = \rho^2\left(1 - \left(\frac{2\phi_0}{\pi}\right)^2\right)$$

$$\rho^2 = \frac{\frac{\alpha}{E}\left(\frac{2\phi_0}{\pi}\right)^2}{1-\left(\frac{2\phi_0}{\pi}\right)^2}$$

Now: $2\phi_0 + \chi = \pi$ for repulsive scattering

$$\rightarrow \frac{2\phi_0}{\pi} = 1 - \frac{\chi}{\pi}$$

$$\rightarrow \left(\frac{2\phi_0}{\pi}\right)^2 = \left(1-\frac{\chi}{\pi}\right)^2$$

$$\rightarrow 1 - \left(\frac{2\phi_0}{\pi}\right)^2 = 1 - \left(1 + \frac{\chi^2}{\pi^2} - \frac{2\chi}{\pi}\right)$$

$$= \frac{2\chi}{\pi} - \frac{\chi^2}{\pi^2}$$

Thus:

$$\rho^2 = \frac{\frac{\alpha}{E}\left(1-\frac{\chi}{\pi}\right)^2}{\left(\frac{2\chi}{\pi} - \frac{\chi^2}{\pi^2}\right)}$$

$$= \frac{\frac{\alpha}{E}(\pi-\chi)^2}{2\pi\chi - \chi^2}$$

Thus,

$$\rho = \sqrt{\frac{\alpha}{E}} \frac{(\pi - \chi)}{\sqrt{2\pi\chi - \chi^2}}$$

Differential cross-section:

$$d\sigma = 2\pi \rho \, d\rho$$
$$= 2\pi \rho \left| \frac{d\rho}{d\chi} \right| d\chi$$

$$d\Omega = 2\pi \sin\chi \, d\chi$$

$$\rightarrow \boxed{d\sigma = \frac{\rho}{\sin\chi} \left| \frac{d\rho}{d\chi} \right| d\Omega}$$

$$\frac{d\rho}{d\chi} = \sqrt{\frac{\alpha}{E}} \left(\frac{-\sqrt{2\pi\chi - \chi^2} - \frac{1}{\cancel{2}\sqrt{}} (\cancel{2}\pi - \cancel{2}\chi)(\pi - \chi)}{2\pi\chi - \chi^2} \right)$$

$$= \sqrt{\frac{\alpha}{E}} \frac{-1}{(2\pi\chi - \chi^2)^{3/2}} \left(\underbrace{2\pi\chi - \chi^2 + (\pi - \chi)^2}_{\cancel{2\pi\chi} - \cancel{\chi^2} + \pi^2 + \cancel{\chi^2} - \cancel{2\pi\chi}} \right)$$

$$= -\sqrt{\frac{\alpha}{E}} \frac{\pi^2}{(2\pi\chi - \chi^2)^{3/2}}$$

$$\text{so } \boxed{\left| \frac{d\rho}{d\chi} \right| = \sqrt{\frac{\alpha}{E}} \frac{\pi^2}{(2\pi\chi - \chi^2)^{3/2}}}$$

Thus,

$$\frac{d\sigma}{d\Omega} = \frac{1}{\sin\chi} \sqrt{\frac{\alpha}{E}} \frac{(\pi - \chi)}{\sqrt{2\pi\chi - \chi^2}} \sqrt{\frac{\alpha}{E}} \frac{\pi^2}{(2\pi\chi - \chi^2)^{3/2}}$$

$$= \frac{1}{\sin\chi} \left(\frac{\alpha}{E} \right) \frac{\pi^2 (\pi - \chi)}{(2\pi\chi - \chi^2)^2}$$

Sec 20, Prob 1:

Start with (18.4):

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \rho^2/r^2 - 2U/mv_\infty^2}}$$

Consider small angle scattering, where

$$\frac{2U}{mv_\infty^2} << 1$$

Then:

$$\frac{1}{\sqrt{1-\rho^2/r^2 - 2U/mv_\infty^2}} = \frac{1}{\sqrt{1-\rho^2/r^2}} \frac{1}{\sqrt{1 - \frac{2U/mv_\infty^2}{1-\rho^2/r^2}}}$$

$$\approx \frac{1}{\sqrt{1-\rho^2/r^2}} \left(1 + \frac{U}{mv_\infty^2}\left(\frac{1}{1-\rho^2/r^2}\right)\right)$$

$$= \frac{1}{\sqrt{1-\rho^2/r^2}} + \frac{U/mv_\infty^2}{(1-\rho^2/r^2)^{3/2}}$$

Now:

$$\int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1-\rho^2/r^2}} = \int_0^{\frac{1}{r_{min}}} \frac{\rho\, du}{\sqrt{1-\rho^2 u^2}} = \int_0^{\pi/2} \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \boxed{\frac{\pi}{2}}$$

let: $u = \frac{1}{r}$

$du = -\frac{1}{r^2}\, dr$

let: $\rho u = \sin\theta$, $\rho\, du = \cos\theta\, d\theta$

$u = \frac{1}{r_{min}} \to \rho = r_{min} \to \theta = \frac{\pi}{2}$

Thus,

$$\phi_0 \approx \frac{\pi}{2} + \frac{1}{mv_\infty^2} \int_{r_{min}}^{\infty} \frac{U(r)\, \rho\, dr/r^2}{(1-\rho^2/r^2)^{3/2}}$$

$$= \frac{\pi}{2} + \frac{1}{mv_\infty^2} \frac{\partial}{\partial\rho}\left[\int_{r_{min}}^{\infty} \frac{U(r)\, dr}{\sqrt{1-\rho^2/r^2}}\right]$$

Now:

$$\int_{r_{min}}^{\infty} \frac{U(r)\, dr}{\sqrt{1-\rho^2/r^2}} = \int_{r_{min}}^{\infty} \frac{U(r)\, r\, dr}{\sqrt{r^2 - \rho^2}}$$

$$\approx \int_{\rho}^{\infty} \frac{U(r)\, r\, dr}{\sqrt{r^2-\rho^2}}$$

$u = U(r) \to du = \frac{dU}{dr}\, dr$

$dv = \frac{r\, dr}{\sqrt{r^2-\rho^2}} = \frac{dx/2}{\sqrt{x}} \to v = \sqrt{x} = \sqrt{r^2-\rho^2}$

(let $x = r^2 - \rho^2 \to dx = 2r\,dr$)

Thus,

$$\int_{r_{min}}^{\infty} \frac{U(r)\, dr}{\sqrt{1-\rho^2/r^2}} \approx \cancel{U(r)\sqrt{r^2-\rho^2}\Big|_{\rho}^{\infty}} - \int_{\rho}^{\infty} dr\, \frac{dU}{dr}\sqrt{r^2-\rho^2}$$

$\to 0$

assume $U(\infty) \to 0$ faster than $\frac{1}{r}$

$$\phi_0 \approx \frac{\pi}{2} - \frac{1}{mv_\infty^2}\frac{\partial}{\partial\rho}\left[\int_\rho^\infty dr \left(\frac{dU}{dr}\right)\sqrt{r^2-\rho^2}\right]$$

$$= \frac{\pi}{2} - \frac{1}{mv_\infty^2}\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\frac{1}{\cancel{2}\sqrt{r^2-\rho^2}}\left(-\cancel{2}\rho\right)$$

$$= \frac{\pi}{2} + \frac{\rho}{mv_\infty^2}\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}$$

Recall:

$$\chi + 2\phi_0 = \pi \quad\rightarrow\quad \phi_0 - \frac{\pi}{2} = -\frac{\chi}{2}$$

Thus,

$$\boxed{\chi = -2\left(\phi_0 - \frac{\pi}{2}\right) \approx \frac{-2\rho}{mv_\infty^2}\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}}$$

Compare to:

$$\theta_1 \approx \frac{-2\rho}{m_1 v_\infty^2}\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}} \quad (20.3)$$

$$= \left(\frac{m_2}{m_1+m_2}\right)\left(\frac{-2\rho}{mv_\infty^2}\right)\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}$$

$$= \left(\frac{m_2}{m_1+m_2}\right)\chi$$

(consistent with $\tan\theta_1 = \frac{m_2 \sin\chi}{m_1+m_2\cos\chi} \rightarrow \theta_1 \approx \frac{m_2\chi}{m_1+m_2}$ for $\chi \ll 1$)

Sec 21, Prob 1

$$x(t) = a\cos(\omega t + \alpha)$$

$$x_0 = a\cos\alpha$$

$$v_0 = \dot{x}(t)\big|_{t=0}$$

$$= -a\omega\sin(\omega t+\alpha)\big|_{t=0}$$

$$= -a\omega\sin\alpha$$

Thus,

$$x_0 = a\cos\alpha$$

$$-\frac{v_0}{\omega} = a\sin\alpha$$

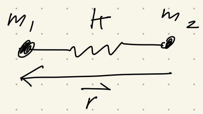
$$\rightarrow a^2 = x_0^2 + \frac{v_0^2}{\omega^2}$$

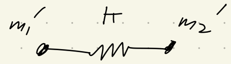
$$\boxed{a = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}}$$

Also:

$$\boxed{\tan\alpha = -\frac{v_0}{\omega x_0}}$$

Sec 21, Prob 2





$$T = \frac{1}{2} m|\dot{\vec{r}}|^2 = \frac{1}{2} m\dot{r}^2 \qquad \text{where} \qquad m = \frac{m_1 m_2}{m_1+m_2}$$

$$U = \frac{1}{2} k r^2$$

$$L = T - U$$
$$= \frac{1}{2} m\dot{r}^2 - \frac{1}{2} k r^2$$

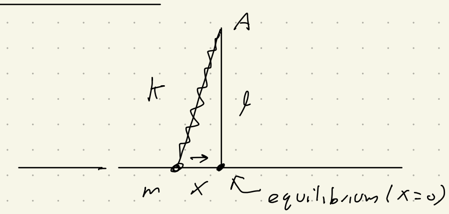
$$\omega = \sqrt{\frac{k}{m}}$$

Similarly, $\omega' = \sqrt{\frac{k}{m'}}$, $m' = \frac{m_1' m_2'}{m_1' + m_2'}$

so $$\frac{\omega'}{\omega} = \sqrt{\frac{k}{m'}}\sqrt{\frac{m}{k}} = \sqrt{\frac{m}{m'}}$$

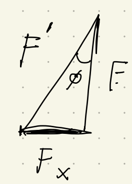
$$= \sqrt{\left(\frac{m_1 m_2}{m_1+m_2}\right)\left(\frac{m_1'+m_2'}{m_1' m_2'}\right)}$$

Sec 21, Prob 3



$$F = kl$$

$$F' = k\sqrt{l^2+x^2} \qquad x \ll l$$
$$= kl\sqrt{1+\left(\frac{x}{l}\right)^2}$$
$$= kl\left(1+\frac{1}{2}\left(\frac{x}{l}\right)^2\right)$$
$$= F\left(1+\frac{1}{2}\left(\frac{x}{l}\right)^2\right)$$

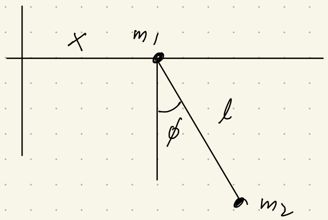


$$F_x = F'\sin\phi$$
$$= F'\frac{x}{l}$$
$$\approx F\left(1+\frac{1}{2}\left(\frac{x}{l}\right)^2\right)\frac{x}{l}$$
$$\approx F\frac{x}{l}$$

$$-F\frac{x}{l} = m\ddot{x} \rightarrow \ddot{x} = -\left(\frac{F}{ml}\right)x, \quad \boxed{\omega = \sqrt{\frac{F}{ml}}}$$

$$U = F\delta l, \quad \delta l = \sqrt{l^2+x^2} - l \qquad \rightarrow U = \frac{1}{2}\frac{F}{l}x^2$$
$$= l\sqrt{1+\left(\frac{x}{l}\right)^2} - l$$
$$\approx l\,\frac{1}{2}\left(\frac{x}{l}\right)^2 = \frac{1}{2}\frac{x^2}{l}$$

Sec 21, Prob 5



From Sec 14, Prob 3

$$E = \frac{1}{2} m_2 l^2 \dot\phi^2 \left(1 - \frac{m_2}{m_1+m_2}\cos^2\phi\right) - m_2 g l \cos\phi$$

stable equilibrium: $\phi = 0$

$$\cos\phi \approx 1 - \frac{\phi^2}{2}$$

$$\cos^2\phi \approx \left(1 - \frac{\phi^2}{2}\right)^2$$

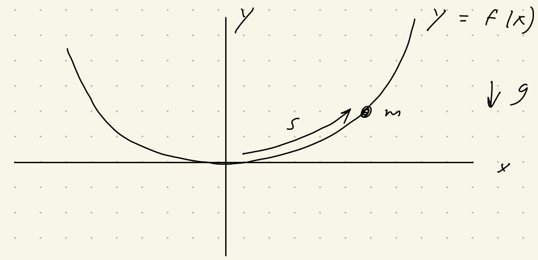
$$\approx 1 - \phi^2$$

$$E \approx \frac{1}{2} m_2 l^2 \dot\phi^2 \underbrace{\left(1 - \frac{m_2}{m_1+m_2}\right)}_{\frac{m_1}{m_1+m_2}} - m_2 g l \left(1 - \frac{\phi^2}{2}\right)$$

$$\rightarrow \quad E + m_2 g l = \frac{1}{2} m l^2 \dot\phi^2 + \frac{1}{2} m_2 g l \phi^2$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{m_2 g l}{m l^2}} = \sqrt{\left(\frac{m_1+m_2}{m_1}\right)\frac{g}{l}}$$

