Quiz # 3:

m_1: hard sphere radius $= a$
m_2: hard sphere radius $= b$

a) Determine $\rho = \rho(\chi)$

b) For $m_1 = m_2 \equiv m$ and $a = b$, find ρ so that $\theta_2 = 60°$?

c) What is $\theta_1 = $?

You can use:

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_2 \cos\chi + m_1}$$

$$\chi + 2\theta_2 = \pi$$

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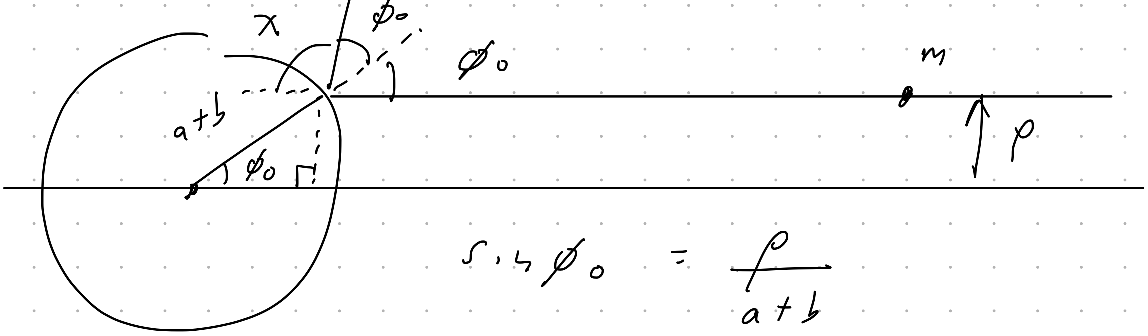
a) Hard sphere interaction $\rightarrow$ $U(r) = \begin{cases} \infty & r < a+b \\ 0 & r > a+b \end{cases}$

$$2\phi_0 + \chi = \pi$$



$$\sin\phi_0 = \frac{\rho}{a+b}$$

Thus, $\rho = (a+b)\sin\phi_0 = (a+b)\sin\left(\frac{\pi}{2} - \frac{\chi}{2}\right) = \boxed{(a+b)\cos\left(\frac{\chi}{2}\right)}$

b) $\rho = (a+b)\cos\left(\frac{\chi}{2}\right)$

$\chi + 2\theta_2 = \pi$

$\theta_2 = 60^\circ \rightarrow \chi = 180^\circ - 120^\circ$
$= 60^\circ$

$a = b \rightarrow \rho = 2a\cos\left(\frac{60^\circ}{2}\right)$

$= 2a\cos 30^\circ$

$= \boxed{a\sqrt{3}}$

c) $\tan\theta_1 = \dfrac{m_2\sin\chi}{m_2\cos\chi + m_1}$

$= \dfrac{\cancel{m}\sin\chi}{\cancel{m}(1+\cos\chi)}$

$= \dfrac{\sin\chi}{1+\cos\chi}$

Now: $\sin\chi = \sin\left(2\cdot\frac{\chi}{2}\right) = 2\sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)$

$\cos\chi = \cos\left(2\cdot\frac{\chi}{2}\right)$

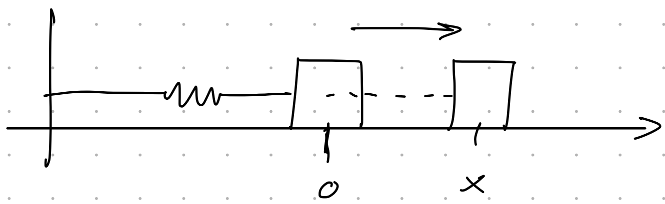
$= \cos^2\left(\frac{\chi}{2}\right) - \sin^2\left(\frac{\chi}{2}\right)$

$= 2\cos^2\left(\frac{\chi}{2}\right) - 1$

$\rightarrow 1 + \cos\chi = 2\cos^2\left(\frac{\chi}{2}\right)$

Thus, $\tan\theta_1 = \dfrac{\cancel{2}\sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)}{\cancel{2}\cos^2\left(\frac{\chi}{2}\right)} = \tan\left(\frac{\chi}{2}\right) \rightarrow \boxed{\theta_1 = \frac{\chi}{2} = 30^\circ}$

Small oscillations



$F = -kx$ (linear restoring force)

$F = m\ddot{x}$ (2nd law)

$$m\ddot{x} = -kx$$
$$\ddot{x} = -\frac{k}{m}x$$

Soln: $x = c_1 \cos \omega t + c_2 \sin \omega t$

$\omega = \sqrt{\frac{k}{m}}$ (angular freq)

Alternative: $x = a \cos(\omega t + \alpha)$
$= \mathrm{re}\left[A e^{i\omega t}\right]$, $A = a e^{i\alpha}$

Lagrangian: $F = -kx = -\frac{dU}{dx} \rightarrow U = \frac{1}{2}kx^2$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

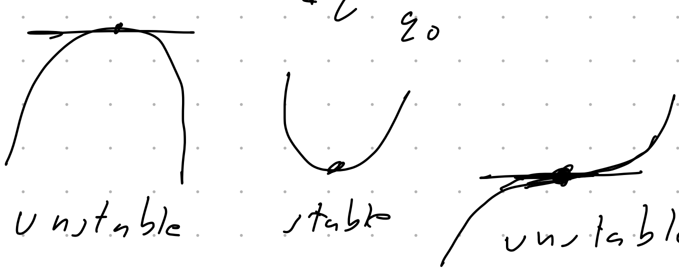
General formalism

$$L = \frac{1}{2}a(q)\dot{q}^2 - U(q)$$

suppose q_0 is a point of stable equilibrium

(i) $\left.\frac{dU}{dq}\right|_{q_0} = 0$, (ii) $\left.\frac{d^2U}{dq^2}\right|_{q_0} > 0$

(stable)



Expand: (about q_0)

$$U(q) = \underbrace{U(q_0)}_{\substack{\text{const} \\ \text{(ignore)}}} + \cancel{\frac{dU}{dq}\Big|_{q_0}}^{\to 0 \text{ (equil)}} (q-q_0) + \frac{1}{2}\frac{d^2U}{dq^2}\Big|_{q_0}(q-q_0)^2 + \cdots$$

Define $x = q - q_0$, $\quad k \equiv \frac{d^2U}{dq^2}\Big|_{q_0}$

$$U(q) = \frac{1}{2}kx^2 \quad (\text{ignore } O(x^3))$$

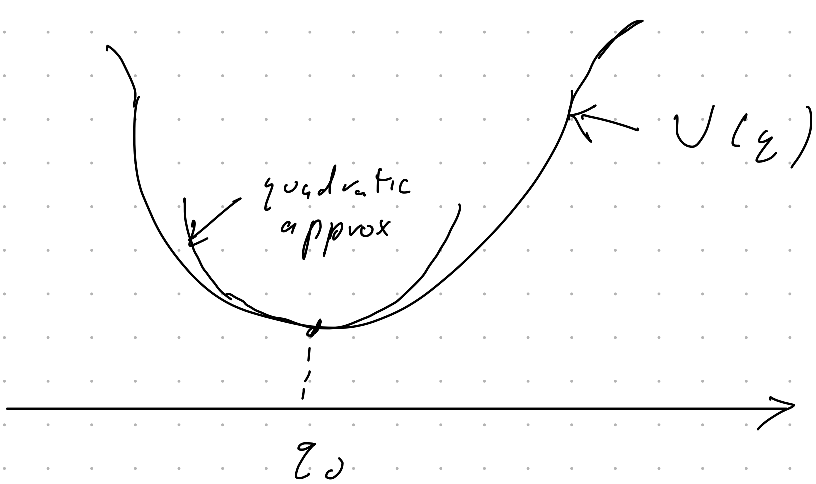
$$T = \frac{1}{2}a(q)\dot{q}^2 = \frac{1}{2}\underbrace{(a(q_0) + \cdots)}_{\text{take } 0^{\text{th}} \text{ order}}\underbrace{\dot{x}^2}_{2^{\text{nd}} \text{ order}}$$

$$= \frac{1}{2}a(q_0)\dot{x}^2$$

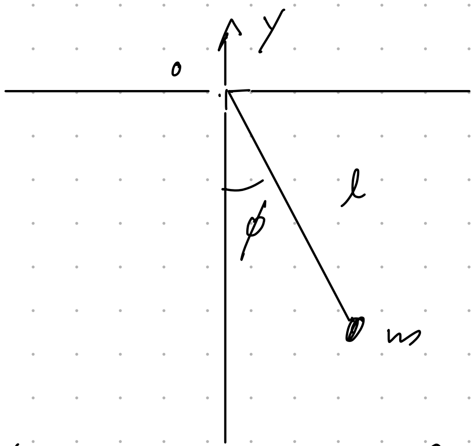
$$= \frac{1}{2}m\dot{x}^2$$

where $m \equiv a(q_0)$

Thus, $\quad L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \quad$ (as before)



Simple pendulum



$$\left(\begin{aligned} U &= mgy \\ &= -mgl\cos\phi \end{aligned} \right)$$

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi$$

Expand $U(\phi)$ around $\phi = 0$

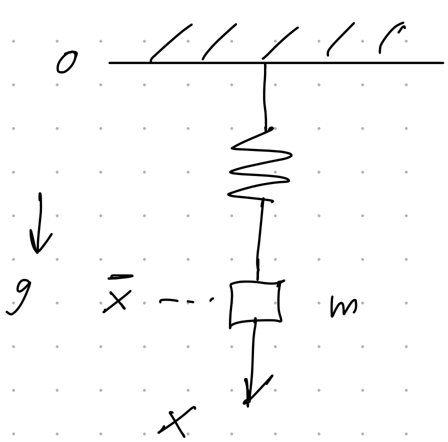
$$U(\phi) = -mgl\left(1 - \frac{\phi^2}{2} + \dots\right)$$

$$\approx \underbrace{-mgl}_{\text{ignore}} + \frac{1}{2} mgl\phi^2$$

$$L \approx \frac{1}{2} \underbrace{ml^2}_{M} \dot{\phi}^2 - \frac{1}{2} \underbrace{mgl}_{k} \phi^2$$

$$\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{\cancel{m}gl}{\cancel{m}l^2}} = \sqrt{\frac{g}{l}}$$

Spring + gravity



$\bar{x}$: unstretched spring

$$U = \frac{1}{2} k (x - \bar{x})^2 - mgx$$

$$-\frac{dU}{dx} = -k(x - \bar{x}) + mg$$

$$-\left.\frac{dU}{dx}\right|_{x_0} = 0 \quad (\text{equilibrium})$$

$$\longrightarrow \quad k(x_0 - \bar{x}) = mg$$

$$k x_0 = mg + k\bar{x}$$

$$\boxed{x_0 = \bar{x} + \frac{mg}{k}}$$

then

$$U(x) = \underbrace{U(x_0)}_{\text{const}} + \cancel{\left.\frac{dU}{dx}\right|_{x_0} (x-x_0)}^{\,0} + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x_0}(x-x_0)^2 + \ldots$$

$$= \frac{1}{2} k (x-x_0)^2$$

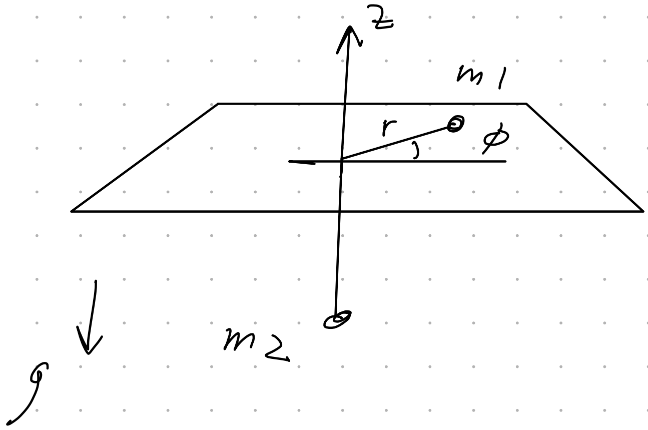
$$= \frac{1}{2} k \eta^2$$

where $\eta \equiv x - x_0$ (deviation from equilibrium)

$$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m \dot{\eta}^2$$

$$L = \frac{1}{2} m \dot{\eta}^2 - \frac{1}{2} k \eta^2 \quad \text{(as before)}$$

