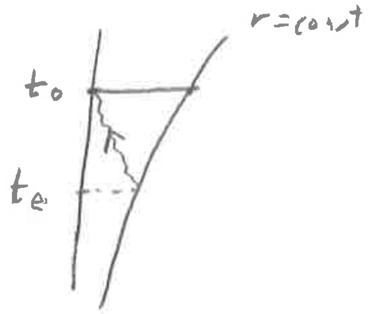
1. rate calculation

total rate: $r = R_0 \cdot \underbrace{\frac{4}{3}\pi d_0^3(z)}_{\text{co-moving volume}}$

where $d_0(z)$ is proper distance today to source which emitted GWs at redshift z

FRW: $ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + S_k^2(r)\,d\Omega^2\right]$

$$d_0(z) = a(t_0)\int_0^r dr'$$
$$= a(t_0)\, r$$
$$= r \qquad \text{for } a(t_0) = 1$$



Radial photon: $ds^2 = 0 = -c^2dt^2 + a^2(t)dr^2$

$$\rightarrow \quad c\,dt = a(t)\,dr$$
$$dr = \frac{c}{a(t)}\,dt$$

Thus, $d_0(z) = \int_0^r dr'$

$$= \int_{t_e}^{t_0} \frac{c\,dt'}{a(t')} \qquad 1+z = \frac{1}{a(t)}$$

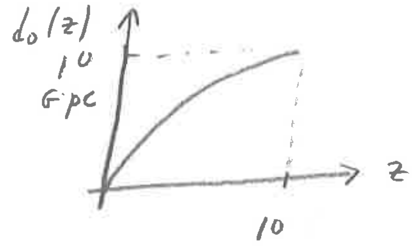
$$= \int_z^0 c(1+z')\left(\frac{dt'}{dz'}\right)dz'$$

$$= c\int_z^0 (1+z')\frac{dt'}{dz'}\,dz'$$

Now: (See Exercise 3) $\frac{dt}{dz} = \frac{-1}{(1+z)H_0 E(z)}$, $\quad E(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}$

$$\rightarrow \quad d_0(z) = \frac{c}{H_0}\int_z^0 \frac{-\cancel{(1+z')}\,dz'}{\cancel{(1+z')}\,E(z')} = \frac{c}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_m(1+z')^3 + \Omega_\Lambda}}$$

Do the integral:



Thus, $d_0(z=10) \approx 10\ \text{Gpc}$

so $r = R_0 \cdot \frac{4}{3}\pi (10\ \text{Gpc})^3$

Now: LIGO rate estimate $R_0 = 10 - 200\ \text{Gpc}^{-3}\ \text{yr}^{-1}$

$R_0 = 10$:

$$r = 10\ \cancel{\text{Gpc}^{-3}}\ \text{yr}^{-1}\ \frac{4}{3}\cancel{\pi}(10\ \cancel{\text{Gpc}})^3$$

$$\cong 4 \times \cancel{10^4}\ \frac{1}{\cancel{\text{yr}}} \underbrace{\left(\frac{1\ \cancel{\text{yr}}}{\pi \times 10^7\ \cancel{s}}\right)\left(\frac{3600\ \cancel{s}}{1\ \text{hr}}\right)}_{\approx\ \cancel{10^{-4}}\ \frac{1}{\text{hr}}}$$

Thus, $r \approx \boxed{4\ \frac{\text{events}}{\text{hr}}}$

$R_0 = 200$: 20 × as large

$\rightarrow \quad r \approx 80\ \frac{\text{events}}{\text{hr}} \approx \boxed{\frac{1\ \text{event}}{\text{minute}}}$

(2) Recall plane-wave expansion:

$$h_{ab}(t,\vec{x}) = \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{n}} \sum_{A=+,\times} h_A(f,\hat{n})\, e^A_{ab}(\hat{n})\, e^{i2\pi f(t+\hat{n}\cdot\vec{x}/c)}$$

~~Thus~~

$$\rightarrow \dot{h}_{ab}(t,\vec{x}) = \int df \int d^2\Omega_{\hat{n}} \sum_{A=+,\times} i2\pi f\, h_A(f,\hat{n})\, e^A_{ab}(\hat{n})\, e^{i2\pi f(t+\hat{n}\cdot\vec{x}/c)}$$

Thus,

$$\rho_{gw} = \frac{c^2}{32\pi G} \langle \dot{h}_{ab}(t,\vec{x})\, \dot{h}^{ab}(t,\vec{x}) \rangle$$

$$= \frac{c^2}{32\pi G} \int df \int df' \int d^2\Omega_{\hat{n}} \int d^2\Omega_{\hat{n}'} \sum_A \sum_{A'} (i2\pi f)(-i2\pi f')$$

$$\langle h_A(f,\hat{n})\, h^*_{A'}(f',\hat{n}') \rangle\, e^A_{ab}(\hat{n})\, e^{A'ab}(\hat{n}')$$

$$e^{i2\pi(f-f')t}\, e^{i2\pi(f\hat{n}-f'\hat{n}')\cdot\vec{x}/c}$$

Use expectation values,

$$\langle h_A(f,\hat{n})\, h^*_{A'}(f',\hat{n}') \rangle = \frac{1}{16\pi} S_h(f)\, \delta(f-f')\, \delta_{AA'}\, \delta^2(\hat{n},\hat{n}')$$

$$\rightarrow \rho_{gw} = \frac{c^2}{32\pi G} \int df \int d^2\Omega_{\hat{n}} \sum_{A=+,\times} (2\pi f)^2 \frac{1}{16\pi} e^A_{ab}(\hat{n})\, e^{Aab}(\hat{n})\, S_h(f)$$

<u>Now:</u>

$$e^+_{ab}(\hat{n})\, e^{+ab}(\hat{n}) = (l_a l_b - m_a m_b)(l^a l^b - m^a m^b)$$

$$= (\hat{l}\cdot\hat{l})^2 + (\hat{m}\cdot\hat{m})^2 - 2\underbrace{(\hat{l}\cdot\hat{m})^2}_{0}$$

$$= 2$$

similarly

$$e^\times_{ab}(\hat{n})\, e^{\times ab}(\hat{n}) = (l_a m_b + m_a l_b)(l^a m^b + m^a l^b)$$

$$= (\hat{l}\cdot\hat{l})(\hat{m}\cdot\hat{m}) + 2\underbrace{(\hat{l}\cdot\hat{m})^2}_{0}$$

$$= 2$$

Thus,

$$\rho_{gw} = \frac{c^2}{32\pi G} \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{n}} \, (2\pi f)^2 \frac{1}{16\pi} (2+2) \, S_h(f)$$

$$= \frac{c^2}{32\pi G} \frac{4\pi^2}{\cancel{16\pi}} \cdot \cancel{4\pi} \cdot \cancel{4} \int_{-\infty}^{\infty} df \, f^2 S_h(f)$$

$$= \frac{\pi c^2}{8G} \int_{-\infty}^{\infty} df \, f^2 S_h(f)$$

<u>Now</u>: $\rho_c \equiv \frac{3H_0^2 c^2}{8\pi G}$ (critical density)

