§23, Prob 1:

$$L = \frac{1}{2}(\dot{x}^2+\dot{y}^2) - \frac{1}{2}\omega_0^2(x^2+y^2) + \alpha x y$$
$$= T - U$$

Let $x_i = (x, y)$ so $x_1 = x$, $x_2 = y$

Then: $T = \frac{1}{2}(\dot{x}^2+\dot{y}^2)$

$$= \frac{1}{2}\sum_{j,k} m_{jk}\dot{x}_j\dot{x}_k$$

where $m_{jk} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$U = \frac{1}{2}\omega_0^2(x^2+y^2) - \alpha x y$$

$$= \frac{1}{2}\sum_{j,k} k_{jk} x_j x_k$$

where $k_{jk} = \begin{bmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{bmatrix}$

EOM:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) = \frac{\partial L}{\partial x_j}$$

$$\frac{d}{dt}\left(\sum_k m_{jk}\dot{x}_k\right) = -\sum_k k_{jk}x_k$$

$$\sum_k (m_{jk}\ddot{x}_k + k_{jk}x_k) = 0$$

Ansatz: $x_k = A_k e^{i\omega t}$

$$\rightarrow \sum_k (-\omega^2 m_{jk}A_k + k_{jk}A_k) = 0$$

$$\sum_{k} \left(K_{jk} - \omega^2 m_{jk} \right) A_k = 0$$

Thus, $\det\left(K_{jk} - \omega^2 m_{jk} \right) = 0$ characteristic equation

$$0 = \det\left(\begin{bmatrix} \omega_0^2 & -\alpha \\ -\alpha & \omega_0^2 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} \omega_0^2 - \omega^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega^2 \end{bmatrix}$$

$$= \left(\omega_0^2 - \omega^2 \right)^2 - \alpha^2$$

Thus, $\omega_0^2 - \omega^2 = \pm \alpha$

$$\omega^2 = \omega_0^2 \mp \alpha$$

$$\omega = \sqrt{\omega_0^2 \mp \alpha} = \omega_1, \omega_2$$

Normal modes:

$\omega_1 = \sqrt{\omega_0^2 - \alpha}$

$$\begin{bmatrix} \omega_0^2 - \omega_1^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega_1^2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $V_1 - V_2 = 0$

$$V_1 = V_2$$

$$\boxed{\vec{V}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \quad \boxed{\omega_1 = \sqrt{\omega_0^2 - \alpha}}$$

$$\omega_2 = \sqrt{\omega_0^2+\alpha} \quad \rightarrow \quad \begin{pmatrix}-\alpha & -\alpha\\ -\alpha & -\alpha\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$v_1 + v_2 = 0$$

$$\rightarrow \quad v_2 = -v_1$$

$$\boxed{\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}}\;,\quad \boxed{\omega_2 = \sqrt{\omega_0^2+\alpha}}$$

NOTE:

$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$ in phase oscillation at $\sqrt{\omega_0^2-\alpha}$

$\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$ out of phase oscillation at $\sqrt{\omega_0^2+\alpha}$

$$\Delta_{K\alpha} = \frac{1}{\sqrt{2}}\begin{array}{c|c} & \rightarrow \alpha \\ \hline \downarrow K & \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\end{array}$$

$\alpha = 1, 2$

$$\begin{pmatrix}x\\ y\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}Q_1\\ Q_2\end{pmatrix}$$

$$x = \frac{1}{\sqrt{2}}(Q_1+Q_2) \qquad\qquad Q_1 = \frac{1}{\sqrt{2}}(x+y)$$
$$\Longleftrightarrow$$
$$y = \frac{1}{\sqrt{2}}(Q_1-Q_2) \qquad\qquad Q_2 = \frac{1}{\sqrt{2}}(x-y)$$

