Part 4

Waveform Analysis and Synthesis

Pure Tones and Complex Tones

Pure tones or sine waves have a single frequency. The sound from musical instruments, voice, and "noise", on the other hand, consists of complex waves. There may be a discernible *pitch* of frequency f, but the sound no longer is that of a simple sine wave.

A sustained complex tone contains many *partial waves*, also called *overtones or harmonics*. (These three designations are synonymous for us at present). We shall use mostly the term *harmonics*. Harmonics are sine waves with frequencies from the *harmonic series* f, 2f, 3f, 4f, 5f, 6f, etc. Here *f* is the so-called *fundamental frequency* that corresponds to the pitch of the tone. The fundamental frequency also is called the first harmonic with harmonic number N = 1. Successive harmonics have the integer *harmonic numbers* N = 1, 2, 3, 4, 5, etc. Each harmonic N is a sine wave of frequency $f_N = N \cdot f$. Each of these sine waves (partial waves) has its own *amplitude*. The relative heights of the amplitudes determine the *quality of sound* or *timbre.*

A sound or tone in air arises from a *displacement* or oscillation of the air molecules about their equilibrium position. The molecules move back and forth about this position while the sound travels through the air. (The sound travels, not the air as a whole. There is no wind blowing as you hear music!)

For those interested in more details:
For a sustained tone in air you can represent the displacement y(t) of the molecules as a function of time. The sound wave then is composed of a sum of harmonics or sine waves, given at a fixed position by

$$\mathbf{y(t) = \sum_N A_N \sin(2\pi N f t - \varphi_N)}$$

The sum is taken over all harmonics of harmonic number N =1, 2, 3, 4….
The quantity A_N is the amplitude of N-th harmonic, $f_N = Nf_1$ the frequency, and φ_N the phase difference between the harmonics. Note that f in this expression is the fundamental frequency f_1 with N = 1.
It is a very remarkable mathematical fact that a tone can be described completely by a sum of sine waves with uniquely determined amplitudes A_N, discrete frequencies f_N, and phases φ_N.
The phases φ_N are counted relative to the first harmonic or fundamental N = 1. Most of the time we are unable to hear phase differences and therefore will not discuss them much in this course. (See also "Ohm's Law of Hearing" later).

Fourier's Theorem

"Any periodic wave of fundamental frequency (pitch) f can be assembled or synthesized as a sum of sine waves, also called partial waves or harmonics, of frequency $f_N = Nf$, where N is the harmonic number N = 1, 2, 3, 4, 5, 6, ... Conversely, any periodic wave can be analyzed with sine waves of fundamental frequency f and harmonics $f_N = Nf$. The amplitudes and phases of the harmonics are uniquely defined by the shape of the wave."

Waveforms and Fourier Spectra

A *waveform* shows the displacement, for instance of air molecules, as a function of time. Its *Fourier spectrum* shows the harmonic amplitues as a function of frequency.

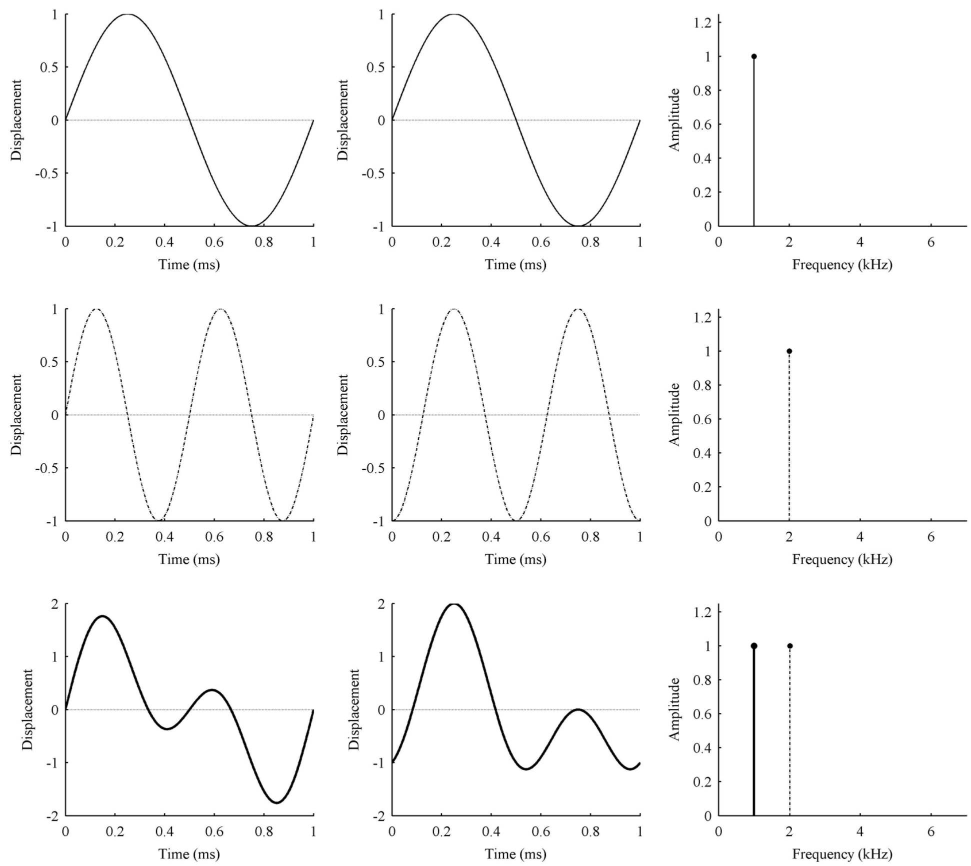


Figure. Waveforms and their Fourier spectra from the combination of two sine waves.
Top: A sine wave corresponding to the 1^{st} harmonic or fundamental (N = 1). The Fourier spectrum is a single spike.
Middle: Two sine waves corresponding to the 2^{nd} harmonic or 1^{st} overtone (N = 2), but 90^0 out-of-phase with respect to each other. The Fourier spectrum is a single spike at twice the frequency of the fundamental.
Bottom: Sum of the two sine waves corresponding to the 1^{st} and 2^{nd} harmonics. The resulting complex waveforms look different, but their Fourier spectra are the same and they also "sound" the same (in accordance with Ohm's law of hearing – see later).

A Complex Waveform

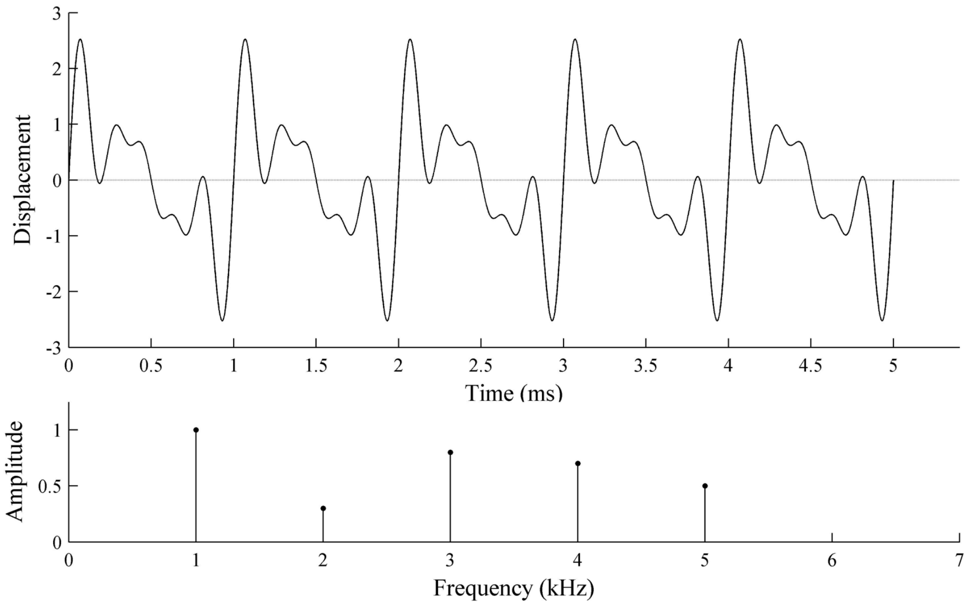


Figure. Top: A complex waveform consisting of the sum of 5 sine waves. Bottom: The Fourier spectrum of the waveform with the relative amplitudes of the 5 harmonics.

