Searches for stochastic gravitational-wave backgrounds

Lecture 1
Les Houches Summer School
19 July 2018

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Connection to other lectures

- Astrophysical sources - I. Mandel (last week)
- ★ Cosmological sources - V. Mandic (starting tomorrow)
- Data analysis - A. Weinstein (last week), J. Veitch (this week)
- Pulsar timing - N. Cornish (next week)
- LISA detector / science - M. Hewitson, A. Sesana (next week)

Some references (not complete)

- ★ B. Allen - “The stochastic gravitational-wave background: sources and detection,” from Les Houches School in Oct 1995
- M. Maggiore - “Gravitational-wave experiments and early universe cosmology” (2000)
- C. Caprini, D. Figueroa - “Cosmological backgrounds of gravitational waves” (2018)
- T. Regimbau - “The astrophysical stochastic gravitational-wave backgrounds” (2011)
- J. Romano, N. Cornish - “Detection methods for stochastic gravitational-wave backgrounds: a unified treatment” (2017)
- R. Smith, E. Thrane - “Optimal search for an astrophysical gravitational-wave background” (2018)
- Plus recent observational papers from LIGO, Virgo, pulsar timing arrays, etc., quoting upper limits on the strength of stochastic gravitational-wave backgrounds

Resources

https://github.com/josephromano/leshouches

- Slides
- Exercises (suggested)
- Solutions
- Code examples (ipython notebooks)

Plan for lectures

Today: Overview / Basics

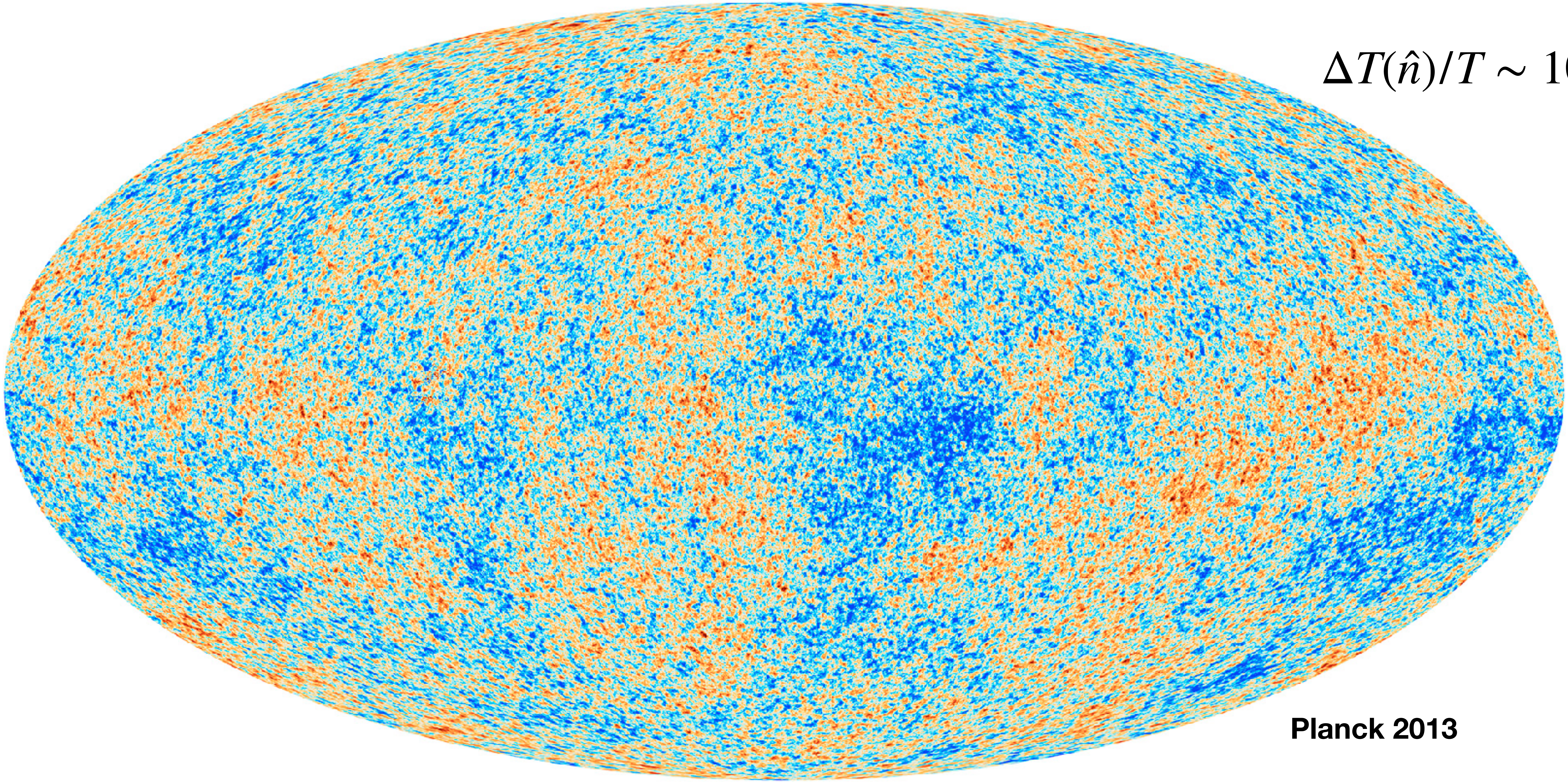
1. Motivation / context
2. Different types of stochastic backgrounds
3. Characterizing a stochastic GW background
4. Correlation methods
5. Some simple examples

Tomorrow: Details / Example

1. Non-trivial detector response
2. Non-trivial correlated response
3. What to do in the absence of correlations (e.g., for LISA)?
4. Frequentist and Bayesian methods
5. Example: searching for the background from BBH mergers

1. Motivation

Ultimate goal: produce GW analogue of CMB sky map



$\Delta T(\hat{n})/T \sim 10^{-5}$

Planck 2013

But there's a long road ahead…

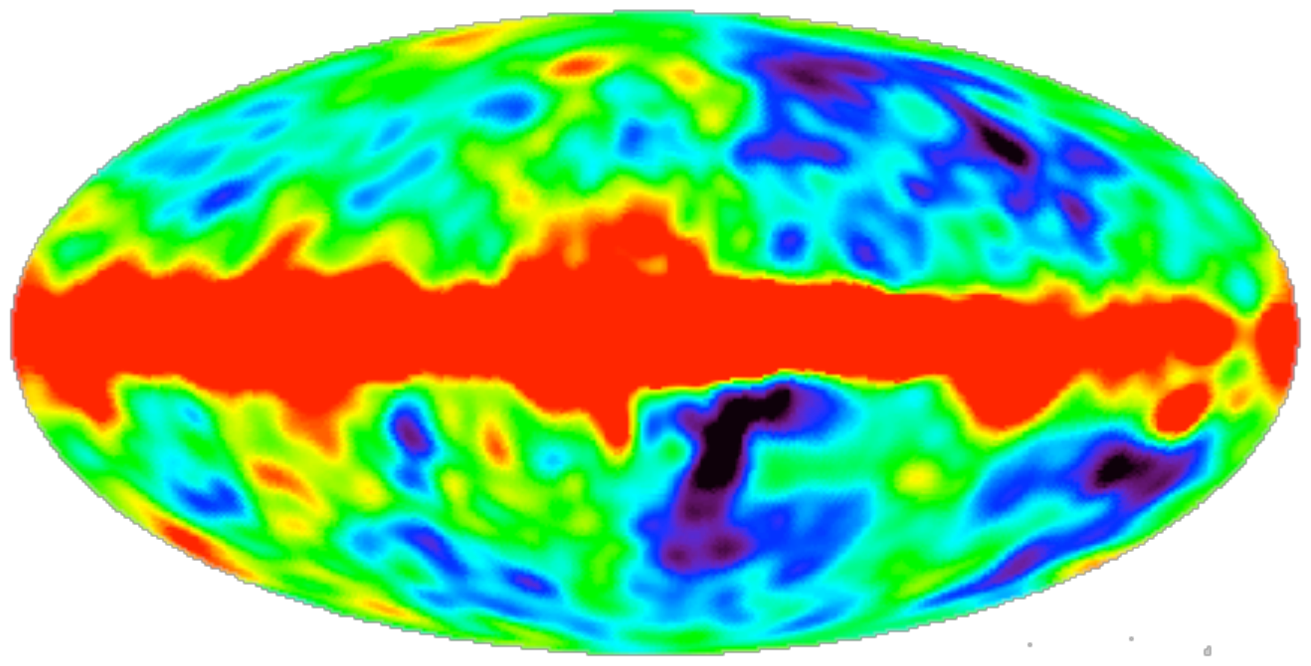
A MEASUREMENT OF EXCESS ANTENNA TEMPERATURE
AT 4080 Mc/s

(4080 Mc/s <-> 7.35 cm)

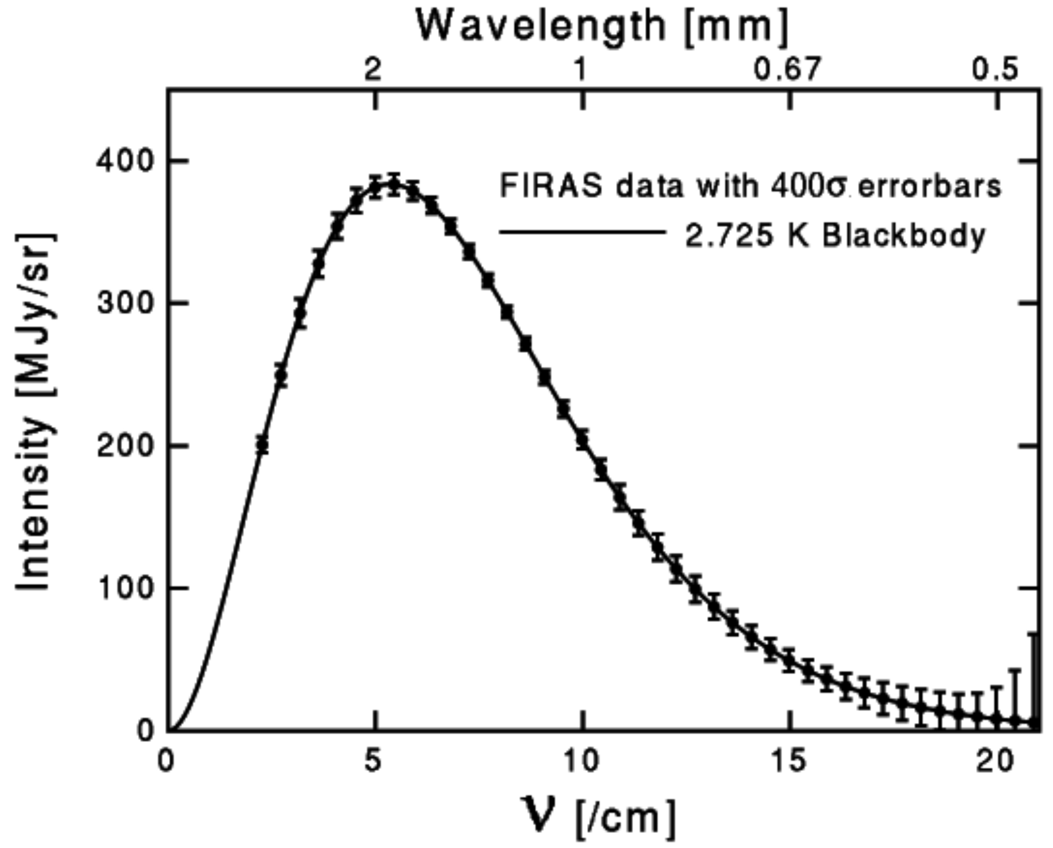
Measurements of the effective zenith noise temperature of the 20-foot horn-reflector antenna (Crawford, Hogg, and Hunt 1961) at the Crawford Hill Laboratory, Holmdel, New Jersey, at 4080 Mc/s have yielded a value about 3.5° K higher than expected. This excess temperature is, within the limits of our observations, isotropic, unpolarized, and free from seasonal variations (July, 1964–April, 1965). A possible explanation for the observed excess noise temperature is the one given by Dicke, Peebles, Roll, and Wilkinson (1965) in a companion letter in this issue.

— **1965: Penzias & Wilson**

— **1992: COBE**



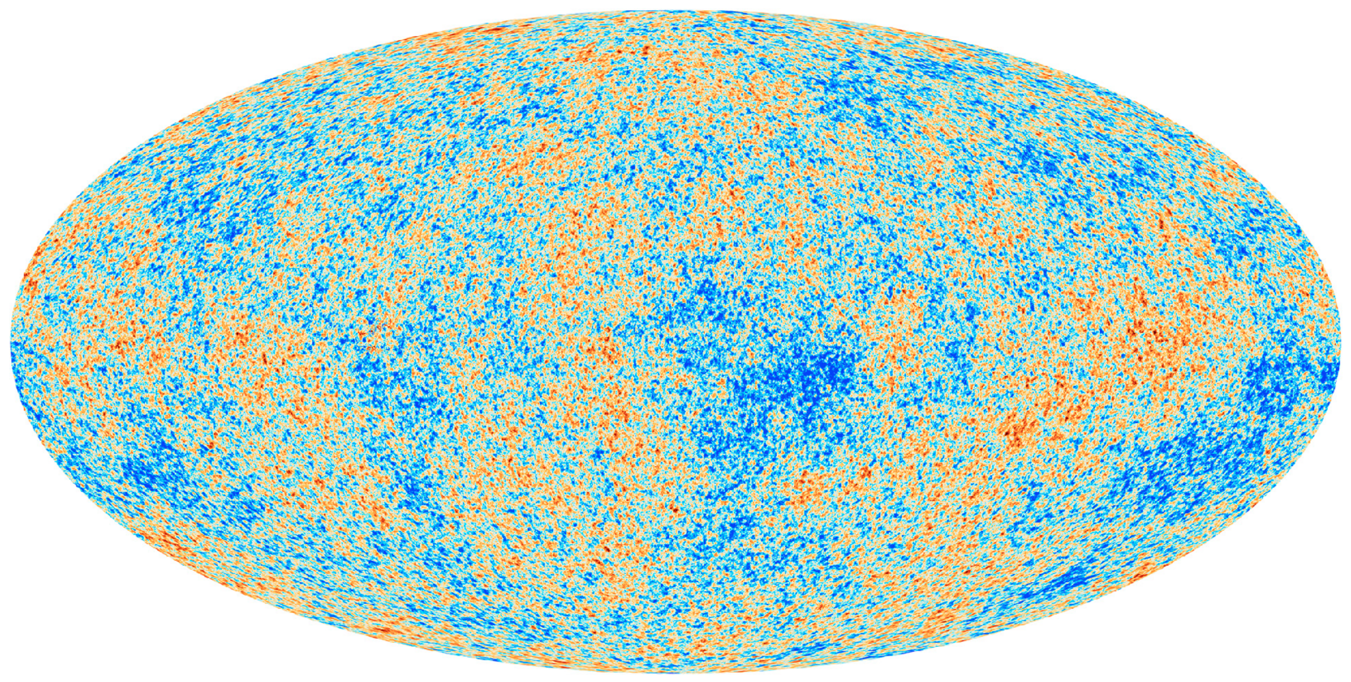
(ang resolution: ~10 degrees)



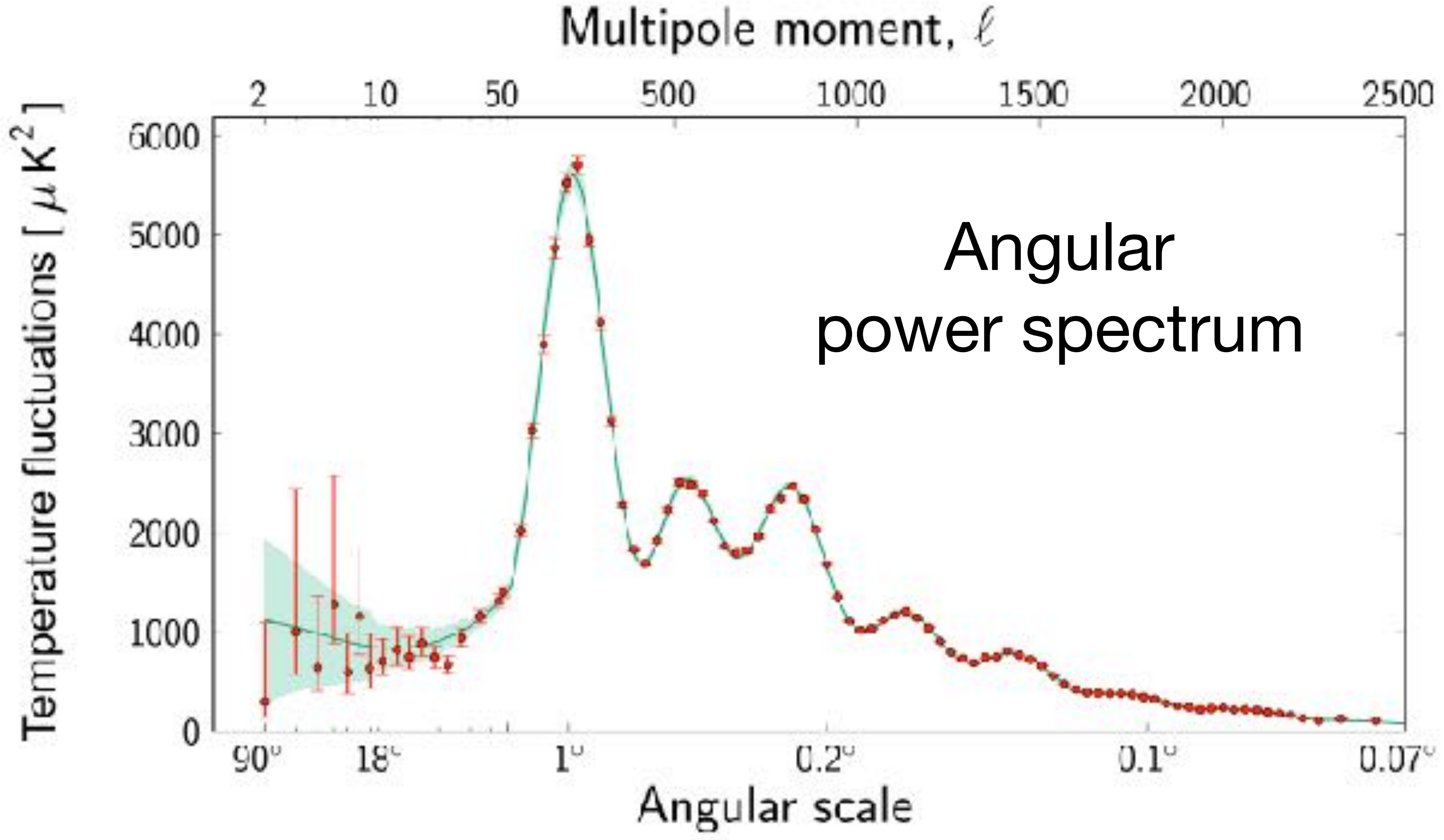
2.725 K blackbody

— **2013: WMAP, Planck**

— **2018:** *And we haven't detected the isotropic component of the GW background yet!*



