Lecture #14: Thu 10/7

Next two lecture — collisions
Following 3 lectures — scattering

1.) Spontaneous disintegration of a single mass $M = m_1 + m_2$ into two particles m_1 and m_2

closed system $\rightarrow$ COM Frame (conservation of total linear momentum)

M

(before)

$\vec{p} = 0$

(after)

m_1 $\vec{p}_1 \equiv \vec{p}_0$

m_2

$\vec{p}_2 = -\vec{p}_0$

$\vec{p}_1 + \vec{p}_2 = 0 \rightarrow \vec{p}_2 = -\vec{p}_1$

Cons. of energy:

COM

$$\rightarrow E_i = E_{1i} + T_{10} + E_{2i} + T_{20}$$

$$= E_{1i} + E_{2i} + \frac{p_0^2}{2m_1} + \frac{p_0^2}{2m_2}$$

$$= E_{1i} + E_{2i} + \frac{p_0^2}{2}\underbrace{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}_{\frac{m_1+m_2}{m_1 m_2} = \frac{1}{m}}$$

E_i: internal energy of M

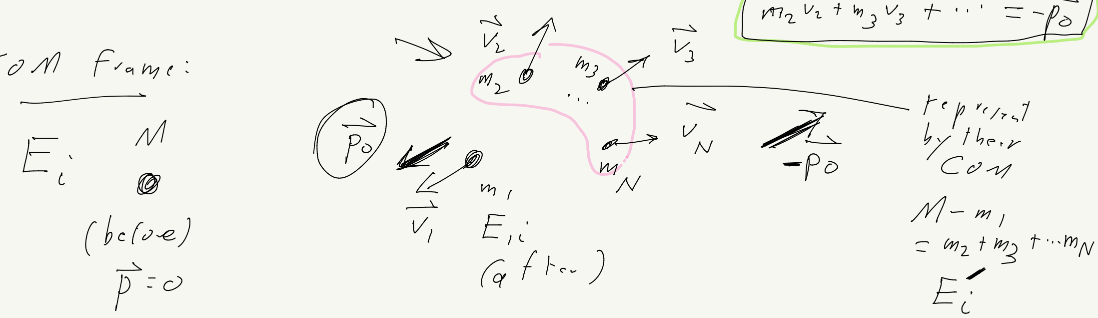
$$\underbrace{E_i - E_{1i} - E_{2i}}_{\epsilon\text{: disintegration energy}} = \frac{p_0^2}{2m}$$

$$\rightarrow p_0 = \sqrt{2m\epsilon} \rightarrow \boxed{v_{10} = \frac{p_0}{m_1}, \quad v_{20} = \frac{p_0}{m_2}}$$

spontaneous disintegration into N masses $m_1, m_2, \dots, m_N$

$$\boxed{m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots = -\vec{p}_0}$$

COM frame:



Upper limit on potential of a single particle m_1

cons. of energy

$$\Rightarrow E_i = \frac{p_0^2}{2m_1} + E_{1i} + \frac{p_0^2}{2(M-m_1)} + E_i'$$

$$= \underline{E_{1i} + E_i'} + \frac{p_0^2}{2}\left(\frac{1}{m_1} + \frac{1}{M-m_1}\right)$$

$$\frac{M}{m_1(M-m_1)}$$

$$E_i - E_{1i} - E_i' = \frac{p_0^2}{2m_1}\left(\frac{M}{M-m_1}\right)$$

$$\underbrace{E_{2i} + E_{3i} + \cdots}_{\text{for maximum KE for } m_1}$$

$$\to \underbrace{\frac{p_0^2}{2m_1}}_{T_{01}} = (E_i - E_{1i} - E_i')\left(\frac{M-m_1}{M}\right)$$

just a number

maximum when E_i' is minimum

$$E_i' + \frac{p_0^2}{2(M-m_1)} = E_{2i} + \frac{1}{2}m_2|\vec{v}_2|^2 + E_{3i} + \frac{1}{2}m_3|\vec{v}_3|^2 + \cdots$$

$$= \underbrace{E_{2i} + E_{3i} + \cdots}_{} + \underbrace{\frac{1}{2}m_2|\vec{v}_2|^2 + \frac{1}{2}m_3|\vec{v}_3|^2 + \cdots}_{}$$

