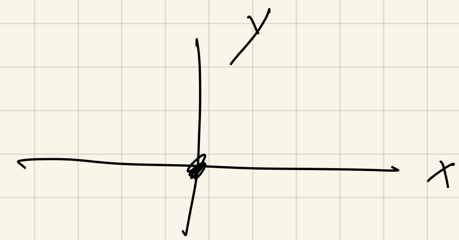
QUIZ # 6 : 27 Oct 2022

Solve for the motion of two bodies subject to the central force associated with $U = \frac{1}{2} k r^2$. What is the trajectory of the orbit in the COM frame?

$$\boxed{L} = \frac{1}{2} m v^2 - \frac{1}{2} k r^2$$

$$= \boxed{\frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2\right) - \frac{1}{2} k r^2}$$



$$= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2} k \left(x^2 + y^2\right)$$

$$= \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2\right) + \left(\frac{1}{2} m \dot{y}^2 - \frac{1}{2} k y^2\right)$$

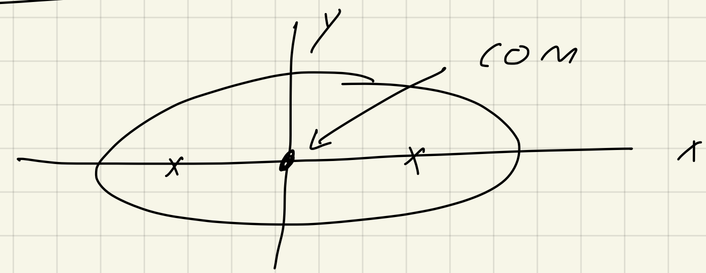
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial x} \longrightarrow m \ddot{x} = -k x \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\boxed{x = c_1 \cos \omega t + d_1 \sin \omega t}$$

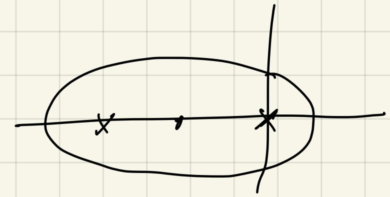
$$\longrightarrow \boxed{y = c_2 \cos \omega t + d_2 \sin \omega t}$$

$$E = T + U$$

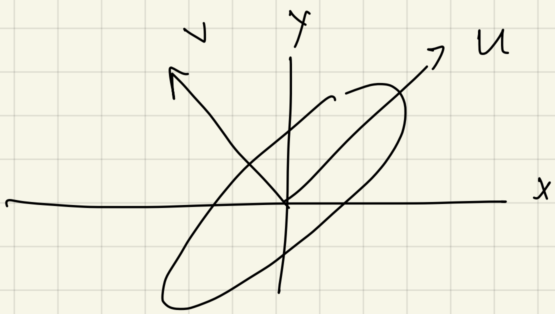
$$= \frac{1}{2} m \dot{r}^2 + \left(\frac{M^2}{2mr^2} + \frac{1}{2} k r^2 \right)$$

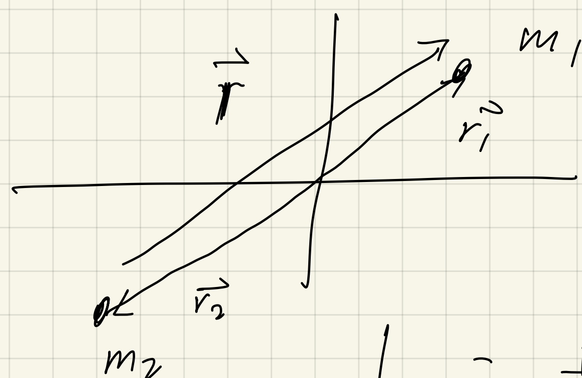


$$U = \frac{1}{2} k r^2$$



$$U = -\frac{\alpha}{r}$$





$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\vec{r}_1 = \frac{m_2 \vec{r}}{m_1 + m_2}$$

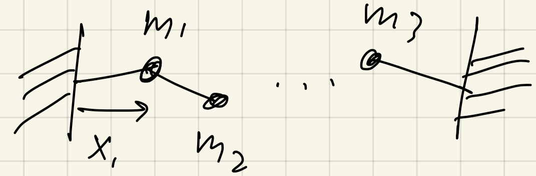
$$\vec{r}_2 = \frac{-m_1 \vec{r}}{m_1 + m_2}$$

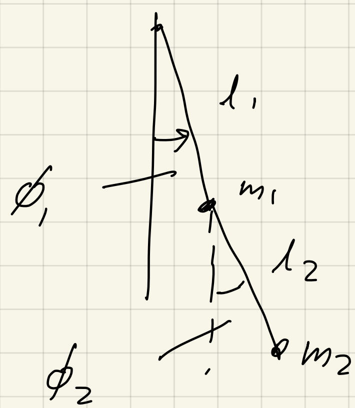
$$L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(|\vec{r}|)$$

