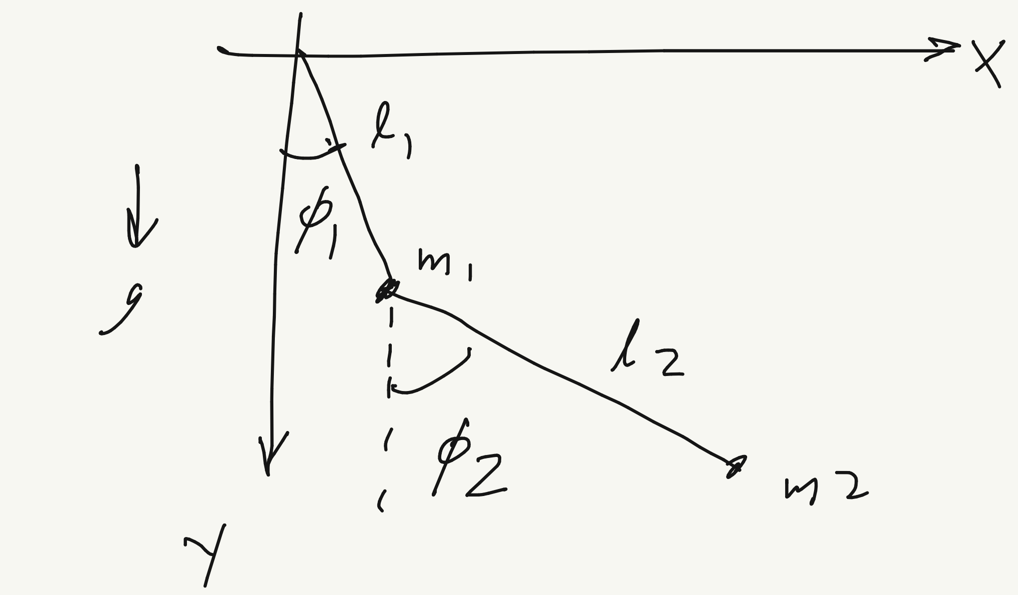
Sec 5, Prob 1 :



$$x_1 = l_1 \sin\phi_1$$
$$y_1 = l_1 \cos\phi_1$$

$$x_2 = x_1 + l_2 \sin\phi_2$$
$$y_2 = y_1 + l_2 \cos\phi_2$$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_1 g l_1 \cos\phi_1 - m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$
$$= -(m_1+m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\dot{x}_1 = l_1 \dot{\phi}_1 \cos\phi_1 \rightarrow \dot{x}_1^2 = l_1^2 \dot{\phi}_1^2 \cos^2\phi_1$$
$$\dot{y}_1 = -l_1 \dot{\phi}_1 \sin\phi_1 \rightarrow \dot{y}_1^2 = l_1^2 \dot{\phi}_1^2 \sin^2\phi_1$$
$$\dot{x}_1^2 + \dot{y}_1^2 = l_1^2 \dot{\phi}_1^2$$

$$\dot{x}_2 = l_1 \dot{\phi}_1 \cos\phi_1 + l_2 \dot{\phi}_2 \cos\phi_2$$
$$\dot{y}_2 = -l_1 \dot{\phi}_1 \sin\phi_1 - l_2 \dot{\phi}_2 \sin\phi_2$$

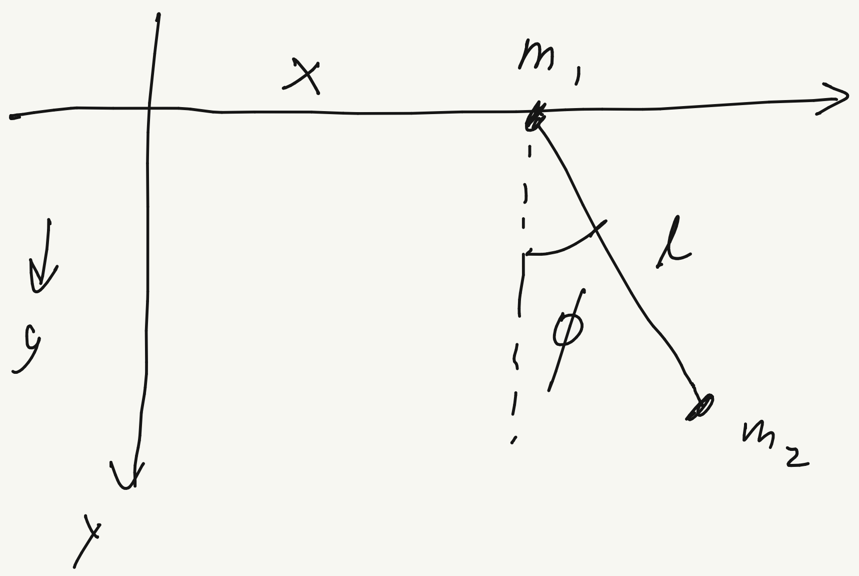
$$\rightarrow \dot{x}_2^2 = l_1^2 \dot{\phi}_1^2 \cos^2\phi_1 + l_2^2 \dot{\phi}_2^2 \cos^2\phi_2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos\phi_1 \cos\phi_2$$
$$\dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 \sin^2\phi_1 + l_2^2 \dot{\phi}_2^2 \sin^2\phi_2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin\phi_1 \sin\phi_2$$

so $\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2)$
$$= l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

Thus,
$$T = \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$
$$= \frac{1}{2} (m_1+m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

$$\rightarrow L = T - U$$
$$= \frac{1}{2} (m_1+m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$
$$+ (m_1+m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2$$

Sec 5, Prob 2:



$$(x_1, y_1) = (x, 0)$$
$$(x_2, y_2) = (x + l\sin\phi,\ l\cos\phi)$$
$$(\dot{x}_1, \dot{y}_1) = (\dot{x}, 0)$$
$$(\dot{x}_2, \dot{y}_2) = (\dot{x} + l\dot{\phi}\cos\phi,\ -l\dot{\phi}\sin\phi)$$

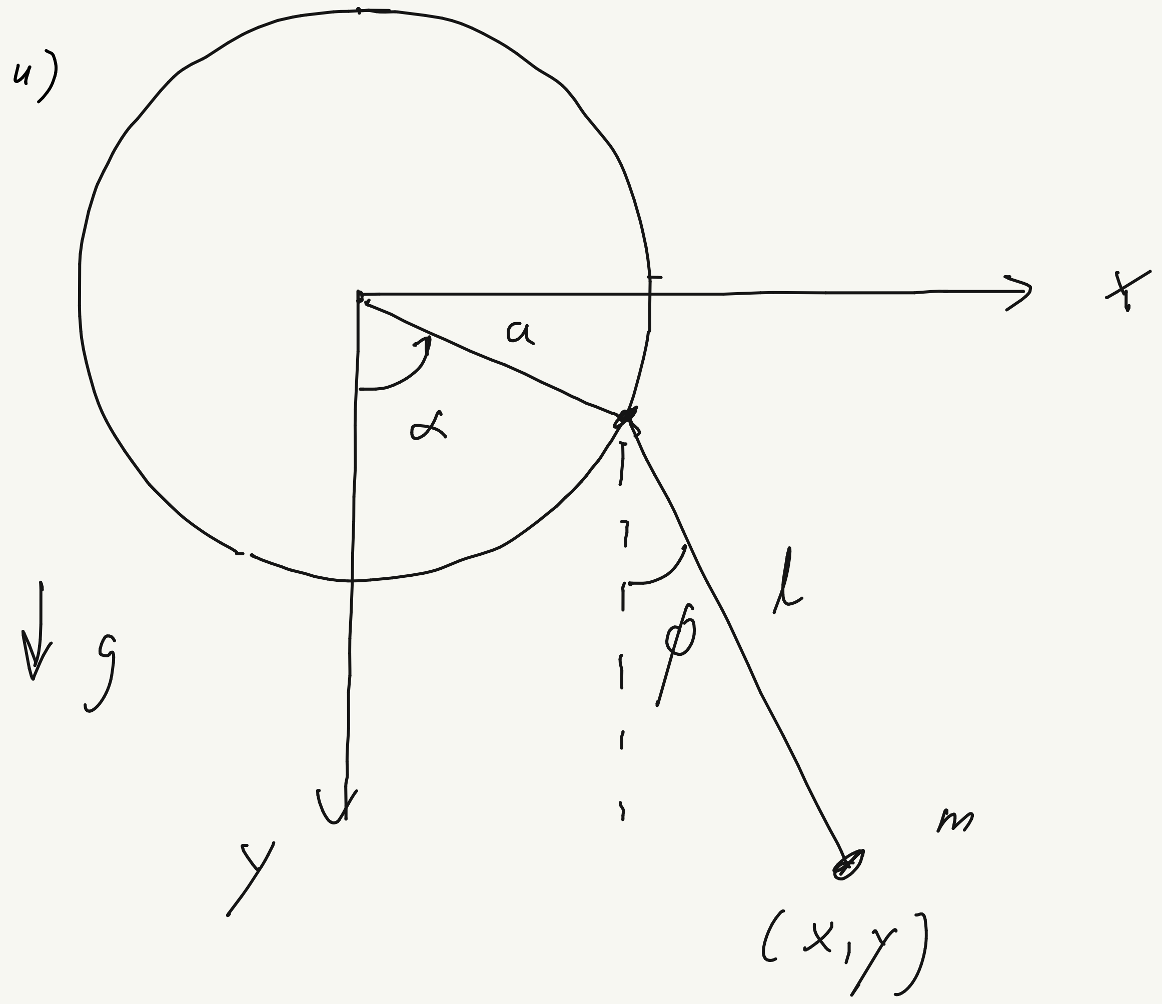
$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$
$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}^2 + l^2\dot{\phi}^2\cos^2\phi + 2 l \dot{x}\dot{\phi}\cos\phi + l^2\dot{\phi}^2\sin^2\phi)$$
$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \dot{x}\dot{\phi}\cos\phi$$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_2 g l \cos\phi$$

$$L = T - U$$
$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \dot{x}\dot{\phi}\cos\phi + m_2 g l \cos\phi$$
$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 (l^2\dot{\phi}^2 + l\dot{x}\dot{\phi}\cos\phi) + m_2 g l \cos\phi$$

Sec 5, Prob 3:

(a)



$$\alpha = \gamma t$$

$$x = a\sin\alpha + l\sin\phi$$
$$y = a\cos\alpha + l\cos\phi$$

$$U = -mgy$$
$$= \underbrace{-mga\cos\alpha}_{\text{prescribed function of time (ignore)}} - mgl\cos\phi$$
$$= -mgl\cos\phi$$

$$\dot{x} = a\gamma\cos\alpha + l\dot{\phi}\cos\phi$$
$$\dot{y} = -a\gamma\sin\alpha - l\dot{\phi}\sin\phi$$

$$\dot{x}^2 = a^2\gamma^2\cos^2\alpha + l^2\dot{\phi}^2\cos^2\phi + 2al\gamma\dot{\phi}\cos\alpha\cos\phi$$
$$\dot{y}^2 = a^2\gamma^2\sin^2\alpha + l^2\dot{\phi}^2\sin^2\phi + 2al\gamma\dot{\phi}\sin\alpha\sin\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2}m\left[a^2\gamma^2 + l^2\dot{\phi}^2 + 2al\gamma\dot{\phi}\cos(\alpha-\phi)\right]$$
$$= \underbrace{\frac{1}{2}ma^2\gamma^2}_{\text{prescribed function of time (ignore)}} + \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\cos(\gamma t-\phi)$$

$$= \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\cos(\gamma t-\phi)$$

$$L = T - U$$
$$= \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\cos(\gamma t-\phi) + mgl\cos\phi$$

Now:

$$\gamma \dot{\phi} \cos(\gamma t - \phi) = \underbrace{\frac{d}{dt}\left[-\gamma \sin(\gamma t - \phi)\right]}_{\text{can ignore since total time derivative.}} + \gamma^2 \cos(\gamma t - \phi)$$

Thus,

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + m a l \gamma^2 \cos(\gamma t - \phi) + m g l \cos\phi$$

NOTE:

EoM should be the same for both Lagrangians:

