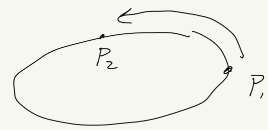
Lecture #1: Aug 24th

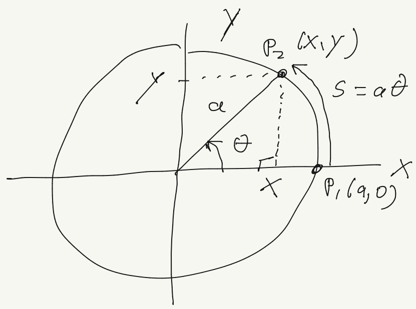
Elliptic functions / integrals:

i) circumference of an ellipse

ii) period of a simple pendulum beyond the small-angle approximation



Circular functions: sines, cosine,

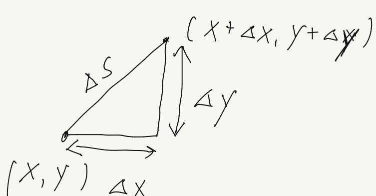


a = radius

$$x^2 + y^2 = a^2$$

$$\theta = \frac{s}{a} = \frac{1}{a}\int_{P_1}^{P_2}\sqrt{dx^2+dy^2} = \int d\theta$$

$$ds^2 = dx^2 + dy^2$$



$$\left.\begin{aligned} \sin\theta &\equiv y/a \\ \cos\theta &\equiv x/a \end{aligned}\right\} \text{definitions}$$

$$x = a\cos\theta$$
$$y = a\sin\theta$$

$$\boxed{x^2 + y^2 = a^2} \rightarrow \cancel{a^2}\cos^2\theta + \cancel{a^2}\sin^2\theta = \cancel{a^2}$$

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

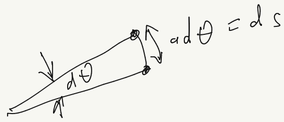
$$\boxed{\frac{d}{d\theta}\sin\theta = \cos\theta}$$

<u>Proof</u>: $\frac{d}{d\theta}\sin\theta = \frac{d}{d\theta}\left(\frac{y}{a}\right)$

$$= \frac{1}{a}\frac{dy}{d\theta}$$

$$= \frac{dy}{\sqrt{dx^2+dy^2}}$$

$$= \frac{1}{\frac{\sqrt{dx^2+dy^2}}{dy}} = \frac{1}{\sqrt{\left(\frac{dx}{dy}\right)^2+1}}$$



$$a\,d\theta = \sqrt{dx^2+dy^2}$$

$$x^2+y^2=a^2 \rightarrow 2x\,dx+2y\,dy=0$$

$$\frac{dx}{dy} = \frac{-y}{x}$$

$$\frac{d}{d\theta}\sin\theta = \frac{1}{\sqrt{\left(\frac{-y}{x}\right)^2+1}} = \frac{1}{\sqrt{\frac{y^2}{x^2}+1}} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{a} = \cos\theta$$

$$\boxed{\frac{d}{d\theta}\sin\theta = \cos\theta}$$

Similarly $\boxed{\frac{d\cos\theta}{d\theta} = -\sin\theta}$

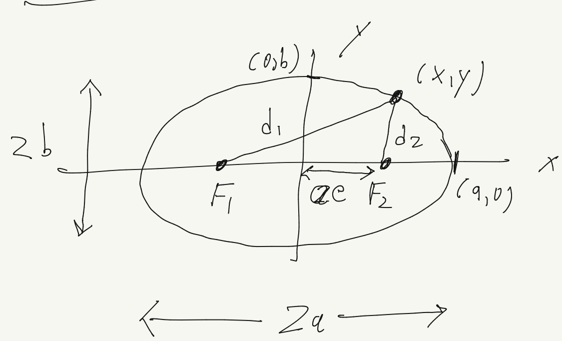
$$\int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta = \theta + \text{const}$$

$t = \sin\theta$

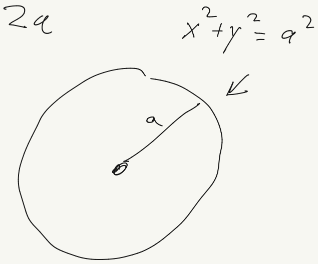
$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-t^2}$

$$\boxed{\int \frac{dt}{\sqrt{1-t^2}} = \theta + \text{const} = \sin^{-1} t + \text{const}}$$

Lec #2: Aug 26th



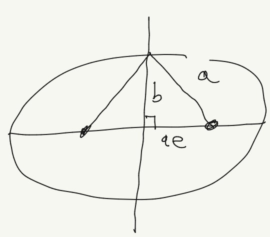
$d_1+d_2 = 2a$



$e \overset{?}{\cancel{=}} \frac{b}{a}$

$e \cancel{=} 1-\frac{b}{a}$

$e^2 = \frac{b^2}{1-a^2}$



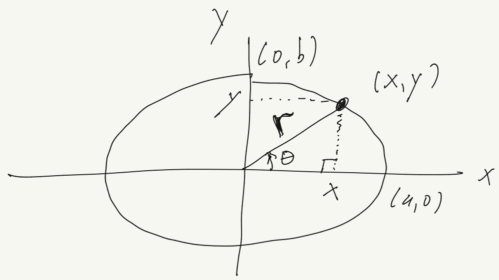
$$\boxed{e = \sqrt{1-\left(\frac{b}{a}\right)^2}}$$

Pf: $a^2 = b^2 + (ae)^2$

$a^2(1-e^2) = b^2$

$1-e^2 = \left(\frac{b}{a}\right)^2$

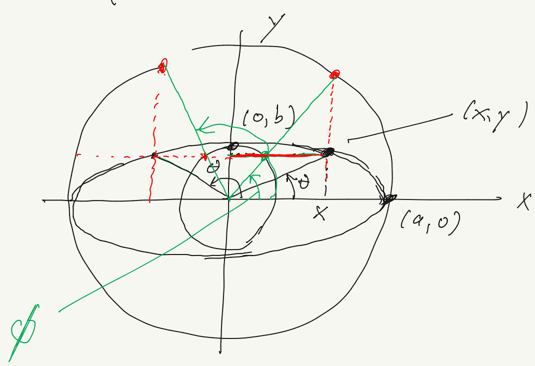
$e = \sqrt{1-\left(\frac{b}{a}\right)^2}$



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$\uparrow$

r changes

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \longleftarrow \begin{cases} x = a\cos\phi \\ y = b\sin\phi \end{cases}$$



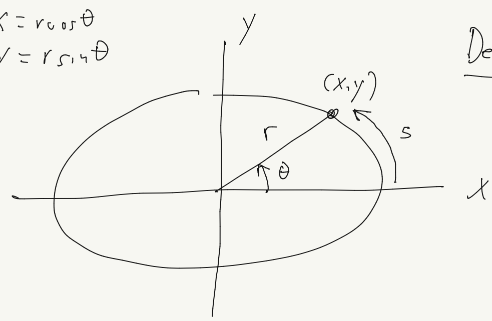
$0 < e < 1$

$e = 0$ circle

$e = 1$ parabola

$e > 1$ hyperbola

$x = r\cos\theta$
$y = r\sin\theta$



Defini:

$$\boxed{\begin{aligned} cn(u;k) &\equiv x/a \\ sn(u;k) &= y/b \\ dn(u;k) &= \frac{r}{a} \end{aligned}}$$

$$u \equiv \frac{1}{b}\int_0^\theta r\,d\theta$$

$\uparrow$

not θ, not arc length

$k = e$, $\underbrace{0 < k < 1}_{\text{modulus}}$

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$dx = dr\cos\theta - r\sin\theta\,d\theta$$
$$dy = dr\sin\theta + r\cos\theta\,d\theta$$

$$ds = \sqrt{dx^2 + dy^2}$$
$$= \sqrt{dr^2 + r^2 d\theta^2}$$

$$bu = \int_0^\theta r\,d\theta < s$$

$cn(u)$ $sn(u)$	pendulum $k = \sin\left(\frac{\phi_0}{2}\right)$

Property:

$$\boxed{cn^2 u + sn^2 u = 1} \leftarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\leftarrow x^2 + y^2 = r^2$$

$$\begin{aligned}
dn^2 u &= cn^2 u + \left(\frac{b}{a}\right)^2 sn^2 u \\
&= 1 - sn^2 u + \left(\frac{b}{a}\right)^2 sn^2 u \\
&= 1 - sn^2 u \left(1 - \left(\frac{b}{a}\right)^2\right) \\
&= 1 - k^2 sn^2 u
\end{aligned}$$

$$\boxed{dn^2 u + k^2 sn^2 u = 1}$$

$$\frac{d}{du} sn\, u = \frac{1}{b}\frac{dy}{du} = \frac{dy}{r\,d\theta}$$

$$u = \frac{1}{b}\int_0^{\theta} r\,d\theta$$

$$du = \frac{r\,d\theta}{b} \rightarrow b\,du = r\,d\theta$$

$x = r\cos\theta$, $y = r\sin\theta$

$$dx = dr\cos\theta - r\sin\theta\,d\theta \rightarrow \sin\theta\,dx = \sin\theta\cos\theta\,dr - r\sin^2\theta\,d\theta$$

$$dy = dr\sin\theta + r\cos\theta\,d\theta \rightarrow -\cos\theta\,dy = -\cos\theta\sin\theta\,dr - r\cos^2\theta\,d\theta$$

add

$$\sin\theta\,dx - \cos\theta\,dy = -r\,d\theta$$

$$\frac{y}{r}dx - \frac{x}{r}dy = -r\,d\theta$$

$$\rightarrow \boxed{r\,d\theta = -\frac{y}{r}dx + \frac{x}{r}dy}$$

$$\frac{d}{du} sn\, u = \frac{dy}{-\frac{y}{r}dx + \frac{x}{r}dy} = \frac{r}{-y\frac{dx}{dy} + x}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad \cancel{2}\frac{x\,dx}{a^2} + \cancel{2}\frac{y\,dy}{b^2} = 0$$

$$\frac{dx}{dy} = -\frac{y}{x}\frac{a^2}{b^2}$$

$$\frac{d\,\mathrm{sn}\,u}{du} = \frac{r}{-y\left(\frac{-y}{x}\right)\frac{a^2}{b^2} + x} = \frac{r\,x}{y^2\left(\frac{a}{b}\right)^2 + x^2}$$

$$= \frac{r}{a}\frac{x}{a}\left(\frac{1}{\left(\frac{y}{b}\right)^2 + \left(\frac{x}{a}\right)^2}\right)$$

$$= \frac{\mathrm{dn}\,u \cdot \mathrm{cn}\,u}{1}$$

$$\boxed{\frac{d}{du}\mathrm{sn}\,u = \mathrm{cn}\,u \cdot \mathrm{dn}\,u}$$

$$\boxed{\frac{d}{du}\boxed{\mathrm{cn}}\,u = -\mathrm{sn}\,u \cdot \mathrm{dn}\,u}$$

$$\boxed{\frac{d}{du}\mathrm{dn}\,u = -k^2\,\mathrm{sn}\,u \cdot \mathrm{cn}\,u}$$

$$\boxed{\frac{d}{du}\mathrm{sn}\,u = \mathrm{cn}\,u \cdot \mathrm{dn}\,u}$$

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1$$

$$\mathrm{dn}^2 u + k^2\,\mathrm{sn}^2 u = 1$$

$$\int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta = \theta \qquad t = \sin\theta$$

$$\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t \qquad \theta = \sin^{-1} t$$

Integrate!

$$\int \frac{d(\mathrm{sn}\,u)}{\mathrm{cn}\,u \cdot \mathrm{dn}\,u} = \int du = u + \text{const}$$

$$t = \mathrm{sn}\,u \qquad \boxed{\int \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}} = \mathrm{sn}^{-1}(t;k) + \text{const}}$$

$$\underbrace{F(\phi,k)}_{\text{incomplete elliptic integral of 1st kind}} \equiv \int_0^{\sin\phi} \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}}$$

