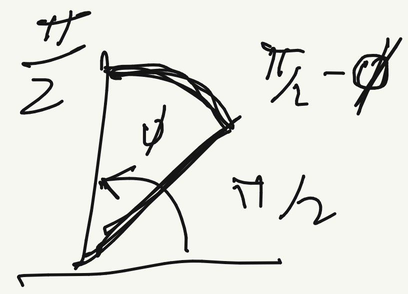
Elliptic functions / integrals:

i) period of a simple pendulum beyond the small-angle approximation
ii) circumference of an ellipse

~~A~~ generalization of definition of circular functions (sines, cosines) to ellipses.



standard notation:

$$\int_0^x \frac{dt}{\sqrt{1-k^2t^2}\sqrt{1-t^2}} = F(\phi, k) = sn^{-1}x$$

$$\int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt = E(\phi, k)$$

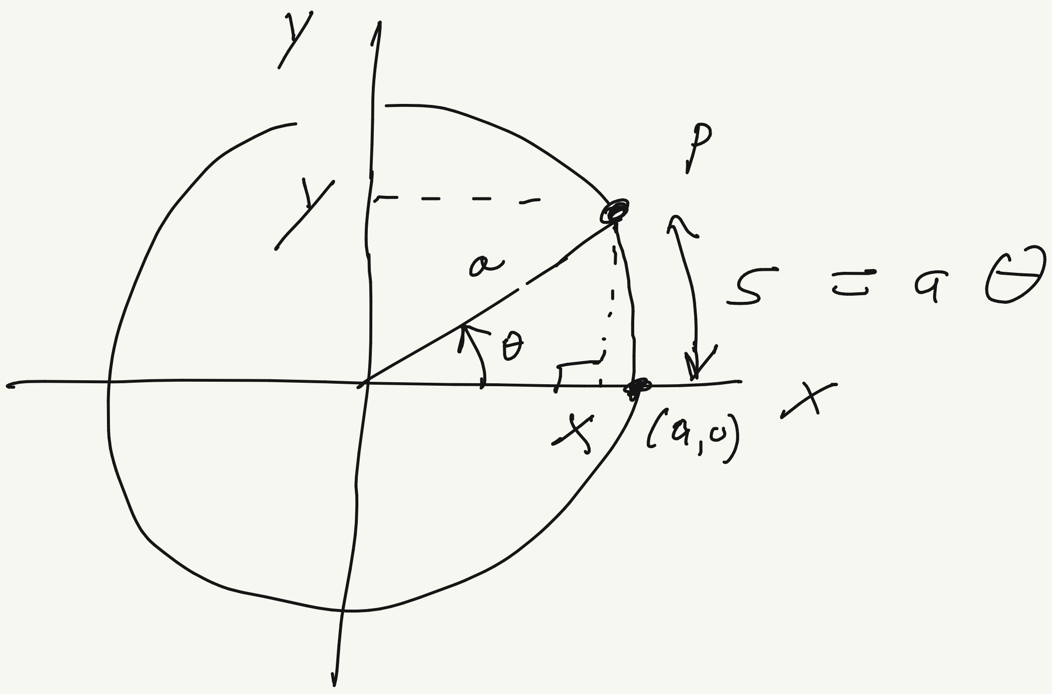
where $x = \sin\phi$ and $0 \le k \le 1$

If we change variables $t \to \sin\theta$ in the integrals then

$$F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$E(\phi, k) = \int_0^\phi \sqrt{1-k^2\sin^2\theta}\, d\theta$$

Circular Functions:



$$\sin\theta \equiv \frac{y}{a}, \qquad \cos\theta \equiv \frac{x}{a}$$

where $\theta \equiv \dfrac{\text{arc length from } (a,0) \text{ to } (x,y)}{a}$

$$= \frac{1}{a}\int_{(a,0)}^{(x,y)} \sqrt{dx^2+dy^2} \qquad \left(= \int d\theta\right)$$

$$x^2+y^2 = a^2 \rightarrow a^2\cos^2\theta + a^2\sin^2\theta = a^2$$

$$\rightarrow \boxed{\cos^2\theta + \sin^2\theta = 1}$$

Derivatives:

$$\frac{d}{d\theta}\sin\theta = \frac{d}{d\theta}\left(\frac{y}{a}\right) = \frac{1}{a}\frac{dy}{d\theta} = \frac{dy}{\sqrt{dx^2+dy^2}}$$

$$= \frac{1}{\sqrt{\left(\frac{dx}{dy}\right)^2+1}}$$

$$= \frac{1}{\sqrt{\left(\frac{-y}{x}\right)^2+1}}$$

Now: $x^2+y^2 = a^2$

$$\rightarrow 2x\,dx + 2y\,dy = 1$$

$$\frac{dx}{dy} = \frac{-y}{x}$$

Thus,

$$\frac{d}{d\theta}\sin\theta = \frac{1}{\sqrt{1+\left(\frac{-y}{x}\right)^2}} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{9} = \cos\theta$$

so $\frac{d\sin\theta}{d\theta} = \cos\theta$

Similarly, $\frac{d\cos\theta}{d\theta} = -\sin\theta$

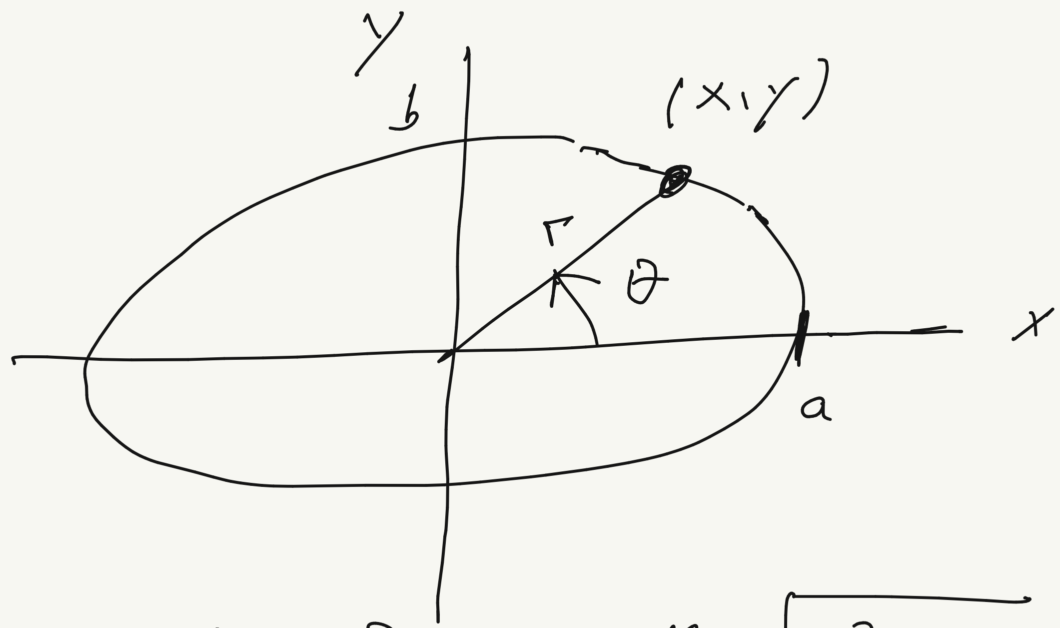
<u>Integrate</u>:

$$\int \frac{d(\sin\theta)}{\cos\theta} = \int d\theta = \theta + \text{const}$$

$t = \sin\theta$
$\cos\theta = \sqrt{1-t^2}$
$dt = d(\sin\theta)$

$$\rightarrow \boxed{\int \frac{dt}{1-t^2} = \theta + \text{const} = \sin^{-1} t + \text{const}}$$

Different parametrization of an ellipse:



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Let $a \geq b$

Eccentricity:

$$e^2 = 1 - \left(\frac{b}{a}\right)^2$$

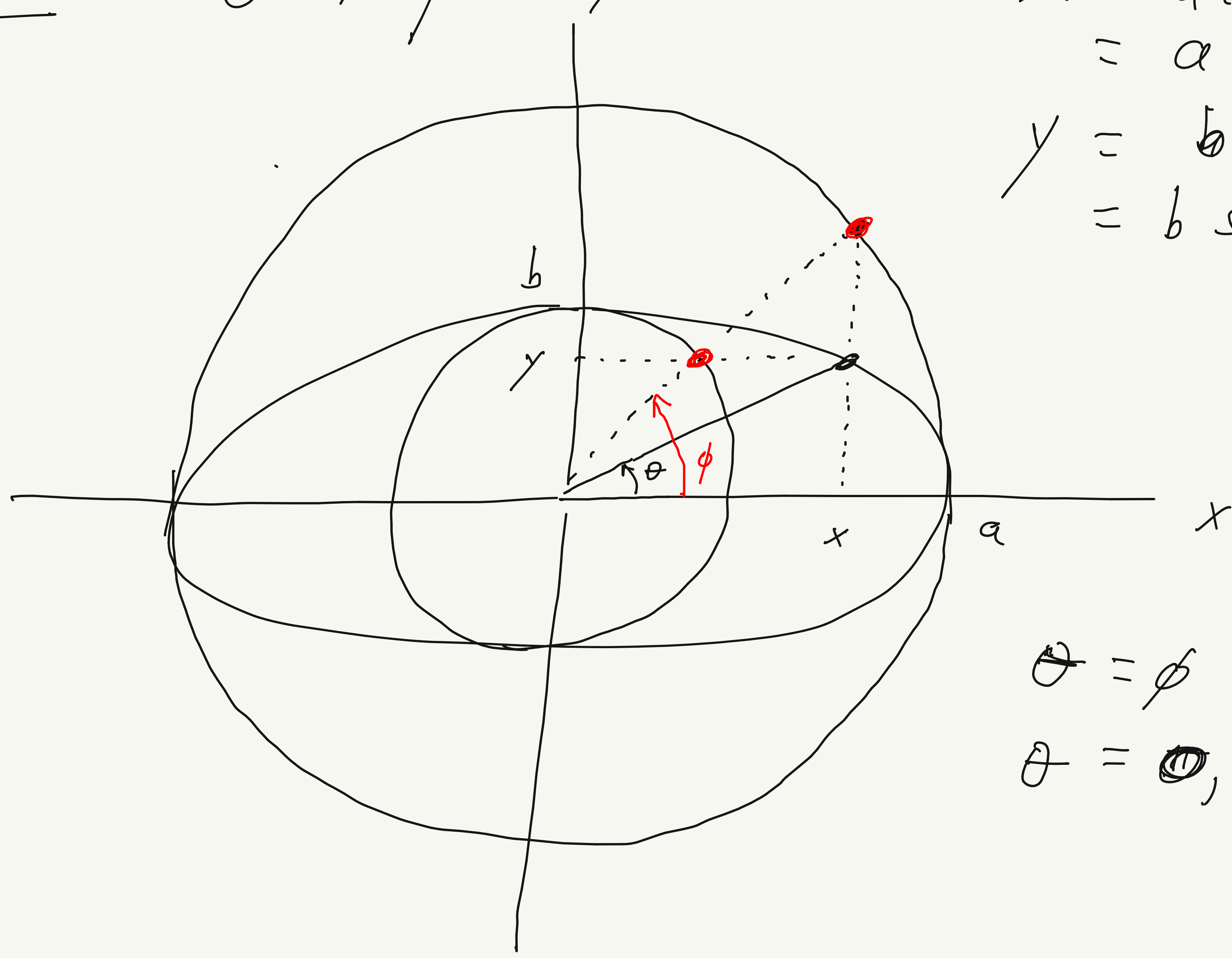
($e = 0$ for a circle)

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad r = \sqrt{x^2+y^2}$$

Another parameterization:

$$\left.\begin{aligned} x &= a\cos\phi \\ y &= b\sin\phi \end{aligned}\right\} \rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

NOTE: $\theta \neq \phi$



$$\begin{aligned} x &= a\cos\phi \\ &= a\,\mathrm{cn}(u;k) \\ y &= b\sin\phi \\ &= b\,\mathrm{sn}(u;k) \end{aligned}$$

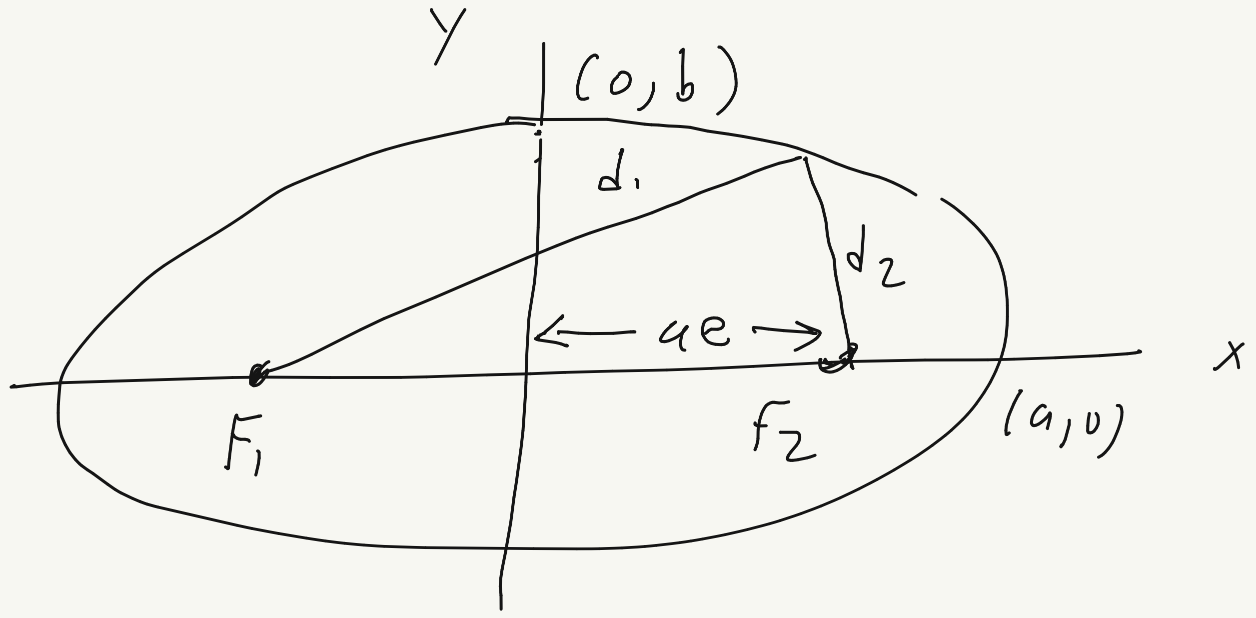
$\theta = \phi$ for

$\theta = 0, \frac{\pi}{2}, \ldots$

$$\tan\theta = \frac{y}{x} = \left(\frac{b}{a}\right)\tan\phi \rightarrow \theta = \arctan\left[\frac{b}{a}\tan\phi\right]$$

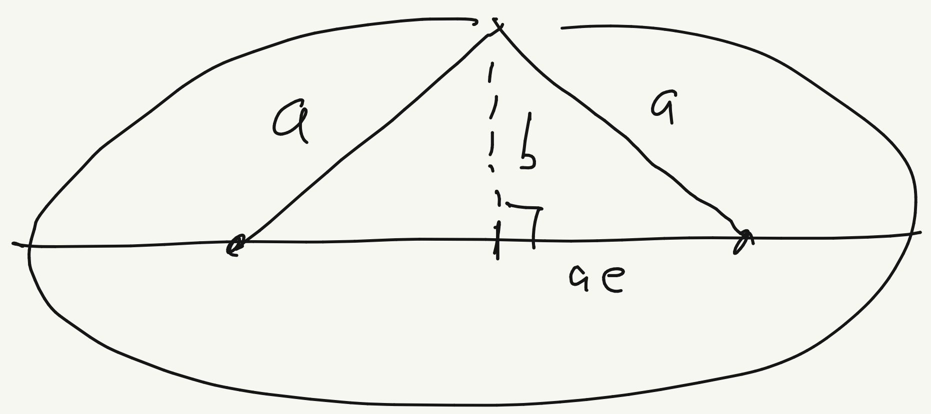
$$\phi = \arctan\left[\frac{a}{b}\tan\theta\right]$$

Elliptic functions:



$$d_1 + d_2 = 2a$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

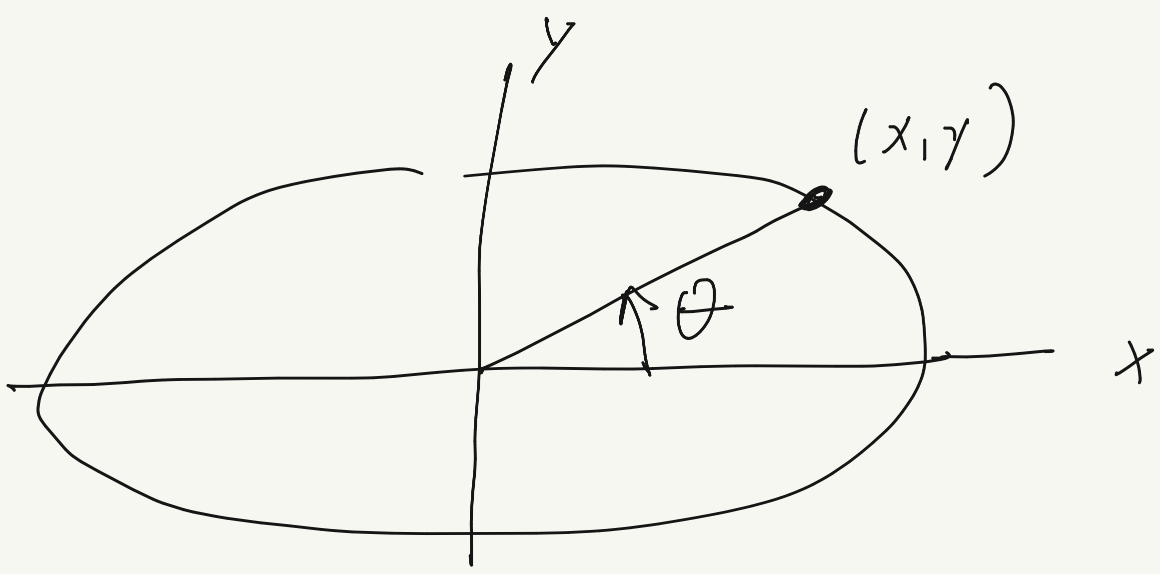


$$a^2 = b^2 + a^2e^2$$

$$a^2(1-e^2) = b^2$$

$$b = a\sqrt{1-e^2}$$

Also, $\left(\frac{b}{a}\right)^2 = 1-e^2 \rightarrow \boxed{e = \sqrt{1-\left(\frac{b}{a}\right)^2}}$



