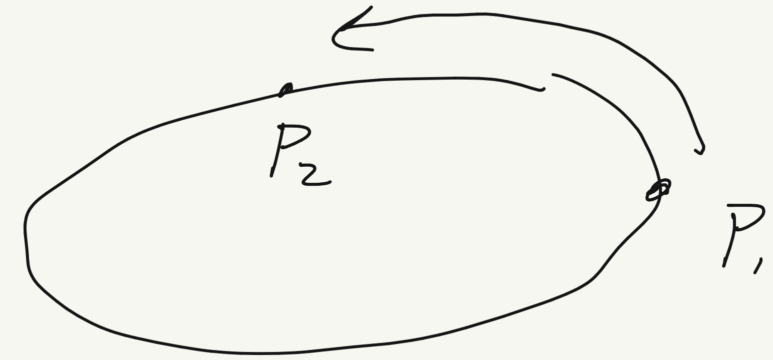
Lecture #1: Aug 24th

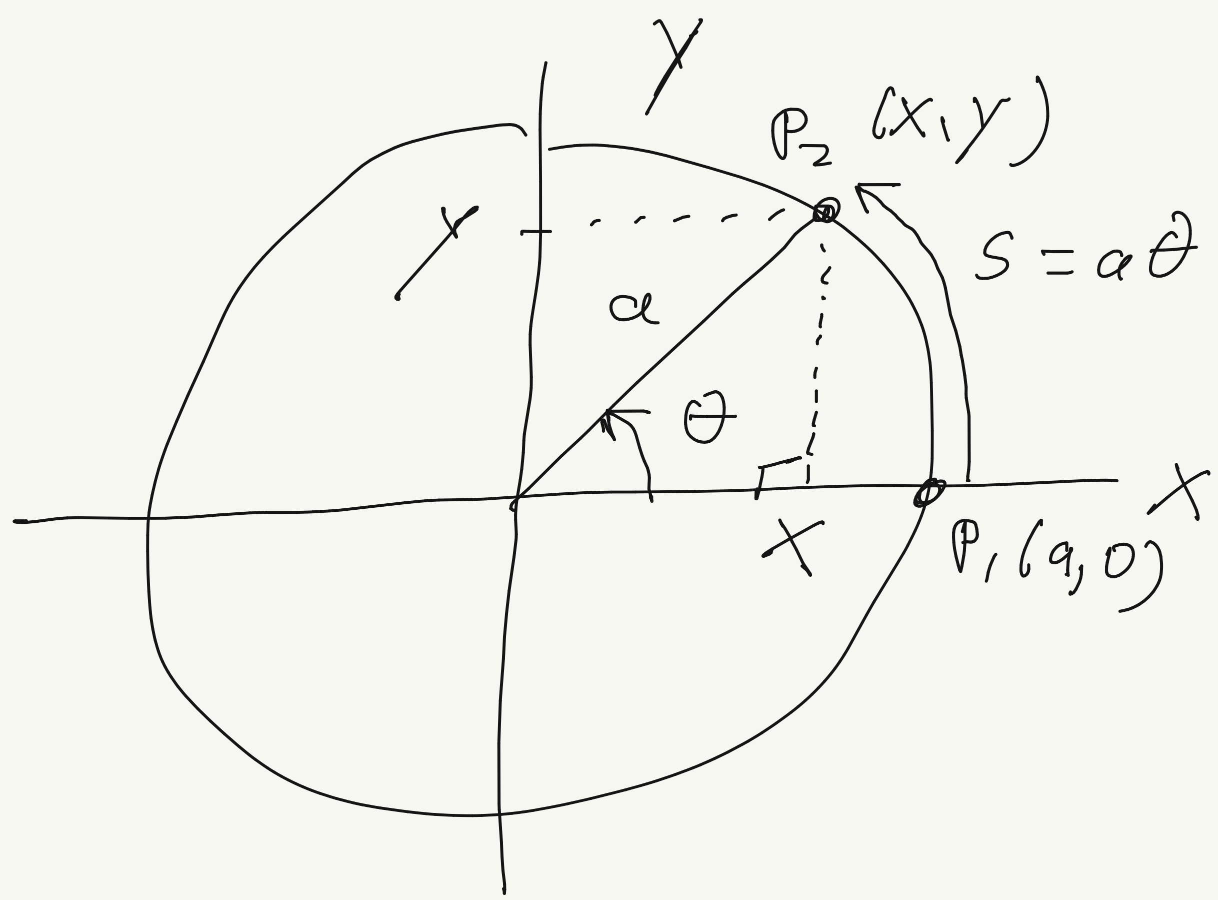
Elliptic functions / integrals:

i) circumference of an ellipse

ii) period of a simple pendulum beyond the small-angle approximation



Circular functions: sines, cosines,



a = radius

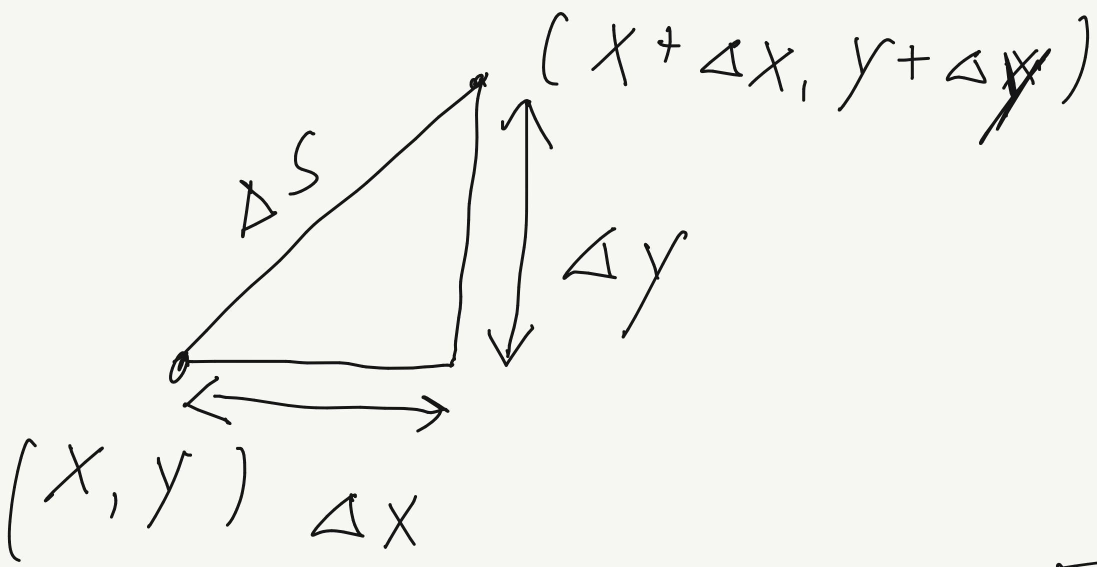
$$x^2 + y^2 = a^2$$

$$\theta = \frac{s}{a}$$

$$= \frac{1}{a}\int_{P_1}^{P_2} \sqrt{dx^2 + dy^2}$$

$$= \int d\theta$$

$$ds^2 = dx^2 + dy^2$$



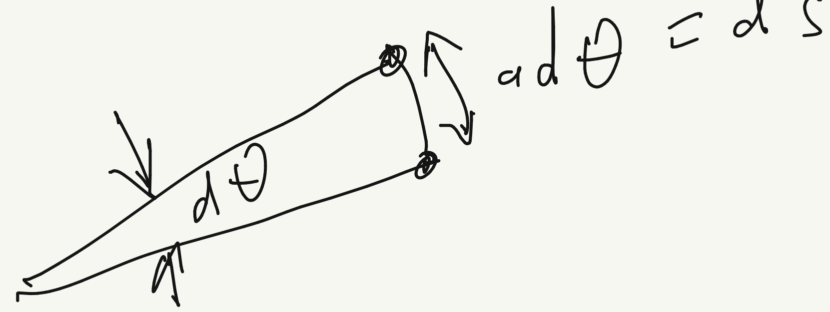
$$\sin\theta \equiv y/a$$
$$\cos\theta \equiv x/a$$
definitions

$$x = a\cos\theta$$
$$y = a\sin\theta$$

$$\boxed{x^2 + y^2 = a^2} \rightarrow \cancel{a^2}\cos^2\theta + \cancel{a^2}\sin^2\theta = \cancel{a^2}$$
$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

$$\boxed{\frac{d}{d\theta}\sin\theta = \cos\theta}$$

Proof:
$$\frac{d}{d\theta}\sin\theta = \frac{d}{d\theta}\left(\frac{y}{a}\right) = \frac{1}{a}\frac{dy}{d\theta} = \frac{dy}{\sqrt{dx^2+dy^2}} = \frac{1}{\frac{\sqrt{dx^2+dy^2}}{dy}} = \frac{1}{\sqrt{\left(\frac{dx}{dy}\right)^2+1}}$$



