

# Searches for stochastic gravitational-wave backgrounds

Lecture 2  
Les Houches Summer School  
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# Plan for lectures

## Yesterday: Overview / Basics

1. Motivation / context
2. Different types of stochastic backgrounds
3. Characterizing a stochastic GW background
4. Correlation methods
5. Some simple examples

## Today: Details / Example

1. Non-trivial detector response
2. Non-trivial correlated response
3. What to do in the absence of correlations (e.g., for LISA)?
4. Frequentist and Bayesian methods
5. Example: searching for the background from BBH mergers

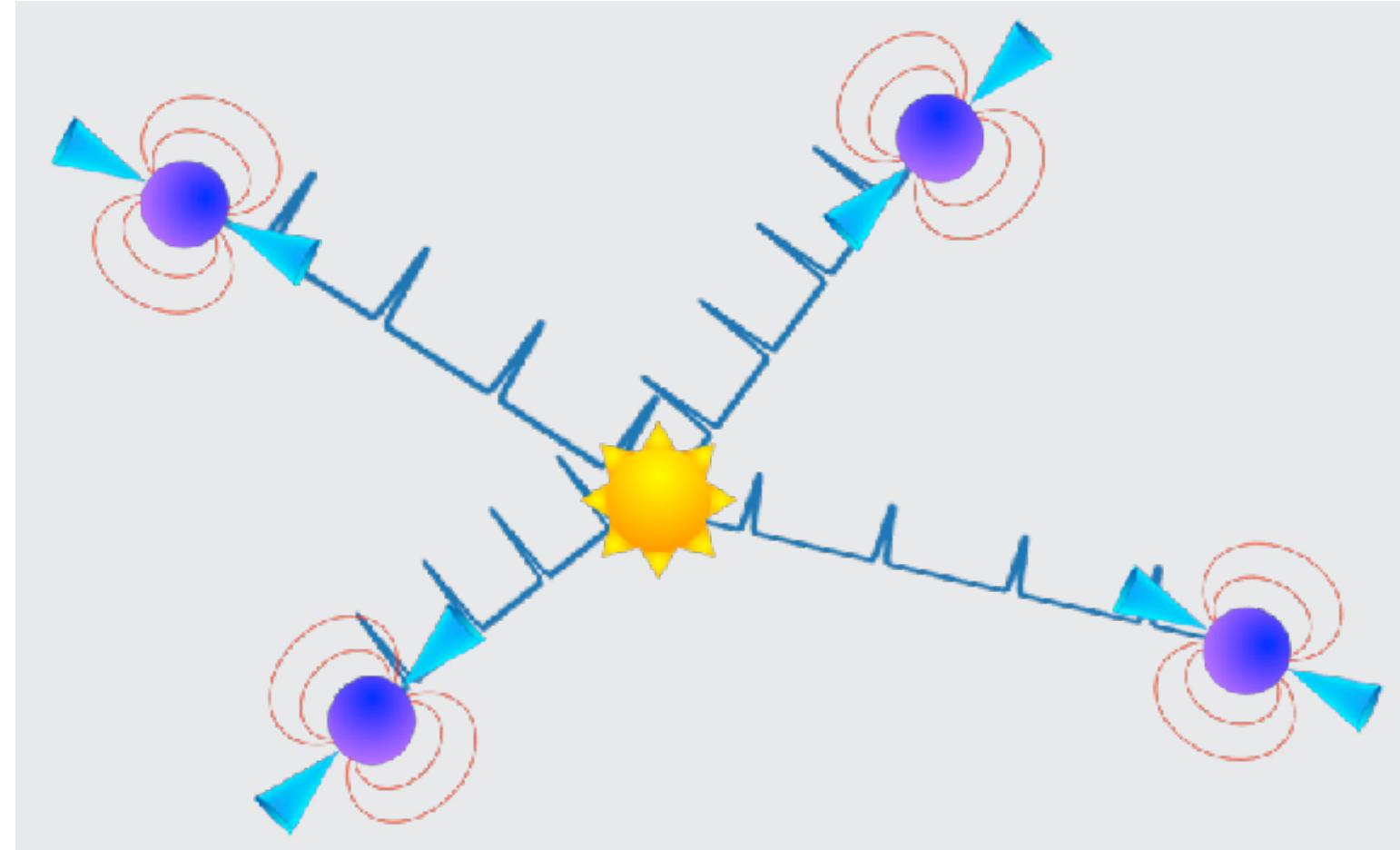
# 1 . Non-trivial detector response

# Beam detectors

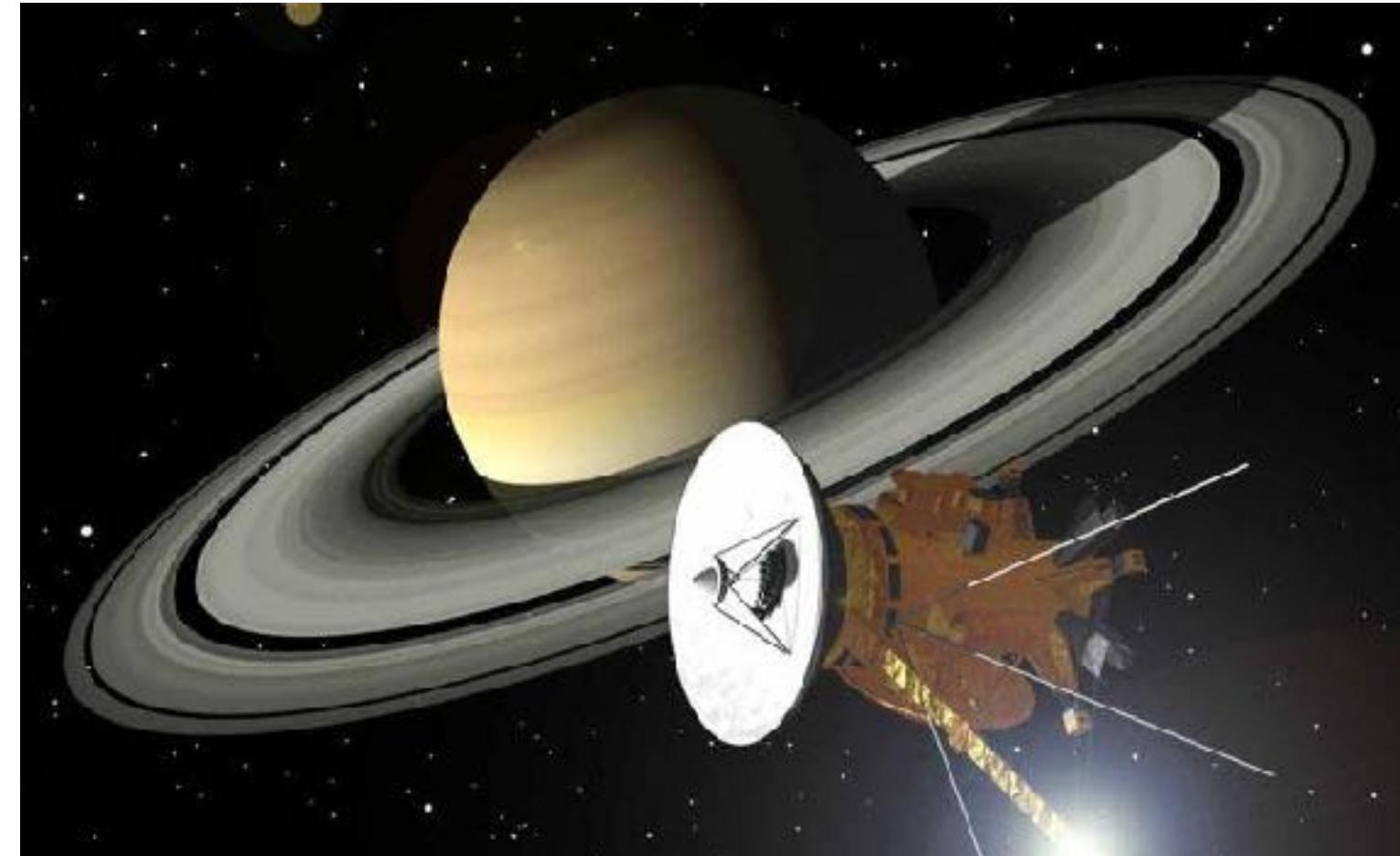
(use electromagnetic radiation to monitor the separation of two or more test masses)

laser interferometers

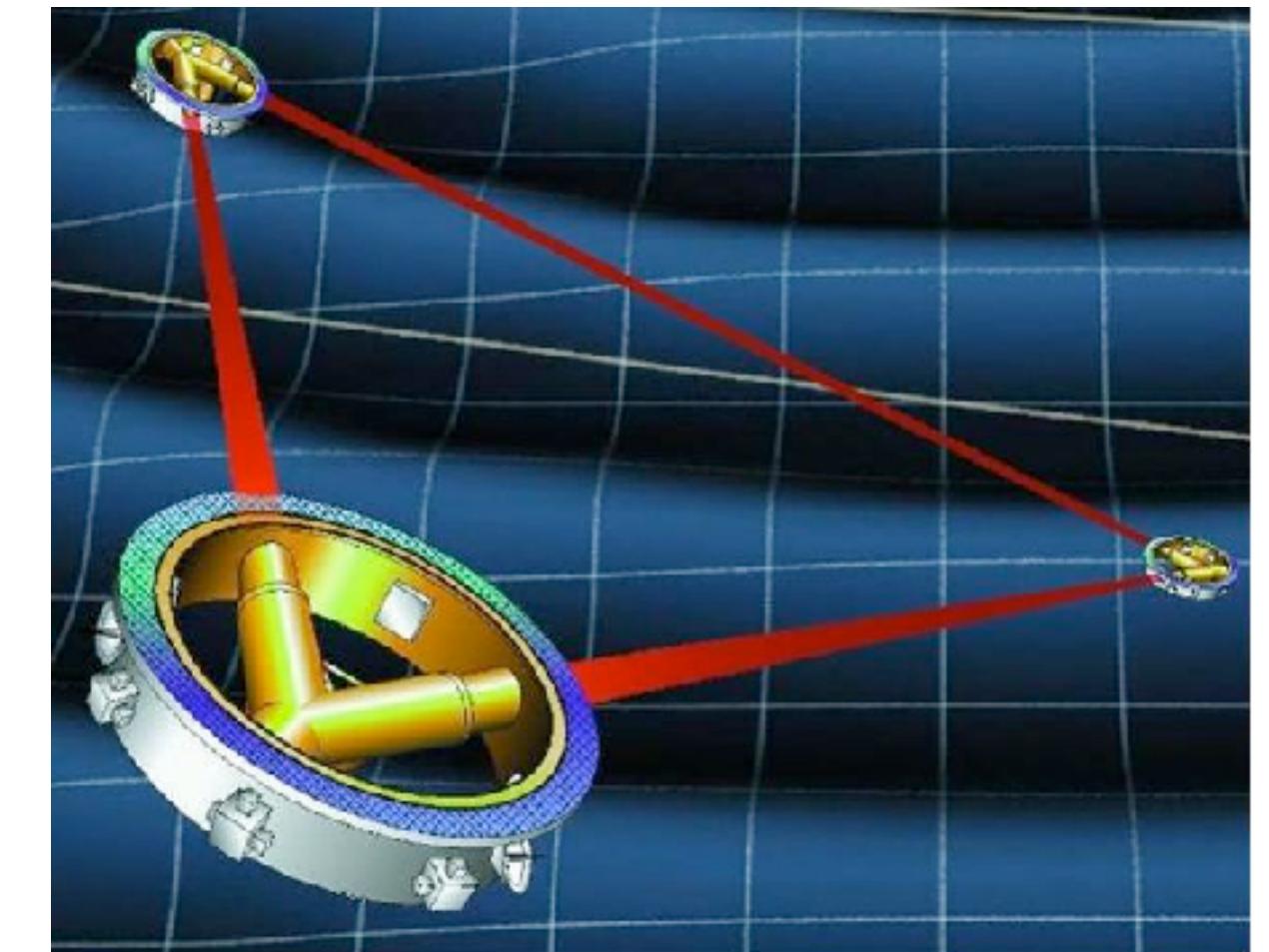
pulsar timing

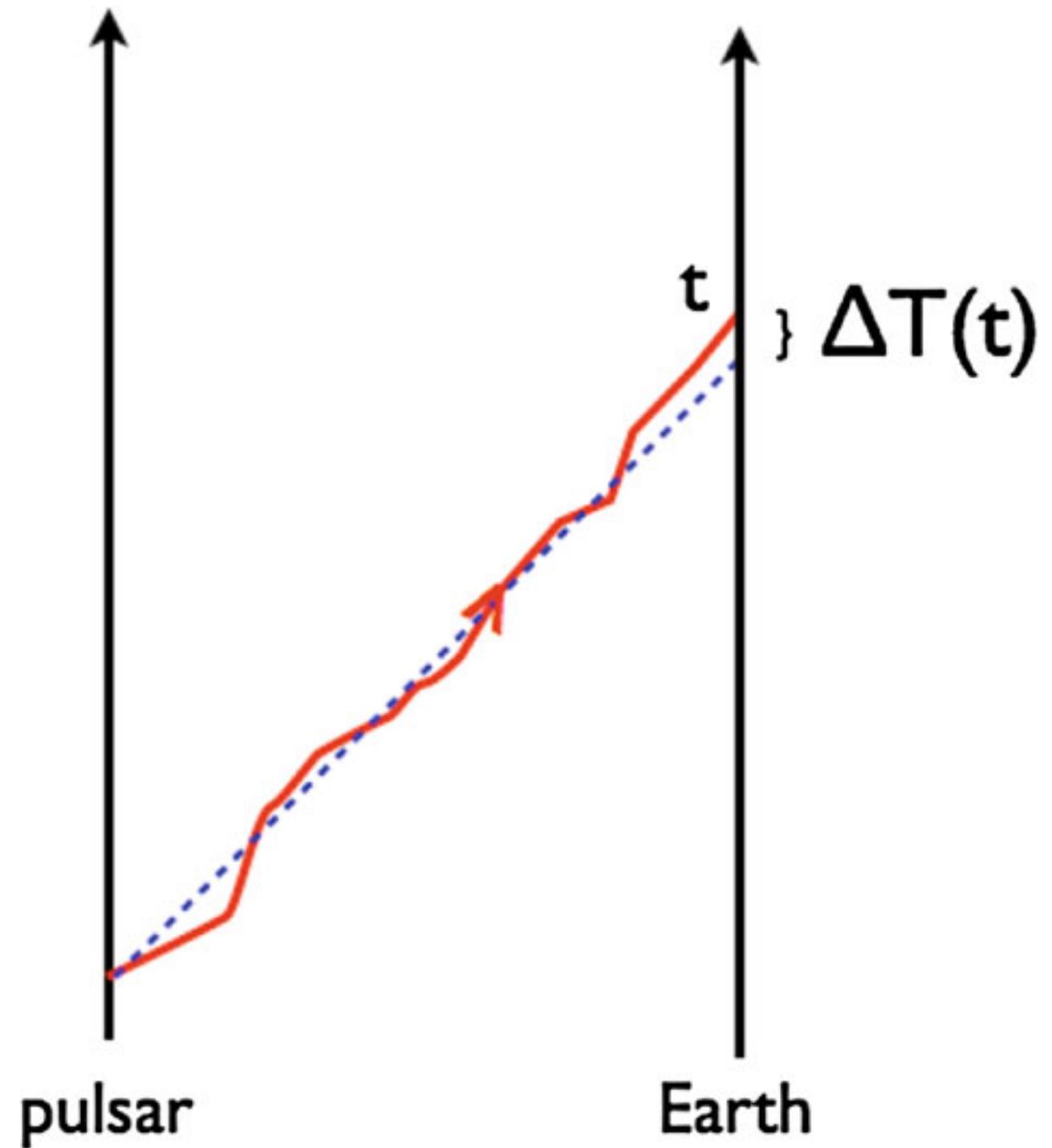
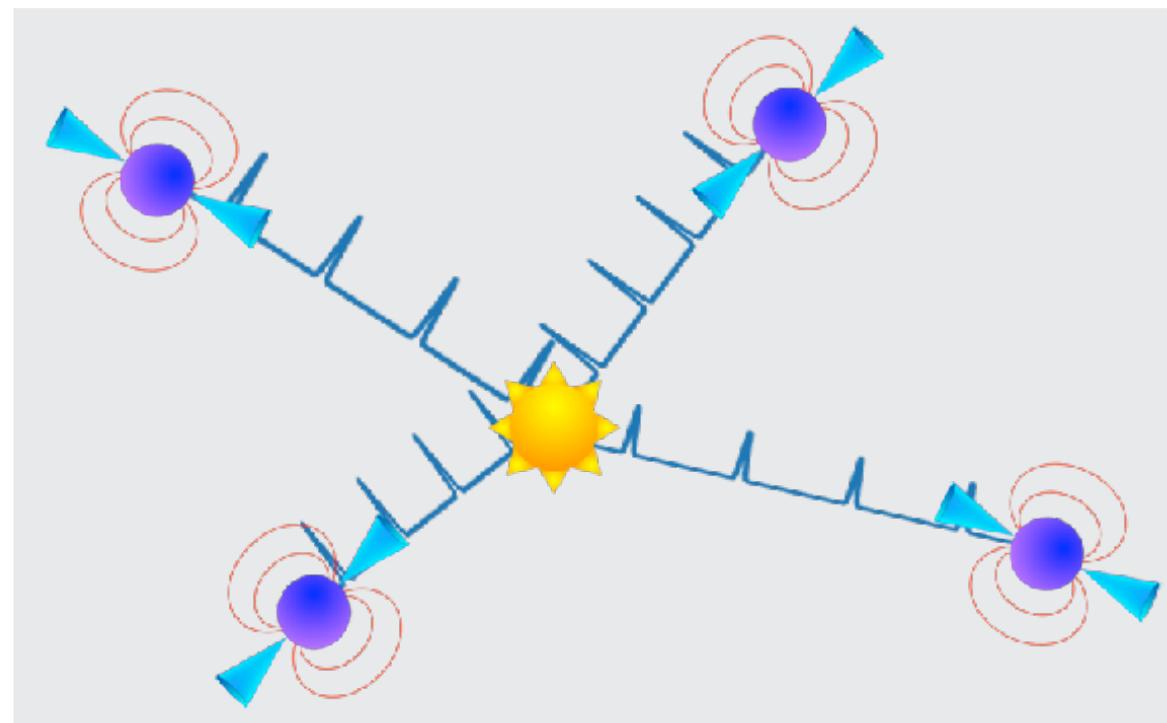


spacecraft Doppler tracking

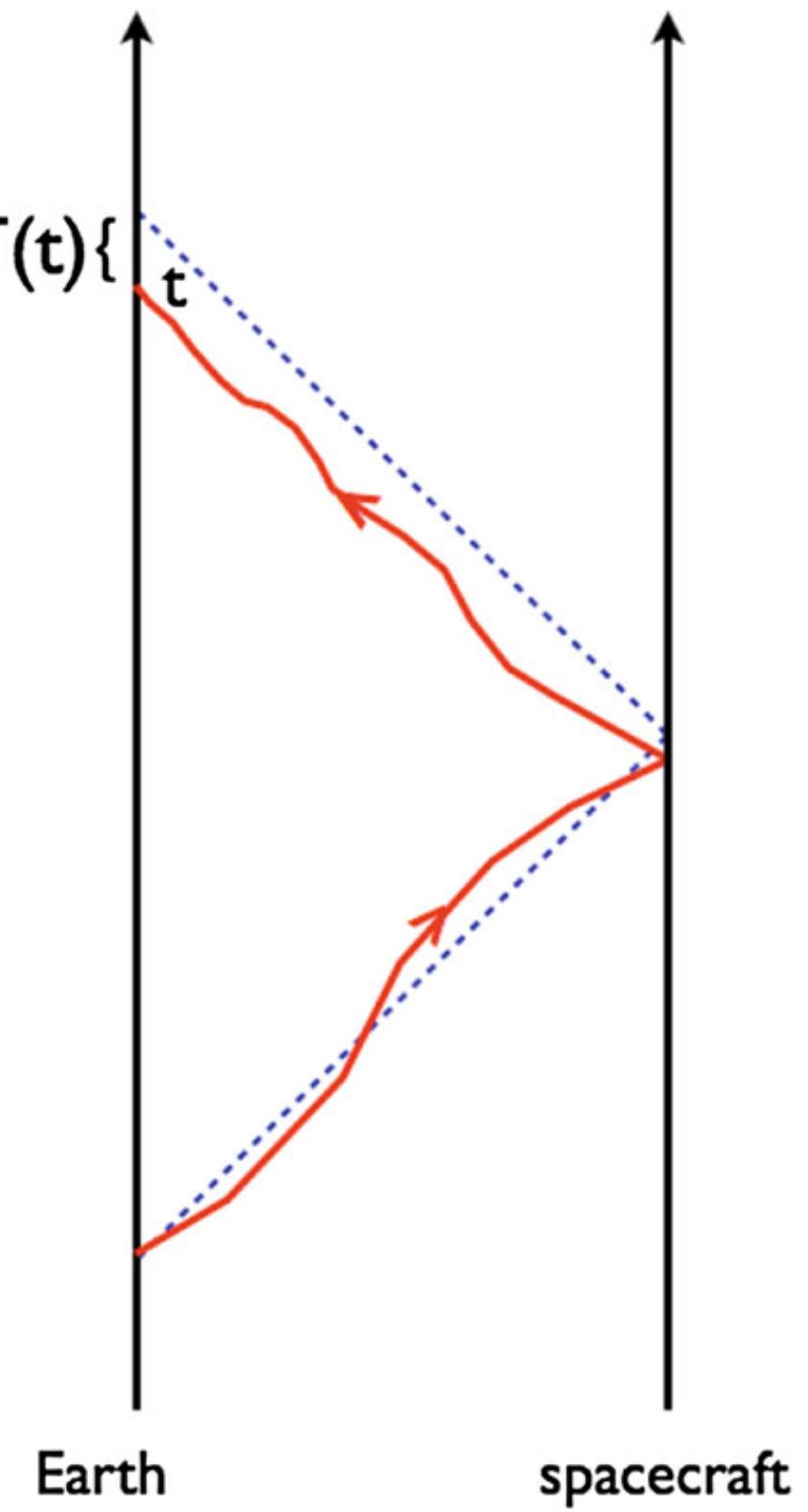
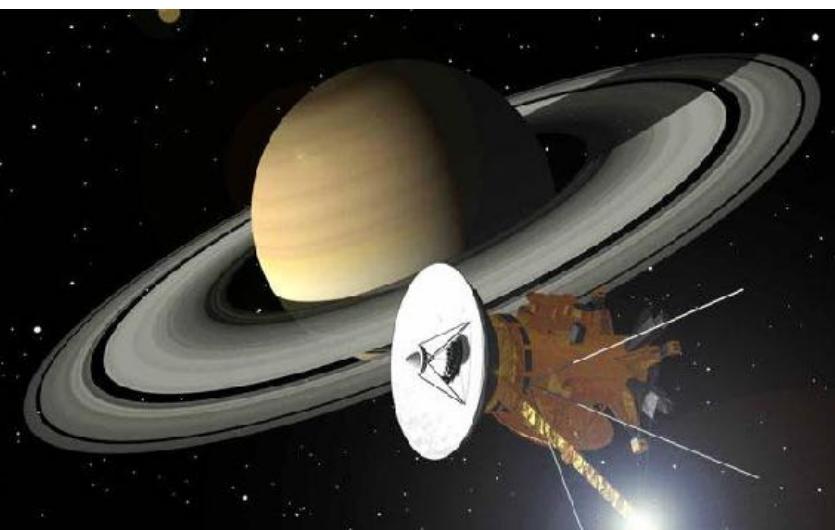


GW perturbs the photon propagation time between the test masses

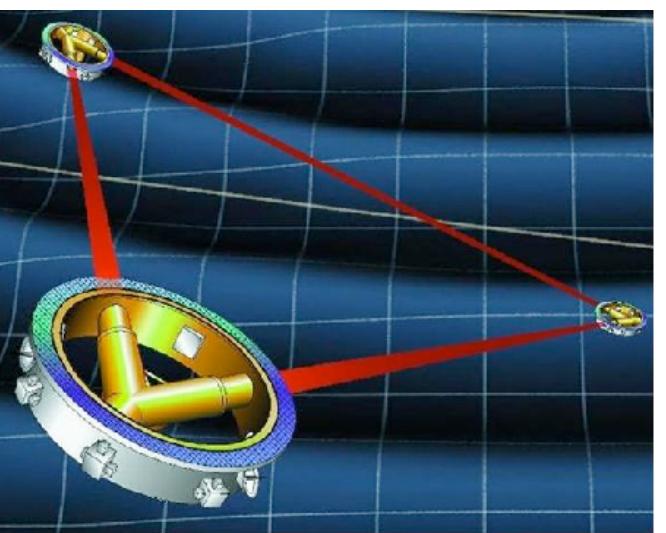




(1-arm, 1-way)



(1-arm, 2-way)



(2-arm, 2-way)

# Different types of response

timing:

$$h(t) \equiv \Delta T(t) \quad (\text{pulsar timing})$$

strain:

$$h(t) \equiv \frac{\Delta L(t)}{L} = \frac{\Delta T(t)}{T} \quad (\text{LIGO, Virgo, ...})$$

Doppler frequency:

$$h(t) \equiv \frac{\Delta\nu(t)}{\nu_0} = \frac{d\Delta T(t)}{dt} \quad (\text{pulsar timing, spacecraft Doppler tracking})$$

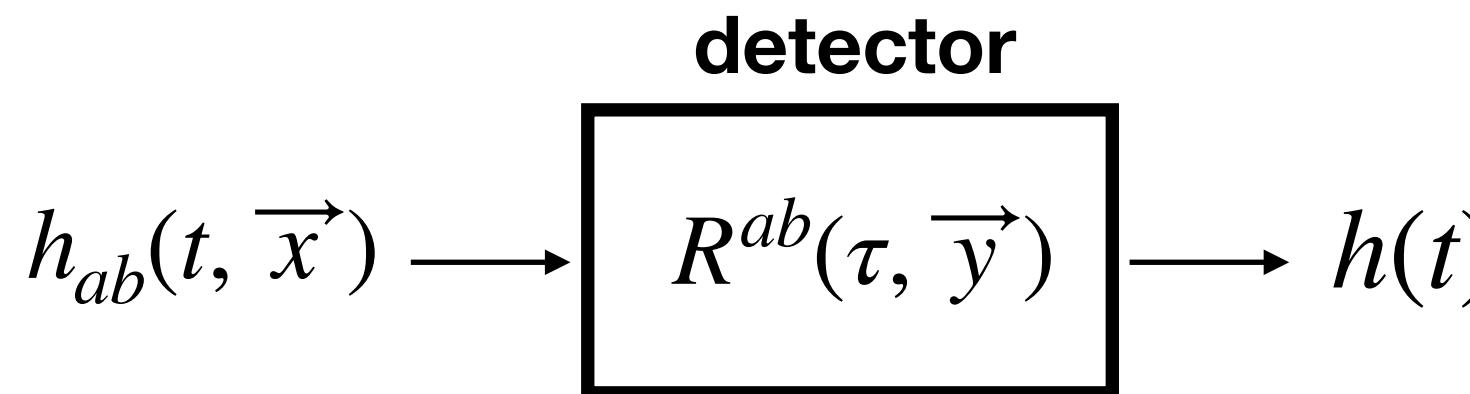
phase:

$$h(t) \equiv \Delta\Phi(t) = 2\pi\nu_0 \Delta T(t) \quad (\text{LISA})$$

All these responses are derivable from the change in light travel time  $\Delta T(t)$

