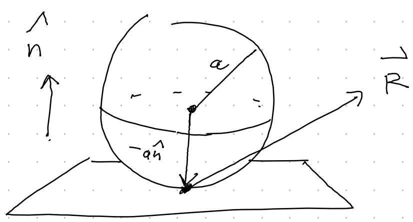
Horizontal turntable



$I = \frac{2}{5} \mu a^2$ for uniform sphere

$\hat{n}$: normal to plane of turntable

$\vec{\omega} = \omega \hat{n}$
$\omega = \text{const}$

EOMs:

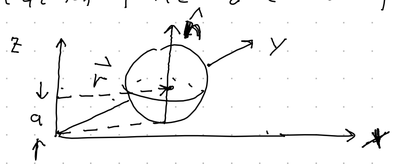
$$\frac{d\vec{P}}{dt} = \vec{F} + \vec{R}$$

$$\frac{d\vec{M}}{dt} = \vec{K} + (-a\hat{n}) \times \vec{R}$$

$\vec{F} = \mu \vec{g} = -\mu g \hat{n}$ (applied force)

$\vec{K} = \vec{0}$

$\vec{R}$: reaction force due to friction.



(x, y, z) : inertial frame

(x, y) : plane of turntable

$\hat{n} = \hat{z}$ for horizontal turntable

EOMs:

$$\mu \ddot{\vec{r}} = -\mu g \hat{n} + \vec{R} \qquad (1)$$

$$I \dot{\vec{\Omega}} = -a\hat{n} \times \vec{R} \qquad (2)$$

Rolling w/out slipping

$$\vec{\omega} \times \vec{r} = \dot{\vec{r}} + \vec{\Omega} \times (-a\hat{n}) \qquad (3)$$

Eliminate reaction force $\vec{R}$ from (1), (2):

$$\vec{R} = \mu \ddot{\vec{r}} + \mu g \hat{n}$$

$$\rightarrow \quad I \dot{\vec{\Omega}} = -a\hat{n} \times (\mu \ddot{\vec{r}} + \mu g \hat{n})$$

$$I \dot{\vec{\Omega}} = -a\mu\, \hat{n} \times \ddot{\vec{r}} \qquad (4)$$

Differentiate constraint equation

$$\vec{\omega} \times \dot{\vec{r}} = \ddot{\vec{r}} - a \dot{\vec{\Omega}} \times \hat{n}$$

$$\rightarrow \quad \ddot{\vec{r}} = \vec{\omega} \times \dot{\vec{r}} + a \dot{\vec{\Omega}} \times \hat{n} \qquad (5)$$

Solve (4) for $\dot{\vec{\Omega}}$ and substitute into (5):

$$\dot{\vec{\Omega}} = -\frac{a\mu}{I} \hat{n} \times \ddot{\vec{r}}$$

$$\rightarrow \quad \ddot{\vec{r}} = \vec{\omega} \times \dot{\vec{r}} - \frac{\mu a^2}{I} (\hat{n} \times \ddot{\vec{r}}) \times \hat{n}$$

Now: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

Thus, $(\hat{n} \times \ddot{\vec{r}}) \times \hat{n} = -\hat{n} \times (\hat{n} \times \ddot{\vec{r}})$

$$= -[\hat{n} \underbrace{(\hat{n} \cdot \ddot{\vec{r}})}_{=0} - \ddot{\vec{r}} \underbrace{(\hat{n} \cdot \hat{n})}_{=1}]$$

$=0$ since $\vec{r}$ in xy plane and $\hat{n} = \hat{z}$

$$= \ddot{\vec{r}}$$

$$\rightarrow \quad \ddot{\vec{r}} = \vec{\omega} \times \dot{\vec{r}} - \frac{\mu a^2}{I} \ddot{\vec{r}}$$

$$\left(1 + \frac{\mu a^2}{I}\right) \ddot{\vec{r}} = \vec{\omega} \times \dot{\vec{r}}$$

