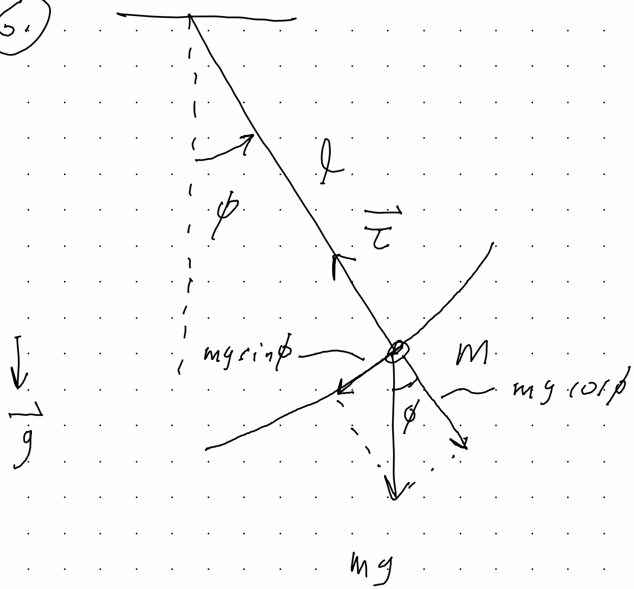
PHYS 5306 8/30

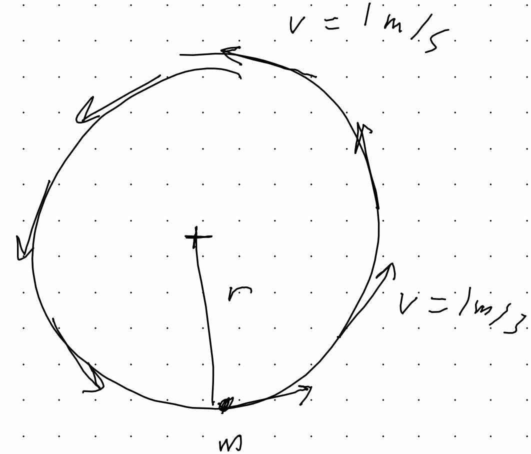
6.

Centripetal force

Centrifugal force

$$m\vec{a} = \vec{F}_{applied} + \vec{F}_{fictitious}$$





$$\cancel{\tau = mg\cos\phi}$$

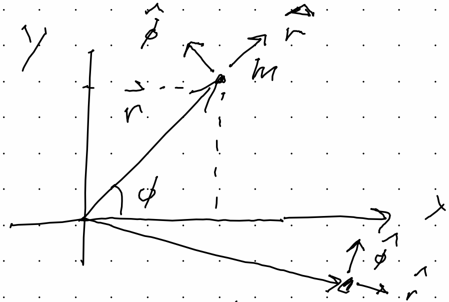
$$\tau - mg\cos\phi = F_{centripetal}$$

$$= m\, a_{centripetal}$$

$$= m\, l\, \dot{\phi}^2$$

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$$

$$a_{centripetal} = \frac{v^2}{r} = \omega^2 r$$



2-d motion:

2-d motion:

$$\vec{r} = x\,\hat{x} + y\,\hat{y}$$

$$\vec{r} = \vec{r}(t)$$

$$\vec{r} = r\,\hat{r}$$

$$\vec{v} = \dot{\vec{r}} = \dot{x}\,\hat{x} + \dot{y}\,\hat{y}$$

$$= \dot{r}\,\hat{r} + r\,\dot{\hat{r}}$$

$$\hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = \underline{\quad}\,\hat{x} \quad \underline{\quad}\,\hat{y}$$

$$\dot{\hat{r}}$$

$$\vec{a} = \ddot{\vec{r}} = (\quad)\,\hat{r} + (\quad)\,\hat{\phi}$$

- Kinetic energy for a single mass m in Cartesian, sph. polar, and polar coordinates.

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

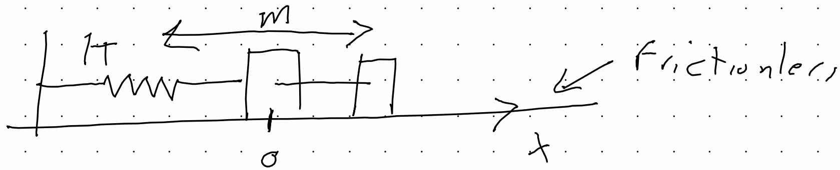
$$= \frac{1}{2} m \left(\qquad \right)$$

