Notes: Thurs 8/27

1) Elliptic functions
2) Simple pendulum

go beyond small angle approx

Elliptic Functions:

$$\int \frac{dx}{\sqrt{1-x^2}} = ? = \sin^{-1}(x) + \text{const}$$

$$\sin^{-1}(x) = \arcsin(x)$$

$\rightarrow \sin^2\theta + \cos^2\theta = 1$

subst: $x = \sin\theta \rightarrow \sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$dx = \cos\theta \, d\theta$

$$\int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \int d\theta = \theta = \sin^{-1}(x)$$

$x = \sin\theta$

$\theta = \sin^{-1}(x)$

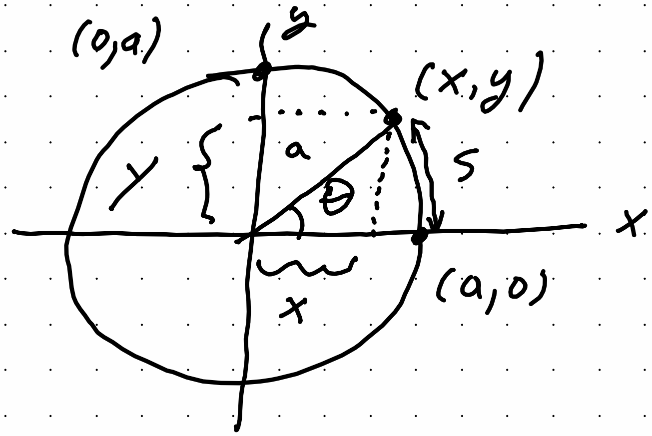
$$\frac{d}{d\theta}\sin\theta = \cos\theta \qquad \lim_{h\to 0}\left(\frac{\sin(\theta+h)-\sin\theta}{h}\right)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

circular functions..

$x^2+y^2=a^2$, a = radius



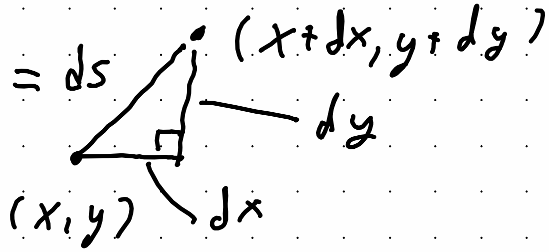
Def: $\sin\theta = \dfrac{y}{a}$, $\cos\theta = \dfrac{x}{a}$

s : arclength from (a,0) to (x,y)

$s = a\theta$

$$\theta = \frac{s}{a} = \frac{1}{a}\int ds = \frac{1}{a}\int\sqrt{dx^2+dy^2} = \int d\theta$$

$\sqrt{dx^2+dy^2} = ds$



Given: $x^2+y^2=a^2$

Follows: (i) $\cancel{a^2}\cos^2\theta + \cancel{a^2}\sin^2\theta = \cancel{a^2} \rightarrow \boxed{\cos^2\theta+\sin^2\theta=1}$

(ii) $\boxed{\dfrac{d}{d\theta}\sin\theta} = \dfrac{1}{a}\dfrac{dy}{d\theta} = \dfrac{\frac{1}{\cancel{a}}\,dy}{\frac{1}{\cancel{a}}\sqrt{dx^2+dy^2}} = \dfrac{1}{\sqrt{\left(\frac{y}{x}\right)^2+1}}$

$\cancel{2}x\,dx + \cancel{2}y\,dy = 0 \rightarrow dx = -\dfrac{y}{x}dy$

$= \dfrac{1}{\sqrt{\dfrac{x^2+y^2}{x^2}}} = \dfrac{x}{a} = \boxed{\cos\theta}$

$$\frac{d(\sin\theta)}{d\theta} = \cos\theta \longrightarrow \int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta$$

