Searches for stochastic gravitational-wave backgrounds

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Abstract

These lecture notes provide a brief introduction to detection methods used to search for a stochastic background of gravitational radiation—a superposition of gravitational-wave signals either too weak or too numerous to individually detect. The lectures are divided into two main pieces: (i) an overview, consisting of a description of different types of gravitational-wave backgrounds and an introduction to the correlation method using multiple detectors; (ii) details, extending the previous discussion to non-trivial detector response, what to do in the absence of correlations, and a recently proposed Bayesian method to search for the gravitational-wave background produced by stellar-mass binary black hole mergers throughout the universe. Some suggested exercises for the reader are given throughout the text.

Contents

1 Motivation

A stochastic background of gravitational radiation is a superposition of gravitational-wave signals either too weak or too numerous to individually detect. The individual signals making up the background are thus *unresolvable*, unlike the large signal-to-noise binary black-hole (BBH) and binary neutron-star (BNS) merger signals recently detected by the advanced LIGO and Virgo detectors. But despite the fact that the individual signals are unresolvable, we shall see below that the detection of a stochastic gravitational-wave background (GWB) will be able to provide information about the statistical properties of the source.

1.1 Gravitational-wave analogue of the CMB

The ultimate goal of gravitational-wave background searches is to produce the GW analogue of Figure 1. This is a sky map of the temperature fluctuations in the CMB blackbody radiation,

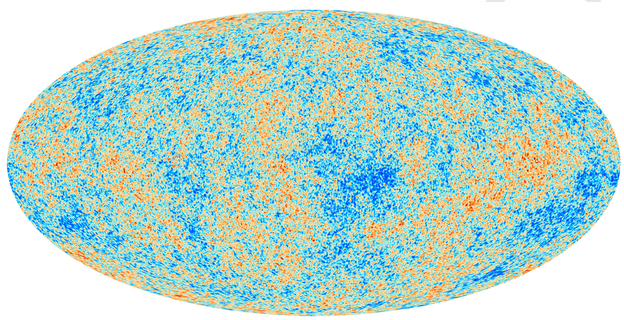

Figure 1: Skymap of $\Delta T/T_0$ for the cosmic microwave background radiation produced by the Planck 2013 mission.

relative to the $T_0 = 2.73$ K isotropic component. (The dipole contribution due to our motion with respect to the cosmic rest frame has also been subtracted out.) Recall that the CMB is a background of electromagnetic radiation, produced at the time of last scattering, roughly 380,000 yr after the Big Bang. At that time, the universe had a temperature of ~ 3000 K, approximately one thousand times larger than the temperature today, but cool enough for neutral hydrogen atoms to first form and photons to propagate freely. The temperature fluctuations in the CMB radiation tell us about the density of matter on the surface of last scattering and also about the integrated ...

For perspective, Figure 1 was produced by the Planck mission in 2013, almost 50 years after the CMB radiation was initially detected by Penzias and Wilson in 1965. It took many years and many more improved experiments (COBE, Boomerang, WMAP, Planck to name a few) to get to the high-precision measurements that we have today. So its somewhat sobering to realize that right now, in 2018, we have yet to detect the isotropic component of the GWB. So we have a long road ahead of us.

1.2 The background of BBH and BNS mergers in the LIGO band

But fortunately, as mentioned above, the advanced LIGO and Virgo detectors have detected other gravitational-wave signals from several individual BBH and BNS mergers. These were very strong signals, having matched-filter signal-to-noise ratios $\gtrsim 10$, and false alarm probabilities $< 2\times 10^{-7}$—i.e., 5-sigma "gold-plated" events. Similar detections are expected in the upcoming observing run O3, which is scheduled to start in early 2019. We also expect that there are many more signals, corresponding to more distant mergers or smaller mass systems, which are individually undetectable (i.e., *subthreshold* events). This weaker background of gravitational radiation is nonetheless detectable *as a collectivity* via the common influence of the gravitational waves on multiple detectors.

To get an idea of the statistical properties of this background, we can estimate the total rate of stellar-mass BBH mergers throughout the universe using the local rate estimate from these first detections, 9-240 $\mathrm{Gpc}^{-3}\,\mathrm{yr}^{-1}$. This leads to a prediction for the total rate of BBH mergers between ~ 1 per minute to a few per hour. (Exercise: 1: Verify these values for the total rate.[1]) Since the duration of BBH merger signals in band is ~ 1 s, which is much smaller than the average duration between successive mergers, the combined signal is highly-nonstationary (or *popcorn*-like). One can do similar calculations for BNS mergers. The predicted total rate for such events is roughly one event every 15 s, while the duration of a BNS signal in band is roughly 100 s. Thus, the signals overlap in time leading to a continuous (or *confusion-limited*) background. Figure 2 is a plot of the expected time-domain signal corresponding the rate estimates calculated above.

