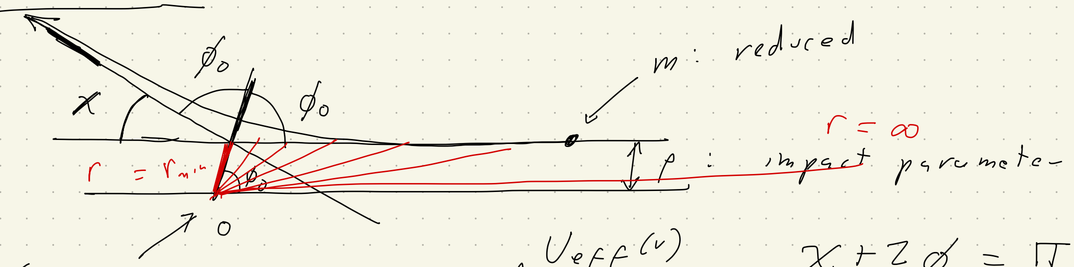
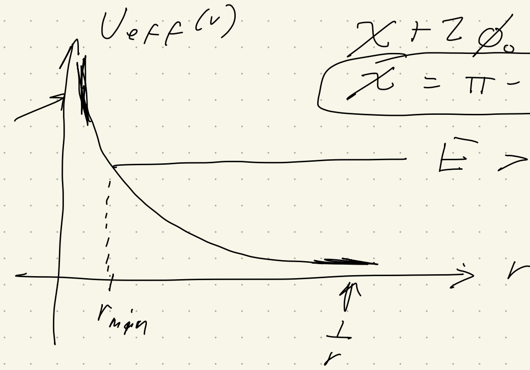
Lecture #16 Thurs 10/14



CoM of original system

$$\chi + 2\phi_0 = \pi$$

$$\boxed{\chi = \pi - 2\phi_0}$$



$U(r) = \frac{\alpha}{r}$, $\alpha > 0$

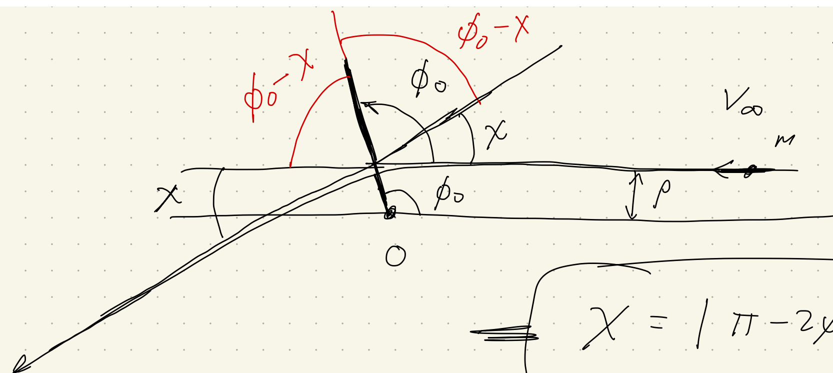
(repulsive)

$$U_{eff}(r) = U(r) + \frac{M_z^2}{2mr^2}$$

$$= \frac{\alpha}{r} + \frac{M_z^2}{2mr^2}$$

$E = U_{eff}(r_{min})$

turning point



$$2(\phi_0 - \chi) + \chi = \pi$$

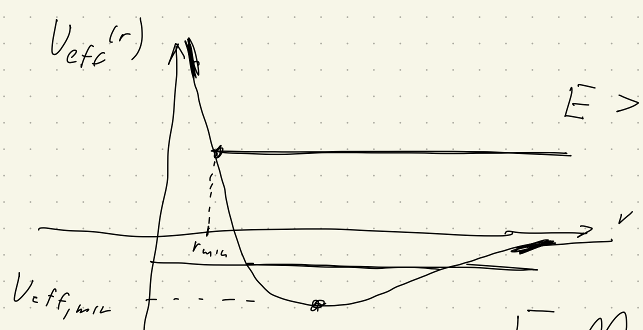
$$2\phi_0 - \chi = \pi$$

$$\boxed{\chi = 2\phi_0 - \pi}$$

$$\boxed{\chi = |\pi - 2\phi_0|}$$

for both repulsive & attractive

$U(r) = -\frac{\alpha}{r}$



$$U_{eff}(r) = -\frac{\alpha}{r} + \frac{M_z^2}{2mr^2}$$

$$\boxed{E = \frac{1}{2} m v_\infty^2 , \quad M = m\rho v_\infty}$$

E, M

ρ, v_∞

Sec 14:

$$t = \pm \int \frac{dr}{\sqrt{}} + \text{const}$$

$$\Rightarrow \quad \phi = \pm \int \frac{M\, dr/r^2}{\sqrt{2m(E - U(r)) - \frac{M^2}{r^2}}} + \text{const}$$

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{M\, dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M^2}{r^2}}}$$

$E = U_{eff}(r_{min})$

$M = m\rho v_\infty$

$E = \frac{1}{2} m v_\infty^2$

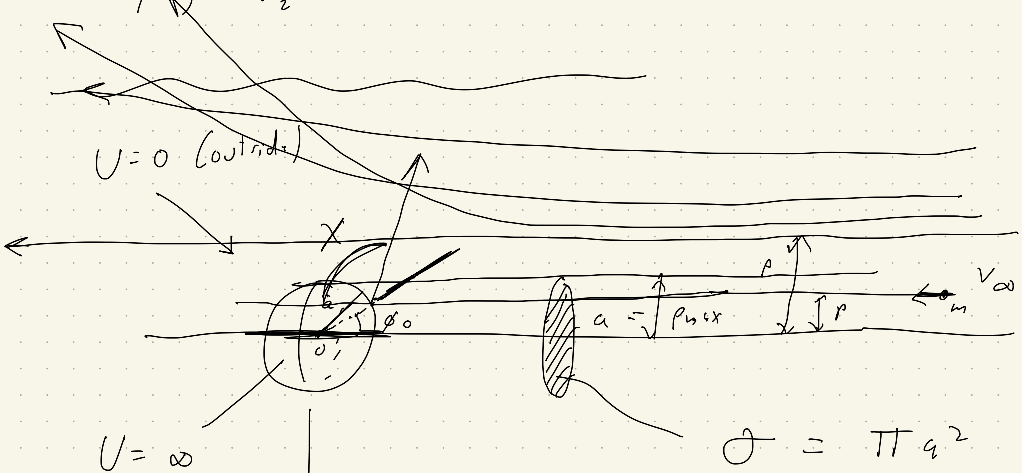
$$= \int_{r_{min}}^{\infty} \frac{m\rho v_\infty\, dr/r^2}{\sqrt{2m\left(\frac{1}{2} m v_\infty^2 - U(r)\right) - \frac{m^2\rho^2 v_\infty^2}{r^2}}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \frac{U(r)}{\frac{1}{2} m v_\infty^2} - \rho^2/r^2}}$$

$$\theta_2 = \frac{1}{2}(\pi - \chi) \qquad \chi = |\pi - 2\phi_0|$$

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}$$

v_1', v_2' = . . .



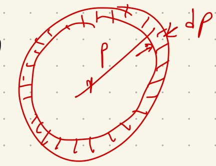
$\sigma = \pi a^2$

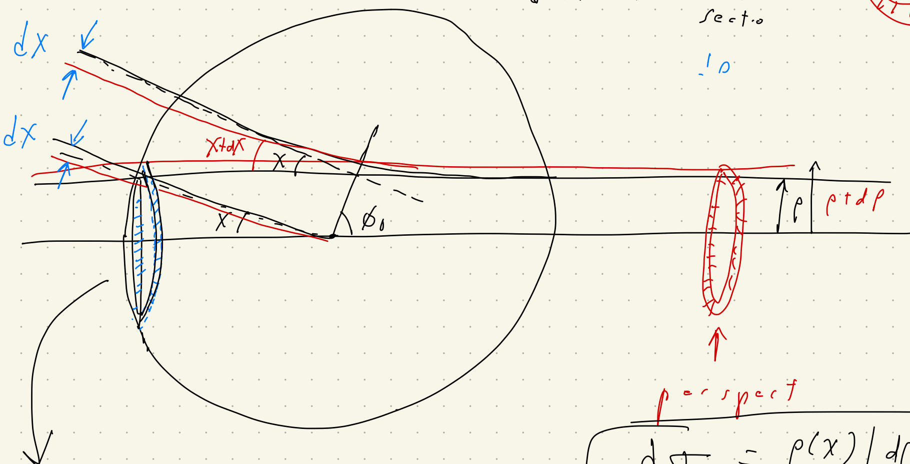
cross-section

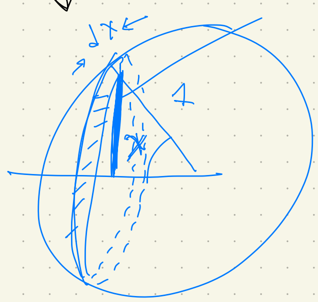
Hard sphere

$$d\sigma = 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi \quad \leftarrow \quad d\sigma = 2\pi\rho\, d\rho$$

differential cross section







$$d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$

$$d\Omega = 2\pi \sin\chi \, d\chi$$

solid angle

$$\frac{d\sigma}{d\Omega} = \frac{\cancel{2\pi}\rho\, d\rho}{\cancel{2\pi} \sin\chi\, d\chi} = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right|$$

$$\frac{d\sigma}{d\Omega} = \frac{\rho}{\sin\chi}\left|\frac{d\rho}{d\chi}\right|$$ CoM frame

$$\frac{d\sigma_1}{d\Omega_1} = \frac{\rho}{\sin\theta_1}\left|\frac{d\rho}{d\theta_1}\right|$$ Lab Frame θ_1

$$\frac{d\sigma_2}{d\Omega_2} = \frac{\rho}{\sin\theta_2}\left|\frac{d\rho}{d\theta_2}\right|$$ Lab Frame θ_2

$$\frac{d\sigma_1}{d\Omega_1} = \frac{\sin\chi}{\sin\theta_1}\left|\frac{d\chi}{d\theta_1}\right| \frac{d\sigma}{d\Omega}$$

$$= \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| \frac{d\sigma}{d\Omega}$$

calculated for the CoM

Lecture #17: Tues Oct 19th

Next week: start small oscillations

Today / Thurs: Q & A (collisions and scattering)

Lecture #18: Thur Oct 21st

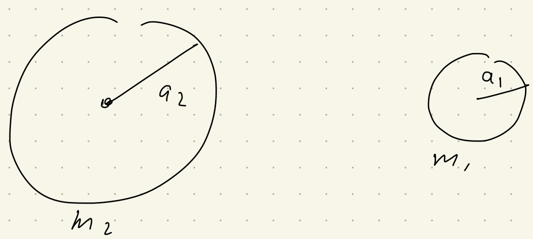
Next week: small oscillations

Today: Q&A, Quiz #3 (last 20 minutes)

$\boxed{\text{Q3\_name.pdf}}$

Quiz #3:

Two hard spheres with masses m_1, m_2 and radii a_1, a_2:



$\rho(\chi) = (a_1 + a_2)\cos\left(\frac{\chi}{2}\right)$

a) Find $\underline{\rho = \rho(\chi)}$ where χ = scattering angle wrt COM frame

b) What value of ρ will give $\theta_2 = 60°$?

$2\theta_2 + \chi = \pi \rightarrow \chi = \pi - 2\theta_2$
$= 60°$

$\rho = (a_1 + a_2)\cos(30°)$
$= \frac{\sqrt{3}}{2}(a_1 + a_2)$

Lec #19: 10/26

Small oscillation: $\boxed{\text{Sec } 21}$, 22, 23

free oscillation in 1-d

more than 1d

Forced oscillation in 1d

Damping: Sec 25 (not covered)

SHM: (simple harmonic motion) is sinusoidal

$$\omega = \frac{2\pi}{T}$$

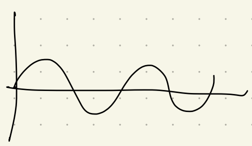
$$x(t) = a \cos(\omega t + \alpha)$$

amplitude

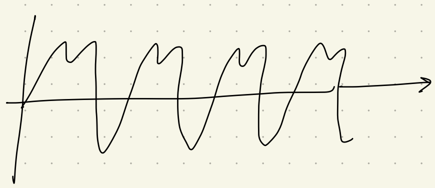
phase

angular freq

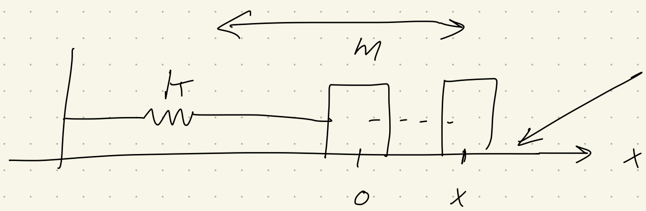
SHM



periodic



Example:



no friction

$$U = \frac{1}{2} k x^2$$

$$F = -\frac{dU}{dx} = -kx$$

$$F = -kx$$

linear

restoring

$$F = m\ddot{x} = -kx \rightarrow \ddot{x} = -\frac{k}{m} x$$

soln: $x = c_1 \cos \omega t + c_2 \sin \omega t$

$$\omega = \sqrt{\frac{k}{m}}$$

Alternative: $x = a \cos(\omega t + \alpha)$

$$x = \mathrm{Re}\left[A e^{i\omega t} \right], \quad A = a e^{i\alpha}$$

$$T = \frac{1}{2} m \dot{x}^2$$
$$U = \frac{1}{2} k x^2$$

$$L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

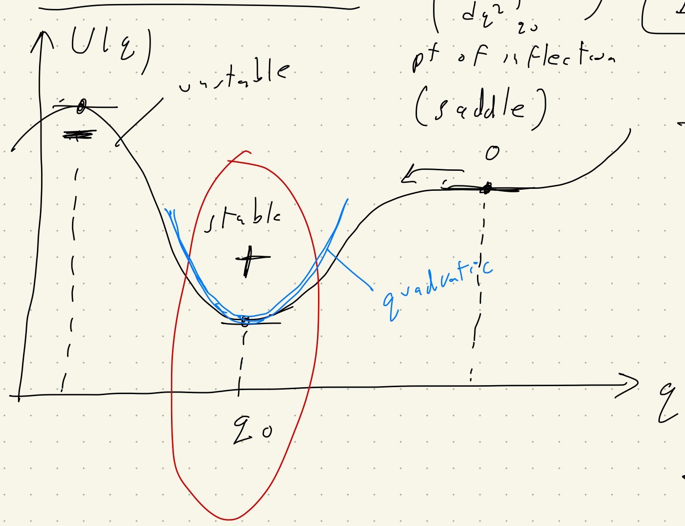
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow m\ddot{x} = -kx$$

