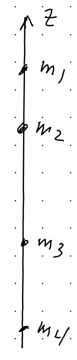
Sec 32, Prob 1:

(a)




$I_3 = 0$ since $x_a = y_a = 0$ for all masses

$$I_1 = \sum_a m_a \left(r_a^2 - \cancel{x_a^2}^{\,0}\right) = \sum_a m_a z_a^2$$

$$I_2 = \sum_a m_a \left(r_a^2 - \cancel{y_a^2}^{\,0}\right) = \sum_a m_a z_a^2$$

$$\to I_1 = I_2 \equiv I = \sum_a m_a z_a^2$$

(assuming COM at $z=0$)

If COM is not at $z=0$, but at $z_{com}$, then:

$$I = \sum_a m_a (z_a - z_{com})^2 \,, \qquad z_{com} = \frac{1}{\mu}\sum_b m_b z_b$$

$$= \sum_a m_a \left(z_a^2 + z_{com}^2 - 2 z_{com} z_a\right)$$

$$= \sum_a m_a z_a^2 + \mu z_{com}^2 - 2 z_{com} \underbrace{\sum_a m_a z_a}_{\mu z_{com}}$$

$$= \sum_a m_a z_a^2 - \mu z_{com}^2$$

This last expression can be rewritten in terms of $l_{ab} \equiv |z_a - z_b|$ as follows:

$$I = \frac{1}{2}\sum_a m_a z_a^2 + \frac{1}{2}\sum_b m_b z_b^2 - \frac{1}{\mu}\left(\sum_a m_a z_a\right)\left(\sum_b m_b z_b\right)$$

$$= \frac{1}{2\mu}\sum_a\sum_b m_b m_a z_a^2 + \frac{1}{2\mu}\sum_a\sum_b m_a m_b z_b^2 - \frac{1}{\mu}\sum_a\sum_b m_a m_b z_a z_b$$

Thus

$$I = \frac{1}{2\mu}\sum_a\sum_b m_a m_b \left(z_a^2 + z_b^2 - 2 z_a z_b\right)$$

$$= \frac{1}{2\mu}\sum_a\sum_b m_a m_b (z_a - z_b)^2$$

$$= \frac{1}{2\mu}\sum_a\sum_b m_a m_b l_{ab}^2$$

NOTE: For just two masses:

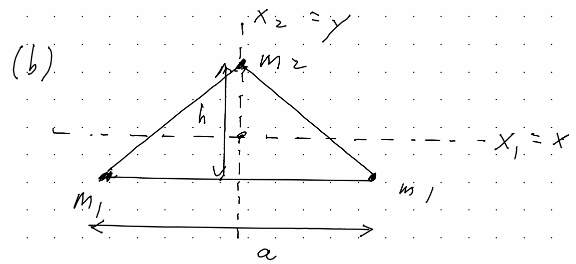
$$I = \frac{1}{2\mu}\left(m_1 m_2 l^2 + m_2 m_1 l^2\right)$$

$$= \frac{m_1 m_2}{\mu} l^2$$

$$= m l^2 \quad \text{where} \quad m \equiv \frac{m_1 m_2}{m_1 + m_2}$$

$$l = |z_1 - z_2|$$

(b)




Assume COM at $(x_1, x_2) = (x, y) = (0,0)$

Then $2m_1 y_1 + m_2 y_2 = 0$

where $y_2 - y_1 = h$

thus, $2m_1 y_1 + m_2(h + y_1) = 0$

$$(2m_1 + m_2) y_1 + m_2 h = 0$$

$$y_1 = \frac{-m_2 h}{\mu}, \quad \mu = 2m_1 + m_2 = \text{total mass}$$

and
$$y_2 = y_1 + h = \frac{-m_2 h}{\mu} + h = \frac{(\mu - m_2) h}{\mu} = \frac{2m_1 h}{\mu}$$

All masses have $z_a = 0$

Thus, $I_3 = \sum_a m_a (r_a^2 - z_a^2) = \sum_a m_a (x_a^2 + y_a^2)$

$$I_1 = \sum_a m_a (r_a^2 - x_a^2) = \sum_a m_a y_a^2$$

$$I_2 = \sum_a m_a (r_a^2 - y_a^2) = \sum_a m_a x_a^2$$

Thus, $I_3 = I_1 + I_2$

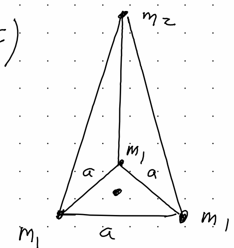
so need to calculate $I_1, I_2$.

$$I_1 = \sum_a m_a y_a^2 = 2m_1 y_1^2 + m_2 y_2^2 = 2m_1 \frac{m_2^2 h^2}{\mu^2} + m_2 \frac{4m_1^2 h^2}{\mu^2} = \frac{2m_1 m_2 h^2}{\mu^2}(m_2 + 2m_1) = \boxed{\frac{2m_1 m_2}{\mu} h^2}$$

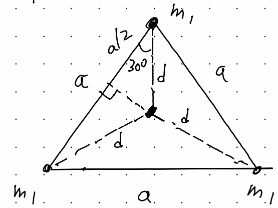
$$I_2 = \sum_a m_a x_a^2 = m_1 \left(\frac{a}{2}\right)^2 + m_1 \left(\frac{-a}{2}\right)^2 = \boxed{\frac{m_1 a^2}{2}}$$

$$I_3 = I_1 + I_2 = \boxed{\frac{2m_1 m_2}{\mu} h^2 + \frac{m_1 a^2}{2}}$$

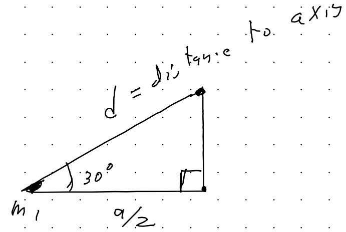
(c)




tetrahedron, height $h$
base: equilateral $\Delta$ with side length $a$




base




$$\cos 30° = \frac{a}{2d} = \frac{\sqrt{3}}{2}$$

$$\rightarrow \quad d = \frac{a}{\sqrt{3}}$$

COM lies on axis of symmetry (z-axis)

Assume COM has $z=0$

Then $0 = m_2 z_2 + 3m_1 z_1$, $\quad z_2 - z_1 = h$

$$= m_2 (z_1 + h) + 3m_1 z_1$$

$$= (3m_1 + m_2) z_1 + m_2 h$$

$$\rightarrow \quad z_1 = \frac{-m_2 h}{3m_1 + m_2} = -\frac{m_2 h}{\mu}$$

$$z_2 = h + z_1 = h - \frac{m_2 h}{\mu} = \frac{3m_1 h}{\mu}$$

Since a tetrahedron has 3-fold rotational symmetry, the $x_1$ principal axes can be chosen arbitrarily in the plane $\perp$ to the symmetry axis ($x_3 \equiv z$). [$x_2$ is $\perp$ to $x_1, x_3$]

Thus, $I_1 = I_2 \equiv I$

$$I_3 = \sum_a m_a (r_a^2 - z_a^2) = \sum_a m_a s_a^2 \quad \text{where } s^2 = r^2 - z^2$$

$$\left.\begin{aligned} I_1 &= \sum_a m_a (r_a^2 - x_a^2) \\ I_2 &= \sum_a m_a (r_a^2 - y_a^2) \end{aligned}\right\} \text{equal } (I_1 = I_2 \equiv I)$$

$$2I = I_1 + I_2$$
$$= \sum_a m_a (2r_a^2 - x_a^2 - y_a^2)$$
$$= \sum_a m_a (2(s_a^2 + z_a^2) - s_a^2)$$
$$= \sum_a m_a s_a^2 + 2\sum_a m_a z_a^2$$
$$= I_3 + 2\sum_a m_a z_a^2$$