$\mathrm{sn}\, u = \sin\phi$

$\frac{y}{b} = \sin\phi$

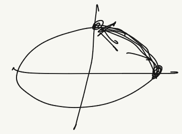
$\mathrm{cn}\, u = \cos\phi$

(angular dependence, period of a simple pendulum)

$$\underbrace{E(\phi,k)}_{\text{incomplete elliptic integral of 2nd kind}} \equiv \int_0^{\sin\phi} \frac{dt\,\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}$$

$\frac{x}{a} = \cos\phi$

(arclength along ellipse)



$\phi = \pi/2$

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} = K(k)$$

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}}\sqrt{1-k^2t^2} = E(k)$$

complete elliptic integrals of 1st and 2nd kind

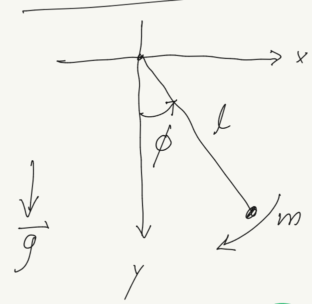
Lec #3: Aug 31st

Simple pendulum: ("Freshman physics")

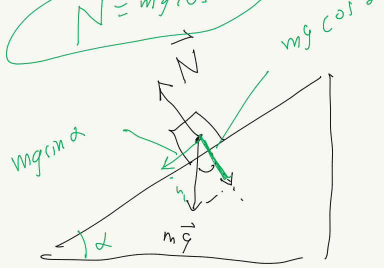
$a_c = \frac{v^2}{r} = \omega^2 r$

$a_t = \alpha r$

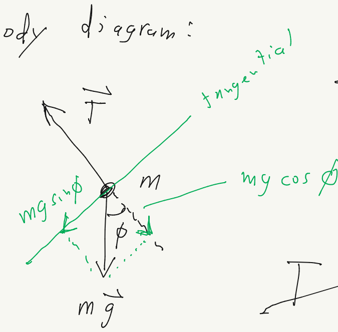
$\alpha = \ddot\phi, \quad \omega = \dot\phi$



$N = mg\cos\alpha$



Free-body diagram:



~~$T = mg\cos\phi$~~

1) $T - mg\cos\phi = m\underbrace{\dot\phi^2 l}_{\text{centripetal acceleration}}$

$$\boxed{T = m\dot\phi^2 l + mg\cos\phi}$$

2) $mg\sin\phi = -m\,a_{tangential}$

$= -ml\ddot\phi$

$$\boxed{\ddot\phi = -\frac{g}{l}\sin\phi}$$ EOM

$$\ddot{\phi} = -\frac{g}{l}\sin\phi$$ (hard to solve; 2nd order non-linear ODE)

<u>Small-angle approx:</u> $\phi << 1$ rad $\cong 57^\circ$ (π radians = 180°)

$$\rightarrow \quad \ddot{\phi} = -\frac{g}{l}\phi \qquad \omega_0 \equiv \sqrt{\frac{g}{l}}$$

complex

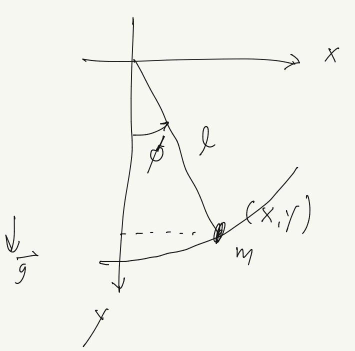
$$\phi(t) = A e^{-i\sqrt{\frac{g}{l}}t}$$

$$= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

$$= a \cos(\omega_0 t + \alpha)$$

<u>Lagrangian formalism:</u>

$$L = T - U$$



$$U = -mgy$$
$$= -mgl\cos\phi$$

$y = l\cos\phi$
$x = l\sin\phi$
$\downarrow$
$\dot{x} = l\dot{\phi}\cos\phi$
$\dot{y} = -l\dot{\phi}\sin\phi$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2}ml^2\dot{\phi}^2$$

$$\boxed{L = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi} \quad \cancel{+U(t)}$$

(x,y) $x = r\cos\phi$
(r,ϕ) $y = r\sin\phi$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi}$$

$$\frac{d}{dt}(ml^2\dot{\phi}) = -mgl\sin\phi$$

$$ml^2\ddot{\phi} = -mgl\sin\phi$$

$$\ddot{\phi} = -\frac{g}{l}\sin\phi$$

L does not depend explicitly on time t $\rightarrow$ E is conserved.

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$$

$$= T + U$$

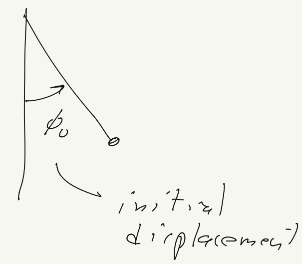
Example:

$$\boxed{E} = \frac{\partial L}{\partial \dot{\phi}}\phi - L$$

$$= ml^2\dot{\phi}\,\dot{\phi} - \left(\frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi\right)$$

$$= \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$$

$$= T + U$$



$$E = -mgl\cos\phi_0$$

$$-\cancel{m}gl\cos\phi_0 = \frac{1}{2}\cancel{m}l^2\dot{\phi}^2 - \cancel{m}gl\cos\phi$$

$$\dot{\phi}^2 = \frac{2}{l^2}\left(gl\cos\phi - gl\cos\phi_0\right)$$

L&L 11.1

$$\dot{\phi}^2 = 2\frac{g}{l}\left(\cos\phi - \cos\phi_0\right)$$

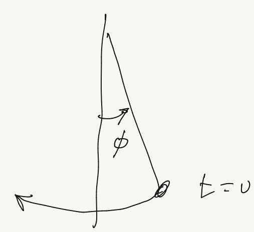
$$\frac{d\phi}{dt} = \dot{\phi} = \pm\sqrt{2}\underbrace{\sqrt{\frac{g}{l}}}_{\omega_0}\sqrt{\cos\phi - \cos\phi_0}$$

$$\int_0^t dt = \int_{\phi_0}^{\phi}\frac{-d\phi}{\sqrt{2}\,\omega_0\sqrt{\cos\phi - \cos\phi_0}}$$

$$\omega_0 t + \text{const} = \pm\int\frac{d\phi}{\sqrt{2}\sqrt{\cos\phi - \cos\phi_0}}$$

$$\int\frac{dt}{\sqrt{1-t^2}} = \sin^{-1}t + \text{const}$$

$$\int\frac{dx}{\sqrt{ax^2+bx+c}} =$$



$$\cos\phi = \cos\left(2\cdot\frac{\phi}{2}\right) = \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right) = 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\cos\phi_0 = 1 - 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$\rightarrow \sqrt{\cos\phi - \cos\phi_0} = \sqrt{2}\sqrt{\sin^2\left(\frac{\phi_0}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)} = \sqrt{2}\left|\sin\left(\frac{\phi_0}{2}\right)\right|\sqrt{1 - \frac{\sin^2(\phi/2)}{\sin^2(\phi_0/2)}}$$

$$k = \left|\sin\left(\frac{\phi_0}{2}\right)\right| \qquad (0 < k < 1)$$

$$x = \frac{\sin\left(\frac{\phi}{2}\right)}{\left|\sin\left(\frac{\phi_0}{2}\right)\right|} = \frac{\sin\left(\frac{\phi}{2}\right)}{k} \longrightarrow dx = \frac{1}{k}\frac{1}{2}\cos\left(\frac{\phi}{2}\right) d\phi$$

$$= \frac{1}{2k}\sqrt{1-\sin^2\left(\frac{\phi}{2}\right)}\, d\phi$$

$$= \frac{1}{2k}\sqrt{1-k^2x^2}\, d\phi$$

$$\boxed{\sqrt{2}\sqrt{\cos\phi - \cos\phi_0} = 2k\sqrt{1-x^2}}$$

$$\boxed{d\phi = \frac{2k\, dx}{\sqrt{1-k^2x^2}}}$$

$$\omega_0 t + \text{const} = \pm\int \frac{\cancel{2k}\, dx}{\sqrt{1-k^2x^2}\ \cancel{2k}\sqrt{1-x^2}}$$

$$= \int_0^x \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

$$= sn^{-1}(x; k)$$

$$\left\{ \begin{array}{l} x = \dfrac{\sin\left(\frac{\phi}{2}\right)}{k} \\ k = \left|\sin\left(\frac{\phi_0}{2}\right)\right| \end{array} \right.$$

$$t = 0 \longleftrightarrow \phi = \phi_0$$

$$\text{const} = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \equiv \underbrace{K}_{}(k)$$

complete elliptic integral of the 1st kind

$$\omega_0 t + K(k) = sn^{-1}\left(\frac{\sin\left(\frac{\phi}{2}\right)}{k};\ k\right) \qquad \Big|\ \underbrace{k \equiv \sin\left(\frac{\phi_0}{2}\right)}$$

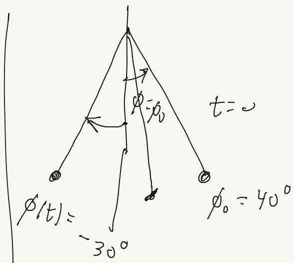
$$sn\left(\omega_0 t + K(k); k\right) = \frac{1}{k}\sin\left(\frac{\phi}{2}\right)$$

$$\longrightarrow \boxed{\phi(t) = 2\arcsin\left(k\, sn(\omega_0 t + K(k); k)\right)}$$

$$P = \frac{4}{\omega_0}K(k) \longrightarrow \omega_0\frac{P}{4} = K(k)$$

Lec #4: 2 Sep 2021

$$\omega_0 \int_0^t dt = -\int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\varnothing}}$$



$$\omega_0 t = -\left[\int_{\phi_0}^{0} + \int_0^{\phi} \right] \frac{d\phi}{\sqrt{\varnothing}}$$

$$= + \int_0^{\phi_0} \frac{d\phi}{\sqrt{\varnothing}} - \int_0^{\phi} \frac{d\phi}{\sqrt{\varnothing}}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} - \int_0^x \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

$$x \equiv \frac{\sin\left(\frac{\phi}{2}\right)}{\left|\sin\left(\frac{\phi_0}{2}\right)\right|}$$

$$k = \left|\sin\left(\frac{\phi_0}{2}\right)\right|$$

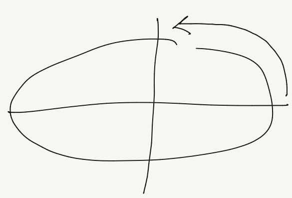
$$= K(k) - sn^{-1}(x; k)$$

$$sn^{-1}(x; k) = K(k) - \omega_0 t$$

$$\omega_0 = \sqrt{\frac{g}{l}}$$

$$\frac{\sin\left(\frac{\phi}{2}\right)}{k} = x = sn\left(K(k) - \omega_0 t ; k \right) = cn(\omega_0 t; k)$$

$$\rightarrow \boxed{\phi(t) = 2 \arcsin\left(k \, sn(K(k) - \omega_0 t; k) \right)}$$



$$\frac{1}{\sqrt{1-k^2x^2}} \approx 1 + \frac{1}{2}k^2x^2$$

$$(1+\epsilon)^p \approx 1 + p\epsilon$$

$$\boxed{\omega_0 \frac{P}{4} = K(k)} \rightarrow \boxed{P = \frac{4}{\omega_0} K(k)}$$

$$K(k) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

$k=0$: $$K(0) = \int_0^1 \frac{dy}{\sqrt{1-x^2}} = \sin^{-1} 1 = \pi/2$$

$$P = \frac{4}{\omega_0}\frac{\pi}{2} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}$$

$0 < k << 1$ $$K(k) = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \left(1 + \frac{1}{2}k^2x^2 \right) = \frac{\pi}{2} + \int_0^1 \frac{dx \, \frac{1}{2}k^2x^2}{\sqrt{1-x^2}}$$

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$$

pf: $$\sin\left(\frac{\pi}{2}-\theta\right) = \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}\cos\theta - \cancel{\cos\left(\frac{\pi}{2}\right)}\sin\theta$$

$$= \cos\theta$$

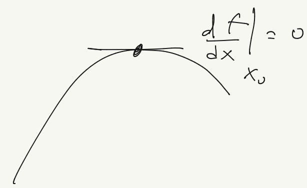
$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta$$

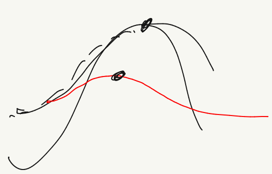
$$\mathrm{sn}(K(k)-x;\, k) = \mathrm{cn}(x)$$

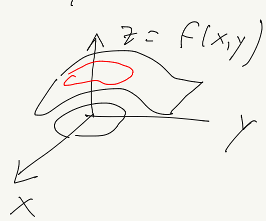
Lagrange multipliers

$f(x)$: max or min ? $\qquad \dfrac{df}{dx} = 0$

$f(x,y)$: " $\qquad \dfrac{\partial f}{\partial x} = 0, \quad \dfrac{\partial f}{\partial y} = 0$







$f(x,y)$: Find extreme value subject to constraint $\varphi(x,y)=0$?