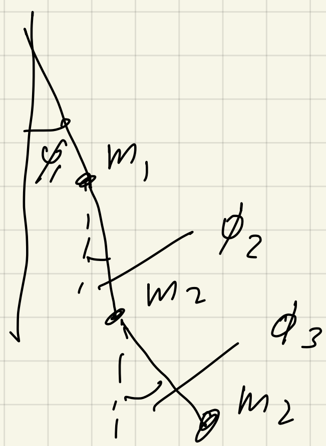
1 Nov 2022:

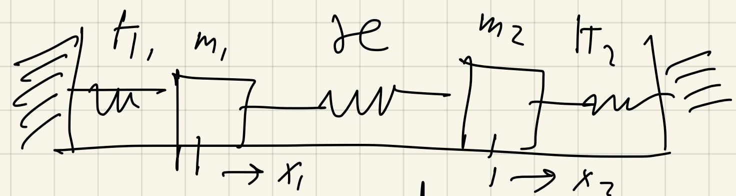
- Rigid body motion — Thurs → Thanksgiving
- Midterm #2 : 11/22 (Tues)
- problem notebook
- Return Midterm #1 (for Kyle)
- SR collisions (last three lectures)

Small oscillations: Multiple bodies









masses and springs

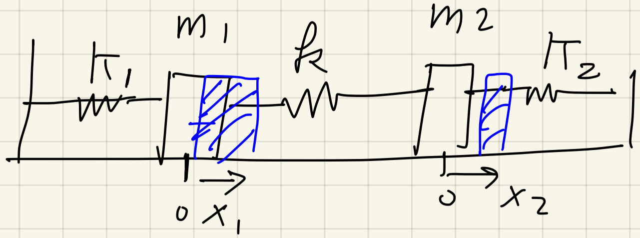
$\downarrow g$ "double pendulum" "triple pendulum"

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$L = \frac{1}{2} \sum_{j,k} m_{jk} \dot{x}_j \dot{x}_k - \frac{1}{2} \sum_{j,k} k_{jk} x_j x_k$$

n : # of mass points

$x_j, \quad j = 1, 2, \cdots, n$



$$k_{jk} = \begin{bmatrix} k_1 + k & -k \\ -k & k_2 + k \end{bmatrix}$$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2} \sum_{j=1}^{2} \sum_{k=1}^{2} m_{jk} \dot{x}_j \dot{x}_k$$

$$= \frac{1}{2} \left(m_{11} \dot{x}_1 \dot{x}_1 + m_{12} \dot{x}_1 \dot{x}_2 + m_{21} \dot{x}_2 \dot{x}_1 + m_{22} \dot{x}_2 \dot{x}_2 \right)$$

$$m_{jk} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \overbrace{\frac{1}{2} k (x_2 - x_1)^2}^{\frac{1}{2} k (x_1^2 + x_2^2 - 2 x_1 x_2)}$$

$$= \frac{1}{2} \left[(k_1 + k) x_1^2 + (k_2 + k) x_2^2 - 2 k x_1 x_2 \right]$$

$$= \frac{1}{2} \left(k_{11} x_1^2 + k_{22} x_2^2 + k_{12} x_1 x_2 + k_{21} x_2 x_1 \right)$$

$$L = \frac{1}{2} \sum_{j,k} m_{jk} \dot{x}_j \dot{x}_k - \frac{1}{2} \sum_{j,k} k_{jk} x_j x_k \Leftarrow$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\ell}\right) = \frac{\partial L}{\partial x_\ell} \qquad \ell = 1, 2$$

$$\frac{\partial L}{\partial \dot{x}_\ell} \cancel{=} 0 \qquad \frac{\partial L}{\partial \dot{x}_\ell} = \frac{\partial}{\partial \dot{x}_\ell}\left(\frac{1}{2} \sum_{j,k} m_{jk} \dot{x}_j \dot{x}_k\right)$$

$$= \frac{1}{2} \sum_{j,k} m_{jk} \frac{\partial}{\partial \dot{x}_\ell}\left(\dot{x}_j \dot{x}_k\right)$$

$$= \frac{1}{2} \sum_{j,k} m_{jk} \left[\underbrace{\frac{\partial \dot{x}_j}{\partial \dot{x}_\ell}}_{\delta_{j\ell}} \dot{x}_k + \dot{x}_j \underbrace{\frac{\partial \dot{x}_k}{\partial \dot{x}_\ell}}_{\delta_{k\ell}} \right]$$

$$= \frac{1}{2} \sum_k m_{\ell k} \dot{x}_k + \frac{1}{2} \sum_j m_{j\ell} \dot{x}_j$$