$$a\,d\theta = \sqrt{dx^2+dy^2}$$

$$x^2+y^2=a^2 \rightarrow 2x\,dx+2y\,dy=0$$

$$\frac{dx}{dy}=-\frac{y}{x}$$

$$\frac{d}{d\theta}\sin\theta=\frac{1}{\sqrt{\left(\frac{-y}{x}\right)^2+1}}=\frac{1}{\sqrt{\frac{y^2}{x^2}+1}}=\frac{x}{\sqrt{x^2+y^2}}=\frac{x}{a}=\cos\theta$$

$$\boxed{\frac{d}{d\theta}\sin\theta=\cos\theta}$$

Similarly $\boxed{\frac{d\cos\theta}{d\theta}=-\sin\theta}$

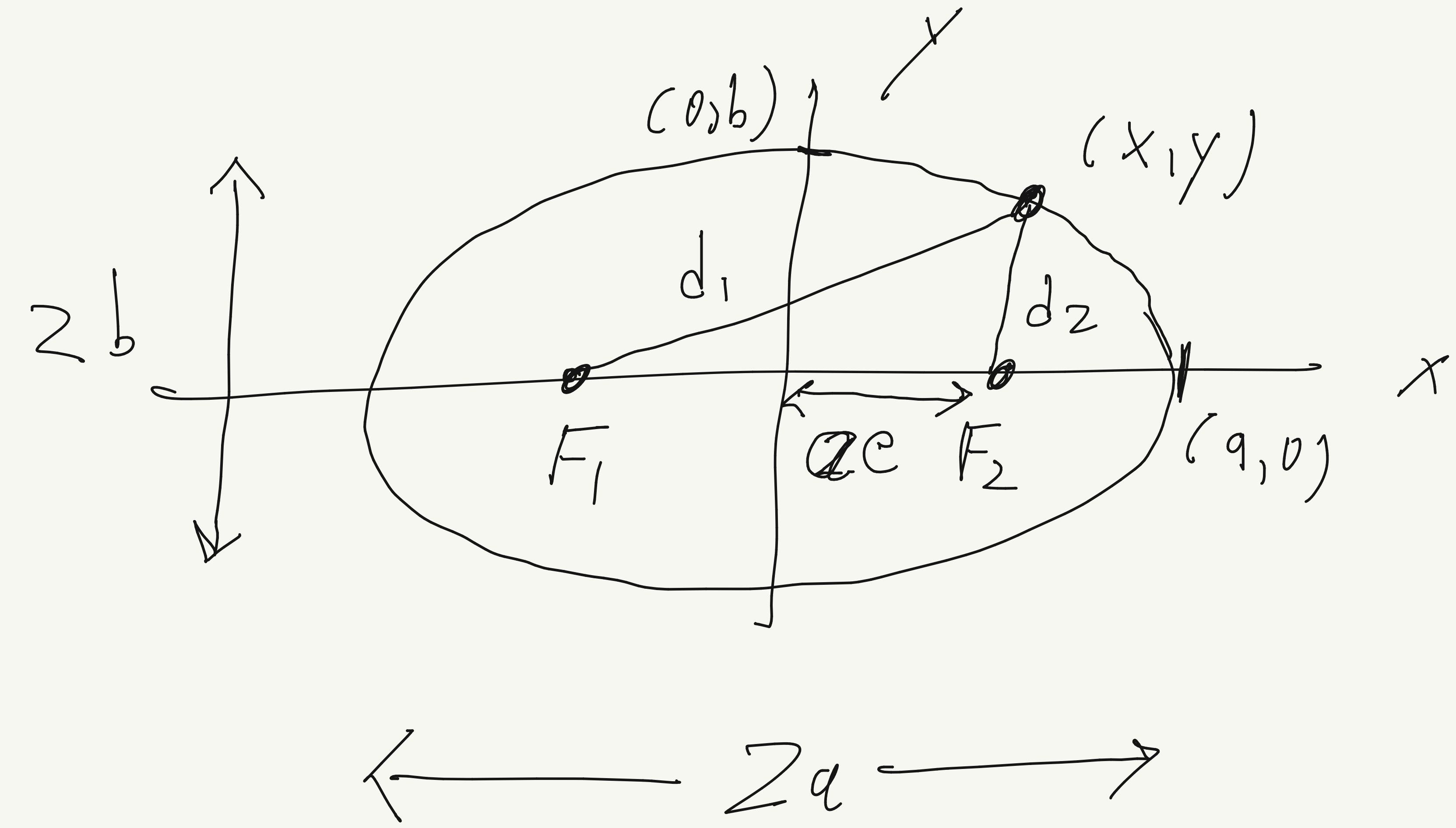
$$\int\frac{d(\sin\theta)}{\cos\theta}=\int d\theta=\theta+\text{const}$$

