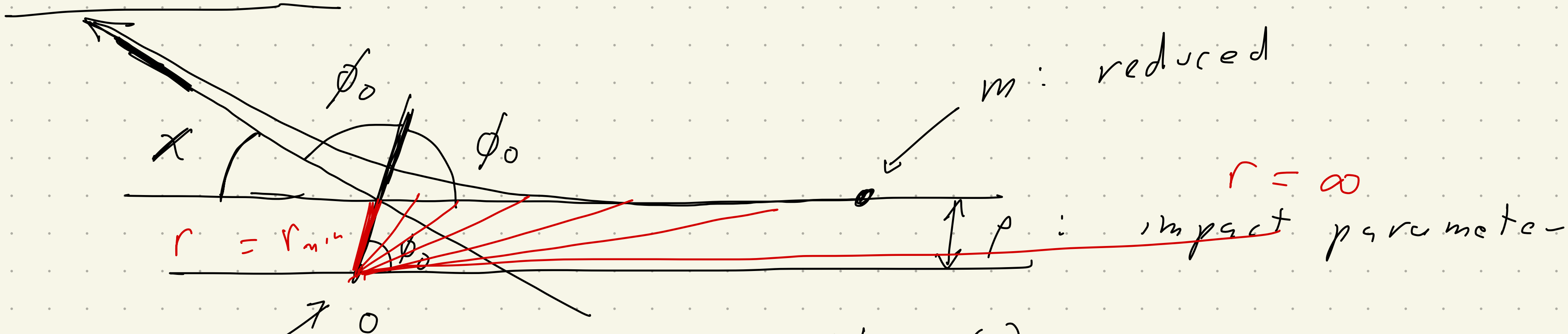
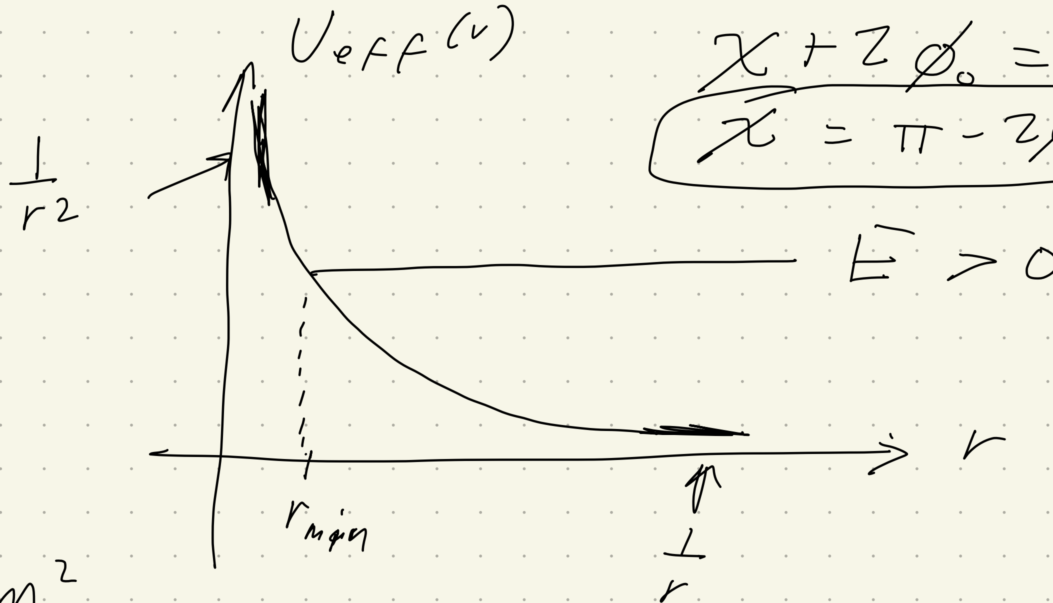
Lecture #16 Thurs 10/14



$\chi + 2\phi_0 = \pi$

$$\boxed{\chi = \pi - 2\phi_0}$$



$U(r) = \frac{\alpha}{r}$, $\alpha > 0$

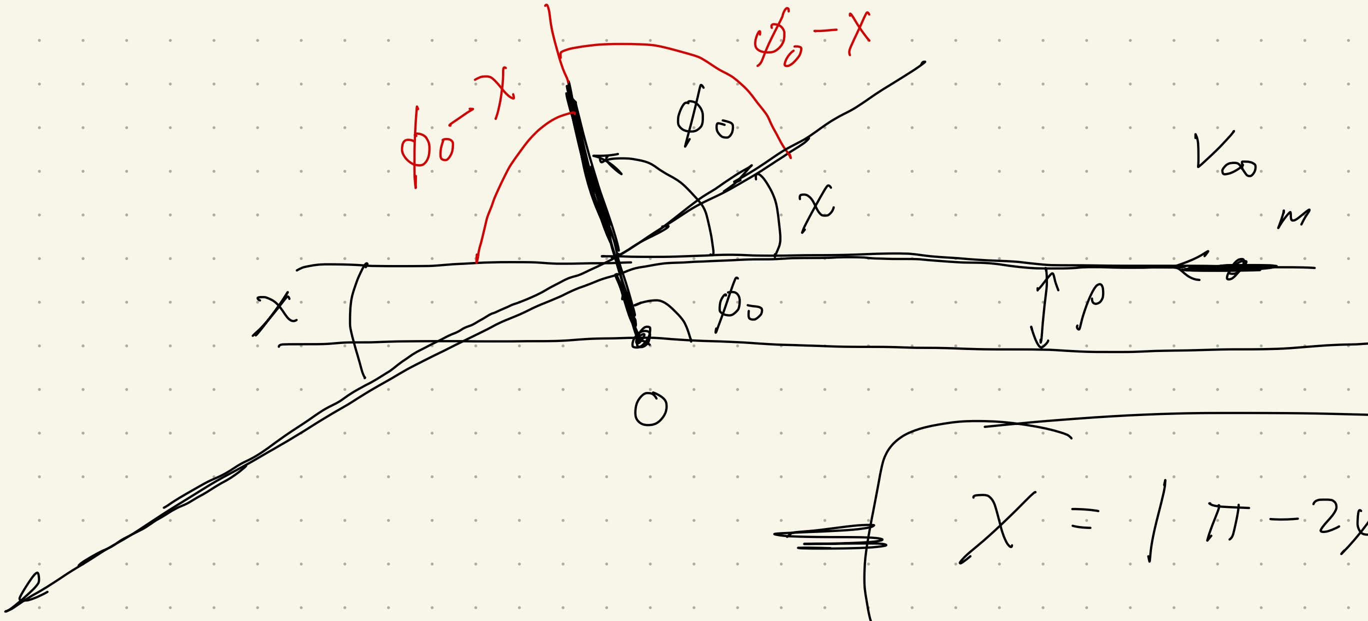
(repulsive)

$$U_{eff}(r) = U(r) + \frac{M_z^2}{2mr^2}$$

$$= \frac{\alpha}{r} + \frac{M_z^2}{2mr^2}$$

$E = U_{eff}(r_{min})$

turning point



$$2(\phi_0-\chi)+\chi=\pi$$

$$2\phi_0-\chi=\pi$$

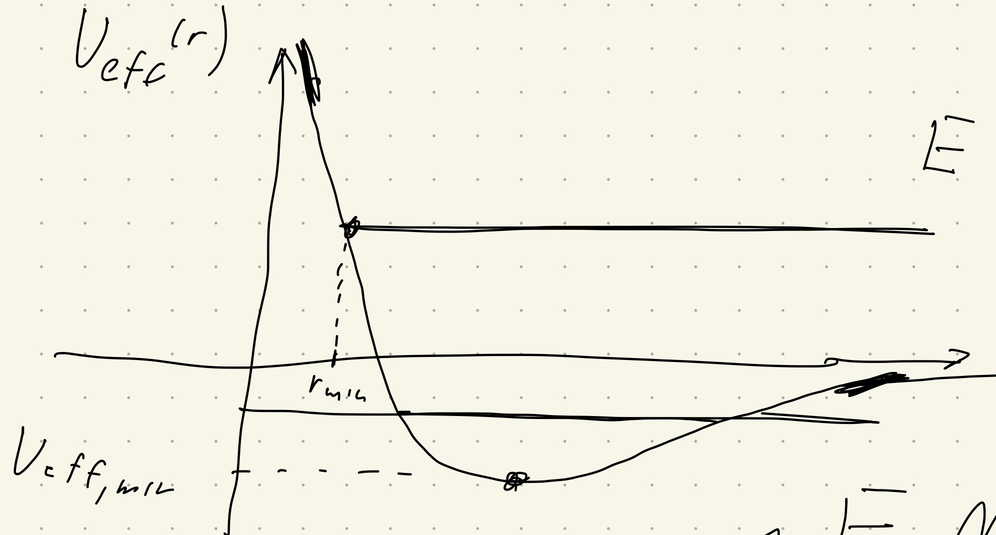
$$\boxed{\chi=2\phi_0-\pi}$$

$$\Rightarrow \boxed{\chi=|\pi-2\phi_0|}$$

for both repulsive & attractive

$$U(r)=-\frac{\alpha}{r}$$

$$V_{eff}(r)=-\frac{\alpha}{r}+\frac{M_t^2}{2mr^2}$$



$E>0$

$$\boxed{E=\frac{1}{2}mv_\infty^2\,,\qquad M=m\rho v_\infty}$$

$$\left(\begin{matrix}E,\ M\\ \rho,\ v_\infty\end{matrix}\right.$$

Sec 14:

$$t = \pm \int \frac{dr}{\sqrt{}} + \text{const}$$

$$\Rightarrow \quad \phi = \pm \int \frac{M\, dr/r^2}{\sqrt{2m(E - U(r)) - \frac{M^2}{r^2}}} + \text{const}$$

$$E = U_{eff}(r_{min})$$

$$\boxed{\phi_0 = \int_{r_{min}}^{\infty} \frac{M\, dr/r^2}{\sqrt{2m(E - U(r)) - \frac{M^2}{r^2}}} = \int_{r_{min}}^{\infty} \frac{m\rho v_\infty\, dr/r^2}{\sqrt{2m\left(\frac{1}{2}mv_\infty^2 - U(r)\right) - \frac{m^2\rho^2 v_\infty^2}{r^2}}} = \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \frac{U(r)}{\frac{1}{2}mv_\infty^2} - \rho^2/r^2}}}$$

