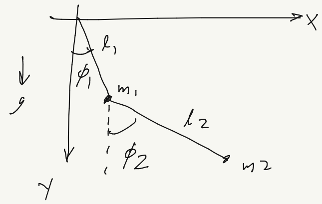
Sec 5, Prob 1:



$$x_1 = l_1 \sin\phi_1$$
$$y_1 = l_1 \cos\phi_1$$

$$x_2 = x_1 + l_2 \sin\phi_2$$
$$y_2 = y_1 + l_2 \cos\phi_2$$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_1 g l_1 \cos\phi_1 - m_2 g (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$
$$= -(m_1+m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

$$T = \tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\dot{x}_1 = l_1 \dot{\phi}_1 \cos\phi_1 \rightarrow \dot{x}_1^2 = l_1^2 \dot{\phi}_1^2 \cos^2\phi_1$$
$$\dot{y}_1 = -l_1 \dot{\phi}_1 \sin\phi_1 \rightarrow \dot{y}_1^2 = l_1^2 \dot{\phi}_1^2 \sin^2\phi_1$$
$$\dot{x}_1^2 + \dot{y}_1^2 = l_1^2 \dot{\phi}_1^2$$

$$\dot{x}_2 = l_1 \dot{\phi}_1 \cos\phi_1 + l_2 \dot{\phi}_2 \cos\phi_2$$
$$\dot{y}_2 = -l_1 \dot{\phi}_1 \sin\phi_1 - l_2 \dot{\phi}_2 \sin\phi_2$$

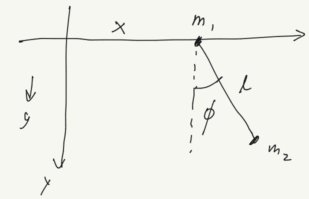
$$\rightarrow \dot{x}_2^2 = l_1^2 \dot{\phi}_1^2 \cos^2\phi_1 + l_2^2 \dot{\phi}_2^2 \cos^2\phi_2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos\phi_1 \cos\phi_2$$
$$\dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 \sin^2\phi_1 + l_2^2 \dot{\phi}_2^2 \sin^2\phi_2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin\phi_1 \sin\phi_2$$

so $\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2)$
$$= l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

Thus,
$$T = \tfrac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \tfrac{1}{2} m_2 l_1^2 \dot{\phi}_1^2 + \tfrac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$
$$= \tfrac{1}{2}(m_1+m_2) l_1^2 \dot{\phi}_1^2 + \tfrac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

$$\rightarrow L = T - U$$
$$= \tfrac{1}{2}(m_1+m_2) l_1^2 \dot{\phi}_1^2 + \tfrac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$
$$+ (m_1+m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2$$

Sec 5, Prob 2:



$$(x_1, y_1) = (x, 0)$$
$$(x_2, y_2) = (x + l \sin\phi, l \cos\phi)$$

$$(\dot{x}_1, \dot{y}_1) = (\dot{x}, 0)$$
$$(\dot{x}_2, \dot{y}_2) = (\dot{x} + l\dot{\phi}\cos\phi, -l\dot{\phi}\sin\phi)$$

$$T = \tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$
$$= \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 (\dot{x}^2 + l^2 \dot{\phi}^2 \cos^2\phi + 2 l \dot{x} \dot{\phi} \cos\phi$$
$$+ l^2 \dot{\phi}^2 \sin^2\phi)$$

$$= \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + \tfrac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \dot{x} \dot{\phi} \cos\phi$$