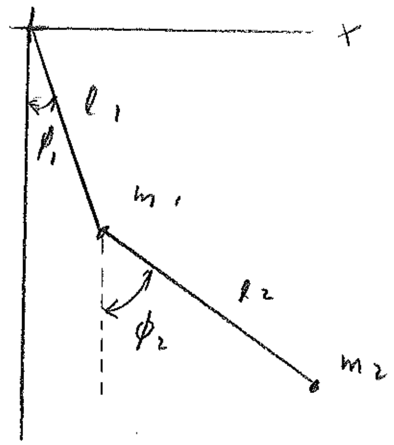
For $\alpha \ll \omega_0^2$:

$$\omega_1 = \sqrt{\omega_0^2-\alpha} = \omega_0\sqrt{1-\frac{\alpha}{\omega_0^2}} \approx \omega_0\left(1-\frac{1}{2}\frac{\alpha}{\omega_0^2}\right) = \omega_0 - \frac{1}{2}\frac{\alpha}{\omega_0}$$

$$\omega_2 = \sqrt{\omega_0^2+\alpha} = \omega_0 + \frac{1}{2}\frac{\alpha}{\omega_0}$$

$$\text{Beat Freq} \equiv |\omega_2-\omega_1| = \boxed{\frac{\alpha}{\omega_0}}$$

§23, Prob 2 : coplanar double pendulum (see §5, Prob 1)



$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)$$
$$+ (m_1+m_2)g l_1\cos\phi_1 + m_2 g l_2 \cos\phi_2$$

Equilibrium: $\phi_1 = \phi_2 = 0$

small oscillations away from equilibrium $|\phi_{1,2}| \ll 1$

$\cos(\phi_1-\phi_2) \simeq 1$ since multiplied by $\dot{\phi}_1\dot{\phi}_2$

$\cos\phi_1 \approx 1 - \frac{1}{2}\phi_1^2$

$\cos\phi_2 \approx 1 - \frac{1}{2}\phi_2^2$

Ignore the constant parts of the potential

$(m_1+m_2)g l_1 + m_2 g l_2$

Thus, for small oscillations

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\phi}_2^2 + m_2 l_1 l_2\dot{\phi}_1\dot{\phi}_2$$
$$+ \frac{1}{2}(m_1+m_2)g l_1\phi_1^2 + \frac{1}{2}m_2 g l_2\phi_2^2$$

$$= \frac{1}{2}\sum_{j,k} m_{jk}\dot{x}_j\dot{x}_k - \frac{1}{2}\sum_{j,k} k_{jk}x_j x_k$$

$x_j = (\phi_1, \phi_2)$

$$m_{jk} = \begin{bmatrix} (m_1+m_2)\,l_1^2 & m_2\, l_1 l_2 \\ m_2\, l_1 l_2 & m_2\, l_2^2 \end{bmatrix}$$

$$K_{jk} = \begin{bmatrix} (m_1+m_2)\,g\,l_1 & 0 \\ 0 & m_2\, g\, l_2 \end{bmatrix}$$

characteristic equation

$$0 = \det\left(K_{jk} - \omega^2 m_{jk}\right)$$

$$= \det \begin{bmatrix} (m_1+m_2)(g l_1 - \omega^2 l_1^2) & -\omega^2 m_2 l_1 l_2 \\ -\omega^2 m_2 l_1 l_2 & m_2 (g l_2 - \omega^2 l_2^2) \end{bmatrix}$$

$$= m_2 (m_1+m_2)\, l_1 l_2\, (g-\omega^2 l_1)(g-\omega^2 l_2) - \omega^4 m_2^2 l_1^2 l_2^2$$

$$= (m_1+m_2)(g-\omega^2 l_1)(g-\omega^2 l_2) - \omega^4 m_2 l_1 l_2$$

$$= \omega^4 - \left(\frac{m_1+m_2}{m_2}\right)\left(\frac{g}{l_1} - \omega^2\right)\left(\frac{g}{l_2} - \omega^2\right)$$

$$= \omega^4 - \left(\frac{m_1+m_2}{m_2}\right)\left[\frac{g}{l_1}\frac{g}{l_2} + \omega^4 - \omega^2\left(\frac{g}{l_1} + \frac{g}{l_2}\right)\right]$$

