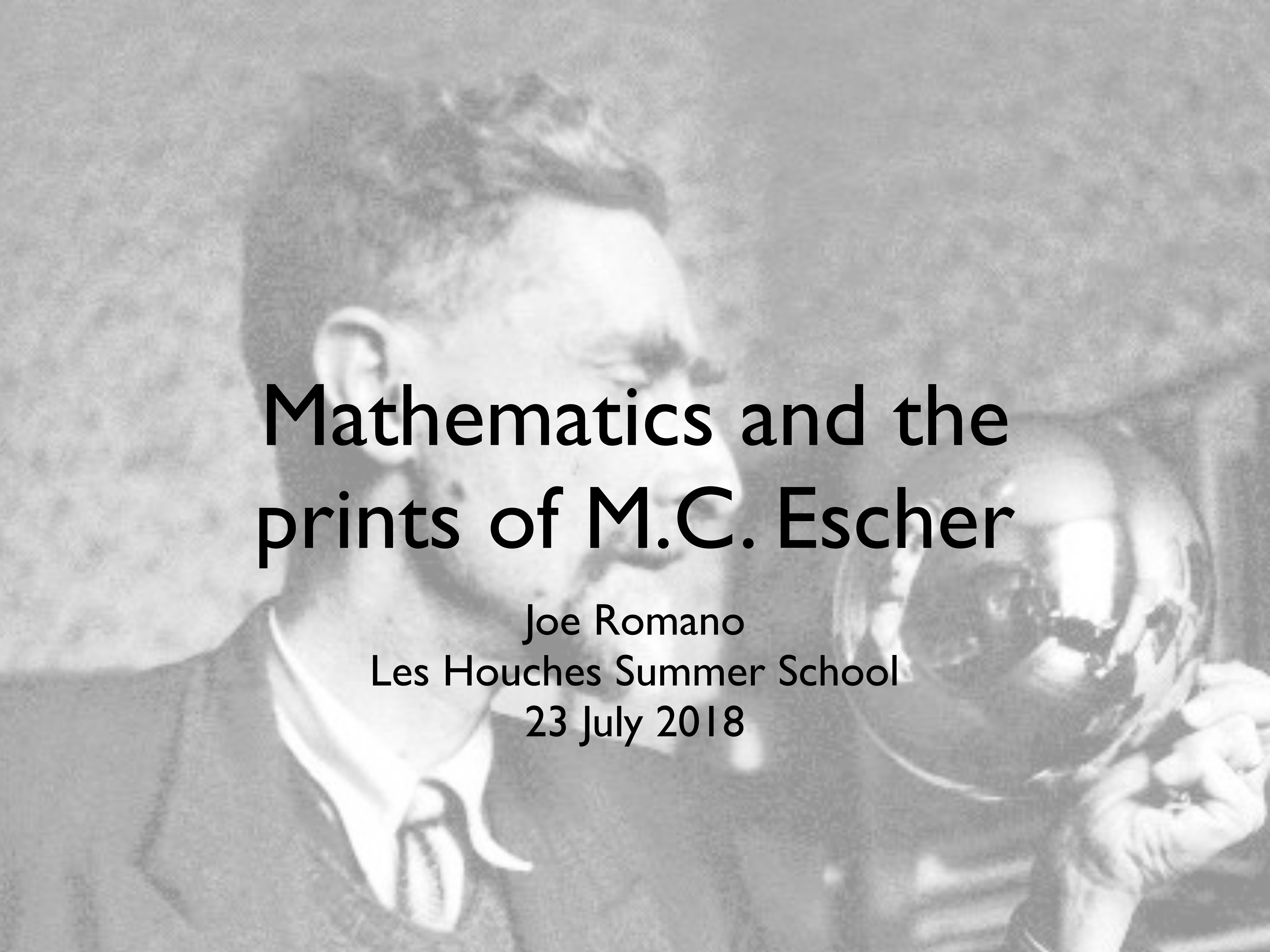


Mathematics and the prints of M.C. Escher

Joe Romano
Les Houches Summer School
23 July 2018

Possible topics

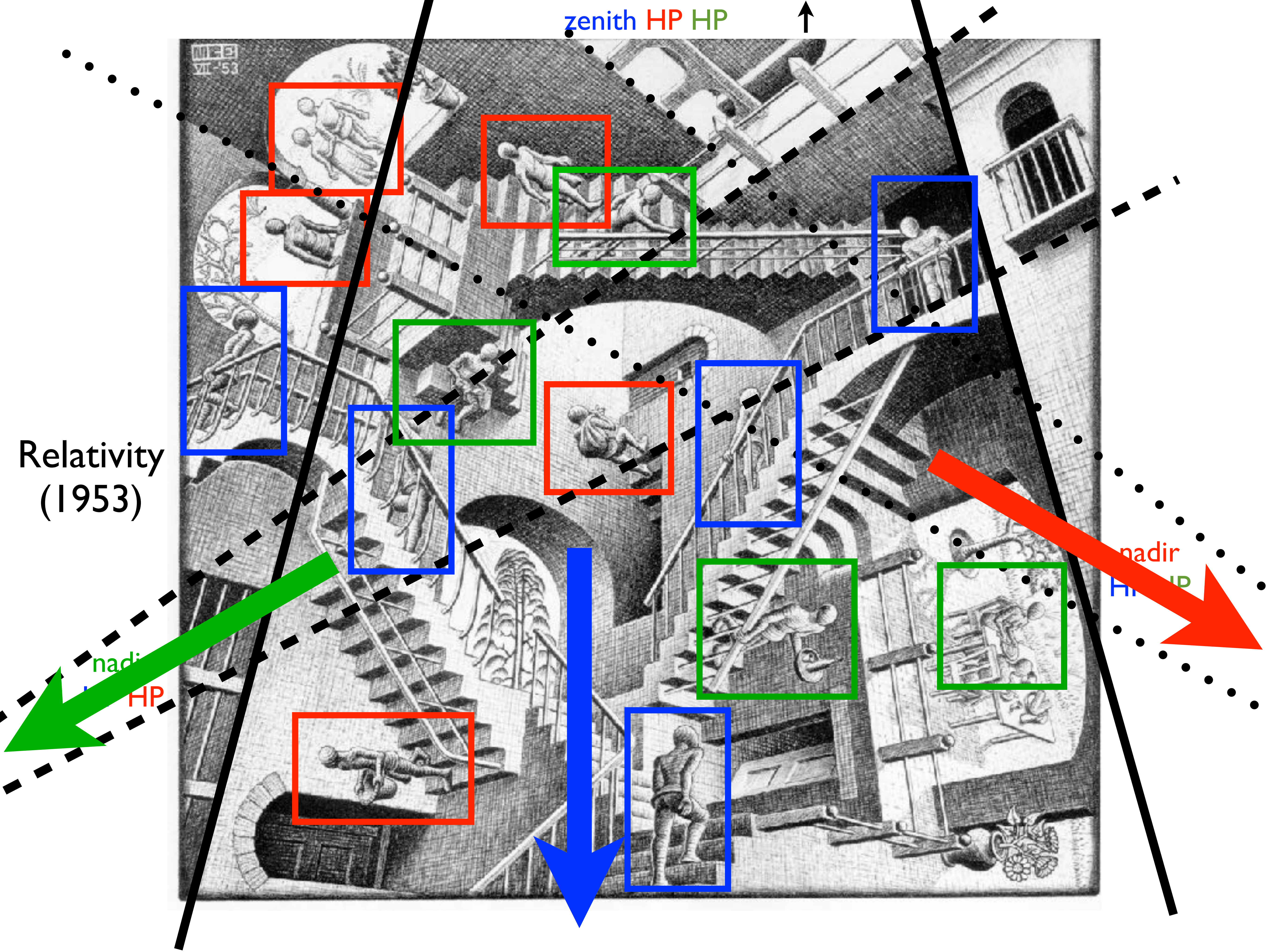
- projective geometry
- non-Euclidean geometry
- topology & knots
- duality & complementarity
- symmetry (periodic tilings)
- representations of infinity
- ambiguous perspective
- impossible objects
- …

Plan of talk

1. Escher - brief biography (connection to R. Penrose)
2. Symmetry transformations - periodic tilings of the plane
3. Periodic tilings using regular polygons (non-Euclidean)

I. Escher the person

- Maurits Cornelis Escher (1898-1972)
- Dutch graphic artist (woodcuts, lithographs)
- Inspired by visits to the Alhambra in Granada, Spain in 1922 and 1936
- From 1937, the topic of periodic tilings of the plane was his "richest source of inspiration"
- He was not a mathematician but thought like a mathematician (corresponded with Coxeter and Penrose)



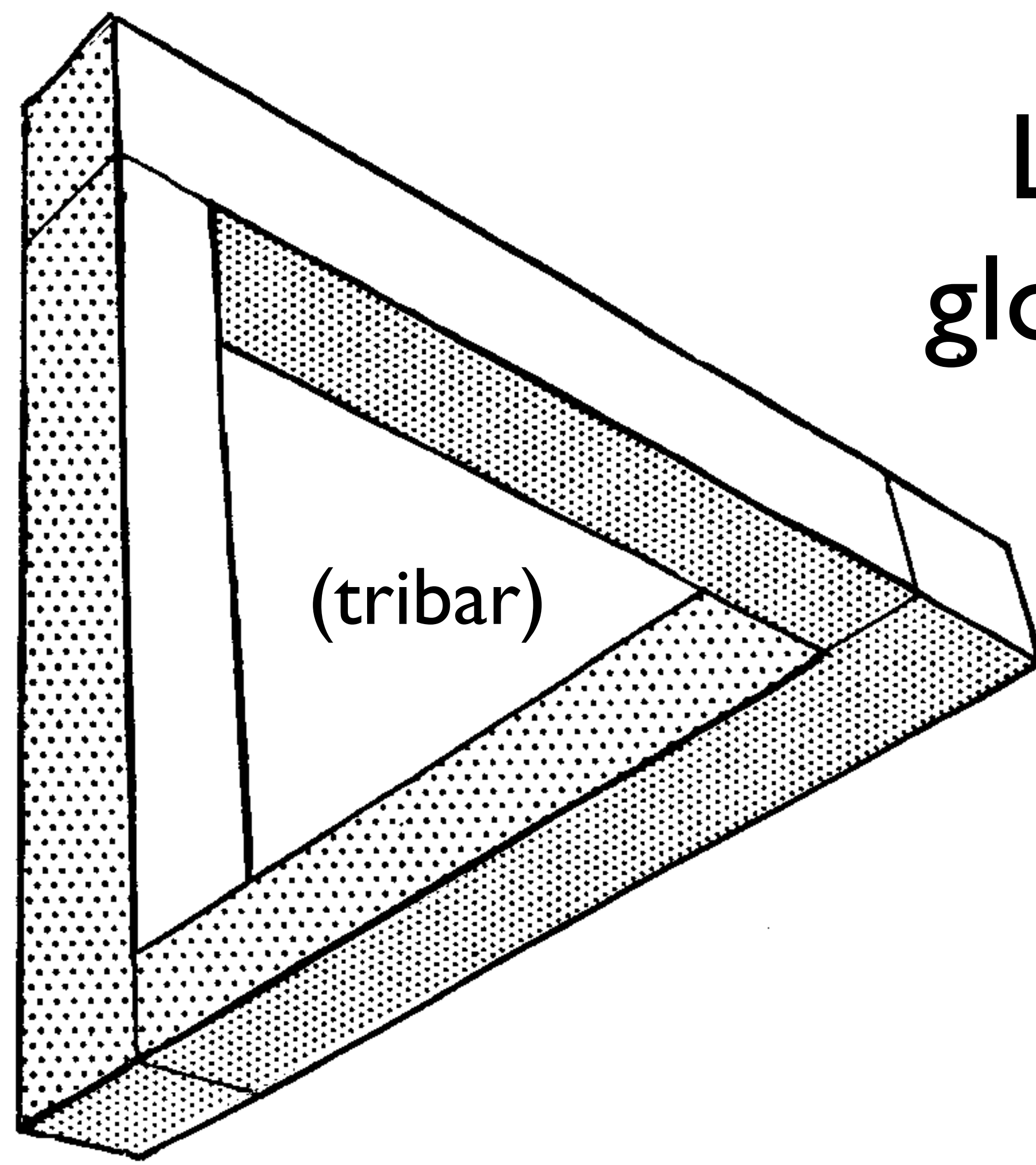
Relativity (1953)

IMPOSSIBLE OBJECTS: A SPECIAL TYPE OF VISUAL ILLUSION

BY L. S. PENROSE AND R. PENROSE

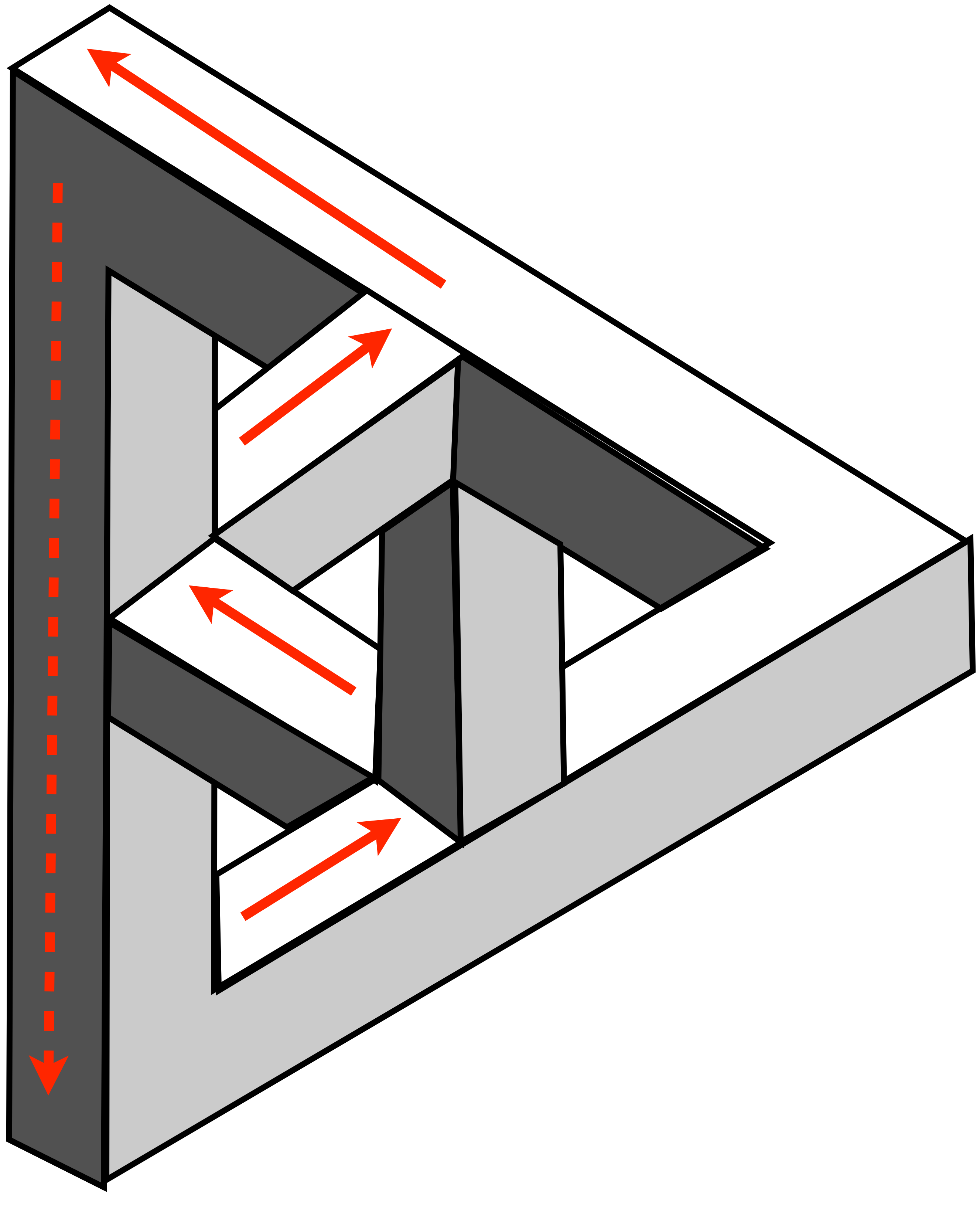
(*University College, London, and Bedford College, London*)