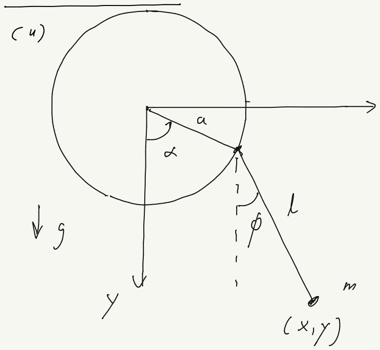
$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_2 g l \cos\phi$$

$$L = T - U$$
$$= \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + \tfrac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \dot{x} \dot{\phi} \cos\phi$$
$$+ m_2 g l \cos\phi$$

$$= \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + \tfrac{1}{2} m_2 (l^2 \dot{\phi}^2 + l \dot{x} \dot{\phi} \cos\phi) + m_2 g l \cos\phi$$

Sec 5, Prob 3:

(a)



$$\alpha = \gamma t$$

$$x = a\sin\alpha + l\sin\phi$$

$$y = a\cos\alpha + l\cos\phi$$

$$U = -mgy$$

$$= \underbrace{-mga\cos\alpha}_{\text{prescribed function of time (ignore)}} - mgl\cos\phi$$

$$= -mgl\cos\phi$$

$$\dot{x} = a\gamma\cos\alpha + l\dot{\phi}\cos\phi$$

$$\dot{y} = -a\gamma\sin\alpha - l\dot{\phi}\sin\phi$$

$$\dot{x}^2 = a^2\gamma^2\cos^2\alpha + l^2\dot{\phi}^2\cos^2\phi + 2al\gamma\dot{\phi}\cos\alpha\cos\phi$$

$$\dot{y}^2 = a^2\gamma^2\sin^2\alpha + l^2\dot{\phi}^2\sin^2\phi + 2al\gamma\dot{\phi}\sin\alpha\sin\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left[a^2\gamma^2 + l^2\dot{\phi}^2 + 2al\gamma\dot{\phi}\cos(\alpha-\phi)\right]$$

$$= \underbrace{\frac{1}{2}ma^2\gamma^2}_{\text{prescribed function of time (ignore)}} + \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\cos(\gamma t-\phi)$$

$$= \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\cos(\gamma t-\phi)$$

$$L = T - U$$

$$= \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\cos(\gamma t-\phi) + mgl\cos\phi$$

Now:

$$\gamma\dot{\phi}\cos(\gamma t-\phi) = \underbrace{\frac{d}{dt}\left[-\gamma\sin(\gamma t-\phi)\right]}_{\text{can ignore since total time derivative.}} + \gamma^2\cos(\gamma t-\phi)$$

Thus,

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos(\gamma t-\phi) + mgl\cos\phi$$

NOTE:

EoM should be the same for both Lagrangians:

(1st): $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi}$

$$\frac{d}{dt}\left(ml^2\dot{\phi} + mal\gamma\cos(\gamma t-\phi)\right) = mal\gamma\dot{\phi}\sin(\gamma t-\phi) - mgl\sin\phi$$

$$ml^2\ddot{\phi} - mal\gamma^2\sin(\gamma t-\phi) + \cancel{mal\gamma\dot{\phi}\sin(\gamma t-\phi)} = \cancel{mal\gamma\dot{\phi}\sin(\gamma t-\phi)} - mgl\sin\phi$$

$$\rightarrow \ddot{\phi} = \frac{a}{l}\gamma^2\sin(\gamma t-\phi) - \frac{g}{l}\sin\phi$$

agree

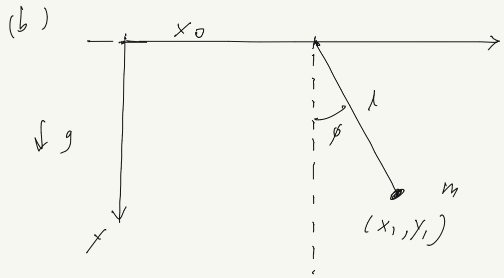
(2nd) $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi}$

$$\frac{d}{dt}\left(ml^2\dot{\phi}\right) = +mal\gamma^2\sin(\gamma t-\phi) - mgl\sin\phi$$

$$ml^2\ddot{\phi} = mal\gamma^2\sin(\gamma t-\phi) - mgl\sin\phi$$

$$\rightarrow \ddot{\phi} = \frac{a}{l}\gamma^2\sin(\gamma t-\phi) - \frac{g}{l}\sin\phi$$