Sec 21, Prob 6



$$T = \frac{1}{2} m(\dot x^2 + \dot y^2) \qquad ds = \sqrt{dx^2+dy^2}$$

$$= \frac{1}{2} m \dot s^2 \qquad = dx\sqrt{1+y'^2}$$

For period (or freq) to be independent of initial amplitude, we need

$$U = \frac{1}{2} k s^2 \quad \text{for some } k > 0$$

Then $L = T - U$

$$= \frac{1}{2} m \dot s^2 - \frac{1}{2} k s^2$$

$$\rightarrow \quad s(t) = a\cos(\omega t + \alpha), \quad \omega = \sqrt{\frac{k}{m}}$$

$$U = \frac{1}{2} k s^2 = m g y \longrightarrow y = \frac{1}{2} A s^2$$

where $A \equiv \frac{k}{mg}$

Differentiate:

$$dy = A s\, ds$$

$$= A\sqrt{\frac{2y}{A}}\sqrt{dx^2+dy^2}$$

$$\cancel{dy} = \sqrt{2Ay}\;\cancel{dy}\sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

$$\rightarrow 1+\left(\frac{dx}{dy}\right)^2 = \frac{1}{2Ay}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{2Ay} - 1$$

$$= \frac{1-2Ay}{2Ay}$$

so $\dfrac{dx}{dy} = \pm\sqrt{\dfrac{1-2Ay}{2Ay}}$

$$\rightarrow \quad x = \int dy\sqrt{\frac{1-2Ay}{2Ay}} + \text{const}$$

Make a change of variables so that

$$2Ay = \frac{1}{2}(1-\cos\theta) = \sin^2\left(\frac{\theta}{2}\right)$$

$$1-2Ay = 1-\sin^2\left(\frac{\theta}{2}\right) = \cos^2\left(\frac{\theta}{2}\right)$$

$$2A\,dy = \cancel{2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\cdot\frac{d\theta}{\cancel{2}} = \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)d\theta$$

$$\cos\theta = \cos^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\theta}{2}\right) = 1-2\sin^2\left(\frac{\theta}{2}\right) = 2\cos^2\left(\frac{\theta}{2}\right)-1$$

Thus,

$$x = \int \frac{1}{2A}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)d\theta\sqrt{\frac{\cos^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}}$$

$$= \int \frac{1}{2A}\cos^2\left(\frac{\theta}{2}\right)d\theta$$

$$= \frac{1}{4A}\int(1+\cos\theta)\,d\theta$$

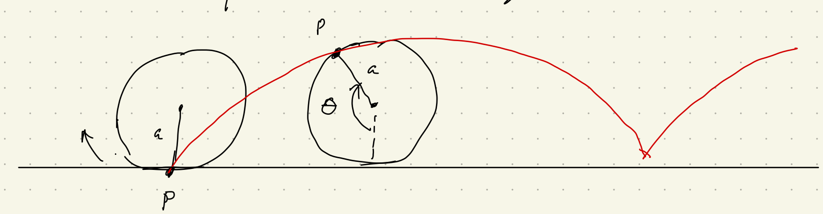
$$= \frac{1}{4A}(\theta+\sin\theta)$$

Summary:

$$x = \frac{1}{4A}(\theta+\sin\theta) = a(\theta+\sin\theta)$$

$$y = \frac{1}{4A}(1-\cos\theta) = a(1-\cos\theta)$$

Parametric equations for a cycloid:



Sec 22, prob 1 a, b, c :

$$\xi = e^{i\omega t}\left[\int_0^t d\bar{t}\, \frac{F(\bar{t})}{m}\, e^{-i\omega\bar{t}} + \xi_0\right]$$

$$\xi = \dot{x} + i\omega x \quad \rightarrow \quad x(t) = \frac{1}{\omega}\,\mathrm{Im}(\xi(t))$$

Assume: $x = \dot{x} = 0$ at $t = 0$

$$\rightarrow \xi_0 = 0$$

a) $F = F_0 = \text{const}$

$$\xi = e^{i\omega t}\int_0^t d\bar{t}\,\frac{F_0}{m}\,e^{-i\omega\bar{t}}$$

$$= \frac{F_0}{m}\,e^{i\omega t}\,\frac{1}{-i\omega}\,e^{-i\omega\bar{t}}\Big|_0^t$$

$$= \frac{iF_0}{m\omega}\,e^{i\omega t}\left[e^{-i\omega t} - 1\right]$$

$$= \frac{iF_0}{m\omega}\left[1 - e^{i\omega t}\right]$$

$$= \frac{iF_0}{m\omega}\left[1 - (\cos(\omega t) + i\sin(\omega t))\right]$$

$$= \frac{iF_0}{m\omega}(1 - \cos\omega t) + \frac{F_0}{m\omega}\sin\omega t$$

$$\rightarrow \boxed{x(t) = \frac{F_0}{m\omega^2}(1 - \cos\omega t)}$$

b) $F(t) = at$

$$\xi = e^{i\omega t}\int_0^t d\bar{t}\,\frac{a\bar{t}}{m}\,e^{-i\omega\bar{t}}$$

$$= \frac{a}{m}\,e^{i\omega t}\int_0^t d\bar{t}\;\bar{t}\,e^{-i\omega\bar{t}}$$

Let: $u = \bar{t}$, $du = d\bar{t}$

$$dv = d\bar{t}\,e^{-i\omega\bar{t}} \rightarrow v = \frac{1}{-i\omega}\,e^{-i\omega\bar{t}}$$

Thus,

$$\xi = \frac{a}{m}\,e^{i\omega t}\left[\frac{\bar{t}}{-i\omega}\,e^{-i\omega\bar{t}}\Big|_0^t - \int_0^t \frac{1}{-i\omega}\,e^{-i\omega\bar{t}}\,d\bar{t}\right]$$

$$= \frac{a}{m}\,e^{i\omega t}\left[i\frac{t}{\omega}\,e^{-i\omega t} + \frac{1}{i\omega}\left(\frac{1}{-i\omega}\right)e^{-i\omega\bar{t}}\Big|_0^t\right]$$

$$= \frac{a}{m}\,e^{i\omega t}\left[i\frac{t}{\omega}\,e^{-i\omega t} + \frac{1}{\omega^2}\left(e^{-i\omega t} - 1\right)\right]$$

$$= \frac{iat}{m\omega} + \frac{a}{m\omega^2}\left(1 - e^{i\omega t}\right)$$

$$= \frac{iat}{m\omega} + \frac{a}{m\omega^2}\left(1 - (\cos\omega t + i\sin\omega t)\right)$$

$$= \frac{a}{m\omega^2}(1 - \cos\omega t) + i\left(\frac{at}{m\omega} - \frac{a}{m\omega^2}\sin\omega t\right)$$

$$= \frac{a}{m\omega^2}(1 - \cos\omega t) + i\frac{a}{m\omega^2}(\omega t - \sin\omega t)$$

$$\rightarrow \boxed{x(t) = \frac{a}{m\omega^3}(\omega t - \sin\omega t)}$$

c) $F = F_0 \exp(-\alpha t)$

$$\xi = e^{i\omega t} \int_0^t d\bar{t}\, \frac{F_0 e^{-\alpha \bar{t}}}{m} e^{-i\omega \bar{t}}$$

$$= \frac{F_0}{m} e^{i\omega t} \int_0^t d\bar{t}\, e^{-(i\omega+\alpha)\bar{t}}$$

$$= \frac{F_0}{m} e^{i\omega t} \frac{1}{-(i\omega+\alpha)} e^{-(i\omega+\alpha)\bar{t}} \Big|_0^t$$

$$= -\frac{F_0}{m} e^{i\omega t} \frac{1}{(i\omega+\alpha)} \left(e^{-(i\omega+\alpha)t} - 1\right)$$

$$= -\frac{F_0}{m} \frac{1}{(i\omega+\alpha)} \left(e^{-\alpha t} - e^{i\omega t}\right)$$

$$= -\frac{F_0}{m} \left(\frac{\alpha - i\omega}{\alpha^2+\omega^2}\right) \left[e^{-\alpha t} - (\cos\omega t + i\sin\omega t)\right]$$

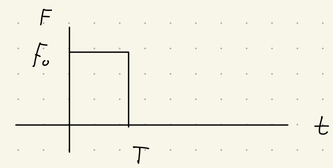
$$= -\frac{F_0}{m} \left(\frac{\alpha - i\omega}{\alpha^2+\omega^2}\right) \left[\left(e^{-\alpha t} - \cos\omega t\right) - i\sin\omega t\right]$$

$\rightarrow$ $x = \dfrac{\mathrm{Im}\,\xi}{\omega}$

$$= -\frac{F_0}{m\omega} \left(\frac{1}{\alpha^2+\omega^2}\right) \left(-\omega\left(e^{-\alpha t} - \cos\omega t\right) - \alpha\sin\omega t\right)$$

$$= \boxed{\frac{F_0}{m} \left(\frac{1}{\alpha^2+\omega^2}\right) \left(e^{-\alpha t} - \cos\omega t + \frac{\alpha}{\omega}\sin\omega t\right)}$$

Sec 22, Prob 3:



$x = 0$, $\dot{x} = 0$ at $t = 0$ $\rightarrow$ $\xi_0 = 0$

$$\xi = e^{i\omega t} \int_0^t d\bar{t}\, \frac{F_0}{m} e^{-i\omega \bar{t}}$$

$$= \frac{F_0}{m} e^{i\omega t} \frac{1}{-i\omega} e^{-i\omega \bar{t}} \Big|_0^t$$

$$= \frac{-F_0}{im\omega} e^{i\omega t} \left(e^{-i\omega t} - 1\right)$$

$$= \frac{-F_0}{im\omega} \left(1 - e^{i\omega t}\right)$$

$$= \frac{-F_0}{im\omega} \left[(1 - \cos\omega t) - i\sin\omega t\right]$$

$$= \frac{F_0}{m\omega} \left[i(1-\cos\omega t) + \sin\omega t\right]$$

$\rightarrow$ $x(t) = \dfrac{\mathrm{Im}\,\xi}{\omega}$

$$= \frac{F_0}{m\omega^2} (1 - \cos\omega t)$$

$$\dot{x}(t) = \frac{F_0}{m\omega} \sin\omega t$$

For $t > T$, general solution is

$$x(t) = c_1 \cos(\omega(t-T)) + c_2 \sin(\omega(t-T))$$

$$\dot{x}(t) = -c_1 \omega \sin(\omega(t-T)) + c_2 \omega \cos(\omega(t-T))$$

Match x and $\dot{x}$ at $t=T$:

$$\rightarrow \quad \frac{F_0}{m\omega^2}(1-\cos\omega T) = c_1$$

$$\rightarrow \quad \frac{F_0}{m\omega} \sin\omega T = c_2 \omega$$

Thus,

$$c_1 = \frac{F_0}{m\omega^2}(1-\cos\omega T)$$

$$c_2 = \frac{F_0}{m\omega^2} \sin\omega T$$

$$x(t) = \frac{F_0}{m\omega^2}(1-\cos\omega T)\cos(\omega(t-T)) + \frac{F_0}{m\omega^2}\sin\omega T \sin(\omega(t-T))$$

$$\rightarrow \quad a = \frac{F_0}{m\omega^2}\sqrt{(1-\cos\omega T)^2 + \sin^2\omega T}$$

$$= \frac{F_0}{m\omega^2}\sqrt{1+\cos^2\omega T - 2\cos\omega T + \sin^2\omega T}$$

$$= \frac{F_0}{m\omega^2}\sqrt{2(1-\cos\omega T)}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$

$$2\sin^2\theta = 1 - \cos 2\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Thus, $1 - \cos\omega T = 2\sin^2\left(\frac{\omega T}{2}\right)$

$$a = \frac{F_0}{m\omega^2}\sqrt{4\sin^2\left(\frac{\omega T}{2}\right)}$$

$$= \frac{2F_0}{m\omega^2}\sin\left(\frac{\omega T}{2}\right)$$

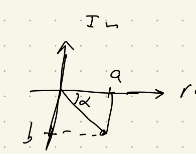
$$a\cos\omega t + b\sin\omega t$$

$$= \mathrm{Re}\left[a e^{i\omega t} - ib\, e^{i\omega t}\right]$$

$$= \mathrm{Re}\left[(a-ib)e^{i\omega t}\right]$$

$$= \mathrm{Re}\left[\sqrt{a^2+b^2}\; e^{i(\omega t-\alpha)}\right]$$

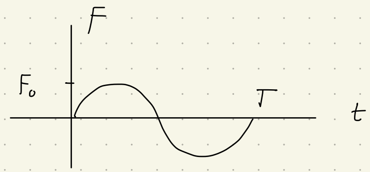
$$= \sqrt{a^2+b^2}\cos(\omega t - \alpha)$$



$$a - ib = \sqrt{a^2+b^2}\; e^{-i\alpha}$$

$$\tan\alpha = \frac{b}{a}$$

Sec 22, Prob 5:



$F = F_0 \sin \omega t$, $\quad T = \frac{2\pi}{\omega}$

$x=0,\ \dot{x}=0$ at $t=0$ $\rightarrow$ $\xi_0 = 0$

$$\xi = e^{i\omega t}\int_0^t d\bar{t}\ \frac{F_0 \sin\omega\bar{t}}{m}\ e^{-i\omega\bar{t}}$$

$$= \frac{F_0}{m} e^{i\omega t}\int_0^t d\bar{t}\ \frac{1}{2i}\left(e^{i\omega\bar{t}} - e^{-i\omega\bar{t}}\right) e^{-i\omega\bar{t}}$$

$$= \frac{F_0}{i2m} e^{i\omega t}\int_0^t d\bar{t}\ \left(1 - e^{-2i\omega\bar{t}}\right)$$

$$= \frac{F_0}{i2m} e^{i\omega t}\left(\bar{t} + \frac{1}{i2\omega} e^{-2i\omega\bar{t}}\right)\Big|_0^t$$

$$= \frac{F_0}{i2m} e^{i\omega t}\left[t + \frac{1}{i2\omega}\left(e^{-2i\omega t} - 1\right)\right]$$

$$= \frac{F_0}{i2m}\left[t e^{i\omega t} + \frac{1}{i2\omega}\left(e^{-i\omega t} - e^{i\omega t}\right)\right]$$

$$= \frac{F_0}{i2m}\left[t e^{i\omega t} - \frac{1}{\omega}\sin\omega t\right]$$

$$= \frac{F_0}{i2m}\left[t\left(\cos\omega t + i\sin\omega t\right) - \frac{1}{\omega}\sin\omega t\right]$$

$$= \frac{F_0}{i2m}\left[\left(t\cos\omega t - \frac{1}{\omega}\sin\omega t\right) + i\,t\sin\omega t\right]$$

$\rightarrow$ $x(t) = \frac{\text{Im}\,\xi}{\omega}$

$$= \frac{-F_0}{2m\omega}\left(t\cos\omega t - \frac{1}{\omega}\sin\omega t\right)$$

$$\dot{x}(t) = \frac{-F_0}{2m\omega}\left(\cancel{\cos\omega t} - \omega t\sin\omega t - \cancel{\cos\omega t}\right)$$

$$= \frac{F_0 t}{2m}\sin\omega t$$

match with (at $t = T = 2\pi/\omega$)

$$x(t) = c_1\cos(\omega(t-T)) + c_2\sin(\omega(t-T))$$

$$\dot{x}(t) = -\omega c_1\sin(\omega(t-T)) + \omega c_2\cos(\omega(t-T))$$

thus, $\frac{-F_0}{2m\omega}\left(T\underbrace{\cos\omega T}_{=1} - \frac{1}{\omega}\underbrace{\sin\omega T}_{0}\right) = c_1$

$$\boxed{c_1 = \frac{-F_0\pi}{m\omega^2}}$$

and $\frac{F_0 T}{2m}\underbrace{\sin\omega T}_{0} = \omega c_2$ $\rightarrow$ $\boxed{c_2 = 0}$