(ang resolution: ~10 arcmin)



Angular power spectrum

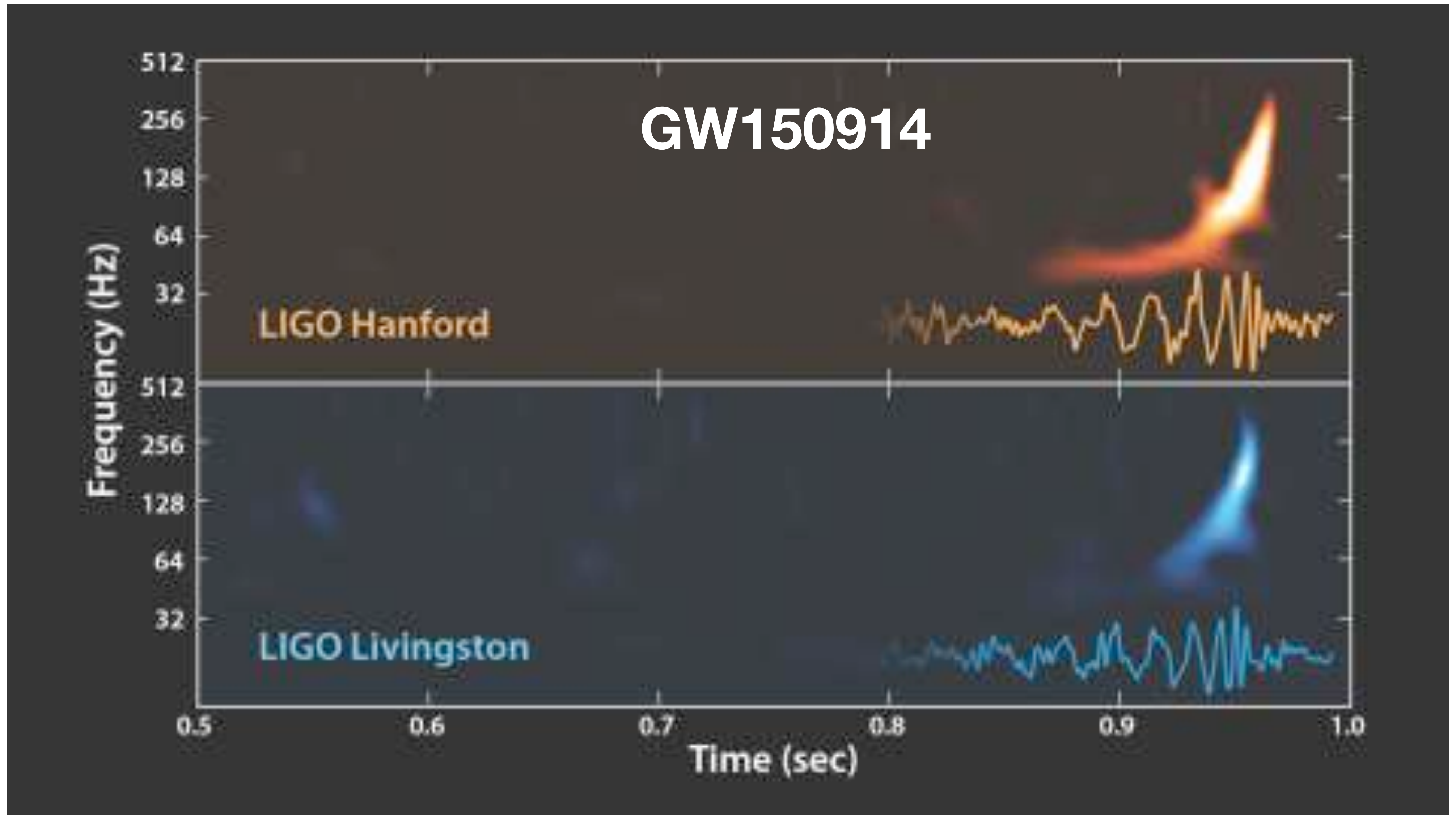
At least we've detected other GW signals…

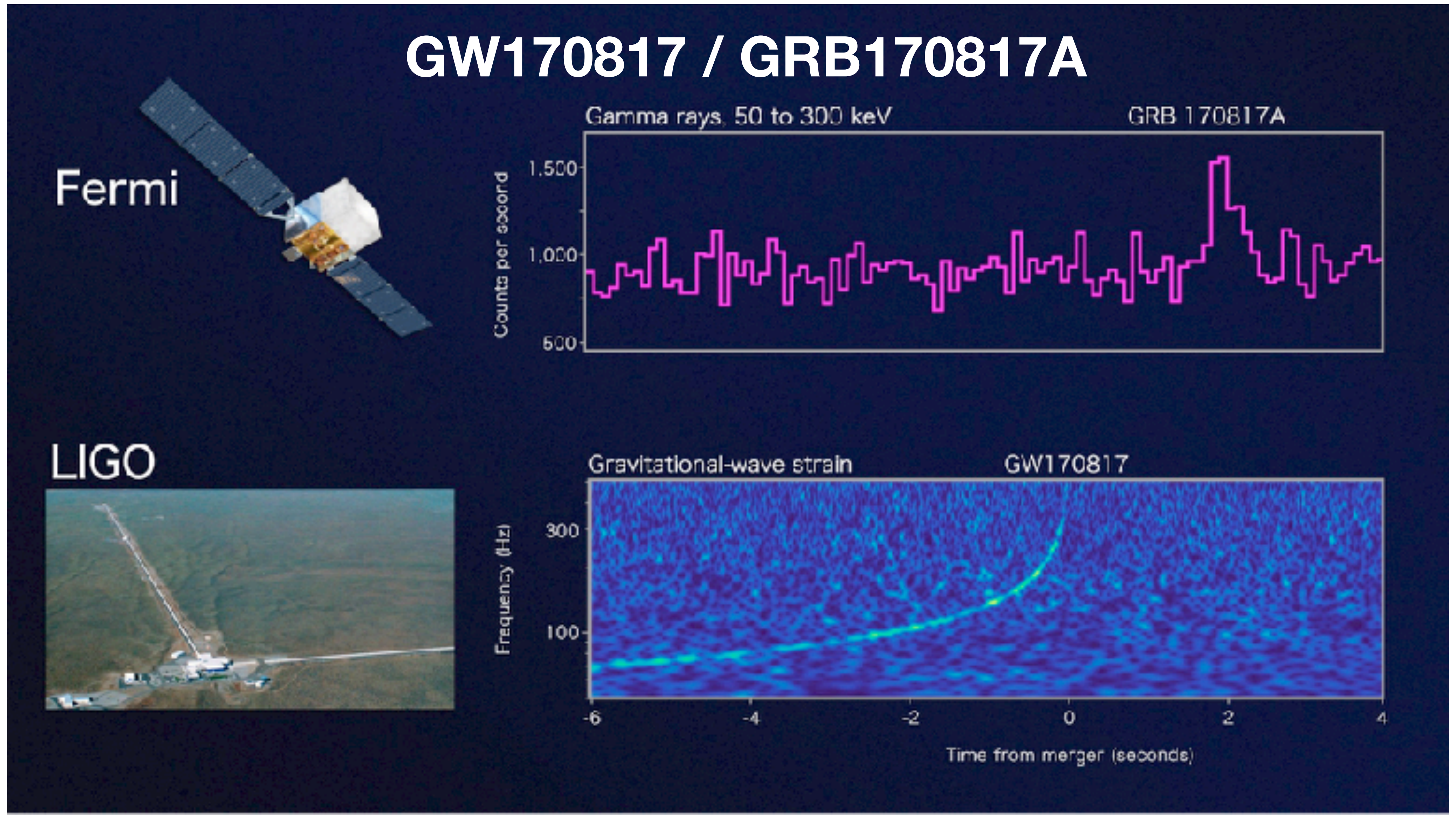
- 5 (or 6) binary black hole (BBH) mergers and 1 binary neutron star (BNS) merger
- with similar detections expected in O3, …
- *very strong events*!!

Observation of Gravitational Waves from a Binary Black Hole Merger

B. P. Abbott *et al.*[*]
(LIGO Scientific Collaboration and Virgo Collaboration)
(Received 21 January 2016; published 11 February 2016)

On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with a peak gravitational-wave strain of 1.0×10^{-21}. It matches the waveform predicted by general relativity for the inspiral and merger of a pair of black holes and the ringdown of the resulting single black hole. The signal was observed with a matched-filter signal-to-noise ratio of 24 and a false alarm rate estimated to be less than 1 event per 203 000 years, equivalent to a significance greater than 5.1σ. The source lies at a luminosity distance of 410^{+160}_{-180} Mpc corresponding to a redshift $z = 0.09^{+0.03}_{-0.04}$. In the source frame, the initial black hole masses are $36^{+5}_{-4} M_\odot$ and $29^{+4}_{-4} M_\odot$, and the final black hole mass is $62^{+4}_{-4} M_\odot$, with $3.0^{+0.5}_{-0.5} M_\odot c^2$ radiated in gravitational waves. All uncertainties define 90% credible intervals. These observations demonstrate the existence of binary stellar-mass black hole systems. This is the first direct detection of gravitational waves and the first observation of a binary black hole merger.



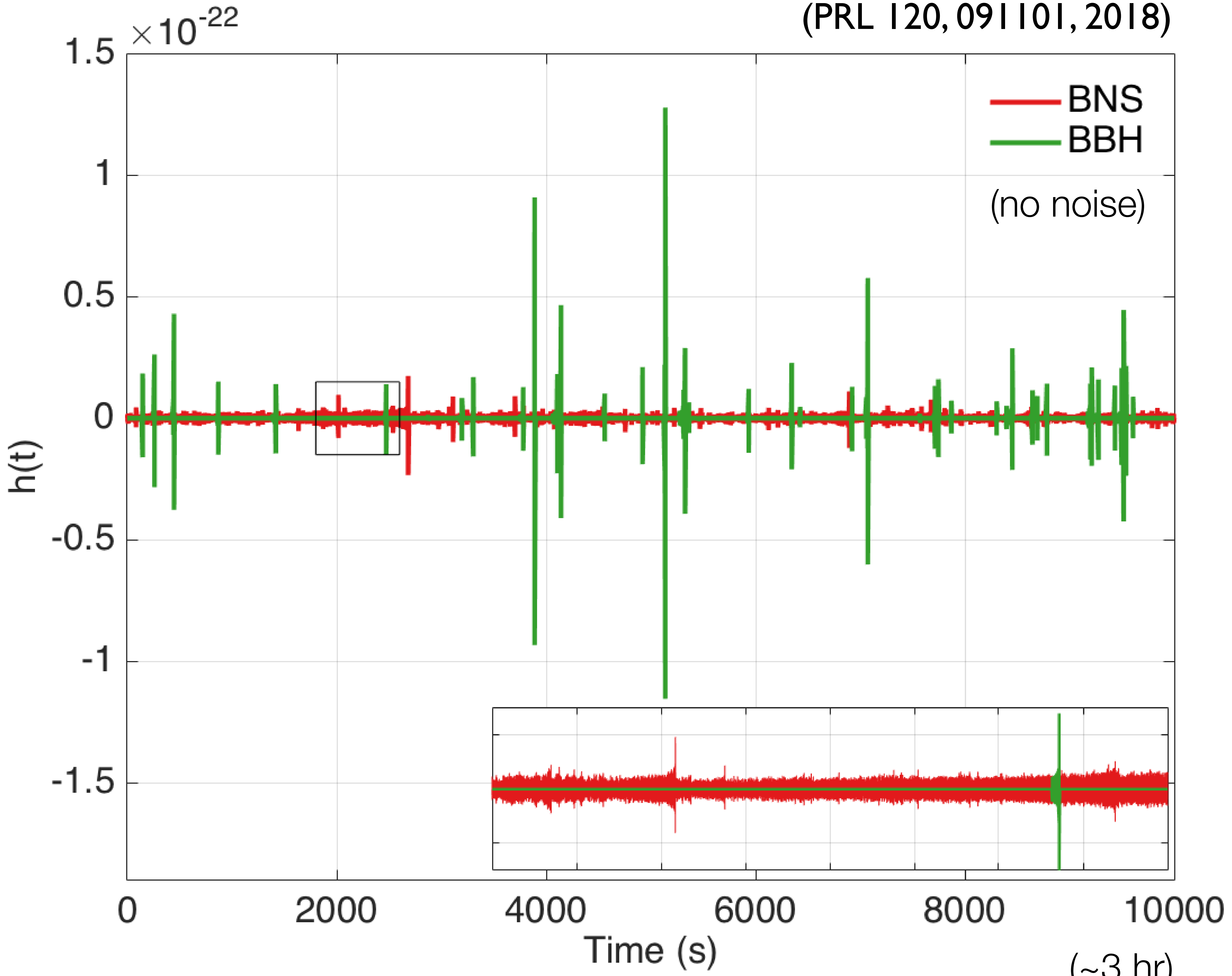


…and we expect many more weaker signals

- individually undetectable (subthreshold)
- but detectable as a collectivity via their common influence on multiple detectors
- combined signal described statistically—*stochastic gravitational-wave background*

<u>Exercise 1</u>: Verify that the expected total rate of stellar-mass BBH mergers is between ~1 per minute and a few per hour.

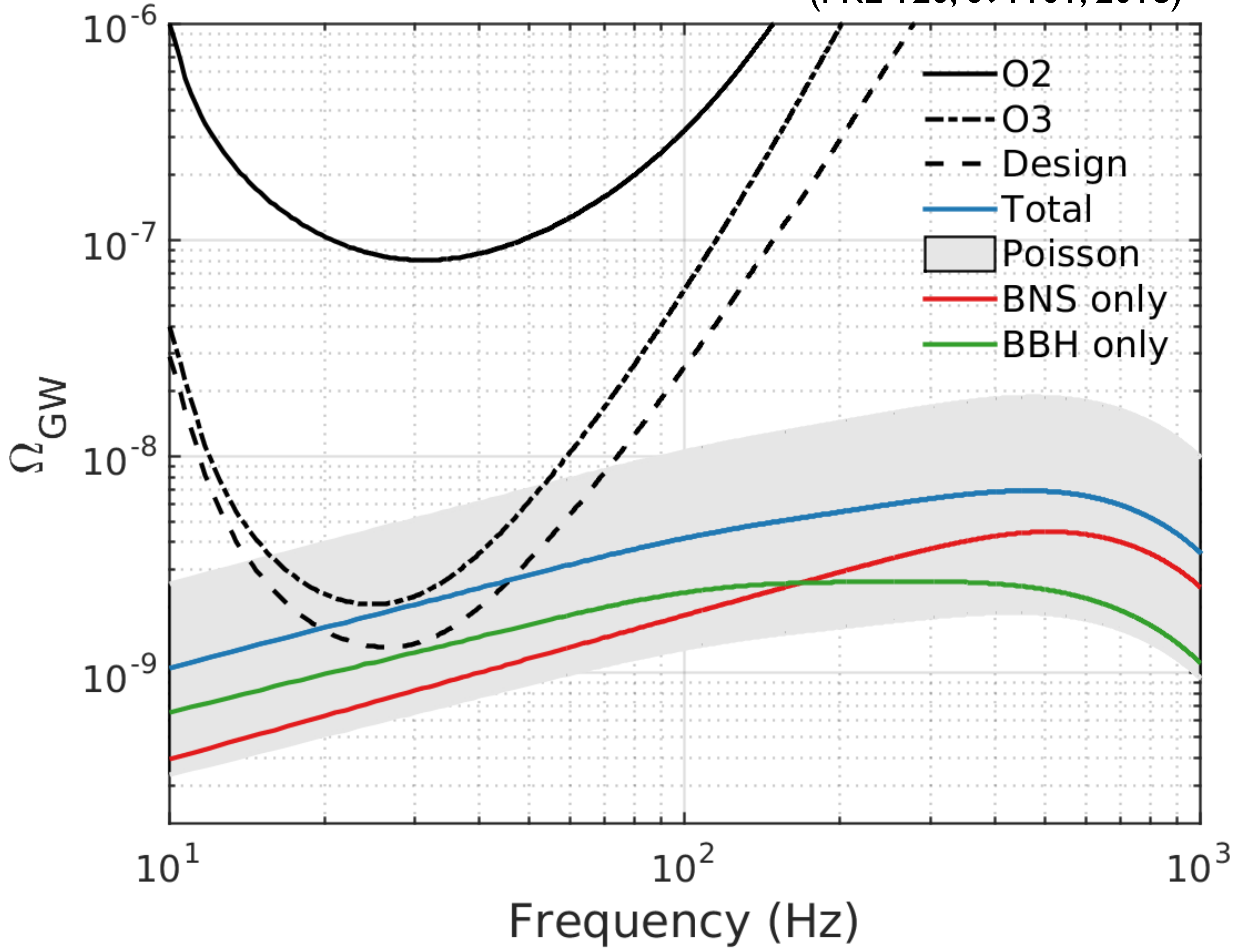
(PRL 120, 091101, 2018)