(b)



$x_0 = a\cos\gamma t$

$\dot{x}_0 = -a\gamma\sin\gamma t$

$x = x_0 + l\sin\phi$

$y = l\cos\phi$

$U = -mgy = -mgl\cos\phi$

$\dot{x} = \dot{x}_0 + l\dot{\phi}\cos\phi$

$= -a\gamma\sin\gamma t + l\dot{\phi}\cos\phi$

$\dot{y} = -l\dot{\phi}\sin\phi$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$= \frac{1}{2}m\left(a^2\gamma^2\sin^2\gamma t + l^2\dot{\phi}^2\cos^2\phi - 2al\gamma\dot{\phi}\sin\gamma t\cos\phi + l^2\dot{\phi}^2\sin^2\phi\right)$

$= \underbrace{\frac{1}{2}ma^2\gamma^2\sin^2\gamma t}_{\text{prescribed function of time (ignore)}} + \frac{1}{2}ml^2\dot{\phi}^2 - mal\underbrace{\gamma\dot{\phi}\sin\gamma t\cos\phi}$

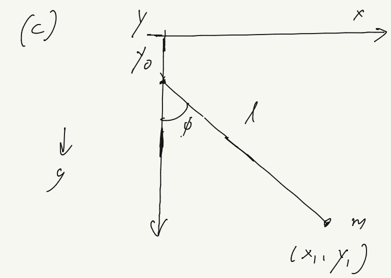
$= \frac{d}{dt}\left(\gamma\sin\gamma t\sin\phi\right) - \gamma^2\cos\gamma t\sin\phi$

ignore

$= \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\sin\phi$

$\rightarrow L = T - U = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\sin\phi + mgl\cos\phi$

(c)



$y_0 = a\cos\gamma t$

$x = l\sin\phi, \quad y = y_0 + l\cos\phi$

$\dot{x} = l\dot{\phi}\cos\phi$

$\dot{y} = \dot{y}_0 + l\dot{\phi}\sin\phi$

$= -a\gamma\sin\gamma t - l\dot{\phi}\sin\phi$

$\dot{x}^2 = l^2\dot{\phi}^2\cos^2\phi$

$\dot{y}^2 = a^2\gamma^2\sin^2\gamma t + l^2\dot{\phi}^2\sin^2\phi + 2al\gamma\dot{\phi}\sin\gamma t\sin\phi$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$= \frac{1}{2}m\left(a^2\gamma^2\sin^2\gamma t + l^2\dot{\phi}^2 + 2al\gamma\dot{\phi}\sin\gamma t\sin\phi\right)$

$= \underbrace{\frac{1}{2}ma^2\gamma^2\sin^2\gamma t}_{\text{prescribed fcn of t (ignore)}} + \frac{1}{2}ml^2\dot{\phi}^2 + mal\underbrace{\gamma\dot{\phi}\sin\gamma t\sin\phi}$

$= \frac{d}{dt}\left(-\gamma\sin\gamma t\cos\phi\right) + \gamma^2\cos\gamma t\cos\phi$

total time derivative (ignore)

$\simeq \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\cos\phi$

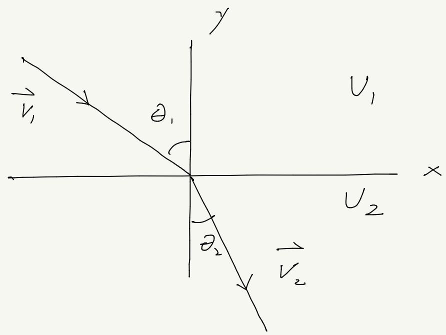
$U = -mgy = -mg(y_0 + l\cos\phi)$

$= \underbrace{-mga\cos\gamma t}_{\text{prescribed fcn of t (ignore)}} - mgl\cos\phi$