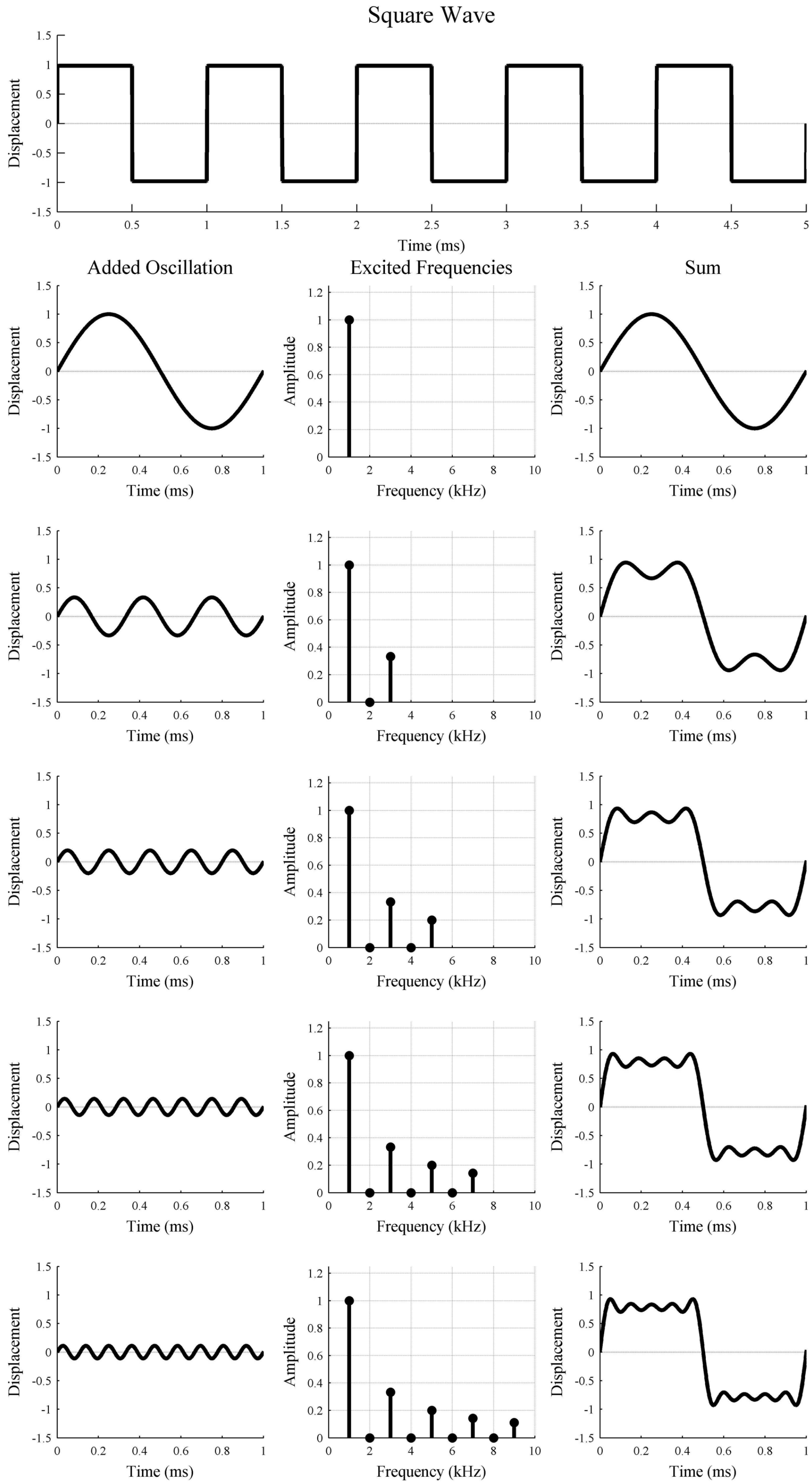
Standard Waveforms

The following pages show a **square wave**, **triangular wave**, and **sawtooth wave**.

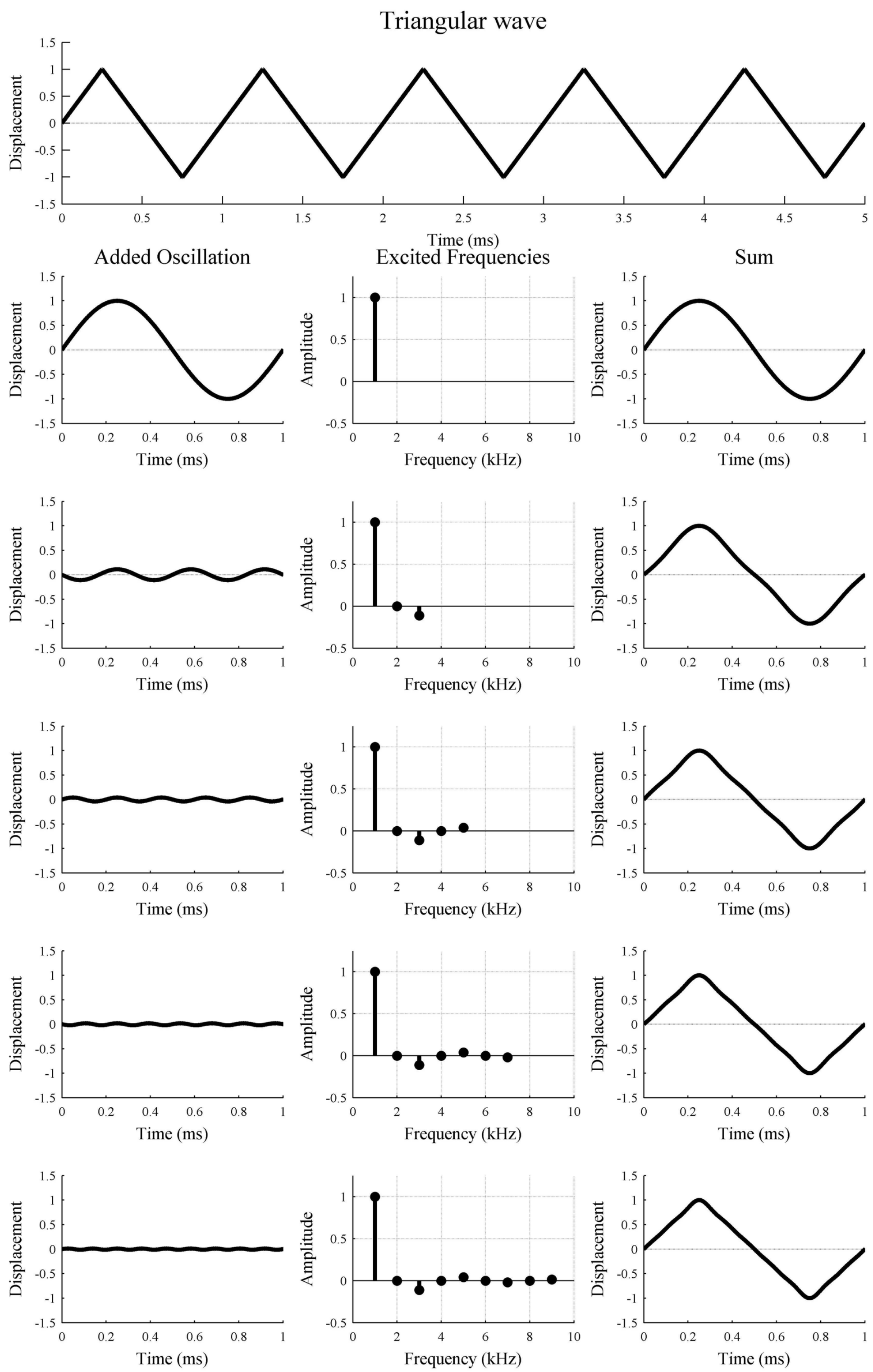
Explanation of the figures:
The Fourier synthesis of the waves is displayed below the original waveforms.
The rows from top to bottom represent the successive inclusion of additional harmonics.
The columns on the left show the waveforms of the individual harmonics.
The columns in the middle show the Fourier spectra up to the N-th harmonic.
The columns on the right show the resulting waveforms up to the N-th harmonic.
It can be seen that, as one progresses down in the right column from top to bottom, the synthesized waveforms resemble more and more the original waveform.
The Fourier synthesis with up to the first 9 harmonics is shown. Of these, only the odd harmonics N = 1, 3, 5, 7, 9 are non-zero.
P. S.: The negative amplitudes A3 and A7 for the triangular wave mean that the harmonics N = 3 and N= 7 start phase-shifted by 180^0 at time t = 0.

For a perfect synthesis of the original waveform, one theoretically would have to include an infinite number of harmonics with decreasing amplitudes.