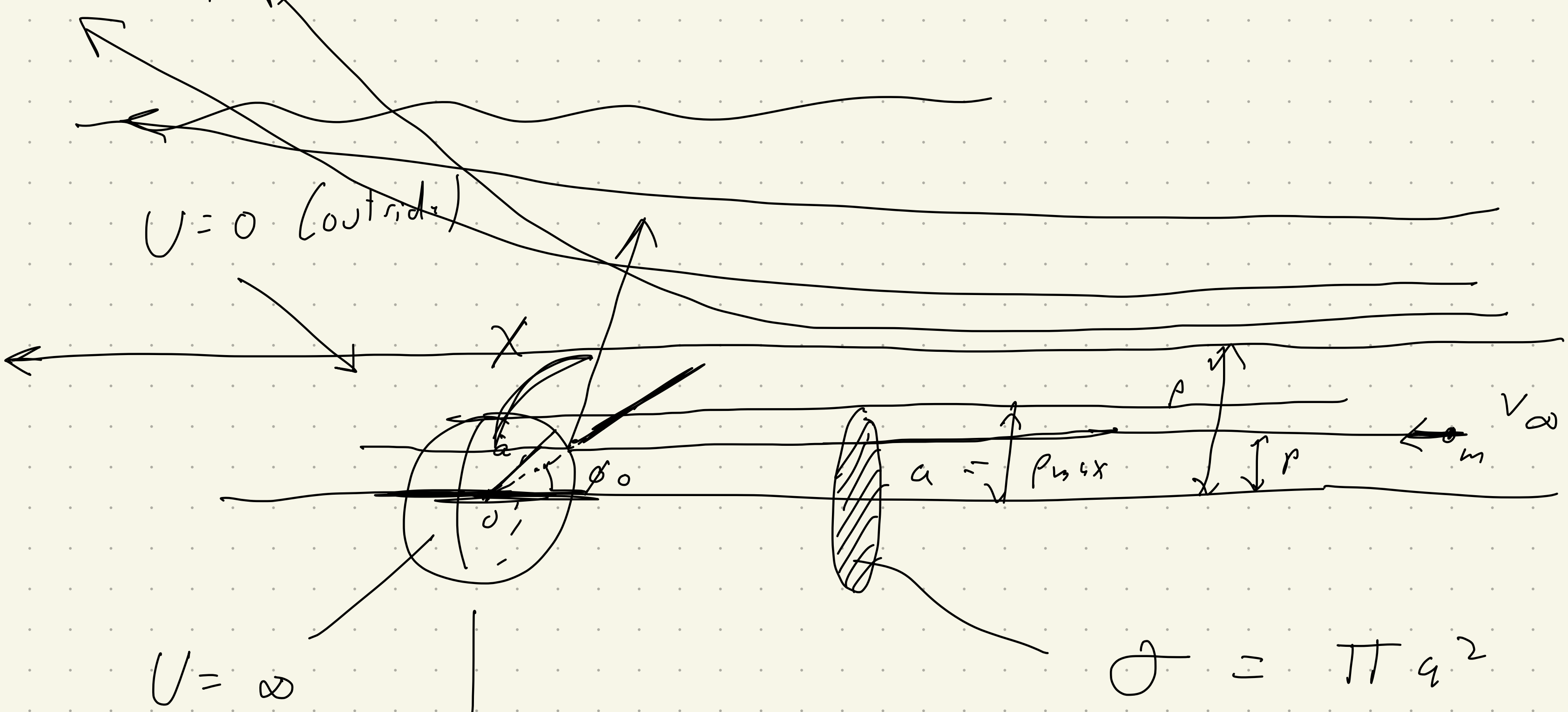
$$M = m\rho v_\infty$$

$$E = \frac{1}{2}mv_\infty^2$$

$$\Theta_2 = \frac{1}{2}(\pi - \chi) \qquad \chi = |\pi - 2\phi_0|$$

$$\tan\Theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}$$

$$v_1' \quad v_2' = \ldots$$



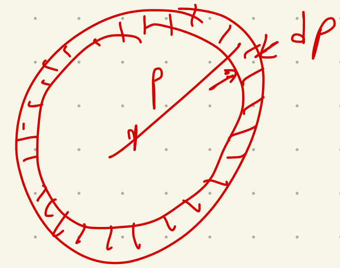
Hard sphere

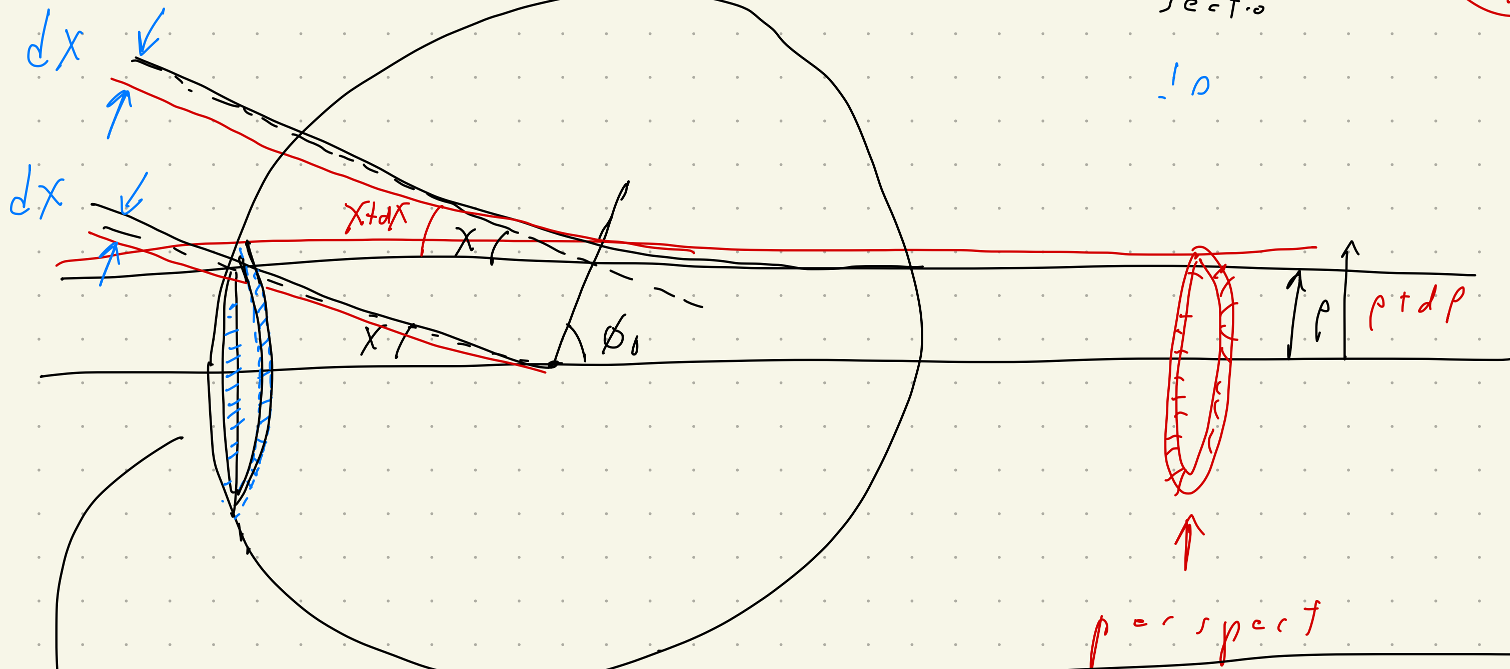
$\sigma = \pi a^2$

cross-section

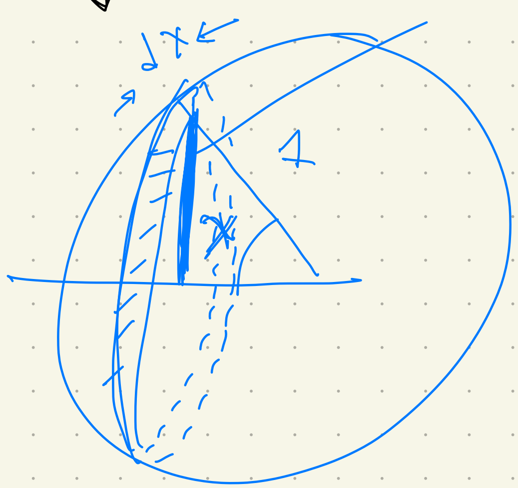
$$\boxed{d\sigma = 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi} \longleftarrow \boxed{d\sigma = 2\pi\rho\, d\rho}$$

differential cross section





$$\boxed{d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega}$$



$$\underbrace{d\Omega}_{\text{solid angle}} = 2\pi \sin\chi \, d\chi$$

$$\frac{d\sigma}{d\Omega} = \frac{\cancel{2\pi}\rho\, d\rho}{\cancel{2\pi}\sin\chi\, d\chi} = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right|$$

$$\frac{d\sigma}{d\Omega} = \frac{\rho}{\sin\chi}\left|\frac{d\rho}{d\chi}\right|$$ COM frame

$$\frac{d\sigma_1}{d\Omega_1} = \frac{\rho}{\sin\theta_1}\left|\frac{d\rho}{d\theta_1}\right|$$ Lab frame θ_1

$$\frac{d\sigma_2}{d\Omega_2} = \frac{\rho}{\sin\theta_2}\left|\frac{d\rho}{d\theta_2}\right|$$ Lab frame θ_2

$$\frac{d\sigma_1}{d\Omega_1} = \frac{\sin\chi}{\sin\theta_1}\left|\frac{d\chi}{d\theta_1}\right| \frac{d\sigma}{d\Omega}$$

$$= \left|\frac{d(\cos\chi)}{d(\cos\theta_1)}\right| \frac{d\sigma}{d\Omega}$$

calculated for the COM

Lecture #17: Tues Oct 19th

Next week: start small oscillations

Today/Thurs: Q & A (collisions and scatterings)

Lecture #18: Thur Oct 21st

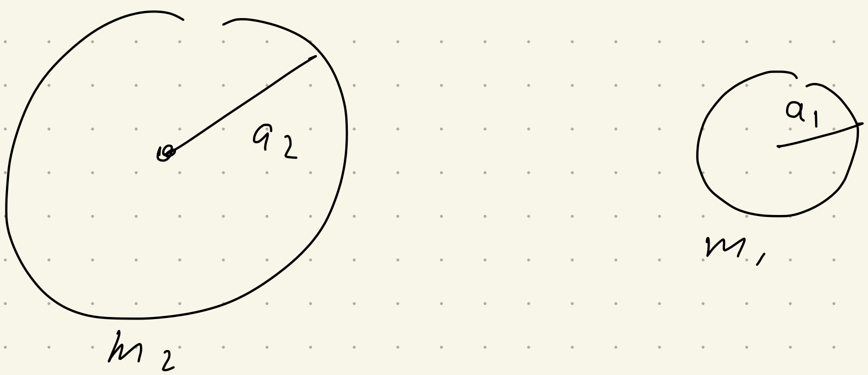
Next week: small oscillations

Today: Q & A, Quiz #3 (last 20 minutes)

Q3_name.pdf

Quiz #3:

Two hard spheres with masses m_1, m_2 and radii a_1, a_2:



$$\rho(\chi) = (a_1 + a_2)\cos\left(\frac{\chi}{2}\right)$$

a) Find $\rho = \rho(\chi)$ where χ = scattering angle wrt COM frame

b) What value of ρ will give $\theta_2 = 60°$?

$$2\theta_2 + \chi = \pi \rightarrow \chi = \pi - 2\theta_2 = 60°$$

$$\rho = (a_1 + a_2)\cos(30°) = \frac{\sqrt{3}}{2}(a_1 + a_2)$$

Lec #19: 10/26

Small oscillation: $\boxed{\text{Sec } 21}$, 22, 23

free oscillation in 1-d

forced oscillation in 1d

more than 1d

Damping: Sec 25 (not covered)