British Journal of Psychology, 49:1 (Feb 1958), p.31

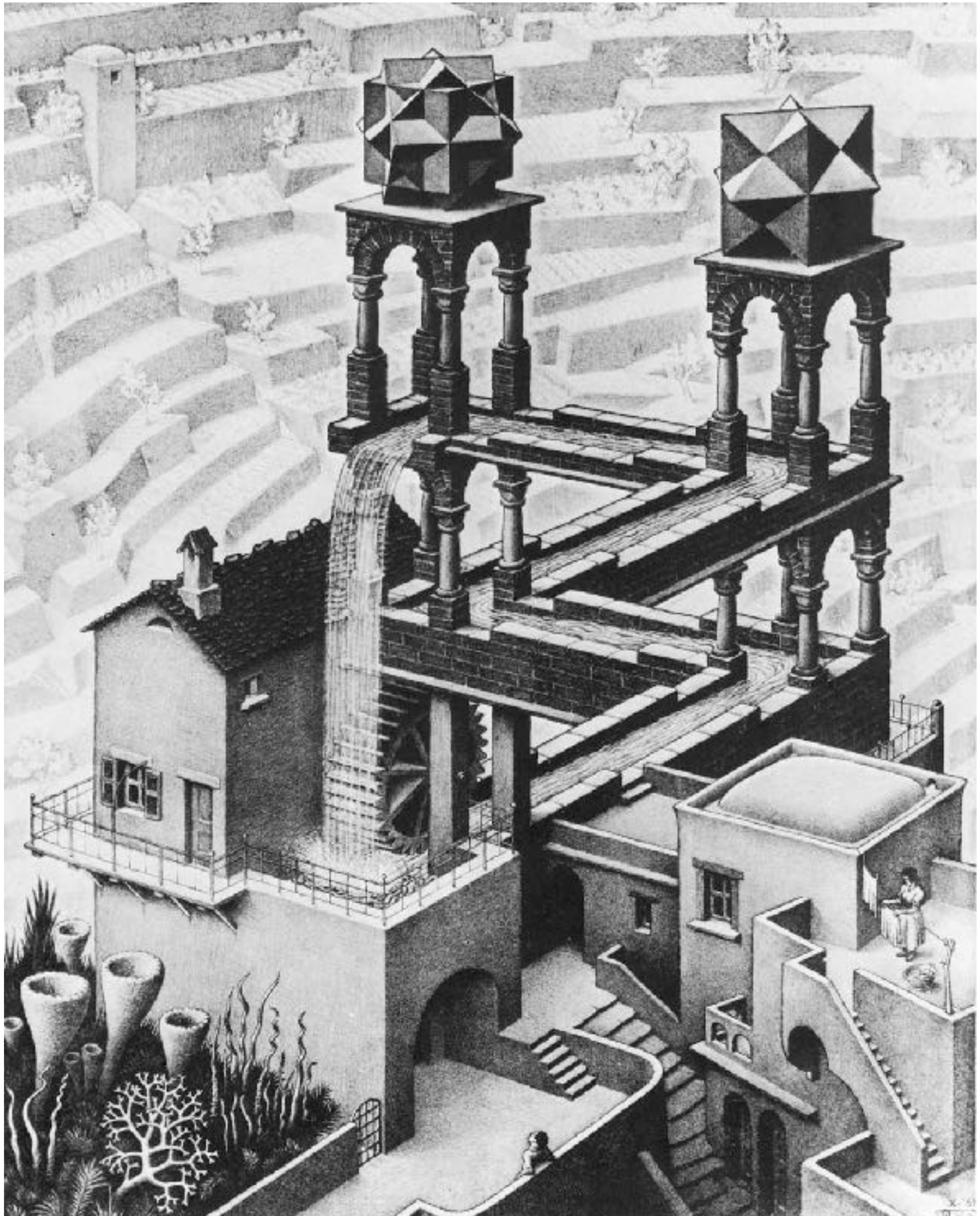


Local possibility, global impossibility!

Fig. 1. Perspective drawing of impossible structure.



Waterfall
(1961)



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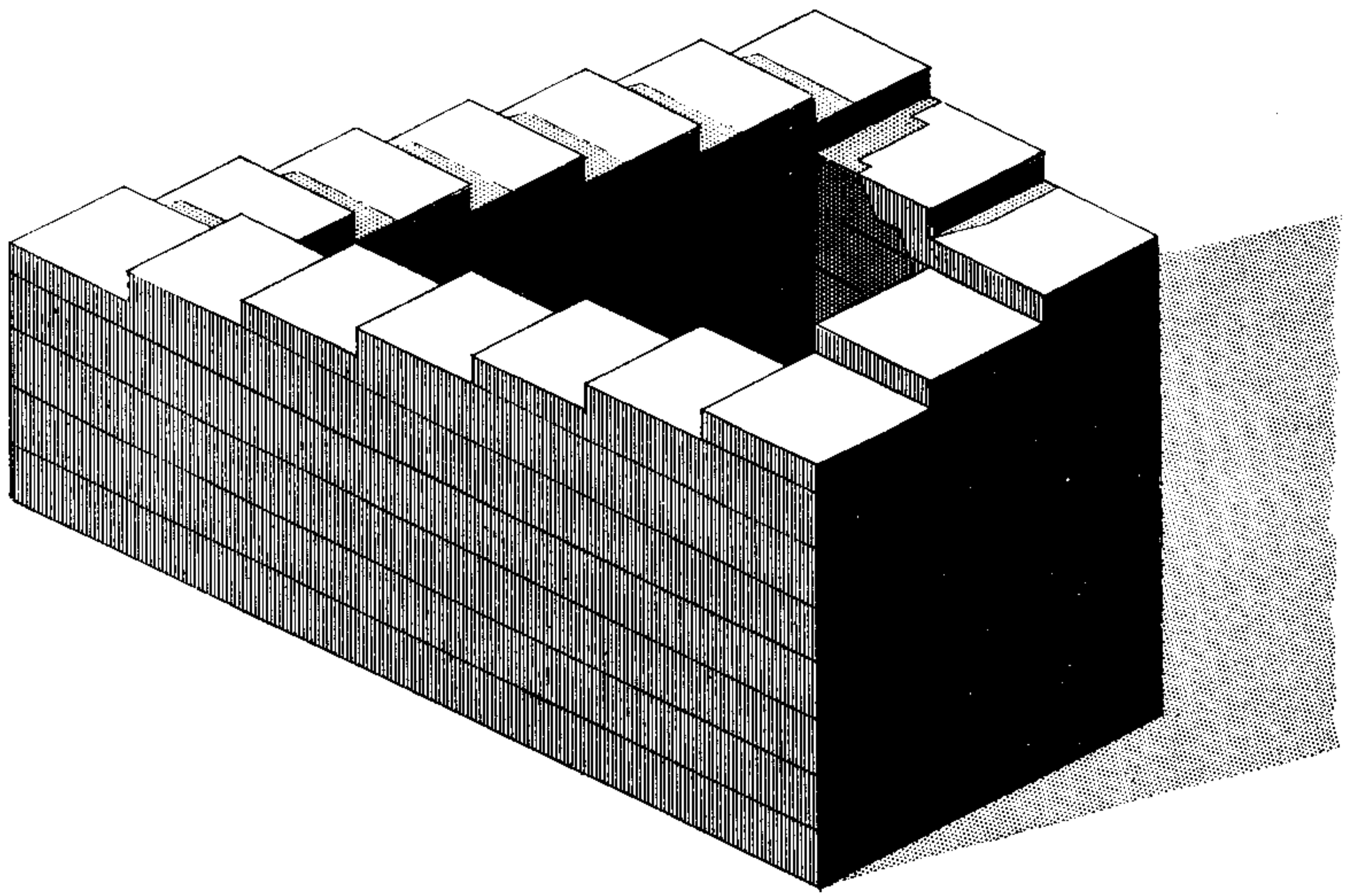
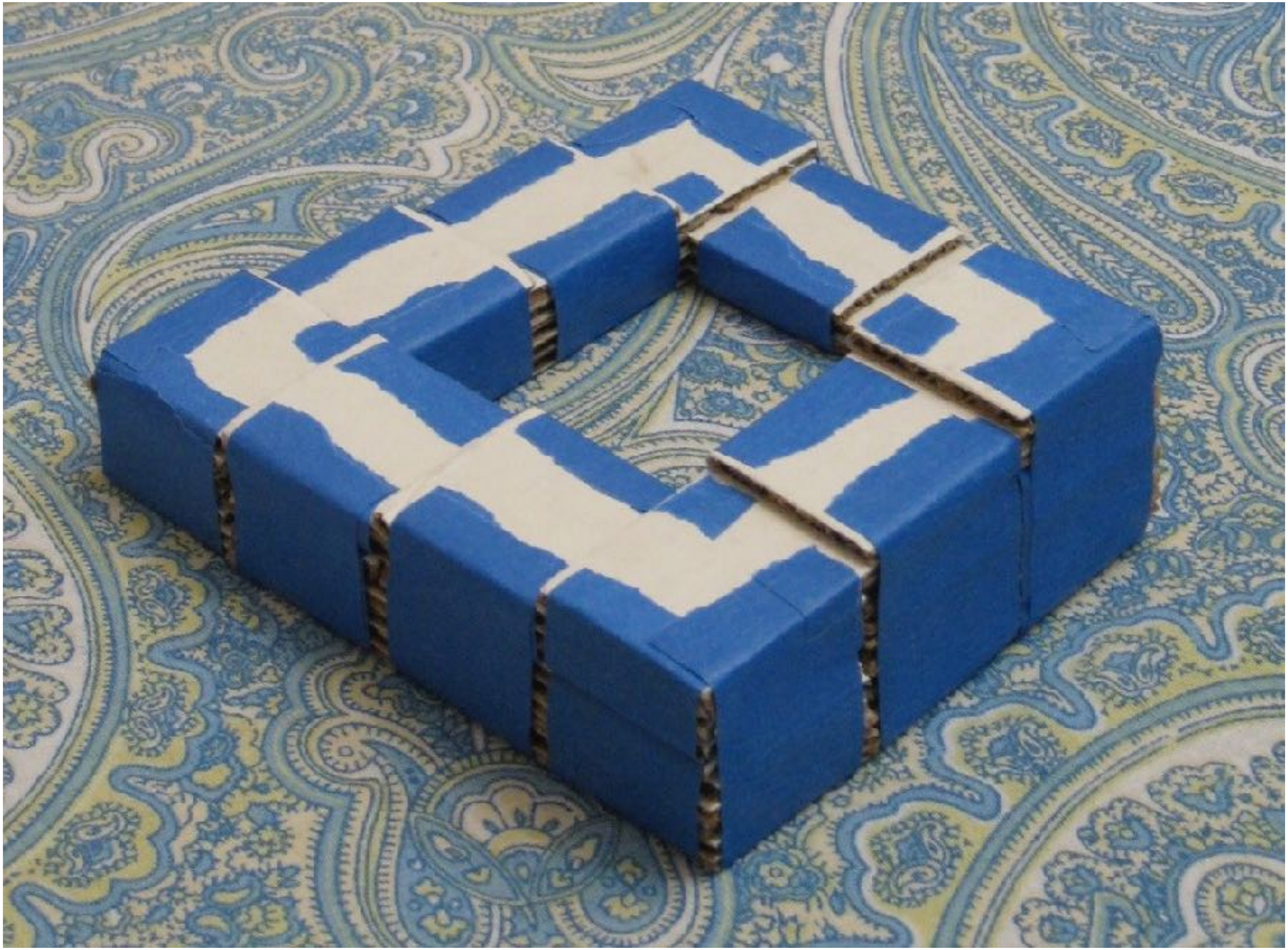
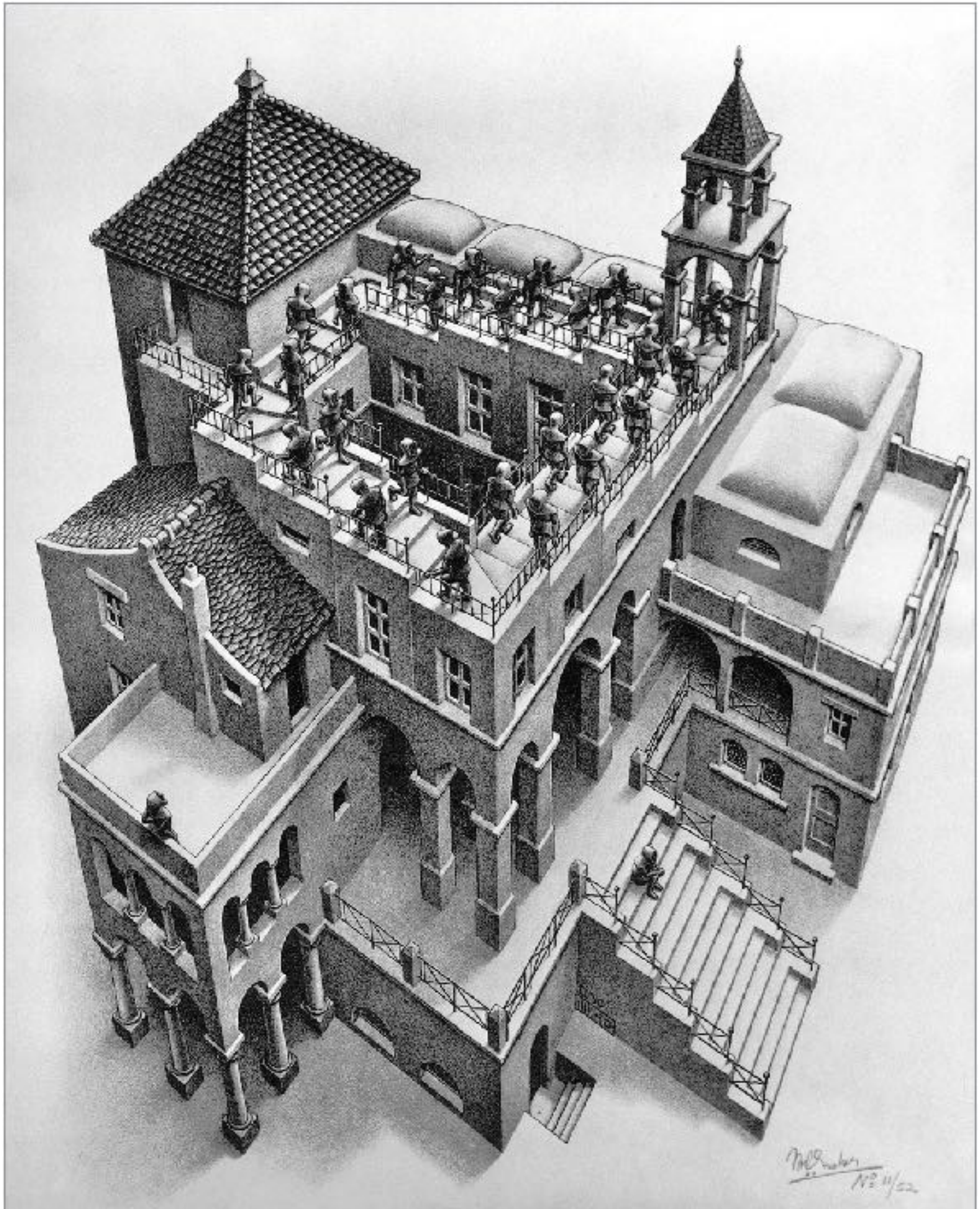


Fig. 3. Continuous flight of steps: shadowed drawing.