Integrate:

$$\left(1+\frac{\mu a^2}{I}\right)\left(\dot{\vec r}-\dot{\vec r}(0)\right) = \vec\omega\times\left(\vec r-\vec r(0)\right)$$

If released from rest, $\dot{\vec r}(0)=0$:

$$\left(1+\frac{\mu a^2}{I}\right)\dot{\vec r} = \vec\omega\times\left(\vec r-\vec r(0)\right)$$

$$\rightarrow \boxed{\dot{\vec r} = \frac{\vec\omega}{\left(1+\frac{\mu a^2}{I}\right)}\times\left(\vec r-\vec r(0)\right)}$$

For a unif sphere, $I = \frac{2}{5}\mu a^2$

$$\rightarrow 1+\frac{\mu a^2}{I} = 1+\frac{\cancel{\mu a^2}}{\frac{2}{5}\cancel{\mu a^2}} = 1+\frac{5}{2} = \frac{7}{2}$$

$$\rightarrow \boxed{\dot{\vec r} = \frac{2}{7}\vec\omega\times\left(\vec r-\vec r(0)\right)}$$

uniform circular motion

$$\vec V = \frac{2}{7}\vec\omega\times\left(\vec r-\vec r(0)\right)$$

$$\vec a = \ddot{\vec r} = \frac{2}{7}\vec\omega\times\dot{\vec r} = \left(\frac{2}{7}\right)^2\vec\omega\times\left(\vec\omega\times(\vec r-\vec r_0)\right)$$

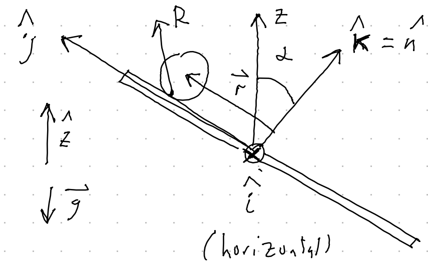
$$\vec\omega\times\left(\vec\omega\times(\vec r-\vec r_0)\right) = \vec\omega\left(\cancelto{0}{\vec\omega\cdot(\vec r-\vec r_0)}\right) - (\vec r-\vec r_0)\,\vec\omega\cdot\vec\omega$$

since $\vec\omega \perp (\vec r-\vec r_0)$

$$= -\omega^2(\vec r-\vec r_0)$$

Thus, $\vec a = \underbrace{-\left(\frac{2}{7}\omega\right)^2(\vec r-\vec r_0)}_{\text{directed radially inward}}$

Tilted rotating turntable:



$(\hat i, \hat j, \hat k)$: orthonormal wrt plane of turntable, but not rotating with turntable

$\hat z$: vertical

$\hat n = \hat k$: $\perp$ to turntable

$\vec r$: position vector of COM

$\hat z = \cos\alpha\,\hat n + \sin\alpha\,\hat j$

$\vec\omega = \omega\hat n$

$$\mu\ddot{\vec r} = -\mu g\hat z + \vec R \qquad (1)$$

$$I\dot{\vec\Omega} = -a\hat n\times\vec R \qquad (2)$$

Rolling without slipping

$$\vec\omega\times\vec r = \dot{\vec r} + \vec\Omega\times(-a\hat n) \qquad (3)$$

Eliminate $\vec R$ from (1) and (2)

$$I\dot{\vec\Omega} = -a\hat n\times\left(\mu\ddot{\vec r} + \mu g\hat z\right)$$

$$I\dot{\vec{\Omega}} = -a\mu\hat{n}\times\ddot{\vec{r}} - a\mu g\,\hat{n}\times\hat{z} \qquad (4)$$

