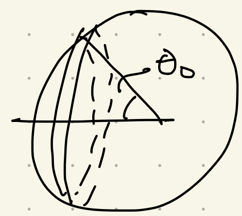
sec 16, prob 2

$dN$ = fraction of particles entering $d\Omega_0$

$$= \frac{d\Omega_0}{4\pi}$$




$$= \frac{2\pi \sin\theta_0 \, d\theta_0}{4\pi}$$

$$= \frac{1}{2} \sin\theta \, d\theta_0$$

$$= -\frac{1}{2} d(\cos\theta_0)$$

Now: (16.6)

$$\cos\theta_0 = -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{1-\frac{V^2}{v_0^2}\sin^2\theta}$$

i) For $V < v_0$, take $+\sqrt{\phantom{x}}$

$$d(\cos\theta_0) = -\frac{2V}{v_0}\sin\theta\cos\theta \, d\theta$$

$$-\sin\theta \, d\theta \sqrt{\phantom{x}}$$

$$+\frac{\cos\theta}{\sqrt{\phantom{x}}}\frac{1}{\cancel{2}}\left(-\frac{V^2}{v_0^2}\right)\cancel{2}\sin\theta\cos\theta \, d\theta$$

$$= -\sin\theta \, d\theta \left\{ \frac{2V}{v_0}\cos\theta + \sqrt{\phantom{x}} + \left(\frac{V}{v_0}\right)^2 \frac{\cos^2\theta}{\sqrt{\phantom{x}}} \right\}$$

Thus,

$$dN = \frac{1}{2}\sin\theta\, d\theta \left\{ \frac{2V}{v_0}\cos\theta + \frac{\left(1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta + \left(\frac{V}{v_0}\right)^2 \cos^2\theta\right)}{\sqrt{\phantom{x}}} \right\}$$

$$= \frac{1}{2}\sin\theta\, d\theta \left\{ \frac{2V}{v_0}\cos\theta + \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right\}$$

where $0 \le \theta \le \pi$

ii) For $V > v_0$, there are two solutions corresponding to the $+\sqrt{\phantom{x}}$ and $-\sqrt{\phantom{x}}$ in (16.6).

- For the $+\sqrt{\phantom{x}}$ we have $d\theta/d\theta_0 > 0$
- For the $-\sqrt{\phantom{x}}$ we have $d\theta/d\theta_0 < 0$

so we should subtact the two contributions

$$dN = dN_+ - dN_-$$

where $dN_+$ = above expression

$$= \frac{1}{2}\sin\theta\, d\theta \left\{ \frac{2V}{v_0}\cos\theta + \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right\}$$

and

$$dN_- = \frac{1}{2}\sin\theta\, d\theta \left\{ \frac{2V}{v_0}\cos\theta - \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right\}$$

Then,

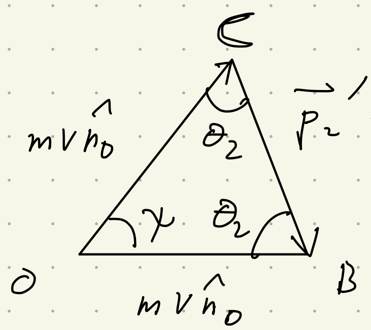
$$dN = \sin\theta \, d\theta \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}}$$

where $\theta \leq \theta \leq \theta_{max}$

Sec 17, Prob 1

Want to determine $v_1'$, $v_2'$ as functions of $\theta_1, \theta_2$ (as opposed to functions of $\chi$)

From Fig 16, triangle OBC:




$$\chi + 2\theta_2 = \pi$$
$$\chi = \pi - 2\theta_2$$

$$(m_2 v_2')^2 = 2(mv)^2 - 2(mv)^2 \cos\chi$$
$$= 2(mv)^2 (1 - \cos\chi)$$

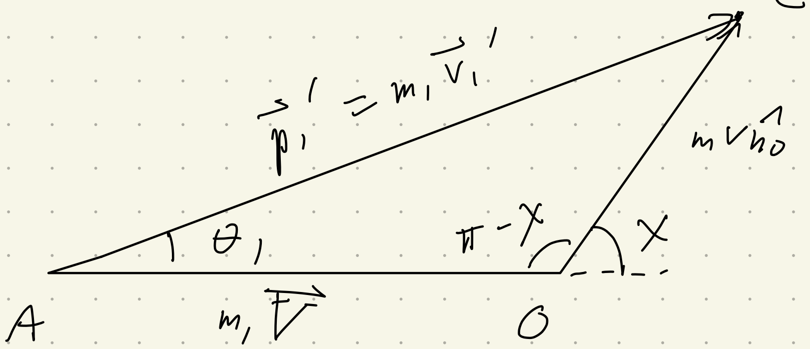
Now: $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$
$\rightarrow 2\sin^2\theta = 1 - \cos 2\theta$

$$\rightarrow (m_2 v_2')^2 = 4(mv)^2 \sin^2\left(\frac{\chi}{2}\right)$$

$$\boxed{v_2'} = 2\left(\frac{mv}{m_2}\right) \sin\left(\frac{\pi}{2} - \theta_2\right)$$

$$= 2\left(\frac{m_1}{m_1+m_2}\right) v \cos\theta_2$$

From Fig 16, $\triangle AOC$




$$\vec{V} = \frac{m_1 \vec{v}_1 + m_2 \cancel{\vec{v}_2}^{\,0}}{m_1 + m_2} = \frac{m_1 \vec{v}_1}{m_1 + m_2} = \frac{m_1 \vec{v}}{m_1 + m_2}$$

