Searches for stochastic gravitational-wave backgrounds: Suggested Exercises

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Abstract

Some suggested exercises accompanying the lectures on searches for stochastic gravitational-wave backgrounds.

Contents

1. *Rate estimate of stellar-mass binary black hole mergers:*

   Estimate the total rate (number of events per time) of stellar-mass binary black hole mergers throughout the universe by multiplying LIGO's O1 local rate estimate $R_0 \sim 10$ - $200\ \mathrm{Gpc}^{-3}\,\mathrm{yr}^{-1}$ by the comoving volume out to some large redshift, e.g., $z = 10$. (For this calculation you can ignore any dependence of the rate density with redshift.) You should find a merger rate of $\sim$1 per minute to a few per hour.

   Hint: You will need to do numerically evaluate the following integral for proper distance today as a function of source redshift:

$$d_0(z) = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')}\,, \qquad E(z) \equiv \sqrt{\Omega_\mathrm{m}(1+z)^3 + \Omega_\Lambda}\,, \tag{1}$$

   with

$$\Omega_\mathrm{m} = 0.31\,, \qquad \Omega_\Lambda = 0.69\,, \qquad H_0 = 68\ \mathrm{km\,s^{-1}\,Mpc^{-1}}\,. \tag{2}$$

   Doing that integral, you should find what's shown in Figure 1, which you can then evaluate at $z = 10$ to convert R_0 (number of events per comoving volume per time) to total rate (number of events per time) for sources out to redshift $z = 10$.

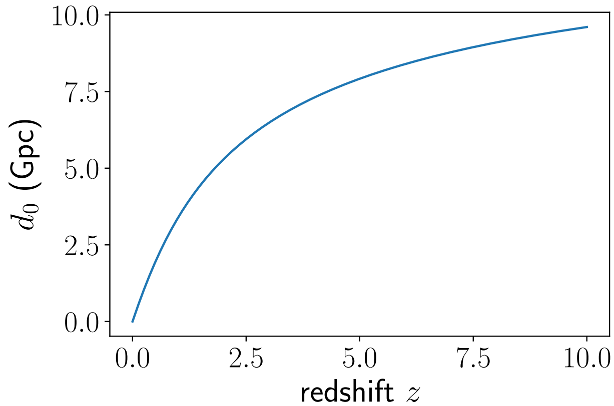

Figure 1:

2. *Relating $S_h(f)$ and $\Omega_\mathrm{gw}(f)$:*

   Derive the relationship

$$S_h(f) = \frac{3H_0^2}{2\pi^2}\frac{\Omega_\mathrm{gw}(f)}{f^3} \tag{3}$$

   between the strain power spectral density $S_h(f)$ and the dimensionless fractional energy density spectrum $\Omega_\mathrm{gw}(f)$. (*Hint*: You will need to use the various definitions of these quantities and also

$$\rho_\mathrm{gw} = \frac{c^2}{32\pi G}\langle \dot{h}_{ab}(t,\vec{x})\dot{h}^{ab}(t,\vec{x})\rangle\,, \tag{4}$$

   which expresses the energy-density in gravitational-waves to the metric perturbations $h_{ab}(t,\vec{x})$.)

3. *Cosmology and the "Phinney formula" for astrophysical backgrounds:*

   (a) Using the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_\mathrm{m}}{a^3} + \Omega_\Lambda\right) \tag{5}$$

for a spatially-flat FRW spacetime with matter and cosmological constant, and the relationship

$$1 + z = \frac{1}{a(t)}\,, \qquad a(t_0) \equiv 1 \quad (t_0 \equiv \text{today})\,, \tag{6}$$

between redshift z and scale factor $a(t)$, derive

$$\frac{\mathrm{d}t}{\mathrm{d}z} = -\frac{1}{(1+z)H_0 E(z)}\,, \qquad E(z) = \sqrt{\Omega_\mathrm{m}(1+z)^3 + \Omega_\Lambda}\,. \tag{7}$$