$$= \omega^4\left[1 - \left(\frac{m_1+m_2}{m_2}\right)\right] + \omega^2\left(\frac{m_1+m_2}{m_2}\right)\left(\frac{g}{l_1} + \frac{g}{l_2}\right) - \left(\frac{m_1+m_2}{m_2}\right)\frac{g}{l_1}\frac{g}{l_2}$$

$$= \omega^4\left(\frac{-m_1}{\cancel{m_2}}\right) + \omega^2\left(\frac{m_1+m_2}{\cancel{m_2}}\right)\left(\frac{g}{l_1} + \frac{g}{l_2}\right) - \left(\frac{m_1+m_2}{\cancel{m_2}}\right)\frac{g}{l_1}\frac{g}{l_2}$$

$$= \omega^4 - \omega^2\left(\frac{m_1+m_2}{m_1}\right)\left(\frac{g}{l_1} + \frac{g}{l_2}\right) + \left(\frac{m_1+m_2}{m_1}\right)\frac{g}{l_1}\frac{g}{l_2}$$

Eigen frequencies

$$\omega_{1,2}^2 = \frac{+\left(\frac{m_1+m_2}{m_1}\right)\left(\frac{g}{l_1}+\frac{g}{l_2}\right) \pm \sqrt{\left(\frac{m_1+m_2}{m_1}\right)^2\left(\frac{g}{l_1}+\frac{g}{l_2}\right)^2 - 4\left(\frac{m_1+m_2}{m_1}\right)\frac{g}{l_1}\frac{g}{l_2}}}{2}$$

$$= \frac{1}{2}\left(\frac{m_1+m_2}{m_1}\right)\frac{g(l_1+l_2)}{l_1 l_2} \pm \frac{1}{2}\left(\frac{m_1+m_2}{m_1}\right)\frac{g}{l_1 l_2}\sqrt{(l_1+l_2)^2 - 4\left(\frac{m_1}{m_1+m_2}\right) l_1 l_2}$$

$$= \frac{g}{2 m_1 l_1 l_2}\left\{(m_1+m_2)(l_1+l_2) \pm \sqrt{m_1+m_2}\sqrt{(m_1+m_2)(l_1+l_2)^2 - 4 m_1 l_1 l_2}\right\}$$

<u>NOTE</u>: In the limit where $m_1 >> m_2$

$$\omega_{1,2}^2 \rightarrow \frac{g}{2\cancel{m_1} l_1 l_2}\left\{\cancel{m_1}(l_1+l_2) \pm \cancel{\sqrt{m_1}}\,\cancel{\sqrt{m_1}}\sqrt{(l_1+l_2)^2 - 4 l_1 l_2}\right\}$$

$$= \frac{g}{2 l_1 l_2}\left\{(l_1+l_2) \pm \sqrt{l_1^2+l_2^2-2 l_1 l_2}\right\}$$

$$= \frac{g}{2 l_1 l_2}\left\{(l_1+l_2) \pm \sqrt{(l_1-l_2)^2}\right\}$$

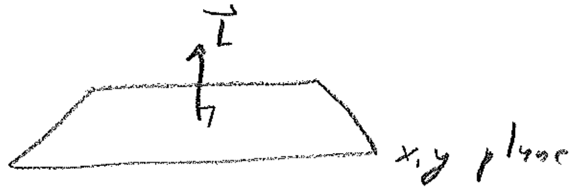
$$= \frac{g}{2 l_1 l_2}\left\{l_1+l_2 \pm |l_1-l_2|\right\}$$

$$= \frac{g}{l_2} \quad \text{or} \quad \frac{g}{l_1}$$

$$\rightarrow \omega_1 = \sqrt{\frac{g}{l_2}}, \quad \omega_2 = \sqrt{\frac{g}{l_1}} \quad \rightarrow$$ indep. oscillations of the two pendula

§23, Prob 3: space oscillator

$U = \frac{1}{2}kr^2$ central potential → cons. of ang. momentum
→ motion in 2-d plane