SHM: (simple harmonic motion) is sinusoidal

$$\omega = \frac{2\pi}{T}$$

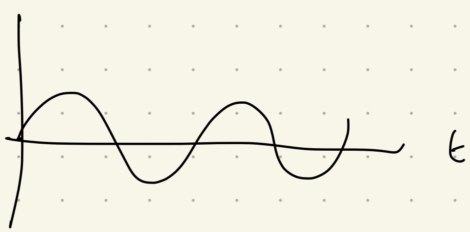
$$x(t) = a \cos(\omega t + \alpha)$$

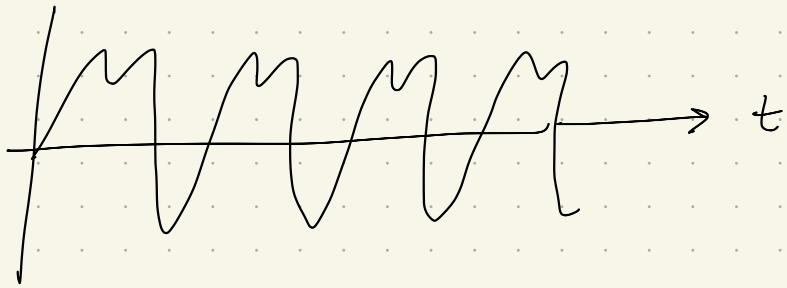
amplitude

phase

angular freq

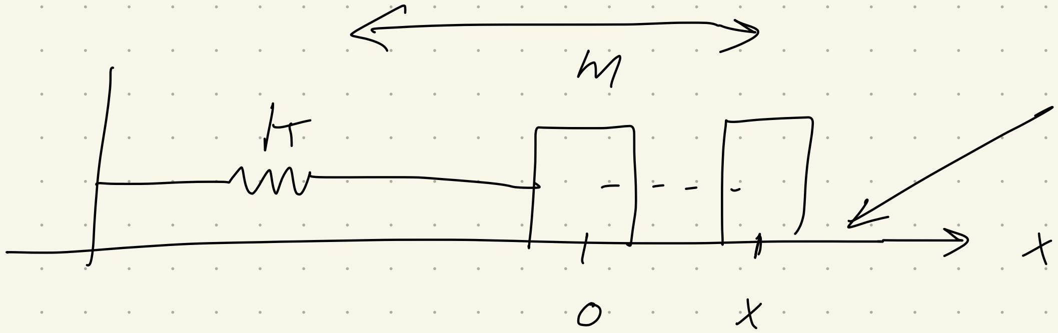
SHM





periodic

Example:



no friction

$$U = \frac{1}{2} k x^2$$

$$F = -\frac{dU}{dx} = -kx$$

$$F = -kx$$

linear

restoring

$$F = m\ddot{x} = -kx \quad \rightarrow \quad \ddot{x} = -\frac{k}{m} x$$

soln:

$$x = c_1 \cos \omega t + c_2 \sin \omega t$$

$$\omega = \sqrt{\frac{k}{m}}$$

Alternative:

$$x = a \cos(\omega t + \alpha)$$

$$x = \mathrm{Re}\left[A e^{i\omega t} \right], \quad A = a e^{i\alpha}$$

$$T = \frac{1}{2} m \dot{x}^2$$
$$U = \frac{1}{2} k x^2$$

$$L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow m\ddot{x} = -kx$$

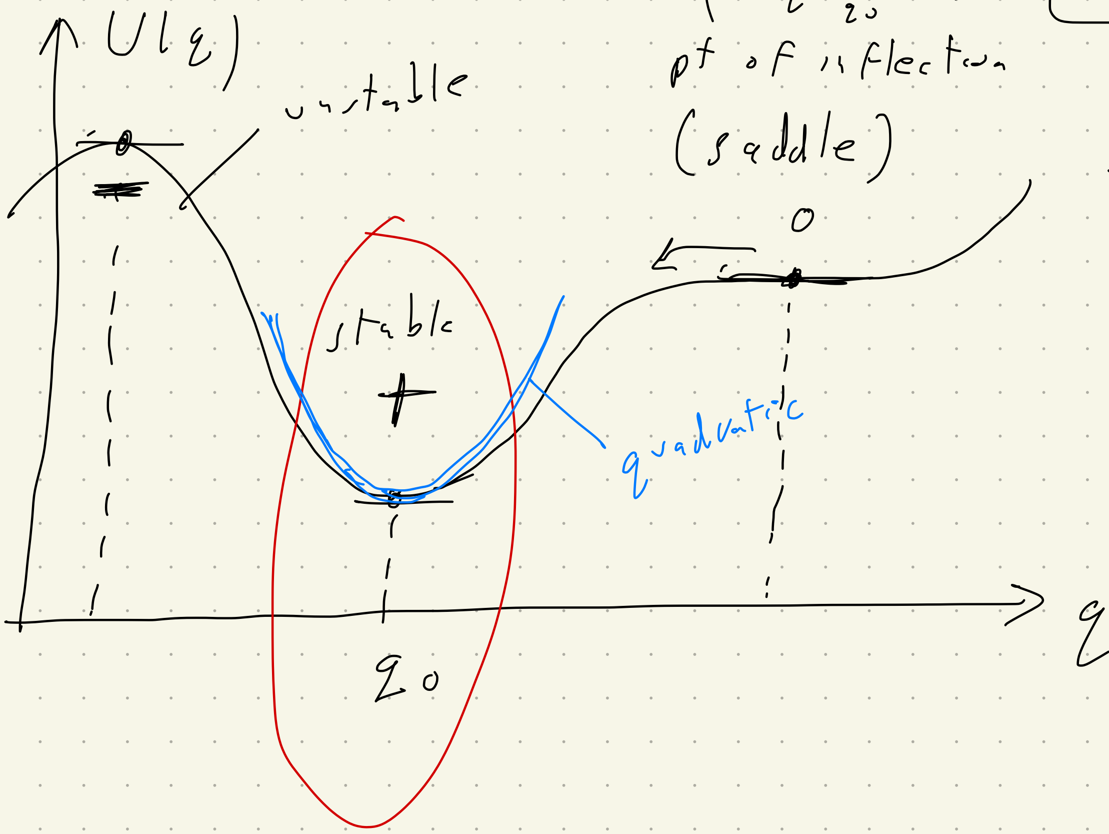
$U(q) = q^3$

$q_0 = 0$

More generally:

$\left(\left.\frac{d^2U}{dq^2}\right|_{q_0} = 0\right)$

pt of inflection (saddle)



$$\boxed{L = \frac{1}{2} a(q)\, \dot{q}^2 - U(q)}$$

Equilibrium: $(q = q_0)$

$$0 = F = -\left.\frac{dU}{dq}\right|_{q_0}$$

stable:

$$\left.\frac{d^2U}{dq^2}\right|_{q_0} > 0$$

$\longrightarrow$ $U = \text{const}$

Expansion: (about q_0)

$$U(q) = \underbrace{U(q_0)}_{\substack{\text{const} \\ \text{(ignore in } L)}} + \cancel{\left.\frac{dU}{dq}\right|_{q_0}}^{\,0 \text{ equilibrium}} \cdot (q - q_0) + \frac{1}{2} \boxed{\left.\frac{d^2U}{dq^2}\right|_{q_0}}^{\,\kappa > 0} (q-q_0)^2$$

$$\underbrace{+ \frac{1}{3!} \left.\frac{d^3U}{dq^3}\right|_{q_0} (q-q_0)^3 + \dots}_{\text{ignore for } q - q_0 \text{ small}}$$

$$\boxed{U(q) \approx \frac{1}{2} \kappa x^2}$$

$$x \equiv q - q_0$$

$$\kappa = \left.\frac{d^2U}{dq^2}\right|_{q_0}$$

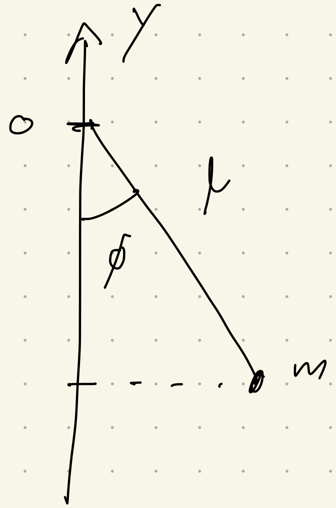
$$T = \frac{1}{2} a(q) \dot{q}^2$$

$$= \frac{1}{2} a(q_0) \dot{x}^2$$

$$\approx \frac{1}{2} m \dot{x}^2$$

$$\boxed{L \approx \frac{1}{2} m \dot{x}^2 - \frac{1}{2} \kappa x^2}$$

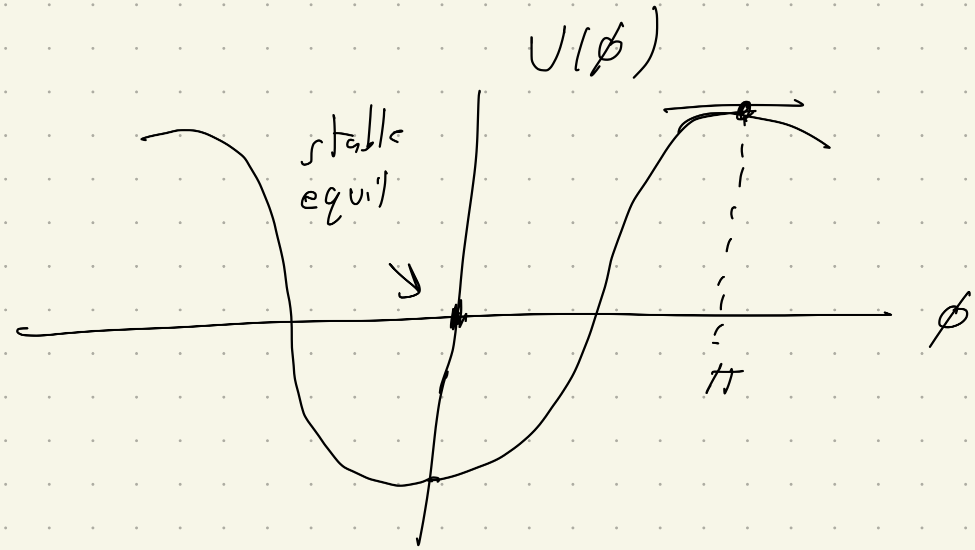
Simple pendulum:



$$T = \frac{1}{2} m l^2 \dot{\phi}^2$$

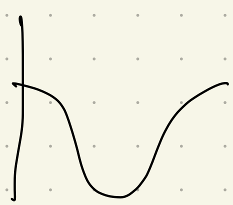
$$U = -mgl\cos\phi$$

$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi$$



$$U = mgy$$
$$= -mgl\cos\phi$$

$$\phi_0 = 0 \quad (\text{equilibrium})$$



$$\cos\phi = 1 - \frac{1}{2}\phi^2 + \cdots$$

$$\cos\phi = \underbrace{\cos 0}_{1} + \cancel{\underbrace{\frac{d(\cos\phi)}{d\phi}}_{\sin\phi}}\Bigg|_{\phi=0} \cdot \phi + \frac{1}{2}\frac{d^2(\cos\phi)}{d\phi^2}\Bigg|_{\phi=0}\phi^2 + \dots$$

