Lecture #14: Thu 10/7

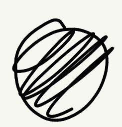
Next two lecture — collisions

Following 3 lectures — Scattering

1.) Spontaneous disintegration of a single mass $M = m_1 + m_2$ into two particles m_1 and m_2

closed system $\longrightarrow$ COM Frame (conservation of total linear momentum)

M



(before)

$\vec{p} = 0$

(after)

m_1 $\quad \vec{p}_1 = \vec{p}_0$

m_2

$\vec{p}_2 = -\vec{p}_0$

$\vec{p}_1 + \vec{p}_2 = 0 \rightarrow \vec{p}_2 = -\vec{p}_1$

Cons. of energy:

$$\Rightarrow \quad E_i = E_{1i} + \underbrace{T_{10}}_{\text{COM}} + E_{2i} + T_{20}$$

E_i : internal energy of M

$$= E_{1i} + E_{2i} + \frac{p_0^2}{2m_1} + \frac{p_0^2}{2m_2}$$

$$= E_{1i} + E_{2i} + \frac{p_0^2}{2}\underbrace{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}_{\frac{m_1+m_2}{m_1 m_2} = \frac{1}{m}}$$

$$\underbrace{E_i - E_{1i} - E_{2i}}_{\mathcal{E}} = \frac{p_0^2}{2m}$$

$\mathcal{E}$: disintegration energy

$$\rightarrow \quad p_0 = \sqrt{2m\mathcal{E}} \quad \rightarrow \quad \boxed{v_{10} = \frac{p_0}{m_1}, \quad v_{20} = \frac{p_0}{m_2}}$$

Spontaneous disintegration into N masses $m_1, m_2, \dots, m_N$

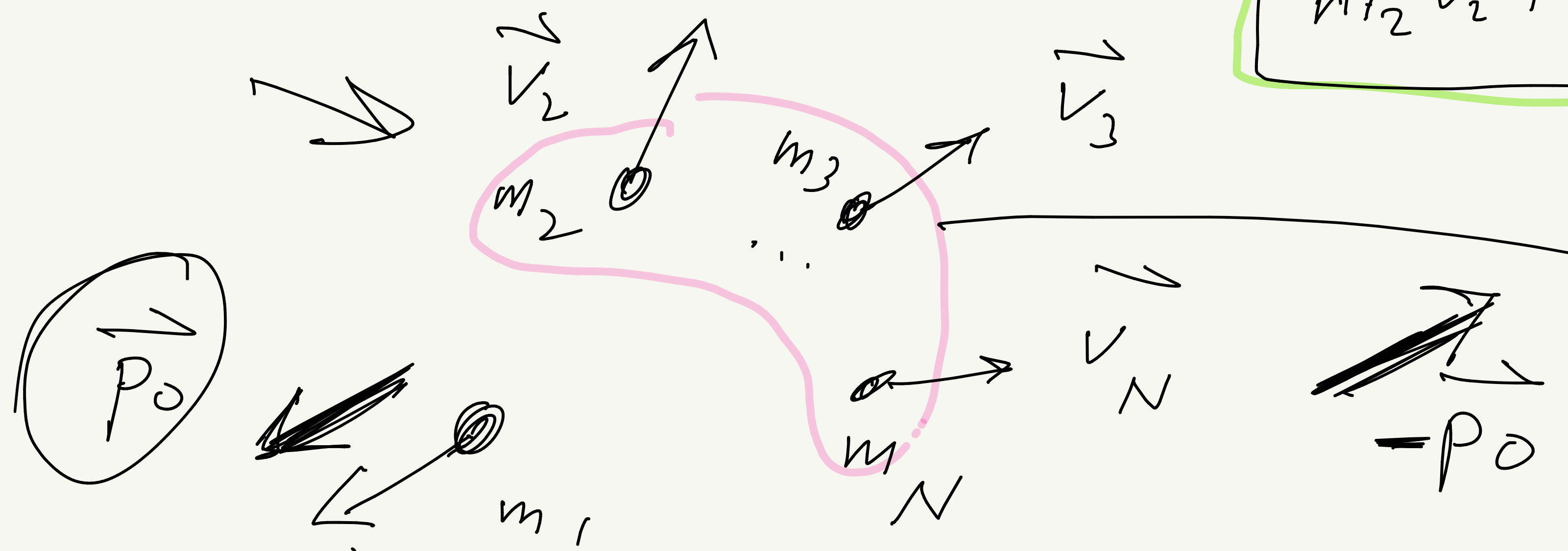
$$\boxed{m_2\vec{v}_2 + m_3\vec{v}_3 + \dots = -\vec{p}_0}$$

COM frame:

E_i M

(before)

$\vec{p} = 0$



$E_{1,i}$

(after)

represent by their COM

$M - m_1$
$= m_2 + m_3 + \dots m_N$

E_i'

Upper limit on potential of a single particle m_1

cons. of energy

$$\Rightarrow E_i = \frac{p_0^2}{2m_1} + E_{1,i} + \frac{p_0^2}{2(M-m_1)} + E_i'$$

$$= \underline{E_{1,i} + E_i'} + \frac{p_0^2}{2}\left(\frac{1}{m_1} + \frac{1}{M-m_1}\right)$$

$$\frac{M}{m_1(M-m_1)}$$

$$E_i - E_{1i} - E_i' = \frac{p_0^2}{2m_1}\left(\frac{M}{M-m_1}\right)$$

$E_{2i}' + E_{3i}' + \cdots$

for maximum KE for m_1

$$\rightarrow \underbrace{\frac{p_0^2}{2m_1}}_{T_{01}} = (E_i - E_{1i} - E_i')\left(\frac{M-m_1}{M}\right)$$

just a number

T_{01} maximum when E_i' is minimum

$$E_i' + \frac{p_0^2}{2(M-m_1)} = E_{2i} + \frac{1}{2}m_2|\vec{v}_2|^2 + E_{3i} + \frac{1}{2}m_3|\vec{v}_3|^2 + \cdots$$

$$= \underbrace{E_{2i} + E_{3i} + \cdots}\ + \underbrace{\frac{1}{2}m_2|\vec{v}_2|^2 + \frac{1}{2}m_3|\vec{v}_3|^2 + \cdots}$$

$$-\vec{p}_0 = m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots \rightarrow p_0^2 = |m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots|^2$$

$$E_i' = \underbrace{E_{2i} + E_{3i}}_{= 0 \text{ possible}} + \cdots + \frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \cdots - \frac{1}{2(m_2 + m_3 + \cdots)} \| m_2 \vec{v}_2 + m_3 \vec{v}_3 + \cdots \|^2$$

$$\| m_2 \vec{v}_2 + m_3 \vec{v}_3 + \cdots \|^2 = m_2^2 |\vec{v}_2|^2 + m_3^2 |\vec{v}_3|^2 + \cdots + 2 m_2 m_3 \vec{v}_2 \cdot \vec{v}_3 + \cdots$$