# Detector response

GWs are **weak** => detector is a **linear system** which converts metric perturbations to detector output



$$h(t) = (\mathbf{R} * \mathbf{h})(t, \vec{x}) \equiv \int_{-\infty}^{\infty} d\tau \int d^3y R^{ab}(\tau, \vec{y}) h_{ab}(t - \tau, \vec{x} - \vec{y})$$

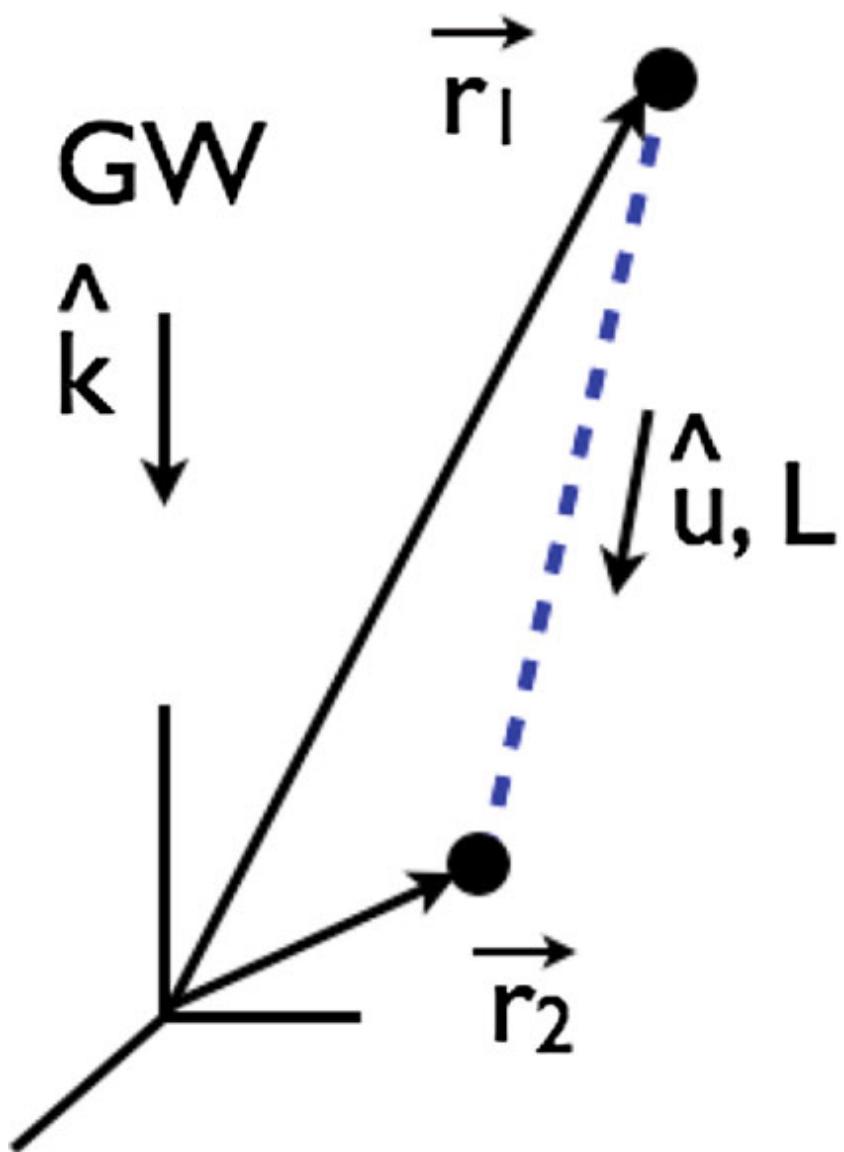
↓ convolution  
↑  
detector output      detector location      ↑ impulse response

$$\Rightarrow \tilde{h}(f) = \int d^2\Omega_{\hat{n}} \sum_A R^A(f, \hat{n}) h_A(f, \hat{n})$$

detector response for a plane-wave  
with frequency f, direction n, polarization A

$$\begin{aligned}
 R^A(f, \hat{n}) &\equiv R^{ab}(f, \hat{n}) e_{ab}^A(\hat{n}) \\
 R^{ab}(f, \hat{n}) &\equiv e^{i2\pi f \hat{n} \cdot \vec{x}/c} \int_{-\infty}^{\infty} d\tau \int d^3y R^{ab}(\tau, \vec{y}) e^{-i2\pi f(\tau + \hat{n} \cdot \vec{y}/c)}
 \end{aligned}$$

# Example: 1-arm, 1-way timing response function (e.g., pulsar timing)



$$h(t) \equiv \Delta T(t) = \frac{1}{2c} u^a u^b \int_0^L ds h_{ab}(t(s), \vec{x}(s))$$

where

$$t(s) = (t - L/c) + s/c, \quad \vec{x}(s) = \vec{r}_1 + s\hat{u}$$

$$R^A(f, \hat{n}) = \frac{1}{i2\pi f} \left[ \frac{1}{2} u^a u^b e^A_{ab}(\hat{n}) \frac{1}{1 + \hat{n} \cdot \hat{u}} \right] \left[ 1 - e^{-\frac{i2\pi f L}{c}(1 + \hat{n} \cdot \hat{u})} \right] e^{i2\pi f \hat{n} \cdot \vec{r}_2 / c}$$

(typically ignored  
for pulsar timing)

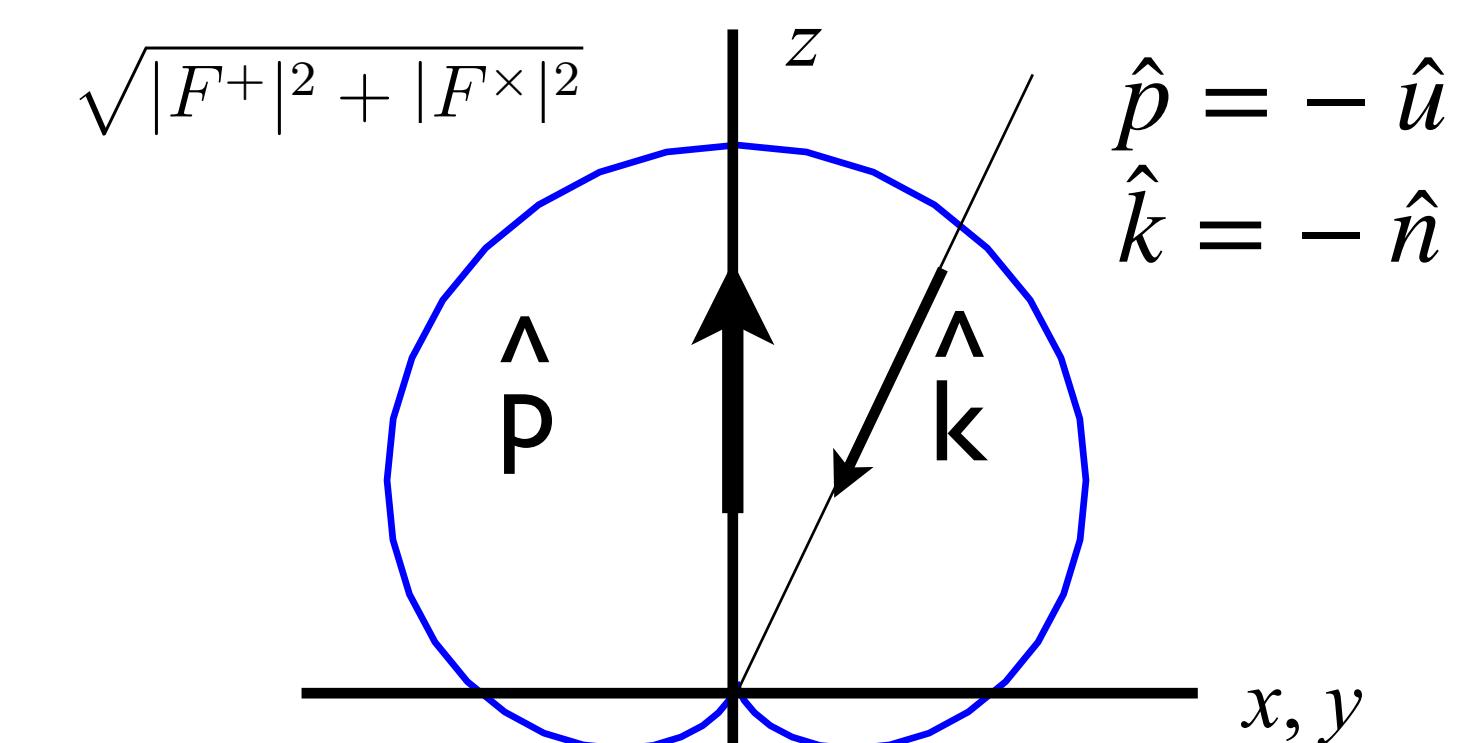
earth term      pulsar term

↓                  ↓

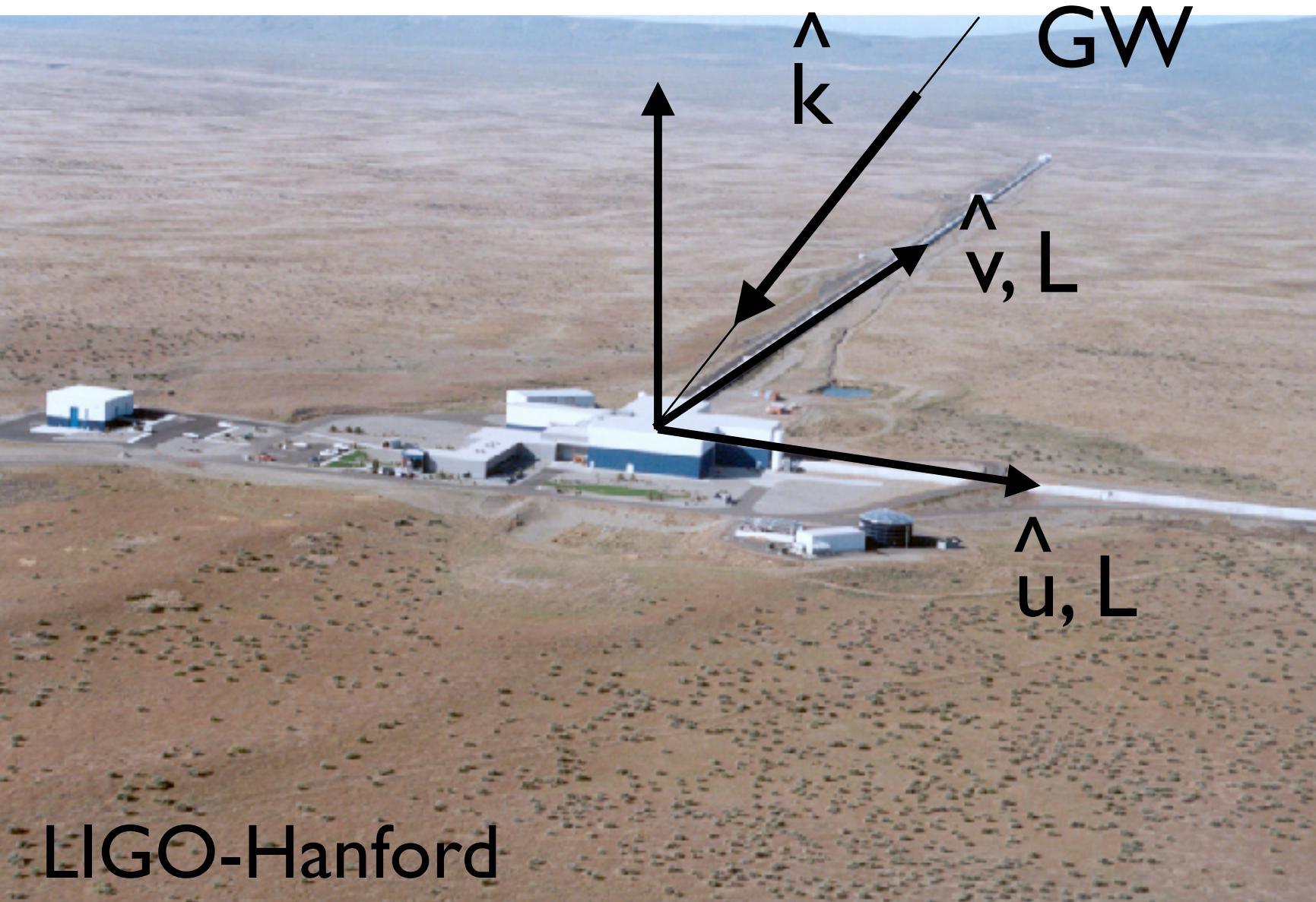
↑                  ↑

=1 for Doppler  
freq measurement       $F^A(\hat{n})$

**Exercise 6:** Derive this expression  
for the response function



# Example: LIGO response (equal-arm, short-antenna limit)

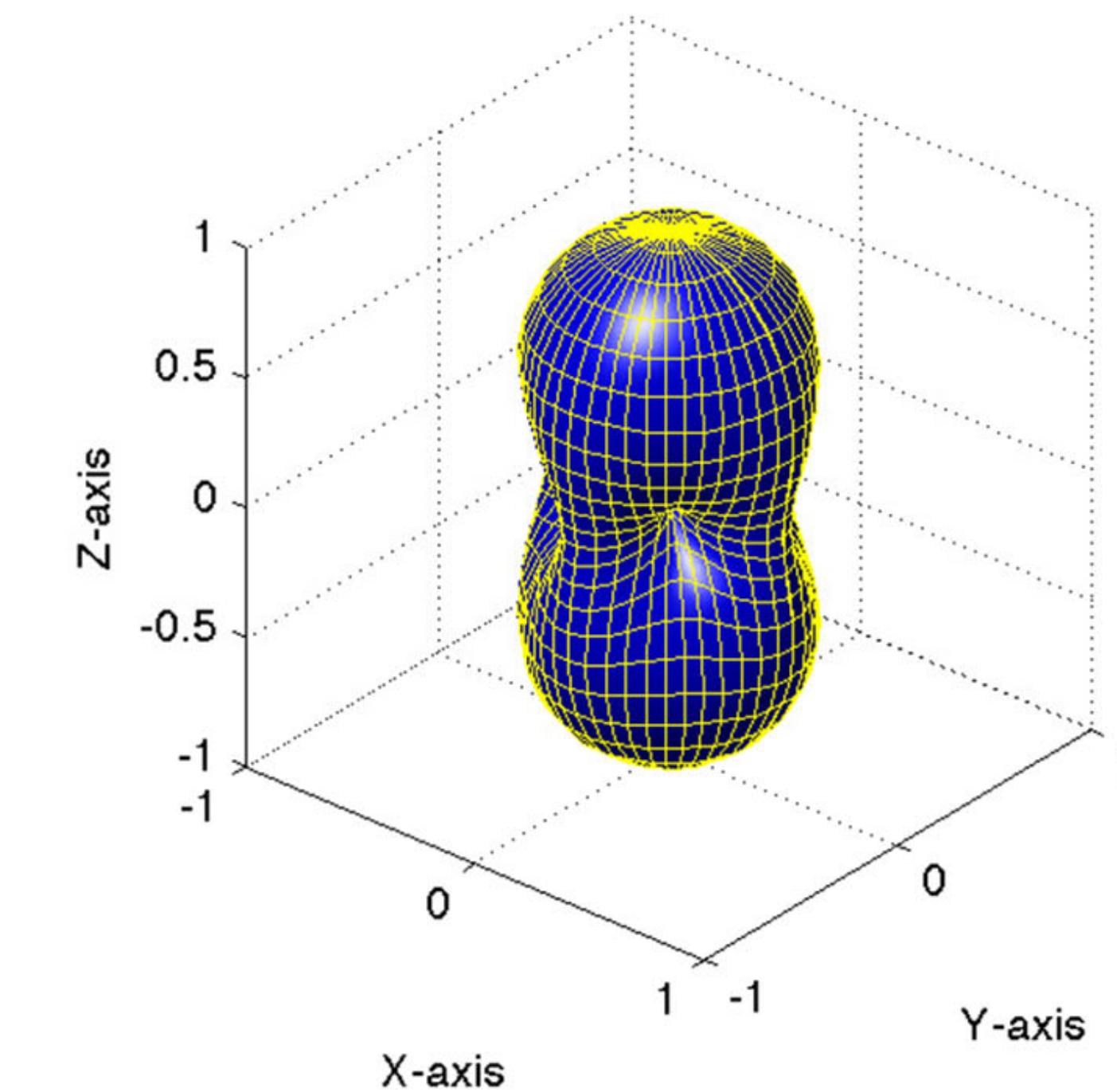


$$h(t) = \frac{1}{2} \left( \frac{\Delta T_{\vec{u}, \text{roundtrip}}(t)}{T} - \frac{\Delta T_{\vec{v}, \text{roundtrip}}(t)}{T} \right)$$

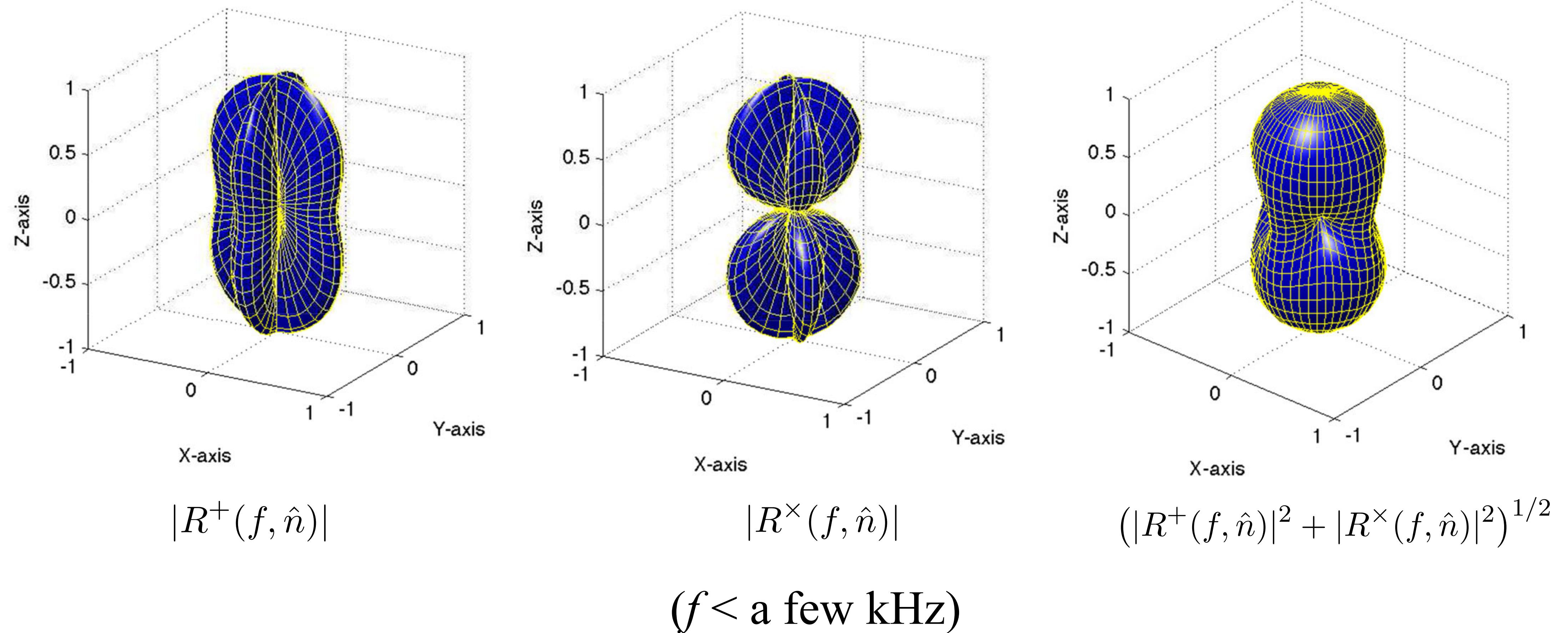
$$R^A(f, \hat{n}) \simeq \boxed{\frac{1}{2} (u^a u^b - v^a v^b)} e_{ab}^A(\hat{n})$$