$$-\vec{p}_0 = m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots \quad \to \quad p_0^2 = |m_2\vec{v}_2 + m_3\vec{v}_2 + \cdots|^2$$

$$E_1' = \underset{\substack{\| \\ 0}}{E_{2i}} + \underset{\substack{\| \\ 0}}{E_{3i}} + \cdots + \frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \cdots - \frac{1}{2(m_2+m_3+\cdots)} \left| m_2 \vec{v}_2 + m_3 \vec{v}_3 + \cdots \right|^2$$

possible

$$|m_2 \vec{v}_2 + m_3 \vec{v}_3 + \cdots|^2 = m_2^2 |\vec{v}_2|^2 + m_3^2 |\vec{v}_3|^2 + \cdots + 2 m_2 m_3 \vec{v}_2 \cdot \vec{v}_3 + \cdots$$

$$\boxed{\underline{\text{Suppose}}: \quad \vec{v}_2 = \vec{v}_3 = \cdots \equiv \vec{v}}$$

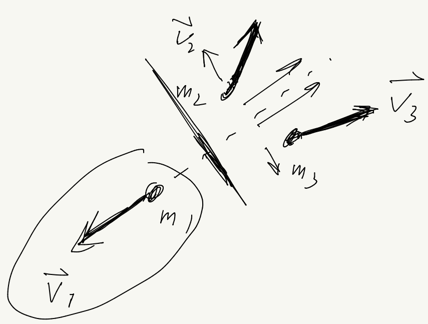
$$\frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \cdots = \frac{1}{2}(m_2 + m_3 + \cdots)|\vec{v}|^2$$

$$- \frac{1}{2(m_2+m_3+\cdots)} (m_2 + m_3 + \cdots)^2 |\vec{v}|^2 = -\frac{1}{2}(m_2 + m_3 + \cdots)|\vec{v}|^2$$

$$\rightarrow E_1' = E_{2i} + E_{3i} + \cdots$$

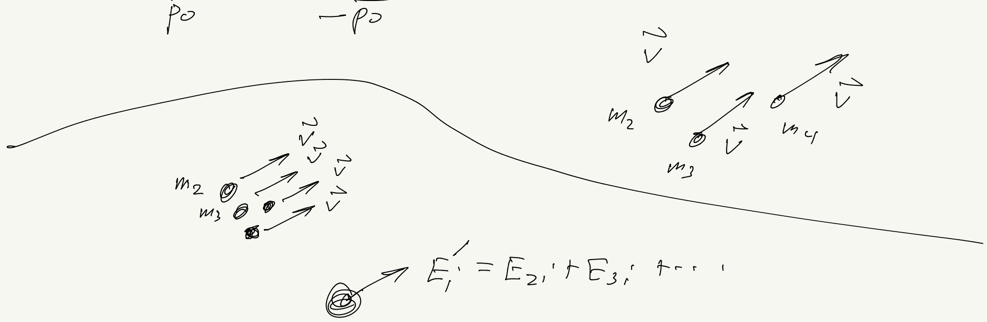
M

$\vec{p} = 0$



$$\frac{p_0^2}{2m_1} = T_{10}$$

$$\underbrace{m_1 \vec{v}_1}_{\vec{p}_0} + \underbrace{m_2 \vec{v}_2 + m_3 \vec{v}_3}_{-\vec{p}_0} = 0$$



$E_1' = E_{2i} + E_{3i} + \cdots$

Added discussion: (meaning of E_i')

Conservation of total energy:

$$E_i = \underbrace{E_{1i} + \tfrac{1}{2} m_1 |\vec{v}_1|^2}_{\text{internal + KE of mass } m_1} + \underbrace{E_{2i} + \tfrac{1}{2} m_2 |\vec{v}_2|^2}_{\text{same for } m_2} + \underbrace{E_{3i} + \tfrac{1}{2} m_3 |\vec{v}_3|^2}_{\text{same for } m_3} + \cdots$$

Let: $T' \equiv \frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \cdots$ (= KE of $m_2, m_3, \cdots$)

$$= \tfrac{1}{2} m_2 |\vec{V}' + \vec{v}_2{}'|^2 + \tfrac{1}{2} m_3 |\vec{V}' + \vec{v}_3{}'|^2 + \cdots$$

where $\vec{V}'$ = velocity of COM of $m_2, m_3, \cdots$

$\vec{v}_2{}'$ = velocity of m_2 with respect to COM of $m_2, m_3, \cdots$

$\vec{v}_3{}'$ = " m_3 "