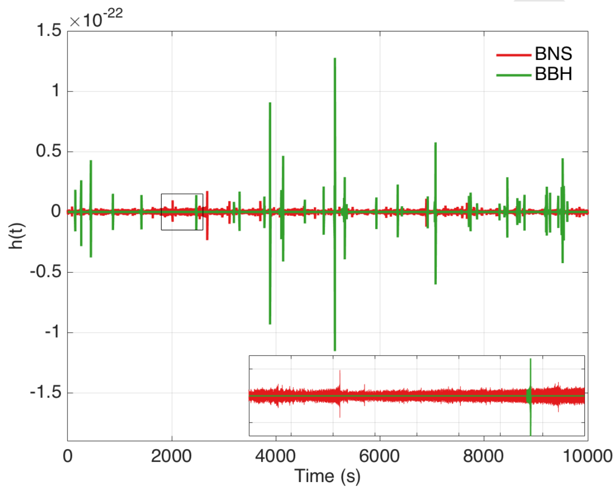

Figure 2: Simulated time-domain signal for the predicted BBH and BNS background. Figure taken from [**?**].

The combined signal from BBH and BNS mergers is potentially detectable with advanced LIGO and Virgo, shortly after reaching design sensitivity. Figure 3-σ, correspond to a 1/1000 false alarm probability).

This estimate of time to detection is based on the standard cross-correlation search, which assumes a Gaussian-stationary background.

But there is a better method, proposed by Smith and Thrane, which we will discuss at the end of these lectures.

2 Different types of stochastic backgrounds

Stochastic backgrounds can be of either astrophysical or cosmological origin:

(i) A potential astrophysical background for the current generation of ground-based interferometers is the combined signal from the population of stellar-mass BBH and BNS mergers throughout the universe. We will discuss the prospects of detecting this potential background throughout these lectures; the last section is devoted to a recently proposed detection method that targets this particular source.

(ii) A potential cosmological background is formed from *relic gravitational waves*—that is, quantum fluctuations in the geometry of space-time, driven to macroscopic scales by a period

[1] A more complete description of this and all other exercises is given in Section 10.

of rapid expansion (e.g., inflation) a mere $\sim 10^{-32}$ s after the Big Bang. This relic background is too weak to be detected by advanced LIGO, Virgo, etc., but is potentially detectable by its effect on the polarization of the cosmic microwave background (CMB) radiation. The Planck satellite and BICEP experiment (located at the South Pole) are searching for this signal.

3 Mathematical characterization of a stochastic background

4 Correlation methods - basic idea

5 Some simple examples

We now apply the above correlation method

6 Non-trivial detector response

7 Non-trivial correlations

8 What to do in the absence of correlations?

9 Searching for the background of binary black-hole mergers

10 Exercises

A more detailed description of the suggested exercises.

1. *Rate estimate of stellar-mass binary black hole mergers:*
   Estimate the total rate (number of events per time) of stellar-mass binary black hole mergers throughout the universe by multiplying LIGO's O1 local rate estimate $R_0 \sim 10$ - $200~\mathrm{Gpc}^{-3}\,\mathrm{yr}^{-1}$ by the comoving volume out to some large redshift, e.g., $z = 10$. (For this calculation you can ignore any dependence of the rate density with redshift.) You should find a merger rate of $\sim$1 per minute to a few per hour.
   Hint: You will need to do numerically evaluate the following integral for proper distance today as a function of source redshift:
   $$d_0(z) = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')}\,, \qquad E(z) \equiv \sqrt{\Omega_\mathrm{m}(1+z)^3 + \Omega_\Lambda}\,, \tag{10.1}$$
   with
   $$\Omega_\mathrm{m} = 0.31\,, \qquad \Omega_\Lambda = 0.69\,, \qquad H_0 = 68~\mathrm{km\,s^{-1}\,Mpc^{-1}}\,. \tag{10.2}$$
   Doing that integral, you should find what's shown in Figure 3, which you can then evaluate at $z = 10$ to convert R_0 (number of events per comoving volume per time) to total rate (number of events per time) for sources out to redshift $z = 10$.