Thus,

$$\rho_{gw} = \frac{\pi c^2}{8G} \left(\frac{3H_0^2}{\pi}\right) \frac{\pi}{3H_0^2} \int_{-\infty}^{\infty} df \, f^2 S_h(f)$$

$$= \frac{\pi^2}{3H_0^2} \rho_c \int_{-\infty}^{\infty} df \, f^2 S_h(f)$$

$$= \frac{2\pi^2}{3H_0^2} \rho_c \int_0^{\infty} df \, f^2 S_h(f)$$

$$= \frac{2\pi^2}{3H_0^2} \rho_c \int_{f=0}^{\infty} \frac{df}{f} \, f^3 S_h(f)$$

$$= \rho_c \int_{f=0}^{\infty} \frac{df}{f} \, \Omega_{gw}(f)$$

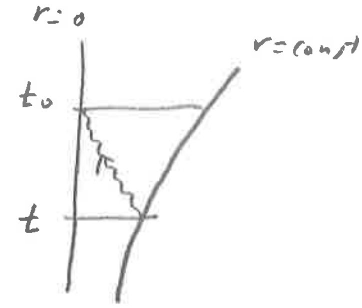
where $\Omega_{gw}(f) = \frac{2\pi^2}{3H_0^2} f^3 S_h(f)$

$$\Longleftrightarrow \quad S_h(f) = \frac{3H_0^2}{2\pi^2} \frac{\Omega_{gw}(f)}{f^3}$$

(3.)

(a) Friedmann equation:

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\left(\frac{\Omega_m}{a^3} + \Omega_\Lambda\right)$$



$$\frac{\dot a}{a} = H_0\sqrt{\frac{\Omega_m}{a^3} + \Omega_\Lambda}$$

$$1+z = \frac{a(t_0)}{a(t)} = \frac{1}{a(t)}$$ where $a(t_0) \equiv 1$ and t corresponds to time of emission

Rewrite Friedmann's equation in terms of z:

$$LHS = \frac{\dot a}{a} = \frac{1}{a}\frac{da}{dt} = (1+z)\frac{d}{dt}\left(\frac{1}{1+z}\right)$$

$$= (1+z)\frac{-1}{(1+z)^2}\frac{dz}{dt}$$

$$= -\frac{1}{(1+z)}\frac{dz}{dt}$$

$$RHS = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda} = H_0 E(z)$$ where $E(z) \equiv \sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}$

Thus, $$-\frac{1}{(1+z)}\frac{dz}{dt} = H_0 E(z)$$

$$\rightarrow \boxed{\frac{dt}{dz} = \frac{-1}{(1+z)H_0 E(z)}}$$

(b) $$\Omega_{gw}(f) = \frac{1}{\rho_c}\int_0^\infty dz\, n(z)\, \frac{1}{1+z}\left(f_s \frac{dE_{gw}}{df_s}\right)\Bigg|_{f_s = f(1+z)}$$

<u>Now</u>: $n(z)\,dz = R(z)\,|dt|$

$$n(z) = R(z)\left|\frac{dt}{dz}\right| = \frac{R(z)}{(1+z)H_0 E(z)}$$

$$\rightarrow \Omega_{gw}(f) = \frac{1}{\rho_c}\int_0^\infty dz\, \frac{R(z)}{H_0 E(z)}\frac{1}{(1+z)^2}\left(f_s\frac{dE_{gw}}{df_s}\right)\Bigg|_{f_s = f(1+z)}$$

$$= \frac{1}{\rho_c H_0}\int_0^\infty dz\, \frac{R(z)}{E(z)}\frac{1}{(1+z)^{\cancel{2}}}\cancel{f(1+z)}\left(\frac{dE_{gw}}{df_s}\right)\Bigg|_{f_s = f(1+z)}$$

$$= \frac{f}{\rho_c H_0}\int_0^\infty dz\, R(z)\frac{1}{(1+z)E(z)}\left(\frac{dE_{gw}}{df_s}\right)\Bigg|_{f_s = f(1+z)}$$

(4) Optimal Filtering:

$$\hat{S}_h \simeq \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \; \delta_T(f-f') \, \tilde{d}_1(f) \, \tilde{d}_2^*(f') \, \tilde{Q}^*(f')$$

Expected value:

$$\mu \equiv \langle \hat{S}_h \rangle$$
$$= \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \; \delta_T(f-f') \, \langle \tilde{d}_1(f) \, \tilde{d}_2^*(f') \rangle \, \tilde{Q}^*(f')$$

For uncorrelated noise:

$$\langle \tilde{d}_1(f) \tilde{d}_2^*(f') \rangle = \langle \tilde{h}_1(f) \tilde{h}_2^*(f') \rangle$$
$$= \frac{1}{2} \delta(f-f') \, \Gamma_{12}(f) \, S_h(f)$$

$$\rightarrow \mu = \frac{1}{2} \int_{-\infty}^{\infty} df \; \underbrace{\delta_T(0)}_{T} \, \Gamma_{12}(f) \, S_h(f) \, \tilde{Q}^*(f)$$

$$= \frac{T}{2} \int_{-\infty}^{\infty} df \; \Gamma_{12}(f) \, S_h(f) \, \tilde{Q}^*(f)$$

Variance:

$$\sigma^2 \equiv \langle (\hat{S}_h - \langle \hat{S}_h \rangle)^2 \rangle$$
$$= \langle \hat{S}_h^2 \rangle - \mu^2$$

$$= \int df \int df' \int dp \int dp' \; \delta_T(f-f') \, \delta_T(p-p') \, \tilde{Q}^*(f') \, \tilde{Q}(p') \Big(\langle \tilde{d}_1(f) \tilde{d}_2^*(f') \tilde{d}_1^*(p) \tilde{d}_2(p') \rangle - \langle \tilde{d}_1(f) \tilde{d}_2^*(f') \rangle \langle \tilde{d}_1^*(p) \tilde{d}_2(p') \rangle \Big)$$

Use: $\langle abcd \rangle = \langle ab \rangle \langle cd \rangle + \langle ac \rangle \langle bd \rangle + \langle ad \rangle \langle bc \rangle$
for Gaussian random variables with zero mean

Then

$$\langle d_1 d_2^* d_1^* d_2 \rangle - \langle d_1 d_2^* \rangle \langle d_1^* d_2 \rangle$$

$$= \langle \tilde{d}_1(f)\, \tilde{d}_1^*(p) \rangle \langle \tilde{d}_2^*(f')\, d_2(p') \rangle + \underbrace{\langle \tilde{d}_1(f)\tilde{d}_2(p') \rangle \langle \tilde{d}_2^*(f')\tilde{d}_1^*(p) \rangle}$$

For uncorrelated noise this term is proportional to S_h^2

$$\approx \frac{1}{2}\delta(f-p)\, P_1(f)\ \frac{1}{2}\delta(f'-p')\, P_2(f')$$

power spectra of the detector output
(contains signal power as well, but $P_1(f) \approx P_{n_1}(f)$ etc., if we assume weak-signal approximation)