Thus, for $t > T$:

$$x(t) = \frac{-F_0 \pi}{m\omega^2} \cos(\omega(t-T))$$

$$\rightarrow \quad a = \frac{F_0 \pi}{m\omega^2} \qquad \text{(amplitude)}$$

Sec 23, Prob 1

$$L = \tfrac{1}{2}(\dot{x}^2+\dot{y}^2) - \tfrac{1}{2}\omega_0^2(x^2+y^2) + \alpha x y$$

$$= \frac{1}{2}\sum_{i,k} m_{ik}\,\dot{x}_i\,\dot{x}_k - \frac{1}{2}\sum_{i,k} k_{ik}\,x_i\,x_k$$

where $m_{ik} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$k_{ik} = \begin{bmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{bmatrix}$$

$$x_i = \begin{bmatrix} x & y \end{bmatrix}$$

Characteristic equation

$$0 = \det(k_{ik} - \omega^2 m_{ik})$$

$$= \det \begin{bmatrix} \omega_0^2 - \omega^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega^2 \end{bmatrix}$$

$$= (\omega_0^2 - \omega^2)^2 - \alpha^2$$

$$\rightarrow (\omega_0^2 - \omega^2) = \pm\alpha$$

$$\omega^2 = \omega_0^2 \mp \alpha$$

so
$$\boxed{\begin{aligned} \omega_+^2 &= \omega_0^2 + \alpha \\ \omega_-^2 &= \omega_0^2 - \alpha \end{aligned}}$$

Eigen vectors:

ω_+^2 : $\begin{bmatrix} \omega_0^2-\omega_+^2 & -\alpha \\ -\alpha & \omega_0^2-\omega_+^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -\alpha & -\alpha \\ -\alpha & -\alpha \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rightarrow \quad A_1 = -A_2$

so $\quad V_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

ω_-^2 : $\begin{bmatrix} \omega_0^2-\omega_-^2 & -\alpha \\ -\alpha & \omega_0^2-\omega_-^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rightarrow \quad A_2 = A_1$

so $\quad V_- = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

General solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathrm{Re}\Big(\sum_{\alpha=+,-} \overbrace{C_\alpha}^{\text{complex constants}} V_\alpha e^{i\omega_\alpha t} \Big)$$

$$= \sum_{\alpha=+,-} V_\alpha \theta_\alpha \quad , \quad \theta_\alpha = \mathrm{Re}(C_\alpha e^{i\omega_\alpha t})$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = V_+\theta_+ + V_-\theta_-$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \theta_+ + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \theta_-$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \theta_+ + \theta_- \\ -\theta_+ + \theta_- \end{bmatrix}$$

For weak coupling ($\alpha \ll \omega_0^2$):

$$\omega_\pm^2 = \omega_0^2 \pm \alpha$$

$$= \omega_0^2 \left(1 \pm \frac{\alpha}{\omega_0^2} \right)$$

$$\rightarrow \omega_\pm = \omega_0 \sqrt{1 \pm \frac{\alpha}{\omega_0^2}}$$

$$\approx \omega_0 \left(1 \pm \frac{1}{2} \frac{\alpha}{\omega_0^2} \right)$$

$$= \begin{cases} \omega_0 + \frac{1}{2}\frac{\alpha}{\omega_0} \\ \omega_0 - \frac{1}{2}\frac{\alpha}{\omega_0} \end{cases}$$

$$\theta_+ + \theta_- = \mathrm{Re}\left(e^{i\omega_+ t} + e^{i\omega_- t} \right)$$

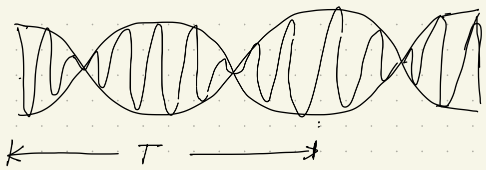
$$= \mathrm{Re}\left[e^{i\omega_0 t} \left(e^{i\frac{1}{2}\frac{\alpha}{\omega_0} t} + e^{-i\frac{1}{2}\frac{\alpha}{\omega_0} t} \right) \right]$$

$$= \mathrm{Re}\left[e^{i\omega_0 t} \, 2\cos\left(\frac{\alpha t}{2\omega_0} \right) \right]$$

$$= 2\cos(\omega_0 t) \cos\left(\frac{\alpha t}{2\omega_0} \right)$$

$$\theta_+ - \theta_- = -2 \sin(\omega_0 t) \sin\left(\frac{\alpha t}{2\omega_0} \right)$$

Amplitude modulation:



$$T_{beat} = \frac{1}{2} T = \frac{1}{2} \frac{2\pi}{(\alpha/2\omega_0)} = \frac{2\pi}{\alpha/\omega_0}$$
$$= \frac{2\pi}{\omega_{beat}}$$

$$\omega_{beat} = \alpha/\omega_0$$
$$= \omega_+ - \omega_-$$

Sec 23, Prob 3:

Space oscillator $U = \frac{1}{2} k r^2$

$$L = T - U$$
$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - \frac{1}{2} k r^2$$
$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k (x^2 + y^2)$$
$$= \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + \frac{1}{2} m \dot{y}^2 - \frac{1}{2} k y^2$$

General solution:

$$x = a \cos(\omega t + \alpha)$$
$$y = b \cos(\omega t + \beta)$$

where

$$\omega_x = \omega_y \equiv \omega = \sqrt{\frac{k}{m}}$$

Now: $$x = a(\cos(\omega t) \cos\alpha - \sin(\omega t) \sin\alpha)$$
$$y = b(\cos(\omega t) \cos\beta - \sin(\omega t) \sin\beta)$$

$$\rightarrow \begin{bmatrix} x/a \\ y/b \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \cos\beta & -\sin\beta \end{bmatrix}}_{M} \begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix}$$

$$\det M = -\cos\alpha \sin\beta + \sin\alpha \cos\beta$$
$$= \sin(\alpha - \beta)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus

$$\begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix} = M^{-1} \begin{bmatrix} x/a \\ y/b \end{bmatrix}$$

$$= \frac{1}{\sin(\alpha-\beta)} \begin{bmatrix} -\sin\beta & \sin\alpha \\ -\cos\beta & \cos\alpha \end{bmatrix} \begin{bmatrix} x/a \\ y/b \end{bmatrix}$$

square and add:

$$\underline{1} = \cos^2\omega t + \sin^2\omega t$$

$$= \frac{1}{\sin^2(\alpha-\beta)} \left(-\sin\beta\left(\frac{x}{a}\right) + \sin\alpha\left(\frac{y}{b}\right)\right)^2$$

$$+ \frac{1}{\sin^2(\alpha-\beta)} \left(-\cos\beta\left(\frac{x}{a}\right) + \cos\alpha\left(\frac{y}{b}\right)\right)^2$$

$$= \frac{1}{\sin^2(\alpha-\beta)} \Bigg[\left(\frac{x}{a}\right)^2 \underbrace{(\sin^2\beta + \cos^2\beta)}_{=1} + \left(\frac{y}{b}\right)^2 (\sin^2\alpha + \cos^2\alpha) - 2\underbrace{(\sin\alpha\sin\beta + \cos\alpha\cos\beta)}_{\cos(\alpha-\beta)}\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\Bigg]$$

$$= \frac{1}{\sin^2(\alpha-\beta)} \left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\cos(\alpha-\beta)\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\right]$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\cos(\alpha-\beta)\left(\frac{x}{a}\right)\left(\frac{y}{b}\right) = \sin^2(\alpha-\beta)$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\cos\delta = \sin^2\delta$$

where $\delta \equiv \alpha - \beta$.

<u>NOTE:</u>

(i) when $\delta = 0$

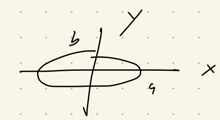
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\left(\frac{x}{a}\right)\left(\frac{y}{b}\right) = 0$$

$$\left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

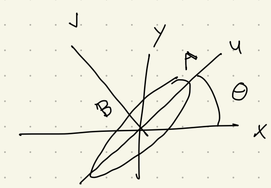
$$\frac{x}{a} = \frac{y}{b} \quad \text{(straight line)}$$

(ii) when $\delta = \pi/2$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



(iii)



(need to determine θ, A, B in terms of δ, a, b)

$$\boxed{\begin{bmatrix} u \\ v \end{bmatrix}} = \boxed{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}} \boxed{\begin{bmatrix} x \\ y \end{bmatrix}} \qquad \boxed{\begin{bmatrix} x \\ y \end{bmatrix}} = \boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \rightarrow \boxed{\begin{bmatrix} u \\ v \end{bmatrix}} = \boxed{\begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}}$$

$$\rightarrow \boxed{\begin{bmatrix} x \\ y \end{bmatrix}} = \boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \boxed{\begin{bmatrix} u \\ v \end{bmatrix}}$$

$$x = u\cos\theta - v\sin\theta$$
$$y = u\sin\theta + v\cos\theta$$

$$\sin^2\delta = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\cos\delta$$

$$= \frac{1}{a^2}\left(u^2\cos^2\theta + v^2\sin^2\theta - 2uv\cos\theta\sin\theta\right)$$
$$+ \frac{1}{b^2}\left(u^2\sin^2\theta + v^2\cos^2\theta + 2uv\sin\theta\cos\theta\right)$$
$$- \frac{2\cos\delta}{ab}\left(u^2\sin\theta\cos\theta - v^2\sin\theta\cos\theta + uv(\cos^2\theta - \sin^2\theta)\right)$$

$$= u^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} - \frac{2}{ab}\cos\delta\sin\theta\cos\theta\right)$$
$$+ v^2\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} + \frac{2}{ab}\cos\delta\sin\theta\cos\theta\right)$$
$$+ 2uv\left(\left(-\frac{1}{a^2} + \frac{1}{b^2}\right)\underbrace{\sin\theta\cos\theta}_{\frac{\sin 2\theta}{2}} - \frac{\cos\delta}{ab}\cos 2\theta\right)$$

Can make the uv term vanish by choosing θ such that:

$$\frac{1}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin 2\theta - \frac{\cos\delta}{ab}\cos 2\theta = 0$$

$$\rightarrow \tan 2\theta = \frac{\frac{\cos\delta}{ab}}{\frac{1}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right)}$$
$$= \frac{2\cos\delta}{ab}\,\frac{a^2b^2}{(a^2 - b^2)}$$
$$= 2\cos\delta\left(\frac{ab}{a^2 - b^2}\right)$$

Thus,

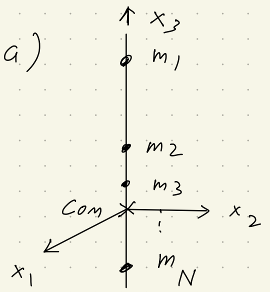
$$\sin^2\delta = u^2\,\oslash + v^2\,\oslash$$
$$1 = \left(\frac{u}{A}\right)^2 + \left(\frac{v}{B}\right)^2$$

where
$$A^2 = \frac{\sin^2\delta}{\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} - \frac{\cos\delta}{ab}\sin 2\theta\right)}$$

$$B^2 = \frac{\sin^2\delta}{\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} + \frac{\cos\delta}{ab}\sin 2\theta\right)}$$

Sec 32, prob 1

a)



$I_3 = 0$ since masses only have a z-component

$$Z_{com} = \frac{1}{M} \sum_a m_a z_a$$

$$I_1 = I_2 \equiv I$$

$$= \sum_a m_a \left(r_a^2 - x_{a1}^2 \right)$$

$$= \sum_a m_a \left(\cancel{x_{a2}^2}^{\,0} + x_{a3}^2 \right)$$

$$= \sum_a m_a x_{a3}^2$$

$$= \sum_a m_a \left(Z_a - Z_{com} \right)^2$$

Suppose we have only two masses

Then:

$$I = m_1 (Z_1 - Z_{com})^2 + m_2 (Z_2 - Z_{com})^2$$

$$Z_{com} = \left(\frac{1}{m_1 + m_2} \right) (m_1 z_1 + m_2 z_2)$$

$$\rightarrow \quad Z_1 - Z_{com} = \frac{(\cancel{m_1} + m_2) Z_1 - (\cancel{m_1 z_1} + m_2 z_2)}{m_1 + m_2}$$

$$= \frac{m_2 (Z_1 - Z_2)}{m_1 + m_2}$$

$$Z_2 - Z_{com} = \frac{(m_1 + \cancel{m_2}) z_2 - (m_1 z_1 + \cancel{m_2 z_2})}{m_1 + m_2}$$

$$= \frac{m_1 (Z_2 - Z_1)}{m_1 + m_2}$$

$$= - \frac{m_1 (Z_1 - Z_2)}{m_1 + m_2}$$

Thus,

$$I = m_1 \frac{m_2^2 (z_1 - z_2)^2}{(m_1 + m_2)^2} + m_2 \frac{m_1^2 (z_1 - z_2)^2}{(m_1 + m_2)^2}$$

$$= \frac{m_1 m_2}{(m_1 + m_2)^{\cancel{2}}} (z_1 - z_2)^2 \cancel{(m_2 + m_1)}$$

$$= \frac{m_1 m_2}{M} l^2 \qquad \text{where } l \equiv |z_1 - z_2|$$

$$= m l^2 \qquad \text{where } m = \text{reduced mass} = \frac{m_1 m_2}{m_1 + m_2}$$

In general:

$$I = \sum_a m_a (z_a - z_{com})^2$$

$$= \sum_a m_a \left(z_a^2 - 2 z_a z_{com} + z_{com}^2 \right)$$

$$= \sum_a m_a z_a^2 - 2\left(\sum_a m_a z_a\right) z_{com} + \mu z_{com}^2$$

$$= \sum_a m_a z_a^2 - \mu z_{com}^2$$

$$= \sum_a m_a z_a^2 - \mu \left(\frac{1}{\mu}\sum_a m_a z_a\right)\left(\frac{1}{\mu}\sum_b m_b z_b\right)$$

$$= \frac{1}{\mu}\sum_a m_a \left(\sum_b m_b\right) z_a^2 - \frac{1}{\mu}\sum_a \sum_b m_a m_b z_a z_b$$