Example:



$$l = r - z$$

$$\rightarrow z = r - l$$

$$\dot{z} = \dot{r}$$

$$T = \frac{1}{2} m_1 (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{1}{2} m_2 \dot{z}^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2$$

$$U = m_2 g z$$

$$= m_2 g (r - l)$$

$$= m_2 g r \underbrace{- m_2 g l}_{\text{constant}}$$

$$L = T - U$$

$$= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r$$

i) $$M_z = \frac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} = \text{const}$$

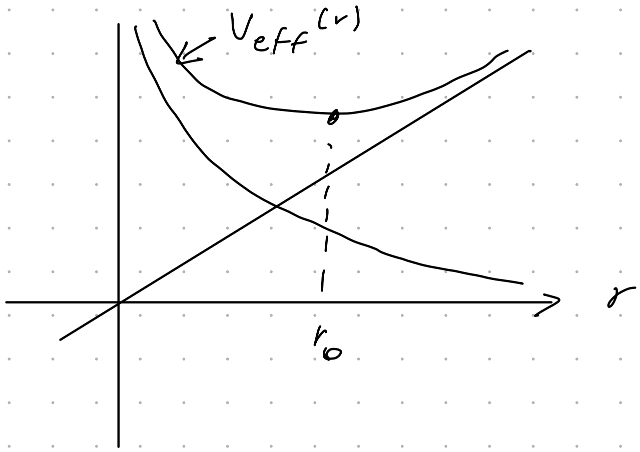
$$\rightarrow \dot{\phi} = \frac{M_z}{m_1 r^2}$$

ii) $$E = T + U = \text{const}$$

$$= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 + m_2 g r$$

$$= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \underbrace{\frac{M_z^2}{2 m_1 r^2} + m_2 g r}_{U_{eff}(r)}$$

$$U_{eff}(r) = \frac{M_z^2}{2m_1 r^2} + m_2 g r$$



$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0}$$

$$= -\frac{M_z^2}{m_1 r_0^3} + m_2 g$$

$$\rightarrow \boxed{M_z^2 = m_1 m_2 g r_0^3}$$

For small oscillation, need

$$k = \left.\frac{d^2 U_{eff}}{dr^2}\right|_{r=r_0}$$

$$= \frac{3M_z^2}{m_1 r_0^4}$$

$$= \frac{3 m_1 m_2 g r_0^3}{m_1 r_0^4} = \boxed{\frac{3 m_2 g}{r_0}}$$

$$\omega_r = \sqrt{\frac{K}{m_1+m_2}} = \sqrt{\frac{3 m_2 g}{(m_1+m_2) r_0}}$$

$$\omega_\phi = \dot{\phi}\Big|_{r=r_0}$$

$$= \frac{M_z}{m_1 r_0^2}$$

$$= \frac{\sqrt{m_1 m_2 g r_0^3}}{m_1 r_0^2}$$

$$= \sqrt{\frac{m_2 g}{m_1 r_0}}$$

<u>NOTE:</u>

$$\omega_r = \omega_\phi \iff \frac{3}{m_1+m_2} = \frac{1}{m_1}$$

$$3 m_1 = m_1 + m_2$$

$$\boxed{2 m_1 = m_2}$$

<u>Demo:</u> animation

Forced oscillations

$$m\ddot{x} = -kx + F(t)$$

$$\rightarrow \quad \ddot{x} + \frac{k}{m}x = \frac{F(t)}{m}$$

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}$$

general solution:

$$x(t) = \underbrace{x_h(t)}_{\substack{\text{homogeneous} \\ (F(t)=0)}} + \underbrace{x_p(t)}_{\text{particular solution to } \frac{F(t)}{m}}$$

$$\boxed{x_h(t) = a\cos(\omega t + \alpha)}$$

Example: suppose $F(t) = f\cos(\gamma t + \beta)$

Guess $x_p(t) = b\cos(\gamma t + \beta)$

Then $\ddot{x}_p + \omega^2 x_p = \dfrac{F(t)}{m}$

$$\rightarrow \quad -\gamma^2 b\cos(\gamma t + \beta) + \omega^2 b\cos(\gamma t + \beta) = \frac{f}{m}\cos(\gamma t + \beta)$$

$$b(\omega^2 - \gamma^2) = \frac{f}{m}$$

$$\rightarrow \quad b = \frac{f}{m(\omega^2 - \gamma^2)}, \quad \boxed{x_p(t) = \frac{f}{m(\omega^2 - \gamma^2)}\cos(\gamma t + \beta)}$$

NOTE:

$x_p(t)$ undefined for $\gamma = \omega$ (resonance)

Can define another particular solution by subtracting off a solution to the homog. equation:

$$x_p(t) = \frac{f}{m(\omega^2-\gamma^2)}\left[\cos(\gamma t+\beta) - \cos(\omega t+\beta)\right]$$

$$\rightarrow \frac{0}{0} \quad \text{when } \gamma \rightarrow \omega$$

$$= \frac{\frac{d}{d\gamma}(\text{num})\Big|_{\gamma=\omega}}{\frac{d}{d\gamma}(\text{den})\Big|_{\gamma=\omega}}$$

L'hopital's rule

$$= \frac{-f t \sin(\omega t+\beta)}{-2m\omega}$$

$$= \frac{f t}{2m\omega}\sin(\omega t+\beta)$$

Thus,

$$\boxed{x(t) = a\cos(\omega t+\alpha) + \frac{f t}{2m\omega}\sin(\omega t+\beta)}$$

after sometime this becomes sufficiently large that x is no longer small

Arbitrary $F(t) = \underbrace{\int_{-\infty}^{\infty} d\gamma \, \tilde{F}(\gamma) \, e^{i\gamma t}}_{\text{Fourier transform}}$

General solution:

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}$$

Let: $\xi = \dot{x} + i\omega x$ (complex)

$\rightarrow \quad \dot{\xi} = \ddot{x} + i\omega \dot{x}$

Thus,
$$i\omega \xi = i\omega \dot{x} - \omega^2 x$$
$$\dot{\xi} = \ddot{x} + i\omega \dot{x}$$

subtract:
$$\dot{\xi} - i\omega \xi = \ddot{x} + \omega^2 x = \frac{F(t)}{m}$$

Thus,
$$\boxed{\dot{\xi} - i\omega \xi = \frac{F(t)}{m}}$$

Homog solution: $\dot{\xi} - i\omega \xi = 0$

$$\rightarrow \quad \xi = A e^{i\omega t}$$

A: complex const

For particular solution, guess

$$\xi_p(t) = \underbrace{A(t)}_{\text{complex function}} e^{i\omega t}$$

$$\rightarrow \dot{\xi}_p - i\omega \xi_p = \frac{F(t)}{m}$$

$$\dot{A} e^{i\omega t} + \cancel{i A \omega e^{i\omega t}} \cancel{- i\omega A e^{i\omega t}} = \frac{F(t)}{m}$$

$$\dot{A} = \frac{F(t) e^{-i\omega t}}{m}$$

Integrate:

$$A(t) = \int d\bar{t} \frac{F(\bar{t})}{m} e^{-i\omega \bar{t}} + \text{const}$$

$$\rightarrow \quad \xi_p(t) = e^{i\omega t}\left[\int d\bar{t}\, \frac{F(\bar{t})}{m} e^{-i\omega \bar{t}} + C\right]$$

so $$\boxed{\xi(t) = e^{i\omega t}\left[\int_0^t d\bar{t}\, \frac{F(\bar{t})}{m} e^{-i\omega \bar{t}} + \xi_0\right]}$$

complex const

(actually the general solution)

Then $$\boxed{x(t) = \frac{1}{\omega} \operatorname{Im} \xi(t)}$$

Free oscillations in 2 or more dimensions:

Recall space oscillator:

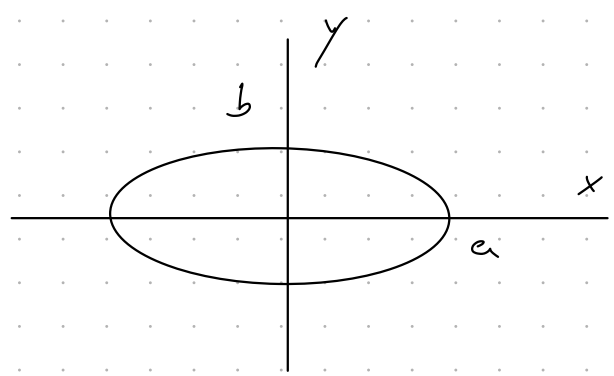
$$U = \frac{1}{2} k r^2 \qquad \text{(motion in plane)}$$
$$= \frac{1}{2} k (x^2 + y^2)$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$$
$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$L = T - U$$
$$= \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + \frac{1}{2} m \dot{y}^2 - \frac{1}{2} k y^2$$

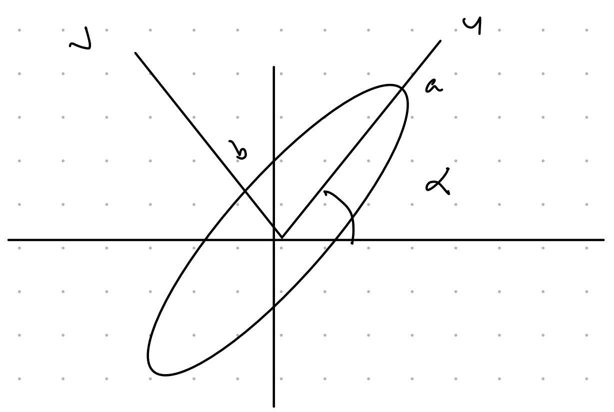
= sum of two independent harmonic oscillators

$$x = a \cos(\omega t + \alpha) \qquad \omega_x = \sqrt{\frac{k}{m}} = \omega_y$$
$$y = b \sin(\omega t + \beta) \qquad \equiv \omega$$



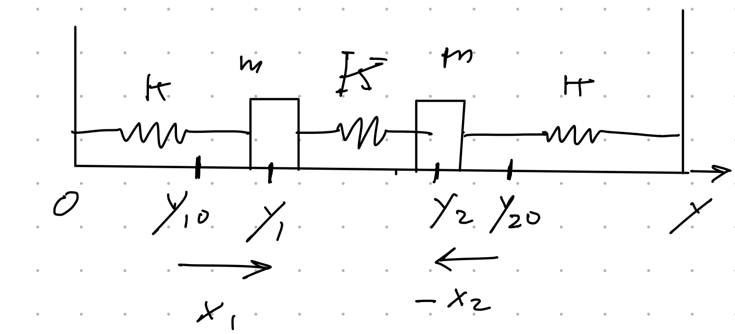
$\alpha = 0$

$\beta = 0$



$\alpha = \beta \neq 0$

Example :



y_{10}, y_{20} : equilibrium positions of the two masses

$m_1 = m_2 = m$

$x_1 \equiv y_1 - y_{10}$

$x_2 \equiv y_2 - y_{20}$

$$T = \frac{1}{2} m (\dot{y}_1^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2)$$

$$= \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k$$

$$m_{ik} = \boxed{\begin{array}{c|c} m & 0 \\ \hline 0 & m \end{array}}$$

$$U = \frac{1}{2} k (y_1 - y_{10})^2 + \frac{1}{2} k (y_2 - y_{20})^2 + \frac{1}{2} \bar{K} (y_2 - y_1)^2$$

$$= \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} \bar{K} (x_2 - x_1)^2$$

$$= \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} \bar{K} (x_1^2 + x_2^2 - 2 x_1 x_2)$$

$$= \frac{1}{2} (k + \bar{K}) x_1^2 + \frac{1}{2} (k + \bar{K}) x_2^2 \underbrace{- \bar{K} x_1 x_2}_{-\frac{1}{2} \bar{K} (x_1 x_2 + x_2 x_1)}$$

$$= \frac{1}{2} \sum_{i,k} k_{ik} x_i x_k$$

$$k_{ik} = \boxed{\begin{array}{c|c} k + \bar{K} & -K \\ \hline -K & k + \bar{K} \end{array}}$$

$$L = T - U$$

$$= \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k - \frac{1}{2} \sum_{i,k} k_{ik} x_i x_k$$