$$= \frac{1}{2} m \left(\qquad \right)$$

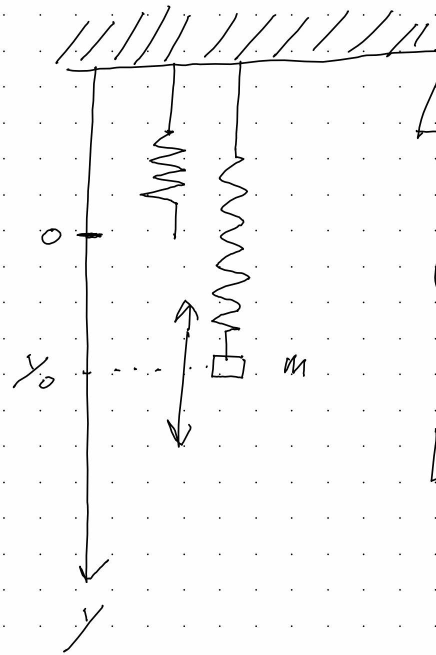
(r, θ, ϕ) sph. polar

(ρ, ϕ, z) cylindrical

$x = r \sin\theta \cos\phi$, etc



$$U = \frac{1}{2} k x^2$$



$$L = T - U$$

$$T = \frac{1}{2} m \dot{y}^2$$

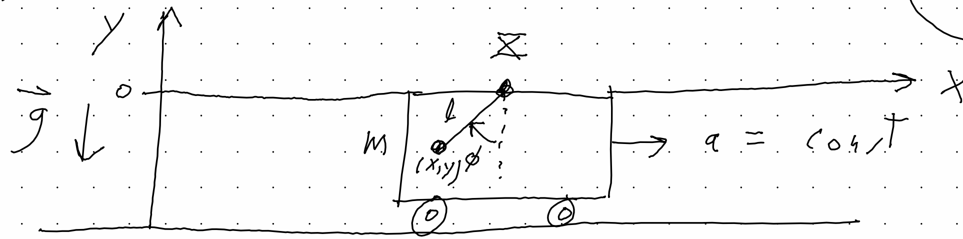
$$U = \underbrace{\frac{1}{2} k y^2}_{U_s} + \underbrace{- m g y}_{U_g}$$

$$E = T + U$$

$$= \frac{1}{2} m \dot{y}^2 + \underbrace{\frac{1}{2} k y^2 - m g y}_{U_{eff}(y)}$$

plot this

9/1: QUIZ #1



$$X = \underset{0}{X_0} + \underset{0}{V_0} t + \frac{1}{2} a t^2$$

$$\dot X = at$$

$$\ddot X = a$$

$$L = ? = T - U$$

$$x = X - l\sin\phi$$

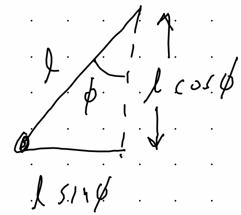
$$y = -l\cos\phi$$

$$U = mgy = -mgl\cos\phi$$

$$T = \frac{1}{2} m(\dot x^2 + \dot y^2)$$

$$\dot x = \dot X - l\dot\phi\cos\phi$$

$$\dot y = l\dot\phi\sin\phi$$



$$\vec r = x\hat x + y\hat y$$

$$\vec g = -g\hat y$$

$$\vec F = m\vec g$$

$$U = -m\vec g\cdot\vec r$$

$$\vec F = -\vec\nabla U = -\frac{\partial U}{\partial \vec r}$$

$$U = -m\vec g\cdot\vec r = mgy$$

$$\dot{x}^2 = \left(\dot{X} - l\dot{\phi}\cos\phi\right)^2$$

$$= \dot{X}^2 + l^2\dot{\phi}^2\cos^2\phi - 2l\dot{X}\dot{\phi}\cos\phi$$

$$\dot{y}^2 = l^2\dot{\phi}^2\sin^2\phi$$

$$\rightarrow \dot{x}^2 + \dot{y}^2 = \dot{X}^2 + l^2\dot{\phi}^2 - 2l\dot{X}\dot{\phi}\cos\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\dot{X}^2 + \frac{1}{2}ml^2\dot{\phi}^2 - ml\dot{X}\dot{\phi}\cos\phi$$

$$U = -mgl\cos\phi$$

$$\boxed{L = \frac{1}{2}m\dot{X}^2 + \frac{1}{2}ml^2\dot{\phi}^2 - ml\dot{X}\dot{\phi}\cos\phi + mgl\cos\phi}$$

$$L = L(\phi, \dot{\phi}, t) \longrightarrow L' = L + \frac{d}{dt}f(\phi, t)$$

$$L = \underbrace{\cancel{\tfrac{1}{2} m \dot{X}^2}}_{\text{prescribed function of time} \longrightarrow \text{ignore}} + \tfrac{1}{2} m l^2 \dot{\phi}^2 - m l \dot{X} \dot{\phi} \cos\phi + m g l \cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$$\dot{X} \dot{\phi} \cos\phi = \underbrace{\frac{d}{dt}\left(\dot{X} \sin\phi\right)}_{\ddot{X} \sin\phi + \dot{X} \dot{\phi} \cos\phi} - \ddot{X} \sin\phi$$

$$-m l \dot{X} \dot{\phi} \cos\phi = \underbrace{\frac{d}{dt}\left(-m l \dot{X} \dot{\phi} \cos\phi\right)}_{\text{ignore this in Lagrangian}} + m l \ddot{X} \sin\phi$$

$$\boxed{L = \tfrac{1}{2} m l^2 \dot{\phi}^2 + m l a \sin\phi + m g l \cos\phi}$$

i) show that both Lagrangians give the same EOMs

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi} \iff \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot\phi}\right) = \frac{\partial L'}{\partial \phi}$$

ii) $L = \frac{1}{2} m l^2 \dot\phi^2 + m a l \sin\phi + m g l \cos\phi$

does not depend explicitly on time

$$\rightarrow \quad E = h = H = \dot\phi \frac{\partial L}{\partial \dot\phi} - L = \text{const}$$