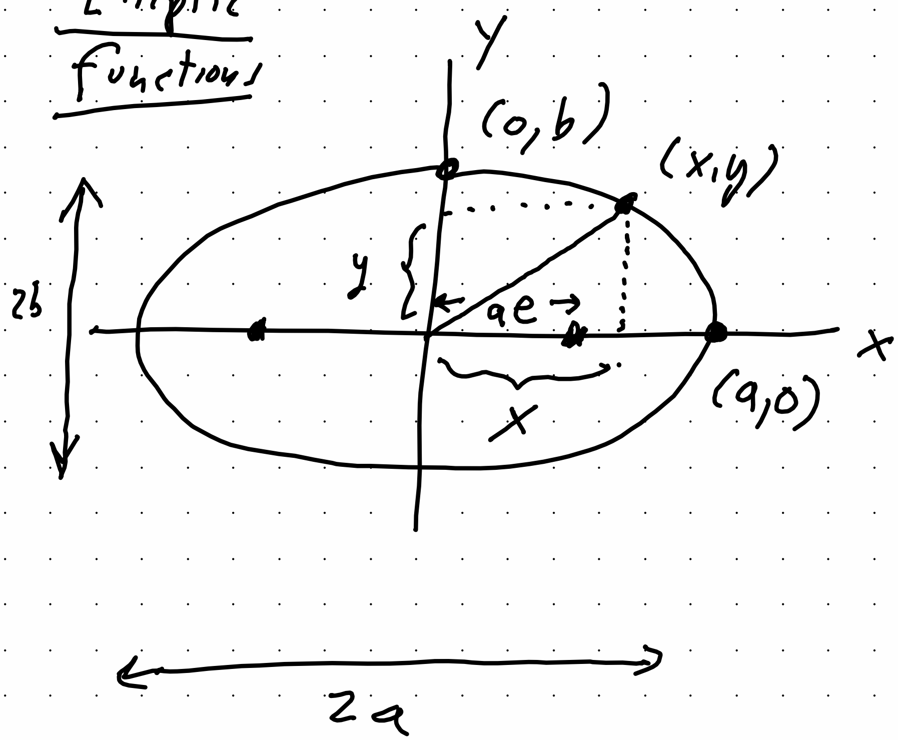
$x = \sin\theta$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$= \sqrt{1-x^2}$

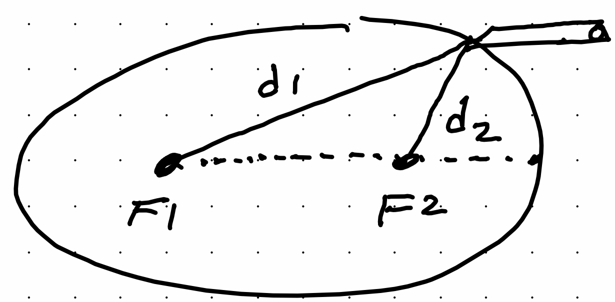
$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \theta = \sin^{-1}(x)}$$

Elliptic functions

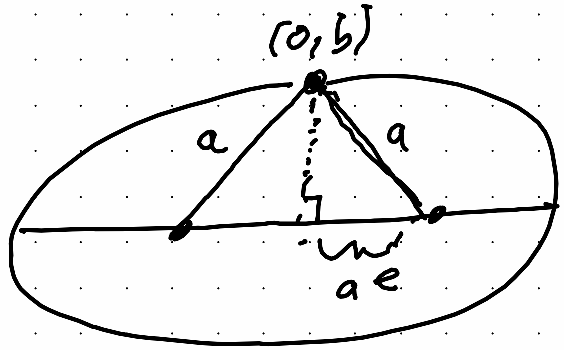


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

eccentricity : $e = 0$ (for circle)



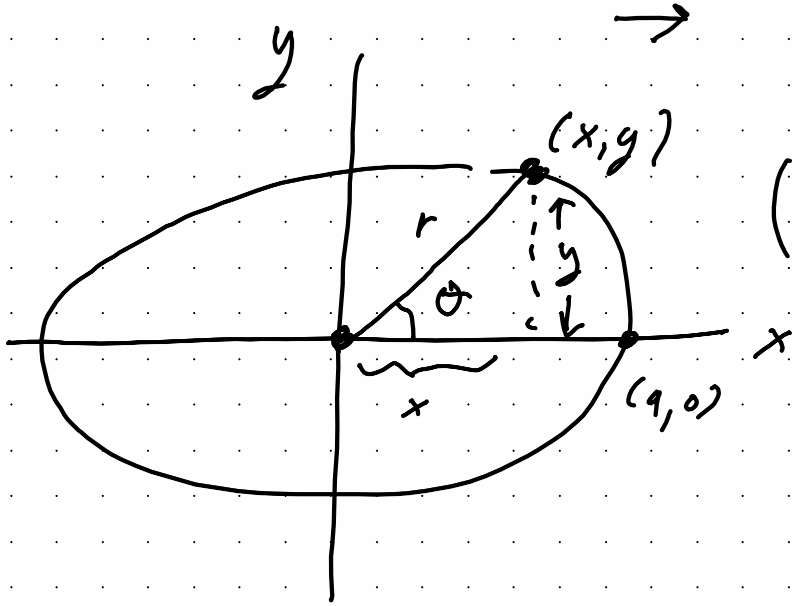
$$d_1 + d_2 = 2a$$



$$(ae)^2 + b^2 = a^2$$
$$a^2e^2 + b^2 = a^2$$
$$e^2 = \frac{a^2 - b^2}{a^2} = 1 - \left(\frac{b}{a}\right)^2$$

$$\rightarrow \quad e = \sqrt{1 - \left(\frac{b}{a}\right)^2} \equiv k$$

notation used in elliptic function



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$x^2 + y^2 = r^2$$
$$x = r\cos\theta, \quad y = r\sin\theta$$