(1st): $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$$\frac{d}{dt}\left(m l^2 \dot{\phi} + m a l \gamma \cos(\gamma t - \phi)\right)$$

$$= m a l \gamma \dot{\phi} \sin(\gamma t - \phi) - m g l \sin\phi$$

$$m l^2 \ddot{\phi} - m a l \gamma^2 \sin(\gamma t - \phi) + \cancel{m a l \gamma \dot{\phi} \sin(\gamma t - \phi)}$$

$$= \cancel{m a l \gamma \dot{\phi} \sin(\gamma t - \phi)} - m g l \sin\phi$$

$$\rightarrow \ddot{\phi} = \frac{a}{l} \gamma^2 \sin(\gamma t - \phi) - \frac{g}{l} \sin\phi$$

agree

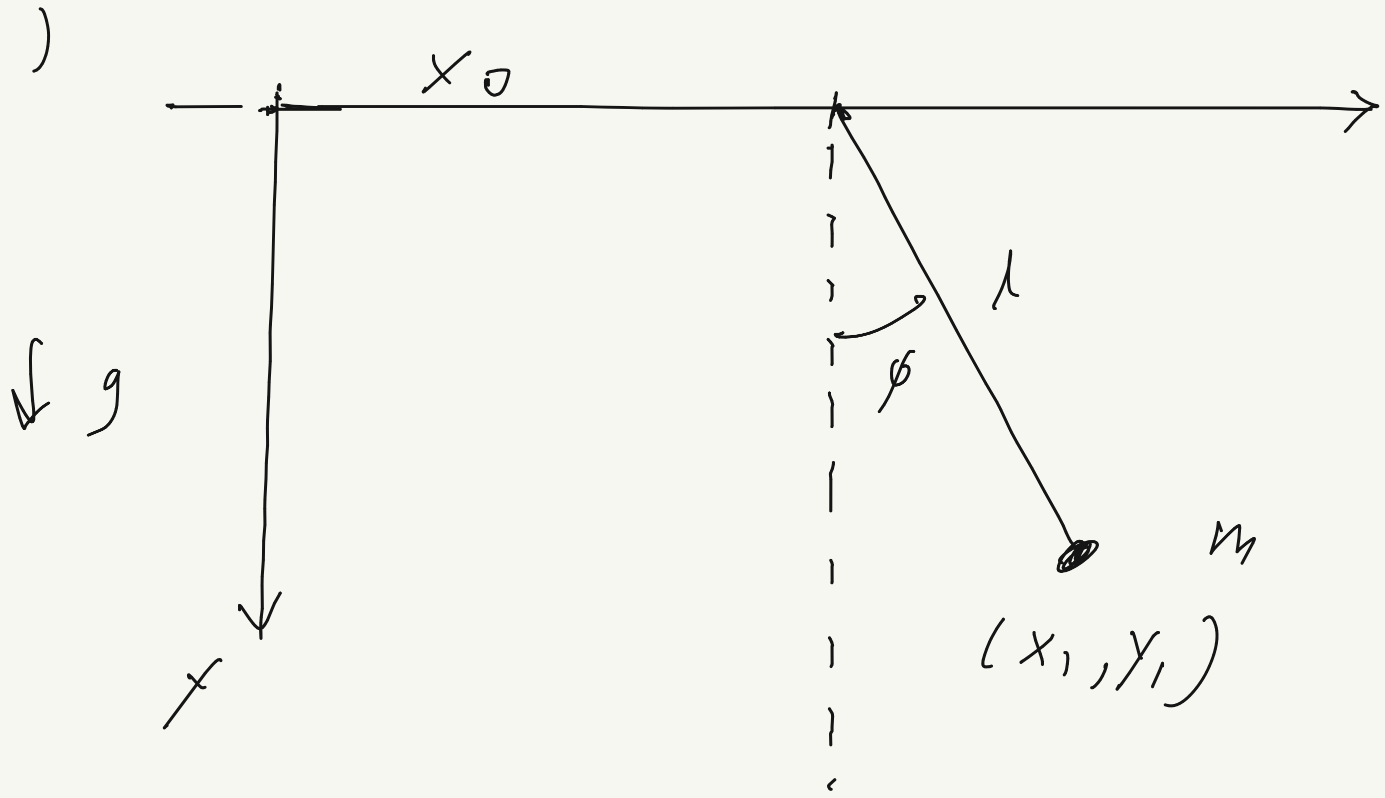
(2nd) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$$\frac{d}{dt}\left(m l^2 \dot{\phi}\right) = + m a l \gamma^2 \sin(\gamma t - \phi) - m g l \sin\phi$$

$$m l^2 \ddot{\phi} = m a l \gamma^2 \sin(\gamma t - \phi) - m g l \sin\phi$$

$$\rightarrow \ddot{\phi} = \frac{a}{l} \gamma^2 \sin(\gamma t - \phi) - \frac{g}{l} \sin\phi$$

(b)



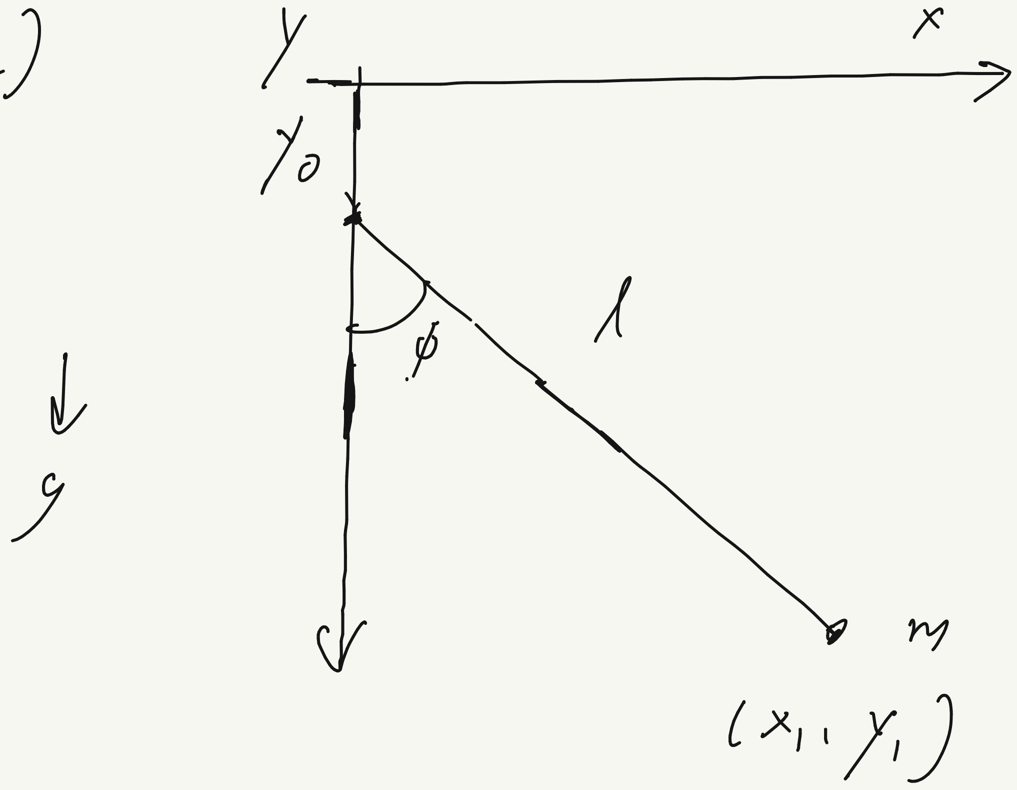
$x_0 = a\cos\gamma t$
$\dot{x}_0 = -a\gamma\sin\gamma t$

$$x = x_0 + l\sin\phi$$
$$y = l\cos\phi$$
$$U = -mgy = -mgl\cos\phi$$
$$\dot{x} = \dot{x} + l\dot{\phi}\cos\phi$$
$$= -a\gamma\sin\gamma t + l\dot{\phi}\cos\phi$$
$$\dot{y} = -l\dot{\phi}\sin\phi$$
$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2}m\left(a^2\gamma^2\sin^2\gamma t + l^2\dot{\phi}^2\cos^2\phi - 2al\gamma\dot{\phi}\sin\gamma t\cos\phi + l^2\dot{\phi}^2\sin^2\phi\right)$$
$$= \underbrace{\frac{1}{2}ma^2\gamma^2\sin^2\gamma t}_{\text{prescribed function of time (ignore)}} + \frac{1}{2}ml^2\dot{\phi}^2 - mal\underbrace{\gamma\dot{\phi}\sin\gamma t\cos\phi}$$
$$= \frac{d}{dt}\left(\gamma\sin\gamma t\sin\phi\right) - \gamma^2\cos\gamma t\sin\phi$$

ignore

$$= \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\sin\phi$$
$$\rightarrow L = T - U = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\sin\phi + mgl\cos\phi$$

(c)



$$y = a\cos\gamma t$$

$$x = l\sin\phi\,, \quad y = y_0 + l\cos\phi$$

$$\dot{x} = l\dot{\phi}\cos\phi$$

$$\dot{y} = \dot{y}_0 + l\dot{\phi}\sin\phi$$

$$= -a\gamma\sin\gamma t - l\dot{\phi}\sin\phi$$

$$\dot{x}^2 = l^2\dot{\phi}^2\cos^2\phi$$

$$\dot{y}^2 = a^2\gamma^2\sin^2\gamma t + l^2\dot{\phi}^2\sin^2\phi + 2al\gamma\dot{\phi}\sin\gamma t\sin\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left(a^2\gamma^2\sin^2\gamma t + l^2\dot{\phi}^2 + 2al\gamma\dot{\phi}\sin\gamma t\sin\phi\right)$$

$$= \underbrace{\frac{1}{2}ma^2\gamma^2\sin^2\gamma t}_{\text{prescribed fcn of } t \text{ (ignore)}} + \frac{1}{2}ml^2\dot{\phi}^2 + m a l\,\underbrace{\gamma\dot{\phi}\sin\gamma t\sin\phi}$$

$$\gamma\dot{\phi}\sin\gamma t\sin\phi = \underbrace{\frac{d}{dt}\left(-\gamma\sin\gamma t\cos\phi\right)}_{\text{total time derivative (ignore)}} + \gamma^2\cos\gamma t\cos\phi$$

$$\simeq \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\cos\phi$$

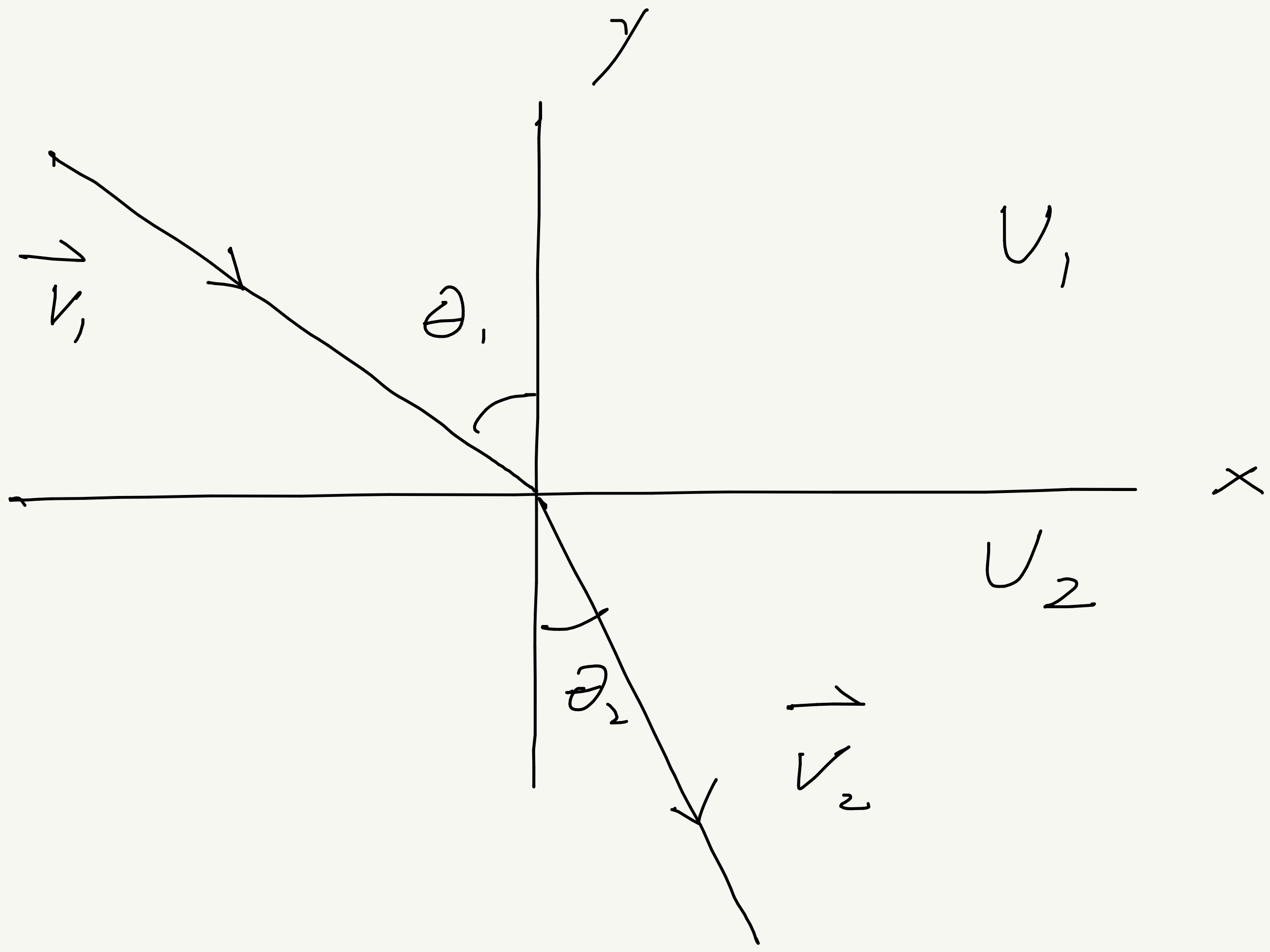
$$U = -mgy = -mg(y + l\cos\phi)$$

$$= \underbrace{-mga\cos\gamma t}_{\text{prescribed fcn of } t \text{ (ignore)}} - mgl\cos\phi$$