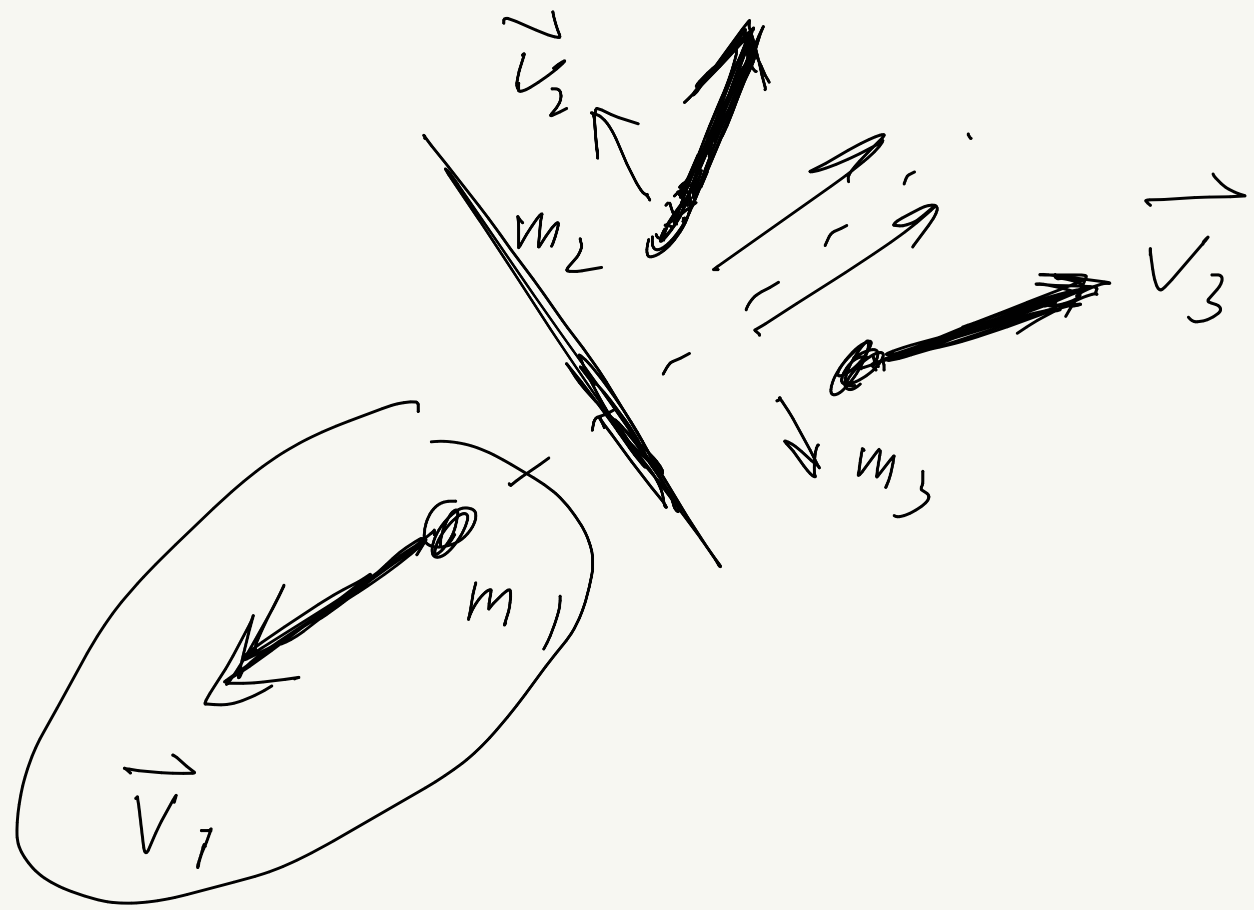
Suppose: $\vec{v}_2 = \vec{v}_3 = \cdots \equiv \vec{v}$

$$\frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \cdots = \frac{1}{2}(m_2 + m_3 + \cdots) |\vec{v}|^2$$

$$-\frac{1}{2(m_2 + m_3 + \cdots)} (m_2 + m_3 + \cdots)^2 |\vec{v}|^2 = -\frac{1}{2}(m_2 + m_3 + \cdots) |\vec{v}|^2$$

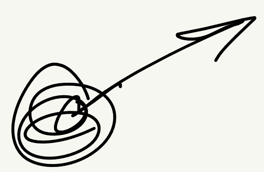
$$\rightarrow \quad E_i' = E_{2i} + E_{3i} + \cdots$$

$\vec{p} = 0$



$$\frac{p_0^2}{2m_1} = T_{10}$$

$$\underbrace{m_1 \vec{v}_1}_{\vec{p}_0} + \underbrace{m_2 \vec{v}_2 + m_3 \vec{v}_2}_{-\vec{p}_0} = 0$$



$$E_1' = E_2' + E_3' + \cdots$$

Added discussion: (meaning of E_i')

Conservation of total energy:

$$E_i = \underbrace{E_{1i} + \tfrac{1}{2} m_1 |\vec{v}_1|^2}_{\text{internal + KE of mass } m_1} + \underbrace{E_{2i} + \tfrac{1}{2} m_2 |\vec{v}_2|^2}_{\text{same for } m_2} + \underbrace{E_{3i} + \tfrac{1}{2} m_3 |\vec{v}_3|^2}_{\text{same for } m_3} + \cdots$$

Let: $T' \equiv \frac{1}{2} m_2 |\vec{v}_2|^2 + \frac{1}{2} m_3 |\vec{v}_3|^2 + \cdots$ $\quad$ (= KE of $m_2, m_3, \ldots$)

$$= \frac{1}{2} m_2 |\vec{V}' + \vec{v}_2'|^2 + \frac{1}{2} m_3 |\vec{V}' + \vec{v}_3'|^2 + \cdots$$

where $\vec{V}'$ = velocity of COM of $m_2, m_3, \ldots$

$\vec{v}_2'$ = velocity of m_2 with respect to COM of $m_2, m_3, \ldots$

$\vec{v}_3'$ = " m_3 "

$\ldots$

$$\rightarrow T' = \frac{1}{2} m_2 \left(|\vec{V}'|^2 + |\vec{v}_2'|^2 + 2\vec{V}' \cdot \vec{v}_2' \right) + \cdots$$

$$= \underbrace{\frac{1}{2}(m_2 + m_3 + \cdots)|\vec{V}'|^2}_{T'_{COM}} + \underbrace{\frac{1}{2} m_2 |\vec{v}_2'|^2 + \frac{1}{2} m_3 |\vec{v}_3'|^2 + \cdots}_{T_0'}$$

$$+ \underbrace{\left(m_2 \vec{v}_2' + m_3 \vec{v}_3' + \cdots \right)}_{=0 \text{ (by definition of COM for } m_2, m_3, \ldots)} \cdot \vec{V}'$$