Define: $\quad cn(u;k) \equiv \frac{x}{a}, \quad sn(u;k) \equiv \frac{y}{b}$

$$dn(u;k) \equiv \frac{r}{a} \quad \text{— } (=1 \text{ for a circle})$$

where $u = \frac{1}{b}\int_0^\theta r\,d\theta$ (=θ for a circle) $\qquad y = r\sin\theta$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\theta^2}$$

Given: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, $x^2+y^2=r^2$ $\qquad dn(u;k) = \frac{r}{a}$

Follow: (i) $cn^2(u;k) + sn^2(u;k) = 1$

(ii) $dn^2(u;k) + k^2 sn^2(u;k) = 1$

$$u = \frac{1}{b}\int_0^\theta r\, d\theta$$

(iii) $\frac{d}{du} sn(u;k) = cn(u;k)\, dn(u;k)$

$\frac{d}{du} cn(u;k) = -sn(u;k)\, dn(u;k)$

$\frac{d}{du} dn(u;k) = -k^2 sn(u;k)\, cn(u;k)$

Integrate: $\frac{d\, sn(u;k)}{du} = cn(u;k)\, dn(u;k)$

$$\int \frac{d\, sn(u;k)}{cn(u;k)\, dn(u;k)} = \int du = u$$

Analogous to

$\frac{d \sin\theta}{d\theta} = \cos\theta$

$\frac{d\sin\theta}{\cos\theta} = d\theta$

$x = \sin\theta$

$\int \frac{dx}{\sqrt{1-x^2}} = \int d\theta = \theta = \sin^{-1} x$

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}} = u + \text{const} = sn^{-1}(x;k) + \text{const} \qquad x \equiv sn(u;k)}$$

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + \text{const}}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \equiv K(k) \longrightarrow$$

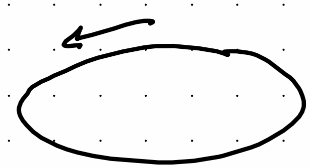
related to Period of a pendulum going beyond small-angle approximation

(complete elliptic integral of 1st kind)

$$\int_0^1 \frac{\sqrt{1-k^2x^2}\,dx}{\sqrt{1-x^2}} \equiv E(k) \longrightarrow$$

circumference around an ellipse

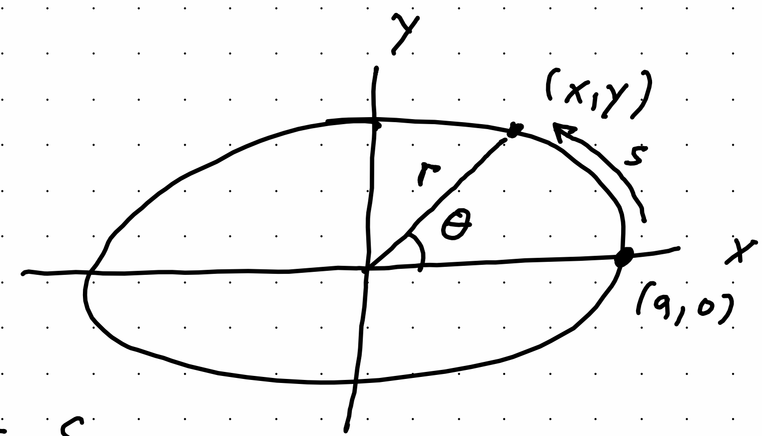
(complete elliptic integral of 2nd kind)