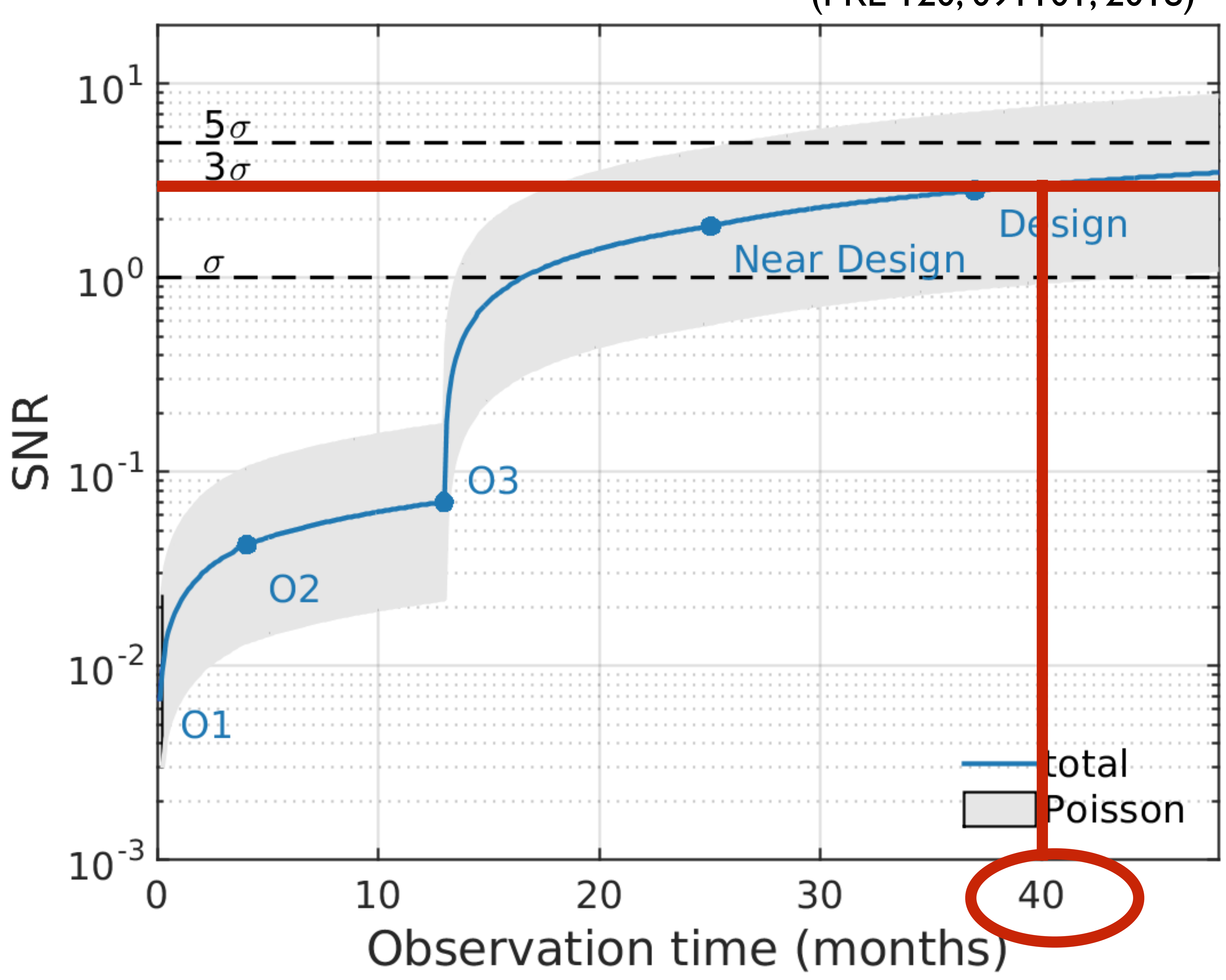
(~3 hr)

Potentially detectable with advanced LIGO/Virgo

(PRL 120, 091101, 2018)

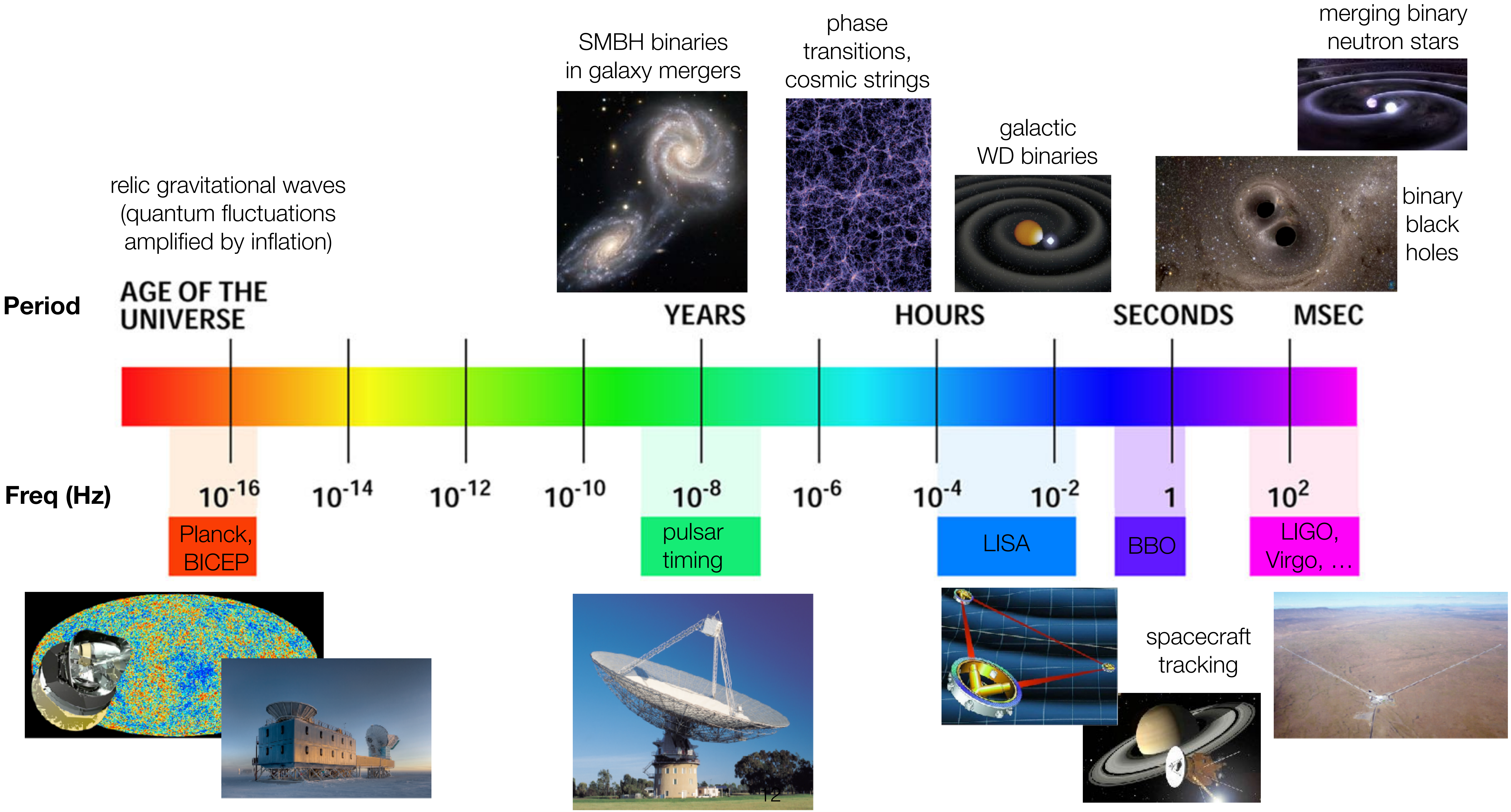


(PRL 120, 091101, 2018)



Based on standard search, but there exists a better method!
(Smith & Thrane, PRX 8, 021019, 2018)

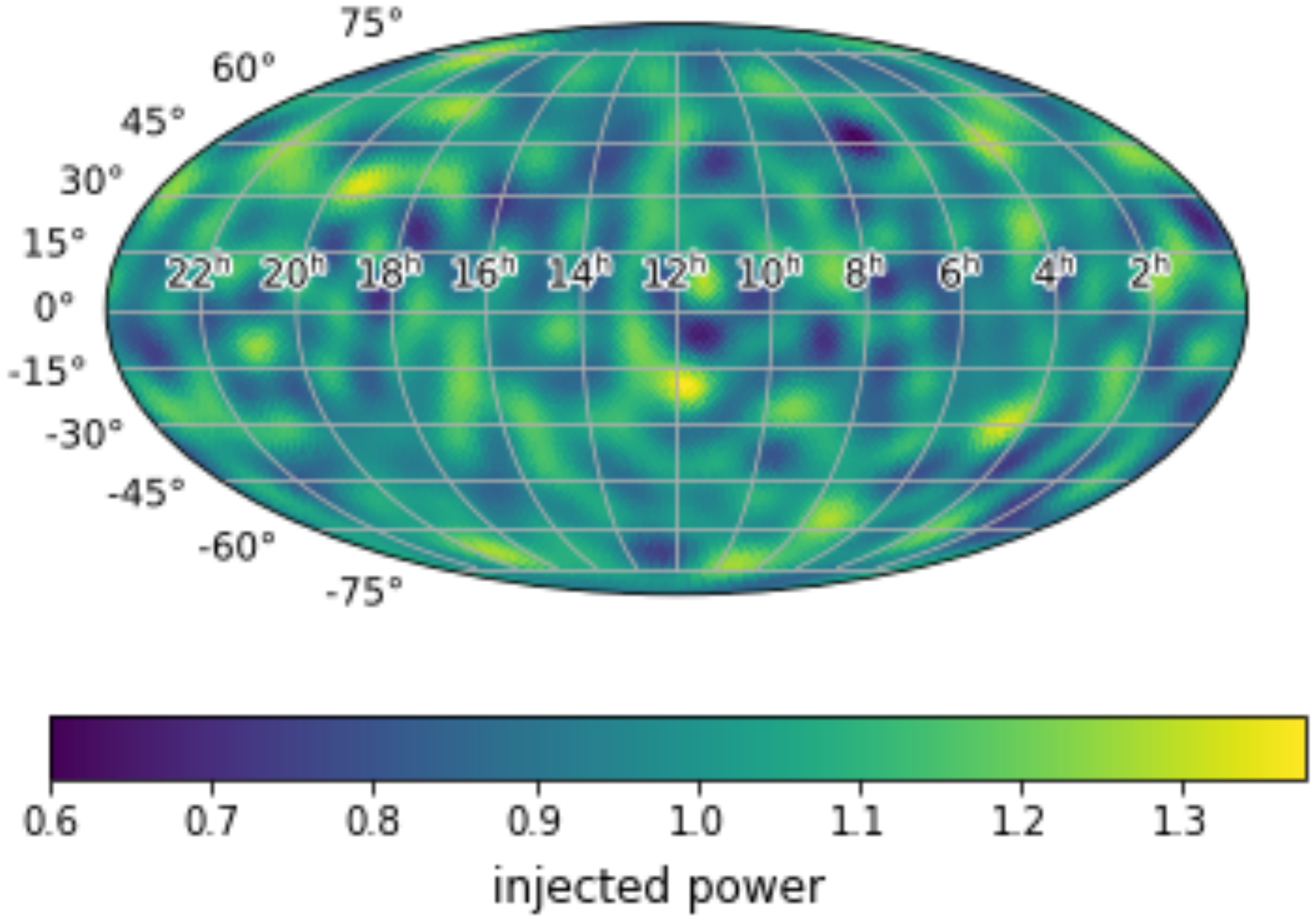
Bigger picture of GWs — sources & detectors



2. Different types of stochastic GW backgrounds

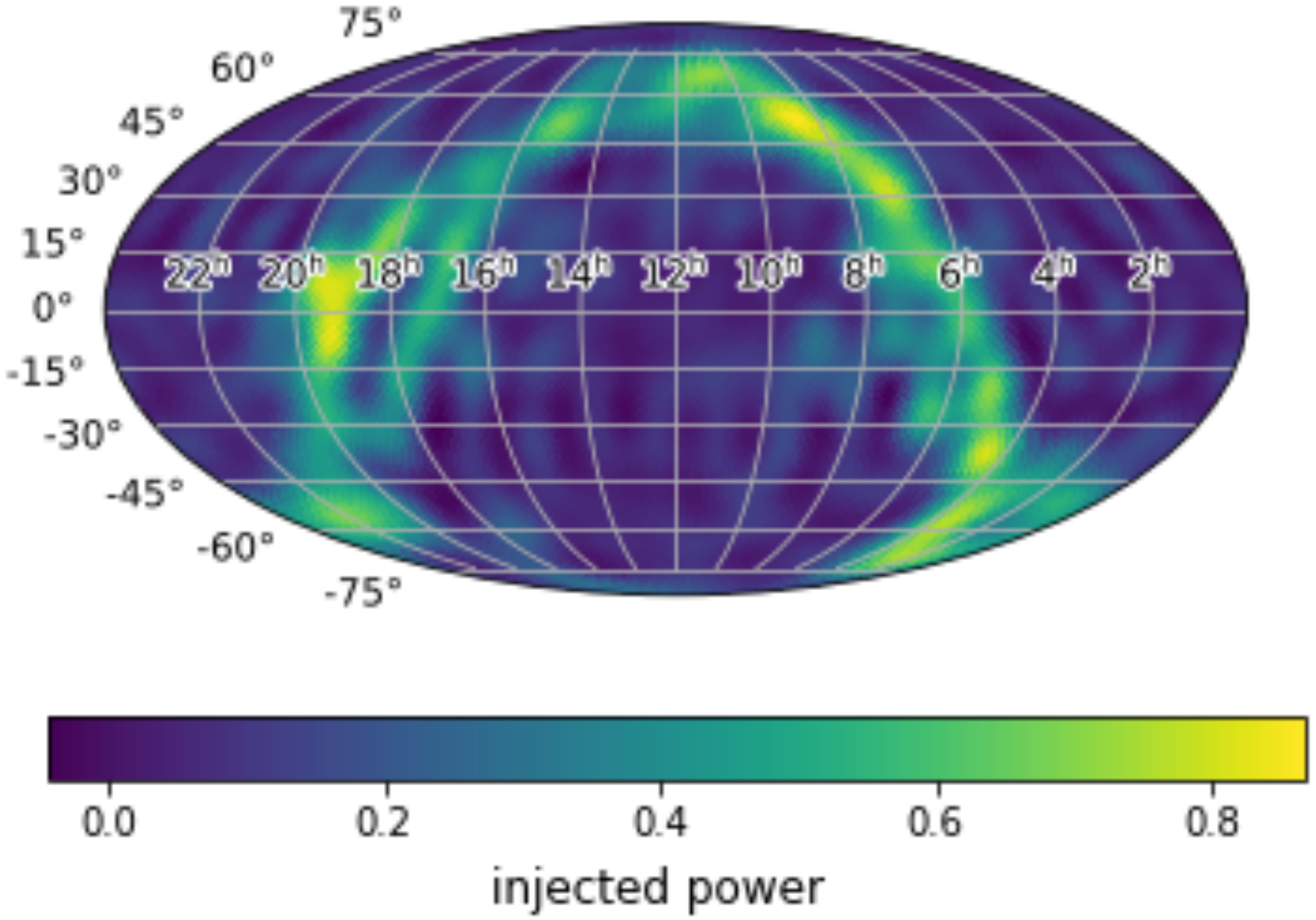
(i) Stochastic backgrounds can differ in spatial distribution

<u>(statistically) isotropic</u>



(like *cosmic microwave background*)