1) "reduced space method":

solve constraint $\varphi(x,y)=0 \rightarrow y = g(x)$

$$F(x) = f(x,y)\Big|_{y=g(x)}$$

e.g., $x^2+y^2=1$

$2x\,dx + 2y\,dy = 0$

$$dx = \frac{-y}{x}\,dy$$

2) "method of Lagrange multipliers"

$$F(x,y,\lambda) = f(x,y) + \lambda\varphi(x,y) \quad \begin{cases} \dfrac{\partial F}{\partial x} = 0 \\ \dfrac{\partial F}{\partial y} = 0, \quad \dfrac{\partial F}{\partial \lambda} = 0 \end{cases}$$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0$$

$$\frac{\partial F}{\partial \lambda} = \varphi(x,y) = 0$$

$F(x,y,\lambda)$
$= f(x,y) + \lambda\varphi(x,y)$

$\varphi(r,\phi) = 0 = r - \ell$

$$\underbrace{L(r,\phi,\dot r,\dot\phi,t) + \lambda(\underbrace{r-\ell}_{\varphi(r,\phi)})}_{} = L'$$

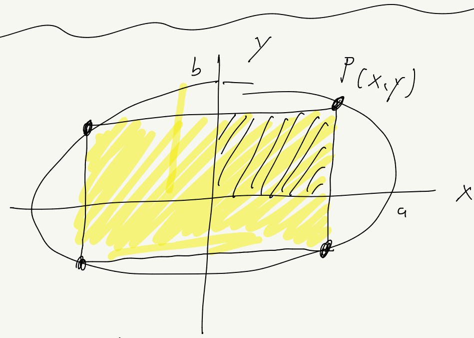
$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot q}\right) = \frac{\partial L'}{\partial q}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q} + \lambda\frac{\partial \varphi}{\partial q}$$

$$\frac{dp}{dt} = -\frac{\partial U}{\partial q} + \lambda\frac{\partial \varphi}{\partial q} = F_{app} + F_{constraint}$$

Lec #5: 9/7

1) Lagrange multipliers
2) Example: 2-d oscillating orbit



Maximize the area of a rectangle whose corners lie on the ellipse

$\rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

$A(x,y) = 4xy$

$\varphi(x,y) = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 0$

$F(x,y,\lambda) = 4xy + \lambda\varphi(x,y)$

$$\boxed{\begin{aligned}\frac{\partial F}{\partial x} &= 4y + \lambda\left(-\frac{2x}{a^2}\right) = 0\\ \frac{\partial F}{\partial y} &= 4x + \lambda\left(-\frac{2y}{b^2}\right) = 0\\ \frac{\partial F}{\partial \lambda} &= 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 0\end{aligned}}$$

$$\frac{\partial F}{\partial x} = 4y + \lambda\left(-\frac{\cancel{2}x}{a^2}\right) = 0 \longrightarrow 4y^2a^2 - 2\lambda xy = 0 \longleftarrow$$

$$\frac{\partial F}{\partial y} = 4x + \lambda\left(-\frac{2y}{b^2}\right) = 0 \longrightarrow \underbrace{4x^2b^2 - 2\lambda yx = 0}_{\text{subtract}}$$

$$\frac{\partial F}{\partial \lambda} = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 0$$

$$4y^2a^2 - 4x^2b^2 = 0$$

$$y^2a^2 = x^2b^2$$

$$\rightarrow \boxed{\frac{y}{b} = \pm\frac{x}{a}}$$

$$0 = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^2$$

$$= 1 - 2\left(\frac{x}{a}\right)^2$$

$$\frac{x}{a} = \frac{1}{\sqrt{2}} \longrightarrow \boxed{x = \frac{a}{\sqrt{2}}}, \ \boxed{y = \frac{b}{\sqrt{2}}} \rightarrow A_{max} = 4xy$$

$$= 4\frac{a}{\sqrt{2}}\frac{b}{\sqrt{2}}$$

$$= \boxed{2ab}$$

— Reduced space method

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$y = b\sqrt{1-\left(\frac{x}{a}\right)^2}$$

$$F(x) = 4xy\Big|_{y=b\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

$$= 4b\,x\sqrt{1-\left(\frac{x}{a}\right)^2}$$

maximize:

$$0 = F'(x) = 4b\sqrt{\ } + 4bx\left(\frac{1}{\cancel{2}}\right)\frac{1}{\sqrt{\ }}\left(-\frac{\cancel{2}x}{a^2}\right)$$

$$= 4b\left(\sqrt{\ } - \left(\frac{x}{a}\right)^2\frac{1}{\sqrt{\ }}\right)$$

$$= \frac{4b}{\sqrt{\ }}\left(1-\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^2\right)$$

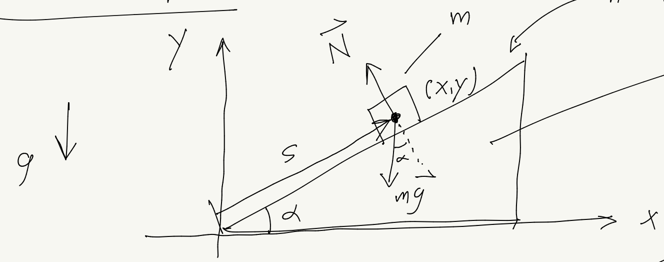
$$= \frac{4b}{\sqrt{\ }}\left(1 - 2\left(\frac{x}{a}\right)^2\right) \longrightarrow \left(\frac{x}{a}\right)^2 = \frac{1}{2}$$

$$x = \pm\frac{a}{\sqrt{2}}$$

Mechanics problem:

no friction

infinitely massive



$N - mg\cos\alpha = 0$

$N = mg\cos\alpha$

Freshman physics analysis:

$m\ddot{s} = -mg\sin\alpha$

$\ddot{s} = -g\sin\alpha$

Lagrangian analysis:

$x = s\cos\alpha, \quad y = s\sin\alpha \qquad \frac{y}{x} = \tan\alpha$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad y = x\tan\alpha$

$U = mgy$

$\boxed{\varphi(x,y) = y - x\tan\alpha = 0}$

$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy$

$L' = L + \lambda\varphi \qquad\qquad F = A + \lambda\varphi$

$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy + \lambda(y - x\tan\alpha)$

$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}}\right) = \frac{\partial L'}{\partial x} \rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} + \lambda\frac{\partial\varphi}{\partial x}}$

$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} + \lambda\frac{\partial\varphi}{\partial y}}$

$\boxed{\frac{dp_x}{dt} = F_x + \lambda\frac{\partial\varphi}{\partial x} \qquad \frac{dp_y}{dt} = F_y + \lambda\frac{\partial\varphi}{\partial y}}$

$\boxed{\varphi(x,y) = y - x\tan\alpha = 0}$

$\boxed{\frac{d\vec{p}}{dt} = \vec{F} + \lambda\vec{\nabla}\varphi}$

$\frac{d}{dt}(m\dot{x}) = 0 + \lambda(-\tan\alpha) \rightarrow m\ddot{x} = -\lambda\tan\alpha$

$\ddot{x} = -\frac{\lambda}{m}\tan\alpha$

$\frac{d}{dt}(m\dot{y}) = -mg + \lambda \longrightarrow \ddot{y} = -g + \frac{\lambda}{m}$

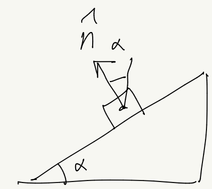
$y - x\tan\alpha = 0$

$$\ddot{y} - \ddot{x}\tan\alpha = 0 \rightarrow \ddot{y} = \ddot{x}\tan\alpha$$

$$\left(-g + \frac{\lambda}{m}\right) = -\frac{\lambda}{m}\tan\alpha \cdot \tan\alpha$$

$$\varphi = y - x\tan\alpha$$

$$\frac{\lambda}{m}\big(\underbrace{1 + \tan^2\alpha}_{\sec^2\alpha = \frac{1}{\cos^2\alpha}}\big) = g$$



$$\frac{\lambda}{m} = g\cos^2\alpha$$

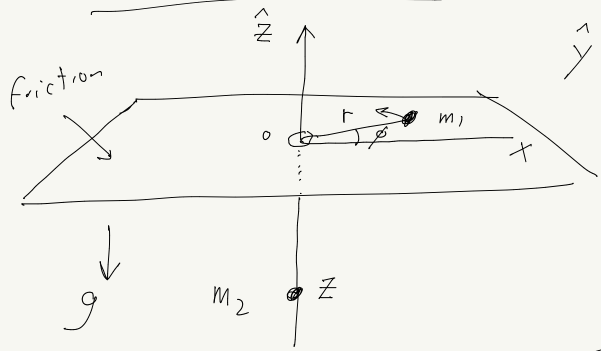
$$\lambda = mg\cos^2\alpha$$

$$\vec{F}_c = \lambda\vec{\nabla}\varphi$$
$$= \lambda(\underbrace{\hat{y} - \hat{x}\tan\alpha}_{\text{unit}})$$

$$\vec{F}_c = \lambda\left(\hat{y} - \hat{x}\frac{\sin\alpha}{\cos\alpha}\right)$$
$$= \frac{\lambda}{\cos\alpha}(\cos\alpha\,\hat{y} - \hat{x}\sin\alpha)$$
$$= \frac{\lambda}{\cos\alpha}\hat{n} = \boxed{mg\cos\alpha\,\hat{n}}$$

2. 2-d example:

string length $= \ell$



$\hat{y}$: into page

g

$$x = r\cos\phi$$
$$y = r\sin\phi$$

$$r - z = \ell$$

$$z = r - \ell$$

$$T = \frac{1}{2}m_1(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m_2\dot{z}^2$$
$$= \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}m_2\dot{r}^2$$
$$= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\phi}^2$$

ignore (const)

$$U = m_2 g z = m_2 g(r - \ell) = m_2 g r \cancel{- m_2 g \ell}$$

$$\boxed{\begin{aligned} L &= T - U \\ &= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\phi}^2 - m_2 g r \end{aligned}}$$

(r, ϕ)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \cancel{\frac{\partial L}{\partial \phi}} \rightarrow 0 \qquad = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = \text{const} = \boxed{m_1 r^2 \dot{\phi} = M_z}$$

No explicit time-dependence:

$$E = \text{const} = \underline{\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L} = T + U$$

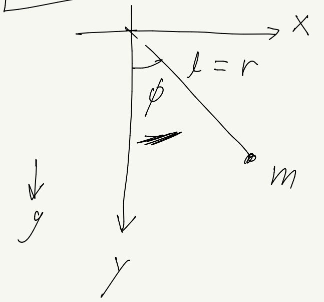
$$\left.\begin{aligned} E &= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\phi}^2 + m_2 g r \\ M_z &= m_1 r^2 \dot{\phi} \quad \longrightarrow \quad \dot{\phi} = \frac{M_z}{m_1 r^2} \end{aligned}\right\} \text{const.}$$

$$\rightarrow E = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\left(\frac{M_z}{m_1 r^2}\right)^2 + m_2 g r$$

$$= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \underbrace{\frac{M_z^2}{2m_1 r^2} + m_2 g r}_{U_{eff}(r)}$$

Lec #6: Sept. 9th

$\boxed{\text{QUIZ \#1}}$



Calculate tension in the string using the method of Lagrange multipliers.