circle: $C = 2\pi a$

Notes: Tuesday 9/1

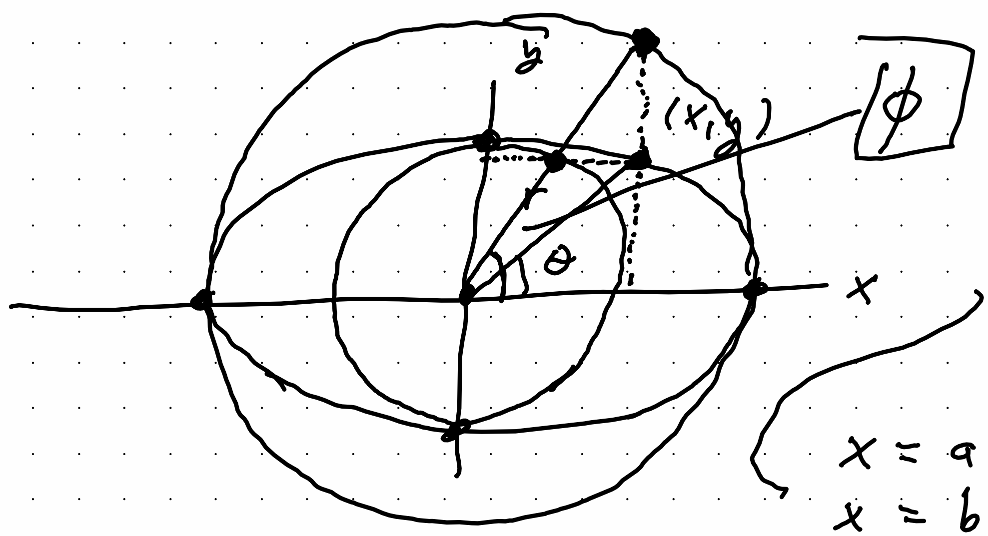
1) Review of elliptic functions
2) Simple pendulum



$$u = \frac{1}{b}\int_{(a,0)}^{(x,y)} r\,d\theta$$

$$bu = \int_0^\theta r\,d\theta \le \int_0^\theta ds = s$$

$$ds = \sqrt{dx^2+dy^2} = \sqrt{dr^2+r^2d\theta^2} \ge r\,d\theta$$



$x = r\cos\theta$

$y = r\sin\theta$

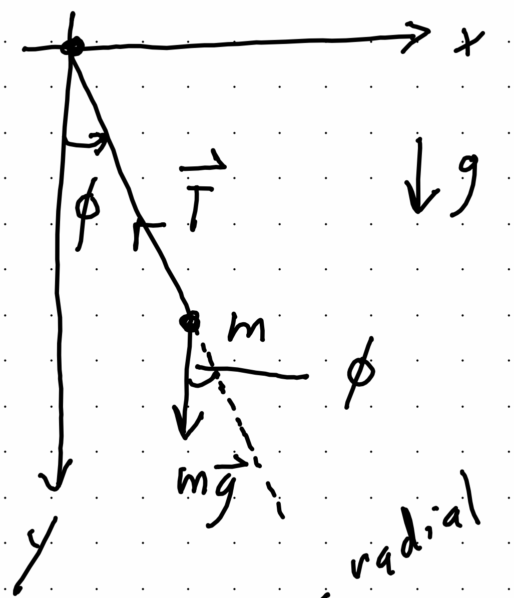
$x = a\cos\phi$

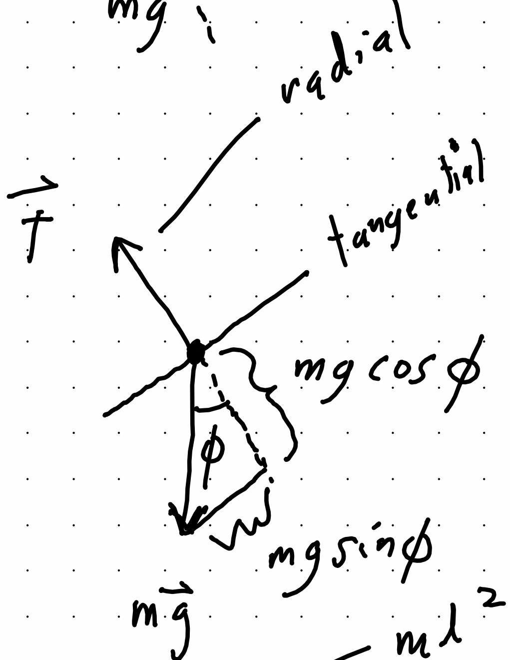
$y = b\sin\phi$

$x = a\cos\phi$, $y = \underline{a}\sin\phi$

$x = b\cos\phi$ $y = b\sin\phi$

simple pendulum:





$$\text{Torque} = \overset{ml^2}{I}\,\overset{\ddot{\phi}}{\alpha}$$

(i) "Freshman physics"

forces, free-body diagram

$\longrightarrow$ EOM, tension

tangential:

$$-mg\sin\phi = m\,a_{tangential}$$

$$-\cancel{m}g\sin\phi = \cancel{m}\,l\,\ddot{\phi}$$

ϕ : angular displacement [rad]

$\dot{\phi}$: angular velocity [rad/sec]