$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{1}{2}k(x^2+y^2)$$

EOM:
$$\left.\begin{aligned} \ddot{x} &= -\frac{k}{m}x \\ \ddot{y} &= -\frac{k}{m}y \end{aligned}\right\} \text{ indep.}$$

sol'n:
$$x(t) = a\cos(\omega t+\alpha)$$
$$y(t) = b\cos(\omega t+\beta)$$

where $\omega \equiv \sqrt{\frac{k}{m}}$

Thus, $\frac{x}{a} = \cos(\omega t+\alpha) = \cos\omega t\cos\alpha - \sin\omega t\sin\alpha$

$\frac{y}{b} = \cos(\omega t+\beta) = \cos\omega t\cos\beta - \sin\omega t\sin\beta$

$$\rightarrow \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \cos\beta & -\sin\beta \end{bmatrix}\begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix} = \begin{bmatrix} x/a \\ y/b \end{bmatrix}$$

Invert:
$$\begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix} = \frac{1}{\cos\beta\sin\alpha - \sin\beta\cos\alpha}\begin{bmatrix} -\sin\beta & \sin\alpha \\ -\cos\beta & \cos\alpha \end{bmatrix}\begin{bmatrix} x/a \\ y/b \end{bmatrix}$$

So
$$\cos\omega t = \frac{1}{\sin(\alpha-\beta)}\left(-\left(\frac{x}{a}\right)\sin\beta + \left(\frac{y}{b}\right)\sin\alpha\right)$$

$$\sin\omega t = \frac{1}{\sin(\alpha-\beta)}\left(-\left(\frac{x}{a}\right)\cos\beta + \left(\frac{y}{b}\right)\cos\alpha\right)$$

so

$$\cos\omega t = \frac{1}{\sin(\alpha-\beta)}\left(-\sin\beta\left(\frac{x}{a}\right) + \sin\alpha\left(\frac{y}{b}\right)\right)$$

$$\sin\omega t = \frac{1}{\sin(\alpha-\beta)}\left(-\cos\beta\left(\frac{x}{a}\right) + \cos\alpha\left(\frac{y}{b}\right)\right)$$

$$1 = \cos^2\omega t + \sin^2\omega t$$

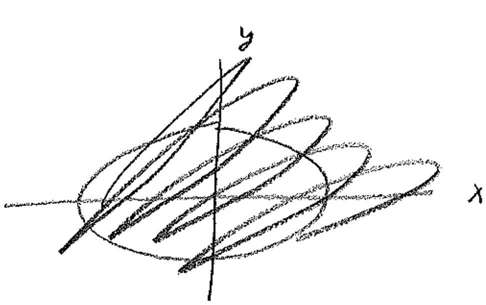
$$= \frac{1}{\sin^2(\alpha-\beta)}\Big[\sin^2\beta\left(\frac{x}{a}\right)^2 + \sin^2\alpha\left(\frac{y}{b}\right)^2 - 2\sin\alpha\sin\beta\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)$$

$$+ \cos^2\beta\left(\frac{x}{a}\right)^2 + \cos^2\alpha\left(\frac{y}{b}\right)^2 - 2\cos\alpha\cos\beta\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\Big]$$

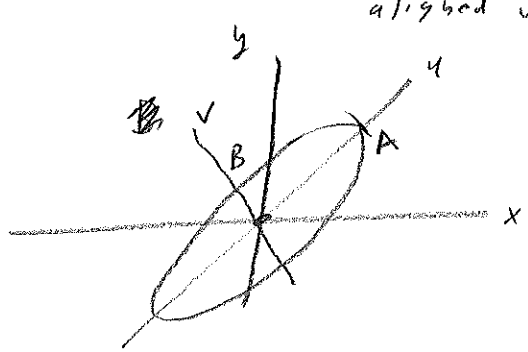
$$= \frac{1}{\sin^2(\alpha-\beta)}\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\cos(\alpha-\beta)\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\right]$$

$\cancel{\sin(\alpha-\beta)}$

$$\boxed{\sin^2(\alpha-\beta) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\cos(\alpha-\beta)\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)}$$