Use law of cosines for OC:

$$OC^2 = AC^2 + AO^2 - 2\,AC \cdot AO \cos\theta_1$$

$$(mv)^2 = (m_1 v_1')^2 + (m_1 V)^2 - 2\, m_1 v_1' m_1 V \cos\theta_1$$

quadratic equation for $v_1'$:

$$0 = (m_1 v_1')^2 - 2\, m_1^2 v_1' V \cos\theta_1 + (m_1 V)^2 - (mv)^2$$

$$= m_1^2 v_1'^2 - 2\frac{m_1^3}{m_1+m_2} v_1' v \cos\theta_1 + \left(\frac{m_1^4}{(m_1+m_2)^2} - \frac{m_1^2 m_2^2}{(m_1+m_2)^2}\right) v^2$$

$$= m_1^2 v^2 \left[ \left(\frac{v_1'}{v}\right)^2 - 2\left(\frac{m_1}{m_1+m_2}\right)\left(\frac{v_1'}{v}\right)\cos\theta_1 + \underbrace{\frac{m_1^2 - m_2^2}{(m_1+m_2)^2}}_{=\left(\frac{m_1 - m_2}{m_1+m_2}\right)} \right]$$

$$\rightarrow \quad \frac{v_1'}{v} = \frac{2\left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \sqrt{4\left(\frac{m_1}{m_1+m_2}\right)^2 \cos^2\theta_1 - 4\left(\frac{m_1 - m_2}{m_1+m_2}\right)}}{2}$$

$$\frac{v_1'}{V} = \left(\frac{m_1}{m_1+m_2}\right)\cos\theta_1 \pm \left(\frac{1}{m_1+m_2}\right)\sqrt{\underbrace{m_1^2\cos^2\theta_1 - (m_1^2-m_2^2)}}$$

$$= m_1^2(\cos^2\theta_1 - 1) + m_2^2$$

$$= m_2^2 - m_1^2\sin^2\theta_1$$

Thus,

$$\boxed{\frac{v_1'}{V} = \frac{m_1}{m_1+m_2}\cos\theta_1 \pm \frac{1}{m_1+m_2}\sqrt{m_2^2 - m_1^2\sin^2\theta_1}}$$

For $m_1 > m_2$, the $\sqrt{\phantom{x}}$ has both $\pm$ signs

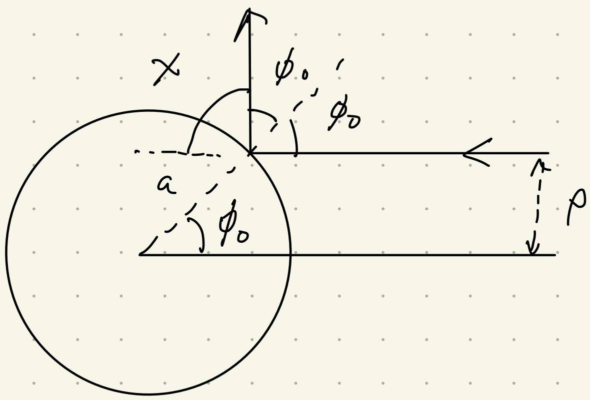
For $m_1 < m_2$, the $\sqrt{\phantom{x}}$ should have the $+$ sign in order for

$$\frac{v_1'}{V} \xrightarrow[\theta_1 \to 0]{} \left(\frac{m_1}{m_1+m_2}\right) + \frac{m_2}{m_1+m_2} = 1$$

Sec. 18, Probl:

Hard sphere

$$U = \begin{cases} 0 & r > a \\ \infty & r < a \end{cases}$$




$$\chi + 2\phi_0 = \pi \quad \rightarrow \quad \phi_0 = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\sin\phi_0 = \rho/a$$

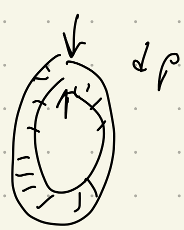
Thus, $\sin\left(\frac{\pi}{2} - \frac{\chi}{2}\right) = \frac{\rho}{a}$

$$\rightarrow \quad \cos\left(\frac{\chi}{2}\right) = \frac{\rho}{a}$$

Effective cross section:

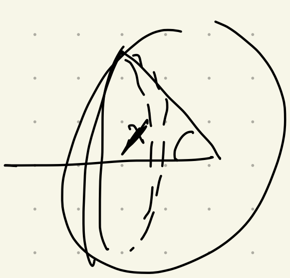
$$d\sigma = 2\pi\rho\, d\rho$$

$$d\Omega = 2\pi \sin\chi\, d\chi$$




$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{\rho\, d\rho}{\sin\chi\, d\chi}$$

$$= \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right|$$



$$\frac{d\sigma}{d\Omega} = \frac{a\cos\left(\frac{\chi}{2}\right)}{\sin\chi}\left|\frac{d}{d\chi}\left(a\cos\left(\frac{\chi}{2}\right)\right)\right|$$

$$= \frac{a^2\cos\left(\frac{\chi}{2}\right)\frac{1}{2}\sin\left(\frac{\chi}{2}\right)}{\sin\chi}$$

$$= \frac{a^2}{2}\,\frac{\cancel{\cos\left(\frac{\chi}{2}\right)\sin\left(\frac{\chi}{2}\right)}}{2\,\cancel{\sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)}}$$

$$= \boxed{\frac{1}{4}a^2}$$

Total cross section

$$\sigma = \int \frac{d\sigma}{d\Omega}\,d\Omega \quad , \qquad d\Omega = 2\pi\sin\chi\,d\chi$$

$$= \frac{1}{4}a^2\int_0^\pi 2\pi\sin\chi\,d\chi$$

$$= \frac{1}{2}\pi a^2\left(-\cos\chi\right)\Big|_0^\pi \qquad \begin{aligned}\cos\pi &= -1\\ \cos 0 &= 1\end{aligned}$$

$$= \frac{1}{2}\pi a^2(-1)(-1-1)$$

$$= \boxed{\pi a^2}$$

Calculate $d\sigma$ wrt $\theta_1$ and $\theta_2$

$$d\sigma = 2\pi\rho\, d\rho = 2\pi\rho\left|\frac{d\rho}{d\chi}\right| d\chi$$

$$d\Omega = 2\pi \sin\chi\, d\chi \quad\rightarrow\quad 2\pi\, d\chi = \frac{d\Omega}{\sin\chi}$$

$$\rightarrow\quad d\sigma = \frac{\rho}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$

Similarly $$d\sigma_1 = \frac{\rho}{\sin\theta_1}\left|\frac{d\rho}{d\theta_1}\right| d\Omega_1$$

$$d\sigma_2 = \frac{\rho}{\sin\theta_2}\left|\frac{d\rho}{d\theta_2}\right| d\Omega_2$$

$$\rightarrow\quad \frac{d\sigma_1}{d\Omega_1} = \frac{\sin\chi\ \ d\chi}{\sin\theta_1\ d\theta_1}\ \frac{d\sigma}{d\Omega}$$

$$= \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| \frac{d\sigma}{d\Omega}$$

and $$\frac{d\sigma_2}{d\Omega_2} = \left|\frac{d(\cos\chi)}{d(\cos\theta_2)}\right| \frac{d\sigma}{d\Omega}$$

where $\dfrac{d\sigma}{d\Omega} = \dfrac{1}{4}a^2$ (for hard sphere)

Now: $2\theta_2 + \chi = \pi$ (always)

$$\chi = \pi - 2\theta_2$$

$$\rightarrow \cos\chi = \cos(\pi - 2\theta_2)$$
$$= -\cos(2\theta_2)$$
$$= -\cos^2\theta_2 + \sin^2\theta_2$$
$$= 1 - 2\cos^2\theta_2$$

$$d(\cos\chi) = -4\cos\theta_2 \, d(\cos\theta_2)$$

so $\left|\dfrac{d(\cos\chi)}{d(\cos\theta_2)}\right| = 4|\cos\theta_2|$

Thus, $$\boxed{\frac{d\sigma_2}{d\Omega_2} = \cancel{4}\cos\theta_2 \frac{1}{\cancel{4}} a^2 = a^2|\cos\theta_2|}$$

Relating $\theta_1$ to $\chi$:

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}$$

Want to find $\cos\chi$ in terms of $\theta_1$

compare to (16.5), (16.6):

$$\tan\theta = \frac{v_0 \sin\theta_0}{v_0\cos\theta_0 + V}$$

$$\cos\theta_0 = -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{1-\frac{V^2}{v_0^2}\sin^2\theta}$$

Thus,

$$\tan\theta_1 = \frac{m_2\sin\chi}{m_2\cos\chi + m_1}$$

$$\rightarrow \boxed{\cos\chi = -\left(\frac{m_1}{m_2}\right)\sin^2\theta_1 \pm \cos\theta_1\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

where $\pm$ sign for $m_1 > m_2$ and $+$ sign for $m_2 > m_1$

(i) For $m_2 > m_1$: (take $+\sqrt{\phantom{x}}$)

$$d(\cos\chi) = -2\left(\frac{m_1}{m_2}\right)\sin\theta_1\cos\theta_1\, d\theta_1 + d(\cos\theta_1)\sqrt{\phantom{x}}$$

$$+\frac{\cos\theta_1}{\sqrt{\phantom{x}}}\left(\frac{1}{\cancel{2}}\right)\left(-\cancel{2}\right)\left(\frac{m_1}{m_2}\right)^2\sin\theta_1\cos\theta_1\, d\theta_1$$