<u>EOMs:</u>

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$

$$\frac{d}{dt}\left(\sum_k m_{ik} \dot{x}_k\right) = -\sum_k k_{ik} x_k$$

$$\sum_k m_{ik} \ddot{x}_k = -\sum_k k_{ik} x_k$$

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = -\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

<u>Trial solution:</u>

$$x_k = A_k e^{i\omega t}$$

$$\ddot{x}_k = -\omega^2 A_k e^{i\omega t}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \sum_k \left(-m_{ik}\omega^2 + k_{ik}\right) A_k \underbrace{e^{i\omega t}}_{\text{can ignore}}$$

$$= \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

if invertible, the $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

so $\begin{bmatrix} & \\ & \end{bmatrix}$ cannot be invertible

$\Leftrightarrow \det \begin{bmatrix} & \\ & \end{bmatrix} = 0$

$$\det\left(K_{ik} - m_{ik}\omega^2\right) = 0$$

$$0 = \det\left(\begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix} - \begin{bmatrix} m\omega^2 & 0 \\ 0 & m\omega^2 \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} k+K-m\omega^2 & -K \\ -K & k+K-m\omega^2 \end{bmatrix}$$

$$= \left((k+K)-m\omega^2\right)^2 - K^2$$

$$= (k+K)^2 - K^2 + m^2\omega^4 - 2m\omega^2(k+K)$$

$$= \left(k^2+2kK\right) + m^2\omega^4 - 2m\omega^2(k+K)$$

$$= \omega^4 - 2\omega^2\left(\frac{k+K}{m}\right) + \left(\frac{k^2+2kK}{m^2}\right)$$

Quadratic:

$$\omega_\pm^2 = \frac{2\left(\frac{k+K}{m}\right) \pm \sqrt{4\left(\frac{k+K}{m}\right)^2 - 4\left(\frac{k^2+2kK}{m^2}\right)}}{2}$$

$$= \frac{k+K}{m} \pm \frac{1}{m}\sqrt{\cancel{k^2} + K^2 + \cancel{2kK} - \cancel{k^2} - \cancel{2kK}}$$

$$= \frac{k+K}{m} \pm \frac{K}{m} = \begin{cases} \frac{k+2K}{m} & (+\text{ sign}) \\ \frac{k}{m} & (-\text{ sign}) \end{cases}$$

$$\omega_+^2 = \frac{k + 2K}{m}, \qquad \omega_-^2 = \frac{k}{m}$$ "normal mode freqs"

<u>Eigenvectors:</u> "normal mode oscillators"

<u>ω_+^2:</u>

$$\begin{bmatrix} k + K - m\omega_+^2 & -K \\ -K & k + K - m\omega_+^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -K & -K \\ -K & -K \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $v_2 = -v_1$

$$\rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = v_+$$

<u>ω_-^2:</u>

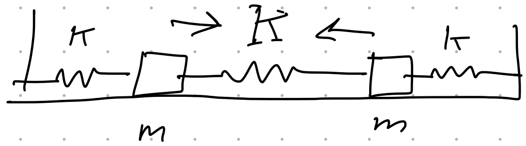
$$\begin{bmatrix} k + K - m\omega_-^2 & -K \\ -K & k + K - m\omega_-^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

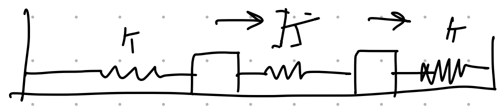
Thus, $v_2 = v_1$

$$\rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = v_-$$

$$v_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



$$v_- = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



middle spring is not compressed

General solution:

$$x_\kappa = \mathrm{Re}\left(\sum_{\alpha=+,-} \Delta_{\kappa\alpha} C_\alpha e^{i\omega_\alpha t} \right)$$

complex, determined by IC's

$$= \sum_\alpha \Delta_{\kappa\alpha} \theta_\alpha$$

where $\Delta_{\kappa\alpha} = \begin{bmatrix} v_+ & v_- \end{bmatrix}$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \leftarrow \text{matrix of eigenvectors}$$

Normal coords:

$$\theta_\alpha = \mathrm{Re}\left[C_\alpha e^{i\omega_\alpha t} \right]$$

$$= (\Delta^{-1})_{\alpha\kappa} x_\kappa$$

$$\Delta^{-1} = \frac{1}{\det\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \Delta^T$$

Lagrangian:

$$L = T - U$$

$$= \frac{1}{2} \sum_{i,k} m_{i,k} \dot{x}_i \dot{x}_k - \frac{1}{2} \sum_{i,k} k_{i,k} x_i x_k$$

$$\sum_{i,k} k_{i,k} x_i x_k = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} & \\ & \end{bmatrix}}_{k_{i,k}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x^T K x$$

$$= (\Delta \theta)^T K \Delta \theta$$

$$= \theta^T (\Delta^T K \Delta) \theta$$

$$\Delta^T K \Delta = \begin{bmatrix} v_+ \\ v_- \end{bmatrix} \begin{bmatrix} K \end{bmatrix} \begin{bmatrix} v_+ & v_- \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} k+\bar{k} & -\bar{k} \\ -\bar{k} & k+\bar{k} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} k+2\bar{k} & k \\ -k-2\bar{k} & k \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2k+4\bar{k} & 0 \\ 0 & 2k \end{bmatrix}$$

$$= \begin{bmatrix} k+2\bar{k} & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} \omega_+^2 m & 0 \\ 0 & \omega_-^2 m \end{bmatrix}$$

$$\sum_{i,k} m_{ik}\, \dot{x}_i\, \dot{x}_k = \dot{x}^T m\, \dot{x}$$

$$= \dot{\theta}^T \left(\Delta^T m \Delta\right) \dot{\theta}$$

$$= \dot{\theta}^T \underbrace{\Delta^T \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \Delta}_{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}} \dot{\theta}$$

$$= m\dot{\theta}_+^2 + m\dot{\theta}_-^2$$

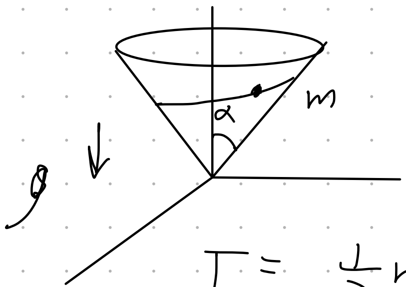
Thus,

$$L = \frac{1}{2} m \dot{\theta}_+^2 + \frac{1}{2} m \dot{\theta}_-^2 - \frac{1}{2}\omega_+^2 m \theta_+^2 - \frac{1}{2}\omega_-^2 m \theta_-^2$$

$$= \left(\frac{1}{2} m \dot{\theta}_+^2 - \frac{1}{2}\omega_+^2 m \theta_+^2\right) + \left(\frac{1}{2} m \dot{\theta}_-^2 - \frac{1}{2}\omega_-^2 m \theta_-^2\right)$$

$$= \left(\frac{1}{2} m \dot{\theta}_+^2 - \frac{1}{2} k_+ \theta_+^2\right) + \left(\frac{1}{2} m \dot{\theta}_-^2 - \frac{1}{2} k_- \theta_-^2\right)$$

Quiz #4



Find freq of small oscillations in the radial direction about a circle orbit $r = r_0$.

$$L = T - U$$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2)$$

$$U = mgz = mgr\cos\alpha$$

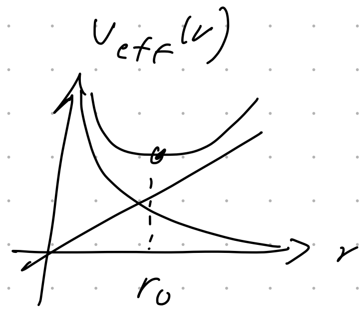
$$M_z = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\alpha\,\dot{\phi} = \text{const}$$

$$E = T + U = \text{const}$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2) + mgr\cos\alpha$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{M_z^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha$$

$$= \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$$



$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0}$$

$$= \frac{-M_z^2}{mr_0^3\sin^2\alpha} + mg\cos\alpha$$

$$M_z^2 = m^2 g r_0^3 \cos\alpha\sin^2\alpha$$

$$k = \left.\frac{d^2U_{eff}}{dr^2}\right|_{r=r_0} = \frac{3M_z^2}{mr_0^4\sin^2\alpha} = \frac{3mg\cos\alpha}{r_0}$$

$$\omega_r = \sqrt{\frac{H}{m}} = \sqrt{\frac{3g\cos\alpha}{r_0}}$$

Compare to ω_ϕ:

$$\omega_\phi = \dot{\phi}\Big|_{r=r_0}$$

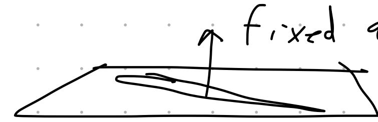
$$= \frac{M_z}{m r_0^2 \sin^2\alpha}$$

$$= \frac{\cancel{m}\sqrt{g}\, r_0^{3/2} \sqrt{\cos\alpha}\, \sin\alpha}{\cancel{m}\, r_0^2 \sin^2\alpha}$$

$$= \sqrt{\frac{g\cos\alpha}{r_0 \sin^2\alpha}}$$

Rigid body motion:

Undergrad physics (rotation in 2-d)



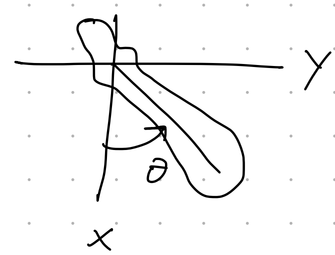
$\theta \,, \quad \omega = \dfrac{d\theta}{dt}$

$L = I\omega$

$T_{rot} = \frac{1}{2} I \omega^2$

$\tau = \dfrac{dL}{dt}$

top view



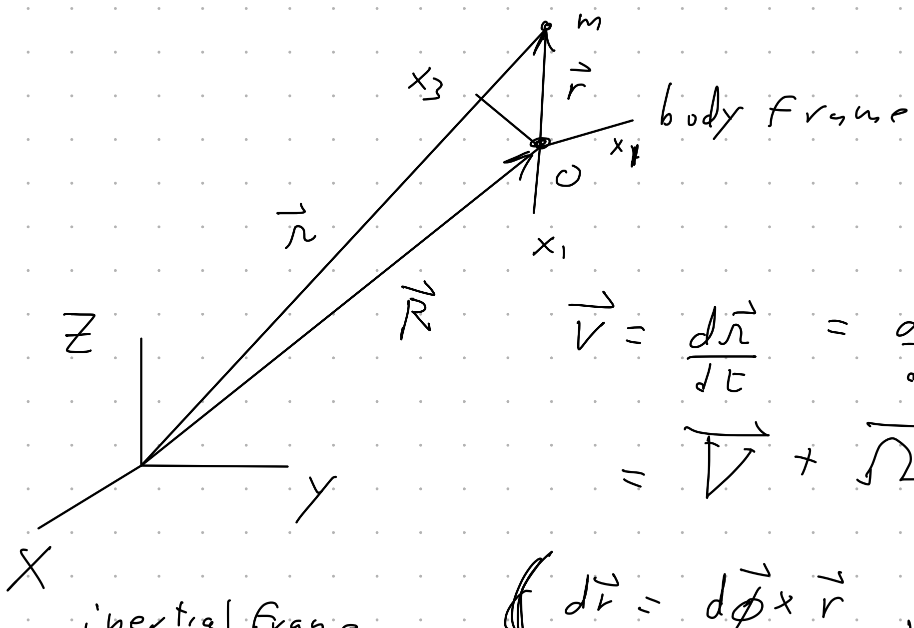
Generalization to 3-d:

$$\vec{\Omega} = \frac{d\vec{\phi}}{dt} \quad \Big(= \underbrace{\vec{\dot{\phi}} + \vec{\dot{\theta}} + \vec{\dot{\psi}}}_{\text{Euler angles}} \Big)$$

$$T_{rot} = \frac{1}{2} I_{jk} \Omega_j \Omega_k$$

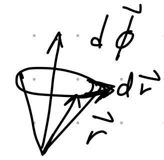
$$M_i = I_{ik} \Omega_k$$

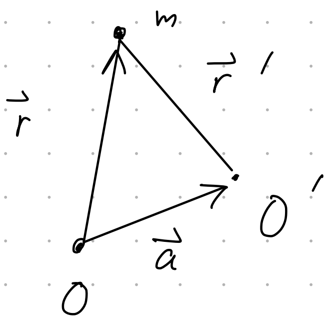
$$\vec{N} = \frac{d\vec{M}}{dt} \,, \quad \vec{F} = \frac{d\vec{P}}{dt} \,, \quad \vec{P} = \mu \vec{V}$$



$$\vec{v} = \frac{d\vec{\mathcal{R}}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}}{dt}$$

$$= \vec{V} + \vec{\Omega} \times \vec{r}$$

$\Big(d\vec{r} = d\vec{\phi} \times \vec{r} \,,$  $\Big)$



$$\vec r = \vec a + \vec r\,'$$
$$\begin{aligned}\vec v &= \vec V + \vec\Omega \times \vec r \\ &= \vec V + \vec\Omega \times (\vec a + \vec r\,') \\ &= \vec V + \vec\Omega \times \vec a + \vec\Omega \times \vec r\,' \\ &= \vec V\,' + \vec\Omega\,' \times \vec r\,'\end{aligned}$$