Differentiate constraint

$$\vec{\omega}\times\dot{\vec{r}} = \ddot{\vec{r}} - a\dot{\vec{\Omega}}\times\hat{n}$$

$$\rightarrow \quad \ddot{\vec{r}} = \vec{\omega}\times\dot{\vec{r}} + a\dot{\vec{\Omega}}\times\hat{n} \qquad (5)$$

substitute $\dot{\vec{\Omega}}$ from (4) into (5):

$$\ddot{\vec{r}} = \vec{\omega}\times\dot{\vec{r}} - \frac{\mu a^2}{I}\left(\hat{n}\times\ddot{\vec{r}}\right)\times\hat{n} - \frac{\mu a^2}{I}g\left(\hat{n}\times\hat{z}\right)\times\hat{n}$$

Now:

$$\left(\hat{n}\times\ddot{\vec{r}}\right)\times\hat{n} = \ddot{\vec{r}} \quad (\text{as before})$$

$$\begin{aligned}
\left(\hat{n}\times\hat{z}\right)\times\hat{n} &= -\hat{n}\times\left(\hat{n}\times\hat{z}\right)\\
&= -\Big[\hat{n}\underbrace{(\hat{n}\cdot\hat{z})}_{\cos\alpha} - \hat{z}\underbrace{(\hat{n}\cdot\hat{n})}_{1}\Big]\\
&= -\hat{n}\cos\alpha + \hat{z}\\
&= \cancel{-\hat{n}\cos\alpha} + \left(\cancel{\cos\alpha\,\hat{n}} + \sin\alpha\,\hat{j}\right)\\
&= \sin\alpha\,\hat{j}
\end{aligned}$$

Thus,

$$\ddot{\vec{r}} = \vec{\omega}\times\dot{\vec{r}} - \frac{\mu a^2}{I}\ddot{\vec{r}} - \frac{\mu a^2}{I}g\sin\alpha\,\hat{j}$$

$$\left(1+\frac{\mu a^2}{I}\right)\ddot{\vec{r}} = \vec{\omega}\times\dot{\vec{r}} - \frac{\mu a^2}{I}g\sin\alpha\,\hat{j}$$

$$\ddot{\vec{r}} = \frac{\vec{\omega}\times\dot{\vec{r}}}{\left(1+\frac{\mu a^2}{I}\right)} - \frac{\left(\frac{\mu a^2}{I}\right)}{\left(1+\frac{\mu a^2}{I}\right)}g\sin\alpha\,\hat{j}$$

For uniform sphere:

$$1+\frac{\mu a^2}{I} = 1+\frac{1}{\left(\frac{2}{5}\right)} = \frac{7}{2}$$

$$\rightarrow \quad \frac{1}{\left(1+\frac{\mu a^2}{I}\right)} = \frac{2}{7}$$

$$\rightarrow \quad \frac{\frac{\mu a^2}{I}}{\left(1+\frac{\mu a^2}{I}\right)} = \frac{2}{7}\times\frac{5}{2} = \frac{5}{7}$$

Thus

$$\boxed{\ddot{\vec{r}} = \frac{2}{7}\vec{\omega}\times\dot{\vec{r}} - \frac{5}{7}g\sin\alpha\,\hat{j}}$$

since $\vec{\omega} = \omega\hat{n} = \omega\hat{k}$ and $\hat{k}\times\hat{i} = \hat{j}$

we can also write:

$$\boxed{\ddot{\vec{r}} = \frac{2}{7}\vec{\omega}\times\Big[\dot{\vec{r}} - \underbrace{\frac{5}{2}\frac{g}{\omega}\sin\alpha\,\hat{i}}_{\text{horizontal direction}}\Big]}$$

Integrate wrt time assuming $\dot{\vec{r}}(0) = 0$

$$\rightarrow \dot{\vec{r}} = \frac{2}{7}\vec{\omega} \times \Big[\underbrace{(\vec{r}-\vec{r}(0)) - \frac{5}{2}\frac{gt\sin\alpha}{\omega}\hat{i}}_{\text{center of circle moves in horizontal direction with constant speed } \frac{5}{7}gt\sin\alpha} \Big]$$