Thus,

$$L = T - U = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\cos\phi + mgl\cos\phi$$

Sec 7, Prob 1 :



- Energy is conserved
- Component of linear momentum in x-direction is also conserved

i) $E = \frac{1}{2} m v_1^2 + U_1 = \frac{1}{2} m v_2^2 + U_2$

$$\frac{1}{2} m v_2^2 = \frac{1}{2} m v_1^2 + (U_1 - U_2)$$

$$v_2 = \sqrt{v_1^2 + \frac{2}{m}(U_1 - U_2)}$$

$$\frac{v_2}{v_1} = \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

ii) $p_x = \cancel{m} v_1 \sin\theta_1 = \cancel{m} v_2 \sin\theta_2$

$$\rightarrow \quad \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1}$$

$$= \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

Sec 8, Prob 1:

$$S[q] = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t)$$

Let inertial frame K' move with velocity $\vec{V}$ w/t inertial frame K.

$$\sum_a m_a \vec{r_a}' = \mu \vec{R}'$$

Then:
$$\vec{v}_a = \vec{v}_a' + \vec{V}$$
$$\vec{r}_a = \vec{r}_a' + \vec{V} \cdot t$$

Thus,

$$L = \frac{1}{2} \sum_a m_a |\vec{v}_a|^2 - U(\vec{r}_1, \vec{r}_2, \dots, t)$$
$$= \frac{1}{2} \sum_a m_a |\vec{v}_a' + \vec{V}|^2 - U$$
$$= \frac{1}{2} \sum_a m_a |\vec{v}_a'|^2 + \frac{1}{2} \sum_a m_a V^2 + \left(\sum_a m_a \vec{v}_a'\right) \cdot \vec{V} - U$$
$$= L' + \vec{p}' \cdot \vec{V} + \frac{1}{2} \mu V^2$$

$$\rightarrow S = \int_{t_1}^{t_2} dt \left(L' + \vec{p}' \cdot \vec{V} + \frac{1}{2} \mu V^2\right)$$
$$= S' + \vec{V} \cdot \sum_a m_a \vec{r}_a' \Big|_{t_1}^{t_2} + \frac{1}{2} \mu V^2 \Big|_{t_1}^{t_2}$$
$$= S' + \mu \vec{V} \cdot \left(\vec{R}'(t_2) - \vec{R}'(t_1)\right) + \frac{1}{2} \mu V^2 (t_2 - t_1)$$

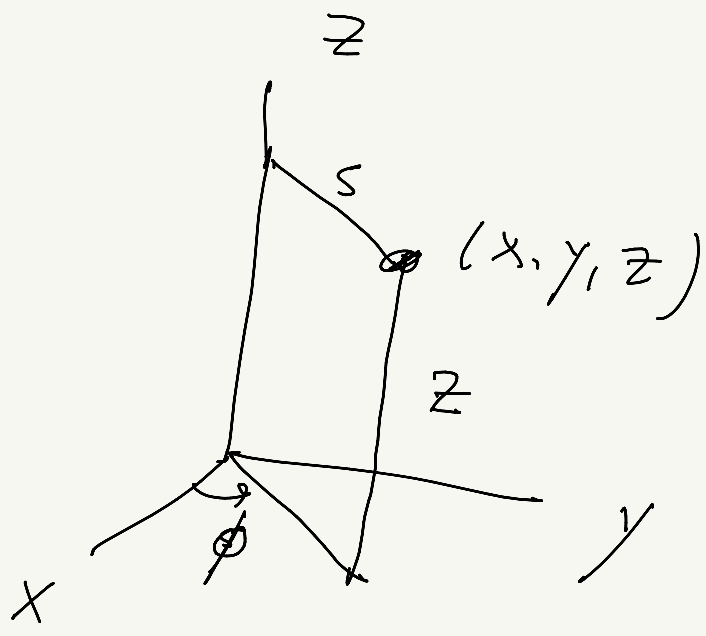
where $\vec{R}'$ is com of system w/t frame K'

Sec 9, Prob 1:

cylindrical coords (s, ϕ, z):

$$x = s\cos\phi, \quad y = s\sin\phi, \quad z = z$$

$$\vec{M} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} = m\vec{r} \times \dot{\vec{r}}$$



Thus,
$$M_x = m(y\dot{z} - z\dot{y})$$
$$M_y = m(z\dot{x} - x\dot{z})$$
$$M_z = m(x\dot{y} - y\dot{x})$$

$$\dot{x} = \dot{s}\cos\phi - s\dot{\phi}\sin\phi$$
$$\dot{y} = \dot{s}\sin\phi + s\dot{\phi}\cos\phi$$
$$z = \dot{z}$$

$$\rightarrow \quad M_x = m\left[s\sin\phi\,\dot{z} - z(\dot{s}\sin\phi + s\dot{\phi}\cos\phi)\right]$$
$$= m\left[\sin\phi(s\dot{z} - z\dot{s}) - zs\dot{\phi}\cos\phi\right]$$

$$M_y = m\left[z(\dot{s}\cos\phi - s\dot{\phi}\sin\phi) - s\cos\phi\,\dot{z}\right]$$
$$= m\left[-\cos\phi(s\dot{z} - z\dot{s}) - zs\dot{\phi}\sin\phi\right]$$

$$M_z = m\left[s\cos\phi(\cancel{\dot{s}\sin\phi} + s\dot{\phi}\cos\phi) - s\sin\phi(\cancel{\dot{s}\cos\phi} - s\dot{\phi}\sin\phi)\right]$$
$$= ms^2\dot{\phi}$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

$$\begin{aligned}
&= m^2 \Big[\sin^2\phi \,(s\dot{z} - z\dot{s})^2 + z^2 s^2 \dot{\phi}^2 \cos^2\phi \\
&\qquad - \cancel{2 z s \dot{\phi} \cos\phi \sin\phi \,(s\dot{z} - z\dot{s})} \\
&\qquad + \cos^2\phi \,(s\dot{z} - z\dot{s})^2 + z^2 s^2 \dot{\phi}^2 \sin^2\phi \\
&\qquad + \cancel{2 z s \dot{\phi} \cos\phi \sin\phi \,(s\dot{z} - z\dot{s})} \\
&\qquad + s^4 \dot{\phi}^2 \Big]
\end{aligned}$$

$$= m^2 \left[(s\dot{z} - z\dot{s})^2 + z^2 s^2 \dot{\phi}^2 + s^4 \dot{\phi}^2 \right]$$

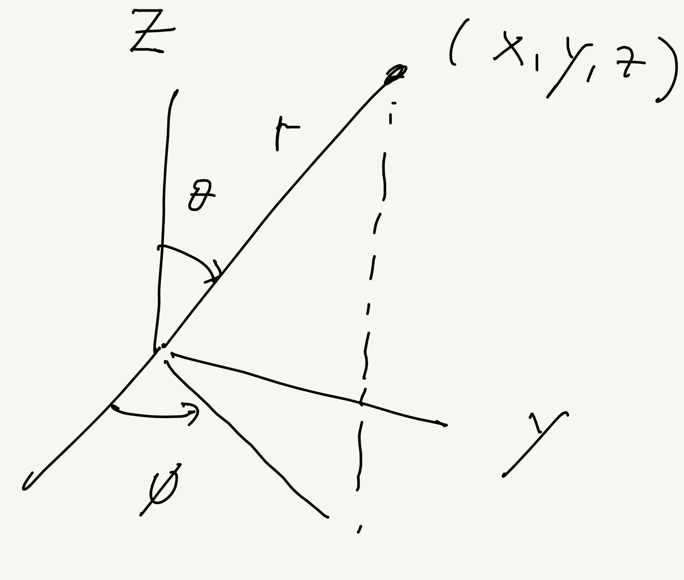
$$= m^2 \left[(s\dot{z} - z\dot{s})^2 + s^2 (z^2 + s^2) \dot{\phi}^2 \right]$$

Sec 9, Prob 2 :

spherical polar coords (r, θ, ϕ) :

$$x = r\sin\theta\cos\phi \, , \quad y = r\sin\theta\sin\phi \, , \quad z = r\cos\theta$$

$$\vec{M} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} = m\vec{r} \times \dot{\vec{r}}$$



Thus,
$$M_x = m(y\dot{z} - z\dot{y})$$
$$M_y = m(z\dot{x} - x\dot{z})$$
$$M_z = m(x\dot{y} - y\dot{x})$$

$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi$$
$$\dot{y} = \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi$$
$$\dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\rightarrow M_x = m\Big[\, r\sin\theta\sin\phi\,(\cancel{\dot{r}\cos\theta} - r\dot{\theta}\sin\theta)$$
$$- r\cos\theta\,(\cancel{\dot{r}\sin\theta\sin\phi} + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi)\Big]$$

$$= m\left[-r^2\dot{\theta}\sin\phi\,(\sin^2\theta + \cos^2\theta) - r^2\dot{\phi}\sin\theta\cos\theta\cos\phi\right]$$

$$= m\left[-r^2\dot{\theta}\sin\phi - r^2\dot{\phi}\sin\theta\cos\theta\cos\phi\right]$$

$$M_y = m\Big[\, r\cos\theta\,(\cancel{\dot{r}\sin\theta\cos\phi} + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi)$$
$$- r\sin\theta\cos\phi\,(\cancel{\dot{r}\cos\theta} - r\dot{\theta}\sin\theta)\Big]$$

$$= m\left[\, r^2\dot{\theta}\cos\phi - r^2\dot{\phi}\sin\theta\cos\theta\sin\phi\right]$$

$$M_z = m\left[r\sin\theta\cos\phi\left(\cancel{\dot r\sin\theta\sin\phi} + r\dot\theta\,\cancel{\cos\theta\sin\phi} + r\dot\phi\sin\theta\cos\phi\right)\right.$$
$$\left. - r\sin\theta\sin\phi\left(\cancel{\dot r\sin\theta\cos\phi} + r\dot\theta\,\cancel{\cos\theta\cos\phi} - r\dot\phi\sin\theta\sin\phi\right)\right]$$
$$= m\left[r^2\dot\phi\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right)\right]$$
$$= m r^2 \dot\phi \sin^2\theta$$

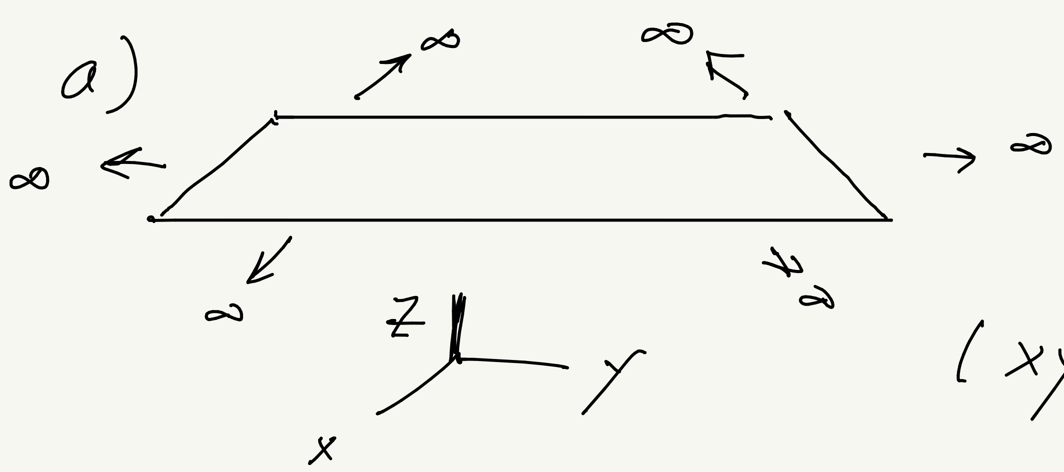
$$M^2 = M_x^2 + M_y^2 + M_z^2$$
$$= m^2\left[r^4\dot\theta^2\sin^2\phi + r^4\dot\phi^2\sin^2\theta\cos^2\theta\cos^2\phi\right.$$
$$\cancel{+2r^4\dot\theta\dot\phi\sin\theta\cos\theta\sin\phi\cos\phi}$$
$$+ r^4\dot\theta^2\cos^2\phi + r^4\dot\phi^2\sin^2\theta\cos^2\theta\sin^2\phi$$
$$\cancel{-2r^4\dot\theta\dot\phi\sin\theta\cos\theta\sin\phi\cos\phi}$$
$$\left. + r^4\dot\phi^2\sin^4\theta\right]$$

$$= m^2 r^4\left[\dot\theta^2 + \dot\phi^2\left(\sin^2\theta\cos^2\theta + \sin^4\theta\right)\right]$$

$$= m r^4\left[\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right]$$

Sec 9, Prob 3:

a)