$\boxed{\text{joseph.d.romano@ttu.edu}}$

---

constraint: $\varphi = l - r = 0$

$U = -mgy$
$= -mgr\cos\phi$

$\boxed{\vec{F_c} = \lambda \vec{\nabla}\varphi}$

$$L = T - U$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + mgr\cos\phi$$

Eric
Niramay
Muhammad

$$(1)\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} + \lambda\frac{\partial \varphi}{\partial r}$$
$$(2)\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} + \lambda\frac{\partial \varphi}{\partial \phi}$$

$\varphi = 0$
$\rightarrow r = l \quad (3)$

$$\frac{d}{dt}(m\dot{r}) = mr\dot{\phi}^2 + mg\cos\phi - \lambda$$
$$\boxed{m\ddot{r} = mr\dot{\phi}^2 + mg\cos\phi - \lambda}$$

$$\frac{d}{dt}(mr^2\dot{\phi}) = -mgr\sin\phi$$
$$\boxed{2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} = -mgr\sin\phi}$$

$\boxed{r = l} \rightarrow \boxed{\dot{r} = 0}, \boxed{\ddot{r} = 0}$

(1) $ml^2\ddot{\phi} = -mgl\sin\phi$

(2) $\ddot{\phi} = \frac{-g}{l}\sin\phi$

(3)

$\varphi = l - r$

$$0 = ml\dot{\phi}^2 + mg\cos\phi - \lambda$$

$\Rightarrow \boxed{\lambda = ml\dot{\phi}^2 + mg\cos\phi}$, $\boxed{\vec{\nabla}\varphi = -\hat{r}}$

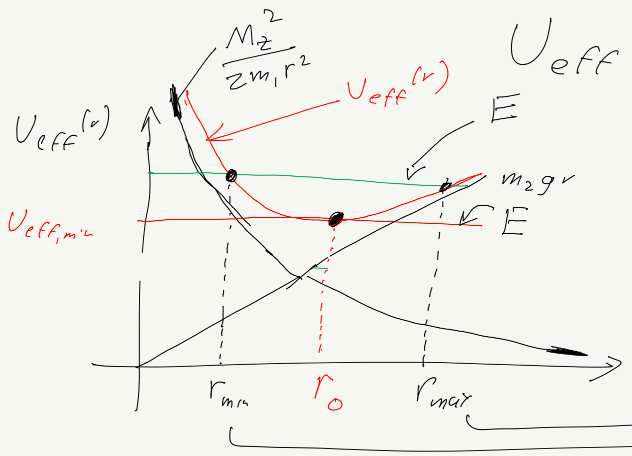
$\boxed{\lambda - mg\cos\phi = ml\dot{\phi}^2}$

$$\boxed{\vec{F_c} = -(ml\dot{\phi}^2 + mg\cos\phi)\hat{r}}$$

Revisit: oscillating orbit example

$$E = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \left(\frac{M_z}{m_1 r^2}\right)^2 + m_2 g r$$

$$= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \underbrace{\frac{M_z^2}{2 m_1 r^2} + m_2 g r}_{U_{eff}(r)}$$



$E \geq U_{eff,min}$

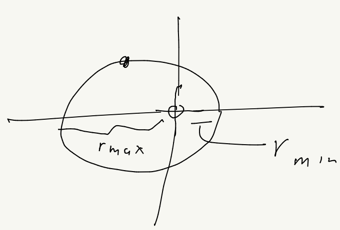
i) $E = U_{eff,min}$

$\rightarrow$ stable circular orbit $r = r_0$

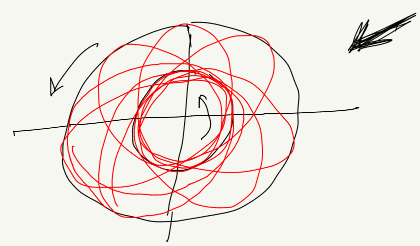
ii) $E > U_{eff,min}$

$\rightarrow$ bounded motion between r_{min} and r_{max}

turning points



closed bound orbit



bound orbit that's not closed

r_0: minimum of the effective potential

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = \left.\left(m_2 g - \frac{M_z^2}{m_1 r^3}\right)\right|_{r=r_0}$$

$$0 = m_2 g - \frac{M_z^2}{m_1 r_0^3}$$

$$\boxed{M_z^2 = m_1 m_2 g r_0^3}$$

$$E = \frac{1}{2}(m_1+m_2)\dot{r}^2 + U_{eff}(r),$$

$$U_{eff}(r) = m_2 g r + \frac{M_z^2}{2m_1 r^2}$$

$$E - U_{eff}(r) = \frac{1}{2}(m_1+m_2)\dot{r}^2$$

$$\frac{dr}{dt} = \dot{r} = \pm\sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E-U_{eff}(r)\right)}$$

$$\boxed{\int dt = \pm\int \frac{dr}{\sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E-U_{eff}(r)\right)}}} \rightarrow t = t(r)$$

$$M_z = m_1 r^2 \dot{\phi} \rightarrow \boxed{\dot{\phi} = \frac{M_z}{m_1 r^2}}$$

$$\frac{dr}{dt} = \pm\sqrt{\oslash}, \quad \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{M_z}{m_1 r^2}\frac{dr}{d\phi}$$

$$\frac{M_z}{m_1 r^2}\frac{dr}{d\phi} = \pm\sqrt{\oslash}$$

orbit equation

$$\boxed{\int \frac{M_z\, dr}{m_1 r^2 \sqrt{\oslash}} = \int \pm\, d\phi} \rightarrow \begin{array}{l} \phi = \phi(r) \\ r = r(\phi) \end{array}$$

$$\oslash = \frac{2}{m_1+m_2}\left(E-U_{eff}(r)\right)$$

$$\boxed{\begin{aligned} \frac{dr}{dt} &= \pm\sqrt{\oslash} \\ \frac{d\phi}{dt} &= \frac{M_z}{m_1 r^2} \end{aligned}}$$

$$\boxed{\begin{aligned} \Delta r &= \pm\sqrt{\oslash}\,\Delta t \\ \Delta\phi &= \frac{M_z}{m_1 r^2}\,\Delta t \end{aligned}}$$

~~start~~ Choose $E, M_z, m_1, m_2,$ etc.

start system off at some values of r and ϕ at $t=0$

step 1: r_1, ϕ_1

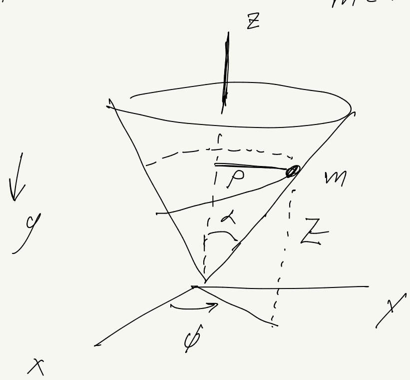
step 2: $r_2 = r_1 + \Delta r$, $\phi_2 = \phi_1 + \Delta\phi$ (repeat)

Lec #7: Sep 14^{th}

Lec #8: Sep 16^{th}

$$\varphi(\rho, \phi, z) = \rho - z\tan\alpha = 0$$

Q & A: Determine constraint force using method of Lagrange multiplier



cylindrical coords (ρ, ϕ, z)

$$\boxed{\tan\alpha = \frac{\rho}{z}} \rightarrow \rho = z\tan\alpha$$

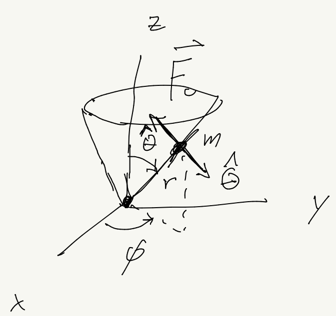
$$\dot{\rho} = \dot{z}\tan\alpha$$

$$T = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) \leftarrow$$

$$= \frac{1}{2}m(\dot{z}^2\tan^2\alpha + z^2\tan^2\alpha\,\dot{\phi}^2 + \dot{z}^2)$$

$$= \frac{1}{2}m(\dot{z}^2(1+\tan^2\alpha) + z^2\tan^2\alpha\,\dot{\phi}^2)$$

$$U = mgz$$



$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$$

$\dot{\theta} \to 0$, $\theta \to \alpha$

$$\boxed{\theta = \alpha} = \text{const}$$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2)$$

$$U = mgz = mgr\cos\alpha$$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$\underbrace{\varphi(r, \theta, \phi) = \alpha - \theta}_{\text{constraint}} = 0$$

$$\boxed{L' = \left(\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - mgr\cos\theta\right) + \lambda(\alpha - \theta)}$$

(r, ϕ) $\quad \phi,\ r = l$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot r}\right) = \frac{\partial L'}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot\theta}\right) = \frac{\partial L'}{\partial \theta}$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot\phi}\right) = \frac{\partial L'}{\partial \phi}$$

$$\varphi = \alpha - \theta = 0 \rightarrow \boxed{\theta = \alpha}$$

$$L' = \underbrace{L}_{T-U} + \lambda\varphi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = \frac{\partial L}{\partial r} + \lambda \frac{\partial \varphi}{\partial r}$$

etc.

Differentiate constraint twice w/t time

$$\dot\varphi = 0 \;, \quad \ddot\varphi = 0$$

$$\dot\theta = 0 \qquad \ddot\theta = 0$$

solve for λ
(algebraic)

$$\vec{F_c} = \lambda \vec\nabla \varphi = -\lambda \hat\theta$$

Hamiltonian:

$$L = T - U$$

$$\underbrace{H}_{} = \left(\sum_i p_i \dot q_i - L\right)$$

$$\sum_i \frac{\partial L}{\partial \dot q_i} \dot q_i - L$$

$$\| $$

$$E(q, \dot q)$$

$$\left(\frac{dE}{dt} = 0 \text{ if } \frac{\partial L}{\partial t} = 0\right)$$

$$\dot q = \dot q(q, p)$$

$$H(q, p, t)$$

$q \equiv q_i$ generalized coord, $i = 1, \cdots, s$

$p \equiv p_i$ generalized momenta $i = 1, \cdots, s$

DOF

$$p_i \equiv \frac{\partial L}{\partial \dot q_i} \text{ (def'n)} = f(q, \dot q)$$

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

Energy conserved ? Yes because $\frac{\partial L}{\partial t} = 0$

$$E = \frac{1}{2} m \dot{x}^2 + U(x)$$

p_x, p_y, p_z

Momentum conjugate to x ?

$$p \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{x} \longrightarrow \dot{x} = p/m$$

$$H = \left(p\dot{x} - L\right)_{\dot{x} = p/m}$$

$$= \left(p\dot{x} - \left(\frac{1}{2} m \dot{x}^2 - U(x)\right)\right)$$

$$= p \cdot p/m - \frac{1}{2} m \left(\frac{p}{m}\right)^2 + U(x)$$

$$= p^2/2m + U(x)$$

Hamilton's equations :

$$\begin{cases} \dot{p}_i = -\dfrac{\partial H}{\partial q_i} & i = 1, \cdots s \; \text{“}3\text{”} \\ \dot{q}_i = \dfrac{\partial H}{\partial p_i} & i = 1, \cdots, s \end{cases}$$

2s 1st-order ordinary differential equations (coupled)

Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

$$i = 1, 2, \cdots, s$$

$$L = L(q, \dot{q})$$

s 2nd order ordinary diff equations (coupled)

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$H = \frac{p^2}{2m} + U(x)$$

EL equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$

$$\boxed{m\ddot{x} = -\frac{\partial U}{\partial x}}$$

$(ma = F)$

Hamilton's equations:

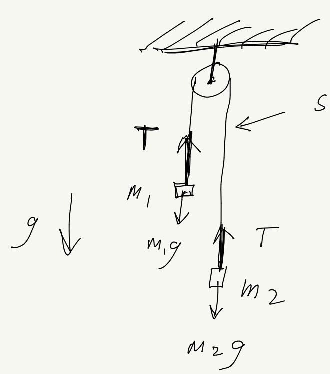
$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$(p = m\dot{x})$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$$

$$\boxed{m\ddot{x} = -\frac{\partial U}{\partial x}}$$

Atwood's machine:



$$a = \frac{g(m_1 - m_2)}{m_1 + m_2}$$

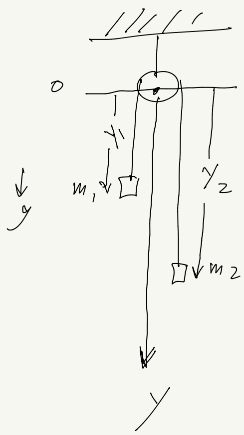
(down)