T'_{COM} = KE of COM of $m_2, m_3, \ldots$

T_0' = KE of $m_2, m_3, \ldots$ w.r.t. COM of $m_2, m_3, \ldots$

so $T' = T'_{com} + T'_0$

$$\rightarrow \quad E_i = E_{1i} + \frac{1}{2} m_1 |\vec{v}_1|^2 + (E_{2i} + E_{3i} + \cdots) + T'_{com} + T'_0$$

$$= E_{1i} + \frac{1}{2} m_1 |\vec{v}_1|^2 + (E_{2i} + E_{3i} + \cdots) + \frac{p_0^2}{2(m_2+m_3+\cdots)} + T'_0$$

compare to

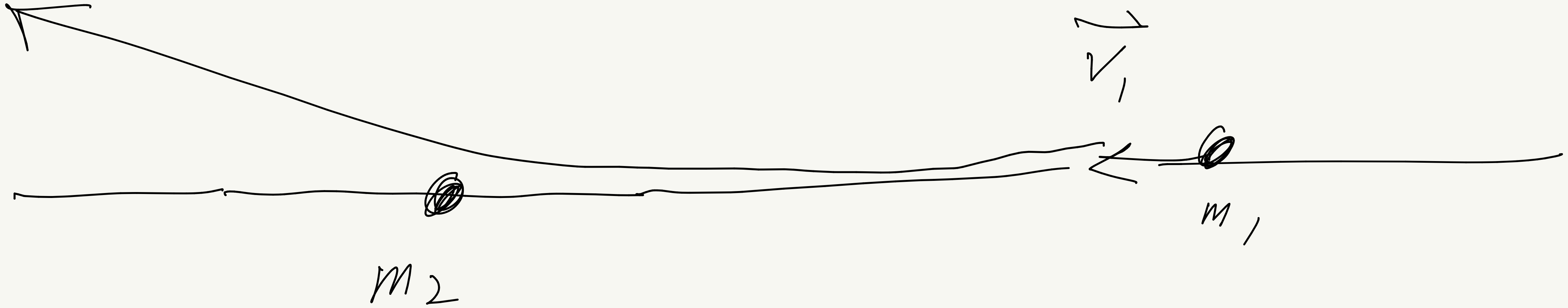
$$E_i = \left(E_{1i} + \frac{1}{2} m_1 |\vec{v}_1|^2\right) + \left(E'_i + \frac{p_0^2}{2(\underbrace{M-m_1}_{m_2+m_3+\cdots})}\right)$$

$$\rightarrow \quad \boxed{\begin{aligned} E'_i &= E_{2i} + E_{3i} + \cdots + T'_0 \\ &= (\text{internal energy of } m_2, m_3, \cdots) + \left(\begin{gathered}\text{KE of } m_2, m_3, \ldots \\ \text{w.r.t COM of} \\ m_2, m_3, \ldots\end{gathered}\right)\end{aligned}}$$

NOTE: when $m_2, m_3, \ldots$ all move with the same velocity, they are moving together with the COM of $m_2, m_3, \ldots$. Then $T'_0 = 0$ and E'_i = internal energy of $m_2, m_3, \ldots$

COM Frame vs. lab Frame

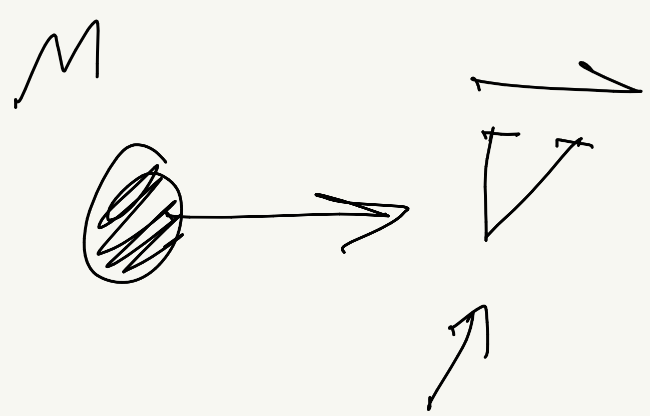
Example:



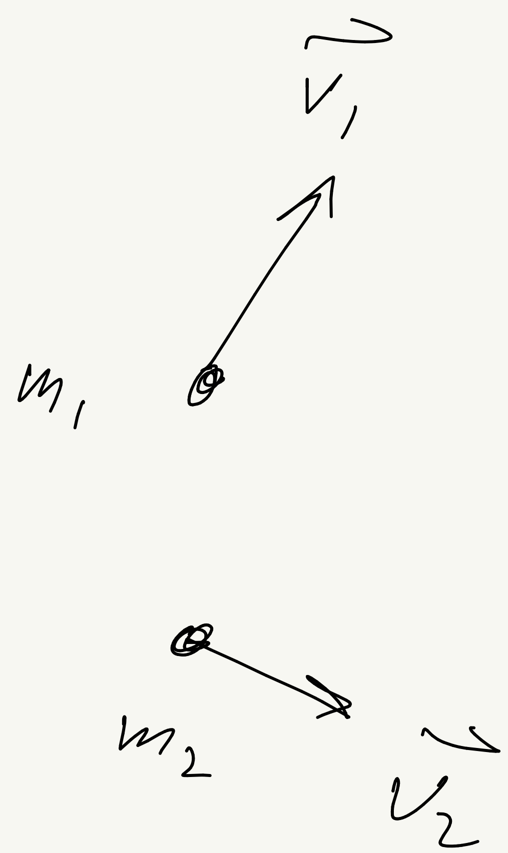
lab Frame

$$\vec{p}_{tot} = m_1 \vec{v}_1 + m_2 \cancel{\vec{v}_2}^{\,0} = m_1 \vec{v}_1 \neq 0$$

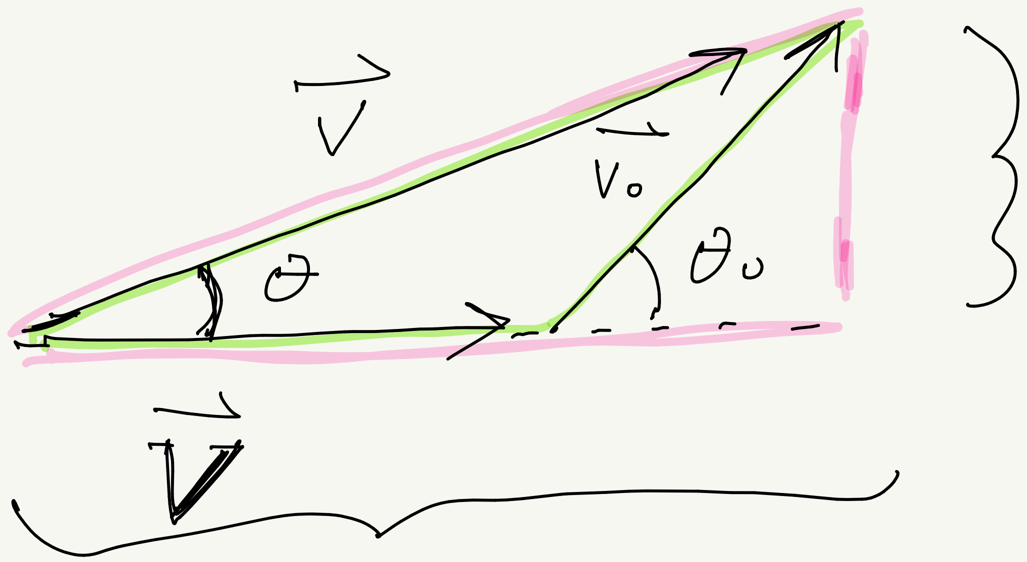
lab frame:



velocity of M w.r.t lab frame



$$M\vec{V} = m_1 \vec{v}_1 + m_2 \vec{v}_2$$



θ : w.r.t lab frame
θ_0 : w.r.t COM frame

$\vec{V_0}$: velocity of m_1 (or m_2) w.r.t COM frame

$\vec{V}$: velocity of M w.r.t lab frame

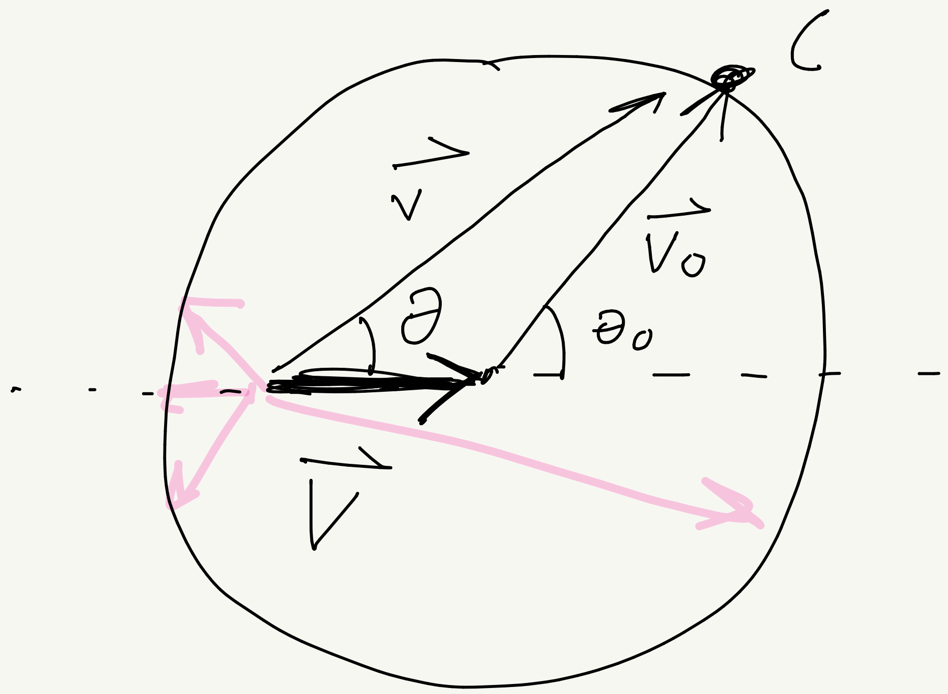
$\vec{v}$: velocity of m_1 (or m_2) wrt lab frame

$$\boxed{v_0^2 = v^2 + V^2 - 2vV\cos\theta}$$

$$v\sin\theta = v_0 \sin\theta_0$$

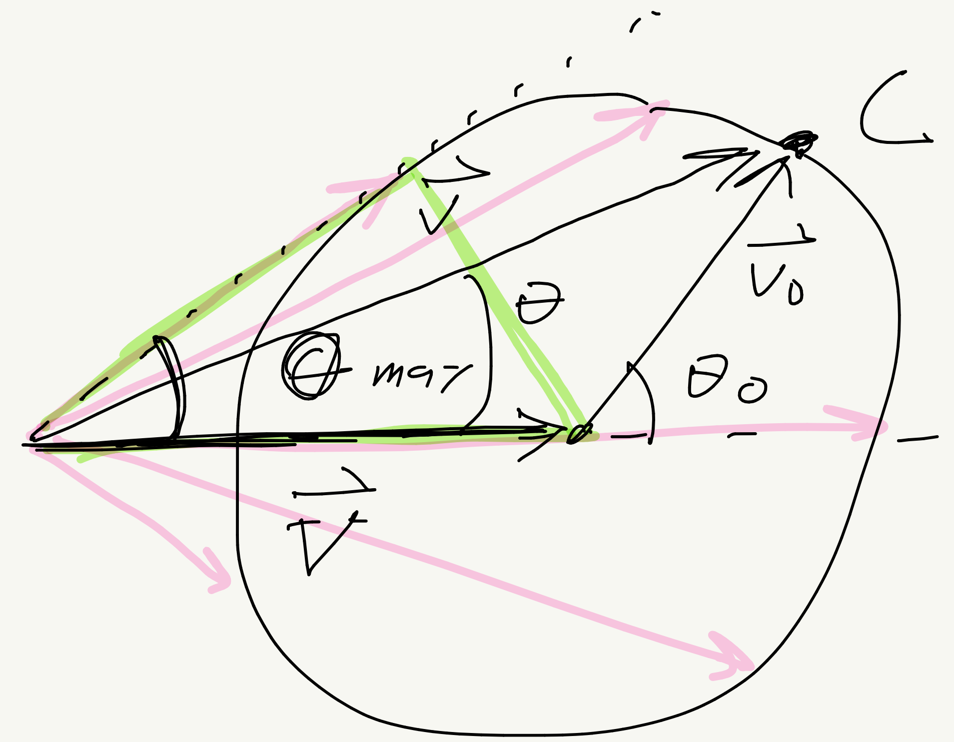
$$v\cos\theta = V + v_0\cos\theta_0$$

$$\div \rightarrow \boxed{\tan\theta = \frac{v_0\sin\theta_0}{V + v_0\cos\theta_0}}$$



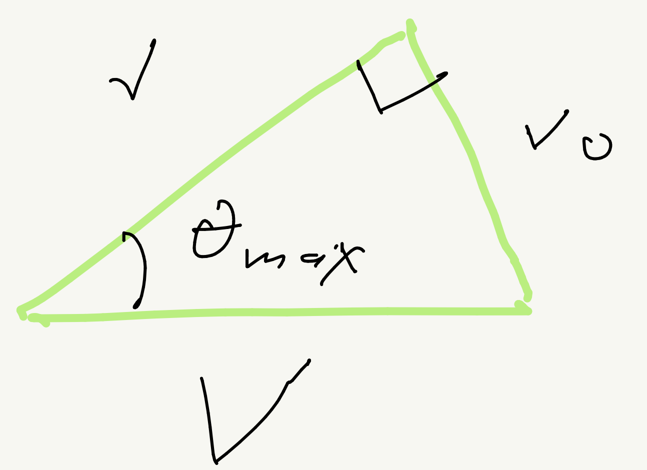
$V < v_0$

$\vec{v}$ can point in any direction



$V > v_0$

$\vec{v}$ can only point in the forward direction



$$\sin\theta_{max} = \frac{v_0}{V}$$

Lec # 15:

grade: max = 20

rescale by $20/16$ (part (b) of long problem # 2)