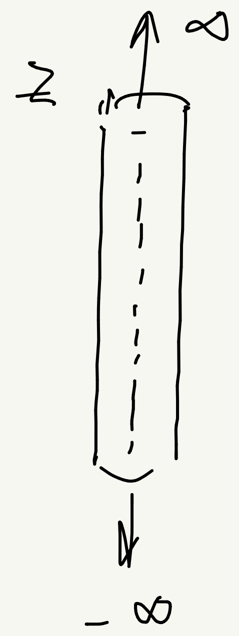


$P_x, P_y = \text{const}$

$M_z = \text{const}$

(xy-plane // to homog. plane)

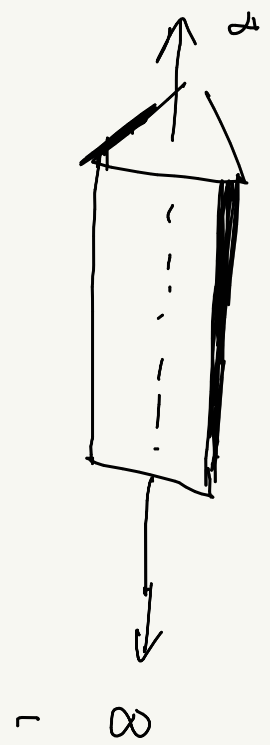
b)



P_z , $M_z = \text{const}$

(z-axis is axis of cylinder)

c)



infinite homog. cylinder

$P_z = \text{const}$

(z-axis is // to edges of prism)

d)

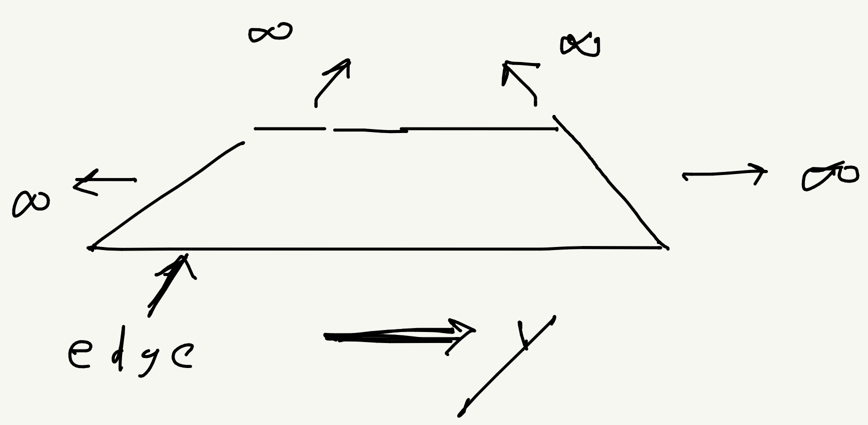
two points

$M_z = \text{const}$

(z-axis passes through the two points)

∞ homog, ½ plane

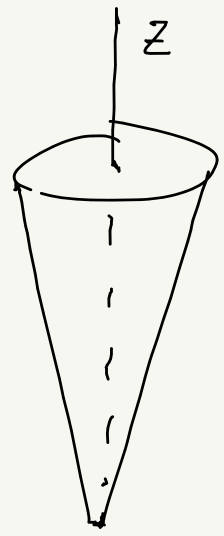
e)



$P_y = \text{const}$

(y axis is // to edge of ½ plane)

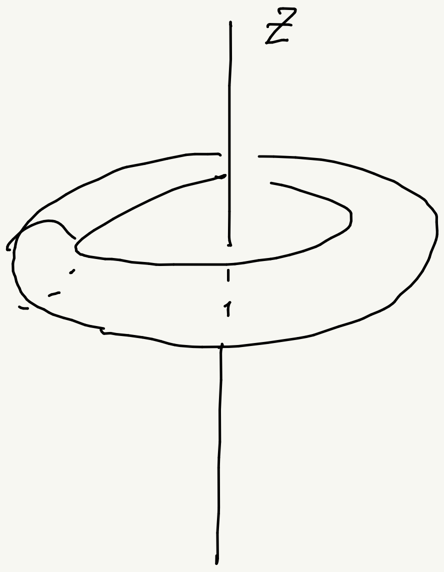
f)



$M_z = \text{const}$

(z-axis is axis of cone)

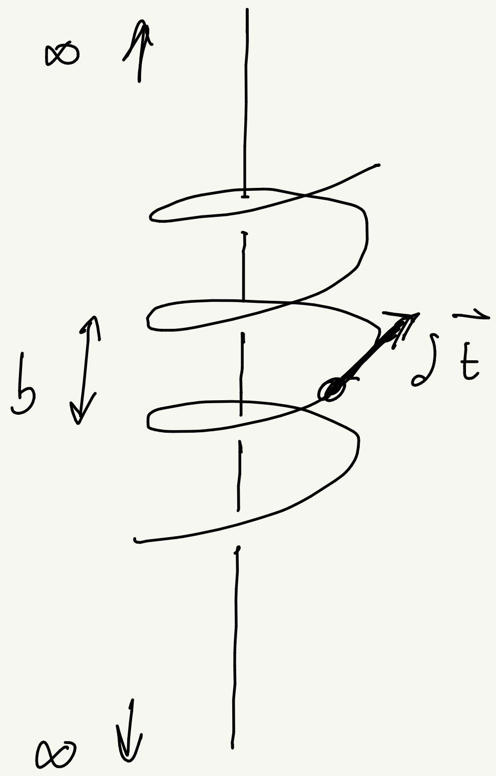
g)



$M_z = \text{const}$

(z-axis is axis of torus)

h)

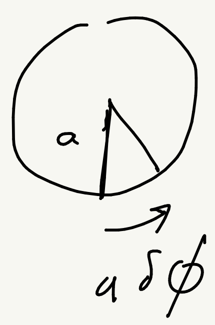


a = radius of helix

b = height between neighboring coils of helix

$h \equiv b/a$ (pitch of helix)

Lagrangian invariant w.r.t translations around the helix:



$$\delta \vec{t} = a\,\delta\phi\,\hat{\phi} + \frac{b\,\delta\phi}{2\pi}\hat{z}$$

$$= a\,\delta\phi\left[\hat{\phi} + \frac{b/a}{2\pi}\hat{z}\right]$$

$$= a\,\delta\phi\left[\hat{\phi} + \frac{h}{2\pi}\hat{z}\right]$$

$$\delta L = \frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial z}\delta z$$

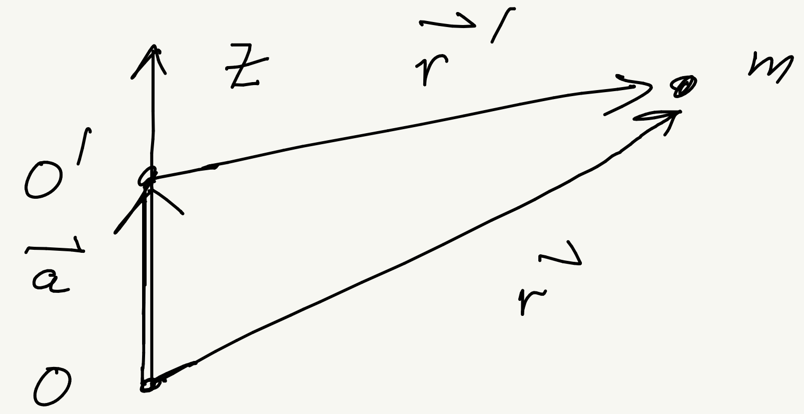
$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right)\delta\phi + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot z}\right)\frac{h}{2\pi}\delta\phi$$

$$= \delta\phi\,\frac{d}{dt}\Big[\underbrace{p_\phi}_{M_z} + p_z\frac{h}{2\pi}\Big] \rightarrow \boxed{M_z + \frac{p_z h}{2\pi} = \text{const}}$$

NOTE: M_z is independent of location of origin on z-axis

$$\vec{M} = \vec{r} \times \vec{p} = m\vec{r} \times \dot{\vec{r}}$$

Change origin by shifting along $\vec{a} = a\hat{z}$



$$\vec{r} = \vec{r}\,' + \vec{a}$$

$$\begin{aligned}
\vec{M} &= m\vec{r} \times \dot{\vec{r}} \\
&= m(\vec{r}\,' + \vec{a}) \times \frac{d}{dt}(\vec{r}\,' + \vec{a}) \\
&= m\vec{r}\,' \times \dot{\vec{r}}\,' + m\vec{a} \times \dot{\vec{r}}\,' \\
&= \vec{M}' + \vec{a} \times \vec{p}\,' \qquad (\text{for arbitrary } \vec{a})
\end{aligned}$$

Thus,

$$\begin{aligned}
M_z &= \vec{M} \cdot \hat{z} \\
&= (\vec{M}' + \vec{a} \times \vec{p}\,') \cdot \hat{z} \\
&= M'_z + a(\hat{z} \times \vec{p}\,') \cdot \hat{z} \\
&= M'_z + a\,\cancelto{0}{(\hat{z} \times \hat{z})} \cdot \vec{p}\,' \\
&= M'_z
\end{aligned}$$

Sec 10, Prob 1:

same path, different masses, same potential energies

$\rightarrow \quad x' = x, \quad m' \neq m, \quad U' = U, \quad t' \neq t$

$$L = T - U = \frac{1}{2} m \dot{x}^2 - U$$

$$L' = \frac{1}{2} m' \left(\frac{dx}{dt'}\right)^2 - U$$

$$= \frac{1}{2} m' \left(\frac{t}{t'}\right)^2 \dot{x}^2 - U$$

Thus, $L' = L \quad \rightarrow \quad m' \left(\frac{t}{t'}\right)^2 = m$

$$\left(\frac{t'}{t}\right)^2 = \frac{m'}{m}$$

$$\rightarrow \quad \frac{t'}{t} = \sqrt{\frac{m'}{m}}$$

Sec 10, Prob 2:

same path, Same mass, potential energies differing by a constant factor ($U' = cU$)

$\rightarrow \quad x = x', \quad m = m', \quad t' \neq t$

$$L = T - U$$
$$= \frac{1}{2} m \dot{x}^2 - U$$
$$L' = \frac{1}{2} m \left(\frac{dx}{dt'}\right)^2 - U'$$
$$= \frac{1}{2} m \left(\frac{t}{t'}\right)^2 \dot{x}^2 - cU$$

Thus, need $\left(\frac{t}{t'}\right)^2 = c$ to get same EOM,

$$\rightarrow \quad \frac{t'}{t} = \sqrt{\frac{1}{c}}$$
$$= \sqrt{\frac{U}{U'}}$$

Sec 40, Prob 1

Hamiltonian for a single particle

$$L = T - U$$

$$= \frac{1}{2} m |\dot{\vec{r}}|^2 - U(\vec{r}, t)$$

Cartesian:

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = p_x/m$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \rightarrow \dot{y} = p_y/m$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$\rightarrow \quad H = \left(\sum_i p_i \dot{q}_i - L \right)_{\dot{q}_i = \dot{q}_i(q, p)}$$

$$= \left(p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(x, y, z, t) \right) \Big|_{\dot{x} = p_x/m,\ etc}$$

$$= \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + U(x, y, z, t)$$

cylindrical: (s, ϕ, z)

$$L = \frac{1}{2}m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2) - U(s,\phi,z,t)$$

$$p_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s} \rightarrow \dot{s} = p_s/m$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ms^2\dot{\phi} \rightarrow \dot{\phi} = p_\phi/ms^2$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$\rightarrow H = \left(p_s\dot{s} + p_\phi\dot{\phi} + p_z\dot{z} - L\right)_{\dot{s}=p_s/m,\ etc}$$

$$= \frac{p_s^2}{m} + \frac{p_\phi^2}{ms^2} + \frac{p_z^2}{m}$$

$$- \frac{1}{2}m\left[\left(\frac{p_s}{m}\right)^2 + s^2\left(\frac{p_\phi}{ms^2}\right)^2 + \left(\frac{p_z}{m}\right)^2\right] + U(s,\phi,z,t)$$

$$= \frac{1}{2m}\left(p_s^2 + \frac{p_\phi^2}{s^2} + p_z^2\right) + U(s,\phi,z,t)$$