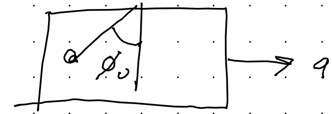
$$= \sum_i \dot q_i \underbrace{\frac{\partial L}{\partial \dot q_i}}_{p_i} - L \quad \text{(more generally)}$$

p_i = momentum conjugate to q_i

in general $\longrightarrow \neq T + U \leftarrow$ total mechanical energy

iii) Equil solution: $\dot{\phi} = 0$

$$\tan\phi_0 = \frac{a}{g}$$



Use EOMs from Lagrangian to show this

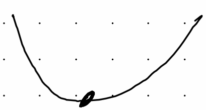
$$\boxed{E = \frac{1}{2} m l^2 \dot{\phi}^2 - m l a \sin\phi - m g l \cos\phi}$$

$$= \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - L$$

$$E = \frac{1}{2} m l^2 \dot{\phi}^2 + U_{eff}(\phi)$$

$$U_{eff}(\phi) = -ml\left(a \sin\phi + g \cos\phi\right)$$

☆ graph it



$$0 = \left.\frac{dU_{eff}}{d\phi}\right|_{\phi=\phi_0}$$

$$U_{eff}(\phi) = U_{eff}(\phi_0) + \cancel{\left.\frac{dU_{eff}}{d\phi}\right|_{\phi_0}}^{\,0}(\phi-\phi_0) + \frac{1}{2}\left.\frac{d^2U_{eff}}{d\phi^2}\right|_{\phi_0}(\phi-\phi_0)^2 + \cdots$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

$$+ \frac{1}{3!}f'''(a)(x-a)^3 + \cdots$$

$\mathcal{H}$

$$E = T + U_{eff}(\phi)$$

$$= T + U_{eff}(\phi_0) + \frac{1}{2}\mathcal{H}(\phi-\phi_0)^2$$

$$= \frac{1}{2}ml^2\underline{\dot{\phi}}^2 + U_{eff}(\phi_0) + \frac{1}{2}\mathcal{H}\,\underline{(\phi-\phi_0)}^2$$

const $\rightarrow$ ignore

$$x = \phi - \phi_0 \qquad |x| \ll 1$$

$$\dot{x} = \dot{\phi}$$

$$E = \frac{1}{2}\underbrace{ml^2}_{M}\dot{x}^2 + \frac{1}{2}\mathcal{H}x^2, \quad \omega = \sqrt{\frac{\mathcal{H}}{M}} = \sqrt{\frac{\mathcal{H}}{ml^2}}$$

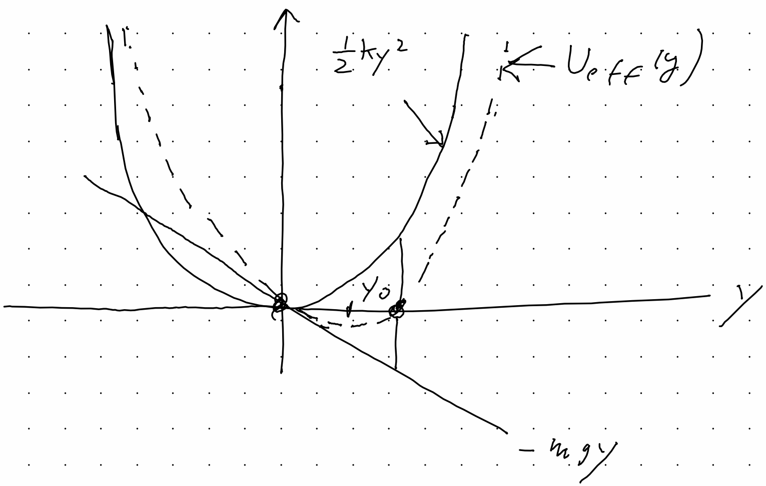
Answers to problems posed at the end of the last class

(1) $\vec{a} = (\underbrace{\ddot{r} - r\dot{\phi}^2}_{\text{centripetal acceleration}})\hat{r} + (2\dot{r}\dot{\phi} + \underbrace{r\ddot{\phi}}_{\text{tangential acceleration}})\hat{\phi}$

(2) $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$

$$= \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$$

(3) plot $U_{eff}(y) = \frac{1}{2}ky^2 - mgy$



$$U_{eff}(y) = U_{eff}(y_0) + \frac{1}{2}\kappa(y - y_0)^2$$

where y_0 is the solution to

$$0 = \left.\frac{dU_{eff}}{dy}\right|_{y_0}$$

and κ is given by

$$\kappa \equiv \left.\frac{d^2U_{eff}}{dy^2}\right|_{y_0}$$

$$0 = \left.\frac{dU_{eff}}{dy}\right|_{y_0} = ky_0 - mg \rightarrow y_0 = \frac{mg}{k}$$

$$\mathcal{H} = \left.\frac{d^2 U_{eff}}{dy^2}\right|_{y_0} = k$$ (so $\mathcal{H}$ is the same as k for this problem)

$$U_{eff}(y_0) = \frac{1}{2}ky_0^2 - mgy_0 = \frac{1}{2}k\left(\frac{mg}{k}\right)^2 - mg\left(\frac{mg}{k}\right) = -\frac{1}{2}\frac{m^2g^2}{k}$$

$$\rightarrow \boxed{U_{eff}(y) = -\frac{1}{2}\frac{m^2g^2}{k} + \frac{1}{2}k\left(y - \frac{mg}{k}\right)^2}$$

You can also obtain the same expression for $U_{eff}(y)$ by completing the square!

$$\begin{aligned} U_{eff}(y) &= \frac{1}{2}ky^2 - mgy \\ &= \frac{1}{2}k\left(y^2 - \frac{2mg}{k}y\right) \\ &= \frac{1}{2}k\left[\left(y - \frac{mg}{k}\right)^2 - \frac{m^2g^2}{k^2}\right] \\ &= \frac{1}{2}k\left(y - \frac{mg}{k}\right)^2 - \frac{1}{2}\frac{m^2g^2}{k} \end{aligned}$$

9/13

Cons. of Energy:

If L does not depend explicitly on time then the energy E (h) is conserved.