Thus,

$$\sigma^2 \approx \frac{1}{4}\int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df'\ \delta_T(f-f')\, \underbrace{\delta_T(f-f')}_{\approx\, \delta(f-f')}\, \tilde{Q}^*(f')\, \tilde{Q}(f')\, P_1(f) P_2(f')$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} df\ \delta_T(0)\, |\tilde{Q}(f)|^2\, P_1(f)\, P_2(f)$$

$$= \frac{T}{4}\int_{-\infty}^{\infty} df\ |\tilde{Q}(f)|^2\, P_1(f)\, P_2(f)$$

$$SNR = \frac{\mu}{\sigma} = \frac{\left(\frac{T}{2}\right)\int_{-\infty}^{\infty} df\ \Gamma_{12}(f)\, S_h(f)\, \tilde{Q}^*(f)}{\sqrt{\frac{T}{4}\int_{-\infty}^{\infty} df\ |\tilde{Q}(f)|^2\, P_1(f)\, P_2(f)}}$$

<u>Define:</u> $(A, B) = \int_{-\infty}^{\infty} df\ A(f)\, B^*(f)\, P_1(f) P_2(f)$

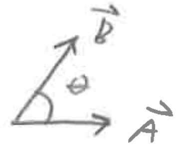
Then

$$SNR = \sqrt{T}\ \frac{\int_{-\infty}^{\infty} df\ \frac{\Gamma_{12}(f) S_h(f)}{P_1(f) P_2(f)}\ \tilde{Q}^*(f)\ P_1(f) P_2(f)}{\sqrt{\int_{-\infty}^{\infty} df\ |\tilde{Q}(f)|^2\, P_1(f)\, P_2(f)}}$$

$$= \sqrt{T}\ \frac{\left(\frac{\Gamma S_h}{P_1 P_2}, Q\right)}{\sqrt{(Q, Q)}} = \sqrt{T}\ \frac{(A, Q)}{\sqrt{(Q,Q)}}$$ where

$$A \equiv \frac{\Gamma_{12}(f)\, S_h(f)}{P_1(f)\, P_2(f)}$$

~~Now~~ Recall: $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$



so $$\frac{\vec{A}\cdot\vec{B}}{\sqrt{\vec{B}\cdot\vec{B}}} = \frac{|\vec{A}|\cancel{|\vec{B}|}\cos\theta}{\cancel{|\vec{B}|}} = |\vec{A}|\cos\theta$$

which is maximized for fixed $\vec{A}$ by choosing $\vec{B}$ to point in the same direction as $\vec{A}$ (i.e., $\theta=0$)

Thus, choose $$\tilde{Q}(f) \propto \frac{\Gamma_{12}(f)\, S_h(f)}{P_1(f)\, P_2(f)}$$
$$= \frac{\Gamma_{12}(f)\; H(f)}{P_1(f)\, P_2(f)}$$

where $H(f)$ is the spectral shape of $S_h(f)$.
[just differ by an overall amplitude]

5. (a) $p(d|a,\sigma) \propto \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\frac{(d_i-a)^2}{\sigma_i^2}\right]$

maximize likelihood $\Leftrightarrow$ maximize $\ln($likelihood$)$

<u>Define:</u> $L(a) = \ln[p(d|a,\sigma)] = -\frac{1}{2}\sum_{i=1}^{N}\frac{(d_i-a)^2}{\sigma_i^2}$

$$\frac{dL}{da} = -\sum_{i=1}^{N}\frac{(d_i-a)}{\sigma_i^2}$$

$$\rightarrow \quad 0 = \left.\frac{dL}{da}\right|_{a=\hat{a}} \qquad \text{(defines ML estimator } \hat{a}\text{)}$$

$$= -\sum_{i=1}^{N}\frac{(d_i-\hat{a})}{\sigma_i^2}$$

$$= -\sum_{i=1}^{N}\frac{d_i}{\sigma_i^2} + \hat{a}\sum_{i=1}^{N}\frac{1}{\sigma_i^2}$$

Thus, $\hat{a} = \sum_{i=1}^{N}\frac{d_i}{\sigma_i^2} \Big/ \sum_{j=1}^{N}\frac{1}{\sigma_j^2}$

(b) $p(d|A,C) \propto \exp\left[-\frac{1}{2}(d-MA)^\dagger C^{-1}(d-MA)\right]$

Dagger ($\equiv$ complex-conjugate) transpose allows for complex data d, parameters A

<u>Define:</u> $L(A, A^\dagger) = \ln[p(d|A,C)]$

$$= -\frac{1}{2}(d-MA)^\dagger C^{-1}(d-MA)$$

Vary L wrt $A, A^\dagger$ treating as independent variables

$$\delta L = \frac{1}{2}\delta A^\dagger M^\dagger C^{-1}(d-MA) + \frac{1}{2}(d-MA)^\dagger C^{-1}M\delta A$$

$\delta L = 0$ for $\delta A^\dagger$:

$$\rightarrow \quad \frac{1}{2}M^\dagger C^{-1}(d-M\hat{A}) = 0$$

$$M^\dagger C^{-1}d - M^\dagger C^{-1}M\hat{A} = 0$$

$$\hat{A} = (M^\dagger C^{-1}M)^{-1}M^\dagger C^{-1}d$$

$$\equiv F^{-1}X$$

where $F \equiv M^\dagger C^{-1}M$ and $X = M^\dagger C^{-1}d$

$\delta L = 0$ for δA:

$$\rightarrow \quad \frac{1}{2}(d - m\hat{A})^\dagger C^{-1} m = 0$$

$$\cancel{(d^\dagger - \hat{A}^\dagger m^\dagger) C^{-1} m = 0}$$

Take $\dagger$ of this equation using $(C^{-1})^\dagger = C^{-1}$:

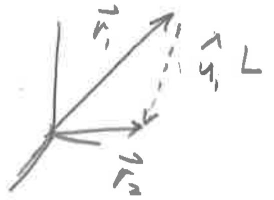
$$m^\dagger C^{-1}(d - m\hat{A}) = 0$$

$$m^\dagger C^{-1} d - m^\dagger C^{-1} m \hat{A} = 0$$

$$\text{so} \quad \hat{A} = (m^\dagger C^{-1} m)^{-1} m^\dagger C^{-1} d$$

(as before)

⑥



$$h(t) \equiv \Delta T(t) = \frac{1}{2c} u^a u^b \int_0^L ds\, h_{ab}(t(s), \vec{x}(s))$$

where $t(s) = \left(t - \frac{L}{c}\right) + \frac{s}{c}$

$$\vec{x}(s) = \vec{r}_1 + s\hat{u}$$

plane wave expansion:

$$h_{ab}(t, \vec{x}) = \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{n}} \sum_{A=+,\times} h_A(f,\hat{n}) e_{ab}^A(\hat{n})\, e^{i2\pi f(t + \hat{n}\cdot\vec{x}/c)}$$

<u>Replace</u>: t by $t(s) = \left(t - \frac{L}{c}\right) + \frac{s}{c}$

$\vec{x}$ by $\vec{x}(s) = \vec{r}_1 + s\hat{u}$

Then exponential becomes

$$e^{i2\pi f(t(s) + \hat{n}\cdot\vec{x}(s)/c)} = e^{i2\pi f\left(t - \frac{L}{c} + \frac{s}{c} + \frac{\hat{n}}{c}\cdot(\vec{r}_1 + s\hat{u})\right)}$$

$$= e^{i2\pi f\left(t - \frac{L}{c} + \frac{\hat{n}\cdot\vec{r}_1}{c}\right)} \underbrace{e^{i2\pi\frac{f}{c}(1+\hat{n}\cdot\hat{u})s}}_{\text{only term that depends on } s}$$

so

$$\int_0^L ds\, e^{i2\pi\frac{f}{c}(1+\hat{n}\cdot\hat{u})s} = \frac{1}{i2\pi\frac{f}{c}(1+\hat{n}\cdot\hat{u})}\left[e^{i2\pi\frac{f}{c}(1+\hat{n}\cdot\hat{u})L} - 1\right]$$

$$= \frac{e^{i2\pi\frac{f}{c}(1+\hat{n}\cdot\hat{u})L}}{i2\pi\frac{f}{c}(1+\hat{n}\cdot\hat{u})}\left[1 - e^{-i2\pi\frac{fL}{c}(1+\hat{n}\cdot\hat{u})}\right]$$