---

spherical polar: (r, θ, ϕ)

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - U(r,\theta,\phi,t)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \longrightarrow \dot{r} = p_r/m$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \rightarrow \dot{\theta} = p_\theta/mr^2$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi} \rightarrow \dot{\phi} = p_\phi/mr^2\sin^2\theta$$

$$\rightarrow \quad H = \left(p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - L \right) \Big|_{\dot{r} = p_r/m,\ \text{etc.}}$$

$$= p_r \left(\frac{p_r}{m} \right) + p_\theta \left(\frac{p_\theta}{m r^2} \right) + p_\phi \left(\frac{p_\phi}{m r^2 \sin^2\theta} \right)$$

$$- \frac{1}{2} m \left(\left(\frac{p_r}{m} \right)^2 + r^2 \left(\frac{p_\theta}{m r^2} \right)^2 + r^2 \sin^2\theta \left(\frac{p_\phi}{m r^2 \sin^2\theta} \right)^2 \right)$$

$$+ U(r, \theta, \phi, t)$$

$$= \frac{1}{2m} \left(p_r^2 + \frac{p_\theta}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta} \right) + U(r, \theta, \phi, t)$$

Sec 40, Prob 2:

For a uniformly rotating ref. frame:

$$L = \frac{1}{2} m v^2 + m\vec{v}\cdot(\vec{\Omega}\times\vec{r}) + \frac{1}{2} m |\vec{\Omega}\times\vec{r}|^2 - U$$

Hamiltonian

$$H = \left(\sum_i p_i \dot{q}_i - L\right)\Big|_{\dot{q} = \dot{q}(q,p)}$$

where

$$\vec{p} \equiv \frac{\partial L}{\partial \vec{v}}$$

$$= m\vec{v} + m(\vec{\Omega}\times\vec{r})$$

$$= m[\vec{v} + \vec{\Omega}\times\vec{r}]$$ — velocity wrt inertial frame

$$\rightarrow \vec{v} = \frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}$$

thus,

$$H = (\vec{p}\cdot\vec{v} - L)\Big|_{\vec{v} = \vec{p}/m - \vec{\Omega}\times\vec{r}}$$

$$= \vec{p}\cdot\left(\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right) - \frac{1}{2} m \left|\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right|^2$$

$$- m\left(\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right)\cdot(\vec{\Omega}\times\vec{r}) - \frac{1}{2} m |\vec{\Omega}\times\vec{r}|^2 + U$$

$$= \frac{p^2}{m} - \cancel{\vec{p}\cdot(\vec{\Omega}\times\vec{r})} - \frac{1}{2} m\left(\frac{p^2}{m^2} + \cancel{|\vec{\Omega}\times\vec{r}|^2} - \cancel{\frac{2}{m}\vec{p}\cdot(\vec{\Omega}\times\vec{r})}\right)$$

$$- \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + \cancel{m|\vec{\Omega}\times\vec{r}|^2} - \cancel{\frac{1}{2} m |\vec{\Omega}\times\vec{r}|^2} + U$$

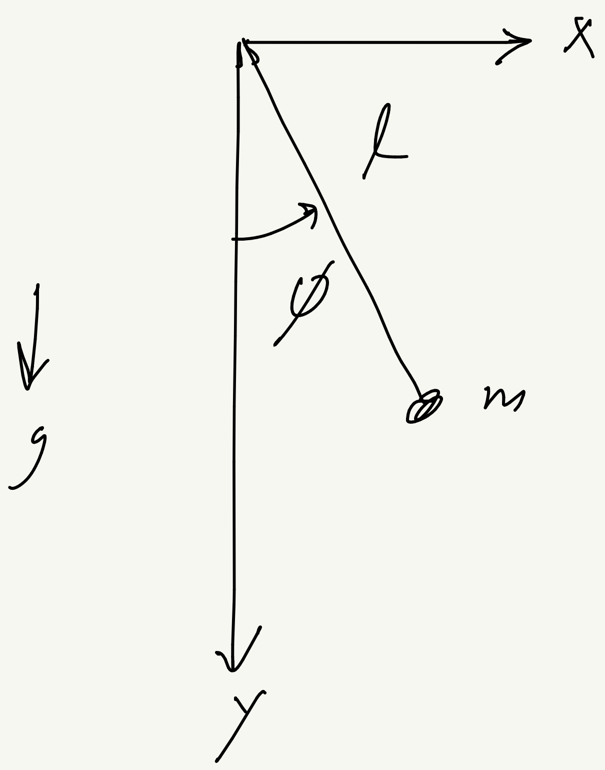
$$H = \frac{p^2}{2m} - \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + U$$

$$= \frac{p^2}{2m} - \vec{\Omega}\cdot(\vec{r}\times\vec{p}) + U$$

$$= \frac{p^2}{2m} - \vec{\Omega}\cdot\vec{M} + U$$

Sec 11, Prob 1:

Simple pendulum:



Cons. of energy:

$$E = \frac{1}{2} m \ell^2 \dot{\phi}^2 - mgl\cos\phi$$
$$= \text{const.}$$

Assume pendulum bob released from rest from $\phi = \phi_0$.

$$\rightarrow E = -mgl\cos\phi_0$$

Thus,
$$-mgl\cos\phi_0 = \frac{1}{2} ml^2\dot{\phi}^2 - mgl\cos\phi$$

$$\frac{d\phi}{dt} \equiv \dot{\phi} = \pm\sqrt{\frac{2g}{\ell}(\cos\phi - \cos\phi_0)}$$
$$= \pm\sqrt{2}\,\omega_0\sqrt{\cos\phi - \cos\phi_0}$$

where $\omega_0 \equiv \sqrt{\frac{g}{l}}$ (angular freq in small-angle approximation)

Separable differential equation:

$$\sqrt{2}\,\omega_0 \int dt = \pm \int \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$

$\frac{1}{4}$ period for $\phi = \phi_0 \rightarrow \phi = 0$

$$\sqrt{2}\,\omega_0 \frac{P}{4} = \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$

$$\rightarrow \quad P = \frac{1}{\omega_0} \frac{4}{\sqrt{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$

<u>substitute</u>: $\cos\phi = \cos^2(\phi/2) - \sin^2(\phi/2)$

$= 1 - 2\sin^2(\phi/2)$

$\cos\phi_0 = 1 - 2\sin^2(\phi_0/2)$

$$\rightarrow \sqrt{} = \sqrt{2}\sqrt{\sin^2(\phi_0/2) - \sin^2(\phi/2)}$$

$$= \sqrt{2}\,\sin(\phi_0/2)\sqrt{1 - \frac{\sin^2(\phi/2)}{\sin^2(\phi_0/2)}}$$

$$= \sqrt{2}\,\sin\left(\frac{\phi_0}{2}\right)\sqrt{1 - x^2}$$

where $x \equiv \dfrac{\sin(\phi/2)}{\sin(\phi_0/2)}$

<u>NOTE</u>: $\phi = 0,\ \phi_0 \rightarrow x = 0, 1$

$$dx = \frac{1}{k}\frac{1}{2}\cos\left(\frac{\phi}{2}\right)d\phi$$

$$= \frac{1}{2k}\sqrt{1 - \sin^2(\phi/2)}\,d\phi$$

$$= \frac{1}{2k}\sqrt{1 - k^2x^2}\,d\phi, \quad k = \sin(\phi_0/2)$$

Thus, $d\phi = \dfrac{2k\,dx}{\sqrt{1-k^2x^2}}$

$$\rightarrow \quad P = \frac{1}{\omega_0}\frac{4}{\cancel{\sqrt{2}}}\int_0^1 \frac{\cancel{2k}\,dx}{\sqrt{1-k^2x^2}}\;\frac{1}{\cancel{\sqrt{2}}\,\cancel{k}\sqrt{1-x^2}}$$

$$= \frac{4}{\omega_0}\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

$$= \frac{4}{\omega_0}\underbrace{K(k)}_{\text{complete elliptic integral of the 1st kind.}}$$

Expand $K(k)$ keeping 1st non-zero correction:

$$k = \sin\left(\frac{\phi_0}{2}\right) \approx \frac{\phi_0}{2}$$

$$K(k) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

$$\simeq \int_0^1 \frac{dx}{\sqrt{1-x^2}}\left(1 + \frac{1}{2}k^2x^2\right)$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2}k^2\int_0^1 \frac{dx\,x^2}{\sqrt{1-x^2}}$$

$$= \sin^{-1}(1) + \frac{1}{2}k^2\int_0^1 \frac{dx\,x^2}{\sqrt{1-x^2}}$$

Now. $\sin^{-1}(1) = \pi/2$

$$\int_0^1 dx \frac{x^2}{\sqrt{1-x^2}} = \int_0^{\pi/2} \frac{\cancel{\cos\theta}\, d\theta \quad \sin^2\theta}{\cancel{\sqrt{1-\sin^2\theta}}}$$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$= \int_0^{\pi/2} \sin^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}(1-\cos 2\theta)$$

(Using $\cos 2\theta = 1 - 2\sin^2\theta \rightarrow \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$)

$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{2}\cancel{\sin 2\theta}\Big|_0^{\pi/2}\right)$$

$$= \frac{\pi}{4}$$

Thus,

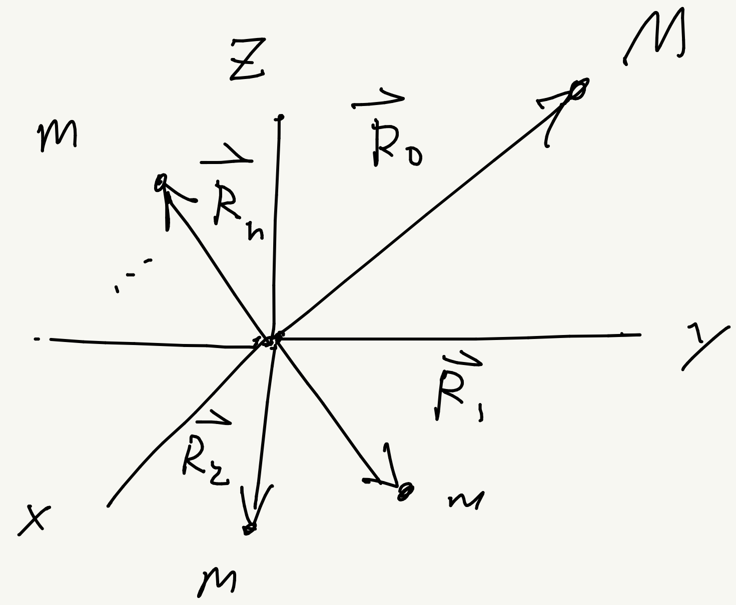
$$K(k) \approx \frac{\pi}{2} + \frac{1}{2}k^2\frac{\pi}{4}$$

$$= \frac{\pi}{2}\left(1 + \frac{1}{4}k^2\right) \qquad k \approx \frac{\phi_0}{2}$$

$$= \frac{\pi}{2}\left(1 + \frac{1}{16}\phi_0^2\right)$$

$$\rightarrow P = \frac{4}{\omega_0}K(k) \simeq \frac{2\pi}{\omega_0}\left(1 + \frac{1}{16}\phi_0^2\right)$$

Sec 13, Prob 1:



$\vec{R}_0$: position vector for mass M

$\vec{R}_i$, $i = 1, 2, \ldots, n$: position vectors for n masses all with mass m

CoM frame: $\vec{0} = M\vec{R}_0 + m\sum_i \vec{R}_i$

Relative position vectors:

$$\vec{r}_i \equiv \vec{R}_i - \vec{R}_0$$

Thus,

$$\vec{0} = M\vec{R}_0 + m\sum_i (\vec{R}_0 + \vec{r}_i)$$

$$= \underbrace{(M + nm)}_{\text{total mass } \mu}\vec{R}_0 + m\sum_i \vec{r}_i$$

$$\rightarrow \quad \vec{R}_0 = -\frac{m}{\mu}\sum_i \vec{r}_i$$

Potential energy:

$$U = U(|\vec{R}_1 - \vec{R}_2|, \ldots, |\vec{R}_1 - \vec{R}_0|, \ldots, |\vec{R}_n - \vec{R}_0|)$$

$$= \underbrace{U(|\vec{r}_1 - \vec{r}_2|, \ldots, |\vec{r}_1|, \ldots, |\vec{r}_n|)}_{\text{depends only on the relative position vectors } \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n}$$

Kinetic energy:

$$T = \frac{1}{2} M |\dot{\vec{R_0}}|^2 + \frac{1}{2} m \sum_i |\dot{\vec{R_i}}|^2$$