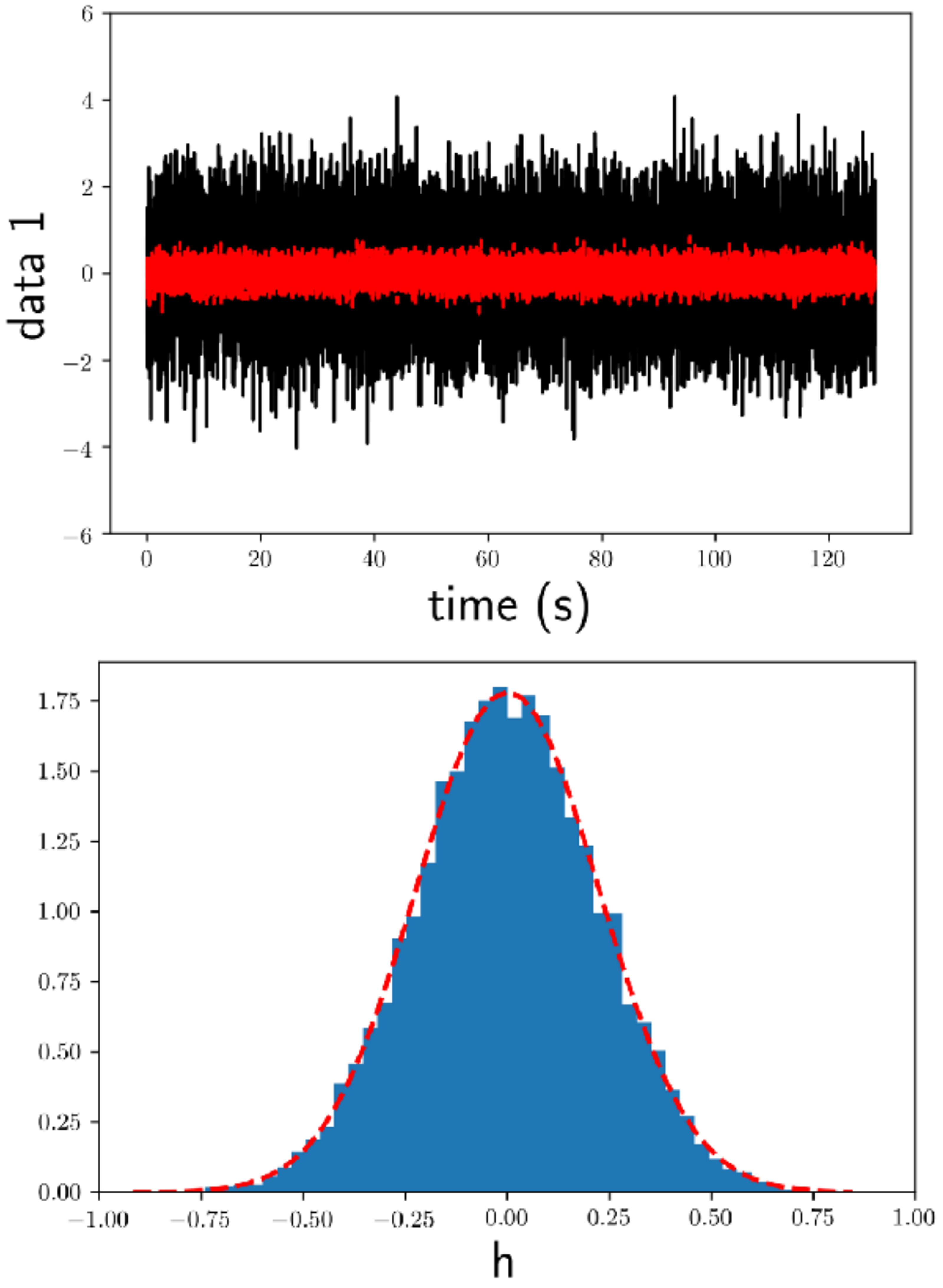
<u>anisotropic</u>



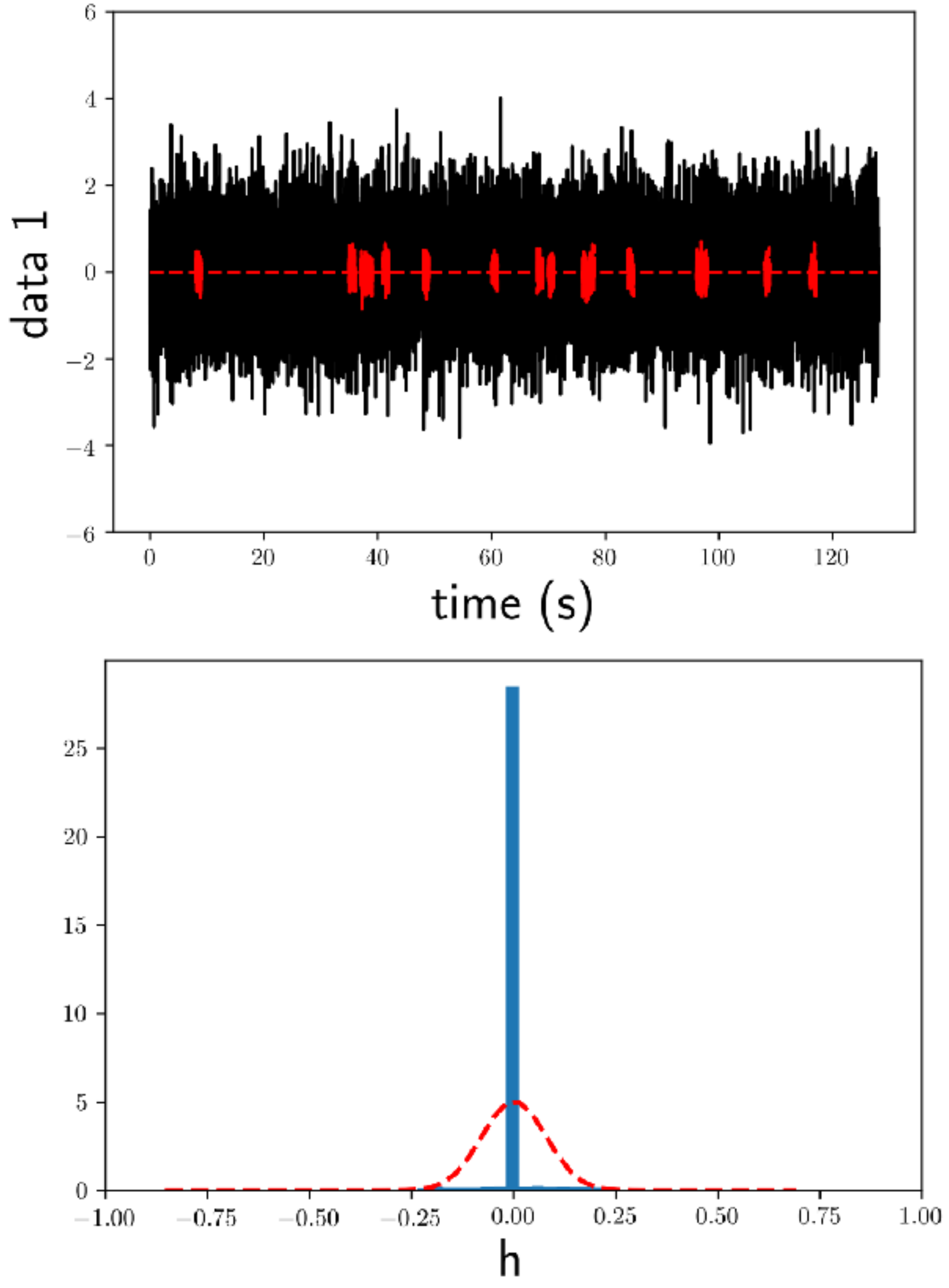
(galactic plane in equatorial coords)

(ii) They can also differ in temporal distribution and amplitude

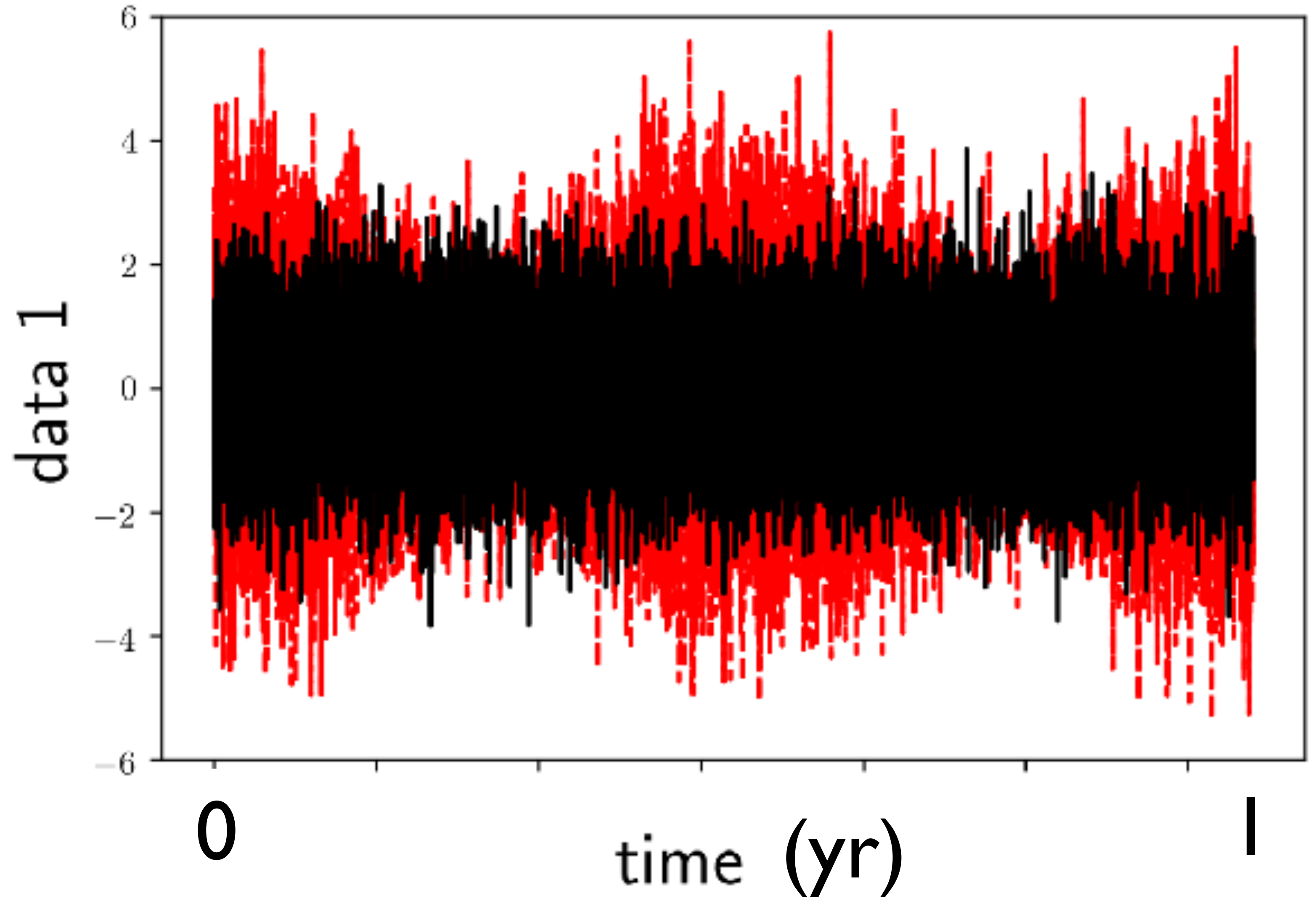
Stationary Gaussian



Non-stationary (non-gaussian)

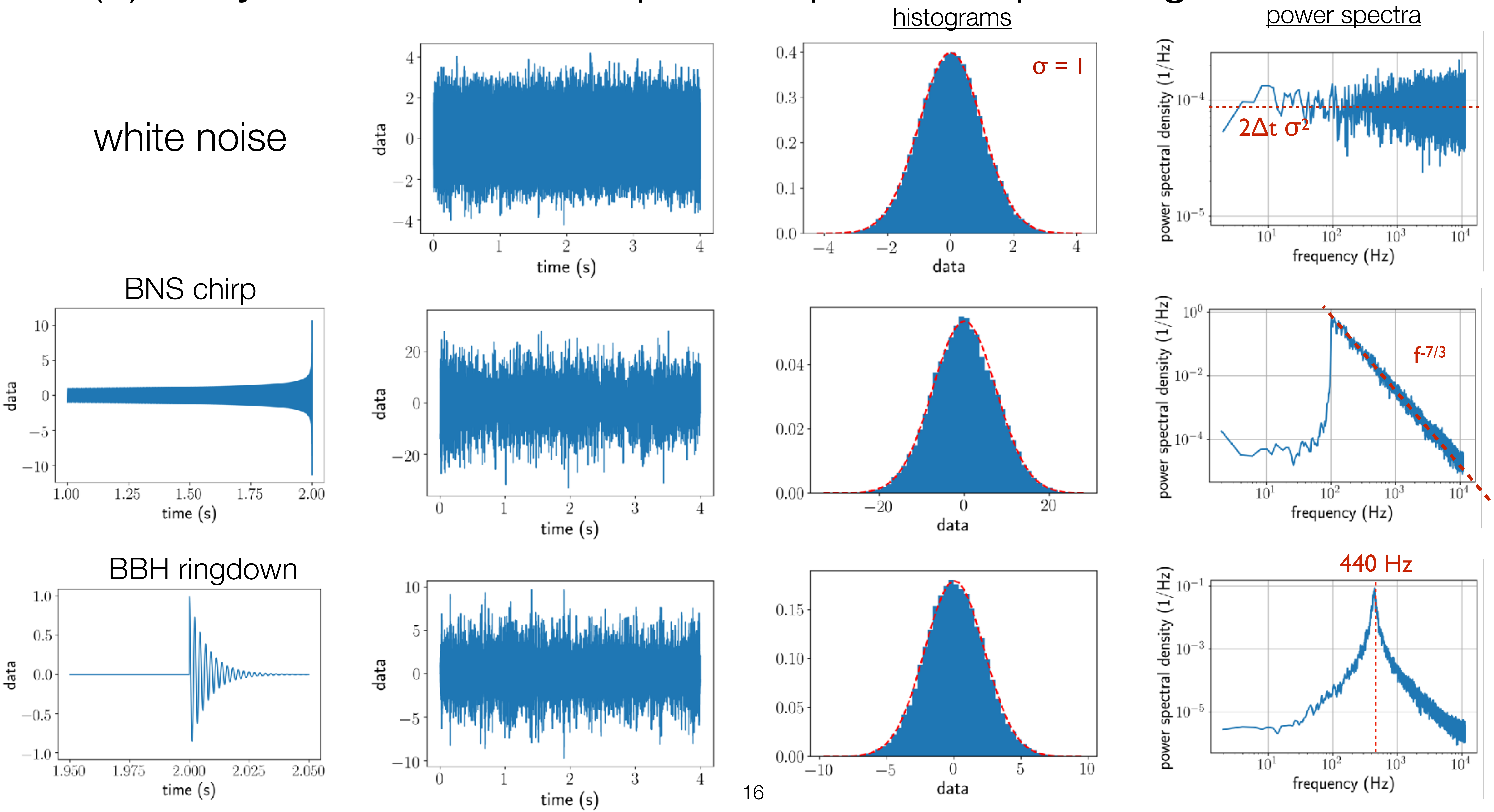


Foreground

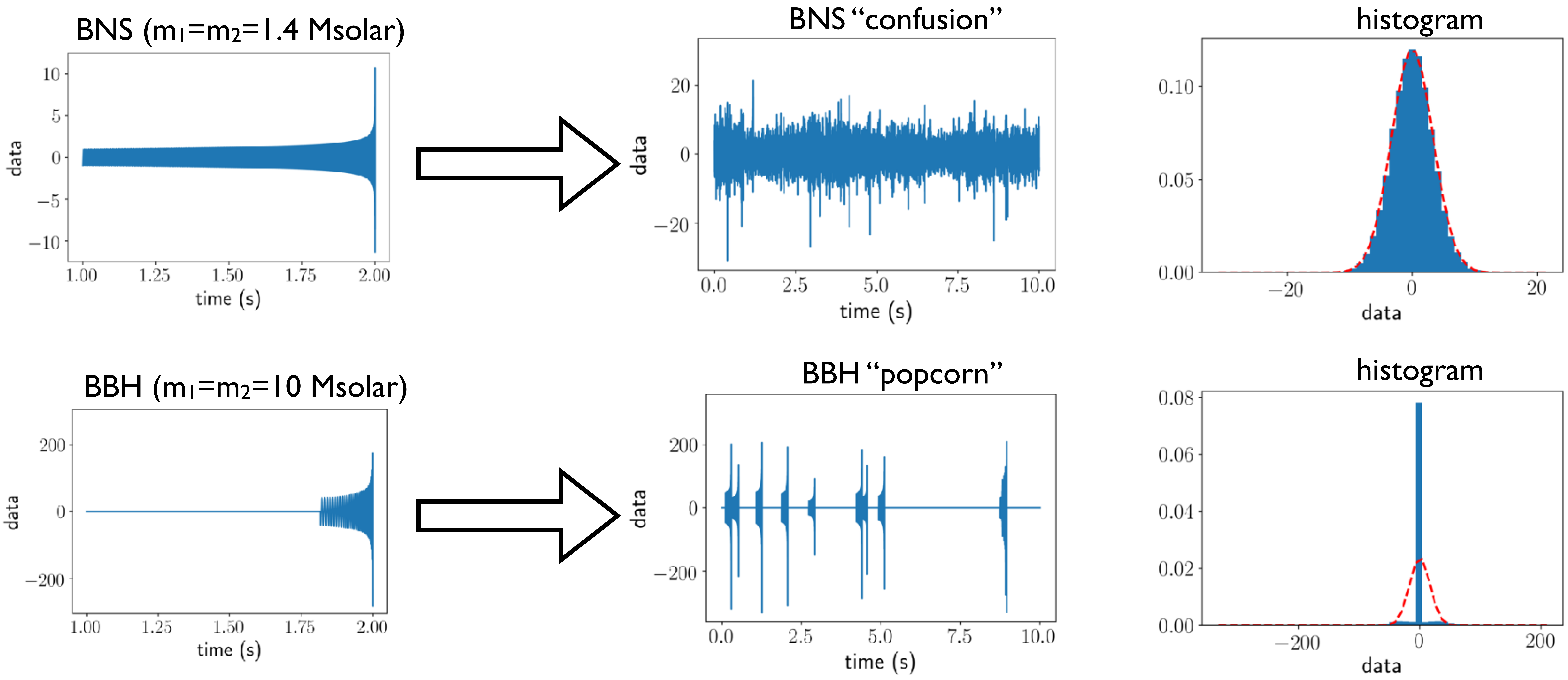


(e.g., from galactic white dwarf binaries; modulated by LISA’s orbital motion)

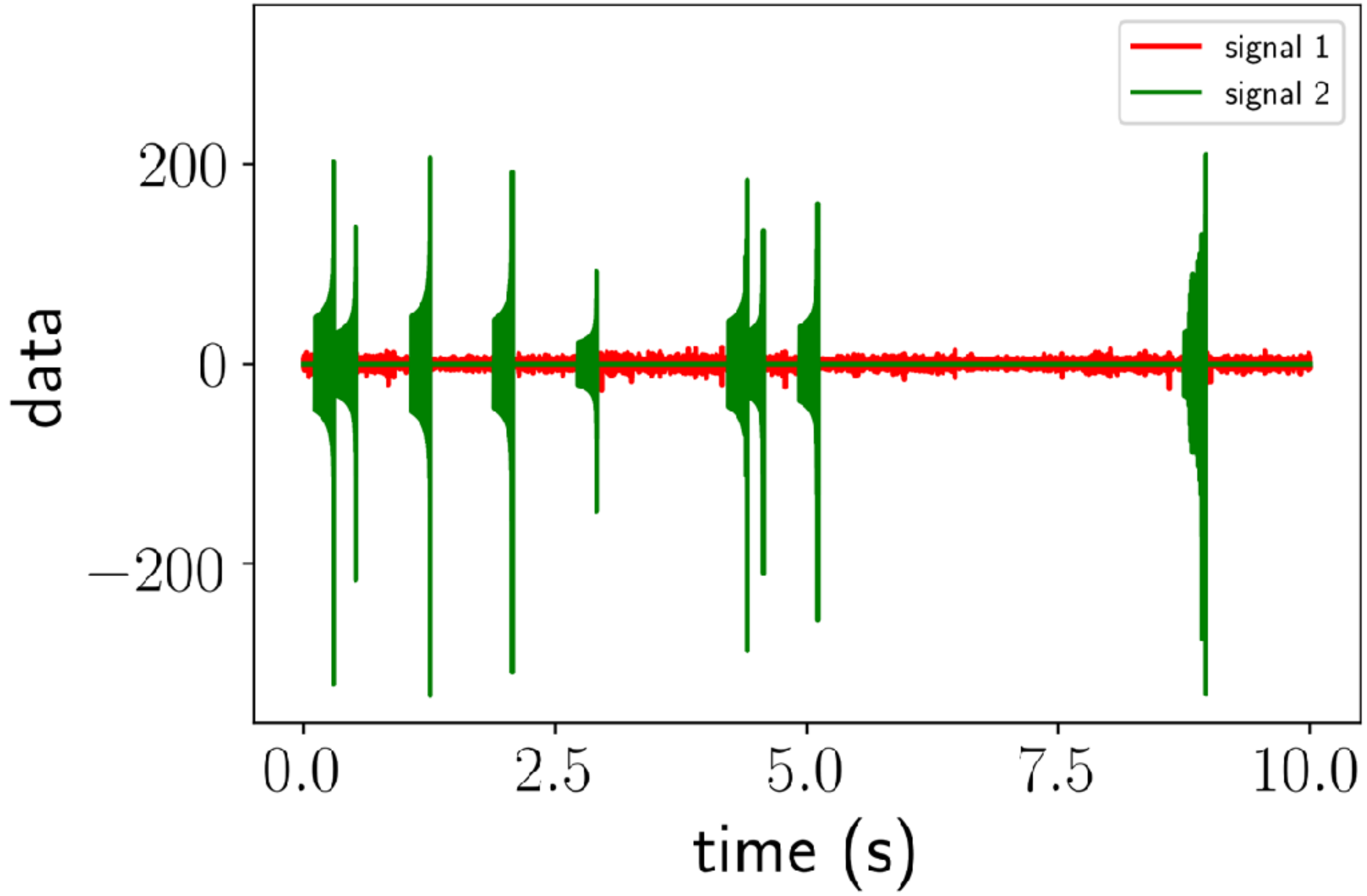
(iii) They can also differ in power spectra depending on source