Combine exponential factors:

$$e^{i2\pi f\left(t - \frac{L}{c} + \hat{n}\cdot\frac{\vec{r}_1}{c}\right)}\, e^{i2\pi\frac{f}{c}(1+\hat{n}\cdot\hat{u})L}$$

$$= \exp\left[i2\pi f\left(t + \hat{n}\cdot\frac{\vec{r}_1}{c} + \hat{n}\cdot\hat{u}\frac{L}{c}\right)\right]$$

$$= \exp\left[i2\pi f\left(t + \hat{n}\cdot\underbrace{(\vec{r}_1 + \hat{u}L)}_{=\vec{r}_2}/c\right)\right]$$

$$= e^{i2\pi f t}\, e^{i2\pi f\, \hat{n}\cdot\vec{r}_2/c}$$

Putting all these results together:

$$h(t) = \frac{1}{2c} u^a u^b \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat n} \sum_{A=+,\times} h_A(f,\hat n)\, e^A_{ab}(\hat n)$$

$$e^{i2\pi f t}\, e^{i2\pi f \hat n\cdot \vec r_2/c}\, \frac{1}{\frac{i2\pi f}{c}(1+\hat n\cdot\hat u)} \left[1 - e^{-\frac{i2\pi f L}{c}(1+\hat n\cdot \hat u)}\right]$$

$$= \int_{-\infty}^{\infty} df\, e^{i2\pi f t} \int d^2\Omega_{\hat n} \sum_{A=+,\times} h_A(f,\hat n)$$

$$\underbrace{e^{\frac{i2\pi f \hat n\cdot\vec r_2}{c}}\, \frac{1}{2c}\, u^a u^b e^A_{ab}(\hat n)\, \frac{1}{\frac{i2\pi f}{c}(1+\hat n\cdot\hat u)} \left[1 - e^{-\frac{i2\pi f L}{c}(1+\hat n\cdot\hat u)}\right]}_{\equiv\ R^A(f,\hat n)}$$

Thus,

$$R^A(f,\hat n) = \underbrace{\left(\frac{1}{i2\pi f}\right)}_{\to 1 \text{ if we calculate Doppler Freq response}} \underbrace{\frac{1}{2}\, \frac{u^a u^b e^A_{ab}(\hat n)}{(1+\hat n\cdot\hat u)}}_{\equiv F^A(\hat n)} \left[1 - e^{-\frac{i2\pi f L}{c}(1+\hat n\cdot\hat u)}\right] \underbrace{e^{\frac{i2\pi f \hat n\cdot\vec r_2}{c}}}_{= 1 \text{ if we choose origin at } \vec r_2}$$

NOTE: Timing transfer function

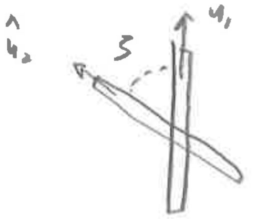
$$\mathcal{T}_{\vec u}(f, \hat n\cdot\hat u) = \frac{1}{i2\pi f}\, \frac{1}{(1+\hat n\cdot\hat u)} \left[1 - e^{-\frac{i2\pi f L}{c}(1+\hat n\cdot\hat u)}\right]$$

$$= \frac{1}{i2\pi f}\, \frac{1}{(1+\hat n\cdot\hat u)}\, e^{-\frac{i\pi f L}{c}(1+\hat n\cdot\hat u)} \underbrace{\left[e^{\frac{i\pi f L}{c}(1+\hat n\cdot\hat u)} - e^{-\frac{i\pi f L}{c}(1+\hat n\cdot\hat u)}\right]}_{2i\sin\left(\frac{\pi f L}{c}(1+\hat n\cdot\hat u)\right)}$$

$$= \frac{1}{\pi f}\, \frac{1}{(1+\hat n\cdot\hat u)}\, e^{-\frac{i\pi f L}{c}(1+\hat n\cdot\hat u)} \sin\left(\frac{\pi f L}{c}(1+\hat n\cdot\hat u)\right)$$

$$= \frac{L}{c}\, e^{-\frac{i\pi f L}{c}(1+\hat n\cdot\hat u)}\, \mathrm{sinc}\left(\frac{\pi f L}{c}(1+\hat n\cdot\hat u)\right) \quad \text{where } \mathrm{sinc}\, x \equiv \frac{\sin x}{x}$$

(7) Electric dipole antennae:



response: $r_I(t) = \hat{u}_I \cdot \vec{E}(t, \vec{x}_0)$

overlap: $\langle \tilde{r}_1(f)\, \tilde{r}_2^*(f') \rangle = \frac{1}{2} \Gamma_{12}(f)\, P(f)\, \delta(f-f')$

$\uparrow$ power in electric field

Equivalent to:

$$\langle r_1(t)\, r_2(t') \rangle = \frac{1}{2}\int_{-\infty}^{\infty} df\; e^{i2\pi f(t-t')}\, \Gamma_{12}(f)\, P(f)$$

plane-wave expansion:

$$\vec{E}(t,\vec{x}) = \int_{-\infty}^{\infty} df \int d^2\Omega_{\hat{n}} \sum_{\alpha=1}^{2} \tilde{E}_\alpha(f,\hat{n})\, \underbrace{\hat{\epsilon}_\alpha(\hat{n})}_{\text{linear polarization vector}}\, e^{i2\pi f(t+\hat{n}\cdot\vec{x}/c)}$$

Quadratic expectation values:

$$\langle \tilde{E}_\alpha(f,\hat{n})\, \tilde{E}^*_{\alpha'}(f',\hat{n}') \rangle = \frac{1}{16\pi} P(f)\, \underbrace{\delta(f-f')\,\delta_{\alpha\alpha'}\,\delta^2(\hat{n},\hat{n}')}_{\text{unpolarized \& isotropic}}$$

Thus,

$$\langle r_1(t)\, r_2(t') \rangle = \int df \int df' \int d^2\Omega_{\hat{n}} \int d^2\Omega_{\hat{n}'} \sum_\alpha \sum_{\alpha'} \langle \tilde{E}_\alpha(f,\hat{n})\, \tilde{E}^*_{\alpha'}(f',\hat{n}') \rangle\; \hat{u}_1 \cdot \hat{\epsilon}_\alpha(\hat{n})\; \hat{u}_2 \cdot \hat{\epsilon}_{\alpha'}(\hat{n}')\; e^{i2\pi(ft-f't')}\, e^{i2\pi(f\hat{n}-f'\hat{n}')\cdot\vec{x}_0/c}$$

$$= \frac{1}{16\pi} \int df \int d^2\Omega_{\hat{n}} \sum_\alpha P(f)\; \hat{u}_1 \cdot \hat{\epsilon}_\alpha(\hat{n})\; \hat{u}_2 \cdot \hat{\epsilon}_\alpha(\hat{n})\; e^{i2\pi f(t-t')}$$