$U(q) = q^3$

$q_0 = 0$

More generally:

$\left(\left.\frac{d^2U}{dq^2}\right|_{q_0} = 0\right)$ pt of inflection (saddle)



$$\boxed{L = \frac{1}{2} a(q)\, \dot{q}^2 - U(q)}$$

Equilibrium: $(q = q_0)$

$$0 = F = -\left.\frac{dU}{dq}\right|_{q_0}$$

stable:

$$\left.\frac{d^2U}{dq^2}\right|_{q_0} > 0$$

$U = \text{const}$

Expansion: (about q_0)

$$U(q) = \underbrace{U(q_0)}_{\substack{\text{const} \\ \text{(ignore in } L)}} + \cancel{\left.\frac{dU}{dq}\right|_{q_0}}^{\,0 \text{ equilibrium}} \cdot (q - q_0) + \frac{1}{2}\overbrace{\left.\frac{d^2U}{dq^2}\right|_{q_0}}^{k>0} (q - q_0)^2$$

$$\underbrace{+ \frac{1}{3!}\left.\frac{d^3U}{dq^3}\right|_{q_0} (q - q_0)^3 + \ldots}_{\text{ignore for } q - q_0 \text{ small}}$$

$$U(q) \approx \frac{1}{2} k x^2$$

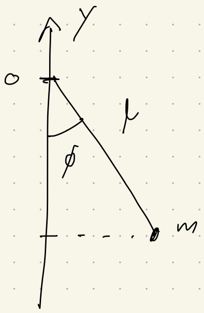
$$x \equiv q - q_0$$

$$k = \left.\frac{d^2U}{dq^2}\right|_{q_0}$$

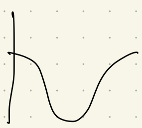
$$T = \frac{1}{2} a(q)\, \dot{q}^2$$
$$= \frac{1}{2} a(q_0)\, \dot{x}^2$$
$$\approx \frac{1}{2} m \dot{x}^2$$

$$L \approx \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

Simple pendulum:



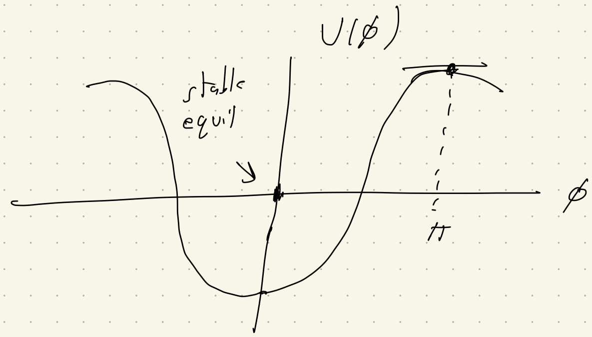
$U = mgy$
$= -mgl\cos\phi$



$T = \frac{1}{2} m l^2 \dot{\phi}^2$

$U = -mgl\cos\phi$

$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi$



$\phi_0 = 0$ (equilibrium)

$\cos\phi = 1 - \frac{1}{2}\phi^2 + \cdots$

$$\cos\phi = \underbrace{\cos 0}_{1} + \underbrace{\cancel{\left.\frac{d(\cos\phi)}{d\phi}\right|_{\phi=0}}}_{-\sin\phi \to 0} \cdot \phi + \frac{1}{2}\left.\frac{d^2(\cos\phi)}{d\phi^2}\right|_{\phi=0} \phi^2 + \cdots$$

$$= 1 - \frac{1}{2}\phi^2 + \cdots$$

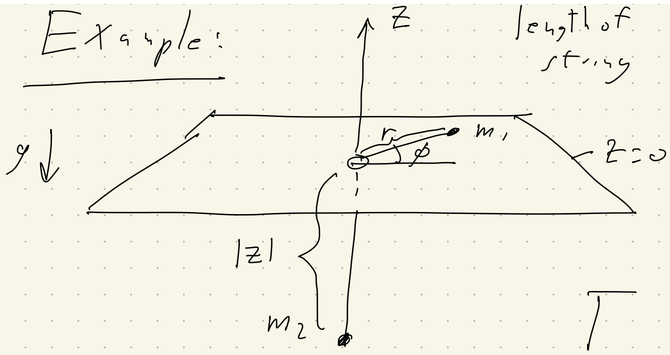
$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\left(1 - \frac{1}{2}\phi^2\right)$$

$$= \frac{1}{2} \underset{M}{\underset{\parallel}{m l^2}} \dot{\phi}^2 - \frac{1}{2} \underset{K}{\underset{\parallel}{mgl}} \phi^2 + \underbrace{mgl}_{\text{ignore}}$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$\omega = \sqrt{\frac{K}{M}} = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}$$

Example:



length of string — $l = r + |z| = r - z$

z: location of m_2

r: for m_1

$$\boxed{z = r - l \rightarrow \dot{z} = \dot{r}}$$

$$T = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}m_2\dot{z}^2$$

$$U = m_2 g z = m_2 g(r - l) = m_2 g r - m_2 g l$$

(ignore) Const

$$\boxed{L = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\phi}^2 - m_2 g r}$$

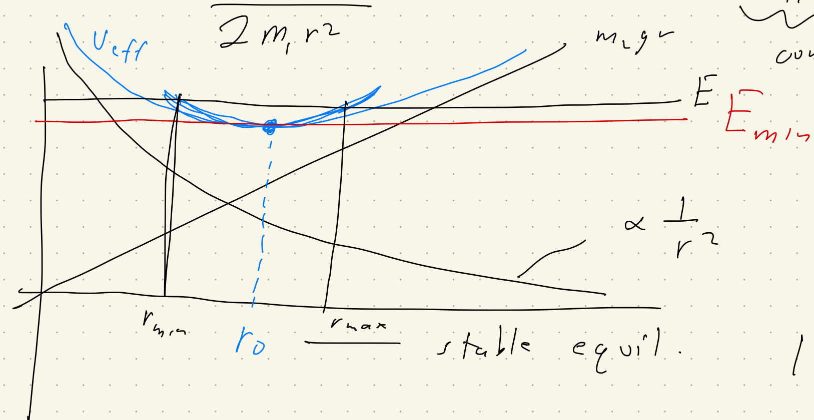
i) $M_z = \frac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{m_1 r^2}$

ii) $E = T + U = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \underbrace{\frac{1}{2}m_1 r^2\dot{\phi}^2}_{\frac{M_z^2}{2m_1 r^2}} + m_2 g r$

$$E = \frac{1}{2}\underbrace{(m_1 + m_2)}_{M}\dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = \frac{M_z^2}{2m_1 r^2} + m_2 g r = \underbrace{U_{eff}(r_0)}_{const} + \frac{1}{2}k(r - r_0)^2$$

$$k = \left.\frac{d^2 U_{eff}}{dr^2}\right|_{r=r_0}$$



stable equil.

$|r - r_0| \ll r_0$

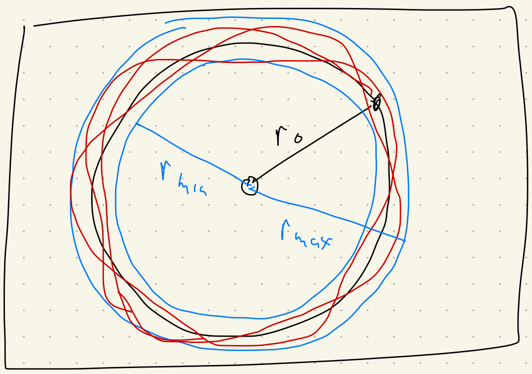
$r_0 = ?$

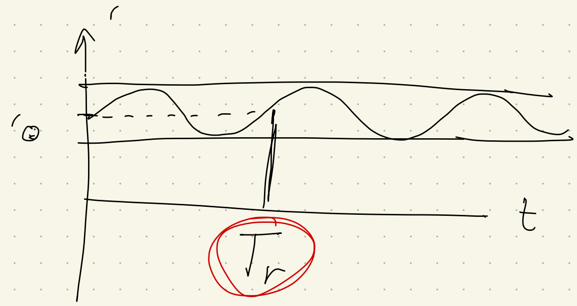
$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0}$$

$$= -\frac{M_z^2}{m_1 r_0^3} + m_2 g \rightarrow \boxed{M_z^2 = m_1 m_2 g r_0^3}$$

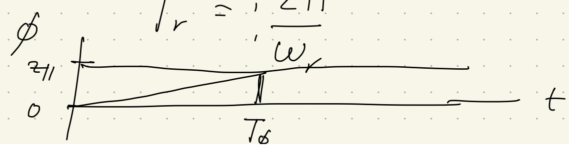
$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r=r_0} = \frac{3M_z^2}{m_1 r_0^4} = \frac{3\cancel{m_1} m_2 g r_0^3}{\cancel{m_1} r_0^4} = \boxed{\frac{3m_2 g}{r_0} = K}$$

$$\omega_r = \sqrt{\frac{K}{\underbrace{m_1+m_2}_{M}}} = \sqrt{\frac{3m_2 g}{(m_1+m_2)r_0}}$$





$$T_r = \frac{2\pi}{\omega_r}$$



$$T_\phi = \frac{2\pi}{\omega_\phi} = \frac{2\pi}{\left.\dot\phi\right|_{r_0}}$$

$$\dot\phi = \frac{M_z}{m_1 r^2}$$

$$\omega_\phi = \left.\dot\phi\right|_{r=r_0} = \frac{M_z}{m_1 r_0^2} = \frac{\sqrt{m_1 m_2 g r_0^3}}{m_1 r_0^2}$$

$$\boxed{\omega_\phi = \sqrt{\frac{m_2 g}{m_1 r_0}}}$$

compare to

$$\boxed{\omega_r = \sqrt{\frac{3m_2 g}{(m_1+m_2)r_0}}}$$

If $\frac{3}{m_1+m_2} = \frac{1}{m_1}$ then $\omega_r = \omega_\phi$

$3m_1 = m_1 + m_2$

$\boxed{2m_1 = m_2}$

Lec #20: 10/28

Forced oscillation:

$$m\ddot{x} = -kx + F(t)$$

$$\ddot{x} + \frac{k}{m}x = \frac{F(t)}{m}$$

$$\boxed{\ddot{x} + \omega^2 x = \frac{F(t)}{m}}, \quad \omega \equiv \sqrt{\frac{k}{m}}$$

$U(x) = \frac{1}{2}kx^2 - x\,F(t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$L = T - U$

$$\frac{\partial L}{\partial x} = -kx + F(t)$$

general sol'n:

$$x(t) = x_h(t) + x_p(t)$$

x_h: homogeneous ($F(t)=0$)

x_p: particular (any solution for $F(t)$)

$$\boxed{x_h(t) = a\cos(\omega t + \alpha)}$$

a, α: two constant (Initial condition)

Suppose: $F(t) = f\cos(\gamma t + \beta)$

$$\ddot{x}_p + \omega^2 x_p = \frac{f}{m} \cos(\gamma t + \beta)$$

Guess!! $\quad x_p(t) = b\cos(\gamma t + \beta)$

$$-b\gamma^2 \cancel{\cos(\gamma t+\beta)} + \omega^2 b \cancel{\cos(\gamma t+\beta)} = \frac{f}{m}\cancel{\cos(\gamma t+\beta)}$$

$$b(\omega^2-\gamma^2) = \frac{f}{m}$$

$$\rightarrow \boxed{b = \frac{f}{m(\omega^2-\gamma^2)}}$$

$$\boxed{x_p(t) = \frac{f}{m(\omega^2-\gamma^2)}\cos(\gamma t+\beta)}$$

$+ \; 10\cos(\omega t)$

$$\boxed{x(t) = a\cos(\omega t+\alpha) + \frac{f}{m(\omega^2-\gamma^2)}\cos(\gamma t+\beta)}$$

$$x(t) = a\cos(\omega t+\alpha) + \underbrace{\frac{f}{m(\omega^2-\gamma^2)}\left[\cos(\gamma t+\beta) - \cos(\omega t+\beta)\right]}_{\frac{0}{0}}$$

L'hopital's

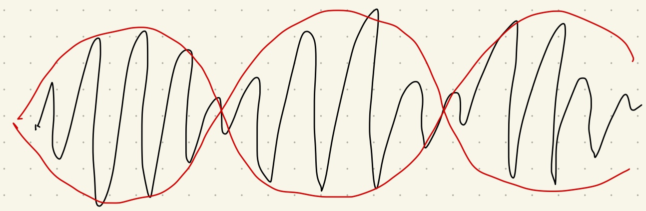
$$= \frac{\frac{d}{d\gamma}(\text{num})\Big|_{\gamma\to\omega}}{\frac{d}{d\gamma}(\text{den})\Big|_{\gamma\to\omega}}$$

$$= \frac{+ft\sin(\omega t+\beta)}{+2m\omega}$$

at resonance $(\gamma=\omega)$

$$= \frac{ft}{2m\omega}\sin(\omega t+\beta)$$

$$\boxed{x(t) = a\cos(\omega t+\alpha) + \frac{ft}{2m\omega}\sin(\omega t+\beta)}$$



$$\frac{\omega+\gamma}{2}$$

$$|\omega-\gamma| = \omega_{beat}$$

General: for arbitrary $F(t)$

$$F(t) = \mathrm{Re}\underbrace{\int_{-\infty}^{\infty} d\gamma\, \tilde{F}(\gamma)\, e^{i\gamma t}}_{\text{Fourier transform}}$$

$$\boxed{\ddot{x} + \omega^2 x = \frac{F(t)}{m}}$$

$$\boxed{y' + P(x)y = Q(x)}$$

Math methods

$y = y(x)$

Let: $\xi = \dot{x} + i\omega x$

complex

$$\dot{\xi} = \ddot{x} + i\omega\dot{x}$$
$$= \ddot{x} + i\omega(\xi - i\omega x)$$
$$= \ddot{x} + i\omega\xi + \omega^2 x$$

$$\rightarrow \quad \dot{\xi} - i\omega\xi = \ddot{x} + \omega^2 x = \frac{F(t)}{m}$$