$$0$$

$$= 1 - \frac{1}{2}\phi^2 + \dots$$

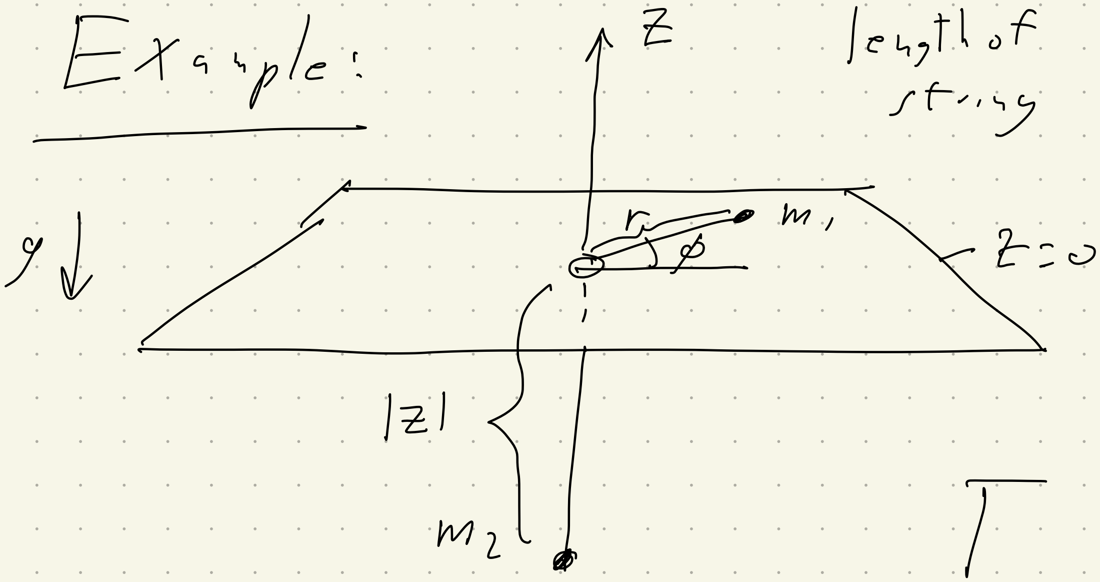
$$L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\left(1 - \frac{1}{2}\phi^2\right)$$

$$= \frac{1}{2}\underbrace{(ml^2)}_{=M}\dot{\phi}^2 - \frac{1}{2}\underbrace{(mgl)}_{=K}\phi^2 + \underbrace{mgl}_{\text{ignore}}$$

$$\omega = \sqrt{\frac{K}{M}} = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2$$

Example:



length of string — $\ell = r + |z| = r - z$

z: location of m_2

r: for m_1

$z = r - \ell \rightarrow \dot{z} = \dot{r}$

$$T = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}m_2\dot{z}^2$$

$$U = m_2 g z = m_2 g(r - \ell) = m_2 g r - m_2 g \ell$$

(ignore) const

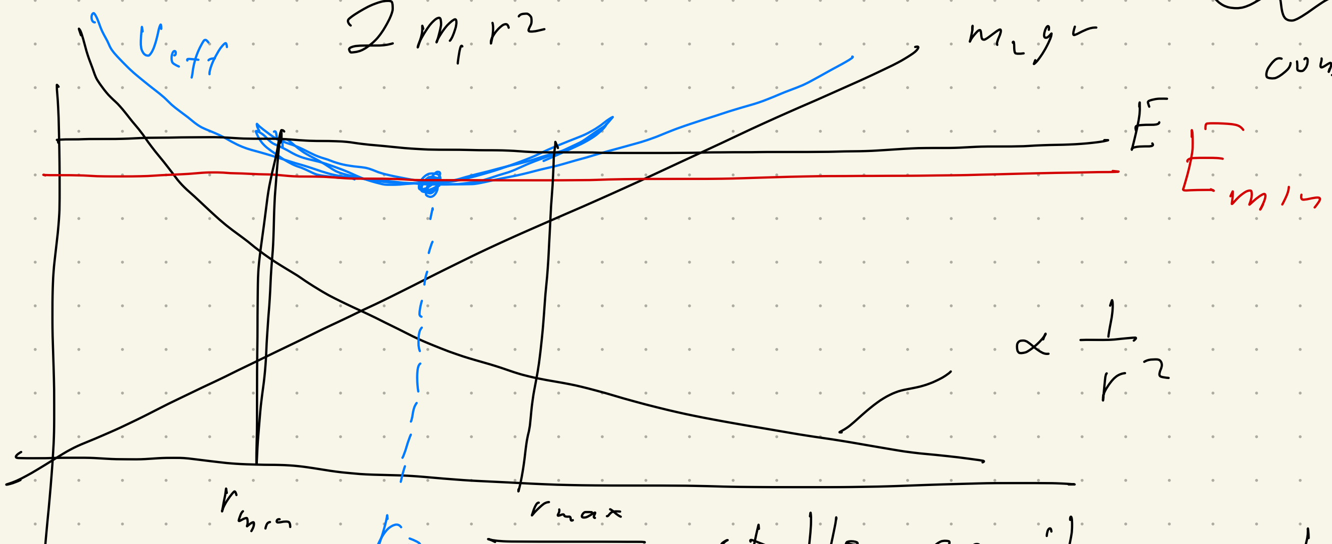
$$\boxed{L = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\phi}^2 - m_2 g r}$$

i) $M_z = \dfrac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} \longrightarrow \dot{\phi} = \dfrac{M_z}{m_1 r^2}$

ii) $E = T + U = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \underbrace{\frac{1}{2}m_1 r^2\dot{\phi}^2}_{\frac{M_z^2}{2m_1 r^2}} + m_2 g r$

$$E = \frac{1}{2}\underbrace{(m_1 + m_2)}_{= M}\dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = \frac{M_z^2}{2 m_1 r^2} + m_2 g r = \underbrace{U_{eff}(r_0)}_{\text{const}} + \frac{1}{2}k(r-r_0)^2$$



stable equil.

$$k = \left.\frac{d^2 U_{eff}}{dr^2}\right|_{r=r_0}$$

$$|r - r_0| \ll r_0$$

$r_0 = ?$

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = -\frac{M_z^2}{m_1 r_0^3} + m_2 g \rightarrow \boxed{M_z^2 = m_1 m_2 g r_0^3}$$

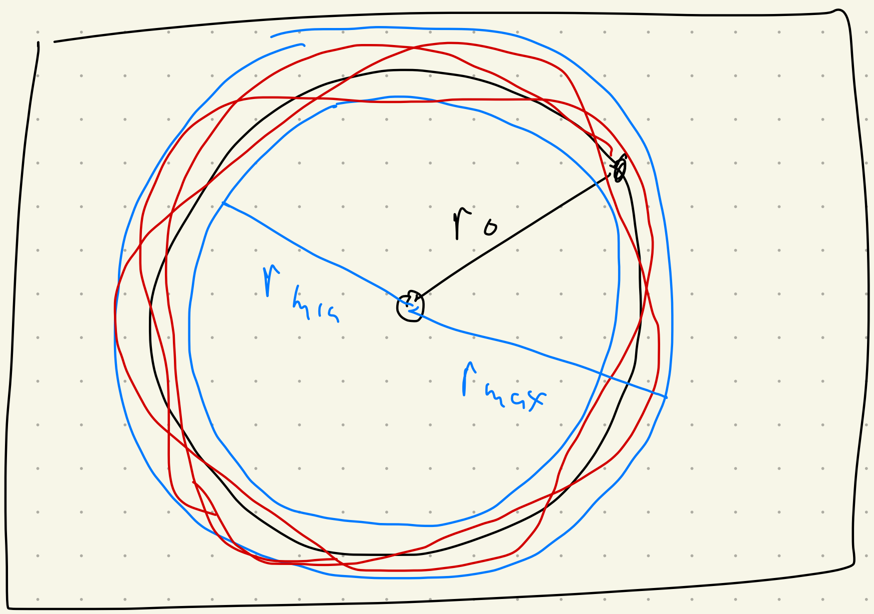
$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r=r_0} = \frac{3M_z^2}{m_1 r_0^4}$$

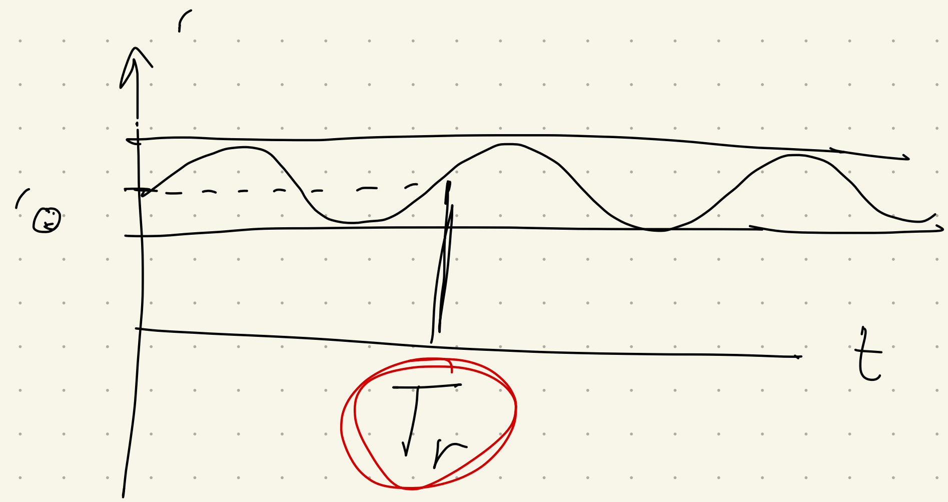
$$= \frac{3\cancel{m_1} m_2 g r_0^3}{\cancel{m_1} r_0^4}$$

$$= \frac{3m_2 g}{r_0} = K$$

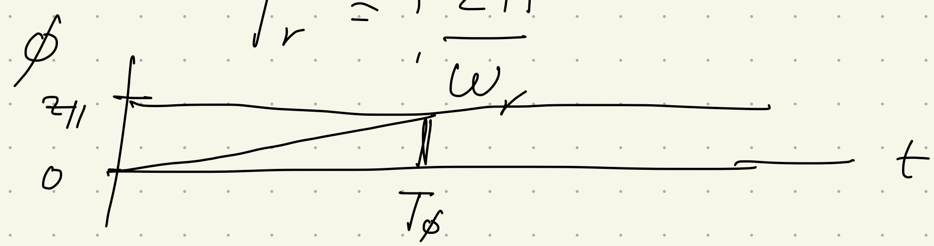
$$\omega_r = \sqrt{\frac{K}{\underbrace{m_1+m_2}_{M}}}$$

$$= \sqrt{\frac{3m_2 g}{(m_1+m_2)r_0}}$$





$$T_r = \frac{2\pi}{\omega_r}$$



$$T_\phi = \frac{2\pi}{\omega_\phi} = \frac{2\pi}{\dot{\phi}\big|_{r_0}}$$

$$\dot{\phi} = \frac{M_z}{m_1 r^2}$$

$$\omega_\phi = \dot{\phi}\Big|_{r=r_0} = \frac{M_z}{m_1 r_0^2} = \frac{\sqrt{m_1 m_2 g r_0^3}}{m_1 r_0^2}$$

$$\boxed{\omega_\phi = \sqrt{\frac{m_2 g}{m_1 r_0}}}$$

compare to

$$\boxed{\omega_r = \sqrt{\frac{3 m_2 g}{(m_1 + m_2) r_0}}}$$

If $\quad \dfrac{3}{m_1 + m_2} = \dfrac{1}{m_1} \quad$ then $\quad \omega_r = \omega_\phi$

$$3m_1 = m_1 + m_2$$

$$\boxed{2m_1 = m_2}$$

Lec #20: 10/28

Forced oscillation:

$$m\ddot{x} = -kx + F(t)$$

$$V(x) = \frac{1}{2}kx^2 - x\,F(t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$L = T - U$$

$$\frac{\partial L}{\partial x} = -kx + F(t)$$

$$\ddot{x} + \frac{k}{m}x = \frac{F(t)}{m}$$

$$\boxed{\ddot{x} + \omega^2 x = \frac{F(t)}{m}}, \quad \omega \equiv \sqrt{\frac{k}{m}}$$

general sol'n:

$$x(t) = x_h(t) + x_p(t)$$

homogeneous ($F(t)=0$)

particular (any solution for $F(t)$)

$$\boxed{x_h(t) = a\cos(\omega t + \alpha)}$$

a, α: two constant (Initial condition)