<u>Now</u>,
$$\hat{\epsilon}_1(\hat{n}) = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} = \hat{\theta}$$
$$\hat{\epsilon}_2(\hat{n}) = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} = \hat{\phi}$$

where $\hat{n}$ points radially outward

To do the calculation, take $\hat{x}, \hat{y}, \hat{z}$ such that $\hat{u}_1 = \hat{z}$ and $\hat{u}_2 = \sin\zeta\,\hat{x} + \cos\zeta\,\hat{z}$ (in $\hat{x}\hat{z}$ plane)

Then:

$$\hat{u}_1 \cdot \hat{\epsilon}_1(\hat{n}) = \hat{z}\cdot\hat{\epsilon}_1(\hat{n}) = -\sin\theta$$

$$\hat{u}_1 \cdot \hat{\epsilon}_2(\hat{n}) = \hat{z}\cdot\hat{\epsilon}_2(\hat{n}) = 0$$

$$\hat{u}_2 \cdot \hat{\epsilon}_1(\hat{n}) = (\sin\zeta\,\hat{x} + \cos\zeta\,\hat{z})\cdot(\cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z})$$

$$= \sin\zeta\cos\theta\cos\phi - \cos\zeta\sin\theta$$

$$\hat{u}_2 \cdot \hat{\epsilon}_2(\hat{n}) = (\sin\zeta\,\hat{x} + \cos\zeta\,\hat{z})\cdot(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$$

$$= -\sin\zeta\sin\phi$$

Thus,

$$\sum_{\alpha=1}^{2} \hat{u}_1\cdot\epsilon_\alpha(\hat{n})\;\hat{u}_2\cdot\epsilon_\alpha(\hat{n}) = \hat{u}_1\cdot\hat{\epsilon}_1(\hat{n})\;\hat{u}_2\cdot\hat{\epsilon}_1(\hat{n}) + \hat{u}_1\cdot\hat{\epsilon}_2(\hat{n})\;\hat{u}_2\cdot\hat{\epsilon}_2(\hat{n})$$

$$= -\sin\theta\,[\sin\zeta\cos\theta\cos\phi - \cos\zeta\sin\theta] + \underbrace{0\cdot[\cancel{-\sin\zeta\sin\phi}]}_{0}$$

$$= -\sin\zeta\sin\theta\cos\theta\cos\phi + \sin^2\theta\cos\zeta$$

Now integrate over the sphere

$$\int d^2\Omega_{\hat{n}} \sum_{\alpha=1}^{2} \hat{u}_1\cdot\hat{\epsilon}_\alpha(\hat{n})\;\hat{u}_2\cdot\hat{\epsilon}_\alpha(\hat{n})$$

$$= \int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi\,[-\sin\zeta\sin\theta\cos\theta\cos\phi + \sin^2\theta\cos\zeta]$$

$$= 0 + 2\pi\cos\zeta\int_0^{\pi}\sin\theta\,d\theta\,\sin^2\theta$$

$$\int_0^{2\pi} d\phi\cos\phi = 0$$

$$= 2\pi\cos\zeta\int_{-1}^{1} d(\cos\theta)\,(1-\cos^2\theta)$$

$$= 2\pi\cos\zeta\left[x - \frac{x^3}{3}\right]\Big|_{-1}^{1}$$

$$= 2\pi\cos\zeta\left(\frac{4}{3}\right)$$

$$= \frac{8\pi}{3}\cos\zeta$$

Thus,

$$\langle r_1(t)\, r_2(t) \rangle = \frac{1}{\cancel{16\pi}\ 2} \int_{-\infty}^{\infty} df \; e^{i2\pi f(t-t')} P(f)\, \frac{8\cancel{\pi}}{3} \cos 5$$

$$= \frac{1}{6} \cos 5 \int_{-\infty}^{\infty} df \; e^{i2\pi f(t-t')} P(f)$$

$$\rightarrow \quad \frac{1}{2} \Gamma_{12}(f) = \frac{1}{6} \cos 5$$

So $\quad \Gamma_{12}(f) = \frac{1}{3} \cos 5 \quad$ [independent of freq]

⑧ Maximum-likelihood estimators:

$$p(d \mid S_{n_1}, S_{n_2}, S_h) = \frac{1}{\sqrt{\det(2\pi C)}} \exp\left[-\frac{1}{2} d^T C^{-1} d\right]$$

where
$$d = \begin{bmatrix} d_1 \\ \hline d_2 \end{bmatrix} = \begin{array}{l} \left.\begin{bmatrix} d_{11} \\ d_{12} \\ \vdots \\ d_{1N} \end{bmatrix}\right\} d_1 \\ \left.\begin{bmatrix} d_{21} \\ d_{22} \\ \vdots \\ d_{2N} \end{bmatrix}\right\} d_2 \end{array}$$

$$C = \left[\begin{array}{c|c} (S_{n_1} + S_h)\, \mathbb{1}_{N\times N} & S_h\, \mathbb{1}_{N\times N} \\ \hline S_h\, \mathbb{1}_{N\times N} & (S_{n_2} + S_h)\, \mathbb{1}_{N\times N} \end{array}\right] = \left[\begin{array}{c|c} S_1\, \mathbb{1}_{N\times N} & S_h\, \mathbb{1}_{N\times N} \\ \hline S_h\, \mathbb{1}_{N\times N} & S_2\, \mathbb{1}_{N\times N} \end{array}\right]$$

where $S_1 \equiv S_{n_1} + S_h$, $S_2 \equiv S_{n_2} + S_h$

$$\rightarrow \quad C^{-1} = \frac{1}{(S_1 S_2 - S_h^2)} \left[\begin{array}{c|c} S_2\, \mathbb{1}_{N\times N} & -S_h\, \mathbb{1}_{N\times N} \\ \hline -S_h\, \mathbb{1}_{N\times N} & S_1\, \mathbb{1}_{N\times N} \end{array}\right]$$

$$= \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)} \left[\begin{array}{c|c} \frac{1}{S_1}\, \mathbb{1}_{N\times N} & -\frac{S_h}{S_1 S_2}\, \mathbb{1}_{N\times N} \\ \hline -\frac{S_h}{S_1 S_2}\, \mathbb{1}_{N\times N} & \frac{1}{S_2}\, \mathbb{1}_{N\times N} \end{array}\right]$$

Thus,

$$p(d \mid S_{n_1}, S_{n_2}, S_h) = \frac{1}{(2\pi)^N (S_1 S_2 - S_h^2)^{N/2}} \exp\left[\frac{-1}{2} \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)} \left(\frac{1}{S_1} \sum_{i=1}^N d_{1i}^2 + \frac{1}{S_2} \sum_{i=1}^N d_{2i}^2 - \frac{2 S_h}{S_1 S_2} \sum_{i=1}^N d_{1i} d_{2i} \right)\right]$$

$$= \frac{1}{(2\pi)^N (S_1 S_2 - S_h^2)^{N/2}} \exp\left[-\frac{N}{2} \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)} \left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2 S_h \hat{C}_{12}}{S_1 S_2} \right)\right]$$