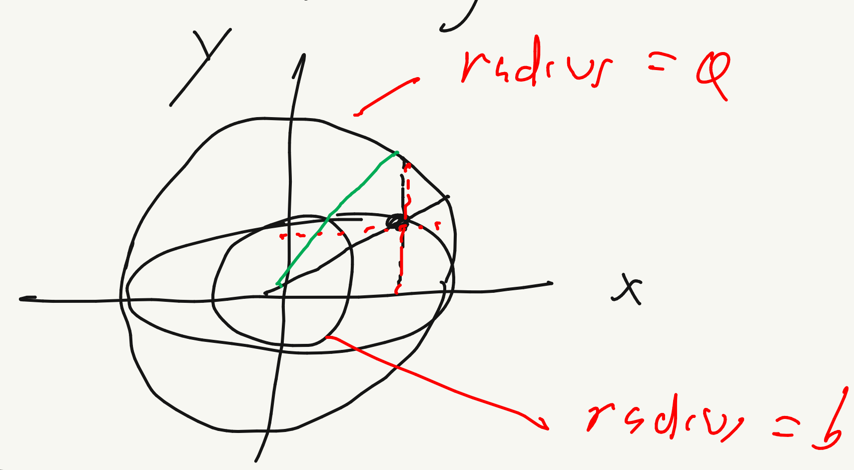
$$x = r\cos\theta$$

$$y = r\sin\theta$$

(but r changes)

$$x = a\cos\phi \qquad \phi \neq \theta$$

$$y = b\sin\phi$$



Define: $cn(u;k) \equiv x/a$ where $k = e$

$sn(u;k) \equiv y/b$ $\quad 0 \leq k \leq 1$

$dn(u;k) \equiv r/a$

~~and~~ where $u = \frac{1}{b}\int_0^\theta r\,d\theta$ — $u = \theta$ for circle

$u \neq$ arc length

since $ds = \sqrt{dr^2 + r^2 d\theta}$

<u>NOTE:</u> $$bu = \int_0^\theta r\,d\theta \leq \int_0^\theta ds$$

$\leq$ arc length from $(a,0)$ to (x,y)

<u>Properties of sn, cn, dn:</u>

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \rightarrow \boxed{cn^2 u + sn^2 u = 1}$$

$$x^2 + y^2 = r^2 \rightarrow a^2 cn^2 u + b^2 sn^2 u = a^2 dn^2 u$$

$$a^2(1 - sn^2 u) + b^2 sn^2 u = a^2 dn^2 u$$

$$1 - sn^2 u + \frac{b^2}{a^2} sn^2 u = dn^2 u$$

$$1 - \left(1 - \frac{b^2}{a^2}\right) sn^2 u = dn^2 u$$

$$1 - k^2 sn^2 u = dn^2 u$$

Thus, $$\boxed{dn^2 u + k^2 sn^2 u = 1}$$

for circle $k = 0$, $dn\, u = 1$

<u>Derivatives:</u>

$$\frac{d}{du} sn\,u = \frac{d}{du}\left(\frac{y}{b}\right) = \frac{dy}{b\,du}$$

<u>Now:</u> $b\,du = r\,d\theta$

$$\rightarrow \frac{d}{du} sn\,u = \frac{dy}{r\,d\theta}$$

$$x = r\cos\theta \quad \rightarrow \quad dx = dr\cos\theta - r\sin\theta\, d\theta$$
$$y = r\sin\theta \qquad\qquad dy = dr\sin\theta + r\cos\theta\, d\theta$$

$$\rightarrow \quad -\sin\theta\, dx = -\sin\theta\cos\theta\, dr + r\sin^2\theta\, d\theta$$
$$+\cos\theta\, dy = \cos\theta\sin\theta\, dr + r\cos^2\theta\, d\theta$$

add: $\cos\theta\, dy - \sin\theta\, dx = r\, d\theta$

$$\rightarrow \quad \frac{x}{r}dy - \frac{y}{r}dx = r\, d\theta$$

thus,

$$\frac{d\,\mathrm{sn}\,u}{du} = \frac{dy}{r\,d\theta} = \frac{\cancel{dy}}{\frac{x}{r}dy - \frac{y}{r}d\cancel{x}}$$

$$= \frac{r}{x - y\frac{dx}{dy}}$$

Also: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \rightarrow \quad \frac{2x\,dx}{a^2} + \frac{2y\,dy}{b^2} = 0$

$$\frac{dx}{dy} = -\frac{y}{x}\left(\frac{a}{b}\right)^2$$

$$\rightarrow \quad \frac{d\,\mathrm{sn}\,u}{du} = \frac{r}{x + \frac{y^2}{x}\left(\frac{a}{b}\right)^2} = \frac{r\,x}{x^2 + y^2\left(\frac{a^2}{b^2}\right)}$$

$$= \frac{r}{a}\,\frac{x}{a}\,\frac{1}{\left(\underbrace{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2}_{1}\right)} = \boxed{\mathrm{cn}\,u \cdot \mathrm{dn}\,u}$$

Then using $\quad sn^2 u + cn^2 u = 1$

we have $\quad \cancel{2}\, sn\, u \dfrac{d}{du} sn\, u + \cancel{2}\, cn\, u \dfrac{d}{du} cn\, u = 0$

$$\rightarrow \quad \frac{d\, cn\, u}{du} = - \frac{sn\, u}{cn\, u} \frac{d}{du} sn\, u$$

$$= - \frac{sn\, u}{\cancel{cn\, u}} \cancel{cn\, u} \cdot dn\, u$$

$$= \boxed{- sn\, u \cdot dn\, u}$$

<u>And using $\quad dn^2 u + k^2 sn^2 u = 1$:</u>

$$\cancel{2}\, dn\, u \frac{d(dn\, u)}{du} + \cancel{2}\, k^2 sn\, u \frac{d}{du}(sn\, u) = 0$$

$$\rightarrow \quad \frac{d}{du}(dn\, u) = - k^2 \frac{sn\, u}{dn\, u} \frac{d(sn\, u)}{du}$$

$$= - k^2 \frac{sn\, u}{\cancel{dn\, u}} cn\, u \cdot \cancel{dn\, u}$$

$$= \boxed{- k^2 sn\, u\, cn\, u}$$

<u>Summary:</u>

$$\frac{d}{du} sn\, u = cn\, u \cdot dn\, u$$

$$\frac{d}{du} cn\, u = - sn\, u \cdot dn\, u$$

$$\frac{d}{du} dn\, u = - k^2 sn\, u\, cn\, u$$

Integrate:

$$\frac{d\, \mathrm{sn}\, u}{du} = \mathrm{cn}\, u \cdot \mathrm{dn}\, u$$

$$\int \frac{d(\mathrm{sn}\, u)}{\mathrm{cn}\, u \cdot \mathrm{dn}\, u} = \int du = u + \text{const}$$

Let $t = \mathrm{sn}\, u \rightarrow \mathrm{cn}\, u = \sqrt{1-t^2}$, $\mathrm{dn}\, u = \sqrt{1-k^2 \mathrm{sn}^2 u}$

$$\rightarrow \int \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} = u + \text{const}$$

But since $t = \mathrm{sn}\, u$:

$$\boxed{\int \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} = \mathrm{sn}^{-1}(t; k) + \text{const}}$$

In handbooks:

$$\boxed{\int_0^{\sin\phi} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} = F(\phi, k)}$$

Incomplete Elliptic integral of the 1st kind

$$\downarrow$$

$$\boxed{\int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\phi, k)}$$

where $t = \sin\theta$.

NOTE: $\boxed{\sin\phi = \mathrm{sn}\, u}$ ☆

Complete elliptic integral of 1^{st} kind:

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}}$$

Connection to simple pendulum:

$$P = \frac{4}{\omega_0} K\left(k \equiv \sin\left(\frac{\phi_0}{2}\right)\right)$$

$$\underset{\sqrt{\frac{g}{l}}}{\underset{\uparrow}{\omega_0}}\, t = sn^{-1}\left(x \equiv \frac{\sin(\phi/2)}{\sin(\frac{\phi_0}{2})},\ k \equiv \sin\left(\frac{\phi_0}{2}\right)\right) + \underset{-K(k)}{\underset{\uparrow}{\text{const}}}$$

for $\phi = \phi_0$
when $t = 0$

$$\phi(t) = 2 \arcsin\left[k\, sn\left(\omega_0\left(t + \frac{P}{4}\right);\ k\right)\right]$$

Elliptic integral of 2nd kind (circumference of ellipse w.r.t. +y-axis)