$\ddot{\phi}$: angular accel [rad/sec^2]

$$\boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi} \quad \text{(EOM)}$$

radial: $T - mg\cos\phi = m\,a_{centripetal}$

$$T - mg\cos\phi = m\dot{\phi}^2 l$$

$$T = mg\cos\phi + m\dot{\phi}^2 l$$

(ii) Small angle approximation:

$$\sin\phi \approx \phi \longleftarrow \underset{\text{radian}}{\phi \ll 1} = 57 \text{ degrees}$$

$$\ddot{\phi} = -\frac{g}{l}\sin\phi \approx -\frac{g}{l}\phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\phi}$$

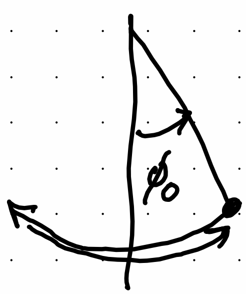
small angle approx.

$$\phi(t) = \boxed{a}\cos(\omega t) + \boxed{b}\sin(\omega t)$$

where $\omega \equiv \sqrt{\frac{g}{l}}$

determined by initial conditions

ICs: If $\phi(0) = \phi_0$ (at rest)



then $\boxed{\phi(t) = \phi_0 \cos(\omega t)}$

Period: $P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$

independent of ϕ_0 !!

(iii) Lagrangian approach

$T \equiv$ Kinetic energy

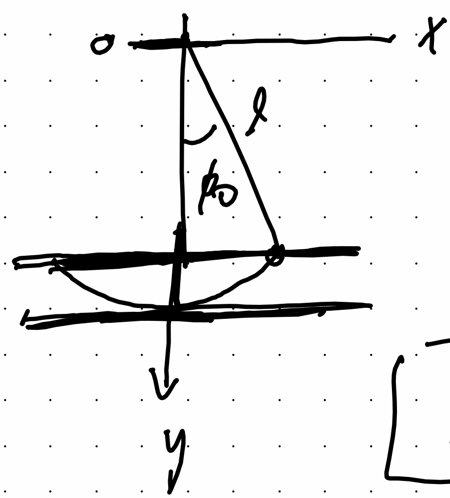
$U \equiv$ Potential energy

$$L = T - U$$

$$\boxed{T = \frac{1}{2} m l^2 \dot{\phi}^2} \quad \left(= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \right)$$

$y = l \cos\phi$

$x = l \sin\phi$



$$\boxed{U = -mgl\cos\phi + \text{const}} \leftarrow U = -mgy$$

$$U = mgl(1 - \cos\phi)$$

$$\boxed{L = \frac{1}{2} m l^2 \dot{\phi}^2 + mgl\cos\phi}$$

action

$$\boxed{S = \int_{t_1}^{t_2} dt\, L(\phi, \dot{\phi}, t)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

Lagrange's equation

$$\frac{d}{dt}\left(m l^2 \dot{\phi}\right) = -mgl\sin\phi$$

$$m l^2 \ddot{\phi} = -mgl\sin\phi \rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi} \quad (\text{EOM})$$

(iv) solving $\ddot{\phi} = -\frac{g}{l}\sin\phi$ (2nd order non-linear ODE) hard!!

$$E = \text{const}$$
$$= T + U$$
$$= \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$$

$$E = 0 - mgl\cos\phi_0$$
$$= -mgl\cos\phi_0$$

release from rest from $\phi = \phi_0$

$$-mgl\cos\phi_0 = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$$
$$-\cancel{m}gl(\cos\phi_0 - \cos\phi) = \frac{1}{2}\cancel{m} l^2 \dot{\phi}^2$$

$$\frac{d\phi}{dt} = \dot{\phi} = \sqrt{-2\frac{g}{l}(\cos\phi_0 - \cos\phi)}$$

$|\phi| \leq \phi_0$

Separable 1st order ODE

$$\int \frac{d\phi}{\sqrt{-\frac{2g}{l}(\cos\phi_0 - \cos\phi)}} = \int dt = t + \text{const}$$

$$t + t_0 = \int \frac{d\phi}{\sqrt{-2\frac{g}{l}\left(\cos\phi_0 - \cos\phi\right)}}$$

$$\frac{1}{\sqrt{a+bx^2}}$$

substitution:

$$\cos\phi = 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\cos\phi_0 = 1 - 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$\cos\phi = \cos\left(2\left(\frac{\phi}{2}\right)\right)$$
$$= \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)$$
$$= 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\rightarrow \cos\phi_0 - \cos\phi = -2\left(\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)\right)$$

$$t + t_0 = \int \frac{d\phi}{2\sqrt{\frac{g}{l}}\sqrt{\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)}}$$

$$|\phi| \leq \phi_0$$

$$= \frac{1}{2\sqrt{\frac{g}{l}}} \int \frac{d\phi}{\sin\left(\frac{\phi_0}{2}\right)\sqrt{1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)}}}$$