my notation

implicit $q(t)$, $\dot{q}(t)$

$L(q, \dot{q}, \cancel{t})$

not the same

$\frac{\partial L}{\partial t} = 0$

$\frac{dL}{dt}$

$L = T - U$

total mechanical energy $= T + U$

$$h \equiv \sum_i p_i \dot{q}_i - L = \text{const}$$

$$= \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$$

$$\frac{dh}{dt} = 0$$

$$H = \left(\sum_i p_i \dot{q}_i - L \right) \Big|_{\dot{q} = \dot{q}(q, p, t)}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = p_i(q, \dot{q}, t)$$

invert

$$\dot{q}_i = \dot{q}_i(q, p, t)$$

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

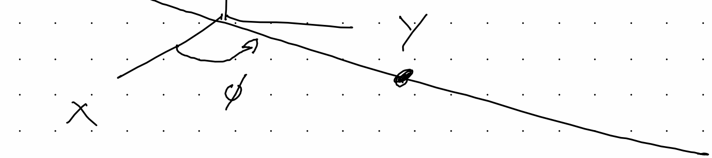
$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \longrightarrow \dot{x} = \frac{p}{m}$$

Example: bead on a ~~spin~~ rod that rotates uniformly in the xy plane.

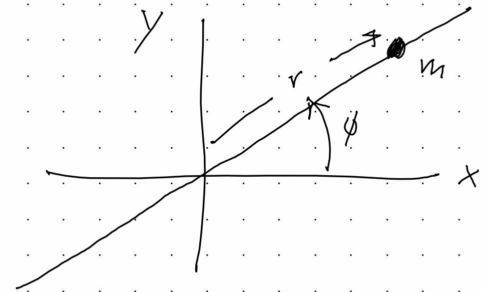
$\boxed{U=0}$

$\phi = \Omega t + \phi_0$

$\dot{\phi} = \Omega$



(top view)



$$L = T - \cancel{U}$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \Omega^2 \right)$$

$$h = \dot{r} \frac{\partial L}{\partial \dot{r}} - L = \text{const}$$

$$= \dot{r}\, m \dot{r} - \frac{1}{2} m \left(\dot{r}^2 + r^2 \Omega^2 \right)$$

$$= \frac{1}{2} m \dot{r}^2 - \frac{1}{2} m r^2 \Omega^2$$

$$= \frac{1}{2} m \left(\dot{r}^2 - r^2 \Omega^2 \right)$$

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right)$$

$$x = r \cos\phi$$

$$y = r \sin\phi$$

$$\phi = \Omega t$$

i) EOM's

ii) Solve EOM

iii) determine the constraint force !

$\phi = \Omega t$

$C = \phi - \Omega t = 0$ constraint

constraint force

$F_{constraint} = \lambda \vec{\nabla} C$

$= \lambda \frac{\partial C}{\partial \vec{r}}$

$L = T$

$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$

$$\boxed{\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= \frac{\partial L}{\partial r} + \lambda \frac{\partial C}{\partial r} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= \frac{\partial L}{\partial \phi} + \lambda \frac{\partial C}{\partial \phi} \\ C = \phi - \Omega t &= 0 \end{aligned}}$$

$\rightarrow$ 3 equations

3 unknowns

$r(t)$, $\phi(t)$

$\lambda(t)$

Solution to bead on rotating rod example:

i) Equation of motion:

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \Omega^2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \rightarrow \frac{d}{dt}(m\dot{r}) = m r \Omega^2$$

$$\cancel{m}\ddot{r} = \cancel{m} r \Omega^2$$

$$\ddot{r} = \Omega^2 r$$

ii) solution of EOM:

$$r(t) = A e^{\Omega t} + B e^{-\Omega t}$$

If initial conditions are

$$r(0) = r_0 , \quad \dot{r}(0) = 0$$

then $r_0 = A + B$

$$\dot{r}(t) = \Omega \left[A e^{\Omega t} - B e^{-\Omega t} \right]$$

$$\dot{r}(0) = 0 = \Omega [A - B]$$

So

$$\begin{aligned} A + B &= r_0 \\ A - B &= 0 \end{aligned} \rightarrow 2A = r_0 \rightarrow A = \frac{r_0}{2}$$

$$B = A = \frac{r_0}{2}$$

Thus,

$$r(t) = \frac{r_0}{2}\left[e^{\Omega t} + e^{-\Omega t}\right]$$

$$= r_0 \cosh(\Omega t)$$

iii) To find constraint force:

Take $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$

constraint: $C \equiv \phi - \Omega t = 0$

<u>EOMs:</u> $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} + \lambda\frac{\partial C}{\partial r} \rightarrow m\ddot{r} = mr\dot{\phi}^2$

$$\boxed{\ddot{r} = r\dot{\phi}^2}$$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} + \lambda\frac{\partial C}{\partial \phi} \rightarrow \frac{d}{dt}\left(mr^2\dot{\phi}\right) = \lambda$

$$\boxed{2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} = \lambda}$$

$\boxed{\phi - \Omega t = 0} \rightarrow \dot{\phi} = \Omega, \; \ddot{\phi} = 0$

3 equations, 3 unknowns $\left(r(t), \phi(t), \lambda(t)\right)$

$$\ddot{r} = r\Omega^2$$

$$\lambda = 2mr\dot{r}\Omega + mr^2\cancel{\ddot{\phi}}^{\to 0} \quad \longrightarrow \quad \lambda = 2mr\dot{r}\Omega$$

Thus, constraint force

$$\vec{F}_c = \lambda\vec{\nabla}C$$

$$= \lambda\left[\cancel{\frac{\partial C}{\partial r}}^{\to 0}\hat{r} + \frac{1}{r}\overset{1}{\frac{\partial C}{\partial \phi}}\hat{\phi}\right]$$

$$= \frac{\lambda}{r}\hat{\phi}$$

$$= 2m\dot{r}\Omega\hat{\phi}$$

so $\boxed{\vec{F}_c = 2m\dot{r}\Omega\hat{\phi}}$

NOTE: $\vec{F}_c = -\vec{F}_{coriolis} = +2m\vec{\Omega}\times\vec{v}'$ — velocity wrt rotating frame