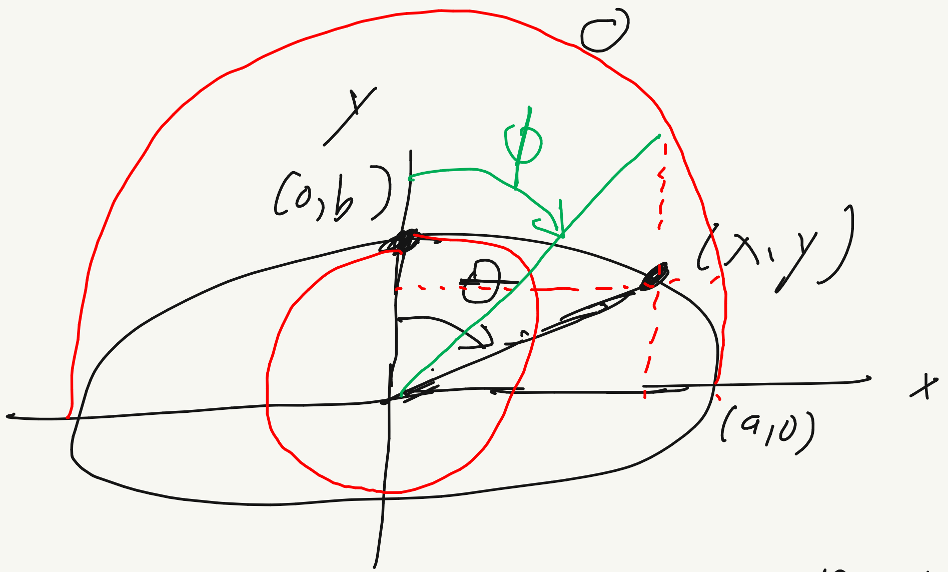
CW

$$E(\phi, k) \equiv \int_0^{\sin\phi} dt \, \frac{\sqrt{1-k^2 t^2}}{\sqrt{1-t^2}}$$

Rewrite: $t = \sin\bar\phi$

$dt = \cos\bar\phi \, d\bar\phi = \sqrt{1-t^2} \, d\bar\phi$

$$E(\phi, k) \equiv \int_0^{\phi} d\bar\phi \sqrt{1-k^2\sin^2\bar\phi} \quad \text{(scipy definition)}$$



$$k = e = \sqrt{1-\left(\frac{b}{a}\right)^2}$$

$x = r\sin\theta = a\sin u = a\sin\phi$

$y = r\cos\theta = b\cos u = b\cos\phi$

} sin, cos interchanged from previous page

$ds = \sqrt{dx^2+dy^2}$

$= \sqrt{a^2\cos^2\phi + b^2\sin^2\phi} \; d\phi$

$= \sqrt{a^2(1-\sin^2\phi) + b^2\sin^2\phi} \; d\phi$

$= a\sqrt{1-\left(1-\left(\frac{b}{a}\right)^2\right)\sin^2\phi} \; d\phi$

$= a\sqrt{1-k^2\sin^2\phi} \; d\phi$

$(\phi \neq \theta)$

$\tan\theta = \frac{a}{b}\tan\phi$

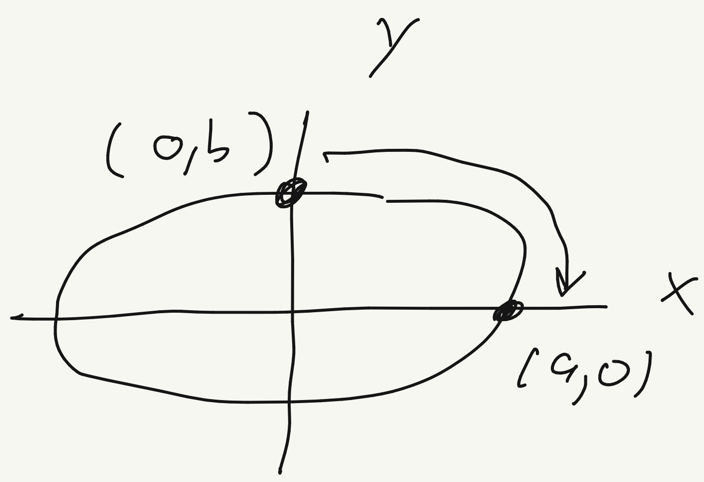
$\theta = \arctan\left(\frac{a}{b}\tan\phi\right)$

$\phi = \arctan\left(\frac{b}{a}\tan\theta\right)$

$$\boxed{S_{(0,b)\to(x,y)} = a\int_0^{\phi}\sqrt{1-k^2\sin^2\bar\phi} \; d\bar\phi} \equiv E(\phi, k)$$

Complete elliptic integral of 2^{nd} kind :

$$E(k) \equiv E\left(\frac{\pi}{2}, k\right)$$
$$= \int_0^{\frac{\pi}{2}} d\phi \sqrt{1 - k^2 \sin^2\phi}$$
$$= \int_0^1 dt \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}$$



Circumference:

$$C = 4a \int_0^1 dt \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}$$

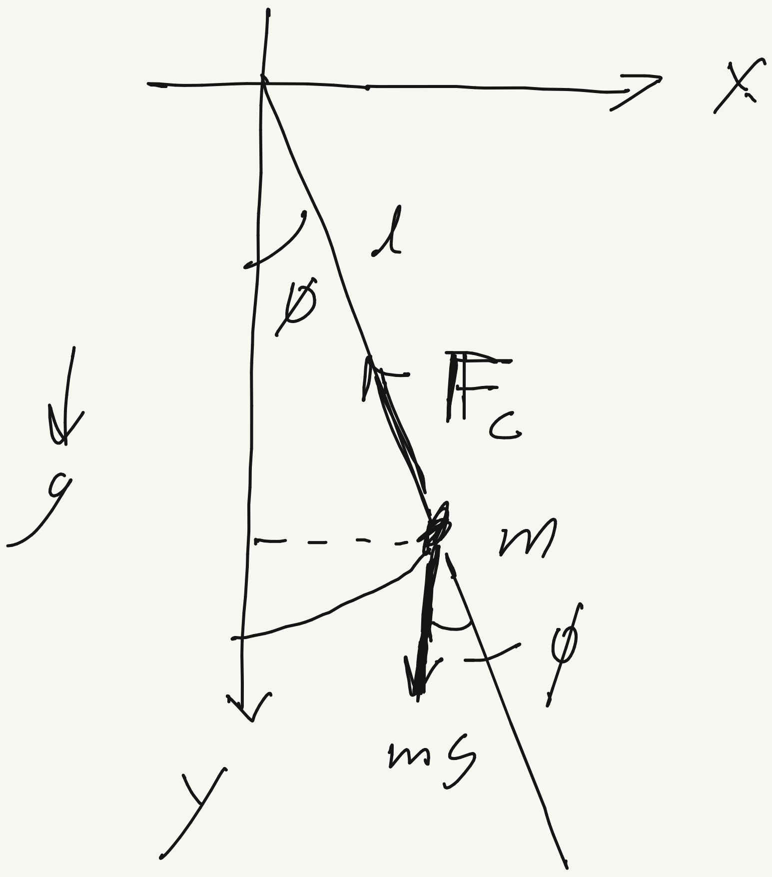
1^{st} order correction (for $k \ll 1$; nearly circular)

$$C \simeq 4a \int_0^1 \frac{dt}{\sqrt{1-t^2}} \left(1 - \frac{1}{2}k^2t^2\right)$$
$$= 4a \left[\underbrace{\int_0^1 \frac{dt}{\sqrt{1-t^2}}}_{\sin^{-1}(1) = \pi/2} - \frac{k^2}{2} \int_0^1 \frac{dt\ t^2}{\sqrt{1-t^2}} \right]$$

$t = \sin x$

$$= 4a \left[\frac{\pi}{2} - \frac{k^2}{2} \int_0^{\pi/2} \frac{\cancel{\cos x}\, dx \sin^2 x}{\cancel{\sqrt{1-\sin^2 x}}} \right]$$
$$= 4a \left[\frac{\pi}{2} - \frac{k^2}{2} \frac{1}{2} \int_0^{\pi/2} (1 - \cancelto{0}{\cos 2x})\, dx \right]$$
$$= 4a \left[\frac{\pi}{2} - \frac{k^2}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right]$$
$$= 2\pi a \left(1 - k^2/4\right)$$

Simple pendulum (Freshman physics analysis):



$$mg\sin\phi = -m a_T$$
$$= -m l \ddot{\phi}$$

$$\rightarrow \ddot{\phi} = -\frac{g}{l}\sin\phi$$

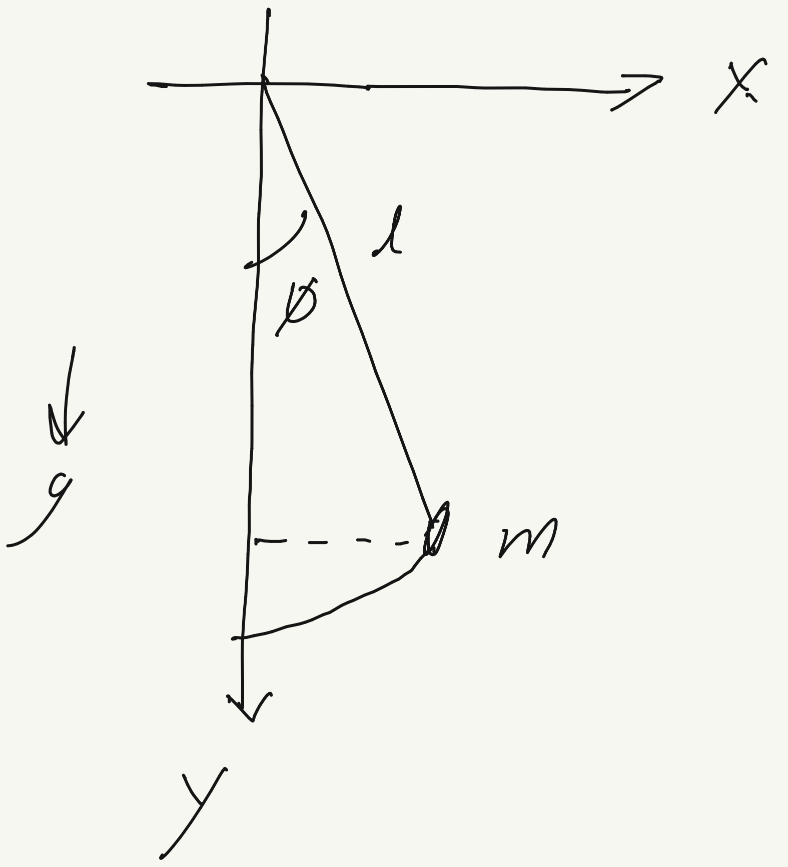
constraint force:

$$F_c - mg\cos\phi = m a_c$$
$$= m\omega^2 l$$
$$= m l \dot{\phi}^2$$

Thus, $F_c = mg\cos\phi + ml\dot{\phi}^2$

(non-zero at turning points as well as in vertical position - i.e., $\theta = 0$)