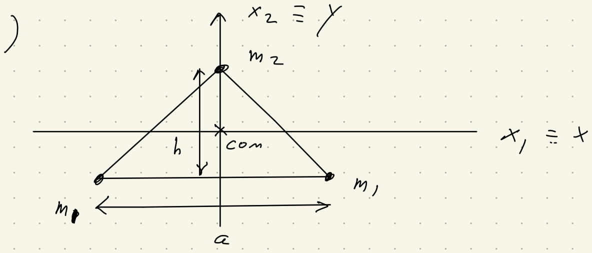
$$= \frac{1}{\mu}\Big[\frac{1}{2}\sum_a\sum_b m_a m_b z_a^2 + \frac{1}{2}\sum_a\sum_b m_a m_b z_b^2 - \sum_a\sum_b m_a m_b z_a z_b \Big]$$

$$= \frac{1}{2\mu}\sum_a\sum_b m_a m_b \underbrace{\left(z_a^2 + z_b^2 - 2 z_a z_b \right)}_{(z_a - z_b)^2}$$

$$= \frac{1}{2\mu}\sum_a\sum_b m_a m_b l_{ab}^2$$

where $l_{ab} \equiv |z_a - z_b|$

b)



x_3 : out of page

To determine x_{12} and x_{22} we require that com be at origin.

$$2 m_1 y_1 + m_2 y_2 = 0$$
$$2 m_1 y_1 + m_2 (y_1 + h) = 0$$
$$(2 m_1 + m_2) y_1 + m_2 h = 0$$

$$\rightarrow \quad y_1 = \frac{-m_2 h}{\mu}, \quad \mu = 2m_1 + m_2$$

$$\rightarrow \quad y_2 = \frac{-2 m_1 y_1}{m_2}$$
$$= -\frac{2 m_1}{\cancel{m_2}}\left(-\frac{\cancel{m_2}}{\mu}\right) h$$
$$= \frac{2 m_1 h}{\mu}$$

$$I_3 = \sum_a m_a (r_a^2 - z_a^2)$$
$$= \sum_a m_a (x_a^2 + y_a^2)$$
$$= 2m_1(x_1^2 + y_1^2) + m_2(\cancelto{0}{x_2^2} + y_2^2)$$
$$= 2m_1\left(\left(\frac{a}{2}\right)^2 + \frac{m_2^2 h^2}{\mu^2}\right) + m_2\left(\frac{2m_1 h}{\mu}\right)^2$$
$$= \frac{m_1 a^2}{2} + \frac{2 m_1 m_2^2 h^2}{\mu^2} + \frac{4 m_2 m_1^2 h^2}{\mu^2}$$
$$= \frac{m_1 a^2}{2} + \frac{2 m_1 m_2 h^2}{\mu^2}(m_2 + 2m_1)$$
$$= \frac{1}{2} m_1 a^2 + \frac{2 m_1 m_2 h^2}{\mu}$$

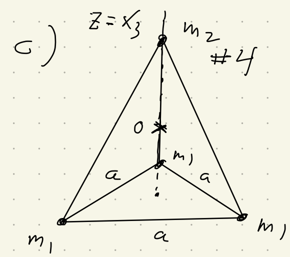
$$I_1 = \sum_a m_a (r_a^2 - x_a^2)$$
$$= \sum_a m_a y_a^2$$
$$= 2m_1 y_1^2 + m_2 y_2^2$$
$$= 2m_1\left(\frac{m_2 h}{\mu}\right)^2 + m_2\left(\frac{2m_1 h}{\mu}\right)^2$$
$$= \frac{2 m_1 m_2^2 h^2}{\mu^2} + \frac{4 m_1 m_2 h^2}{\mu^2}$$
$$= \frac{2 m_1 m_2 h^2}{\mu}$$

$$I_2 = \sum_a m_a (r_a^2 - y_a^2)$$
$$= \sum_a m_a x_a^2$$
$$= 2m_1\left(\frac{a}{2}\right)^2 + m_2 \cancelto{0}{x_2^2}$$
$$= \frac{1}{2} m_1 a^2$$

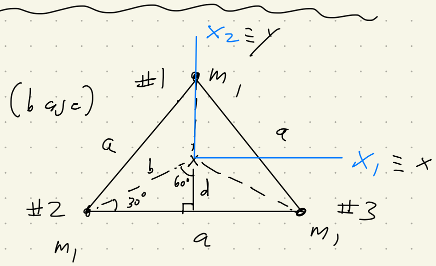
$\rightarrow$

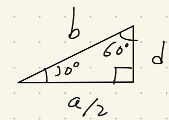
NOTE: $I_3 = I_1 + I_2$

c)



(base)





$$\sin 60° = \frac{a/2}{b}$$
$$\frac{\sqrt{3}}{2} = \frac{a}{2b}$$
$$\rightarrow b = \frac{a}{\sqrt{3}}$$

$$\sin 30° = \frac{d}{b}$$
$$\frac{1}{2} = \frac{d}{a/\sqrt{3}}$$
$$d = \frac{a}{2\sqrt{3}}$$

Thus, $(x_1, y_1) = \left(0, \frac{a}{\sqrt{3}}\right)$

$(x_2, y_2) = \left(-\frac{a}{2}, -\frac{a}{2\sqrt{3}}\right)$

$(x_3, y_3) = \left(\frac{a}{2}, -\frac{a}{2\sqrt{3}}\right)$

Assume origin at COM of system.

$$0 = m_1(z_1 + z_2 + z_3) + m_2 z_4$$

$$= 3m_1 z_1 + m_2 z_4 , \quad z_4 = h + z_1$$

h = height of tetrahedron

$$= 3m_1 z_1 + m_2(h + z_1)$$

$$= (3m_1 + m_2) z_1 + m_2 h$$

$$\rightarrow z_1 = -\frac{m_2}{\mu} h \quad (= z_2 = z_3)$$

$$z_4 = h + z_1 = h - \frac{m_2}{\mu} h$$

$$= \frac{h}{\mu}(\mu - m_2)$$

$$= \frac{h}{\mu} 3m_1$$

$$= \frac{3m_1 h}{\mu}$$

Thus,

$$(x_1, y_1, z_1) = \left(0, \frac{a}{\sqrt{3}}, -\frac{m_2 h}{\mu}\right)$$

$$(x_2, y_2, z_2) = \left(\frac{-a}{2}, \frac{-a}{2\sqrt{3}}, \frac{-m_2 h}{\mu}\right)$$

$$(x_3, y_3, z_3) = \left(\frac{a}{2}, \frac{-a}{2\sqrt{3}}, \frac{-m_2 h}{\mu}\right)$$

$$(x_4, y_4, z_4) = \left(0, 0, \frac{3m_1 h}{\mu}\right)$$

$$I_3 = \sum_a m_a (r_a^2 - z_a^2)$$

$$= \sum_a m_a (x_a^2 + y_a^2)$$

$$= 3m_1 b^2 + \cancel{m_2 0^2}^{\,0}$$

$$= \cancel{3} m_1 \frac{a^2}{\cancel{3}}$$

$$= \boxed{m_1 a^2}$$

$$I_1 = \sum_a m_a (r_a^2 - x_a^2)$$

$$= \sum_a m_a (y_a^2 + z_a^2)$$

$$= m_1\left(\left(\frac{a}{\sqrt{3}}\right)^2 + \left(\frac{-m_2 h}{\mu}\right)^2\right)$$

$$+ m_1\left(\left(\frac{-a}{2\sqrt{3}}\right)^2 + \left(\frac{-m_2 h}{\mu}\right)^2\right)$$

$$+ m_1\left(\left(\frac{-a}{2\sqrt{3}}\right)^2 + \left(\frac{-m_2 h}{\mu}\right)^2\right)$$

$$+ m_2\left(\cancel{0^2} + \left(\frac{3m_1 h}{\mu}\right)^2\right)$$

$$= m_1\left(\frac{a^2}{3} + 2\left(\frac{a^2}{4 \cdot 3}\right) + 3\frac{m_2^2 h^2}{\mu^2}\right) + m_2 \frac{9m_1^2 h^2}{\mu^2}$$

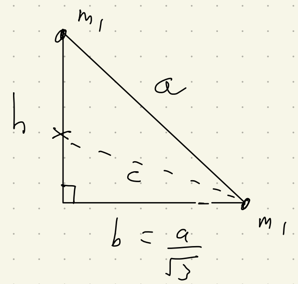
$$= m_1\left[\frac{a^2}{3}\left(1 + \frac{1}{2}\right) + 3\frac{m_2^2 h^2}{\mu^2}\right] + \frac{9m_1^2 m_2 h^2}{\mu^2}$$

$$I_1 = \frac{1}{2} m_1 a^2 + \frac{3 m_1 m_2}{\mu^2} h^2 (\underbrace{m_2 + 3m_1}_{=\mu})$$

$$= \frac{1}{2} m_1 a^2 + \frac{3 m_1 m_2 h^2}{\mu}$$

$I_2 = I_1$ (since equilateral base → symmetric top)

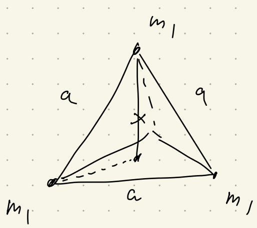
NOTE: A regular tetrahedron ($m_1 = m_2 \to \mu = 4m_1$)



$$a^2 = h^2 + \frac{a^2}{3}$$

$$\to h^2 = \frac{2}{3} a^2$$

so $h = \sqrt{\frac{2}{3}}\, a$



Thus,

$$I_1 = \frac{1}{2} m_1 a^2 + \frac{\cancel{3} m_1^2}{\mu} \frac{2}{\cancel{3}} a^2$$

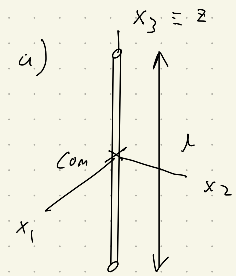
$$= \frac{1}{2} m_1 a^2 + \frac{m_1^2}{4m_1} 2a^2$$

$$= m_1 a^2$$

$$\boxed{\text{so } I_1 = I_2 = I_3 = m_1 a^2}$$

Sec 32, Prob 2:

a)



$I_3 = 0$

$I_1 = I_2 \equiv I$

$$= \int \rho\, dV (r^2 - x^2)$$

$$= \int \rho\, dV (\cancel{y^2}^{\,0} + z^2)$$

$$= \int \rho\, dV\, z^2$$

$$\rho = \frac{\mu}{l} \delta(x) \delta(y)$$

$$I = \frac{\mu}{l} \iiint dx\,dy\,dz\ \delta(x)\delta(y)\ z^2$$

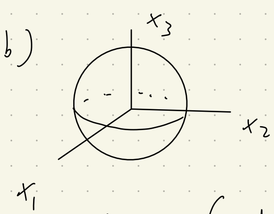
$$= \frac{\mu}{l} \int_{-l/2}^{l/2} z^2\, dz$$

$$= \frac{\mu}{l} \frac{z^3}{3} \Big|_{-l/2}^{l/2}$$

$$= \frac{\mu}{l} \frac{2}{3} \frac{l^3}{8}$$

$$= \frac{1}{12} \mu l^2 = I_1 = I_2$$

b)



$I_1 = I_2 = I_3 \equiv I$

$I_1 = \int \rho \, dV \, (r^2 - x^2) \quad = \int \rho \, d$

$= \int \rho \, dV \, (y^2 + z^2)$

$I_2 = \int \rho \, dV \, (x^2 + z^2)$

$I_3 = \int \rho \, dV \, (x^2 + y^2)$

$\rightarrow \quad I = \frac{1}{3}\left(I_1 + I_2 + I_3\right)$

$= \frac{1}{3} \int \rho \, dV \cdot 2\,(x^2 + y^2 + z^2)$

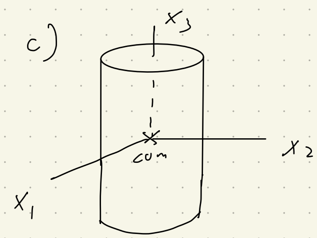
$= \frac{2}{3} \rho \int dV \; r^2$

$= \frac{2}{\cancel{3}} \left(\frac{M}{\frac{4}{\cancel{3}} \pi R^3} \right) \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{R} r^2 \sin\theta \, dr \, d\theta \, d\phi \quad r^2$

$= \frac{M}{2 \cancel{\pi} R^3} \cdot 4 \cancel{\pi} \int_0^R r^4 \, dr$

$= \frac{2M}{R^3} \frac{R^5}{5} \quad = \boxed{\frac{2}{5} M R^2}$

c)



height : h

radius : R

$\rho = \frac{M}{\pi R^2 h}$

$I_1 = I_2 \equiv I$

$I_3 = \int \rho \, dV \, (r^2 - z^2)$

$= \int \rho \, dV \, (x^2 + y^2)$

$= \int \rho \, dV \, s^2$

where (s, ϕ, z) are cylindrical coords.