Homog: $\dot{\xi}_h - i\omega\xi_h = 0 \rightarrow \xi_h(t) = A e^{i\omega t}$

complex constant

replace

Guess: $\boxed{\xi_p(t) = A(t)\, e^{i\omega t}}$

$$\boxed{\dot{\xi}_p - i\omega \xi_p = \frac{F(t)}{m}}, \quad \xi_p = A(t) e^{i\omega t}$$

$$\rightarrow \dot{A} e^{i\omega t} + \cancel{iA\omega e^{i\omega t}} \cancel{-i\omega A e^{i\omega t}} = \frac{F(t)}{m}$$

$$\dot{A} = \frac{e^{-i\omega t} F(t)}{m}$$

$$A(t) = \int dt \frac{F(t)}{m} e^{-i\omega t} + \text{const}$$

$$\boxed{\xi(t) = e^{i\omega t}\left[\int_0^t d\bar{t}\, \frac{F(\bar{t})}{m} e^{-i\omega \bar{t}} + \xi_0\right]}$$

Complex constant (I.C's)

$$\xi = \dot{x} + i\omega x \rightarrow \boxed{x(t) = \frac{1}{\omega} \operatorname{Im}(\xi(t))}$$

$$y' + P(x) y = Q(x) \qquad f(x)\,dx = g(y)\,dy$$

$$\frac{dy}{dx} + P(x) y - Q(x) = 0$$

$$\boxed{1\,dy + (P(x)y - Q(x))\,dx = 0} \neq dU$$

$$dU(x,y) = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

$$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y}$$

M. Boas ~~Mathemat~~

$$\boxed{\mu(x)\left[dy + (P(x)y - Q(x))dx\right] = dU}$$

$$\frac{d\mu}{dx} = \frac{\partial}{\partial y}\left(P(x)\, y - Q(x) \right)\mu(x)$$

$$\frac{d\mu}{dx} = P(x)\,\mu(x)$$

$$\int \frac{d\mu}{\mu} = \int P(x)\,dx$$

$$\ln \mu = \int P(x)\,dx$$

$$\mu(x) = e^{\int P(x)\,dx}$$

Lec #21: Nov. 2nd

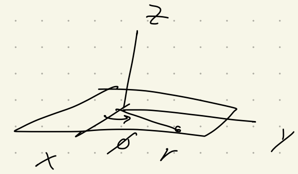
Today: Sec 23 Free oscillation in 2 or more dimensions.

Thurs: Rigid body motion

EXAM 2: Nov ~~18th~~ $\rightarrow$ Nov 23rd

QUIZ #4: Today (at end of class)

Example: space oscillator



$$U = \frac{1}{2} k r^2 = \frac{1}{2} k (x^2 + y^2)$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$L = T - U$$

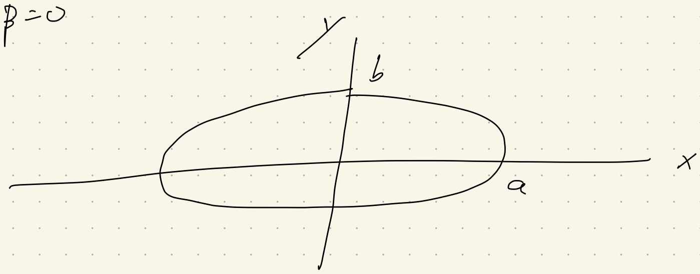
$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k (x^2 + y^2)$$

$$= \underbrace{\left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2\right)}_{\omega_x = \sqrt{\frac{k}{m}}} + \underbrace{\left(\frac{1}{2} m \dot{y}^2 - \frac{1}{2} k y^2\right)}_{\omega_y = \sqrt{\frac{k}{m}}}$$

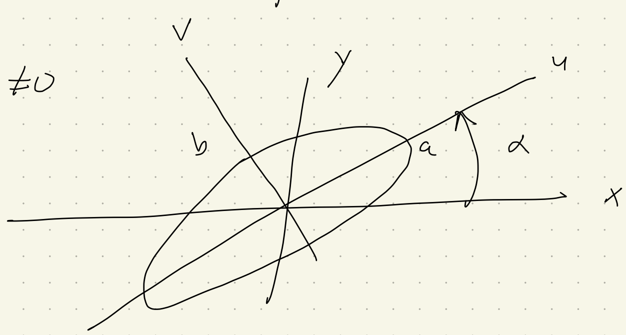
$$\omega = \sqrt{\frac{k}{m}}$$

$$\boxed{\begin{aligned} x &= a \cos(\omega t + \alpha) \\ y &= b \sin(\omega t + \beta) \end{aligned}}$$ — general solution

$\alpha = \beta = 0$

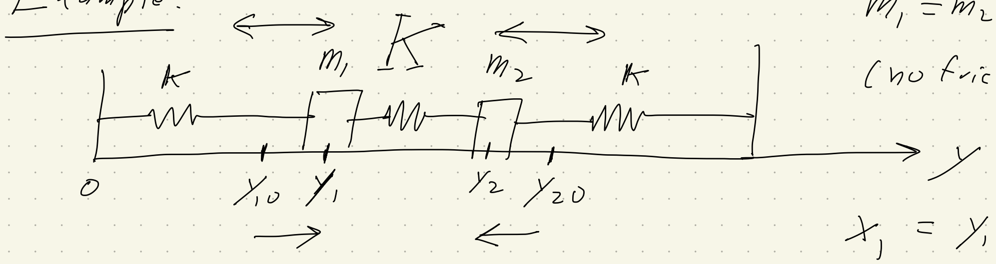


$\alpha = \beta \neq 0$



$\alpha \neq \beta$ — ellipse

Example:



$m_1 = m_2 \equiv m$

(no friction)

$x_1 = y_1 - y_{10}$

$x_2 = y_2 - y_{20}$

$$L = T - U$$

$$T = \frac{1}{2} m \dot{y}_1^2 + \frac{1}{2} m \dot{y}_2^2 = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} K (x_2 - x_1)^2$$

$$= \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} K (x_2^2 + x_1^2 - 2 x_1 x_2)$$

$$= \frac{1}{2} (k + K) x_1^2 + \frac{1}{2} (k + K) x_2^2 - K x_1 x_2$$

$-\frac{1}{2} K x_1 x_2$

$-\frac{1}{2} K x_2 x_1$

$$T = \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k, \quad m_{ik} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$U = \frac{1}{2} \sum_{i,k} k_{ik} x_i x_k, \quad k_{ik} = \begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix}$$

$$\boxed{L = \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k - \frac{1}{2} \sum_{i,k} k_{ik} x_i x_k}$$

general

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) = \frac{\partial L}{\partial x_j} \qquad j = 1, 2, \cdots$$

$$\frac{d}{dt}\left(\sum_k m_{jk} \dot{x}_k\right) = -\sum_k k_{jk} x_k$$

$$\underbrace{\frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k}_{} = \frac{1}{2} \Big(m_{11} \dot{x}_1^2 + m_{12} \dot{x}_1 \dot{x}_2 + m_{13} \dot{x}_1 \dot{x}_3$$
$$+ m_{21} \dot{x}_2 \dot{x}_1 + m_{22} \dot{x}_2^2 + m_{23} \dot{x}_2 \dot{x}_3$$
$$+ m_{31} \dot{x}_3 \dot{x}_1 + m_{32} \dot{x}_3 \dot{x}_2 + m_{33} \dot{x}_3^2 \Big)$$

$$\frac{\partial L}{\partial \dot{x}_2} = \frac{1}{2} \left(m_{12} \dot{x}_1 + m_{21} \dot{x}_1 + 2 m_{22} \dot{x}_2 + m_{23} \dot{x}_3 + m_{32} \dot{x}_3 \right)$$
$$= \frac{1}{\cancel{2}} \left(\cancel{2} m_{21} \dot{x}_1 + \cancel{2} m_{22} \dot{x}_2 + \cancel{2} m_{23} \dot{x}_3 \right)$$
$$= \sum_k m_{2k} \dot{x}_k$$

$$\sum_k m_{jk}\,\ddot{x}_k = -\sum_k k_{jk}\,x_k$$

<u>Guess:</u> $x_k = A_k e^{i\omega t}$

$\rightarrow \ddot{x}_k = -\omega^2 A_k e^{i\omega t}$

$$\sum_k -\omega^2 m_{jk} A_k \cancel{e^{i\omega t}} = \underbrace{-\sum_k k_{jk} A_k \cancel{e^{i\omega t}}}_{\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} \\ \end{bmatrix}}$$

$$\boxed{\sum_k \underbrace{\left(k_{jk} - \omega^2 m_{jk}\right)}_{\begin{bmatrix} & \\ & \end{bmatrix}} A_k = 0} \text{ — 0 vector}$$

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If invertible $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\underline{\underline{M}} \cdot \vec{v} = \lambda \vec{v}$$

$$\underbrace{\left(\underline{\underline{M}} - \lambda \underline{\underline{\mathbb{1}}}\right)}_{\det = 0} \cdot \vec{v} = 0$$

$$\underbrace{0 = \det\left(K_{j,k} - \omega^2 m_{j,k}\right)}_{\text{characteristic equation}}$$

polynomial equation $(\omega^2)^N$

Normal mode freqs, characteristic freqs, eigen freqs.

ω_α^2 label the N different eigen freqs.

$$0 = \det\left(\begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\right)$$

$$= \det \begin{bmatrix} k+K-\omega^2 m & -K \\ -K & k+K-\omega^2 m \end{bmatrix}$$

$$= \left((k+K)-\omega^2 m\right)^2 - K^2$$

$$\left((k+K) - \omega^2 m\right)^2 = K^2$$

$$(k+K) - \omega^2 m = \pm K$$

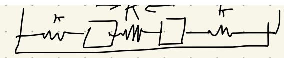
$$\rightarrow \quad \omega^2 = \frac{(k+K) \pm K}{m}$$

$$\boxed{\omega_+^2 = \frac{k+2K}{m}, \quad \omega_-^2 = \frac{k}{m}}$$

eigen freqs.

solve for eigenvectors:

$$\omega_+^2 = \frac{k+2\kappa}{m} \qquad V_+ = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

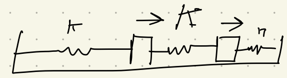


$$\omega_-^2 = \frac{k}{m} \qquad V_- = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{normal mode oscillations}}$$

$$\Delta_{k\alpha} = \begin{array}{c|c} & \begin{array}{cc} + & - \end{array} \ \to \alpha \\ \hline \begin{array}{c}1\\2\\ \downarrow k\end{array} & \begin{bmatrix} V_+ & V_- \end{bmatrix} \end{array}$$

matrix of eigenvector



$$\boxed{\begin{aligned} x_k &= \mathrm{Re}\Big[\sum_\alpha \Delta_{k\alpha} C_\alpha e^{i\omega_\alpha t}\Big] \\ &= \sum_\alpha \Delta_{k\alpha}\,\theta_\alpha \\ \theta_\alpha &\equiv \mathrm{Re}\big[C_\alpha e^{i\omega_\alpha t}\big] \end{aligned}}$$

complex const determined by ICs

normal coords

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}\sum_{i,k} m_{ik}\,\dot{x}_i\,\dot{x}_k - \frac{1}{2}\sum_{i,k} k_{ik}\,x_i\,x_k \\ &= \frac{1}{2}\dot{x}^T m\,\dot{x} - \frac{1}{2}x^T k\,x \\ &= \frac{1}{2}\dot{\theta}^T(\Delta^T m\,\Delta)\,\dot{\theta} - \frac{1}{2}\theta^T(\Delta^T k\,\Delta)\,\theta \\ &= \frac{1}{2}\sum_\alpha \mathcal{M}_\alpha\,\dot{\theta}_\alpha^2 - \frac{1}{2}\sum_\alpha \mathcal{K}_\alpha\,\theta_\alpha^2 \end{aligned}$$

where $\mathcal{M}$ = diagonal matrix $= \Delta^T m\,\Delta$

$\mathcal{K}$ = " $= \Delta^T k\,\Delta$

☆☆ eigenvectors diagonalize both m_{ik} and k_{ik} with

$$\boxed{\mathcal{K}_\alpha = \mathcal{M}_\alpha\,\omega_\alpha^2}$$

CHECK: $\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}m&0\\0&m\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\-1&1\end{bmatrix}$

$$= \frac{m}{2}\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\-1&1\end{bmatrix}$$

$$= \frac{m}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix} \to \mathcal{M}_+ = m,\ \mathcal{M}_- = m$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}k+K & -K\\ -K & k+K\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}k+2K & k\\ -k-2K & k\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}2k+4K & 0\\ 0 & 2k\end{bmatrix}$$

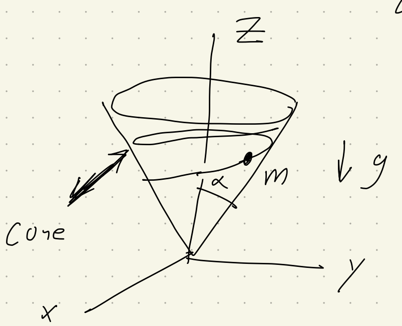
$$= \begin{bmatrix}k+2K & 0\\ 0 & k\end{bmatrix}$$

$$= \begin{bmatrix}\omega_+^2 m & 0\\ 0 & \omega_-^2 m\end{bmatrix} \quad \rightarrow \quad \begin{aligned}\varkappa_+ &= \omega_+^2 m\\ \varkappa_- &= \omega_-^2 m\end{aligned}$$

Thus, $$L = \frac{1}{2} m\left(\dot{\theta}_+^2 + \dot{\theta}_-^2\right) - \frac{1}{2}\left(m\omega_+^2\theta_+^2 + m\omega_-^2\theta_-^2\right)$$
$$= \left(\frac{1}{2} m\dot{\theta}_+^2 - \frac{1}{2} m\omega_+^2\theta_+^2\right) + \left(\frac{1}{2} m\dot{\theta}_-^2 - \frac{1}{2} m\omega_-^2\theta_-^2\right)$$

Quiz #4:

Calculate freq of small oscillations around the circular orbit solution $r = r_0$



α fixed

(sph. polar coords)

a) Lagrangian?

b) Determine r_0 (relationship between r_0 and M_z)

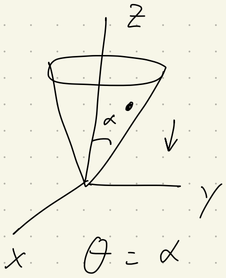
c) Determine ω_r

~~q~~ q4_name.pdf

joseph.d.romano@ttu.edu

Lec #22: Thurs Nov 4th

- Next 5 lectures rigid body motion 31-36, 38
- EXAM 2 - Tues 11/23 (not Thur 11/18)