Thus,

$$I = \frac{1}{2} I_3 + \sum_a m_a z_a^2$$

Now

$$I_3 = \sum_a m_a s_a^2 = 3 m_1 d^2 = \cancel{3} m_1 \frac{a^2}{\cancel{3}} = \boxed{m_1 a^2}$$

Also,

$$\sum_a m_a z_a^2 = 3m_1 z_1^2 + m_2 z_2^2$$

$$= 3m_1\left(-\frac{m_2 h}{\mu}\right)^2 + m_2\left(\frac{3m_1 h}{\mu}\right)^2$$

$$= \frac{3m_1 m_2^2 h^2}{\mu^2} + 9\frac{m_1^2 m_2 h^2}{\mu}$$

$$= \frac{3m_1 m_2 h^2}{\mu^2}\underbrace{(m_2 + 3m_1)}_{\mu}$$

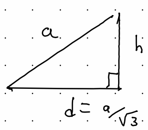
$$= \frac{3m_1 m_2 h^2}{\mu}$$

Thus,

$$I = \frac{1}{2} I_3 + \sum_a m_a z_a^2$$

$$= \boxed{\frac{1}{2} m_1 a^2 + \frac{3m_1 m_2 h^2}{\mu}}$$

Regular tetrahedron: $m_1 = m_2$




$$h^2 + \frac{a^2}{3} = a^2 \rightarrow h = \sqrt{\frac{2}{3}}\, a$$

$$\mu = 4m_1$$

$$I_3 = m_1 a^2$$

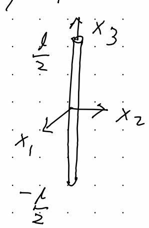
$$I = \frac{1}{2} m_1 a^2 + \frac{\cancel{3}\cancel{m_1} m_1}{4\cancel{m_1}}\left(\frac{2}{\cancel{3}}\right) a^2$$

$$= m_1 a^2 \quad (= I_1 = I_2)$$

so $I_1 = I_2 = I_3 = m_1 a^2$

Sec 32, Prob 2:

(a) Thin rod of length $l$:




$I_3 = \boxed{0}$

and $I_1 = I_2 \equiv I$

$$I = \int \rho\, dV\,(r^2 - x^2)$$

$$= \int \rho\, dV\, z^2$$

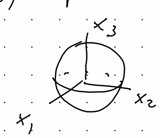
$$= \int_{-l/2}^{l/2} dz\left(\frac{\mu}{l}\right) z^2$$

$$= \frac{\mu}{l}\,\frac{z^3}{3}\Big|_{-l/2}^{l/2}$$

$$= \frac{\mu}{l}\,\frac{2}{3}\,\frac{l^3}{8}$$

$$= \boxed{\frac{1}{12}\mu l^2}$$

(b) Sphere of radius $R$:




$$I_1 = I_2 = I_3 = I$$

$$I = \frac{1}{3}(I_1 + I_2 + I_3)$$

$$= \frac{1}{3}\left[\int \rho\, dV\,(r^2 - x^2) + \int \rho\, dV\,(r^2 - y^2) + \int \rho\, dV\,(r^2 - z^2)\right]$$

$$= \frac{1}{3}\int \rho\, dV\left[3r^2 - x^2 - y^2 - y^2\right]$$

$$= \frac{2}{3}\int \rho\, dV\, r^2$$

$$I = \frac{2}{3}\int \rho\, dV\, r^2 \qquad \rho = \frac{M}{\frac{4}{3}\pi R^3}$$

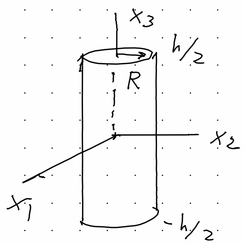
$$= \frac{\cancel{2}}{\cancel{3}} \frac{M}{\frac{4}{\cancel{3}}\pi R^3} \int_0^R r^4 dr \underbrace{\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi}_{4\pi}$$

$$= \frac{M}{2\pi R^3} \cdot 4\pi \int_0^R r^4 dr$$

$$= \frac{2M}{R^3}\,\frac{R^5}{5}$$

$$= \boxed{\frac{2}{5} M R^2}$$

(c)




$$\rho = \frac{M}{\pi R^2 \cdot h}$$

$$dV = ds\, s\, d\phi\, dz$$

where $s^2 = x^2 + y^2$

$$I_1 = I_2 \equiv I$$

$$2I = I_1 + I_2$$

$$= \int \rho\, dV (r^2 - x^2) + \int \rho\, dV (r^2 - y^2)$$

$$= \int \rho\, dV (2r^2 - s^2)$$

$$= \int \rho\, dV \left(2(s^2 + z^2) - s^2\right)$$

$$= \int \rho\, dV\, s^2 + 2\int \rho\, dV\, z^2$$

$$\rightarrow \quad I = \frac{1}{2} I_3 + \int \rho\, dV \cdot z^2$$

$$I_3 = \int \rho\, dV\, s^2$$

$$= \frac{M}{\pi R^2 h} \int_0^R s^3 ds \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz$$

$$= \frac{M}{\cancel{\pi} R^2 \cancel{h}} \, \frac{R^4}{4} \cdot 2\cancel{\pi} \cdot \cancel{h}$$

$$= \boxed{\frac{1}{2} M R^2}$$

$$I = \frac{1}{2} I_3 + \int \rho\, dV\, z^2$$

$$\int \rho\, dV\, z^2 = \frac{M}{\pi R^2 h} \int_0^R s\, ds \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} z^2 dz$$

$$= \frac{M}{\cancel{\pi R^2} h} \, \frac{\cancel{R^2}}{\cancel{2}} \cdot \cancel{2\pi} \cdot \left.\frac{z^3}{3}\right|_{-h/2}^{h/2}$$

$$= \frac{M}{h}\,\frac{2}{3}\,\frac{h^3}{8}$$

$$= \frac{1}{12} M h^2$$

Thus,

$$I = \frac{1}{2}\left(\frac{1}{2} M R^2\right) + \frac{1}{12} M h^2$$

$$= \frac{1}{4} M R^2 + \frac{1}{12} M h^2$$

$$= \boxed{\frac{1}{4} M \left(R^2 + \frac{1}{3} h^2\right)}$$

NOTE: special limiting case:

(i) Thin rod $(R \to 0)$

$$I_3 = 0$$

$$I_1 = I_2 = \frac{1}{12} M h^2$$

(ii) Thin disk $(h \to 0)$

$$I_3 = \frac{1}{2} M R^2$$

$$I_1 = I_2 = \frac{1}{4} M R^2$$

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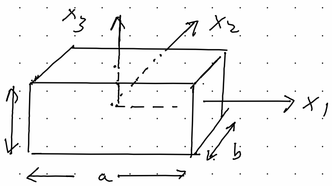
(d)




$$\rho = \frac{M}{abc}$$

$$dV = dx\,dy\,dz$$

$$I_1 = \int \rho\, dV\, (r^2 - x^2)$$

$$= \int \rho\, dV\, (y^2 + z^2)$$

$$= \frac{M}{abc} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz\, (y^2 + z^2)$$

$$= \frac{M}{\cancel{a}bc} \cancel{a} \int_{-b/2}^{b/2} dy \left( y^2 z + \frac{z^3}{3} \right) \Big|_{-c/2}^{c/2}$$

$$= \frac{M}{bc} \int_{-b/2}^{b/2} dy \left( c y^2 + \frac{2}{3} \frac{c^3}{8} \right)$$

$$= \frac{M}{bc} \left[ c \frac{y^3}{3} \Big|_{-b/2}^{b/2} + \frac{1}{12} c^3 y \Big|_{-b/2}^{b/2} \right]$$

$$I_1 = \frac{M}{bc} \left[ c \frac{2}{3} \frac{b^3}{8} + \frac{1}{12} b c^3 \right]$$

$$= \frac{M}{12 bc} \left[ c b^3 + b c^3 \right]$$

$$= \frac{M}{12} (b^2 + c^2)$$

$$I_2 = \frac{M}{12} (c^2 + a^2)$$

$$I_3 = \frac{M}{12} (a^2 + b^2)$$

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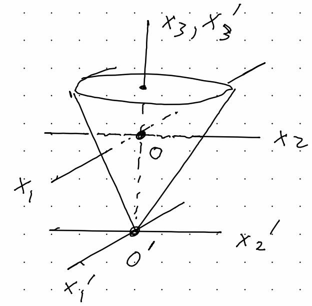
(e)




First calculate $I_{ij}'$ (w.r.t. $x_1', x_2', x_3'$).

Then calculate $I_{ij}$ via

$$I_{ij} = I_{ij}' - M(a^2 \delta_{ij} - a_i a_j)$$

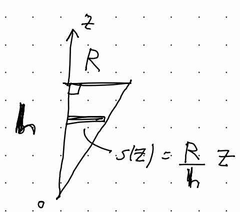
where $\vec{a} = (0, 0, -d)$

d: height of com above O'

Volume of cone:

$$V = \int dV$$

$$= \int_0^h dz \int_0^{2\pi} d\phi \int_0^{\frac{R}{h} z} s\, ds$$




$$= \cancel{2}\pi \int_0^h dz\, \frac{s^2}{\cancel{2}} \Big|_0^{Rz/h}$$

$$V = \pi \int_0^h dz \, \frac{R^2 z^2}{h^2}$$

$$= \frac{\pi R^2}{h^2} \frac{z^3}{3} \Big|_0^h$$

$$= \frac{\pi R^2}{h^2} \frac{h^3}{3}$$

$$= \boxed{\frac{1}{3} \pi R^2 h}$$

Thus, $\rho = \frac{M}{V} = \boxed{\frac{M}{\frac{1}{3}\pi R^2 h}}$ (mass density)

$$I_3' = \int \rho \, dV \, (r^2 - z^2)$$

$$= \int \rho \, dV \, s^2$$

$$= \rho \int_0^h dz \int_0^{2\pi} d\phi \int_0^{Rz/h} s^3 \, ds$$

$$= \frac{\rho}{4} \cdot 2\pi \int_0^h dz \, s^4 \Big|_0^{\frac{Rz}{h}}$$

$$= \frac{\rho \pi}{2} \int_0^h dz \left(\frac{R^4}{h^4}\right) z^4$$

$$= \frac{\rho \pi R^4}{2 h^4} \frac{h^5}{5}$$

$$= \frac{\rho \pi}{10} R^4 h$$

Thus,

$$I_3' = \frac{M}{\frac{1}{3}\cancel{\pi} R^2 \cancel{h}} \frac{\cancel{\pi}}{10} R^4 \cancel{h}$$

$$= \boxed{\frac{3}{10} M R^2}$$

Similar to the cylinder, we have $I_1' = I_2' \equiv I'$ where

$$I' = \frac{1}{2} I_3' + \int \rho \, dV \, z^2$$

Now:

$$\int \rho \, dV z^2 = \rho \int_0^h dz \, z^2 \int_0^{2\pi} d\phi \int_0^{Rz/h} s \, ds$$

$$= \cancel{2} \pi \rho \int_0^h dz \, z^2 \frac{s^2}{\cancel{2}} \Big|_0^{Rz/h}$$

$$= \pi \rho \frac{R^2}{h^2} \int_0^h dz \, z^4$$

$$= \pi \rho \frac{R^2}{h^2} \frac{h^5}{5}$$

$$= \frac{\pi}{5} \rho R^2 h^3$$

$$= \frac{\cancel{\pi}}{5} \left( \frac{M}{\frac{1}{3}\cancel{\pi} \cancel{R^2} h} \right) \cancel{R^2} h^3$$

$$= \frac{3}{5} M h^2$$

So

$$I' = \frac{1}{2}\left(\frac{3}{10}\mu R^2\right) + \frac{3}{5}\mu h^2$$

$$= \boxed{\frac{3}{5}\mu\left(\frac{R^2}{4} + h^2\right)} = I_1' = I_2'$$

Need to find location of COM:

$$d = \frac{1}{\mu}\int \rho\, dV\, z$$

$$= \frac{1}{\mu}\rho \int_0^h z\,dz \int_0^{2\pi} d\phi \int_0^{Rz/h} s\,ds$$

$$= \frac{1}{\mu}\rho \cdot \cancel{2}\pi \int_0^h z\,dz\ \frac{1}{\cancel{2}}\left(\frac{R}{h}\right)^2 z^2$$

$$= \frac{\pi\rho}{\mu}\frac{R^2}{h^2}\frac{z^4}{4}\Big|_0^h$$

$$= \frac{\pi\rho}{4\mu} R^2 h^2$$

$$= \frac{\cancel{\pi}}{4\cancel{\mu}} \cdot \frac{\cancel{\mu}}{\frac{1}{3}\cancel{\pi}\cancel{R^2}h}\cancel{R^2}h^2$$

$$= \boxed{\frac{3}{4}h}$$

Thus,

$$I_{ij} = I_{ij}' - \mu(a^2\delta_{ij} - a_i a_j)$$

where $\vec{a} = (0, 0, -\frac{3}{4}h) \rightarrow a^2 = \frac{9}{16}h^2$

$$\rightarrow I_1 = I_1' - \mu a^2$$

$$= \frac{3}{5}\mu\left(\frac{R^2}{4} + h^2\right) - \mu\frac{9}{16}h^2$$

$$= \frac{3}{20}\mu R^2 + \mu h^2 \Big(\underbrace{\frac{3}{5} - \frac{9}{16}}_{\frac{48-45}{80} = \frac{3}{80}}\Big)$$

$$= \boxed{\frac{3}{20}\mu\left(R^2 + \frac{h^2}{4}\right)}$$

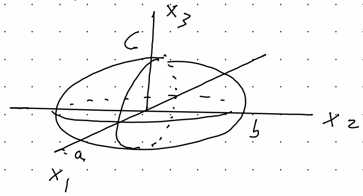
Also, $I_2 = I_1$

Finally, $I_3 = I_3' - \mu(\underbrace{a^2 - a^2}_{\to 0})$

$$= I_3'$$

$$= \boxed{\frac{3}{10}\mu R^2}$$

(f) Ellipsoid with semi-axes $a, b, c$




$(a,b,c) \leftrightarrow (x_1, x_2, x_3)$

Define rescaled coordinates:

$$(u,v,w) \equiv \left(\frac{x_1}{a}, \frac{x_2}{b}, \frac{x_3}{c}\right)$$

So that boundary of ellipsoid

$$1 = \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 + \left(\frac{x_3}{c}\right)^2 = \underbrace{u^2+v^2+w^2}_{\text{unit 2-sphere}}$$