Now: $\sin\theta_1\cos\theta_1\, d\theta_1 = -\cos\theta_1\, d(\cos\theta_1)$

$$d(\cos\chi) = d(\cos\theta_1)\left[2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \sqrt{\phantom{x}} + \frac{\cos^2\theta_1}{\sqrt{\phantom{x}}}\left(\frac{m_1}{m_2}\right)^2\right]$$

$$\frac{d(\cos\chi)}{d(\cos\theta_1)} = 2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1 + \cos^2\theta_1\left(\frac{m_1}{m_2}\right)^2}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

$$= 2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

$$\rightarrow \boxed{\frac{d\sigma_1}{d\Omega_1} = \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right|\frac{d\sigma}{d\Omega} = \frac{1}{4}a^2\left[2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\right]}$$

for $0 \le \theta_1 \le \pi$

(ii) <u>For $m_1 > m_2$</u> : contribution from both $\pm$ signs

$+$ sign: $\frac{d\chi}{d\theta_1} > 0$

$-$ sign: $\frac{d\chi}{d\theta_1} < 0$

so need to subtract

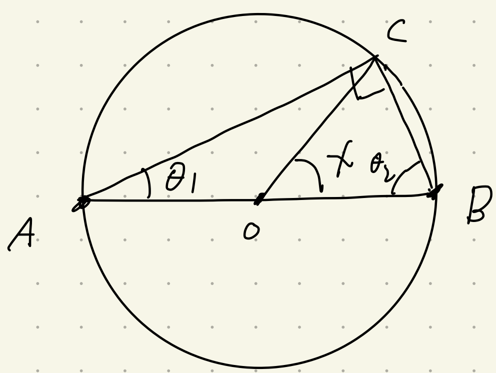
$$\frac{d(\cos\chi)}{d(\cos\theta_1)} = 2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1}{\sqrt{\phantom{x}}} - \left(2\left(\frac{m_1}{m_2}\right)\cos\theta_1 - \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1}{\sqrt{\phantom{x}}}\right)$$

$$\rightarrow \quad \frac{d(\cos\chi)}{d(\cos\theta_1)} = 2\,\frac{\left(1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1\right)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}$$

$$\boxed{\frac{d\sigma_1}{d\Omega_1} = \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| \frac{d\sigma}{d\Omega} \leftarrow \frac{1}{4}a^2 \\ = \frac{1}{2}a^2\,\frac{\left(1+\left(\frac{m_1}{m_2}\right)^2\cos(2\theta_1)\right)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}}$$

for $0 \le \theta_1 \le \theta_{max}$

(iii) For $m_1 = m_2$, $\quad \theta_1 + \theta_2 = \pi/2$




$$2\theta_2 + \chi = \pi \rightarrow \theta_2 = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\rightarrow \theta_1 = \frac{\pi}{2} - \theta_2$$
$$= \frac{\pi}{2} - \left(\frac{\pi}{2} - \frac{\chi}{2}\right)$$
$$= \frac{\chi}{2}$$

Thus, $$\frac{d(\cos\chi)}{d(\cos\theta_1)} = \frac{d(\cos(2\theta_1))}{d(\cos\theta_1)}$$

But $\cos(2\theta_1) = 2\cos^2\theta_1 - 1$

$$\rightarrow \quad \frac{d(\cos\chi)}{d(\cos\theta_1)} = \frac{4\cos\theta_1 \cancel{d(\cos\theta_1)}}{\cancel{d\cos\theta_1}}$$

$$= 4\cos\theta_1$$

Thus,

$$\frac{d\sigma_1}{d\Omega_1} = \left| \frac{d(\cos\chi)}{d(\cos\theta_1)} \right| \frac{d\sigma}{d\Omega}$$

$$= 4|\cos\theta_1| \frac{1}{4} a^2$$

$$= a^2|\cos\theta_1|$$

Sec 18, Prob 2:

From problem 1,
$$d\sigma = \frac{1}{4} a^2 d\Omega$$
$$= \frac{1}{4} a^2 \, 2\pi \sin\chi \, d\chi$$
$$= \frac{1}{2} \pi a^2 \sin\chi \, d\chi$$

Want to replace $\sin\chi \, d\chi$ by some function involving $\epsilon$, the energy lost by the scattered particle.