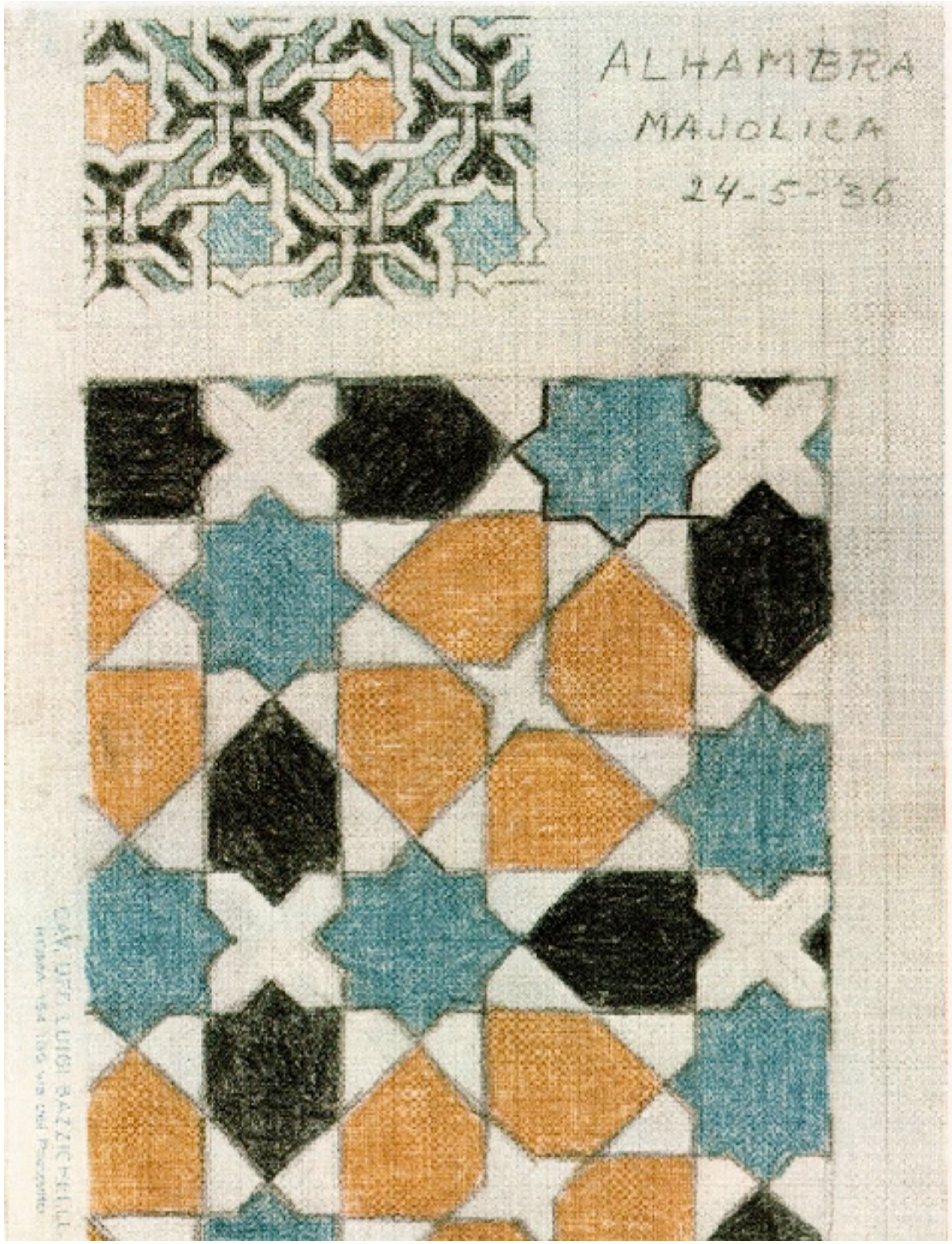
Ascending and Descending (1960)

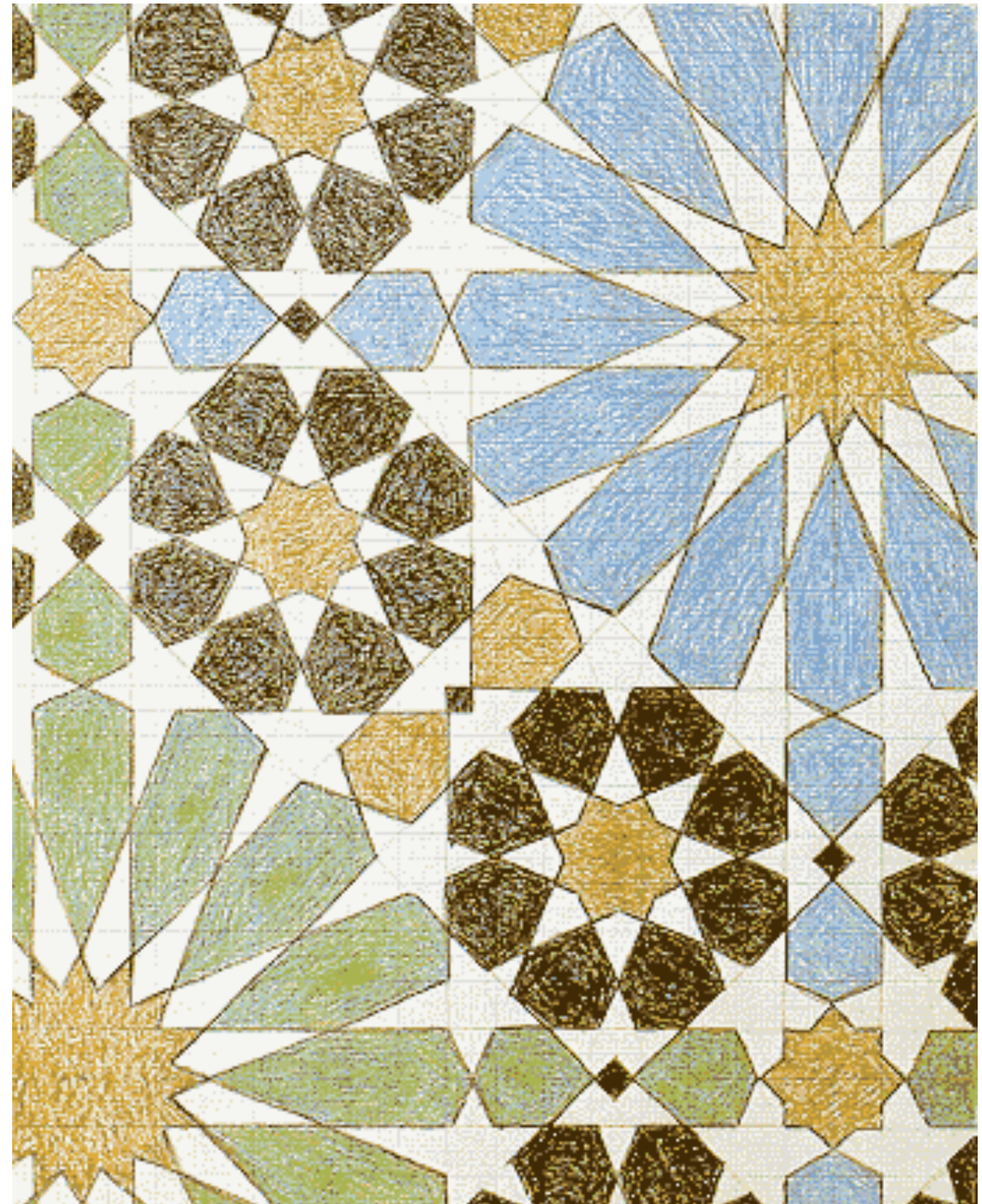


2. Symmetry transformations


The Alhambra
(Granada, Spain)
fortress (889)
palace (1333)

Escher's sketches

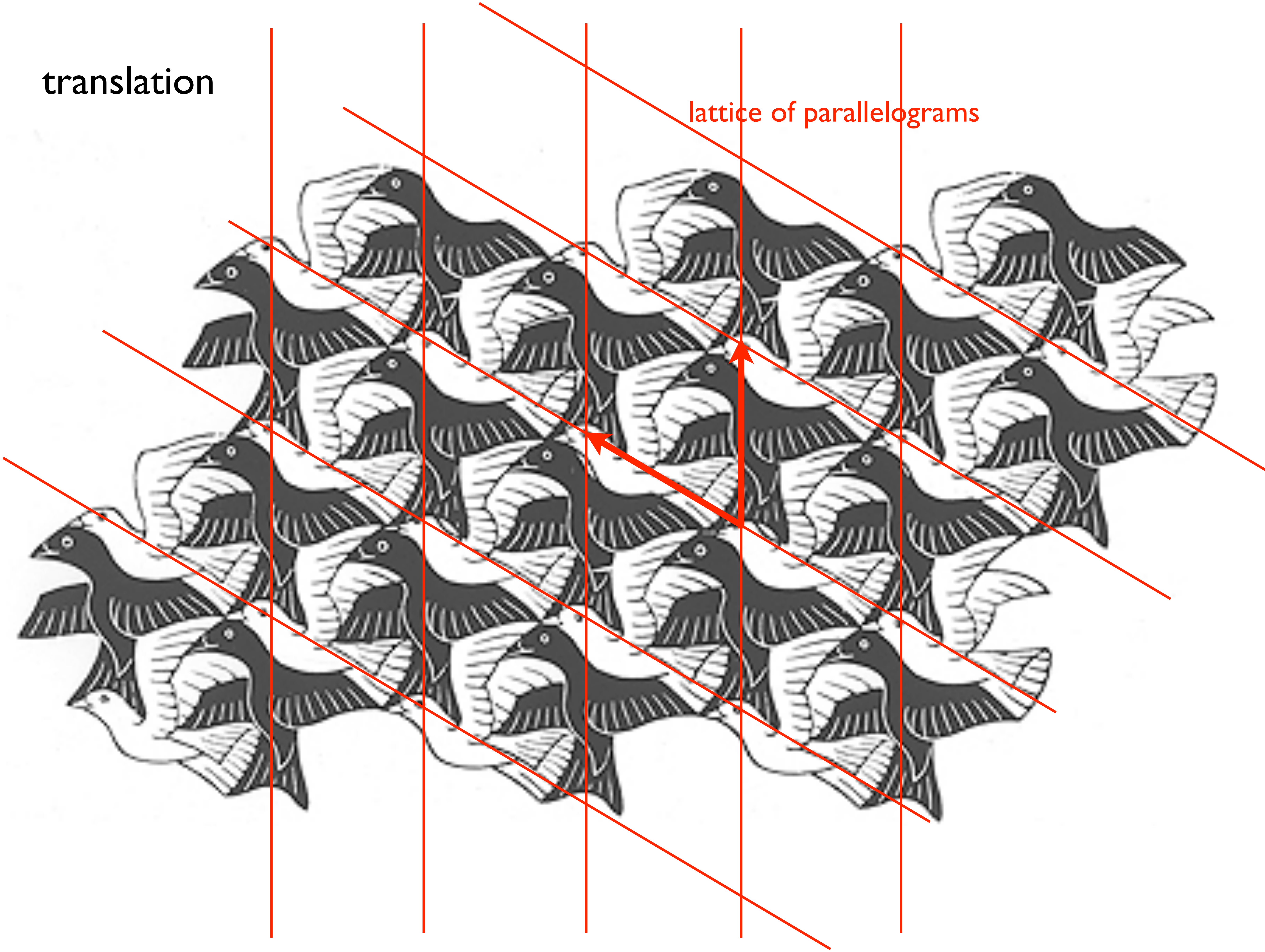




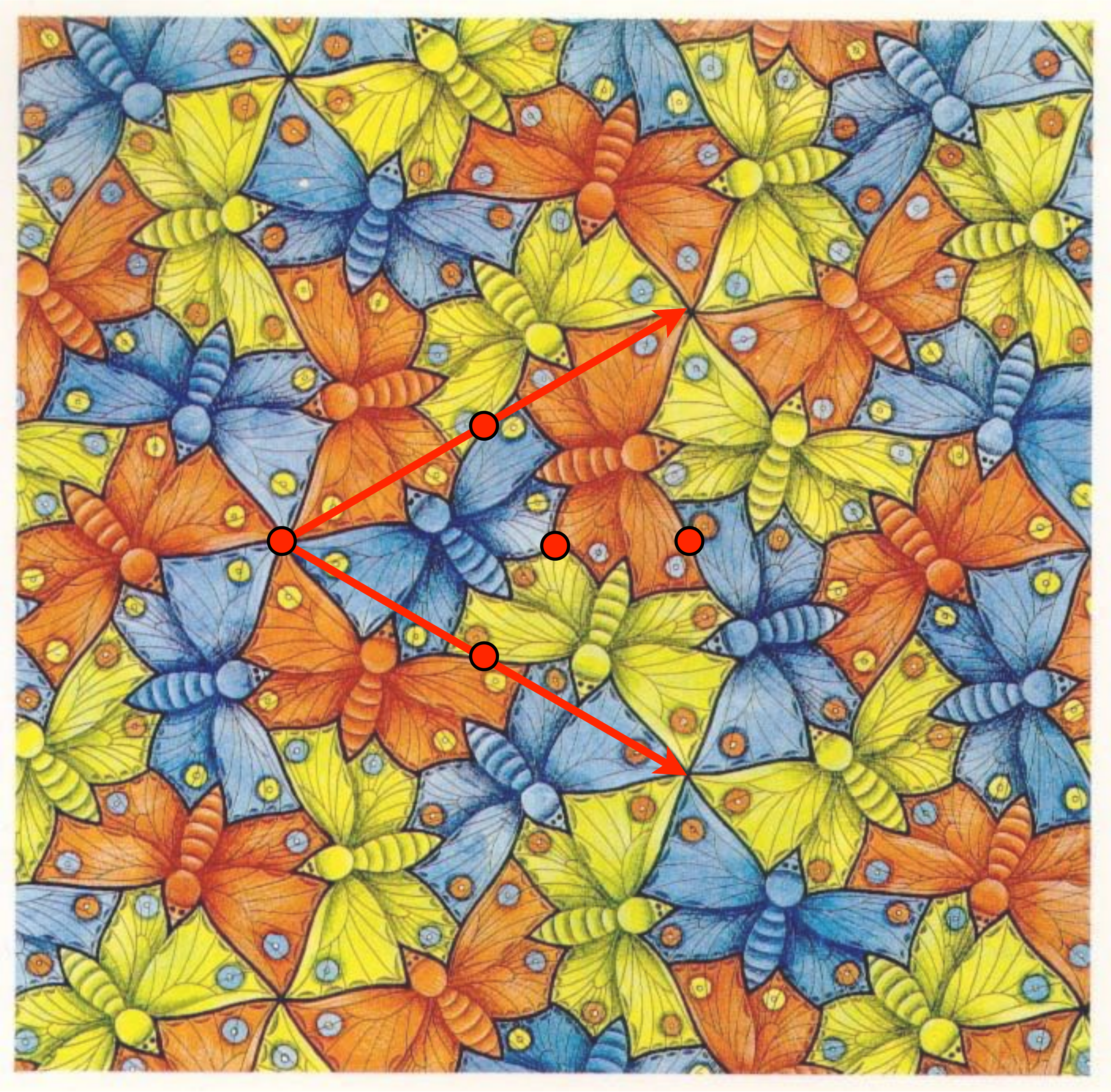
Some observations

- Alhambra wall tilings are periodic tilings of the plane (complete covering of a flat 2-d surface using repeated congruent figures)
- Instead of geometric shapes, Escher wanted to use recognizable figures (e.g., birds, fish, lizards, ...), forbidden by religion of Moors
- Congruent figures (same size & shape) limits allowed transformations (translations, rotations, reflections, glide reflections)
- Symmetric with respect to a particular transformation means it is unchanged after applying that transformation