detector tensor

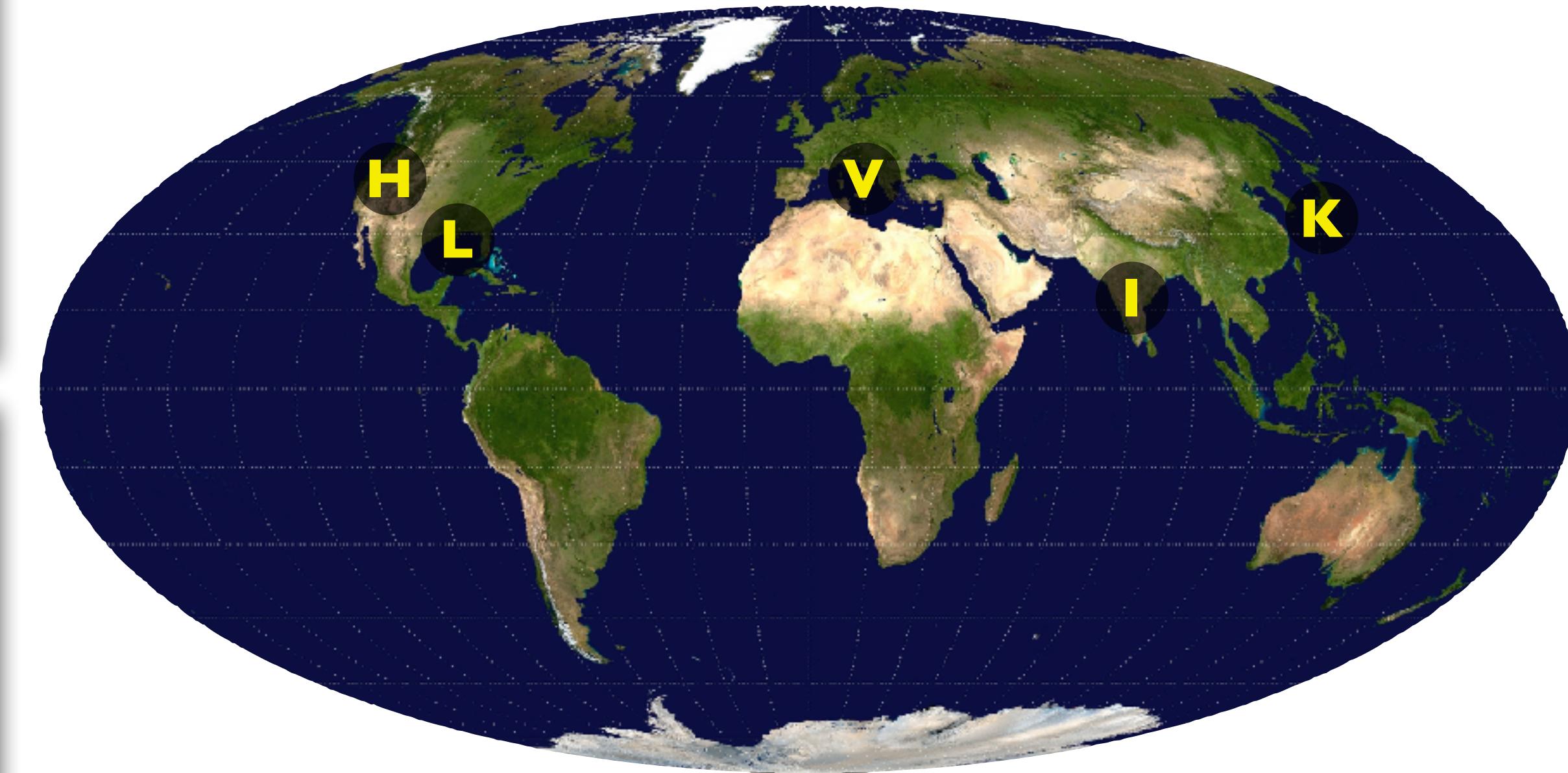
$$\sqrt{|R^+|^2 + |R^\times|^2}$$



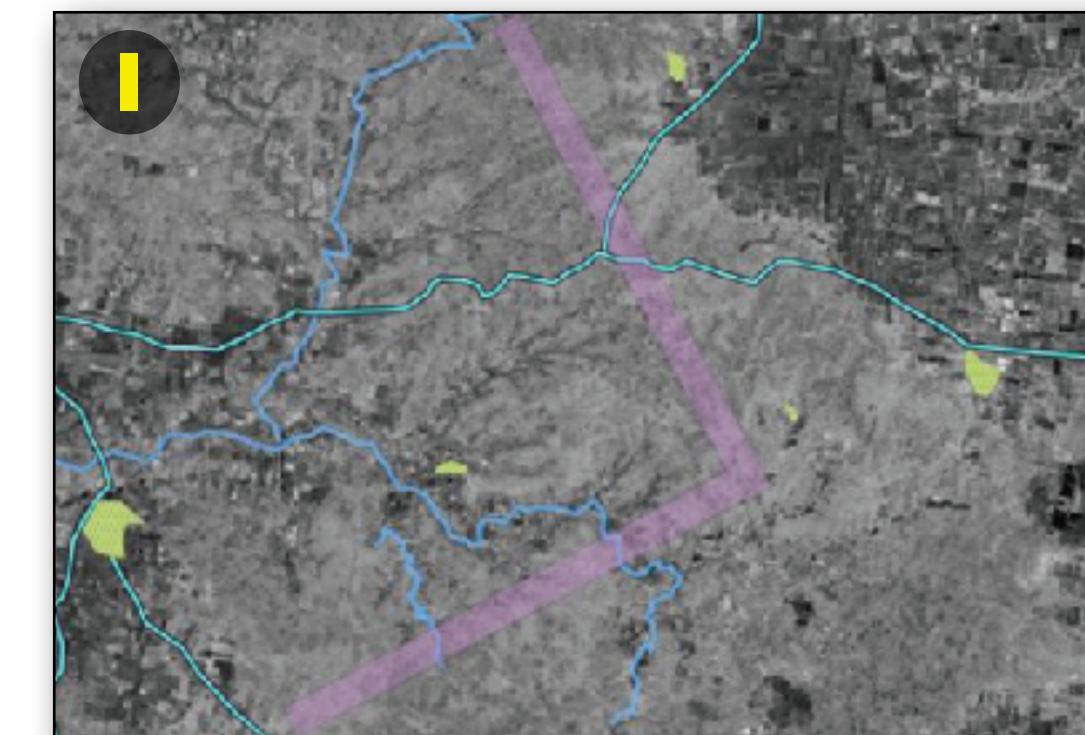
# Beam pattern functions



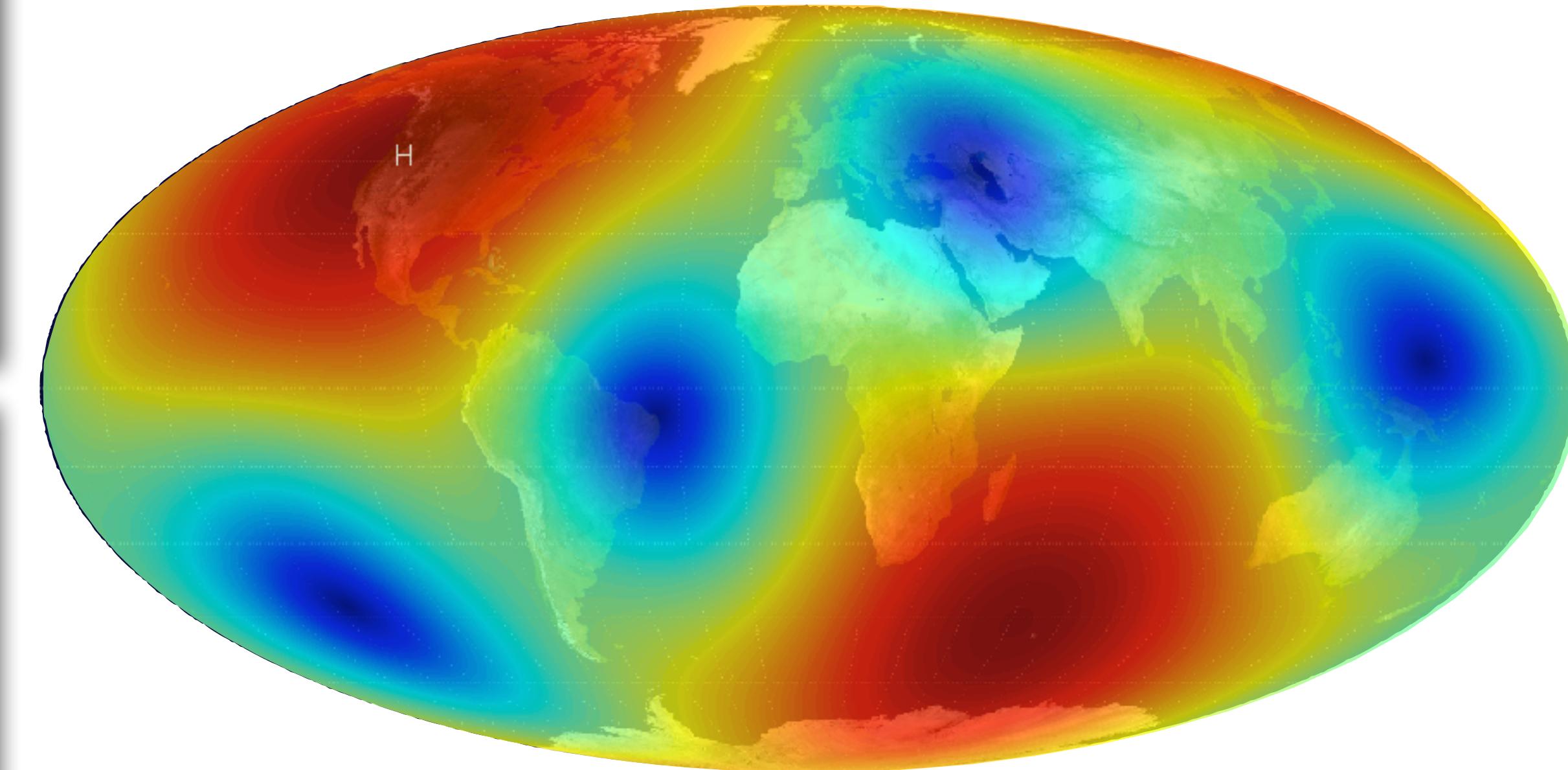
# Terrestrial Network



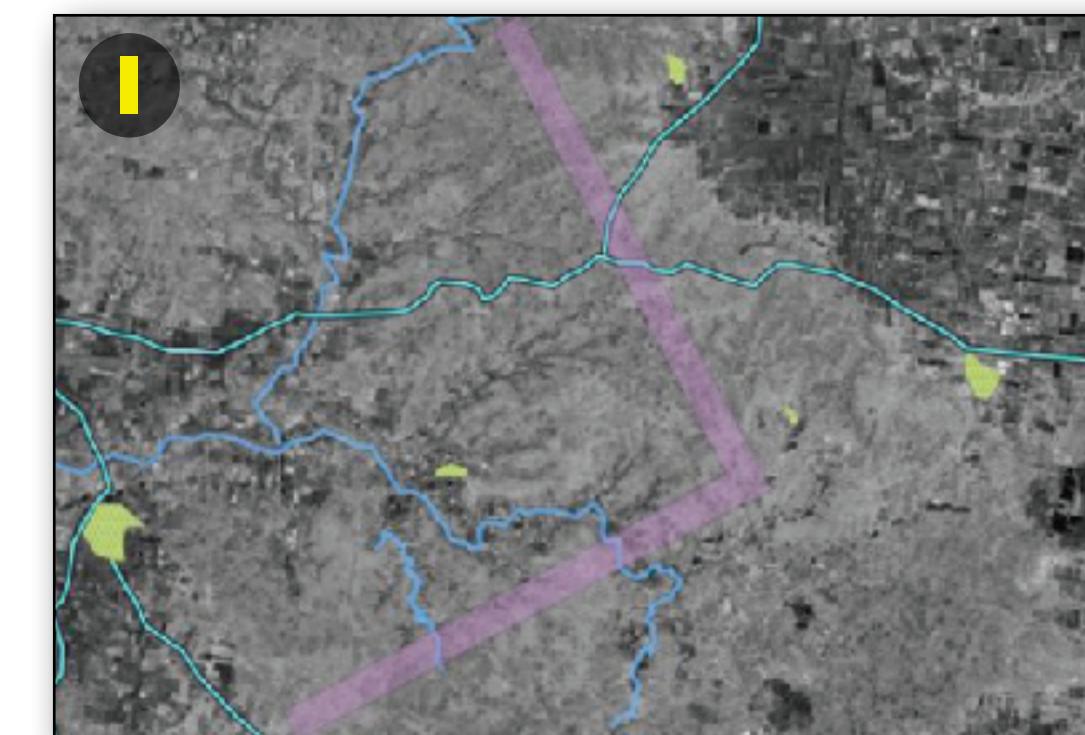
(credit: N. Cornish)



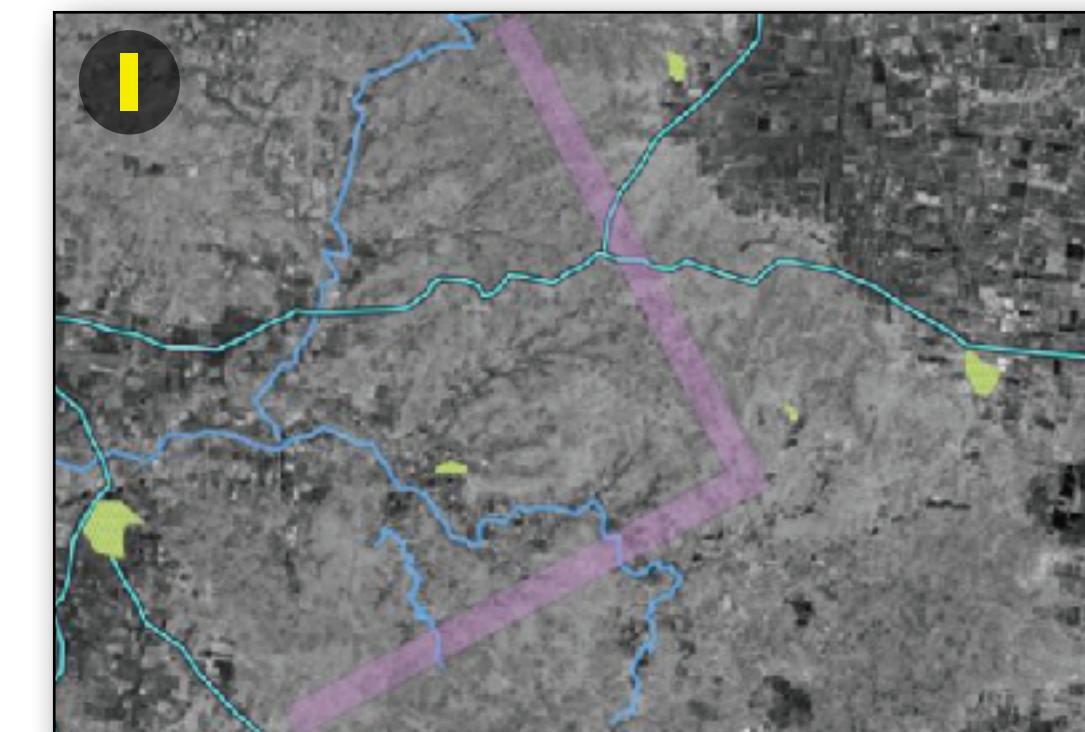
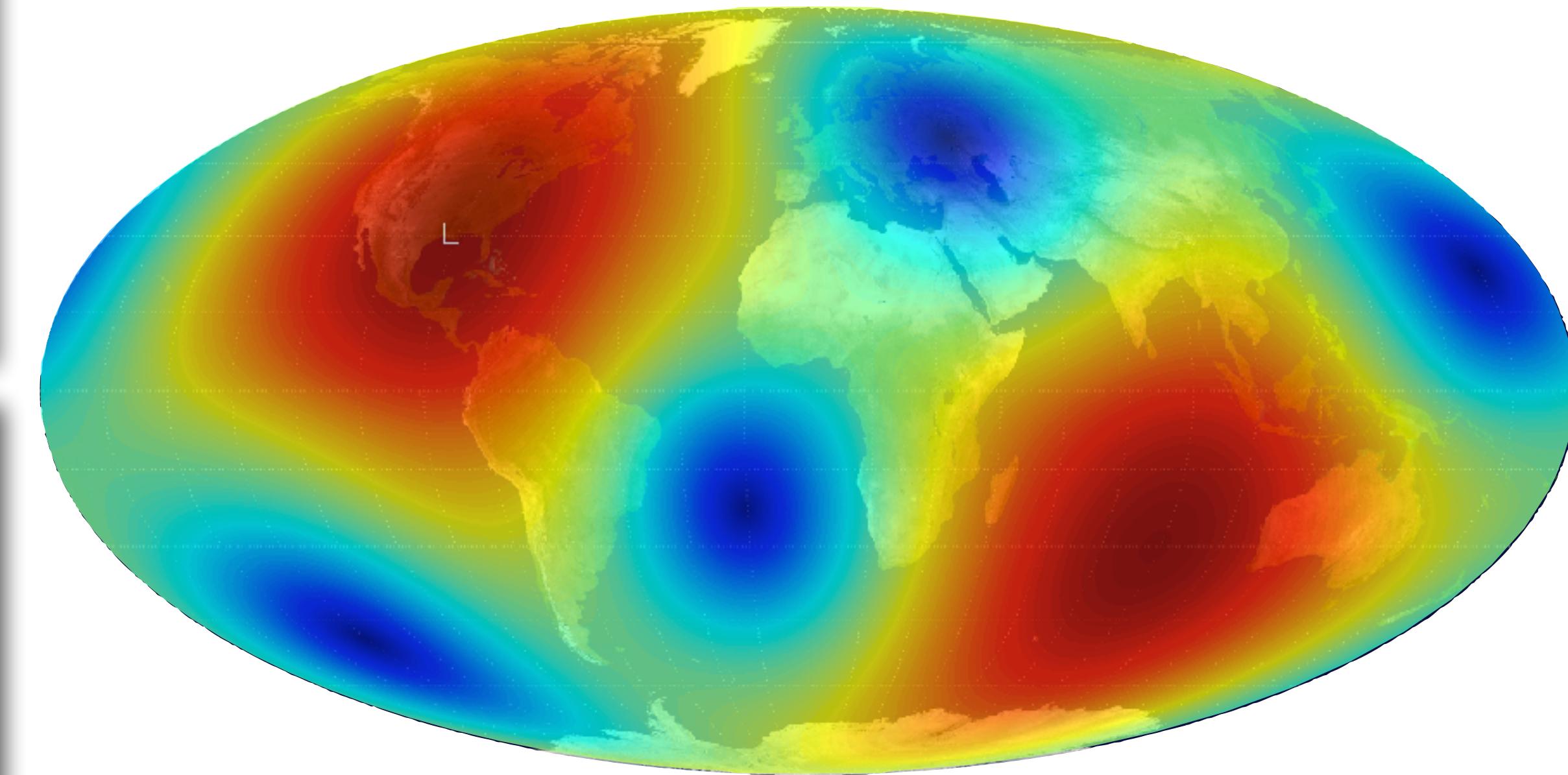
# Terrestrial Network



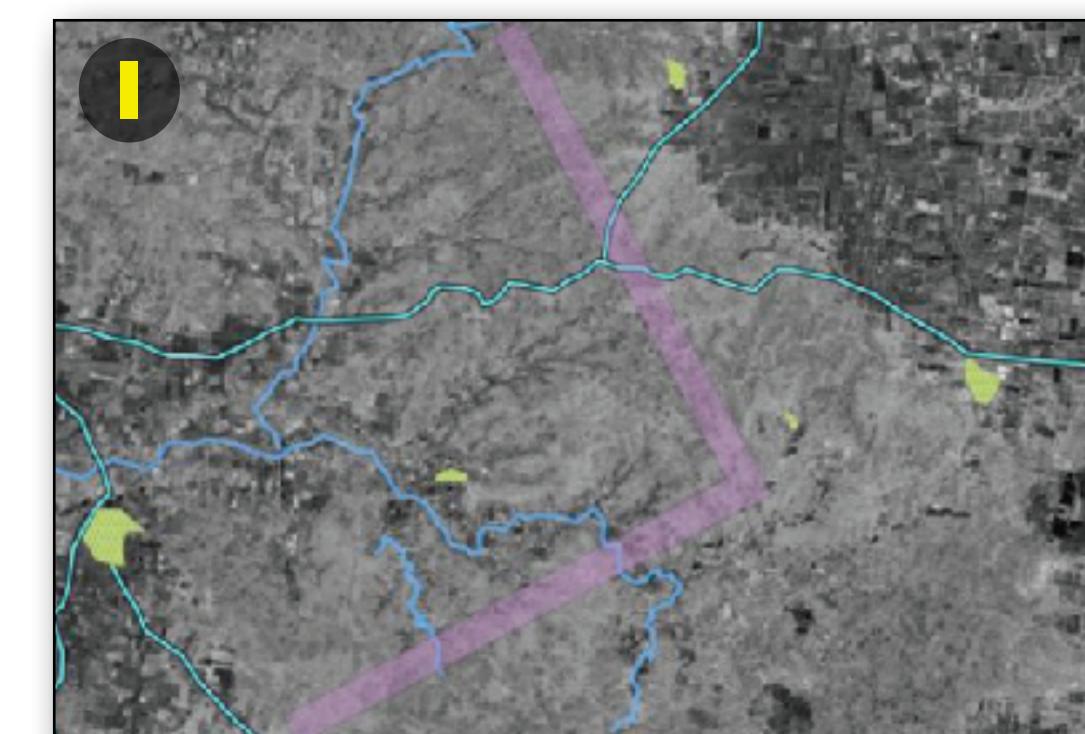
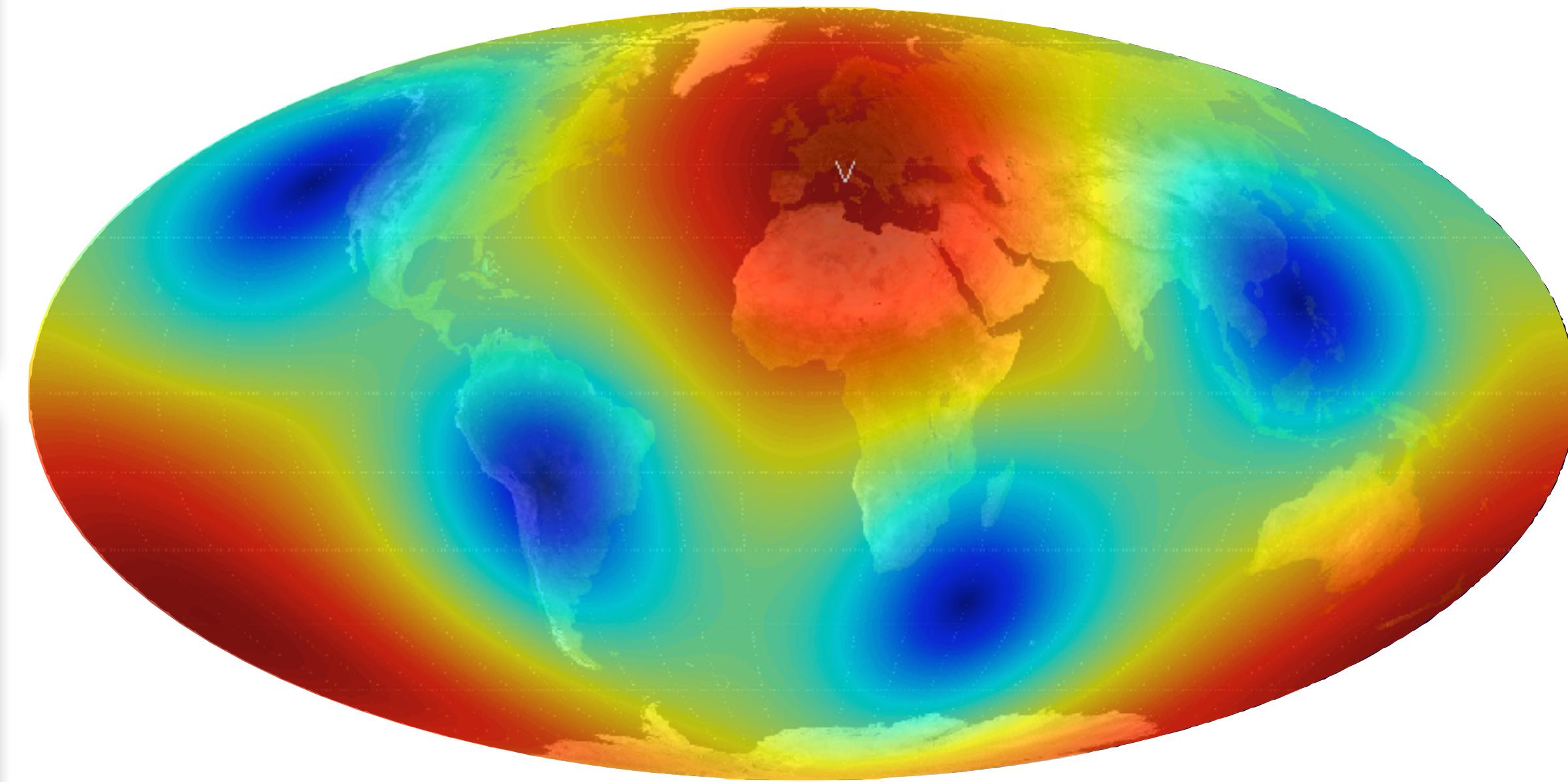
(credit: N. Cornish)



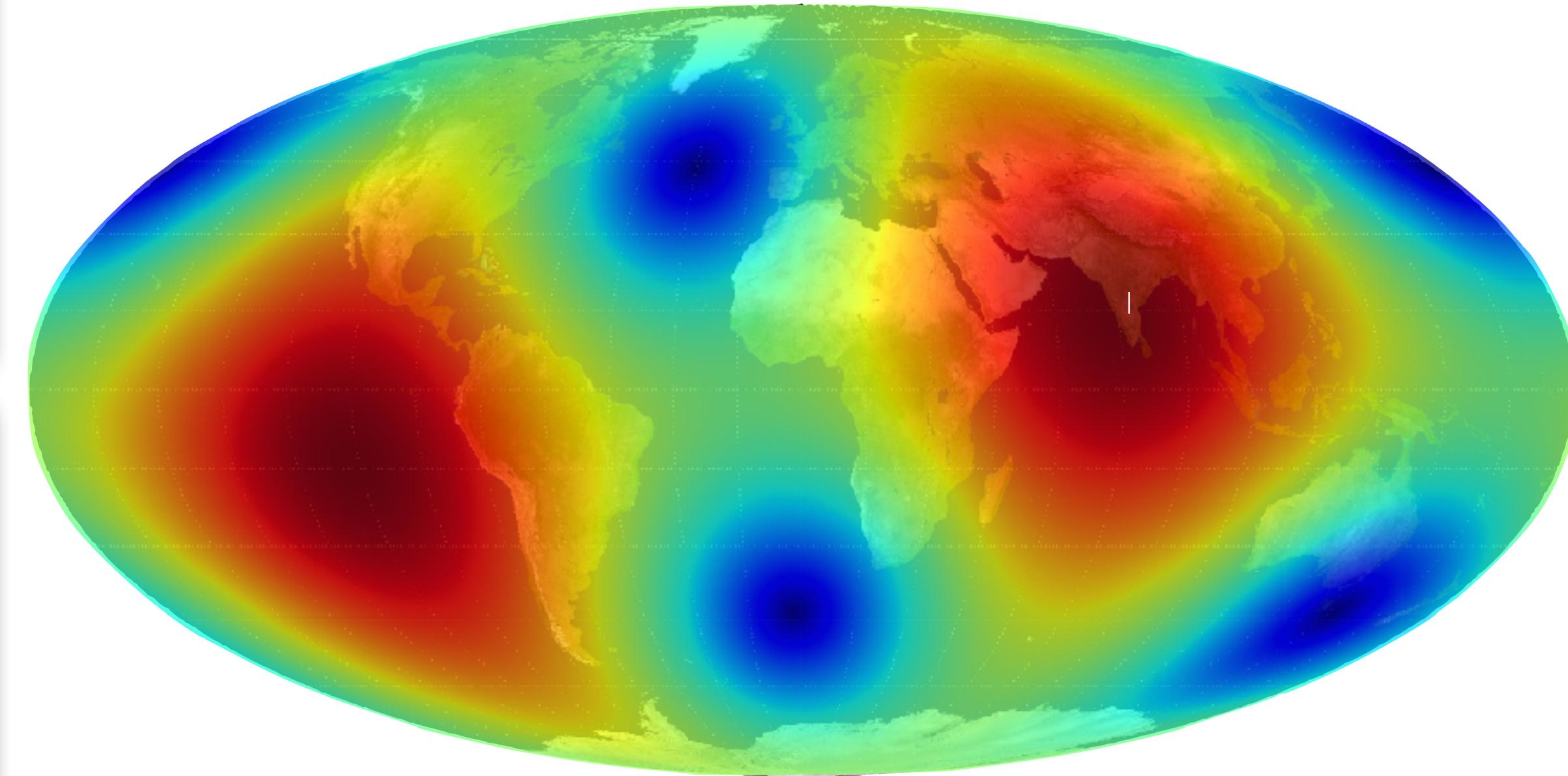
# Terrestrial Network



# Terrestrial Network



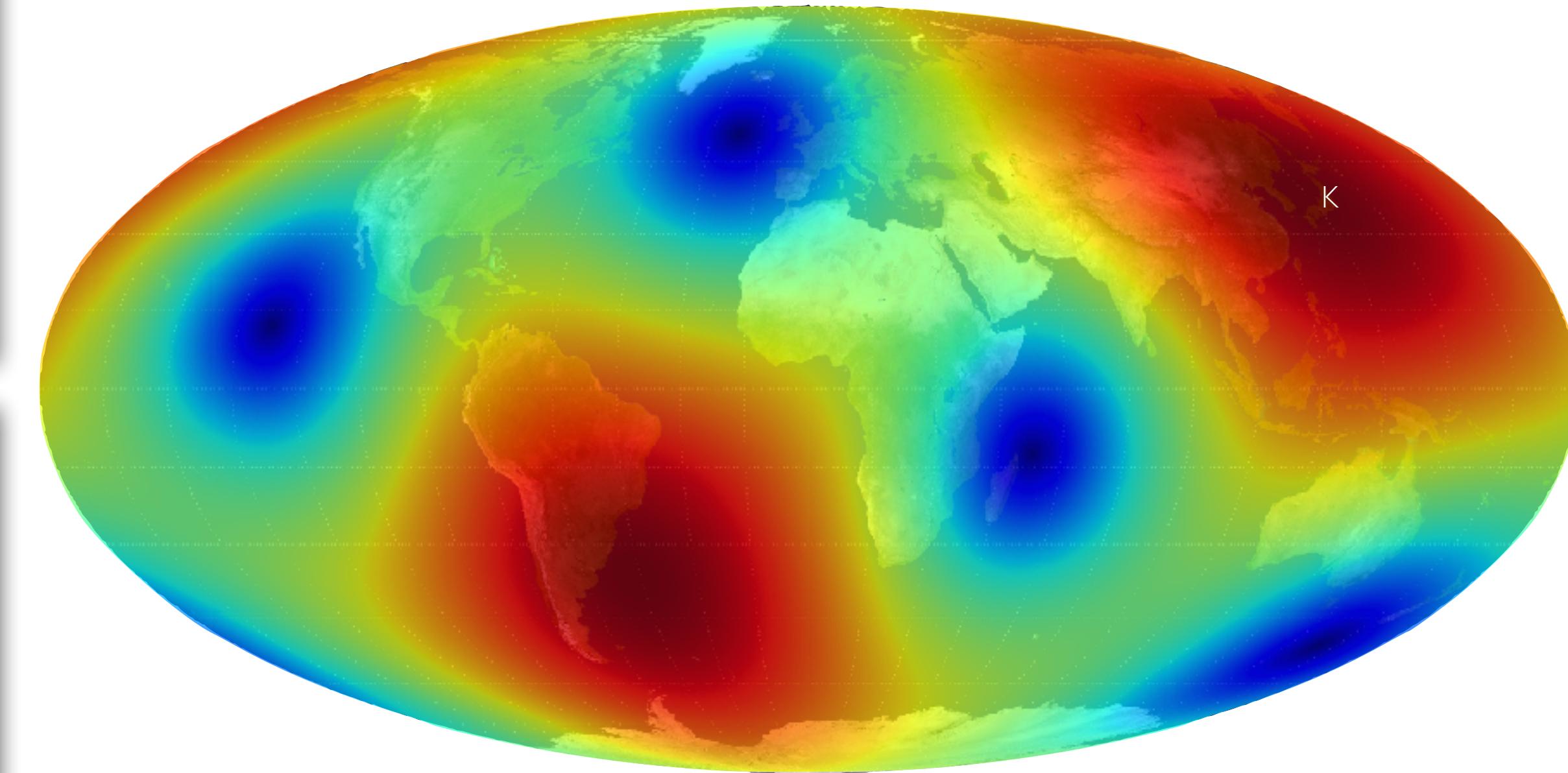
# Terrestrial Network



(credit: N. Cornish)