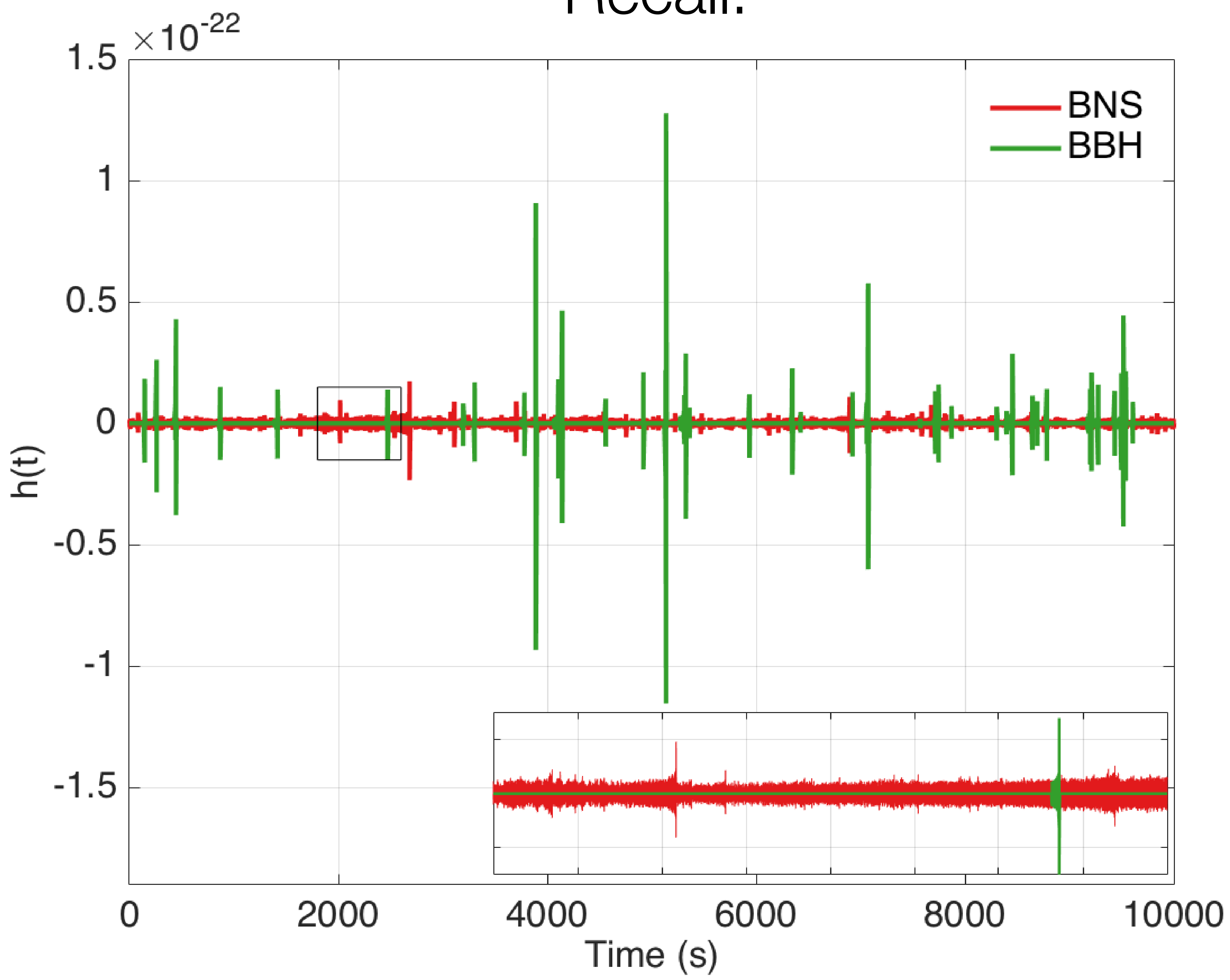
Example: Rate estimates and signal durations imply BNS “confusion” & BBH “popcorn” for LIGO / Virgo



Combined BBH / BNS background signal



Recall:

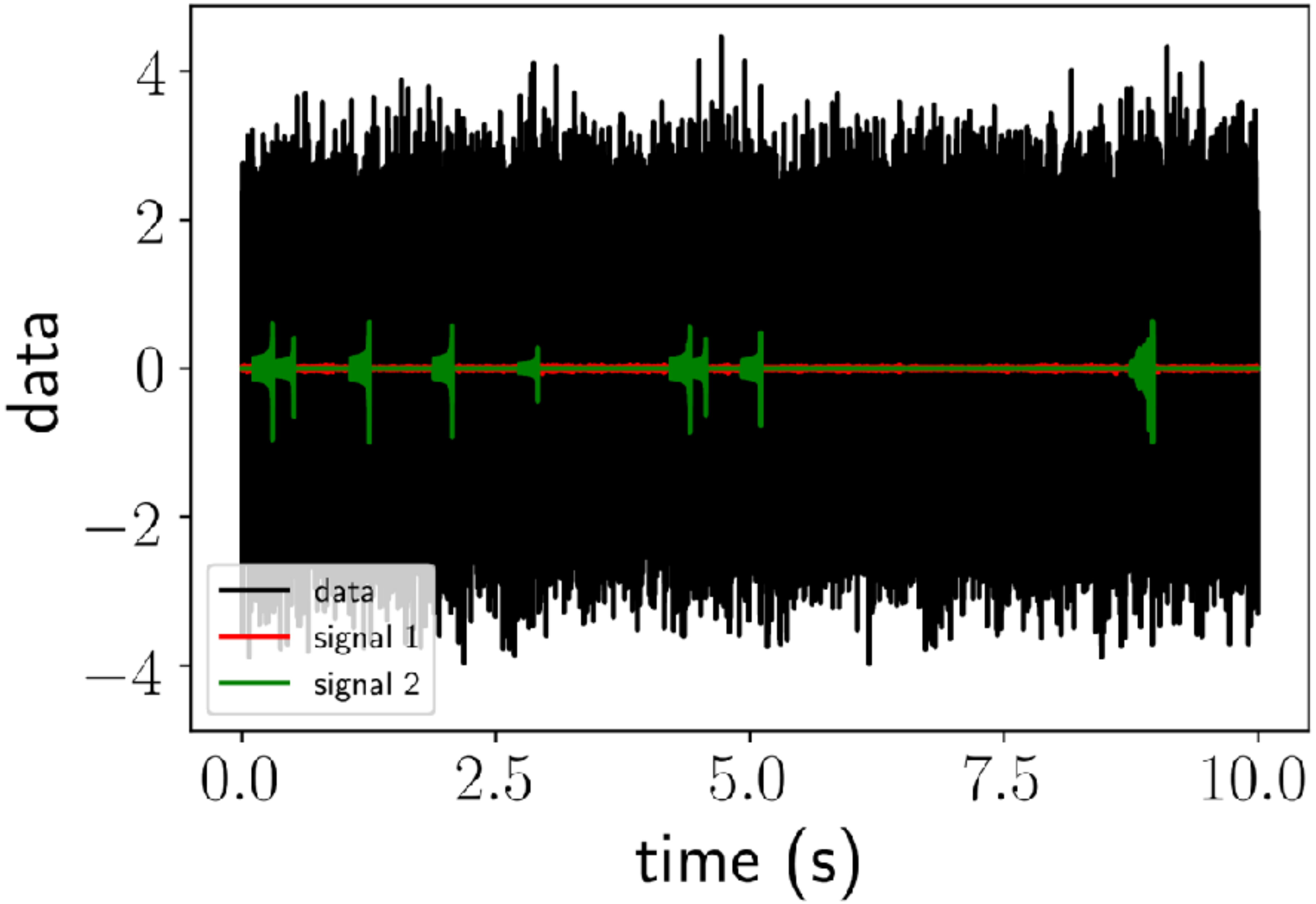


(PRL 120, 091101, 2018)

3. Characterizing a stochastic GW background

Definition of a stochastic background

- Superposition of signals too weak or too numerous to individually detect
- Looks like noise in a single detector
- Characterized statistically in terms of moments (ensemble averages) of the metric perturbations

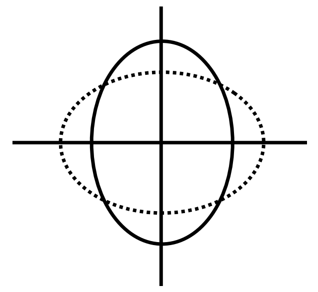


Plane wave expansion of metric perturbations

$$h_{ab}(t, \vec{x}) = \int_{-\infty}^{\infty} \mathrm{d}f \int \mathrm{d}^2\Omega_{\hat{n}} \sum_{A=+,\times} h_A(f,\hat{n}) e^A_{ab}(\hat{n})\, e^{i2\pi f(t+\hat{n}\cdot\vec{x}/c)}$$

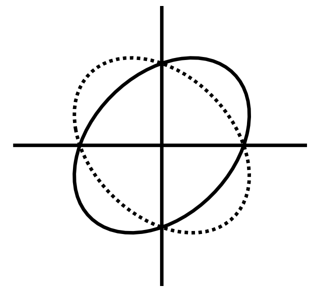
Polarization tensors:

$$e^{+}_{ab}(\hat{n}) = \hat{l}_a\hat{l}_b - \hat{m}_a\hat{m}_b$$



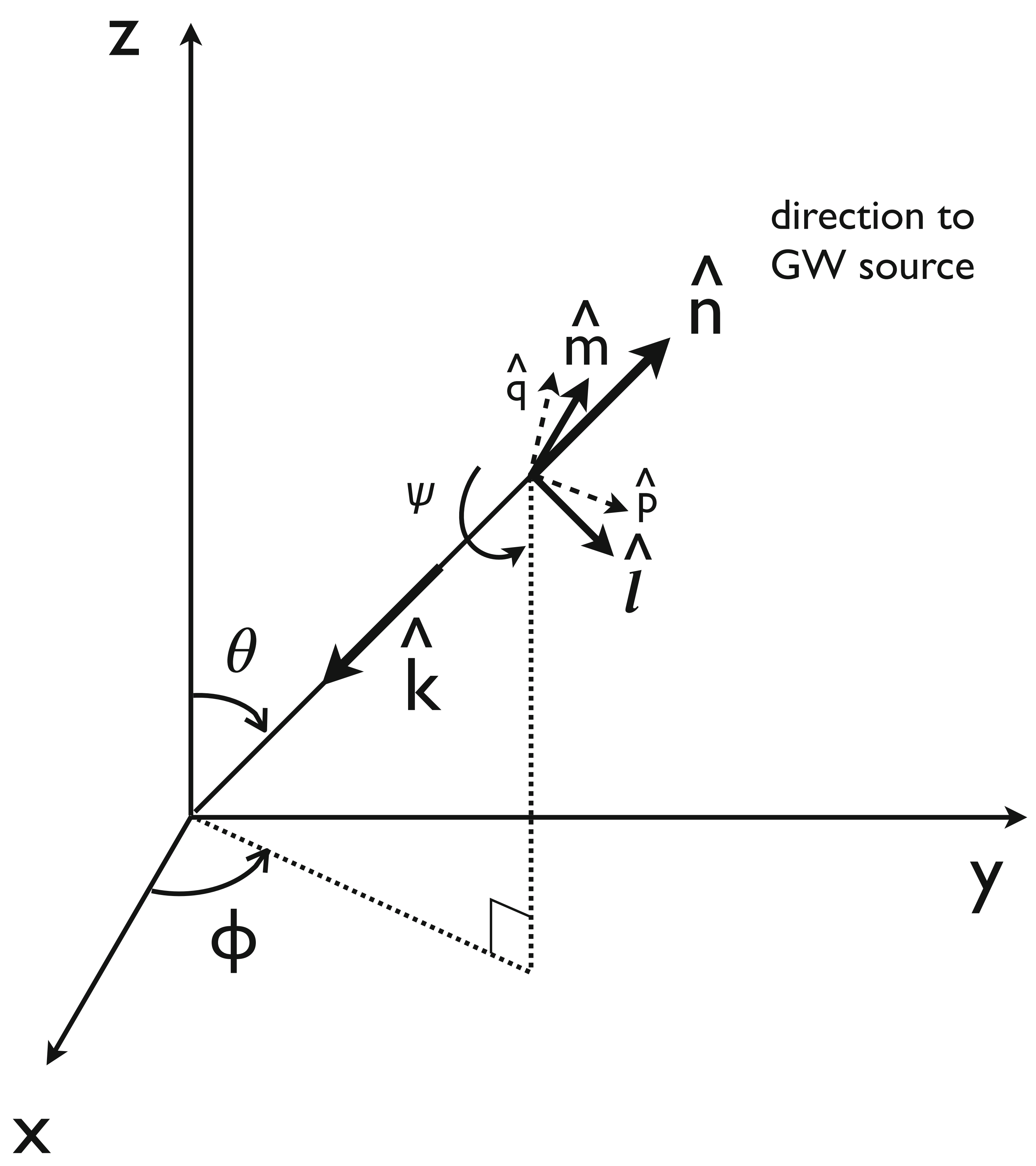
A=+

$$e^{+}_{ab}(\hat{n}) = \hat{l}_a\hat{m}_b + \hat{m}_a\hat{l}_b$$



A=x

$$(\hat{l} = \hat{\theta}\,, \quad \hat{m} = \hat{\phi})$$



Statistical properties encoded in:

$$\langle h_A(f,\hat{n})\rangle\,, \quad \langle h_A(f,\hat{n})h_{A'}(f',\hat{n}')\rangle\,, \quad \langle h_A(f,\hat{n})h_{A'}(f',\hat{n}')h_{A''}(f'',\hat{n}'')\rangle\,, \quad \cdots$$

0

(no loss of generality)

in terms of quadratic expectation values

(if Gaussian)

Quadratic expectation values specify different types of Gaussian stochastic backgrounds

Unpolarized, stationary isotropic:

$$\langle h_A(f,\hat{n}) h^*_{A'}(f',\hat{n}')\rangle = \frac{1}{16\pi} S_h(f)\delta(f-f')\delta_{AA'}\delta^2(\hat{n},\hat{n}')$$

Unpolarized, stationary, anisotropic:

$$\langle h_A(f,\hat{n}) h^*_{A'}(f',\hat{n}')\rangle = \frac{1}{4}\mathscr{P}(f,\hat{n})\delta(f-f')\delta_{AA'}\delta^2(\hat{n},\hat{n}')$$

where $$S_h(f) = \int \mathrm{d}^2\Omega_{\hat{n}}\, \mathscr{P}(f,\hat{n})$$

power spectral density (Hz^{-1})

$$S_h(f) = \frac{3H_0^2}{2\pi^2}\frac{\Omega_{\rm gw}(f)}{f^3}$$

Exercise 2: Derive the above relationship.

energy density spectrum (dimensionless)

$$\Omega_{\rm gw}(f) \equiv \frac{1}{\rho_c}\frac{\mathrm{d}\rho_{\rm gw}}{\mathrm{d}\ln f} = \frac{f}{\rho_c}\frac{\mathrm{d}\rho_{\rm gw}}{\mathrm{d}f}$$

$$\rho_c \equiv \frac{3H_0^2c^2}{8\pi G} \qquad \rho_{gw} = \frac{c^2}{32\pi G}\langle \dot{h}_{ab}(t,\vec{x})\dot{h}^{ab}(t,\vec{x})\rangle$$

characteristic strain (dimensionless)

$$h_c(f) \equiv \sqrt{fS_h(f)} = A_\alpha\left(\frac{f}{f_{\rm ref}}\right)^\alpha$$

“Phinney formula”: Calculating $\Omega_{\rm gw}(f)$ for an astrophysical background

astro-ph/0108028

A practical theorem on gravitational wave backgrounds

E.S. Phinney⋆
Theoretical Astrophysics, 130-33 Caltech, Pasadena, CA 91125, USA

ABSTRACT
There is an extremely simple relationship between the spectrum of the gravitational wave background produced by a cosmological distribution of discrete gravitational wave sources, the total time-integrated energy spectrum of an individual source, and the present-day comoving number density of remnants. Stated in this way, the background is entirely independent of the cosmology, and only weakly dependent on the evolutionary history of the sources. This relationship allows one easily to compute the amplitude and spectrum of cosmic gravitational wave backgrounds from a broad range of astrophysical sources, and to evaluate the uncertainties therein.