Volume:

$$V = \int dx_1 \int dx_2 \int dx_3$$

$$= abc \int du \int dv \int dw$$

$$= abc \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^1 r^2 dr$$

$$= abc \; 2\pi \cdot 2 \cdot \frac{r^3}{3}\Big|_0^1$$

$$= \boxed{\frac{4}{3}\pi abc}$$

$$\longrightarrow \rho = \frac{M}{\frac{4}{3}\pi abc}$$

$$I_3 = \int \rho \, dV (r^2 - z^2)$$

$$= \int \rho \, dV (x^2+y^2)$$

$$= \frac{M}{\frac{4}{3}\pi abc} \iiint dx\, dy\, dz \, (x^2+y^2)$$

$$= \frac{M}{\frac{4}{3}\pi \cancel{abc}} \cancel{abc} \iiint du\, dv\, dw \, (a^2u^2 + b^2v^2)$$

$$= \frac{M}{\frac{4}{3}\pi} \int_0^1 r^2 dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \, \big(a^2 r^2 \sin^2\theta \cos^2\phi$$

$$+ b^2 r^2 \sin^2\theta \sin^2\phi\big)$$

Now: $$\int_0^{\pi} \sin\theta d\theta \, \sin^2\theta = \int_{-1}^{1} d(\cos\theta)(1-\cos^2\theta)$$

$$= \int_{-1}^{1} dx (1-x^2)$$

$$= \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$= 2 \cdot \frac{2}{3} = \boxed{\frac{4}{3}}$$

$$\int_0^1 r^4 dr = \frac{r^5}{5}\Big|_0^1 = \boxed{\frac{1}{5}}$$

$$\int_0^{2\pi} d\phi \begin{Bmatrix} \sin^2\phi \\ \cos^2\phi \end{Bmatrix} = 2\pi \cdot \frac{1}{2} = \boxed{\pi}$$

Thus,

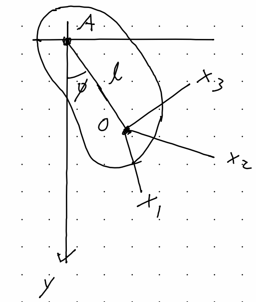
$$I_3 = \frac{M}{\cancel{\frac{4}{3}}\cancel{\pi}}\left(a^2\, \cancel{\tfrac{4}{3}} \cdot \tfrac{1}{5} \cdot \cancel{\pi} + b^2\, \cancel{\tfrac{4}{3}} \cdot \tfrac{1}{5} \cdot \cancel{\pi}\right)$$

$$= \boxed{\frac{M}{5}\left(a^2+b^2\right)}$$

cyclically permuting $a, b, c$ $\rightarrow$

$$\boxed{\begin{aligned} I_1 &= \frac{M}{5}\left(b^2+c^2\right) \\ I_2 &= \frac{M}{5}\left(c^2+a^2\right) \end{aligned}}$$

Sec 32, Prob 3:




com at O

rotation axis at A, out of page

$$\vec{\Omega} = \dot{\phi}\,\hat{n}$$

$$U = Mgl\left(1-\cos\phi\right)$$

$$\approx \frac{1}{2} Mgl\phi^2 \quad \text{for } \phi \ll 1$$

$$L = T - U$$

$$T = \frac{1}{2} M V^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$\vec{\Omega} = \dot{\phi}\,\hat{n} \rightarrow \begin{aligned} \Omega_1 &= \dot{\phi}\,\hat{n}\cdot\hat{x}_1 = \dot{\phi}\cos\alpha \\ \Omega_2 &= \dot{\phi}\,\hat{n}\cdot\hat{x}_2 = \dot{\phi}\cos\beta \\ \Omega_3 &= \dot{\phi}\,\hat{n}\cdot x_3 = \dot{\phi}\cos\gamma \end{aligned}$$

$$V = l\dot{\phi}$$

Thus,

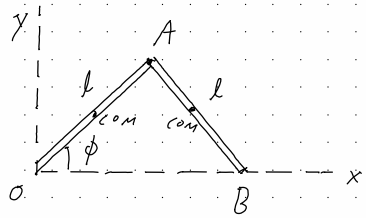
$$T = \frac{1}{2} M l^2 \dot{\phi}^2 + \frac{1}{2}\dot{\phi}^2\left(I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)$$

so

$$L = \frac{1}{2}\dot{\phi}^2\left(Ml^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right) - \frac{1}{2} Mgl\phi^2$$

$$\rightarrow \omega = \sqrt{\frac{Mgl}{Ml^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma}}$$

## Sec. 32, Prob 4




two uniform rods

$I_{com} = \frac{1}{12} M l^2$

$I_{end} = \frac{1}{3} M l^2$

$$T = T_1 + T_2$$

$$T_1 = \frac{1}{2} M \left(\frac{l}{2}\right)^2 \dot{\phi}^2 + \frac{1}{2} I_{com} \dot{\phi}^2$$

$$= \frac{1}{8} M l^2 \dot{\phi}^2 + \frac{1}{24} M l^2 \dot{\phi}^2$$

$$= \left(\frac{1}{8} + \frac{1}{24}\right) M l^2 \dot{\phi}^2$$

$$= \frac{1}{6} M l^2 \dot{\phi}^2$$

$$T_2 = \frac{1}{2} M V^2 + \frac{1}{2} I_{com} \dot{\phi}^2$$

Now: $V^2 = \dot{x}^2 + \dot{y}^2$

$$x = \frac{3}{2} l \cos\phi, \quad y = \frac{l}{2} \sin\phi$$

$$\dot{x} = -\frac{3}{2} l \sin\phi\, \dot{\phi}, \quad \dot{y} = \frac{l}{2} \cos\phi\, \dot{\phi}$$

$$\rightarrow V^2 = \frac{9}{4} l^2 \sin^2\phi\, \dot{\phi}^2 + \frac{l^2}{4} \cos^2\phi\, \dot{\phi}^2$$

$$= 2 l^2 \sin^2\phi\, \dot{\phi}^2 + \frac{l^2}{4} \dot{\phi}^2$$

$$= 2 l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{8}\right)$$

so

$$T_2 = \frac{1}{\cancel{2}} M \cancel{2} l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{8}\right) + \frac{1}{24} M l^2 \dot{\phi}^2$$

$$= M l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{8} + \frac{1}{24}\right)$$

$$= M l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{6}\right)$$

Thus,

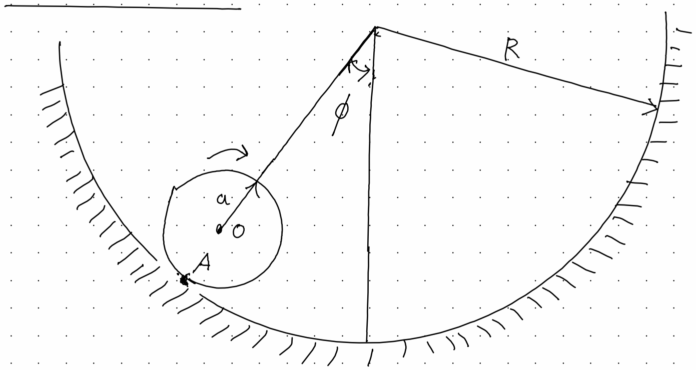
$$T = T_1 + T_2$$

$$= \frac{1}{6} M l^2 \dot{\phi}^2 + M l^2 \dot{\phi}^2 \left(\sin^2\phi + \frac{1}{6}\right)$$

$$= M l^2 \dot{\phi}^2 \left(\frac{1}{3} + \sin^2\phi\right)$$

$$= \frac{1}{3} M l^2 \dot{\phi}^2 \left(1 + 3 \sin^2\phi\right)$$

## Sec 32, Prob 6:




Homogeneous cylinder of radius $a$, mass $\mu$:

$I_3 = \frac{1}{2}\mu a^2$ (about com)

$\vec{V} = -(R-a)\dot{\phi}$ (velocity of com)

Instantaneous axis of rotation at $A$:

$0 = \vec{V} + \vec{\Omega} \times (-a\hat{n})$, $\vec{\Omega}$ into page

so $V = \Omega a$

Thus, $\Omega a = (R-a)|\dot{\phi}|$

$\Omega = \left(\frac{R-a}{a}\right)|\dot{\phi}|$

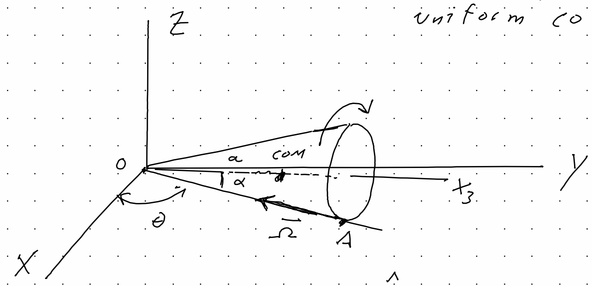
$T = \frac{1}{2}\mu V^2 + \frac{1}{2} I_3 \Omega^2$

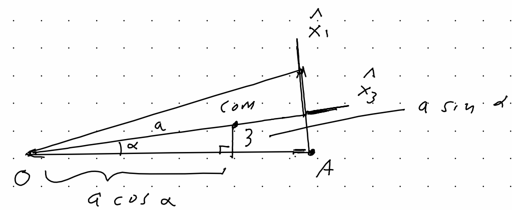
$= \frac{1}{2}\mu\left(\frac{R-a}{\cancel{a}}\right)^2\cancel{a^2}\dot{\phi}^2 + \frac{1}{4}\mu\cancel{a^2}\left(\frac{R-a}{\cancel{a}}\right)^2\dot{\phi}^2$

$= \mu(R-a)^2\dot{\phi}^2\left(\frac{1}{2}+\frac{1}{4}\right) = \boxed{\frac{3}{4}\mu(R-a)^2\dot{\phi}^2}$

## Sec 32, Prob 7.

radius $R$, height $h$
uniform cone




$V$ = velocity of COM
$= a\cos\alpha\,\dot{\theta}$

OA: instantaneous axis of rotation

$0 = \vec{V} - a\sin\alpha\,\vec{\Omega}\times\hat{n}$   $(\hat{n} = \hat{z})$

$0 = V - a\sin\alpha\,\Omega$

Thus, $a\sin\alpha\,\Omega = V = a\cos\alpha\,\dot{\theta}$

$$\boxed{\Omega = \cot\alpha\,\dot{\theta}}$$

$\vec{\Omega}$: directed from $A$ to $O$

$\Omega_3 = -\Omega\cos\alpha = -\frac{\cos^2\alpha}{\sin\alpha}\dot{\theta}$

$\Omega_1 = \Omega\sin\alpha = \cos\alpha\,\dot{\theta}$

Thus,

$$T = \frac{1}{2}\mu v^2 + \frac{1}{2}\left(I_1 \Omega_1^2 + \overset{(I_2 = I_1)}{I_1''}\, \Omega_2^2 + I_3 \Omega_3^2\right)$$

$$= \frac{1}{2}\mu\, a^2 \cos^2\alpha\, \dot\theta^2 + \frac{1}{2}\left(I_1 \cos^2\alpha\, \dot\theta^2 + I_3 \frac{\cos^4\alpha}{\sin^2\alpha}\dot\theta^2\right)$$

Now: $I_1 = I_2 = \frac{3}{20}\mu\left(R^2 + \frac{1}{4}h^2\right)$

$I_3 = \frac{3}{10}\mu R^2$

Also: $a = \frac{3}{4}h$

$\tan\alpha = \frac{R}{h} \rightarrow R = h\tan\alpha$

Thus,

$$T = \frac{1}{2}\mu\left(\frac{9}{16}\right)h^2 \cos^2\alpha\, \dot\theta^2$$

$$+ \frac{1}{2}\left[\frac{3}{20}\mu\left(h^2\tan^2\alpha + \frac{1}{4}h^2\right)\cos^2\alpha\, \dot\theta^2\right.$$

$$\left. + \frac{3}{10}\mu\, h^2 \tan^2\alpha \frac{\cos^4\alpha}{\sin^2\alpha}\dot\theta^2\right]$$

$$= \mu h^2 \dot\theta^2\left[\frac{9}{32}\cos^2\alpha + \frac{3}{40}\overset{(1-\cos^2\alpha)}{\sin^2\alpha} + \frac{3}{160}\cos^2\alpha\right.$$

$$\left. + \frac{3}{20}\cos^2\alpha\right]$$

$$= \mu h^2 \dot\theta^2\left[\frac{3}{40} + \cos^2\alpha\left(\frac{9}{32} - \frac{3}{40} + \frac{3}{160} + \frac{3}{20}\right)\right]$$

Now: $\frac{9}{32} - \frac{3}{40} + \frac{3}{160} + \frac{3}{20}$

$$= \frac{1}{160}\left[45 - 12 + 3 + 24\right]$$

$$= \frac{60}{160}$$

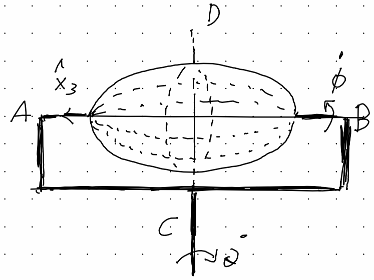
$$= \frac{15}{40}$$

Thus,

$$T = \mu h^2 \dot\theta^2\left[\frac{3}{40} + \frac{15}{40}\cos^2\alpha\right]$$

$$= \boxed{\frac{3}{40}\mu h^2 \dot\theta^2\left[1 + 5\cos^2\alpha\right]}$$

## Sec 32, Prob 9.




homogeneous ellipsoid with principal moments of inertia $I_1, I_2, I_3$

$$\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta}$$

Now: $\vec{\dot\phi} = \dot\phi\,\hat{x}_3$

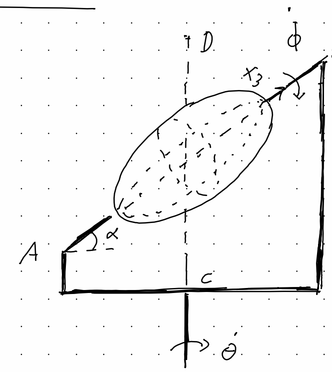
$$\vec{\dot\theta} = \dot\theta\left[\cos\phi\,\hat{x}_1 + \sin\phi\,\hat{x}_2\right]$$

so $\vec{\Omega} = \dot\theta\cos\phi\,\hat{x}_1 + \dot\theta\sin\phi\,\hat{x}_2 + \dot\phi\,\hat{x}_3$

$$T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$= \frac{1}{2}\left[\left(I_1\cos^2\phi + I_2\sin^2\phi\right)\dot\theta^2 + I_3\dot\phi^2\right]$$