Thus, $\vec V\,' = \vec V + \vec\Omega \times \vec a$
$\vec\Omega\,' = \vec\Omega$

$\vec\Omega$: property of rigid body, not choice of frame within the rigid body

KE: (take O at COM)

$$\begin{aligned} T &= \sum_a \frac{1}{2} m_a |\vec v_a|^2 \\ &= \sum_a \frac{1}{2} m_a |\vec V + \vec\Omega \times \vec r_a|^2 \\ &= \sum_a \frac{1}{2} m_a \Big(|\vec V|^2 + 2\vec V \cdot (\vec\Omega \times \vec r_a) \\ &\qquad + (\vec\Omega \times \vec r_a)\cdot(\vec\Omega \times \vec r_a)\Big)\end{aligned}$$

$$\begin{aligned} 1^{st}\text{ term} &= \sum_a \frac{1}{2} m_a |\vec V|^2 \\ &= \frac{1}{2}\mu |\vec V|^2\end{aligned}$$

2^{nd} term $= \sum_a m_a \vec{V} \cdot (\vec{\Omega} \times \vec{r}_a)$

$$= \left(\sum_a m_a \vec{r}_a\right) \cdot (\vec{V} \times \vec{\Omega})$$

$$= \mu \underbrace{\vec{R}}_{\overset{\shortparallel}{0} \text{ (if O at com)}} \cdot (\vec{V} \times \vec{\Omega})$$

$$= 0$$

3^{rd} term $= \frac{1}{2} \sum_a m_a (\vec{\Omega} \times \vec{r}_a) \cdot (\vec{\Omega} \times \vec{r}_a)$

$$= \frac{1}{2} \sum_a m_a \vec{\Omega} \cdot (\vec{r}_a \times (\vec{\Omega} \times \vec{r}_a))$$

$$= \frac{1}{2} \sum_a m_a \vec{\Omega} \cdot (\vec{\Omega} |\vec{r}_a|^2 - \vec{r}_a (\vec{r}_a \cdot \vec{\Omega}))$$

$$= \frac{1}{2} \sum_a m_a \left(\Omega^2 |\vec{r}_a|^2 - (\vec{\Omega} \cdot \vec{r}_a)^2\right)$$

$$= \frac{1}{2} \sum_a m_a \Big(\delta_{jk} \Omega_j \Omega_k r_a^2$$

$$- \Omega_j \Omega_k x_{aj} x_{ak} \Big)$$

$$= \frac{1}{2} \sum_a m_a (r_a^2 \delta_{jk} - x_{aj} x_{ak}) \Omega_j \Omega_k$$

$$= \frac{1}{2} I_{jk} \Omega_j \Omega_k$$

summation over j, k implied.

$$I_{jk} = \sum_a m_a (r_a^2 \delta_{jk} - x_{aj} x_{ak})$$

Angular momentum (about com)

$$\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a$$

$$= \sum_a m_a \vec{r}_a \times \vec{V}_a$$

$$= \sum_a m_a \vec{r}_a \times (\cancelto{0}{\vec{V}} + \vec{\Omega} \times \vec{r}_a)$$

$$= \sum_a m_a \vec{r}_a \times (\vec{\Omega} \times \vec{r}_a)$$

$$= \sum_a m_a \left(\vec{\Omega}\, r_a^2 - \vec{r}_a (\vec{r}_a \cdot \vec{\Omega})\right)$$

$$M_i = \sum_a m_a \left(\Omega_i r_a^2 - x_{ai} x_{aj} \Omega_j\right)$$

$$= \sum_a m_a \left(\delta_{ij} r_a^2 - x_{ai} x_{aj}\right) \Omega_j$$

$$= I_{ij} \Omega_k$$

$$I_{ij} = \sum_a m_a (\delta_{ij} r_a^2 - x_{ai} x_{aj})$$

$$= \int_{\text{body}} \rho\, dV\, (\delta_{ij} r^2 - x_i x_j)$$

$I(\hat{n}) = I_{ij} n_i n_j$ = moment of inertia about $\vec{n}$

$$I_{11} = \int \rho\, dV (r^2 - x_1^2) = \int \rho\, dV (x_2^2 + x_3^2)$$

etc.

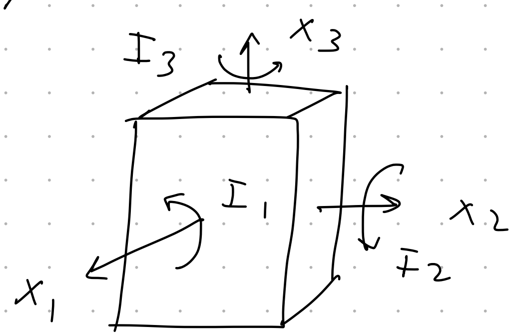
I_{ij} : real, symmetric matrix

$\Longrightarrow$ can diagonalize

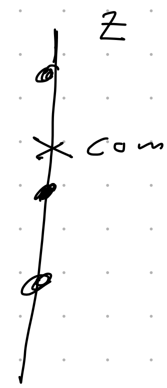
$$I_{ij} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

$(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ that diagonalize I are called "principle axes"

Symmetry $\rightarrow$ principle axes

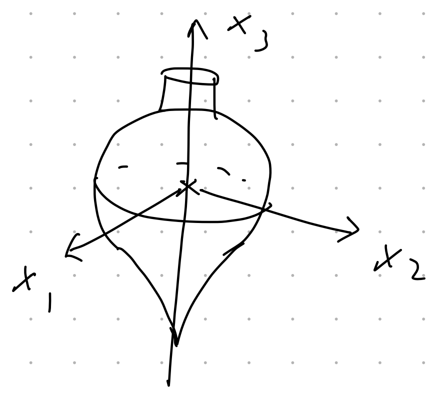


In general $I_1 \neq I_2 \neq I_3$



If $I_3 = 0$ $\longleftrightarrow$ "rotator"

If $I_1 = I_2$ $\Longleftrightarrow$ "symmetric top"



Directions of $\hat{x}_1, \hat{x}_2$ are arbitrary in the plane $\perp$ to x_3

$$I_3 = \int \rho \, dV \, (r^2 - x_3^2)$$
$$= \int \rho \, dV \, (x_1^2 + x_2^2)$$

$$I_1 = \int \rho \, dV \, (x_2^2 + x_3^2)$$
$$I_2 = \int \rho \, dV \, (x_3^2 + x_1^2)$$

suppose $I_1 = I_2 \equiv I$

$$I = \frac{1}{2}(I_1 + I_2)$$

$$= \frac{1}{2} \int \rho \, dV \, (2x_3^2 + x_1^2 + x_2^2)$$

$$= \int \rho \, dV \, x_3^2 + \frac{1}{2} \int \rho \, dV \, (x_1^2 + x_2^2)$$

$$= \frac{1}{2} I_3 + \int \rho \, dV \, x_3^2$$

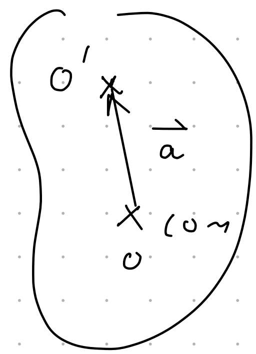
For simplicity, I will use x, y, z for (x_1, x_2, x_3)

w.r.t principle axes:

$$T = \frac{1}{2}(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2)$$

$$M_i = I_i \Omega_i$$

Change of origin: (generalized "parallel axis theorem")



$$\vec{r} = \vec{a} + \vec{r}\,'$$

$$\vec{r}\,' = \vec{r} - \vec{a}$$

$$r'^2 = r^2 + a^2 - 2\vec{a}\cdot\vec{r}$$

$$x_i' x_j' = (x_i - a_i)(x_j - a_j)$$

$$= x_i x_j + a_i a_j - a_i x_j - a_j x_i$$

$$I_{ij}' = \int \rho\, dV \left(\delta_{ij} r'^2 - x_i' x_j' \right)$$

$$= \int \rho\, dV \Big[\delta_{ij} (r^2 + a^2 - 2\vec{a}\cdot\vec{r})$$

$$- (x_i x_j + a_i a_j - a_i x_j - a_j x_i) \Big]$$

$$= \int \rho\, dV \left(\delta_{ij} r^2 - x_i x_j \right)$$

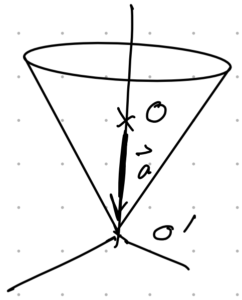
$$+ \underbrace{\int \rho\, dV}_{=M} \left(\delta_{ij} a^2 - a_i a_j \right)$$

$$- 2\vec{a} \cdot \underbrace{\int \rho\, dV\, \vec{r}}_{=0 \text{ (com)}} - a_i \underbrace{\int \rho\, dV\, x_j}_{=0}$$

$$- a_j \underbrace{\int \rho\, dV\, x_i}_{=0}$$

$$= I_{ij} + M\left(\delta_{ij} a^2 - a_i a_j \right)$$

Sec 32, Prob 2e



Cone

simpler to calculate I'_{ij} (about tip)

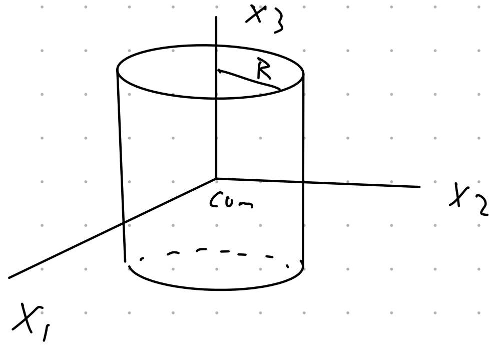
Then use

$$I'_{ij} = I_{ij} + \mu (\delta_{ij} a^2 - a_i a_j)$$

to find I_{ij} (about com).