Max likelihood $\leftrightarrow$ maximize $\ln$ (likelihood)

$$\mathcal{L}(S_1, S_2, S_h) \equiv \ln \left[p(d | S_{n_1}, S_{n_2}, S_h) \right]$$

$$= -N \ln 2\pi - \frac{N}{2} \ln(S_1 S_2 - S_h^2) - \frac{N}{2} \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)} \left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2 S_h \hat{C}_{12}}{S_1 S_2} \right)$$

$$= \text{const} - \frac{N}{2} \left[\ln(S_1 S_2 - S_h^2) + \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)} \left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2 S_h \hat{C}_{12}}{S_1 S_2} \right) \right]$$

---

want to show that $\hat{C}_{11}$, $\hat{C}_{22}$, $\hat{C}_{12}$ are the ML estimators of S_1, S_2, S_h.

$$0 = \frac{\partial \mathcal{L}}{\partial S_1}$$

$$= -\frac{N}{2} \left[\left(\frac{1}{S_1 S_2 - S_h^2} \right) S_2 - \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)^2} \frac{S_h^2}{S_1^2 S_2} \left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2 S_h \hat{C}_{12}}{S_1 S_2} \right) \right.$$

$$\left. + \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)^2} \left(\frac{-\hat{C}_{11}}{S_1^2} + \frac{2 S_h \hat{C}_{12}}{S_1^2 S_2} \right) \right]$$

$$= -\frac{N}{2} \left[\left(\frac{1}{S_1 S_2 - S_h^2} \right) S_2 - \frac{1}{(S_1 S_2 - S_h^2)^2} S_2 S_h^2 \left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2 S_h \hat{C}_{12}}{S_1 S_2} \right) \right.$$

$$\left. + \frac{1}{(S_1 S_2 - S_h^2)} S_2 \left(\frac{-\hat{C}_{11}}{S_1} + \frac{2 S_h \hat{C}_{12}}{S_1 S_2} \right) \right]$$

multiply through by $-\frac{2}{N} (S_1 S_2 - S_h^2)^2 \frac{1}{S_2}$:

$$0 = (S_1 S_2 - S_h^2) - S_h^2 \left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2 S_h \hat{C}_{12}}{S_1 S_2} \right) + (S_1 S_2 - S_h^2) \left(\frac{-\hat{C}_{11}}{S_1} + \frac{2 S_h \hat{C}_{12}}{S_1 S_2} \right)$$

$$= (S_1 S_2 - S_h^2) \left(1 - \frac{\hat{C}_{11}}{S_1} + \frac{2 S_h \hat{C}_{12}}{S_1 S_2} \right) - S_h^2 \left(\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2 S_h \hat{C}_{12}}{S_1 S_2} \right)$$

substitute $\hat{c}_{11}, \hat{c}_{22}, \hat{c}_{12}$ for s_1, s_2, s_h on RHS:

$$\text{RHS} = \left(\hat{c}_{11}\hat{c}_{22} - \hat{c}_{12}^2\right)\left(\cancel{1 - \frac{\hat{c}_{11}}{\hat{c}_{11}}} + \frac{2\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right) - \hat{c}_{12}^2\left(\frac{\hat{c}_{11}}{\hat{c}_{11}} + \frac{\hat{c}_{22}}{\hat{c}_{22}} - \frac{2\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right)$$

$$= \left(\hat{c}_{11}\hat{c}_{22} - \hat{c}_{12}^2\right) 2\frac{\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}} - \hat{c}_{12}^2\left(2 - 2\frac{\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right)$$

$$= 2\hat{c}_{12}^2\left(1 - \frac{\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right) - 2\hat{c}_{12}^2\left(1 - \frac{\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right)$$

$$= 0 \checkmark$$

---

same analysis with $s_2 \leftrightarrow s_1$ gives $\left.\frac{\partial \mathcal{L}}{\partial s_2}\right|_{\substack{s_1 = \hat{c}_{11} \\ s_2 = \hat{c}_{22} \\ s_h = \hat{c}_{12}}} = 0$

Finally, consider

$$0 = \frac{\partial \mathcal{L}}{\partial s_h}$$

$$= -\frac{N}{2}\left[\left(\frac{1}{s_1 s_2 - s_h^2}\right)(-2s_h) - \frac{1}{\left(1 - \frac{s_h^2}{s_1 s_2}\right)^2}\left(\frac{-2s_h}{s_1 s_2}\right)\left(\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2s_h\hat{c}_{12}}{s_1 s_2}\right) + \frac{1}{\left(1 - \frac{s_h^2}{s_1 s_2}\right)}\left(\frac{-2\hat{c}_{12}}{s_1 s_2}\right)\right]$$

multiply through by $\frac{-2}{N}\left(s_1 s_2 - s_h^2\right)^2\left(-\frac{1}{2}\right)$

$$0 = \left(s_1 s_2 - s_h^2\right)s_h - s_h s_1 s_2\left(\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2s_h\hat{c}_{12}}{s_1 s_2}\right) + \left(s_1 s_2 - s_h^2\right)\hat{c}_{12}$$

$$= \left(s_1 s_2 - s_h^2\right)\left(s_h + \hat{c}_{12}\right) - s_h s_1 s_2\left(\frac{\hat{c}_{11}}{s_1} + \frac{\hat{c}_{22}}{s_2} - \frac{2s_h\hat{c}_{12}}{s_1 s_2}\right)$$

substitute $\hat{c}_{11}, \hat{c}_{22}, \hat{c}_{12}$ for s_1, s_2, s_h on RHS:

$$\text{RHS} = \left(\hat{c}_{11}\hat{c}_{22} - \hat{c}_{12}^2\right)\left(\hat{c}_{12} + \hat{c}_{12}\right) - \hat{c}_{12}\hat{c}_{11}\hat{c}_{22}\left(\underbrace{\frac{\hat{c}_{11}}{\hat{c}_{11}} + \frac{\hat{c}_{22}}{\hat{c}_{22}}}_{2} - \frac{2\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right)$$

$$= 2\hat{c}_{12}\left(\hat{c}_{11}\hat{c}_{22} - \hat{c}_{12}^2\right) - 2\hat{c}_{12}\hat{c}_{11}\hat{c}_{22}\left(1 - \frac{\hat{c}_{12}^2}{\hat{c}_{11}\hat{c}_{22}}\right)$$

$$= 0 \checkmark$$

Thus, $\hat{c}_{11}, \hat{c}_{22}, \hat{c}_{12}$ are ML estimators of s_1, s_2, s_h.