$\rightarrow \quad dV = s \, ds \, d\phi \, dz$

thus, $I_3 = \frac{M}{\pi R^2 h} \iiint s \, ds \, d\phi \, dz \quad s^2$

$= \frac{M}{\pi R^2 h} \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz \int_0^R ds \, s^3$

$= \frac{M}{\cancel{\pi} R^2 \cancel{h}} \cdot 2\cancel{\pi} \cdot \cancel{h} \cdot \frac{R^4}{4}$

$= \boxed{\frac{1}{2} M R^2}$

$$I_1 = I_2 \equiv I$$

$$I = \frac{1}{2}(I_1 + I_2)$$

$$= \frac{1}{2}\left[\int \rho\, dV (y^2+z^2) + \int \rho\, dV (x^2+z^2)\right]$$

$$= \frac{1}{2} \int \rho\, dV (x^2+y^2+2z^2)$$

$$= \frac{1}{2} \int \rho\, dV\, s^2 + \int \rho\, dV\, z^2$$

$$= \frac{1}{2} I_3 + \int \rho\, dV\, z^2$$

Now: $\displaystyle\int \rho\, dV\, z^2 = \rho \iiint s\, ds\, d\phi\, dz\, z^2$

$$= \frac{M}{\pi R^2 h} \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz\, z^2 \int_0^R ds\, s$$

$$= \frac{M}{\cancel{\pi} R^2 h} \cdot 2\cancel{\pi}\, \frac{z^3}{3}\Big|_{-h/2}^{h/2} \frac{s^2}{2}\Big|_0^R$$

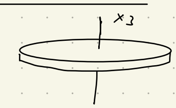
$$= \frac{2M}{\cancel{R^2} h} \frac{\cancel{2}}{3}\left(\frac{h}{2}\right)^3 \frac{\cancel{R^2}}{\cancel{2}}$$

$$= \frac{1}{12} M h^2$$

$$\rightarrow I = \frac{1}{2}\left(\frac{1}{2} M R^2\right) + \frac{1}{12} M h^2$$

$$= \boxed{\frac{1}{4} M\left(R^2 + \frac{h^2}{3}\right)}$$

Limiting case:

i) 

thin disk ($h \to 0$)

$$I_3 = \frac{1}{2} M R^2$$

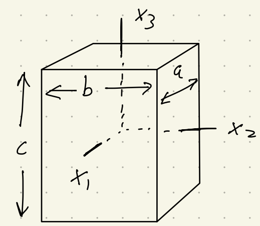
$$I_1 = I_2 = \frac{1}{4} M R^2$$

ii) 

thin rod ($R \to 0$)

$$I_3 = 0$$

$$I_1 = I_2 = \frac{1}{12} M h^2$$

d) 

$$\rho = \frac{M}{abc}$$

$$I_1 = \int \rho\, dV (y^2+z^2)$$

$$= \frac{M}{abc} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz\, (y^2+z^2)$$

$$= \frac{M}{abc}\left(ac\, \frac{y^3}{3}\Big|_{-b/2}^{b/2} + ab\, \frac{z^3}{3}\Big|_{-c/2}^{c/2}\right)$$

$$= \frac{M}{abc}\left(ac\, \frac{2}{3}\left(\frac{b}{2}\right)^3 + ab\, \frac{2}{3}\left(\frac{c}{2}\right)^3\right)$$

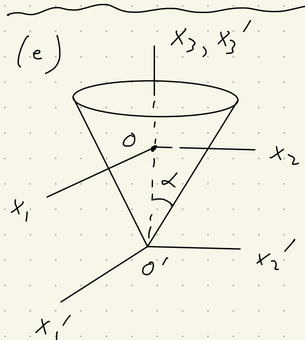
$$= \frac{1}{12} M (b^2 + c^2)$$

Similarly,

$$I_2 = \frac{1}{12} M (c^2 + a^2)$$

$$I_3 = \frac{1}{12} M (a^2 + b^2)$$

(e)

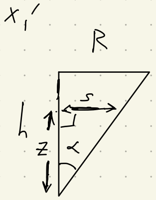


base radius : R

height : h

$$vol = \int dV$$

$$= \int_0^{2\pi} d\phi \int_0^h dz \int_0^{zR/h} s\, ds$$



$$\tan\alpha = \frac{s}{z} = \frac{R}{h}$$

$$\longrightarrow \quad s = z\frac{R}{h}$$

Thus, $$vol = \cancel{2}\pi \int_0^h dz \left.\frac{s^2}{\cancel{2}}\right|_0^{zR/h}$$

$$= \pi \int_0^h dz\, z^2 \frac{R^2}{h^2}$$

$$= \frac{\pi R^2}{h^2} \left.\frac{z^3}{3}\right|_0^h$$

$$= \frac{1}{3}\pi R^2 h$$

$$\rho = \frac{M}{Vol} = \frac{3M}{\pi R^2 h}$$

$$I_3' = \int \rho\, dV (x^2 + y^2)$$

$$= \rho \int s\, ds\, d\phi\, dz\; s^2$$

$$= \rho \int_0^{2\pi} d\phi \int_0^h dz \int_0^{zR/h} ds\, s^3$$

$$= \rho\, 2\pi \int_0^h dz \left.\frac{s^4}{4}\right|_0^{zR/h}$$

$$= \rho \frac{\pi}{2} \int_0^h dz\; z^4 \left(\frac{R}{h}\right)^4$$

$$= \rho \frac{\pi}{2} \left(\frac{R}{h}\right)^4 \left.\frac{z^5}{5}\right|_0^h$$

$$= \frac{3M}{\cancel{\pi} R^2 \cancel{h}} \frac{\cancel{\pi}}{2} \frac{R^4}{\cancel{h^4}} \frac{\cancel{h^5}}{5}$$

$$= \boxed{\frac{3}{10} M R^2}$$

$$I_1' = I_2' \equiv I'$$

$$I' = \frac{1}{2}(I_1' + I_2')$$

$$= \frac{1}{2} I_3' + \int \rho\, dV z^2 \quad \text{(like cylinder)}$$

$$\int \rho\, dV\, z^2 = \rho \int s\, ds\, d\phi\, dz\, z^2$$

$$= \rho \int_0^{2\pi} d\phi \int_0^h dz\, z^2 \int_0^{zR/h} s\, ds$$

$$= \rho\, \cancel{2}\pi \int_0^h dz\, z^2\, \frac{s^2}{\cancel{2}} \Big|_0^{zR/h}$$

$$= \rho\, \pi \left(\frac{R}{h}\right)^2 \int_0^h dz\, z^4$$

$$= \rho\, \pi \left(\frac{R}{h}\right)^2 \frac{z^5}{5} \Big|_0^h$$

$$= \frac{3\mu}{\cancel{\pi} \cancel{R^2} h}\, \cancel{\pi} \left(\frac{\cancel{R}}{h}\right)^2 \frac{h^5}{5}$$

$$= \frac{3}{5} \mu h^2$$

$$I' = \frac{1}{2} I_3' + \frac{3}{5} \mu h^2$$

$$= \frac{1}{2} \frac{3}{10} \mu R^2 + \frac{3}{5} \mu h^2$$

$$= \boxed{\frac{3}{5} \mu \left(h^2 + \frac{R^2}{4}\right)}$$

To find location of COM:

$$z'_{com} = \frac{1}{\mu} \int \rho\, dV\, z$$

$$= \frac{1}{\cancel{\mu}} \frac{3\cancel{\mu}}{\pi R^2 h} \int_0^{2\pi} d\phi \int_0^h dz\, z \int_0^{zR/h} ds\, s$$

$$= \frac{3}{\cancel{\pi} R^2 h}\, \cancel{2}\cancel{\pi} \int_0^h dz\, z\, \frac{s^2}{\cancel{2}} \Big|_0^{zR/h}$$

$$= \frac{3}{R^2 h} \left(\frac{R}{h}\right)^2 \int_0^h dz\, z^3$$

$$= \frac{3}{R^2 h} \left(\frac{R}{h}\right)^2 \frac{z^4}{4} \Big|_0^h$$

$$= \frac{3}{4} \frac{1}{\cancel{R^2} h} \left(\frac{\cancel{R}}{h}\right)^2 h^4$$

$$= \boxed{\frac{3}{4} h}$$

Thus, $\vec{a} = -\frac{3}{4} h\, \hat{x}_3$

$$I_{ij} = I'_{ij} - \mu\left(\delta_{ij}\, a^2 - a_i a_j\right)$$

$$\boxed{I_3 = I_3' - \mu\left(\cancel{a^2 - a^2}\right)^0 = \frac{3}{10} \mu R^2}$$

$$I_1 = I_1' - \mu(a^2 - 0 \cdot 0)$$

$$= I_1' - \mu\left(\frac{3}{4}h\right)^2$$

$$= \frac{3}{5}\mu\left(h^2 + \frac{R^2}{4}\right) - \frac{9}{16}\mu h^2$$

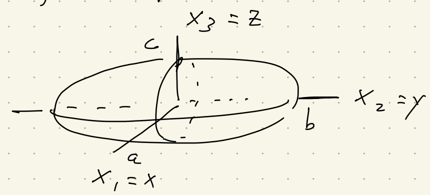
$$= \mu\left(\left(\frac{3}{5} - \frac{9}{16}\right)h^2 + \frac{3}{20}R^2\right)$$

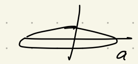
$$= \mu\left(\left(\frac{48-45}{80}\right)h^2 + \frac{3}{20}R^2\right)$$

$$= \mu\left(\frac{3}{80}h^2 + \frac{3}{20}R^2\right)$$

$$= \boxed{\frac{3}{20}\mu\left(R^2 + \frac{h^2}{4}\right)} = I_2$$

(f) ellipsoid of semi-axes a, b, c





$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

$$u^2 + v^2 + w^2 = 1 \qquad u = \frac{x}{a},\ v = \frac{y}{b},\ w = \frac{z}{c}$$

$$\text{Vol} = \int dV$$

$$= \iiint dx\, dy\, dz$$

$$= abc \iiint du\, dv\, dw$$

$$= abc \underbrace{\iiint r^2 \sin\theta\, dr\, d\theta\, d\phi}_{\text{unit sphere}}$$

$$= abc\, 4\pi \left.\frac{r^3}{3}\right|_0^1$$

$$= \frac{4}{3}\pi abc$$

$$\rho = \frac{\mu}{\frac{4}{3}\pi abc}$$

$$I_3 = \rho \int dV \, (x^2+y^2)$$
$$= \rho \int dV \, (a^2u^2 + b^2v^2)$$
$$= a^2\rho \int dV \, u^2 \quad + \quad b^2\rho \int dV \, v^2$$

$u = r\sin\theta\cos\phi$ $\qquad\qquad\qquad r^2\sin\theta$

$v = r\sin\theta\sin\phi$

$$\int dV \, u^2 = abc \iiint r^2 \sin\theta \, dr\, d\theta\, d\phi \; r^2 \sin^2\theta \cos^2\phi$$
$$= abc \int_0^1 r^4 dr \int_0^{2\pi} \cos^2\phi \, d\phi \int_{-1}^{1} d(\cos\theta)(1-\cos^2\theta)$$
$$= abc \left.\frac{r^5}{5}\right|_0^1 \int_0^{2\pi} \frac{1}{2}(1+\underbrace{\cancel{\cos 2\phi}}_{0})\, d\phi \int_{-1}^{1} dx\,(1-x^2)$$
$$= abc \, \frac{1}{5} \, \frac{1}{\cancel{2}} \cdot \cancel{2}\pi \left. \left(x - \frac{x^3}{3}\right)\right|_{-1}^{1}$$
$$= abc \, \frac{\pi}{5} (2)\left(1-\frac{1}{3}\right)$$
$$= abc \, \frac{4\pi}{3}\left(\frac{1}{5}\right)$$

Thus, $I_3 = \cancel{\frac{4\pi}{3}} \cancel{abc} \left(\frac{1}{5}\right) \frac{M}{\cancel{\frac{4\pi}{3}abc}} (a^2+b^2) = \frac{1}{5}(a^2+b^2)$

$\cos 2\phi = \cos^2\phi - \sin^2\phi = 2\cos^2\phi - 1$

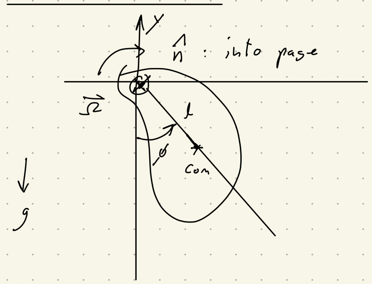
$\cos^2\phi = \frac{1}{2}(1+\cos 2\phi)$

Thus, $I_3 = \frac{1}{5}(a^2+b^2)$

$I_1 = \frac{1}{5}(b^2+c^2)$

$I_2 = \frac{1}{5}(c^2+a^2)$

Sec 32, Prob 3:



$U = \mu g y \quad , \quad y = l \cos\phi$

Small oscillations about $\phi = 0$:

$$y = -l \cos\phi$$
$$\cong -l\left(1 - \frac{\phi^2}{2}\right)$$

$$\to U \cong -\mu g l\left(1 - \frac{\phi^2}{2}\right)$$
$$= \frac{1}{2}\mu g l \phi^2 + \text{const}$$

$$T = \frac{1}{2}\mu |\vec{V}|^2 + \frac{1}{2} I_{ik}\Omega_i \Omega_k$$
$$= \frac{1}{2}\mu l^2 \dot{\phi}^2 + \frac{1}{2}\left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2\right)$$

$$\vec{\Omega} = \dot{\phi}\,\hat{n}$$

$$\Omega_1 = \vec{\Omega}\cdot\hat{x}_1 = \dot{\phi}\,\hat{n}\cdot\hat{x}_1 = \dot{\phi}\cos\alpha$$
$$\Omega_2 = \vec{\Omega}\cdot\hat{x}_2 = \dot{\phi}\,\hat{n}\cdot\hat{x}_2 = \dot{\phi}\cos\beta$$
$$\Omega_3 = \vec{\Omega}\cdot\hat{x}_3 = \dot{\phi}\,\hat{n}\cdot\hat{x}_3 = \dot{\phi}\cos\gamma$$

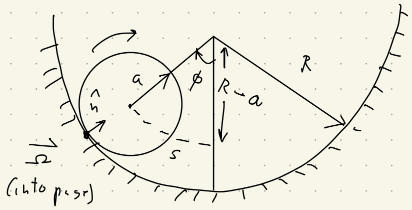
Thus, $T = \frac{1}{2}\mu l^2 \dot{\phi}^2 + \frac{1}{2}\dot{\phi}^2\left(I_1 \cos^2\alpha + I_2 \cos^2\beta + I_3 \cos^2\gamma\right)$

$$L = T - U$$
$$= \frac{1}{2}\dot{\phi}^2\left(\mu l^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)$$
$$- \frac{1}{2}\mu g l \phi^2$$

$$E = T + U$$
$$= \frac{1}{2}\dot{\phi}^2\left(\mu l^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)$$
$$+ \frac{1}{2}\mu g l \phi^2$$

$$\omega = \sqrt{\frac{k}{m}}$$
$$= \sqrt{\frac{\mu g l}{\mu l^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma}}$$
$$= \sqrt{\frac{g}{l}\,\frac{1}{\left(I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)/\mu l^2}}$$

Sec 32 Prob 6:



$s = (R-a)\phi$

homog cylinder: μ, a, b

$I_3 = \frac{1}{2}\mu a^2$

$V = \dot s$
$\quad = (R-a)\dot\phi$

Point of contact with surface has $\vec v = 0$

$0 = \vec V + \vec\Omega \times \vec r$
$\quad = \vec V + \vec\Omega \times (-a\hat n)$