Suppose: $F(t) = f\cos(\gamma t + \beta)$

$$\ddot{x}_p + w^2 x_p = \frac{f}{m}\cos(\gamma t + \beta)$$

Guess!! $\quad x_p(t) = b\cos(\gamma t + \beta)$

$$\to -b\gamma^2 \cancel{\cos(\gamma t+\beta)} + w^2 b \cancel{\cos(\gamma t+\beta)} = \frac{f}{m}\cancel{\cos(\gamma t+\beta)}$$

$$b(w^2-\gamma^2) = \frac{f}{m}$$

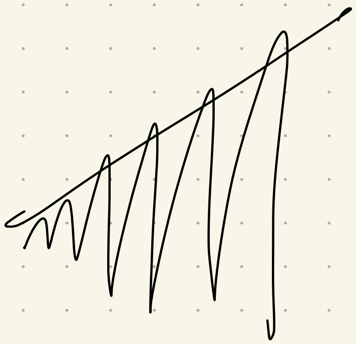
$$\to \boxed{b = \frac{f}{m(w^2-\gamma^2)}}$$

$$\boxed{x_p(t) = \frac{f}{m(w^2-\gamma^2)}\cos(\gamma t + \beta)}$$

$+ \; 10\cos(wt)$

$$\boxed{x(t) = a\cos(wt+\alpha) + \frac{f}{m(w^2-\gamma^2)}\cos(\gamma t + \beta)}$$

$$x(t) = a\cos(\omega t+\alpha) + \underbrace{\frac{f}{m(\omega^2-\gamma^2)}\left[\cos(\gamma t+\beta) - \cos(\omega t+\beta)\right]}_{\frac{0}{0}}$$



L'hopital's

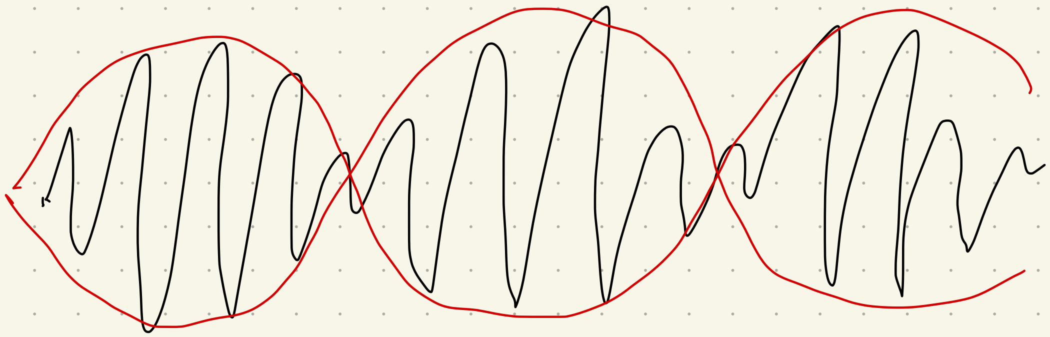
$$= \frac{\left.\frac{d}{d\gamma}(\text{num})\right|_{\gamma\to\omega}}{\left.\frac{d}{d\gamma}(\text{den})\right|_{\gamma\to\omega}}$$

$$= \frac{+\, f t \sin(\omega t+\beta)}{+\, 2m\omega}$$

$$= \frac{ft}{2m\omega}\sin(\omega t+\beta)$$

at resonance $(\gamma=\omega)$

$$\boxed{x(t) = a\cos(\omega t+\alpha) + \frac{ft}{2m\omega}\sin(\omega t+\beta)}$$



$$\frac{\omega + \gamma}{2}$$

$$|\omega - \gamma| = \omega_{beat}$$

General: for arbitrary $F(t)$

$$F(t) = \mathrm{Re} \underbrace{\int_{-\infty}^{\infty} d\gamma\, \tilde{F}(\gamma) e^{i\gamma t}}_{\text{Fourier transform}}$$

$$\boxed{\ddot{x} + w^2 x = \frac{F(t)}{m}}$$

$$\boxed{y' + P(x)y = Q(x)}$$

Math methods

$y = y(x)$

Let: $\xi = \dot{x} + iwx$

complex

$$\dot{\xi} = \ddot{x} + iw\dot{x}$$

$$= \ddot{x} + iw(\xi - iwx)$$

$$= \ddot{x} + iw\xi + w^2 x$$

$$\rightarrow \quad \dot{\xi} - iw\xi = \ddot{x} + w^2 x = \frac{F(t)}{m}$$

Homog: $\dot{\xi}_h - iw\xi_h = 0 \quad \rightarrow \quad \xi_h(t) = Ae^{iwt}$

complex constant

replace

Guess: $\boxed{\xi_p(t) = A(t)e^{iwt}}$

$$\boxed{\dot{\xi}_p - i\omega \xi_p = \frac{F(t)}{m}}, \quad \xi_p = A(t)e^{i\omega t}$$

$$\rightarrow \dot{A}e^{i\omega t} + \cancel{iA\omega e^{i\omega t}} \cancel{- i\omega A e^{i\omega t}} = \frac{F(t)}{m}$$

$$\dot{A} = \frac{e^{-i\omega t}F(t)}{m}$$

$$A(t) = \int dt \frac{F(t)}{m} e^{-i\omega t} + \text{const}$$

$$\boxed{\xi(t) = e^{i\omega t}\left[\int_0^t d\bar{t} \frac{F(\bar{t})}{m} e^{-i\omega \bar{t}} + \xi_0\right]}$$

$$\xi = \dot{x} + i\omega x \rightarrow \boxed{x(t) = \frac{1}{\omega}\operatorname{Im}(\xi(t))}$$

Complex constant (I.C.'s)

$$\cancel{y'} + P(x)\cancel{y} = Q(x) \qquad f(x)\,dx = g(y)\,d\cancel{y}$$

$$\frac{d\cancel{y}}{dx} + P(x)\cancel{y} - Q(x) = 0$$

$$\boxed{1\,d\cancel{y} + (P(x)\cancel{y} - Q(x))\,dx = 0} \neq dU$$

$$dU(x,y) = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}d\cancel{x}$$

$$\frac{\partial^2 U}{\partial y\,\partial x} = \frac{\partial^2 U}{\partial x\,\partial y}$$

M. Boas

~~Mathemat~~

$$\boxed{\mu(x)\left[dy + (P(x)y - Q(x))dx\right] = dU}$$

$$\frac{d\mu}{dx} = \frac{\partial}{\partial \cancel{y}}\left(P(x)\cancel{y} - Q(x)\right)\mu(x)$$

$$\frac{d\mu}{dx} = P(x)\mu(x)$$

$$\int \frac{d\mu}{\mu} = \int P(x)\,dx$$

$$\ln \mu = \int P(x)\,dx$$

$$\mu(x) = e^{\int P(x)\,dx}$$

Lec #21: Nov. 2nd

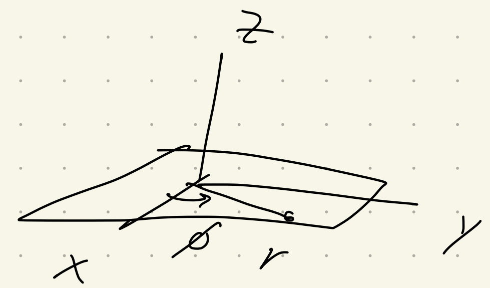
Today: Sec 23 Free oscillation in 2 or more dimensions.

Thurs: Rigid body motion

EXAM 2: Nov 18th → Nov 23rd

QUIZ #4: Today (at end of class)

Example: space oscillator



$$U = \frac{1}{2} k r^2 = \frac{1}{2} k (x^2 + y^2)$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$L = T - U$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k (x^2 + y^2)$$

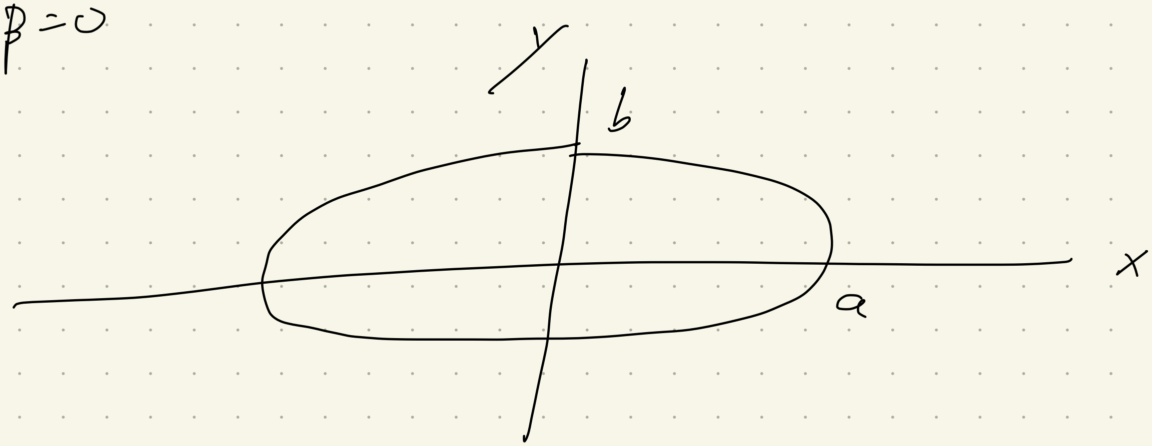
$$= \underbrace{\left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2\right)}_{\omega_x = \sqrt{\frac{k}{m}}} + \underbrace{\left(\frac{1}{2} m \dot{y}^2 - \frac{1}{2} k y^2\right)}_{\omega_y = \sqrt{\frac{k}{m}}}$$