(b) Using this result for $\mathrm{d}t/\mathrm{d}z$, show that

$$\Omega_\mathrm{gw}(f) = \frac{f}{\rho_\mathrm{c} H_0}\int_0^\infty \mathrm{d}z\, R(z)\,\frac{1}{(1+z)E(z)}\left(\frac{\mathrm{d}E_\mathrm{gw}}{\mathrm{d}f_\mathrm{s}}\right)\bigg|_{f_\mathrm{s}=f(1+z)} \tag{8}$$

in terms of the rate density $R(z)$ as measured in the source frame (number of events per comoving volume per time interval in the source frame). (*Hint*: The expression for $\mathrm{d}t/\mathrm{d}z$ from part (a) will allow you to go from the "Phinney formula" for $\Omega_\mathrm{gw}(f)$ written in terms of the number density $n(z)$,

$$\Omega_\mathrm{gw}(f) = \frac{1}{\rho_c}\int_0^\infty \mathrm{d}z\, n(z)\,\frac{1}{1+z}\left(f_\mathrm{s}\,\frac{\mathrm{d}E_\mathrm{gw}}{\mathrm{d}f_\mathrm{s}}\right)\bigg|_{f_\mathrm{s}=f(1+z)}\,, \tag{9}$$

to one in terms of the rate density $R(z)$, where $n(z)\,\mathrm{d}z = R(z)\,|\mathrm{d}t|_{t=t(z)}$. Note: Both of the above expressions for $\Omega_\mathrm{gw}(f)$ assume that there is only one type of source, described by some set of average source parameters. If there is more than one type of source, one must sum the contributions of each source to $\Omega_\mathrm{gw}(f)$.)

4. *Optimal filtering for the cross-correlation statistic:*
Verify the form

$$\tilde{Q}(f) \propto \frac{\Gamma_{12}(f)H(f)}{P_1(f)P_2(f)}\,, \tag{10}$$

of the optimal filter function in the weak-signal limit, where $H(f)$ is the assumed spectral shape of the gravitational-wave background, $\Gamma_{12}(f)$ is the overlap function, and $P_1(f)$, $P_2(f)$ are the power spectral densities of the outputs of the two detectors (which are approximately equal to $P_{n_1}(f)$, $P_{n_2}(f)$, respectively). Recall that the optimal filter $\tilde{Q}(f)$ maximizes the signal-to-noise ratio of the cross-correlation statistic. (*Hint*: Introduce an inner product on the space of functions of frequency $A(f)$, $B(f)$:

$$(A, B) \equiv \int df\, A(f)B^*(f)P_1(f)P_2(f)\,. \tag{11}$$

This inner product has all of the properties of the familiar dot product of vectors in 3-dimensional space. The signal-to-noise ratio of the cross-correlation statistic can be written in terms of this inner product.)

5. *Maximum-likelihood estimators for single and multiple parameters:*
(a) Show that the maximum-likelihood estimator $\hat{a}$ of the single parameter a in the likelihood function

$$p(d|a,\sigma) \propto \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\frac{(d_i - a)^2}{\sigma_i^2}\right] \tag{12}$$

is given by the noise-weighted average

$$\hat{a} = \sum_i \frac{d_i}{\sigma_i^2}\bigg/\sum_j \frac{1}{\sigma_j^2}\,. \tag{13}$$

(b) Extend the previous calculation to the likelihood

$$p(d|A,C) \propto \exp\left[-\frac{1}{2}(d-MA)^\dagger C^{-1}(d-MA)\right] , \tag{14}$$

where $A \equiv A_\alpha$ is a vector of parameters, $C \equiv C_{ij}$ is the noise covariance matrix, and $M \equiv M_{i\alpha}$ is the response matrix mapping A_α to data samples, $MA \equiv \sum_\alpha M_{i\alpha}A_\alpha$. For this more general case you should find:

$$\hat{A} = F^{-1}X , \tag{15}$$

where

$$F \equiv M^\dagger C^{-1} M , \qquad X \equiv M^\dagger C^{-1} d . \tag{16}$$

In general, the matrix F (called the *Fisher* matrix) is not invertible, so some sort of regularization is needed to do the matrix inversion.

6. *Timing-residual response for a 1-arm, 1-way detector:*
Derive the timing residual reponse function

$$R^A(f,\hat{n}) = \frac{1}{2}u^a u^b e^A_{ab}(\hat{n})\frac{1}{i2\pi f}\frac{1}{1+\hat{n}\cdot\hat{u}}\left[1-e^{-\frac{i2\pi f L}{c}(1+\hat{n}\cdot\hat{u})}\right] \tag{17}$$

for a single-link (i.e., a one-arm, one-way detector like that for pulsar timing). Here $\hat{u}$ is the direction of propagation of the electromagnetic pulse, and $\hat{n}$ is the direction to the GW source (the direction of wave propagation is $\hat{k} \equiv -\hat{n}$ and the direction to the pulsar is $\hat{p} \equiv -\hat{u}$). The origin of coordinates is taken to be at the position of the detector.