Net force down:

$$m_1 g - T = m_1 a$$
$$m_2 g - T = -m_2 a$$

subtract $(m_1 - m_2) g = (m_1 + m_2) a$

$$\boxed{a = \frac{(m_1 - m_2) g}{m_1 + m_2}}$$



$$L = T - U$$

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 \qquad y_2 = l - y_1$$

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_1^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{y}_1^2$$

$$\varphi = l - (y_1 + y_2)$$

$$U = -m_1 g y_1 - m_2 g y_2$$

$$= -m_1 g y_1 - m_2 g (l - y_1)$$

$$= -(m_1 - m_2) g y_1 \underbrace{- m_2 g l}_{\text{const ignore}}$$

$$y_1 + y_2 = l$$

$$y_2 = l - y_1$$

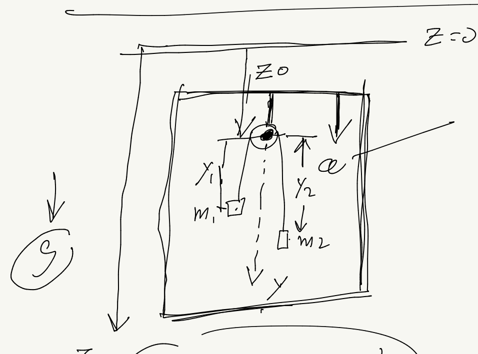
$$\dot{y}_2 = -\dot{y}_1$$

$$L = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 + (m_1 - m_2) g y_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) = \frac{\partial L}{\partial y_1} \rightarrow (m_1 + m_2)\ddot{y}_1 = (m_1 - m_2) g$$

$$\boxed{\ddot{y}_1 = \frac{(m_1 - m_2) g}{m_1 + m_2}}$$

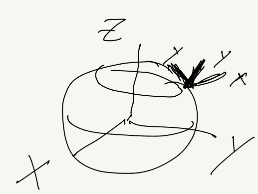
Analyze Atwood's machine in an accelerating reference frame



given acceleration

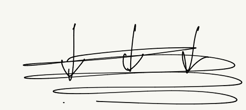
$$\boxed{Z_0 = \frac{1}{2} a t^2}$$

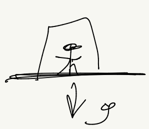
$$\boxed{\ddot{y}_1 = ??}$$

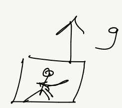


$L = T - U$ is valid only w.r.t an inertial reference frame

$\vec{F} = m\vec{a}$ (only w.r.t inertial frame)







Lec #9: Sep 21st

$$\vec{F} = m\vec{a} \qquad \text{(valid only in an inertial Frame)}$$

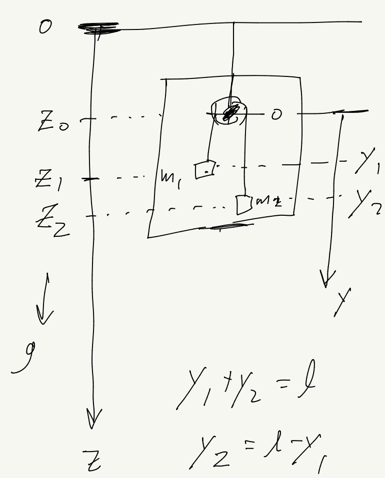
$$\vec{F} + \underbrace{\vec{F}_{\text{fictitious}}}_{} = m\vec{a} \qquad \text{wrt a non-inertial Frame}$$

coriolis Force ←
+ centrifugal force ←
+ linear acceleration
+ angular acceleration

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

same form of EOMs in inertial and non-inertial ref Frame

$L = T - U$ —— valid only in an inertial frame



$$z_1 = z_0 + y_1$$
$$z_2 = z_0 + y_2$$

$$\underbrace{z_0 = \frac{1}{2} a t^2}_{\text{unif acceleration}}$$

$$y_1 + y_2 = l$$
$$y_2 = l - y_1$$

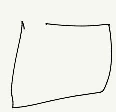
$$T = \frac{1}{2} m_1 \dot{z}_1^2 + \frac{1}{2} m_2 \dot{z}_2^2$$
$$= \frac{1}{2} m_1 (\dot{y}_1^2 + a^2 t^2 + 2at\dot{y}_1)$$
$$+ \frac{1}{2} m_2 (\dot{y}_2^2 + a^2 t^2 + 2at\dot{y}_2)$$

$$\dot{z}_1 = \dot{z}_0 + \dot{y}_1$$
$$= at + \dot{y}_1$$
$$\dot{z}_2 = at + \dot{y}_2$$

$$= \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 + \frac{1}{2}(m_1 + m_2) a^2 t^2$$
$$+ \underbrace{at(m_1 \dot{y}_1 + m_2 \dot{y}_2)}$$

$$= \underbrace{\frac{d}{dt}\left(at(m_1 y_1 + m_2 y_2)\right)}_{\text{ignore}} - a(m_1 y_1 + m_2 y_2)$$

(ignore prescribed function of time)



↓

$$L = \underbrace{T - U}_{\text{wrt inertial frame}}$$

$$\tilde{T} = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 - a(m_1 y_1 + m_2 y_2)$$

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 - a(m_1 y_1 + m_2 y_2)$$

$$y_2 = \ell - y_1 \quad \rightarrow \quad \dot{y}_2 = -\dot{y}_1$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - a(m_1 y_1 + m_2 \ell - m_2 y_1)$$

$$\boxed{T = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - a(m_1 - m_2) y_1} \underbrace{- a m_2 \ell}_{\text{ignore}}$$

$$U = -m_1 g z_1 - m_2 g z_2$$

$$= -m_1 g (z_0 + y_1) - m_2 g (z_0 + y_2)$$

$$= -m_1 g y_1 - m_2 g y_2 \underbrace{- (m_1 + m_2) g z_0}_{\text{ignore}}$$

$$\boxed{U = -(m_1 - m_2) g y_1} \underbrace{- m_2 g \ell}_{\text{ignore}}$$

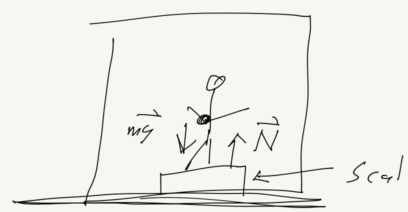
$$L = T - U = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - a(m_1 - m_2) y_1 + (m_1 - m_2) g y_1$$

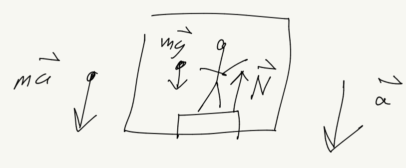
$$\boxed{L = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 + (m_1 - m_2)(g - a) y_1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) = \frac{\partial L}{\partial y_1}$$

$$(m_1 + m_2)\ddot{y}_1 = (m_1 - m_2)(g - a)$$

$$\rightarrow \boxed{\ddot{y}_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)(g - a)}$$





$$m\vec{a} = \vec{F}$$
$$= m\vec{g} + \vec{N}$$

$$\vec{N} = -m(\vec{g} - \vec{a})$$

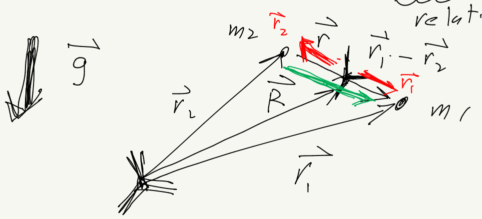
$$\vec{N} = 0 \quad \text{if } \vec{a} = \vec{g}$$

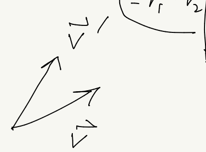
Central Force motion:

Two masses, closed system, interact via a central potential

$$U = U(\underbrace{|\vec{r}_1 - \vec{r}_2|})$$

relative separation vector $\left(\vec{r} = \vec{r}_1 - \vec{r}_2\right)$



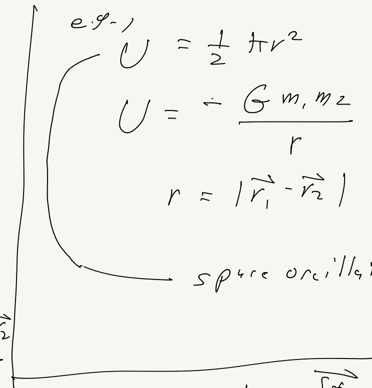


e.g.,

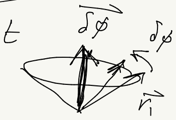
$U = \frac{1}{2} kr^2$ — spring oscillator

$U = -\frac{G m_1 m_2}{r}$

$r = |\vec{r}_1 - \vec{r}_2|$



$t \to t + \delta t$



$$L = T - U$$

$$= \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 - U(|\vec{r}_1 - \vec{r}_2|)$$

(1) no explicit time dependence $\to E = T + U = \text{const}$

(2) translational invariance $\vec{r}_a \to \vec{r}_a + \delta\vec{x}$: $\vec{P} = \sum_a m_a \vec{v}_a = \text{const}$

$\begin{pmatrix} \vec{r}_1 + \delta\vec{x} \\ \vec{r}_2 + \delta\vec{x} \end{pmatrix}$

(3) rotational invariance : $\begin{pmatrix} \vec{r}_a \to \vec{r}_a + \vec{\delta\phi} \times \vec{r}_a \\ \vec{v}_a \to \vec{v}_a + \vec{\delta\phi} \times \vec{v}_a \end{pmatrix}$

$\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a = \text{const}$

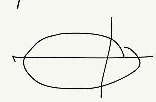
$\underbrace{L + \delta L}_{L'}$

$$\vec{M} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 \leftarrow$$

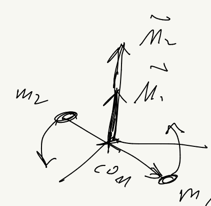
$$0 = \dot{\vec{M}} = \cancel{\vec{v}_1 \times \vec{p}_1}^{\,0} + \vec{r}_1 \times \dot{\vec{p}}_1 + \cancel{\vec{v}_2 \times \vec{p}_2}^{\,0} + \vec{r}_2 \times \dot{\vec{p}}_2$$

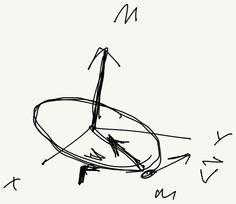
$$\vec{p}_1 = m_1 \vec{v}_1$$

$$= \vec{r}_1 \times \dot{\vec{p}}_1 + \vec{r}_2 \times \dot{\vec{p}}_2$$



two-body



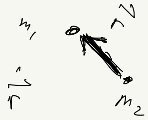


$$\vec{M} = \vec{M}_1 + \vec{M}_2$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2$$

$$= m_1 \vec{v}_1 + m_2 \vec{v}_2$$

$$\dot{\vec{P}} = 0$$



$$\vec{M} = m\vec{r} \times \vec{v} \qquad \dot{\vec{r}}$$

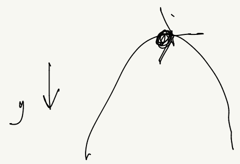
$$0 = \dot{\vec{M}} = \underline{m\dot{\vec{r}} \times \vec{v}} + m\vec{r} \times \dot{\vec{v}}$$

$$\boxed{0 = m\vec{r} \times \dot{\vec{v}}}$$

$\vec{P}$ = const   COM moves with const velocity

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2$$

$$= \frac{d}{dt}\underbrace{\left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}\right)}_{\vec{R}} \cdot (m_1+m_2)$$

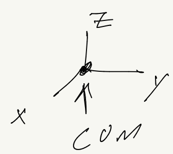


$$= (m_1+m_2)\boxed{\frac{d\vec{R}}{dt}}$$

We can go to the COM Frame.

$$\boxed{m_1\vec{r}_1 + m_2\vec{r}_2 = 0}$$

$$\boxed{\vec{r} = \vec{r}_1 - \vec{r}_2}$$



$(\vec{P} = 0$

$\frac{d\vec{R}}{dt} = 0$

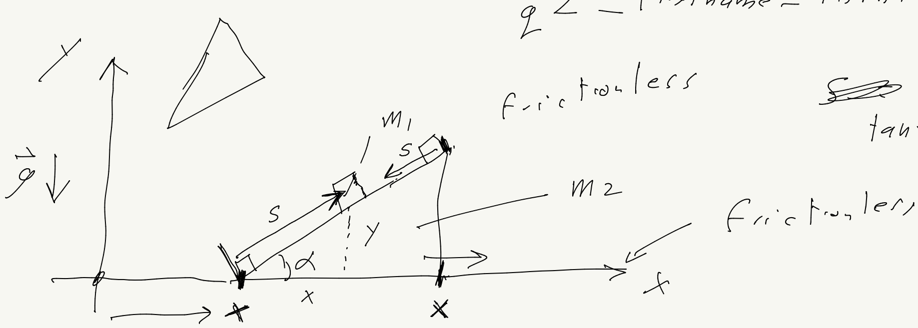
$\vec{R}$ = const $= \vec{0}$

Origin

QUIZ #2:

joseph.d.romano@ttu.edu

q2_Firstname_lastname.pdf



$\tan\alpha = \frac{y}{x}$

1) Write down Lagrangian in terms of generalized coords

2) What quantities (if any) are conserved? Why?