$(m_{jk}$ symmetric$)$

$$= \frac{1}{2} \sum_k m_{\ell k} \dot{x}_k + \frac{1}{2} \sum_j m_{\ell j} \dot{x}_j \qquad k$$

$$= \sum_k m_{\ell k} \dot{x}_k$$

$$\sum_k m_{j,k}\ddot{x}_k = -\sum_k k_{j,k}\, x_k \qquad j=1,2$$

$$\sum_k \left(m_{j,k}\ddot{x}_k + k_{j,k}\, x_k \right) = 0$$

Ansatz
"Guess"

$$x_k = A_k e^{i\omega t}$$

$$\ddot{x}_k = -\omega^2 A_k e^{i\omega t} = -\omega^2 x_k$$

$$\sum_k \left(-\omega^2 m_{j,k} + k_{j,k} \right) A_k \cancel{e^{i\omega t}} = 0$$

vector

$$\sum_k M_{j,k} v_k = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$\det\left(-\omega^2 m_{j,k} + k_{j,k} \right) = 0$$

characteristic equation

real positive for ω^2

$\omega_\alpha^2 \; : \; \alpha = 1,2$

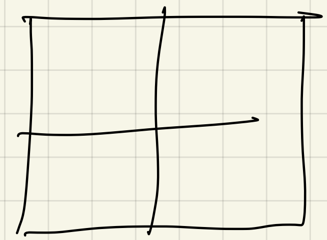
$$\underline{\underline{M}} \cdot \underline{v} = \lambda \, \underline{\underline{v}} = \lambda \, \underline{\underline{\mathbb{1}}} \, v$$

eigenvalue

identity

$$\left(\underline{\underline{M}} - \lambda \underline{\underline{\mathbb{1}}} \right) \cdot \underline{v} = \underline{0}$$

$$\det \left(\underline{M} - \lambda \underline{\mathbb{1}} \right) = 0$$

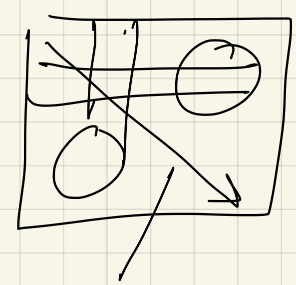


Hermitian

symmetric

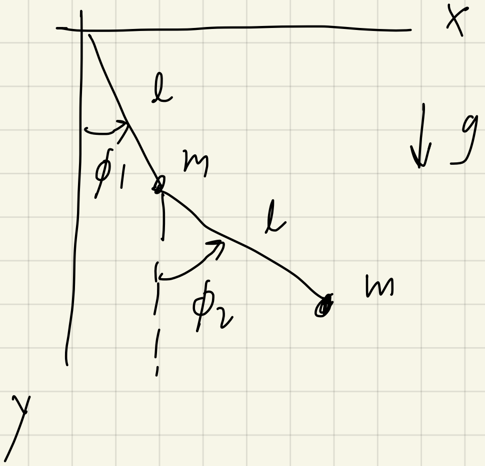
$$\boxed{\sum_k \left(M_{jk} - \lambda \delta_{jk} \right) v_k = 0}$$

$$\det \left(M_{jk} - \lambda \delta_{jk} \right) = 0$$



eigenvalues

11/3/2022



$$L = \frac{1}{2}(2m)\,l^2\dot{\phi}_1^2 + \frac{1}{2}m\,l^2\dot{\phi}_2^2 + m l^2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)$$
$$+ 2mgl\cos\phi_1 + mgl\cos\phi_2$$

$\phi_1 = 0,\ \phi_2 = 0 \leftarrow$ stable equilibrium

$|\phi_1| \ll 1$, $|\phi_2| \ll 1$

$$\left.\begin{aligned} \cos\phi_1 &\approx 1 - \frac{1}{2}\phi_1^2 \\ \cos\phi_2 &\approx 1 - \frac{1}{2}\phi_2^2 \\ \cos(\phi_1-\phi_2) &\approx 1 \end{aligned}\right\} \rightarrow \begin{aligned} L = {} & \frac{1}{2}(2m)\,l^2\dot{\phi}_1^2 + \frac{1}{2}m\,l^2\dot{\phi}_2^2 \\ & + \frac{1}{2}(2m)\,l^2\dot{\phi}_1\dot{\phi}_2 \\ & - \frac{1}{2}2mgl\,\phi_1^2 \\ & - \frac{1}{2}mgl\,\phi_2^2 \end{aligned}$$

$$L = \frac{1}{2}\sum_{j,k} m_{j,k}\,\dot{\phi}_j\dot{\phi}_k - \frac{1}{2}\sum_{j,k} k_{j,k}\,\phi_j\phi_k$$

$$m_{j,k} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} m l^2 \quad , \quad k_{j,k} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} mgl$$

$$0 = \det\left(K_{jk} - \omega^2 M_{jk}\right)$$

$$= \det\left(mgl \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \omega^2 m l^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

$$= \det\left(mgl \left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \frac{\omega^2}{g/l} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right)\right)$$

$$\frac{\omega^2}{g/l} = \frac{\omega^2}{\omega_0^2} \equiv y$$

$$0 = \det\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - y \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} 2(1-y) & -y \\ -y & 1-y \end{bmatrix}$$

$$0 = 2(1-y)^2 - y^2$$

$$= 2(1 + y^2 - 2y) - y^2$$

$$= 2 + 2y^2 - 4y - y^2$$

$$0 = y^2 - 4y + 2$$

$$\boxed{y_\pm = \frac{4 \pm \sqrt{16 - 4\cdot 2}}{2} = 2 \pm \sqrt{2}}$$

$$\omega_+^2 = (2+\sqrt{2})\frac{g}{l}$$

$$\omega_-^2 = (2-\sqrt{2})\frac{g}{l}$$

$$\omega_+^2 = (2+\sqrt{2})\frac{g}{l}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2(1-y_+) & -y_+ \\ -y_+ & 1-y_+ \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2(1-2-\sqrt{2}) & -2-\sqrt{2} \\ -2-\sqrt{2} & 1-2-\sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