$\epsilon$ = energy lost by scattered particle
= energy gained by scattering particle
$= \frac{1}{2} m_2 v_2'^2$

<u>Now</u>: $v_2' = \frac{2 m_1 v}{m_1 + m_2} \sin\left(\frac{\chi}{2}\right)$ , $v = v_1 - \cancel{v_2}^{\,0} = v_\infty$

$$= 2 \frac{m v_\infty}{m_2} \sin\left(\frac{\chi}{2}\right)$$

$$\rightarrow \epsilon = \frac{1}{2} m_2 \frac{4 m^2 v_\infty^2}{m_2^2} \sin^2\left(\frac{\chi}{2}\right)$$
$$= 2 \frac{m^2}{m_2} v_\infty^2 \sin^2\left(\frac{\chi}{2}\right) = 2 \frac{m_1^2 m_2 v_\infty^2}{(m_1 + m_2)^2} \sin^2\left(\frac{\chi}{2}\right)$$

---

$\cos 2\theta = 1 - 2\sin^2\theta \longrightarrow \sin^2 \frac{\chi}{2} = \frac{1 - \cos\chi}{2}$

$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$$\epsilon = \frac{m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 (1-\cos\chi)$$

$$d\epsilon = \frac{m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 \sin\chi \, d\chi$$

Thus, $$d\sigma = \frac{1}{2}\pi a^2 \sin\chi \, d\chi$$

$$= \frac{1}{2}\pi a^2 \frac{(m_1+m_2)^2}{m_1^2 m_2 v_\infty^2} d\epsilon$$

NOTE: $$\epsilon = \frac{m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 (1-\cos\chi)$$

has $$\epsilon_{max} = \frac{2 m_1^2 m_2}{(m_1+m_2)^2} v_\infty^2 \quad \text{when } \chi = \pi$$

$$\rightarrow \quad \boxed{d\sigma = \frac{\pi a^2}{\epsilon_{max}} d\epsilon}$$

(Uniform distribution in $\epsilon$ between $0$ and $\epsilon_{max}$)

Sec 18, Prob 4:

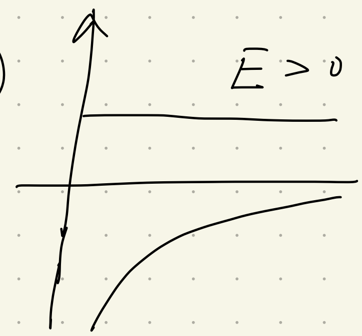
Effective cross section for particle to fall to center of potential $U = -\frac{\alpha}{r^2}$

$$U_{eff}(r) = \frac{M^2}{2mr^2} + U(r)$$

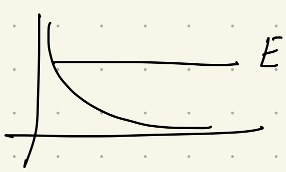
$$= \frac{M^2}{2mr^2} - \frac{\alpha}{r^2}$$

$$= \frac{1}{r^2}\left(\frac{M^2}{2m} - \alpha\right)$$

Need $\alpha \geq \frac{M^2}{2m}$ so that $\rightarrow U_{eff}(r)$



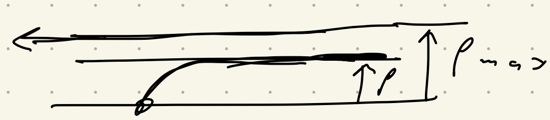

otherwise for $\alpha < \frac{M^2}{2m}$, the effective potential is repulsive




and particle can't reach the center.

$$\alpha \geq \frac{M^2}{2m} = \frac{m^2 \rho^2 v_\infty^2}{2m} = \rho^2\left(\frac{1}{2} m v_\infty^2\right)$$

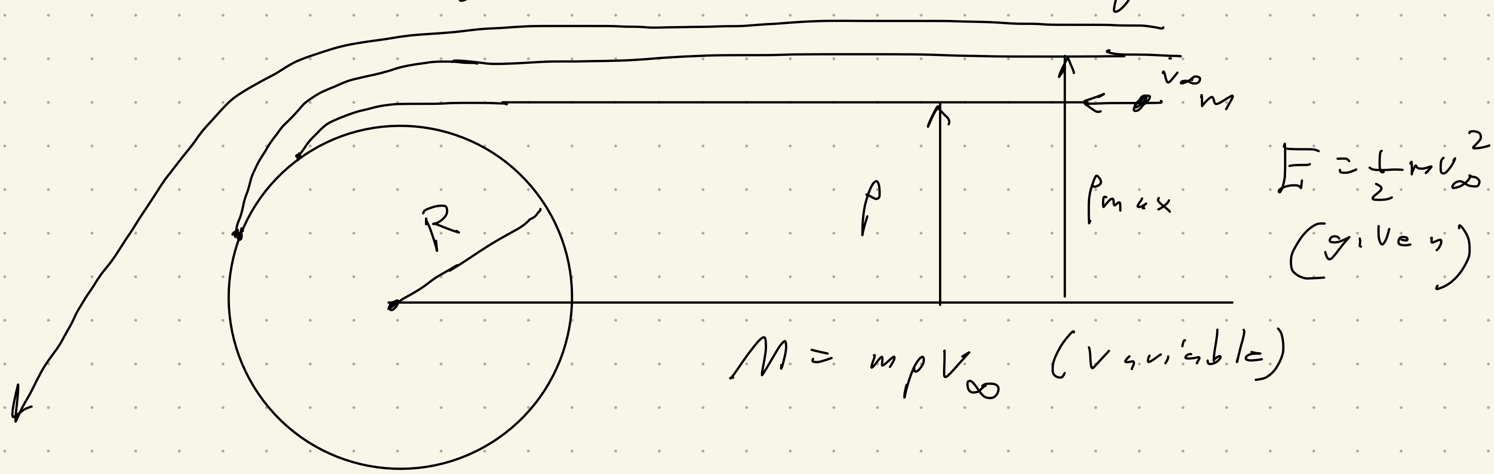
$$\rho \leq \sqrt{\frac{\alpha}{\frac{1}{2} m v_\infty^2}} = \rho_{max}$$