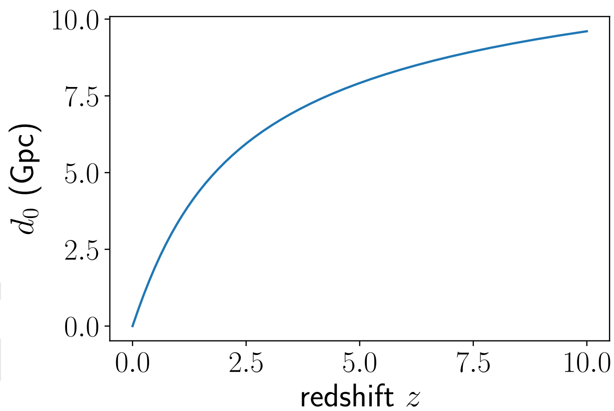

Figure 3

2. *Relating $S_h(f)$ and $\Omega_\mathrm{gw}(f)$:*
   Derive the relationship
   $$S_h(f) = \frac{3H_0^2}{2\pi^2}\frac{\Omega_\mathrm{gw}(f)}{f^3} \tag{10.3}$$
   between the strain power spectral density $S_h(f)$ and the dimensionless fractional energy density spectrum $\Omega_\mathrm{gw}(f)$. (*Hint*: You will need to use the various definitions of these quantities and also
   $$\rho_\mathrm{gw} = \frac{c^2}{32\pi G}\langle \dot{h}_{ab}(t,\vec{x})\dot{h}^{ab}(t,\vec{x})\rangle\,, \tag{10.4}$$
   which expresses the energy-density in gravitational-waves to the metric perturbations $h_{ab}(t,\vec{x})$.)
3. *Cosmology and the "Phinney formula" for astrophysical backgrounds:*
   (a) Using the Friedmann equation
   $$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_\mathrm{m}}{a^3} + \Omega_\Lambda\right) \tag{10.5}$$

for a spatially-flat FRW spacetime with matter and cosmological constant, and the relationship

$$1 + z = \frac{1}{a(t)}\,, \qquad a(t_0) \equiv 1 \quad (t_0 \equiv \text{today})\,, \tag{10.6}$$

between redshift z and scale factor $a(t)$, derive

$$\frac{\mathrm{d}t}{\mathrm{d}z} = -\frac{1}{(1+z)H_0 E(z)}\,, \qquad E(z) = \sqrt{\Omega_{\mathrm{m}}(1+z)^3 + \Omega_\Lambda}\,. \tag{10.7}$$

(b) Using this result for $\mathrm{d}t/\mathrm{d}z$, show that

$$\Omega_{\mathrm{gw}}(f) = \frac{f}{\rho_{\mathrm{c}} H_0} \int_0^\infty \mathrm{d}z\, R(z)\, \frac{1}{(1+z)E(z)} \left(\frac{\mathrm{d}E_{\mathrm{gw}}}{\mathrm{d}f_{\mathrm{s}}} \right) \bigg|_{f_{\mathrm{s}} = f(1+z)} \tag{10.8}$$

in terms of the rate density $R(z)$ as measured in the source frame (number of events per comoving volume per time interval in the source frame). (*Hint*: The expression for $\mathrm{d}t/\mathrm{d}z$ from part (a) will allow you to go from the "Phinney formula" for $\Omega_{\mathrm{gw}}(f)$ written in terms of the number density $n(z)$,

$$\Omega_{\mathrm{gw}}(f) = \frac{1}{\rho_c} \int_0^\infty \mathrm{d}z\, n(z)\, \frac{1}{1+z} \left(f_{\mathrm{s}} \frac{\mathrm{d}E_{\mathrm{gw}}}{\mathrm{d}f_{\mathrm{s}}} \right) \bigg|_{f_{\mathrm{s}} = f(1+z)}\,, \tag{10.9}$$

to one in terms of the rate density $R(z)$, where $n(z)\,\mathrm{d}z = R(z)\,|\mathrm{d}t|_{t=t(z)}$. Note: Both of the above expressions for $\Omega_{\mathrm{gw}}(f)$ assume that there is only one type of source, described by some set of average source parameters. If there is more than one type of source, one must sum the contributions of each source to $\Omega_{\mathrm{gw}}(f)$.)