Average: $12/20$ $9/20$
(with rescaling)

Example: $\dfrac{14}{20} \longrightarrow 14 \times \dfrac{20}{\cancel{16}} = 14 \times \dfrac{10}{8}$

$= \dfrac{140}{8}$

$= 17. \ldots$

Quiz # 3 — next week sometime

Elastic collisions: (Sec 17)

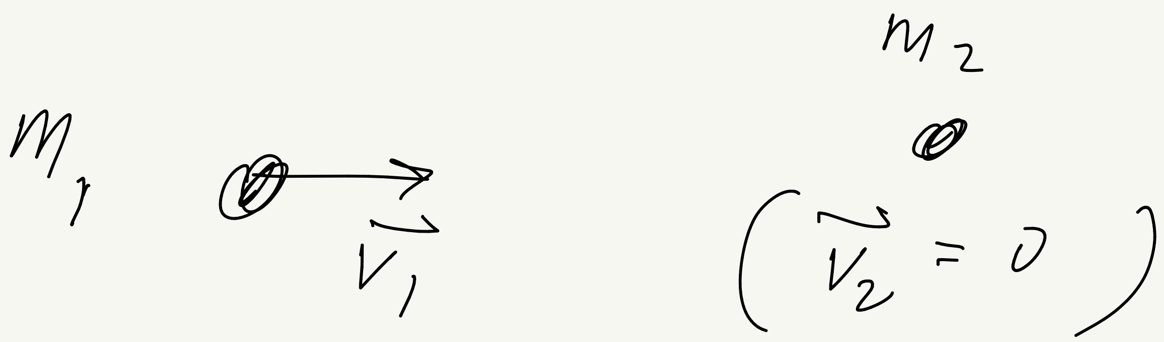
motion was in a 2-d plane

– cons. of linear momentum, angular momentum, total energy

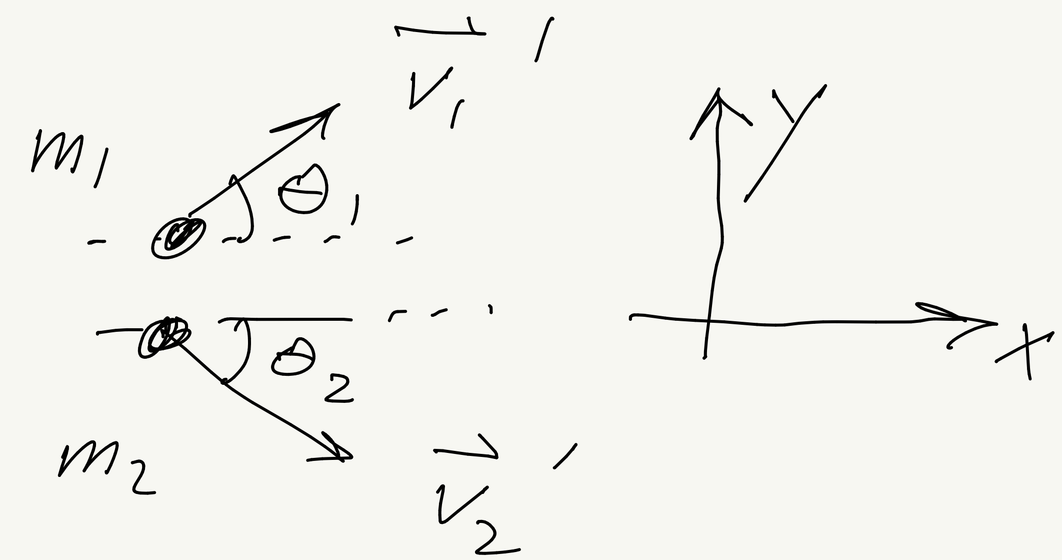
elastic collision — no change in internal energy

→ **Kinetic energy** is conserved

specify masses m_1, m_2 and initial velocities $\vec{v}_1, \vec{v}_2$



(before)



(after)

(lab frame)

Unknowns: (4)

$v_1', v_2', \theta_1, \theta_2$

Equations:

$\Rightarrow T_{before} = T_{after}$ (1 equation)

$\vec{P}_{before} = \vec{P}_{after}$ (2 equations)

3 equations

We normally specify θ_2 in order to solve for v_1', v_2', θ_1

Numerical:

$m_1 = 1\,kg$

$m_2 = 2\,kg$

$v_1 = 1\,m/s$

$v_2 = 0$

$\theta_2 = 60^\circ$

$v_1' = ?? = \frac{1}{3}\,m/s$

$v_2' = ?? = \frac{\sqrt{7}}{3}\,m/s$

$\theta_1 = ?? \approx 41^\circ$

$$\frac{1}{2} m_1 |\vec{v}_1|^2 = \frac{1}{2} m_1 |\vec{v}_1'|^2 + \frac{1}{2} m_2 |\vec{v}_2'|^2$$

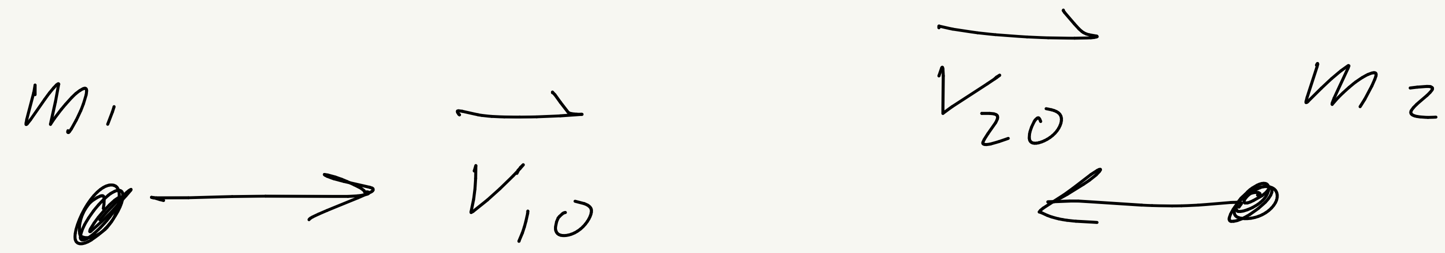
$$\boxed{m_1 v_1^2 = m_1 (v_1')^2 + m_2 (v_2')^2}$$

v_1', v_2', θ_1

$$\underline{x:} \quad \boxed{m_1 v_1 = m_1 v_1' \cos\theta_1 + m_2 v_2' \cos\theta_2}$$

$$\underline{y:} \quad \boxed{0 = m_1 v_1' \sin\theta_1 - m_2 v_2' \sin\theta_2}$$