$$t=\sin\theta$$

$$\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-t^2}$$

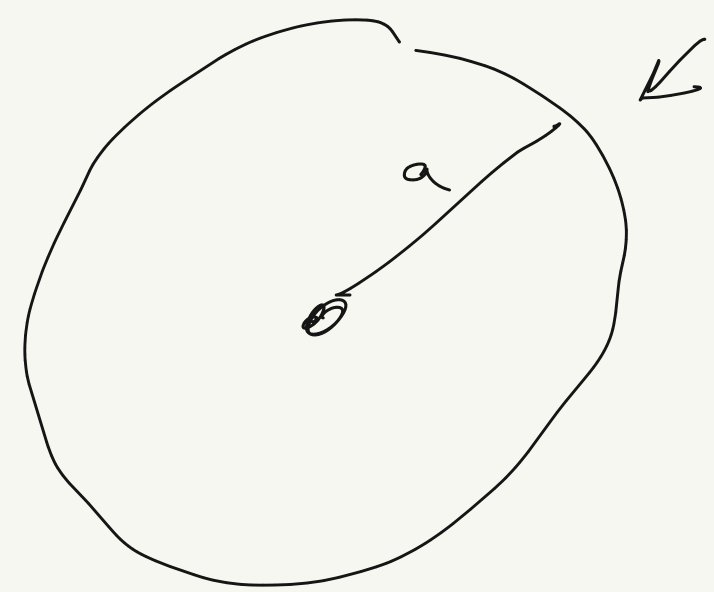
$$\boxed{\int\frac{dt}{\sqrt{1-t^2}}=\theta+\text{const}=\sin^{-1}t+\text{const}}$$