# Terrestrial Network



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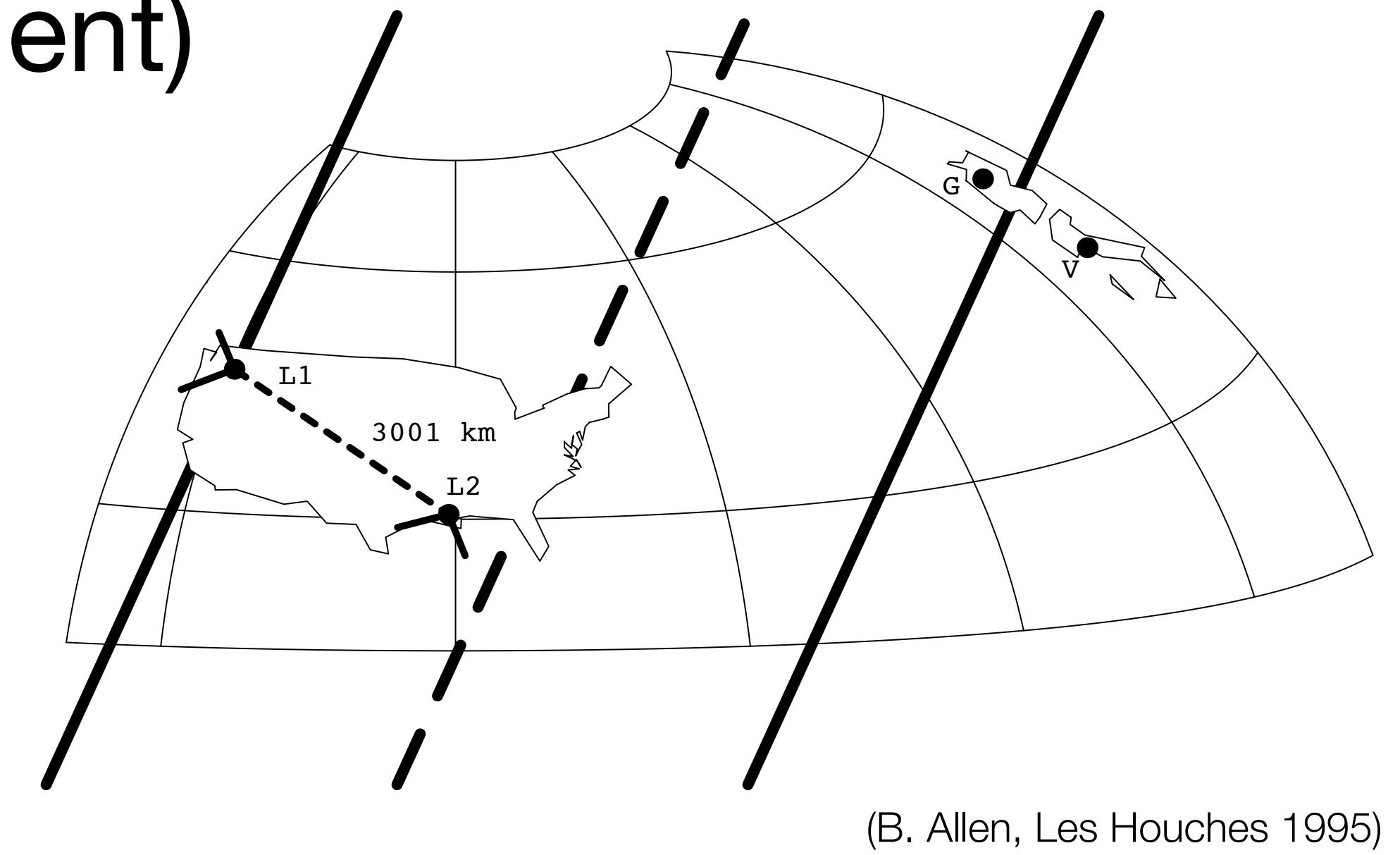


## 2. Non-trivial correlated response

# Overlap function (correlation coefficient)

- Detectors in **different locations** and with **different orientations** respond differently to a passing GW.
- Overlap function encodes reduction in sensitivity of a cross-correlation analysis due to **separation** and **misalignment** of the detectors

Expected correlation:



(B. Allen, Les Houches 1995)

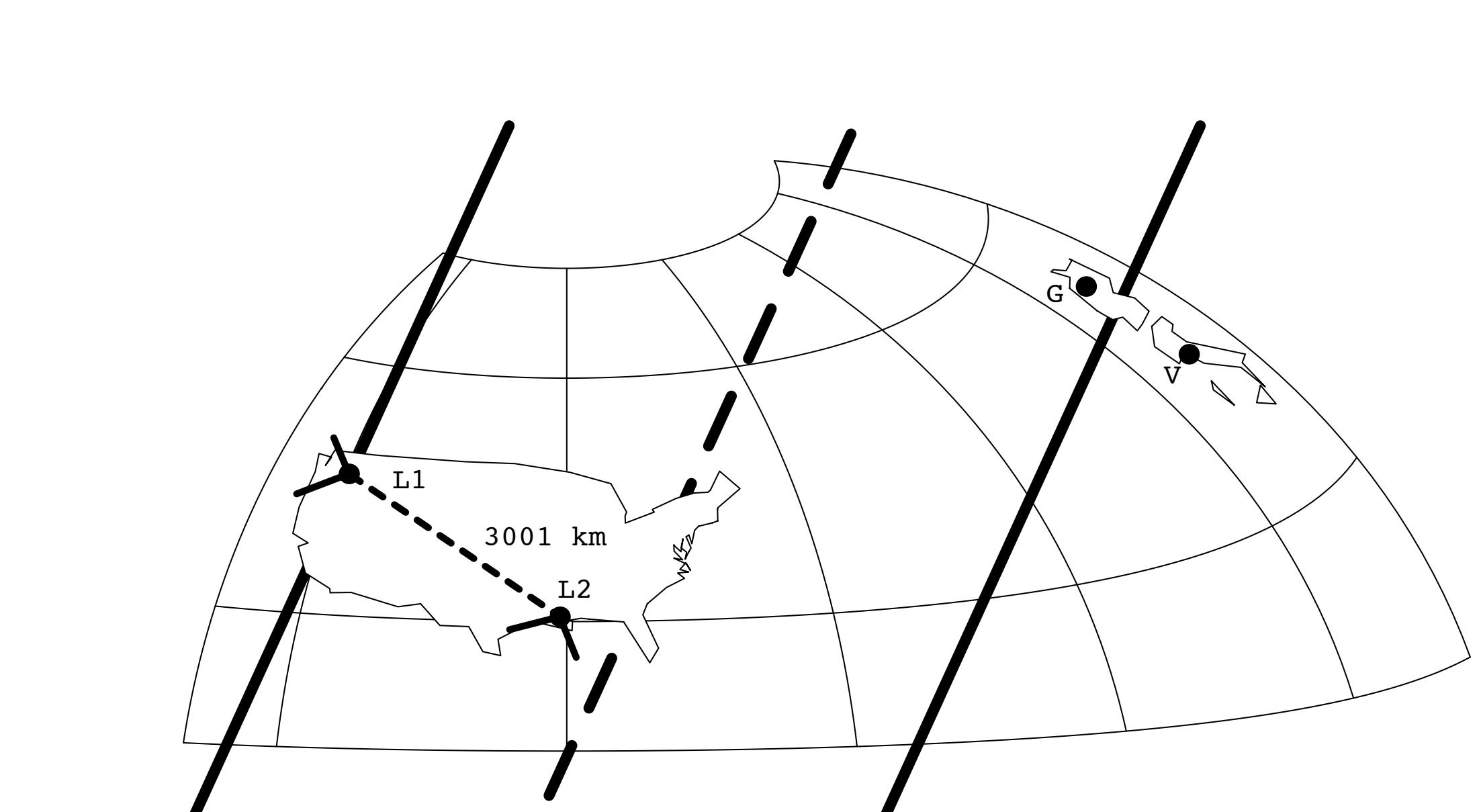
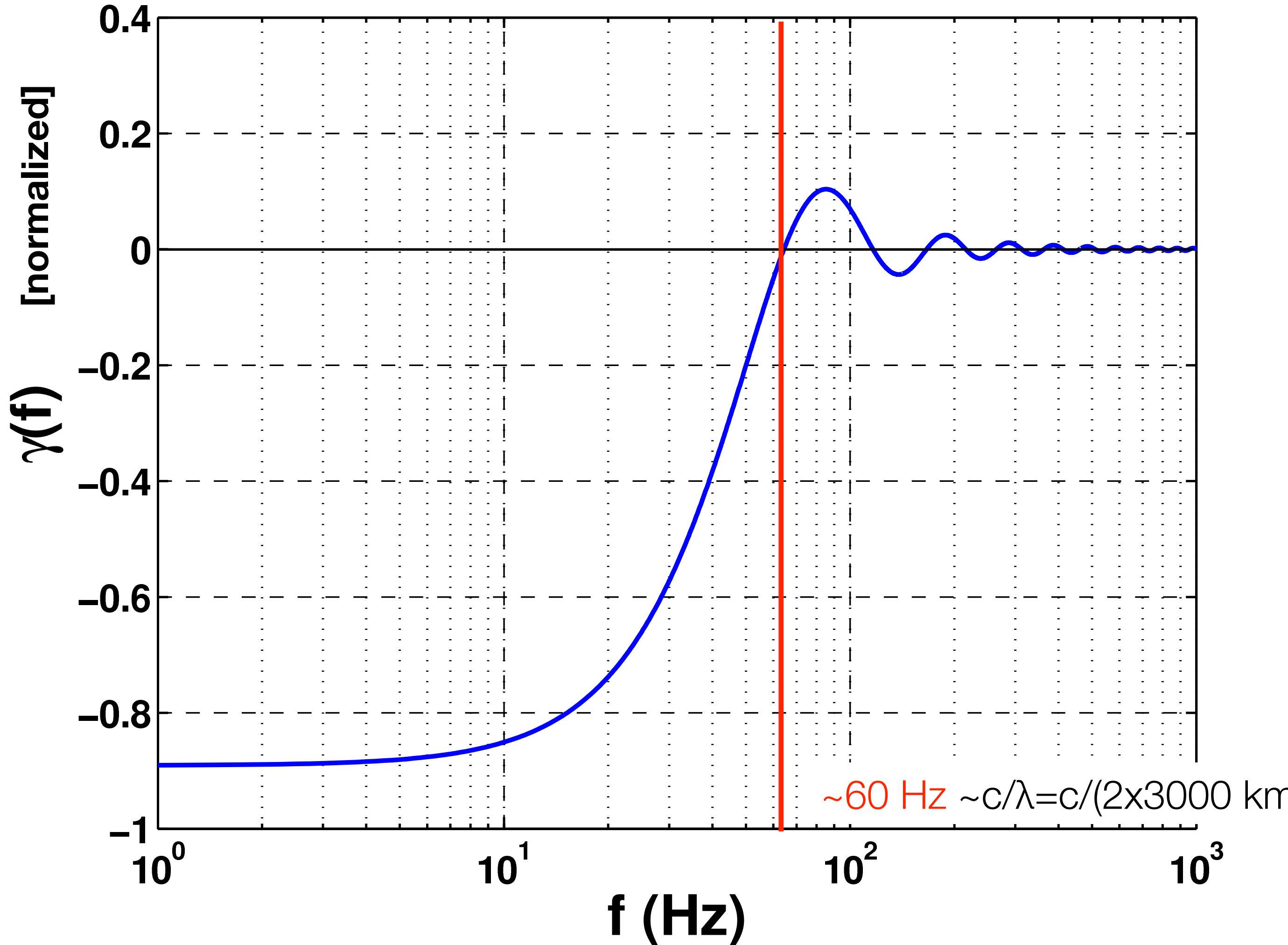
$$\langle h_I(t)h_J(t') \rangle = \frac{1}{2} \int_{-\infty}^{\infty} df e^{i2\pi f(t-t')} \Gamma_{IJ}(f) S_h(f) \quad \Leftrightarrow \quad \langle \tilde{h}_I(f)\tilde{h}_J^*(f') \rangle = \frac{1}{2} \delta(f-f') \Gamma_{IJ}(f) S_h(f)$$

$$\Gamma_{IJ}(f) = \frac{1}{8\pi} \int d^2\Omega_{\hat{n}} \sum_A R_I^A(f, \hat{n}) R_J^{A*}(f, \hat{n})$$

**(unpolarized, stationary, isotropic background)**

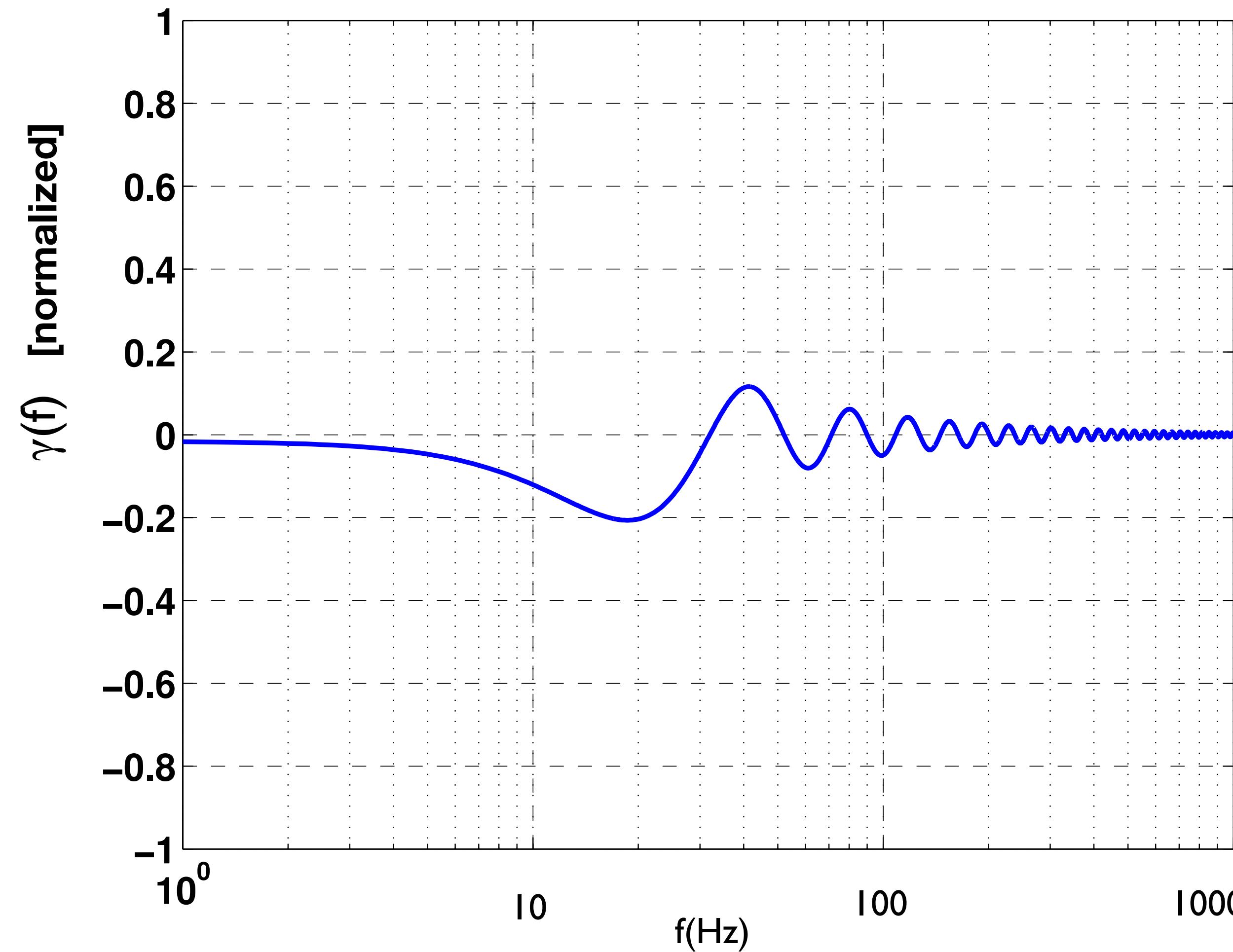
$\Gamma_{IJ}(f)$  is the **transfer function** between GW power and detector cross-power; **integrand** of  $\Gamma_{IJ}(f)$  is important for **anisotropic** stochastic backgrounds

# LIGO Hanford-LIGO Livingston overlap function (small-antenna approximation)

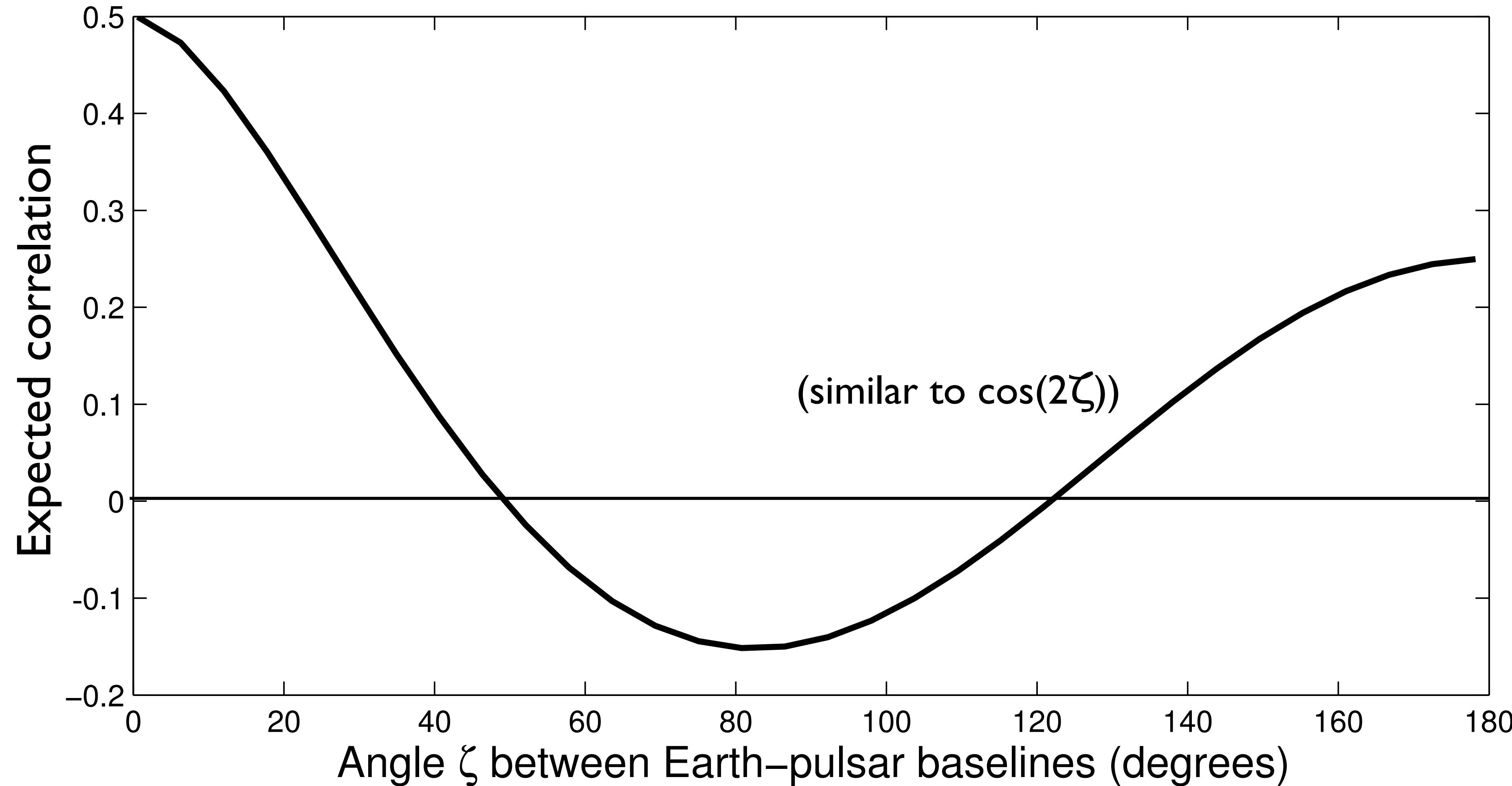


(B. Allen, Les Houches 1995)

# LIGO Hanford-Virgo overlap function (small-antenna approximation)



Pulsar timing correlations (Hellings & Downs curve)  
(correlation for an isotropic, unpolarized GW background in GR)

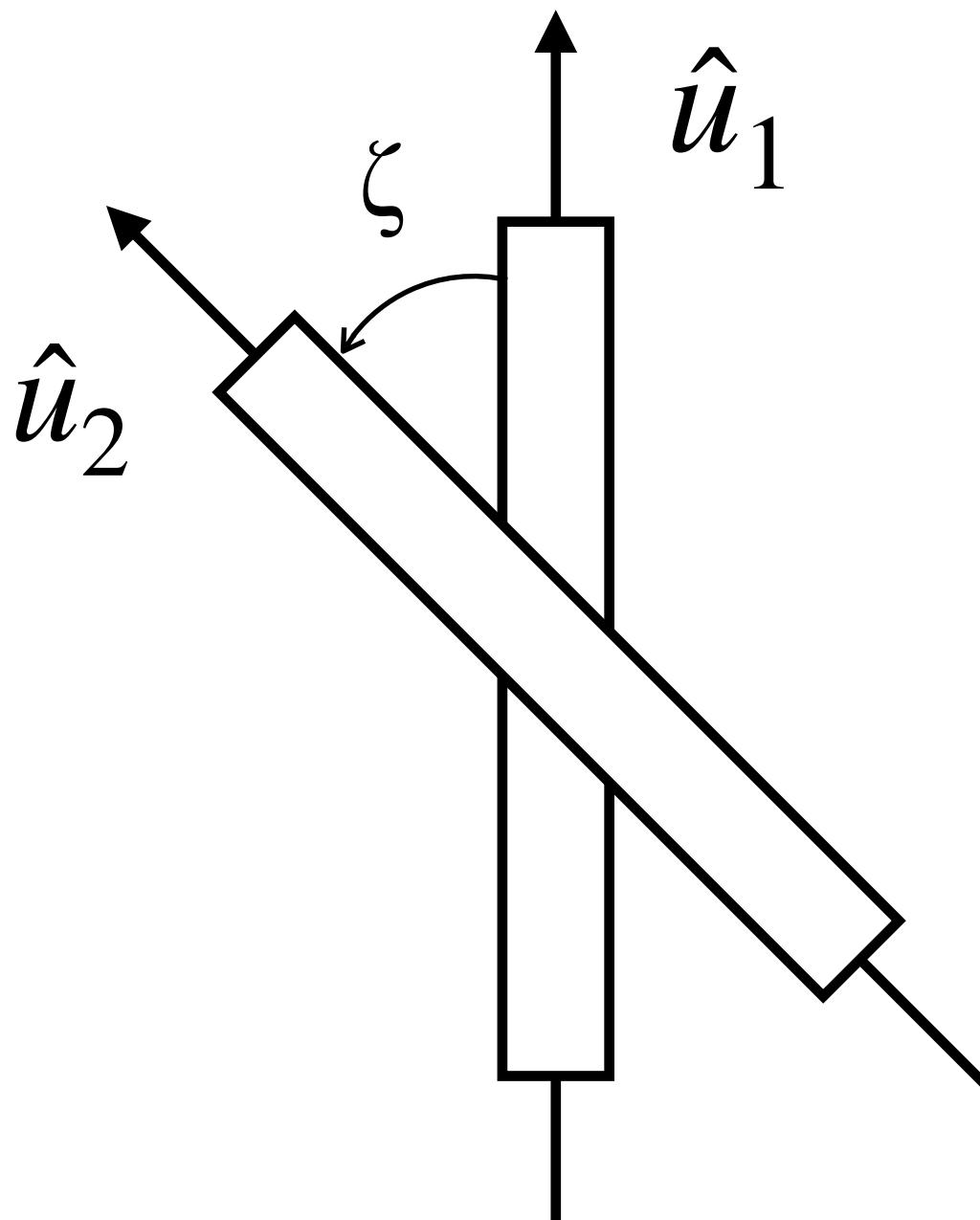


**Exercise 7: Show that the overlap function for a pair of short, colocated electric dipole antennae pointing in direction  $\hat{u}_1$  and  $\hat{u}_2$  is given by**

$$\Gamma_{12} \propto \hat{u}_1 \cdot \hat{u}_2 \equiv \cos \zeta$$

**for an unpolarized, isotropic electromagnetic field.**

Jenet and Romano, AJP 83 (7), 2015



Hint:  $r_I(t) = \hat{u}_I \cdot \vec{E}(t, \vec{x}_0)$

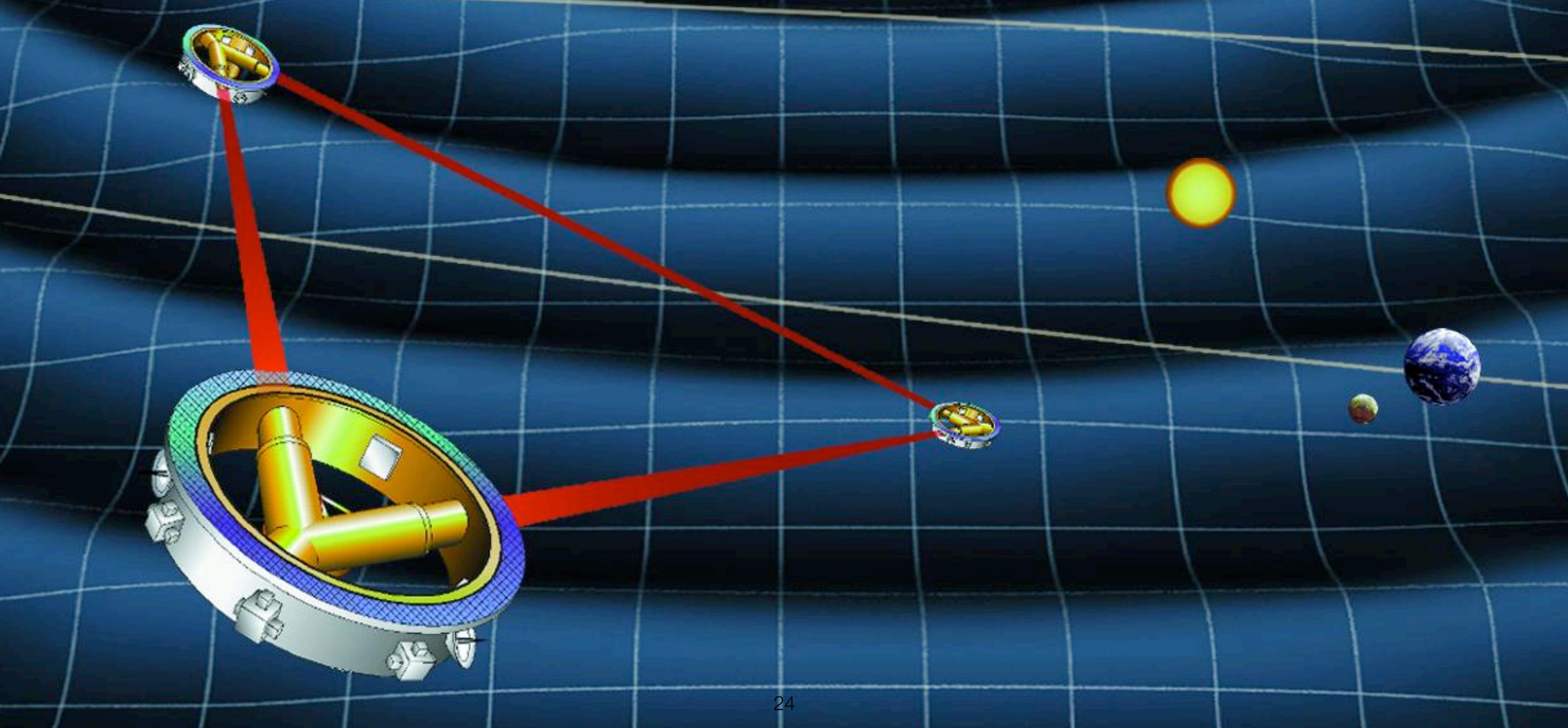
$$\vec{E}(t, \vec{x}) = \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{n}} \sum_{\alpha=1}^2 \tilde{E}_{\alpha}(f, \hat{n}) \hat{e}_{\alpha}(\hat{n}) e^{i2\pi f(t + \hat{n} \cdot \vec{x}/c)}$$

$$\hat{e}_1(\hat{n}) = \hat{\theta}, \quad \hat{e}_2(\hat{n}) = \hat{\phi}$$

etc. ...