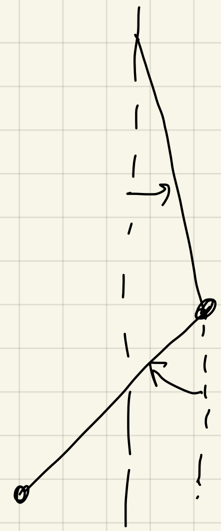
$$0 = -2(1+\sqrt{2})v_1 - (2+\sqrt{2})v_2$$

$$v_2 = \frac{-2(1+\sqrt{2})v_1}{2+\sqrt{2}}\left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right) = \frac{-2(\cancel{2}-\cancel{2}-\sqrt{2}+2\sqrt{2})v_1}{4-2}$$

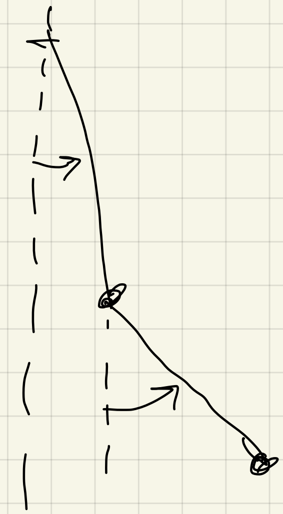
$$= -\sqrt{2}\,v_1$$

$$e_+ = N_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \qquad e_- = N_- \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

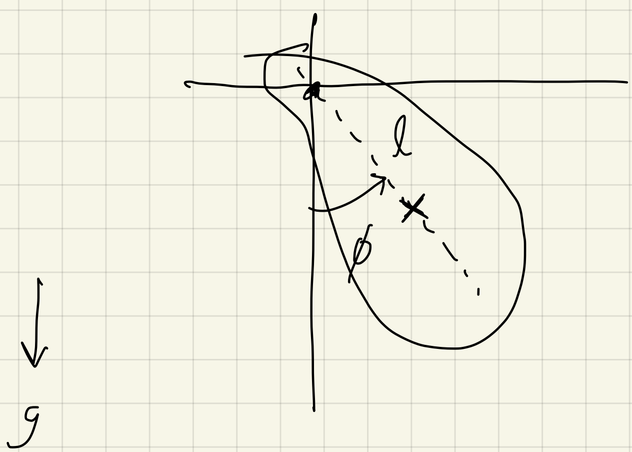
$\underline{\omega_+^2}$: higher freq



$\underline{\omega_-^2}$: lower freq



Rigid body motion:



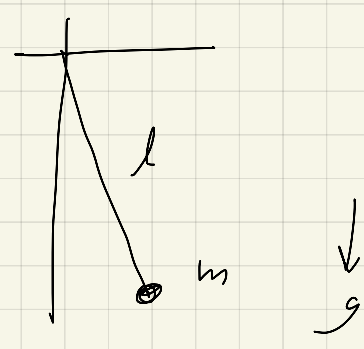
U

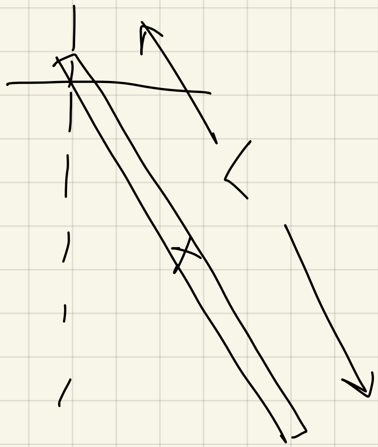
$$T = \frac{1}{2} I \Omega^2$$

$$= \frac{1}{2} I \dot{\phi}^2$$

$$I = M R^2$$

$$\omega = \sqrt{\frac{g}{\ell}}$$

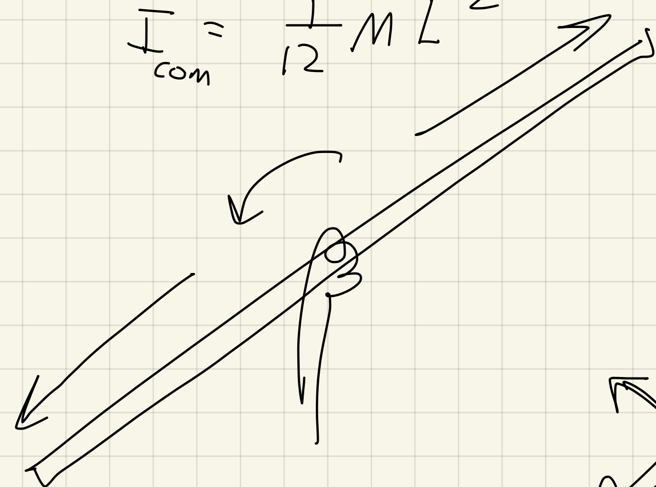




M

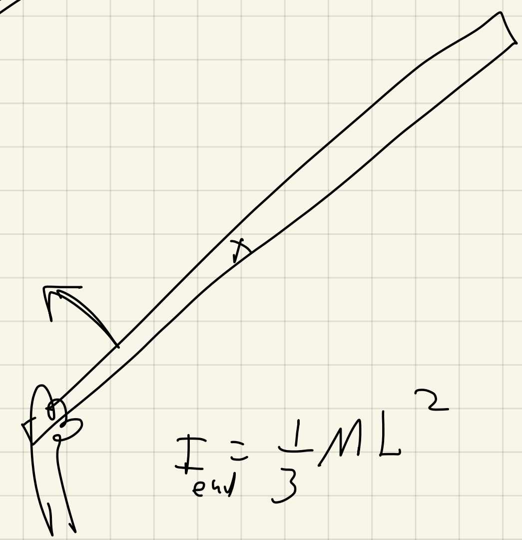
$\omega = \sqrt{}$??

$I_{com} = \frac{1}{12} M L^2$



I =

uniform
thin rod of
length L



$I_{end} = \frac{1}{3} M L^2$

I

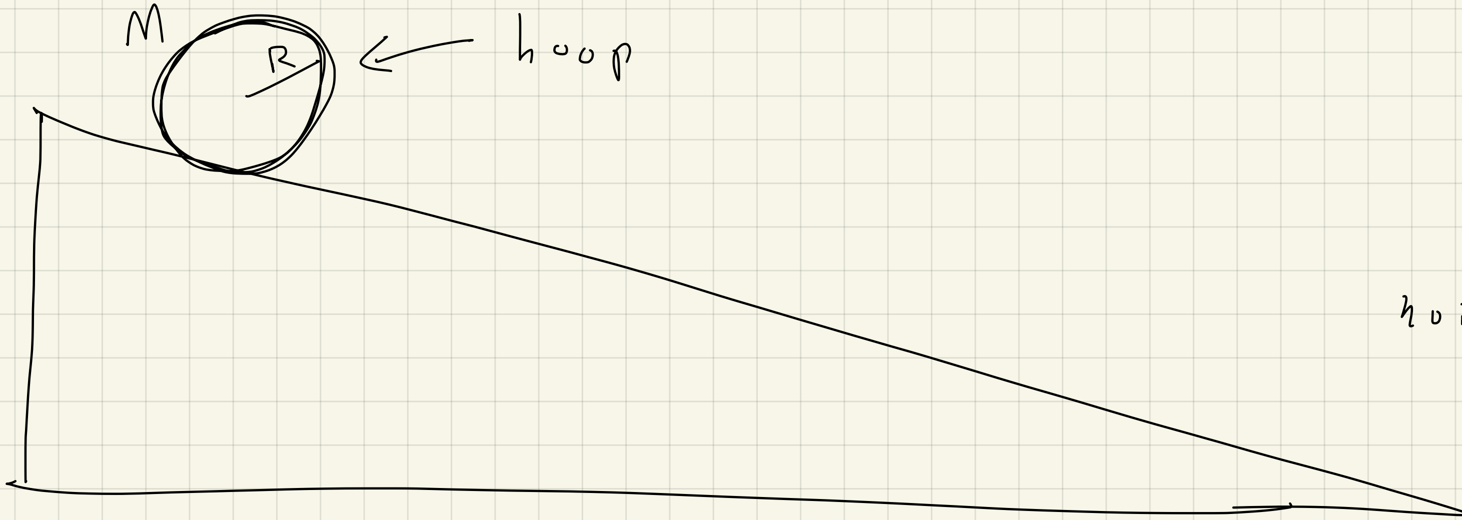
I_{ij}

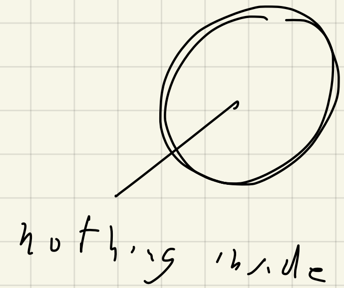
$$I_{end} = I_{com} + M d^2 \qquad d = \frac{L}{2}$$

$$= \frac{1}{12} ML^2 + M \frac{L^2}{4}$$

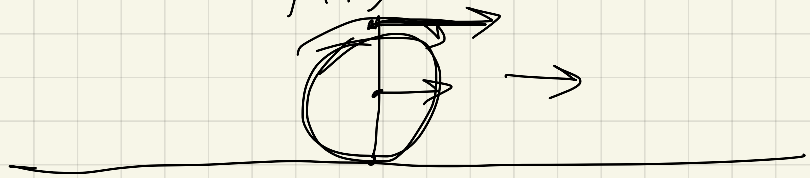
$$= \left(\frac{1}{12} + \frac{1}{4}\right) ML^2$$