## Sec 32, Prob 10.




Uniform ellipsoid with $I_1 = I_2$ (circular cross section)

$$\vec{\Omega} = \vec{\dot\phi} + \vec{\dot\theta}$$

$$\vec{\dot\phi} = \dot\phi\,\hat{x}_3$$

$$\vec{\dot\theta} = \dot\theta\left[\cos\left(\tfrac{\pi}{2}-\alpha\right)\hat{x}_3 + \sin\left(\tfrac{\pi}{2}-\alpha\right)\left(\cos\phi\,\hat{x}_1 + \sin\phi\,\hat{x}_2\right)\right]$$

Now: $\cos\left(\frac{\pi}{2}-\alpha\right) = \overset{0}{\cancel{\cos\left(\tfrac{\pi}{2}\right)}}\cos\alpha + \sin\left(\frac{\pi}{2}\right)\sin\alpha$
$= \sin\alpha$

$\sin\left(\frac{\pi}{2}-\alpha\right) = \sin\frac{\pi}{2}\cos\alpha - \overset{0}{\cancel{\cos\left(\tfrac{\pi}{2}\right)}}\sin\alpha$
$= \cos\alpha$

so $\vec{\dot\theta} = \dot\theta\left[\sin\alpha\,\hat{x}_3 + \cos\alpha\cos\phi\,\hat{x}_1 + \cos\alpha\sin\phi\,\hat{x}_2\right]$

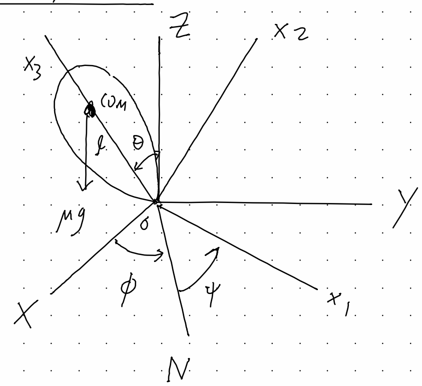
$\rightarrow \vec{\Omega} = \dot\theta\cos\alpha\left(\cos\phi\,\hat{x}_1 + \sin\phi\,\hat{x}_2\right) + \left(\dot\phi + \dot\theta\sin\alpha\right)\hat{x}_3$

$$T = \frac{1}{2}\left(I_1\Omega_1^2 + I_1\Omega_2^2 + I_3\Omega_3^2\right)$$

$$= \frac{1}{2}\left[I_1\dot\theta^2\cos^2\alpha\left(\cos^2\phi + \sin^2\phi\right) + I_3\left(\dot\phi + \dot\theta\sin\alpha\right)^2\right]$$

$$= \frac{1}{2}\left[I_1\cos^2\alpha\,\dot\theta^2 + I_3\left(\dot\phi + \dot\theta\sin\alpha\right)^2\right]$$

Sec 35, prob 1:




symmetrical top:
$I_1 = I_2$ , $I_3$ (wrt principal axes passing through COM)

$I_1' = I_1 + Ml^2$
$I_2' = I_1'$
$I_3' = I_3$
(wrt axes passing through O, which is displaced from the COM by $l$ in the $-x_3$ direction)

$L = T - U$

$U = mgZ = mgl\cos\theta$

$T = \frac{1}{2}\left( I_1'\Omega_1^2 + I_2'\Omega_2^2 + I_3'\Omega_3^2 \right)$

$= \frac{1}{2}\left[ I_1'(\Omega_1^2+\Omega_2^2) + I_3\Omega_3^2 \right]$

Now: (from (35.1))

$\Omega_1 = \dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi$

$\Omega_2 = \dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi$

$\Omega_3 = \dot\phi\cos\theta + \dot\psi$

so $\Omega_3^2 = (\dot\phi\cos\theta + \dot\psi)^2$

$\Omega_1^2 = \dot\phi^2\sin^2\theta\sin^2\psi + \dot\theta^2\cos^2\psi + 2\dot\theta\dot\psi\sin\theta\sin\psi\cos\psi$

$\Omega_2^2 = \dot\phi^2\sin^2\theta\cos^2\psi + \dot\theta^2\sin^2\psi - 2\dot\theta\dot\psi\sin\theta\cos\psi\sin\psi$

$\rightarrow \Omega_1^2+\Omega_2^2 = \dot\phi^2\sin^2\theta + \dot\theta^2$

Thus,

$$L = \frac{1}{2}\left[ I_1'(\dot\phi^2\sin^2\theta + \dot\theta^2) + I_3(\dot\phi\cos\theta+\dot\psi)^2 \right] - Mgl\cos\theta$$

1) No explicit $t$-dependence
$E = T + U = \text{const}$

2) No explicit $\phi$ dependence:

$$p_\phi \equiv \frac{\partial L}{\partial\dot\phi} = I_1'\sin^2\theta\,\dot\phi + I_3(\dot\phi\cos\theta+\dot\psi)\cos\theta = \text{const}$$

3) No explicit $\psi$ dependence:

$$p_\psi \equiv \frac{\partial L}{\partial\dot\psi} = I_3(\dot\phi\cos\theta+\dot\psi) = \text{const}$$

We can solve $p_\phi$, $p_\psi$ for $\dot\phi, \dot\psi$:

$$p_\phi = I_1' \sin^2\theta\,\dot\phi + I_3(\dot\phi\cos\theta+\dot\psi)\cos\theta$$

$$p_\psi = I_3(\dot\phi\cos\theta+\dot\psi)$$

$$\rightarrow p_\phi = I_1'\sin^2\theta\,\dot\phi + p_\psi\cos\theta$$

so $I_1'\sin^2\theta\,\dot\phi = p_\phi - p_\psi\cos\theta$

$$\boxed{\dot\phi = \frac{p_\phi - p_\psi\cos\theta}{I_1'\sin^2\theta}}$$

And:

$$\frac{p_\psi}{I_3} = \dot\phi\cos\theta+\dot\psi$$

$$\rightarrow \boxed{\dot\psi = \frac{p_\psi}{I_3} - \dot\phi\cos\theta = \frac{p_\psi}{I_3} - \left(\frac{p_\phi-p_\psi\cos\theta}{I_1'\sin^2\theta}\right)\cos\theta}$$

Also:

$$E = T+U = \frac{1}{2}\left[I_1'(\sin^2\theta\,\dot\phi^2+\dot\theta^2) + I_3(\dot\phi\cos\theta+\dot\psi)^2\right] + \mu g l\cos\theta$$

can be rewritten as

$$E = \frac{1}{2}I_1'\dot\theta^2 + \frac{1}{2}I_1'\sin^2\theta\,\dot\phi^2 + \frac{1}{2}I_3\left(\frac{p_\psi}{I_3}\right)^2 + \mu g l\cos\phi$$

$$= \frac{1}{2}I_1'\dot\theta^2 + \frac{1}{2}I_1'\sin^2\theta\frac{(p_\phi-p_\psi\cos\theta)^2}{I_1'^2\sin^4\theta} + \frac{1}{2}\frac{p_\psi^2}{I_3} + \mu g l\cos\phi$$

$$= \frac{1}{2}I_1'\dot\theta^2 + \frac{1}{2}\frac{p_\psi^2}{I_3} + \frac{1}{2}\frac{(p_\phi-p_\psi\cos\theta)^2}{I_1'\sin^2\theta} + \mu g l\cos\theta$$