...

$$\rightarrow T' = \tfrac{1}{2} m_2 \left(|\vec{V}'|^2 + |\vec{v}_2{}'|^2 + 2 \vec{V}' \cdot \vec{v}_2{}' \right) + \cdots$$

$$= \tfrac{1}{2}(m_2 + m_3 + \cdots) |\vec{V}'|^2 + \tfrac{1}{2} m_2 |\vec{v}_2{}'|^2 + \tfrac{1}{2} m_3 |\vec{v}_3{}'|^2 + \cdots$$

$$+ \underbrace{(m_2 \vec{v}_2{}' + m_3 \vec{v}_3{}' + \cdots)}_{=0 \text{ (by definition of COM for } m_2, m_3, \cdots)} \cdot \vec{V}'$$

T'_{com} = KE of COM of $m_2, m_3, \cdots$

T_0' = KE of $m_2, m_3, \cdots$ w.r.t COM of $m_2, m_3, \cdots$

so $T' = T'_{com} + T_0'$

$$\rightarrow E_i = E_{1i} + \tfrac{1}{2} m_1 |\vec{v}_1|^2 + (E_{2i} + E_{3i} + \cdots) + T'_{com} + T_0'$$

$$= E_{1i} + \tfrac{1}{2} m_1 |\vec{v}_1|^2 + (E_{2i} + E_{3i} + \cdots) + \frac{p_0^2}{2(m_2 + m_3 + \cdots)} + T_0'$$

compare to

$$E_i = \left(E_{1i} + \tfrac{1}{2} m_1 |\vec{v}_1|^2\right) + \left(E_i' + \frac{p_0^2}{2(M - m_1)}\right)$$

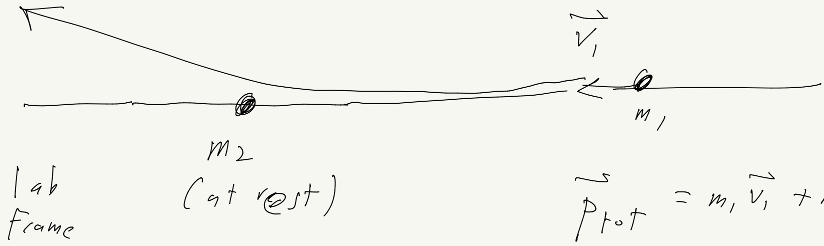
$M - m_1$: $m_2 + m_3 + \cdots$

$$\rightarrow \boxed{\begin{aligned} E_i' &= E_{2i} + E_{3i} + \cdots + T_0' \\ &= (\text{internal energy of } m_2, m_3, \cdots) + (\text{KE of } m_2, m_3, \cdots \text{ w.r.t COM of } m_2, m_3, \cdots) \end{aligned}}$$

NOTE: when $m_2, m_3, \cdots$ all move with the same velocity, they are moving together with the COM of $m_2, m_3, \cdots$. Then $T_0' = 0$ and E_i' = internal energy of $m_2, m_3, \cdots$

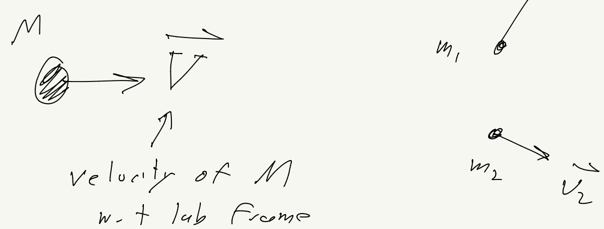
COM Frame vs. lab Frame

Example:

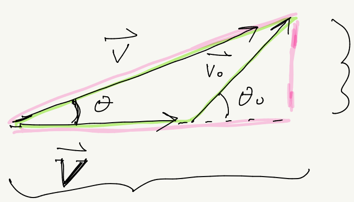


$$\vec{p}_{tot} = m_1 \vec{v}_1 + m_2 \cancel{\vec{v}_2}^{\,0} = m_1 \vec{v}_1 \neq 0$$

lab frame:



$$M\vec{V} = m_1 \vec{v}_1 + m_2 \vec{v}_2$$



θ : w.r.t lab Frame

θ_0 : w.r.t COM Frame

$\vec{v}_0$: velocity of m_1 (or m_2) w.r.t COM Frame

$\vec{V}$: velocity of M w.r.t lab frame

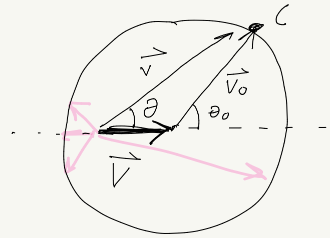
$\vec{v}$: velocity of m_1 (or m_2) w.r.t lab Frame

$$\boxed{v_0^2 = v^2 + V^2 - 2vV\cos\theta}$$

$$v\sin\theta = v_0 \sin\theta_0$$

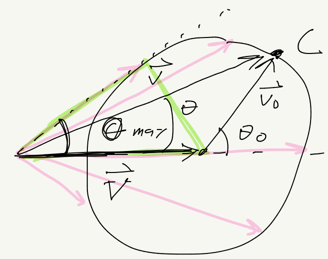
$$v\cos\theta = V + v_0\cos\theta_0$$

$$\div \rightarrow \boxed{\tan\theta = \frac{v_0\sin\theta_0}{V + v_0\cos\theta_0}}$$



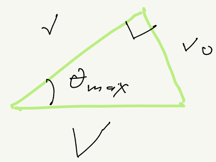
$V < v_0$

$\vec{v}$ can point in any direction



$V > v_0$

$\vec{V}$ can only point in the forward direction



$$\sin\theta_{max} = \frac{v_0}{V}$$

Lec # 15:

grade: max = 20

rescale by $20/16$ (part (b) of long problem # 2)

Average: $12/20$ $9/20$
(with rescaling)

Example: $\frac{14}{20} \longrightarrow 14 \times \frac{20}{16} = 14 \times \frac{10}{8}$

$= \frac{140}{8}$

$= 17. \ldots$

Quiz # 3 — next week sometime

Elastic collisions: (Sec 17)

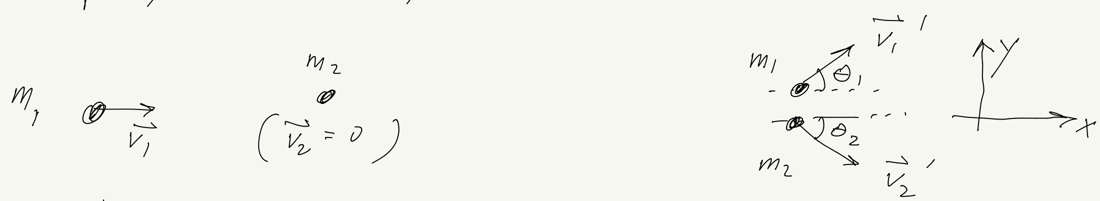
- cons. of linear momentum, angular momentum, total energy

motion was in a 2-d plane

elastic collision — no change in internal energy

→ **Kinetic energy** is conserved

specify masses m_1, m_2 and initial velocities $\vec{v}_1, \vec{v}_2$



(before) (after)

(lab frame)

Unknowns: (4)

$v_1', v_2', \theta_1, \theta_2$

Equations:

$$T_{before} = T_{after} \quad (1 \text{ equation})$$

$$\vec{P}_{before} = \vec{P}_{after} \quad (2 \text{ equations})$$

3 equations

We normally specify θ_2 in order to solve for v_1', v_2', θ_1

Numerical:

$m_1 = 1\,kg$

$m_2 = 2\,kg$

$v_1 = 1\,m/s$

$v_2 = 0$

$\theta_2 = 60^\circ$

$v_1' = ?? = \frac{1}{3}\ m/s$

$v_2' = ?? = \frac{\sqrt{7}}{3}\ m/s$

$\theta_1 = ?? \approx 41^\circ$

$$\frac{1}{2} m_1 |\vec{v}_1|^2 = \frac{1}{2} m_1 |\vec{v}_1\,'|^2 + \frac{1}{2} m_2 |\vec{v}_2\,'|^2$$