Now: $\dot{\vec{R_i}} = \dot{\vec{r_i}} + \dot{\vec{R_0}}$

$$\rightarrow |\dot{\vec{R_i}}|^2 = |\dot{\vec{r_i}}|^2 + |\dot{\vec{R_0}}|^2 + 2\,\dot{\vec{r_i}} \cdot \dot{\vec{R_0}}$$

and $\dot{\vec{R_0}} = -\frac{m}{\mu} \sum_i \dot{\vec{r_i}}$

$$\rightarrow |\dot{\vec{R_0}}|^2 = \frac{m^2}{\mu^2} \Big|\sum_i \dot{\vec{r_i}}\Big|^2$$

thus,

$$T = \frac{1}{2} \frac{M m^2}{\mu^2} \Big|\sum_i \dot{\vec{r_i}}\Big|^2 + \frac{1}{2} m \sum_i |\dot{\vec{r_i}}|^2 + \frac{1}{2} m n |\dot{\vec{R_0}}|^2 + m \sum_i \dot{\vec{r_i}} \cdot \dot{\vec{R_0}}$$

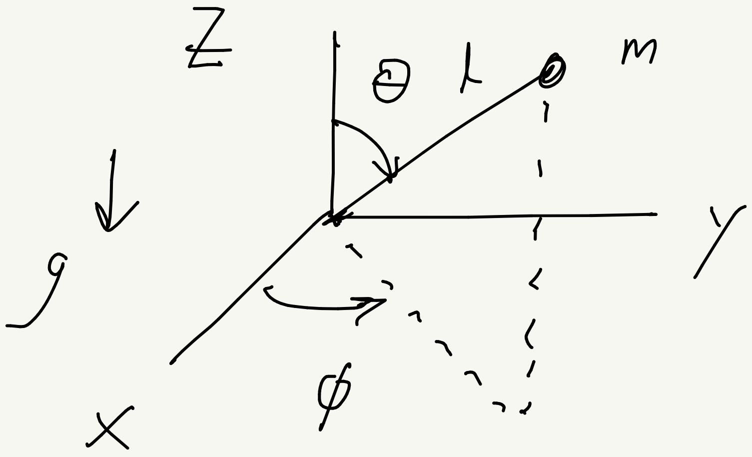
$$= \frac{1}{2} m \sum_i |\dot{\vec{r_i}}|^2 + \frac{1}{2} \frac{M m^2}{\mu^2} \Big|\sum_i \dot{\vec{r_i}}\Big|^2 + \frac{1}{2} m n \frac{m^2}{\mu^2} \Big|\sum_i \dot{\vec{r_i}}\Big|^2 - \frac{m^2}{\mu} \Big|\sum_i \dot{\vec{r_i}}\Big|^2$$

$$= \frac{1}{2} m \sum_i |\dot{\vec{r_i}}|^2 + \frac{1}{2} \frac{m^2}{\mu^2} \Big|\sum_i \dot{\vec{r_i}}\Big|^2 \big(\underbrace{M + mn}_{\mu} - 2\mu\big)$$

$$= \frac{1}{2} m \sum_i |\dot{\vec{r_i}}|^2 - \frac{1}{2} \frac{m^2}{\mu} \Big|\sum_i \dot{\vec{r_i}}\Big|^2$$

$$\rightarrow \boxed{L = \frac{1}{2} m \sum_i |\dot{\vec{r_i}}|^2 - \frac{1}{2} \frac{m^2}{\mu} \Big|\sum_i \dot{\vec{r_i}}\Big|^2 - U(\vec{r_1}, \vec{r_2}, \dots, \vec{r_n})}$$

Sec 14, Prob 1



$$x = l\sin\theta\cos\phi$$
$$y = l\sin\theta\sin\phi$$
$$z = l\cos\theta$$

$$U = mgz = mgl\cos\theta$$
$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$= \frac{1}{2}m(l^2\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\phi}^2)$$

$$L = T - U$$
$$= \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) - mgl\cos\theta$$

No explicit t dependence or ϕ dependence

$$\longrightarrow \quad E \equiv T + U = \text{const}, \quad M_z \equiv p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \text{const}$$

$$E = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + mgl\cos\theta$$

$$M_z = ml^2\sin^2\theta\,\dot{\phi} \longrightarrow \dot{\phi} = \frac{M_z}{ml^2\sin^2\theta}$$

Thus,

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\sin^2\theta\left(\frac{M_z^2}{m^2l^4\sin^4\theta}\right) + mgl\cos\theta$$

$$= \frac{1}{2}ml^2\dot{\theta}^2 + \underbrace{\frac{M_z^2}{2ml^2\sin^2\theta} + mgl\cos\theta}_{\text{effective potential} \equiv U_{eff}(\theta)}$$

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + U_{eff}(\theta)$$

$$\rightarrow \dot{\theta} = \pm \sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}$$

so
$$\frac{d\theta}{\pm\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}} = dt$$

$$\rightarrow \boxed{t = \int \frac{d\theta}{\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}} + \text{const}}$$

$$\longrightarrow t = t(\theta)$$

~~Trajectory~~ :

To find $\theta = \theta(\phi)$ write

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d\theta}{d\phi}\dot{\phi} = \frac{d\theta}{d\phi}\frac{M_z}{ml^2 \sin^2\theta}$$

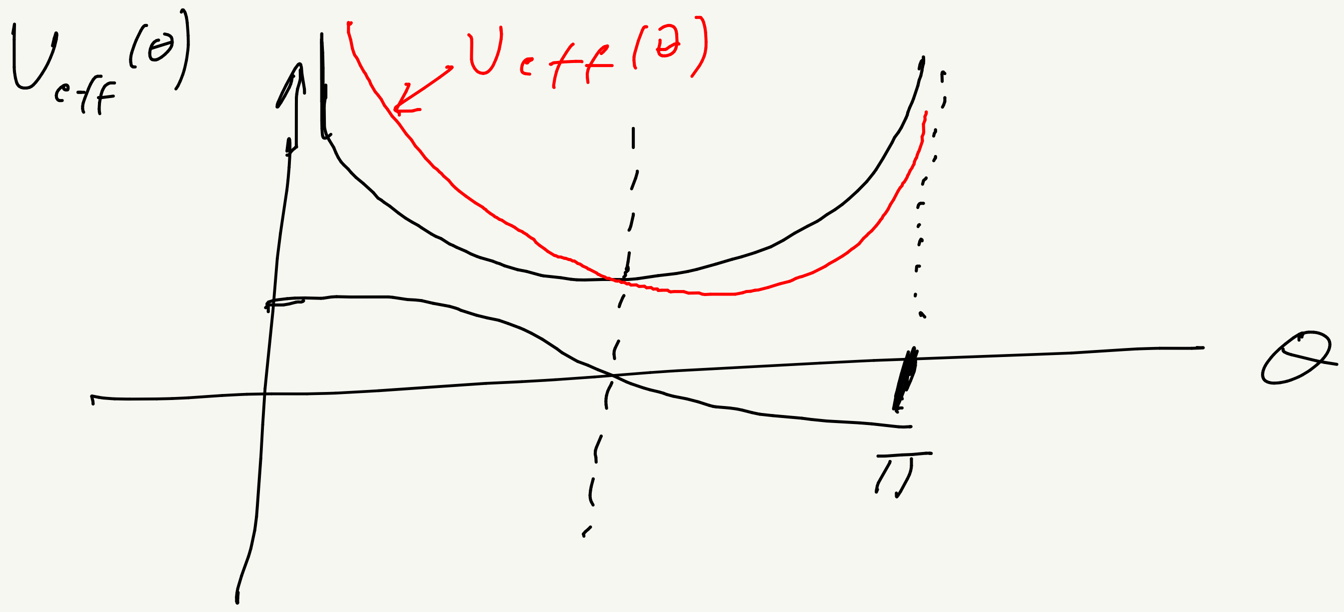
so
$$\frac{d\theta}{d\phi}\frac{M_z}{ml^2\sin^2\theta} = \pm\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}$$

$$\frac{d\theta / \sin^2\theta}{\pm\frac{ml^2}{M_z}\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}} = d\phi$$

$$\rightarrow \boxed{\phi = \int \frac{d\theta/\sin^2\theta}{\sqrt{\frac{2ml^2}{M_z^2}\left(E - U_{eff}(\theta)\right)}} + \text{const}}$$

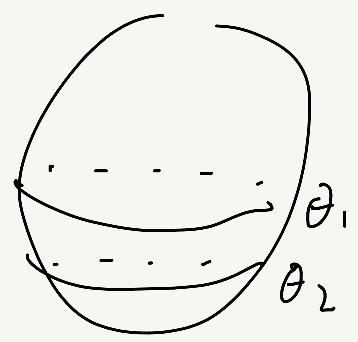
$$U_{eff}(\theta) = \frac{M_z^2}{2ml^2\sin^2\theta} + mgl\cos\theta$$

$$= \frac{M_z^2}{2ml^2(1-\cos^2\theta)} + mgl\cos\theta$$



Turning points: $\theta = \theta_1, \theta_2$ for which

$$E - U_{eff}(\theta) = 0$$



$$E = U_{eff}(\theta)$$

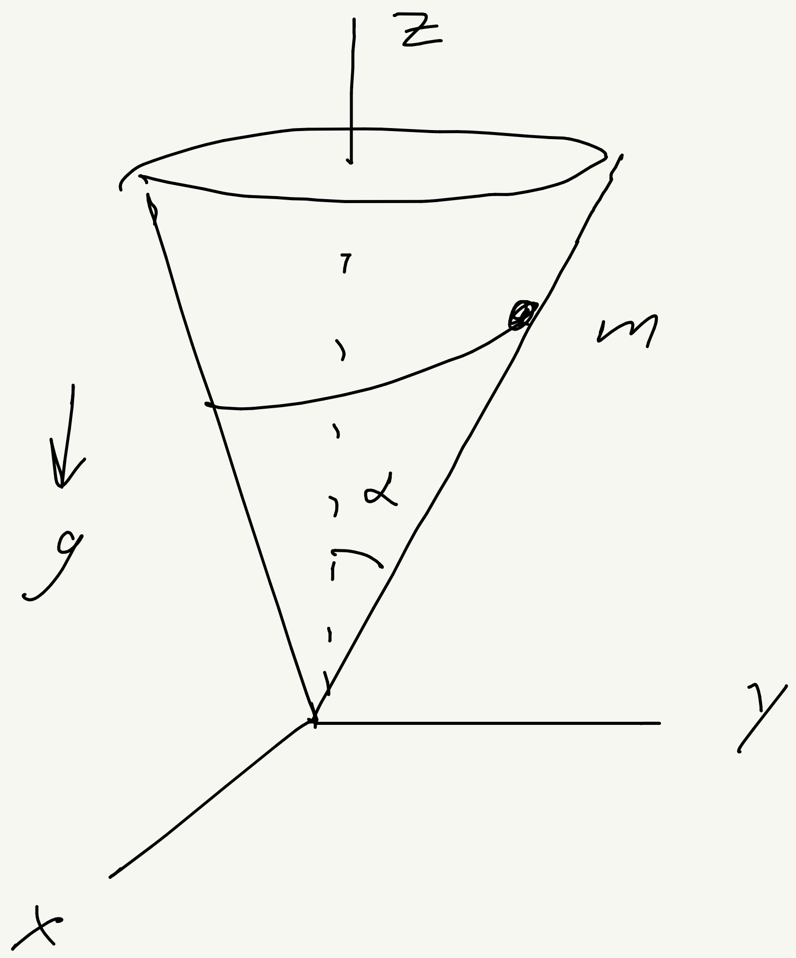
$$= \frac{M_z^2}{2ml^2(1-\cos^2\theta)} + mgl\cos\theta$$

$$2ml^2E(1-\cos^2\theta) = M_z^2 + 2m^2gl^3\cos\theta(1-\cos^2\theta)$$

$$2ml^2E - 2ml^2E\cos^2\theta = M_z^2 + 2m^2gl^3\cos\theta - 2m^2gl^3\cos^3\theta$$

$$\rightarrow \underbrace{(2ml^2E - M_z^2) - 2m^2gl^3\cos\theta - 2ml^2E\cos^2\theta + 2m^2gl^3\cos^3\theta = 0}_{\text{Cubic equation for } \cos\theta}$$

Sec 14, Prob 2



spherical coords: (r, θ, ϕ)

$$\theta = \alpha$$

$$\rightarrow \quad x = r \sin\alpha \cos\phi$$

$$y = r \sin\alpha \sin\phi$$

$$z = r \cos\alpha$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2)$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \sin^2\alpha\, \dot{\phi}^2)$$

$$U = mgz = mgr\cos\alpha$$

$$L = T - U$$

No explicit t, ϕ dependence $\rightarrow$

$$E = T + U = \text{const}$$

$$M_z \equiv p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\alpha\, \dot{\phi} = \text{const}$$