Period of a simple pendulum:



$$U = -mgy$$

where $y = l\cos\phi$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2}ml^2\dot{\phi}^2$$

$$L = T - U$$
$$= \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$$\frac{d}{dt}\left(ml^2\dot{\phi}\right) = -mgl\sin\phi$$

$$ml^2\ddot{\phi} = -mgl\sin\phi$$

$$\rightarrow \boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi}$$ (same as before)

Small-angle approx: $\sin\phi \simeq \phi$

$$\Longrightarrow \dot{\phi} \simeq -\frac{g}{l}\phi = -\omega^2\phi$$

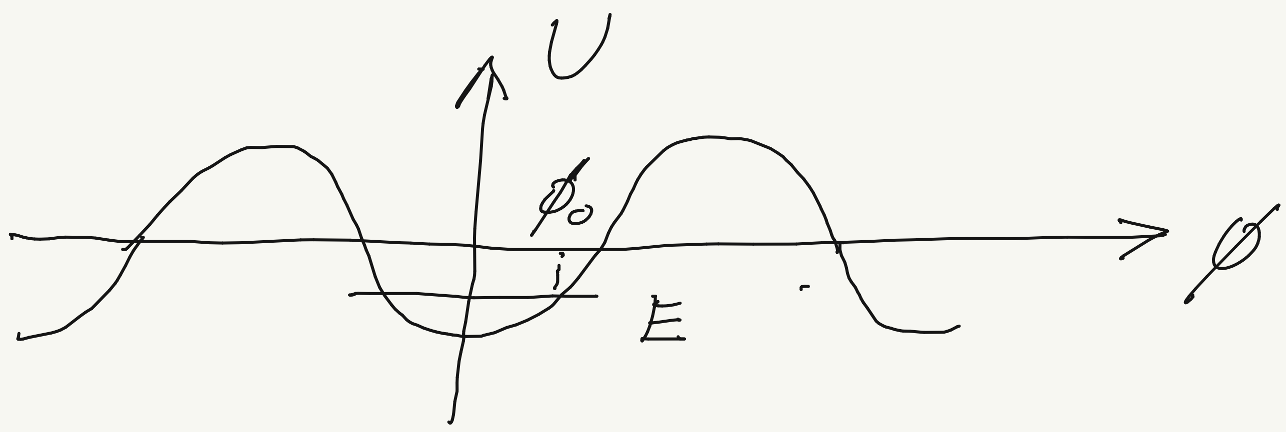
sol'n: $\phi(t) = \underset{\text{complex}}{A}e^{i\omega t}$, $\omega \equiv \sqrt{\frac{g}{l}}$, $P = \frac{2\pi}{\omega}$

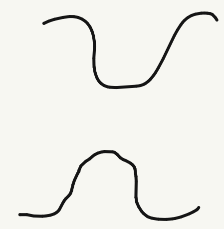
Beyond small-angle approx:

Cons. of energy:

$$E = T + U$$
$$= \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$$

$E = \text{const} = -mgl\cos\phi_0$ at turning points





Thus,

$$-mgl\cos\phi_0 = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi$$

$$\frac{1}{2} m l^2 \dot{\phi}^2 = +mgl\underbrace{(\cos\phi - \cos\phi_0)}_{\geq 0 \text{ since } \phi \leq \phi_0}$$

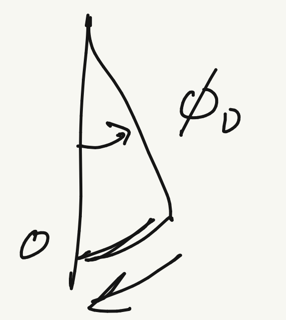
$$\rightarrow \quad \dot{\phi} = \pm\sqrt{2\frac{g}{l}(\cos\phi - \cos\phi_0)}$$

$$\int dt = \pm\int \frac{d\phi}{\sqrt{2\frac{g}{l}}\sqrt{\cos\phi - \cos\phi_0}}$$

$$\rightarrow \boxed{t = \frac{\pm 1}{\sqrt{2}\,\omega_0}\int \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}} + \text{const}}, \quad \omega_0 \equiv \sqrt{\frac{g}{l}}$$

Period:

$$P = \frac{4}{\sqrt{2}\,\omega_0} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$



$\frac{1}{4}$ th of a complete cycle

Now: $\cos\phi = \cos\left(2\,\frac{\phi}{2}\right)$

$$= \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{\phi}{2}\right)$$

$$\rightarrow \quad \cos\phi - \cos\phi_0 = -2\sin^2\left(\frac{\phi}{2}\right) + 2\sin^2\left(\frac{\phi_0}{2}\right)$$

$$= 2\sin^2\left(\frac{\phi_0}{2}\right)\left[1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_0}{2}\right)}\right]$$

Let: $x \equiv \dfrac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi_0}{2}\right)} \rightarrow \quad dx = \dfrac{1}{\sin\left(\frac{\phi_0}{2}\right)}\,\dfrac{1}{2}\cos\left(\frac{\phi}{2}\right) d\phi$

define $k \equiv \sin\left(\frac{\phi_0}{2}\right)$

$$= \frac{1}{2\sin\left(\frac{\phi_0}{2}\right)} \sqrt{1 - \sin^2\left(\frac{\phi}{2}\right)}\; d\phi$$

$$= \frac{1}{2k}\sqrt{1 - k^2x^2}\; d\phi$$

Thus,

$$P = \frac{4}{\cancel{\sqrt{2}}\,\omega_0} \int_0^1 \frac{dx\;\cancel{2}\cancel{k}}{\sqrt{1-k^2x^2}}\;\frac{1}{\cancel{\sqrt{2}}\cancel{k}\sqrt{1-x^2}}$$

$$= \boxed{\frac{4}{\omega_0}\int_0^1 \frac{dx}{\sqrt{1-k^2x^2}\sqrt{1-x^2}}} = \frac{4}{\omega_0}K(k)$$

complete elliptic integral of 1st kind

Leading-order correction to period:

$$P = \frac{4}{\omega_0}\int_0^1 \frac{dx}{\sqrt{1-k^2x^2}\,\sqrt{1-x^2}}$$

suppose $\phi_0 << 1 \rightarrow k \equiv \sin\left(\frac{\phi_0}{2}\right) \approx \frac{\phi_0}{2} << 1$

Thus, $$\frac{1}{\sqrt{1-k^2x^2}} \approx 1 + \frac{1}{2}k^2x^2$$
$$= 1 + \frac{1}{2}\left(\frac{\phi_0^2}{4}\right)x^2$$
$$= 1 + \frac{1}{8}\phi_0^2x^2$$

$$\rightarrow P \approx \frac{4}{\omega_0}\int_0^1 \frac{dx}{\sqrt{1-x^2}}\left[1 + \frac{1}{8}\phi_0^2x^2\right]$$
$$= \frac{4}{\omega_0}\int_0^1\frac{dx}{\sqrt{1-x^2}} + \frac{\phi_0^2}{2\omega_0}\int_0^1\frac{dx\,x^2}{\sqrt{1-x^2}}$$
$$= \frac{4}{\omega_0}\sin^{-1}(1) + \frac{\phi_0^2}{2\omega_0}\int_0^{\pi/2}\frac{\cancel{\cos\theta}\,d\theta\,\sin^2\theta}{\cancel{\sqrt{1-\sin^2\theta}}}$$
$$= \frac{4}{\omega_0}\cdot\frac{\pi}{2} + \frac{\phi_0^2}{2\omega_0}\int_0^{\pi/2}d\theta\,\frac{1}{2}(1-\cos 2\theta)$$
$$= \frac{2\pi}{\omega_0} + \frac{\phi_0^2}{4\omega_0}\left(\frac{\pi}{2} - \frac{1}{2}\underbrace{\cancel{\sin 2\theta\,\Big|_0^{\pi/2}}}_{0}\right)$$
$$= \boxed{\frac{2\pi}{\omega_0}\left(1 + \frac{\phi_0^2}{16}\right)}$$