(9.) Maximum-likelihood ratio detection statistic

$$p(d \mid S_{n_1}, S_{n_2}, M_0) = \frac{1}{\sqrt{\det(2\pi C_n)}} \exp\left[-\frac{1}{2} d^T C_n^{-1} d\right]$$

$$p(d \mid S_{n_1}, S_{n_2}, S_h, M_1) = \frac{1}{\sqrt{\det(2\pi C)}} \exp\left[-\frac{1}{2} d^T C^{-1} d\right]$$

where
$$d \equiv \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} d_{11} \\ d_{12} \\ \vdots \\ d_{1N} \\ d_{21} \\ d_{22} \\ \vdots \\ d_{2N} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} d_1 \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} d_2 \end{matrix}, \quad C_n = \begin{bmatrix} S_{n_1} \mathbb{1}_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & S_{n_2} \mathbb{1}_{N\times N} \end{bmatrix}$$

$$C = \begin{bmatrix} (S_{n_1}+S_h)\mathbb{1}_{N\times N} & S_h \mathbb{1}_{N\times N} \\ S_h \mathbb{1}_{N\times N} & (S_{n_2}+S_h)\mathbb{1}_{N\times N} \end{bmatrix} = \begin{bmatrix} S_1 \mathbb{1}_{N\times N} & S_h \mathbb{1}_{N\times N} \\ S_h \mathbb{1}_{N\times N} & S_2 \mathbb{1}_{N\times N} \end{bmatrix}$$

NOTE:
$$C_n^{-1} = \begin{bmatrix} \frac{1}{S_{n_1}} \mathbb{1}_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & \frac{1}{S_{n_2}} \mathbb{1}_{N\times N} \end{bmatrix}$$

$$C^{-1} = \frac{1}{S_1 S_2 - S_h^2} \begin{bmatrix} S_2 \mathbb{1}_{N\times N} & -S_h \mathbb{1}_{N\times N} \\ -S_h \mathbb{1}_{N\times N} & S_1 \mathbb{1}_{N\times N} \end{bmatrix}$$

$$= \frac{1}{\left(1 - \frac{S_h^2}{S_1 S_2}\right)} \begin{bmatrix} \frac{1}{S_1} \mathbb{1}_{N\times N} & -\frac{S_h}{S_1 S_2} \mathbb{1}_{N\times N} \\ -\frac{S_h}{S_1 S_2} \mathbb{1}_{N\times N} & \frac{1}{S_2} \mathbb{1}_{N\times N} \end{bmatrix}$$

Arguments of exponential:

$$-\frac{1}{2} d^T C_n^{-1} d = -\frac{1}{2}\left(\frac{1}{S_{n_1}} \sum_{i=1}^N d_{1i}^2 + \frac{1}{S_{n_2}} \sum_{i=1}^N d_{2i}^2\right)$$

$$= -\frac{N}{2}\left(\frac{1}{S_{n_1}}\left(\frac{1}{N}\sum_{i=1}^N d_{1i}^2\right) + \frac{1}{S_{n_2}}\left(\frac{1}{N}\sum_{i=1}^N d_{2i}^2\right)\right)$$

$$= -\frac{N}{2}\left(\frac{\hat{C}_{11}}{S_{n_1}} + \frac{\hat{C}_{22}}{S_{n_2}}\right)$$

Similarly

$$-\frac{1}{2} d^T C^{-1} d = -\frac{1}{2} \frac{1}{\left(1-\frac{S_h^2}{S_1 S_2}\right)} \left[\frac{1}{S_1} \sum_{i=1}^{N} d_{1i}^2 + \frac{1}{S_2} \sum_{i=1}^{N} d_{2i}^2 - \frac{2 S_h}{S_1 S_2} \sum_{i=1}^{N} d_{1i} d_{2i} \right]$$

$$= -\frac{N}{2} \frac{1}{\left(1-\frac{S_h^2}{S_1 S_2}\right)} \left[\frac{1}{S_1}\left(\frac{1}{N}\sum_{i=1}^{N} d_{1i}^2\right) + \frac{1}{S_2}\left(\frac{1}{N}\sum_{i=1}^{N} d_{2i}^2\right) - \frac{2 S_h}{S_1 S_2}\left(\frac{1}{N}\sum_{i=1}^{N} d_{1i} d_{2i}\right)\right]$$

$$= -\frac{N}{2} \frac{1}{\left(1-\frac{S_h^2}{S_1 S_2}\right)} \left[\frac{\hat{C}_{11}}{S_1} + \frac{\hat{C}_{22}}{S_2} - \frac{2 S_h}{S_1 S_2} \hat{C}_{12} \right]$$

when $\hat{C}_{11} \equiv \frac{1}{N}\sum_{i=1}^{N} d_{1i}^2$, $\hat{C}_{22} \equiv \frac{1}{N}\sum_{i=1}^{N} d_{2i}^2$, $\hat{C}_{12} \equiv \frac{1}{N}\sum_{i=1}^{N} d_{1i} d_{2i}$

__NOTE__: As shown in Exercise 8, these data combinations are the maximum-likelihood estimators of S_{n_1} and S_{n_2} for the noise-only model and ML estimators of S_1, S_2, S_h for the signal + noise model

__Thus__, for M_0: $\hat{S}_{n_1} = \hat{C}_{11}$, $\hat{S}_{n_2} = \hat{C}_{22}$

for M_1: $\hat{S}_1 = \hat{C}_{11}$, $\hat{S}_2 = \hat{C}_{22}$, $\hat{S}_h = \hat{C}_{12}$

$$\hat{S}_{n_1} = \hat{S}_1 - \hat{S}_h = \hat{C}_{11} - \hat{C}_{12}$$
$$\hat{S}_{n_2} = \hat{S}_2 - \hat{S}_h = \hat{C}_{22} - C_{12}$$

Detection statistic:

$$\Lambda_{ML}(d) = \frac{\max_{S_{n_1}, S_{n_2}, S_h} p(d \mid S_{n_1}, S_{n_2}, S_h, M_1)}{\max_{S_{n_1}, S_{n_2}} p(d \mid S_{n_1}, S_{n_2}, M_0)}$$

$$\text{numerator} = \frac{1}{(2\pi)^N \left(\hat{C}_{11}\hat{C}_{22} - \hat{C}_{12}^2\right)^{N/2}} \exp\left[-\frac{1}{2} \frac{N}{\left(1-\frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{12}}\right)} \underbrace{\left(\frac{\hat{C}_{11}}{\hat{C}_{11}} + \frac{\hat{C}_{22}}{\hat{C}_{22}} - 2\frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}\right)}_{2\left(1-\frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}\right)}\right]$$

$$= \frac{1}{(2\pi)^N \left(\hat{C}_{11}\hat{C}_{22} - \hat{C}_{12}^2\right)^{N/2}} \exp\left[-N\right]$$

denominator $= \dfrac{1}{(2\pi)^N (\hat{C}_{11}\hat{C}_{22})^{N/2}} \exp\left[-\dfrac{N}{2}\left(\underbrace{\dfrac{\hat{C}_{11}}{\hat{C}_{11}} + \dfrac{\hat{C}_{22}}{\hat{C}_{22}}}_{2}\right)\right]$

$$= \frac{1}{(2\pi)^N (\hat{C}_{11}\hat{C}_{22})^{N/2}} \exp[-N]$$

Thus, $\Lambda_{ML}(d) = \dfrac{\dfrac{1}{\cancel{(2\pi)^N}(\hat{C}_{11}\hat{C}_{22} - \hat{C}_{12}^2)^{N/2}} \cancel{\exp[-N]}}{\dfrac{1}{\cancel{(2\pi)^N}(\hat{C}_{11}\hat{C}_{22})^{N/2}} \cancel{\exp[-N]}}$

$$= \left(\frac{\hat{C}_{11}\hat{C}_{22}}{\hat{C}_{11}\hat{C}_{22} - \hat{C}_{12}^2}\right)^{N/2}$$

$$= \frac{1}{\left(1 - \dfrac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}\right)^{N/2}}$$

$$2 \ln \Lambda_{ML}(d) = 2 \ln\left[\left(1 - \frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}\right)^{-N/2}\right]$$

$$= -N \ln\left(1 - \frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}\right)$$

$$\approx N \frac{\hat{C}_{12}^2}{\hat{C}_{11}\hat{C}_{22}}$$

since $\ln(1+x) \approx x$ for small x

(10.) Perform marginalization integral

$$p(d \mid S_{n_1}, S_{n_2}, S_h) = \int_{-\infty}^{\infty} dh\, p_n(d-h \mid S_{n_1}, S_{n_2})\, p(h \mid S_h)$$

where $$p_n(d-h \mid S_{n_1}, S_{n_2}) = \frac{1}{2\pi\sqrt{S_{n_1} S_{n_2}}} \exp\left[-\frac{1}{2}\left(\frac{(d_1-h)^2}{S_{n_1}} + \frac{(d_2-h)^2}{S_{n_2}}\right)\right]$$

$$p(h \mid S_h) = \frac{1}{\sqrt{2\pi S_h}} \exp\left[-\frac{1}{2}\frac{h^2}{S_h}\right]$$