Analyze in COM frame instead



$$m_1 \vec{v}_{10} + m_2 \vec{v}_{20} = 0$$

$\vec{p}_0 = 0$ (before and after the collision)

$$\vec{V} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$ (velocity of COM in the lab frame)

$$\vec{v}_1 = \vec{V} + \vec{v}_{10}, \quad \vec{v}_2 = \vec{V} + \vec{v}_{20}$$

$$\vec{v}_1{}' = \vec{V} + \vec{v}_{10}{}', \quad \vec{v}_2{}' = \vec{V} + \vec{v}_{20}{}'$$

$\uparrow$ lab frame ($\vec{v}_1$) $\qquad \uparrow$ COM frame ($\vec{v}_{10}$)

$$\underbrace{\begin{aligned} \vec{r} &= \vec{r}_{10} - \vec{r}_{20} \\ &= \vec{r}_1 - \vec{r}_2 \\ \vec{v} &= \vec{v}_{10} - \vec{v}_{20} \\ &= \vec{v}_1 - \vec{v}_2 \end{aligned}}_{\text{relative vector}}$$

$\vec{R}$

$$\boxed{T_0 = \frac{1}{2} m_1 |\vec{v}_{10}|^2 + \frac{1}{2} m_2 |\vec{v}_{20}|^2 = \frac{1}{2} m |\vec{v}|^2}, \quad m = \frac{m_1 m_2}{m_1 + m_2}$$

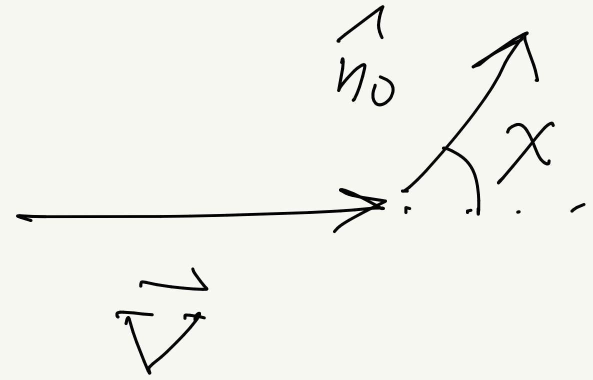
$$\vec{r}_{10} = \frac{m_2}{m_1+m_2}\vec{r}, \quad \vec{r}_{20} = -\frac{m_1}{m_1+m_2}\vec{r}$$

$$\boxed{\vec{v}_{10} = \frac{m_2}{m_1+m_2}\vec{v}}, \quad \boxed{\vec{v}_{20} = -\frac{m_1}{m_1+m_2}\vec{v}}$$

$$T_0' = \underbrace{\frac{1}{2} m |\vec{v}'|^2 = T_0 = \frac{1}{2} m |\vec{v}|^2}$$

$$\boxed{|\vec{v}'| = |\vec{v}|}, \quad \boxed{\begin{aligned} v' &= v \\ v_{10}' &= v_{10} \\ v_{20}' &= v_{20} \end{aligned}}$$

$\vec{v}' = v\,\hat{n}_0$, $\hat{n}_0$: unit vector makes an angle χ w.r.t com velocity



$$\vec{v} = \vec{v}_{10} - \vec{v}_{20} = \vec{v}_1 - \vec{v}_2$$

w.r.t lab frame

w.r.t com frame

$$\vec{v}_1' = \vec{V} + \vec{v}_{10}' = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} + \frac{m_2}{m_1+m_2}\,v\,\hat{n}_0$$

$$\vec{v}_2' = \vec{V} + \vec{v}_{20}' = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} - \frac{m_1}{m_1+m_2}\,v\,\hat{n}_0$$

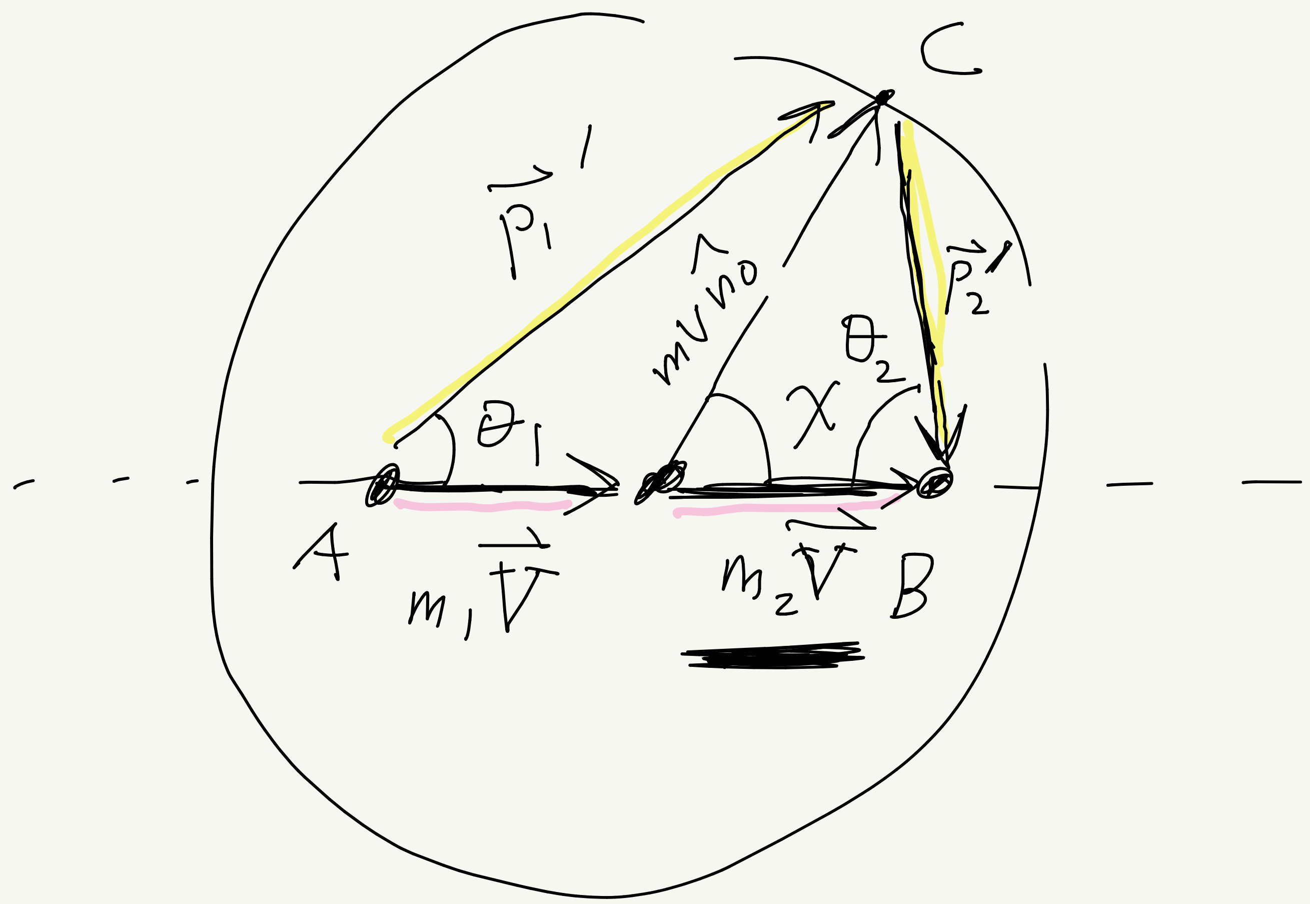
$$\boxed{\begin{aligned}\vec{p}_1' &= m_1\vec{V} + m\,v\,\hat{n}_0\\ \vec{p}_2' &= m_2\vec{V} - m\,v\,\hat{n}_0\end{aligned}}$$