4. *Optimal filtering for the cross-correlation statistic:*
   Verify the form

$$\tilde{Q}(f) \propto \frac{\Gamma_{12}(f) H(f)}{P_1(f) P_2(f)}\,, \tag{10.10}$$

of the optimal filter function in the weak-signal limit, where $H(f)$ is the assumed spectral shape of the gravitational-wave background, $\Gamma_{12}(f)$ is the overlap function, and $P_1(f)$, $P_2(f)$ are the power spectral densities of the outputs of the two detectors (which are approximately equal to $P_{n_1}(f)$, $P_{n_2}(f)$, respectively). Recall that the optimal filter $\tilde{Q}(f)$ maximizes the signal-to-noise ratio of the cross-correlation statistic. (*Hint*: Introduce an inner product on the space of functions of frequency $A(f)$, $B(f)$:

$$(A, B) \equiv \int df\, A(f) B^*(f) P_1(f) P_2(f)\,. \tag{10.11}$$

This inner product has all of the properties of the familiar dot product of vectors in 3-dimensional space. The signal-to-noise ratio of the cross-correlation statistic can be written in terms of this inner product.)

5. *Maximum-likelihood estimators for single and multiple parameters:*
   (a) Show that the maximum-likelihood estimator $\hat{a}$ of the single parameter a in the likelihood function

$$p(d|a,\sigma) \propto \exp\left[-\frac{1}{2} \sum_{i=1}^N \frac{(d_i - a)^2}{\sigma_i^2} \right] \tag{10.12}$$

is given by the noise-weighted average

$$\hat{a} = \sum_i \frac{d_i}{\sigma_i^2} \bigg/ \sum_j \frac{1}{\sigma_j^2}\,. \tag{10.13}$$

(b) Extend the previous calculation to the likelihood

$$p(d|A,C) \propto \exp\left[-\frac{1}{2}(d-MA)^\dagger C^{-1}(d-MA)\right], \tag{10.14}$$

where $A \equiv A_\alpha$ is a vector of parameters, $C \equiv C_{ij}$ is the noise covariance matrix, and $M \equiv M_{i\alpha}$ is the response matrix mapping A_α to data samples, $MA \equiv \sum_\alpha M_{i\alpha}A_\alpha$. For this more general case you should find:

$$\hat{A} = F^{-1}X\,, \tag{10.15}$$

where

$$F \equiv M^\dagger C^{-1} M\,, \qquad X \equiv M^\dagger C^{-1} d\,. \tag{10.16}$$

In general, the matrix F (called the *Fisher* matrix) is not invertible, so some sort of regularization is needed to do the matrix inversion.

6. *Timing-residual response for a 1-arm, 1-way detector:*
Derive the timing residual reponse function

$$R^A(f,\hat{n}) = \frac{1}{2}u^a u^b e^A_{ab}(\hat{n})\frac{1}{i2\pi f}\frac{1}{1+\hat{n}\cdot\hat{u}}\left[1-e^{-\frac{i2\pi fL}{c}(1+\hat{n}\cdot\hat{u})}\right] \tag{10.17}$$

for a single-link (i.e., a one-arm, one-way detector like that for pulsar timing). Here $\hat{u}$ is the direction of propagation of the electromagnetic pulse, and $\hat{n}$ is the direction to the GW source (the direction of wave propagation is $\hat{k} \equiv -\hat{n}$ and the direction to the pulsar is $\hat{p} \equiv -\hat{u}$). The origin of coordinates is taken to be at the position of the detector.

7. *Overlap function for colocated electric dipole antennae:*
Show that the overlap function for a pair of (short) colocated electric dipole antennae pointing in directions $\hat{u}_1$ and $\hat{u}_2$ is given by

$$\Gamma_{12} \propto \hat{u}_1 \cdot \hat{u}_2 \equiv \cos\zeta \tag{10.18}$$

for the case of an unpolarized, isotropic electromagnetic field. (*Hint*: "short" means that the phase of the electric field can be taken to be constant over of the lengths of the dipole antennae, so that the reponse of antenna $I = 1,2$ to the field is given by $r_I(t) = \hat{u}_I \cdot \vec{E}(t,\vec{x}_0)$, where $\vec{x}_0$ is the common location of the two antenna.)