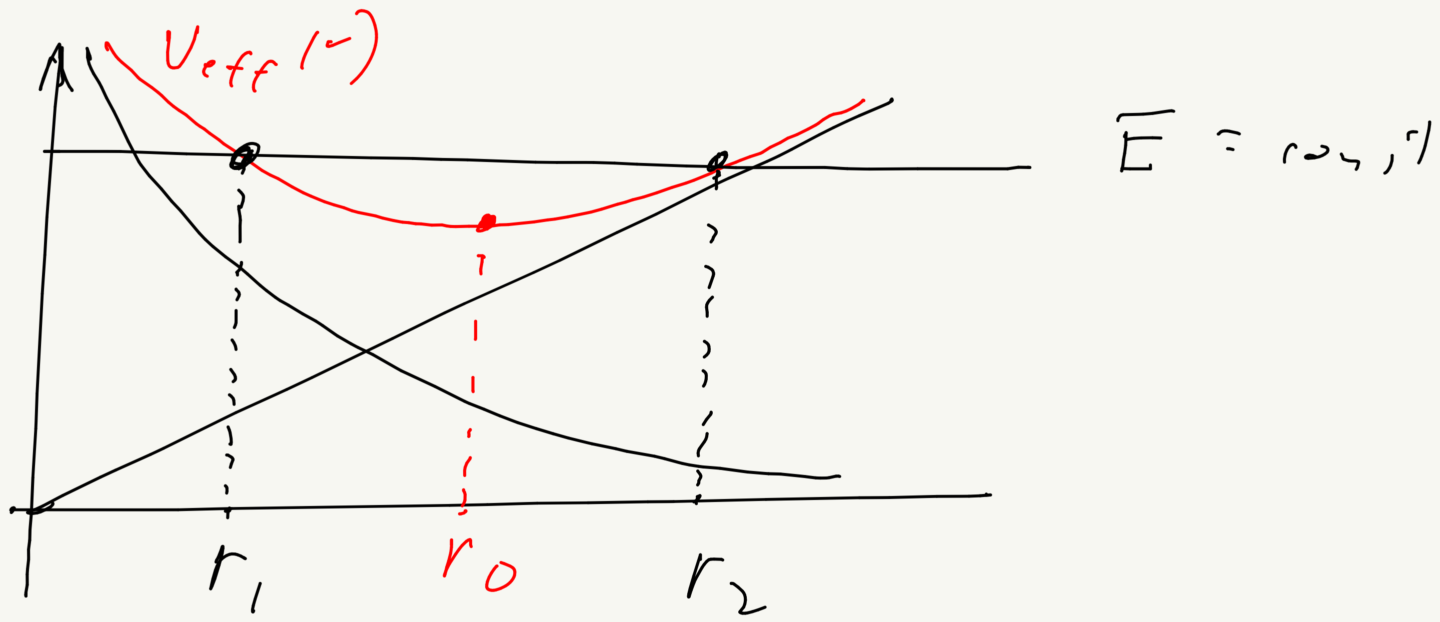
$$\rightarrow \quad \dot{\phi} = \frac{M_z}{m r^2 \sin^2\alpha}$$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \sin^2\alpha \left(\frac{M_z}{m r^2 \sin^2\alpha}\right)^2 + mgr\cos\alpha$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{M_z^2}{2 m r^2 \sin^2\alpha} + mgr\cos\alpha$$

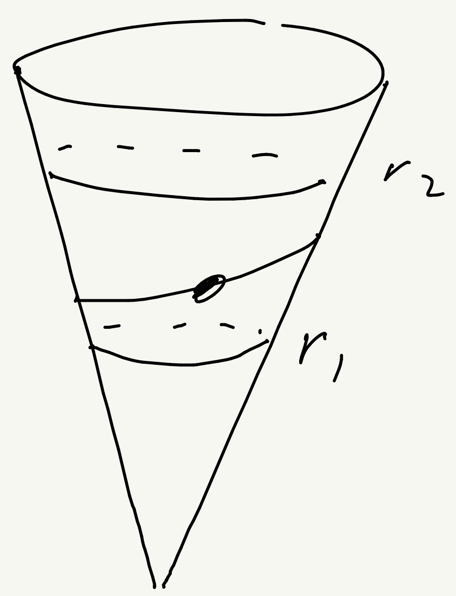
$$= \frac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = \frac{M_z^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha$$



Turning points: $r = r_1, r_2$ when $E = U_{eff}(r)$

$$E = \frac{M_z^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha$$



$$2mEr^2\sin^2\alpha = M_z^2 + 2m^2gr^3\sin^2\alpha\cos\alpha$$

$$\underbrace{0 = M_z - 2mEr^2\sin^2\alpha - 2m^2gr^3\sin^2\alpha\cos\alpha}_{\text{cubic equation for } r}$$

$$r_1 \leq r \leq r_2$$

<u>Integrals for $t = t(r)$, $\phi = \phi(r)$:</u>

$$E = \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$$

$$\pm\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)} = \frac{dr}{dt}$$

$$\rightarrow \boxed{t = \pm\int\frac{dr}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}} + \text{const}}$$

Orbit equation:

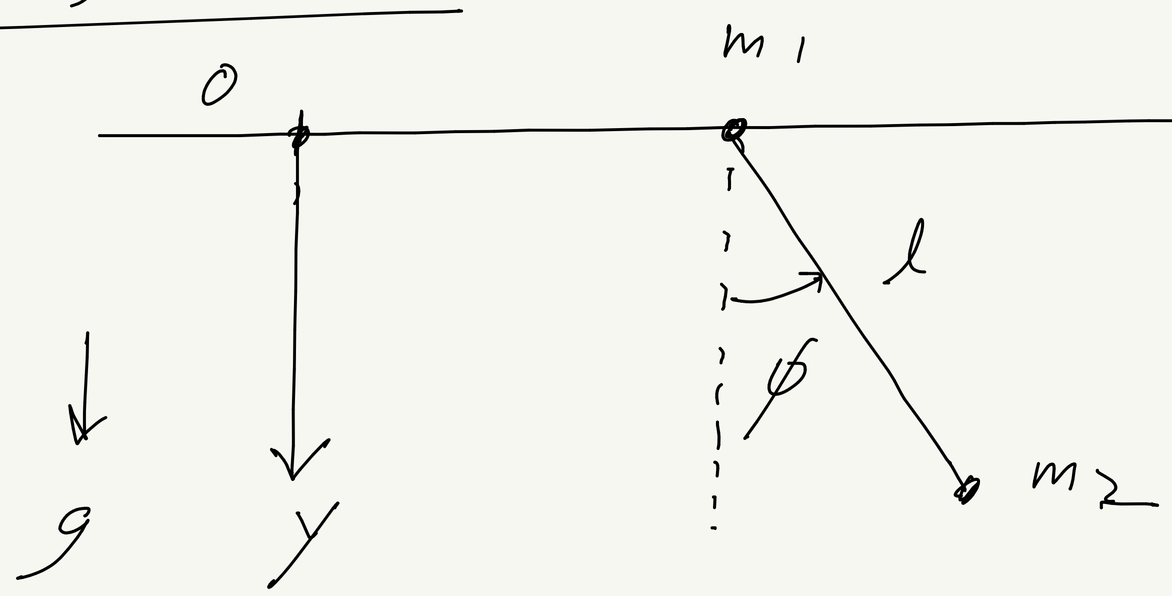
$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M_z}{mr^2\sin^2\alpha}$$

so
$$\frac{dr}{d\phi}\frac{M_z}{mr^2\sin^2\alpha} = \pm\sqrt{\frac{2}{m}\left(E - V_{eff}(r)\right)}$$

$$\frac{dr/r^2}{\frac{m\sin^2\alpha}{M_z}\sqrt{\quad}} = \pm\, d\phi$$

$$\rightarrow \quad \boxed{\phi = \frac{\pm M_z}{\sin^2\alpha}\int \frac{dr/r^2}{\sqrt{2m\left(E - V_{eff}(r)\right)}} + \text{const}}$$

Sec 14, Prob 3:



$x_1 = x$
$y_1 = 0$
$x_2 = x + l\sin\phi$
$y_2 = l\cos\phi$

$$U = -mgy_2$$
$$= -mgl\cos\phi$$

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$
$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left[(\dot{x} + l\dot{\phi}\cos\phi)^2 + l^2\dot{\phi}^2\sin^2\phi\right]$$
$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left[\dot{x}^2 + 2l\dot{x}\dot{\phi}\cos\phi + l^2\dot{\phi}^2\cos^2\phi + l^2\dot{\phi}\sin^2\phi\right]$$
$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2 l\dot{x}\dot{\phi}\cos\phi$$

$$L = T - U$$
$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2 l\dot{x}\dot{\phi}\cos\phi + mgl\cos\phi$$

No explicit t, x dependence $\longrightarrow$

$$E = T + U = \text{const}$$
$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} + m_2 l\dot{\phi}\cos\phi = \text{const}$$

Since $p_x = \text{const}$, we can work in the x-com frame where $x_{com} = 0$ and $p_x = 0$

In this frame:

$$0 = x_{com}$$
$$= \frac{m_1 x + m_2 (x + l \sin\phi)}{m_1 + m_2}$$
$$= \frac{(m_1 + m_2) x + m_2 l \sin\phi}{m_1 + m_2}$$
$$= x + \left(\frac{m_2}{m_1 + m_2}\right) l \sin\phi$$

$$\rightarrow \boxed{x = -\left(\frac{m_2}{m_1 + m_2}\right) l \sin\phi}$$

Thus, $\dot{x} = -\left(\frac{m_2}{m_1 + m_2}\right) l \dot{\phi} \cos\phi$

$$\rightarrow E = \frac{1}{2}(m_1 + m_2) \frac{m_2^2}{(m_1 + m_2)^2} l^2 \dot{\phi}^2 \cos^2\phi + \frac{1}{2} m_2 l^2 \dot{\phi}^2$$
$$- \frac{m_2^2 l^2}{m_1 + m_2} \dot{\phi}^2 \cos^2\phi - m_2 g l \cos\phi$$

$$= \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left(1 - \left(\frac{m_2}{m_1 + m_2}\right) \cos^2\phi\right) - m_2 g l \cos\phi$$
$$= \frac{1}{2}\left(\frac{m_2}{m_1 + m_2}\right) l^2 \dot{\phi}^2 \left(m_1 + m_2 \sin^2\phi\right) - m_2 g l \cos\phi$$

Solve for $\dot{\phi}$:

$$E + m_2 g l \cos\phi = \frac{1}{2}\left(\frac{m_2}{m_1+m_2}\right) l^2 \dot{\phi}^2 \left(m_1 + m_2 \sin^2\phi\right)$$

$$\pm\sqrt{\frac{\frac{2}{l^2}\left(\frac{m_1+m_2}{m_2}\right)\left(E + m_2 g l \cos\phi\right)}{m_1 + m_2 \sin^2\phi}} = \frac{d\phi}{dt}$$

$$\rightarrow \boxed{t = \pm\sqrt{\frac{l^2}{2}\left(\frac{m_2}{m_1+m_2}\right)} \int d\phi \sqrt{\frac{m_1 + m_2 \sin^2\phi}{E + m_2 g l \cos\phi}} + \text{const}}$$

NOTE: In the x-com frame $x = -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi$

$$\begin{aligned} x_2 &= x + l \sin\phi \\ &= -\left(\frac{m_2}{m_1+m_2}\right) l \sin\phi + l \sin\phi \\ &= \left(\frac{m_1}{m_1+m_2}\right) l \sin\phi \\ &\equiv b \sin\phi \end{aligned}$$

$$y_2 = l \cos\phi \equiv a \cos\phi$$

Thus,

$$\left(\frac{x_2}{b}\right)^2 + \left(\frac{y_2}{a}\right)^2 = \sin^2\phi + \cos^2\phi = 1$$

So m_2 traces out an ellipse in the x-com frame.