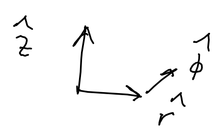
$\vec{v}' = \dot{r}\hat{r}$

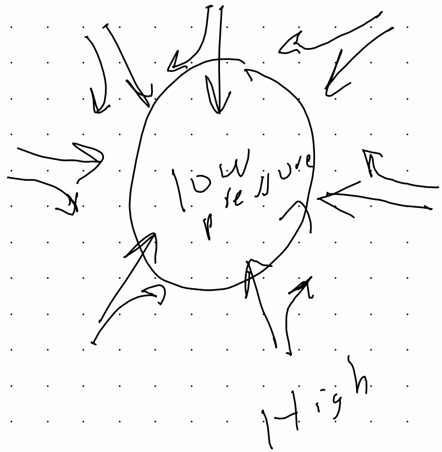
check: $RHS = 2m\Omega\hat{z}\times(\dot{r}\hat{r})$

$= 2m\dot{r}\Omega\hat{z}\times\hat{r}$

$= 2m\dot{r}\Omega\hat{\phi}$

$= F_c$





$\vec{\Omega} \times \vec{v}$

velocity in the non-inertial

$$H(q,p,t) = \left[\sum_i p_i \dot{q}_i - L(q,\dot{q},t) \right] \Bigg|_{\dot{q} = \dot{q}(q,p,t)}$$

Hamilton's equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, 2, \cdots, n$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad i = 1, 2, \cdots, n$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \qquad i = 1, 2, \cdots, n$$

Example: $L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x,y,z)$

i) Find $H(q,p,t)$

ii) Write down Hamilton's EOMs

iii) Show Hamilton's EOMs are equivalent to Lagrange equation

Solution:

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - U(x,y,z)$$

i) To find $H(q,p,t)$:

$$H = \left(\sum_i p_i \dot{q}_i - L \right) \Big|_{\dot{q} = \dot{q}(q,p,t)}$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = p_x/m$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \rightarrow \dot{y} = p_y/m$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

Thus,
$$H = \left(p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) + U(x,y,z) \right) \Bigg|_{\dot{q} = \dot{q}(q,p,t)}$$

$$= p_x \frac{p_x}{m} + p_y \frac{p_y}{m} + p_z \frac{p_z}{m} - \frac{1}{2} m \left(\left(\frac{p_x}{m}\right)^2 + \left(\frac{p_y}{m}\right)^2 + \left(\frac{p_z}{m}\right)^2 \right) + U(x,y,z)$$

$$= \frac{1}{2m} \left(p_x^2 + p_y^2 + p_z^2 \right) + U(x,y,z)$$

ii) Hamilton's EOMs:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

$$\rightarrow \quad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \rightarrow p_x = m\dot{x}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \rightarrow p_y = m\dot{y}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \rightarrow p_z = m\dot{z}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -\frac{\partial U}{\partial y}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{\partial U}{\partial z}$$

6 1st-order equations

iii) Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow m\ddot{x} = -\frac{\partial U}{\partial x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} \rightarrow m\ddot{y} = -\frac{\partial U}{\partial y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{\partial L}{\partial z} \rightarrow m\ddot{z} = -\frac{\partial U}{\partial z}$$

3 2nd-order equations

<u>NOTE</u>: Hamilton's equations can be combined to reproduce Lagrange's equations

$$p_x = m\dot{x}, \quad \dot{p}_x = -\frac{\partial U}{\partial x} \longrightarrow \frac{d}{dt}(m\dot{x}) = -\frac{\partial U}{\partial x}$$

substitute for $p_x = m\dot{x}$ into $\dot{p}_x = -\frac{\partial U}{\partial x}$

$$\underbrace{m\ddot{x} = -\frac{\partial U}{\partial x}}_{\text{which is the Lagrange equation for } x}$$

Similarly for y, z:

$$p_y = m\dot{y}, \quad \dot{p}_y = -\frac{\partial U}{\partial y} \longrightarrow m\ddot{y} = -\frac{\partial U}{\partial y}$$

$$p_z = m\dot{z}, \quad \dot{p}_z = -\frac{\partial U}{\partial z} \longrightarrow m\ddot{z} = -\frac{\partial U}{\partial z}$$

$$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot q} d\dot q + \frac{\partial L}{\partial t} dt$$

$$= \dot p \, dq + p \, d\dot q + \frac{\partial L}{\partial t} dt$$

$$= \dot p \, dq + d(p\dot q) - \dot q \, dp + \frac{\partial L}{\partial t} dt$$

$$\underbrace{d(p\dot q) - dL}_{= d(p\dot q - L) = dH} = \dot q \, dp - \dot p \, dq - \frac{\partial L}{\partial t} dt$$

$$\boxed{dH = -\dot p \, dq + \dot q \, dp - \frac{\partial L}{\partial t} dt}$$

$$f\,dg = d(fg) - g\,df$$

$$d(fg) = f\,dg + g\,df$$

$$\boxed{\begin{aligned} \frac{\partial H}{\partial q} &= -\dot p \\ \frac{\partial H}{\partial p} &= \dot q \\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} \end{aligned}}$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

Quiz #2:

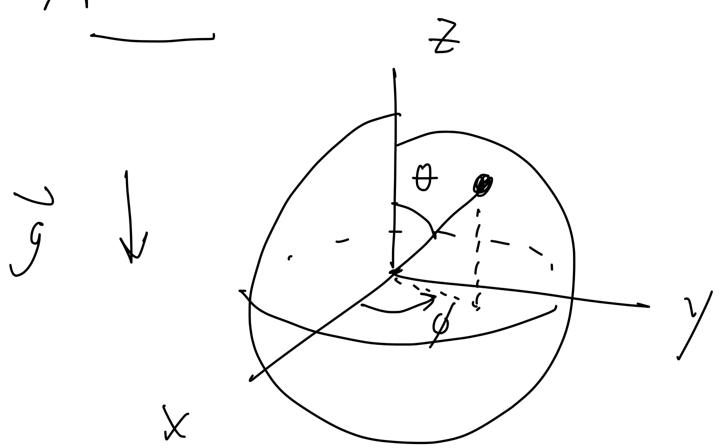
Particle of mass m is constrained to move on the surface of a sphere of radius R in a uniform gravitational field $\vec{g}$ (pointing downward).

a) write down Lagrangian L —— 0.5

b) " Hamiltonian H —— 0.5

c) " Hamilton's equations —— 1

Answer:



$r = R \qquad q_i = (\theta, \phi)$

$$\boxed{T = \frac{1}{2} m R^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2)}$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2)$$

$$U = mgz = \boxed{mgR\cos\theta}$$

$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$

$$L = \frac{1}{2} m R^2 \left(\dot\theta^2 + \sin^2\theta\, \dot\phi^2\right) - m g R \cos\theta$$

b) $$H = \left(\sum_i p_i \dot q_i - L\right)\Big|_{\dot q = \dot q(q,p,t)}$$

p_θ, p_ϕ

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = m R^2 \dot\theta \longrightarrow \dot\theta = \frac{p_\theta}{m R^2}$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = m R^2 \sin^2\theta\, \dot\phi \longrightarrow \dot\phi = \frac{p_\phi}{m R^2 \sin^2\theta}$$

$$\boxed{H} = p_\theta \frac{p_\theta}{m R^2} + p_\phi \frac{p_\phi}{m R^2 \sin^2\theta} - \frac{1}{2} m R^2 \left(\frac{p_\theta^2}{m^2 R^4} + \sin^2\theta \frac{p_\phi^2}{m^2 R^4 \sin^4\theta}\right) + m g R \cos\theta$$

$$\boxed{= \frac{1}{2 m R^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + m g R \cos\theta}$$

c) $$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m R^2}, \quad \dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m R^2 \sin^2\theta}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{m R^2 \sin^3\theta} + mgR\sin\theta$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \quad \rightarrow \quad p_\phi = \text{const}$$

$$L = \frac{1}{2} m R^2 \left(\dot\theta^2 + \sin^2\theta\, \dot\phi^2\right) - mgR\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{\partial L}{\partial \theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi} = 0 \quad \rightarrow \quad \boxed{p_\phi \equiv \frac{\partial L}{\partial \dot\phi} = m R^2 \sin^2\theta\, \dot\phi} = \text{const}$$

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = m R^2 \dot\theta$$

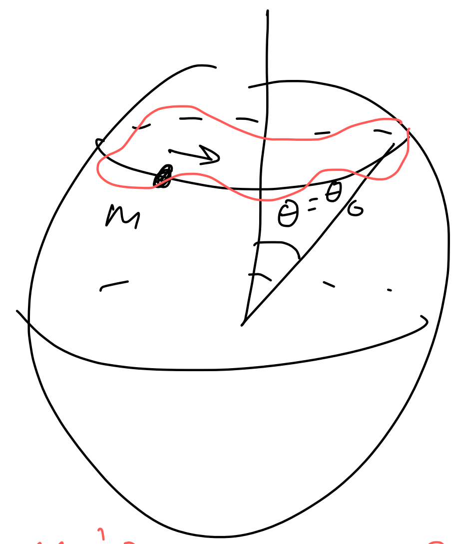
$$\boxed{h = p_\theta \dot\theta + p_\phi \dot\phi - L = T + U = \frac{1}{2} m R^2 \left(\dot\theta^2 + \sin^2\theta\, \dot\phi^2\right) + mgR\cos\theta} \equiv E$$

$$E = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \sin^2\theta \, \frac{p_\phi^2}{m^2 R^4 \sin^4\theta} + mgR\cos\theta$$

$$= \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} \frac{p_\phi^2}{m R^2 \sin^2\theta} + mgR\cos\theta$$

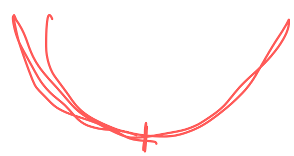
$$= \frac{1}{2} m R^2 \dot{\theta}^2 + U_{eff}(\theta)$$

$$U_{eff}(\theta) = \frac{1}{2} \frac{p_\phi^2}{m R^2 \sin^2\theta} + mgR\cos\theta$$



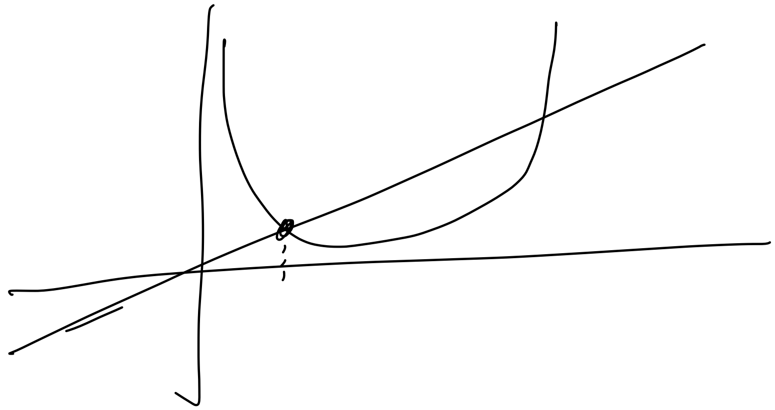
$\dot{\theta} = 0$ (stationary)