Sec 32, Prob 2c

uniform cylinder, radius R, height h, mass μ



$$I_3 = \int \rho \, dV \, (r^2 - x_3^2)$$

$$= \int \rho \, dV \, (x^2 + y^2)$$

$$= \frac{\mu}{\pi R^2 h} \int s \, ds \, d\phi \, dz \; s^2$$

$$= \frac{\mu}{\pi R^2 h} \int_0^R s^3 ds \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz$$

$$= \frac{\mu}{\cancel{\pi} R^2 h} \cdot 2\cancel{\pi} h \; \frac{s^4}{4} \Big|_0^R$$

$$= \frac{1}{2} \mu R^2$$

Let $s = \sqrt{x^2 + y^2}$

(s, ϕ, z): cylindrical coords

$$dV = s \, ds \, d\phi \, dz$$

$$\rho = \frac{\mu}{\text{volume}} = \frac{\mu}{\pi R^2 h}$$

$$I \equiv I_1 = I_2$$

$$= \frac{1}{2} I_3 + \int \rho \, dV \, z^2$$

$$= \frac{1}{2}\left(\frac{1}{2} M R^2\right) + \frac{M}{\pi R^2 h} \int s \, ds \, d\phi \, dz \, z^2$$

$$= \frac{1}{4} M R^2 + \frac{M}{\pi R^2 h} \int_0^{2\pi} d\phi \int_0^R s \, ds \int_{-h/2}^{h/2} z^2 \, dz$$

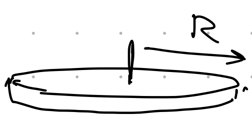
$$= \frac{1}{4} M R^2 + \frac{M}{\cancel{\pi} R^2 h} \cancel{2}\cancel{\pi} \left.\frac{s^2}{\cancel{2}}\right|_0^R \left.\frac{z^3}{3}\right|_{-h/2}^{h/2}$$

$$= \frac{1}{4} M R^2 + \frac{M}{\cancel{R^2} h} \cdot \cancel{R^2} \, \frac{2}{3} \, \frac{h^3}{8}$$

$$= \frac{1}{4} M R^2 + \frac{1}{12} M h^2$$

$$= \frac{1}{4} M \left(R^2 + \frac{1}{3} h^2\right)$$

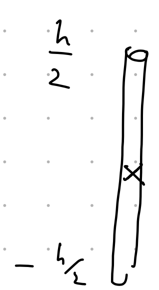
Limiting behavior:



thin disk ($h \to 0$)

$$I_3 = \frac{1}{2} M R^2$$

$$I_1 = I_2 = \frac{1}{4} M R^2$$



thin rod

($R \to 0$)

$$I_3 = 0$$

$$I_1 = I_2 = \frac{1}{12} M h^2$$

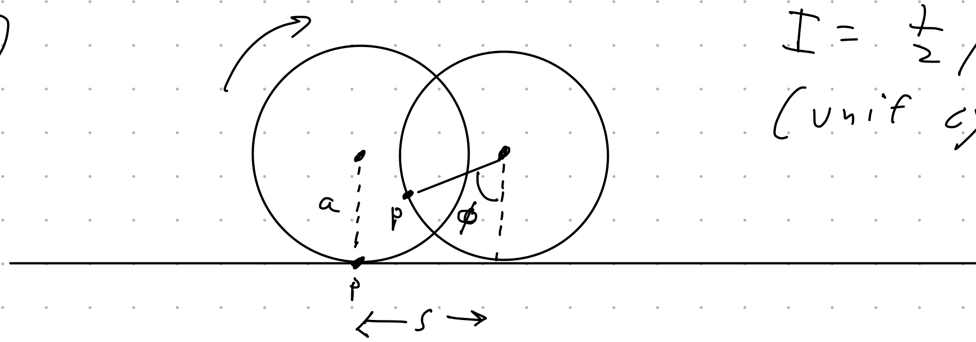
Examples: (sec 32)

$$T = \tfrac{1}{2}\mu V^2 + \tfrac{1}{2} I \Omega^2$$

$$I = \tfrac{1}{2}\mu a^2$$

(unif cylinder radius $= a$)

(1.)



$$s = a\phi$$

$$V = \dot{s} = a\dot{\phi}$$

$$\vec{V} = \vec{\Omega} \times \vec{a} \qquad \vec{\Omega} \text{ into page}$$

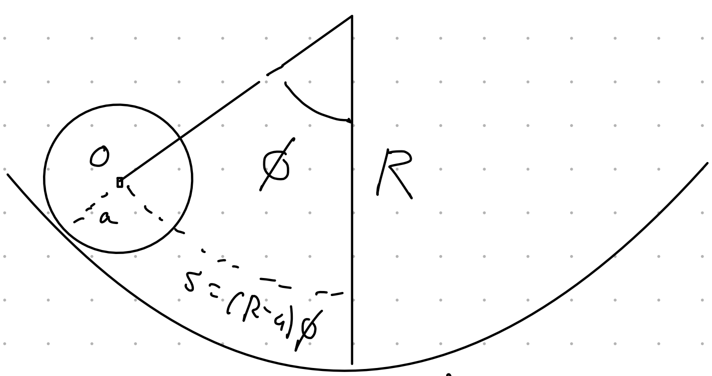
$$= \Omega a \text{ (to right)} \qquad \vec{a} = a\hat{n}$$

so $\Omega = \dot{\phi}$

$$\rightarrow \quad T = \tfrac{1}{2}\mu a^2 \dot{\phi}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}\mu a^2\right)\dot{\phi}^2$$

$$= \tfrac{3}{4}\mu a^2 \dot{\phi}^2$$

(2)



$$V = \dot{s}$$

$$= (R-a)\dot{\phi}$$

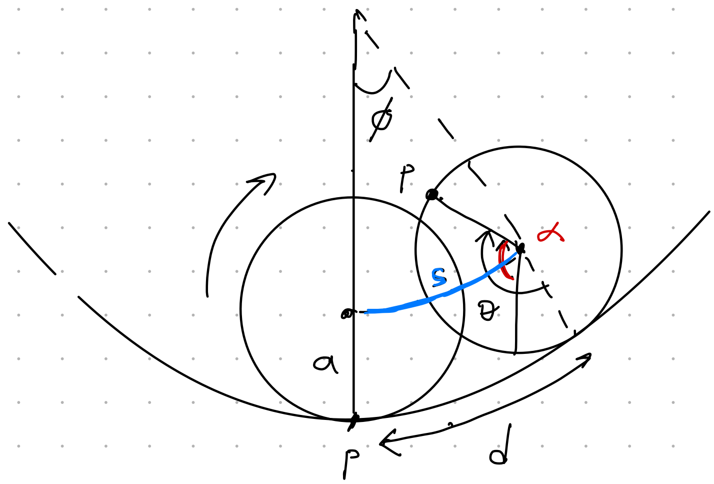
$$V = \Omega a$$

$$\longrightarrow \quad \Omega = \left(\frac{R-a}{a}\right)\dot{\phi}$$

$$T = \tfrac{1}{2}\mu (R-a)^2 \dot{\phi}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}\mu a^2\right)\left(\frac{R-a}{a}\right)^2 \dot{\phi}^2$$

$$= \tfrac{3}{4}\mu (R-a)^2 \dot{\phi}^2$$

Alternative calculation:



α : wrt vertical

θ : wrt normal

$\alpha = \theta - \phi$

$d = a\theta = R\phi \longrightarrow \theta = \frac{R}{a}\phi$

$s = (R-a)\phi$

$V = \dot{s} = (R-a)\dot{\phi}$

$V = a\Omega$

$\rightarrow \Omega = \frac{(R-a)}{a}\dot{\phi}$

NOTE:

$$
\begin{aligned}
\dot{\alpha} &= \dot{\theta} - \dot{\phi} \\
&= \frac{R}{a}\dot{\phi} - \dot{\phi} \\
&= \frac{(R-a)}{a}\dot{\phi} \\
&= \Omega
\end{aligned}
$$

EOMs:

$$\frac{d\vec{P}}{dt} = \vec{F} = \sum \vec{f}$$

$$\frac{d\vec{M}}{dt} = \vec{K} = \sum \vec{r} \times \vec{f}$$

proof:

$$L = \frac{1}{2}\mu |\vec{V}|^2 + \frac{1}{2} I_{ik}\Omega_i \Omega_k - U(\vec{r})$$

Generalized coords: $(\vec{R}, \vec{\phi})$

" velocities: $(\vec{V}, \vec{\Omega})$, $\vec{V} = \frac{d\vec{R}}{dt}$

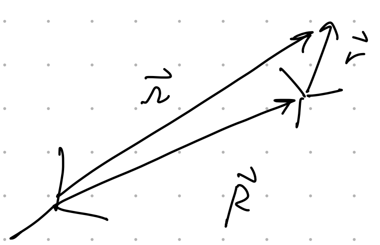
$$\vec{\Omega} = \frac{d\vec{\phi}}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{V}}\right) = \frac{\partial L}{\partial \vec{R}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{\Omega}}\right) = \frac{\partial L}{\partial \vec{\phi}}$$

$$\frac{\partial L}{\partial \vec{V}} = \mu \vec{V} = \vec{P}$$

$$\frac{\partial L}{\partial \vec{\Omega}} = I_{ik}\Omega_k = M_i = \vec{M}$$



$$\delta\vec{r} = \delta\vec{R} + \delta\vec{r}$$

$$= \delta\vec{R} + \delta\vec{\phi} \times \vec{r}$$

$$\delta U = \sum \frac{\partial U}{\partial \vec{r}} \cdot \left(\delta\vec{R} + \delta\vec{\phi} \times \vec{r}\right)$$

$$\frac{\partial U}{\partial \vec{r}} = -\vec{f}$$

$$\delta U = -\sum \vec{f} \cdot (\delta \vec{R} + \delta \vec{\phi} \times \vec{r})$$

$$= -\delta \vec{R} \cdot \sum \vec{f} \cdot \quad \delta \vec{\phi} \cdot \sum (\vec{r} \times \vec{f})$$

thus, $\frac{\partial L}{\partial \vec{R}} = -\frac{\partial U}{\partial \vec{R}} = \sum \vec{f}$

$$\frac{\partial L}{\partial \vec{\phi}} = -\frac{\partial U}{\partial \vec{\phi}} = \sum \vec{r} \times \vec{f}$$

$$\rightarrow \quad \frac{d\vec{P}}{dt} = \sum \vec{f} \equiv \vec{F}$$

$$\frac{d\vec{M}}{dt} = \sum \vec{r} \times \vec{f} \equiv \vec{K}$$

Euler's equation for rigid-body motion:

$\rightarrow$ above equations in RB frame

use $\underbrace{\frac{d\vec{A}}{dt}}_{\substack{\text{wrt} \\ \text{inertial} \\ \text{frame}}} = \underbrace{\frac{d'\vec{A}}{dt}}_{\substack{\text{wrt} \\ \text{rigid} \\ \text{body} \\ \text{frame}}} + \vec{\Omega} \times \vec{A}$

$$\left(\frac{d'\vec{A}}{dt}\right)_i = \frac{dA_i}{dt}, \quad \text{where } \vec{A} = A_i \hat{x}_i$$

$$\frac{d\vec{A}}{dt} = \frac{d}{dt}\left(\sum_i A_i \hat{x}_i\right)$$

$$= \sum_i \frac{dA_i}{dt}\hat{x}_i + \sum_i A_i \frac{d\hat{x}_i}{dt}$$

$$= \frac{d'\vec{A}}{dt} + \vec{\Omega}\times\vec{A}$$

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$= \frac{d'\vec{p}}{dt} + \vec{\Omega}\times\vec{p}$$

$$F_1 = \frac{dP_1}{dt} + (\vec{\Omega}\times\vec{p})_1$$

$$= \dot{P}_1 + \Omega_2 P_3 - \Omega_3 P_2$$

$$= \mu\dot{V}_1 + \mu(\Omega_2 V_3 - \Omega_3 V_2)$$

and similar equations for F_2, F_3

$$\vec{K} = \frac{d\vec{m}}{dt}$$

$$= \frac{d'\vec{m}}{dt} + \vec{\Omega}\times\vec{m}$$

$$K_1 = \dot{m}_1 + \Omega_2 m_3 - \Omega_3 m_2$$

$$= I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_2)$$

and similiar equations for K_2, K_3

Summary:

$$F_1 = m\dot{V}_1 + m(\Omega_2 V_3 - \Omega_3 V_2)$$
$$K_1 = I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_2)$$

+ cyclic permutations $1 \to 2 \to 3$

Torque free motion: ($\vec{K} = 0 \to \vec{M} = $ const)

$$0 = I_1\dot{\Omega}_1 + \Omega_2\Omega_3(I_3 - I_2)$$
$$0 = I_2\dot{\Omega}_2 + \Omega_3\Omega_1(I_1 - I_3)$$
$$0 = I_3\dot{\Omega}_3 + \Omega_1\Omega_2(I_2 - I_1)$$

Special case: torque free with constant $\vec{\Omega}$

$$\frac{d\vec{\Omega}}{dt} = \frac{d'\vec{\Omega}}{dt} + \vec{\Omega}\times\vec{\Omega}$$

so $\dfrac{d\vec{\Omega}}{dt} = \dfrac{d'\vec{\Omega}}{dt} \to \dot{\Omega}_1 = 0$, etc.