Recall:

$$\Omega_{\rm gw}(f) \equiv \frac{1}{\rho_c}\frac{{\rm d}\rho_{\rm gw}}{{\rm d}\ln f} = \frac{f}{\rho_c}\frac{{\rm d}\rho_{\rm gw}}{{\rm d}f}$$

For a collection of sources:

$$\Omega_{\rm gw}(f) = \frac{1}{\rho_c}\int_0^\infty {\rm d}z\, n(z)\,\frac{1}{1+z}\left(f_{\rm s}\frac{{\rm d}E_{\rm gw}}{{\rm d}f_{\rm s}}\right)\Bigg|_{f_{\rm s}=f(1+z)}$$

$$f_s = f(1+z)$$

In terms of event rate:

$$n(z)\,dz = R(z)\,|\,dt\,|_{\rm s}$$

$$\left|\frac{{\rm d}t}{{\rm d}z}\right|_s = \frac{1}{(1+z)H_0E(z)}$$

$$E(z) = \sqrt{\Omega_{\rm m}(1+z)^3+\Omega_\Lambda} \longleftarrow$$ **cosmology**

<u>Exercise 3</u>: Verify the above expression for |dt/dz| as well as the “Phinney formula” in terms of the rate R(z).

$$\Omega_{\rm gw}(f) = \frac{f}{\rho_{\rm c}H_0}\int_0^\infty {\rm d}z\, R(z)\,\frac{1}{(1+z)E(z)}\left(\frac{{\rm d}E_{\rm gw}}{{\rm d}f_{\rm s}}\right)\Bigg|_{f_{\rm s}=f(1+z)}$$

Simple example: circular binaries

Units: $G = c = 1$

Kepler's law: $\omega^2 r^3 = GM \implies r \sim M^{1/3}\omega^{-2/3}, \quad \dot{r} \sim -r\dot{\omega}/\omega$

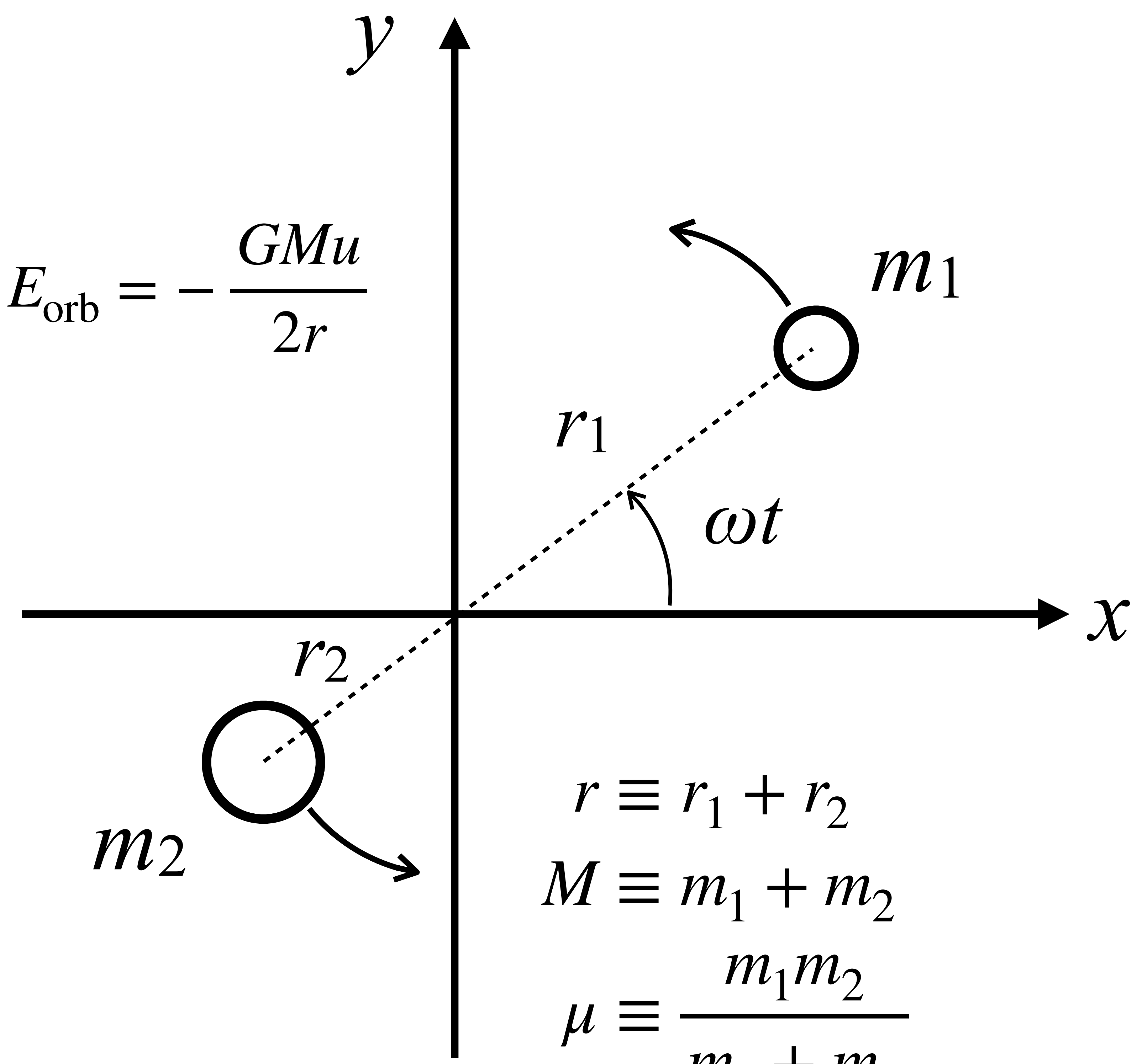
Energy balance: $\dfrac{dE_{\rm gw}}{dt} = -\dfrac{dE_{\rm orb}}{dt}$

$$\implies \frac{dE_{\rm gw}}{dt} \sim -M\mu\dot{r}/r^2 \sim \mathscr{M}_c^{5/3}\omega^{-1/3}\dot{\omega}$$

$$\implies \frac{dE_{\rm gw}}{df} = \frac{dt}{df}\frac{dE_{\rm gw}}{dt} \sim \frac{1}{\dot{\omega}}\frac{dE_{\rm gw}}{dt} \sim \mathscr{M}_c^{5/3} f^{-1/3}$$

$$\implies \boxed{\Omega_{\rm gw}(f) \propto f^{2/3}, \quad h_c(f) \propto f^{-2/3}}$$

$$E_{\rm orb} = -\frac{GMu}{2r}$$



$$r \equiv r_1 + r_2$$

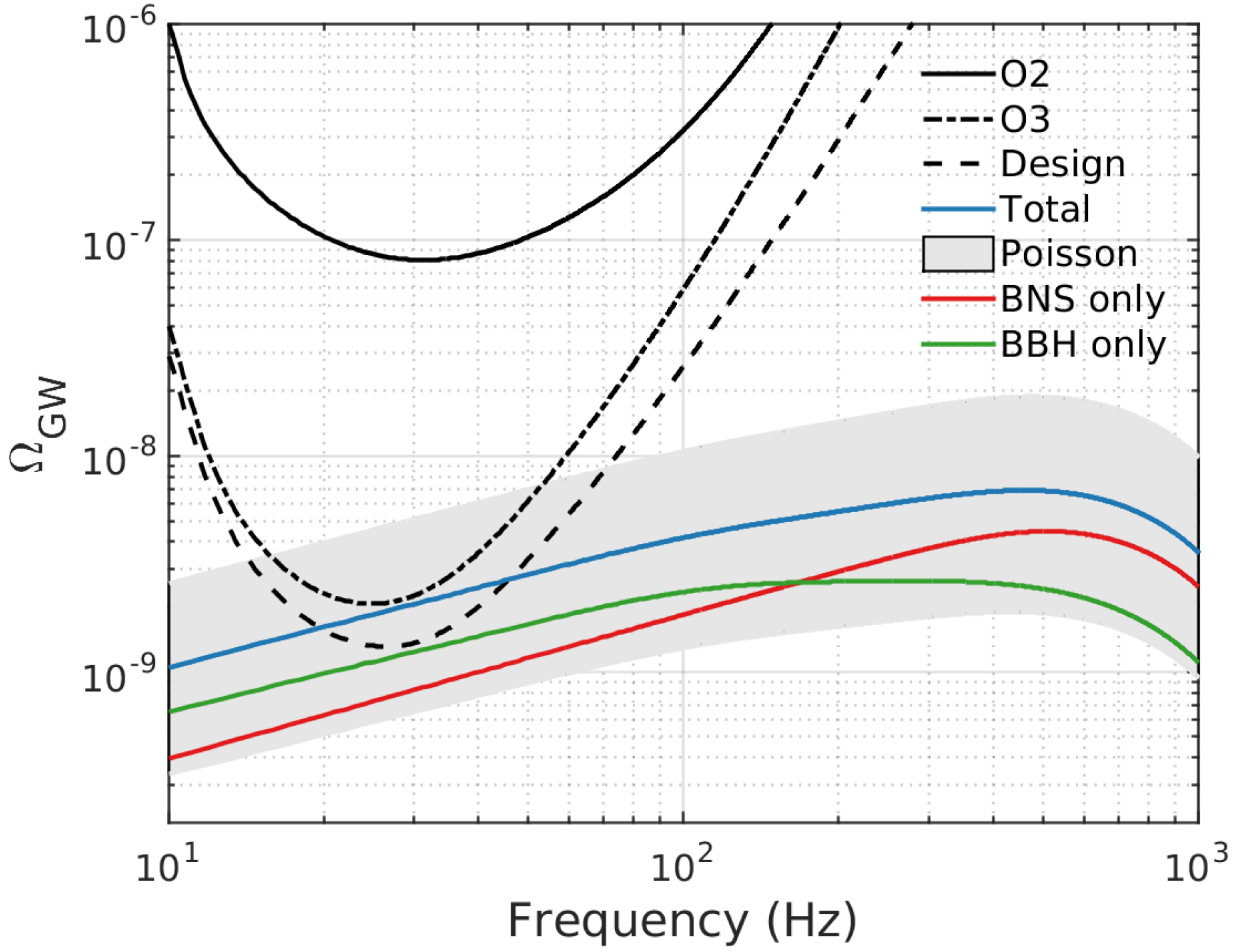
$$M \equiv m_1 + m_2$$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

$$\mathscr{M}_c \equiv \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} = \mu^{3/5} M^{2/5}$$

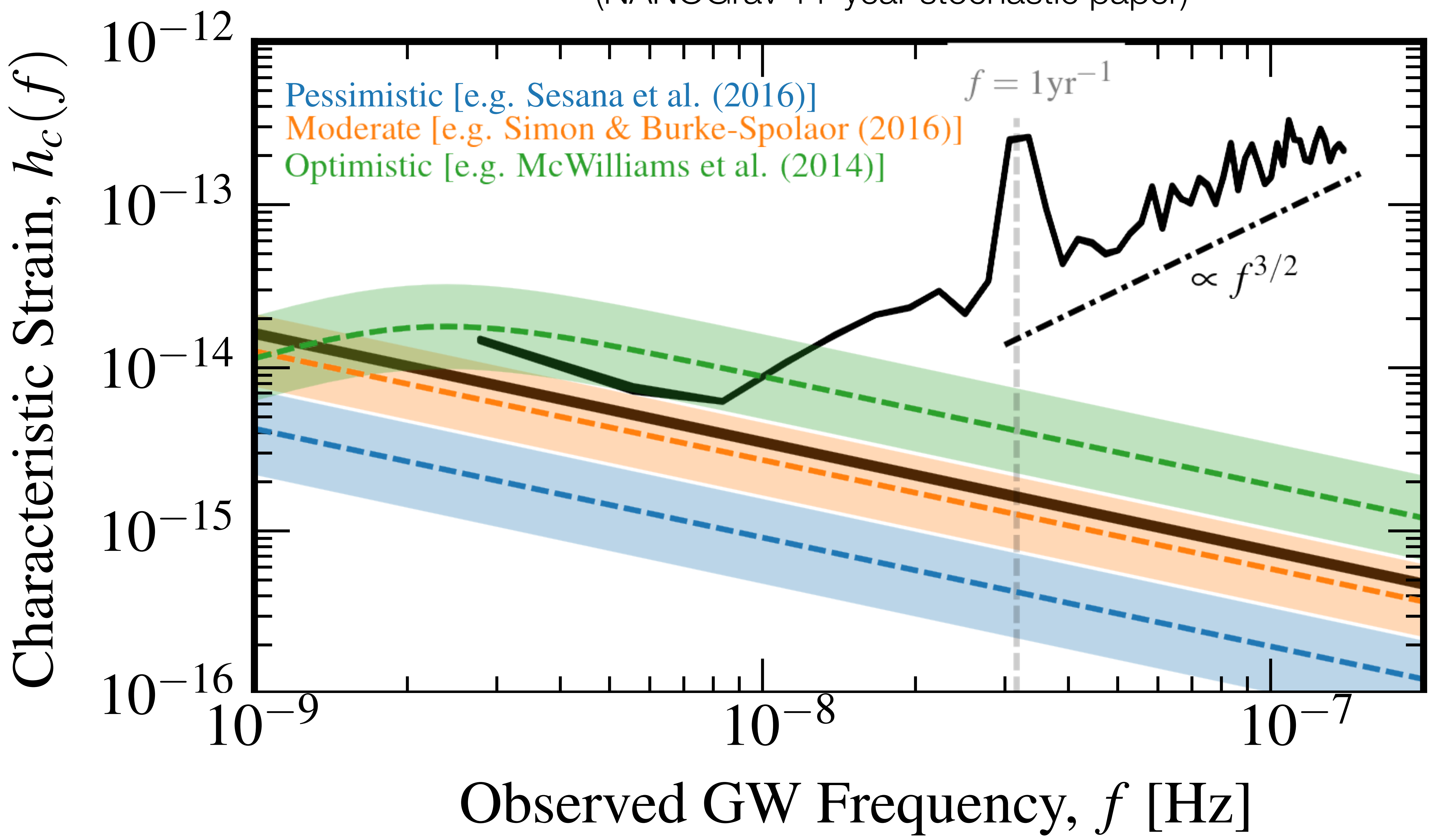
LIGO

(GW170817 stochastic implications paper)



Pulsar Timing

(NANOGrav 11-year stochastic paper)



$$\boxed{\Omega_{\mathrm{gw}}(f) \propto f^{2/3}, \quad h_c(f) \propto f^{-2/3}}$$

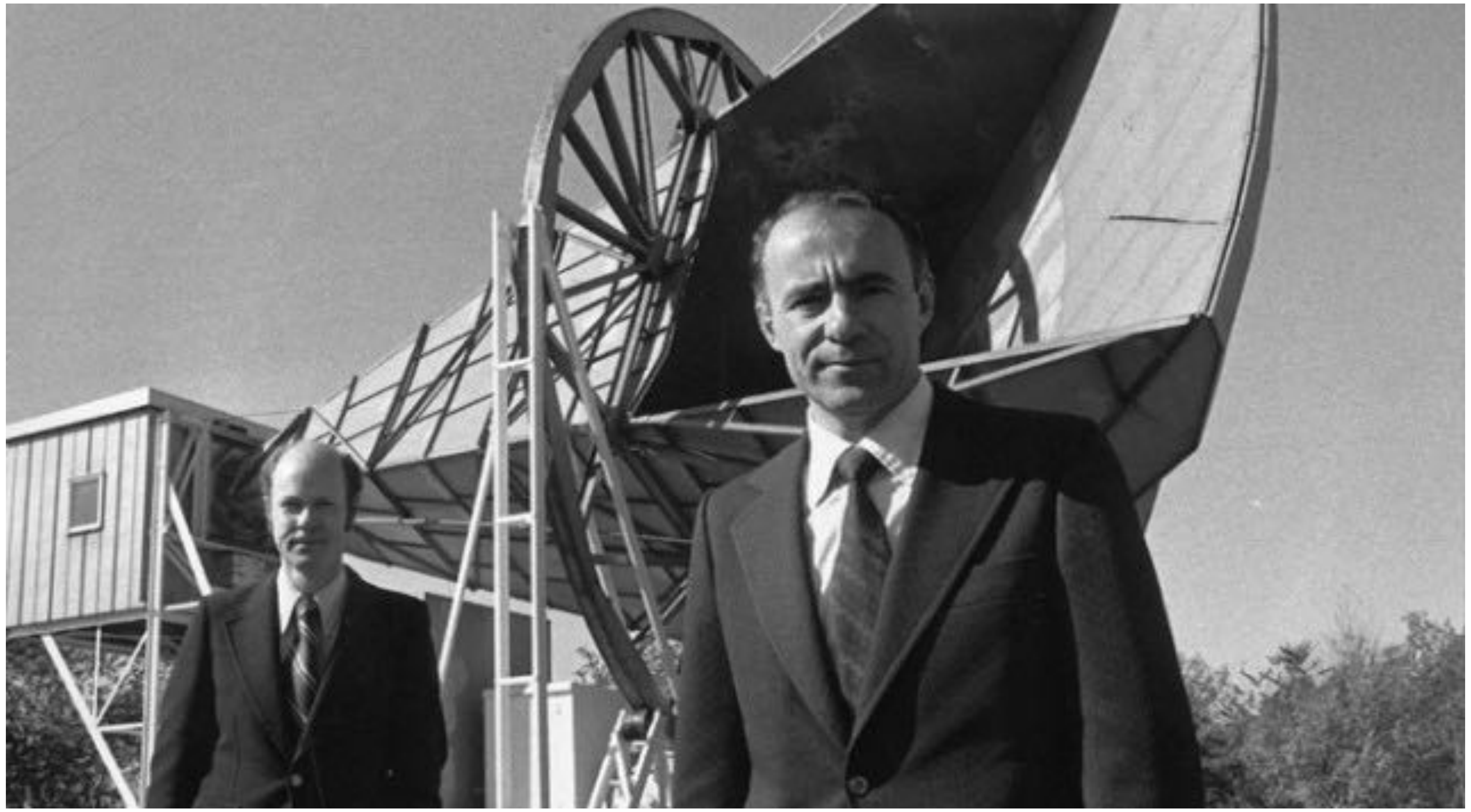
4. Correlation methods

Single vs. multiple detectors

- Initial discovery of the CMB was in a single detector (excess noise that could not be attributed to any known noise source)
- Ground-based detectors currently aren't sensitive enough for expected GW backgrounds to stand out above instrumental noise (LISA is another story!)
- Instead, look for evidence of a common disturbance in multiple detectors -> cross-correlation
- Signal might be weak, but you can build up SNR by correlating for long periods of time

> A MEASUREMENT OF EXCESS ANTENNA TEMPERATURE AT 4080 Mc/s
>
> Measurements of the effective zenith noise temperature of the 20-foot horn-reflector antenna (Crawford, Hogg, and Hunt 1961) at the Crawford Hill Laboratory, Holmdel, New Jersey, at 4080 Mc/s have yielded a value about 3.5° K higher than expected. This excess temperature is, within the limits of our observations, isotropic, unpolarized, and free from seasonal variations (July, 1964–April, 1965). A possible explanation for the observed excess noise temperature is the one given by Dicke, Peebles, Roll, and Wilkinson (1965) in a companion letter in this issue.