Thus,

$L = T - U = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos\gamma t\cos\phi + mgl\cos\phi$

Sec 7, Prob 1:



- Energy is conserved
- Component of linear momentum in x-direction is also conserved

i) $E = \frac{1}{2} m v_1^2 + U_1 = \frac{1}{2} m v_2^2 + U_2$

$$\frac{1}{2} m v_2^2 = \frac{1}{2} m v_1^2 + (U_1 - U_2)$$

$$v_2 = \sqrt{v_1^2 + \frac{2}{m}(U_1 - U_2)}$$

$$\frac{v_2}{v_1} = \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

ii) $p_x = \cancel{m} v_1 \sin\theta_1 = \cancel{m} v_2 \sin\theta_2$

$$\rightarrow \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1}$$

$$= \sqrt{1 + \frac{(U_1 - U_2)}{\frac{1}{2} m v_1^2}}$$

Sec 8, Prob 1:

$$S[q] = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t)$$

Let inertial frame K' move with velocity $\vec{V}$ w/t inertial frame K.

Then:
$$\vec{v}_a = \vec{v}_a{}' + \vec{V}$$
$$\vec{r}_a = \vec{r}_a{}' + \vec{V} \cdot t$$

Thus,

$$L = \frac{1}{2} \sum_a m_a |\vec{v}_a|^2 - U(\vec{r}_1, \vec{r}_2, \dots, t)$$
$$= \frac{1}{2} \sum_a m_a |\vec{v}_a{}' + \vec{V}|^2 - U$$
$$= \frac{1}{2} \sum_a m_a |\vec{v}_a{}'|^2 + \frac{1}{2} \sum_a m_a V^2 + \left(\sum_a m_a \vec{v}_a{}'\right) \cdot \vec{V} - U$$
$$= L' + \vec{p}\,' \cdot \vec{V} + \frac{1}{2} \mu V^2$$

$$\rightarrow S = \int_{t_1}^{t_2} dt \left(L' + \vec{p}\,' \cdot \vec{V} + \frac{1}{2} \mu V^2\right)$$
$$= S' + \vec{V} \cdot \sum_a m_a \vec{r}_a{}' \Big|_{t_1}^{t_2} + \frac{1}{2} \mu V^2$$
$$= S' + \mu \vec{V} \cdot \left(\vec{R}'(t_2) - \vec{R}'(t_1)\right) + \frac{1}{2} \mu V^2 (t_2 - t_1)$$

where $\vec{R}'$ is com of system w/t frame K'

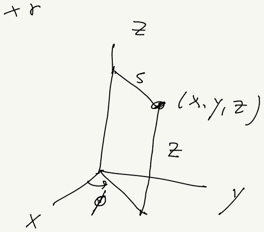
Sec 9, Prob 1:

cylindrical coords (s, ϕ, z):

$$x = s\cos\phi, \quad y = s\sin\phi, \quad z = z$$

$$\vec{M} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v} = m\vec{r}\times\dot{\vec{r}}$$

Thus,
$$M_x = m(y\dot z - z\dot y)$$
$$M_y = m(z\dot x - x\dot z)$$
$$M_z = m(x\dot y - y\dot x)$$



$$\dot x = \dot s\cos\phi - s\dot\phi\sin\phi$$
$$\dot y = \dot s\sin\phi + s\dot\phi\cos\phi$$
$$\dot z = \dot z$$

$$\rightarrow M_x = m\left[s\sin\phi\,\dot z - z(\dot s\sin\phi + s\dot\phi\cos\phi)\right]$$
$$= m\left[\sin\phi(s\dot z - z\dot s) - zs\dot\phi\cos\phi\right]$$