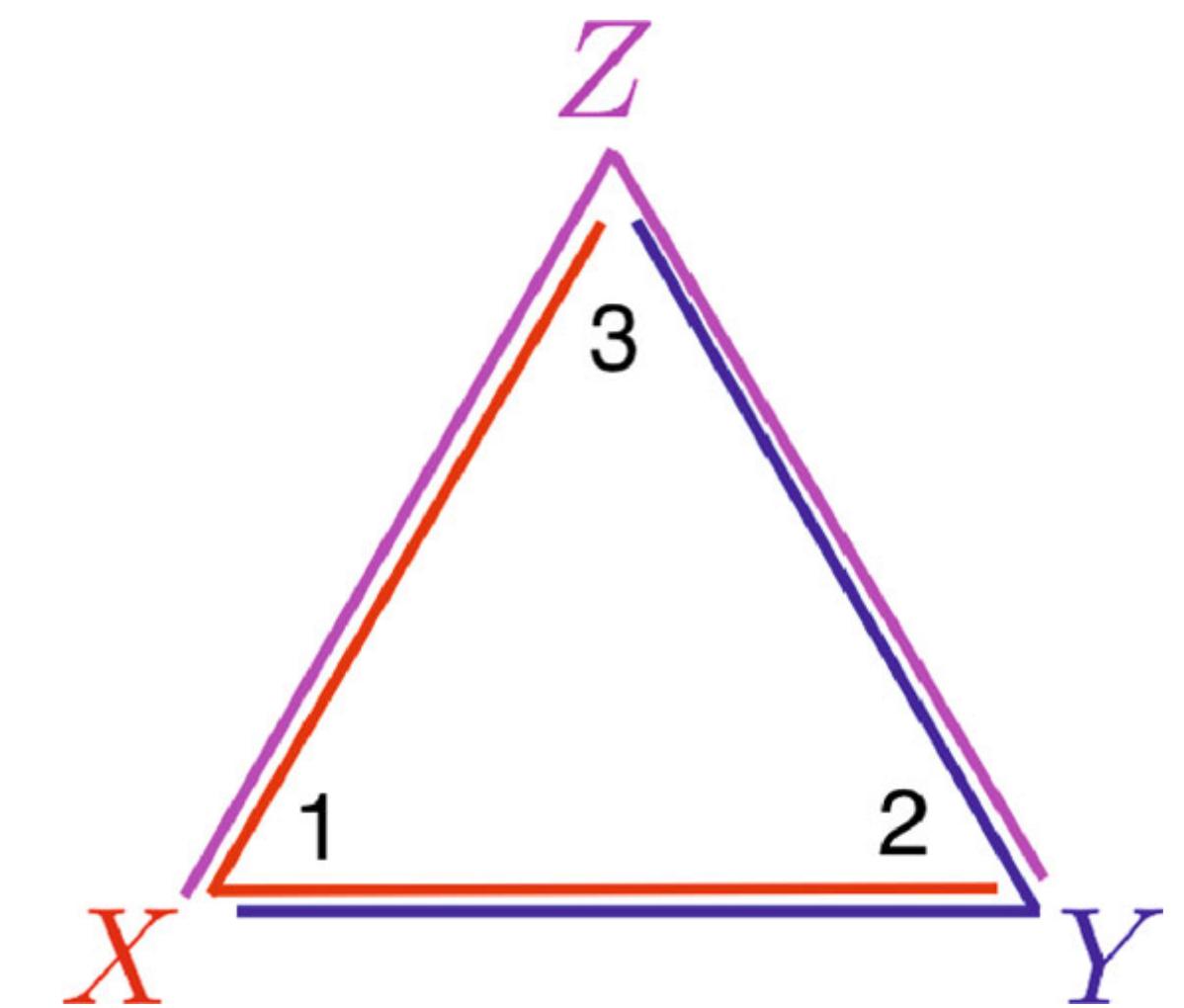
3. What to do in the absence of correlations (e.g., for LISA)?

# LISA (Laser Interferometer Space Antenna)



# Cross-correlation is not an option for LISA (at least for low frequencies)

- Although there are 3 Michelson combinations ( $X, Y, Z$ ), they have **common noise** (since they share arms)
- Can diagonalize the noise covariance matrix to obtain **noise-orthogonal** combinations ( $A, E, T$ ), which also turn out to be **signal orthogonal**
  - $A, E$ : two Michelsons **rotated by 45 degrees**
  - $T$ : relatively insensitive to GW (**null channel**)
- Nonetheless, **proper modeling** of **instrumental noise**, astrophysical **foregrounds** (galactic WD binaries), and **GWB** allows you to discriminate all three components (Adams & Cornish, 2010, 2014)

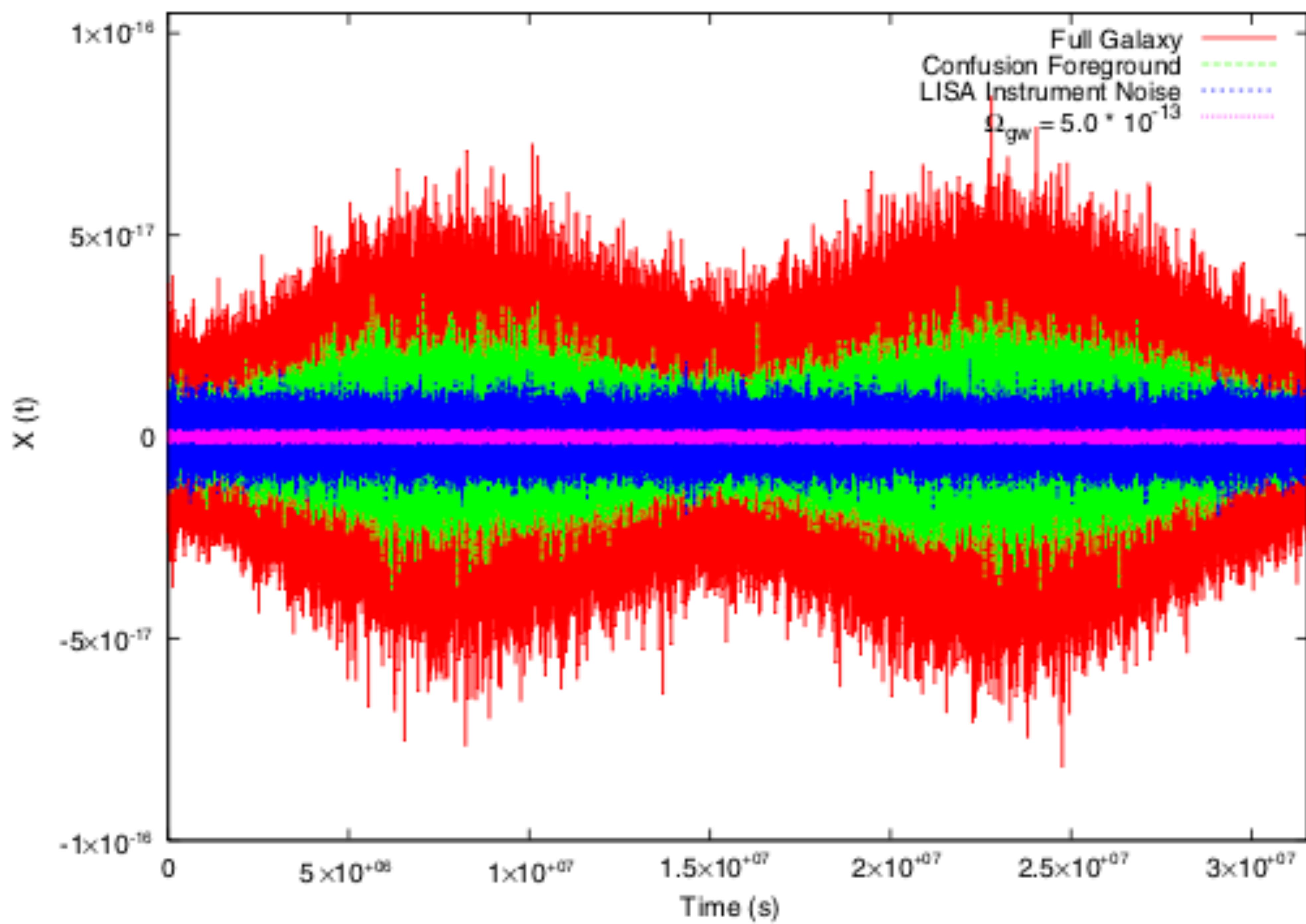


$$A \equiv \frac{1}{3}(2X - Y - Z),$$

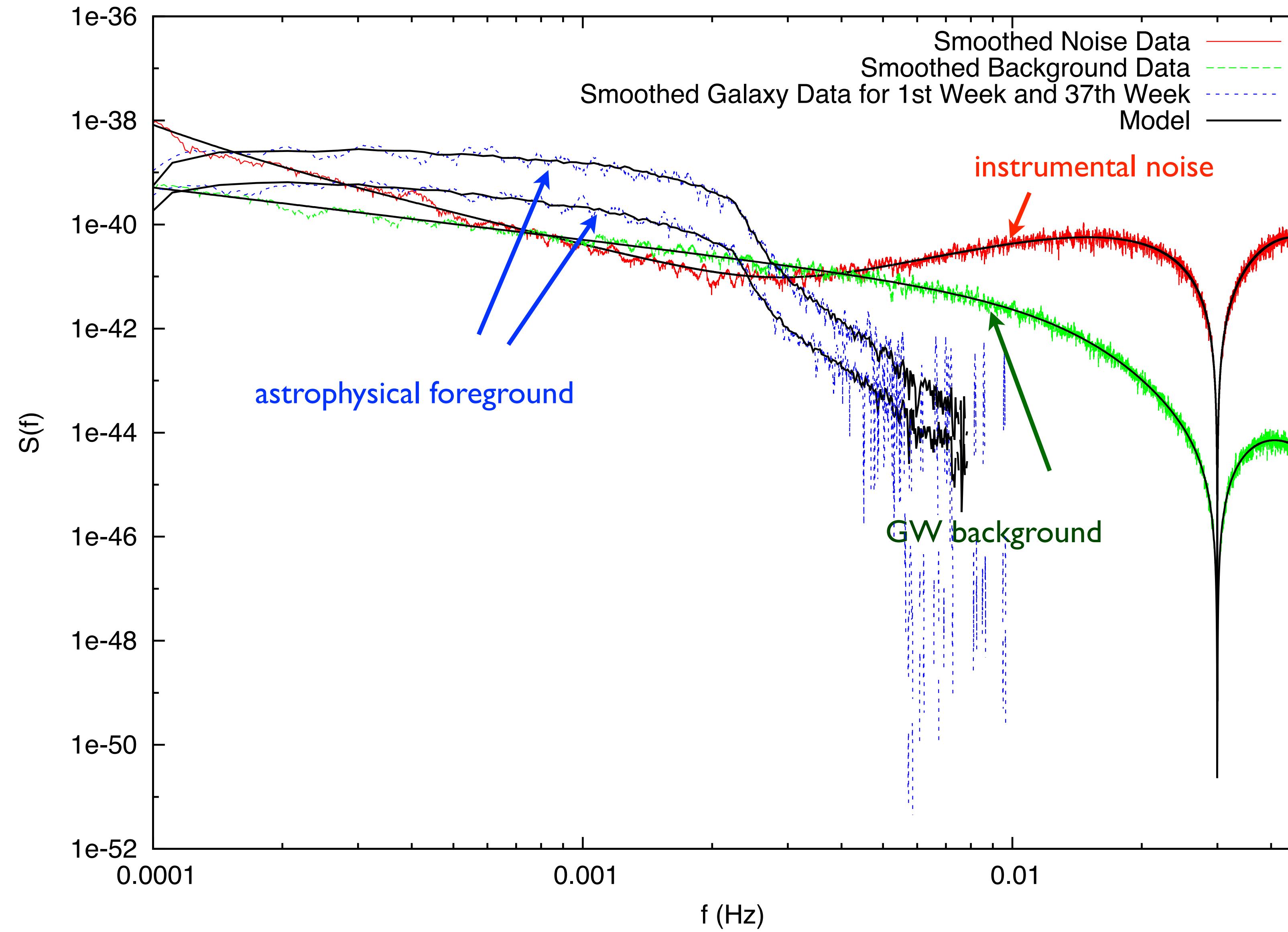
$$E \equiv \frac{1}{\sqrt{3}}(Z - Y),$$

$$T \equiv \frac{1}{3}(X + Y + Z).$$

**Detailed questions? Ask Neil when he arrives!**



# Different spectra => differentiate different noise components



# 4. Frequentist and Bayesian methods

Frequentist statistics	Bayesian inference
Probabilities are <b>long-run relative occurrences</b> of outcomes of repeatable expts —> can't be assigned to hypotheses	Probabilities are <b>degree of belief</b> —> can be assigned to hypotheses
Usually start with a <b>likelihood function</b> $p(d H)$	Same as frequentist
Construct <b>statistics</b> for parameter estimation / hypothesis testing	Specify <b>priors</b> for parameters and hypotheses
Calculate probability distribution of the statistics (e.g., using time slide)	Use <b>Bayes' theorem</b> to update degree of belief
Calculates <b>confidence intervals</b> and <b>p-values</b>	Construct <b>postriors</b> and <b>odds ratios (Bayes factors)</b>

# Likelihood function

Starting point for both frequentist & Bayesian analyses:

$$\text{likelihood} = p(\text{data} \mid \text{parameters, model})$$

Gaussian detector noise and GWB:

$$p(d \mid C_n, \mathcal{M}_0) = \frac{1}{\sqrt{\det(2\pi C_n)}} \exp \left[ -\frac{1}{2} d^T C_n^{-1} d \right] \quad (\text{noise-only model})$$

$$p(d \mid C_n, S_h, \mathcal{M}_1) = \frac{1}{\sqrt{\det(2\pi C)}} \exp \left[ -\frac{1}{2} d^T C^{-1} d \right] \quad (\text{signal+noise model})$$

N samples of white noise, white GWB, in two colocated and coaligned detectors:

$$C_n = \begin{bmatrix} S_{n_1} \mathbf{1}_{N \times N} & \mathbf{0}_{N \times N} \\ \mathbf{0}_{N \times N} & S_{n_2} \mathbf{1}_{N \times N} \end{bmatrix} \quad \& \quad C = \begin{bmatrix} (S_{n_1} + S_h) \mathbf{1}_{N \times N} & S_h \mathbf{1}_{N \times N} \\ S_h \mathbf{1}_{N \times N} & (S_{n_2} + S_h) \mathbf{1}_{N \times N} \end{bmatrix}$$

# Frequentist analysis

Use maximum-likelihood (ML) ratio for detection, and maximum-likelihood parameter values as estimators

Maximum-likelihood detection statistic:

$$\Lambda_{\text{ML}}(d) \equiv \frac{\max_{S_{n_1}, S_{n_2}, S_h} p(d | S_{n_1}, S_{n_2}, S_h, \mathcal{M}_1)}{\max_{S_{n_1}, S_{n_2}} p(d | S_{n_1}, S_{n_2}, \mathcal{M}_0)}$$
$$\Lambda(d) \equiv 2 \ln(\Lambda_{\text{ML}}(d)) \simeq \frac{\hat{S}_h^2}{\hat{S}_{n_1} \hat{S}_{n_2} / N} \quad \longleftarrow \text{SNR}^2$$

Maximum-likelihood estimators:

$$\hat{S}_h \equiv \frac{1}{N} \sum_{i=1}^N d_{1i} d_{2i} \quad \longleftarrow \text{cross-correlation statistic}$$

$$\hat{S}_{n_1} \equiv \frac{1}{N} \sum_{i=1}^N d_{1i}^2 - \hat{S}_h$$

$$\hat{S}_{n_2} \equiv \frac{1}{N} \sum_{i=1}^N d_{2i}^2 - \hat{S}_h$$

**Exercise 8: Verify the expressions for the ML estimators.**

**Exercise 9: Verify the expression for the detection statistic**  $2 \ln(\Lambda_{\text{ML}}(d))$

# Bayesian analysis

Use Bayes' theorem to calculate posterior distributions for parameter estimation and odds ratios (Bayes factors) for model selection

# Bayes' theorem:

$$p(H|d) = \frac{p(d|H)p(H)}{p(d)}$$

likelihood

prior

normalization factor

## Posteriors:

$$p(S_{n_1}, S_{n_2}, S_h | d, \mathcal{M}_1) = \frac{p(d | S_{n_1}, S_{n_2}, S_h, \mathcal{M}_1)p(S_{n_1}, S_{n_2}, S_h | \mathcal{M}_1)}{p(d | \mathcal{M}_1)}$$

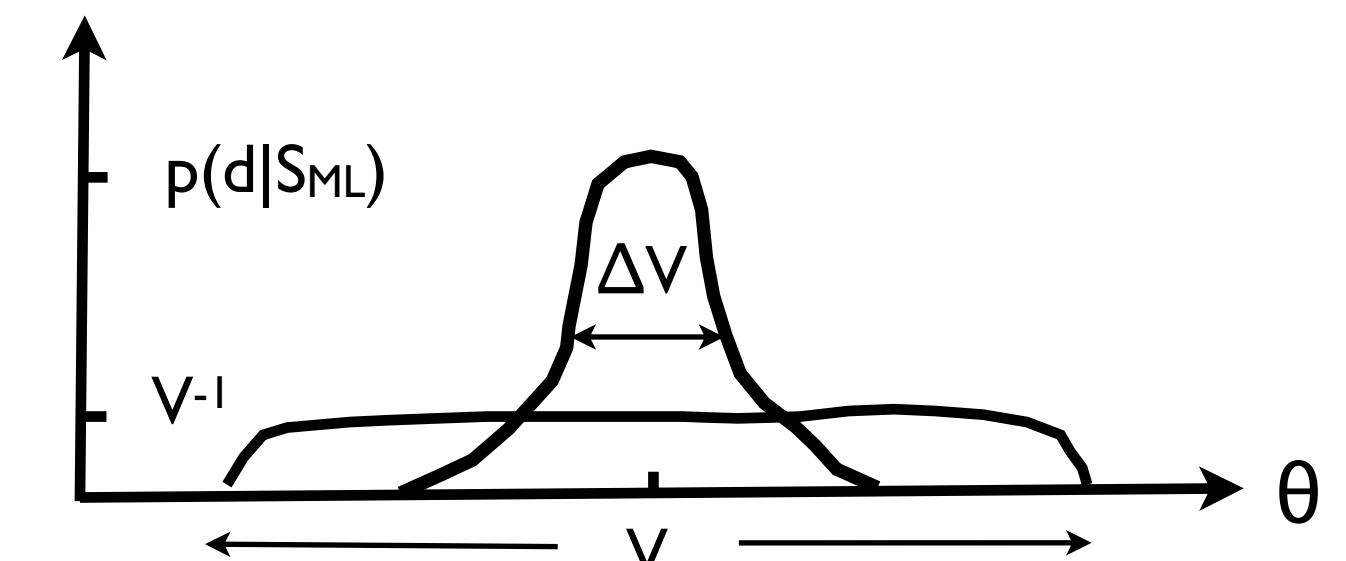
## Model selection:

$$\frac{p(\mathcal{M}_1 | d)}{p(\mathcal{M}_0 | d)} = \frac{p(d | \mathcal{M}_1) p(\mathcal{M}_1)}{p(d | \mathcal{M}_0) p(\mathcal{M}_0)}$$

Bayes factor  $\mathcal{B}_{10}(d)$

# Relationship to frequentist approach:

$$\mathcal{B}_{10}(d) \equiv \frac{p(d | \mathcal{M}_1)}{p(d | \mathcal{M}_0)} = \frac{\int dS_{n_1} \int dS_{n_2} \int dS_h p(d | S_{n_1}, S_{n_2}, S_h, \mathcal{M}_1) p(S_{n_1}, S_{n_2}, S_h | \mathcal{M}_1)}{\int dS_{n_1} \int dS_{n_2} p(d | S_{n_1}, S_{n_2}, \mathcal{M}_0) p(S_{n_1}, S_{n_2} | \mathcal{M}_0)} \simeq \Lambda_{\text{ML}}(d) \frac{\Delta V_1 / V_1}{\Delta V_0 / V_0}$$



# Example: Derivation of standard stochastic likelihood by marginalizing over a stochastic signal prior

Generic likelihood:

$$p(d | C_n, h) \equiv p_n(d - h | C_n) = \frac{1}{\sqrt{\det(2\pi C_n)}} \exp \left[ -\frac{1}{2} (d - h)^T C_n^{-1} (d - h) \right]$$

↑  
signal model

↑  
covariance matrix for noise, e.g.,  $C_n = \begin{bmatrix} S_{n_1} & 0 \\ 0 & S_{n_2} \end{bmatrix}$

stochastic signal model:

$$p(h | S_h) = \frac{1}{\sqrt{2\pi S_h}} \exp \left[ -\frac{1}{2} \frac{h^2}{S_h} \right]$$

Marginalized likelihood:

$$p(d | C_n, S_h) = \int dh p_n(d - h | C_n) p(h | S_h) = \frac{1}{\sqrt{\det(2\pi C)}} \exp \left[ -\frac{1}{2} d^T C^{-1} d \right]$$

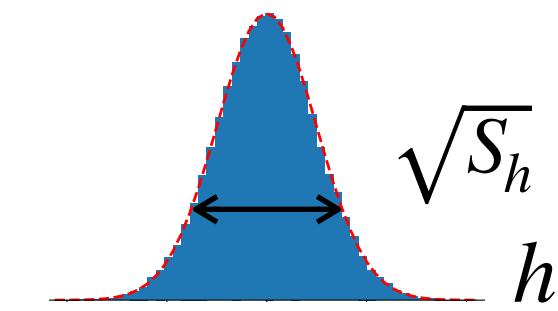
covariance matrix  
for signal + noise  $\longrightarrow C = \begin{bmatrix} S_{n_1} + S_h & S_h \\ S_h & S_{n_2} + S_h \end{bmatrix}$

**Exercise 10: Do the marginalization over h to obtain this final result.**

# Signal priors define the signal model...

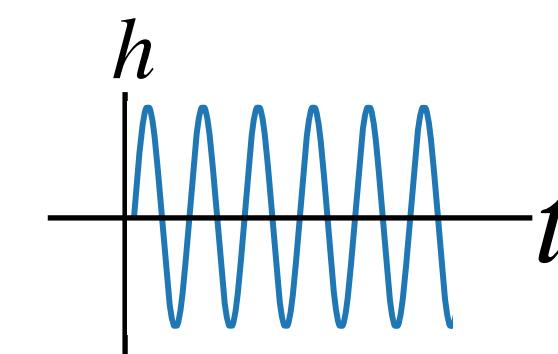
stochastic:

$$p(h | S_h) = \frac{1}{\sqrt{2\pi S_h}} \exp \left[ -\frac{1}{2} \frac{h^2}{S_h} \right]$$