$V = -\Omega a$

$\rightarrow \Omega = -V/a$
$\quad = -\frac{(R-a)}{a}\dot\phi$

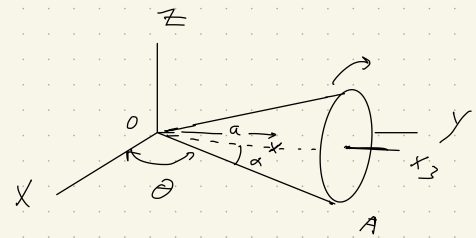
$T = \frac{1}{2}\mu|\vec V|^2 + \frac{1}{2} I_3 \Omega^2$

$= \frac{1}{2}\mu (R-a)^2\dot\phi^2 + \frac{1}{2}\left(\frac{1}{2}\mu a^2\right)\left(\frac{R-a}{a}\right)^2\dot\phi^2$

$= \left(\frac{1}{2} + \frac{1}{4}\right)\mu (R-a)^2\dot\phi^2$

$= \frac{3}{4}\mu (R-a)^2\dot\phi^2$

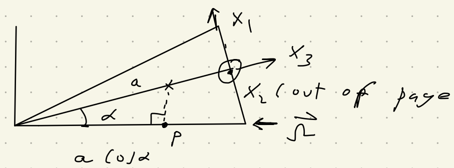
Sec 32, Prob 7:



homogeneous cone:
α: $\frac{1}{2}$ angle
h: height
R: radius

$\tan\alpha = \frac{R}{h}$

$\vec\Omega$: along AO



$\downarrow \vec r = -a\sin\alpha\,\hat Z$

$s = a\cos\alpha\,\theta$

$V = \dot s = a\cos\alpha\,\dot\theta$

$\vec\Omega = \Omega\cos\alpha\,\hat x_3 - \Omega\sin\alpha\,\hat x_1$

Instantaneous velocity of $P = 0$:

$0 = \vec v = \vec V + \vec\Omega\times\vec r$

$0 = a\cos\alpha\,\dot\theta - \Omega\, a\sin\alpha$

$\rightarrow \Omega = +\cot\alpha\,\dot\theta$

$\vec\Omega = -\Omega\cos\alpha\,\hat x_3 + \Omega\sin\alpha\,\hat x_1$

$\rightarrow \Omega_3 = -\Omega\cos\alpha = -\frac{\cos^2\alpha}{\sin\alpha}\dot\theta$

$\Omega_1 = \Omega\sin\alpha = \cos\alpha\,\dot\theta$

$$T = \frac{1}{2}M|\vec{V}|^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega^2 + I_3\Omega_3^2\right)$$

$$= \frac{1}{2}M\, a^2\cos^2\alpha\,\dot\theta^2$$

$$+ \frac{1}{2}I_1\cos^2\alpha\,\dot\theta^2 + \frac{1}{2}I_3\frac{\cos^4\alpha}{\sin^2\alpha}\dot\theta^2$$

$$= \frac{1}{2}\dot\theta^2\cos^2\alpha\left[Ma^2 + I_1 + I_3\cot^2\alpha\right]$$

$$= \frac{1}{2}\dot\theta^2\cos^2\alpha\left[Ma^2 + \frac{3}{20}M\left(R^2 + \frac{1}{4}h^2\right) + \frac{3}{10}M R^2\frac{\cos^2\alpha}{\sin^2\alpha}\right]$$

Recall : $\tan\alpha = \frac{R}{h}$, $a = \frac{3}{4}h$

$\rightarrow$ $Ma^2 = \frac{9}{16}Mh^2$

$$R^2 + \frac{1}{4}h^2 = h^2\tan^2\alpha + \frac{1}{4}h^2$$
$$= h^2\left(\frac{1}{4} + \tan^2\alpha\right)$$

$$R^2\frac{\cos^2\alpha}{\sin^2\alpha} = h^2\cancel{\tan^2\alpha}\,\cancel{\frac{\cos^2\alpha}{\sin^2\alpha}} = h^2$$

Thus,

$$[\ \ ] = \frac{9}{16}Mh^2 + \frac{3}{20}Mh^2\left(\frac{1}{4} + \tan^2\alpha\right) + \frac{3}{10}Mh^2$$

$$[\ \ ] = Mh^2\left(\frac{9}{16} + \frac{3}{20}\left(\frac{1}{4} + \tan^2\alpha\right) + \frac{3}{10}\right)$$

$$= Mh^2\left(\frac{9}{16} + \frac{3}{80} + \frac{3}{10} + \frac{3}{20}\tan^2\alpha\right)$$

$$= Mh^2\left(\frac{45}{80} + \frac{3}{80} + \frac{24}{80} + \frac{3}{20}\tan^2\alpha\right)$$

$$= Mh^2\left(\frac{9}{10} + \frac{3}{20}\tan^2\alpha\right)$$

Thus,

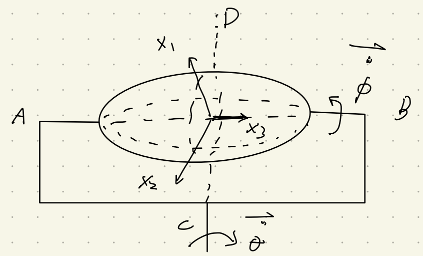
$$T = \frac{1}{2}\dot\theta^2\cos^2\alpha\, Mh^2\left(\frac{9}{10} + \frac{3}{20}\tan^2\alpha\right)$$

$$= \frac{3}{40}Mh^2\dot\theta^2\left(\sin^2\alpha + 6\cos^2\alpha\right)$$

$$= \frac{3}{40}Mh^2\dot\theta^2\left(\sin^2\alpha + \cos^2\alpha + 5\cos^2\alpha\right)$$

$$= \frac{3}{40}Mh^2\dot\theta^2\left(1 + 5\cos^2\alpha\right)$$

Sec 32, Prob 9:



$(\hat{x}_1, \hat{x}_2, \hat{x}_3)$: principal axes

$$\vec{\dot\phi} = \dot\phi\, \hat{x}_3$$

$$\vec{\dot\theta} = \dot\theta \left(\cos\phi\, \hat{x}_1 + \sin\phi\, \hat{x}_2\right)$$

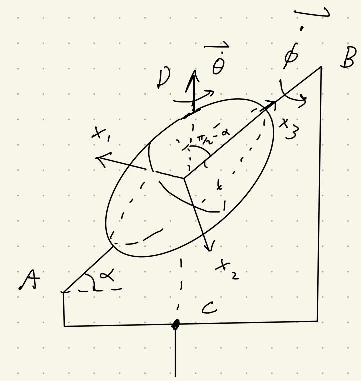
$$\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta}$$

$$= \dot\theta\cos\phi\, \hat{x}_1 + \dot\theta\sin\phi\, \hat{x}_2 + \dot\phi\, \hat{x}_3$$

$$\Omega_1 = \dot\theta\cos\phi$$
$$\Omega_2 = \dot\theta\sin\phi$$
$$\Omega_3 = \dot\phi$$

$$T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$
$$= \frac{1}{2}\left[\left(I_1\cos^2\phi + I_2\sin^2\phi\right)\dot\theta^2 + I_3\dot\phi^2\right]$$

Sec 32, Prob 10:



symmetric ellipsoid

$$I_1 = I_2 \neq I_3$$

$$\vec{\Omega} = \vec{\dot\theta} + \vec{\dot\phi}$$

Decompose along x_1, x_2, x_3

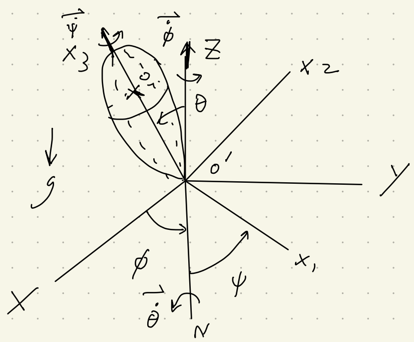
$$\vec{\dot\phi} = \dot\phi\, \hat{x}_3$$

$$\vec{\dot\theta} = \dot\theta \left[\sin\alpha\, \hat{x}_3 + \cos\alpha\cos\phi\, \hat{x}_1 + \cos\alpha\sin\phi\, \hat{x}_2\right)$$

Thus,
$$\Omega_1 = \dot\theta\cos\alpha\cos\phi$$
$$\Omega_2 = \dot\theta\cos\alpha\sin\phi$$
$$\Omega_3 = \dot\theta\sin\alpha + \dot\phi$$

$$\rightarrow\ T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$
$$= \frac{1}{2}\left(I_1\dot\theta^2\cos^2\alpha\cos^2\phi + I_1\dot\theta^2\cos^2\alpha\sin^2\phi + I_3\left(\dot\theta\sin\alpha + \dot\phi\right)^2\right)$$
$$= \frac{1}{2}\left(I_1\dot\theta^2\cos^2\alpha + I_3\left(\dot\theta\sin\alpha + \dot\phi\right)^2\right)$$

Sec 35, Prob 1:



Heavy symmetrical top with lowest point fixed.

$I_1 = I_2$, μ = total mass

l = distance from O to COM

$$U = \mu g l \cos\theta$$

$$T = \frac{1}{2}\mu V^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$\Omega_1 = \dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi$$

$$\Omega_2 = \dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi \qquad (35.1)$$

$$\Omega_3 = \dot\phi\cos\theta + \dot\psi$$

pt O at rest

$$0 = \vec V + \vec\Omega \times \vec r$$

$$= \vec V - \vec\Omega \times l\hat x_3$$

$$\rightarrow \vec V = l\vec\Omega \times \hat x_3$$

$$= l\left(\Omega_2\hat x_1 - \Omega_1\hat x_2\right)$$

$$|\vec V|^2 = l^2\left(\Omega_2^2 + \Omega_1^2\right)$$

$$\vec A \times \vec B = \hat x_1\left(A_2B_3 - A_3B_2\right) + \cdots$$

$$\Omega_1^2 + \Omega_2^2 = \dot\phi^2\sin^2\theta\sin^2\psi + \dot\theta^2\cos^2\psi$$

$$+ \cancel{2\dot\phi\dot\theta\sin\theta\sin\psi\cos\psi}$$

$$+ \dot\phi^2\sin^2\theta\cos^2\psi + \dot\theta^2\sin^2\psi$$

$$\cancel{-2\dot\phi\dot\theta\sin\theta\sin\psi\cos\psi}$$

$$= \dot\phi^2\sin^2\theta + \dot\theta^2$$

Thus,

$$L = T - U$$

$$= \frac{1}{2}\mu l^2\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right)$$

$$+ \frac{1}{2} I_1\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right) + \frac{1}{2} I_3\left(\dot\phi\cos\theta + \dot\psi\right)^2$$

$$- \mu g l\cos\theta$$

$$= \frac{1}{2}\left(I_1 + \mu l^2\right)\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right)$$

$$+ \frac{1}{2} I_3\left(\dot\phi\cos\theta + \dot\psi\right)^2 - \mu g l\cos\theta$$

$$= \frac{1}{2} I_1'\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right) + \frac{1}{2} I_3'\left(\dot\phi\cos\theta + \dot\psi\right)^2$$

$$- \mu g l\cos\theta$$

where $\left.\begin{array}{l} I_1' = I_2' = I_1 + \mu l^2 \\ I_3' = I_3 \end{array}\right\}$ moments of inertia wrt origin at O'

E, p_ϕ, p_ψ conserved since no explicit t dependence, ϕ dependence, or ψ dependence

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1' \sin^2\theta\, \dot\phi + I_3'(\dot\phi \cos\theta + \dot\psi)\cos\theta$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3'(\dot\phi\cos\theta + \dot\psi)$$

NOTE: $p_\phi = M_Z$, $p_\psi = M_3 = I_3'\Omega_3$

Proof: $M_Z = \vec{M} \cdot \hat{Z}$

$$= (M_1 \hat{x}_1 + M_2 \hat{x}_2 + M_3 \hat{x}_3) \cdot \hat{Z}$$

Now: $\hat{x}_3 \cdot \hat{Z} = \cos\theta$

$\hat{x}_2 \cdot \hat{Z} = \sin\theta \cos\psi$

$\hat{x}_1 \cdot \hat{Z} = \sin\theta \sin\psi$

$$\rightarrow M_Z = M_1 \sin\theta \sin\psi + M_2 \sin\theta\cos\psi + M_3\cos\theta$$

$$= I_1'\Omega_1 \sin\theta\sin\psi + I_1'\Omega_2 \sin\theta\cos\psi + I_3'\Omega_3\cos\theta$$

$$M_Z = I_1'(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)\sin\theta\sin\psi + I_1'(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)\sin\theta\cos\psi + I_3'(\dot\phi\cos\theta + \dot\psi)\cos\theta$$

$$= I_1'\dot\phi\sin^2\theta + I_3'(\dot\phi\cos\theta + \dot\psi)\cos\theta$$

$$= p_\phi$$

$$E = T + U$$

$$= \frac{1}{2} I_1'(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3'(\dot\phi\cos\theta + \dot\psi)^2 + Mgl\cos\theta$$

Can invert

$$M_3 = I_3'(\dot\phi\cos\theta + \dot\psi)$$

$$M_Z = I_1'\dot\phi\sin^2\theta + I_3'(\dot\phi\cos\theta + \dot\psi)\cos\theta$$

to express $\dot\phi, \dot\psi$ in terms of M_3, M_Z.

$$\rightarrow M_Z = I_1'\dot\phi\sin^2\theta + M_3\cos\theta$$

$$\boxed{\dot\phi = \frac{M_Z - M_3\cos\theta}{I_1'\sin^2\theta}}$$

$$\frac{M_3}{I_3'} = \dot\phi\cos\theta + \dot\psi$$

$$\rightarrow \boxed{\dot\psi = \frac{M_3}{I_3'} - \dot\phi\cos\theta = \frac{M_3}{I_3'} - \left(\frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}\right)\cos\theta}$$