Ellipse with center at the origin, but not necessarily aligned with x,y axes



If $\alpha - \beta = \frac{\pi}{2}$:

$\cos(\alpha-\beta) = 0$, $\sin(\alpha-\beta) = 1$

$$\rightarrow \quad 1 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

[ellipse centred at the origin, with semi-major axis a, semi-minor axis b]

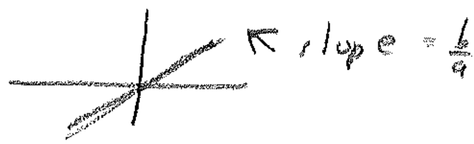
If $\alpha - \beta = 0$:

$\cos(\alpha-\beta) = 1$, $\sin(\alpha-\beta) = 0$

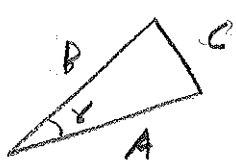
$$0 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)$$

$$0 = \left(\frac{x}{a} - \frac{y}{b}\right)^2$$

$$\rightarrow \quad \frac{x}{a} = \frac{y}{b}$$



$$\rightarrow \quad y = \frac{b}{a}x \quad \text{(straight line)}$$



$C^2 = A^2 + B^2 - 2AB\cos\gamma$

$A = \frac{x}{a}$, $B = \frac{y}{b}$, $C = \sin(\alpha-\beta)$

$\gamma = \alpha - \beta$

If $\alpha - \beta = \pi$:

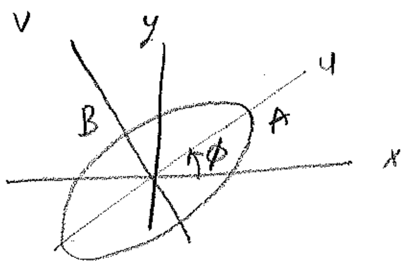
$\cos(\alpha-\beta) = -1$, $\sin(\alpha-\beta) = 0$

$$0 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + 2\left(\frac{x}{a}\right)\left(\frac{y}{b}\right) = \left(\left(\frac{x}{a}\right) + \left(\frac{y}{b}\right)\right)^2$$

$$\rightarrow \quad \frac{x}{a} = -\frac{y}{b} \quad \rightarrow \quad y = -\frac{b}{a}x$$

Given: $\sin^2\delta = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 2\cos\delta\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)$

where $\delta = \alpha - \beta$, Find rotation angle ϕ to (u,v) axes where the ellipse has semi-major and semi-minor axes along (u, v).



$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Thus, $\sin^2\delta = \frac{1}{a^2}\left(u\cos\phi - v\sin\phi\right)^2 + \frac{1}{b^2}\left(u\sin\phi + v\cos\phi\right)^2$

$$- 2\cos\delta \, \frac{1}{a}(u\cos\phi - v\sin\phi)\frac{1}{b}(u\sin\phi + v\cos\phi)$$

$$= \frac{1}{a^2}\left(u^2\cos^2\phi + v^2\sin^2\phi - 2uv\sin\phi\cos\phi\right)$$
$$+ \frac{1}{b^2}\left(u^2\sin^2\phi + v^2\cos^2\phi + 2uv\sin\phi\cos\phi\right)$$
$$- \frac{2}{ab}\cos\delta\left(u^2\sin\phi\cos\phi - v^2\sin\phi\cos\phi + uv\cos 2\phi\right)$$

$$= u^2\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2} - \frac{2\cos\delta\sin\phi\cos\phi}{ab}\right)$$
$$+ v^2\left(\frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{b^2} + \frac{2\cos\delta\sin\phi\cos\phi}{ab}\right)$$
$$+ 2uv\left(\frac{-\sin\phi\cos\phi}{a^2} + \frac{\sin\phi\cos\phi}{b^2} - \frac{\cos\delta}{ab}\cos 2\phi\right)$$