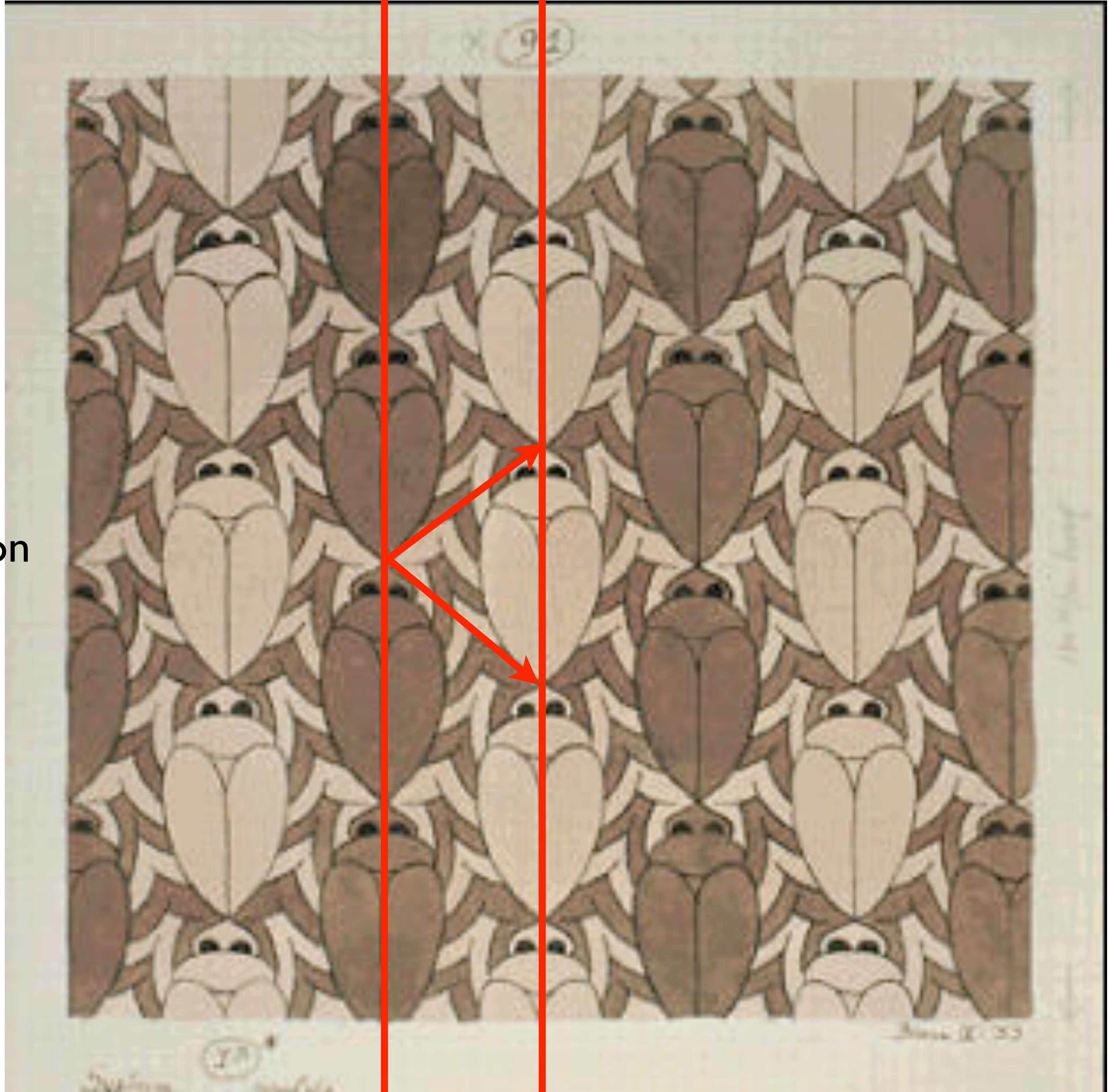
translation



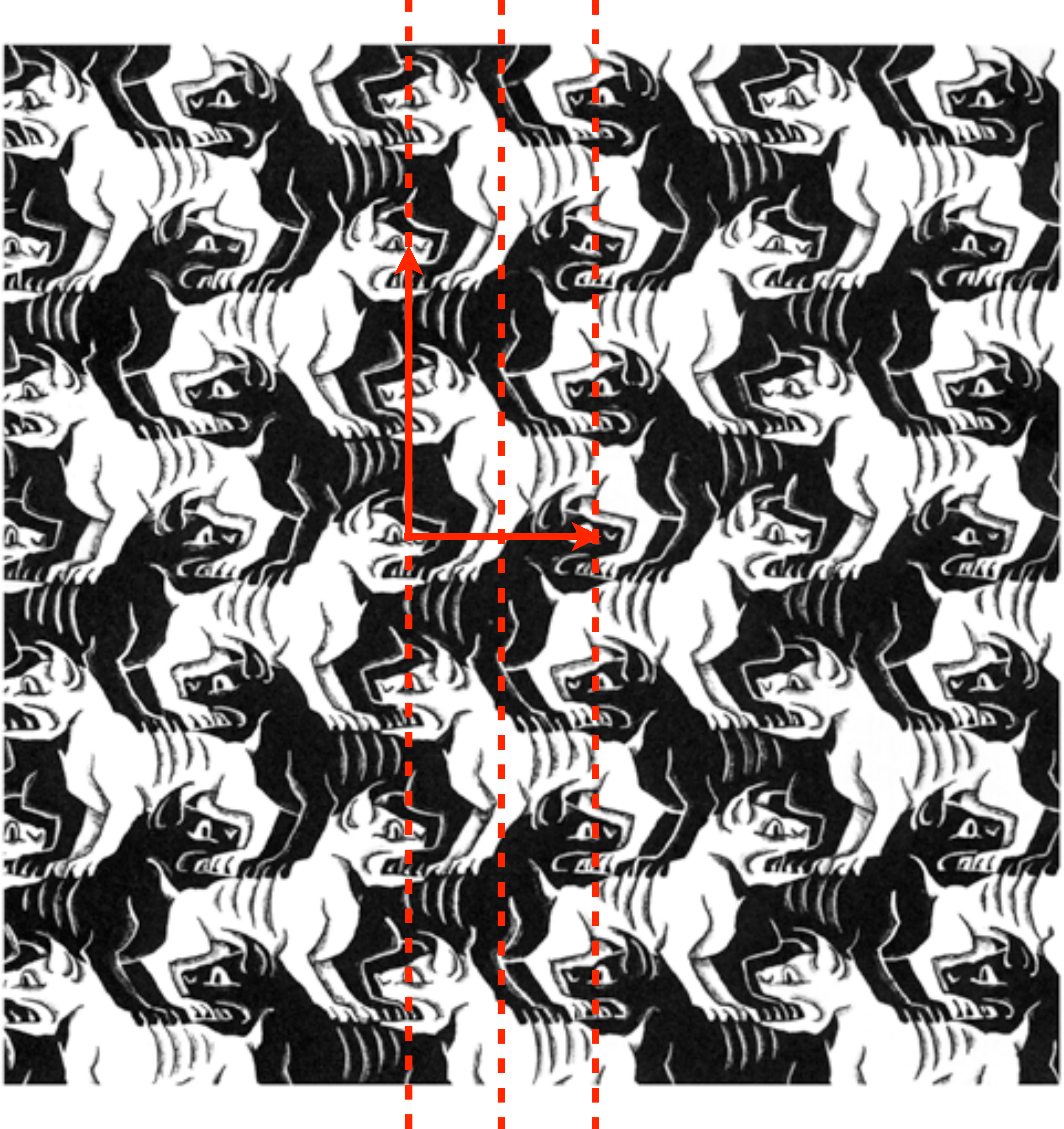
rotation
(2, 3, 6)



reflection



glide
reflection



translation,
rotation,
reflection, and
glide reflection

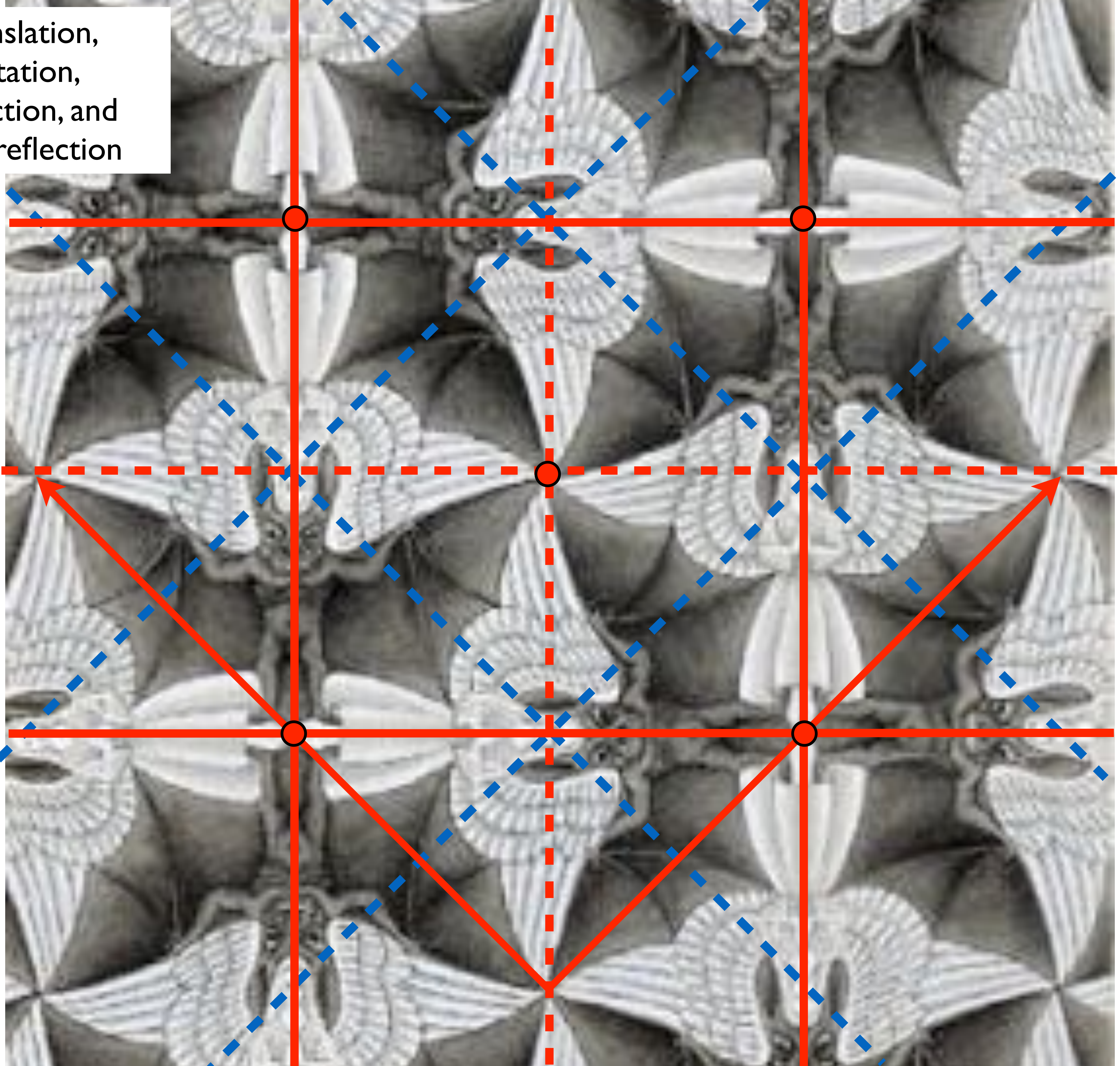
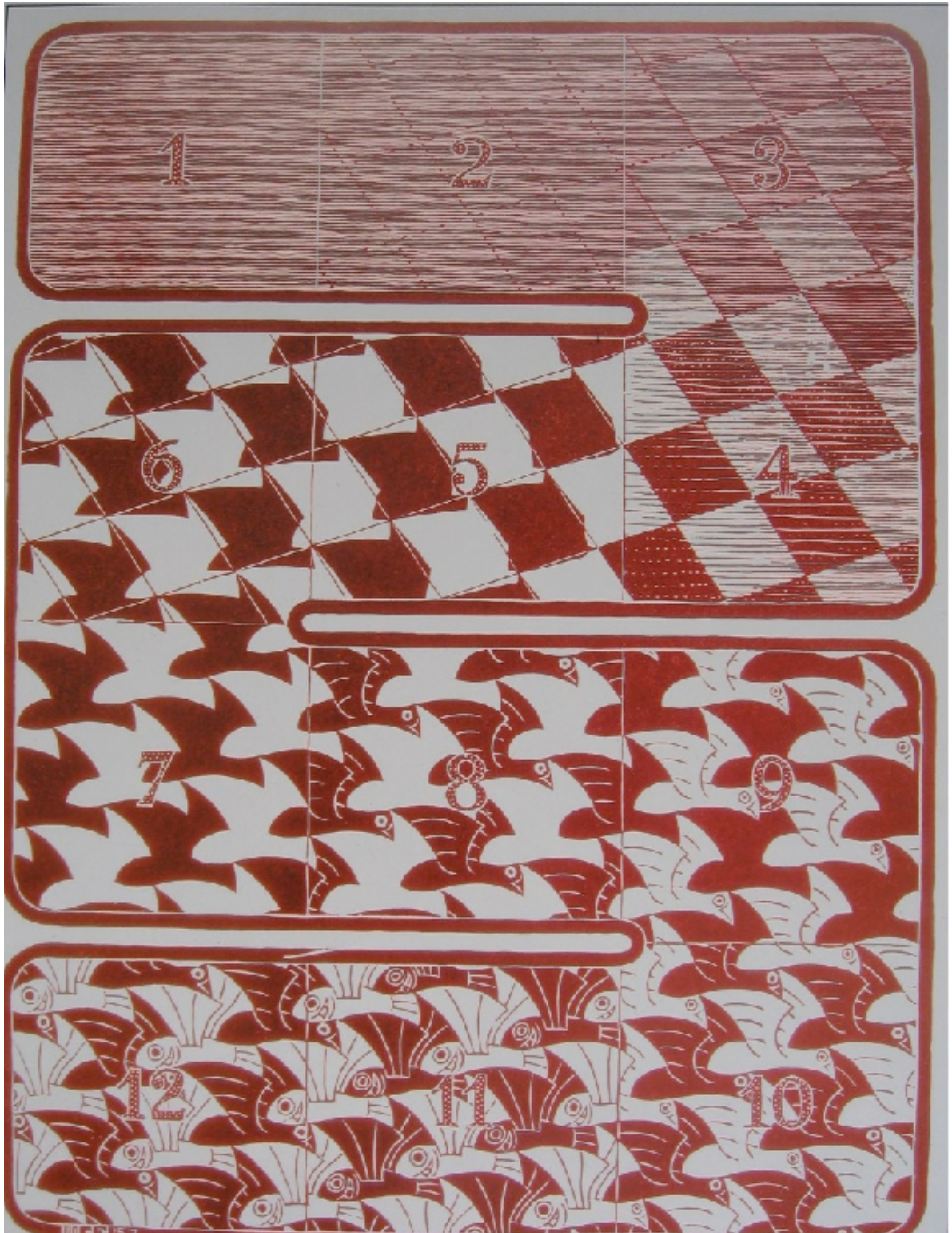
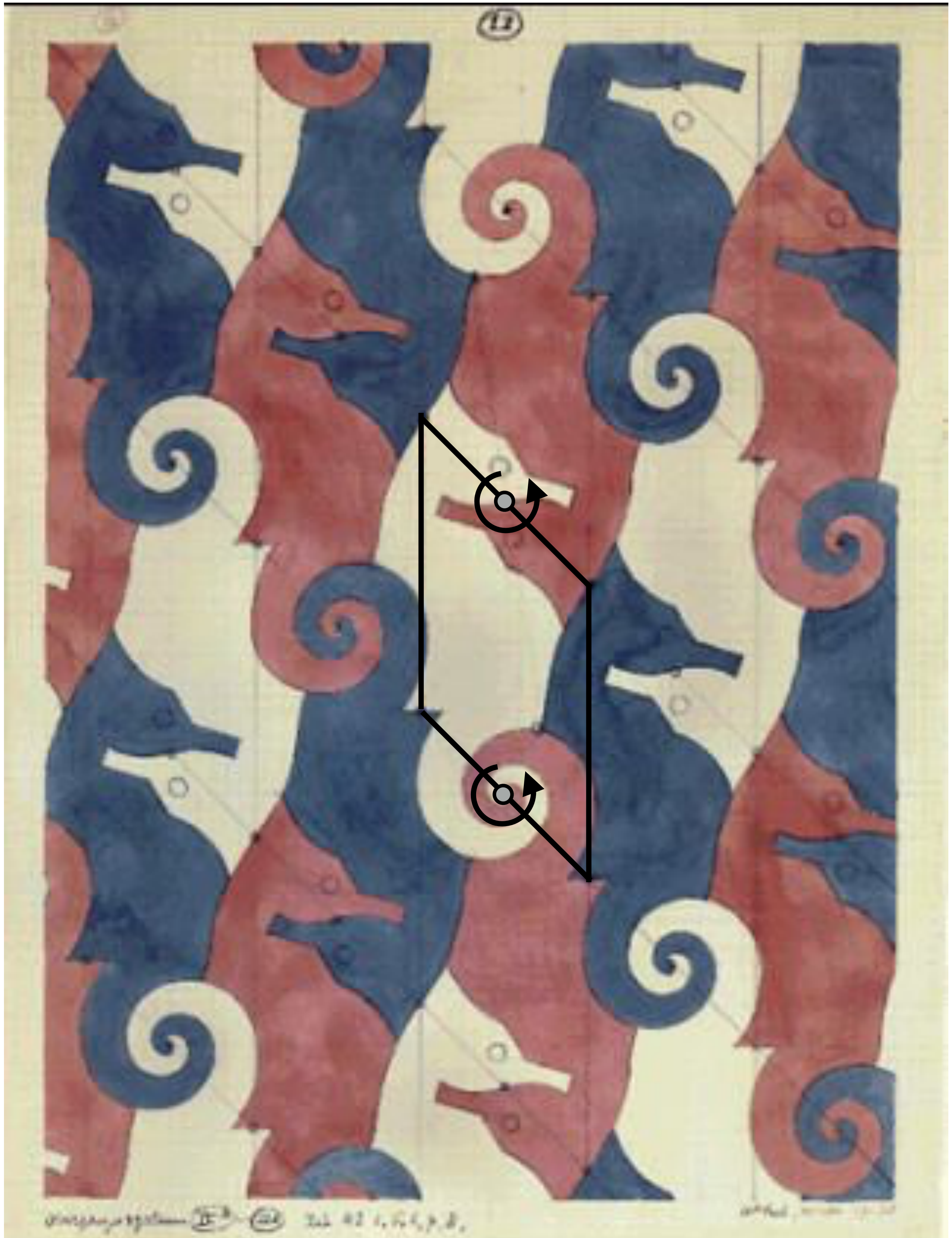