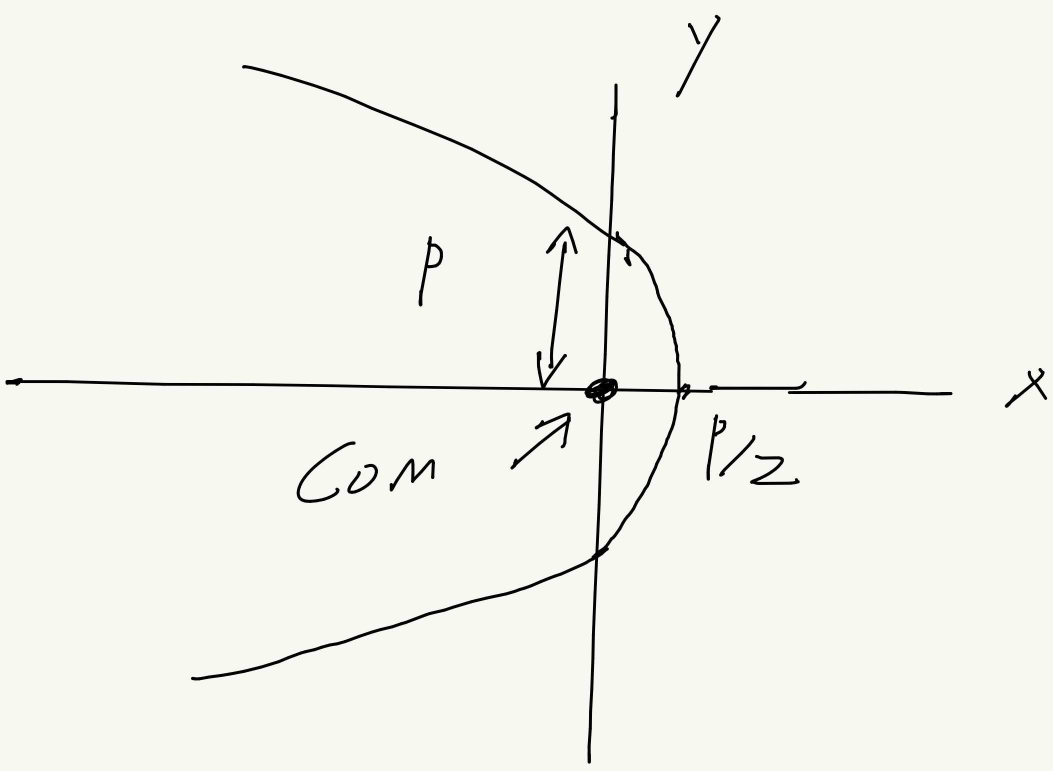
Sec 15, Prob 1:

$E = 0$ (parabola), $e = 1$

$U = -\alpha/r$

$$\left.\begin{aligned} \frac{p}{r} &= 1 + e\cos\phi \\ &= 1 + \cos\phi \end{aligned}\right\}$$

$$\phi = 0 \;\rightarrow\; \frac{p}{r} = 2 \;\rightarrow\; r_{min} = \frac{p}{2}$$



Time dependence:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U(r)\right) - \frac{M^2}{m^2 r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{2}{m}\left(\cancel{E}^{\,0} + \frac{\alpha}{r}\right) - \frac{M^2}{m^2 r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{M^2}{m^2 r^2}}} + \text{const}$$

Recall: $p = \frac{M^2}{m\alpha} \rightarrow M^2 = m\alpha p$

$$\rightarrow t = \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{m\alpha p}{m^2 r^2}}} + \text{const}$$

$$= \int \frac{dr}{\sqrt{\frac{\alpha}{m}}\sqrt{\frac{2}{r} - \frac{p}{r^2}}} + \text{const}$$

$$= \sqrt{\frac{m}{\alpha p}} \int \frac{r\,dr}{\sqrt{\frac{2r}{p} - 1}} + \text{const}$$

Let: $\frac{2r}{p} - 1 = \xi^2$ $\left(\xi = 0 \rightarrow \frac{2r}{p} - 1 = 0 \rightarrow r = p/2 \right)$

Thus, $\frac{\cancel{2}dr}{p} = \cancel{2}\xi\, d\xi$

$$dr = p\xi\, d\xi$$

Also: $\frac{2r}{p} = 1 + \xi^2 \rightarrow r = \frac{p}{2}(1+\xi^2)$

$$\rightarrow t = \sqrt{\frac{m}{\alpha p}} \int \frac{\frac{p}{2}(1+\xi^2)\, p\cancel{\xi}\, d\xi}{\cancel{\sqrt{\xi^2}}} + \text{const}$$

$$= \frac{p^2}{2}\sqrt{\frac{m}{\alpha p}} \int (1+\xi^2)\, d\xi + \text{const}$$

so $\quad t = \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}}\left(\xi + \frac{\xi^3}{3}\right) + \text{const}$

choose const $= 0$ so that $t = 0 \longleftrightarrow \xi = 0$

Thus,

$$\boxed{\begin{aligned} r &= \frac{p}{2}\left(1+\xi^2\right) \\ t &= \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}}\left(\xi + \frac{1}{3}\xi^3\right) \end{aligned}}$$

Recall:
$$\frac{p}{r} = 1 + \cos\phi$$

so
$$1+\cos\phi = \frac{2}{1+\xi^2}$$

$$\boxed{\cos\phi} = \frac{2}{1+\xi^2} - 1$$

$$= \boxed{\frac{1-\xi^2}{1+\xi^2}}$$

Cartesian:
$$x = r\cos\phi$$
$$= \frac{p}{2}\left(\cancel{1+\xi^2}\right)\left(\frac{1-\xi^2}{\cancel{1+\xi^2}}\right)$$
$$= \frac{p}{2}\left(1-\xi^2\right)$$

$$y = r\sin\phi = \frac{p}{2}\left(1+\xi^2\right)\sqrt{1 - \left(\frac{1-\xi^2}{1+\xi^2}\right)^2}$$

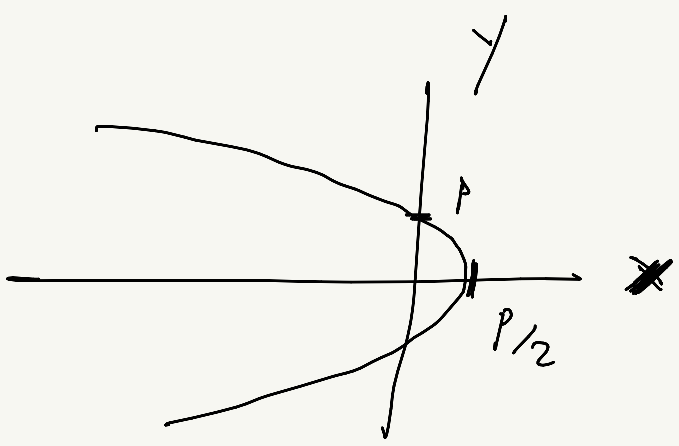
$$y = \frac{p}{2}\cancel{(1+\xi^2)}\,\frac{1}{\cancel{(1+\xi^2)}}\sqrt{(1+\xi^2)^2-(1-\xi^2)^2}$$

$$= \frac{p}{2}\sqrt{\cancel{1}+\cancel{\xi^4}+2\xi^2-(\cancel{1}+\cancel{\xi^4}-2\xi^2)}$$

$$= \frac{p}{2}\sqrt{4\xi^2}$$

$$= p\,\xi$$

Thus,
$$\boxed{\begin{aligned} x &= \frac{p}{2}(1-\xi^2) \\ y &= p\,\xi \end{aligned}}$$



NOTE:

$$x = \frac{p}{2} - \frac{p}{2}\xi^2$$

$$= \frac{p}{2} - \frac{y^2}{2p}$$

parabola

$x = y^2$

$\underline{y=0}:\quad x = p/2$

$\underline{x=0}:\quad y = \pm p$

Sec 15, Prob 3:

$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}} \qquad (14.10)$$

Consider a small perturbation $\delta U(r)$ to the potential energy:

$$\to \Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U-\delta U) - M^2/r^2}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - M^2/r^2 - 2m\delta U}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}\sqrt{1 - \frac{2m\delta U}{2m(E-U) - M^2/r^2}}}$$

$$\approx 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}}\left(1 + \frac{m\delta U}{2m(E-U) - M^2/r^2}\right)$$

$$= \underbrace{2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - M^2/r^2}}}_{\Delta\phi_0} + \underbrace{2\int_{r_{min}}^{r_{max}} \frac{Mm\,\delta U\,dr/r^2}{\left(2m(E-U) - M^2/r^2\right)^{3/2}}}_{\delta\phi}$$

Now:
$$\frac{\partial}{\partial M}\left(\frac{1}{\sqrt{2m(E-U)-M^2/r^2}}\right) = -\frac{1}{2}\frac{(-2M/r^2)}{\left(2m(E-U)-M^2/r^2\right)^{3/2}}$$
$$= \frac{M/r^2}{\left(2m(E-U)-M^2/r^2\right)^{3/2}}$$

Thus,
$$\delta\phi = \frac{\partial}{\partial M}\left(\int_{r_{min}}^{r_{max}} \frac{2m\,\delta U\,dr}{\sqrt{2m(E-U)-M^2/r^2}}\right)$$

Consider the case $U(r) = -\alpha/r$

Then: $\Delta\phi_0 = 2\pi$ (since a bound orbit is an ellipse, which is closed)

Also: $$\delta\phi = \frac{\partial}{\partial M}\left(\int_{r_{min}}^{r_{max}} \frac{2m\,\delta U\,dr}{\sqrt{2m(E+\alpha/r)-M^2/r^2}}\right)$$

In the integral, we can use the solution for the unperturbed motion:
$$\frac{p}{r} = 1 + e\cos\phi \longrightarrow -\frac{p}{r^2}dr = -e\sin\phi\,d\phi$$
$$\text{so}\quad dr = \frac{e}{p}r^2\sin\phi\,d\phi$$
$$r = r_{min}, r_{max} \longleftrightarrow \phi = 0, \pi$$

$$\sqrt{\ } = \sqrt{2m\left(E+\frac{\alpha}{r}\right) - \frac{M^2}{r^2}}$$

$$= \sqrt{2mE + \frac{2m\alpha(1+e\cos\phi)}{p} - M^2\frac{(1+e\cos\phi)^2}{p^2}}$$

$$= \frac{1}{p}\sqrt{2mEp^2 + 2m\alpha p(1+e\cos\phi) - M^2(1+e\cos\phi)^2}$$

Recall: $$p = \frac{M^2}{m\alpha}$$

$$e = \sqrt{1+\frac{2EM^2}{m\alpha^2}}$$

thus, $$2mEp^2 = 2mE\frac{M^4}{m^2\alpha^2} = \left(\frac{2EM^2}{m\alpha^2}\right)M^2$$

$$2m\alpha p = 2M^2$$

$$\rightarrow \sqrt{\ } = \frac{m\alpha}{M^2}\sqrt{\left(\frac{2EM^2}{m\alpha^2}\right)M^2 + 2M^2(1+\cancel{e\cos\phi}) - M^2(1+e^2\cos^2\phi + \cancel{2e\cos\phi})}$$

$$= \frac{m\alpha}{M}\sqrt{\underbrace{\frac{2EM^2}{m\alpha^2}+1}_{e^2} - e^2\cos^2\phi}$$

$$= \frac{m\alpha e}{M}\sin\phi$$

Thus,

$$\delta\phi = \frac{\partial}{\partial M}\left(\int_0^{\pi} \frac{2\cancel{m}\,\delta U\, r^2 \frac{\cancel{e}}{p}\cancel{\sin\phi}\, d\phi}{\frac{\cancel{m}\alpha\cancel{e}}{M}\cancel{\sin\phi}}\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2M}{\alpha p}\int_0^{\pi} r^2\,\delta U\, d\phi\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m}{M}\int_0^{\pi} r^2\,\delta U\, d\phi\right) \qquad \left(\text{using } p = \frac{M^2}{m\alpha}\right)$$

where $\frac{p}{r} = 1 + e\cos\phi$

(a) $\delta U = \frac{\beta}{r^2}$

$$\rightarrow \delta\phi = \frac{\partial}{\partial M}\left(\frac{2m}{M}\int_0^{\pi} \cancel{r^2}\frac{\beta}{\cancel{r^2}}\, d\phi\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m}{M}\beta\pi\right)$$

$$= -\frac{2\cancel{\pi} m\beta}{M^2} \quad \boxed{= \frac{-2\pi\beta}{\alpha p}}$$

(b) $\delta U = \dfrac{\gamma}{r^3}$

$$\rightarrow \quad \delta\phi = \frac{\partial}{\partial M}\left(\frac{2m}{M}\int_0^{\pi} r^2 \frac{\gamma}{r^3}\, d\phi\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m\gamma}{M}\int_0^{\pi} \frac{d\phi}{r}\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m\gamma}{M}\frac{1}{p}\int_0^{\pi} (1+e\cos\phi)\, d\phi\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2m\gamma}{Mp}\left[\pi + \underbrace{e\sin\phi\Big|_0^{\pi}}_{0}\right]\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2\pi m\gamma}{Mp}\right)$$

$$= \frac{\partial}{\partial M}\left(\frac{2\pi m^2 \gamma\alpha}{M^3}\right) \longleftarrow \text{using } p = \frac{M^2}{m\alpha}$$

$$= -\frac{6\pi m^2 \gamma\alpha}{M^4}$$

$$= \frac{-6\pi \cancel{m^2}\gamma\alpha}{p^2 \cancel{m^2}\alpha^2}$$

$$= \boxed{\frac{-6\pi\gamma}{p^2\alpha}}$$