let $x = \frac{\sin(\phi/2)}{\sin\left(\frac{\phi_0}{2}\right)}$

$$\boxed{x = \frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)}} \longrightarrow dx = \frac{1}{\sin\left(\frac{\phi_0}{2}\right)} \frac{1}{2} \cos\left(\frac{\phi}{2}\right) d\phi$$

$$\sqrt{1 - x^2}$$

denominator

$$d\phi = \frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\cos\left(\frac{\phi}{2}\right)} = \frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1 - \sin^2\left(\frac{\phi}{2}\right)}} = \frac{2 \sin\left(\frac{\phi_0}{2}\right) dx}{\sqrt{1 - \underbrace{\sin^2\left(\frac{\phi_0}{2}\right)}_{k^2} x^2}}$$

Find this out

1. $\phi(t) =$
2. Period = ??
3. Redo the analysis using Lagrange multiplier for find tension in string

$$t + t_0 = \int \bigcirc$$ — integrand for $sn^{-1}(x; k)$

$$k = \sin\left(\frac{\phi_0}{2}\right)$$

Lec #4 : Thurs 9/3

$$\phi(t) = 2\sin^{-1}\left[k\,\mathrm{sn}\left(\omega_0\left(t+\frac{P}{4}\right);k\right)\right] \quad ☆$$

$$k \equiv \sin\left(\frac{\phi_0}{2}\right), \qquad \omega_0 = \sqrt{\frac{g}{l}}$$

$$P = 4\sqrt{\frac{l}{g}}\,K(k) = 4\sqrt{\frac{l}{g}}\int_0^1 \frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}}$$

small angle approx

$$P_0 = 2\pi\sqrt{\frac{l}{g}}$$

$$\frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)} = x = \mathrm{sn}\left[\sqrt{\frac{g}{l}}(t+t_0);k\right]$$

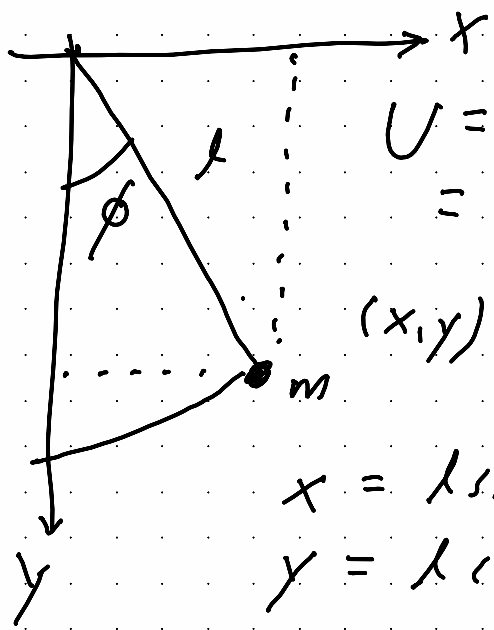
$$\sqrt{\frac{l}{g}}\int\frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}} = \sqrt{\frac{l}{g}}\,\mathrm{sn}^{-1}(x;k) = t+t_0$$

$$\mathrm{sn}^{-1}(x;k) = \sqrt{\frac{g}{l}}(t+t_0)$$

$$P = 4\sqrt{\frac{l}{g}}\,K(k) = 2\pi\sqrt{\frac{l}{g}}\underbrace{\left(1 + \frac{1}{16}\phi_0^2 + \dots\right)}_{\text{problem Landau 11.1}}$$

Lagrange multiplier:

$$\underbrace{\varphi(x,y)}_{\text{constraint function}} = x^2 + y^2 - l^2 = 0$$



$$U = -mgy$$
$$= -mgl\cos\phi$$

$$x^2 + y^2 = l^2\sin^2\phi + l^2\cos^2\phi$$
$$= l^2$$

$$x = l\sin\phi$$
$$y = l\cos\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2}ml^2\dot{\phi}^2$$

$$L = T - U + \lambda \varphi$$

Lagrange multiplier

$\lambda, \cancel{\dot{\lambda}}$

$L(x, \dot{x}, y, \dot{y}, t) \quad q = (x, y) \quad \dot{q} = (\dot{x}, \dot{y})$

$L(r, \dot{r}, \phi, \dot{\phi}, t) \quad q = (r, \phi) \quad \dot{q} = (\dot{r}, \dot{\phi})$