$$M_y = m\left[z(\dot s\cos\phi - s\dot\phi\sin\phi) - s\cos\phi\,\dot z\right]$$
$$= m\left[-\cos\phi(s\dot z - z\dot s) - zs\dot\phi\sin\phi\right]$$

$$M_z = m\left[s\cos\phi(\cancel{\dot s\sin\phi} + s\dot\phi\cos\phi) - s\sin\phi(\cancel{\dot s\cos\phi} - s\dot\phi\sin\phi)\right]$$
$$= ms^2\dot\phi$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$
$$= m^2\big[\sin^2\phi(s\dot z - z\dot s)^2 + z^2s^2\dot\phi^2\cos^2\phi$$
$$\cancel{-2zs\dot\phi\cos\phi\sin\phi(s\dot z - z\dot s)}$$
$$+\cos^2\phi(s\dot z - z\dot s)^2 + z^2s^2\dot\phi^2\sin^2\phi$$
$$\cancel{+2zs\dot\phi\cos\phi\sin\phi(s\dot z - z\dot s)}$$
$$+ s^4\dot\phi^2\big]$$

$$= m^2\left[(s\dot z - z\dot s)^2 + z^2s^2\dot\phi^2 + s^4\dot\phi^2\right]$$
$$= m^2\left[(s\dot z - z\dot s)^2 + s^2(z^2+s^2)\dot\phi^2\right]$$

Sec 9, Prob 2:

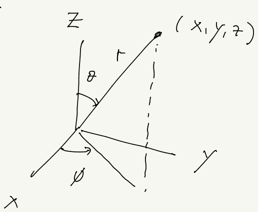
spherical polar coords (r, θ, ϕ):

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$\vec{M} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v} = m\vec{r}\times\dot{\vec{r}}$

Thus,

$$M_x = m(y\dot z - z\dot y)$$
$$M_y = m(z\dot x - x\dot z)$$
$$M_z = m(x\dot y - y\dot x)$$



$$\dot x = \dot r\sin\theta\cos\phi + r\dot\theta\cos\theta\cos\phi - r\dot\phi\sin\theta\sin\phi$$
$$\dot y = \dot r\sin\theta\sin\phi + r\dot\theta\cos\theta\sin\phi + r\dot\phi\sin\theta\cos\phi$$
$$\dot z = \dot r\cos\theta - r\dot\theta\sin\theta$$

$$\rightarrow M_x = m\left[r\sin\theta\sin\phi(\cancel{\dot r\cos\theta} - r\dot\theta\sin\theta) - r\cos\theta(\cancel{\dot r\sin\theta\sin\phi} + r\dot\theta\cos\theta\sin\phi + r\dot\phi\sin\theta\cos\phi)\right]$$

$$= m\left[-r^2\dot\theta\sin\phi(\sin^2\theta + \cos^2\theta) - r^2\dot\phi\sin\theta\cos\theta\cos\phi\right]$$

$$= m\left[-r^2\dot\theta\sin\phi - r^2\dot\phi\sin\theta\cos\theta\cos\phi\right]$$

$$M_y = m\left[r\cos\theta(\cancel{\dot r\sin\theta\cos\phi} + r\dot\theta\cos\theta\cos\phi - r\dot\phi\sin\theta\sin\phi) - r\sin\theta\cos\phi(\cancel{\dot r\cos\theta} - r\dot\theta\sin\theta)\right]$$

$$= m\left[r^2\dot\theta\cos\phi - r^2\dot\phi\sin\theta\cos\theta\sin\phi\right]$$

$$M_z = m\left[r\sin\theta\cos\phi(\cancel{\dot r\sin\theta\sin\phi} + r\dot\theta\cancel{\cos\theta\sin\phi} + r\dot\phi\sin\theta\cos\phi) - r\sin\theta\sin\phi(\cancel{\dot r\sin\theta\cos\phi} + r\dot\theta\cancel{\cos\theta\cos\phi} - r\dot\phi\sin\theta\sin\phi)\right]$$

$$= m\left[r^2\dot\phi\sin^2\theta(\cos^2\phi + \sin^2\phi)\right]$$

$$= m\,r^2\dot\phi\sin^2\theta$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