$$\omega = \sqrt{\frac{k}{m}}$$

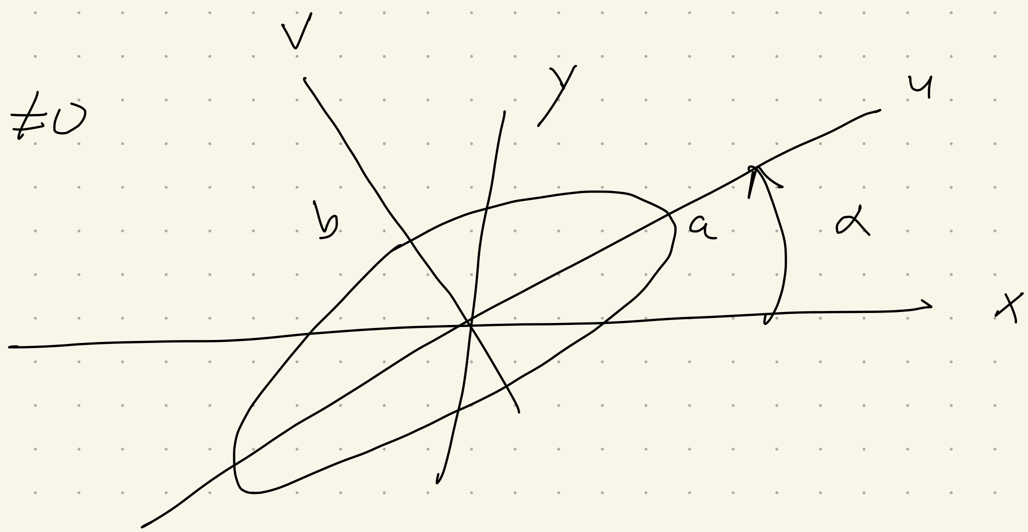
$$\boxed{\begin{aligned} x &= a \cos(\omega t + \alpha) \\ y &= b \sin(\omega t + \beta) \end{aligned}}$$

general solution

$\alpha = \beta = 0$



$\alpha = \beta \neq 0$

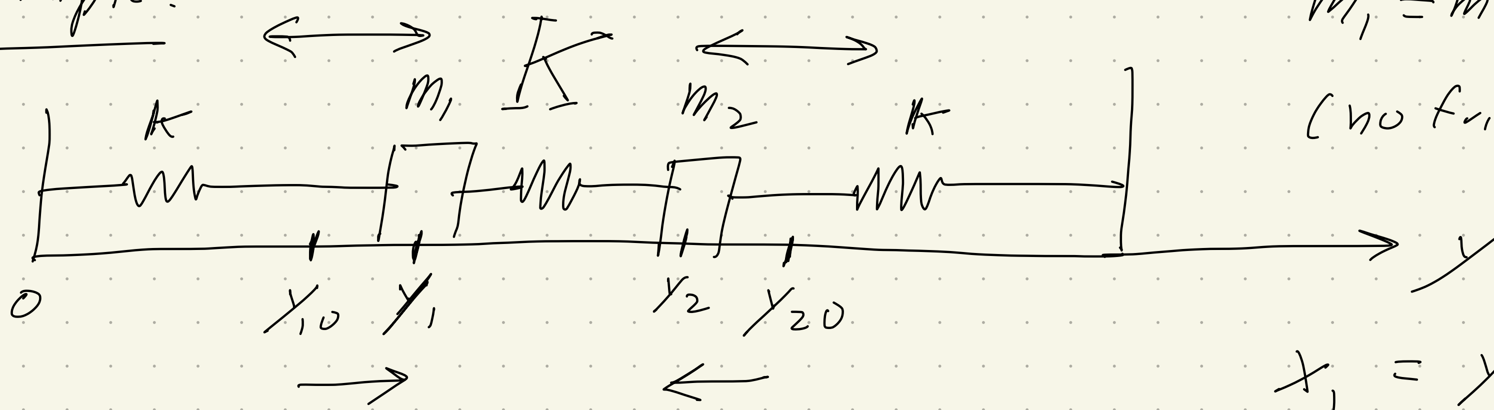


$\alpha \neq \beta$ —— ellipse

Example:

$m_1 = m_2 \equiv m$

(no friction)



$x_1 = y_1 - y_{10}$

$x_2 = y_2 - y_{20}$

$$L = T - U$$

$$T = \frac{1}{2} m \dot{y}_1^2 + \frac{1}{2} m \dot{y}_2^2 = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} K (x_2 - x_1)^2$$

$$= \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} K (x_2^2 + x_1^2 - 2 x_1 x_2)$$

$$= \frac{1}{2} (k + K) x_1^2 + \frac{1}{2} (k + K) x_2^2 - K x_1 x_2$$

$-\frac{1}{2} K x_1 x_2$

$-\frac{1}{2} K x_2 x_1$

$$T = \frac{1}{2} \sum_{i,k} m_{ik} \dot{x}_i \dot{x}_k, \quad m_{ik} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$U = \frac{1}{2} \sum_{i,k} K_{ik} x_i x_k, \quad k_{ik} = \begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix}$$

$$L = \frac{1}{2}\sum_{i,k} m_{ik}\dot{x}_i\dot{x}_k - \frac{1}{2}\sum_{i,k} k_{ik} x_i x_k$$ general

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) = \frac{\partial L}{\partial x_j} \qquad j = 1, 2, \cdots$$

$$\frac{d}{dt}\left(\sum_k m_{jk}\dot{x}_k\right) = -\sum_k k_{jk} x_k$$

$$\frac{1}{2}\underbrace{\sum_{i,k} m_{ik}\dot{x}_i\dot{x}_k}_{} = \frac{1}{2}\Big(m_{11}\dot{x}_1^2 + m_{12}\dot{x}_1\dot{x}_2 + m_{13}\dot{x}_1\dot{x}_3$$
$$+ m_{21}\dot{x}_2\dot{x}_1 + m_{22}\dot{x}_2^2 + m_{23}\dot{x}_2\dot{x}_3$$
$$+ m_{31}\dot{x}_3\dot{x}_1 + m_{32}\dot{x}_3\dot{x}_2 + m_{33}\dot{x}_3^2\Big)$$

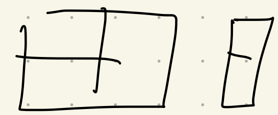
$$\frac{\partial L}{\partial \dot{x}_2} = \frac{1}{2}\left(m_{12}\dot{x}_1 + m_{21}\dot{x}_1 + 2m_{22}\dot{x}_2 + m_{23}\dot{x}_3 + m_{32}\dot{x}_3\right)$$
$$= \cancel{\frac{1}{2}}\left(\cancel{2}m_{21}\dot{x}_1 + \cancel{2}m_{22}\dot{x}_2 + \cancel{2}m_{23}\dot{x}_3\right)$$
$$= \sum_k m_{2k}\dot{x}_k$$

$$\sum_k m_{j,k}\,\ddot{x}_k = -\sum_k k_{j,k}\,x_k$$

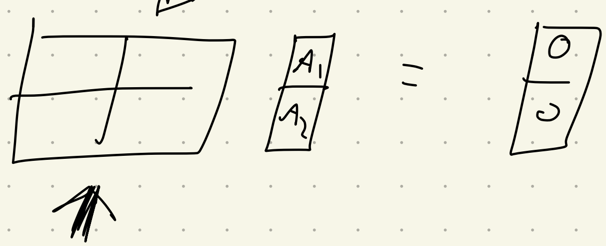
$$\underline{\underline{\text{Guess:}}}\quad x_k = A_k e^{i\omega t}$$

$$\rightarrow \ddot{x}_k = -\omega^2 A_k e^{i\omega t}$$

$$\sum_k -\omega^2 m_{j,k} A_k \cancel{e^{i\omega t}} = \underbrace{-\sum_k k_{j,k} A_k \cancel{e^{i\omega t}}}$$



$$\boxed{\sum_k \underbrace{\left(k_{j,k} - \omega^2 m_{j,k}\right)} A_k = 0}\ \text{— 0 vector}$$



If invertible

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{\underline{M}} \cdot \vec{v} = \lambda \vec{v}$$

$$\underbrace{\left(\underline{\underline{M}} - \lambda \underline{\underline{\mathbb{1}}} \right)}_{\det = 0} \cdot \vec{v} = 0$$

$$\underbrace{0 = \det\left(k_{j,k} - \omega^2 m_{j,k}\right)}_{\text{characteristic equation}}$$

polynomial equation $(\omega^2)^N$

Normal mode freqs, characteristic freqs, eigen freqs.

$$0 = \det\left(\begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\right)$$

ω_α^2 — α label the N different eigen freqs.

$$= \det \begin{bmatrix} k+K-\omega^2 m & -K \\ -K & k+K-\omega^2 m \end{bmatrix}$$

$$= \left((k+K)-\omega^2 m\right)^2 - K^2$$

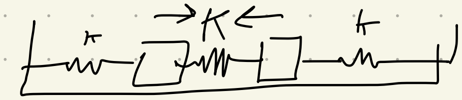
$$\left((k+K) - \omega^2 m \right)^2 = K^2$$

$$(k+K) - \omega^2 m = \pm K$$

$$\longrightarrow \quad \omega^2 = \frac{(k+K) \pm K}{m}$$

$$\boxed{\omega_+^2 = \frac{k+2K}{m} \,, \quad \omega_-^2 = \frac{k}{m}} \leftarrow \text{eigen freqs.}$$

solve for eigenvectors:



$$\omega_+^2 = \frac{k+2K}{m} \qquad V_+ = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

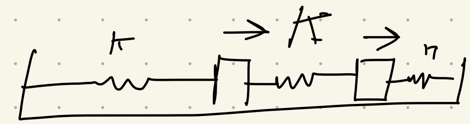
$$\omega_-^2 = \frac{k}{m} \qquad V_- = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

normal mode oscillation

$$\Delta_{k\alpha} = \begin{array}{c|cc} & + & - \\ \hline 1 & & \\ 2 & \multicolumn{2}{c}{\begin{bmatrix} V_+ & V_- \end{bmatrix}} \end{array}$$

$\downarrow k$ $\quad \rightarrow \alpha$

matrix of eigenvector



$$\boxed{\begin{aligned} x_k &= \mathrm{Re}\left[\sum_\alpha \Delta_{k\alpha}\, C_\alpha\, e^{i\omega_\alpha t}\right] \\ &= \sum_\alpha \Delta_{k\alpha}\, \Theta_\alpha \\ \Theta_\alpha &= \mathrm{Re}\left[C_\alpha e^{i\omega_\alpha t}\right] \end{aligned}}$$

complex const determined by ICs

normal coords

$$L = T - U$$

$$= \frac{1}{2}\sum_{i,k} m_{ik}\,\dot{x}_i\,\dot{x}_k - \frac{1}{2}\sum_{i,k} k_{ik}\,x_i\,x_k$$

$$= \frac{1}{2}\,\dot{x}^T m\,\dot{x} - \frac{1}{2}\,x^T k\,x$$

$$= \frac{1}{2}\,\dot{\theta}^T(\Delta^T m\,\Delta)\,\dot{\theta} - \frac{1}{2}\,\theta^T(\Delta^T k\,\Delta)\,\theta$$

$$= \frac{1}{2}\sum_\alpha \mathcal{M}_\alpha\,\dot{\theta}_\alpha^2 - \frac{1}{2}\sum_\alpha \mathcal{K}_\alpha\,\theta_\alpha^2$$