$\theta = \alpha$ (sph. polar)

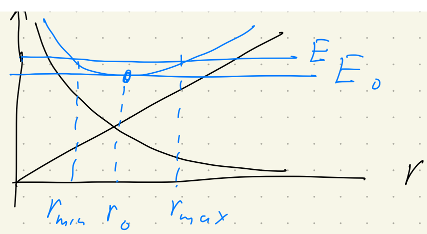
$$T = \frac{1}{2} m (\dot r^2 + r^2 \sin^2\alpha \, \dot\phi^2)$$

$$U = mgz = mgr\cos\alpha$$

$$L = T - U$$
$$= \frac{1}{2} m(\dot r^2 + r^2 \sin^2\alpha \, \dot\phi^2) - mgr\cos\alpha$$

$$M_z = m r^2 \sin^2\alpha \, \dot\phi \rightarrow \dot\phi = \frac{M_z}{m r^2 \sin^2\alpha}$$

$$E = T + U = \sum_i \left(\dot q_i \frac{\partial L}{\partial \dot q_i} - L \right)$$
$$= \frac{1}{2} m \dot r^2 + \underbrace{\frac{M_z^2}{2 m r^2 \sin^2\alpha} + mgr\cos\alpha}_{U_{eff}(r)}$$



$$0 = \left. \frac{d U_{eff}}{dr} \right|_{r = r_0}$$

$$M_z^2 = m^2 g r_0^3 \sin^2\alpha \cos\alpha$$

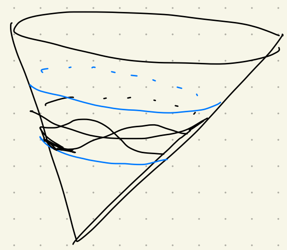
$$\omega_r = \sqrt{\frac{k}{m}}$$

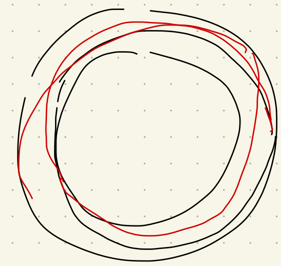
$$k = \left. \frac{d^2 U_{eff}}{dr^2} \right|_{r = r_0}$$
$$= \frac{3 mg\cos\alpha}{r_0}$$

$$\rightarrow \omega_r = \sqrt{\frac{k}{m}} = \sqrt{\frac{3 g\cos\alpha}{r_0}}$$

$$\omega_\phi = \left. \dot\phi \right|_{r = r_0} = \sqrt{\frac{g\cos\alpha}{r_0 \sin^2\alpha}}$$

don't agree (not closed)

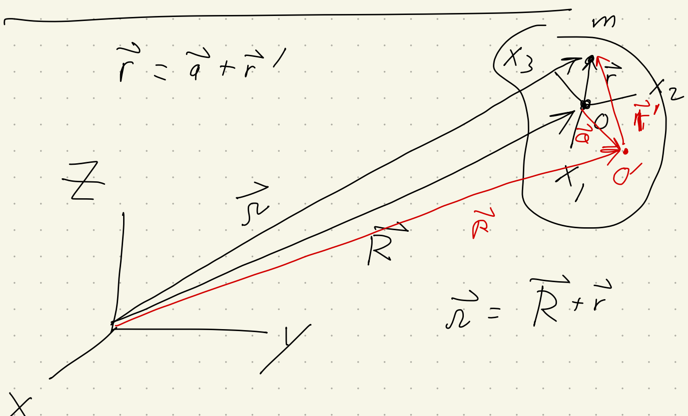




Rigid body motion:

$x_i = (x_1, x_2, x_3)$
$= (x, y, z)$

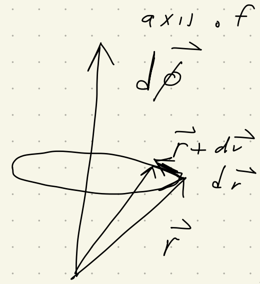
$\vec r = \vec a + \vec r\,'$



fixed inertial frame

$$\boxed{\vec v = \frac{d\vec\rho}{dt} = \frac{d\vec R}{dt} + \frac{d\vec r}{dt} = \vec V + \vec\Omega \times \vec r}$$

$\vec\rho = \vec R + \vec r$



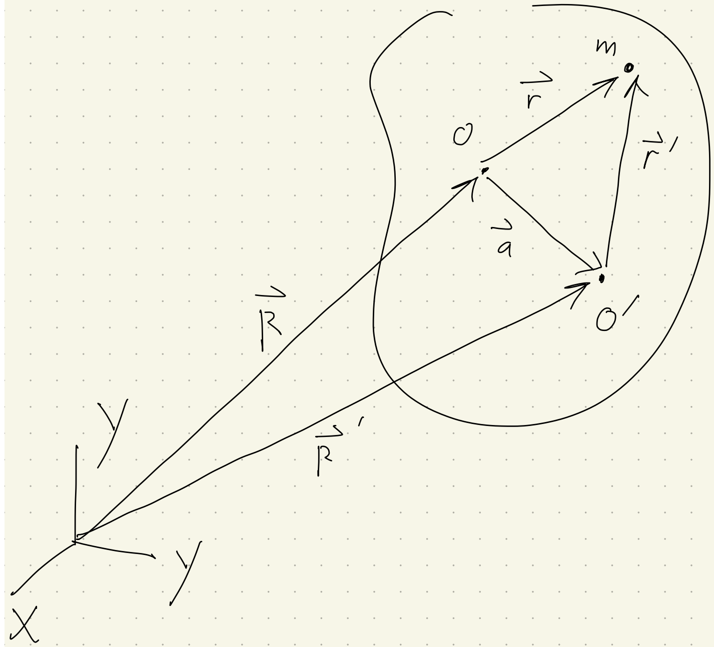
$|d\vec\phi| = d\phi$

$d\vec r = d\vec\phi \times \vec r$

most convenient if O is at com of RB.

$$\frac{d\vec r}{dt} = \frac{d\vec\phi}{dt} \times \vec r = \vec\Omega \times \vec r$$

$\vec r = \vec a + \vec r' \longrightarrow \vec v = \vec V + \vec\Omega \times \vec r$



$$\boxed{\vec v = \vec V' + \vec\Omega' \times \vec r'}$$

$$\vec v = \vec V + \vec\Omega \times (\vec a + \vec r')$$

$$\boxed{= \vec V + \vec\Omega \times \vec a + \vec\Omega \times \vec r'}$$

$$\vec V' \doteq \vec V + \vec\Omega \times \vec a$$

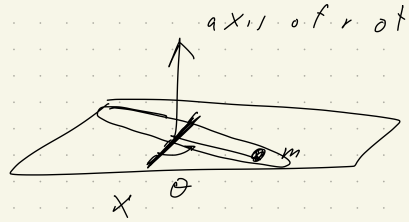
$$\vec\Omega' = \vec\Omega$$

2-d rotational motion:

$$\boxed{\vec F = \frac{d\vec p}{dt}} \quad \vec P = m\vec V$$

θ, $\frac{d\theta}{dt} = \omega$, $T_{rot} = \frac{1}{2} I \omega^2$

moment of inertia



3-d

I_{ij} : inertia tensor

(3x3 symmetric matrix)

$$L = I\omega$$

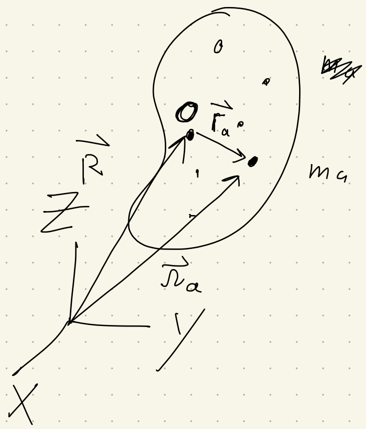
$$M_i = \sum_k I_{ik}\Omega_k$$

$$\vec M = \underline{\underline{I}}\,\vec\Omega$$

$$\boxed{\vec N = \frac{d\vec M}{dt}}$$

KE :

m_a — a : labels the mass point in the rigid body



$$T = \sum_a \frac{1}{2} m_a |\vec{v}_a|^2$$

$$\vec{v}_a = \vec{V} + \vec{\Omega} \times \vec{r}_a$$

$$|\vec{v}_a|^2 = (\vec{V} + \vec{\Omega}\times\vec{r}_a)\cdot(\vec{V}+\vec{\Omega}\times\vec{r}_a)$$

$$= \underline{|\vec{V}|^2 + 2\vec{V}\cdot(\vec{\Omega}\times\vec{r}_a)}$$

$$\underline{+ (\vec{\Omega}\times\vec{r}_a)\cdot(\vec{\Omega}\times\vec{r}_a)}$$

1^{st} term $\quad \sum_a \frac{1}{2} m_a |\vec{V}|^2 = \frac{1}{2}\mu|\vec{V}|^2$

$\mu = \sum_a m_a$ = total mass

2^{nd} term $= \sum_a \cancel{\frac{1}{2}}\,\cancel{2}\, m_a \vec{V}\cdot(\vec{\Omega}\times\vec{r}_a)$

$$= \left(\sum_a m_a \vec{r}_a\right)\cdot(\vec{V}\times\vec{\Omega}) = 0 \quad \left(\text{if origin is at COM}\right)$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

3^{rd} term $= (\vec{\Omega}\times\vec{r}_a)\cdot(\vec{\Omega}\times\vec{r}_a)$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$= \vec{\Omega}\cdot(\vec{r}_a\times(\vec{\Omega}\times\vec{r}_a))$$

$$= \vec{\Omega}\cdot(\vec{\Omega}|\vec{r}_a|^2 - \vec{r}_a(\vec{r}_a\cdot\vec{\Omega}))$$

$$= \vec{\Omega}\cdot\vec{\Omega}\, r_a^2 - (\vec{\Omega}\cdot\vec{r}_a)(\vec{\Omega}\cdot\vec{r}_a)$$

$$T = \frac{1}{2}\mu|\vec{V}|^2 + \sum_a \frac{1}{2} m_a \left(\vec{\Omega}\cdot\vec{\Omega}\, r_a^2 - \underbrace{(\vec{\Omega}\cdot\vec{r}_a)(\vec{\Omega}\cdot\vec{r}_a)}_{\sum_i \Omega_i x_{ai} \sum_j \Omega_j x_{aj}} \right)$$

$$\sum_i \Omega_i^2 = \sum_{i,j} \Omega_i \Omega_j \delta_{ij}$$

$$\sum_i \Omega_i x_{ai} \sum_j \Omega_j x_{aj}$$

$$= \frac{1}{2}\mu|\vec{V}|^2 + \sum_a \frac{1}{2} m_a \sum_{i,j} \left(r_a^2 \delta_{ij} - x_{ai} x_{aj} \right) \Omega_i \Omega_j$$

$$= \frac{1}{2}\mu|\vec{V}|^2 + \frac{1}{2} \sum_{i,j} \underbrace{\left(\sum_a m_a \left(r_a^2 \delta_{ij} - x_{ai} x_{aj} \right) \right)}_{I_{ij}} \Omega_i \Omega_j$$

$$= \boxed{\frac{1}{2}\mu|\vec{V}|^2 + \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j}$$

Lec #23: Nov 9th

Quiz #5: next Tuesday

EXAM 2: Tuesday 11/23

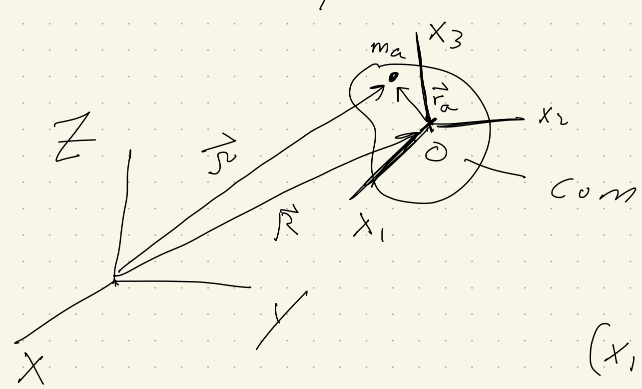
Today: Inertia tensor

Thurs: Equation of motion

Last time:

$$T = \frac{1}{2}\mu|\vec{V}|^2 + \frac{1}{2} \underbrace{I_{ij}\Omega_i\Omega_j}$$

$$\underbrace{\sum_{i=1}^{3}\sum_{j=1}^{3}}_{\text{"drop"}} I_{ij}\Omega_i\Omega_j$$



$$(x_1, x_2, x_3) = (x, y, z)$$

$$I_{ij} = \sum_a m_a \left(r_a^2 \delta_{ij} - x_{ai} x_{aj} \right)$$

$$r_a^2 = \sum_i x_{ai}^2$$

$dm = \rho \, dV$

mass density

$$\boxed{I_{ij} = \int \rho \, dV \left(r^2 \delta_{ij} - x_i x_j \right)}$$

Angular momentum: (wrt COM)

$$\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a \qquad \vec{p}_a = m_a \vec{v}_a$$

$$= \sum_a m_a \vec{r}_a \times \vec{v}_a$$

$$= \sum_a m_a \vec{r}_a \times \left(\cancel{\vec{V}}^{\,0} + \vec{\Omega} \times \vec{r}_a \right)$$

$$= \sum_a m_a \vec{r}_a \times \left(\vec{\Omega} \times \vec{r}_a \right)$$

$$\vec{M} = \sum_a m_a \left(\vec{\Omega} \, r_a^2 - \vec{r}_a \left(\vec{\Omega} \cdot \vec{r}_a \right) \right)$$

$$M_i = \sum_a m_a \left(\underset{\substack{\| \\ \delta_{ij} \Omega_j}}{\Omega_i} r_a^2 - x_{ai} \left(\Omega_j x_{aj} \right) \right)$$

$$= \sum_a m_a \left(\delta_{ij} r_a^2 - x_{ai} x_{aj} \right) \Omega_j$$

$$= I_{ij} \Omega_j$$

$$\boxed{\vec{M} = \underline{\underline{I}} \cdot \vec{\Omega}}$$

$$\boxed{M_i = I_{ij} \Omega_j \quad , \quad T = \frac{1}{2} \mu V^2 + \frac{1}{2} I_{ij} \Omega_i \Omega_j}$$

COM frame

Properties of I_{ij}:

inertia tensor (3x3 matrix)