Lec #2: Aug 26th



$$d_1 + d_2 = 2a$$

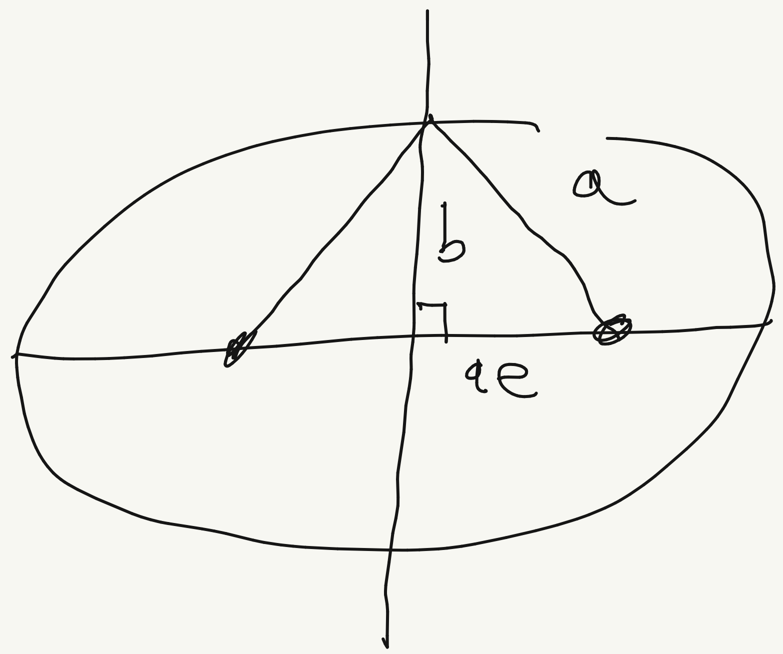
$$x^2 + y^2 = a^2$$



$$e \stackrel{?}{\neq} \frac{b}{a}$$

$$e \neq 1 - \frac{b}{a}$$

$$e^2 = \frac{b^2}{1-a^2}$$



$$\boxed{e = \sqrt{1 - \left(\frac{b}{a}\right)^2}}$$

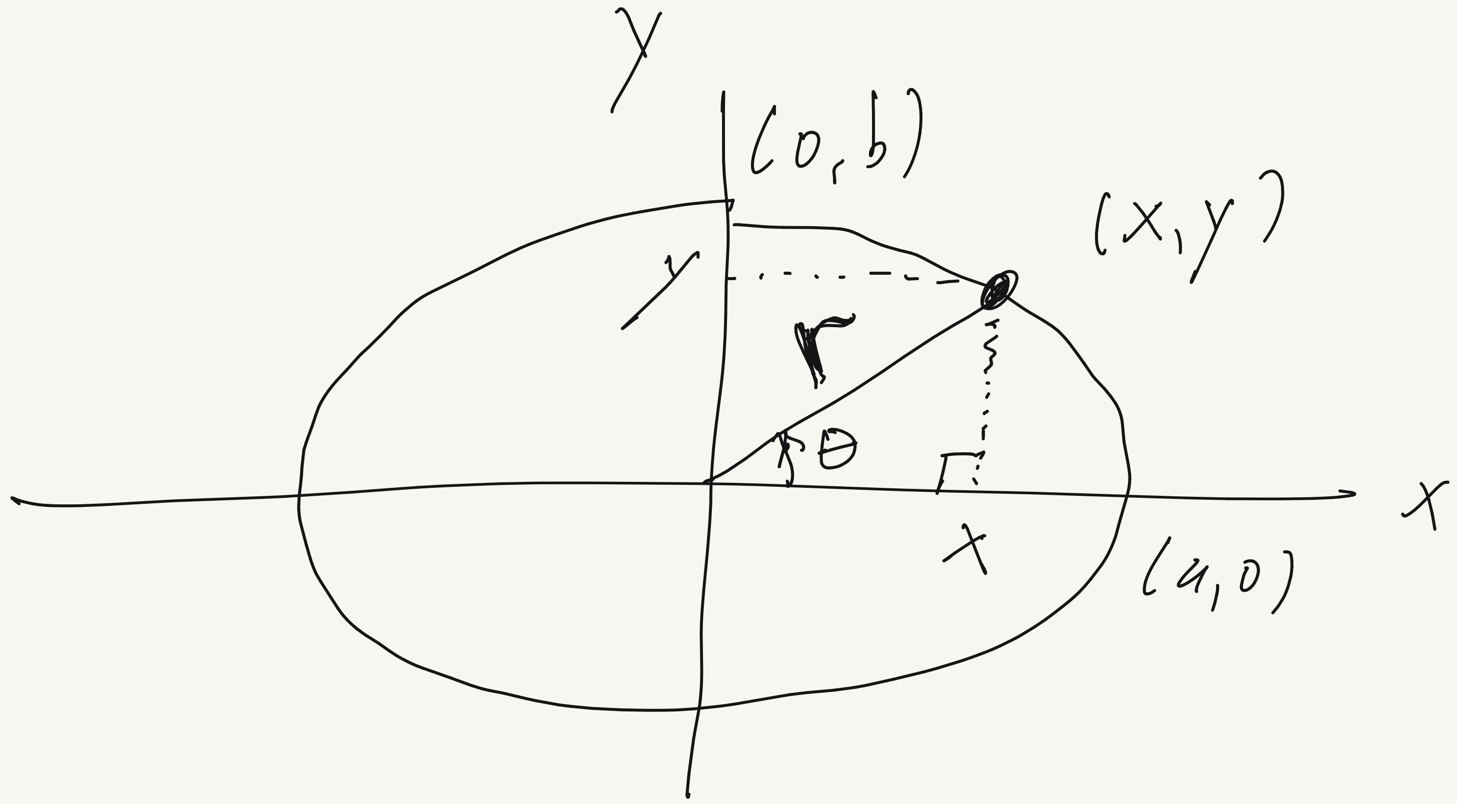
Pf:

$$a^2 = b^2 + (ae)^2$$

$$a^2(1-e^2) = b^2$$

$$1 - e^2 = \left(\frac{b}{a}\right)^2$$

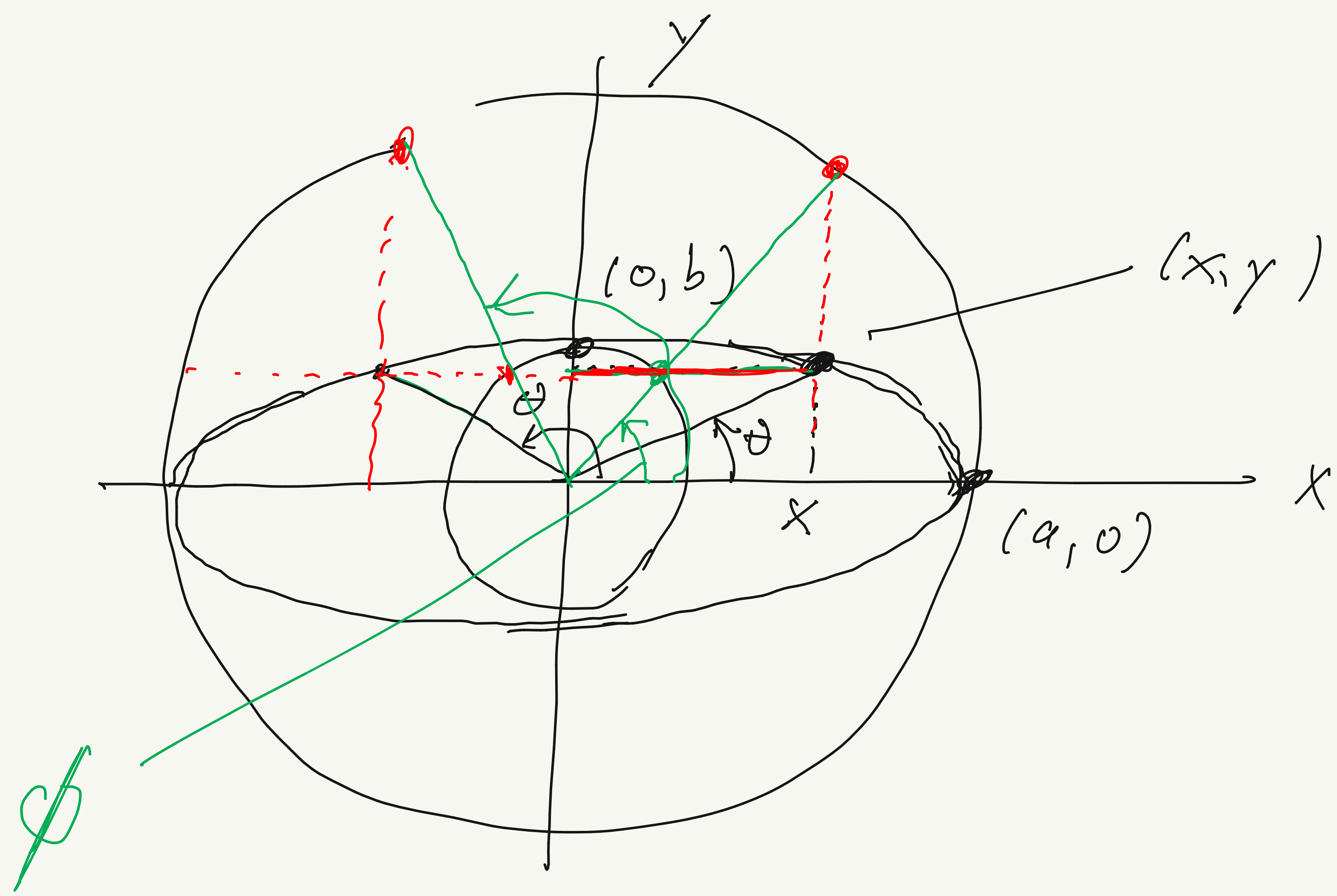
$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$



$$x = r\cos\theta$$
$$y = r\sin\theta$$

r changes

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \longleftarrow \begin{cases} x = a\cos\phi \\ y = b\sin\phi \end{cases}$$