(Penzias & Wilson, 1965)



Cross-correlation: basic idea

Data from two detectors:

$$d_1 = h + n_1$$
$$d_2 = h + n_2$$

common GW signal component

Expected value of cross-correlation:

$$\langle \hat{C}_{12} \rangle = \langle d_1 d_2 \rangle = \langle h^2 \rangle + \langle n_1 n_2 \rangle + \cancelto{0}{\langle h n_2 \rangle} + \cancelto{0}{\langle n_1 h \rangle} = \langle h^2 \rangle + \langle n_1 n_2 \rangle$$

Assuming detector noise is uncorrelated:

$$\boxed{\langle \hat{C}_{12} \rangle = \langle h^2 \rangle \equiv S_h}$$

Worked example: N samples, white GWB in white detector noise

$$\hat{S}_h \equiv \hat{C}_{12} = \frac{1}{N}\sum_{i=1}^{N} d_{1i}d_{2i} \qquad d_{1i} = h_i + n_{1i}\,, \quad d_{2i} = h_i + n_{2i}$$

(common GW signal, uncorrelated detector noise)

Expected value:

$$\mu \equiv \langle \hat{C}_{12} \rangle = \frac{1}{N}\sum_{i=1}^{N} \langle d_{1i}d_{2i} \rangle = \frac{1}{N}\sum_{i=1}^{N} \langle h_i^2 \rangle = S_h$$

Variance:

$$\sigma^2 \equiv \langle \hat{C}_{12}^2 \rangle - \langle \hat{C}_{12} \rangle^2 = \left(\frac{1}{N}\right)^2 \sum_{i=1}^{N}\sum_{j=1}^{N} \Big(\langle d_{1i}d_{2i}d_{1j}d_{2j} \rangle - \langle d_{1i}d_{2i} \rangle \langle d_{1j}d_{2j} \rangle \Big)$$

$$\downarrow \ \langle abcd \rangle = \langle ab \rangle \langle cd \rangle + \langle ac \rangle \langle bd \rangle + \langle ad \rangle \langle bc \rangle$$

$$= \left(\frac{1}{N}\right)^2 \sum_{i=1}^{N}\sum_{j=1}^{N} \Big(\langle d_{1i}d_{1j} \rangle \langle d_{2i}d_{2j} \rangle + \langle d_{1i}d_{2j} \rangle \langle d_{2i}d_{1j} \rangle \Big)$$

$$= \left(\frac{1}{N}\right)^2 \sum_{i=1}^{N}\sum_{j=1}^{N} \Big(S_1\delta_{ij}\, S_2\delta_{ij} + S_h\delta_{ij}\, S_h\delta_{ij} \Big)$$

where

$$S_1 = S_{n_1} + S_h$$
$$S_2 = S_{n_2} + S_h$$

$$= \left(\frac{1}{N}\right)^2 \sum_{i=1}^{N} \left(S_1 S_2 + S_h^2\right) = \frac{1}{N}\left(S_1 S_2 + S_h^2\right) \simeq \frac{1}{N} S_1 S_2$$

SNR:

$$\rho \equiv \frac{\mu}{\sigma} = \frac{S_h}{\sqrt{S_1 S_2 / N}} \quad \Longrightarrow \quad \boxed{\rho = \sqrt{N}\frac{S_h}{S_n}}$$

where $\sqrt{S_1 S_2} \approx \sqrt{S_{n_1} S_{n_2}} \equiv S_n$

Cross-correlation estimators / optimal filtering

More generally:

$$\hat{S}_h = \int_{-T/2}^{T/2} \mathrm{d}t \int_{-T/2}^{T/2} \mathrm{d}t'\, d_1(t) d_2(t') Q(t,t')$$

if stationary
$Q(t,t') = Q(t-t')$

$$\hat{S}_h \simeq \int_{-\infty}^{\infty} \mathrm{d}f \int_{-\infty}^{\infty} \mathrm{d}f'\, \delta_T(f-f') \tilde{d}_1(f) \tilde{d}_2^*(f') \tilde{Q}^*(f')$$

Choose Q to maximize SNR for fixed spectral shape:

$$\tilde{Q}(f) \propto \frac{\Gamma_{12}(f) H(f)}{P_1(f) P_2(f)}$$

expected
signal spectrum

correlation coeff (overlap)
between two detectors

$$\langle \tilde{h}_1(f) \tilde{h}_2^*(f') \rangle = \frac{1}{2} \delta(f-f') \Gamma_{12}(f) S_h(f)$$

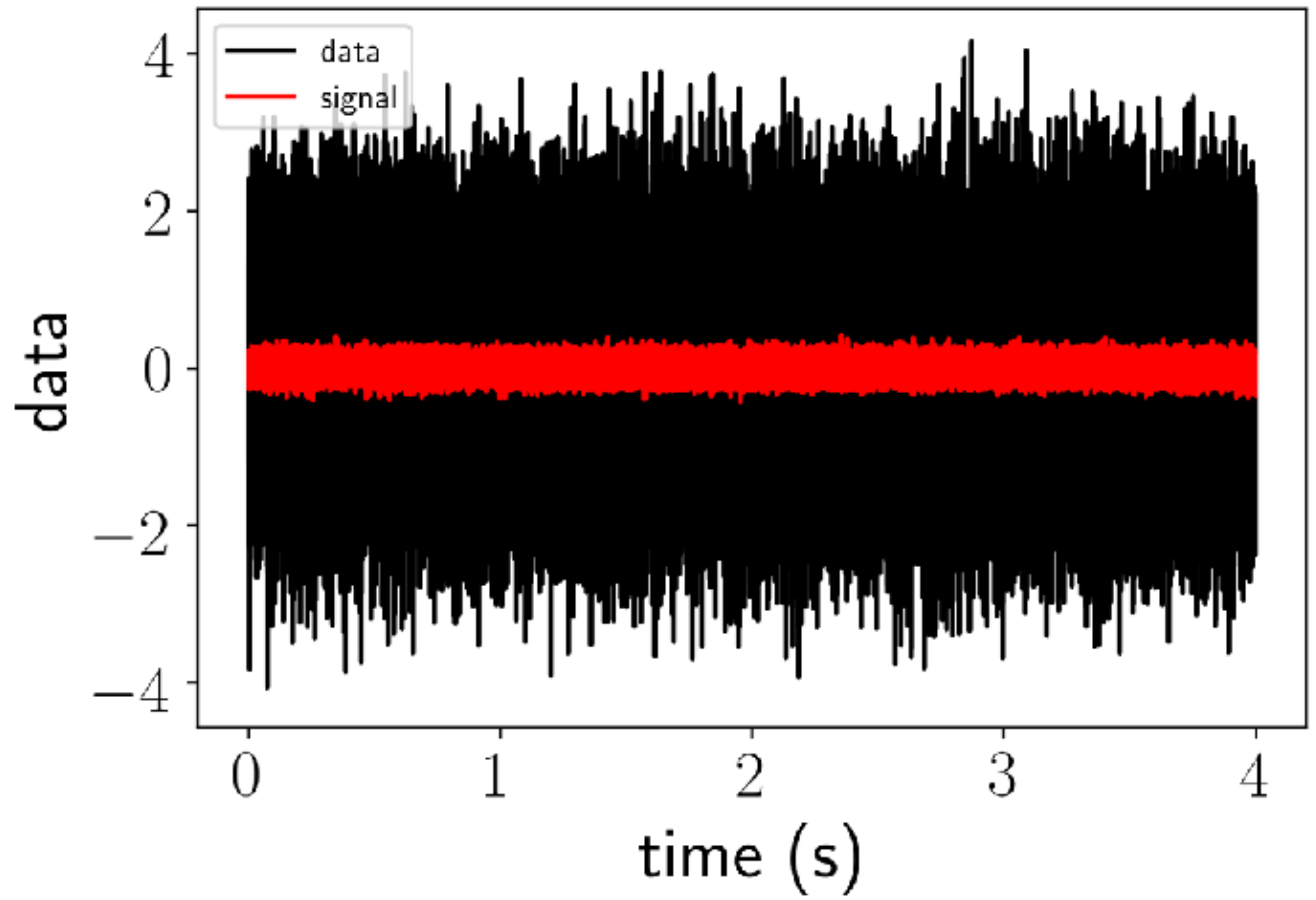
de-weight correlation
when noise is large or
overlap is small

<u>Exercise 4</u>: Verify the expression for the optimal filter function Q(f).

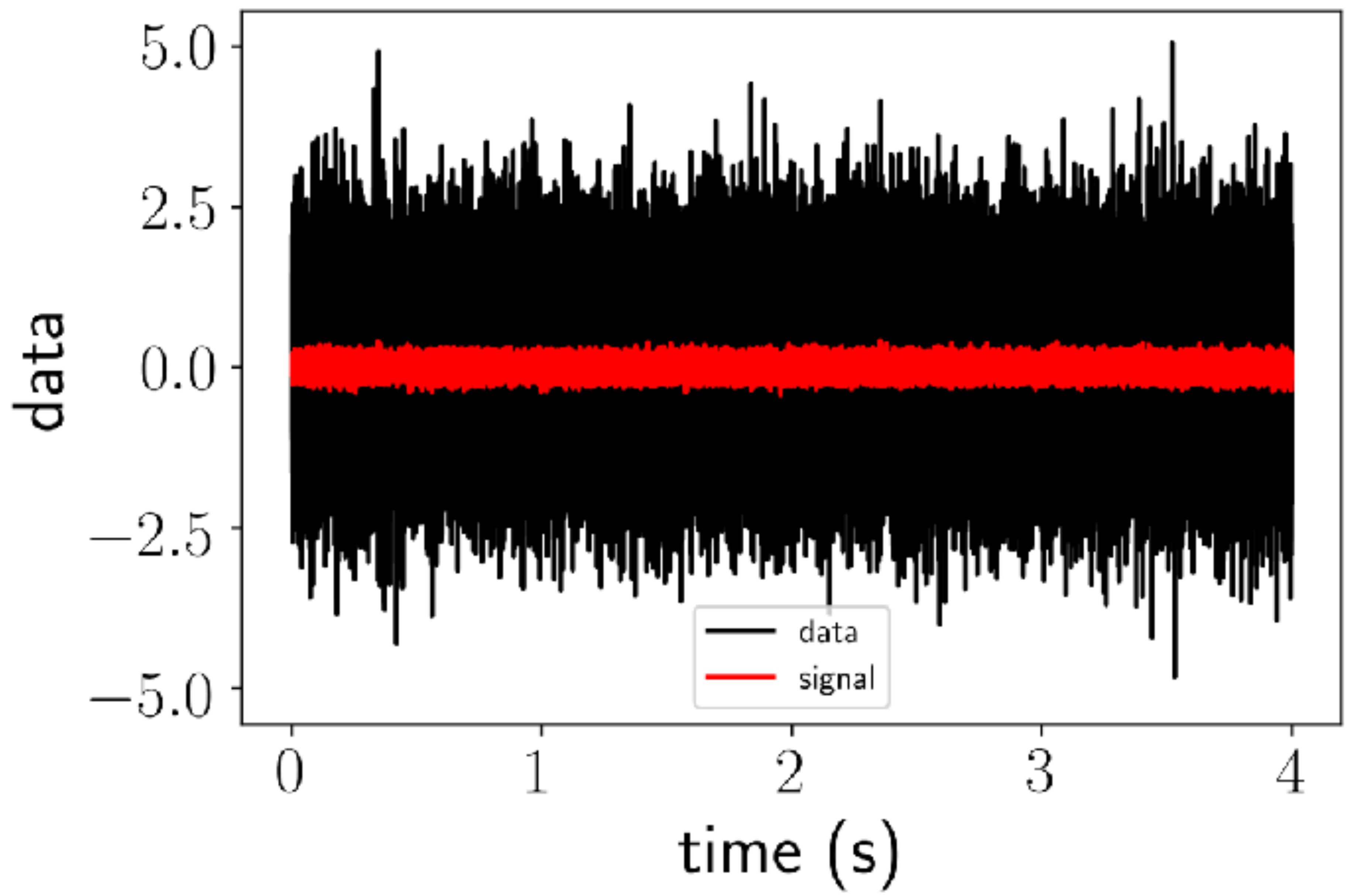
5. Some simple examples

(code to simulate data and do the analyses are on github repository)

(i) White GWB in white detector noise



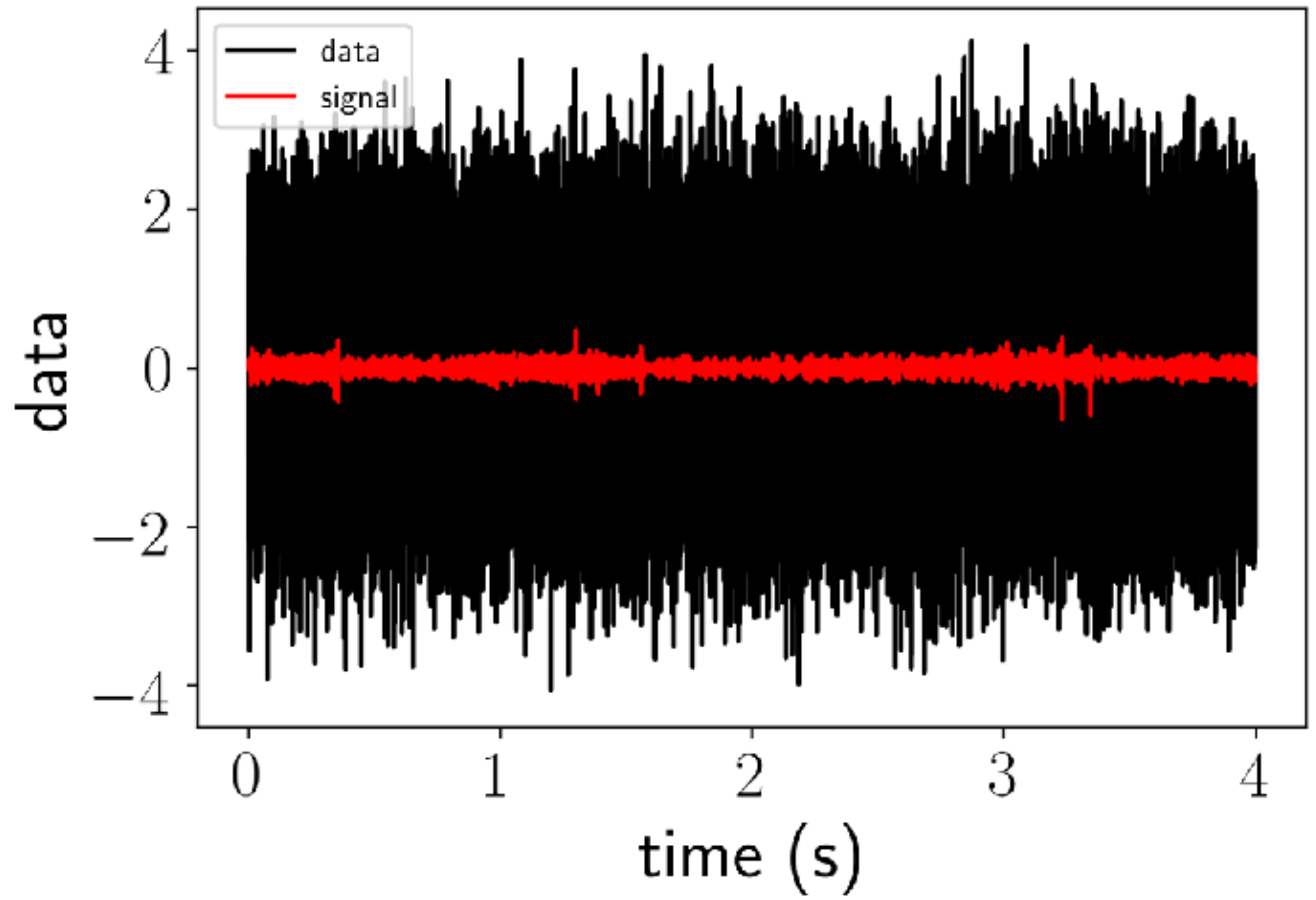
$$H(f) = 1$$



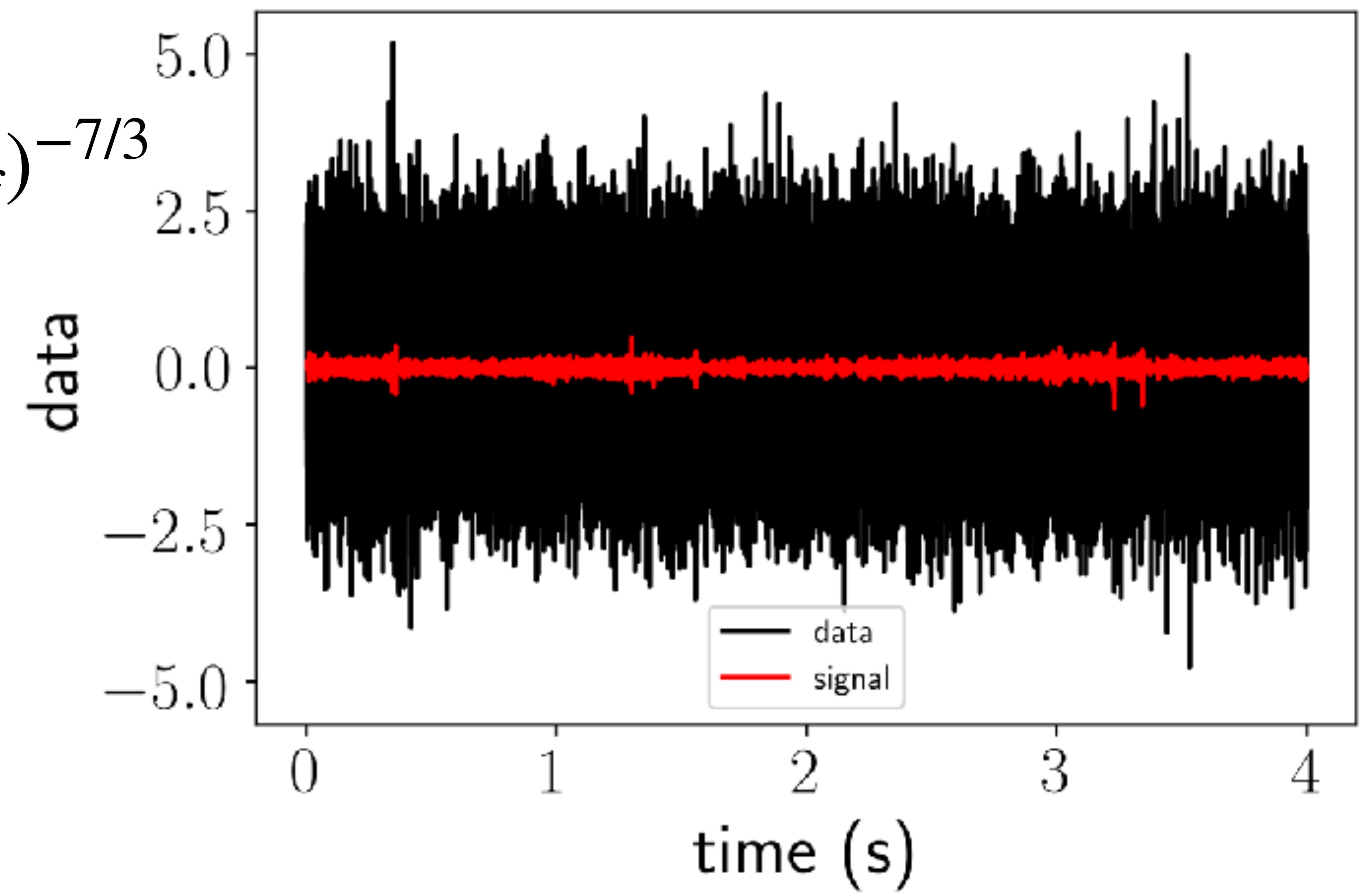
expected and estimated values of power agree to 3.5%, within 1 sigma