$\lambda(t)$
$r(t)$
$\phi(t)$

$L(\phi, \dot{\phi}, t) \quad \varphi(x, y) = x^2 + y^2 - l^2 = 0$

$$\boxed{\varphi(r, \phi) = r - l = 0}$$

$$\boxed{L = \frac{1}{2} m (\dot{r}^2 + \underline{r^2 \dot{\phi}^2}) + mg \underbrace{r\cos\phi}_{y} + \lambda (r - l)}$$

$\underline{r}: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \rightarrow \boxed{\cancel{m\ddot{r}} = mr\dot{\phi}^2 + mg\cos\phi + \lambda}$

$\underline{\phi}: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \rightarrow \frac{d}{dt}(mr^2\dot{\phi}) = -mgr\sin\phi$

$\underline{\lambda}: \quad r - l = 0 \quad \checkmark \qquad \cancel{2m\dot{r}r\dot{\phi}} + mr^2\ddot{\phi} = -mgr\sin\phi$

$$r - l = 0 \longrightarrow r = l$$

$$\dot{r} = 0$$

$$\ddot{r} = 0$$

$$\longrightarrow \quad \ddot{\phi} = -\frac{g}{l}\sin\dot{\phi}$$

$$m l \dot{\phi}^2 + m g \cos\phi + \lambda = 0$$

$$\boxed{\lambda = -\underbrace{\left(m g \cos\phi + m l \dot{\phi}^2\right)}_{T}}$$

$$\lambda = -T$$

$$L = T - U + \lambda \varphi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}(m\dot{x}) = -\frac{\partial U}{\partial x} + \lambda \frac{\partial \varphi}{\partial x}$$

$U = U(x,t)$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$\varphi(x, y, t)$

$U(x, y, t)$

$$\frac{d}{dt}(m\dot{y}) = -\frac{\partial U}{\partial y} + \lambda \frac{\partial \varphi}{\partial y}$$

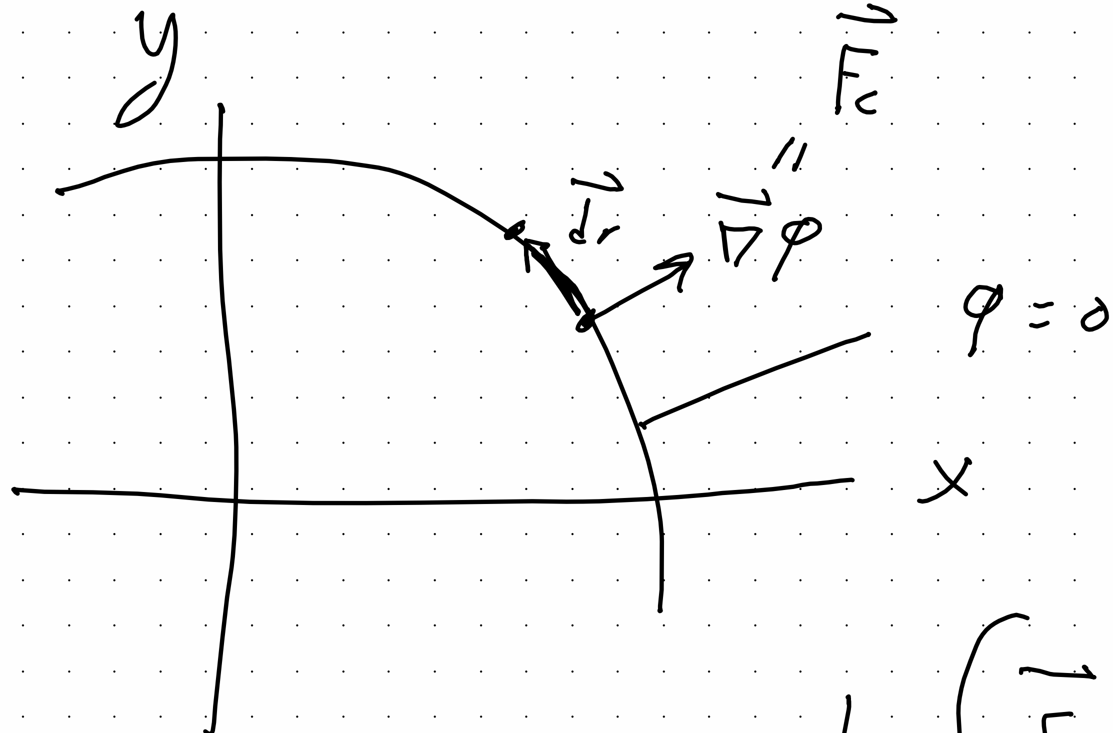
$$\boxed{\frac{d\vec{p}}{dt} = \vec{F}_{net}}$$

applied force force constraint

$$\left.\begin{aligned} \frac{dp_x}{dt} &= F_x + \lambda \frac{\partial \varphi}{\partial x} \\ \frac{dp_y}{dt} &= F_y + \lambda \frac{\partial \varphi}{\partial y} \end{aligned}\right\}$$

$$\boxed{\frac{d\vec{p}}{dt} = \vec{F} + \lambda \vec{\nabla}\varphi}$$

$$\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) = \vec{\nabla}\varphi$$