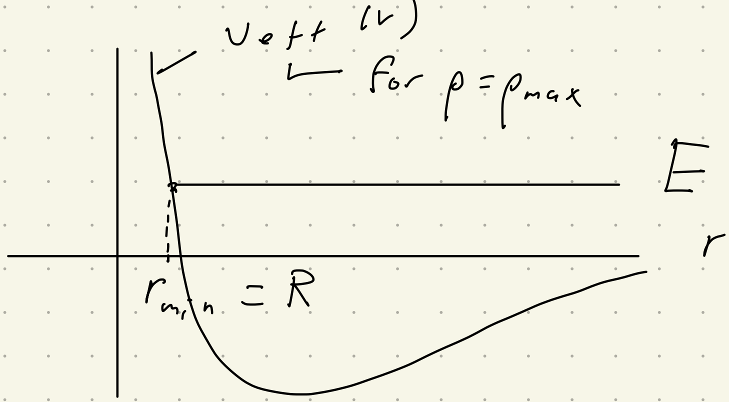

$$\boxed{\sigma = \pi \rho_{max}^2 = \frac{\pi \alpha}{\frac{1}{2} m v_\infty^2}}$$

Sec 18, Prob 6:

Effective cross section for particle of mass $m_1$ to strike a sphere of mass $m_2$ and radius $R$ subject to Newtonian gravity




outside sphere $U = -\dfrac{G m_1 m_2}{r} = -\dfrac{\alpha}{r}$




$$\frac{1}{2} m V_\infty^2 = U_{eff}(r_{min})$$

$$= U_{eff}(R)$$

$$= -\frac{\alpha}{R} + \frac{M^2}{2 m R^2}$$

$$= -\frac{\alpha}{R} + \frac{m^2 \rho_{max}^2 V_\infty^2}{2 m R^2}$$

$$= -\frac{\alpha}{R} + \left(\frac{1}{2} m V_\infty^2\right) \frac{\rho_{max}^2}{R^2}$$

Thus, $E = -\frac{\alpha}{R} + E \frac{\rho_{max}^2}{R^2}$

$$\rightarrow \frac{E + \alpha/R}{E} = \frac{\rho_{max}^2}{R^2}$$

$$\rightarrow \sigma = \pi \rho_{max}^2$$

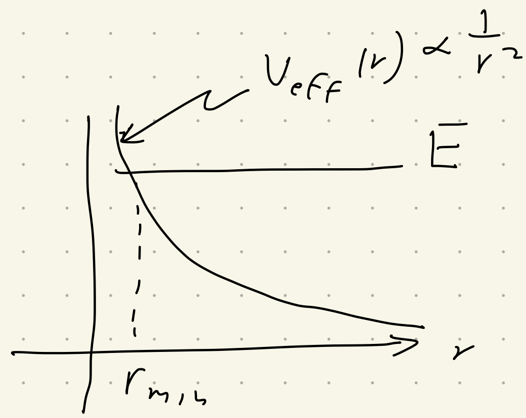
$$= \pi R^2 \left( 1 + \frac{\alpha}{R} \cdot \frac{1}{E} \right)$$

$$= \pi R^2 \left( 1 + \frac{G \cancel{m_1 m_2}}{R} \frac{1}{\frac{1}{2} \frac{\cancel{m_1 m_2}}{(m_1 + m_2)} v_\infty^2} \right)$$

$$= \pi R^2 \left( 1 + \frac{2 G (m_1 + m_2)}{R v_\infty^2} \right)$$

Sec 19, Prob 1: $\quad U = \dfrac{\alpha}{r^2}, \quad \alpha > 0$

$$U_{eff}(r) = U(r) + \frac{M^2}{2mr^2}$$
$$= \frac{\alpha}{r^2} + \frac{\rho^2 m^2 v_\infty^2}{2mr^2}$$
$$= \frac{1}{r^2}\left(\alpha + \frac{1}{2} m v_\infty^2 \cdot \rho^2\right)$$




$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \rho^2/r^2 - 2U/mv_\infty^2}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \left(\rho^2 + \frac{2\alpha}{mv_\infty^2}\right)\frac{1}{r^2}}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \beta^2/r^2}}$$

$$\beta^2 \equiv \rho^2 + \frac{2\alpha}{mv_\infty^2} = \rho^2 + \frac{\alpha}{E}$$