8. *Maximum-likelihood estimators for the standard cross-correlation statistic:*
Verify that

$$\hat{C}_{11} \equiv \frac{1}{N}\sum_{i=1}^{N} d_{1i}^2\,, \qquad \hat{C}_{22} \equiv \frac{1}{N}\sum_{i=1}^{N} d_{2i}^2\,, \qquad \hat{C}_{12} \equiv \frac{1}{N}\sum_{i=1}^{N} d_{1i}d_{2i} \tag{10.19}$$

are maximum-likelihood estimators of

$$S_1 \equiv S_{n_1} + S_h\,, \quad S_2 \equiv S_{n_2} + S_h\,, \quad S_h\,, \tag{10.20}$$

for the case of N samples of a white GWB in uncorrelated white detector noise, for a pair of colocated and coaligned detectors. Recall that the likelihood function is

$$p(d|S_{n_1},S_{n_2},S_h) = \frac{1}{\sqrt{\det(2\pi C)}}\exp\left[-\frac{1}{2}d^T C^{-1} d\right], \tag{10.21}$$

where

$$C = \begin{bmatrix} (S_{n_1}+S_h)\,1_{N\times N} & S_h\,1_{N\times N} \\ S_h\,1_{N\times N} & (S_{n_2}+S_h)\,1_{N\times N} \end{bmatrix} \tag{10.22}$$

and

$$d^T C^{-1} d \equiv \sum_{I,J=1}^{2}\sum_{i,j=1}^{N} d_{Ii}\left(C^{-1}\right)_{Ii,Jj} d_{Jj}\,. \tag{10.23}$$

9. *Derivation of the maximum-likelihood ratio detection statistic:*
Verify that twice the log of the maximum-likelihood ratio for the standard stochastic likelihood function goes like the square of the (power) signal-to-noise ratio,

$$2\ln\Lambda_{\rm ML}(d) \simeq \frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}/N}\,, \tag{10.24}$$

in the weak-signal approximation. (*Hint:* For simplicity, do the calculation in the context of N samples of a white GWB in uncorrelated white detector noise, for a pair of colocated and coaligned detectors, using the results of Exercise 8.)

10. *Standard cross-correlation likelihood by marginalizing over stochastic signal prior:*
Derive the standard form of the likelihood function for stochastic background searches

$$p(d|S_{n_1}, S_{n_2}, S_h) = \frac{1}{\sqrt{\det(2\pi C)}}\exp\left[-\frac{1}{2}\sum_{I,J=1}^{2} d_I\left(C^{-1}\right)_{IJ} d_J\right]\,, \tag{10.25}$$

where

$$C \equiv \begin{bmatrix} S_{n_1}+S_h & S_h \\ S_h & S_{n_2}+S_h\end{bmatrix}\,, \tag{10.26}$$

by marginalizing

$$p_n(d-h|S_{n_1}, S_{n_2}) = \frac{1}{2\pi\sqrt{S_{n_1}S_{n_2}}}\exp\left[-\frac{1}{2}\left\{\frac{(d_1-h)^2}{S_{n_1}} + \frac{(d_2-h)^2}{S_{n_2}}\right\}\right] \tag{10.27}$$

over the signal samples h for the *stochastic* signal prior

$$p(h|S_h) = \frac{1}{\sqrt{2\pi S_h}}\exp\left[-\frac{1}{2}\frac{h^2}{S_h}\right]\,. \tag{10.28}$$

In other words, show that

$$p(d|S_{n_1}, S_{n_2}, S_h) = \int_{-\infty}^{\infty} {\rm d}h\, p_n(d-h|S_{n_1}, S_{n_2})p(h|S_h)\,. \tag{10.29}$$

(*Hint*: You'll have to complete the square in the argument of the exponential in the marginalization integral.)

References

A few references, which you might find helpful. *Disclaimer: This list is not in anyway complete, and there may be better references that you already know of.*

1. B. Allen - "The stochastic gravitational-wave background: sources and detection," from Les Houches School in Oct 1995
2. M. Maggiore - "Gravitational-wave experiments and early universe cosmology" (2000)
3. C. Caprini, D. Figueroa - "Cosmological backgrounds of gravitational waves" (2018)
4. T. Regimbau - "The astrophysical stochastic gravitational-wave backgrounds" (2011)
5. J. Romano, N. Cornish - "Detection methods for stochastic gravitational-wave backgrounds: a unified treatment" (2017)
6. R. Smith, E. Thrane - "Optimal search for an astrophysical gravitational-wave background" (2018)
7. Plus recent observational papers from LIGO, Virgo, pulsar timing arrays, etc., quoting upper limits on the strength of stochastic gravitational-wave backgrounds