☆☆

where $\mathcal{M}$ = diagonal matrix $= \Delta^T m\,\Delta$

$\mathcal{K}$ = " $= \Delta^T k\,\Delta$

eigenvectors diagonalize both m_{ik} and k_{ik} with

$$\boxed{\mathcal{K}_\alpha = \mathcal{M}_\alpha\,\omega_\alpha^2}$$

CHECK: $\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$

$$= \frac{m}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$

$$= \frac{m}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \rightarrow \mathcal{M}_+ = m,\ \mathcal{M}_- = m$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} k+K & -K \\ -K & k+K \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} k+2K & k \\ -k-2K & k \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2k+4K & 0 \\ 0 & 2k \end{pmatrix}$$

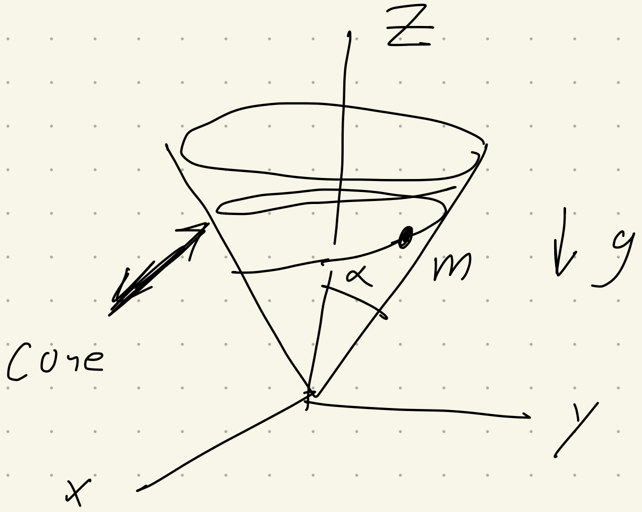
$$= \begin{pmatrix} k+2K & 0 \\ 0 & k \end{pmatrix}$$

$$= \begin{pmatrix} \omega_+^2 m & 0 \\ 0 & \omega_-^2 m \end{pmatrix} \longrightarrow \begin{aligned} \mathcal{K}_+ &= \omega_+^2 m \\ \mathcal{K}_- &= \omega_-^2 m \end{aligned}$$

Thus,
$$L = \frac{1}{2} m\left(\dot{\theta}_+^2 + \dot{\theta}_-^2\right) - \frac{1}{2}\left(m\omega_+^2 \theta_+^2 + m\omega_-^2 \theta_-^2 \right)$$
$$= \left(\frac{1}{2} m \dot{\theta}_+^2 - \frac{1}{2} m \omega_+^2 \theta_+^2\right) + \left(\frac{1}{2} m \dot{\theta}_-^2 - \frac{1}{2} m \omega_-^2 \theta_-^2\right)$$

Quiz #4:

Calculate freq of small oscillations around the circular orbit solution $r = r_0$



α fixed

(Sph. polar coords)

a) Lagrangian ?

b) Determine r_0

(relationship between r_0 and M_z)

c) Determine ω_r

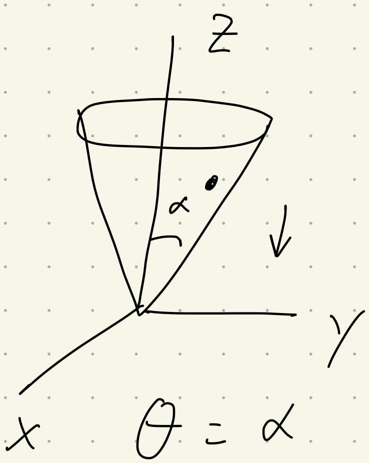
q~~uiz~~

q4_name.pdf

joseph.d.romano@ttu.edu

Lec #22 : Thurs Nov 4th

- Next 5 lectures rigid body motion 31-36, 38
- EXAM 2 - Tues 11/23 (not Thur 11/18)



$\theta = \alpha$

(sph. polar)

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \sin^2\alpha\, \dot{\phi}^2)$$

$$U = mgz = mgr\cos\alpha$$

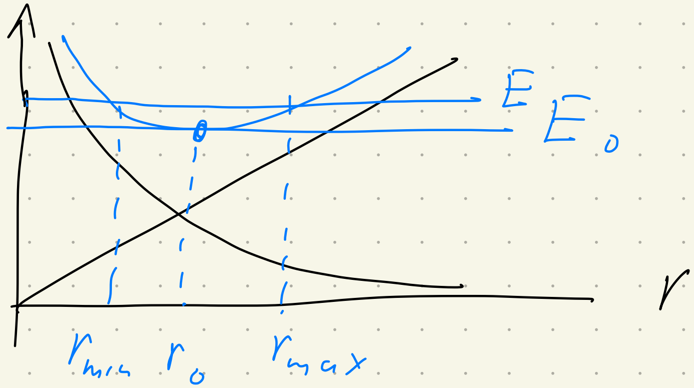
$$L = T - U$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \sin^2\alpha\, \dot{\phi}^2) - mgr\cos\alpha$$

$$M_z = mr^2 \sin^2\alpha\, \dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{mr^2\sin^2\alpha}$$

$$E = T + U = \sum_i \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right)$$

$$= \frac{1}{2} m\dot{r}^2 + \underbrace{\frac{M_z^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha}_{U_{eff}(r)}$$



$$\omega_r = \sqrt{\frac{K}{m}}$$

$$K = \left.\frac{d^2 U_{eff}}{dr^2}\right|_{r=r_0}$$

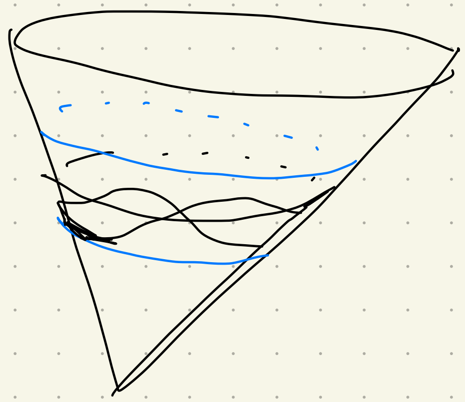
$$= \frac{3\, mg \cos\alpha}{r_0}$$

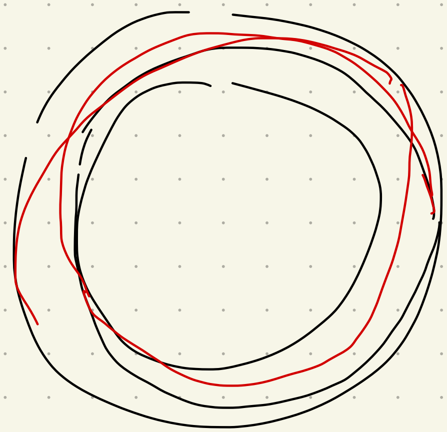
$$\underbrace{0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0}}_{M_z^2 = m^2 g r_0^3 \sin^2\alpha \cos\alpha}$$

$$\rightarrow \quad \omega_r = \sqrt{\frac{K}{m}} = \sqrt{\frac{3g\cos\alpha}{r_0}}$$

$$\omega_\phi = \left.\dot\phi\right|_{r=r_0} = \sqrt{\frac{g\cos\alpha}{r_0 \sin^2\alpha}}$$

don't agree (not closed)

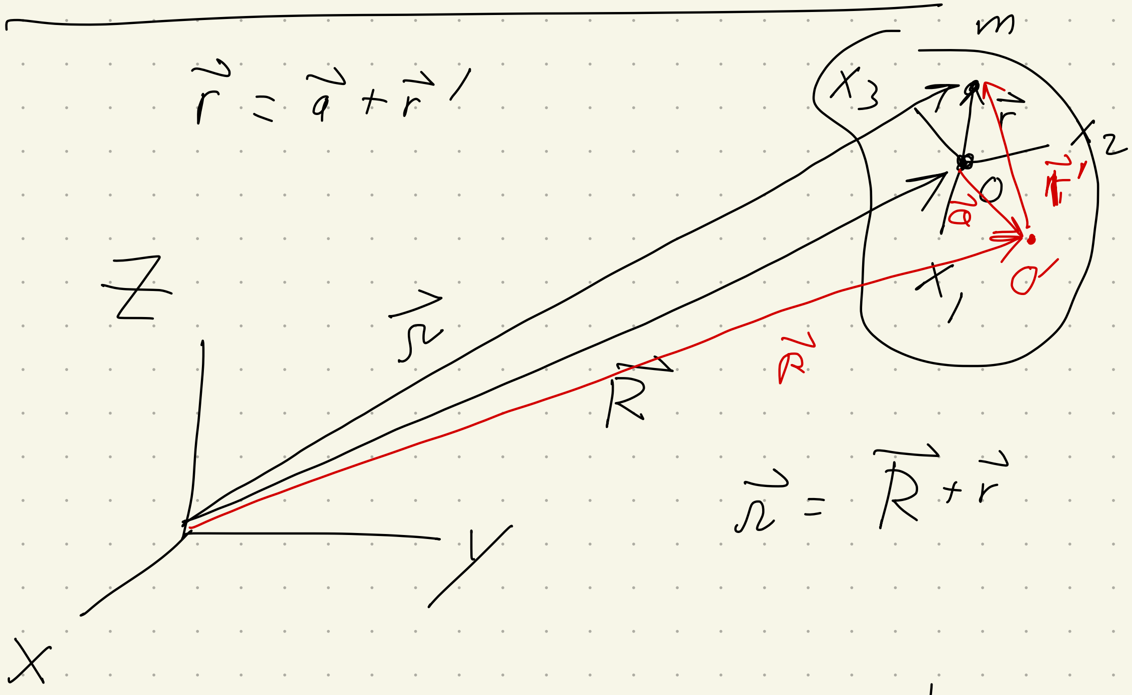




Rigid body motion:

$$x_i = (x_1, x_2, x_3) = (x, y, z)$$

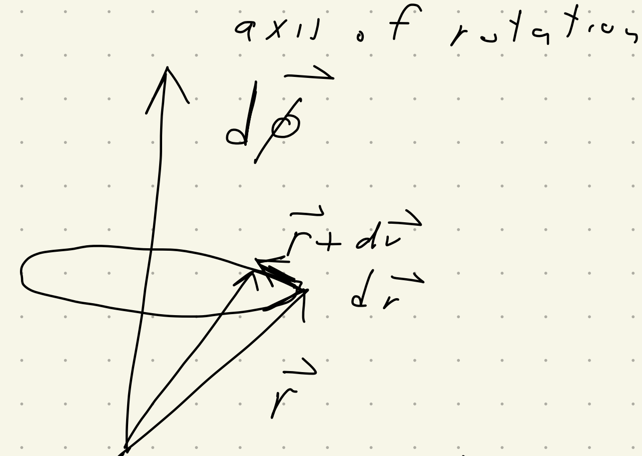
$$\vec{r} = \vec{a} + \vec{r}\,'$$



fixed inertial frame

$$\vec{\mathcal{R}} = \vec{R} + \vec{r}$$

$$\boxed{\vec{v} = \frac{d\vec{\mathcal{R}}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}}{dt} = \vec{V} + \vec{\Omega} \times \vec{r}}$$