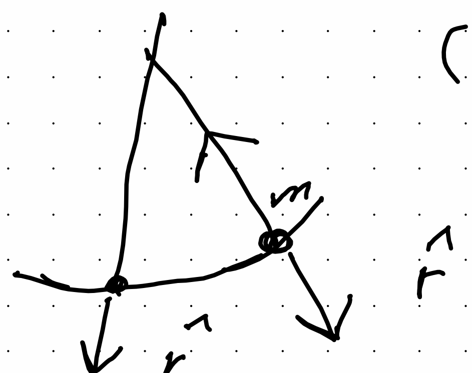
$\vec{\nabla}\varphi \perp \varphi = \text{const}$

$\int \underbrace{\vec{F}_c \cdot d\vec{r}}_{0} = \text{Work}$

$\varphi = r - l$

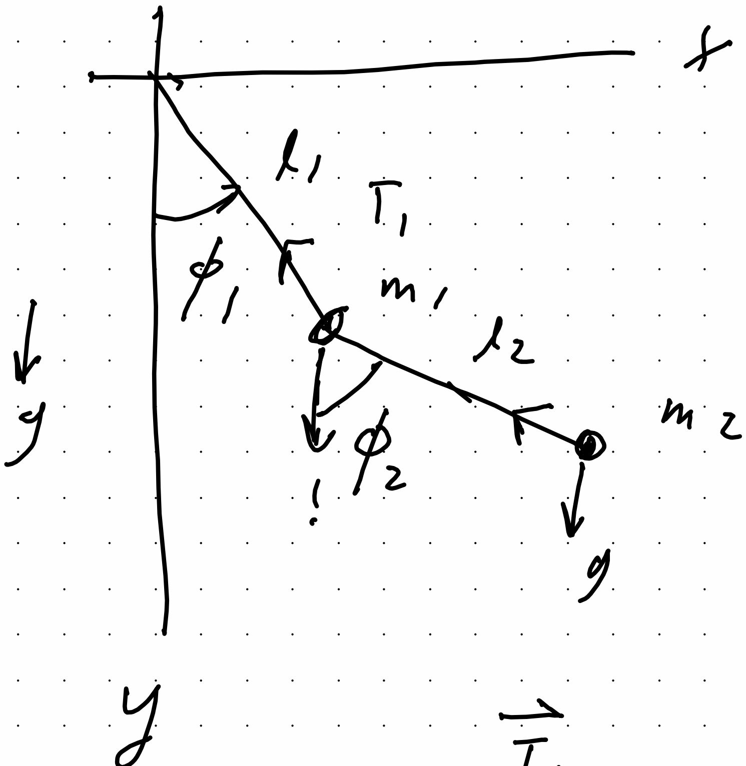
$\vec{\nabla}\varphi = \hat{r}$

$\lambda \vec{\nabla}\varphi$



(r, ϕ)

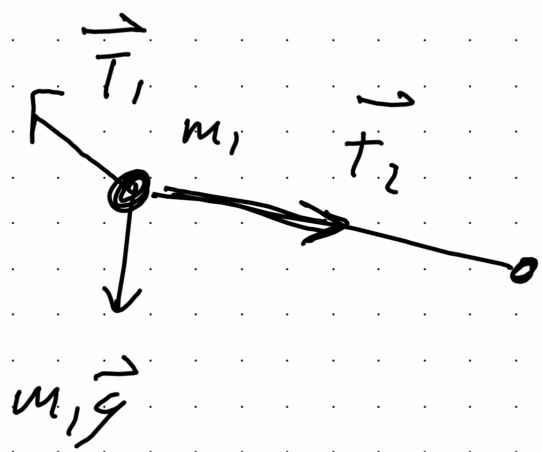
$\hat{r}, \hat{\phi}$



$$U = -m_1 g y_1 - m_2 g y_2$$

$$\ddot{\phi}_1, \ \ddot{\phi}_2$$

EOM



$\lambda \varphi$

$$\left(\lambda_1 \varphi_1 + \lambda_2 \varphi_2 + \ldots \right)$$

$L \longrightarrow$ EoM

$$L \longrightarrow \underbrace{L + \frac{d}{dt}\left(f(q,t)\right)}_{L'}$$

$f(q, \cancel{\dot{q}}, t)$