<u>Now</u>: we can make the factor multiply uv equal to zero,

$$0 = \sin\phi\cos\phi\left(-\frac{1}{a^2}+\frac{1}{b^2}\right) - \frac{\cos\delta}{ab}\cos 2\phi$$

$$= \sin 2\phi \ \frac{1}{2}\left(\frac{a^2-b^2}{a^2b^2}\right) - \frac{ab\cos\delta\cos 2\phi}{a^2b^2}$$

$$= \frac{1}{2a^2b^2}\left[(a^2-b^2)\sin 2\phi - 2ab\cos\delta\cos 2\phi\right]$$

Thus, set $(a^2-b^2)\sin 2\phi = 2ab\cos\delta\cos 2\phi$

$$\boxed{\tan 2\phi = \left(\frac{2ab}{a^2-b^2}\right)\cos\delta}$$ assuming $a \neq b$

If $a=b$, then we simply set $\cos 2\phi = 0 \rightarrow \boxed{\phi = \frac{\pi}{4}}$

Sec 23, Prob 3:

space oscillator: $U(r) = \frac{1}{2} k r^2$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r)$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi} = \text{const} \rightarrow \dot{\phi} = \frac{p_\phi}{m r^2}$$

$$E = \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} + \dot{r} \frac{\partial L}{\partial \dot{r}} - L = \text{const}$$

$$= T + U$$

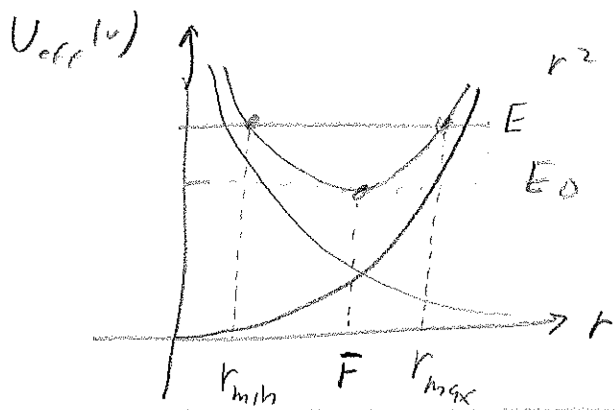
$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{1}{2} k r^2$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \left(\frac{p_\phi^2}{m^2 r^4} \right) + \frac{1}{2} k r^2$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{p_\phi^2}{2 m r^2} + \frac{1}{2} k r^2$$

$$= \frac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

$$U_{eff}(r) = \frac{p_\phi^2}{2 m r^2} + \frac{1}{2} k r^2$$



$$0 = \frac{dU_{eff}}{dr} = -\frac{p_\phi^2}{m r^3} + k r$$

$$\frac{p_\phi^2}{m r^3} = k r \rightarrow \bar{r} = \left(\frac{p_\phi^2}{k m} \right)^{\frac{1}{4}}$$

$$\rightarrow E_0 = U_{eff}(\bar{r}) = \frac{p_\phi^2}{2m \sqrt{\frac{p_\phi^2}{km}}} + \frac{1}{2} k \left(\frac{p_\phi^2}{km} \right)^{\frac{1}{2}}$$

$$E_0 = \frac{1}{2}\sqrt{\frac{p_\phi^2 \cdot k}{m}} + \frac{1}{2}\sqrt{\frac{k p_\phi^2}{m}}$$

$$= \sqrt{\frac{p_\phi^2 k}{m}}$$

thus, $E_0^2 = \frac{p_\phi^2 k}{m}$

so $E \geq E_0$

$$\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)} = \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{p_\phi}{mr^2}$$

Thus,

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}}$$

$$\boxed{t = \sqrt{\frac{m}{2}}\int \frac{dr}{\sqrt{E - U_{eff}(r)}} + \text{const}}$$

and

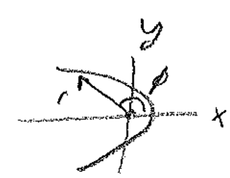
$$d\phi = \frac{p_\phi}{m}\frac{dr}{r^2}\frac{1}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}}$$

$$= \frac{p_\phi}{\sqrt{2m}}\frac{dr}{r^2\sqrt{E - U_{eff}(r)}}$$

$$\longrightarrow \boxed{\phi = \frac{p_\phi}{\sqrt{2m}}\int \frac{dr/r^2}{\sqrt{E - U_{eff}(r)}} + \text{const}}$$