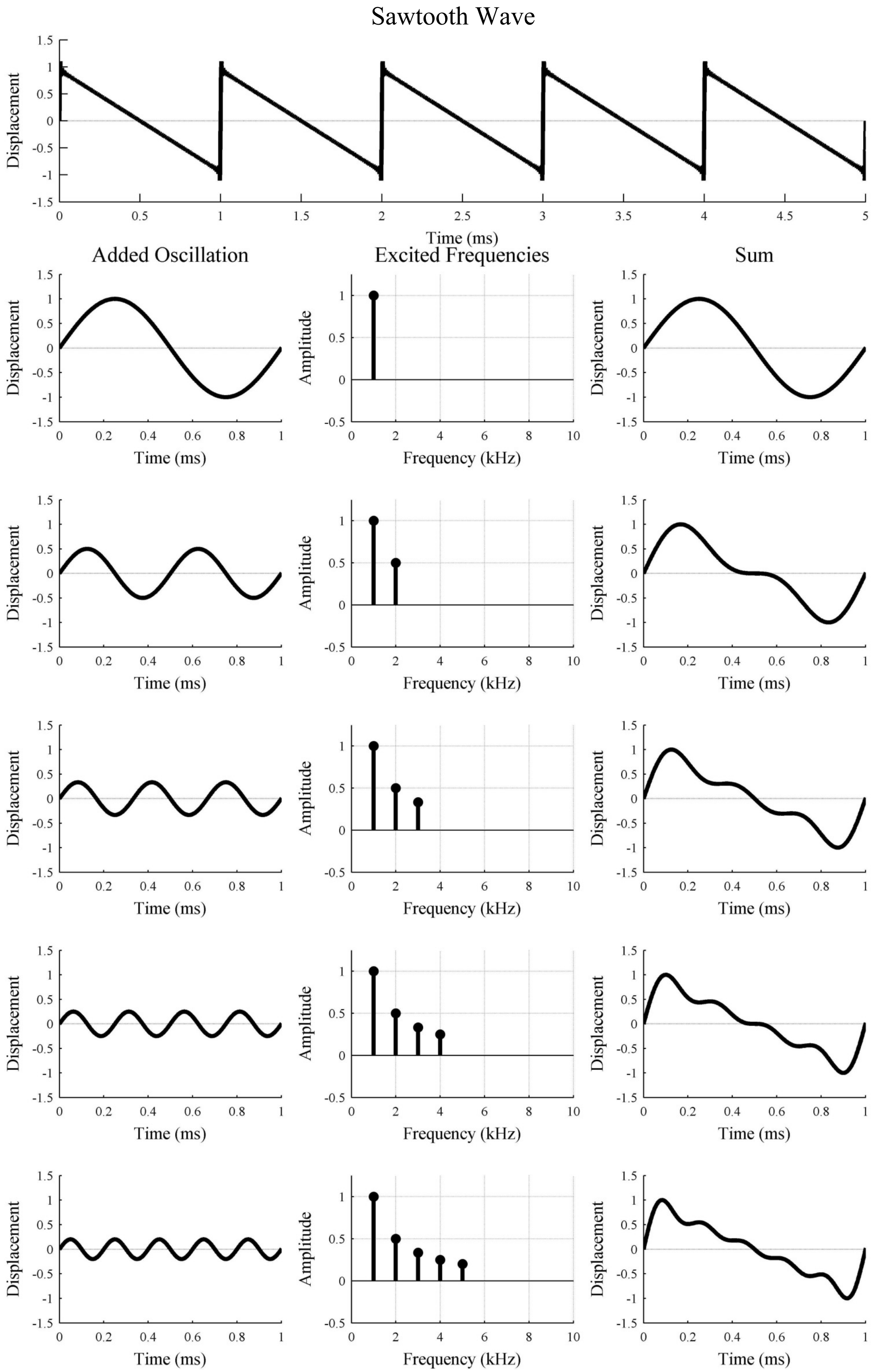
Square Wave and Fourier Synthesis



Triangular Wave and Fourier Synthesis



Sawtooth Wave and Fourier Synthesis



Standard Waveforms continued

Fourier spectrum of a triangular wave Fourier spectrum of a square wave

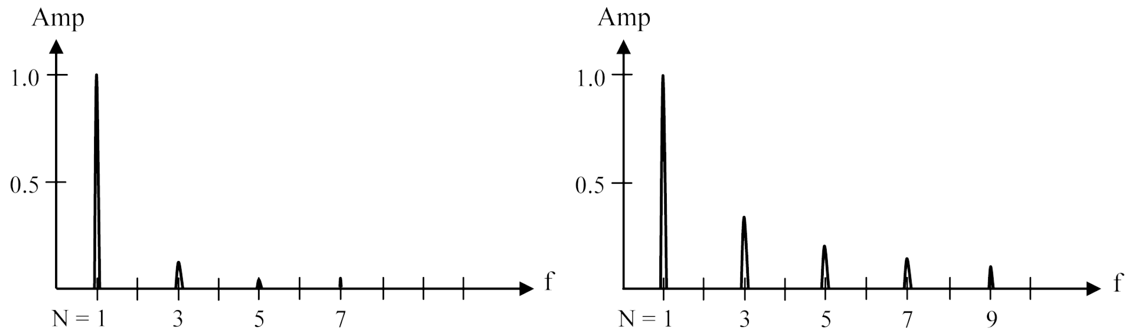


Figure. Relative amplitudes (Fourier spectra) of a triangular wave and a square wave, revisited. Only the odd harmonics contribute in these two cases.

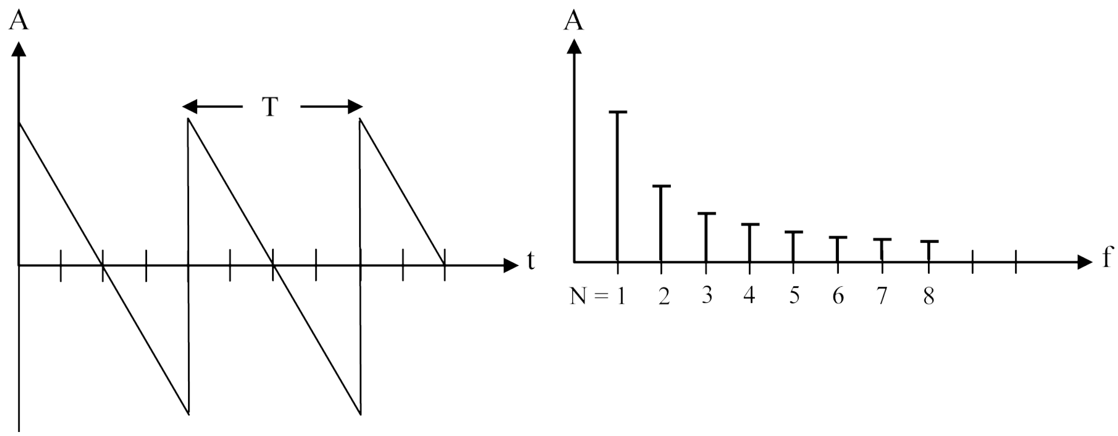


Figure. The sawtooth wave revisited. Left: Waveform. Right: Amplitude spectrum. The amplitudes decrease according to $A_N = A_1/N$, see also table below.

Table. Relative amplitudes of the first 10 harmonics of five standard waveforms. The amplitude of the first harmonic in all cases, i.e. fundamental N = 1, is set to $A_1 = 1$.

Wave Type						Amplitudes						
	N	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	
Sine		1	0	0	0	0	0	0	0	0	0	Only N=1
Triangle		1	0	$-\frac{1}{9}$	0	$\frac{1}{25}$	0	$-\frac{1}{49}$	0	$\frac{1}{81}$	0	Odd N
Square		1	0	$\frac{1}{3}$	0	$\frac{1}{5}$	0	$\frac{1}{7}$	0	$\frac{1}{9}$	0	Odd N
Sawtooth		1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	All N
Pulse Train		1	1	1	1	1	1	1	1	1	1	All N

Demonstrations of Waveforms and Fourier Spectra

Show some standard waveforms and their Fourier spectra (amplitude spectra).

1. Listen to some standard waveforms from a frequency generator or keyboard.
Sine wave
Triangular wave
Square wave
Saw tooth wave (ramp)
Pulse train

2. Show some of the waveforms.
Remember: A *waveform* represents the displacement **y(t)** of air molecules from their equilibrium position as a function of time t.

3. Observe the spectra of some of the waveforms.
Remember: The *Fourier or amplitude spectrum* of a waveform shows the amplitude **A(f)** as a function of frequency f.

4. Show *waveforms* and *amplitude spectra* of the sound from some musical instruments. Use actual instruments or use the keyboard to simulate their sound.

Question
Which of the two displays, waveform y(t) or amplitude spectrum A(f), do you find more informative?
Answer: ____

5. Play an instrument such as a didgeridoo, Indian flute, violin, trombone etc. Observe the amplitude spectra. Synthesize one of the spectra (e.g. for a didgeridoo) with a software-based sound generator or with sine wave generators. Select the frequencies of the harmonics as multiples of the fundamental frequency. Adjust the amplitudes until they look close to the recorded spectrum. Is the synthesized sound similar to the one from the instrument?

6. Use 5 frequency generators and set them to the first five harmonics 100, 200, 300, 400, 500 Hz. Vary the amplitudes and listen to the change in the quality of the sound (change in timbre). Do the resulting sounds resemble any instrument?