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$\left\{\begin{aligned} x_2 &= X \\ y_2 &= 0 \\ x_1 &= X + s\cos\alpha \\ y_1 &= s\sin\alpha \end{aligned}\right.$$

Lec #10: (Th 9/23)

Midterm 1: Tues Oct 5th

Q2: $\dot{x}_1 = \dot{x} + \dot{s}\cos\alpha$ $\qquad x_1 = x + s\cos\alpha$

$\dot{y}_1 = \dot{s}\sin\alpha$

$\dot{x}_2 = \dot{x}$

$\dot{y}_2 = 0$

$$T = \frac{1}{2} m_1\left(\left(\dot{x}^2 + \dot{s}^2\cos^2\alpha + 2\dot{x}\dot{s}\cos\alpha\right) + \dot{s}^2\sin^2\alpha\right) + \frac{1}{2} m_2 \dot{x}^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2} m_1 \dot{s}^2 + m_1\dot{x}\dot{s}\cos\alpha$$

$$U = m_1 g y_1 = m_1 g s \sin\alpha$$

$L = T - U$, $\quad E = T + U$ conserved (no explicit t dependence)

$\frac{\partial L}{\partial x} = 0 \rightarrow \frac{\partial L}{\partial \dot{x}} = P_x$ = conserved (no x dependence)

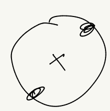
COM: $m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \quad (1)$

$\vec{r}_1 - \vec{r}_2 = \vec{r} \quad (2) \longrightarrow \vec{r}_2 = \vec{r}_1 - \vec{r}$

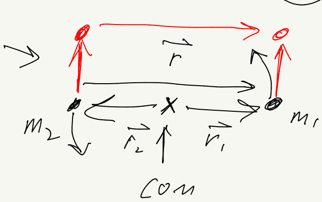
$m_1\vec{r}_1 + m_2(\vec{r}_1 - \vec{r}) = 0$

$(m_1+m_2)\vec{r}_1 - m_2\vec{r} = 0$

$\vec{r} = \vec{r}_1 - \vec{r}_2$



$\dot{\vec{r}} \neq 0$ for rotational motion $\rightarrow$



$$\boxed{\vec{r}_1 = \left(\frac{m_2}{m_1+m_2}\right)\vec{r}}$$

$$\boxed{\vec{r}_2 = -\left(\frac{m_1}{m_1+m_2}\right)\vec{r}}$$

in COM frame

$\dot{\vec{r}}_1 = \left(\frac{m_2}{m_1+m_2}\right)\dot{\vec{r}}$, $\quad \dot{\vec{r}}_2 = -\left(\frac{m_1}{m_1+m_2}\right)\dot{\vec{r}}$

$$T = \frac{1}{2} m_1|\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2|\dot{\vec{r}}_2|^2$$

...

$$= \frac{1}{2} m|\dot{\vec{r}}|^2 \cancel{+ \frac{1}{2}(m_1+m_2)|\vec{V}|^2}$$

$= 0$ in COM frame

$$\boxed{m \equiv \frac{m_1 m_2}{m_1+m_2}}$$

define

reduced mass of system

$$U = U(|\vec{r}_1 - \vec{r}_2|) = U(|\vec{r}|) = U(r)$$

$$\boxed{L = T - U = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(r)}$$

(effective one-body problem)

ii) cons. of angular momentum

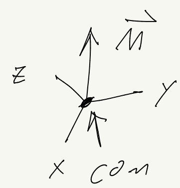
$$\vec{M} = \text{const}$$

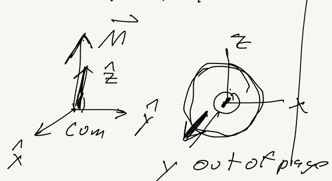
$$= m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2$$

$$\vdots$$

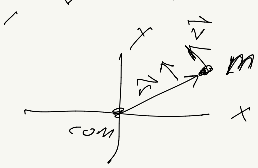
$$= m \vec{r} \times \vec{v}$$

$$= m \vec{r} \times \dot{\vec{r}}$$





$\vec{r}$, $\vec{v}$ must lie in the xy plane



$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(r)$$

$$= \frac{1}{2} m (\underbrace{\dot{r}^2 + r^2\dot{\phi}^2}_{\dot{x}^2 + \dot{y}^2}) - U(r)$$

using plane polar coordinates (r, ϕ)

$$\frac{\partial L}{\partial \phi} = 0 \rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{const} = m r^2 \dot{\phi}$$

$$\frac{\partial L}{\partial \dot{\phi}} \equiv p_\phi = M = M_z$$

$$\boxed{M_z = m r^2 \dot{\phi}} = \text{const}$$

not total mass

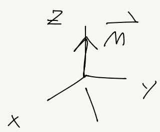
magnitude of angular momentum vector $\vec{M}$.

$$\frac{d\vec{M}}{dt} = 0$$

i) cons. of linear momentum $\rightarrow$ COM Frame

$$L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(r) \qquad \vec{r} = \vec{r}_1 - \vec{r}_2, \quad m = \frac{m_1 m_2}{m_1 + m_2}$$

ii) cons. of angular momentum $\rightarrow$ chose $\hat{z}$-axis to point along $\vec{M}$



COM

$$\vec{M} = \vec{r} \times \vec{p} = m \vec{r} \times \dot{\vec{r}} \quad \rightarrow \quad \text{motion } (\vec{r}, \dot{\vec{r}}) \text{ lies in the x-y plane}$$

Using plane-polar coord (r, ϕ):

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r)$$

$$\frac{\partial L}{\partial \phi} = 0 \quad \rightarrow \quad \frac{\partial L}{\partial \dot{\phi}} = \text{const} = \boxed{m r^2 \dot{\phi} = M_z}$$

$$\boxed{\dot{\phi} = \frac{M_z}{m r^2}}$$

$$\frac{\partial L}{\partial \dot{\phi}} = p_\phi = M_z = M$$

iii) cons of energy $\quad E = T + U = \text{const}$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 + U(r) = \frac{1}{2} m \dot{r}^2 + \underbrace{\left(\frac{M_z^2}{2 m r^2} + U(r) \right)}_{U_{eff}(r)}$$

$$\boxed{E = \frac{1}{2} m \dot{r}^2 + U_{eff}(r)} \quad \text{where} \quad \underbrace{U_{eff}(r) = U(r) + \frac{M_z^2}{2 m r^2}}$$

$$E - U_{eff}(r) = \frac{1}{2} m \dot{r}^2$$

$$\pm \sqrt{\frac{2}{m} (E - U_{eff}(r))} = \dot{r} = \frac{dr}{dt} \quad \leftarrow \quad \frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt}$$

$$\pm \sqrt{} = \frac{dr}{d\phi} \frac{M_z}{m r^2} \rightarrow$$

next page

$$\int dt = \pm \int \frac{dr}{\sqrt{\frac{2}{m} (E - U_{eff}(r))}}$$

$$\boxed{t = \pm \int \frac{dr}{\sqrt{\frac{2}{m} (E - U(r)) - \frac{M_z^2}{m^2 r^2}}} + \text{const}}$$

(14.6) L&L

$\rightarrow t = t(r)$

$r = r(t)$

$$\frac{d\phi}{dt} = \frac{M_z}{m r^2} \quad \rightarrow \quad \int d\phi = \int \frac{M_z}{m r^2(t)} dt$$

$$\rightarrow \quad \phi = \int \frac{M_z}{m r^2(t)} dt + \text{const}$$

$$\int d\phi = \int \frac{\pm M_z \; dr/r^2}{m\sqrt{\frac{2}{m}\left(E-U(r)\right) - \frac{M_z^2}{m^2 r^2}}}$$

(14.7) L&L

$$\boxed{\phi = \pm M_z \int \frac{dr/r^2}{\sqrt{2m\left(E-U(r)\right) - \frac{M_z^2}{r^2}}} + \text{const}}$$

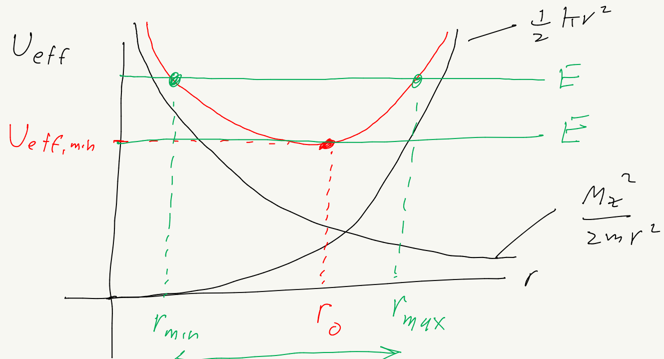
$$\phi = \phi(r) \longleftrightarrow r = r(\phi)$$

☆☆ $U(r) = \frac{1}{2} k r^2$ (space oscillator)

$U(r) = -\frac{\alpha}{r} = -\frac{G m_1 m_2}{r}$ (Newtonian gravity)

$$U_{eff}(r) = U(r) + \frac{M_z^2}{2mr^2} = \frac{1}{2} k r^2 + \frac{M_z^2}{2mr^2}$$

$$\left.\frac{dU_{eff}}{dr}\right|_{r_0} = 0$$



$E = U_{eff,min} \longrightarrow$ stable circular orbit with $r = r_0$

$E > U_{eff,min} \longrightarrow$ bounded motion between r_{min} and r_{max} ($r_{min} \le r \le r_{max}$)

$$\phi = M_z \int \frac{dr/r^2}{\sqrt{2m(E-U(r)) - \frac{M_z^2}{r^2}}} + \text{const}$$

$$= M_z \int \frac{dr/r^2}{\sqrt{2m(E-\frac{1}{2}kr^2) - \frac{M_z^2}{r^2}}} + \text{const}$$

$$= M_z \int \frac{dr/r^2}{\sqrt{2mE - mkr^2 - M_z^2/r^2}} + \text{const}$$

$$\sqrt{} = \sqrt{a + bx + cx^2}$$

e.g. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + \text{const}$

$x = \sin\theta \rightarrow dx = \cos\theta \, d\theta$

$\sqrt{} = \sqrt{1-\sin^2\theta} = \cos\theta$

$\sin^{-1} x = \theta = \int d\theta$

$$\boxed{u = \frac{1}{r}} \rightarrow du = -\frac{1}{r^2} dr \rightarrow dr/r^2 = -du$$

$$\sqrt{} = \sqrt{2mE - \frac{mk}{u^2} - M_z^2 u^2}$$

$$= \frac{1}{u}\sqrt{2mEu^2 - mk - M_z^2 u^4}$$

$$\phi = -M_z \int \frac{u\,du}{\sqrt{2mEu^2 - mk - M_z^2 u^4}} + \text{const}$$

$$\boxed{v = u^2} \rightarrow dv = 2u\,du \rightarrow u\,du = \frac{dv}{2}$$

$$\sqrt{} = \sqrt{2mEv - mk - M_z^2 v^2}$$

$$\boxed{\phi = -\frac{M_z}{2} \int \frac{dv}{\sqrt{2mEv - mk - M_z^2 v^2}}} \boxed{+ \text{const}}$$