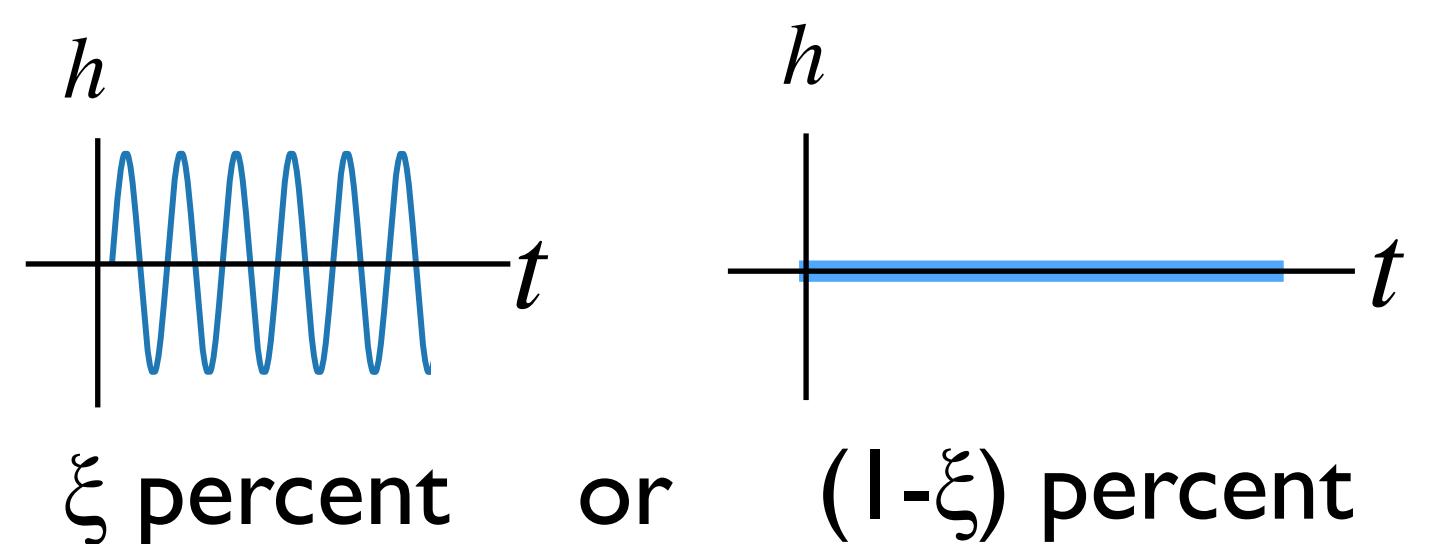
deterministic:

$$p(h | A, t_0, f_0) = \delta \left( h - A \sin[2\pi f_0(t - t_0)] \right)$$



hybrid:

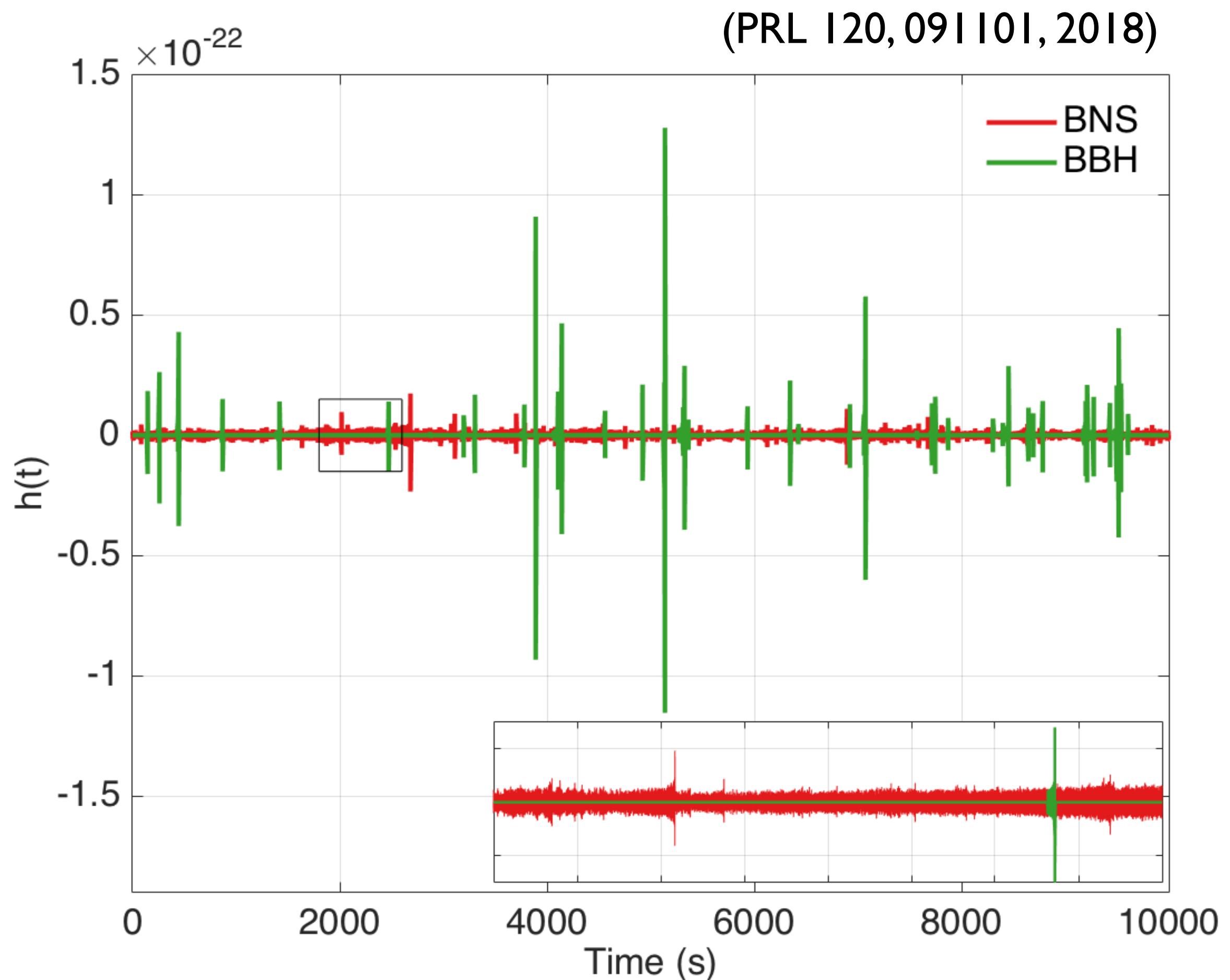
$$p(h | \xi, A, t_0, f_0) = \xi \delta \left( h - A \sin[2\pi f_0(t - t_0)] \right) + (1 - \xi) \delta(h)$$



# 5. Example: searching for the background from BBH mergers

# Recall: Non-stationary background from BBH mergers is a potential signal for advanced LIGO,Virgo

- Recent detections of **BBH** and **BNS** mergers by advanced LIGO, Virgo imply the existence of a stochastic background of weaker events
- Smith & Thrane (PRX 8, 021019,2018) have proposed an alternative method to search for the BBH component, optimally suited for the **non-stationarity**
- Describe BBH background with a **hybrid signal model**
- Average over chirp parameters to **infer only rate** of mergers
- Use **two detectors** to discriminate against **glitches**



# Mathematical details

Split data in short (e.g., 4 sec) segments, which should contain at most 1 BBH merger.

For each segment we have:

Likelihood:

$$p(d | C_n, h) \equiv p_n(d - h | C_n)$$

Hybrid signal model:

$$p(h | \xi, \vec{\lambda}) = \xi \delta\left(h - \text{chirp}(\vec{\lambda})\right) + (1 - \xi) \delta(h)$$

Marginalized likelihoods:

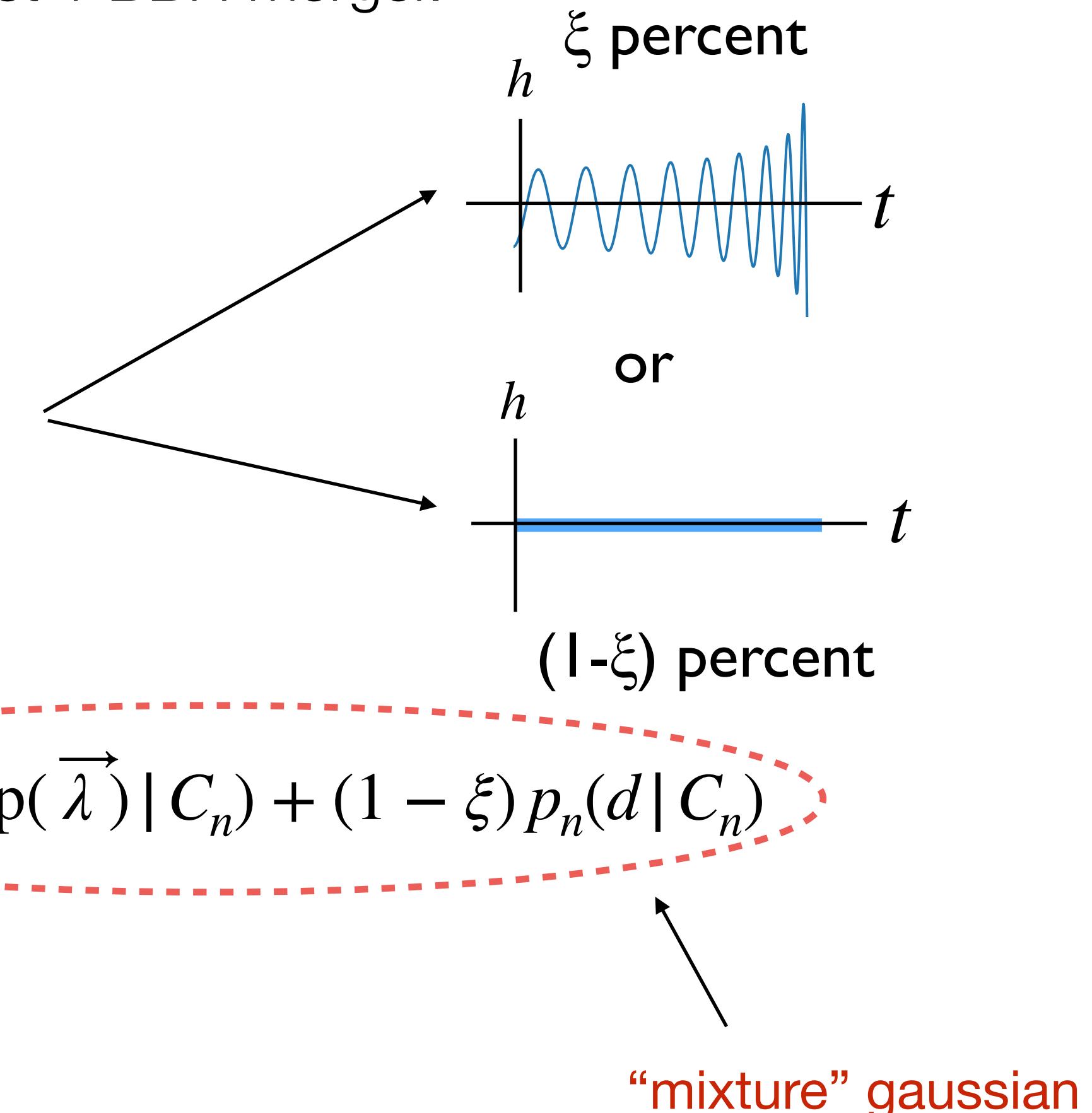
$$p(d | \xi, \vec{\lambda}) = \int dh p(d | C_n, h) p(h | \xi, \vec{\lambda}) = \xi p_n(d - \text{chirp}(\vec{\lambda}) | C_n) + (1 - \xi) p_n(d | C_n)$$

$$p(d | \xi) = \int d \vec{\lambda} p(d | \xi, \vec{\lambda}) p(\vec{\lambda}) = (S - N)\xi + N$$

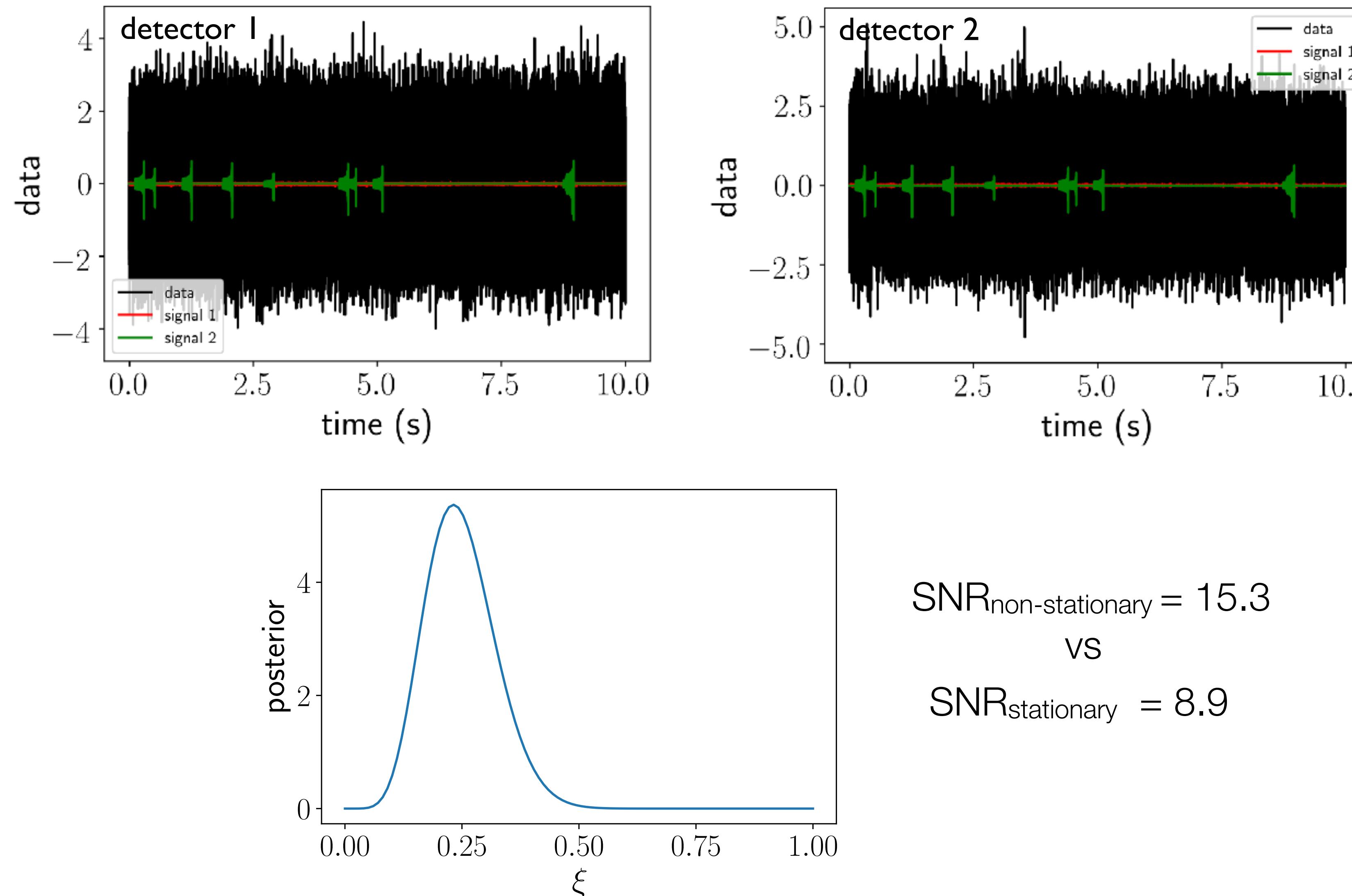
Posterior:

$$p(\xi | d) = \frac{p(d | \xi) p(\xi)}{p(d)}$$

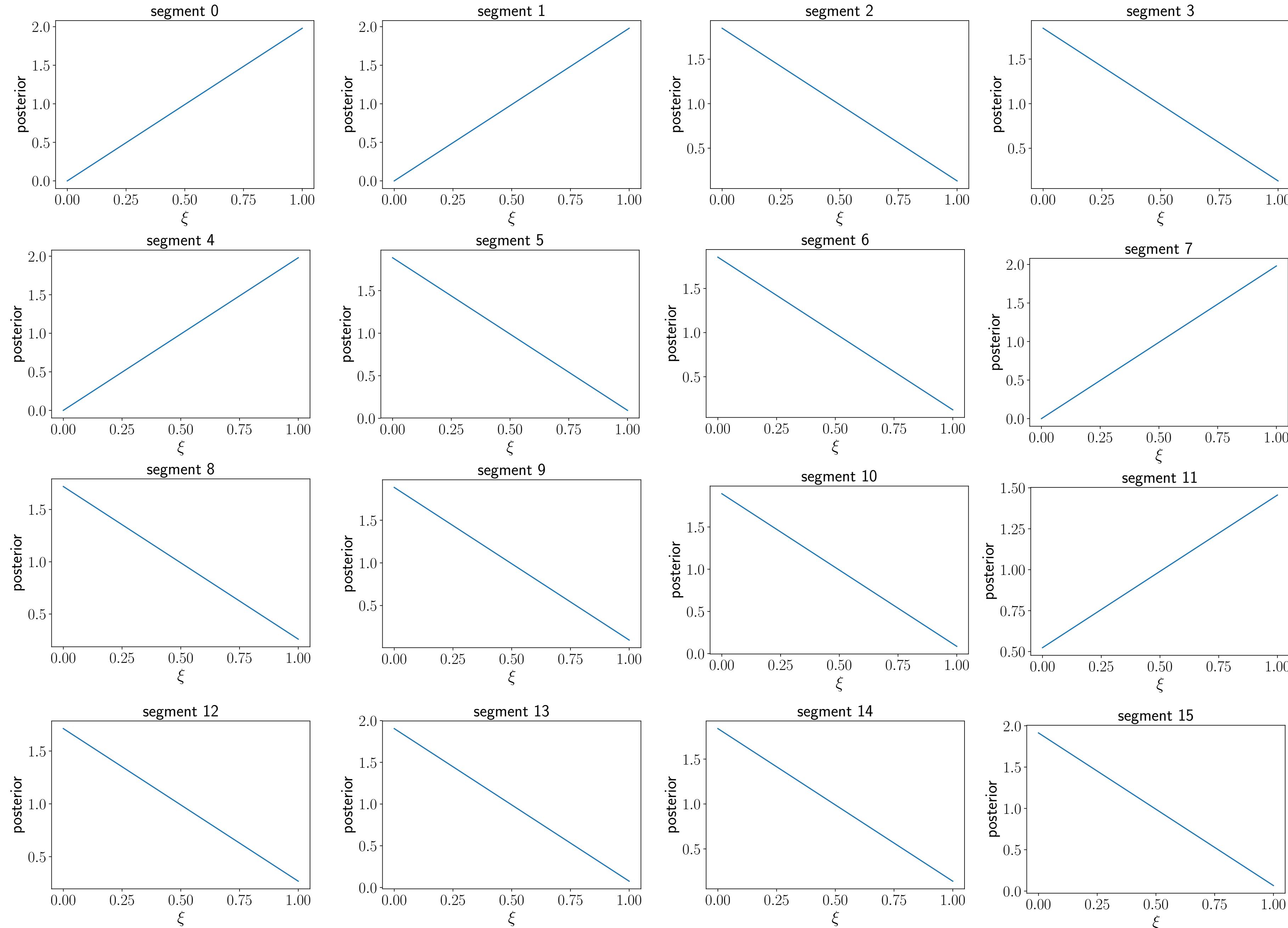
Combine segments by multiplying likelihoods, ...



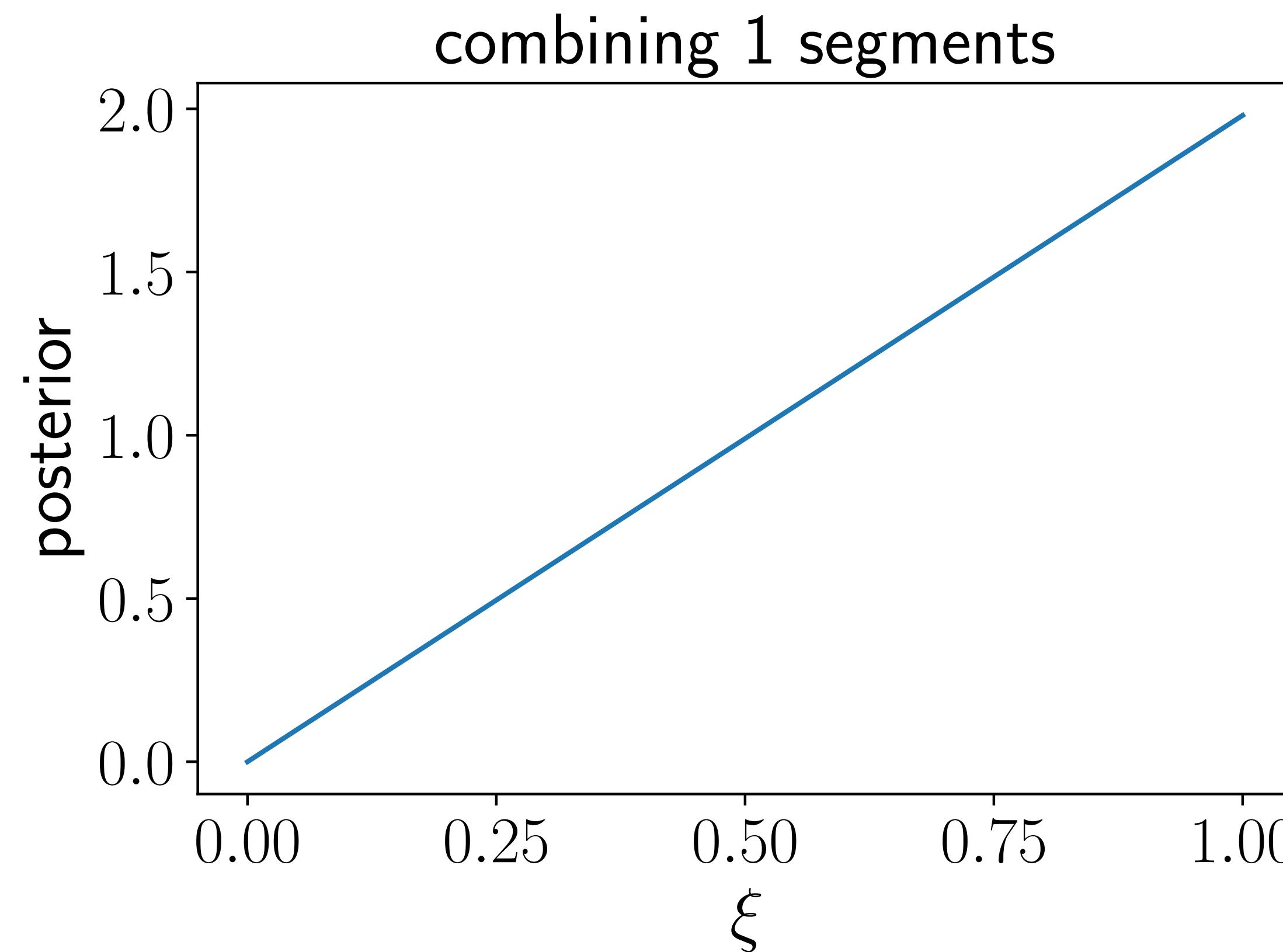
# Example: Simulated BBH background in white detector noise and confusion-limited BNS background



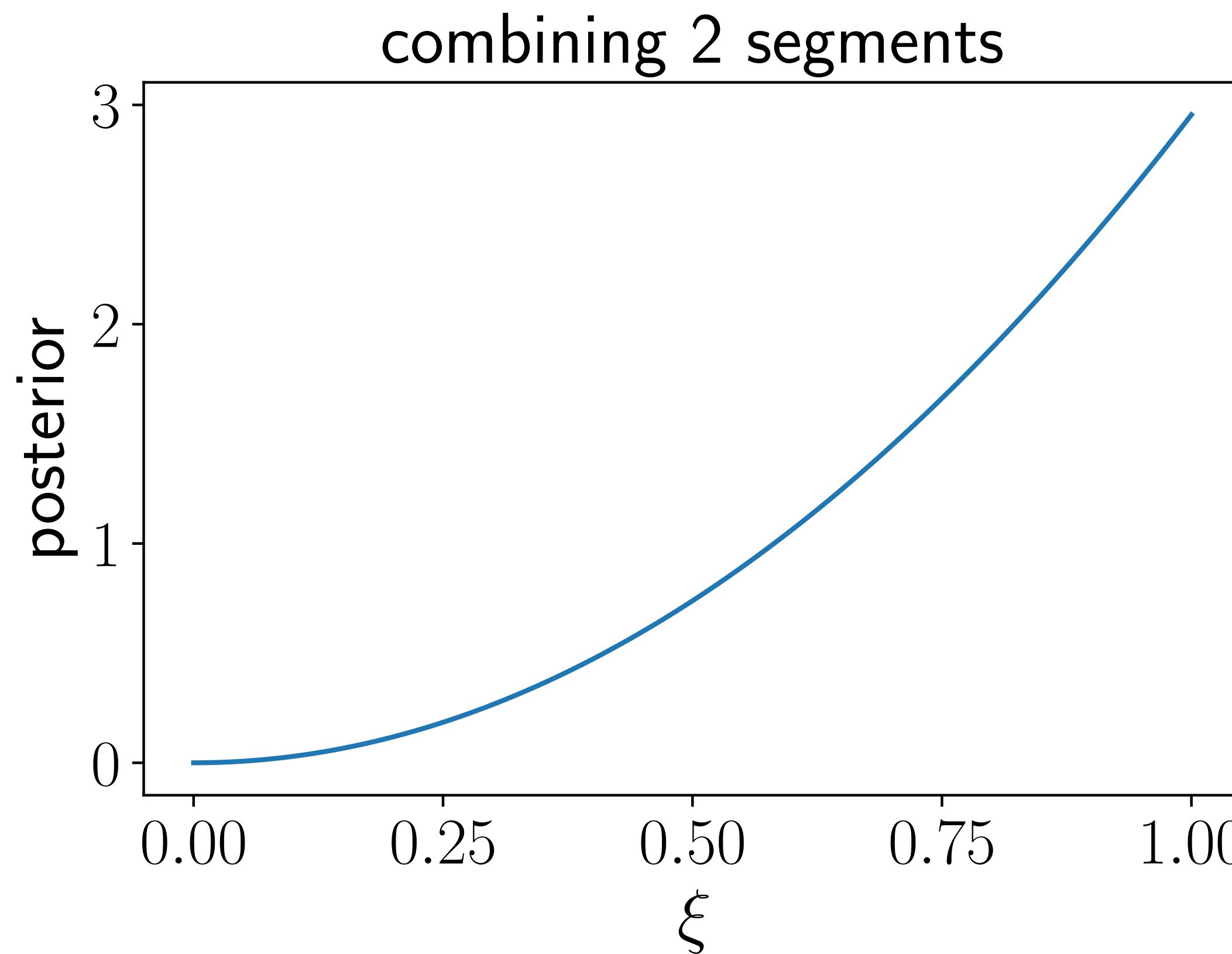
# Posteriors $p(\xi|d)$ for individual segments