Let: $u = \dfrac{1}{r} \quad \rightarrow \quad du = -\dfrac{1}{r^2}\, dr$

$$\phi_0 = \int_0^{\frac{1}{r_{min}}} \frac{\rho\, du}{\sqrt{1 - \beta^2 u^2}}, \qquad 1 - \frac{\beta^2}{r_{min}^2} = 0 \rightarrow \boxed{\beta = r_{min}}$$

$$\phi_0 = \int_0^{\frac{1}{\beta}} \frac{\rho \, du}{\sqrt{1-\beta^2 u^2}}$$

Let: $\beta u = \sin\theta \rightarrow du = \frac{1}{\beta}\cos\theta \, d\theta$

$$\sqrt{1-\beta^2 u^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$

$$u = 0 \rightarrow \theta = 0$$

$$u = \frac{1}{\beta} \rightarrow \sin\theta = 1 \rightarrow \theta = \pi/2$$

$$\rightarrow \phi_0 = \int_0^{\pi/2} \frac{\rho}{\beta} \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}}$$

$$= \frac{\pi}{2} \frac{\rho}{\beta}$$

$$= \frac{\pi}{2} \frac{\rho}{\sqrt{\rho^2 + \frac{\alpha}{E}}}$$

Square both sides

$$\left(\frac{2\phi_0}{\pi}\right)^2 = \frac{\rho^2}{\rho^2 + \frac{\alpha}{E}}$$

$$\left(\rho^2 + \frac{\alpha}{E}\right)\left(\frac{2\phi_0}{\pi}\right)^2 = \rho^2$$

$$\frac{\alpha}{E}\left(\frac{2\phi_0}{\pi}\right)^2 = \rho^2\left(1 - \left(\frac{2\phi_0}{\pi}\right)^2\right)$$

$$\rho^2 = \frac{\frac{\alpha}{E}\left(\frac{2\phi_0}{\pi}\right)^2}{1-\left(\frac{2\phi_0}{\pi}\right)^2}$$

<u>Now:</u> $2\phi_0 + \chi = \pi$ for repulsive scattering

$$\rightarrow \quad \frac{2\phi_0}{\pi} = 1 - \frac{\chi}{\pi}$$

$$\rightarrow \quad \left(\frac{2\phi_0}{\pi}\right)^2 = \left(1-\frac{\chi}{\pi}\right)^2$$

$$\rightarrow \quad 1-\left(\frac{2\phi_0}{\pi}\right)^2 = 1 - \left(1 + \frac{\chi^2}{\pi^2} - \frac{2\chi}{\pi}\right)$$

$$= \frac{2\chi}{\pi} - \frac{\chi^2}{\pi^2}$$

<u>Thus:</u>

$$\rho^2 = \frac{\frac{\alpha}{E}\left(1-\frac{\chi}{\pi}\right)^2}{\left(\frac{2\chi}{\pi}-\frac{\chi^2}{\pi^2}\right)}$$

$$= \frac{\frac{\alpha}{E}(\pi-\chi)^2}{2\pi\chi-\chi^2}$$

Thus,

$$\rho = \sqrt{\frac{\alpha}{E}} \; \frac{(\pi - \chi)}{\sqrt{2\pi\chi - \chi^2}}$$

Differential cross-section!

$$d\sigma = 2\pi \rho \, d\rho$$
$$= 2\pi \rho \left| \frac{d\rho}{d\chi} \right| d\chi$$

$$d\Omega = 2\pi \sin\chi \, d\chi$$

$$\rightarrow \boxed{d\sigma = \frac{\rho}{\sin\chi} \left| \frac{d\rho}{d\chi} \right| d\Omega}$$

$$\frac{d\rho}{d\chi} = \sqrt{\frac{\alpha}{E}} \left( \frac{-\sqrt{2\pi\chi - \chi^2} - \frac{1}{\cancel{2}\sqrt{\phantom{x}}}(\cancel{2}\pi - \cancel{2}\chi)(\pi - \chi)}{2\pi\chi - \chi^2} \right)$$

$$= \sqrt{\frac{\alpha}{E}} \; \frac{-1}{(2\pi\chi - \chi^2)^{3/2}} \left( \underbrace{2\pi\chi - \chi^2 + (\pi - \chi)^2}_{\cancel{2\pi\chi} - \cancel{\chi^2} + \pi^2 + \cancel{\chi^2} - \cancel{2\pi\chi}} \right)$$

$$= -\sqrt{\frac{\alpha}{E}} \; \frac{\pi^2}{(2\pi\chi - \chi^2)^{3/2}}$$

so $$\boxed{\left| \frac{d\rho}{d\chi} \right| = \sqrt{\frac{\alpha}{E}} \; \frac{\pi^2}{(2\pi\chi - \chi^2)^{3/2}}}$$

Thus,

$$\frac{d\sigma}{d\Omega} = \frac{1}{\sin\chi}\sqrt{\frac{\alpha}{E}}\,\frac{(\pi-\chi)}{\sqrt{2\pi\chi-\chi^2}}\,\sqrt{\frac{\alpha}{E}}\,\frac{\pi^2}{(2\pi\chi-\chi^2)^{3/2}}$$

$$= \frac{1}{\sin\chi}\left(\frac{\alpha}{E}\right)\frac{\pi^2(\pi-\chi)}{(2\pi\chi-\chi^2)^2}$$