Fourier Spectra of a Clarinet and Viola

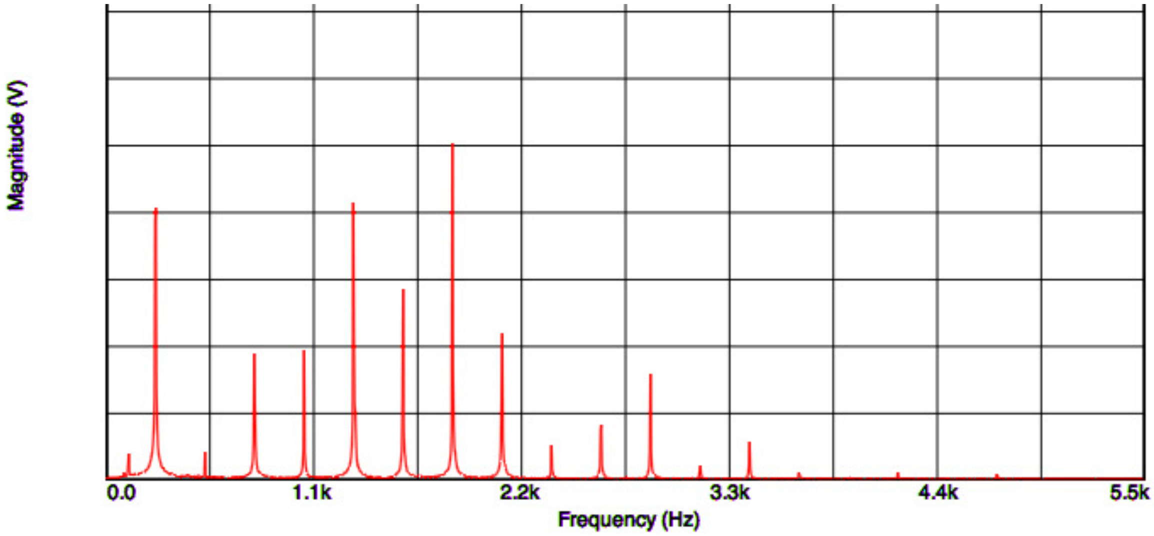


Figure. Harmonic spectrum from a clarinet playing middle C4 = 261.6 Hz. (Played by Professor David Shea, School of Music, Texas Tech University.)

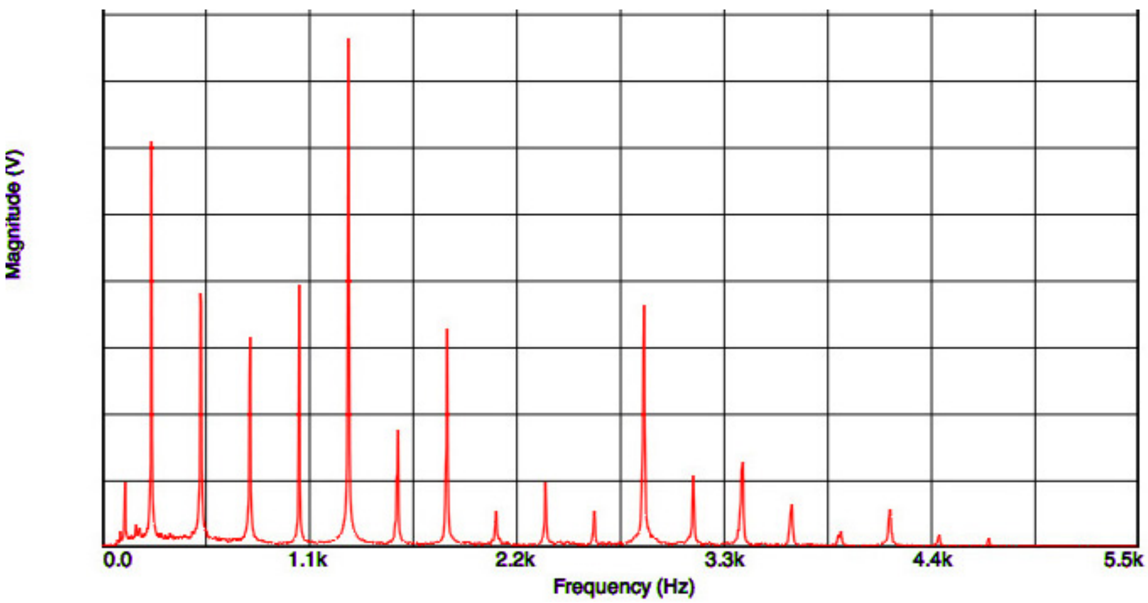


Figure. Harmonic spectrum from a viola playing middle C4 = 261.6 Hz. (Played by Professor Renee Skerik, School of Music, Texas Tech University.)

Exercise
Highlight the dominant frequency regions (formants) in both figures.
Question
Why do the clarinet and the viola sound different?
Answer: ____

Fourier Analysis of Sound continued, Harmonic and Inharmonic Spectra

Remember that the Fourier spectrum of a complex tone in a Cartesian coordinate system shows the amplitudes of the harmonics on the ordinate (the y-axis) as a function of the frequency of the harmonics on the abscissa (the x-axis).

Demonstrations
1. Show additional Fourier spectra for a variety of sounds.
2. Play "middle C4" on some instruments or simulate them on the keyboard synthesizer.
3. Listen to the *quality of sound* or *timbre* and observe the Fourier spectra.

Questions
1. Can you see a relationship between *timbre* and the Fourier spectrum?
Answer: Pronounced harmonics in a spectrum and their groupings of them called "formants", determine timbre.
2.Which instruments sound "plain"? Which produce a rich sound? How is this related to the number of harmonics present?
Answer: ____

Types of Sound Spectra
a) Harmonic frequency spectra
A sustained tone from a string or wind instrument has a nearly perfect *harmonic* frequency spectrum. The vibrating medium is highly elastic and satisfies Hooke's law. The spectrum is *discrete* and shows the harmonics as sharp spikes. The harmonics are equally spaced at the frequencies f_1, $2f_1$, $3f_1$, $4f_1$, $5f_1$, …, where f_1 is the fundamental frequency that determines the pitch of a tone.

b) Inharmonic frequency spectra
The frequencies in the sound spectra from chimes, rods, marimba, cymbals, etc. are not evenly spaced and are not harmonics of a fundamental. The vibrating medium is stiff and rigid. Inharmonicities also occur with drums, where the medium is fairly elastic but stretches in two dimensions. We still may observe a *discrete* frequency spectrum but with non-equidistant separations between the frequency spikes. The complex sound is still made up of a series of individual waves, which, however, are not harmonics of the fundamental. We call these waves *partial waves* or *overtones* in distinction to *harmonics.* Non-equidistant frequency spacings are a sign of *inharmonicities* in the sound.

For a recording of an inharmonic spectrum see following page. It shows the partials from a brass rod struck at one end with a hammer. As it is clearly seen, the frequencies of the partial waves are not equally spaced and are not harmonics.

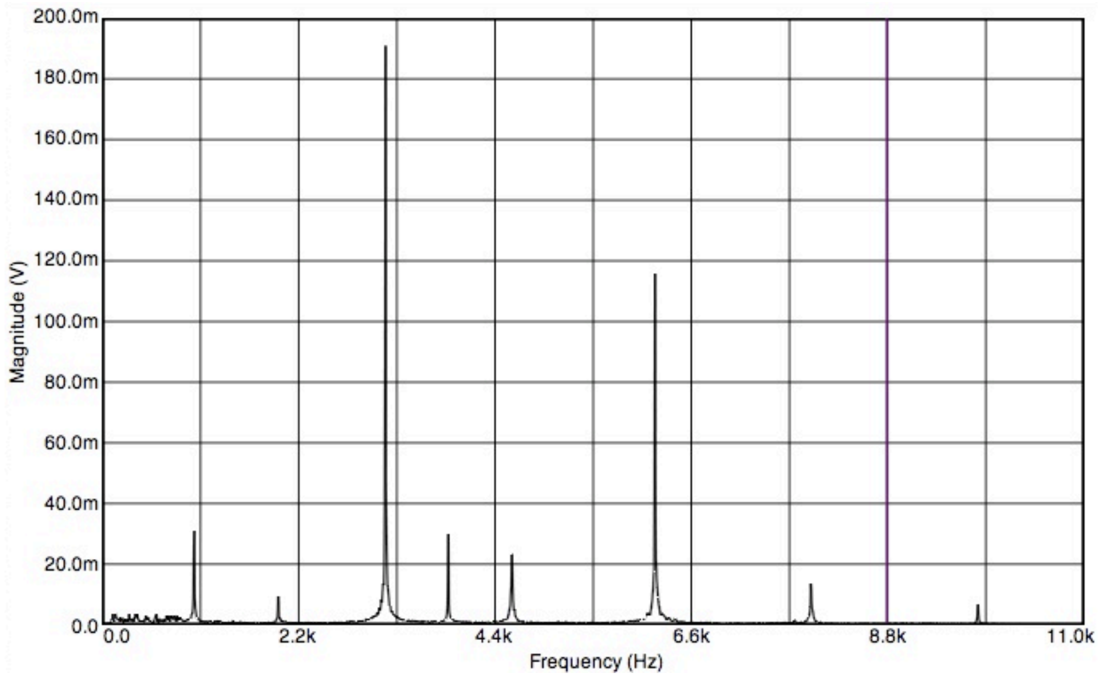


Figure. An inharmonic amplitude spectrum from a brass rod containing 8 non-equidistant partials, which no longer are harmonics. (Length of rod 46.0 cm, diameter 2.54 cm.)

Question:
What causes such inharmonicities in the vibrating medium? (Remember Hooke's law.)
Answer: For vibrations to be harmonics, the restoring force has to be proportional to the displacement of the medium (Hooke's law). This condition is not satisfied for the rigid medium of bars, bells, cymbals, etc.

c) Quasi-continuous frequency spectrum (make some noise)
A spectrum may have so many closely spaced frequencies that it looks almost continuous. The resulting sound generally is "noisy" and not "musical".