optimally filtered CC SNR = 2.9

(ii) Confusion-limited BNS GWB in white detector noise



$$H(f) = (f/f_{\text{ref}})^{-7/3}$$

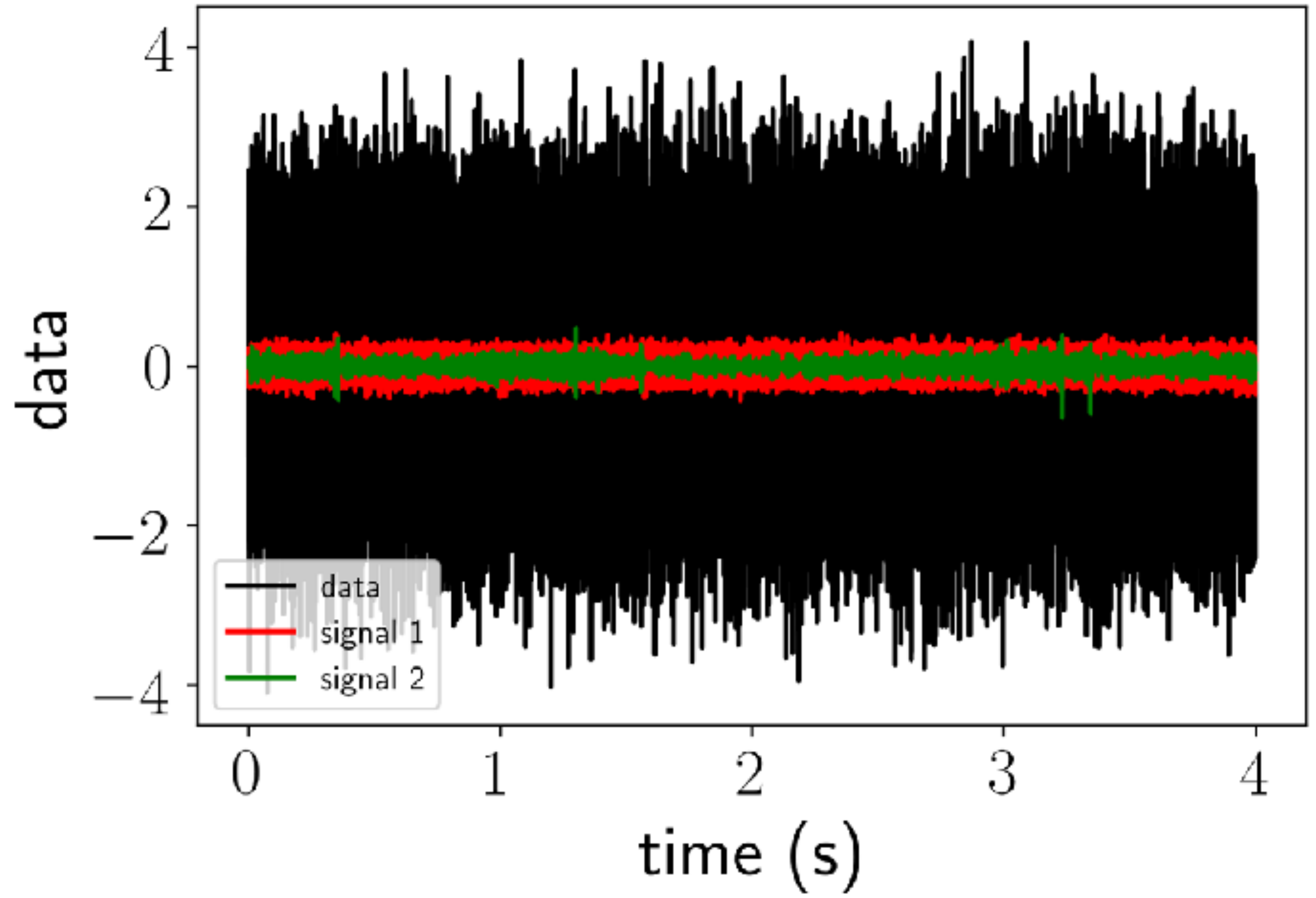


expected and estimated values of
power agree to 2.7%, within 1 sigma

optimally filtered CC SNR = 12

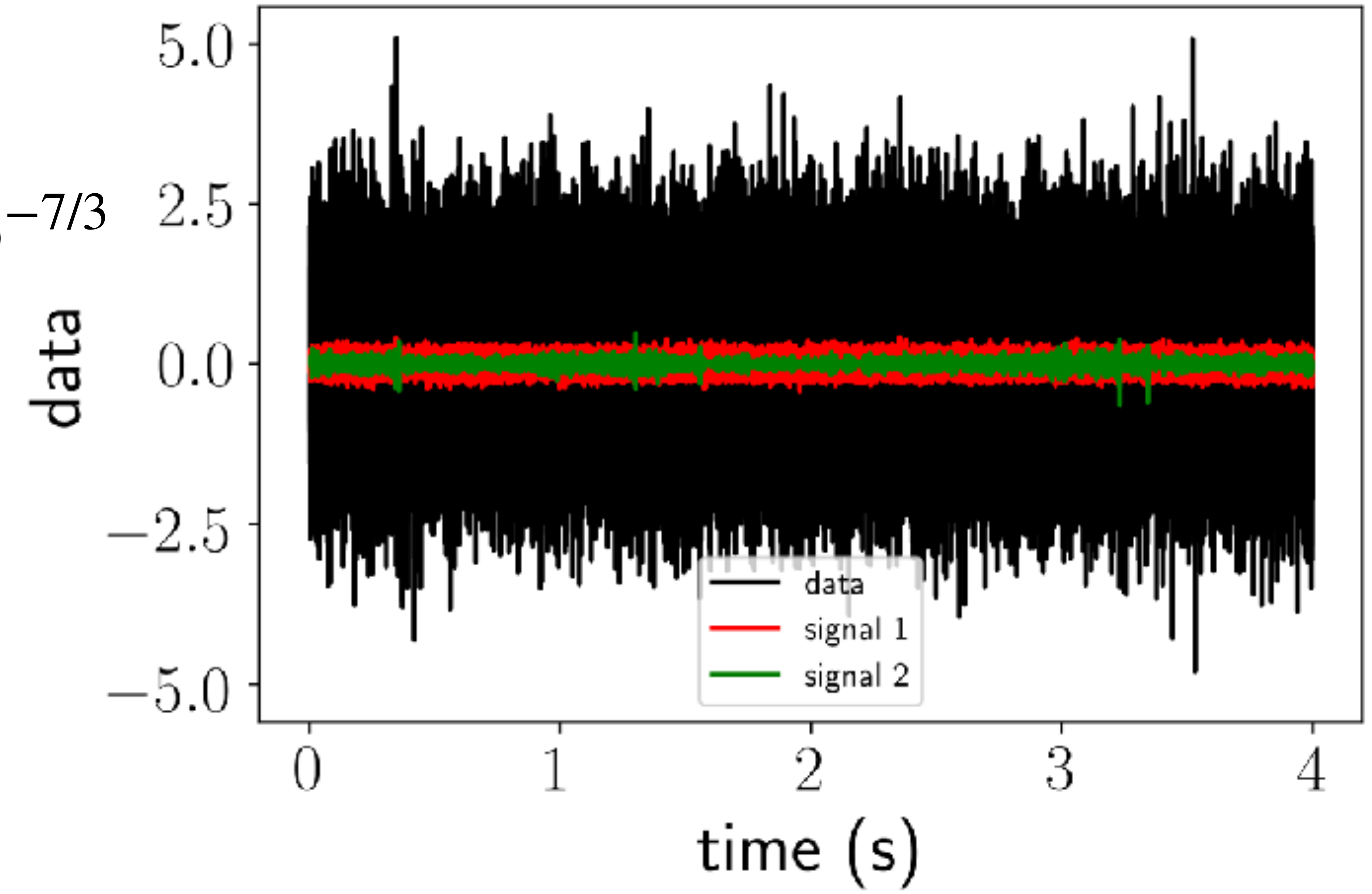
(iii) Two-component GWB in white detector noise

(GWB = white GWB + confusion-limited BNS GWB)



$$H_1(f) = 1$$

$$H_2(f) = (f/f_{\rm ref})^{-7/3}$$



<u>optimal-filtering for each component separately</u>:
white GWB: 48% overestimate, > 1 sigma
BNS GWB: 6.9% overestimate, within 1 sigma

<u>joint multi-component analysis:</u>
white GWB: agreement to 7.3%, SNR=1.4
BNS GWB: agreement to 3.8%, SNR=6.0

separate analyses overestimate strength of each GWB component and underestimate error bars

joint analysis properly takes into account the overlap (covariance) between the component spectral shapes

Parida et al, JCAP 024, 2106

Joint multi-component analysis

Data are cross-correlation estimates:

$$\hat{C}_{12}(f) \equiv \frac{2}{T}\tilde{d}_1(f)\tilde{d}_2^*(f)$$

spectral shapes

Expectation value:

$$\langle \hat{C}_{12}(f) \rangle = \sum_\alpha A_\alpha \Gamma_{12}(f) H_\alpha(f) \equiv \sum_\alpha M_\alpha(f) A_\alpha$$

amplitudes

Noise covariance matrix:

$$N_{12}(f,f') \equiv \langle \hat{C}_{12}(f)\hat{C}_{12}^*(f') \rangle - \langle \hat{C}_{12}(f) \rangle \langle \hat{C}_{12}^*(f') \rangle$$
$$\approx \delta_{ff'} P_1(f) P_2(f)$$

Likelihood function:

$$p(\hat{C}|A,N) \propto \exp\left[-\frac{1}{2}(\hat{C}-MA)^\dagger N^{-1}(\hat{C}-MA)\right]$$

Maximum-likelihood estimators:

$$\hat{A} = F^{-1}X$$

$$F \equiv M^\dagger N^{-1} M\,, \qquad X \equiv M^\dagger N^{-1}\hat{C}$$

<u>Exercise 5</u>: Verify the expression for the ML estimators.

$$F_{\alpha\beta} = \int_{-\infty}^{\infty} \mathrm{d}f\, \frac{H_\alpha(f)\Gamma_{12}^2(f)H_\beta(f)}{P_1(f)P_2(f)}$$

(noise-weighted inner product of spectral shapes; inverse covariance matrix for A)

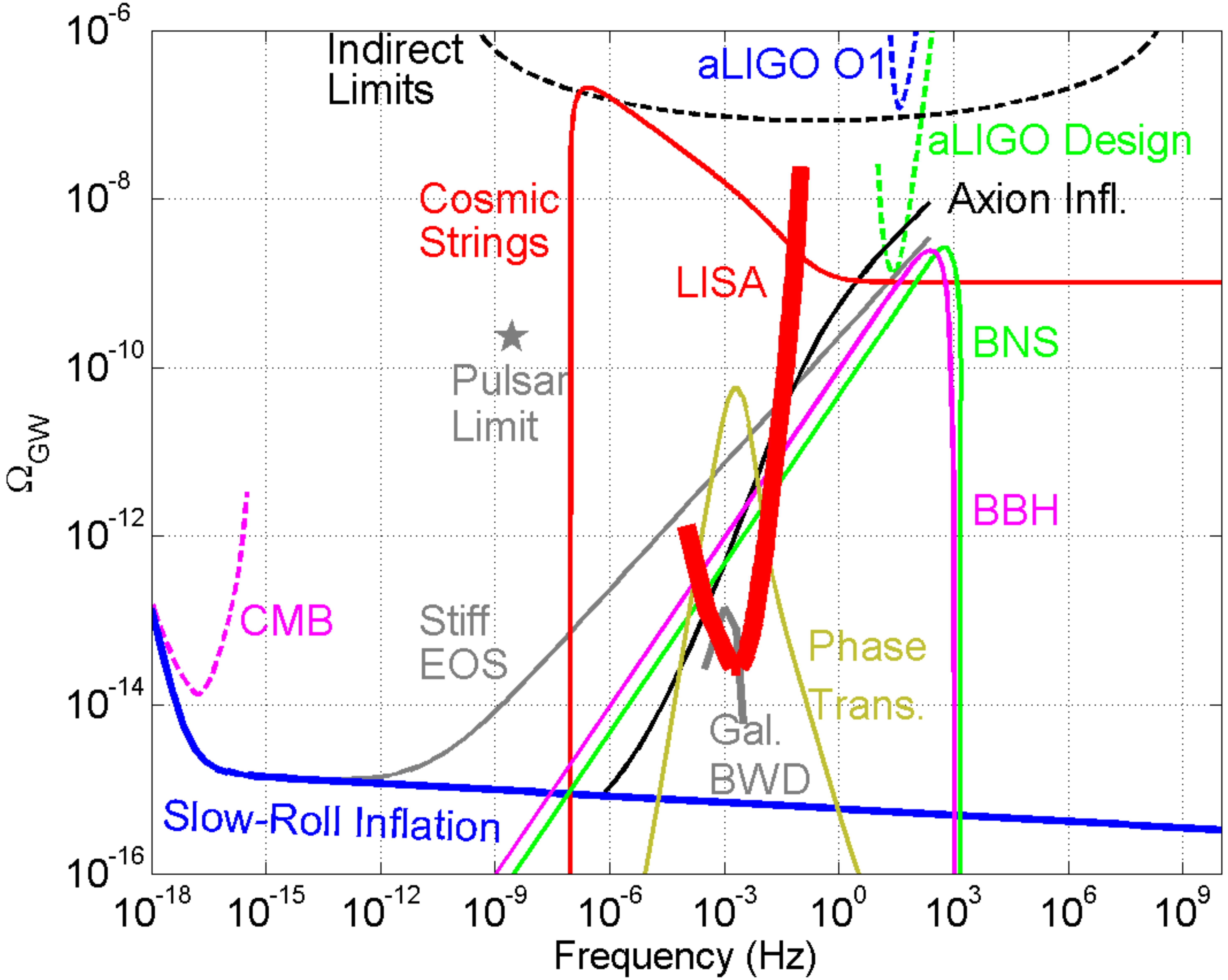
Fisher matrix

end lecture 1

extra slides

What are the prospects for detection?

No detections yet; only upper limits on strength of background in different freq bands.



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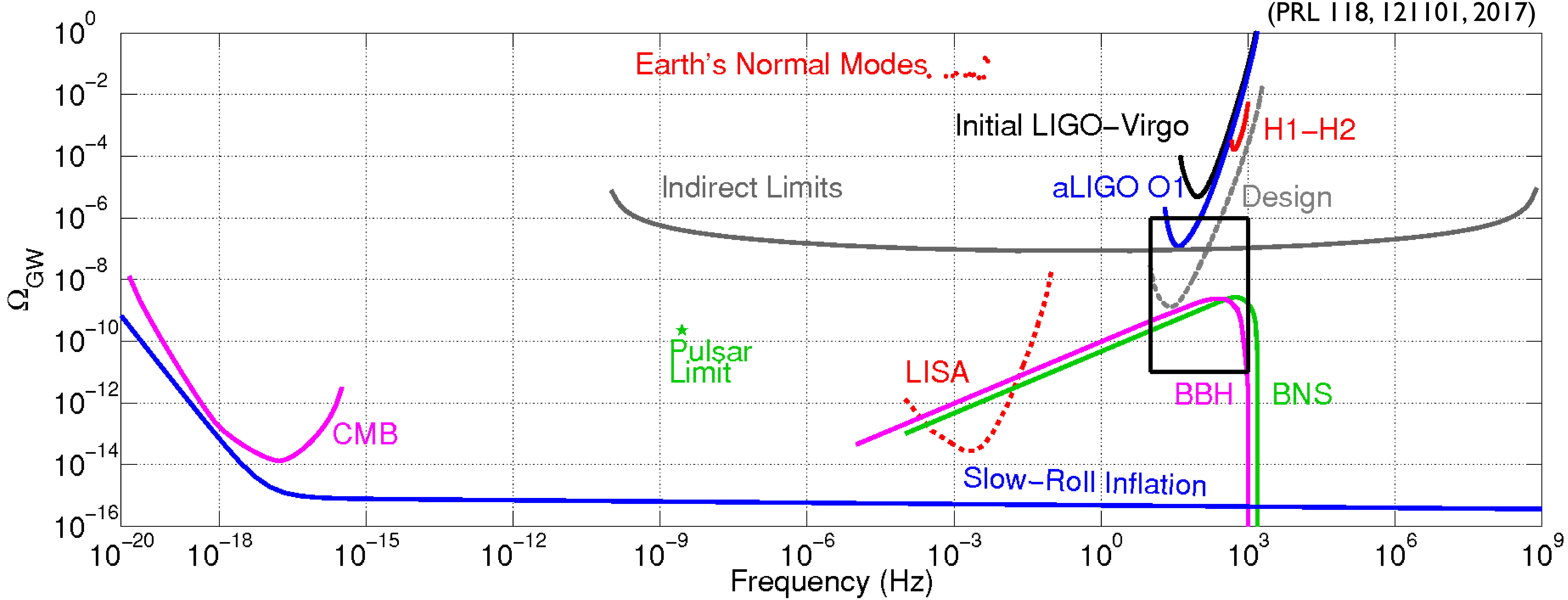
(PRL 118, 121101, 2017)



Cosmological model

Friedmann-Robertson-Walker line element:

$$ds^2 = -c^2\,dt^2 + a^2(t)\left(dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\right)$$

Redshifted frequencies and time-intervals:

$$1+z = \frac{a(t_0)}{a(t_\mathrm{s})} \quad\longrightarrow\quad 1+z = \frac{1}{a(t_\mathrm{s})}\,, \quad a(t_0) \equiv 1$$

$$f_\mathrm{s} = (1+z) f_0$$

$$\Delta t_0 = (1+z)\,\Delta t_\mathrm{s}$$

$$\left|\frac{dt}{dz}\right| = \frac{1}{(1+z)H_0 E(z)}$$

Friedmann equation

$$\frac{\dot a}{a} \equiv H(t) = H_0\,E(z)$$

$$E(z) = \sqrt{\Omega_\mathrm{m}(1+z)^3 + \Omega_\Lambda}$$

cosmology