Figure 69
(Plane Tessellations)





28 different ways of constructing asymmetric tiles (Heesch group)

3. Periodic tilings using regular polygons

Q: How many ways can you tile 2-d flat space using regular polygons?

A. Zero

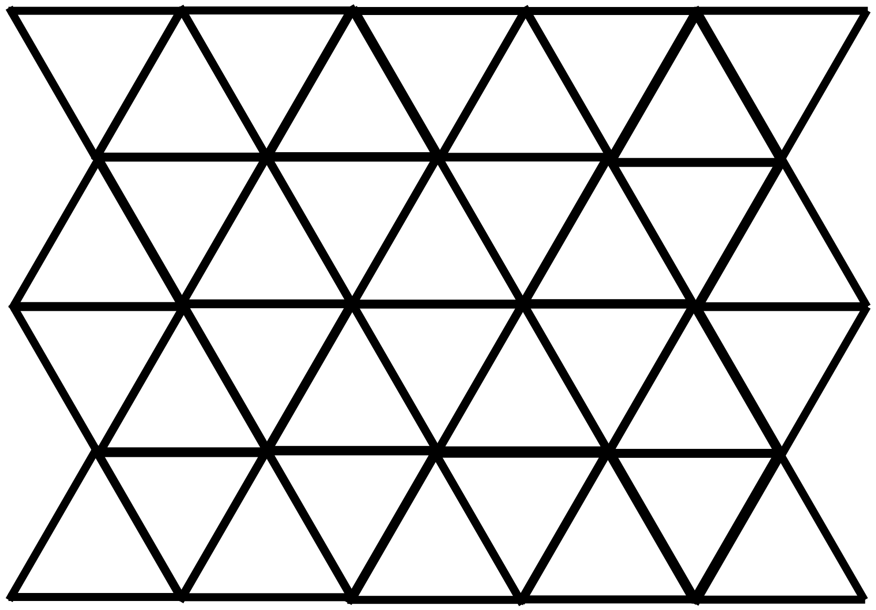
B. Three

C. Five

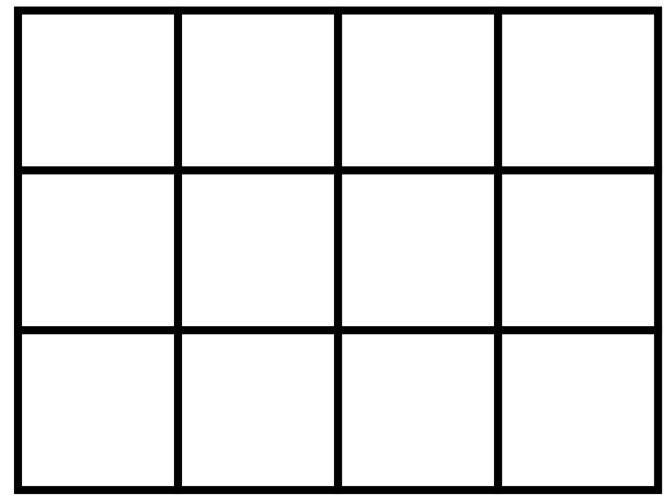
D. Infinity

Answer: Three (equilateral triangles, squares, or hexagons)

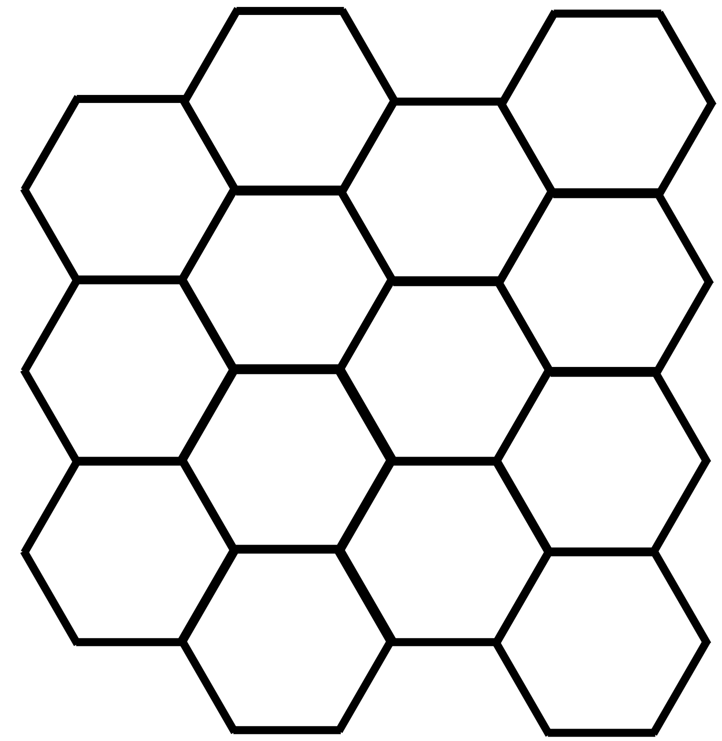
Tilings of 2-d flat space (i.e., plane) using regular polygons



$\{p,q\} = \{3,6\}$



$\{4,4\}$



$\{6,3\}$

Q: But how do you prove that these are the only three?

A: Sum of the angles around each vertex = 360°

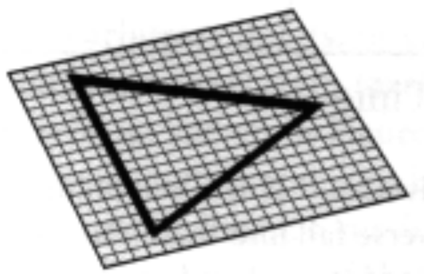
Opening angle of a regular p-gon = $(p-2) * 180^\circ / p$

$q * (p-2) * 180^\circ / p = 360^\circ$

$\boxed{\text{Tiling condition for q p-gons meeting at a vertex: } 1/p + 1/q = 1/2}$

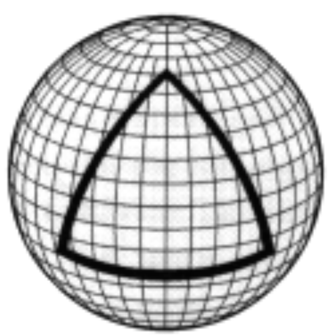
Extension to 2-d sphere and hyperbolic space

Straight lines in a curved space are the shortest distance curves



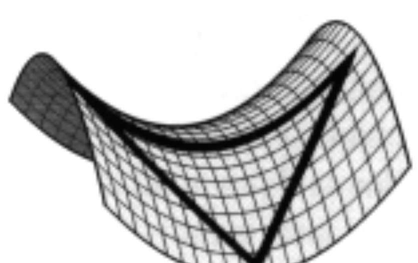
flat (Euclidean)

1. Parallel lines never intersect
2. Sum of angles of a triangle =180°
3. C = 2πr



positively curved
(spherical, elliptic)

1. Initially parallel lines intersect
2. Sum of angles of a triangle >180°
3. C < 2πr



negatively curved
(hyperbolic)

1. Initially parallel lines diverge
2. Sum of angles of a triangle <180°
3. C > 2πr

Q: How many ways can you tile a 2-d sphere using regular polygons?

A. Zero

B. Three

C. Five

D. Infinity

$\boxed{\text{Answer: 5 or infinity}}$

Proof: Tiling condition for 2-d sphere is $1/p + 1/q > 1/2$

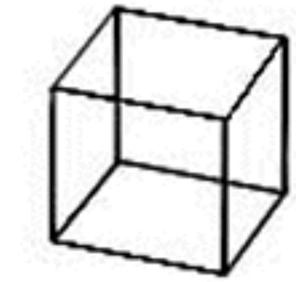
$\{p,q\} = \{3,3\}, \{4,3\}, \{3,4\}, \{5,3\}, \{3,5\}, \{2,n\}, \{n,2\}$

Tilings of the sphere using regular polygons: $1/p + 1/q > 1/2$

{3,3}

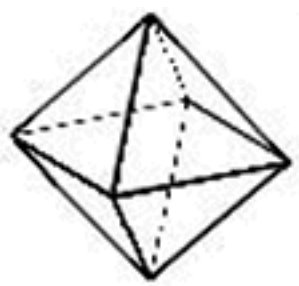
tetrahedron

{4,3}



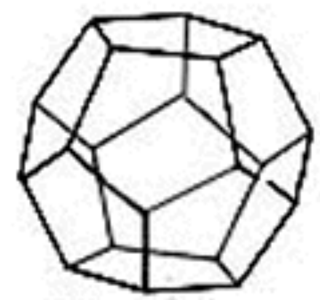
cube

{3,4}



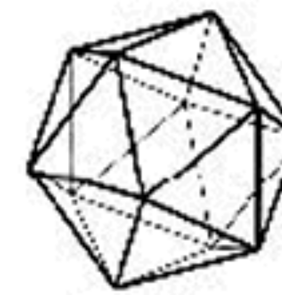
octahedron

{5,3}



dodecahedron

{3,5}



icosahedron

Degenerate tessellations

{2,n}

(biangles)

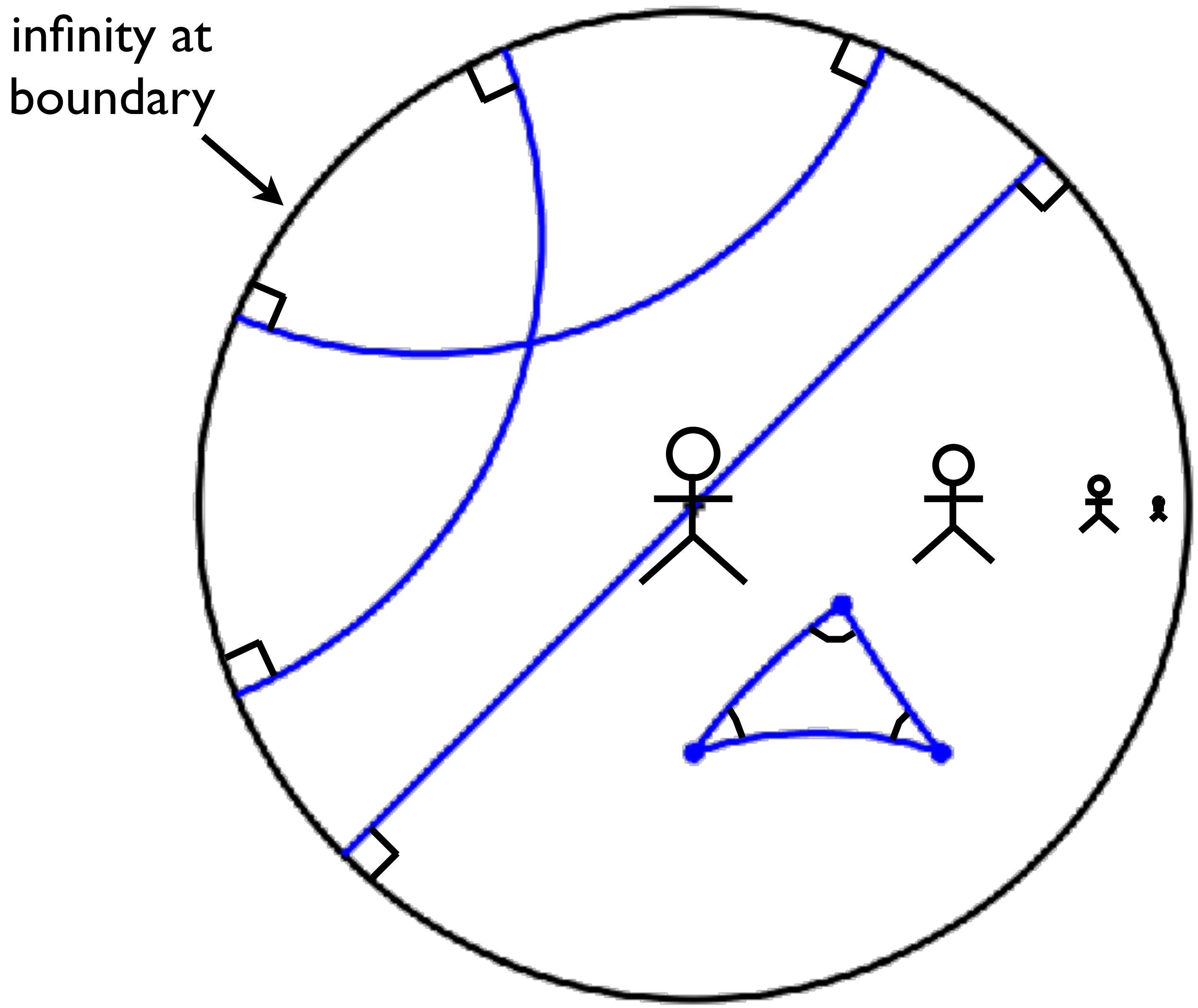
{n,2}

(from EscherMath)