Now: $\frac{1}{2}\frac{p_\psi^2}{I_3} = \text{const}$

Also: $\mu g l\cos\theta = -\mu g l(1-\cos\theta) + \mu g l$

so

$$\underbrace{E - \frac{1}{2}\frac{p_\psi^2}{I_3} - \mu g l}_{E'} = \frac{1}{2}I_1'\dot\theta^2 + \frac{1}{2}\frac{(p_\phi-p_\psi\cos\theta)^2}{I_1'\sin^2\theta} - \mu g l(1-\cos\theta)$$

$$\rightarrow E' = \frac{1}{2}I_1'\dot\theta^2 + U_{eff}(\theta)$$

where $\boxed{U_{eff}(\theta) = \frac{1}{2}\frac{(p_\phi-p_\psi\cos\theta)^2}{I_1'\sin^2\theta} - \mu g l(1-\cos\theta)}$

EOM:

$$E' = \frac{1}{2} I_1' \dot{\theta}^2 + U_{eff}(\theta)$$

$$\dot{\theta}^2 = \frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)$$

$$\frac{d\theta}{dt} = \sqrt{\frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)}$$

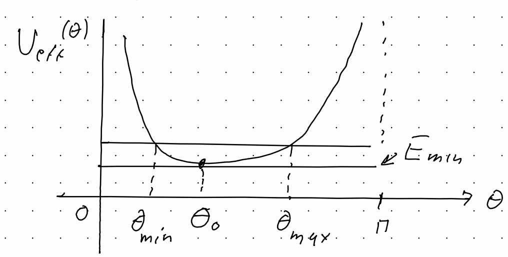
$$\rightarrow \int dt = \int \frac{d\theta}{\sqrt{\frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)}}$$

$$\boxed{t = \int \frac{d\theta}{\sqrt{\frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)}} + \text{const}}$$

Effective potential:

$$U_{eff}(\theta) = \frac{1}{2}\frac{(p_\phi - p_\psi \cos\theta)^2}{I_1' \sin^2\theta} - \mu g l (1 - \cos\theta)$$

For $p_\phi \neq p_\psi$, $U_{eff}(\theta) \to \infty$ as $\theta \to 0, \pi$




To find $\theta_0$:

$$0 = \left.\frac{dU_{eff}}{d\theta}\right|_{\theta_0}$$

$$= \frac{(p_\phi - p_\psi \cos\theta_0)\, p_\psi \sin\theta_0}{I_1' \sin^2\theta_0} - \frac{(p_\phi - p_\psi \cos\theta_0)^2 \cos\theta_0}{I_1' \sin^3\theta_0} - \mu g l \sin\theta_0$$

$$= \beta \frac{p_\psi}{I_1' \sin\theta_0} - \beta^2 \frac{\cos\theta_0}{I_1' \sin^3\theta_0} - \mu g l \sin\theta_0$$

Multiply through by $-I_1' \sin^3\theta_0$:

$$0 = \beta^2 \cos\theta_0 - \beta p_\psi \sin^2\theta_0 + \mu g l I_1' \sin^4\theta_0$$

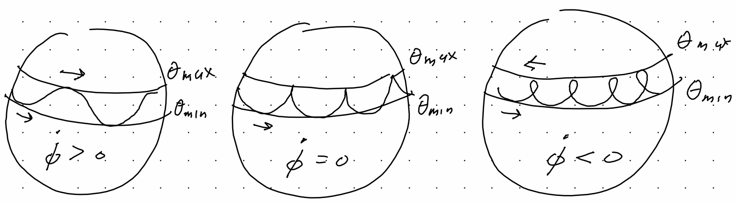
Quadratic equation for $\beta$:

$$\beta_\pm = \frac{p_\psi \sin^2\theta_0 \pm \sqrt{p_\psi^2 \sin^4\theta_0 - 4\mu g l I_1' \sin^4\theta \cos\theta_0}}{2\cos\theta_0}$$

$$= \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \sqrt{\underbrace{1 - \frac{4\mu g l I_1' \cos\theta_0}{p_\psi^2}}_{\text{need this to be } \geq 0 \text{ for a real solution.}}}\right)$$

The last equation is a transcendental equation for $\theta_0$ since $\beta = p_\phi - p_\psi \cos\theta$

For $E > E_{min}$, $\theta$ varies between $\theta_{min}$ and $\theta_{max}$. The motion of the $x_3$-axis of the top can have the following three forms depending on the sign of $\dot\phi$ when $\theta = \theta_{max}$.




This motion is called nutation.

Sec 3.5, Prob 2

For rotation of a top around a vertical axis to be stable, we need $\left.\dfrac{d^2 U_{eff}}{d\theta^2}\right|_{\theta=0} > 0$

Now:

$$U_{eff}(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1' \sin^2\theta} - \mu g l (1-\cos\theta)$$

Also:

$$p_\phi = I_1' \sin^2\theta\, \dot\phi + I_3(\dot\phi \cos\theta + \dot\psi)\cos\theta$$

$$p_\psi = I_3(\dot\phi\cos\theta + \dot\psi)$$

In the limit $\theta \to 0$

$$\left.\begin{aligned} p_\phi &\simeq I_3(\dot\phi + \dot\psi) \\ p_\psi &\simeq I_3(\dot\phi + \dot\psi) \end{aligned}\right\} \text{ so they are equal in this limit}$$

Thus,

$$U_{eff}(\theta) \simeq \frac{p_\phi^2 (1-\cos\theta)^2}{2 I_1' \sin^2\theta} - \mu g l (1-\cos\theta)$$

$$\simeq \frac{p_\phi^2 \left(\frac{\theta^2}{2}\right)^2}{2 I_1' \theta^2} - \mu g l \frac{\theta^2}{2}$$

$$= \left(\frac{1}{8}\frac{p_\phi^2}{I_1'} - \frac{1}{2}\mu g l\right)\theta^2$$

Thus

$$U_{eff}(\theta) \simeq \left(\frac{1}{8}\frac{p_\phi}{I_1'} - \frac{1}{2}\mu g l\right)\theta^2$$

$\longrightarrow$ $U_{eff}(0) = 0$

$$\left.\frac{dU_{eff}}{d\theta}\right|_{\theta=0} = 2\left(\frac{1}{8}\frac{p_\phi}{I_1'} - \frac{1}{2}\mu g l\right)\theta\Big|_{\theta=0} = 0$$

$$\left.\frac{d^2U_{eff}}{d\theta}\right|_{\theta=0} = 2\left(\frac{1}{8}\frac{p_\phi}{I_1'} - \frac{1}{2}\mu g l\right)$$

Need $\left.\frac{d^2U_{eff}}{d\theta}\right|_{\theta=0} > 0$ for stable rotation!

$$\frac{1}{8}\frac{p_\phi^2}{I_1'} - \frac{1}{2}\mu g l > 0$$

$$\boxed{p_\phi^2 > 4\mu g l I_1'}$$

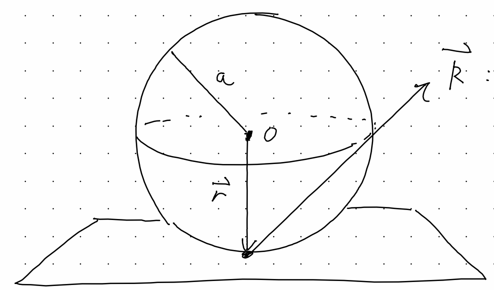
Since $p_\phi = p_\psi = I_3\Omega_3$, we can also write this condition as

$$I_3^2\Omega_3^2 > 4\mu g l I_1'$$

$$\boxed{\Omega_3^2 > \frac{4\mu g l I_1'}{I_3^2}}$$

Sec. 38, Prob. 1:

Homogeneous sphere (radius $a$) rolling without slipping on a horizontal surface, subject to applied force $\vec{F}$ and torque $\vec{K}$.