θ_0



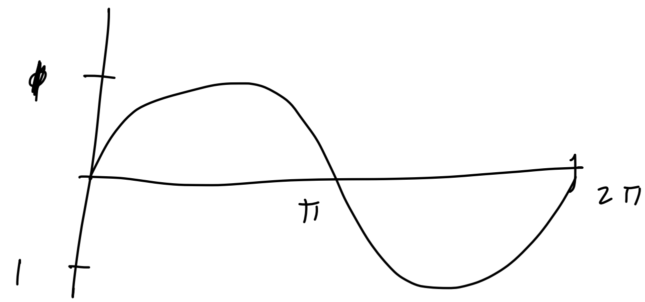
$\frac{1}{2} M \dot{x}^2 - \frac{1}{2} K x^2$

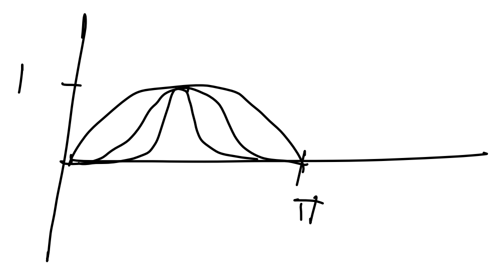
1) Plot $U_{eff}(\theta)$
2) Determine θ_0
3) Determine freq of small oscillations about θ_0



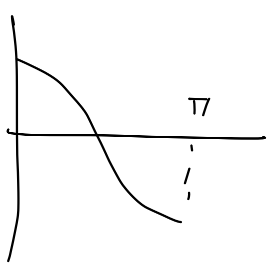
$\sin^4\theta_0 = -\cos\theta_0$

$\theta \in [0, \pi]$





cos θ



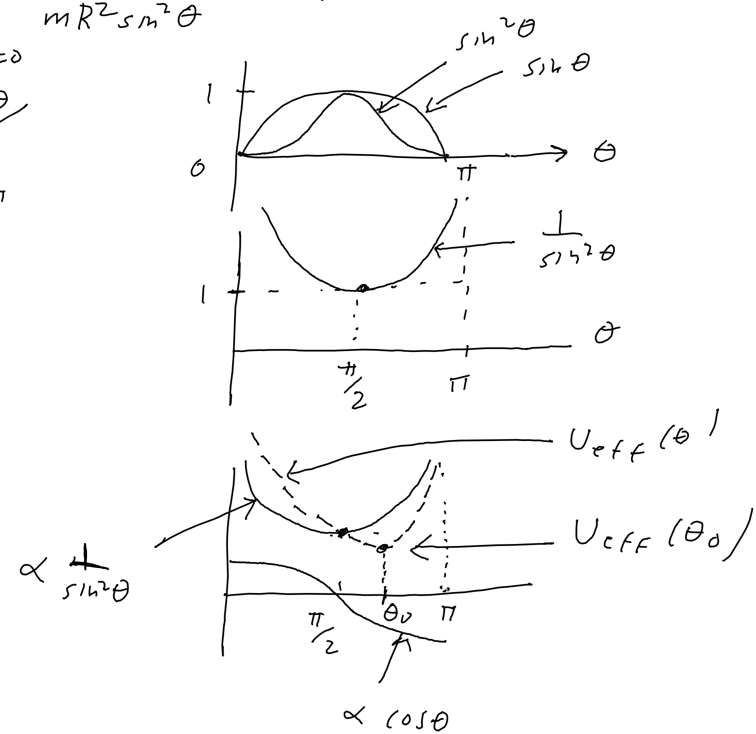
$$U_{eff}(\theta) = U_{eff}(\theta_0) + \left.\frac{dU_{eff}}{d\theta}\right|_{\theta_0} (\theta - \theta_0)$$

$$+ \frac{1}{2} \left.\frac{d^2 U_{eff}}{d\theta^2}\right|_{\theta_0} (\theta - \theta_0)^2 + \dots$$

sol'n:

i) plot $U_{eff}(\theta) = \frac{1}{2}\frac{p_\phi^2}{mR^2\sin^2\theta} + mgR\cos\theta$

$\theta \in [0, \pi]$



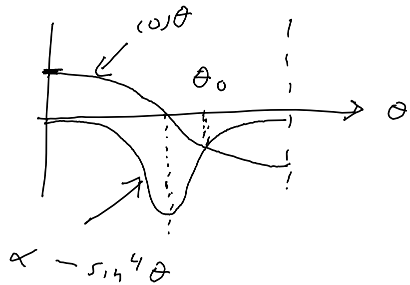
ii) Determine θ_0

$$0 = \left.\frac{dU_{eff}}{d\theta}\right|_{\theta_0} = -\frac{p_\phi^2\cos\theta_0}{mR^2\sin^3\theta_0} - mgR\sin\theta_0$$

$$-\frac{p_\phi^2 \cos\theta_0}{mR^2\sin^3\theta_0} = mgR\sin\theta_0$$

$$\Rightarrow \boxed{p_\phi^2\cos\theta_0 = -m^2 g R^3 \sin^4\theta_0}$$

graphical solution:



iii) Freq of small oscillations

$$L = \frac{1}{2}mR^2(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) - mgR\cos\theta$$