<u>Now:</u> $E - U_{eff}(r) = E - \frac{p_\phi^2}{2mr^2} - \frac{1}{2}kr^2$

Let $\phi = 0$ when $r = r_0$ (closest approach)



Then $\phi = \frac{p_\phi}{\sqrt{2m}}\int_{r_0}^{r}\frac{dr/r^2}{\sqrt{E - U_{eff}(r)}}$ (r_0 is a turning point so $E - U_{eff}(r_0) = 0$)

$$\phi = \frac{p_\phi}{\sqrt{2m}} \int_{r_0}^{r} \frac{dr/r^2}{\sqrt{E - \frac{p_\phi^2}{2mr^2} - \frac{1}{2}kr^2}}$$

Let: $u = \frac{1}{r}$ $\quad du = -\frac{1}{r^2}\,dr$

$r = r_0, r \Leftrightarrow u = \frac{1}{r_0}, \frac{1}{r}$

$$\phi = \frac{p_\phi}{\sqrt{2m}} \int_{\frac{1}{r_0}}^{\frac{1}{r}} \frac{-du}{\sqrt{E - \frac{p_\phi^2}{2m}u^2 - \frac{1}{2}\frac{k}{u^2}}}$$

$$= \frac{p_\phi}{\sqrt{2m}} \int_{\frac{1}{r}}^{\frac{1}{r_0}} \frac{u\,du}{\sqrt{Eu^2 - \frac{p_\phi^2}{2m}u^4 - \frac{1}{2}k}}$$

$$= \int_{\frac{1}{r}}^{\frac{1}{r_0}} \frac{u\,du}{\sqrt{-u^4 + \frac{2mE}{p_\phi^2}u^2 - \frac{mk}{p_\phi^2}}}$$

Let: $x = u^2$, $\quad dx = 2u\,du$

$u = \frac{1}{r}, \frac{1}{r_0} \rightarrow x = \frac{1}{r^2}, \frac{1}{r_0^2}$

$$\phi = \frac{1}{2} \int_{\frac{1}{r^2}}^{\frac{1}{r_0^2}} \frac{dx}{\sqrt{-x^2 + \frac{2mE}{p_\phi^2}x - \frac{mk}{p_\phi^2}}}$$

$$= \frac{1}{2} \int_{\frac{1}{r^2}}^{\frac{1}{r_0^2}} \frac{dx}{\sqrt{a + bx + cx^2}}$$

where

$a = \frac{-mk}{p_\phi^2} < 0$

$b = \frac{2mE}{p_\phi^2} > 0$

$c = -1$

Alternatively, work in terms of roots

$$-x^2 + \frac{2mE}{p_\phi^2}x - \frac{mH}{p_\phi^2} = (x - x_1)(x_2 - x_1)$$

where $x_1 < x_2$ are the two roots of this equation

$$x = \frac{-\frac{2mE}{p_\phi^2} \pm \sqrt{\left(\frac{2mE}{p_\phi^2}\right)^2 - 4\left(\frac{mH}{p_\phi^2}\right)}}{-2}$$

$$= \frac{mE}{p_\phi^2} \mp \frac{mE}{p_\phi^2}\sqrt{1 - \frac{H p_\phi^2}{mE^2}}$$

$$= \frac{mE}{p_\phi^2}\left[1 \mp \sqrt{1 - \frac{H p_\phi^2}{mE^2}}\right]$$

$$x_1 = \frac{mE}{p_\phi^2}\left[1 - \sqrt{\ }\right], \quad x_2 = \frac{mE}{p_\phi^2}\left[1 + \sqrt{\ }\right]$$

$$\phi = \frac{1}{2}\int_{\frac{1}{r^2}}^{\frac{1}{r_{min}^2}} \frac{dx}{(x - x_1)(x_2 - x_1)}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2(x - x_1)}{(x_2 - x_1)} - 1\right)\Bigg|_{\frac{1}{r^2}}^{\frac{1}{r_{min}^2}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2\left(\frac{1}{r_{min}^2} - x_1\right)}{(x_2 - x_1)} - 1\right) - \frac{1}{2}\sin^{-1}\left(\frac{2\left(\frac{1}{r^2} - x_1\right)}{(x_2 - x_1)} - 1\right)$$