Thus,
$$0 = \Omega_2\Omega_3(I_3 - I_2)$$
$$0 = \Omega_3\Omega_1(I_1 - I_3)$$
$$0 = \Omega_1\Omega_2(I_2 - I_1)$$

$$\left.\begin{aligned} \Omega_1 &= \text{const}, \ \Omega_2 = \Omega_3 = 0 \\ \Omega_2 &= \text{const}, \ \Omega_3 = \Omega_1 = 0 \\ \Omega_3 &= \text{const}, \ \Omega_1 = \Omega_2 = 0 \end{aligned}\right\} \text{rotation about a principal axis.}$$

Suppose $I_1 < I_2 < I_3$

Turns out that $\Omega_1 = \text{const}$, $\Omega_2 = \Omega_3 = 0$

$\Omega_3 = \text{const}$, $\Omega_1 = \Omega_2 = 0$

are stable but $\Omega_2 = \text{const}$, $\Omega_1 = \Omega_3 = 0$ is unstable.

Proof:

$$\Omega_1 = \text{const} + \epsilon_1$$
$$\Omega_2 = \epsilon_2$$
$$\Omega_3 = \epsilon_3$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ are small time dependent quantities.

Then $$\frac{d\Omega_1}{dt} = -\frac{\Omega_2 \Omega_3}{I_1}(I_3 - I_2)$$

$$\rightarrow \frac{d}{dt}(\text{const} + \epsilon_1) = -\frac{\epsilon_2 \epsilon_3}{I_1}(I_3 - I_2)$$

$$\frac{d\epsilon_1}{dt} \approx 0$$

so $\epsilon_1 = \text{const}$

Also $$\frac{d\Omega_2}{dt} = -\frac{\Omega_3 \Omega_1}{I_2}(I_1 - I_3)$$

$$\frac{d\epsilon_2}{dt} = -\Omega_1 \epsilon_3 \left(\frac{I_1 - I_3}{I_2}\right)$$

$$\frac{d\Omega_3}{dt} = -\frac{\Omega_1\Omega_2}{I_3}(I_2 - I_1)$$

$$\frac{d\epsilon_3}{dt} = -\frac{\Omega_1(I_2 - I_1)}{I_3}\epsilon_2$$

so
$$\dot{\epsilon}_2 = -\Omega_1\frac{(I_1 - I_3)}{I_2}\epsilon_3$$
$$\dot{\epsilon}_3 = -\Omega_1\frac{(I_2 - I_1)}{I_3}\epsilon_2$$

$$\rightarrow \quad \ddot{\epsilon}_2 = -\Omega_1\frac{(I_1 - I_3)}{I_2}\dot{\epsilon}_3$$
$$= +\underbrace{\Omega_1^2\frac{(I_1 - I_3)(I_2 - I_1)}{I_2 I_3}}_{<0}\epsilon_2$$

so $\epsilon_2(t)$ = sinusoidal

$$\rightarrow \quad \ddot{\epsilon}_3 = -\Omega_1\frac{(I_2 - I_1)}{I_3}\dot{\epsilon}_2$$
$$= +\underbrace{\Omega_1^2\frac{(I_2 - I_1)(I_1 - I_3)}{I_2 I_3}}_{<0}\epsilon_3$$

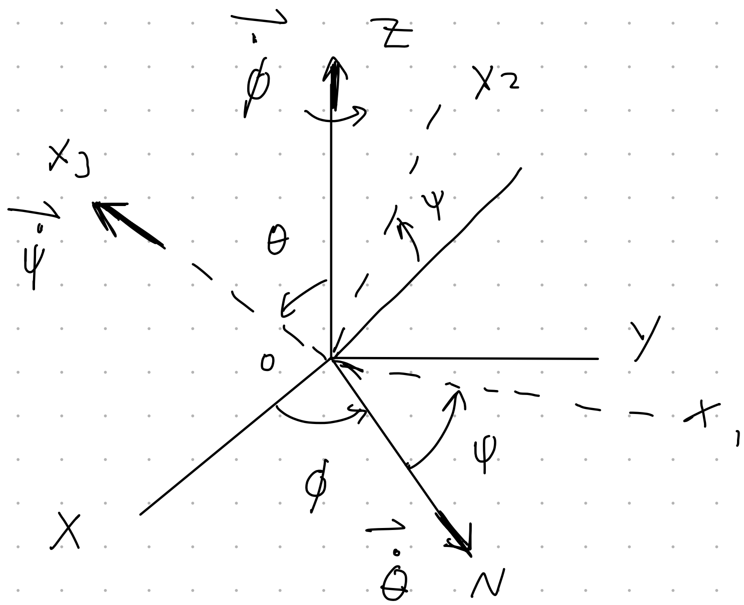
so $\epsilon_3(t)$ = sinusoidal

Same behavior for Ω_3 = const, but for Ω_2 = const have

$$\ddot{\epsilon}_1 = \underbrace{+\Omega_2^2 \frac{(I_3 - I_2)(I_2 - I_1)}{I_3 I_1}}_{>0 \,=\, \lambda^2} \epsilon_1$$

so $e^{\pm\lambda t}$ solutions with $e^{+\lambda t}$ growing with time.

Euler angles:



$$\vec{\dot\psi} = \dot\psi\, \hat{x}_3$$

$$\vec{\dot\phi} = \dot\phi \left(\cos\theta\, \hat{x}_3 + \sin\theta \cos\psi\, \hat{x}_2 + \sin\theta \sin\psi\, \hat{x}_1 \right)$$

$$\vec{\dot\theta} = \dot\theta \left(\cos\psi\, \hat{x}_1 - \sin\psi\, \hat{x}_2 \right)$$

$$\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta} + \vec{\dot\psi}$$

$$= \left(\dot\theta \cos\psi + \dot\phi \sin\theta \sin\psi \right) \hat{x}_1$$

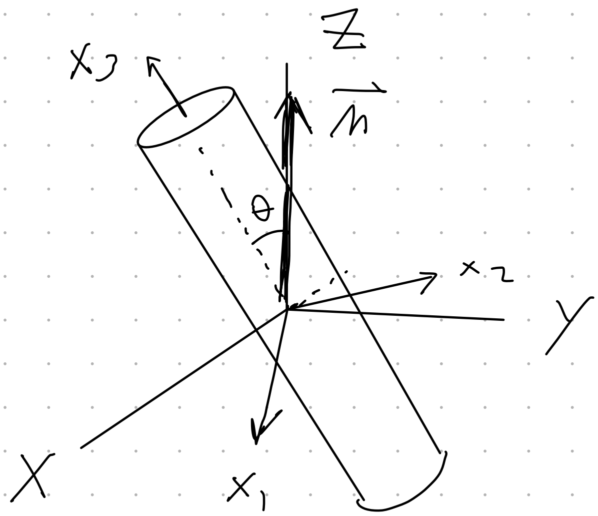
$$+ \left(-\dot\theta \sin\psi + \dot\phi \sin\theta \cos\psi \right) \hat{x}_2$$

$$+ \left(\dot\phi \cos\theta + \dot\psi \right) \hat{x}_3$$

If $\hat{x}_1$ along ON, then $\psi = 0$

$$\Omega_1 = \dot\theta, \quad \Omega_2 = \dot\phi \sin\theta, \quad \Omega_3 = \dot\phi \cos\theta + \dot\psi$$

Torque-free motion of a symmetric top:



$\hat{x}_1$ in XY plane

$I_1 = I_2$

$$\vec{M} = M\cos\theta\,\hat{x}_3 + M\sin\theta\,\hat{x}_2$$

$$M_1 = I_1\Omega_1\,,\quad M_2 = I_2\Omega_2\,,\quad M_3 = I_3\Omega_3$$

For $\psi = 0$,
$$\Omega_1 = \dot{\theta}$$
$$\Omega_2 = \dot{\phi}\sin\theta$$
$$\Omega_3 = (\dot{\phi}\cos\theta + \dot{\psi})$$

Thus,
$$M_1 = 0 = I_1\Omega_1 = I_1\dot{\theta}$$
$$\rightarrow \dot{\theta} = 0 \rightarrow \boxed{\theta = \theta_0}$$

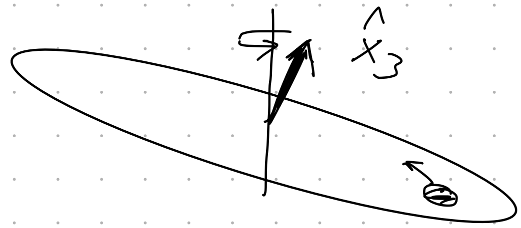
$$M_2 = M\cancel{\sin\theta} = I_1\dot{\phi}\cancel{\sin\theta} \rightarrow \dot{\phi} = \frac{M}{I_1} = \text{const}$$
$$\boxed{\phi = at + b}$$

$$M_3 = M\cos\theta = I_3\Omega_3$$

$$\rightarrow \Omega_3 = \frac{M\cos\theta_0}{I_3} = \text{const} = \dot{\phi}\cos\theta_0 + \dot{\psi}$$

$$\rightarrow \boxed{\dot{\psi} = \text{const}}$$

Feynman dinner plate



wobble freq : $\dot{\phi}$

spin freq : Ω_3 (component of $\vec{\Omega}$ along $\hat{x}_3$)

Feynman said: "spin = 2x wobble freq"

<u>Check</u>: $\dot{\phi} = \dfrac{M}{I_1}$

$$\Omega_3 = \frac{M \cos\theta_0}{I_3}$$

$\theta_0 \approx 0$

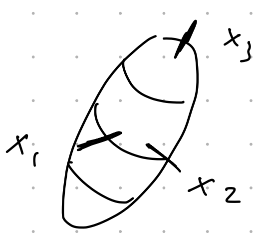
$$\rightarrow \frac{\Omega_3}{\dot{\phi}} = \frac{\cancel{M}\cos\theta_0}{I_3}\,\frac{I_1}{\cancel{M}} \approx \frac{I_1}{I_3}$$

<u>For dinner plate:</u> (this disk)

$$\left.\begin{array}{l} I_3 = \frac{1}{2} M R^2 \\ I_1 = I_2 = \frac{1}{4} M R^2 \end{array}\right\} \quad \frac{I_1}{I_2} = \boxed{\frac{1}{2}} \;!!$$

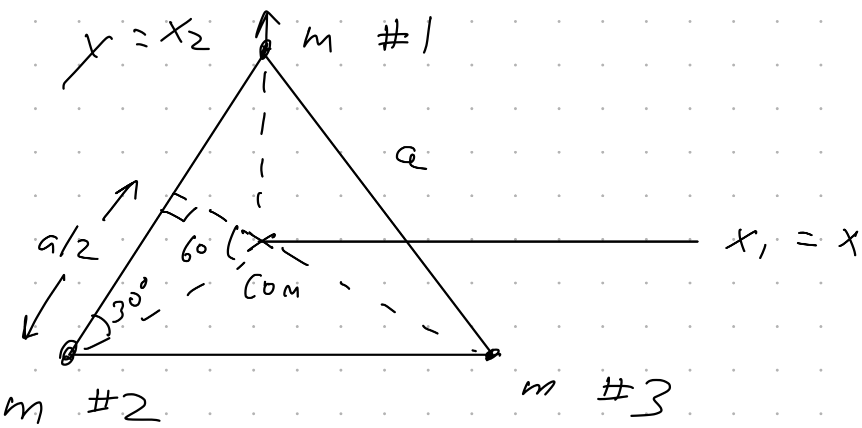
<u>For football:</u>

$I_3 < I_2$ can have spin freq larger than wobble freq.