$$|d\vec{\phi}| = d\phi$$

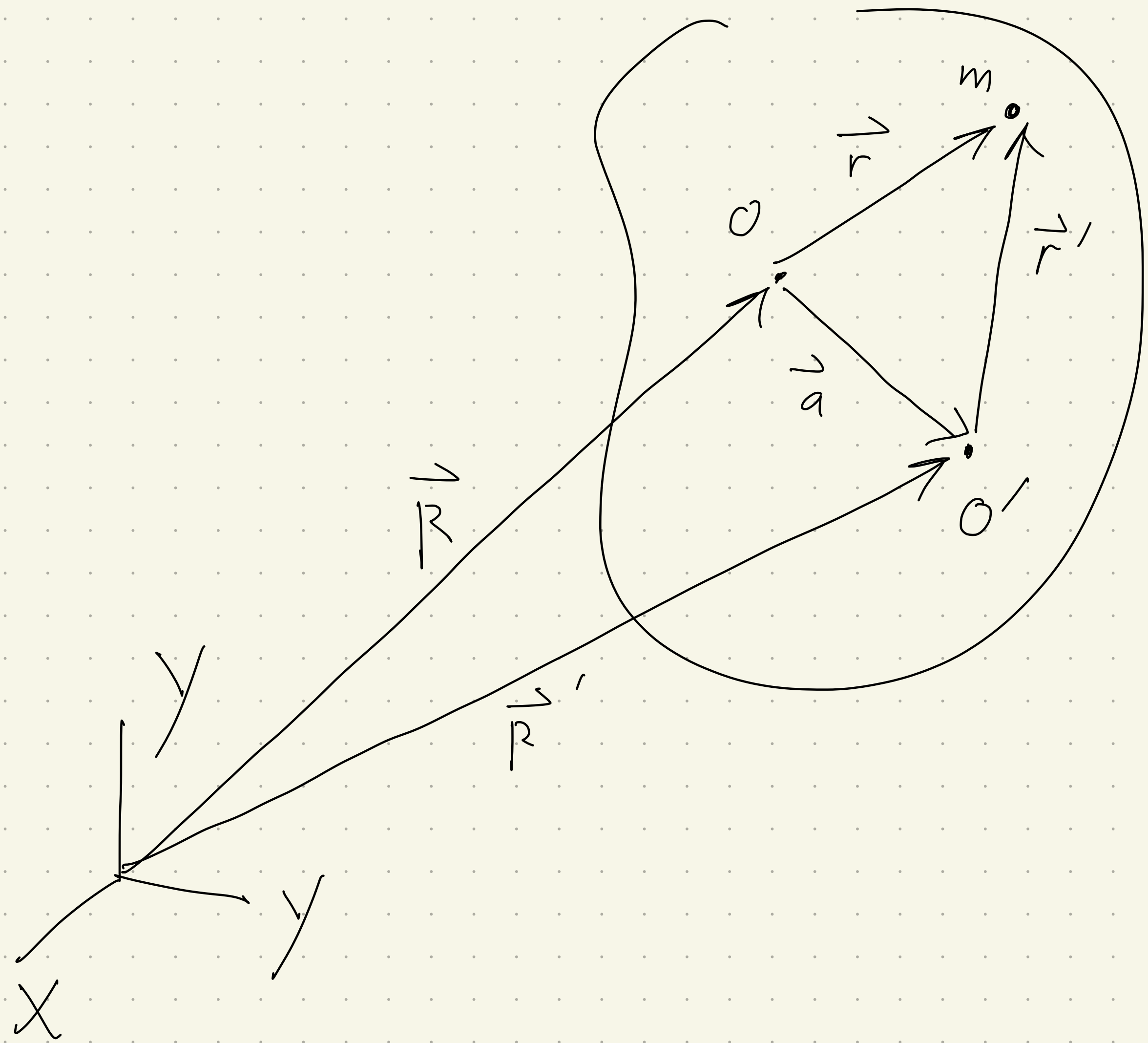
$$d\vec{r} = d\vec{\phi} \times \vec{r}$$

most convenient if O is at com of RB.

$$\frac{d\vec{r}}{dt} = \frac{d\vec{\phi}}{dt} \times \vec{r} = \vec{\Omega} \times \vec{r}$$

$$\vec{r} = \vec{a} + \vec{r}\,' \longrightarrow \vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$$

$$\vec{v} = \vec{V}\,' + \vec{\Omega}\,' \times \vec{r}\,'$$



$$\vec{v} = \vec{V} + \vec{\Omega} \times (\vec{a} + \vec{r}\,')$$

$$= \vec{V} + \vec{\Omega} \times \vec{a} + \vec{\Omega} \times \vec{r}\,'$$

$$\vec{V}\,' = \vec{V} + \vec{\Omega} \times \vec{a}$$

$$\vec{\Omega}\,' = \vec{\Omega}$$

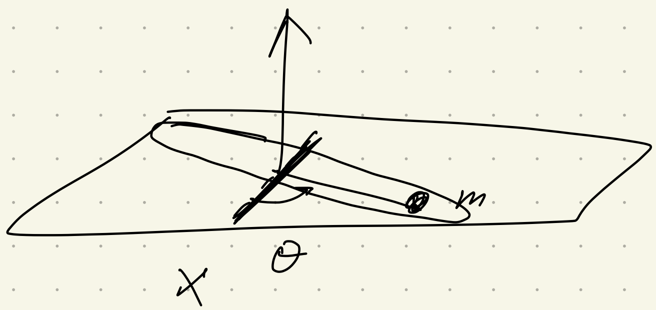
2-d rotational motion:

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt}} \qquad \vec{p} = m\vec{v}$$

$$\theta, \quad \frac{d\theta}{dt} = \omega, \quad T_{rot} = \frac{1}{2} I \omega^2$$

moment of inertia

axis of rotation (fixed



3-d

$$L = I\omega$$

I_{ij} : inertia tensor

(3x3 symmetric matrix)

$$M_i = \sum_k I_{ik} \Omega_k$$

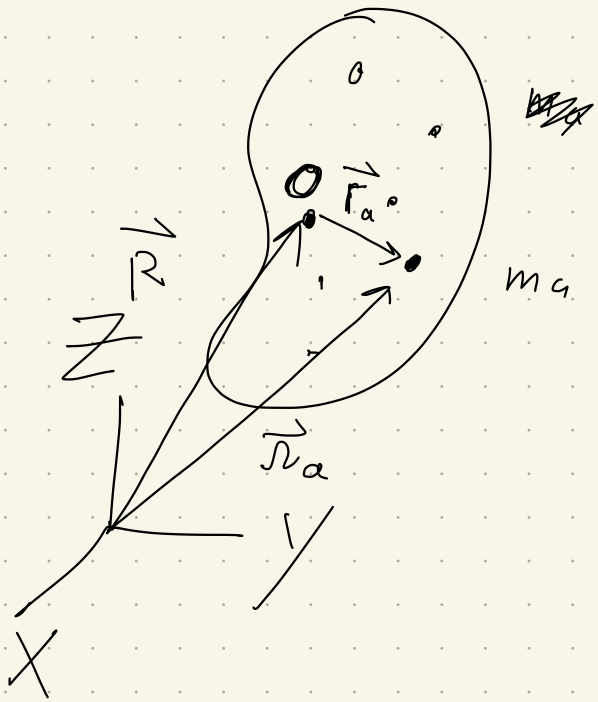
$$\vec{M} = \underline{\underline{I}}\, \vec{\Omega}$$

$$\boxed{\vec{N} = \frac{d\vec{M}}{dt}}$$

KE :

m_a —— a : labels the mass point in the rigid body

$$T = \sum_a \frac{1}{2} m_a |\vec{v}_a|^2$$



$$\vec{v}_a = \vec{V} + \vec{\Omega} \times \vec{r}_a$$

$$|\vec{v}_a|^2 = (\vec{V} + \vec{\Omega} \times \vec{r}_a) \cdot (\vec{V} + \vec{\Omega} \times \vec{r}_a)$$

$$= |\vec{V}|^2 + 2\vec{V} \cdot (\vec{\Omega} \times \vec{r}_a)$$

$$+ (\vec{\Omega} \times \vec{r}_a) \cdot (\vec{\Omega} \times \vec{r}_a)$$

1^{st} term $\quad \sum_a \frac{1}{2} m_a |\vec{V}|^2 = \frac{1}{2} \mu |\vec{V}|^2$

$\mu = \sum_a m_a$ = total mass

$$2^{nd}\text{ term} = \sum_a \cancel{\tfrac{1}{2}}\,\cancel{2}\, m_a \vec{V}\cdot(\vec{\Omega}\times\vec{r}_a)$$

$$= \left(\sum_a m_a \vec{r}_a\right)\cdot(\vec{V}\times\vec{\Omega}) = 0 \left(\begin{array}{l}\text{if}\\ \text{origin}\\ \text{is at}\\ \text{COM}\end{array}\right)$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$3^{rd}\text{ term} = (\vec{\Omega}\times\vec{r}_a)\cdot(\vec{\Omega}\times\vec{r}_a)$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$= \vec{\Omega}\cdot(\vec{r}_a\times(\vec{\Omega}\times\vec{r}_a))$$

$$= \vec{\Omega}\cdot(\vec{\Omega}\,|\vec{r}_a|^2 - \vec{r}_a(\vec{r}_a\cdot\vec{\Omega}))$$

$$= \vec{\Omega}\cdot\vec{\Omega}\; r_a^2 - (\vec{\Omega}\cdot\vec{r}_a)(\vec{\Omega}\cdot\vec{r}_a)$$

$$T = \frac{1}{2} M \left|\vec{V}\right|^2 + \sum_a \frac{1}{2} m_a \left(\underbrace{\vec{\Omega}\cdot\vec{\Omega}}_{\sum_i \Omega_i^2 = \sum_{i,j} \Omega_i \Omega_j \delta_{ij}} r_a^2 - \underbrace{\left(\vec{\Omega}\cdot\vec{r_a}\right)\left(\vec{\Omega}\cdot\vec{r_a}\right)}_{\sum_i \Omega_i x_{ai} \sum_j \Omega_j x_{aj}} \right)$$

$$= \frac{1}{2} M \left|\vec{V}\right|^2 + \sum_a \frac{1}{2} m_a \sum_{i,j} \left(r_a^2 \delta_{ij} - x_{ai} x_{aj} \right) \Omega_i \Omega_j$$

$$= \frac{1}{2} M \left|\vec{V}\right|^2 + \frac{1}{2} \sum_{i,j} \underbrace{\left(\sum_a m_a \left(r_a^2 \delta_{ij} - x_{ai} x_{aj} \right) \right)}_{I_{ij}} \Omega_i \Omega_j$$

$$= \boxed{\frac{1}{2} M \left|\vec{V}\right|^2 + \frac{1}{2} \sum_{i,j} I_{ij} \Omega_i \Omega_j}$$