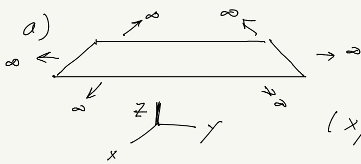
$$= m^2\left[r^4\dot\theta^2\sin^2\phi + r^4\dot\phi^2\sin^2\theta\cos^2\theta\cos^2\phi \cancel{+ 2r^4\dot\theta\dot\phi\sin\theta\cos\theta\sin\phi\cos\phi} + r^4\dot\theta^2\cos^2\phi + r^4\dot\phi^2\sin^2\theta\cos^2\theta\sin^2\phi \cancel{- 2r^4\dot\theta\dot\phi\sin\theta\cos\theta\sin\phi\cos\phi} + r^4\dot\phi^2\sin^4\theta\right]$$

$$= m^2r^4\left[\dot\theta^2 + \dot\phi^2(\sin^2\theta\cos^2\theta + \sin^4\theta)\right]$$

$$= m\,r^4\left[\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right]$$

Sec 9, Prob 3:

a)

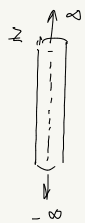


p_x, p_y = const

M_z = const

(xy-plane // to homog. plane)

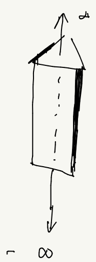
b)



p_z , M_z = const

(z-axis is axis of cylinder)

c)



infinite homog. cylinder

p_z = const

(z-axis is // to edges of prism)

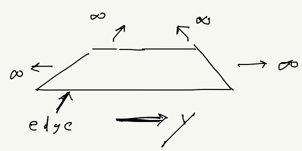
d)

two points

M_z = const

(z-axis passes through the two points)

e)

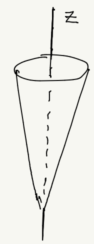


∞ homog. $\frac{1}{2}$ plane

p_y = const

(y axis is // to edge of $\frac{1}{2}$ plane)

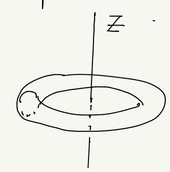
f)



M_z = const

(z-axis is axis of cone)

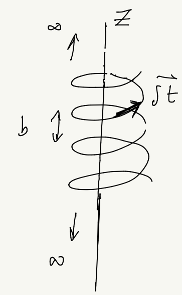
g)



M_z = const

(z-axis is axis of torus)

h)



a = radius

b = height between neighboring coils

$h = b/a$

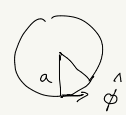
Invariance w.r.t. translations around the helix