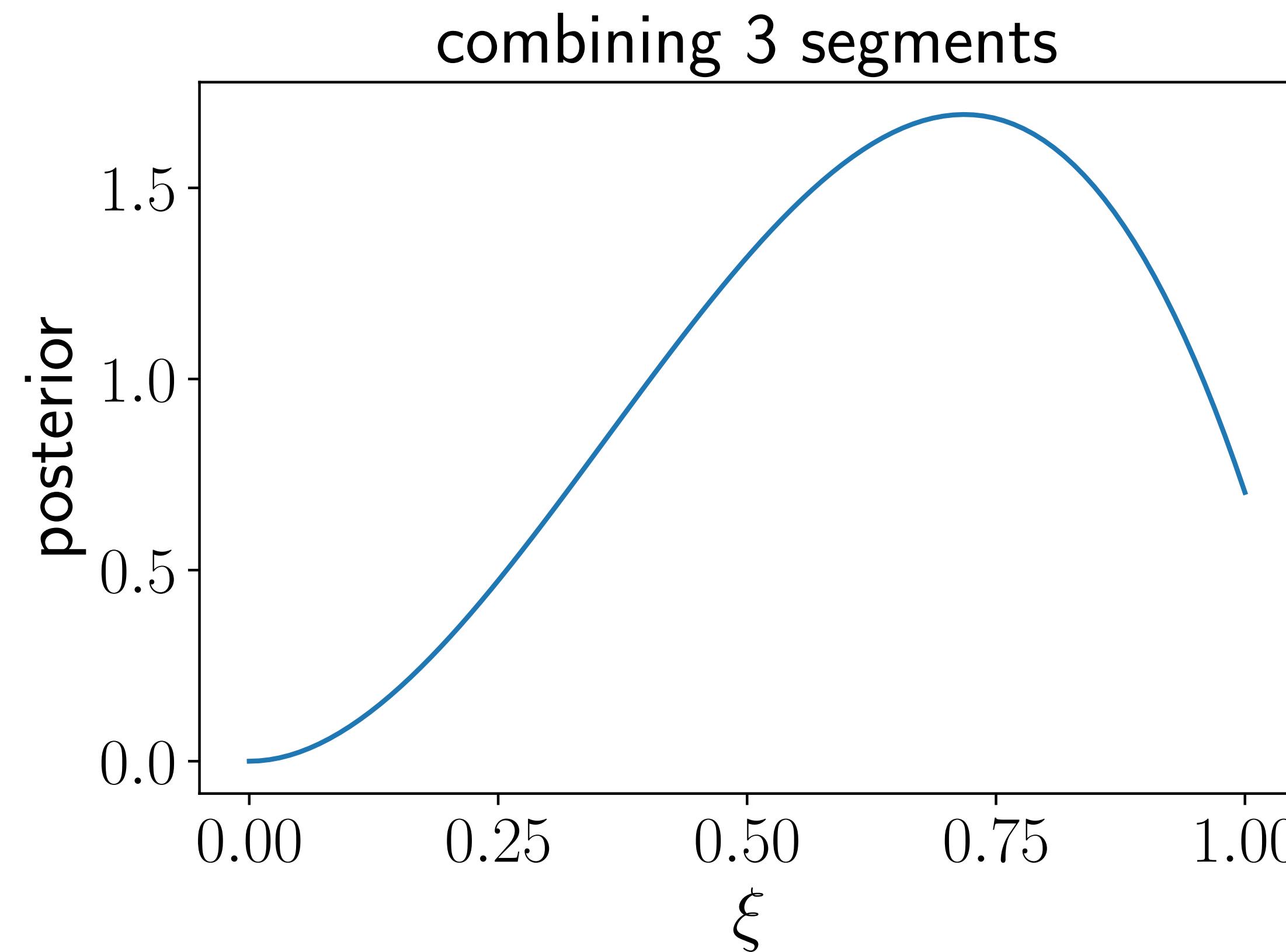
# Combining segment posteriors



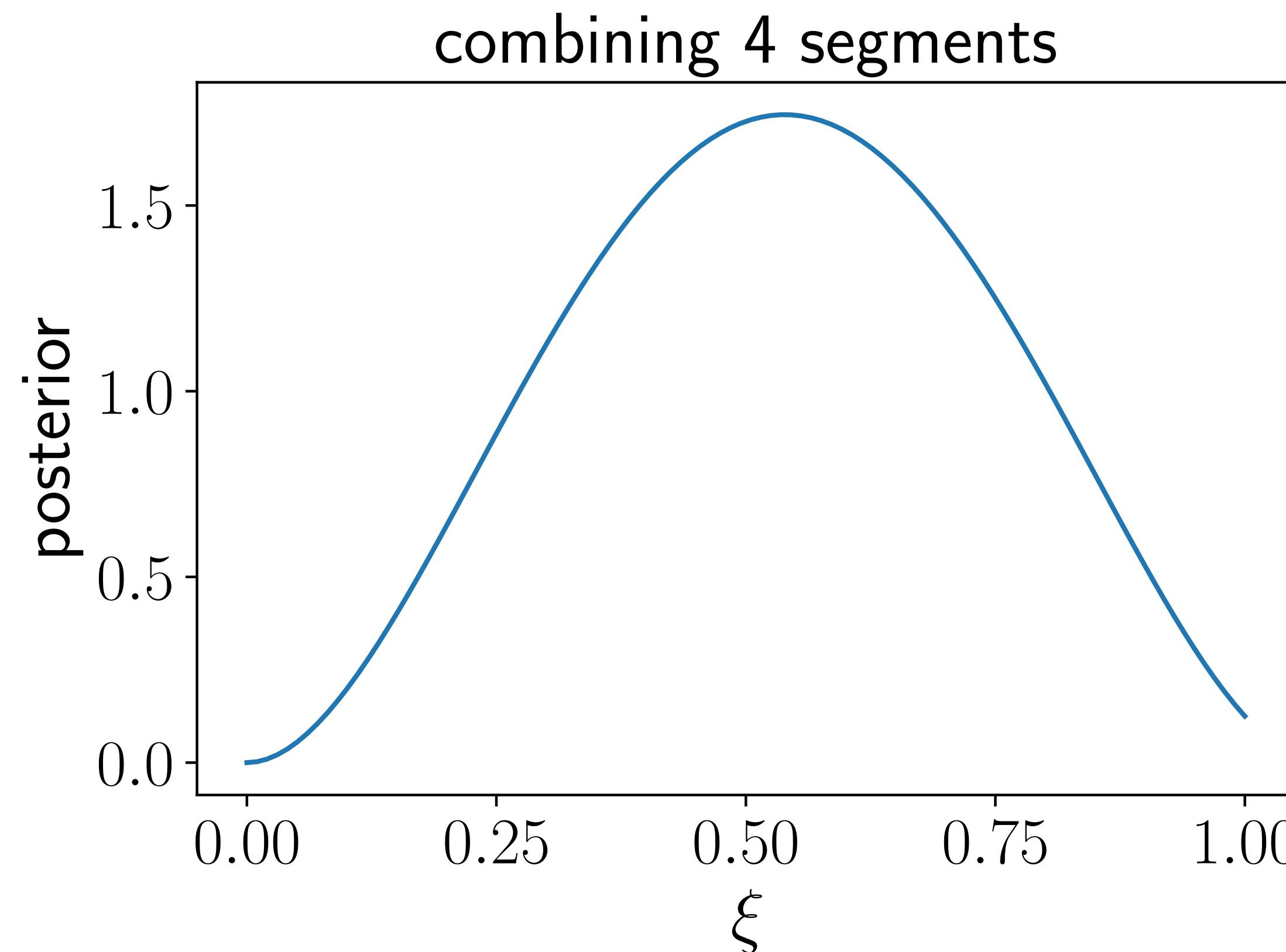
# Combining segment posteriors



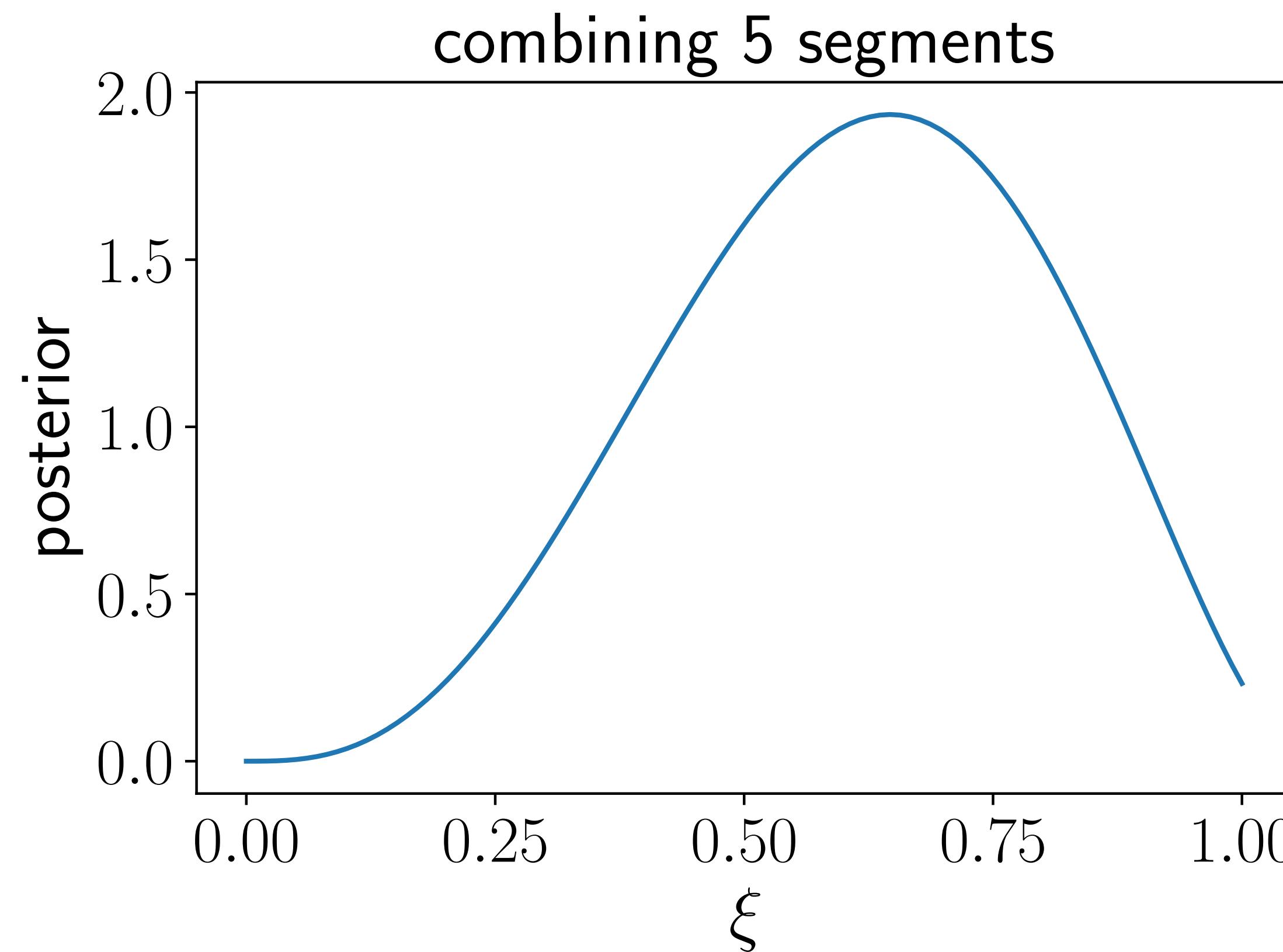
# Combining segment posteriors



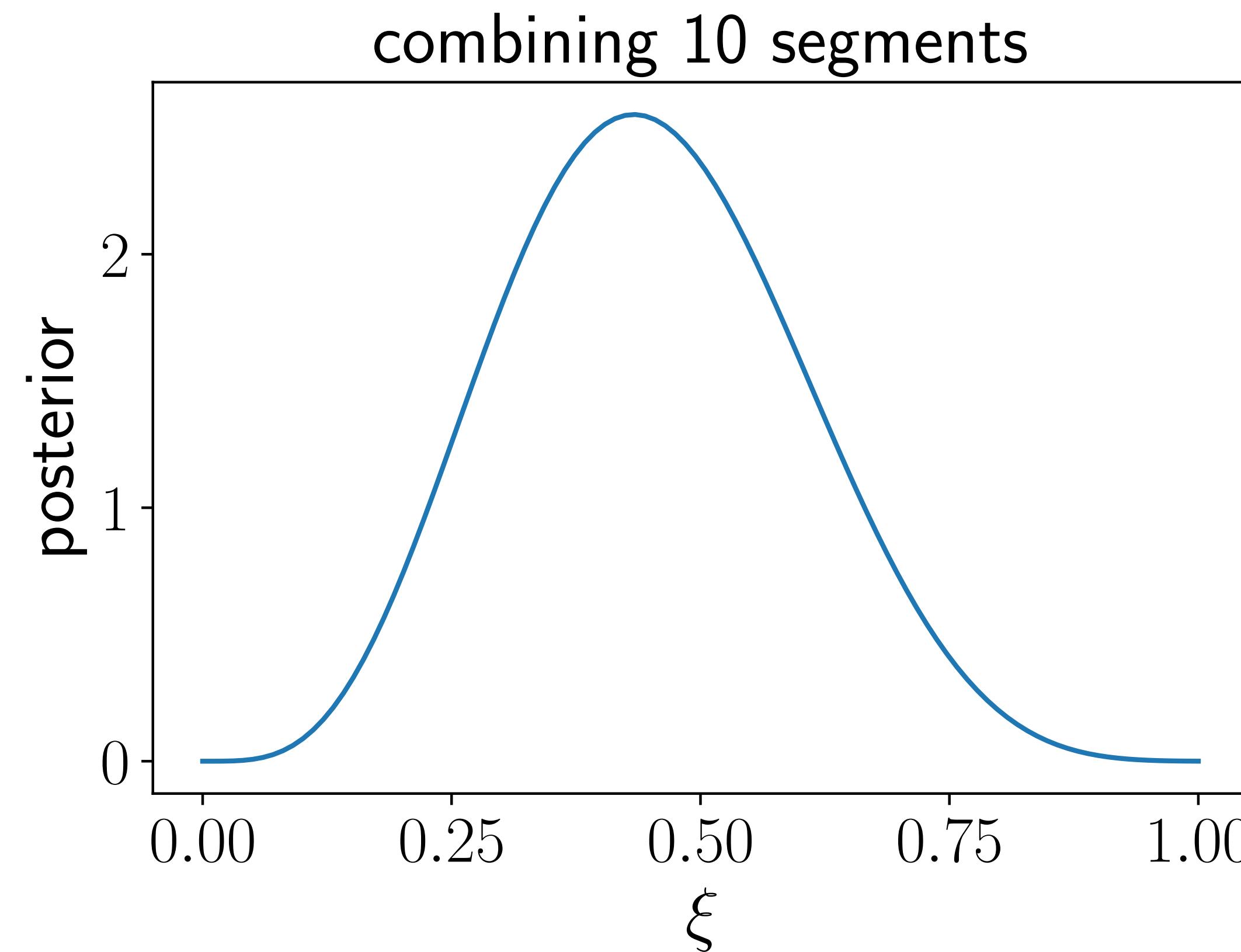
# Combining segment posteriors



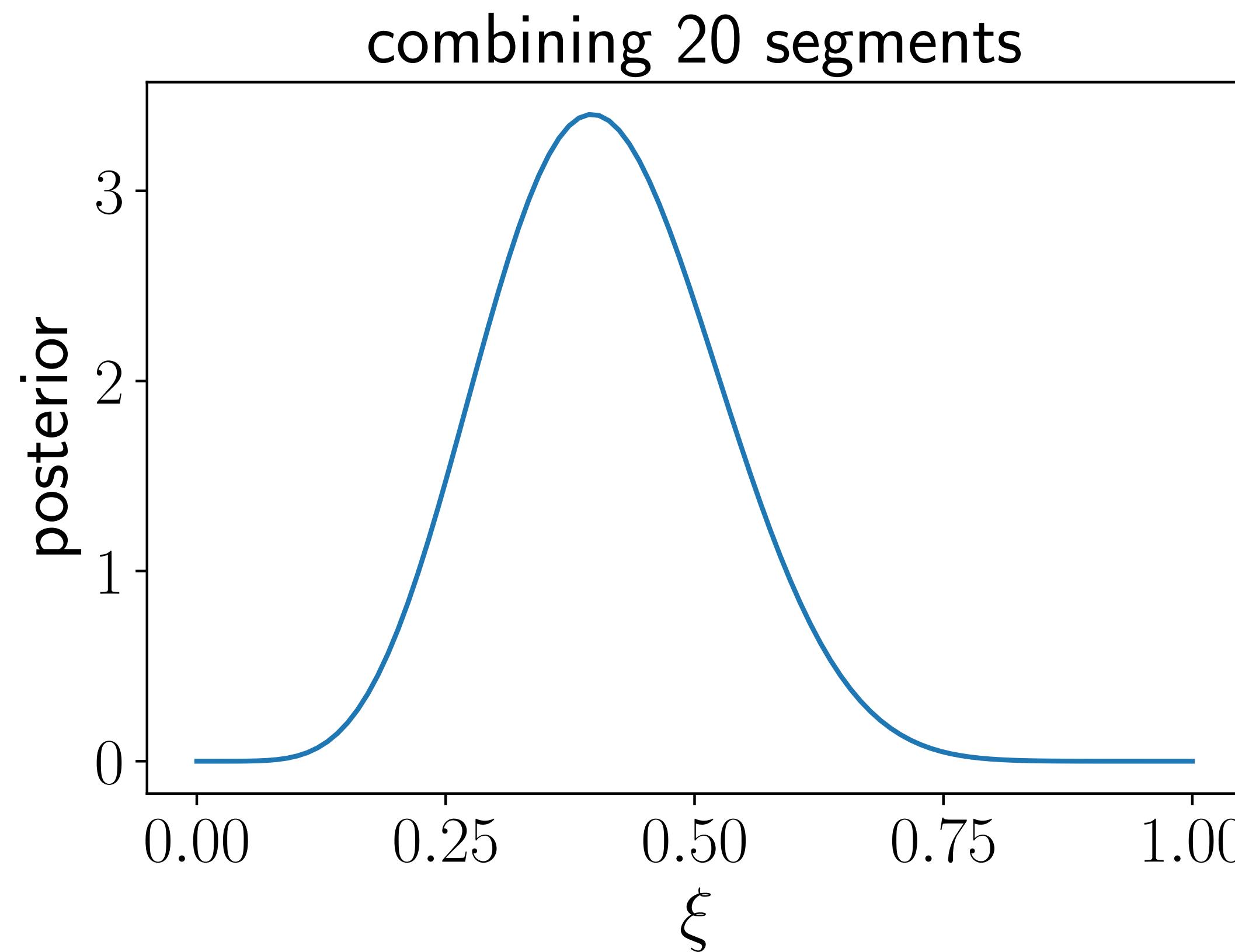
# Combining segment posteriors



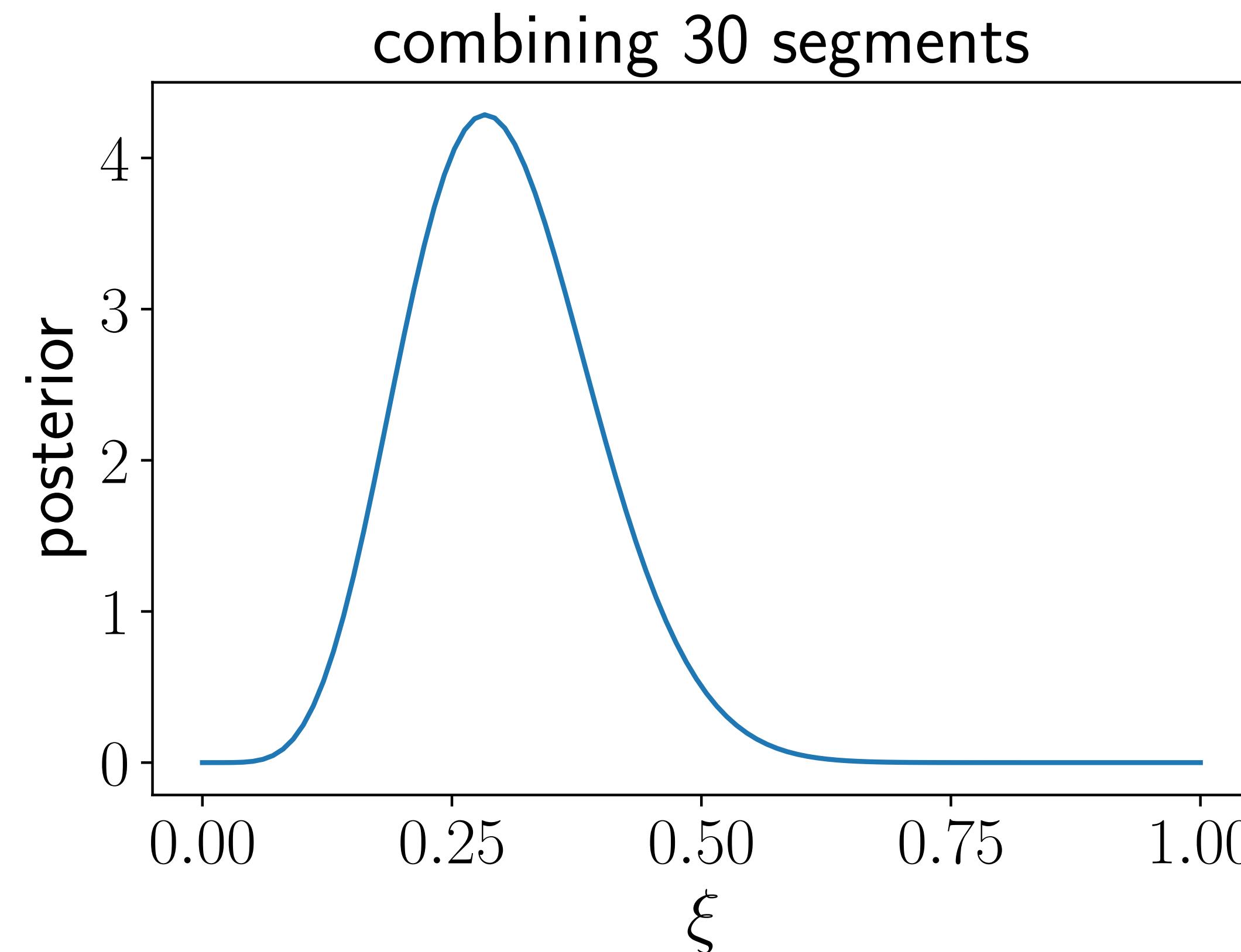
# Combining segment posteriors



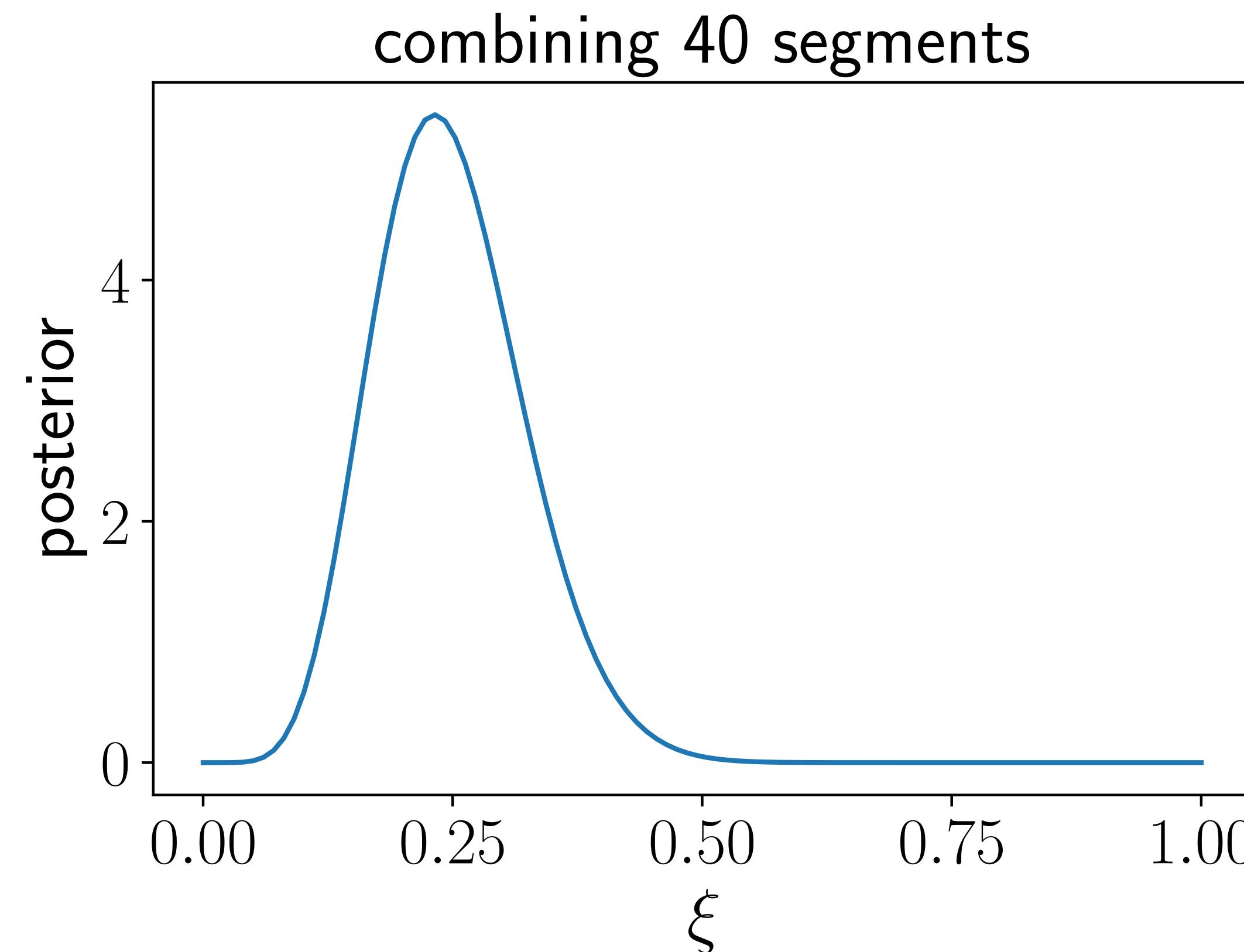
# Combining segment posteriors



# Combining segment posteriors



# Combining segment posteriors



# The optimal analysis reduces time to detection because...

- All segments contribute to estimating probability parameter  $\xi$
- BBH chirp signal is deterministic and not stochastic

$$\frac{\text{SNR}_{\text{non-stationary}}}{\text{SNR}_{\text{stationary}}} \sim \sqrt{\frac{N_{\text{cycles}}}{\xi}}$$