General time dependence of oscillation:

$$\int_0^t dt = \frac{-1}{2w_0}\int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}}$$ where $\phi(0) = \phi_0$ and $\frac{d\phi}{dt} < 0$

$$\rightarrow t = -\frac{1}{w_0}\int_1^x \frac{dx}{\sqrt{1-k^2x^2}\sqrt{1-x^2}}$$

$$= -\frac{1}{w_0}\left[\int_1^0 + \int_0^x\right] \frac{dx}{\sqrt{1-k^2x^2}\sqrt{1-x^2}}$$

$$= +\frac{1}{w_0}\int_0^1 \frac{dx}{\sqrt{1-k^2x^2}\sqrt{1-x^2}} - \frac{1}{w_0}\int_0^x \frac{dx}{\sqrt{1-k^2x^2}\sqrt{1-x^2}}$$

$$= \frac{1}{w_0}\left[K(k) - sn^{-1}(x;k) \right]$$

Thus, $sn^{-1}(x;k) = K(k) - w_0 t$

$$\rightarrow \quad x = sn\Big(\underbrace{K(k)}_{\frac{w_0 P}{4}} - w_0 t \,;\, k\Big) = cn(w_0 t; k)$$

similar to $\cos\theta = \sin(\frac{\pi}{2} - \theta)$

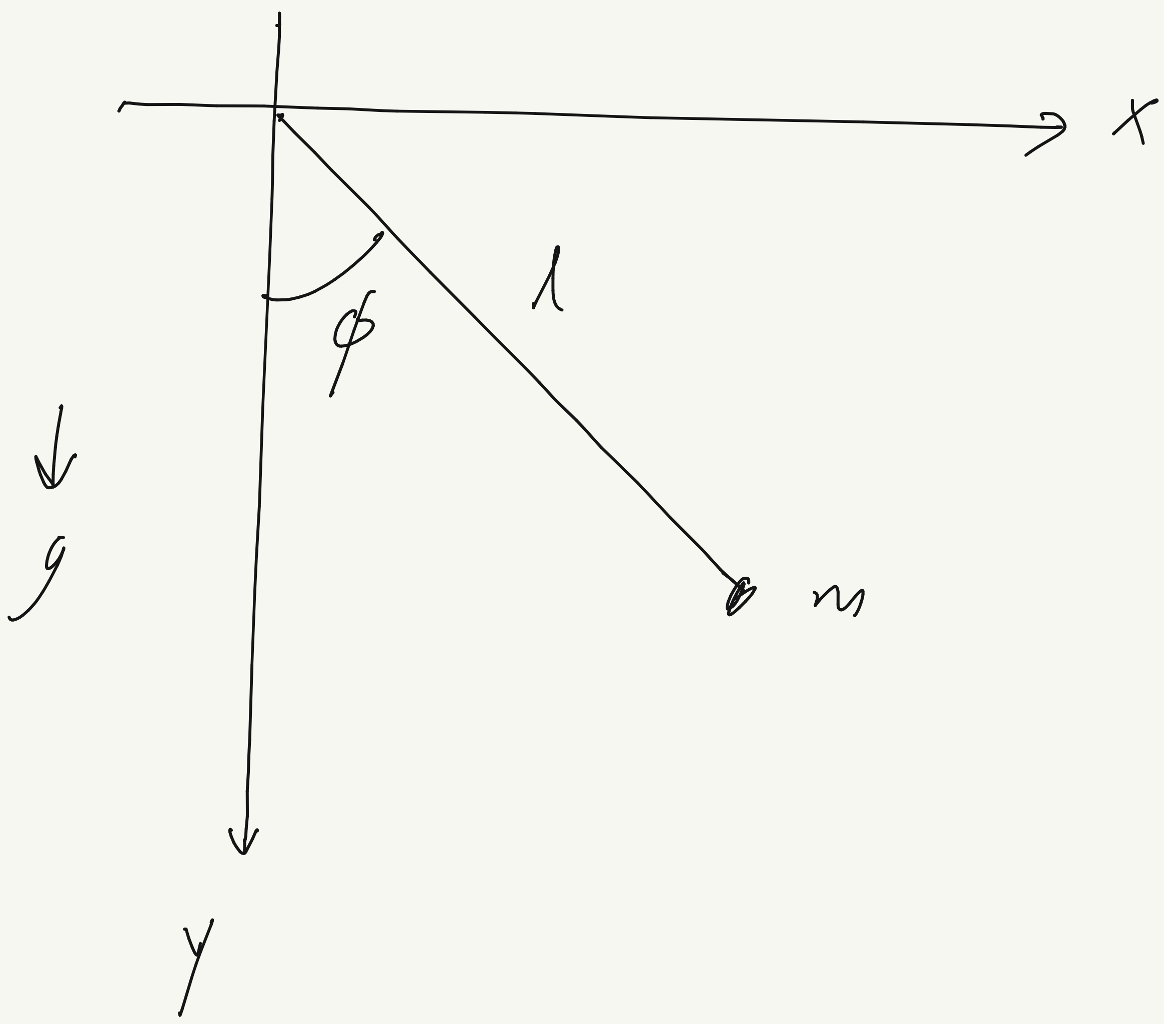
But $x = \sin(\phi/2)/\sin(\phi_0/2) = \frac{1}{k}\sin(\phi/2)$

$$\rightarrow \quad \sin\left(\frac{\phi}{2}\right) = k\, cn(w_0 t; k)$$

$$\boxed{\phi(t) = 2 \arcsin\left[k \; cn(w_0 t; k)\right]}$$

where $k = \sin\left(\frac{\phi_0}{2}\right)$

Lagrange multipliers:



constraint:

$$l = r = \sqrt{x^2+y^2}$$

$$U = -mgy$$
$$= -mgr\cos\phi$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$
$$= \frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2)$$

$$\varphi(r,\phi) \equiv l - r = 0$$

$$L = T - U + \lambda\varphi$$
$$= \frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2) + mgr\cos\phi + \lambda(l-r)$$

EOMs:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \qquad (2)$$

$$\varphi = 0 \qquad (3)$$

$$\frac{d}{dt}(m\dot{r}) = mr\dot{\phi}^2 + mg\cos\phi - \lambda$$

$$\boxed{\ddot{r} = r\dot{\phi}^2 + g\cos\phi - \frac{\lambda}{m}} \qquad (1)$$

$$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = -mgr\sin\phi$$

$$2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} = -mgr\sin\phi$$

$$\boxed{\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = -\frac{g}{r}\sin\phi} \qquad (2)$$

$$\boxed{l - r = 0} \qquad (3)$$

Differentiate constraint twice w.r.t time:

$\rightarrow \dot{r} = 0 \;,\; \ddot{r} = 0$

Substitute for $\ddot{r}$ using (1):

$$0 = \ddot{r} = r\dot{\phi}^2 + g\cos\phi - \frac{\lambda}{m}$$

$$\rightarrow \lambda = mr\dot{\phi}^2 + mg\cos\phi$$

$$= ml\dot{\phi}^2 + mg\cos\phi$$

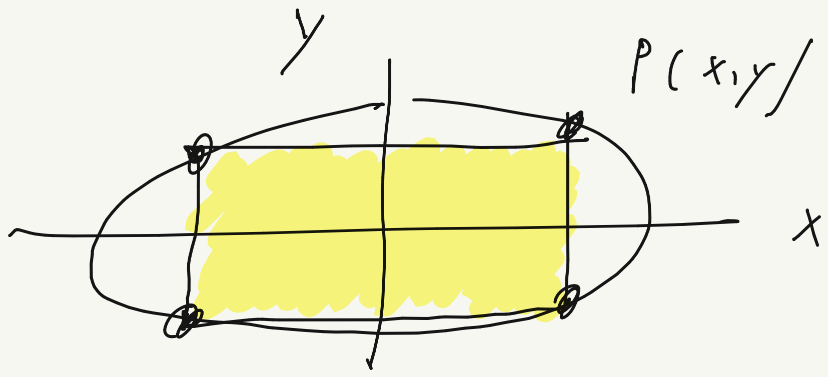
Compare with freshman physics calculation

$$F_c = mg\cos\phi + ml\dot{\phi}^2 \qquad (\text{in } -\hat{r} \text{ direction})$$

Thus,

$$\vec{F}_c = \lambda\vec{\nabla}\varphi \qquad (\text{since } \varphi = l - r \rightarrow \vec{\nabla}\varphi = -\hat{r})$$

Another example:



Suppose you want to maximize the area of a rectangle subject to the constraint that its corners lie on an ellipse $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$ (with a, b given).