Sec 20, Prob 1:

Start with (18.4):

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \rho^2/r^2 - 2U/mv_\infty^2}}$$

Consider small angle scattering, where

$$\frac{2U}{mv_\infty^2} << 1$$

Then:

$$\frac{1}{\sqrt{1-\rho^2/r^2 - 2U/mv_\infty^2}} = \frac{1}{\sqrt{1-\rho^2/r^2}} \frac{1}{\sqrt{1 - \frac{2U/mv_\infty^2}{1-\rho^2/r^2}}}$$

$$\approx \frac{1}{\sqrt{1-\rho^2/r^2}} \left(1 + \frac{U}{mv_\infty^2}\left(\frac{1}{1-\rho^2/r^2}\right)\right)$$

$$= \frac{1}{\sqrt{1-\rho^2/r^2}} + \frac{U/mv_\infty^2}{(1-\rho^2/r^2)^{3/2}}$$

Now:

$$\int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1-\rho^2/r^2}} = \int_0^{\frac{1}{r_{min}}} \frac{\rho\, du}{\sqrt{1-\rho^2u^2}} = \int_0^{\pi/2} \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \boxed{\frac{\pi}{2}}$$

let: $u = \frac{1}{r}$

$$du = -\frac{1}{r^2}\, dr$$

let: $\rho u = \sin\theta$, $\rho\, du = \cos\theta\, d\theta$

$$u = \frac{1}{r_{min}} \rightarrow \rho = r_{min} \rightarrow \theta = \frac{\pi}{2}$$

Thus,

$$\phi_0 \approx \frac{\pi}{2} + \frac{1}{mv_\infty^2} \int_{r_{min}}^{\infty} \frac{U(r)\,\rho\, dr/r^2}{(1-\rho^2/r^2)^{3/2}}$$

$$= \frac{\pi}{2} + \frac{1}{mv_\infty^2} \frac{\partial}{\partial \rho}\left[\int_{r_{min}}^{\infty} \frac{U(r)\,dr}{\sqrt{1-\rho^2/r^2}}\right]$$

Now:

$$\int_{r_{min}}^{\infty} \frac{U(r)\,dr}{\sqrt{1-\rho^2/r^2}} = \int_{r_{min}}^{\infty} \frac{U(r)\,r\,dr}{\sqrt{r^2-\rho^2}}$$

$$\approx \int_{\rho}^{\infty} \frac{U(r)\,r\,dr}{\sqrt{r^2-\rho^2}}$$

$$u = U(r) \rightarrow du = \frac{dU}{dr}\,dr$$

$$dv = \frac{r\,dr}{\sqrt{r^2-\rho^2}} = \frac{dx/2}{\sqrt{x}} \rightarrow v = \sqrt{x} = \sqrt{r^2-\rho^2}$$

$$(\text{let } x = r^2-\rho^2 \rightarrow dx = 2r\,dr)$$

Thus,

$$\int_{r_{min}}^{\infty} \frac{U(r)\,dr}{\sqrt{1-\rho^2/r^2}} \approx \underbrace{U(r)\sqrt{r^2-\rho^2}\,\Big|_{\rho}^{\infty}}_{0} - \int_{\rho}^{\infty} dr\,\frac{dU}{dr}\sqrt{r^2-\rho^2}$$

assume $U(\infty) \to 0$
faster than $\frac{1}{r}$

$$\phi_0 \approx \frac{\pi}{2} - \frac{1}{mv_\infty^2}\frac{\partial}{\partial\rho}\left[\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\sqrt{r^2-\rho^2}\right]$$

$$= \frac{\pi}{2} - \frac{1}{mv_\infty^2}\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\frac{1}{\cancel{2}\sqrt{r^2-\rho^2}}\,\cancel{-2}\rho\,]$$

$$= \frac{\pi}{2} + \frac{\rho}{mv_\infty^2}\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}$$

Recall:

$$\chi + 2\phi_0 = \pi \longrightarrow \phi_0 - \frac{\pi}{2} = -\frac{\chi}{2}$$

Thus,

$$\boxed{\chi = -2\left(\phi_0 - \frac{\pi}{2}\right) \approx \frac{-2\rho}{mv_\infty^2}\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}}$$

Compare to:

$$\theta_1 \approx \frac{-2\rho}{m_1 v_\infty^2}\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}} \qquad (20.3)$$

$$= \left(\frac{m_2}{m_1+m_2}\right)\left(\frac{-2\rho}{mv_\infty^2}\right)\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}$$

$$= \left(\frac{m_2}{m_1+m_2}\right)\chi$$

(consistent with $\tan\theta_1 = \dfrac{m_2\sin\chi}{m_1+m_2\cos\chi} \longrightarrow \theta_1 \approx \dfrac{m_2\chi}{m_1+m_2}$ for $\chi \ll 1$)