Complete the square:

$$-M_z^2 v^2 + 2mEv - mK = -M_z^2\left(v^2 - \frac{2mE}{M_z^2}v + \frac{mK}{M_z^2}\right)$$

$$= -M_z^2\left(\left(v - \frac{mE}{M_z^2}\right)^2 - \frac{m^2E^2}{M_z^4} + \frac{mK}{M_z^2}\right)$$

$$= -M_z^2\left((v-A)^2 - B^2\right)$$

$$= M_z^2\left(B^2 - (v-A)^2\right)$$

$$\boxed{A = \frac{mE}{M_z^2} \qquad B^2 \equiv A^2 - \frac{mK}{M_z^2}}$$

$E \geq U_{eff,min}$

$$\phi = -\frac{\cancel{M_z}}{2}\int \frac{dv}{\sqrt{\cancel{M_z^2}\left(B^2 - (v-A)^2\right)}} + \text{const}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x$$

Let: $v - A = B\sin\theta$

$dv = B\cos\theta\, d\theta$

$$B^2 - (v-A)^2 = B^2 - B^2\sin^2\theta = B^2\cos^2\theta$$

$$\sqrt{} = B\cos\theta$$

$$\phi = -\frac{1}{2}\int d\theta + \text{const}$$

$$= -\frac{1}{2}\theta + \text{const}$$

$$= -\frac{1}{2}\sin^{-1}\left(\frac{v-A}{B}\right) + \text{const}$$

$$= -\frac{1}{2}\sin^{-1}\left(\frac{1/r^2 - A}{B}\right) + \text{const}$$

$v - A = B\sin\theta$

$\sin\theta = \frac{v-A}{B}$

$\theta = \sin^{-1}\left(\frac{v-A}{B}\right)$

$v = u^2 = 1/r^2$

$u = \frac{1}{r}$

Choose const so that $\phi = 0 \leftrightarrow r = r_{max}$

$$0 = -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r_{max}^2} - A}{B}\right) + \text{const}$$

At a turning point ($r = r_{max}$ or $r = r_{min}$)

$$B^2 - (v-A)^2 = 0 \rightarrow v - A = \pm B$$

$$0 = -\frac{1}{2}\sin^{-1}\left(\frac{-B}{B}\right) + \text{const}$$

$$= -\frac{1}{2}\sin^{-1}(-1) + \text{const}$$

$$= -\frac{1}{2}\left(-\frac{\pi}{2}\right) + \text{const}$$

$$= \frac{\pi}{4} + \text{const}$$

$$\text{const} = -\frac{\pi}{4}$$

$$\frac{1}{r_{min}^2,\ r_{max}^2} = A \pm B$$

$$\frac{1}{r_{max}^2} = A - B$$

$$\frac{1}{r_{min}^2} = A + B$$

$$A = \frac{mE}{M_z^2}$$

$$B^2 = A^2 - \frac{mk}{M_z^2}$$

$$\phi = -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2} - A}{B}\right) - \frac{\pi}{4}$$

$$\phi + \frac{\pi}{4} = -\frac{1}{2}\sin^{-1}\left(\frac{\frac{1}{r^2} - A}{B}\right)$$

$$-2\left(\phi + \frac{\pi}{4}\right) = \sin^{-1}\left(\frac{\frac{1}{r^2} - A}{B}\right)$$

$$-\sin\left(2\left(\phi + \frac{\pi}{4}\right)\right) = \frac{\frac{1}{r^2} - A}{B}$$

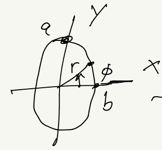
$\sin(-x) = -\sin x$

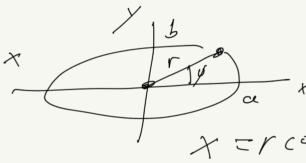
$$\text{LHS} = -\sin\left(2\phi + \pi/2\right)$$

$$= -\left(\sin(2\phi)\cancelto{0}{\cos(\pi/2)} + \cos(2\phi)\underbrace{\sin(\pi/2)}_{=1}\right)$$

$$= -\cos 2\phi$$

$$-\cos 2\phi = \frac{\frac{1}{r^2} - A}{B}$$





$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$x = r\cos\phi$

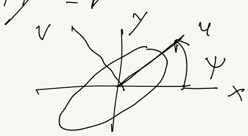
$y = r\sin\phi$

$$-B\cos 2\phi = \frac{1}{r^2} - A$$

$$\boxed{\frac{1}{r^2} = A - B\cos 2\phi}$$

ellipse with center of ellipse at origin

$x^2 + y^2 = r^2$



$$1 = Ar^2 - B\underbrace{r^2\cos 2\phi}_{\cos^2\phi - \sin^2\phi}$$

$$= Ar^2 - Br^2(\cos^2\phi - \sin^2\phi)$$

$$= A(x^2+y^2) - B(x^2 - y^2)$$

$$= (A-B)x^2 + (A+B)y^2$$

$$= \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

$$\frac{1}{a^2} = A - B$$

$$\frac{1}{b^2} = A + B$$

$$a = \frac{1}{\sqrt{A-B}}, \quad b = \frac{1}{\sqrt{A+B}}$$

Lec #11: Tues ~~9~~/28

- Midterm #1, Tues, Oct 5^{th} (In class, MCom 269)
  - 8 short answer - 40%
  - 2 longer problems - 60%

notify me if you are not feeling well

- Blackboard Zoom recordings
  - i) deriving $t = t(r)$, $\phi = \phi(r)$ integrals
  - ii) solving for $\phi = \phi(r)$ for $U = \frac{1}{2}kr^2$
- Today
  - i) clear up constant calculation
  - ii) complete space oscillator potential problem
- Thursday: Q & A

$$\frac{1}{a^2} = A - B \quad , \quad \frac{1}{b^2} = A + B$$

$$A = \frac{mE}{M_z^2} \longleftarrow$$

$$B^2 = A^2 - \frac{mk}{M_z^2}$$

$$a^2 + b^2 = \frac{1}{A-B} + \frac{1}{A+B}$$

$$= \frac{A + \cancel{B} + A - \cancel{B}}{A^2 - B^2}$$

$$= \frac{2A}{A^2 - B^2} = \frac{2\cancel{m}E/\cancel{M_z^2}}{\cancel{m}k/\cancel{M_z^2}} = \frac{2E}{k}$$

$$\boxed{E = \frac{1}{2}k(a^2+b^2)} \longleftarrow E = \frac{1}{2}k x_0^2$$

max displacement with zero initial velocity

$$\frac{mk}{M_z^2} = A^2 - B^2 = (A-B)(A+B)$$

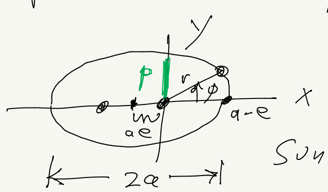
$$= \frac{1}{a^2}\frac{1}{b^2}$$

$$\frac{mk}{M_z^2} = \frac{1}{a^2 b^2} \longrightarrow \boxed{M_z = \sqrt{mk}\; ab}$$

$$U = -\frac{\alpha}{r} = -\frac{G m_1 m_2}{r}$$

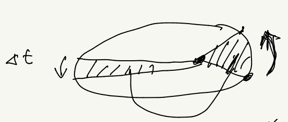
(Kepler's law)

(1.) $\boxed{E = \frac{-\alpha}{2a}}$



planet orbits the sun in an elliptical orbit with Sun at focal point

(2) Equal areas in equal times



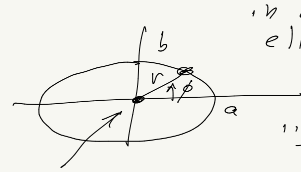
area are equal

(3). $\boxed{P^2 \propto a^3}$ $\quad \frac{P^2}{a^3} = \text{const}$ $\quad \boxed{\frac{P_1^2}{a_1^3} = \frac{P_2^2}{a_2^3}}$

period

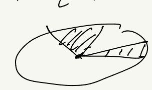
$$U = \frac{1}{2}kr^2$$

(1.) "Planet" moves around "Sun" in an elliptical orbit with "Sun" at center



$\boxed{E = \frac{1}{2}k(a^2+b^2)}$

(2.) Equal areas in equal time

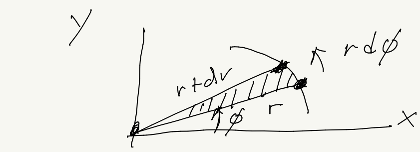


(3.) $P = \frac{2\pi}{\omega}$, $\omega \equiv \sqrt{\frac{k}{m}}$

indep. of the size of ellipse

semi-latus rectum

$a, b \qquad a, e \qquad a, p$



$$dA = \tfrac{1}{2}\ \text{base} \cdot \text{height}$$
$$\simeq \frac{1}{2}\, r\,d\phi \cdot (r+dr)$$
$$= \underline{\frac{1}{2} r^2\, d\phi} + \frac{1}{2}\, r\,d\phi \cdot dr$$

2nd order small

ignore

$$M_z = m r^2 \dot{\phi} \rightarrow \dot{\phi} = \frac{M_z}{m r^2}$$

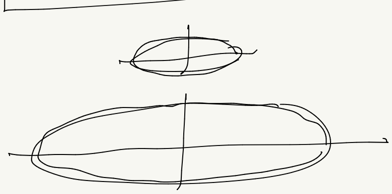
$$\dot{A}(t) = \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt}$$
$$= \frac{1}{2} \cancel{r^2}\, \frac{M_z}{m \cancel{r^2}}$$
$$= \frac{1}{2} \frac{M_z}{m}$$
$$= \text{const}$$

$$\underline{dA} = \text{const} \cdot \underline{dt}$$

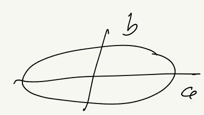
$$\boxed{\frac{dA}{dt} = \frac{1}{2} \frac{M_z}{m}}$$

Equal areas in equal times is a consequence of cons. of angular momentum (holds for all central potentials)

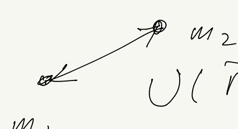
$$\frac{dA}{dt} = \frac{1}{2} \frac{M_z}{m}$$
$$\int dA = \int \frac{1}{2} \frac{M_z}{m}\, dt$$
$$A = \frac{1}{2} \frac{M_z}{m} P$$
$$\pi \cancel{ab} = \frac{1}{2} \frac{\sqrt{mk}\, \cancel{ab}}{m} P$$
$$\pi = \frac{1}{2} \sqrt{\frac{k}{m}}\, P$$
$$\boxed{\frac{2\pi}{\omega} = P}$$



$$M_z = \sqrt{mk}\ ab$$

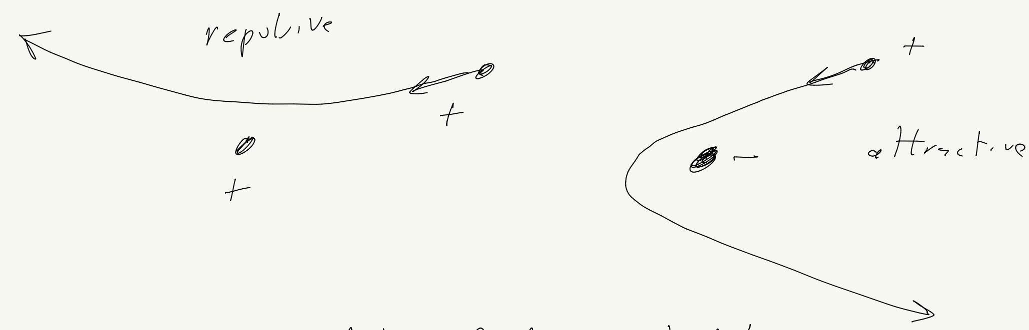


$$\sqrt{\frac{k}{m}} = \omega$$



$$U(\vec{r}_1, \vec{r}_2)$$
$$\underline{U(|\vec{r}_1 - \vec{r}_2|)}$$

cons. of angular momentum



Solving for $r = r(t)$, $\phi(t)$, $x(t)$, $y(t)$: (for $U = \frac{1}{2}kr^2$)

$$t = -\int \frac{dr}{\sqrt{\frac{2}{m}\left(E - \frac{1}{2}kr^2\right) - \frac{M_z^2}{m^2 r^2}}} + \text{const}$$