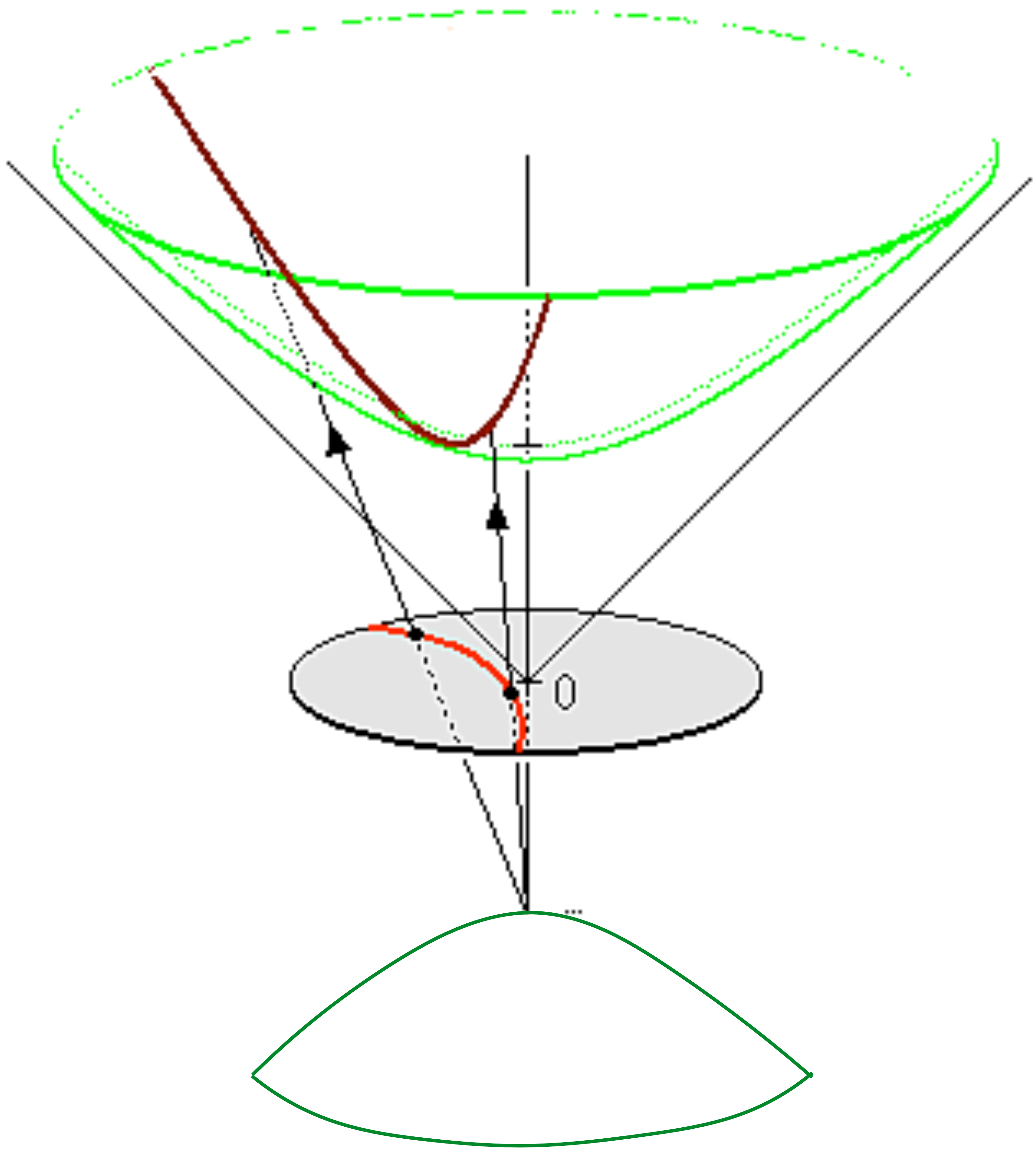
Sphere with Angels and Devils (1942)

(regular quadrilaterals meeting 3 at a vertex)

2-d hyperbolic space with constant negative curvature (Poincaré disk model)



- finite representation of an infinite space with const negative curvature
- straight lines are arcs of circles that intersect the boundary at right angles
- sum of the angles of a triangle $< 180^{o}$
- conformal representation (angles preserved)
- distances are distorted as you move toward the boundary: $1-(\rho/R)^2$



(from wikipedia)

Q: How many ways can you tile 2-d hyperbolic space using regular polygons?

A. Zero
B. Three
C. Five
D. Infinity

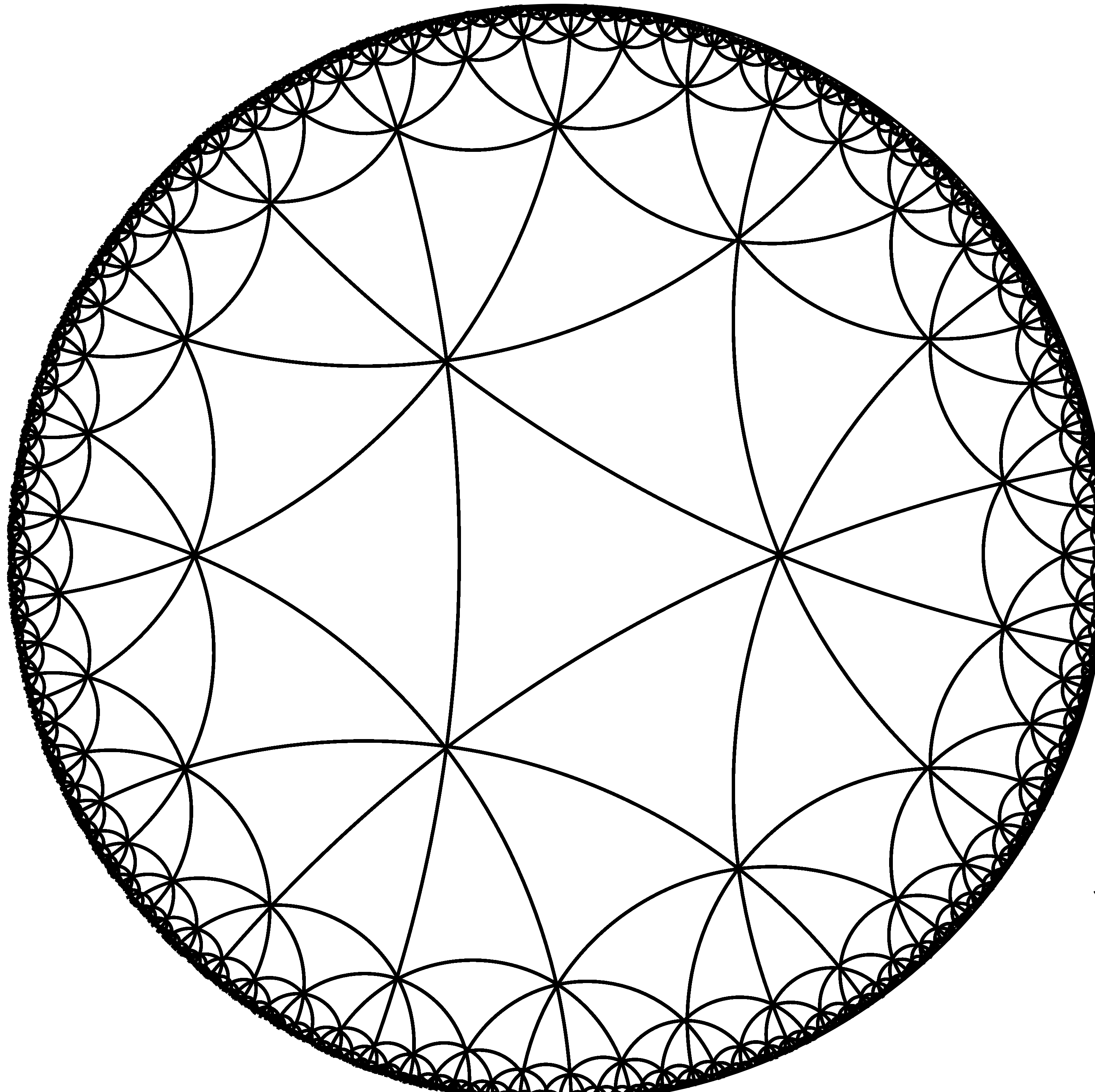
$\boxed{\text{Answer: } 1/p + 1/q < 1/2 \text{ implies infinity!!}}$

$$1/p + 1/q \gtreqless 1/2$$

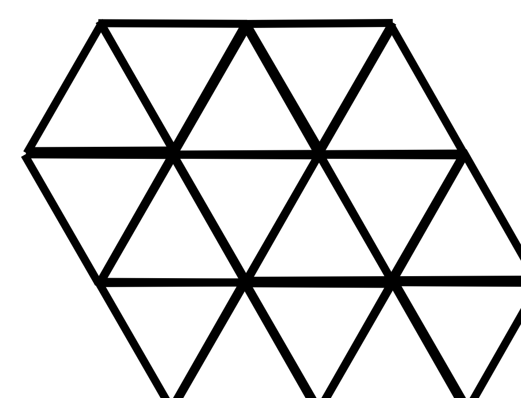
{p,q}	2	3	4	5	6	7
2	DS	DS	DS	DS	DS	DS
3	DS	S	S	S	F	H
4	DS	S	F	H	H	H
5	DS	S	H	H	H	H
6	DS	F	H	H	H	H
7	DS	H	H	H	H	H

Tiling of Poincaré disk by 45-45-45 triangles

{3,8} tiling

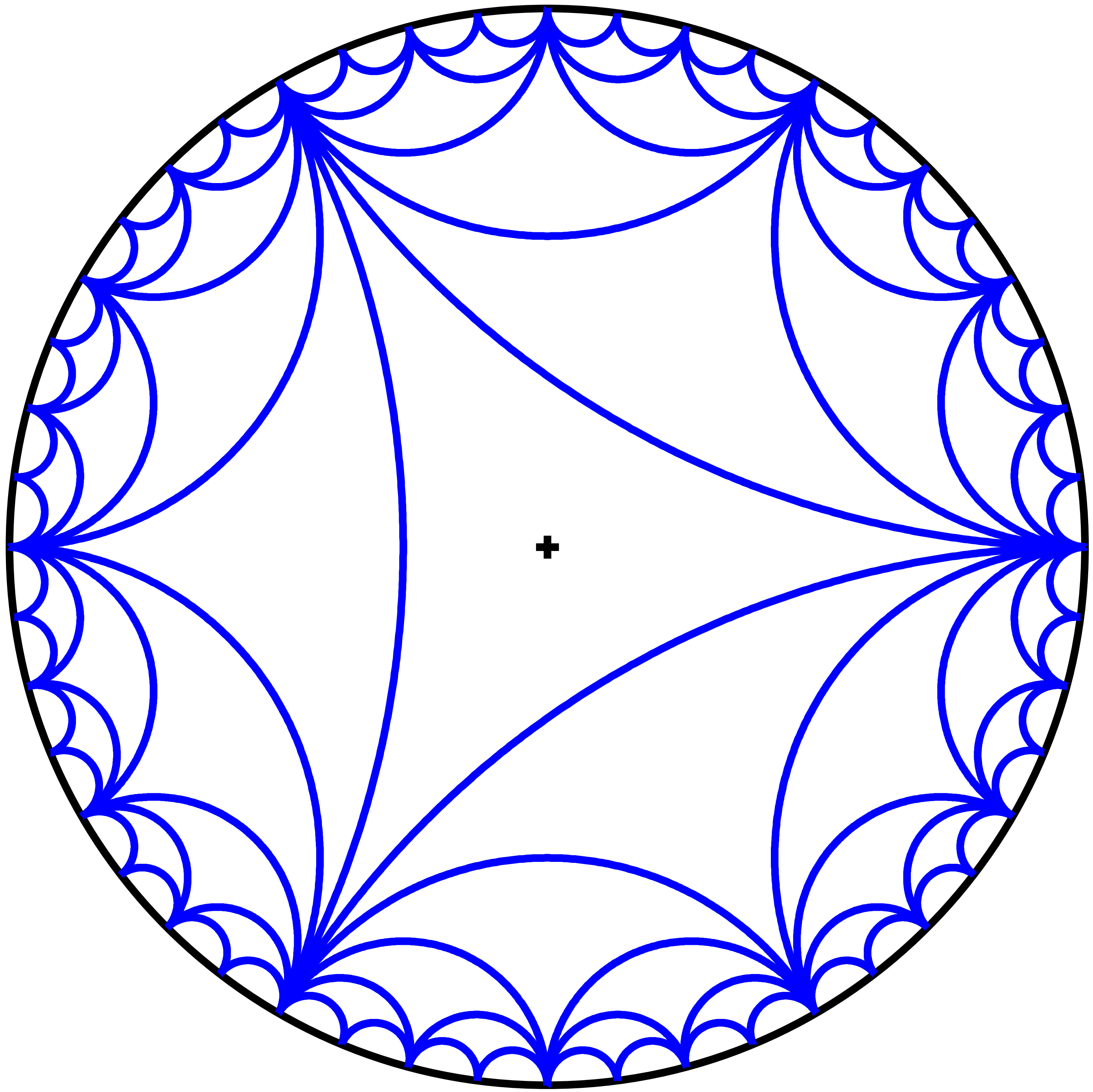


tiling of flat space by
60-60-60 triangles



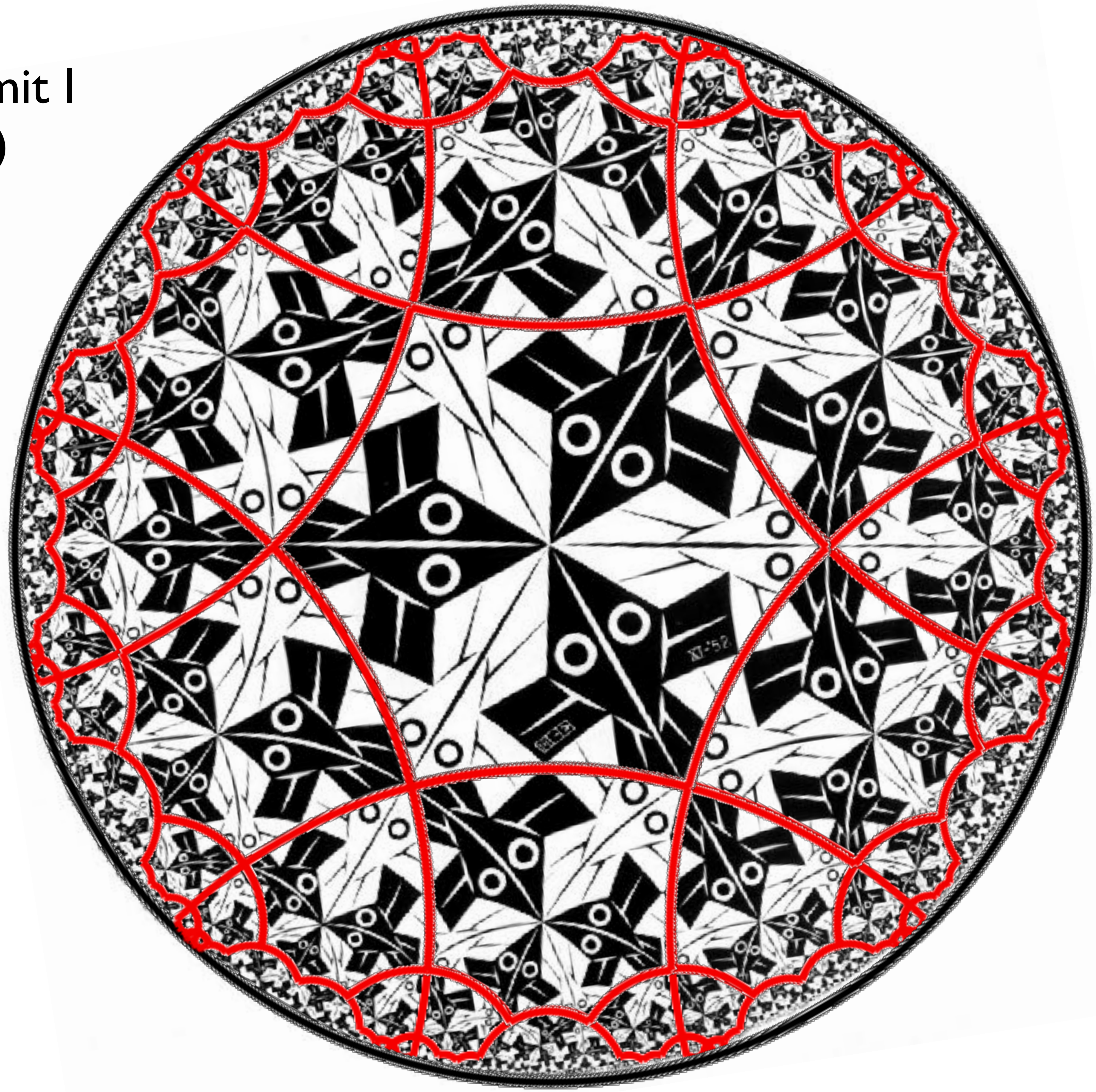
Tiling of Poincaré disk by 0-0-0 triangles!!