$I = \frac{2}{5}\mu a^2$ (homog. sphere about com)

$$\frac{d\vec{P}}{dt} = \vec{F} + \vec{R} \longrightarrow \mu\frac{d\vec{V}}{dt} = \vec{F} + \vec{R}$$

$$\frac{d\vec{M}}{dt} = \vec{K} + \vec{r}\times\vec{R} \rightarrow I\frac{d\vec{\Omega}}{dt} = \vec{K} - a\hat{z}\times\vec{R}$$

Rolling without slipping:

$$0 = \vec{V} + \vec{\Omega}\times\vec{r}$$
$$= \vec{V} - a\vec{\Omega}\times\hat{z}$$

So $\vec{V} = a\vec{\Omega}\times\hat{z}$

Using $(\vec{A}\times\vec{B})_1 = A_2B_3 - A_3B_2$, etc

$$\boxed{\begin{aligned} V_x &= a\Omega_y \\ V_y &= -a\Omega_x \\ V_z &= 0 \end{aligned}}$$

no motion off surface

Combine:

$$\mu \frac{d\vec{V}}{dt} = \vec{F} + \vec{R} \qquad (1)$$

$$I \frac{d\vec{\Omega}}{dt} = \vec{K} - a\hat{z} \times \vec{R} \qquad (2)$$

$$V_x = a\Omega_y, \; V_y = -a\Omega_x, \; V_z = 0 \quad \text{(constraint)}$$

Take time derivative of constraint equation:

$$\frac{dV_x}{dt} = a\frac{d\Omega_y}{dt}, \quad \frac{dV_y}{dt} = -a\frac{d\Omega_x}{dt}$$

substitute from (1), (2) into these two equations

$$\frac{1}{\mu}\left(F_x + R_x\right) = \frac{a}{I}\left(K_y - aR_x\right)$$

$$\frac{1}{\mu}\left(F_y + R_y\right) = \frac{-a}{I}\left(K_x + aR_y\right)$$

Thus,

$$F_x + R_x = \frac{\mu a}{I} K_y - \frac{\mu a^2}{I} R_x$$

$$R_x\left(\frac{I + \mu a^2}{I}\right) = \frac{\mu a}{I} K_y - F_x$$

$$R_x \frac{\frac{7}{5}\cancel{\mu a^2}}{\frac{2}{5}\cancel{\mu a^2}} = \frac{\cancel{\mu a}}{\frac{2}{5}\mu a^2} K_y - F_x$$

$$\rightarrow \boxed{R_x = \frac{5}{7}\frac{K_y}{a} - \frac{2}{7}F_x}$$

Similarly,

$$F_y + R_y = -\frac{\mu a}{I} K_x - \frac{\mu a^2}{I} R_y$$

$$R_y\left(\frac{I + \mu a^2}{I}\right) = -\frac{\mu a}{I} K_x - F_y$$

$$R_y \frac{\frac{7}{5}\cancel{\mu a^2}}{\frac{2}{5}\cancel{\mu a^2}} = -\frac{\cancel{\mu a}}{\frac{2}{5}\mu a^2} K_x - F_y$$

$$\rightarrow \boxed{R_y = -\frac{5}{7}\frac{K_x}{a} - \frac{2}{7}F_y}$$

Also,

$$\mu \frac{d\cancelto{0}{V_z}}{dt} = F_z + R_z$$

$$\rightarrow \boxed{R_z = -F_z}$$

Using these expression for $R_x, R_y$ we can write down EOMs for $V_x, V_y$

$$\boxed{\mu \frac{dV_x}{dt} = F_x + R_x = F_x + \frac{5}{7}\frac{K_y}{a} - \frac{2}{7}F_x = \frac{5}{7}\left(F_x + \frac{K_y}{a}\right)}$$

Similarly

$$\boxed{\mu \frac{dV_y}{dt} = F_y + R_y = F_y - \frac{5}{7}\frac{K_x}{a} - \frac{2}{7}F_y = \frac{5}{7}\left(F_y - \frac{K_x}{a}\right)}$$

Summary:

$$R_x = \frac{5}{7}\frac{K_y}{a} - \frac{2}{7}F_x$$

$$R_y = -\frac{5}{7}\frac{K_x}{a} - \frac{2}{7}F_y$$

$$R_z = -F_z$$

$$\mu \frac{dV_x}{dt} = \frac{5}{7}\left(F_x + \frac{K_y}{a}\right)$$

$$\mu \frac{dV_y}{dt} = \frac{5}{7}\left(F_y - \frac{K_x}{a}\right)$$

$$V_z = 0$$

$$\Omega_x = -\frac{V_y}{a}$$

$$\Omega_y = \frac{V_x}{a}$$

$$I \frac{d\Omega_z}{dt} = K_z$$

Example:

Suppose: $\vec{F} = -\mu g \hat{z} + F_0 \hat{x}$

$\vec{K} = 0$

Then: $F_x = F_0$, $F_y = 0$, $F_z = -\mu g$

Solution from previous page gives:

$$R_x = -\frac{2}{7}F_0$$

$$R_y = 0$$

$R_z = \mu g$ (normal force upward)

$$\mu \frac{dV_x}{dt} = \frac{5}{7}F_0 \rightarrow V_x = \frac{5}{7}\frac{F_0}{\mu}t + V_{x0}$$

$$\mu \frac{dV_y}{dt} = 0 \rightarrow V_y = \text{const} = V_{y0}$$

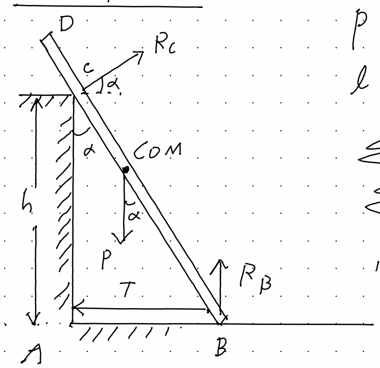
$$V_z = 0$$

$$\Omega_x = -\frac{V_{y0}}{a}$$

$$\Omega_y = \frac{V_x}{a} = \frac{5}{7}\frac{F_0}{\mu a}t + \frac{V_{x0}}{a}$$

$$I \frac{d\Omega_z}{dt} = 0 \rightarrow \Omega_z = \text{const} = \Omega_{z0}$$

Sec 38, Prob 2




$P$ = weight = $\mu g$

$l$ : length of uniform rod

$\sum \vec{F} = 0$

$\sum \vec{r} \times \vec{F} = 0$

including reaction force

$T$, $R_c$, $R_B$

horizontal direction

$$-T + R_c \cos\alpha = 0 \qquad (1)$$

vertical direction

$$R_c \sin\alpha - P + R_B = 0 \qquad (2)$$

torques around D :

$$\frac{l}{2} \sin\alpha \, P - \frac{h}{\cos\alpha} R_c = 0 \qquad (3)$$


$\cos\alpha = \frac{h}{d}$

$\rightarrow d = \frac{h}{\cos\alpha}$

Thus,

$$\frac{h}{\cos\alpha} R_c = \frac{l}{2} \sin\alpha \, P$$

$$\boxed{R_c = \frac{lP}{2h} \sin\alpha \cos\alpha}$$

$$\boxed{T = R_c \cos\alpha}$$

$$\boxed{R_B = P - R_c \sin\alpha}$$