$$\vec{p}_1' = m_1\vec{v}_1'$$
$$\vec{p}_2' = m_2\vec{v}_2'$$

$$\vec{p}_1' + \vec{p}_2' = (m_1+m_2)\vec{V} = \vec{P} = \vec{p}_1 + \vec{p}_2$$

$\vec{v}_1, \vec{v}_2$ are arbitrary

AB: $\vec{P}$



$\underline{\vec{v}_2 = 0}$: (standard lab frame)

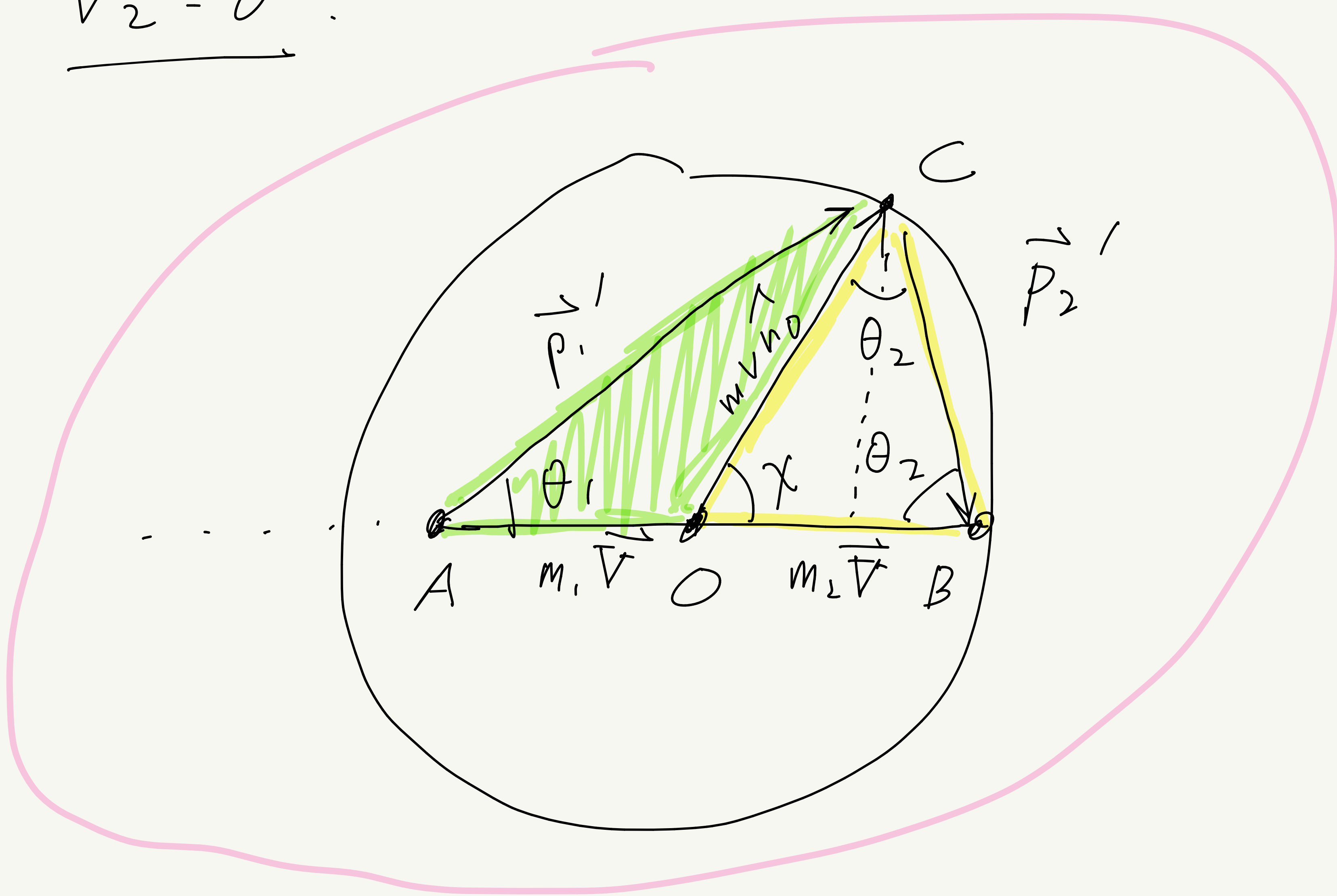
$$\vec{v} = \vec{v}_1 - \cancelto{0}{\vec{v}_2} = \vec{v}_1$$

$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} = \frac{m_1\vec{v}_1}{m_1+m_2} = \frac{m_1\vec{v}}{m_1+m_2}$$

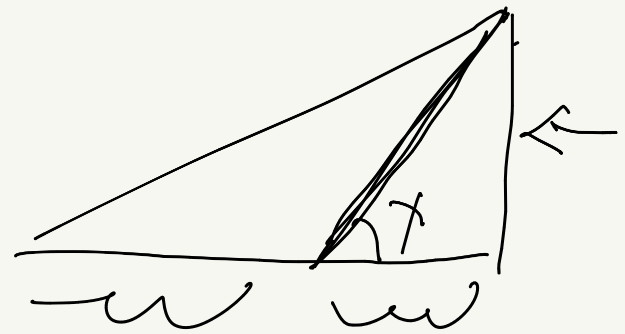
$$m_2\vec{V} = \frac{m_2 m_1\vec{v}}{m_1+m_2} = m\vec{v}$$