No explicit t dependence $\rightarrow h = \sum_i \frac{\partial L}{\partial \dot q_i}\dot q_i - L = \text{const}\ (\equiv E)$

No ϕ dependence $\rightarrow \frac{\partial L}{\partial\dot\phi} = \text{const}\ (\equiv p_\phi)$

$$E = mR^2\dot\theta\,\dot\theta + mR^2\sin^2\theta\,\dot\phi\,\dot\phi - \frac{1}{2}mR^2(\dot\theta^2+\sin^2\theta\,\dot\phi^2) + mgR\cos\theta$$

$$= \frac{1}{2}mR^2(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) + mgR\cos\theta$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = mR^2\sin^2\theta\,\dot\phi \rightarrow \dot\phi = \frac{p_\phi}{mR^2\sin^2\theta}$$

$$\rightarrow E = \frac{1}{2}mR^2\dot\theta^2 + \frac{1}{2}mR^2\sin^2\theta\left(\frac{p_\phi^2}{m^2R^4\sin^4\theta}\right) + mgR\cos\theta$$

$$= \frac{1}{2}mR^2\dot\theta^2 + \frac{p_\phi^2}{2mR^2\sin^2\theta} + mgR\cos\theta$$

$$= \frac{1}{2}mR^2\dot\theta^2 + U_{eff}(\theta) \quad \leftarrow \text{effective 1-d motion}$$

$$U_{eff}(\theta) = \frac{p_\phi^2}{2mR^2\sin^2\theta} + mgR\cos\theta$$

Expand $U_{eff}(\theta)$ about $\theta = \theta_0$

$$U_{eff}(\theta) = \underbrace{U_{eff}(\theta_0)}_{\substack{=\text{const} \\ \text{(can ignore)}}} + \cancelto{0}{\left.\frac{dU_{eff}}{d\theta}\right|_{\theta=\theta_0}}(\theta-\theta_0) + \frac{1}{2}\left.\frac{d^2U_{eff}}{d\theta^2}\right|_{\theta=\theta_0}(\theta-\theta_0)^2 + \underbrace{\ldots}_{\substack{\text{can} \\ \text{ignore if} \\ |\theta-\theta_0| \ll 1}}$$

so

$$U_{eff}(\theta) \approx \frac{1}{2}\left.\frac{d^2U_{eff}}{d\theta^2}\right|_{\theta=\theta_0}(\theta-\theta_0)^2$$

Define: $x = \theta - \theta_0 \quad \rightarrow \quad \ddot{\theta} = \ddot{x}$

$$\varkappa = \left.\frac{d^2 U_{eff}}{d\theta^2}\right|_{\theta=\theta_0}$$

$$mR^2 = M$$

Then

$$E = \frac{1}{2} mR^2 \dot{\theta}^2 + U_{eff}(\theta)$$

$$\approx \frac{1}{2} M \dot{x}^2 + \frac{1}{2}\varkappa x^2 \qquad (SHM)$$

with $\omega = \sqrt{\dfrac{\varkappa}{M}} = \sqrt{\dfrac{1}{mR^2} \left.\dfrac{d^2 U_{eff}}{d\theta^2}\right|_{\theta=\theta_0}}$ (= angular freq)

Now: $U_{eff}(\theta) = \dfrac{p_\phi^2}{2mR^2 \sin^2\theta} + mgR\cos\theta$

$$\frac{dU_{eff}}{d\theta} = \frac{-p_\phi^2 \cos\theta}{mR^2 \sin^3\theta} - mgR\sin\theta$$

$$\frac{d^2 U_{eff}}{d\theta^2} = \frac{p_\phi^2 \sin\theta}{mR^2 \sin^3\theta} + 3\frac{p_\phi^2 \cos\theta}{mR^2} \frac{\cos\theta}{\sin^4\theta} - mgR\cos\theta$$

$$\left.\frac{d^2U_{eff}}{d\theta^2}\right|_{\theta=\theta_0} = \frac{p_\phi^2}{mR^2\sin^2\theta_0} + \frac{3p_\phi^2\cos^2\theta_0}{mR^2\sin^4\theta_0} - mgR\cos\theta_0$$

$$= \frac{p_\phi^2}{mR^2\sin^2\theta_0}\left[1 + \frac{3\cos^2\theta_0}{\sin^2\theta_0}\right] - mgR\cos\theta_0$$

Recall: $p_\phi^2\cos\theta_0 = -m^2gR^3\sin^4\theta_0$

$$\left.\frac{d^2U_{eff}}{d\theta^2}\right|_{\theta=\theta_0} = \frac{p_\phi^2}{mR^2\sin^2\theta_0}\left[1 + \frac{3m^4g^2R^6\sin^8\theta_0}{p_\phi^4\sin^2\theta_0}\right] + \frac{m^3gR^4\sin^4\theta_0}{p_\phi^2}$$

$$= \frac{p_\phi^2}{mR^2\sin^2\theta_0} + \frac{3m^3gR^4\sin^4\theta_0}{p_\phi^2} + \frac{m^3gR^4\sin^4\theta_0}{p_\phi^2}$$

$$= \boxed{\frac{p_\phi^2}{mR^2\sin^2\theta_0} + \frac{4mg\,m^2R^4\sin^4\theta_0}{p_\phi^2}}$$

$$\rightarrow \boxed{\omega = \sqrt{\frac{1}{mR^2}\left.\frac{d^2U_{eff}}{d\theta^2}\right|_{\theta=\theta_0}} = \sqrt{\frac{p_\phi^2}{m^2R^4\sin^2\theta_0} + \frac{4mg\,mR^2\sin^4\theta_0}{p_\phi^2}}}$$

where θ_0 is determined from graphical solution.