{3,∞} tiling

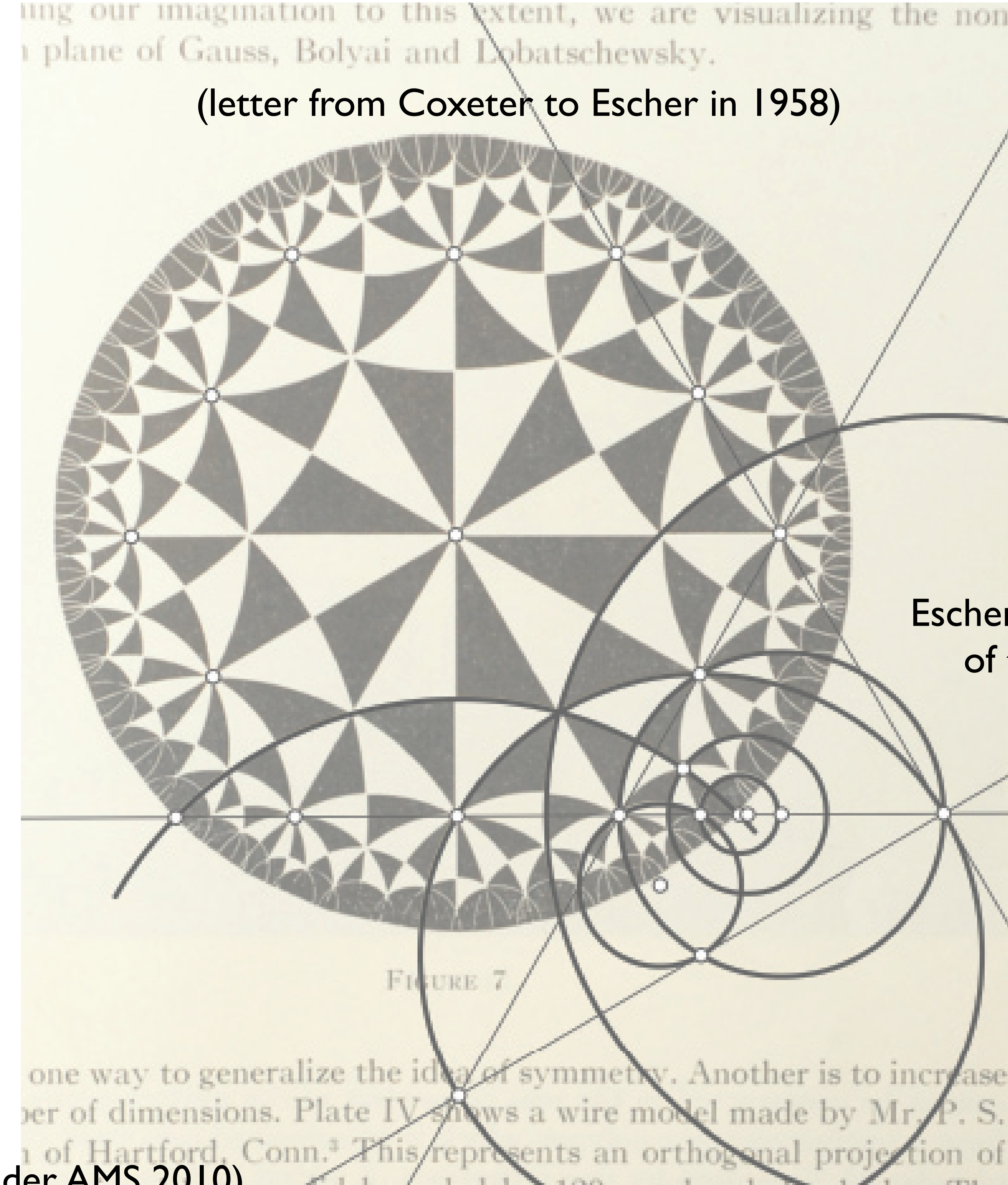


Circle Limit I
(1958)

{6,4} tiling



(letter from Coxeter to Escher in 1958)

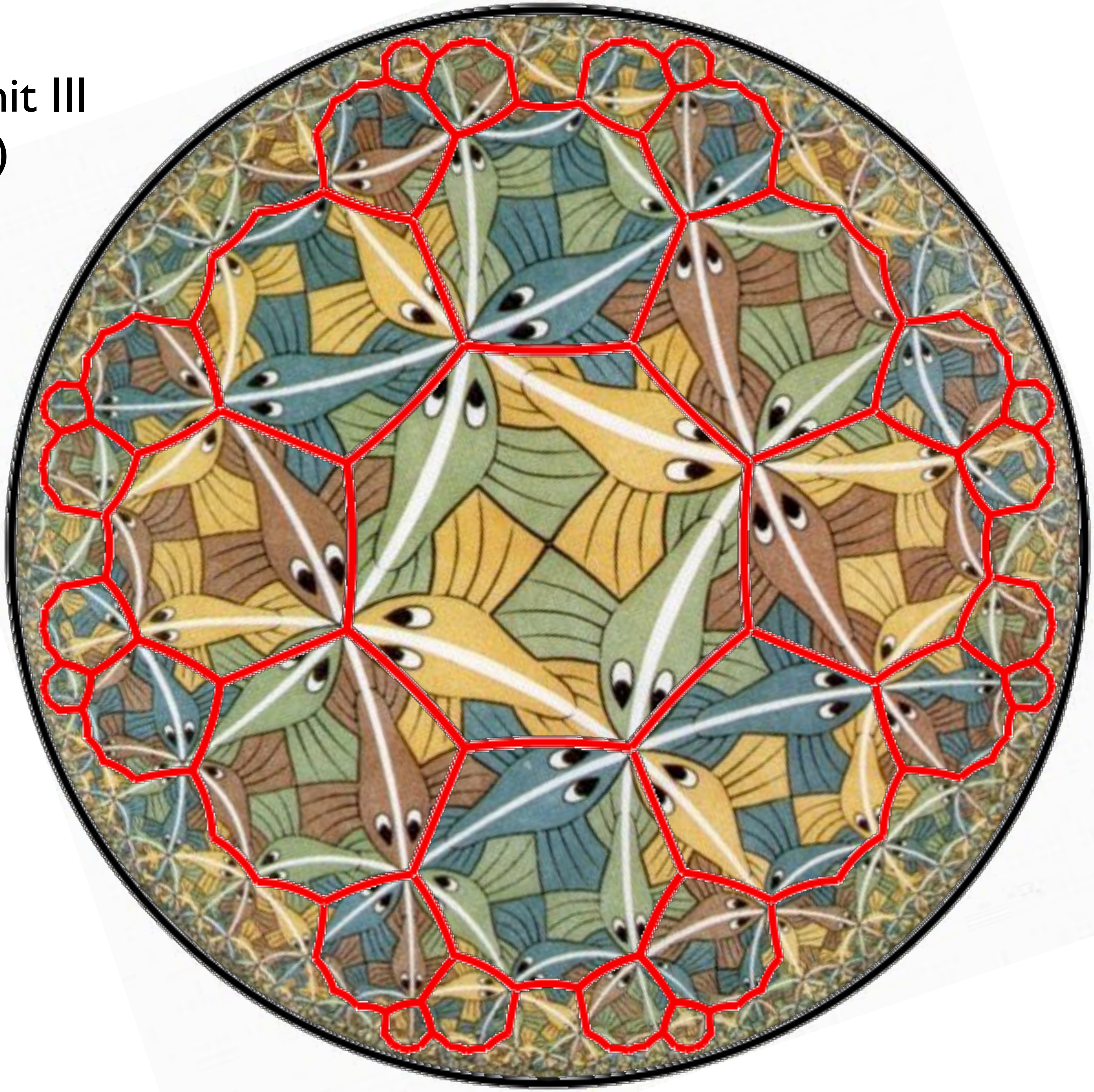


Escher's reconstruction of the scaffolding

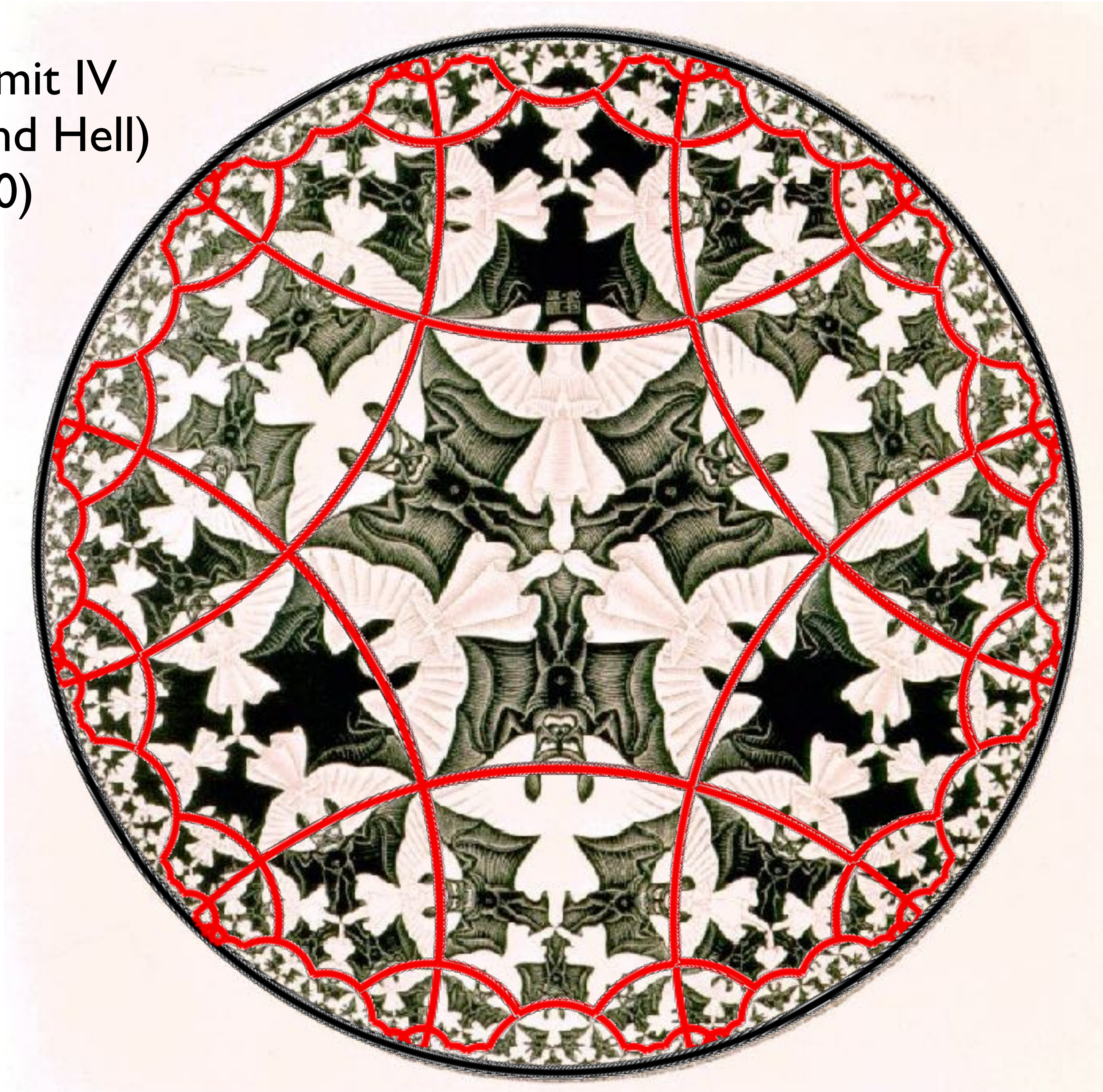
(from Schattschneider, AMS 2010)

Circle Limit III
(1959)

{8,3} tiling

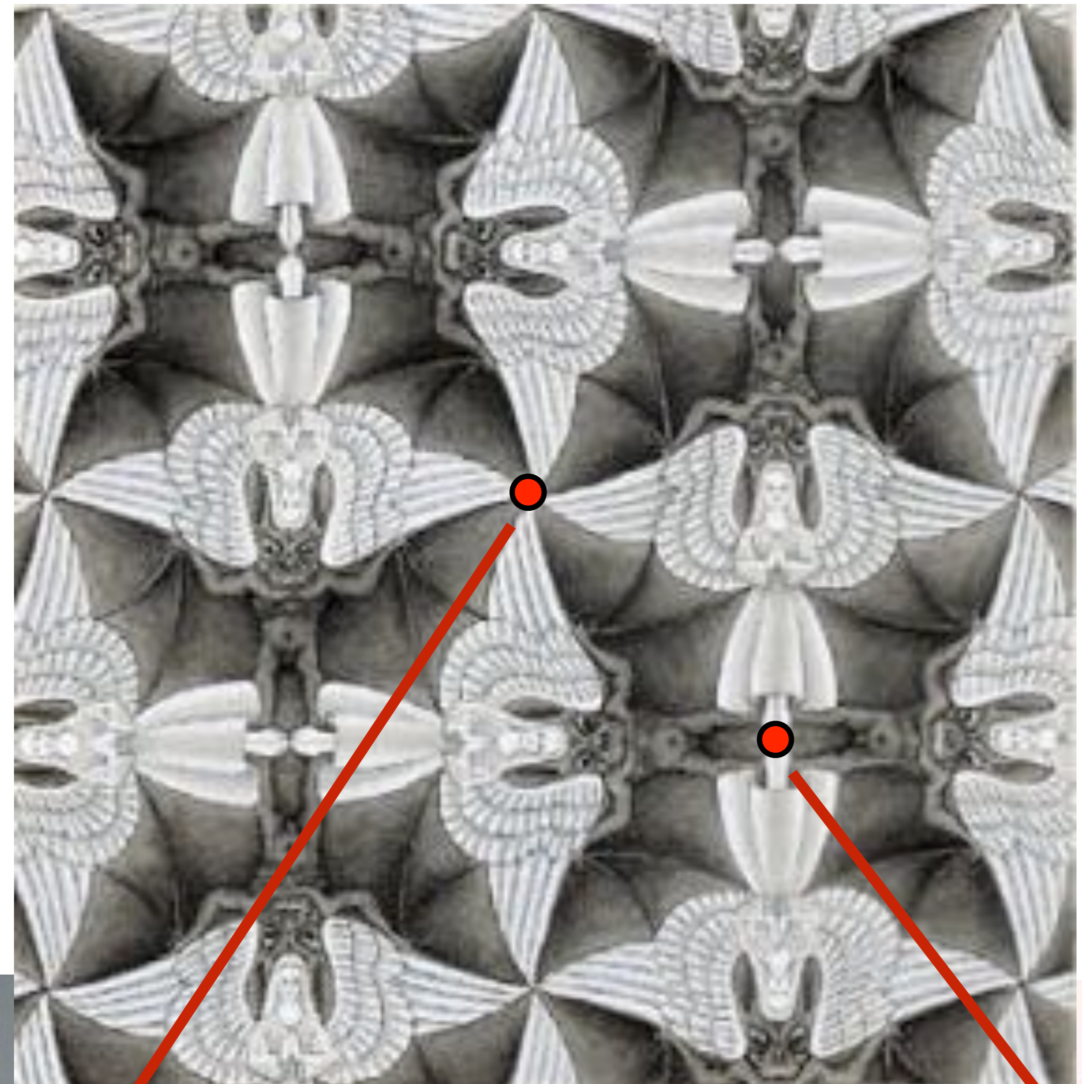


Circle Limit IV
(Heaven and Hell)
(1960)