Demonstrations
1. Observe the sound spectra when striking an aluminum or brass rod.
2. Marimba bars
3. Chromatic bell set, Glockenspiel
4. Cowbell, other bells
5. Show piano wires and note their stiffness.
6. Djembe (a membrane drum)
7. Strike the brass rod gently with a hammer, place the rod on an open styrofoam box. Notice the low fundamental frequency that was inaudible before. Hear the higher overtones with the rod on and off the box.
8. Triangle
9. Cymbals

Factors Affecting Tone Quality (Timbre)

Consider a *sustained* tone. Its timbre is determined by the harmonic content of the tone, or in other words: the relative amplitudes in the spectrum determine timbre.

(a) Amplitude spectrum
More overtones produce a richer tone. Compare a violin with a flute.
(b) Attack and decay transients
Compare a harpsichord with a church organ. A harpsichord has a short attack and slow decay transient, an organ has a long attack transient and a short decay transient.
(c) Inharmonicities
Deviations from the harmonic series f, 2f, 3f, 4f, 5f, … result in inharmonicities and may allow identification of the instrument, for instance percussion instruments.
(d) Formants
These are frequency regions where harmonics are pronounced. An example is a bassoon as compared with a flute. A flute has only very few harmonics while the bassoon has harmonics grouped in *formant regions.* The human voice has a rich and unique tone quality that originates in the resonant cavities of the vocal tract (vocal folds, larynx, pharynx, mouth, nasal cavity). These account for several vocal formant regions.
(e) Vibrato and tremolo
Vibrato means slight changes in *frequency*, *tremolo* means changes in *amplitude*.
You can produce vibrato with string instruments, but not with the plucked strings of a harp. Singers can produce tremolo and vibrato separately or simultaneously. A trombone exhibits pitch vibrato, a flute tremolo, the latter also called "diaphragm vibrato".
(f) Chorus effect
The chorus effect results from several instruments of the same kind playing together. The phase differences and slight frequency differences result in a characteristic orchestral timbre that differs from a single instrument played at the same loudness level. This allows one to distinguish the solo violin from the violin section in a violin concerto.

Transient Sound
The amplitude spectrum here is of limited use at the first instant when a note is struck, for instance when a piano key is pressed or a violin bow touches the strings.
Then the **attack transients** and **decay transients** in the overall sound play a significant role in defining the timbre.
Demonstrations
Listen to attack and decay transients and observe them on sonograms, which display frequencies as a function of time, with the intensity of the sound in color:
1. Strike a piano (keyboard) key.
2. Strike a percussion instrument.
3. Pluck a string on a violin and listen to the attack and decay transients.
Question
When does a tone sound "musical" in the presence of attack and decay transients?
Answer: When the tone is much longer in duration than the attack and decay transients.

Timbre From Contrabasses With Gut Strings and Steel Strings

The two figures below show the harmonics of the note A1 from two contrabasses, one with gut strings (top figure) and the other with steel strings (bottom figure). The gut strings gave a softer and more subdued timbre, while the steel strings with their more pronounced higher harmonics sounded more brilliant.
Both spectra show missing fundamentals, and formants below 270 Hz, between 270 and 550 Hz, and between 550 and 800 Hz, respectively.

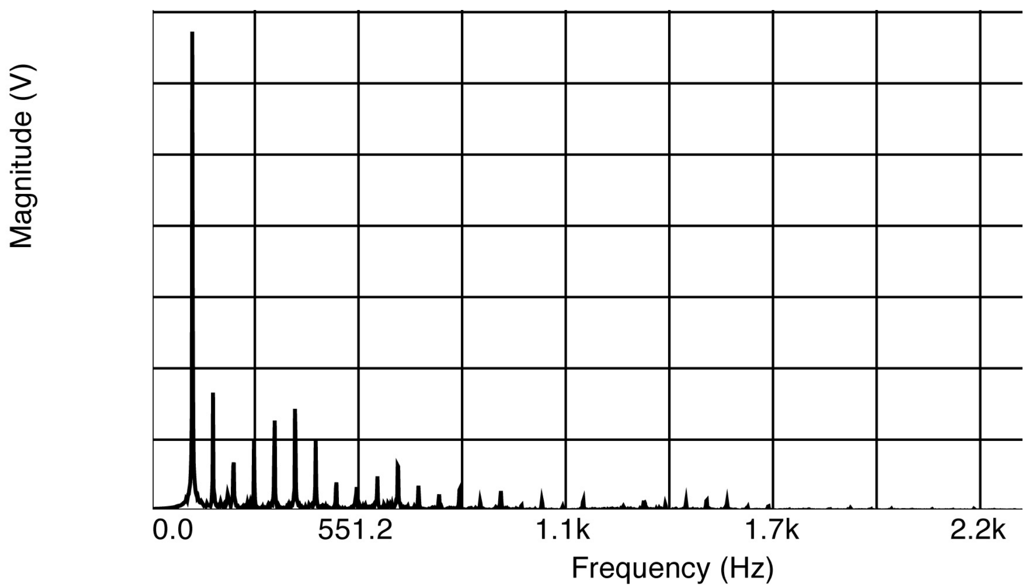


Figure. 18th century contrabass strung with gut strings, played by Mark Morton.

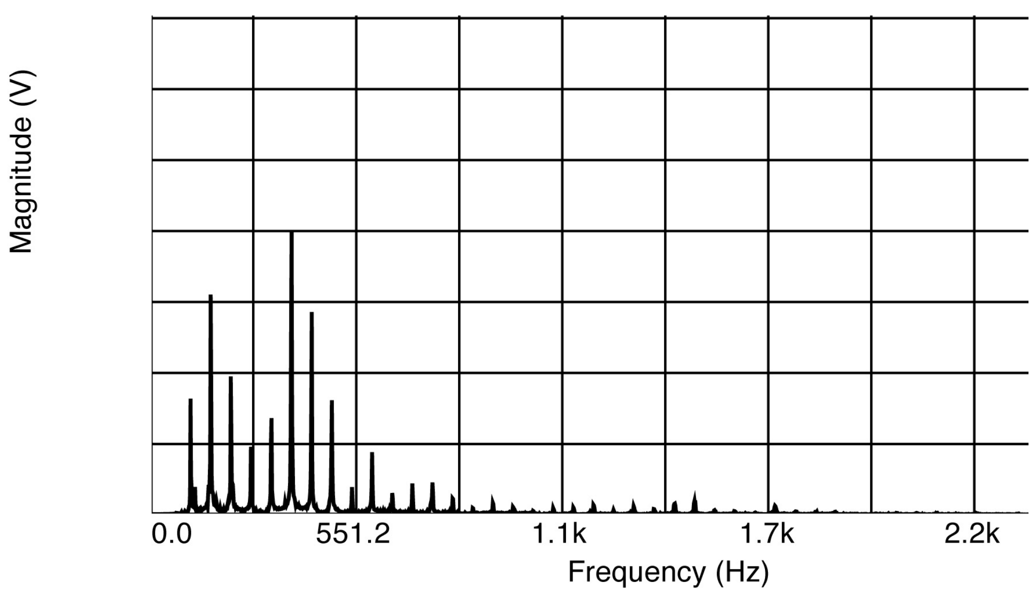


Figure. Steel strings on a reproduction of the contrabass above, played by Trevor Drew.

The Piano and Inharmonicities

The inherent stiffness of the piano wires gives rise to inharmonicities in the sound. The stiffness produces a region with no motion on the string near the nodes of the harmonics, while for an elastic string the nodes are point-like. This effect shortens the effective string length for the higher harmonics and increases their frequency. Thus, the higher the vibrational mode, the higher the frequency deviation from true harmonics. The overtones become progressively sharper resulting in the "spicy" sound of the piano.

Exercise

Draw a piano string vibrating in its first two near-harmonic modes. Show the shortening of the effective string length near the node of the 2nd harmonic due to stiffness of the string.

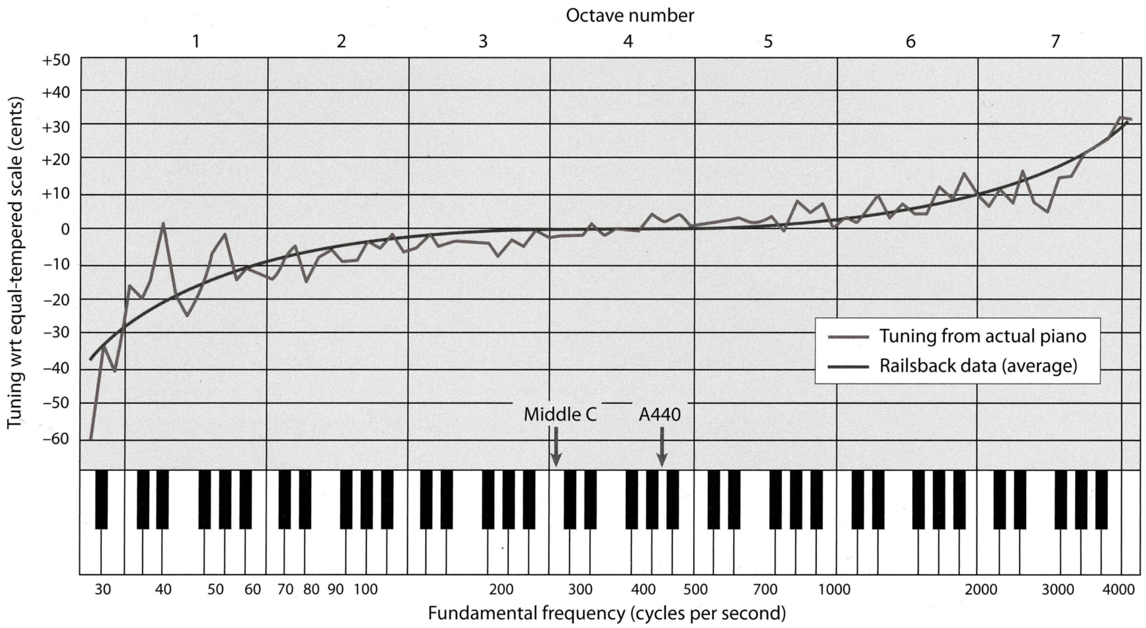


Figure. The Railsback curve showing the difference in cents between the usual piano tuning and equal temperament (100 cents = 1 semitone or half-step). The smooth curve is an average from many pianos, the jagged curve is for an individual piano. Without inharmonicities, the curve would be a horizontal line at 0 cents. (From: Eric J.Heller, Why You Hear What You Hear, Fig. 19.3, 395, Princeton University Press, 2013.)