Thus,

$$E = \frac{1}{2} I_1' \left(\dot\phi^2 \sin^2\theta + \dot\theta^2\right) + \frac{1}{2} I_3' \frac{M_3^2}{(I_3')^2} + \mu g l \cos\theta$$

$$= \frac{1}{2} I_1' \left(\left(\frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}\right)^2 \sin^2\theta + \dot\theta^2 \right) + \frac{1}{2}\frac{M_3^2}{I_3'} + \mu g l \cos\theta$$

$$= \frac{1}{2} I_1' \left(\frac{(M_z - M_3\cos\theta)^2}{(I_1')^2 \sin^2\theta} + \dot\theta^2 \right) + \frac{1}{2}\frac{M_3^2}{I_3'} + \mu g l \cos\theta$$

$$= \frac{1}{2} I_1' \dot\theta^2 + \frac{1}{2}\frac{(M_z - M_3\cos\theta)^2}{I_1'\sin^2\theta} + \underbrace{\frac{1}{2}\frac{M_3^2}{I_3'}}_{\text{const}} - \mu g l (1 - \cos\theta) + \underbrace{\mu g l}_{\text{const}}$$

$$\underbrace{E - \mu g l - \frac{1}{2}\frac{M_3^2}{I_3'}}_{E'} = \frac{1}{2} I_1' \dot\theta^2 + \frac{1}{2}\frac{(M_z - M_3\cos\theta)^2}{I_1'\sin^2\theta} - \mu g l (1 - \cos\theta)$$

$$E' = \frac{1}{2} I_1' \dot\theta^2 + \frac{1}{2}\frac{(M_z - M_3\cos\theta)^2}{I_1'\sin^2\theta} - \mu g l (1 - \cos\theta)$$

$$= \frac{1}{2} I_1' \dot\theta^2 + U_{eff}(\theta)$$

where

$$U_{eff}(\theta) = \frac{(M_z - M_3\cos\theta)^2}{2 I_1'\sin^2\theta} - \mu g l (1 - \cos\theta)$$

Thus, $\pm\sqrt{\frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)} = \dot\theta = \frac{d\theta}{dt}$

$$\boxed{t = \pm\int \frac{d\theta}{\sqrt{\frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)}} + \text{const}}$$

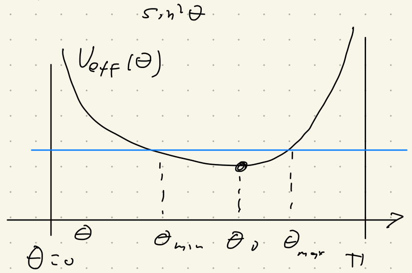
(solution via quadrature)

$$V_{eff}(\theta) = \frac{(M_z - M_3\cos\theta)^2}{2I_1'\sin^2\theta} - \mu g l(1-\cos\theta)$$

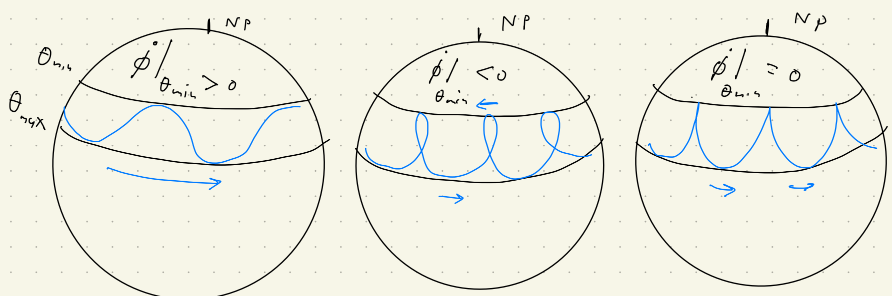
$$= \frac{1}{2I_1'\sin^2\theta}\Big[(M_z - M_3\cos\theta)^2 - 2\mu g l I_1'\sin^2\theta(1-\cos\theta)\Big]$$

At $\theta = 0, \pi$: $1-\cos\theta = 0$

but $V_{eff}(\theta) \to \infty$, if $M_z \neq M_3$ due to $\frac{1}{\sin^2\theta}$ factor



$$\dot\phi = \frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}$$ (form of motion depends on the sign of $M_z - M_3\cos\theta_{min}$)



"nutation"

Sec 35, Prob 2:

For stability of the top's motion about the vertical, need $\theta \to 0$ to be a minimum of $V_{eff}(\theta)$. So $\left.\frac{d^2 V_{eff}(\theta)}{d\theta^2}\right|_{\theta=0} > 0$

$$V_{eff}(\theta) = \frac{(M_z - M_3\cos\theta)^2}{2I_1'\sin^2\theta} - \mu g l(1-\cos\theta)$$

Also:

$$M_z = I_1'\sin^2\theta\,\dot\phi + I_3'(\dot\phi\cos\theta + \dot\psi)\cos\theta$$

$$M_3 = I_3'(\dot\phi\cos\theta + \dot\psi)$$

In limit $\theta \to 0$:

$$M_z \simeq I_3'(\dot\phi + \dot\psi)$$
$$M_3 \simeq I_3'(\dot\phi + \dot\psi)$$

So $M_z = M_3$ in this limit

$$\to V_{eff}(\theta) \simeq \left.\left(\frac{M_3^2(1-\cos\theta)^2}{2I_1'\sin^2\theta} - \mu g l(1-\cos\theta)\right)\right|_{\theta\to 0}$$

$$\simeq \frac{M_3^2(\theta^2/2)^2}{2I_1'\theta^2} - \mu g l\frac{\theta^2}{2}$$

$$U_{eff}(\theta) \simeq \frac{1}{8}\frac{M_3^2}{I_1'}\theta^2 - \frac{1}{2}\mu g l \theta^2$$

$$= \frac{1}{2}\Big[\underbrace{\frac{M_3^2}{4I_1'} - \mu g l}_{\equiv K}\Big]\theta^2$$

Need $K \equiv \left.\dfrac{d^2 U_{eff}}{d\theta^2}\right|_{\theta=0} > 0$

Thus, $\dfrac{M_3^2}{4I_1'} - \mu g l > 0$

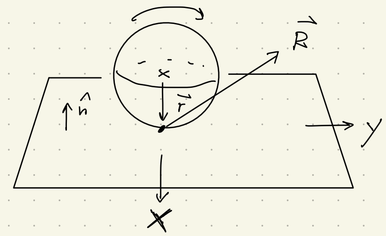
$$M_3^2 > 4 I_1' \mu g l$$

$$\boxed{M_3 > 2\sqrt{I_1' \mu g l}}$$

since $M_3 = I_3 \Omega_3$

$$\rightarrow \quad \Omega_3 > 2\sqrt{\frac{I_1' \mu g l}{(I_3')^2}}$$

Sec 38, prob 1:



Find EOM of a homogeneous sphere (radius a, mass μ) rolling without slipping on a horizontal surface under external force $\vec{F}$ and torque $\vec{K}$.

Rolling without slipping

$$0 = \vec{V} + \vec{\Omega}\times\vec{r} \qquad (\hat{n} = \hat{z})$$
$$= \vec{V} - a\vec{\Omega}\times\hat{n}$$

$\vec{\Omega}$: angular velocity vector (in XY plane)

$\vec{R}$: reaction force (in arbitrary direction for rolling without slipping)

$$\frac{d\vec{P}}{dt} = \sum \vec{f} = \vec{F} + \vec{R}$$

$$\frac{d\vec{M}}{dt} = \sum \vec{r}\times\vec{f} = \vec{K} - a\hat{n}\times\vec{R}$$

$\vec{K}$: applied torque around CoM

$\vec{F}$: applied force

$\vec{M} = I\vec{\Omega}$, $\quad I = \frac{2}{5}\mu a^2$

$V_z = 0$ (since CoM has $z = a = $ const)

$$\mu \frac{d\vec{V}}{dt} = \vec{F} + \vec{R} \qquad (1)$$

$$I \frac{d\vec{\Omega}}{dt} = \vec{K} - a\,\hat{z} \times \vec{R} \qquad (2)$$

$$\vec{V} = a\,\vec{\Omega} \times \hat{z} \qquad \text{(constraint)}$$

$$V_x = a(\vec{\Omega} \times \hat{z})_x = a\Omega_y$$

$$V_y = a(\vec{\Omega} \times \hat{z})_y = -a\Omega_x$$

$$0 = a(\vec{\Omega} \times \hat{z})_z = 0 \qquad \text{(no new information)}$$

Thus, $\boxed{\Omega_x = -\frac{V_y}{a}}$, $\boxed{\Omega_y = \frac{V_x}{a}}$, $\boxed{V_z = 0}$

Take time derivative of constraint equation:

$$\frac{d\vec{V}}{dt} = a \frac{d\vec{\Omega}}{dt} \times \hat{z}$$

$$\rightarrow \mu \frac{d\vec{V}}{dt} = \mu a \frac{d\vec{\Omega}}{dt} \times \hat{z}$$

$$\rightarrow \vec{F} + \vec{R} = \mu a \left(\frac{1}{I} \vec{K} - \frac{a}{I} \hat{z} \times \vec{R} \right) \times \hat{z}$$

Thus

$$F_x + R_x = \mu a \left(\frac{1}{I} K_y - \frac{a}{I} (\hat{z} \times \vec{R})_y \right)$$

$$= \frac{\mu a}{I} K_y - \frac{\mu a^2}{I} R_x$$

$$= \frac{\mu a}{\frac{2}{5}\mu a^2} K_y - \frac{\mu a^2}{\frac{2}{5}\mu a^2} R_x$$

$$= \frac{5}{2} \frac{K_y}{a} - \frac{5}{2} R_x$$

$$\rightarrow \frac{7}{2} R_x = \frac{5}{2} \frac{K_y}{a} - F_x$$

$$\boxed{R_x = \frac{5}{7} \frac{K_y}{a} - \frac{2}{7} F_x}$$

Similarly,

$$F_y + R_y = \mu a \left(-\frac{K_x}{I} + \frac{a}{I} (\hat{z} \times \vec{R})_x \right)$$

$$= -\frac{\mu a K_x}{I} - \frac{\mu a^2}{I} R_y$$

$$= -\frac{5}{2} \frac{K_x}{a} - \frac{5}{2} R_y$$

$$\rightarrow \frac{7}{2} R_y = -\frac{5}{2} \frac{K_x}{a} - F_y$$

$$\boxed{R_y = -\frac{5}{7} \frac{K_x}{a} - \frac{2}{7} F_y}$$

$F_z + R_z = 0$ (since no z-component of any $\vec{A} \times \hat{z}$)

Thus, $\boxed{R_z = -F_z}$

Summary: $R_x = \frac{5}{7}\frac{K_y}{a} - \frac{2}{7} F_x$

$R_y = -\frac{5}{7}\frac{K_x}{a} - \frac{2}{7} F_y$

$R_z = -F_z$

$$\mu \frac{dV_x}{dt} = F_x + R_x = \frac{5}{7} F_x + \frac{5}{7}\frac{K_y}{a}$$

$$\mu \frac{dV_y}{dt} = F_y + R_y = \frac{5}{7} F_y - \frac{5}{7}\frac{K_x}{a}$$

$$V_z = 0$$

$$\Omega_x = \frac{-V_y}{a}$$

$$\Omega_y = +\frac{V_x}{a}$$

$$I \frac{d\Omega_z}{dt} = K_z$$

Example:

suppose: $\vec{F} = \underbrace{-\mu g \hat{z}}_{\text{gravity}} + F_o \hat{x}$ — const

$$\vec{K} = a G_o \hat{z}$$

Then $K_x = K_y = 0$, $K_z = a G_o$

$F_x = F_o$, $F_y = 0$, $F_z = -\mu g$

$$\mu \frac{dV_x}{dt} = \frac{5}{7}\left(F_x + \cancelto{0}{\frac{K_y}{a}}\right) = \frac{5}{7} F_o$$

$$\rightarrow \boxed{V_x = V_{xo} + \frac{5}{7}\frac{F_o}{\mu} t}$$

$$\mu \frac{dV_y}{dt} = \frac{5}{7}\left(\cancelto{0}{F_y} - \cancelto{0}{\frac{K_x}{a}}\right) = 0 \rightarrow \boxed{V_y = V_{yo}}$$

$$\boxed{V_z = 0}$$

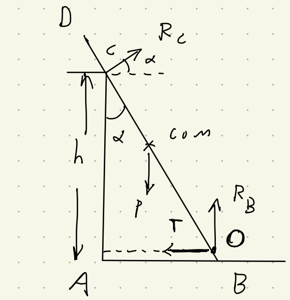
$$I \frac{d\Omega_z}{dt} = a G_o \rightarrow \Omega_z = \Omega_{zo} + \frac{a G_o}{I} t$$

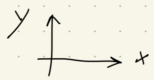
$$\boxed{\Omega_z = \Omega_{zo} + \frac{5}{2}\frac{G_o}{\mu a} t}$$

$$\boxed{\Omega_x = -\frac{V_y}{a}, \quad \Omega_y = \frac{V_x}{a}}$$

$\boxed{R_x = -\frac{2}{7} F_o}$, $\boxed{R_y = 0}$, $\boxed{R_z = \mu g}$ ← normal force

Sec 38, Prob 2:





Uniform thin rod, weight P, length l

$$I = \frac{1}{12} m l^2 \quad \text{(about com)}$$

$$= \frac{1}{12}\frac{P}{g} l^2$$

EOMs:

$$\sum \vec{F} = 0$$

$$\sum \vec{r}\times\vec{F} = 0$$

including reaction forces

1) $0 = -T + R_C \cos\alpha$ (x-component of forces)

2) $0 = R_C \sin\alpha - P + R_B$ (y-component of force)

3) $0 = P\frac{l}{2}\sin\alpha - R_C \frac{h}{\cos\alpha}$ (ccw torque about O)

Using $\cos\alpha = \frac{h}{d}$

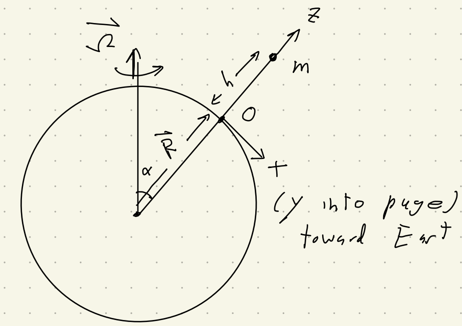
Thus,

$$R_C = P\frac{l}{2}\frac{\sin\alpha\cos\alpha}{h} = \frac{Pl}{4h}\sin 2\alpha$$

$$T = R_C\cos\alpha$$

$$R_B = P - R_C\sin\alpha$$

Sec 39, Prob 1:



$h << R$

$\vec{r}$ measured wrt O

$$m\vec{a} = m\vec{g} - m\vec{W} - 2m\vec{\Omega}\times\vec{v} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\vec{W} = \vec{\Omega}\times(\vec{\Omega}\times\vec{R}) \quad \text{(acceleration of O w.r.t inertial frame)}$$