$$\vec{\delta t} = a\,\delta\phi\,\hat{\phi} + \delta z\,\hat{z}$$

$$= a\,\delta\phi\,\hat{\phi} + \frac{b}{2\pi}\,\delta\phi\,\hat{z}$$

$$= a\,\delta\phi\left[\hat{\phi} + \frac{b/a}{2\pi}\,\hat{z}\right]$$

$$= a\,\delta\phi\left[\hat{\phi} + \frac{h}{2\pi}\,\hat{z}\right]$$



$$\frac{\delta\phi}{2\pi} = \frac{\delta z}{b}$$

$$\rightarrow \delta z = \frac{b}{2\pi}\,\delta\phi$$

Thus, $\vec{p}\cdot\vec{\delta t} = p_\phi + \frac{h}{2\pi}\,p_z$

$= M_z + \frac{h}{2\pi}\,p_z$

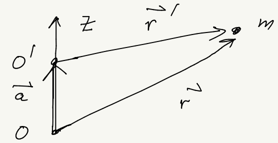
$=$ const

NOTE: M_z is independent of location of origin on z-axis

$$\vec{M} = \vec{r}\times\vec{p} = m\vec{r}\times\dot{\vec{r}}$$

Change origin by shifting along $\vec{a} = a\hat{z}$

$$\vec{r} = \vec{r}\,' + \vec{a}$$



$$\begin{aligned}
\vec{M} &= m\vec{r}\times\dot{\vec{r}} \\
&= m(\vec{r}\,' + \vec{a})\times\frac{d}{dt}(\vec{r}\,' + \vec{a}) \\
&= m\vec{r}\,'\times\dot{\vec{r}}\,' + m\vec{a}\times\dot{\vec{r}}\,' \\
&= \vec{M}' + \vec{a}\times\vec{p}\,' \quad (\text{for arbitrary } \vec{a})
\end{aligned}$$

Thus,

$$\begin{aligned}
M_z &= \vec{M}\cdot\hat{z} \\
&= (\vec{M}' + \vec{a}\times\vec{p}\,')\cdot\hat{z} \\
&= M'_z + a(\hat{z}\times\vec{p}\,')\cdot\hat{z} \\
&= M'_z + a\,\cancelto{0}{(\hat{z}\times\hat{z})}\cdot\vec{p}\,' \\
&= M'_z
\end{aligned}$$

Sec 10, Prob 1:

same path, different masses, same potential energies

$\rightarrow\ x' = x,\quad m' \neq m,\quad U' = U,\quad t' \neq t$

$$\begin{aligned}
L &= T - U = \tfrac{1}{2}m\dot{x}^2 - U \\
L' &= \tfrac{1}{2}m'\left(\frac{dx}{dt'}\right)^2 - U \\
&= \tfrac{1}{2}m'\left(\frac{t}{t'}\right)^2\dot{x}^2 - U
\end{aligned}$$

Thus, $L' = L \rightarrow$

$$m'\left(\frac{t}{t'}\right)^2 = m$$

$$\left(\frac{t'}{t}\right)^2 = \frac{m'}{m}$$

$$\rightarrow\ \frac{t'}{t} = \sqrt{\frac{m'}{m}}$$

Sec 10, Prob 2:

Same path, Same mass, potential energies differing by a constant factor ($U' = c\,U$)

$\rightarrow \quad x = x', \quad m = m', \quad t' \neq t$

$$L = T - U$$
$$= \tfrac{1}{2} m \dot{x}^2 - U$$
$$L' = \tfrac{1}{2} m \left(\frac{dx}{dt'}\right)^2 - U'$$
$$= \tfrac{1}{2} m \left(\frac{t}{t'}\right)^2 \dot{x}^2 - c\,U$$

Thus, need $\left(\frac{t}{t'}\right)^2 = c$ to get same EOM,

$$\rightarrow \quad \frac{t'}{t} = \sqrt{\frac{1}{c}}$$
$$= \sqrt{\frac{U}{U'}}$$

Sec 40, Prob 1

Hamiltonian for a single particle

$$L = T - U$$
$$= \tfrac{1}{2} m |\dot{\vec{r}}|^2 - U(\vec{r}, t)$$

Cartesian:

$$L = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = p_x/m$$
$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \rightarrow \dot{y} = p_y/m$$
$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$\rightarrow \quad H = \left(\sum_i p_i \dot{q}_i - L\right)_{\dot{q}_i = \dot{q}_i(q,p)}$$
$$= \left(p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(x,y,z,t)\right)\Big|_{\dot{x} = p_x/m,\ \text{etc}}$$
$$= \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + U(x,y,z,t)$$

cylindrical: (s, ϕ, z)

$$L = \frac{1}{2} m (\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2) - U(s,\phi,z,t)$$

$$p_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s} \rightarrow \dot{s} = p_s/m$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m s^2 \dot{\phi} \rightarrow \dot{\phi} = p_\phi / m s^2$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = p_z/m$$