1) I_{ij} real, symmetric $(I_{ij} = I_{ji})$

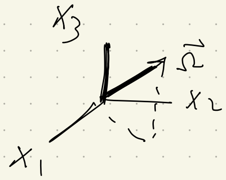
can always diagonalize:

$$I_{ij} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} = \delta_{ij} I_i \quad \text{(no summation)}$$

(x_1, x_2, x_3) for such a basis are called "principle axes"

$$T = \frac{1}{2}\mu V^2 + \frac{1}{2} I_i \Omega_i^2$$

$$M_i = I_i \Omega_i \qquad (M_1 = I_1 \Omega_1,\ M_2 = I_2 \Omega_2,\ M_3 = I_3 \Omega_3)$$

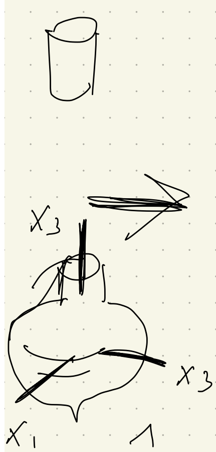


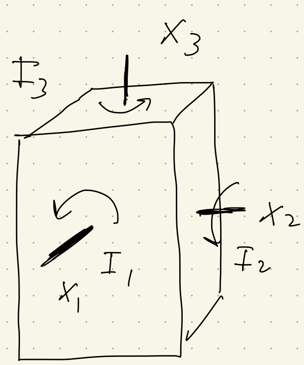
$\vec{\Omega} = (1,1,1)$

$I_1 = 1,\ I_2 = 1,\ I_3 = 2$

$\vec{M} = (1,1,2)$

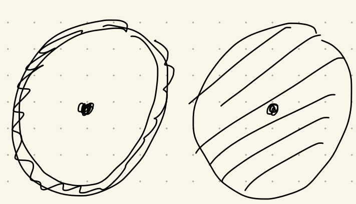
$I_1 \neq I_2 \neq I_3$ in general



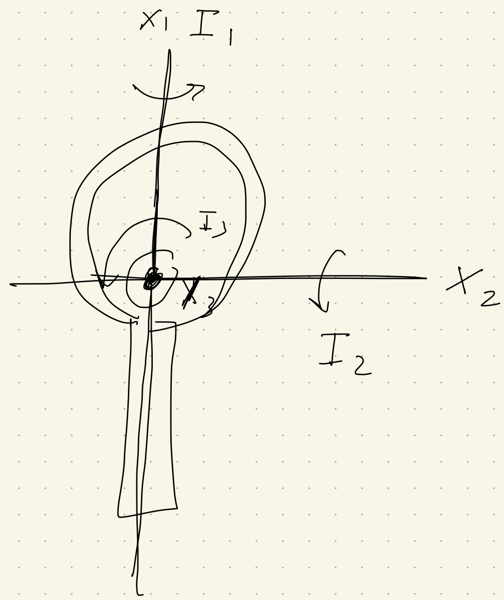


$$\underbrace{I_1 > I_2 > I_3}_{\text{moment of inertia about axis } x_1, x_2, x_3}$$

$$\hat{n}: \quad \underbrace{I(\hat{n}) = I_{ij} n_i n_j}_{\text{moment of inertia about } \hat{n}}$$



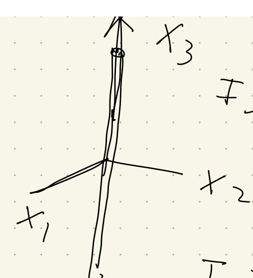
$I_1 < I_2 < I_3$



$I_3 = 0$ "rotator"

$I_1 = I_2 = I_3$ "sphere"

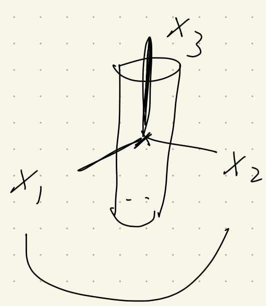
"cube"



$I_3 = 0$

$I_1 = I_2 \neq 0$

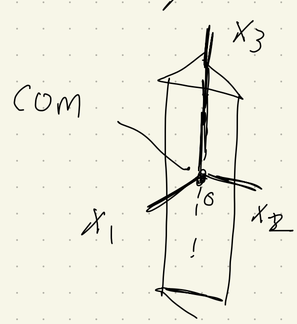
$I_1 = I_2$, "symmetric top"



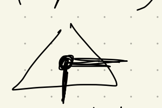
arbitrarily chosen in plane $\perp$ to x_3

$I_1 \neq I_2 \neq I_3$

"asymmetric top"

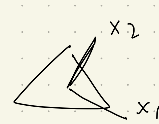


(top view)

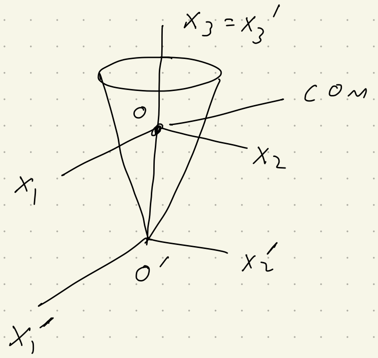


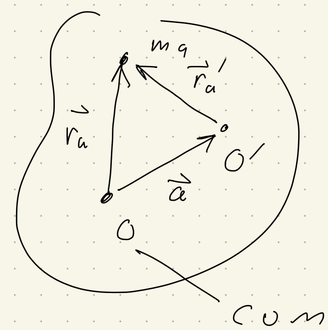
equilateral triangle

$I_1 \overset{??}{=} I_2$



Shift the origin:





$$\vec{r}_a = \vec{a} + \vec{r}_a'$$

$$\vec{r}_a' = \vec{r}_a - \vec{a}$$

$$I'_{ij} = \sum_a m_a \Big(\delta_{ij} \underbrace{|\vec{r}_a'|^2}_{|\vec{r}_a|^2 + |\vec{a}|^2 + 2\vec{r}_a \cdot \vec{a}} - \underbrace{x'_{ai}\, x'_{aj}} \Big)$$

$$(x_{ai} - a_i)(x_{aj} - a_j) = x_{ai} x_{aj} + a_i a_j - a_i x_{aj} - a_j x_{ai}$$

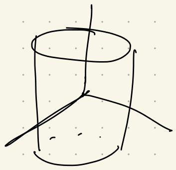
$$I'_{ij} = \underbrace{\sum_a m_a \left(\delta_{ij} |\vec{r}_a|^2 - x_{ai} x_{aj} \right)}_{I_{ij}}$$

$$+ \underbrace{\sum_a}_{\mu} m_a \left(\delta_{ij} |\vec{a}|^2 - a_i a_j \right)$$

$$+ 2 \underbrace{\cancel{\sum_a m_a \vec{r}_a}}_{\to 0} \cdot \vec{a} - \underbrace{\cancel{\sum_a m_a x_{aj}}}_{\to 0} a_i - \underbrace{\cancel{\sum_a m_a x_{ai}}}_{\to 0} a_j$$

$$\mu \vec{R}_{COM} = 0$$

$$I'_{ij} = I_{ij} + \mu \left(\delta_{ij} a^2 - a_i a_j \right)$$



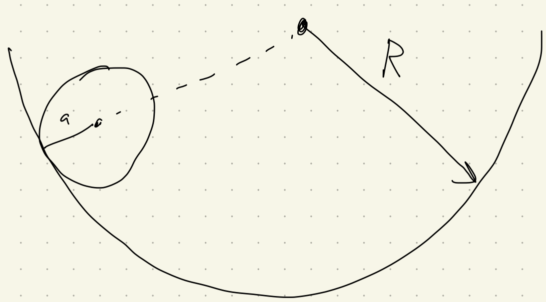
I_1, I_2, I_3

unif cylinder : μ, R, h

Sec 32, Prob 2c

rolling without slipping

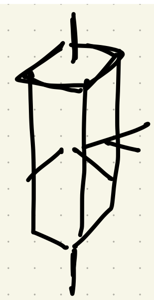
$T = ?$



$T = ?$

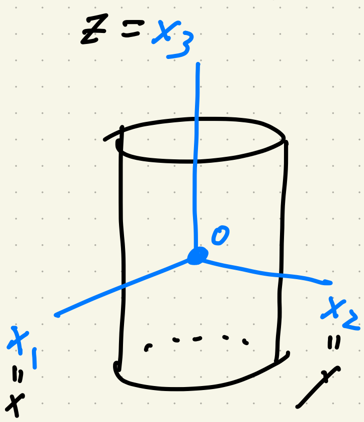
Lecture #24: Thur Nov 11th

Quiz #5 – Tuesday Nov 16th
Exam #2 – Tuesday Nov 23rd



Inertia tensor:

$$I_{ij} = \int \rho \, dV \left(r^2 \delta_{ij} - x_i x_j \right)$$

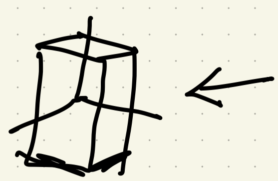


unif cylinder
mass: μ
radius: R
height: h

$$\rho = \frac{\mu}{\pi R^2 \cdot h}$$

$I_3 =$

$I = I_1 = I_2$



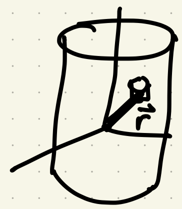
O : com of the cylinder

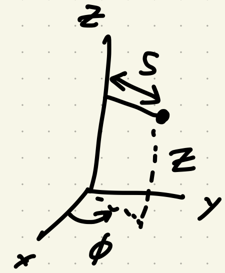
$$\sum_{i,j} I_{ij}\, n_i n_j = I(\hat{n})$$

$$I_3 = \sum_{i,j} I_{ij}\, (x_3)_i (x_3)_j = I_{33}$$

$$I_1 = I_{11}$$

$$I_2 = I_{22}$$





$$I_3 = \int \rho\, dV \left(r^2 \delta_{33} - x_3^2 \right)$$

$$= \int \rho\, dV \left(r^2 - z^2 \right)$$

sph. polar radius

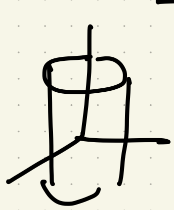
$$= \int \rho\, dV \left(\underbrace{x^2 + y^2}_{s^2} \right)$$

$$dV = s\, ds\, d\phi\, dz$$

$$I_3 = \rho \int_{s=0}^{R} \int_{\phi=0} \int_{z=-\frac{h}{2}}^{h/2} s\, ds\, d\phi\, dz\; s^2$$

$$= \frac{M}{\pi R^2 h}\, 2\pi\, h \underbrace{\int_{s=0}^{R} s^3\, ds}_{\left.\frac{s^4}{4}\right|_0^R = \frac{R^4}{4}}$$

$$= \boxed{\frac{1}{2} M R^2}$$



$$I = I_1 = \rho \int dV \, (\underbrace{r^2 - x_1^2}_{y^2+z^2})$$

(s, ϕ, z)

$y = s \sin\phi$

$x = s \cos\phi$

$$= \rho \int dV \left(y^2 + z^2\right)$$

$$I = I_2 = \rho \int dV \left(x^2 + z^2\right)$$

$$2I = I_1 + I_2$$

$$= \rho \int dV \left(x^2 + y^2 + 2z^2\right)$$

$$= \rho \int dV \, s^2 + 2\rho \int dV \, z^2$$

$$\rightarrow \boxed{I = \frac{1}{2} I_3 + \rho \int dV \, z^2} \leftarrow$$

$$I_1 = I_2 = \frac{1}{2} I_3 + \rho \int dV \, z^2$$

$$= \frac{1}{4} M R^2 + \frac{M}{\pi R^2 h} \int_{s=0}^{R} \int_{z=-\frac{h}{2}}^{\frac{h}{2}} \int_{\phi=0}^{2\pi} s \, ds \, d\phi \, dz \; z^2$$

$$= \frac{1}{4} M R^2 + \frac{M}{\pi R^2 h} \, 2\pi \underbrace{\left(\frac{z^3}{3}\right)\Bigg|_{-h/2}^{h/2}} \; \frac{s^2}{2}\Bigg|_0^R$$

$$+ \frac{M}{\cancel{\pi} \cancel{R^2} h} \cancel{2} \cancel{\pi} \, \frac{2}{3} \left(\frac{h}{2}\right)^3 \frac{\cancel{R^2}}{\cancel{2}}$$

$\frac{1}{12}$

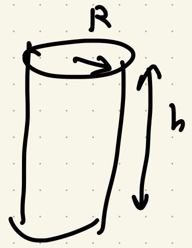
$\frac{1}{12} M h^2$

$I_1 = I_2 =$

$$= \frac{1}{4} M R^2 + \frac{1}{12} M h^2 = \boxed{\frac{1}{4} M \left(R^2 + \frac{h^2}{3}\right)}$$

$$I_1 = I_2 = \frac{1}{4} M\left(R^2 + \frac{1}{3} h^2\right)$$

$$I_3 = \frac{1}{2} M R^2$$



i) thin disk $(h \to 0)$

$$\begin{cases} I_3 = \frac{1}{2} M R^2 \\ I_1 = I_2 = \frac{1}{4} M R^2 \end{cases}$$

ii) $(R \to 0)$

thin rod

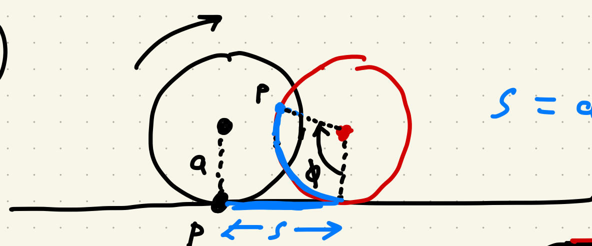
$$\begin{cases} I_3 = 0 \\ I_1 = I_2 = \frac{1}{12} M h^2 \end{cases}$$

Calculate: KE of a rolling cylinder (rolling w/out slipping)

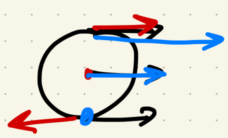
$(M, a, \text{height} = h)$

$$T = \frac{1}{2} M |\vec{V}|^2 + \frac{1}{2} I_{ij} \Omega_i \Omega_j$$
$$= \frac{1}{2} M |\vec{V}|^2 + \frac{1}{2}\left(\cancel{I_1 \Omega_1^2} + \cancel{I_2 \Omega_2^2} + I_3 \Omega_3^2\right)$$

i)