$\sqrt{\emptyset}$

$E = \frac{1}{2}k(a^2+b^2)$

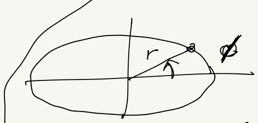
$M_k = \sqrt{mk}\, ab$

$$\emptyset = \frac{\cancel{2}}{m}\left(\frac{1}{\cancel{2}}k(a^2+b^2) - \frac{1}{\cancel{2}}kr^2\right) - \frac{\cancel{m}k a^2 b^2}{m^{\cancel{2}} r^2}$$

$$= \frac{k}{m}\left(a^2+b^2 - r^2 - \frac{a^2b^2}{r^2}\right)$$

$$\sqrt{\emptyset} = \sqrt{\frac{1}{r^2}\frac{k}{m}\left(r^2(a^2+b^2) - r^4 - a^2b^2\right)}$$

$$= \frac{1}{r}\,\omega\sqrt{-(r^2-a^2)(r^2-b^2)}$$



$x = r\cos\phi$

$y = r\sin\phi$

$$\to\ t = -\int \frac{dr}{\frac{\omega}{r}\sqrt{-(r^2-a^2)(r^2-b^2)}} + \text{const}$$

$$\omega t = -\int \frac{r\,dr}{\sqrt{-(r^2-a^2)(r^2-b^2)}} + \text{const}$$

Let: $r^2 = a^2\cos^2\xi + b^2\sin^2\xi$

$\cancel{2}\, r\,dr = \cancel{2}a^2\cos\xi(-\sin\xi)\,d\xi + \cancel{2}b^2\sin\xi\cos\xi\,d\xi$

$r\,dr = -(a^2-b^2)\sin\xi\cos\xi\,d\xi$

in numerator

$r^2 = x^2+y^2$

$x = a\cos\xi$

$y = b\sin\xi$

parameter:

$\xi \neq \phi$

$$-(r^2-a^2)(r^2-b^2) = (a^2-b^2)^2\sin^2\xi\cos^2\xi \qquad r^2 = a^2\cos^2\xi + b^2\sin^2\xi$$

$$\boxed{\begin{aligned} r^2-a^2 &= a^2(\cos^2\xi-1) + b^2\sin^2\xi \\ &= -(a^2-b^2)\sin^2\xi \\ r^2-b^2 &= a^2\cos^2\xi + b^2(\sin^2\xi-1) \\ &= (a^2-b^2)\cos^2\xi \end{aligned}}$$

$$\sqrt{\quad} = (a^2-b^2)\sin\xi\cos\xi$$

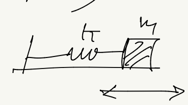
$$r\,dr = -(a^2-b^2)\sin\xi\cos\xi\,d\xi$$

$$\rightarrow \quad \omega t = \cancel{+}\int d\xi + \text{const} \quad \rightarrow \quad \omega t = \xi + \underbrace{\text{const}}_{=0}$$

$$\omega t = \xi$$

$$\boxed{r^2 = a^2\cos^2(\omega t) + b^2\sin^2(\omega t)} \leftarrow r = r(t)$$

$$= x^2 + y^2$$



$$\boxed{\begin{aligned} x &= a\cos(\omega t) \\ y &= b\sin(\omega t) \end{aligned}} \longleftarrow x(t),\ y(t)$$

$$\omega = \sqrt{\frac{k}{m}} = \text{const}$$

$$U = \frac{1}{2}kr^2$$

$$\boxed{\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}} \quad \div \quad \tan\phi = \frac{y}{x} = \frac{b}{a}\tan(\omega t)$$

$$\boxed{\phi = \arctan\left(\frac{b}{a}\tan(\omega t)\right)} \leftarrow \phi = \phi(t)$$

$$M_z = mr^2\dot\phi = \sqrt{mk}\,ab$$

$$r^2\dot\phi = \sqrt{\frac{k}{m}}\,ab \quad \rightarrow \quad \boxed{\dot\phi = \frac{\omega\,ab}{r^2}}$$

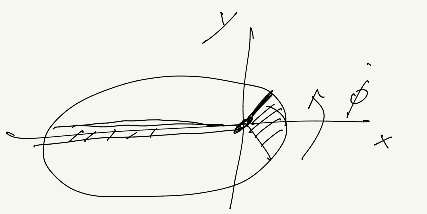
circle
$a = b = r$

Angular velocity: (r, ϕ)

$\dot{\phi}$

$$M_z = mr^2\dot{\phi}$$

$$\dot{\phi} = \frac{M_z}{mr^2}$$

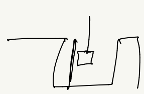


Lecture #12: Thursday 9/30

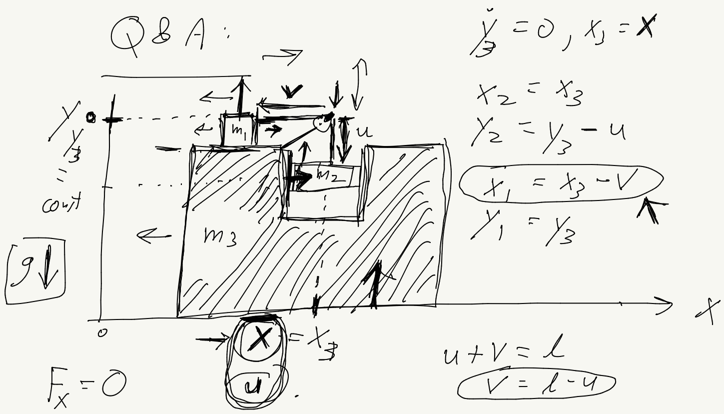
EXAM1: Tues Oct 5^{th}, MCOM 269

(* If you are feeling sick, please contact me before the class to make arrangements for a remote exam.)

8 short answer questions — 8 pts
2 longer problems — 12 pts
20pts (total)



Q&A:



$F_x = 0$

$\ddot{y}_3 = 0$, $x_3 = X$

$x_2 = x_3$

$y_2 = y_3 - u$

$x_1 = x_3 - v$

$y_1 = y_3$

$u + v = l$

$v = l - u$

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}m_3(\dot{x}_3^2 + \dot{y}_3^2)$$

$$U = m_1 g y_1 + m_2 g y_2 + m_3 g y_3$$

$$= \underbrace{\cancel{m_1 g y_3}}_{\text{const}} + m_2 g y_2 + \underbrace{\cancel{m_3 g y_3}}_{\text{const}}$$

$$= m_2 g (y_3 - u)$$

$$= \underbrace{\cancel{m_2 g y_3}}_{\text{ignore}} \boxed{- m_2 g u} = U$$

$$\rightarrow x_1 = x_3 - v = x - (l - u) \quad \rightarrow \quad \dot{x}_1 = \dot{x} + \dot{u} \quad \leftarrow$$

$$x_2 = x_3 = x \quad \rightarrow \quad \dot{x}_2 = \dot{x}$$

$$x_3 = x \quad \rightarrow \quad \dot{x}_3 = \dot{x}$$

$$y_1 = y_3 = \text{const} \quad \rightarrow \quad \dot{y}_1 = 0$$

$$y_2 = y_3 - u \quad \rightarrow \quad \dot{y}_2 = -\dot{u}$$

$$y_3 = \text{const} \quad \rightarrow \quad \dot{y}_3 = 0$$

$$T = \frac{1}{2} m_1 (\dot{x} + \dot{u})^2 + \frac{1}{2} m_2 (\dot{x}^2 + \dot{u}^2) + \frac{1}{2} m_3 \dot{x}^2$$

$$= \frac{1}{2} m_1 (\dot{x}^2 + \dot{u}^2 + 2\dot{x}\dot{u}) + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_3 \dot{x}^2 + \frac{1}{2} m_2 \dot{u}^2$$

$$= \frac{1}{2} (m_1 + m_2 + m_3) \dot{x}^2 + \frac{1}{2} m_1 \dot{u}^2 + \boxed{m_1 \dot{x} \dot{u}} + \frac{1}{2} m_2 \dot{u}^2$$

$$= \frac{1}{2} (m_1 + m_2 + m_3) \dot{x}^2 + \frac{1}{2} (m_1 + m_2) \dot{u}^2 + m_1 \dot{x} \dot{u}$$

$$\boxed{U = -m_2 g u}$$

$$L = T - U \qquad E = T + U = \text{const}$$

No x-dependence $\rightarrow 0 = p_x = \text{const}$

$$\boxed{0 = (m_1 + m_2 + m_3)\dot{x} + m_1 \dot{u}}$$

$$x_{com} = 0$$

$$(m_1 + m_2 + m_3) X_{com} = m_1 x_1 + m_2 x_2 + m_2 x_3 = 0$$

$x_1 = x - l + u$ $\qquad x \qquad x$

$$= (m_1 + m_2 + m_3) x - m_1 l + m_1 u$$

$$= 0$$

$$(m_1 + m_2 + m_3) x = m_1 (l - u)$$

---

q_i doesn't appear in Lagrangian

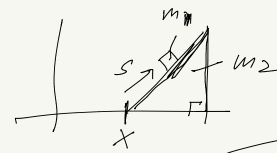
$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)}_{\text{const}} = \cancel{\frac{\partial L}{\partial q_i}}^{\,0} = 0 \qquad p_i = \frac{\partial L}{\partial \dot{q}_i} = \text{const}$$

Constraints: $y_3 = \text{const} = 0$

$y_1 = \text{const}$

$x_2 = x_3 = x$

$x_1 = x_3 - (l - u) = x_3 - l + y_2 \rightarrow x_1 - x_3 + y_2 = -l$

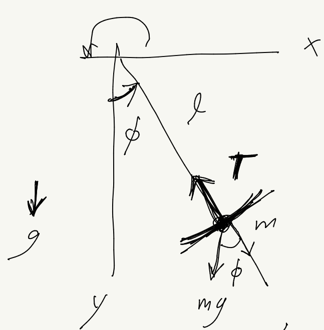


generalized → $y_2 = \cancel{y_3}^{\,0} - u = -u$

generalized coords → $x_3 = x$

6 DOF
−4 constraints
2 DOF

$$\boxed{y_3 = 0, \quad y_1 = \text{const}, \quad x_2 - x_3 = 0, \quad x_1 - x_3 + y_2 = -l}$$



Radial: $T - mg\cos\phi = m a_{radial}$

tangential: $mg\sin\phi = m a_{tangential}$

$-l\ddot{\phi} = a_{tangential}$

$l\dot{\phi}^2 = a_{radial}$

$L = T - U + \lambda\varphi$

$\varphi = r - l = 0$

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$

$U = -mgy$

$= -mgr\cos\phi$

$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \quad \phi, \quad \varphi = r - l = 0}$

$\boxed{T = mg\cos\phi + ml\dot{\phi}^2}$

$\cancel{m}g\sin\phi = -\cancel{m}l\ddot{\phi}$

$\boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi}$

don't do work

$\vec{F}_c = \lambda\vec{\nabla}\varphi$

$\vec{F}_c = -T\hat{r}$

tension

$$\vec{F}_c = -(mg\cos\phi + ml\dot{\phi}^2)\hat{r}$$

$$= -\vec{\nabla} U_{const}$$

$$\text{Work} = \int_{path} \vec{F}_c \cdot d\vec{l}$$

$$\frac{d\vec{l}}{dt}\, dt$$

$$\vec{v}\, dt$$

$$= \int_{path} \vec{F}_c \cdot \vec{v}\, dt$$

$$= 0$$

$$\vec{F}_c \perp \vec{v}$$