{6,4} tiling

2-d flat space



2-d sphere

$\boxed{1/p + 1/q \gtreqless 1/2}$

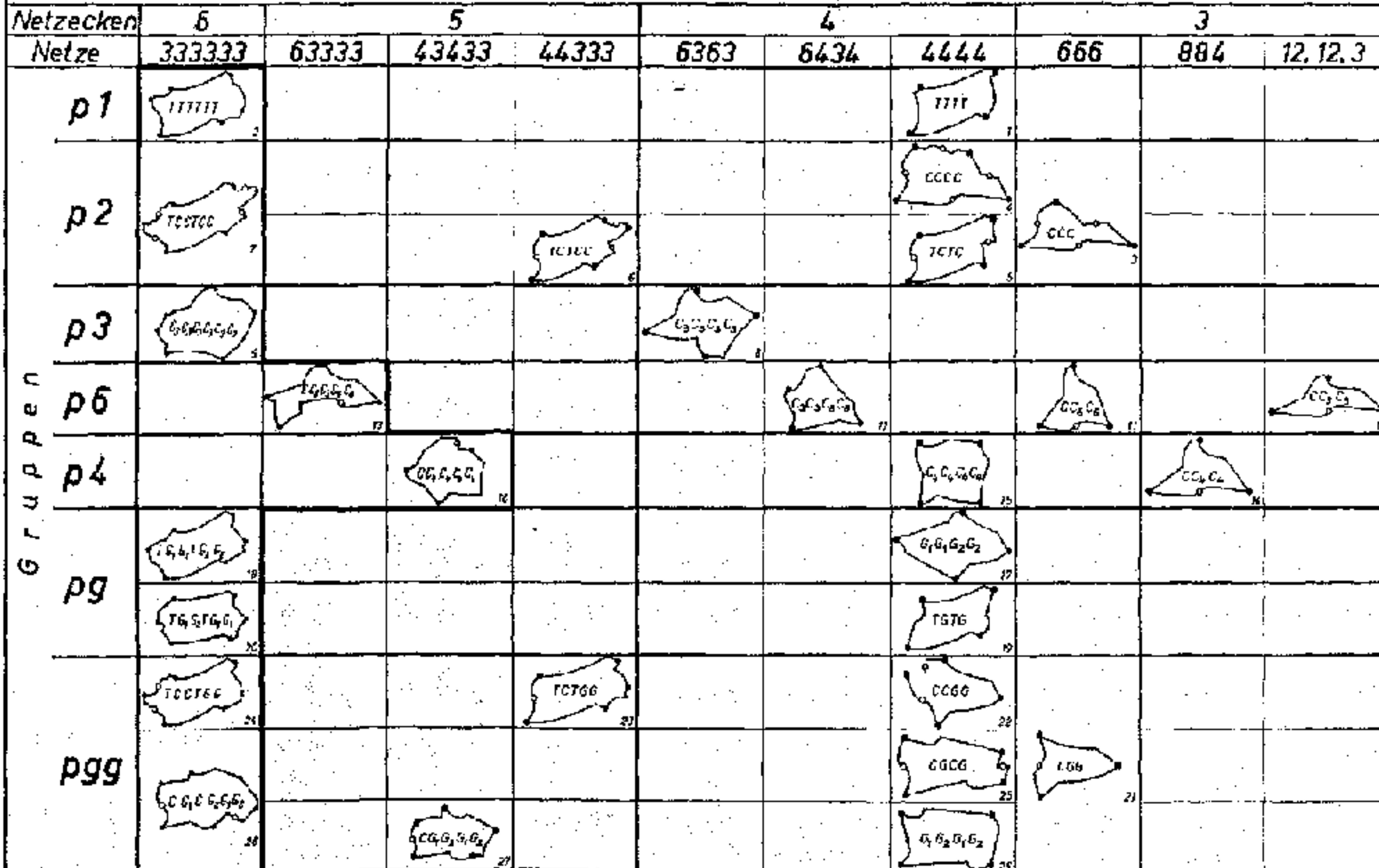
2-d hyperbolic space

References / credits

1. http://www.mcescher.com/ (Official M.C. Escher website)
2. ☆ http://mathcs.slu.edu/escher/ (EscherMath)
3. "M.C. Escher, His Life and Complete Graphic Work," by F.H. Bool et al.
4. ☆ "The Magic Mirror of M.C. Escher" by Bruno Ernst
5. ☆ "M.C. Escher, Visions of Symmetry," by Doris Schattschneider
6. Several mathematics articles by Doris Schattschneider and Douglas Dunham
7. ... and numerous other websites!

extra slides

Tafel 10. *Die 28 Grundtypen des Flächenschlusses*



Die starke Umrandung umfaßt die 9 Haupttypen, von denen die anderen durch Schrumpfung von Linien oder Linienpaaren entstanden gedacht werden können.

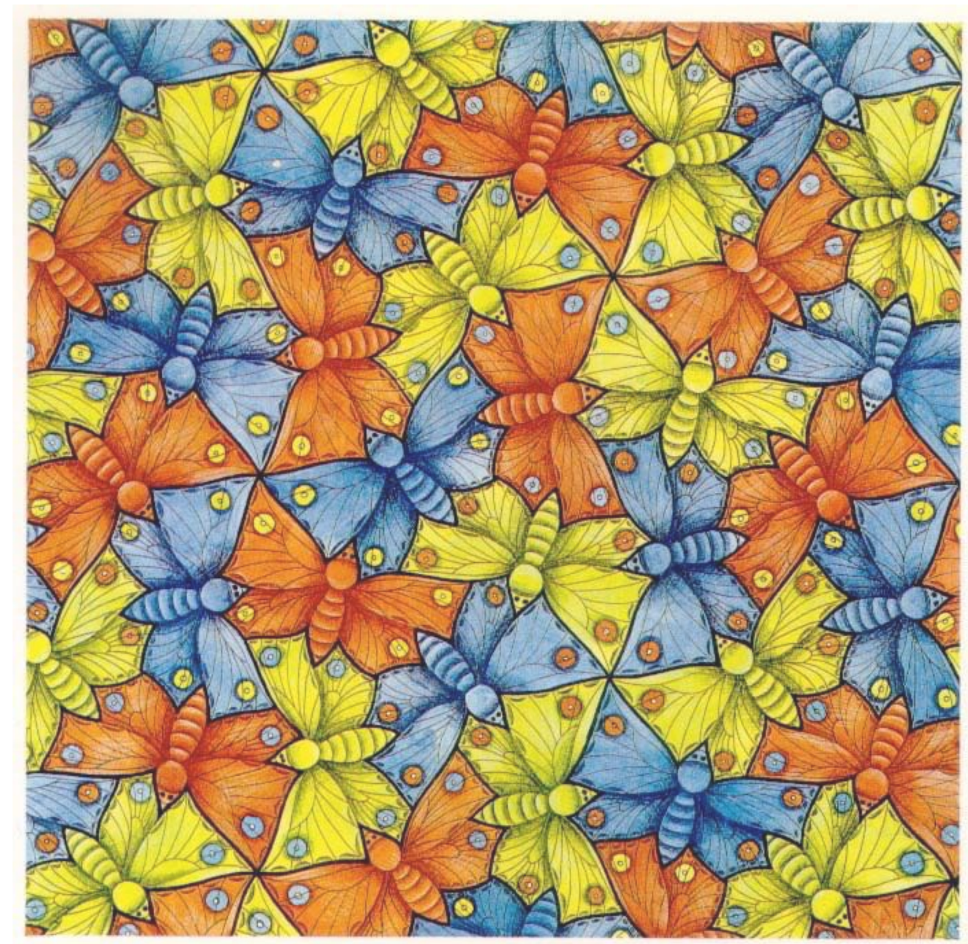
Die Nummer rechts unten in jedem Feld ist die Nummer des zugehörigen Einzelblattes, S. 44 bis 72.

Netzecke —o— Drehpunkt einer C-Linie

3. Classifying periodic tilings

Q: How many different periodic wall tilings are there?

?
=

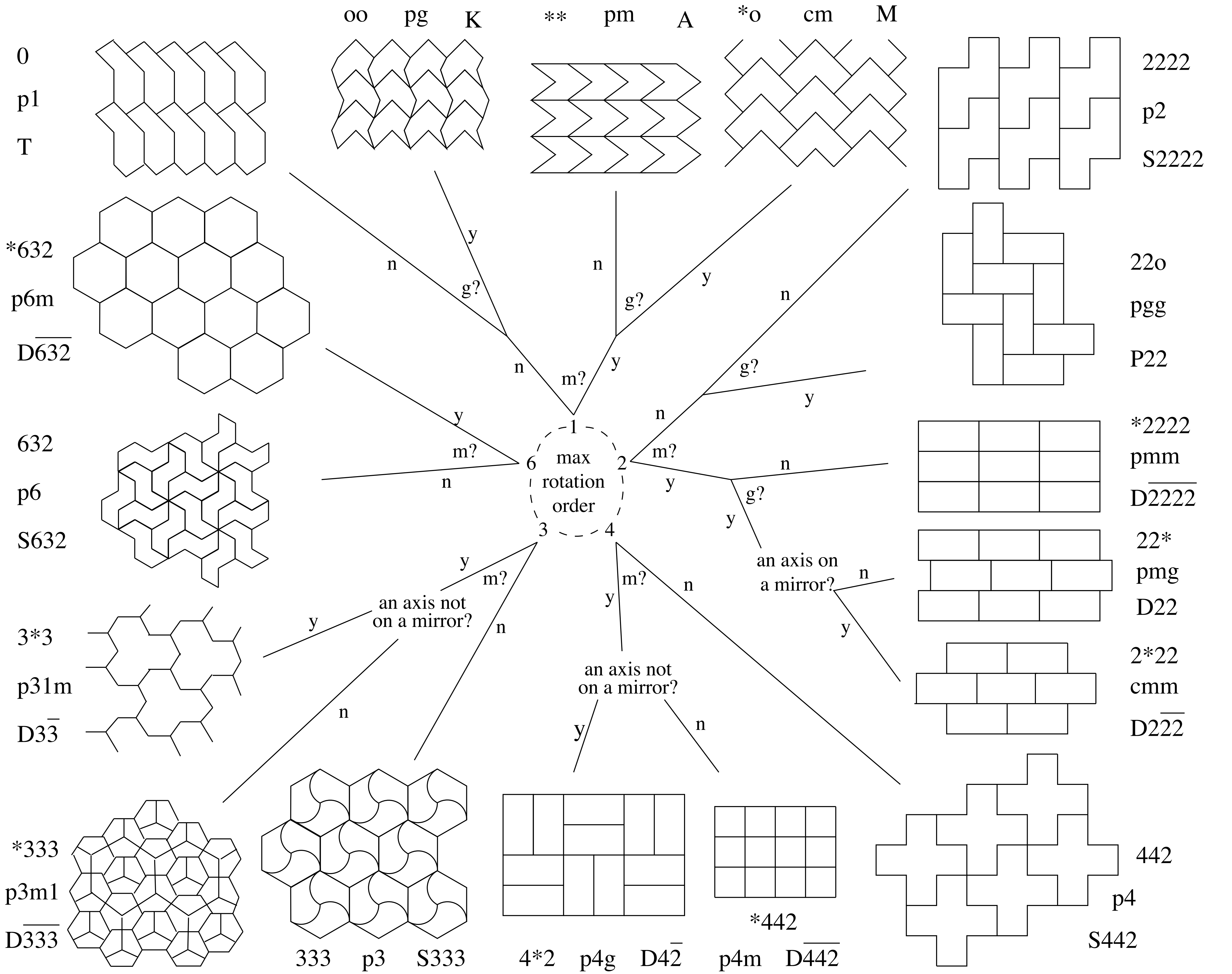


A) Three B) Seven C) Seventeen D) 230 E) Infinity

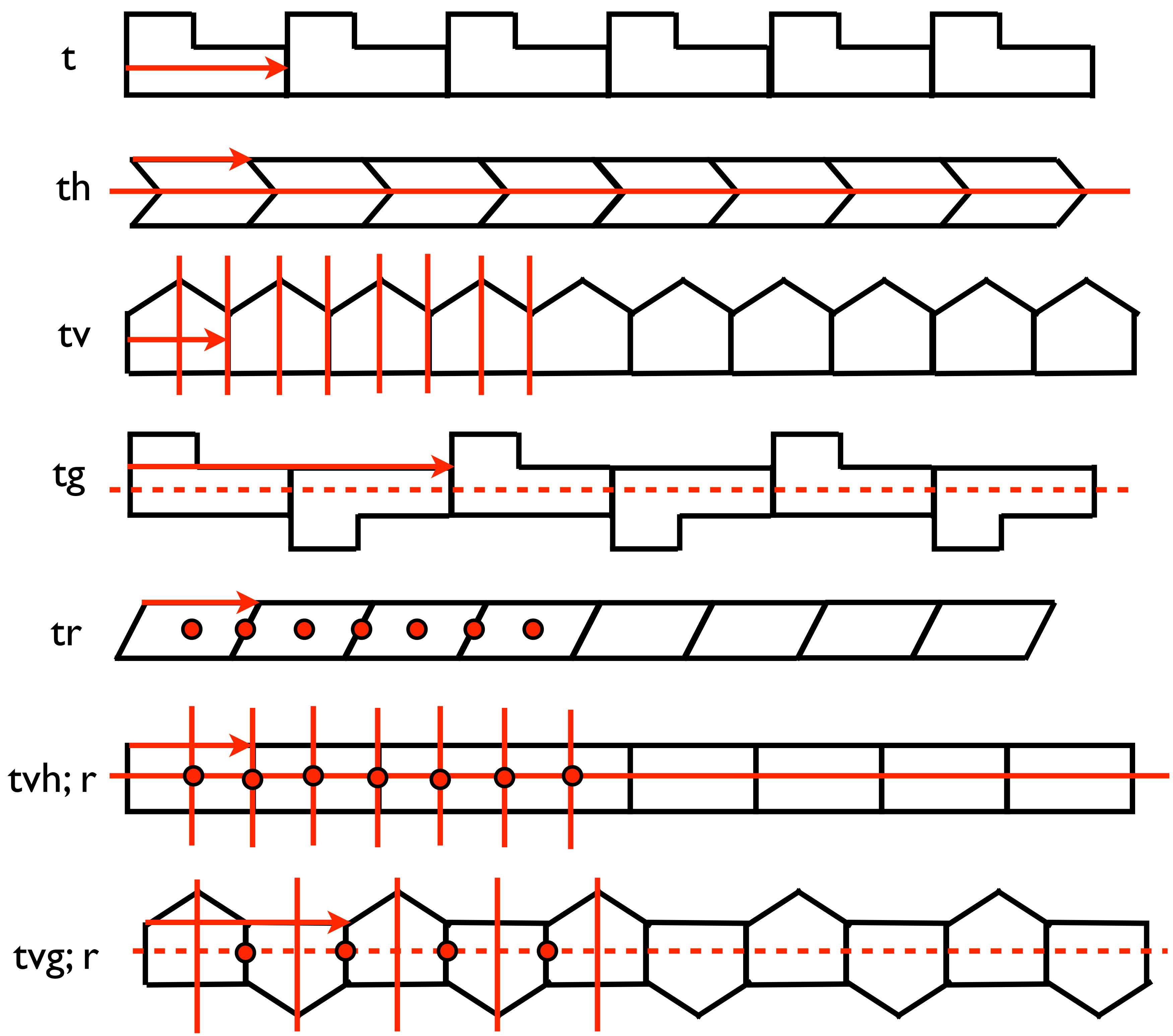
Answer: 17 different 2-d wall tilings (Fedorov 1891, Polya 1924)

17 wallpaper symmetry patterns

(from Brian Sanderson's webpage)



7 border patterns



border symmetries