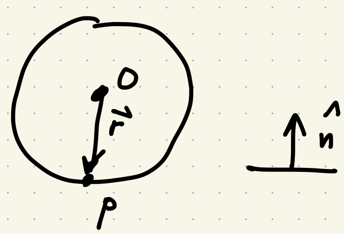
$s = a\phi$



$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$$

$$V = \dot{s} = a\dot{\phi}$$

$$\Omega = \frac{V}{a} = \frac{a\dot{\phi}}{a} = \dot{\phi}$$

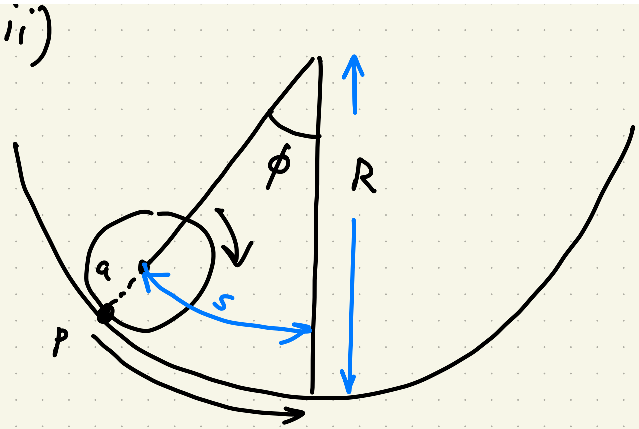


$$0 = \vec{V} + \underbrace{\vec{\Omega}}_{\text{into page}} \times (-a\hat{n})$$

$$\boxed{V = \Omega a}$$

$$T = \frac{1}{2} M a^2 \dot{\phi}^2 + \frac{1}{2}\left(\frac{1}{2} M a^2\right) \dot{\phi}^2$$
$$= \boxed{\frac{3}{4} M a^2 \dot{\phi}^2}$$

ii)

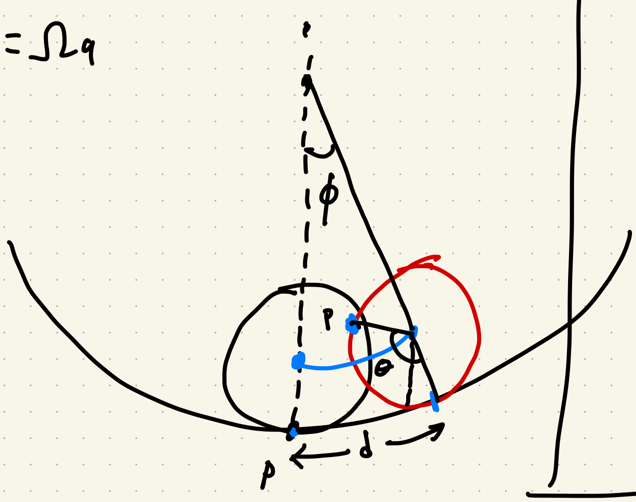


$$s = (R-a)\phi$$

$$V = \Omega a$$

$$V = \dot{s}$$
$$= (R-a)\dot{\phi}$$
$$V = \Omega a$$
$$\Omega = \left(\frac{R-a}{a}\right)\dot{\phi} \rightarrow \boxed{T = \frac{3}{4}M(R-a)^2\dot{\phi}^2}$$

$$V = \Omega a$$



$$\alpha = \theta - \phi$$
$$\dot{\alpha} = \dot{\theta} - \dot{\phi}$$
$$= \frac{R}{a}\dot{\phi} - \dot{\phi}$$
$$= \left(\frac{R}{a} - 1\right)\dot{\phi}$$
$$= \left(\frac{R-a}{a}\right)\dot{\phi}$$
$$= \Omega$$

$$a\theta = d = R\phi \rightarrow \theta = \frac{R}{a}\phi$$

$$s = (R-a)\phi$$

$$V = \dot{s} = (R-a)\dot{\phi} = \Omega a \rightarrow \boxed{\Omega = \left(\frac{R-a}{a}\right)\dot{\phi}}$$

Lecture #25: Nov 16th

EXAM 2 – Next tuesday 11/23

(Zoom review session: 7pm – 8pm, Sunday)

Today: – Quiz #5

– RB motion

Thursday – RB motion / statics ??
(Sec 38)

Tues: 11/23 EXAM 2

Thurs: Thanksgiving

Tues: 11/30: non-inertial ref frames

Sat 12/4: 1:30 – 4:00pm $\left\{ \begin{array}{l} \text{sat} \\ \text{sun} \\ \text{mon} \end{array} \right.$

EOMs:

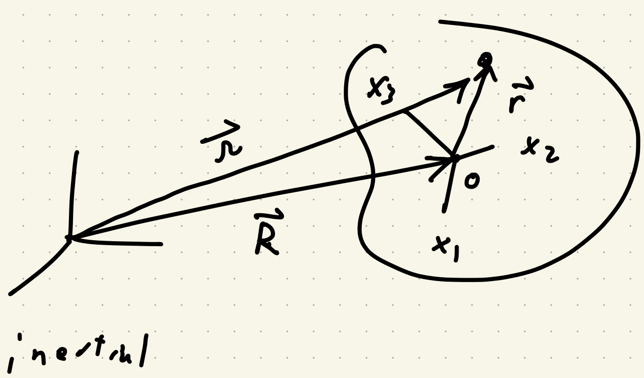
$$\frac{d\vec{M}}{dt} = \vec{K} \equiv \sum \vec{r} \times \vec{f}$$

$$\uparrow \sum_a \vec{r}_a \times \vec{f}_a$$

$$\frac{d\vec{P}}{dt} = \vec{F} = \sum \vec{f}$$

$$6 \text{ DOF's}: \left(\vec{R}, \vec{\phi}\right) = q_i$$

$$\left(\vec{V}, \vec{\Omega}\right) = \dot{q}_i$$



$$\vec{v} = \frac{d\vec{\mathfrak{r}}}{dt}$$

$$= \frac{d}{dt}\left(\vec{R} + \vec{r}\right)$$

$$= \frac{d\vec{R}}{dt} + \frac{d\vec{r}}{dt}$$

$$= \vec{V} + \vec{\Omega} \times \vec{r}$$

Derive:

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}\left(\sum \vec{p}\right) = \sum \frac{d\vec{p}}{dt} = \sum \vec{f}$$

$$\frac{d\vec{M}}{dt} = \frac{d}{dt}\left(\sum \vec{r}\times\vec{p}\right) = \sum\left(\cancelto{0}{\dot{\vec{r}}\times\vec{p}} + \vec{r}\times\dot{\vec{p}}\right)$$

$$\dot{\vec{p}} = \vec{f}$$

$$\vec{p} = m\vec{v} = m\frac{d\vec{\Omega}}{dt} = m\frac{d\vec{r}}{dt} = m\dot{\vec{r}}$$

$$= \sum \vec{r}\times\vec{f}$$

$$\dot{\vec{r}} = \vec{\Omega}\times\vec{r}$$

Alternative derivation:

$\sum_{i,j}$ implied

$$L = \underbrace{\frac{1}{2}\mu|\vec{V}|^2 + \frac{1}{2}I_{ij}\Omega_i\Omega_j}_{T} - U(\vec{\mathcal{R}})$$

"are"

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{V}}\right) = \frac{\partial L}{\partial \vec{R}} \longleftrightarrow \frac{d\vec{P}}{dt} = \sum \vec{f}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{\Omega}}\right) = \frac{\partial L}{\partial \vec{\phi}} \longleftrightarrow \frac{d\vec{M}}{dt} = \sum \vec{r}\times\vec{f}$$

$$\frac{\partial L}{\partial \vec{V}} = \mu\vec{V} = \vec{P}$$

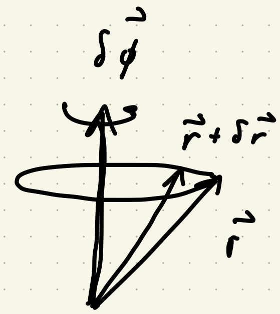
$$\frac{\partial L}{\partial \vec{\Omega}} = \vec{M}$$

$$\frac{\partial L}{\partial \Omega_i} = \sum_j I_{ij}\Omega_j = M_i$$

$$\frac{1}{2} I_{ij}\,\Omega_i\,\Omega_j = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$\frac{\partial L}{\partial \Omega_3} = I_3\,\Omega_3 = M_3$$

$$\delta U(\vec{r}) = \sum_a \frac{\partial U}{\partial \vec{r}_a}\cdot\delta\vec{r}_a$$



$$\vec{r} = \vec{R} + \vec{r}$$

$$\delta\vec{r} = \delta\vec{R} + \delta\vec{r}$$

$$= \delta\vec{R} + \delta\vec{\phi}\times\vec{r}$$

$$\delta U = \sum \underbrace{\frac{\partial U}{\partial \vec{r}}}_{-\vec{f}}\cdot\left(\delta\vec{R} + \delta\vec{\phi}\times\vec{r}\right)$$

$$\delta U = -\left(\sum \vec{f}\right)\cdot\delta\vec{R} - \sum \underbrace{\vec{f}\cdot\left(\delta\vec{\phi}\times\vec{r}\right)}_{\delta\vec{\phi}\cdot\sum(\vec{r}\times\vec{f})}$$

$$\frac{\partial U}{\partial \vec{R}} = -\sum\vec{f} = -\vec{F}\,,\qquad \frac{\partial U}{\partial \phi} = -\sum\vec{r}\times\vec{f} = -\vec{K}$$

$$\frac{\partial L}{\partial \vec{R}} = -\frac{\partial U}{\partial \vec{R}} = \vec{F}\,,\qquad \frac{\partial L}{\partial \vec{\phi}} = -\frac{\partial U}{\partial \vec{\phi}} = \vec{K}$$

$U(x,y)$ $$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

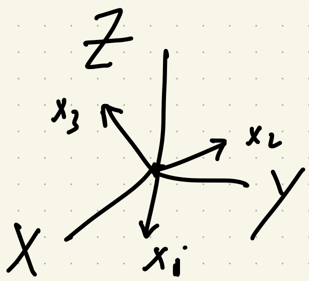
$$\delta U = \frac{\partial U}{\partial \vec{R}}\cdot\delta\vec{R} + \frac{\partial U}{\partial \vec{\phi}}\cdot\delta\vec{\phi}$$

Euler's equations for RB motion:

$$\underbrace{\frac{d\vec{A}}{dt}}_{\text{wrt inertial frame}} = \underbrace{\frac{d'\vec{A}}{dt}}_{\text{wrt rotating frame}} + \vec{\Omega} \times \vec{A}$$

$\vec{\Omega}$: ang. velocity of the rotating frame

$$\vec{A} = \sum_i A_i \hat{x}_i \qquad \left(= A_X \hat{X} + A_Y \hat{Y} + A_Z \hat{Z}\right)$$



$$\frac{d\vec{A}}{dt} = \underbrace{\sum_i \frac{dA_i}{dt}\hat{x}_i}_{\frac{d'\vec{A}}{dt}} + \underbrace{\sum_i A_i \frac{d\hat{x}_i}{dt}}_{\vec{\Omega}\times\vec{A}}$$

$$\left(\frac{d'\vec{A}}{dt}\right)_i = \frac{dA_i}{dt}$$

$$\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\Omega}\times\vec{A}$$

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d'\vec{P}}{dt} + \vec{\Omega}\times\vec{P}$$

$P_i = \mu V_i$

$$F_1 = \left(\frac{d'\vec{P}}{dt}\right)_1 + (\vec{\Omega}\times\vec{P})_1$$

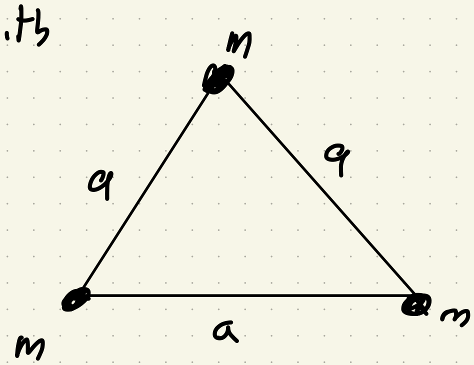
$$= \frac{dP_1}{dt} + \Omega_2 P_3 - \Omega_3 P_2$$

$$= \mu\left(\frac{dV_1}{dt} + \Omega_2 V_3 - \Omega_3 V_2\right)$$

+ cyclic

q5_ lastname :

Calculate the principal moments of inertia for a rigid body which consists of 3 equal masses at the corners of an equilateral triangle with side = a



Solution: take points in xy plane. Since masses are equal, and an equilateral Δ has 120° rotational symmetry $\rightarrow$ symmetric top (so $I_1 = I_2 \equiv I$)

Also, because points don't have a z-component:

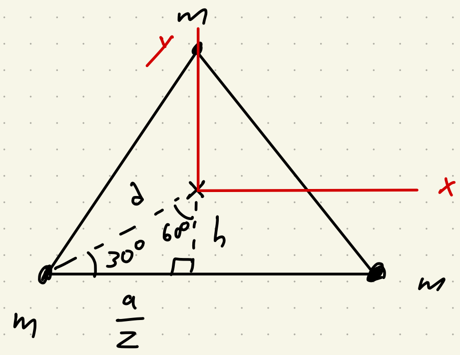
$$I_1 = \sum_a m_a (y_a^2 + \underbrace{z_a^2}_{0}) = \sum_a m_a y_a^2$$

$$I_2 = \sum_a m_a (\underbrace{z_a^2}_{0} + x_a^2) = \sum_a m_a x_a^2$$

$$\rightarrow \quad I_3 = \sum_a m_a (x_a^2 + y_a^2) = I_1 + I_2 = 2I$$

Thus, $I = \frac{1}{2} I_3$

To find I_3, note that $x_a^2 + y_a^2 = \perp$ distance of mass points from z-axis



$$\sin 30^\circ = \frac{h}{d} = \frac{1}{2}$$

$$\sin 60^\circ = \frac{a/2}{d} = \frac{\sqrt{3}}{2}$$

$$\rightarrow \boxed{\begin{aligned} d &= \frac{a}{\sqrt{3}} \\ h &= \frac{d}{2} = \frac{a}{2\sqrt{3}} \end{aligned}}$$