$$= \left(\frac{1}{12} + \frac{3}{12}\right) ML^2 = \frac{1}{3} ML^2$$

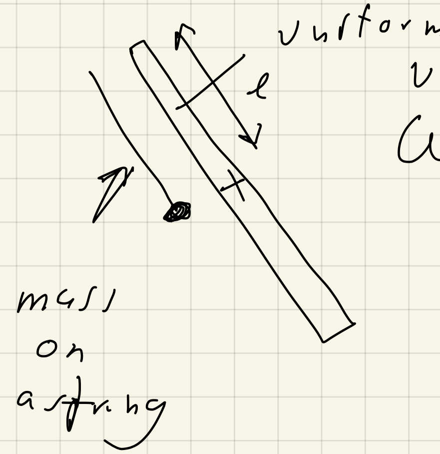




disk reach bottom first

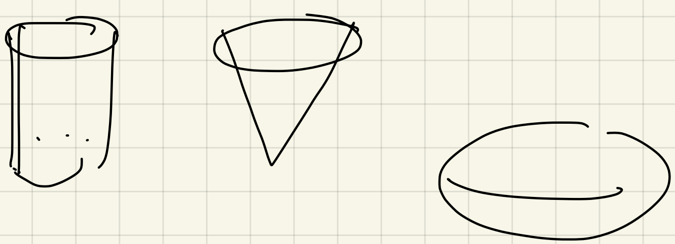


$$I_{hoop} > I_{disk}$$
$$\| \qquad\qquad \|$$
$$MR^2 \qquad \frac{1}{2}MR^2$$



uniform rod

$$\omega = \sqrt{\frac{3}{4}\frac{g}{l}}$$



Derive:

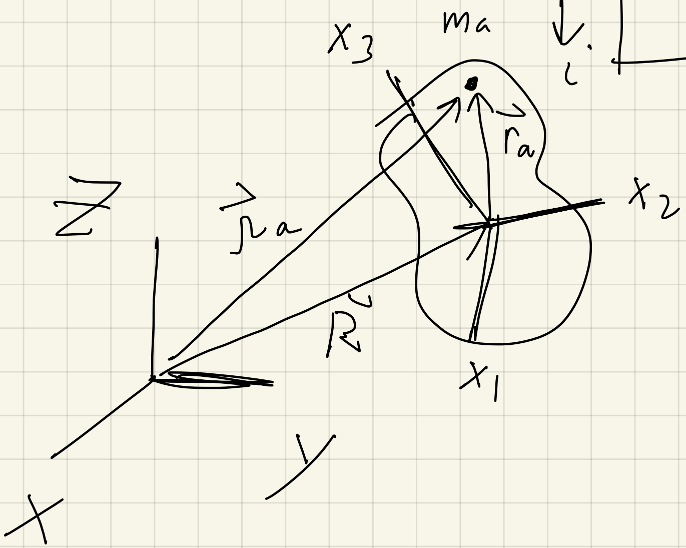
$$I_{ij} = \sum_a m_a \left(r_a^2 \delta_{ij} - r_{ai} r_{aj} \right)$$

$$I = I_{ij} n_i n_j = \sum_{i,j} I_{ij} n_i n_j$$

rotational inertia about axis $\hat{n} = (n_1, n_2, n_3)$

$$I_{ij} = \begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \rightarrow j \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \\ \downarrow \\ i \end{array} & \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \end{array}$$

symmetric



w.r.t principle axes

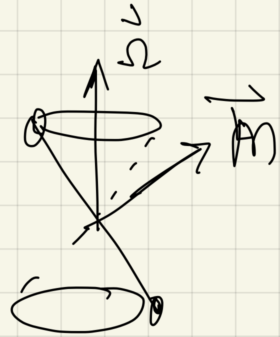
$$I_{ij} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$T = \frac{1}{2}\sum_{i,j} I_{ij}\,\Omega_i\,\Omega_j + \frac{1}{2}\mu V^2$$

μ: total mass

$$M_i = \sum_j I_{ij}\,\Omega_j$$

about CoM in Rigid body



$\vec{M} = \vec{r} \times \vec{p}$

$\vec{p} = m\vec{V}$

$\vec{M} = I\,\vec{\omega}$

need not point in same

$$I_{ij} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$\vec{\Omega}_i = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}, \quad M_i = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$$

$$= \begin{bmatrix} I_1\Omega_1 \\ I_2\Omega_2 \\ I_3\Omega_3 \end{bmatrix}$$

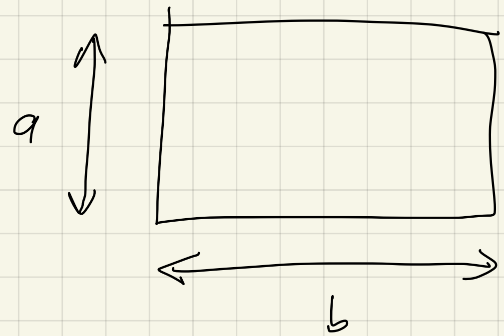
$\vec{\Omega}$

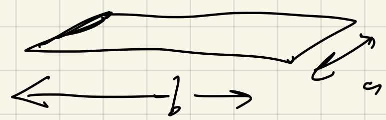
Quiz #8

10 Nov 2022

(2-d)