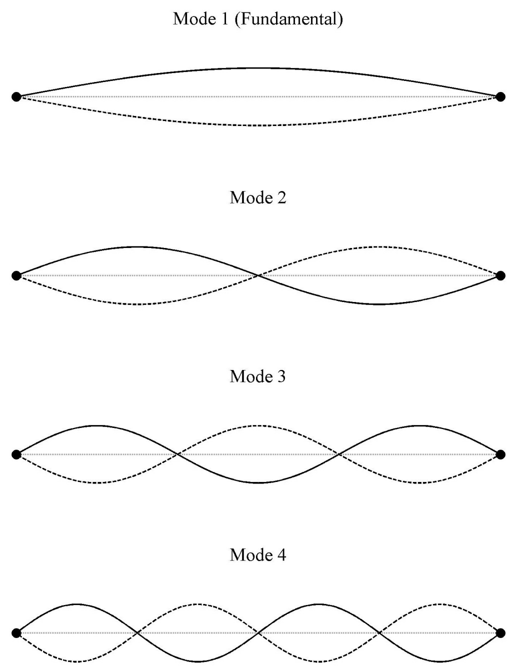
The piano sound is a complex combination of:
1. Inharmonicities
2. Attack transients
3. Decay transients
4. Tuning, where the high notes are tuned slightly high and the low notes slightly low.

Challenge Question

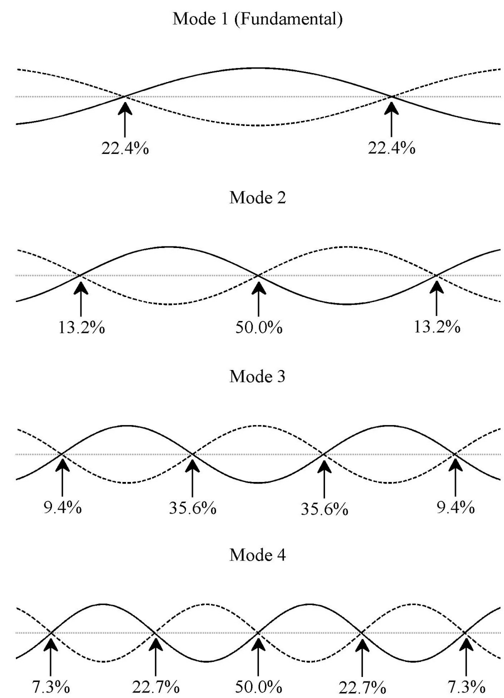
Why do the usual string instruments such as a violin produce harmonic spectra and not inharmonic spectra?

Vibrating Strings and Bars

Harmonic Vibrational Modes



Inharmonic Vibrational Modes



Left figure. The first 4 vibrational modes of a stretched string fastened at its ends.
Right figure. The first 4 transverse vibrational modes of a bar, rod, or rigid tube, free to move at its un-clamped ends.

Mode number	Frequency	Interval
1	f_1	unison
2	$2f_1$	octave
3	$3f_1$	twelfth
4	$4f_1$	2 octaves

Mode number	Frequency	Interval
1	f_1	unison
2	$2.756\ f_1$	1st partial
3	$5.404\ f_1$	2nd partial
4	$8.933\ f_1$	3rd partial

Demonstrations

1. Show an aluminum tubing and demonstrate its first three vibrational modes this way: Strike the tubing and let it glide down between your fingers to the nodal points. Listen to the individual modes as they become alive. You will still hear a mix of the modes, but you should hear a dominant one.
2. Play chimes from the "TTU Physics Chime Set" and observe the amplitude spectrum. Note the non-equidistant spacing of the partials, representative of inharmonicities.
3. Excite longitudinal (and transverse) modes in a long aluminum rod or a brass rod with a mallet, slide the rod vertically between your fingers. Note the vibrational modes.

Ohm's Law of Hearing

The quality or timbre of a tone is not affected much even by large changes in the relative *phases* of the harmonics. What matters primarily are the *amplitudes* of the harmonics.

We have seen in several demonstrations that the timbre of the sound from an instrument is given by the relative amplitudes of the harmonics. We have not spoken yet about the phases of the harmonics. We have done this for good reason, which can be summarized in the empirical *"Ohm's law of hearing"*, said here in colloquial form:

What you hear is the *amplitude spectrum* of a tone, *not the phases* between the harmonics.

For the mathematically interested, remember that you can write a complex wave as the function

$$\mathbf{y(t) = \sum_{N} A_N \sin(2\pi N f t - \varphi_N)}$$

where y(t) is the displacement from equilibrium of the air molecules (or parts of a string) as a function of time t at a given position. The sum extends over all harmonic numbers N. The angle φ_N is the relative phase angle of the Nth harmonic with respect top the phase of the fundamental.

Ohm's law of hearing then says that you hear the same no matter what the relative phases are between the harmonics. You can even leave all phases off ($\varphi_N = 0$) and you hear the same, with the sound wave now represented by

$$\mathbf{y(t) = \sum_{N} A_N \sin(2\pi N f t)}$$

However, you do need the phases φ_N for a mathematically correct representation of the waveform, but you do not need them for correctly hearing the sound. We more directly sense or "hear" the amplitude spectrum of a sound rather than its waveform. Our auditory system thus is a spectrum analyzer (with a few exotic exceptions).

Demonstrations

1. Start two sine waves of different frequencies, one after the other at arbitrary times, i.e. arbitrary phases between them. Note that it does not matter at what time the second sine wave is turned on. It always sounds the same, and the timbre is the same.
2. Add two sine waves with sound processing software. Shift the phase of one of them with respect to the other. Can you hear any changes in timbre? If you can't, your auditory system is obeying Ohm's law.

Resonance and Noise

When you play an instrument, resonances are set up in the air, strings, solid material, or membranes of the instrument. Resonators, such as the body of string instruments, drums, or loudspeakers, enhance certain frequency regions of the emitted sound. In order to excite these resonances, some sort of broadband noise with many frequencies is required. The resonator then responds favorably with its own resonance frequencies to that noise. It selects the frequencies which it “likes best” and “resonates” preferably at these frequencies.

Examples

1. The bow of string instruments produces a broadband spectrum through its “slip and stop” motion on the string. The string then responds and vibrates with its characteristic harmonic frequencies.
2. The reed of a bassoon produces a broadband frequency spectrum or “white noise”. The air column in the instrument responds selectively by sounding its resonance frequencies. As a result, the sound from the instrument is “musical”. An interplay exists between the instrument and its reed. The player notices how the instrument reacts with its resonance frequencies to the buzz from the reed.

Resonance Curves

A resonance curve of a musical instrument is a summary of its acoustical properties. It tells us about the possible frequencies and overtones that could be present and in what strength and therefore is a physical description of the resonating system.

Fourier Spectrum

In distinction to the resonance curve of an acoustical system, the Fourier spectrum of an emitted tone gives us a description of the amplitude/frequency structure of that particular tone.

Types of Noise

When the spectrum consists of many densely spaced spikes, we have a quasi-continuous spectrum. An important feature of such spectra is that they have no discernible periodicity. The sound is aperiodic and not musical.

White Noise

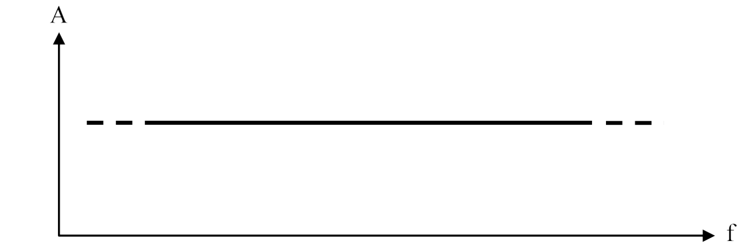


Figure. White noise. The spectrum has equal intensity at all frequencies and is nearly continuous.

Colored or Filtered Noise

An example is the howling wind with a low frequency maximum. The maximum may be perceived as a low rumble. A microphone is sensitive to this wind noise. To protect against it, a bulky windscreen of fuzzy material can be placed over the microphone.