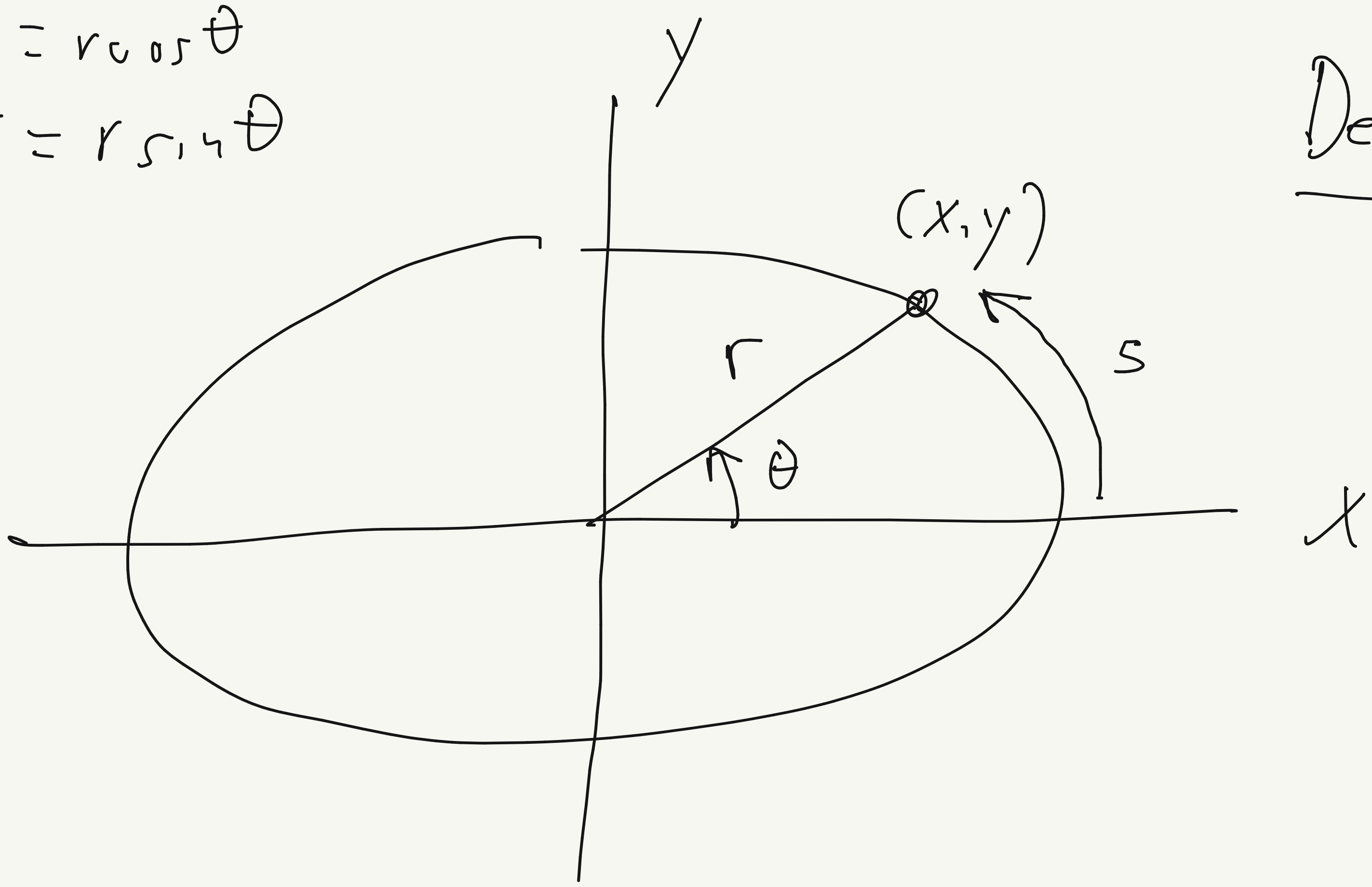
$0 < e < 1$

$e = 0$ circle

$e = 1$ parabola

$e > 1$ hyperbola

$x = r\cos\theta$
$y = r\sin\theta$



Define:

$$\boxed{\begin{aligned} \mathrm{cn}(u;k) &\equiv x/a \\ \mathrm{sn}(u;k) &\equiv y/b \\ \mathrm{dn}(u;k) &= \frac{r}{a} \end{aligned}}$$

$$u \equiv \frac{1}{b}\int_0^\theta r\,d\theta$$

not θ, not arc length

$$k = e, \qquad \underbrace{0 < k < 1}_{\text{modulus}}$$

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$dx = dr\cos\theta - r\sin\theta\,d\theta$$
$$dy = dr\sin\theta + r\cos\theta\,d\theta$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\theta^2}$$

$$bu = \int_0^\theta r\,d\theta < s$$

$\mathrm{cn}(u)$	pendulum
$\mathrm{sn}(u)$	$k = \sin\left(\frac{\phi_0}{2}\right)$

Property:

$$\boxed{cn^2 u + sn^2 u = 1} \quad \leftarrow \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\leftarrow \quad x^2 + y^2 = r^2$$

$$\begin{aligned} dn^2 u &= cn^2 u + \left(\frac{b}{a}\right)^2 sn^2 u \\ &= 1 - sn^2 u + \left(\frac{b}{a}\right)^2 sn^2 u \\ &= 1 - sn^2 u \left(1 - \left(\frac{b}{a}\right)^2\right) \\ &= 1 - k^2 sn^2 u \end{aligned}$$

$$\boxed{dn^2 u + k^2 sn^2 u = 1}$$

$$\frac{d}{du}\,\mathrm{sn}\,u = \frac{1}{b}\frac{dy}{du}$$

$$= \frac{dy}{r\,d\theta}$$

$$u = \frac{1}{b}\int_0^\theta r\,d\theta$$

$$du = \frac{r\,d\theta}{b} \rightarrow b\,du = r\,d\theta$$

$x = r\cos\theta\,,\quad y = r\sin\theta$

$dx = dr\cos\theta - r\sin\theta\,d\theta \longrightarrow \sin\theta\,dx = \sin\theta\cos\theta\,dr - r\sin^2\theta\,d\theta$