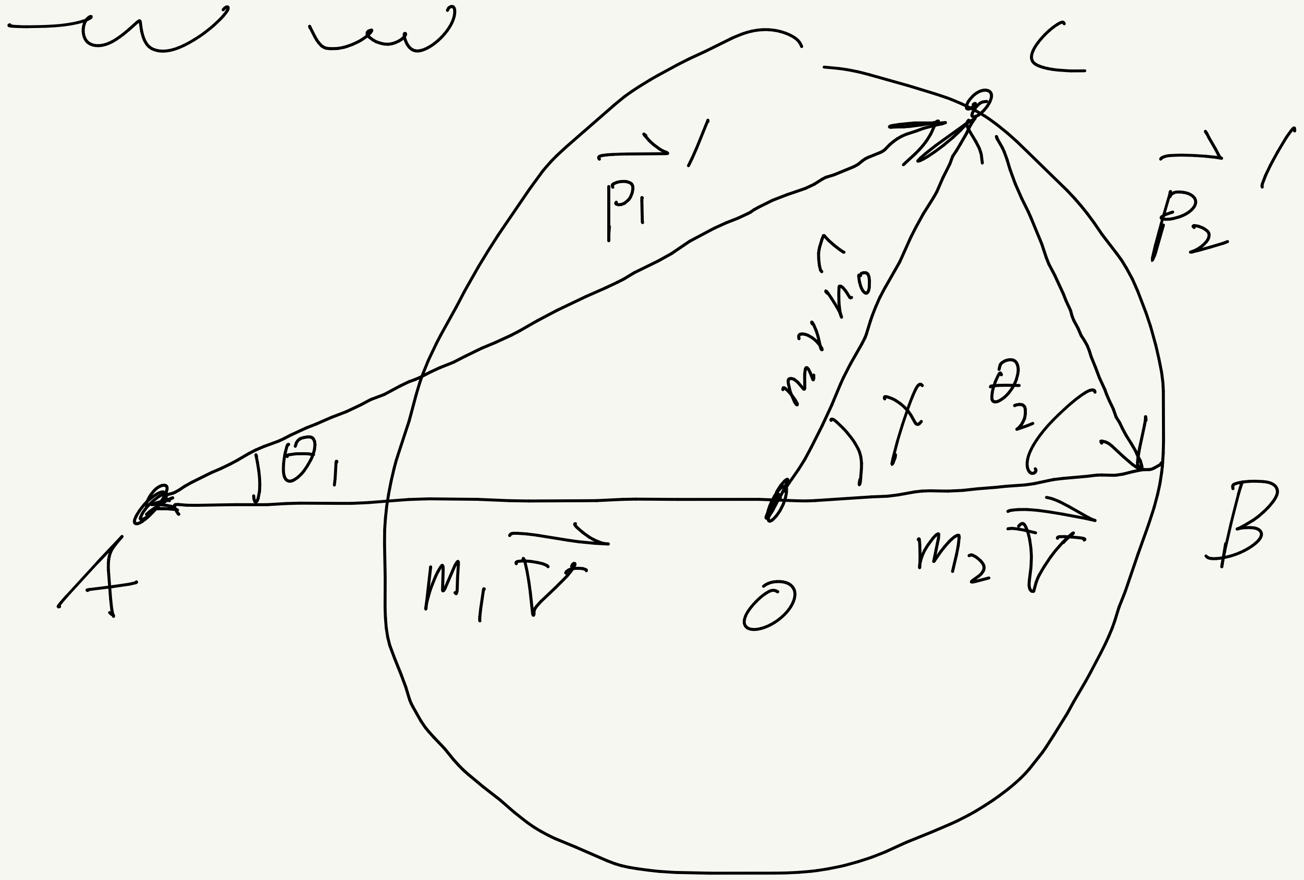
$\longrightarrow$ B lies on the circle

$\vec{V}_2 = 0$:



$m_1 < m_2$





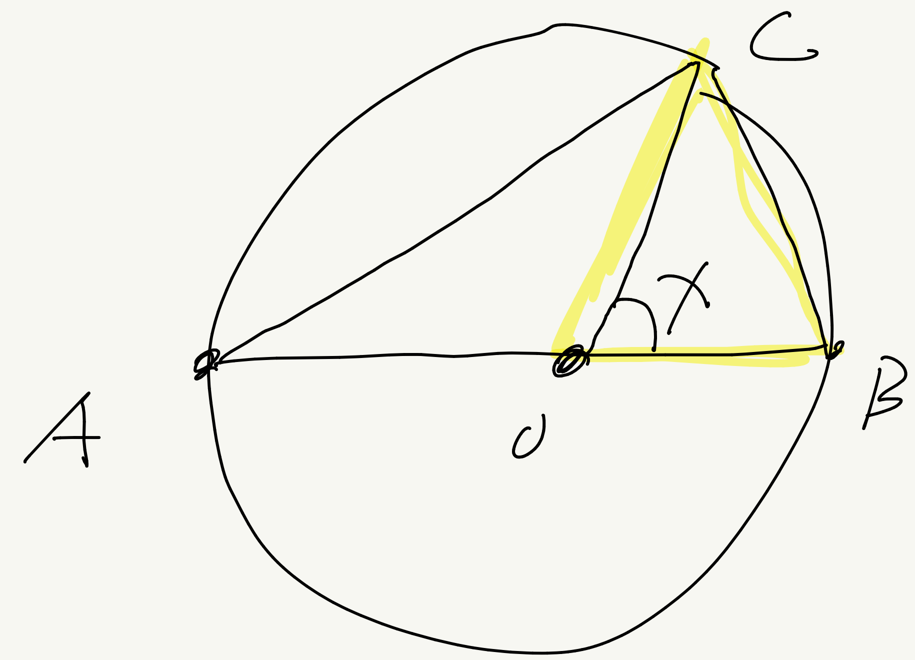
$m_1 > m_2$

$$\chi + 2\theta_2 = \pi$$

$$\boxed{\theta_2 = \frac{\pi}{2} - \frac{\chi}{2}}$$

$$\tan\theta_1 = \frac{opp}{adj} = \frac{m v \sin\chi}{m_1 V + m v \cos\chi}$$

$$\boxed{\tan\theta_1 = \frac{\left(\frac{\cancel{m_1}\, m_2}{\cancel{m_1+m_2}}\right)\cancel{v}\,\sin\chi}{\cancel{m_1}\left(\frac{m_1 \cancel{v}}{\cancel{m_1+m_2}}\right) + \left(\frac{\cancel{m_1}\, m_2}{\cancel{m_1+m_2}}\right)\cancel{v}\,\cos\chi} = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}}$$



$m_1 = m_2$

$$\cancel{m_2^2}\,(v_2')^2 = 2m^2v^2 - 2m^2v^2\cos\chi$$

$$= 2m^2v^2(1-\cos\chi)$$

$$= 2\,\frac{m_1^2\,\cancel{m_2^2}}{(m_1+m_2)^2}\,v^2\; 2\sin^2\!\left(\frac{\chi}{2}\right)$$

$$\boxed{v_2' = \left(\frac{2\,m_1 v}{m_1+m_2}\right)\sin\left(\frac{\chi}{2}\right)}$$

$$\cos\chi = \cos\left(2\,\frac{\chi}{2}\right)$$

$$= \cos^2\left(\frac{\chi}{2}\right) - \sin^2\left(\frac{\chi}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{\chi}{2}\right)$$

$$1 - \cos\chi = 2\sin^2\left(\frac{\chi}{2}\right)$$

$$\boxed{v_1' = \frac{v}{m_1 + m_2}\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos\chi}}$$