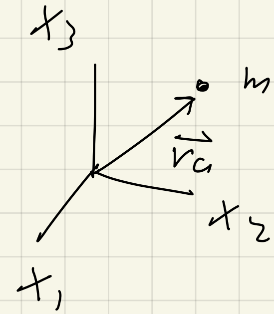
Uniform rectangular plate with sides a and b



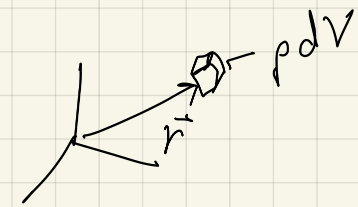


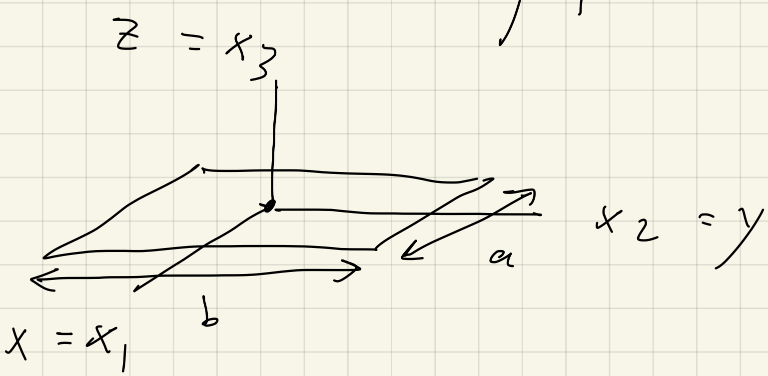
Calculate principal moments of inertia for this object.

$$I_{ij} = \sum_a \overbrace{m_a}^{\rho\, dV} \left(r_a^2 \delta_{ij} - r_{ai} r_{aj} \right)$$



$$= \int \rho\, dV \left(r^2 \delta_{ij} - r_i r_j \right)$$





$$\rho\, dV = \sigma\, dx\, dy$$

$$\boxed{\sigma = \frac{M}{ab}}$$

$$\rho = \sigma\, \delta(z)$$

$$I_{xx} = \frac{M}{ab}\int dx \int dy \,\left(r^2 \underbrace{\delta_{xx}}_{1} - x^2\right)$$

$$= \frac{M}{ab}\int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \; y^2$$

$$= \frac{M}{\cancel{a}b} \cdot \cancel{a} \cdot \frac{\cancel{2}}{3}\left(\frac{b}{2}\right)^3$$

$$= \boxed{\frac{1}{12} M b^2}$$

$$I_{yy} = \boxed{\frac{1}{12} M a^2}$$

$$I_{zz} = \frac{M}{ab}\int dx \int dy \,\underbrace{\left(r^2 \delta_{zz} - z^2\right)}_{x^2+y^2}$$

$$= I_{xx} + I_{yy}$$

$$= \boxed{\frac{1}{12} M (a^2 + b^2)}$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

Scalar triple product

$$-\sum_a \frac{\partial U}{\partial \vec{r}_a} \cdot (\delta\vec{\phi} \times \vec{r}_a)$$

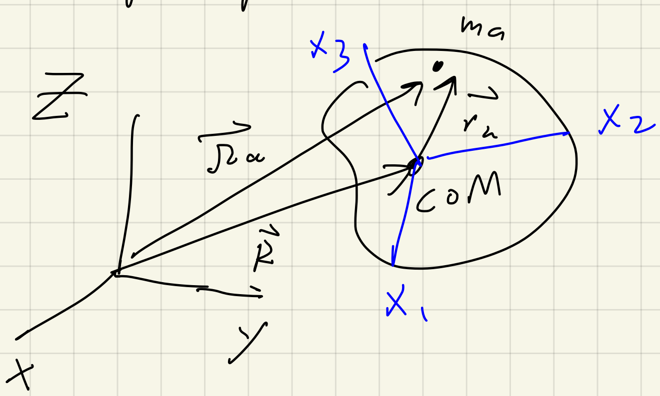
$$= -\sum_a \delta\vec{\phi} \cdot \left(\vec{r}_a \times \frac{\partial U}{\partial \vec{r}_a}\right)$$

$$= \delta\vec{\phi} \cdot \sum_a \left(\vec{r}_a \times \left(-\frac{\partial U}{\partial \vec{r}_a}\right)\right)$$

$$= \delta\vec{\phi} \cdot \sum_a (\vec{r}_a \times \vec{f}_a)$$

$$= \delta\vec{\phi} \cdot \vec{N}$$

total torque



$\vec{A}$, A_i

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\vec{\Omega} \times (\vec{r}_a \times \vec{\Omega}) = \vec{r}_a (\vec{\Omega} \cdot \vec{\Omega}) - \vec{\Omega}(\vec{\Omega} \cdot \vec{r}_a)$$

$$= \vec{r}_a \Omega^2 - \vec{\Omega}\, \Omega_j r_{aj}$$

$$(\vec{\Omega} \times (\vec{r}_a \times \vec{\Omega}))_i = r_{ai} \Omega^2 - \Omega_i \Omega_j r_{aj}$$