7. *Overlap function for colocated electric dipole antennae:*
Show that the overlap function for a pair of (short) colocated electric dipole antennae pointing in directions $\hat{u}_1$ and $\hat{u}_2$ is given by

$$\Gamma_{12} \propto \hat{u}_1 \cdot \hat{u}_2 \equiv \cos\zeta \tag{18}$$

for the case of an unpolarized, isotropic electromagnetic field. (*Hint*: "short" means that the phase of the electric field can be taken to be constant over of the lengths of the dipole antennae, so that the reponse of antenna $I=1,2$ to the field is given by $r_I(t) = \hat{u}_I \cdot \vec{E}(t,\vec{x}_0)$, where $\vec{x}_0$ is the common location of the two antenna.)

8. *Maximum-likelihood estimators for the standard cross-correlation statistic:*
Verify that

$$\hat{C}_{11} \equiv \frac{1}{N}\sum_{i=1}^N d_{1i}^2 , \qquad \hat{C}_{22} \equiv \frac{1}{N}\sum_{i=1}^N d_{2i}^2 , \qquad \hat{C}_{12} \equiv \frac{1}{N}\sum_{i=1}^N d_{1i}d_{2i} \tag{19}$$

are maximum-likelihood estimators of

$$S_1 \equiv S_{n_1} + S_h , \quad S_2 \equiv S_{n_2} + S_h , \quad S_h , \tag{20}$$

for the case of N samples of a white GWB in uncorrelated white detector noise, for a pair of colocated and coaligned detectors. Recall that the likelihood function is

$$p(d|S_{n_1},S_{n_2},S_h) = \frac{1}{\sqrt{\det(2\pi C)}}\exp\left[-\frac{1}{2}d^T C^{-1} d\right] , \tag{21}$$

where

$$C = \begin{bmatrix} (S_{n_1}+S_h)\,1_{N\times N} & S_h\,1_{N\times N} \\ S_h\,1_{N\times N} & (S_{n_2}+S_h)\,1_{N\times N} \end{bmatrix} \tag{22}$$

and

$$d^T C^{-1} d \equiv \sum_{I,J=1}^2 \sum_{i,j=1}^N d_{Ii}\left(C^{-1}\right)_{Ii,Jj} d_{Jj} . \tag{23}$$

9. *Derivation of the maximum-likelihood ratio detection statistic:*

   Verify that twice the log of the maximum-likelihood ratio for the standard stochastic likelihood function goes like the square of the (power) signal-to-noise ratio,

$$2 \ln \Lambda_{\rm ML}(d) \simeq \frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}/N}\,, \tag{24}$$

   in the weak-signal approximation. (*Hint:* For simplicity, do the calculation in the context of N samples of a white GWB in uncorrelated white detector noise, for a pair of colocated and coaligned detectors, using the results of Exercise 8.)

10. *Standard cross-correlation likelihood by marginalizing over stochastic signal prior:*

    Derive the standard form of the likelihood function for stochastic background searches

$$p(d|S_{n_1}, S_{n_2}, S_h) = \frac{1}{\sqrt{\det(2\pi C)}} \exp\left[-\frac{1}{2}\sum_{I,J=1}^{2} d_I \left(C^{-1}\right)_{IJ} d_J\right]\,, \tag{25}$$

    where

$$C \equiv \begin{bmatrix} S_{n_1}+S_h & S_h \\ S_h & S_{n_2}+S_h \end{bmatrix}\,, \tag{26}$$

    by marginalizing

$$p_n(d-h|S_{n_1}, S_{n_2}) = \frac{1}{2\pi\sqrt{S_{n_1}S_{n_2}}} \exp\left[-\frac{1}{2}\left\{\frac{(d_1-h)^2}{S_{n_1}} + \frac{(d_2-h)^2}{S_{n_2}}\right\}\right] \tag{27}$$

    over the signal samples h for the *stochastic* signal prior

$$p(h|S_h) = \frac{1}{\sqrt{2\pi S_h}} \exp\left[-\frac{1}{2}\frac{h^2}{S_h}\right]\,. \tag{28}$$

    In other words, show that

$$p(d|S_{n_1}, S_{n_2}, S_h) = \int_{-\infty}^{\infty} {\rm d}h\, p_n(d-h|S_{n_1}, S_{n_2}) p(h|S_h)\,. \tag{29}$$

    (*Hint*: You'll have to complete the square in the argument of the exponential in the marginalization integral.)