Integrand on RHS

$$= \frac{1}{(2\pi)^{3/2}\sqrt{S_{n_1} S_{n_2} S_h}} \exp\left[-\frac{1}{2}\left(\frac{(d_1-h)^2}{S_{n_1}} + \frac{(d_2-h)^2}{S_{n_2}} + \frac{h^2}{S_h}\right)\right]$$

Now: $$[\;] = -\frac{1}{2}\left(\frac{(d_1^2+h^2-2d_1h)}{S_{n_1}} + \frac{(d_2^2+h^2-2d_2h)}{S_{n_2}} + \frac{h^2}{S_h}\right)$$

$$= -\frac{1}{2}\left[h^2\left(\frac{1}{S_{n_1}} + \frac{1}{S_{n_2}} + \frac{1}{S_h}\right) - 2h\left(\frac{d_1}{S_{n_1}} + \frac{d_2}{S_{n_2}}\right) + \left(\frac{d_1^2}{S_{n_1}} + \frac{d_2^2}{S_{n_2}}\right)\right]$$

$$= -\frac{1}{2}\left[Ah^2 - 2hB + D\right]$$

$$= -\frac{A}{2}\left[h^2 - 2h\frac{B}{A} + \frac{D}{A}\right]$$

where $$A \equiv \frac{1}{S_{n_1}} + \frac{1}{S_{n_2}} + \frac{1}{S_h}$$

$$= \frac{S_{n_2}S_h + S_{n_1}S_h + S_{n_1}S_{n_2}}{S_{n_1}S_{n_2}S_h}$$

$$= \frac{S_{n_1}S_{n_2} + S_h(S_{n_1}+S_{n_2})}{S_{n_1}S_{n_2}S_h}$$

$$= \frac{\det C}{\det C_n \cdot S_h}$$

where $$C = \begin{vmatrix} S_{n_1}+S_h & S_h \\ S_h & S_{n_2}+S_h \end{vmatrix}$$

$$C_n = \begin{vmatrix} S_{n_1} & 0 \\ 0 & S_{n_2} \end{vmatrix}$$

$$B \equiv \frac{d_1}{S_{n1}} + \frac{d_2}{S_{n2}}$$

$$D \equiv \frac{d_1^2}{S_{n1}} + \frac{d_2^2}{S_{n2}}$$

complete the square:

$$[\ ] = -\frac{A}{2}\left[\left(h-\frac{B}{A}\right)^2 - \frac{B^2}{A^2} + \frac{D}{A}\right]$$

$$= -\frac{A}{2}\left[\left(h-\frac{B}{A}\right)^2 + \left(\frac{B^2 - AD}{A^2}\right)\right]$$

Now:

$$\int_{-\infty}^{\infty} dh \, \exp\left[-\frac{A}{2}\left(h-\frac{B}{A}\right)^2\right] = \sqrt{2\pi}\,\frac{1}{\sqrt{A}}$$

using $\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} dx \, e^{-\frac{(x-\mu)^2}{2\sigma^2}} = 1$ for a Gaussian probability distribution.

Thus,

$$p(d \,|\, S_{n1}, S_{n2}, S_h) = \frac{1}{(2\pi)^{3/2}\sqrt{S_{n1}S_{n2}S_h}}\,\cancel{\sqrt{2\pi}}\,\frac{1}{\sqrt{A}}\exp\left[+\frac{A}{2}\left(\frac{B^2 - AD}{A^2}\right)\right]$$

$$= \frac{1}{2\pi\sqrt{\det C}}\exp\left[-\frac{1}{2}\left(\frac{AD - B^2}{A}\right)\right]$$

Argument of the exponential

$$-\frac{1}{2}\left(\frac{AD-B^2}{A}\right) = -\frac{1}{2}\left(\frac{S_{n1}S_{n2}S_h}{\det C}\right)\left(\frac{S_{n1}S_{n2} + S_h(S_{n1}+S_{n2})}{S_{n1}S_{n2}S_h}\left(\frac{d_1^2}{S_{n1}} + \frac{d_2^2}{S_{n2}}\right) - \left(\frac{d_1}{S_{n1}} + \frac{d_2}{S_{n2}}\right)^2\right)$$

$$= -\frac{1}{2}\left(\frac{1}{\det C}\right)\left(\left(S_{n1}S_{n2} + S_h(S_{n1}+S_{n2})\right)\left(\frac{d_1^2}{S_{n1}} + \frac{d_2^2}{S_{n2}}\right) - S_{n1}S_{n2}S_h\left(\frac{d_1^2}{S_{n1}^2} + \frac{d_2^2}{S_{n2}^2} + \frac{2d_1d_2}{S_{n1}S_{n2}}\right)\right)$$

$$= -\frac{1}{2}\left(\frac{1}{\det C}\right)\left(d_1^2\left(S_{n2} + S_h + \cancel{\frac{S_h S_{n2}}{S_{n1}}} - \cancel{\frac{S_h S_{n2}}{S_{n1}}}\right) + d_2^2\left(S_{n1} + S_h + \cancel{\frac{S_h S_{n1}}{S_{n2}}} - \cancel{\frac{S_h S_{n1}}{S_{n2}}}\right) - 2S_h d_1 d_2\right)$$

$$= -\frac{1}{2}\left(\frac{1}{\det C}\right)\left(d_1^2 (S_{n_2}+S_h) + d_2^2 (S_{n_1}+S_h) - 2S_h\, d_1 d_2 \right)$$

$$= -\frac{1}{2}\left(d_1^2 \left(\frac{S_{n_2}+S_h}{\det C}\right) + d_2^2 \left(\frac{S_{n_1}+S_h}{\det C}\right) + 2\, d_1 d_2 \left(\frac{-S_h}{\det C}\right)\right)$$

$$= -\frac{1}{2} \sum_{I,J=1}^{2} d_I \left(C^{-1}\right)_{IJ}\, d_J$$

where $(C^{-1})_{IJ}$ are the matrix components of the inverse to the matrix $C = \begin{bmatrix} S_{h_1}+S_h & S_h \\ S_h & S_{n_2}+S_h \end{bmatrix}$

i.e. $C^{-1} = \left(\frac{1}{\det C}\right) \begin{bmatrix} S_{h_2}+S_h & -S_h \\ -S_h & S_{n_1}+S_h \end{bmatrix}$