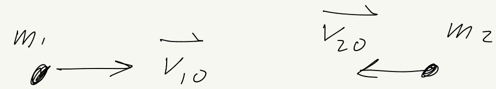
$$\boxed{m_1 v_1^2 = m_1 (v_1')^2 + m_2 (v_2')^2}$$

v_1', v_2', θ_1

$\underline{x:}$ $$\boxed{m_1 v_1 = m_1 v_1' \cos\theta_1 + m_2 v_2' \cos\theta_2}$$

$\underline{y:}$ $$\boxed{0 = m_1 v_1' \sin\theta_1 - m_2 v_2' \sin\theta_2}$$

Analyze in COM frame instead



$$m_1 \vec{v}_{10} + m_2 \vec{v}_{20} = 0$$

$\vec{p}_0 = 0$ (before and after the collision)

$$\vec{V} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$ (velocity of COM in the lab frame)

$\vec{v}_1 = \vec{V} + \vec{v}_{10}$, $\vec{v}_2 = \vec{V} + \vec{v}_{20}$

$\vec{v}_1\,' = \vec{V} + \vec{v}_{10}\,'$, $\vec{v}_2\,' = \vec{V} + \vec{v}_{20}\,'$

lab frame   COM frame

$\vec{r} = \vec{r}_{10} - \vec{r}_{20}$
$= \vec{r}_1 - \vec{r}_2$

$\vec{v} = \vec{v}_{10} - \vec{v}_{20}$
$= \underbrace{\vec{v}_1 - \vec{v}_2}_{\text{relative vector}}$

$\vec{R}$

$$\boxed{T_0 = \frac{1}{2} m_1 |\vec{v}_{10}|^2 + \frac{1}{2} m_2 |\vec{v}_{20}|^2 = \frac{1}{2} m |\vec{v}|^2} \quad , \qquad m = \frac{m_1 m_2}{m_1 + m_2}$$

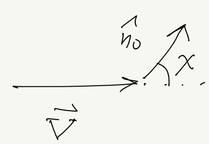
$$\vec{r}_{10} = \frac{m_2}{m_1+m_2}\vec{r} \quad , \qquad \vec{r}_{20} = -\frac{m_1}{m_1+m_2}\vec{r}$$

$$\boxed{\vec{v}_{10} = \frac{m_2}{m_1+m_2}\vec{v}} \quad , \qquad \boxed{\vec{v}_{20} = -\frac{m_1}{m_1+m_2}\vec{v}}$$

$$T_0' = \underbrace{\frac{1}{2} m |\vec{v}'|^2 = T_0 = \frac{1}{2} m |\vec{v}|^2}$$

$$\boxed{|\vec{v}'| = |\vec{v}|} \quad , \qquad \boxed{\begin{aligned} v' &= v \\ v_{10}' &= v_{10} \\ v_{20}' &= v_{20} \end{aligned}}$$

$\vec{v}' = v\hat{n}_0$, $\hat{n}_0$: unit vector makes an angle χ w.r.t com velocity



$$\vec{v} = \vec{v}_{10} - \vec{v}_{20} = \vec{v}_1 - \vec{v}_2$$

w.r.t lab frame

w.r.t com frame

$$\vec{v}_1{}' = \vec{V} + \vec{v}_{10}{}' = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} + \frac{m_2}{m_1+m_2} v\hat{n}_0$$

$$\vec{v}_2{}' = \vec{V} + \vec{v}_{20}{}' = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} - \frac{m_1}{m_1+m_2} v\hat{n}_0$$

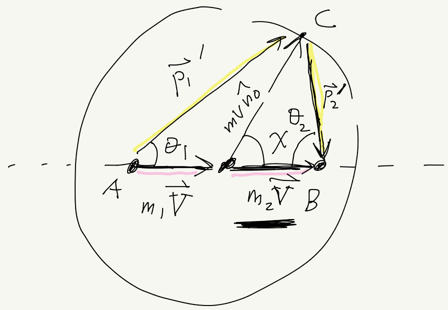
$$\vec{p}_1{}' = m_1\vec{v}_1{}'$$
$$\vec{p}_2{}' = m_2\vec{v}_2{}'$$

$$\boxed{\begin{aligned} \vec{p}_1{}' &= m_1\vec{V} + m v\hat{n}_0 \\ \vec{p}_2{}' &= m_2\vec{V} - m v\hat{n}_0 \end{aligned}}$$

$$\vec{p}_1{}' + \vec{p}_2{}' = (m_1+m_2)\vec{V} = \vec{P} = \vec{p}_1 + \vec{p}_2$$