Quiz # 5:

Calculate principal moments of inertia for three equal masses m at the corners of an equilateral triangle of side $= a$



$z = x_3$: out of page

mass points do not have z components

$$I_1 = \sum_a m_a (r_a^2 - x_a^2)$$

$$= \sum_a m_a (y_a^2 + \cancel{z_a^2})$$

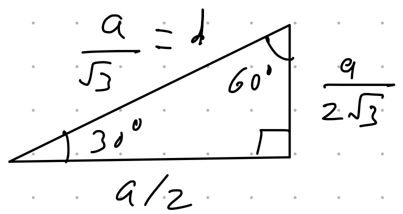
$$= \sum_a m_a y_a^2$$

$$\rightarrow \quad I_2 = \sum_a m_a x_a^2$$

$$I_3 = I_1 + I_2$$

But $I_1 = I_2 = I \quad \longrightarrow \quad I_3 = 2I$

$$\longrightarrow \quad I = \frac{1}{2} I_3$$



$$\sin 30^\circ = \frac{1}{2}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{o}{d}$$

$$\rightarrow \quad d = \frac{o}{\sqrt{3}/2} = \frac{a/2}{\sqrt{3}/2} = \boxed{\frac{a}{\sqrt{3}}}$$

$$I_3 = \sum_a m_a \left(x_a^2 + y_a^2\right) = \sum_a m_a \left(\frac{a}{\sqrt{3}}\right)^2 = 3m\,\frac{a^2}{3} = \boxed{ma^2}$$

$$I_1 = I_2 = \boxed{\frac{1}{2}\,m a^2}$$

Alternative approach:

#1: $x_1 = 0, \quad y_1 = \dfrac{a}{\sqrt{3}}$

#2: $x_2 = -a/2, \quad y_2 = -\dfrac{a}{2\sqrt{3}}$

#3: $x_3 = a/2, \quad y_3 = \dfrac{a}{2\sqrt{3}}$

$$I_1 = \sum_a m_a y_a^2$$

$$= m\left(\left(\frac{a}{\sqrt{3}}\right)^2 + \left(\frac{-a}{2\sqrt{3}}\right)^2 + \left(\frac{a}{2\sqrt{3}}\right)^2 \right)$$

$$= ma^2 \left(\frac{1}{3} + \frac{1}{4\cdot 3} + \frac{1}{4\cdot 3} \right)$$

$$= \frac{1}{12} ma^2 \left(4 + 1 + 1 \right)$$

$$= \boxed{\frac{1}{2} ma^2}$$

$$I_2 = \sum_a m_a x_a^2$$

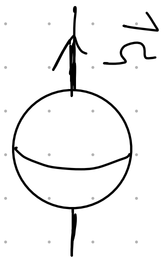
$$= m\left(0^2 + \left(-\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2 \right)$$

$$= ma^2 \left(\frac{1}{4} + \frac{1}{4} \right)$$

$$= \boxed{\frac{1}{2} ma^2}$$

Where is $\vec{M}$?

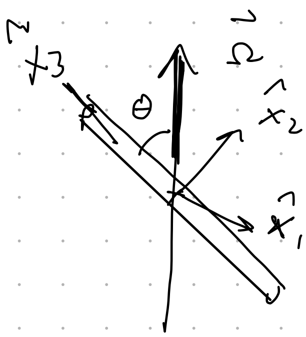
(1.)



$$I_1 = I_2 = I_3$$

$$\vec{M} = I\vec{\Omega}$$

(2.)



(in XY plane)

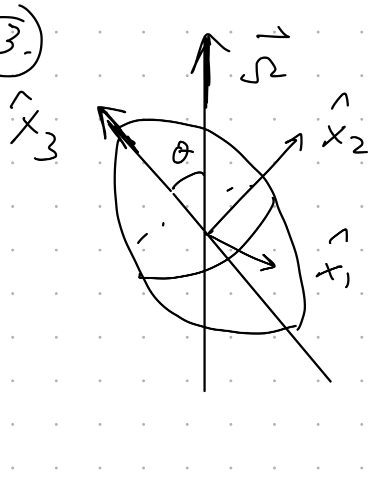
$$\Omega_1 = 0$$
$$\Omega_2 = \Omega \sin\theta$$
$$\Omega_3 = \Omega \cos\theta$$

thin rod

$$M_1 = I_1 \Omega_1 = I_1 \cdot 0 = 0$$
$$M_2 = I_2 \Omega_2 = I_2 \Omega \sin\theta$$
$$M_3 = I_3 \Omega_3 = 0 \cdot \Omega \cos\theta = 0$$

so $\vec{M} = I_2 \Omega \sin\theta \; \hat{x}_2$

(3.)



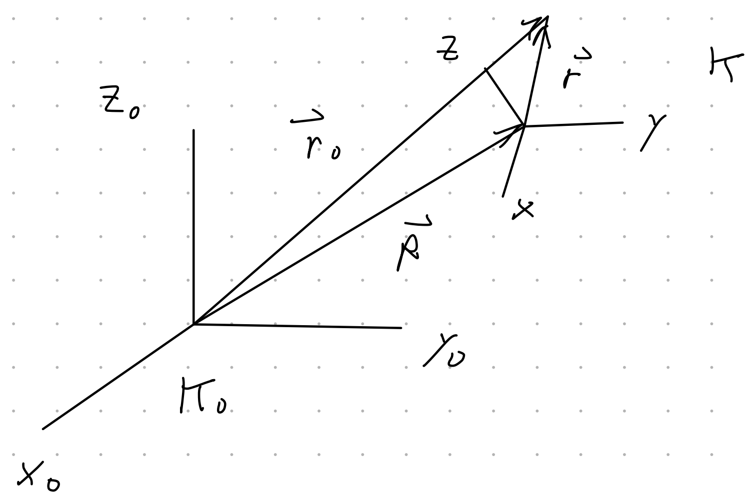
$$\Omega_1 = 0$$
$$\Omega_2 = \Omega \sin\theta$$
$$\Omega_3 = \Omega \cos\theta$$

$\rightarrow$ $M_1 = 0$

$$M_2 = I_2 \Omega \sin\theta$$
$$M_3 = I_3 \Omega \cos\theta$$

so $\vec{M} = M_2 \hat{x}_2 + M_3 \hat{x}_3$

Motion in a non-inertial reference frame:



$\vec{r}$: need not be fixed in K.

$$\vec{r}_0 = \vec{R} + \vec{r}$$

$$\vec{v}_0 = \left.\frac{d\vec{r}_0}{dt}\right|_0 = \left.\frac{d\vec{R}}{dt}\right|_0 + \left.\frac{d\vec{r}}{dt}\right|_0$$

$$= \vec{V} + \left(\frac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{r}\right)$$ holds $\forall$ vectors

$$= \vec{V} + \vec{v} + \vec{\Omega}\times\vec{r}$$

$$\vec{a}_0 = \left.\frac{d\vec{v}_0}{dt}\right|_0$$

$$= \left.\frac{d\vec{V}}{dt}\right|_0 + \left.\frac{d\vec{v}}{dt}\right|_0 + \left.\frac{d\vec{\Omega}}{dt}\right|_0\times\vec{r}$$

$$+ \vec{\Omega}\times\left.\frac{d\vec{r}}{dt}\right|_0$$

$$= \vec{W} + \left(\frac{d\vec{v}}{dt} + \vec{\Omega}\times\vec{v}\right) + \dot{\vec{\Omega}}\times\vec{r}$$

$$+ \vec{\Omega}\times\left(\vec{v} + \vec{\Omega}\times\vec{r}\right)$$

$$= \vec{W} + \vec{a} + 2\vec{\Omega}\times\vec{v} + \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

Newton's 2nd law: (valid only in inertial frames)

$$m\vec{a}_0 = \vec{F}$$

$$\rightarrow \quad m\vec{W} + m\vec{a} + 2m\vec{\Omega}\times\vec{v} + m\dot{\vec{\Omega}}\times\vec{r} + m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = \vec{F}$$

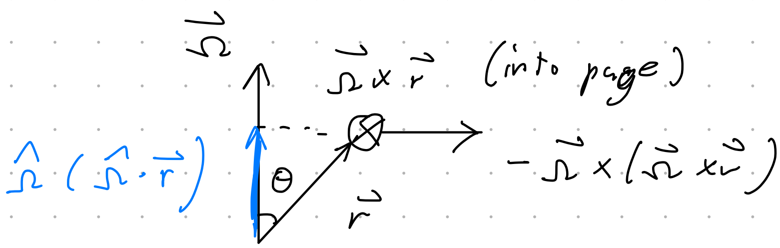
$$\rightarrow \quad \boxed{m\vec{a} = \vec{F} - m\vec{W} - m\dot{\vec{\Omega}}\times\vec{r} - 2m\vec{\Omega}\times\vec{v} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$$

linear acceleration

rotational acceleration

Coriolis

Centrifugal



$$|\vec{\Omega}\times\vec{r}| = \Omega r \sin\theta = \Omega r_\perp$$

$$\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = \vec{\Omega}(\vec{\Omega}\cdot\vec{r}) - \vec{r}\,\Omega^2$$

$$= \Omega^2\left(\hat{\Omega}(\hat{\Omega}\cdot\vec{r}) - \vec{r}\right)$$

$$= -\Omega^2\,\vec{r}_\perp$$

points away from $\vec{\Omega}$ with magnitude $r\sin\theta$

Thus, $\vec{F}_{centrifugal} = -m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = m\Omega^2\,\vec{r}_\perp$

Derivation from Lagrangian:

$$L = T_0 - U(\vec{r}_0)$$
$$= \frac{1}{2} m |\vec{v}_0|^2 - U(\vec{r}_0) \qquad \vec{r}_0 = \vec{R} + \vec{r}$$
$$= \frac{1}{2} m (\vec{V} + \vec{v} + \vec{\Omega} \times \vec{r}) \cdot (\vec{V} + \vec{v} + \vec{\Omega} \times \vec{r})$$
$$- U(\vec{r})$$

$$= \frac{1}{2} m |\vec{V}|^2 + \frac{1}{2} m |\vec{v}|^2 + \frac{1}{2} m (\vec{\Omega} \times \vec{r}) \cdot (\vec{\Omega} \times \vec{r})$$
$$+ m \vec{V} \cdot \vec{v} + m \vec{V} \cdot (\vec{\Omega} \times \vec{r}) + m \vec{v} \cdot (\vec{\Omega} \times \vec{r})$$

Now:

(i) $\frac{1}{2} m |\vec{V}|^2$ = prescribed function of time (ignore)

(ii) $m \vec{V} \cdot \vec{v} + m \vec{V} \cdot (\vec{\Omega} \times \vec{r}) = m \vec{V} \cdot (\vec{v} + \vec{\Omega} \times \vec{r})$
$$= m \vec{V} \cdot \left.\frac{d\vec{r}}{dt}\right|_0$$
$$= m \left.\frac{d}{dt}\right|_0 (\vec{V} \cdot \vec{r}) - m \left.\frac{d\vec{V}}{dt}\right|_0 \cdot \vec{r}$$
$$= \underbrace{\frac{d}{dt}(m \vec{V} \cdot \vec{r})}_{\text{total time derivative (ignore)}} - m \vec{W} \cdot \vec{r}$$

$\left.\frac{d}{dt}\right|_0 = \frac{d}{dt}$ for a scalar thus, ignoring total time derivatives:

$$L = \frac{1}{2}m|\vec{v}|^2 - m\vec{W}\cdot\vec{r} + \frac{1}{2}m(\vec{\Omega}\times\vec{r})\cdot(\vec{\Omega}\times\vec{r})$$
$$+ m\vec{v}\cdot(\vec{\Omega}\times\vec{r}) - U(\vec{r})$$

so we need to include extra "potential" terms:

$$-m\vec{W}\cdot\vec{r} + \frac{1}{2}m(\vec{\Omega}\times\vec{r})\cdot(\vec{\Omega}\times\vec{r}) + m\vec{v}\cdot(\vec{\Omega}\times\vec{r})$$