Ignore terms that are $O(\Omega^2)$

$\rightarrow \quad m\vec{a} \approx m\vec{g} - 2m\vec{\Omega}\times\vec{v}$

$\rightarrow \quad \vec{a} = \vec{g} - 2\vec{\Omega}\times\vec{v}$, $\vec{g}$, $\vec{\Omega}$ are constants

$$\frac{d\vec{v}}{dt} = \vec{g} - 2\vec{\Omega}\times\vec{v}$$

Write $\vec{v} = \vec{v}_1 + \vec{v}_2$ where $\vec{v}_2$ is perturbation.

i.e, $\frac{d\vec{v}_1}{dt} = \vec{g} \quad \rightarrow \quad \vec{v}_1 = \vec{v}_0 + \vec{g}t$

$\vec{v}_0$: initial velocity

$$\rightarrow \quad \frac{d\vec{v}_2}{dt} = -2\vec{\Omega}\times(\vec{v}_1 + \vec{v}_2)$$

$$\approx -2\vec{\Omega}\times\vec{v}_1 \quad \text{(ignore the smaller } \vec{v}_2 \text{ term)}$$

$$= -2\vec{\Omega}\times(\vec{v}_0 + \vec{g}t)$$

Thus, $\vec{v}_2 \approx -2\vec{\Omega}\times\vec{v}_0\, t - \vec{\Omega}\times\vec{g}\, t^2$

$$\to \quad \vec{v} = \vec{v}_0 + \vec{g}\,t - 2\vec{\Omega}\times\left(\vec{v}_0\, t + \tfrac{1}{2}\vec{g}\,t^2\right)$$

Also $\vec{v} = \dfrac{d\vec{r}}{dt}$

$$\to \quad \boxed{\vec{r} = \vec{h} + \vec{v}_0\, t + \tfrac{1}{2}\vec{g}\,t^2 - \vec{\Omega}\times\vec{v}_0\, t^2 - \tfrac{1}{3}(\vec{\Omega}\times\vec{g})\,t^3}$$

Take
$$\vec{h} = h\hat{z}$$
$$\vec{v}_0 = 0$$
$$\vec{g} = -g\hat{z}$$
$$\vec{\Omega} = \Omega\cos\alpha\,\hat{z} - \Omega\sin\alpha\,\hat{x}$$

Then
$$\vec{r} = h\hat{z} - \tfrac{1}{2}g\hat{z}\,t^2 + \tfrac{1}{3}g\Omega\sin\alpha\,\hat{y}\,t^3$$

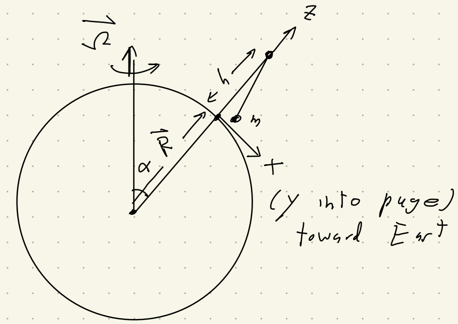
when the body hits the ground

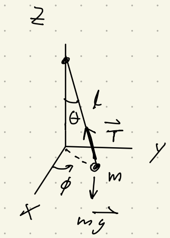
$$h - \tfrac{1}{2}gT^2 = 0 \quad \to \quad T = \sqrt{\frac{2h}{g}}$$

$$\to \quad \delta\vec{r} = \tfrac{1}{3}g\Omega\sin\alpha\left(\frac{2h}{g}\right)^{3/2}\hat{y}$$
$$= \frac{\sqrt{8}}{3}\sqrt{\frac{h^3}{g}}\,\Omega\sin\alpha\,\hat{y}$$

(deflection to the East)

Sec 39, Prob 3:



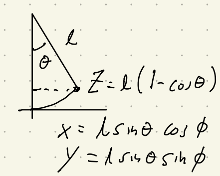


Foucault's pendulum: small oscillations ($\theta \ll 1$)

Again ignore terms that are 2^{nd} order in $\vec{\Omega}$.

$$m\vec{a} = \vec{T} + m\vec{g} - 2m(\vec{\Omega}\times\vec{v})$$

Now:



$$x = \ell\sin\theta\cos\phi \approx \ell\theta\cos\phi$$
$$y = \ell\sin\theta\sin\phi \approx \ell\theta\sin\phi$$
$$z = \ell(1-\cos\theta) \approx \ell\frac{\theta^2}{2} \approx \boxed{0}$$
$$\boxed{\ddot{z} = 0}$$

Consider motion in xy plane

$$T_z = -T\cos\theta \approx -T$$
$$T_x = -T\sin\theta\cos\phi = -T\frac{x}{\ell}$$
$$T_y = -T\sin\theta\sin\phi = -T\frac{y}{\ell}$$

$$\vec{g} = -g\hat{z}$$
$$\vec{\Omega} = \Omega\cos\alpha\,\hat{z} - \Omega\sin\alpha\,\hat{x} \equiv \Omega_z\hat{z} + \Omega_x\hat{x}$$
$$\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \cancel{\dot{z}}^{=0}\hat{z} \approx \dot{x}\hat{x} + \dot{y}\hat{y}$$

$$(\vec{\Omega}\times\vec{v})_x = \Omega_y \cancel{\dot{z}}^{\,0} - \Omega_z \dot{y} = -\Omega_z \dot{y}$$

$$(\vec{\Omega}\times\vec{v})_y = \Omega_z \dot{x} - \cancel{\Omega_x \dot{z}}^{\,0} = \Omega_z \dot{x}$$

$$(\vec{\Omega}\times\vec{v})_z = \Omega_x \dot{y} - \cancel{\Omega_y}^{\,0} \dot{x} = \Omega_x \dot{y}$$

Thus,

$$m\ddot{x} \approx -T\frac{x}{\ell} + 2m\Omega_z \dot{y}$$

$$m\ddot{y} \approx -T\frac{y}{\ell} - 2m\Omega_z \dot{x}$$

$$0 \approx m\ddot{z} = -T - mg - 2m\Omega_x \dot{y}$$

Now: $\Omega_x \dot{y} \sim \Omega \frac{D}{P}$

$$= \frac{\Omega D \omega}{2\pi}$$

$$\ll \frac{\omega^2 \ell}{2\pi}$$

$$= \frac{g}{\cancel{\ell}}\frac{\cancel{\ell}}{2\pi}$$

$$= g/2\pi$$

D: displacement in xy plane

$(D \ll \ell)$

P: period of oscillation

$= \frac{2\pi}{\omega}$, $\omega = \sqrt{\frac{g}{\ell}}$

$(\Omega \ll \omega)$

Thus $m\Omega_x \dot{y} \ll mg$

$$\rightarrow 0 \approx -T - mg \rightarrow \boxed{T \approx mg}$$

Rewrite equations:

$$m\ddot{x} \approx -mg\frac{x}{\ell} + 2m\Omega_z \dot{y}$$

$$\rightarrow \ddot{x} \approx -\frac{g}{\ell}x + 2\Omega_z \dot{y}$$

$$\boxed{\ddot{x} = -\omega^2 x + 2\Omega_z \dot{y}}$$

Similarly,

$$\boxed{\ddot{y} = -\omega^2 y - 2\Omega_z \dot{x}}$$

Define: $\xi = x + iy$

$$\rightarrow \dot{\xi} = \dot{x} + i\dot{y}$$

$$\rightarrow \ddot{\xi} = \ddot{x} + i\ddot{y}$$

Thus,

$$\ddot{\xi} = -\omega^2 \xi + 2\Omega_z(\dot{y} - i\dot{x})$$

$$\ddot{\xi} = -\omega^2 \xi - 2i\Omega_z \dot{\xi}$$

so $\ddot{\xi} + 2i\Omega_z \dot{\xi} + \omega^2 \xi = 0$

Guess: $\xi = Ae^{i\lambda t}$

$$\rightarrow -\lambda^2 \cancel{\xi} + 2i\Omega_z i\lambda \cancel{\xi} + \omega^2 \cancel{\xi} = 0$$

$$\lambda^2 + 2\Omega_z \lambda - \omega^2 = 0$$

$$\rightarrow \lambda_\pm = \frac{-2\Omega_z \pm \sqrt{4\Omega_z^2 + 4\omega^2}}{2}$$

Use $\Omega^2 << \omega^2$ to approximate

$$\sqrt{4\Omega_z^2 + 4\omega^2} \approx 2\omega$$

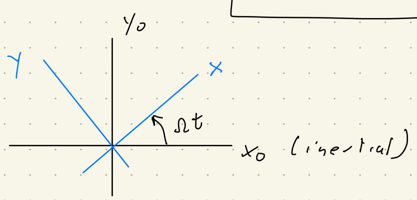
Thus, $$\lambda_\pm \approx \frac{-2\Omega_z \pm 2\omega}{2} = -\Omega_z \pm \omega$$

General solution:

$$\xi(t) = A e^{i\lambda_+ t} + B e^{i\lambda_- t}$$
$$= e^{-i\Omega_z t}\left(A e^{i\omega t} + B e^{-i\omega t}\right)$$

Note:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \Omega_z t & \sin \Omega_z t \\ -\sin \Omega_z t & \cos \Omega_z t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$



$$\rightarrow \quad \xi = x + iy$$
$$= x_0 \cos(\Omega_z t) + y_0 \sin(\Omega_z t)$$
$$\quad + i\left(-x_0 \sin(\Omega_z t) + y_0 \cos(\Omega_z t)\right)$$
$$= (x_0 + iy_0)(\cos \Omega_z t - i \sin \Omega_z t)$$
$$= \xi_0 e^{-i\Omega_z t}$$

Thus, motion in inertial frame is

$$\xi_0(t) = x_0(t) + iy_0(t)$$
$$= \underbrace{A e^{i\omega t} + B}_{} e^{-i\omega t}$$

complex, determined by initial conditions.

Example:

Suppose
$x(0) = D$
$y(0) = 0$
$\dot{x}(0) = 0$
$\dot{y}(0) = 0$
} released from rest

Then $\xi(0) = D$, $\dot{\xi}(0) = 0$

$$\xi(t) = e^{-i\Omega_z t}\left(A e^{i\omega t} + B e^{-i\omega t}\right)$$

$$\rightarrow \quad \xi(0) = \boxed{A + B = D}$$

$$\dot{\xi}(t) = -i\Omega_z e^{-i\Omega_z t}\left(A e^{i\omega t} + B e^{-i\omega t}\right)$$
$$+ e^{-i\Omega_z t}\left(i\omega A e^{i\omega t} - i\omega B e^{-i\omega t}\right)$$

$$\rightarrow \quad \dot{\xi}(0) = -i\Omega_z (A + B) + i\omega (A - B) = 0$$

$$\rightarrow \quad \boxed{A + B = \frac{\omega}{\Omega_z}(A - B)}$$

$A = a_1 + i a_2$
$B = b_1 + i b_2$

$A + B = \underset{\uparrow \text{ real}}{D} \rightarrow \quad a_1 + b_1 = D$
$a_2 + b_2 = 0$

$A + B = \frac{\omega}{\Omega_z}(A - B) \rightarrow \quad D = \frac{\omega}{\Omega_z}(a_1 - b_1)$

$0 = \frac{\omega}{\Omega_z}(a_2 - b_2)$

$\rightarrow \; a_2 - b_2 = 0$
and $\; a_2 + b_2 = 0 \quad \Big\} \quad \boxed{a_2 = 0} \; \boxed{b_2 = 0}$

$a_1 + b_1 = D$
$a_1 - b_1 = D\frac{\Omega_z}{\omega} \quad \Big\} \quad \boxed{a_1 = \frac{D}{2}\left(1 + \frac{\Omega_z}{\omega}\right)}$

$b_1 = D - \frac{D}{2}\left(1 + \frac{\Omega_z}{\omega}\right)$

$\boxed{b_1 = \frac{D}{2}\left(1 - \frac{\Omega_z}{\omega}\right)}$

Thus, $\xi_0(t) = x_0(t) + i y_0(t)$

$= a_1 e^{i\omega t} + b_1 e^{-i\omega t}$

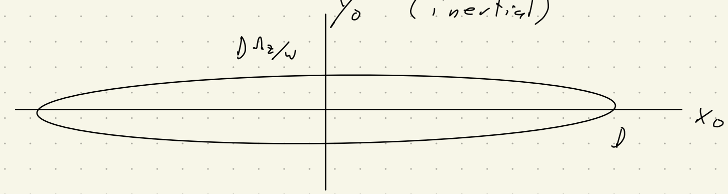
$= \frac{D}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) + \frac{D}{2}\frac{\Omega_z}{\omega}\left(e^{i\omega t} - e^{-i\omega t}\right)$

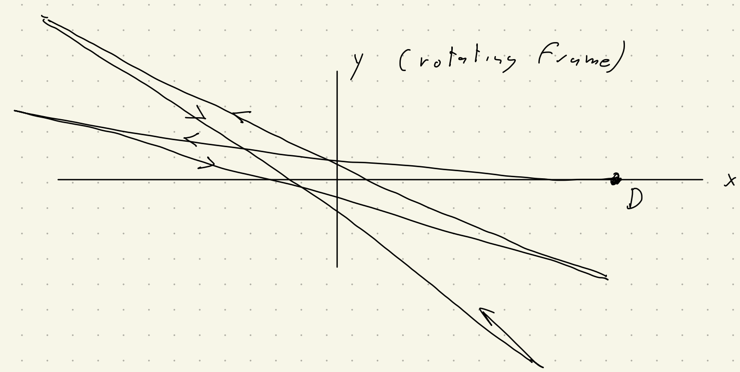
$\xi_0(t) = D\cos\omega t + i D\left(\frac{\Omega_z}{\omega}\right)\sin\omega t$

$= x_0(t) + i y_0(t)$

Thus, $x_0(t) = D\cos\omega t$
$y_0(t) = D\left(\frac{\Omega_z}{\omega}\right)\sin\omega t$

$$\underbrace{\frac{x_0^2}{D^2} + \frac{y_0^2}{\left(D\left(\frac{\Omega_z}{\omega}\right)\right)^2} = 1}_{\text{ellipse}}$$





Precessional Frequency:

$$\Omega_z = \Omega \cos\alpha$$

$$\text{Period} = \frac{2\pi}{\Omega_z}$$

$$= \frac{2\pi}{\Omega \cos\alpha}$$

$$= \frac{\cancel{2\pi}}{\left(\frac{\cancel{2\pi}}{24\text{hrs}}\right)\cos\alpha}$$

$$= \frac{24\text{ hrs}}{\cos\alpha}$$

$\alpha = 0$ (NP) $\rightarrow$ Period = 24 hrs

$\alpha = \frac{\pi}{2}$ (equator) $\rightarrow$ Period = ∞

(no precession)