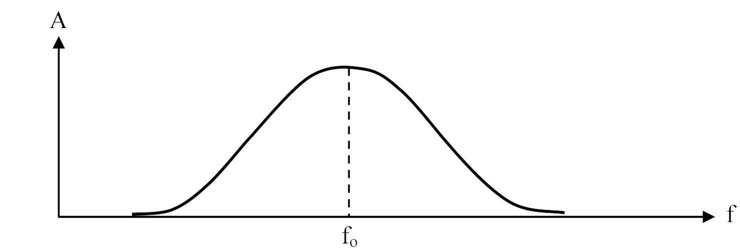


Figure. The frequency spectrum is broadband with a maximum at some frequency f_0.

Pink Noise

This is a special case where the sound intensity decreases uniformly with increasing frequency over the audible range. Specifically, the intensity decreases by a factor of 2, i.e. 3 dB per octave. Pink noise is called “1/f-noise” and is used for testing audio equipment. A 1/f-decrease in sound intensity closely approximates the time-averaged energy distribution of the sound in orchestral music. The time-averaged spectrum is a noise spectrum. (However, the instantaneous sound may contain discrete frequencies.)

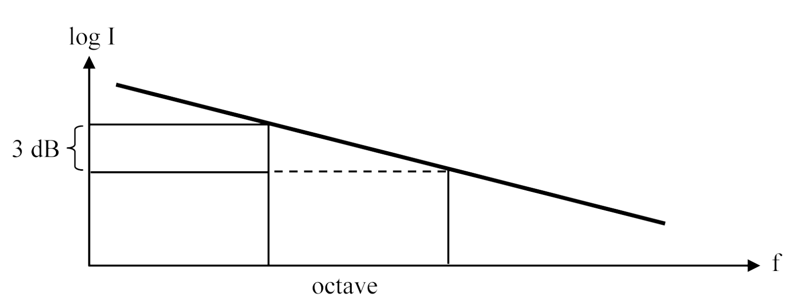


Figure. Pink noise with a 3 dB decrease per octave.

Helmholtz Resonators

Hermann Ludwig Ferdinand von Helmholtz (1821–1894) used a series of spherical resonators of different sizes to perform frequency analyses of musical instruments.

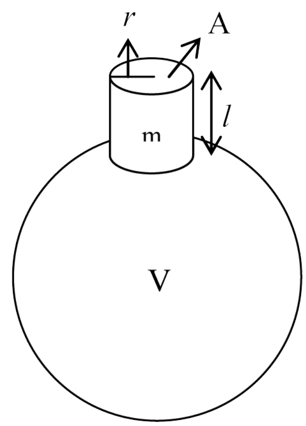


Figure. Prototype Helmholtz resonator. The neck has area $A = \pi r^2$, where r is the neck radius, m is the mass of the vibrating air column, l the length of the bottleneck, and V the volume of air in the resonator. (The volume does not have to be of spherical shape.)

The resonator acts as a harmonic oscillator, where the air mass m in the neck is the "weight" that moves harmonically a small distance in and out. The air in the volume V acts as the "spring". One can derive a spring constant for this system, and from that the resonance frequency f. The result is given by

$$f = \frac{v}{2\pi}\sqrt{\frac{A}{\ell_{eff} V}}$$

In this formula, v is the speed of sound, A area of the neck opening, l_{eff} the length of the neck after adding the corrections at the open ends, and V is the volume of the resonator**.**

Numerical Example – Glass Flask
Calculate the resonance frequency for a spherical glass flask having the following dimensions: volume V = 500 cm^3, average neck radius = 0.65 cm, neck length l = 3.0 cm.
Answer: $l_{eff} = l + 2\cdot 0.6\cdot r = 3.0 + 1.2\cdot r = 3.0 + 1.2\cdot 0.65 = 3.78$ cm = 0.0378 m.
Frequency $f = (v/2\pi)(A/l_{eff}V)^{1/2} = (346/2\pi)[\pi\cdot 0.0065^2/(0.0378\cdot 0.0005)]^{1/2} = 145.9$ Hz.

The following figure shows the Helmholtz resonance of this glass flask excited by blowing air across the neck opening. Note that the measured resonance frequency is f = 146.7 Hz. This is in very good agreement with the calculated frequency above.

P.S.: More strictly, the neck of a flask or bottle is an unbaffled opening on the outside, with a length correction of $0.61\cdot r$, and a baffled opening to the inside, with a correction of $0.85\cdot r$. This results in an effective length $l_{eff} = l + 0.61\cdot r + 0.85\cdot r = l + 1.46\cdot r = 0.0395$ m However, this gives no improvement in the calculated resonance frequency.

The resonance curve of a Helmholtz resonator has a prominent peak.

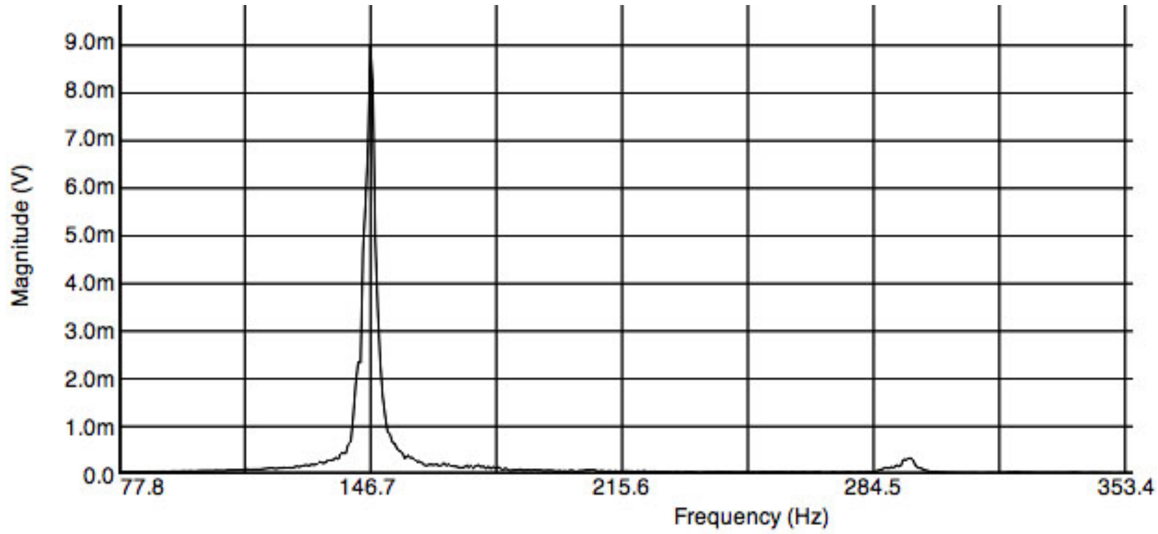


Figure. Resonance curve of a spherical Helmholtz resonator consisting of 500 cm^3 spherical glass flask. The measured peak is at f = 146.7 Hz.

The assumption in the formula for the resonance frequency is that f is low (or wavelength λ large) so that there are no significant nodes or antinodes in the neck and resonator volume. This means that $l_{eff} << \lambda$, with $\lambda = v/f$. This condition generally is satisfied. For our example of the spherical flask we had f = 146 Hz $\rightarrow \lambda = 346/146 = 2.37$ m, while l_{eff} = 0.0378 m is much smaller than this, and so is the diameter of the flask (D = 0.0985 m).

Further Numerical Examples and Demonstrations

1. Wine bottle. Volume V = 1.5 liter, diameter of opening d = 1.87 cm, length of bottle neck l = 7.7 cm, i.e. effective length $l_{eff} = l + 2\cdot 0.6\cdot R$ = 8.822 cm = 0.08822 m.
Calculate resonance frequency from formula and verify: f = 79.3 Hz ± 5.0%
Resonance frequency measured with spectrum analyzer: f = 81.6 Hz ± 0.5%
This is very good agreement between theory and experiment (better than expected).

2. Metal bar mounted on a rectangular resonator box of volume V = 677.4 cm^3.
We treat this as a Helmholtz resonator with a hole but without an explicit neck.
The opening has a thickness l = 0.60 cm of the surrounding wood.
The opening here is an oval of area of area A = 25.9 cm^2.
We approximate this oval with a circular opening of the same area A having an effective diameter $d_{eff} = 2(A/\pi)^{1/2}$ = 5.80 cm.
For the length l_{eff} in the formula for the frequency we take $l_{eff} = l + 0.85\cdot d_{eff}$ = 5.53 cm.
The correction factor here of 0.85 is for an opening baffled on both sides. This replaces the factor of 0.61 for a non-baffled opening such as an open cylindrical pipe.
Then the calculated resonance frequency is f = 456 Hz.
Compare this with the frequency of f = 440 Hz of the bar tuned to “concert A”.
The agreement is quite good considering the assumption of a circular opening.