Two methods:

$$\text{Area} = 4x\cdot y$$

$$\text{Constraint:} \quad \varphi(x,y) = 1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2$$

1. Define:

$$F(x,y,\lambda) = 4x\cdot y - \lambda\,\varphi(x,y)$$

$$= 4xy - \lambda\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)$$

require:

$$\frac{\partial F}{\partial x}=0 \qquad (1)$$

$$\frac{\partial F}{\partial y}=0 \qquad (2)$$

$$\frac{\partial F}{\partial \lambda}=0 \qquad (3)$$

$$\left.\begin{aligned}
(1)\quad 0 &= 4y + \frac{2\lambda x}{a^2}\\
(2)\quad 0 &= 4x + \frac{2\lambda y}{b^2}\\
(3)\quad 0 &= 1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2
\end{aligned}\right\}$$

3 equations, 3 unknowns (x, y, λ)

$$0 = \frac{4y^2}{b^2} + \frac{2\lambda xy}{a^2b^2}$$

$$0 = \frac{4x^2}{a^2} + \frac{2\lambda yx}{b^2a^2}$$

subtract: $0 = 4\left[\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right]$

$$\rightarrow \quad \frac{x}{a} = \pm\frac{y}{b}$$

substitute into (3): $\quad 0 = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^2$

$$2\left(\frac{x}{a}\right)^2 = 1$$

$$\frac{x}{a} = \frac{1}{\sqrt{2}}$$

$$\boxed{x = \frac{a}{\sqrt{2}}} \rightarrow \boxed{y = \frac{b}{\sqrt{2}}}$$

(2.) Reduced space method

$F(x) = 4xy \;\Big|\; y = b\sqrt{1 - \left(\frac{x}{a}\right)^2}$

$$= 4xb\sqrt{1 - \left(\frac{x}{a}\right)^2}$$

Solve $F'(x) = 0$

$$0 = F'(x)$$

$$= 4b\sqrt{1-\left(\frac{x}{a}\right)^2} + 4xb\,\frac{1}{\cancel{2}}\,\frac{1}{\sqrt{}}\left(-\frac{\cancel{2}x}{a^2}\right)$$

$$= 4b\left(\sqrt{1-\left(\frac{x}{a}\right)^2} - \left(\frac{x}{a}\right)^2\frac{1}{\sqrt{}}\right)$$

$$= \frac{4b}{\sqrt{}}\left[1-\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^2\right]$$

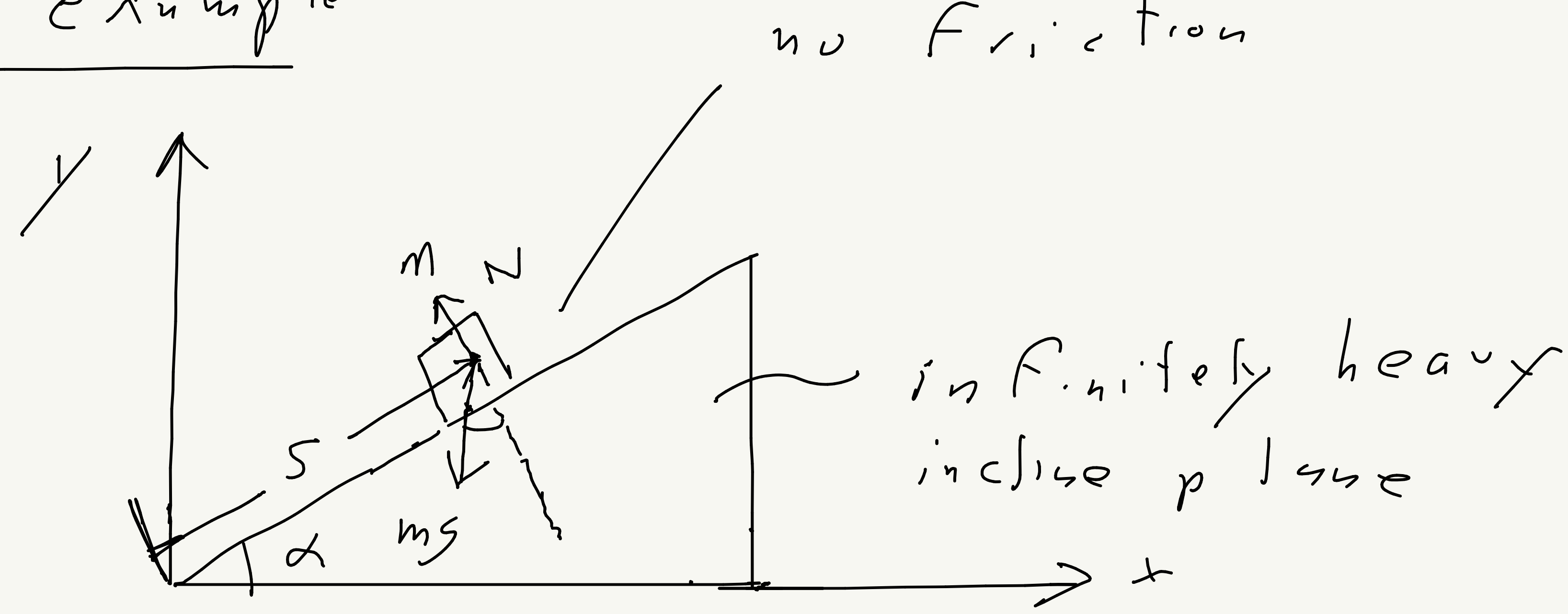
$$= \frac{4b}{\sqrt{}}\left[1 - 2\left(\frac{x}{a}\right)^2\right]$$

$$\rightarrow \left(\frac{x}{a}\right)^2 = \frac{1}{2} \rightarrow \boxed{x = \frac{a}{\sqrt{2}}}$$

$$\rightarrow \boxed{y = b\sqrt{1-\left(\frac{x}{a}\right)^2} = \frac{b}{\sqrt{2}}} \quad \Big| \quad \frac{x}{a} = \frac{1}{\sqrt{2}}$$

(same result as Lagrange multiplier method)

Another example



$$x = s\cos\alpha \qquad m\ddot{s} = -mg\sin\alpha$$
$$y = s\sin\alpha \qquad \boxed{\ddot{s} = -g\sin\alpha} \quad \text{EOM}$$

$$N - mg\cos\alpha = 0$$

$$\rightarrow \boxed{N = mg\cos\alpha} \quad \text{constraint force (normal force)}$$

Using Lagrange multipliers:

(x, y) degrees of freedom

Constraint: $\dfrac{y}{x} = \tan\alpha \rightarrow \varphi \equiv y - x\tan\alpha = 0$

Lagrangian:

$$L = T - U$$
$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

Enforce constraint

$$L' = L + \lambda\varphi(x, y)$$
$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(y - x\tan\alpha)$$

EOMs:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} \qquad (2)$$

$$\varphi(x,y) = 0 \qquad (3)$$

(1) $m\ddot{x} = -\lambda \tan\alpha$

$\rightarrow \ddot{x} = -\frac{\lambda}{m}\tan\alpha$

(2) $m\ddot{y} = -mg + \lambda$

$\rightarrow \ddot{y} = -g + \frac{\lambda}{m}$

(3) $y - x\tan\alpha = 0$

$y = x\tan\alpha$

Differentiate contraint twice w.r.t t:

$$\ddot{y} = \ddot{x}\tan\alpha$$

Substitute from (1) and (2):

$$-g + \frac{\lambda}{m} = -\frac{\lambda}{m}\tan\alpha \cdot \tan\alpha$$

$$\frac{\lambda}{m}\left(1 + \tan^2\alpha\right) = g$$

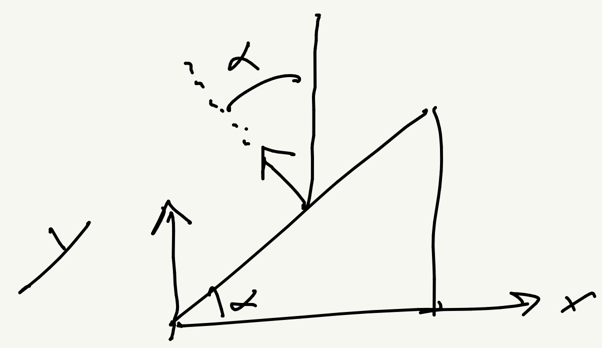
$s^2 + c^2 = 1$

$1 + \tan^2 = \sec^2\alpha$

$$\lambda = \frac{mg}{\sec^2\alpha} = \boxed{mg\cos^2\alpha}$$

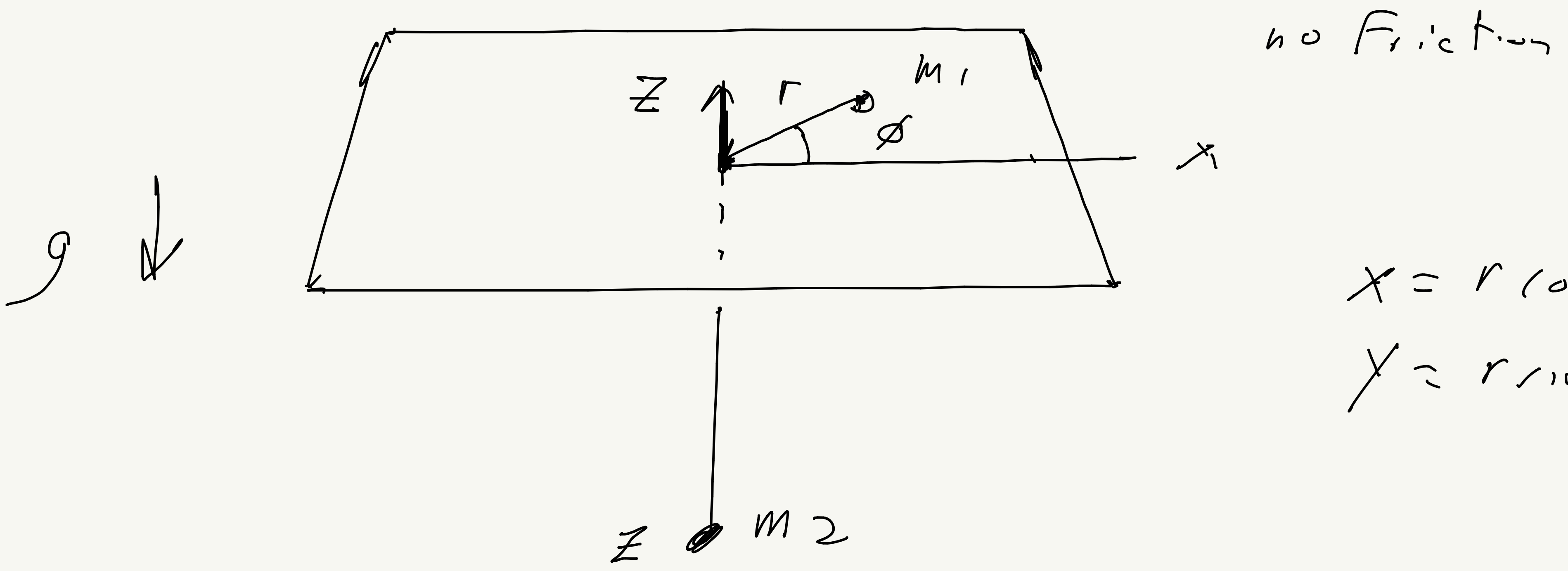
Constraint force:

$$
\begin{aligned}
\vec{F}_c &= \lambda \vec{\nabla} \phi \\
&= mg\cos^2\alpha \left(\hat{y} - \hat{x}\tan\alpha \right) \\
&= mg\cos\alpha \left(\cos\alpha\,\hat{y} - \sin\alpha\,\hat{x} \right) \\
&= mg\cos\alpha\,\hat{n} \\
&= N\hat{n}
\end{aligned}
$$



where $\hat{n} = -\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}$ is unit vector $\perp$ to surface of incline plane

Conservation of momentum example



$x = r\cos\phi$
$y = r\sin\phi$

Constraint: $r - Z = l$

$$T = \frac{1}{2} m_1 (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m_2 \dot{Z}^2$$

$$= \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2} m_2 \dot{r}^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2$$

$$U = m_2 g Z = m_2 g (r - l) = m_2 g r \underbrace{+ \text{const}}_{\text{ignore}}$$

$$L = T - U$$

$$= \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 - m_2 g r$$

No t-dependence:

$$E = T + U = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\phi}^2 + m_2 g r$$

$$= \text{const}$$

No ϕ-dependence

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} = \text{const} \equiv M_Z$$

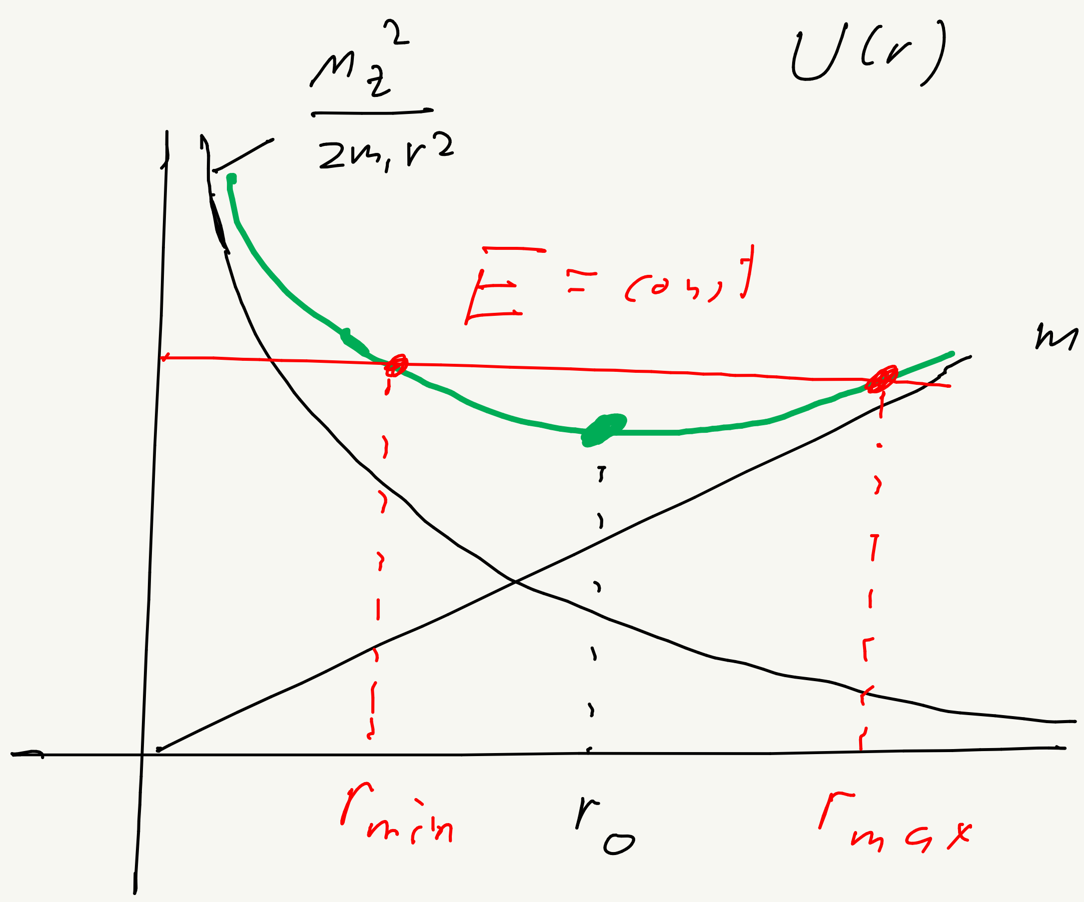
Thus, $\dot{\phi} = \dfrac{M_z}{m_1 r^2}$

$$E = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \left(\frac{M_z^2}{m_1^2 r^4}\right) + m_2 g r$$

$$= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{M_z^2}{2m_1 r^2} + m_2 g r$$

$$= \frac{1}{2}(m_1+m_2)\dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = \frac{M_z^2}{2m_1 r^2} + \underbrace{m_2 g r}_{U(r)}$$



r_{min}, r_{max} : turning point

$E = U_{eff}(r_0) \rightarrow$ stable circular orbit $(r = r_0)$

$E \geq U_{eff}(r_0)$

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0}$$

$$= \left.\left(-\frac{M_z^2}{m_1 r^3} + m_2 g\right)\right|_{r=r_0}$$

$$= \frac{-M_z^2}{m_1 r_0^3} + m_2 g$$

$$M_z^2 = m_1 m_2 g r_0^3$$

so $\boxed{M_z = \sqrt{m_1 m_2 g r_0^3}}$

angular momentum ~~necessary~~ to give ~~you~~ a circular orbit at $r = r_0$

$$U_{eff,min} = U_{eff}(r_0)$$

$$= \left.\left(\frac{M_z^2}{2 m_1 r_0^2} + m_2 g r_0\right)\right|_{M_z = \sqrt{m_1 m_2 g r_0^3}}$$

$$= \frac{\cancel{m_1} m_2 g r_0^{\cancel{3}}}{2 \cancel{m_1} \cancel{r_0^2}} + m_2 g r_0$$

$$= \frac{3}{2} m_2 g r_0$$

EoM's:

$$E = \frac{1}{2}(m_1+m_2)\dot{r}^2 + U_{eff}(r)$$

$$= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{M_z^2}{2m_1 r^2} + m_2 g r$$

$$\rightarrow \frac{2}{(m_1+m_2)}\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right) = \dot{r}^2$$

$$\frac{dr}{dt} = \pm\sqrt{\left(\frac{2}{m_1+m_2}\right)\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}$$

$$\boxed{\int \pm\, dt = \int \frac{dr}{\sqrt{\frac{2}{m_1+m_2}\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}}}$$

Integrate to get $t = t(r)$

Orbit equation:

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{M_z}{m_1 r^2}$$

Thus,
$$\frac{dr}{d\phi}\frac{M_z}{m_1 r^2} = \pm\sqrt{\circledcirc}$$

$$\frac{dr}{r^2} = \pm\frac{m_1}{M_z}\sqrt{\circledcirc}\, d\phi$$

$$\boxed{\int \pm\, d\phi = \int \frac{dr/r^2}{\frac{m_1}{M_z}\sqrt{\frac{2}{m_1+m_2}\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)}}}$$

Can try to evaluate these integrals analytically making substitutions of the type

$$u = \frac{1}{r} \rightarrow du = -\frac{1}{r^2}\, dr \quad , \text{etc.}$$

Numerically:

$$dr = \pm\sqrt{\bigcirc}\; dt \,, \quad \bigcirc = \frac{2}{m_1+m_2}\left(E - \frac{M_z^2}{2m_1 r^2} - m_2 g r\right)$$

$$d\phi = \frac{M_z}{m_1 r^2}\, dt$$

So start the system at time $t = t_0$ with some values of r and ϕ for given E, M_z

$\underline{t = t_0}$: $\quad r(t_0)\,, \quad \phi(t_0)$

$\underline{t = t_0 + \Delta t}$: $\quad r(t_0) + \Delta r\,, \quad \phi(t_0) + \Delta\phi$

where $\Delta r = \pm\sqrt{\bigcirc}\,\Delta t\,, \quad \Delta\phi = \frac{M_z}{m_1 r(t_0)}\,\Delta t$

⋮

(repeat)