$\vec{v}_1, \vec{v}_2$ are arbitrary

AB: $\vec{P}$



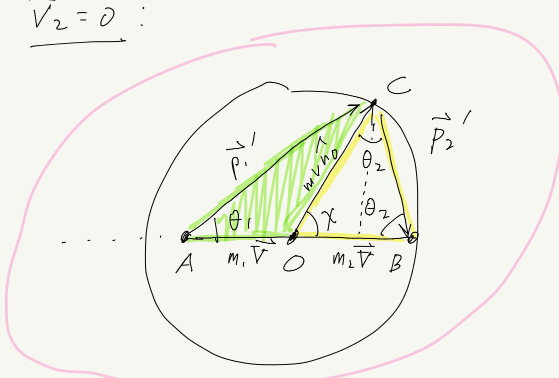
$\underline{\vec{v}_2 = 0}$: (standard lab frame)

$$\vec{v} = \vec{v}_1 - \cancel{\vec{v}_2}^{\,0} = \vec{v}_1$$

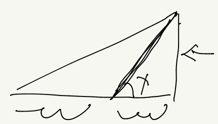
$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} = \frac{m_1\vec{v}_1}{m_1+m_2} = \frac{m_1\vec{v}}{m_1+m_2}$$

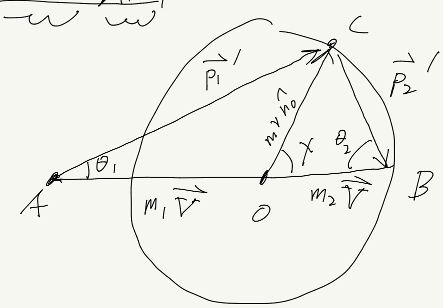
$$m_2\vec{V} = \frac{m_2 m_1 \vec{v}}{m_1+m_2} = m\vec{v} \longrightarrow B \text{ lies on the Circle}$$

$\underline{\vec{v}_2 = 0}$:



$m_1 < m_2$





$m_1 > m_2$

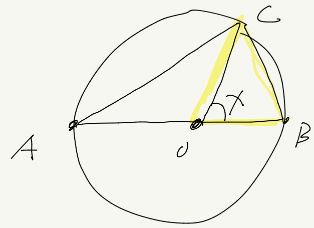
$$\chi + 2\theta_2 = \pi$$

$$\boxed{\theta_2 = \frac{\pi}{2} - \frac{\chi}{2}}$$

$$\tan\theta_1 = \frac{opp}{adj} = \frac{m v \sin\chi}{m_1 V + m v\cos\chi}$$

$$\tan\theta_1 = \frac{\left(\frac{\cancel{m_1} m_2}{\cancel{m_1+m_2}}\right)\cancel{v}\sin\chi}{\cancel{m_1}\left(\frac{m_1 \cancel{v}}{\cancel{m_1+m_2}}\right) + \left(\frac{\cancel{m_1} m_2}{\cancel{m_1+m_2}}\right)\cancel{v}\cos\chi}$$

$$= \frac{m_2\sin\chi}{m_1 + m_2\cos\chi}$$



$m_1 = m_2$

$$\cancel{m_2^2}|v_2'|^2 = 2m^2v^2 - 2m^2v^2\cos\chi$$

$$= 2m^2v^2(1-\cos\chi)$$

$$= 2\frac{m_1^2\cancel{m_2^2}}{(m_1+m_2)^2}v^2\, 2\sin^2\left(\frac{\chi}{2}\right)$$

$$\boxed{v_2' = \left(\frac{2m_1 v}{m_1+m_2}\right)\sin\left(\frac{\chi}{2}\right)}$$

$$\cos\chi = \cos\left(2\frac{\chi}{2}\right)$$

$$= \cos^2\left(\frac{\chi}{2}\right) - \sin^2\left(\frac{\chi}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{\chi}{2}\right)$$

$$1 - \cos\chi = 2\sin^2\left(\frac{\chi}{2}\right)$$

$$\boxed{v_1' = \frac{v}{m_1+m_2}\sqrt{m_1^2 + m_2^2 + 2m_1m_2\cos\chi}}$$