Thus,

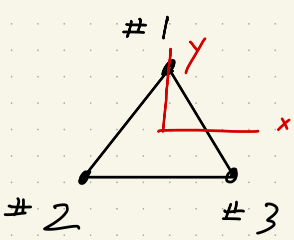
$$I_3 = \sum_a m_a (x_a^2 + y_a^2)$$

$$= \sum_a m_a d_a^2 = 3m\left(\frac{a}{\sqrt{3}}\right)^2 = \boxed{ma^2}$$

$$\rightarrow I_1 = I_2 = \frac{1}{2} I_3 = \boxed{\frac{1}{2} ma^2}$$

Alternative: (explicit calculation of I_1, I_2)

Take (x_1, x_2) to be (x, y)



Then $\#1: \; (x,y) = \left(0, \frac{a}{\sqrt{3}}\right)$

$\#2: \; (x,y) = \left(-\frac{a}{2}, -\frac{a}{2\sqrt{3}}\right)$

$\#3: \; (x,y) = \left(\frac{a}{2}, -\frac{a}{2\sqrt{3}}\right)$

Then $I_1 = \sum_a m_a y_a^2 = m\left(\left(\frac{a}{\sqrt{3}}\right)^2 + \left(\frac{-a}{2\sqrt{3}}\right)^2 + \left(\frac{-a}{2\sqrt{3}}\right)^2\right)$

$$= ma^2\left(\frac{1}{3} + \frac{1}{12} + \frac{1}{12}\right)$$

$$= \boxed{\frac{1}{2} ma^2}$$

$$I_2 = \sum_a m_a x_a^2 = m\left(0^2 + \left(\frac{-a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right)$$

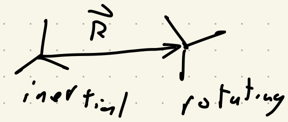
$$= \boxed{\frac{1}{2} ma^2}$$

$$\rightarrow I_3 = I_1 + I_2 = \boxed{ma^2}$$

Recall:

$$\underbrace{\frac{d\vec{A}}{dt}}_{\substack{\text{wrt}\\ \text{inertial}\\ \text{frame}}} = \underbrace{\frac{d'\vec{A}}{dt}}_{\substack{\text{wrt}\\ \text{rotating}\\ \text{frame}}} + \vec{\Omega} \times \vec{A}$$

$\vec{\Omega}$: angular velocity of rotating frame



$$\vec{A} = \sum_i A_i \hat{x}_i$$

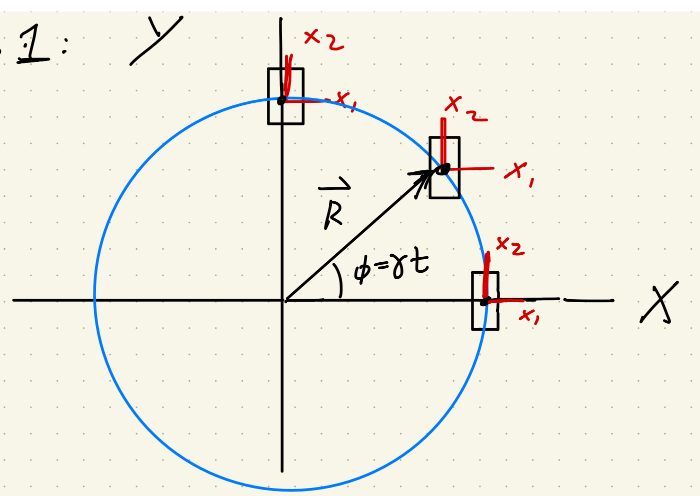
$\hat{x}_i$: basis vectors (time-dependent) for rotating frame

$$\frac{d\vec{A}}{dt} = \underbrace{\sum_i \frac{dA_i}{dt}\hat{x}_i}_{\frac{d'\vec{A}}{dt}} + \underbrace{\sum_i A_i \frac{d\hat{x}_i}{dt}}_{\vec{\Omega}\times\vec{A}}$$

NOTE: i) $\frac{d\vec{A}}{dt}$, $\frac{d'\vec{A}}{dt}$ involve the same vector

ii) Inertial frame and rotating frame do not have to have the same origin.

Example 1:



Com move in circle of radius R with unit angular velocity γ.

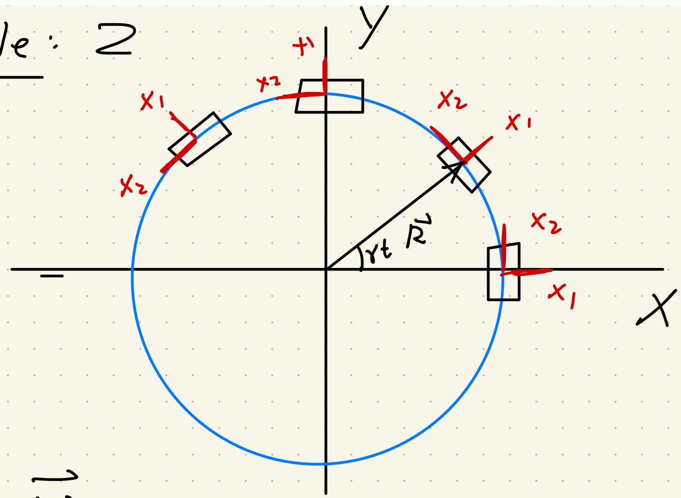
Thus, $$\vec{R} = R\cos(\gamma t)\hat{X} + R\sin(\gamma t)\hat{Y}$$

$$\vec{V} = \frac{d\vec{R}}{dt} = R\gamma\left(-\sin(\gamma t)\hat{X} + \cos(\gamma t)\hat{Y}\right)$$

Since $(\hat{x}_1, \hat{x}_2)$ point in same direction as $(\hat{X}, \hat{Y})$

then $\frac{d\vec{A}}{dt} = \frac{d\vec{A}'}{dt}$ for any $\vec{A}$, including $\vec{V}$

Example: 2



Now the rigid body frame axes are rotating relative to the fixed inertial frame

$\vec{R}$, $\vec{V}$ same as for example 1.

$$\hat{x}_1 = \cos(\gamma t)\,\hat{X} + \sin(\gamma t)\,\hat{Y}$$

$$\hat{x}_2 = -\sin(\gamma t)\,\hat{X} + \cos(\gamma t)\,\hat{Y}$$

$$\frac{d\hat{x}_1}{dt} = -\gamma\sin(\gamma t)\,\hat{X} + \gamma\cos(\gamma t)\,\hat{Y} = \gamma\,\hat{x}_2$$

$$\frac{d\hat{x}_2}{dt} = -\gamma\cos(\gamma t)\,\hat{X} - \gamma\sin(\gamma t)\,\hat{Y} = -\gamma\,\hat{x}_1$$

Thus

$$\sum_i A_i \frac{d\hat{x}_i}{dt} = \gamma A_1 \hat{x}_2 - \gamma A_2 \hat{x}_1$$

$$= \vec{\Omega}\times\vec{A}$$

where $\vec{\Omega} = \gamma\hat{Z} = \gamma\hat{x}_3$

check:

$$(\vec{\Omega}\times\vec{A})_1 = \cancel{\Omega_2}^{\,0} A_3 - \Omega_3 A_2 = -\gamma A_2$$

$$(\vec{\Omega}\times\vec{A})_2 = \Omega_3 A_1 - \cancel{\Omega_1}^{\,0} A_3 = \gamma A_1$$

$$(\vec{\Omega}\times\vec{A})_3 = \cancel{\Omega_1}^{\,0} A_2 - \cancel{\Omega_2}^{\,0} A_1 = 0$$

so $\dfrac{d\vec{A}}{dt} = \dfrac{d'\vec{A}}{dt} + \vec{\Omega}\times\vec{A}$ as claimed

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NOTE: In rotating frame $\vec{R} = R\hat{x}_1$, $\vec{V} = R\gamma\hat{x}_2$

so $\dfrac{d'\vec{R}}{dt} = 0$, $\dfrac{d'\vec{V}}{dt} = 0$ $\Longrightarrow \vec{V} = \dfrac{d\vec{R}}{dt} = 0 + \vec{\Omega}\times\vec{R}$

$$= -\gamma R\sin(\gamma t)\,\hat{x}_1 + \gamma R\cos(\gamma t)\,\hat{x}_2$$

Lecture #26:

EXAM 2 - next Tuesday

Review session (optional) - Sunday, 7pm

Q&A (via Zoom)

Last class: 11/30 (Tuesday) — non-inertial ref frame

scattering, small oscillations, RB motion

Euler's equations of RB motion:

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d'\vec{P}}{dt} + \vec{\Omega} \times \vec{P}$$

$$\vec{K} = \frac{d\vec{M}}{dt} = \frac{d'\vec{M}}{dt} + \vec{\Omega} \times \vec{M}$$

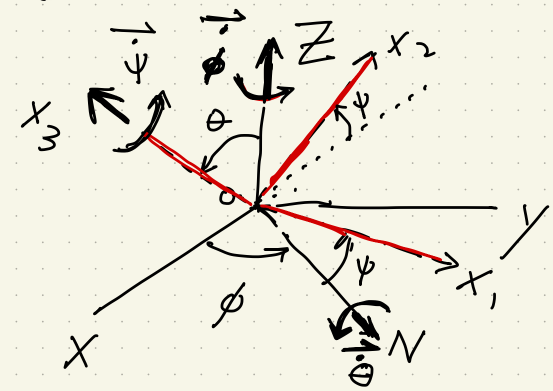
$$\boxed{F_1 = \mu\left(\frac{dV_1}{dt} + \Omega_2 V_3 - \Omega_3 V_2\right)}$$

similar for 2, 3

$$\boxed{K_1 = I_1 \frac{d\Omega_1}{dt} + \Omega_2 \Omega_3 (I_3 - I_2)}$$

similar for 2, 3

Euler angles: $\vec{\phi}$ $\quad (\phi, \theta, \psi)$



$$\boxed{\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta} + \vec{\dot\psi}}$$

$\Omega_1, \Omega_2, \Omega_3$

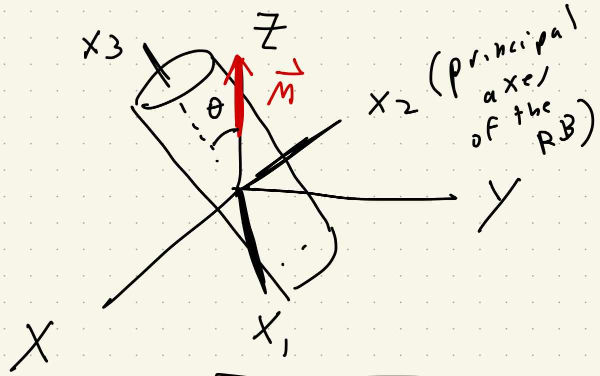
$$\vec{\dot\psi} = \dot\psi\, \hat{x}_3$$

$$\vec{\dot\theta} = \dot\theta\left(\cos\psi\, \hat{x}_1 - \sin\psi\, \hat{x}_2\right)$$

$$\vec{\dot\phi} = \dot\phi\left(\cos\theta\, \hat{x}_3 + \sin\theta\cos\psi\, \hat{x}_2 + \sin\theta\sin\psi\, \hat{x}_1\right)$$

ON: line of nodes

$$\boxed{\vec{\Omega} = \left(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi\right)\hat{x}_1 + \left(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi\right)\hat{x}_2 + \left(\dot\phi\cos\theta + \dot\psi\right)\hat{x}_3}$$



x_1 in (X, Y) plane)

$\psi = 0$

Torque-free motion of a symmetric top

$$\vec{K} = 0 = \frac{d\vec{M}}{dt}$$

$\vec{M} = \text{const}$ (choose to point along $\hat{Z}$)

$$\boxed{\begin{aligned} \Omega_1 &= \dot\theta \\ \Omega_2 &= \dot\phi\sin\theta \\ \Omega_3 &= \dot\phi\cos\theta + \dot\psi \end{aligned}}$$

(when $\psi = 0$)

$M_1 = 0$
$M_2 = M\sin\theta$
$M_3 = M\cos\theta$

$M_1 = I_1\Omega_1$, $M_2 = I_2\Omega_2$, $M_3 = I_3\Omega_3$

$0 = I_1\dot\theta \rightarrow \dot\theta = 0$

$\theta = \text{const}$

$M\cancel{\sin\theta} = I_2\dot\phi\cancel{\sin\theta} \rightarrow \boxed{\dot\phi = \frac{M}{I_2}} = \text{const}$

$$M_3 = I_3 \Omega_3 \qquad \Omega_3 = \frac{M_3}{I_3} = \frac{M\cos\theta}{I_3}$$

$$\Omega_3 = \text{const} \longrightarrow \dot\psi = \text{const}$$

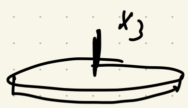
$$\underbrace{\dot\phi}_{=\,\text{const}\,=\,\frac{M}{I_2}}\ \underbrace{\cos\theta}_{\theta=\theta_0} + \dot\psi = \underbrace{\frac{M\cos\theta}{I_3}}_{\theta=\theta_0}$$

---

spin freq $= \Omega_3$

wobble freq $= \dot\phi$

$$\frac{\text{spin}}{\text{wobble}} = \frac{\Omega_3}{\dot\phi} = \frac{\cancel{M}\cos\theta}{I_3}\frac{I_2}{\cancel{M}} = \underbrace{\cos\theta}_{\theta\approx 0}\frac{I_2}{I_3} \simeq \frac{I_2}{I_3} = \boxed{\frac{1}{2}}$$



$$I_3 = \frac{1}{2} M R^2, \quad I_1 = I_2 = \frac{1}{4} M R^2$$

Torque-free motion with $\vec\Omega$ = const:

$\vec M = \text{const}$

$$0 = \Omega_2 \Omega_3 (I_3 - I_2)$$
$$0 = \Omega_3 \Omega_1 (I_1 - I_3)$$
$$0 = \Omega_1 \Omega_2 (I_2 - I_1)$$

$$\left.\begin{aligned} &\Omega_1 = \text{const}, \quad \Omega_2 = 0, \quad \Omega_3 = 0 \\ &\Omega_2 = \text{const}, \quad \Omega_3 = 0, \quad \Omega_1 = 0 \\ &\Omega_3 = \text{const}, \quad \Omega_1 = 0, \quad \Omega_2 = 0 \end{aligned}\right\} \text{rotation about a single principal axis}$$

$$I_1 < I_2 < I_3$$

$\underline{I_1}$: $\quad \Omega_1 = \text{const}, \quad \Omega_2 = 0, \quad \Omega_3 = 0$

$$\Rightarrow \boxed{\Omega_1 = \text{const} + \epsilon_1}, \quad \Omega_2 = \epsilon_2, \quad \Omega_3 = \epsilon_3$$