Exercise
Verify the resonance frequencies calculated in the two preceding examples. Fill in the blanks with your answers:
Answer: Wine bottle f = __________Hz   Marimba bar f = ___________Hz

More Examples for Helmholtz Resonators
1. Loudspeakers with ducted ports.
2. String instruments. The air volume in the body is a complex Helmholtz resonator. The tone quality and loudness for low notes are improved. (Note that the so-called “wood” or body resonances are not Helmholtz resonances but come from the vibrating solid material)

Demonstrations With Helmholtz Resonators
1. White Zinfandel wine bottle, V = 1.5 liter, and other bottles. Blow over the top and listen to the Helmholtz resonance.
2. Hold a metal bar over a resonator box, with and without a cover over its opening.
3. Hold a seashell, empty can, or drinking glass close to your ear and listen to the sound. Where does the perceived “sound of the ocean” come from?
4. Move a can towards and away from the ear. Does the “pitch” change?
5. Make a cup with one hand and place it over your ear. Listen to the sound.
6. Make a larger cup with both hands close together and hold them over an ear. How does the frequency change compared to the smaller “cup”?
7. Take two cans and hold one over each ear for a “stereo effect”.
8. Play a “cajon”. This is a rhythm box acting as a Helmholtz resonator.
9. Play a “djembe”, an African drum. Point the opening towards the audience.
10. Have students applaud in the classroom. Place a coffee can over a microphone. Observe the filtered noise due to the Helmholtz resonance in the can.

Questions
1. Describe how the bottle sound is excited. Compare the sound with a simple sine wave.
2. Does the frequency or pitch increase or decrease as you move a can farther away from your ear?
3. Qualitatively explain this frequency change with Helmholtz’s formula. Hint: Discuss how the resonance frequency depends on the area A, length l_{eff}, and volume V of the resonator.

Remark
Hermann von Helmholtz used a series of spherical resonators of different sizes to determine the Fourier spectra of musical instruments, all done in the second half of the 19^{th} century. This was an ingenious early way to analyze the timbre of instruments before the advent of electronic spectrum analyzers.

Indian Rudra Vina. Other Helmholtz Resonators

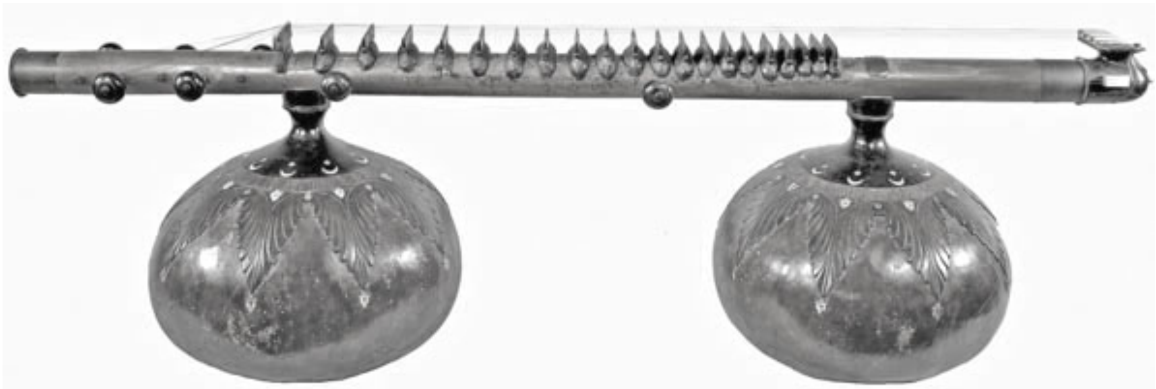


Figure. A rudra vina string instrument from India with two Helmholtz resonators. The resonators are made of gourds and connected to a bridge holding the strings.

Figure. Helmholtz resonators. Father John A. Zahm, Professor of Physics of Notre Dame University, Indiana, bought this set of seven Helmholtz resonators and gave them to St. Mary's College for Women. This set is one of many at St. Mary's.

Figure. A cylindrical Helmholtz resonator made of two tubes for changing the volume and thus changing the resonance frequency. The frequencies and corresponding notes can be engraved on the sides of the cylinder. This is one of the tunable Helmholtz resonators at the University of Vermont.

More About Musical Instruments

WIND INSTRUMENTS
One end is an “open end”, the other end can be open or closed.

Instruments with “both ends open”
Standing waves of all harmonics are allowed.
The harmonics are f_1, $2f_1$, $3f_1$, $4f_1$, $5f_1$, $6f_1$, $7f_1$, $8f_1$,……
Examples
Orchestral flute: Two open ends, including the embouchure hole (mouth piece).
Recorder family: Two open ends, including the wedge-like mouthpiece (fipple).
Question
When you “overblow” a flute or recorder, what is the first existing overtone? What is the musical interval between it and the fundamental?
Answer: $N = 2$, with frequency $2f_1$. The interval is an “octave” with ratio 2:1.

Instruments with “one end closed”
Only standing waves of odd harmonics theoretically are allowed.
The harmonics are f_1, $3f_1$, $5f_1$, $7f_1$, …
In practice, this is true approximately for the lowest harmonics of instruments having a *cylindrical* bore.
Examples
Reed instruments such as the clarinet with a straight *cylindrical* bore.
The reed acts as a closed end, because it is “slapped” closed much of the time.
Question
When you “overblow” a clarinet, what is the first existing overtone? What is the musical interval between it and the fundamental?
Answer: $N = 3$, with frequency $3f_1$. The interval is an “octave + fifth” with ratio 3:1.

Many reed instruments have conical bores and curved tubes of varying diameter. These admit all harmonics, even and odd, whether one end of the instrument can be considered “closed” or “open”.
Examples
Bassoon, oboe, saxophone

BRASS INSTRUMENTS
The player buzzes his lips directly into the mouthpiece (e.g. without a reed).
Brass instruments also have a tube of varying diameter and a bell shape at the end.
Question
Do you expect all harmonics or only odd harmonics from a brass instrument such as a trumpet or tuba?
Answer: All harmonics because of the complex non-cylindrical shape of the instrument.

Demonstrations
Recorder, American Indian flute, didgeridoo, clarinet, trumpet, trombone, harmonica.

Harmonic and Non-Harmonic Vibrations. Pitch, Timbre, and Noise

Vibration Type	Pitch and Timbre	Examples and Demonstrations
Sine wave, single frequency (fundamental only)	Definite pitch, boring, bland tone	Sine wave from synthesizer, whistle, short narrow tubing
Fundamental and a few harmonic overtones	Definite pitch, timbre depends on the relative amplitudes of harmonics	Some woodwind instruments, recorders, Indian flute, tuned (carved out) marimba bars, Helmholtz resonators, bottles
Fundamental plus many harmonics	Definite pitch, full and clear tone, spiciness of tone determined by strong higher harmonics	Well-made string instruments woodwinds and brass, bassoon, krummhorn, harmonica
Fundamental plus poorly tuned harmonics	Discernible pitch, but may sound slightly off the fundamental	Poorly made instruments, strings stiff, piano, inadequate tension
Fundamental with inharmonic widely spaced partials	Defined pitch, spicy timbre	Marimba bars (not tuned), bells, chimes, tubings, glockenspiel
Closely spaced non-harmonic frequencies	Different pitches may be heard, chordal, muddy, or jangly sound	Irregular strings, some gongs and triangles, scrap metal
Many closely spaced non-harmonic frequencies	No dominant pitch, tone quality may vary widely	Cymbals, some gongs and triangles, long thin metal rods struck longitudinally
Distinct fundamental with many inharmonic partials and noise	Rough pitch, discernible timbre depending on mix of partials	Snare drums, djembe, membranes, rattle, electric razor, kitchen blender, lawn mower, scraping sound
Disordered vibrational pattern, quasi-continuous	Noise, sound frequencies over a broad band, some wide peaks, no isolated strong frequencies	Many people randomly clapping hands, radio static, rushing wind, rain, maracas

(Some parts adapted from Bart Hopkin, Musical Instrument Design, 6^{th} printing, page 4, Sharp press, Tucson, 2007.)

Sachs-Hornbostel Categorization of Musical Instruments (1914)

According to Curt Sachs and Erich von Hornbostel there are 4 broad instrument classifications according to what the vibrating medium is. There are 9 additional sub-classifications (not shown here).
The categories 5) and 6) below have been named more recently.

1) **Idiophones**
A s*olid material* vibrates with its own stiffness without being stretched.
Free-bar instruments: Marimbas, chimes
Rods fixed at one end: Kalimbas, tongue drums
Tuning forks
Bells, cymbals, gongs

2) **Membranophones**
A s*tretched membrane* vibrates.
Drums, djembe

3) **Chordophones**
The initial vibrator is a *stretched string*.
String instruments: Violin, viola, cello, double bass, guitar
Harp, zither, lyre, lute

4) **Aerophones**
The initial vibrator is *air*, enclosed in a chamber, or free.
Wind instruments: Flute, clarinet, oboe, bassoon, recorder, panpipe
Brass instruments: Trumpet, trombone, French horn, tuba, euphonium
Plosive Aerophone
Siren

5) **Electrophones**
Electrons vibrate in a wire or circuit.
Synthesizer
Electric guitar
Theremin

6) **Hydrophones**
The initial vibrator is a *liquid*.
Droplets falling into water can make a musical sound.

(Partly taken from Bart Hopkin, Musical Instrument Design, 6th printing, Sidebar 4-1, p. 30, See Sharp Press, Tucson, 2007.)
See also “Musical Instrument Categorization Systems” in the Appendix of this Course Guide.