$$\rightarrow H = \left(p_s \dot{s} + p_\phi \dot{\phi} + p_z \dot{z} - L\right)_{\dot{s} = p_s/m, \text{ etc}}$$

$$= \frac{p_s^2}{m} + \frac{p_\phi^2}{m s^2} + \frac{p_z^2}{m}$$

$$- \frac{1}{2} m \left[\left(\frac{p_s}{m}\right)^2 + s^2 \left(\frac{p_\phi}{m s^2}\right)^2 + \left(\frac{p_z}{m}\right)^2\right] + U(s,\phi,z,t)$$

$$= \frac{1}{2m}\left(p_s^2 + \frac{p_\phi^2}{s^2} + p_z^2\right) + U(s,\phi,z,t)$$

Spherical polar: (r, θ, ϕ)

$$L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2) - U(r,\theta,\phi,t)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \longrightarrow \dot{r} = p_r/m$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \rightarrow \dot{\theta} = p_\theta / m r^2$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\theta\, \dot{\phi} \rightarrow \dot{\phi} = p_\phi / m r^2 \sin^2\theta$$

$$\rightarrow H = \left(p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - L\right)\Big|_{\dot{r} = p_r/m, \text{ etc.}}$$

$$= p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{m r^2}\right) + p_\phi\left(\frac{p_\phi}{m r^2 \sin^2\theta}\right)$$

$$- \frac{1}{2} m \left(\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{m r^2}\right)^2 + r^2 \sin^2\theta \left(\frac{p_\phi}{m r^2 \sin^2\theta}\right)^2\right)$$

$$+ U(r,\theta,\phi,t)$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + U(r,\theta,\phi,t)$$

Sec 40, Prob 2:

For a uniformly rotating ref. frame:

$$L = \frac{1}{2}mv^2 + m\vec{v}\cdot(\vec{\Omega}\times\vec{r}) + \frac{1}{2}m|\vec{\Omega}\times\vec{r}|^2 - U$$

Hamiltonian

$$H = \left(\sum_i p_i \dot{q}_i - L\right)\Big|_{\dot{q}=\dot{q}(q,p)}$$

where

$$\vec{p} = \frac{\partial L}{\partial \vec{v}}$$

$$= m\vec{v} + m(\vec{\Omega}\times\vec{r})$$

$$= m[\vec{v} + \vec{\Omega}\times\vec{r}]$$ ——— velocity wrt inertial frame

$$\rightarrow \vec{v} = \frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}$$

thus,

$$H = \left(\vec{p}\cdot\vec{v} - L\right)\Big|_{\vec{v}=\vec{p}/m - \vec{\Omega}\times\vec{r}}$$

$$= \vec{p}\cdot\left(\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right) - \frac{1}{2}m\left|\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right|^2$$

$$- m\left(\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right)\cdot(\vec{\Omega}\times\vec{r}) - \frac{1}{2}m|\vec{\Omega}\times\vec{r}|^2 + U$$

$$= \frac{p^2}{m} - \cancel{\vec{p}\cdot(\vec{\Omega}\times\vec{r})} - \frac{1}{2}m\left(\frac{p^2}{m^2} + \cancel{|\vec{\Omega}\times\vec{r}|^2} - \cancel{\frac{2}{m}\vec{p}\cdot(\vec{\Omega}\times\vec{r})}\right)$$

$$- \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + \cancel{m|\vec{\Omega}\times\vec{r}|^2} - \cancel{\frac{1}{2}m|\vec{\Omega}\times\vec{r}|^2} + U$$

$$H = \frac{p^2}{2m} - \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + U$$

$$= \frac{p^2}{2m} - \vec{\Omega}\cdot(\vec{r}\times\vec{p}) + U$$

$$= \frac{p^2}{2m} - \vec{\Omega}\cdot\vec{M} + U$$