$$T = \frac{1}{2} \sum_a m_a |\dot{\vec{r}}_a|^2$$

$$\vec{r}_a = \vec{R} + \vec{r}_a$$

$$\dot{\vec{r}}_a = \dot{\vec{R}} + \dot{\vec{r}}_a$$

$$= \vec{V} + \underbrace{\vec{v}_a}_{\vec{\Omega} \times \vec{r}_a}$$

$$= \frac{1}{2} \sum_a m_a \left| \vec{V} + (\vec{\Omega} \times \vec{r}_a) \right|^2$$

$$= \frac{1}{2} \sum_a m_a \left(V^2 + |\vec{\Omega} \times \vec{r}_a|^2 + 2 \vec{V} \cdot (\vec{\Omega} \times \vec{r}_a) \right)$$

$$= \frac{1}{2} \mu V^2 + \frac{1}{2} \sum_a m_a |\vec{\Omega} \times \vec{r}_a|^2$$

$$+ \underbrace{\sum_a m_a \vec{V} \cdot (\vec{\Omega} \times \vec{r}_a)}_{\vec{V} \cdot \vec{\Omega} \times \cancel{\left(\sum_a m_a \vec{r}_a\right)} \to 0}$$

$$T = \frac{1}{2} \mu V^2 + \frac{1}{2} I_{ij} \Omega_i \Omega_j$$

$$\vec{V}_a = \dot{\vec{r}}_a$$

$$= \vec{V} + \vec{\Omega} \times \vec{r}_a$$

$$\frac{1}{2} \vec{\Omega}^T \cdot \underline{\underline{I}} \cdot \vec{\Omega}$$

$$L = T - U = \frac{1}{2} \mu V^2 + \frac{1}{2} I_{ij} \Omega_i \Omega_j - U(\vec{r}_a)$$

$$\frac{1}{2}\sum_a m_a \left|\vec{\Omega}\times\vec{r_a}\right|^2$$

$$\overset{\vec{A}}{\left(\vec{\Omega}\times\vec{r_a}\right)}\cdot\left(\overset{\vec{B}}{\vec{\Omega}}\times\overset{\vec{C}}{\vec{r_a}}\right)$$

$$= \vec{\Omega}\cdot\left(\vec{r_a}\times\left(\vec{\Omega}\times\vec{r_a}\right)\right)$$

$$= \vec{\Omega}\cdot\left[\vec{\Omega}\left(\vec{r_a}\cdot\vec{r_a}\right) - \vec{r_a}\left(\vec{r_a}\cdot\vec{\Omega}\right)\right]$$

$$= \Omega^2 r_a^2 - \left(\vec{\Omega}\cdot\vec{r_a}\right)^2$$

$$= \Omega_i\Omega_j\,\delta_{ij}\,r_a^2 - \left(\Omega_i r_{ai}\right)\left(\Omega_j r_{aj}\right)$$

$$= \Omega_i\Omega_j\left[r_a^2\delta_{ij} - r_{ai}r_{aj}\right]$$

$$\frac{1}{2}\sum_a m_a\left|\vec{\Omega}\times\vec{r_a}\right|^2 = \frac{1}{2}\sum_a m_a\,\Omega_i\Omega_j\left[r_a^2\delta_{ij} - r_{ai}r_{aj}\right]$$

$$= \frac{1}{2}\Omega_i\Omega_j\underbrace{\sum_a m_a\left(r_a^2\delta_{ij} - r_{ai}r_{aj}\right)}_{I_{ij}}$$

$$= \frac{1}{2}I_{ij}\Omega_i\Omega_j$$

$$\underbrace{\vec{R},\ \vec{\phi}}_{6\ DOF} \qquad \underbrace{\vec{V},\ \vec{\Omega}}$$

$$q_i \qquad \dot{q}_i$$

$$\vec{P} = \mu \vec{V}$$

$$\vec{M} = \underline{\underline{I}}\,\vec{\Omega}$$

$$(M_i = I_{ij}\Omega_j)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{V}}\right) = \frac{\partial L}{\partial \vec{R}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{\Omega}}\right) = \frac{\partial L}{\partial \vec{\phi}}$$

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt}$$

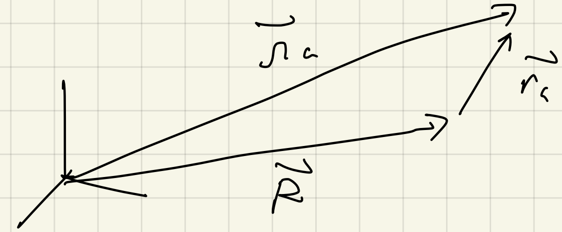
$$\frac{d\vec{P}}{dt} = \sum_a f_a = \vec{F}$$

$$\frac{d\vec{M}}{dt} = \sum_a (\vec{r}_a \times \vec{f}_a) = \vec{N}$$

$$\vec{\rho}_a = \vec{R} + \vec{r}_a$$

$$\delta\vec{\rho}_a = \delta\vec{R} + \delta\vec{r}_a$$

$$= \delta\vec{R} + \delta\vec{\phi} \times \vec{r}_a$$



$\delta\vec{\phi}$ —— direction

$|\delta\vec{\phi}| = \delta\phi$