$\epsilon_1, \epsilon_2, \epsilon_3$: functions of time

$|\epsilon_1| << 1$, etc.

$$K_1 = 0 = I_1 \dot{\Omega}_1 + \Omega_2 \Omega_3 (I_3 - I_2)$$

$$\dot{\Omega}_1 = - \Omega_2 \Omega_3 \left(\frac{I_3 - I_2}{I_1} \right)$$

$$\frac{d}{dt} (\cancel{\text{const}}^{\,0} + \epsilon_1) = - \epsilon_2 \epsilon_3 \left(\frac{I_3 - I_2}{I_1} \right)$$

$$\dot{\epsilon}_1 = - \epsilon_2 \epsilon_3 \left(\frac{I_3 - I_2}{I_1} \right) = O(\epsilon^2) \approx 0$$

$$\boxed{\epsilon_1 = \text{const}} \quad \rightarrow \boxed{\Omega_1 = \text{const}}$$

$$\rightarrow 0 = I_2 \dot{\Omega}_2 + \Omega_3 \Omega_1 (I_1 - I_3)$$

$$\dot{\Omega}_2 = - \Omega_3 \Omega_1 \frac{(I_1 - I_3)}{I_2}$$

$$\boxed{\dot{\epsilon}_2 = - \epsilon_3 \underset{}{\overset{\text{const}}{\Omega_1}} \frac{(I_1 - I_3)}{I_2}}$$

$$\rightarrow 0 = I_3 \dot{\Omega}_3 + \Omega_1 \Omega_2 (I_2 - I_1)$$

$$\dot{\Omega}_3 = - \underset{\text{const}}{\Omega_1} \Omega_2 \frac{(I_2 - I_1)}{I_3}$$

$$\boxed{\dot{\epsilon}_3 = - \epsilon_2 \Omega_1 \frac{(I_2 - I_1)}{I_3}}$$

$$\ddot{\epsilon}_2 = -\Omega_1 \frac{(I_1 - I_3)}{I_2} \boxed{\epsilon_3}$$

$$= +\Omega_1 \frac{(I_1 - I_3)}{I_2} \overbrace{\Omega_1 \frac{(I_2 - I_1)}{I_3} \epsilon_2}$$

$$= +\Omega_1^2 \underbrace{\frac{\overset{<0}{(I_1 - I_3)}\overset{>0}{(I_2 - I_1)}}{I_2 I_3}}_{<0} \epsilon_2$$

$(I_1 < I_2 < I_3)$

$$\boxed{\ddot{\epsilon}_2 = -\lambda^2 \epsilon_2}$$

similarly

$$\boxed{\ddot{\epsilon}_3 = -\lambda^2 \epsilon_3}$$

Sinusoidal evolution

$$\epsilon_3, \epsilon_2 = a\cos(\lambda t) + b\sin(\lambda t)$$

$\Rightarrow$ perturbations remain small

$\Downarrow$

$\Omega_1 =$ const solution

$\Omega_2 = 0$

$\Omega_3 = 0$

stable

Similarly:

$\boxed{\vec{I}_3}$: $\Omega_3 = \text{const}$, $\Omega_1 = 0$, $\Omega_2 = 0$

$\rightarrow$ sinusoidal behavior for ϵ_1, ϵ_2

$\rightarrow$ stable

However:

$I_1 < I_2 < I_3$

$\boxed{\vec{I}_2}$: $\Omega_2 = \text{const}$, $\Omega_1 = 0$, $\Omega_3 = 0$

$$\Omega_2 = \text{const} + \epsilon_2, \quad \Omega_1 = \epsilon_1, \quad \Omega_3 = \epsilon_3$$

$$\ddot{\epsilon}_1 = \Omega_2^2 \frac{\overset{>0}{(I_2 - I_1)}\overset{>0}{(I_3 - I_2)}}{I_3 I_1} \epsilon_1 = \lambda^2 \epsilon_1$$

$$\ddot{\epsilon}_1 = \lambda^2 \epsilon_1$$

$$\underbrace{\epsilon_1(t)} = a \underbrace{e^{\lambda t}}_{\text{grows exponentially with time.}} + b \underbrace{e^{-\lambda t}}_{\text{decays to zero}}$$

$\Omega_1 = 0$

$$\Longrightarrow \boxed{\Omega_2 = \text{const}, \ \Omega_1 = 0, \ \Omega_3 = 0 \text{ is \underline{unstable} to small perturbations.}}$$

Lec # 28 : Tues 11/30

- Today : last class (non-inertial frames) Sec 39
- Saturday, Sunday : oral Finals
  12/4 12/5
- EXAM 2 graded (avg = 11.5/20)
- EXAM 2, Quiz 5 : to be returned today
  (dropped lowest quiz score)

- Please turn in course evaluation

Motion in a non-inertial ref frame:

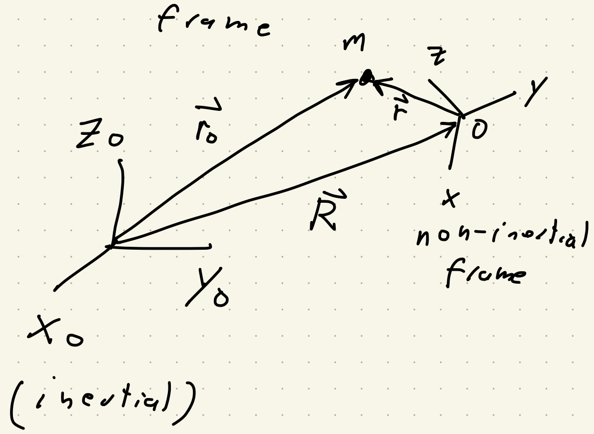
$\vec F = m\vec a_0$ valid only in an inertial frame

$$\left.\frac{d\vec A}{dt}\right|_{0} = \frac{d\vec A}{dt} + \vec\Omega\times\vec A$$

$$\underbrace{\left.\frac{d\vec A}{dt}\right|_{0}}_{\text{inertial}} \qquad \underbrace{\frac{d\vec A}{dt}}_{\text{rotating}}$$

$$m\vec a = \vec F + \vec F_{\text{fictitious}}$$

in a non-inertial frame



$$\vec r_0 = \vec R + \vec r$$

$$\boxed{\vec v_0} = \left.\frac{d\vec r_0}{dt}\right|_{0}$$

$$= \left.\frac{d\vec R}{dt}\right|_{0} + \left.\frac{d\vec r}{dt}\right|_{0}$$

$$= \vec V + \frac{d\vec r}{dt} + \vec\Omega\times\vec r$$

$$= \boxed{\vec V + \vec v + \vec\Omega\times r}$$

$$\boxed{\vec a_0} = \left.\frac{d\vec v_0}{dt}\right|_{0}$$

$$= \left.\frac{d\vec V}{dt}\right|_{0} + \underbrace{\left.\frac{d\vec v}{dt}\right|_{0}} + \left.\frac{d\vec\Omega}{dt}\right|_{0}\times\vec r + \vec\Omega\times\underbrace{\left.\frac{d\vec r}{dt}\right|_{0}}$$

$$= \vec W + \left(\frac{d\vec v}{dt} + \vec\Omega\times\vec v\right) + \dot{\vec\Omega}\times\vec r + \vec\Omega\times\left(\frac{d\vec r}{dt} + \vec\Omega\times\vec r\right)$$

$$= \boxed{\vec W + \vec a + 2\vec\Omega\times\vec v + \dot{\vec\Omega}\times\vec r + \vec\Omega\times(\vec\Omega\times\vec r)}$$

acceleration of the non-inertial

$$\left.\frac{d\vec A}{dt}\right|_{0} = \left.\frac{d}{dt}\right|_{0}\left(\sum_i A_i\hat x_i\right) = \underbrace{\sum_i \frac{dA_i}{dt}\hat x_i}_{=\,\frac{d\vec A}{dt}} + \underbrace{\sum_i A_i\frac{d\hat x_i}{dt}}_{+\,\vec\Omega\times\vec A}$$

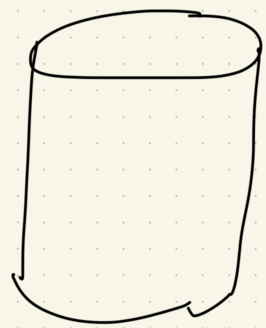
$$\vec{F} = m\vec{a}_0$$

$$= m\left(\vec{a} + \vec{W} + \dot{\vec{\Omega}}\times\vec{r} + 2\vec{\Omega}\times\vec{v} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})\right)$$

$$\rightarrow \boxed{m\vec{a} = \vec{F} - \underbrace{m\vec{W}}_{\text{trans acceleration}} - \underbrace{m\dot{\vec{\Omega}}\times\vec{r}}_{\text{rotational acceleration}} - \underbrace{2m\vec{\Omega}\times\vec{v}}_{\text{coriolis}} - \underbrace{m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}_{\text{centrifugal}}}$$

Newton's 2^{nd} law in a non-inertial frame

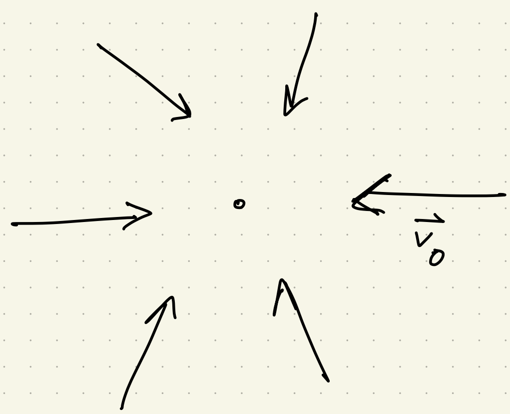
$$\underbrace{\vec{\Omega}\times(\vec{\Omega}\times\vec{r})} = -\Omega^2\,\vec{r}_\perp$$

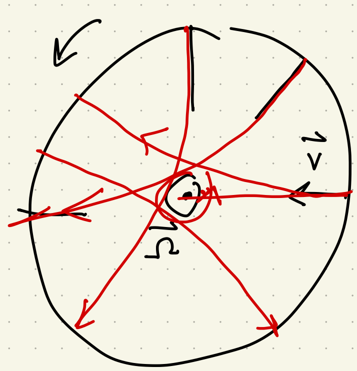


$$= \vec{\Omega}(\vec{\Omega}\cdot\vec{r}) - \vec{r}\,\Omega^2$$

$$= \Omega^2\left(\hat{\Omega}(\hat{\Omega}\cdot\vec{r}) - \vec{r}\right)$$

$$= -\Omega^2\left(\underbrace{\vec{r} - \hat{\Omega}(\hat{\Omega}\cdot\vec{r})}_{\vec{r}_\perp}\right)$$





cyclonic action

$\vec{\Omega} = \Omega\hat{z}$ (out of page

$$\underbrace{L = T - U}_{\text{valid only in an inertial frame}}$$

$$-\frac{\partial U}{\partial \vec{r}} = \vec{F}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) = \frac{\partial L}{\partial \vec{r}}$$

$\longrightarrow$ same EOMs as before

$$L = T_0 - U(\vec{r}_0)$$

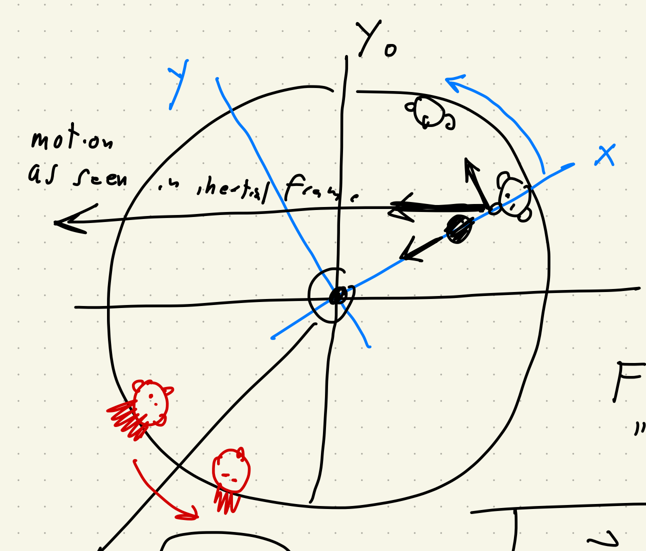
$$= \frac{1}{2} m |\vec{v}_0|^2 - U(\vec{r}_0)$$

$$= \frac{1}{2} m \,|\vec{V} + \vec{v} + \vec{\Omega}\times\vec{r}|^2 - U(\vec{r})$$

$\cancel{\frac{1}{2} m|\vec{V}|^2} \rightarrow$ (ignore) ⋮ throw away total time derivatives

$$= \frac{1}{2} m|\vec{v}|^2 + \frac{1}{2} m\,|\vec{\Omega}\times\vec{r}|^2 - m\,\vec{W}\cdot\vec{r} + m\vec{v}\cdot(\vec{\Omega}\times\vec{r}) - U(\vec{r})$$

Example: Merry-go-round $\equiv$ rotating platform



(non-inertial) fixed to the rotating platform

x_0 (inertial)

Frictionless surface "Hockey puck"

$$\vec{\Omega} = \Omega\,\hat{z} \quad \text{constant}$$

$$\hat{z} = \hat{z}_0$$

$$\boxed{m\vec{a} = -2m(\vec{\Omega}\times\vec{v}) - m\,\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$$

$$\vec{r} = x\,\hat{x} + y\,\hat{y} \qquad \vec{\Omega} = \Omega\,\hat{z}$$

$$\vec{v} = \dot{x}\,\hat{x} + \dot{y}\,\hat{y}$$

$$\vec{a} = \ddot{x}\,\hat{x} + \ddot{y}\,\hat{y}$$

...

$$\boxed{\begin{aligned} \ddot{x} &= 2\dot{y} + \Omega^2 x \\ \ddot{y} &= -2\dot{x} + \Omega^2 y \end{aligned}}$$

A, B: complex constants determined by IC's

"Trick": $\xi = x + iy$

$$\boxed{\ddot{\xi} + 2i\dot{\xi} - \Omega^2 \xi = 0}$$

~~Solve~~: Assume: $\xi = e^{i\lambda t} \longrightarrow \lambda = -\Omega$ (double root)

Solution: $$\boxed{\xi(t) = \underbrace{(A + Bt)}_{\xi_0(t) = x_0(t) + iy_0(t)} e^{-i\Omega t}}$$