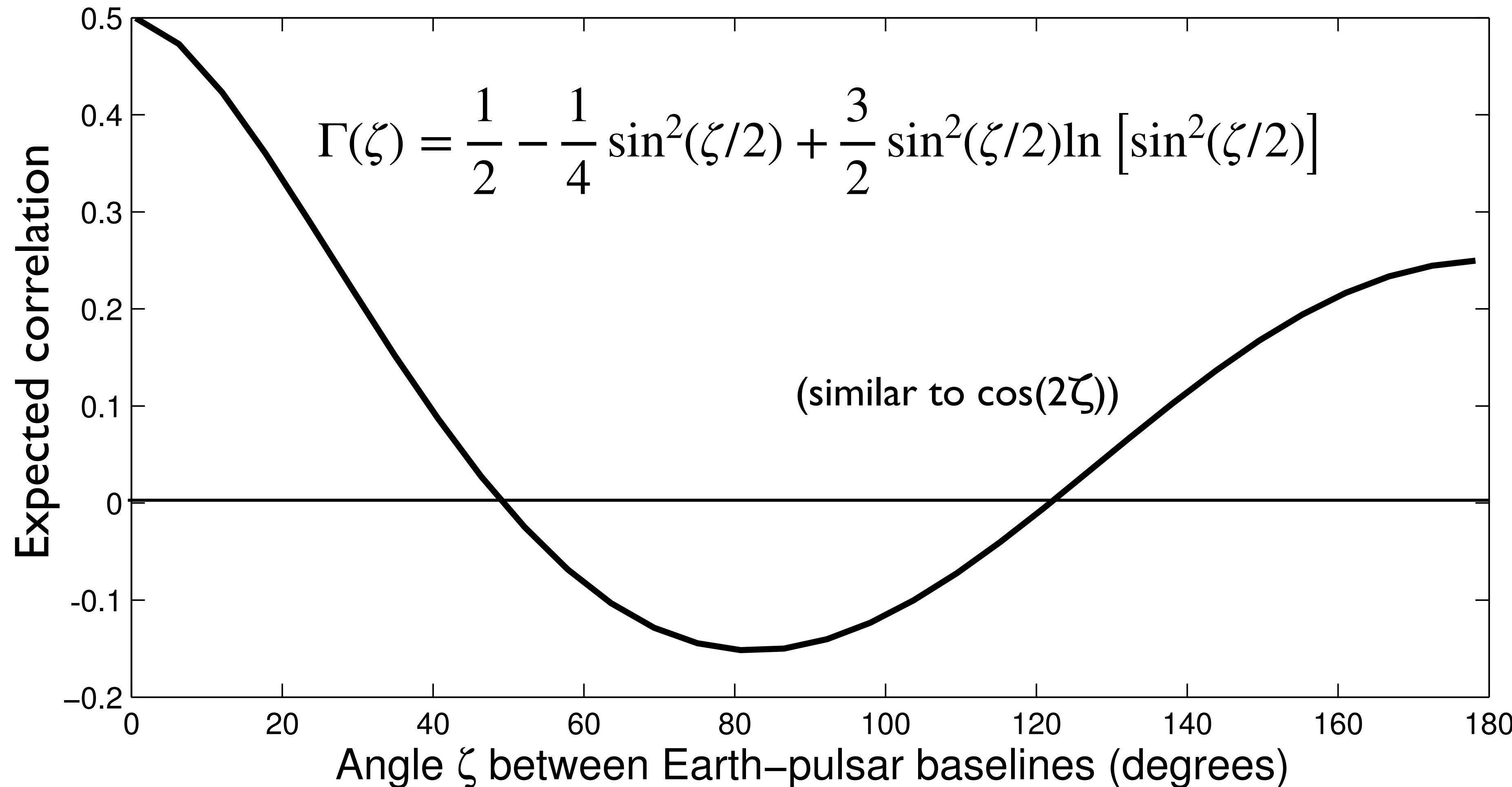
haven't been able to rigorously prove the  $N_{\text{cycles}}$  part!!

~40 months of observation reduces to ~1 day!!

*So stay tuned!!*

# extra slides

Pulsar timing correlations (Hellings & Downs curve)  
(correlation for an isotropic, unpolarized GW background in GR)



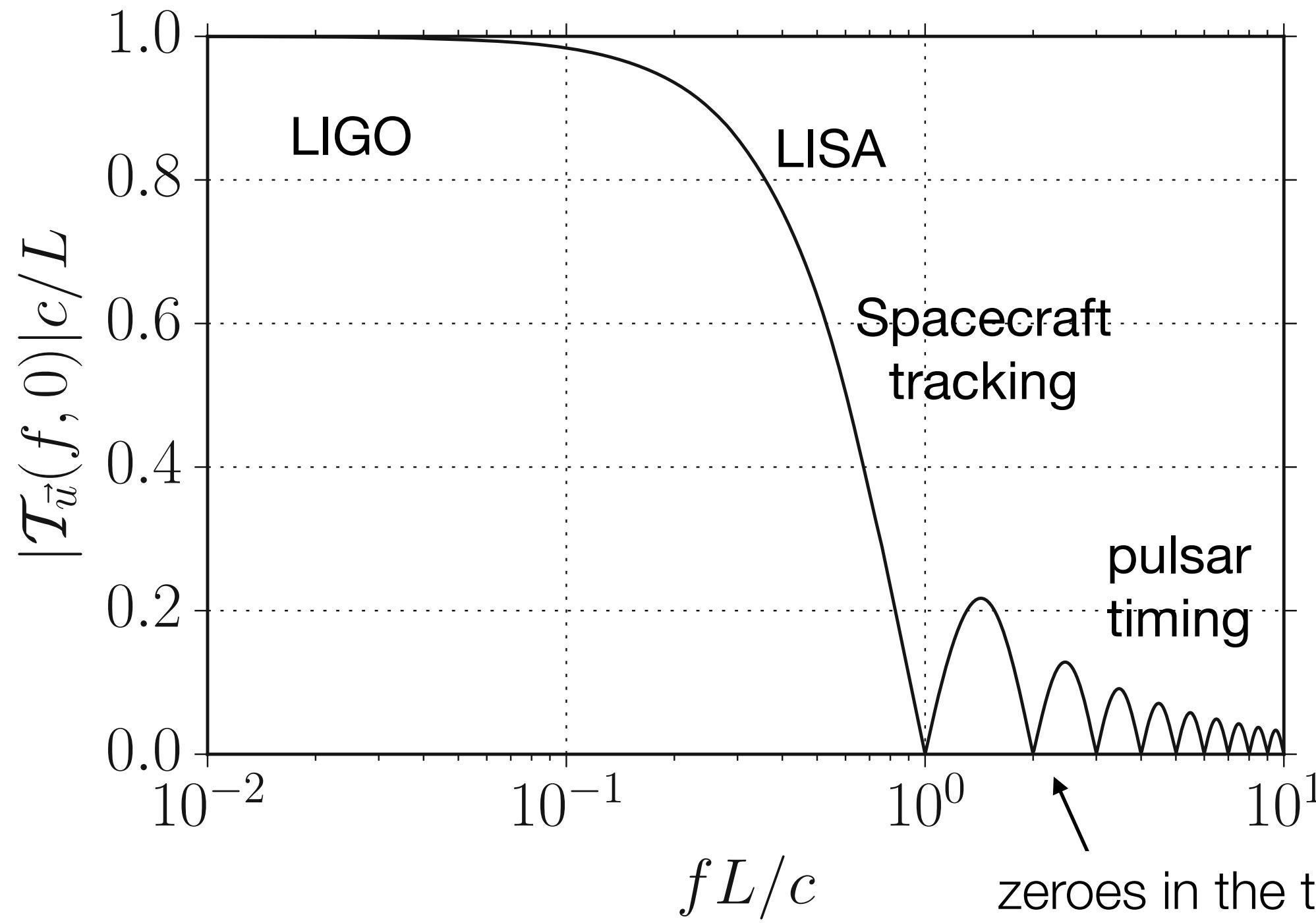
# Beyond the short-antenna limit

LISA, spacecraft Doppler tracking and pulsar timing all operate outside of the short-antenna limit

Recall response function:  
(one-arm, one-way)

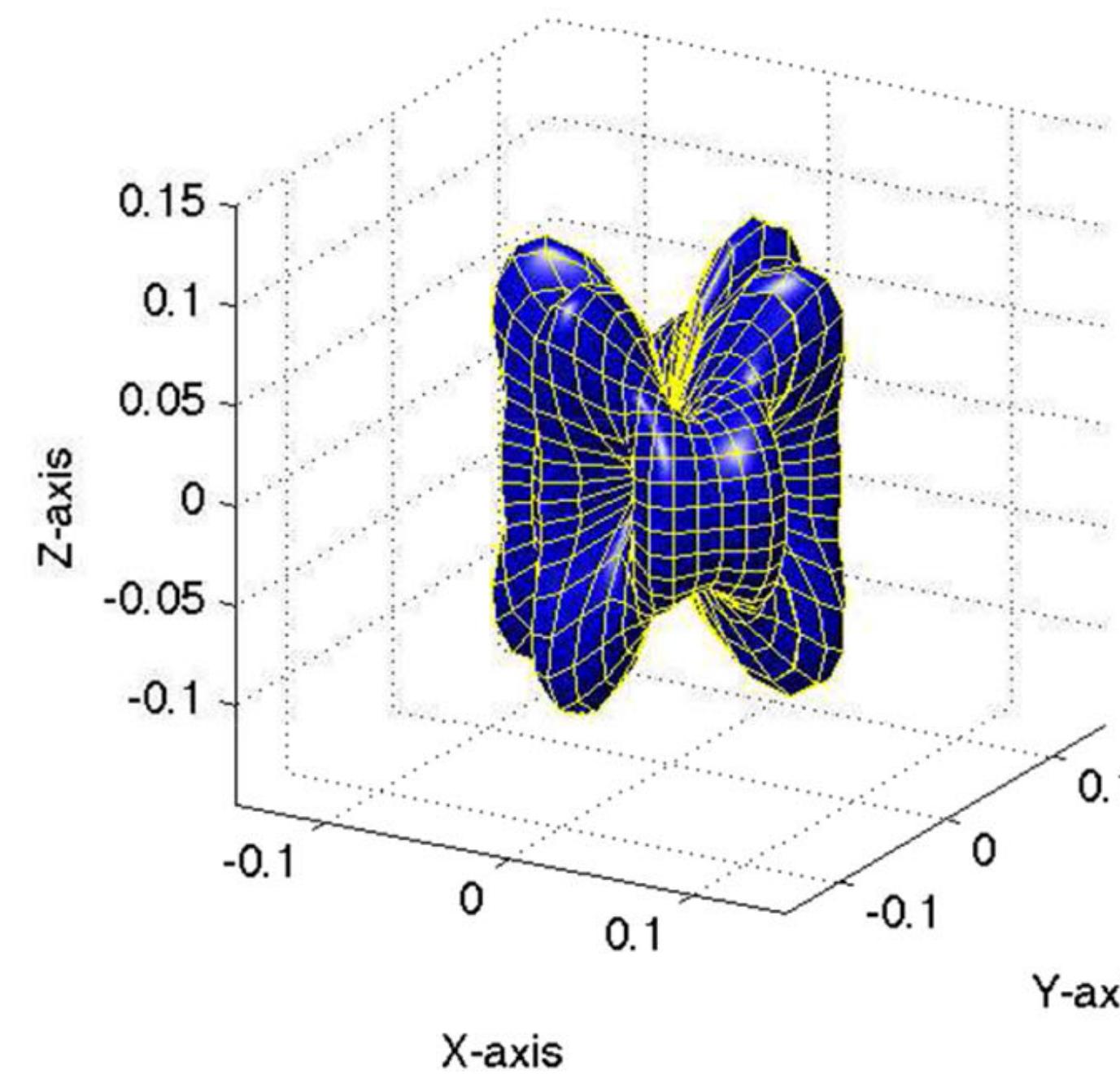
$$R^A(f, \hat{n}) = \frac{1}{2} u^a u^b e_{ab}^A(\hat{n}) \frac{1}{i2\pi f} \frac{1}{1 + \hat{n} \cdot \hat{u}} \left[ 1 - e^{-\frac{i2\pi f L}{c}(1 + \hat{n} \cdot \hat{u})} \right] e^{i2\pi f \hat{n} \cdot \vec{r}_2 / c}$$

$$\mathcal{T}_{\hat{u}}(f, \hat{n} \cdot \hat{u}) \equiv \frac{1}{i2\pi f} \frac{1}{1 + \hat{n} \cdot \hat{u}} \left[ 1 - e^{-\frac{i2\pi f L}{c}(1 + \hat{n} \cdot \hat{u})} \right] = \frac{L}{c} e^{-\frac{i\pi f L}{c}(1 + \hat{n} \cdot \hat{u})} \text{sinc}\left(\frac{\pi f L}{c}[1 + \hat{n} \cdot \hat{u}]\right)$$

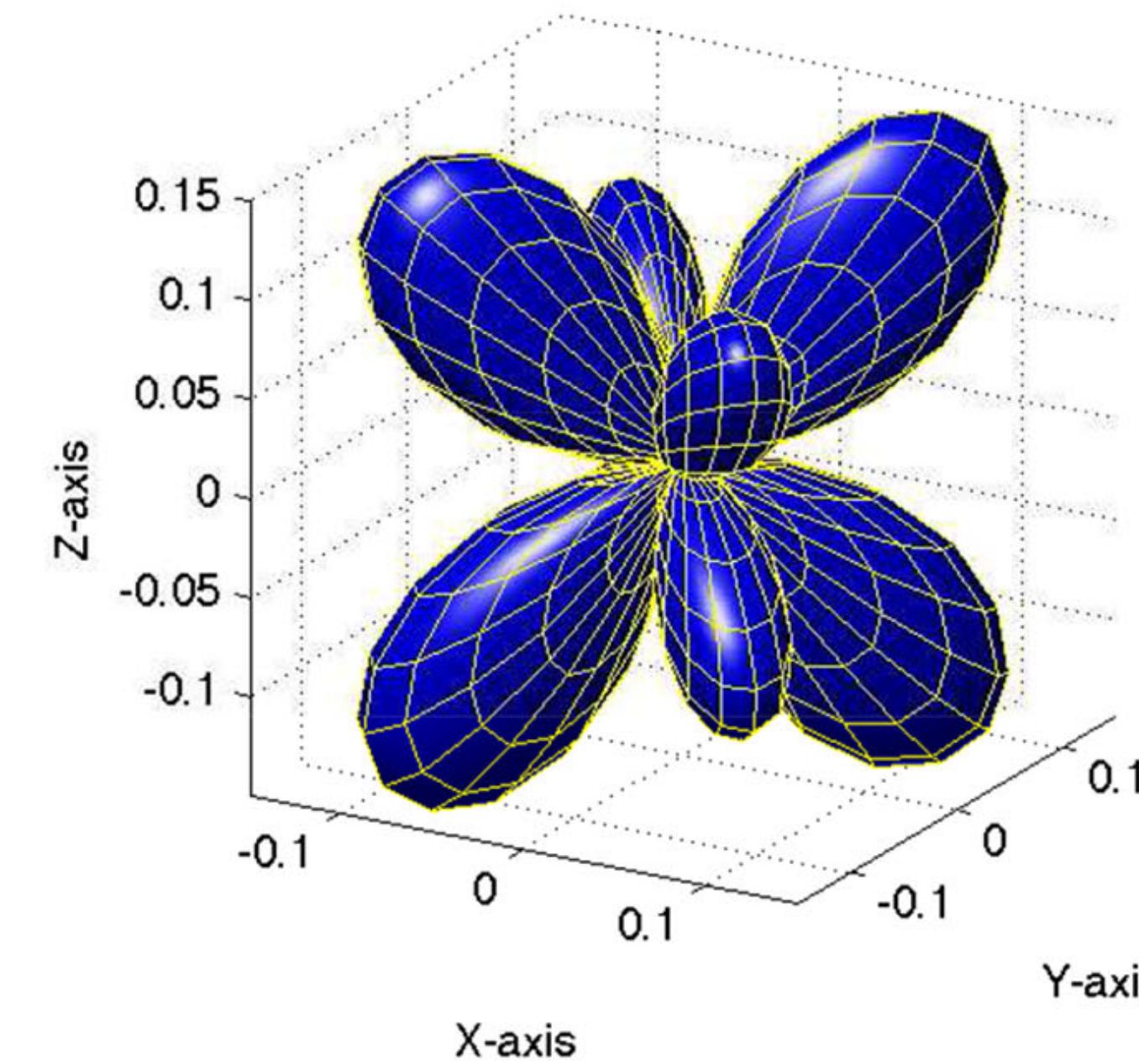


Beam detector	$L$ (km)	$f_*$ (Hz)	$f$ (Hz)	$f/f_*$	Relation
Ground-based interferometer	$\sim 1$	$\sim 10^5$	$10 - 10^4$	$10^{-4} - 10^{-1}$	$f \ll f_*$
Space-based interferometer	$\sim 10^6$	$\sim 10^{-1}$	$10^{-4} - 10^{-1}$	$10^{-3} - 1$	$f \lesssim f_*$
Spacecraft Doppler tracking	$\sim 10^9$	$\sim 10^{-4}$	$10^{-6} - 10^{-3}$	$10^{-2} - 10$	$f \sim f_*$
Pulsar timing	$\sim 10^{17}$	$\sim 10^{-12}$	$10^{-9} - 10^{-7}$	$10^3 - 10^5$	$f \gg f_*$

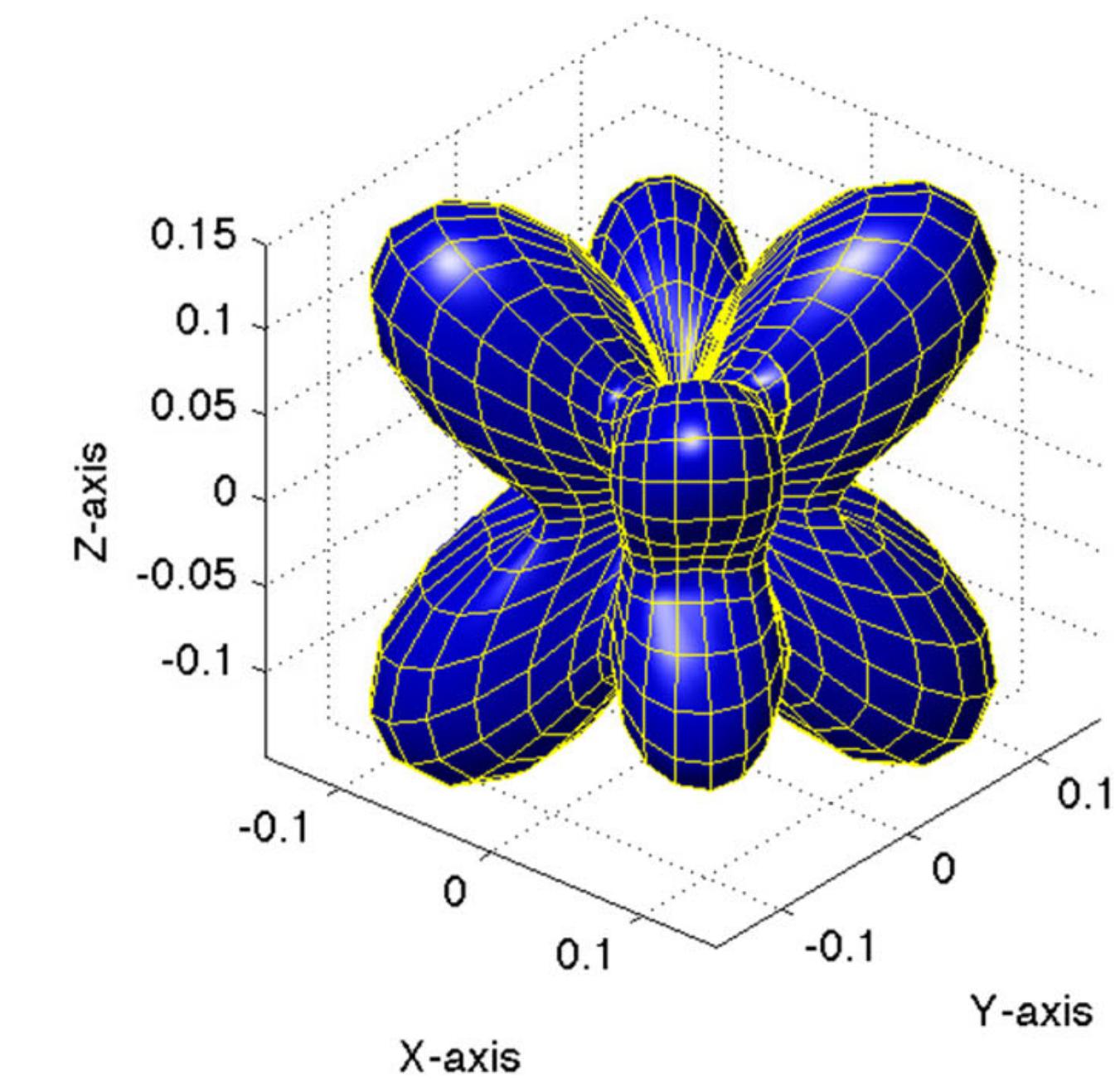
# Beam pattern functions



$$|R^+(f, \hat{n})|$$

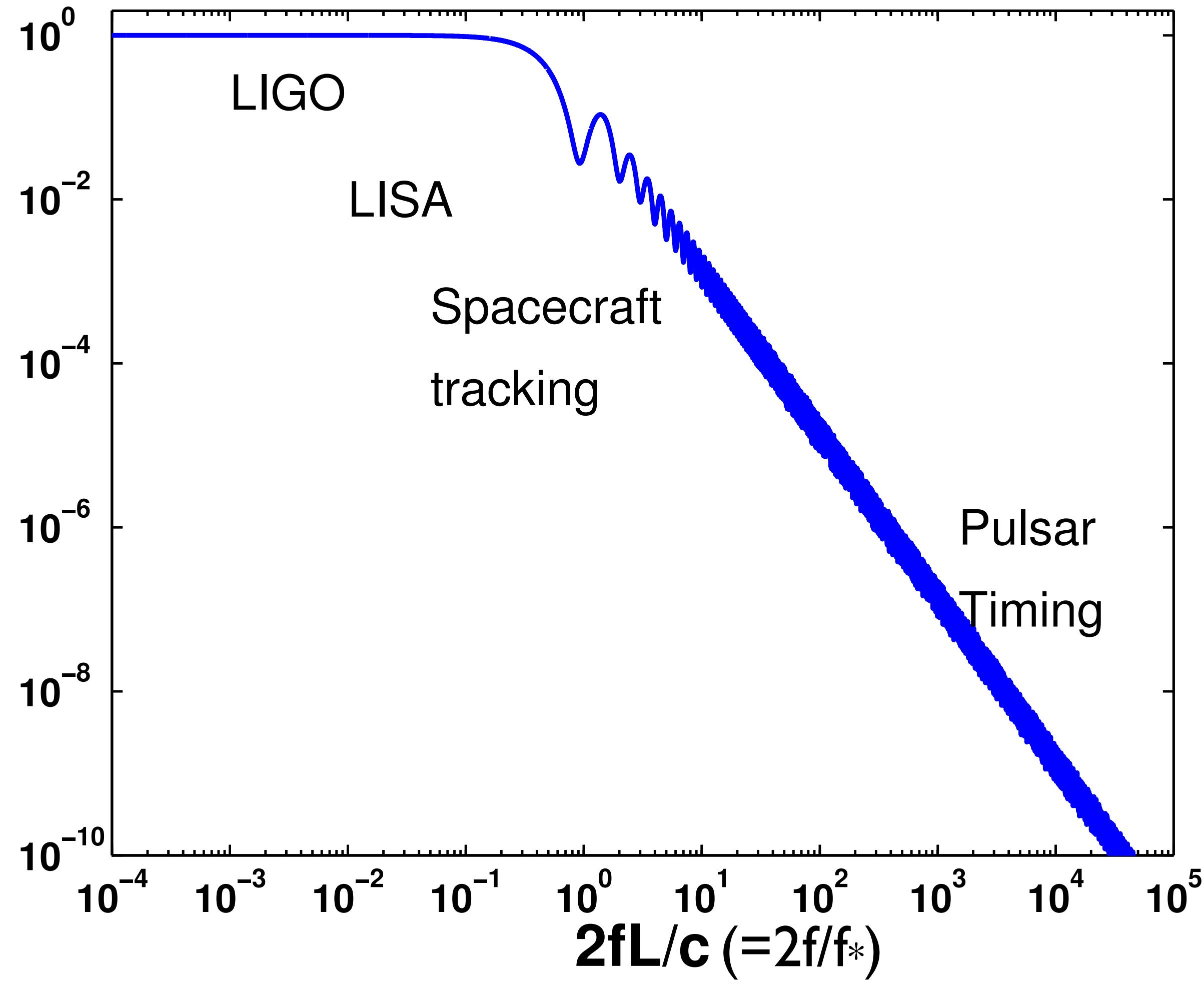


$$|R^\times(f, \hat{n})|$$



$$(|R^+(f, \hat{n})|^2 + |R^\times(f, \hat{n})|^2)^{1/2}$$

$$(f = c/(2L) = 37.5 \text{ kHz})$$



Beam detector	$L$ (km)	$f_*$ (Hz)	$f$ (Hz)	$f/f_*$	Relation
Ground-based interferometer	$\sim 1$	$\sim 10^5$	$10 - 10^4$	$10^{-4} - 10^{-1}$	$f \ll f_*$
Space-based interferometer	$\sim 10^6$	$\sim 10^{-1}$	$10^{-4} - 10^{-1}$	$10^{-3} - 1$	$f \lesssim f_*$
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Pulsar timing	$\sim 10^{17}$	$\sim 10^{-12}$	$10^{-9} - 10^{-7}$	$10^3 - 10^5$	$f \gg f_*$

$\mathcal{B}_{\alpha\beta}(d)$	$2 \ln \mathcal{B}_{\alpha\beta}(d)$	Evidence for model $\mathcal{M}_\alpha$ relative to $\mathcal{M}_\beta$
<1	<0	Negative (supports model $\mathcal{M}_\beta$ )
1–3	0–2	Not worth more than a bare mention
3–20	2–6	Positive
20–150	6–10	Strong
>150	>10	Very strong

Adapted from [Kass and Raftery \(1995\)](#)

# Matched-filtering determination of measured TOAs

$$C(\Delta t) = \mathcal{N} \int dt y(t)p(t - \Delta t)$$