EOMs:

$$\delta L = m\vec{v}\cdot\delta\vec{v} - m\vec{W}\cdot\delta\vec{r} + m(\vec{\Omega}\times\vec{r})\cdot(\vec{\Omega}\times\delta\vec{r})$$
$$+ m\delta\vec{v}\cdot(\vec{\Omega}\times\vec{r}) + m\vec{v}\cdot(\vec{\Omega}\times\delta\vec{r}) - \frac{\partial U}{\partial\vec{r}}\cdot\delta\vec{r}$$

$$= m\vec{v}\cdot\delta\vec{v} - m\vec{W}\cdot\delta\vec{r} + m\delta\vec{r}\cdot((\vec{\Omega}\times\vec{r})\times\vec{\Omega})$$
$$+ m\delta\vec{v}\cdot(\vec{\Omega}\times\vec{r}) + m\delta\vec{r}\cdot(\vec{v}\times\vec{\Omega}) - \frac{\partial U}{\partial\vec{r}}\cdot\delta\vec{r}$$

$$= m\delta\vec{v}\cdot(\vec{v}+\vec{\Omega}\times\vec{r})$$
$$+ \delta\vec{r}\cdot\left(-m\vec{W} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) - m(\vec{\Omega}\times\vec{v}) - \frac{\partial U}{\partial\vec{r}}\right)$$

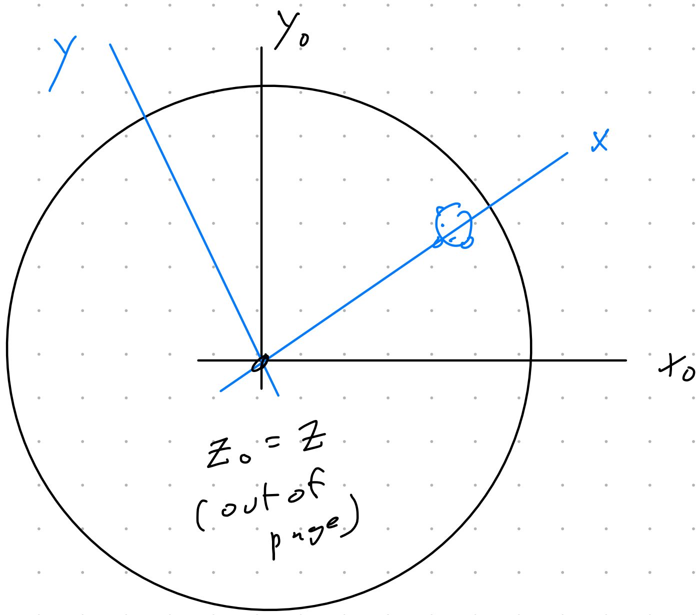
so

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\vec{v}}\right) = \frac{\partial L}{\partial\vec{r}}$$

$$\frac{d}{dt}\left(m(\vec{v}+\vec{\Omega}\times\vec{r})\right) = -\frac{\partial U}{\partial\vec{r}} - m\vec{W} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) - m(\vec{\Omega}\times\vec{v})$$

$$\Rightarrow \boxed{m\vec{a} = -\frac{\partial U}{\partial\vec{r}} - m\vec{W} - m\dot{\vec{\Omega}}\times\vec{r} - 2m(\vec{\Omega}\times\vec{v}) - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$$

Example: Merry go-round (assume $\vec{\Omega} = \Omega \hat{z}_0$)
$\Omega = \text{const}$



Describe motion of a "hockey puck" w/t rotating frame. Assume frictionless surface and no external force.

Thus,

$$m\vec{a} = -2m(\vec{\Omega} \times \vec{v}) - m\,\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$\left.\begin{aligned} \vec{r} &= x\hat{x} + y\hat{y} \\ \vec{v} &= \dot{x}\hat{x} + \dot{y}\hat{y} \\ \vec{a} &= \ddot{x}\hat{x} + \ddot{y}\hat{y} \end{aligned}\right\} \quad \text{no } \hat{z} \text{ component}$$

$$\vec{\Omega} = \Omega \hat{z}_0 = \Omega z$$

$$\begin{aligned} \vec{\Omega} \times \vec{r} &= \Omega\, \hat{z} \times (x\hat{x} + y\hat{y}) \\ &= \Omega(x\hat{y} - y\hat{x}) \end{aligned}$$

$$\begin{aligned} \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) &= \Omega^2\, \hat{z} \times (x\hat{y} - y\hat{x}) \\ &= \Omega^2(-x\hat{x} - y\hat{y}) \\ &= -\Omega^2 \vec{r} \end{aligned}$$

$$\vec{\Omega} \times \vec{v} = \Omega \hat{z} \times (\dot{x}\hat{x} + \dot{y}\hat{y})$$
$$= \Omega(\dot{x}\hat{y} - \dot{y}\hat{x})$$

Thus,

$$m\ddot{x} = +2m\dot{y} + m\Omega^2 x$$
$$m\ddot{y} = -2m\dot{x} + m\Omega^2 y$$

$$\rightarrow \quad \ddot{x} = 2\dot{y} + \Omega^2 x$$
$$\ddot{y} = -2\dot{x} + \Omega^2 y$$

<u>Define:</u> $\xi = x + iy$
$$\dot{\xi} = \dot{x} + i\dot{y}$$
$$\ddot{\xi} = \ddot{x} + i\ddot{y}$$

$$\rightarrow \ddot{\xi} = 2(\dot{y} - i\dot{x}) + \Omega^2(x + iy)$$
$$= -2i(\dot{x} + i\dot{y}) + \Omega^2(x + iy)$$
$$= -2i\dot{\xi} + \Omega^2 \xi$$

so $\boxed{\ddot{\xi} + 2i\dot{\xi} - \Omega^2 \xi = 0}$

<u>Soln:</u> guess: $\xi = Ae^{i\lambda t}$

$$\rightarrow \ddot{\xi} = -\lambda^2 \xi, \quad \dot{\xi} = i\lambda \xi$$

Thus, $-\lambda^2 \xi - 2\lambda \xi - \Omega^2 \xi = 0$

$$+\lambda^2 + 2\lambda + \Omega^2 = 0$$

$$(\lambda + \Omega)^2 = 0$$

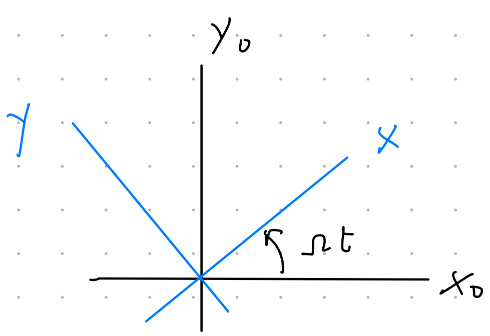
$$\lambda = -\Omega \quad (\text{double root})$$

General sol'n:

$$\xi(t) = (A + Bt)\, e^{-i\Omega t}$$

complex constants (determined by I.C.s)

Relate motion in inertial and non-inertial frames.



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\begin{aligned} \xi &= x + iy \\ &= (x_0 \cos\Omega t + y_0 \sin\Omega t) + i(-x_0 \sin\Omega t + y_0 \cos\Omega t) \\ &= (x_0 + i y_0)(\cos\Omega t - i \sin\Omega t) \end{aligned}$$

$$\xi = (x_0 + i y_0)\, e^{-i\Omega t}$$
$$= \xi_0\, e^{-i\Omega t}$$

compare to $\xi = (A + Bt)\, e^{-i\Omega t}$

$\rightarrow \quad \xi_0 \equiv x_0 + i y_0 = A + Bt$

so motion in inertial frame is linear.

$$x_0 + i y_0 = (a_1 + i a_2) + (b_1 + i b_2)\, t$$
$$= (a_1 + b_1 t) + i (a_2 + b_2 t)$$

$$\rightarrow \boxed{\begin{aligned} x_0 &= a_1 + b_1 t \\ y_0 &= a_2 + b_2 t \end{aligned}}$$

Specific example:

$x(0) = R$, $\quad y(0) = 0$
$\dot{x}(0) = -V_i$, $\quad \dot{y}(0) = 0$

Thus, $\xi(0) = x(0) + i y(0) = (A + B\cdot 0)\, \cancel{e^{-i\Omega\cdot 0}}^{\,1}$
$$R + i\cdot 0 = A$$
$$\boxed{A = R}$$

$$\dot{\xi}(0) = \dot{x}(0) + i\dot{y}(0) = B\, \cancel{e^{-i\Omega 0}}^{\,1} + (A + B\cdot 0)\, (-i\Omega)\cdot 1$$
$$= -V_i + i\, 0 = B - i\Omega A$$

$$\boxed{B = i\Omega R - v_i}$$

Thus,

$$x_0(0) = a_1 = R$$

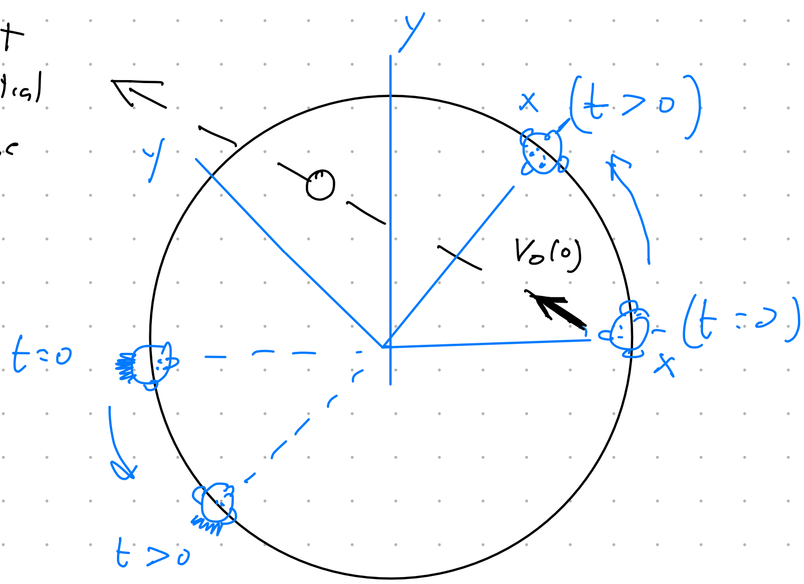
$$y_0(0) = a_2 = 0$$

$$\dot{x}_0(0) = b_1 = -v_i$$

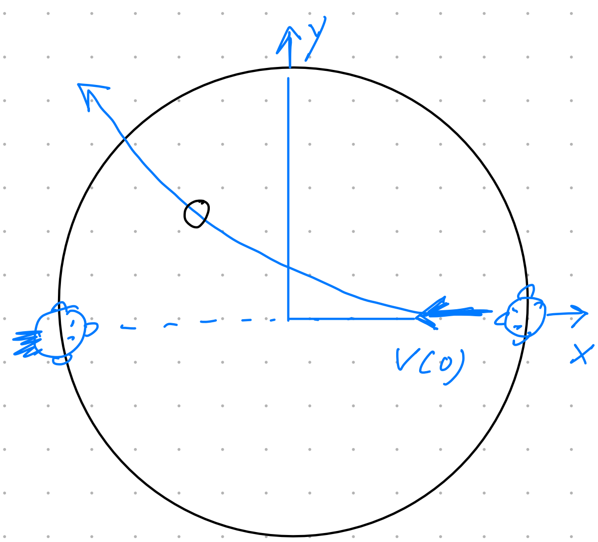
$$\dot{y}_0(0) = b_2 = \Omega R$$

so non-zero velocity in y_0 direction due to rotation of merry-go-round

w.r.t inertial frame



w.r.t rotating frame



motion is deflected to the right.

"sideway force"

≡ Coriolis force