$dy = dr\sin\theta + r\cos\theta\,d\theta \longrightarrow -\cos\theta\,dy = -\cos\theta\sin\theta\,dr - r\cos^2\theta\,d\theta$

add

$\sin\theta\,dx - \cos\theta\,dy = -r\,d\theta$

$\frac{y}{r}dx - \frac{x}{r}dy = -r\,d\theta$

$$\rightarrow \boxed{r\,d\theta = -\frac{y}{r}dx + \frac{x}{r}dy}$$

$$\frac{d}{du}\,\mathrm{sn}\,u = \frac{dy}{-\frac{y}{r}dx + \frac{x}{r}dy} = \frac{r}{-y\frac{dx}{dy} + x}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad \cancel{2}\,\frac{x\,dx}{a^2} + \cancel{2}\,\frac{y\,dy}{b^2} = 0$$

$$\frac{dx}{dy} = -\frac{y}{x}\,\frac{a^2}{b^2}$$

$$\frac{d\,\mathrm{sn}\,u}{du} = \frac{r}{-y\left(\frac{-y}{x}\right)\frac{a^2}{b^2} + x} = \frac{r\,x}{y^2\left(\frac{a}{b}\right)^2 + x^2}$$

$$= \frac{r}{a}\,\frac{x}{a}\left(\frac{1}{\left(\frac{y}{b}\right)^2 + \left(\frac{x}{a}\right)^2}\right)$$

$$= \frac{\mathrm{dn}\,u \cdot \mathrm{cn}\,u}{1}$$

$$\boxed{\frac{d}{du}\,\mathrm{sn}\,u = \mathrm{cn}\,u \cdot \mathrm{dn}\,u}$$

$$\frac{d}{du}\,\boxed{cn}\,u = -\,sn\,u \cdot dn\,u$$

$$\frac{d}{du}\,dn\,u = -k^2\,sn\,u \cdot cn\,u$$

$$\frac{d}{du}\,sn\,u = cn\,u \cdot dn\,u$$

$$sn^2 u + cn^2 u = 1$$

$$dn^2 u + k^2 sn^2 u = 1$$

$$\int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta = \theta$$

$t = \sin\theta$

$\theta = \sin^{-1} t$

$$\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t$$

Integrate!

$$\int \frac{d(sn\,u)}{cn\,u \cdot dn\,u} = \int du = u + \text{const}$$

$t = sn\,u$

$$\boxed{\int \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}} = \underline{sn^{-1}(t;k)} + \text{const}}$$

$$\underbrace{F(\phi, k)}_{\text{incomplete elliptic integral of } 1^{st} \text{ kind}} \equiv \int_0^{\sin\phi} \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}}$$

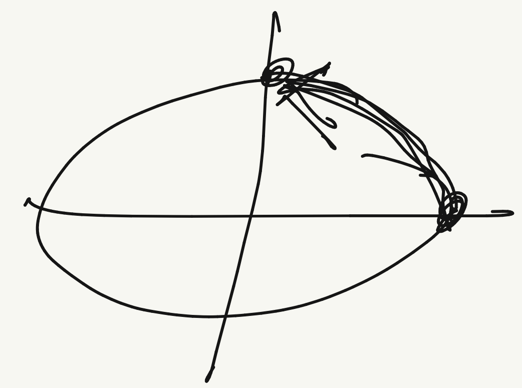
(angular dependence, period of a simple pendulum)

$$\mathrm{sn}\, u = \sin\phi \qquad \frac{y}{b} = \sin\phi$$

$$\mathrm{cn}\, u = \cos\phi \qquad \frac{x}{a} = \cos\phi$$

$$\underbrace{E(\phi, k)}_{\text{incomplete elliptic integral of } 2^{nd} \text{ kind}} \equiv \int_0^{\sin\phi} \frac{dt\,\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}$$

(arclength along ellipse)



$$\phi = \pi/2$$

$$\left.\begin{aligned} \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} &= K(k) \\ \int_0^1 \frac{dt}{\sqrt{1-t^2}}\sqrt{1-k^2t^2} &= E(k) \end{aligned}\right\} \text{complete elliptic integrals of } 1^{st} \text{ and } 2^{nd} \text{ kind}$$