Now: $\frac{1}{r_{min}^2} = x_2$, $\frac{1}{r_{max}^2} = x_1$

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$$\phi = \frac{1}{2}\sin^{-1}\left(\frac{\cancel{2\left(\frac{1}{r_{min}^2}-\frac{1}{r_{max}^2}\right)}}{\cancel{\left(\frac{1}{r_{min}^2}-\frac{1}{r_{max}^2}\right)}}-1\right) - \frac{1}{2}\sin^{-1}\left(\frac{2\left(\frac{1}{r^2}-\frac{1}{r_{max}^2}\right)}{\left(\frac{1}{r_{min}^2}-\frac{1}{r_{max}^2}\right)}-1\right)$$

$$= \frac{1}{2}\underbrace{\sin^{-1}(1)}_{\frac{\pi}{2}} - \frac{1}{2}\sin^{-1}\left(\frac{2\left(\frac{1}{r^2}-\frac{1}{r_{max}^2}\right)}{\left(\frac{1}{r_{min}^2}-\frac{1}{r_{max}^2}\right)}-1\right)$$

$$2\phi = \frac{\pi}{2} - \sin^{-1}(\quad)$$

$$2\phi - \frac{\pi}{2} = -\sin^{-1}(\quad)$$

$$\cancel{\sin\left(2\phi - \frac{\pi}{2}\right) = \frac{2\left(\frac{1}{r^2}-\frac{1}{r_{max}^2}\right)}{\frac{1}{r_{min}^2}-\frac{1}{r_{max}^2}}} + 1$$

$$\sin^{-1}(\quad) = \frac{\pi}{2} - 2\phi$$

$$\frac{2\left(\frac{1}{r^2}-\frac{1}{r_{max}^2}\right)}{\left(\frac{1}{r_{min}^2}-\frac{1}{r_{max}^2}\right)} - 1 = \sin\left(\frac{\pi}{2}-2\phi\right)$$

$$= \sin\frac{\pi}{2}\cos(2\phi) - \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0}\sin(2\phi)$$

$$= \cos(2\phi)$$

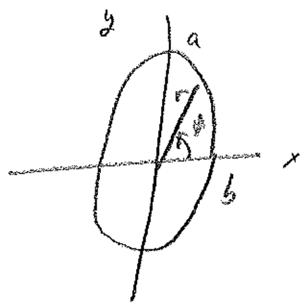
$$\frac{2\left(\frac{1}{r^2}-\frac{1}{r_{max}^2}\right)}{\frac{1}{r_{min}^2}-\frac{1}{r_{max}^2}} = 1+\cos 2\phi = 2\left(\frac{1+\cos 2\phi}{2}\right) = 2\cos^2\phi$$

Thus,

$$\frac{1}{r^2} - \frac{1}{r_{max}^2} = \left(\frac{1}{r_{min}^2} - \frac{1}{r_{max}^2}\right)\cos^2\phi$$

$$\rightarrow \boxed{\frac{1}{r^2} = \frac{1}{r_{max}^2} + \left(\frac{1}{r_{min}^2} - \frac{1}{r_{max}^2}\right)\cos^2\phi}$$

which is an ellipse with $r_{max} = a$, $r_{min} = b$ centered at the origin.



<u>Check:</u> $x = r\cos\phi$, $y = r\sin\phi$

$$\left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$$

$$r^2\frac{\cos^2\phi}{b^2} + r^2\frac{\sin^2\phi}{a^2} = 1$$

$$\frac{\cos^2\phi}{b^2} + \frac{1-\cos^2\phi}{a^2} = \frac{1}{r^2}$$

$$\frac{1}{r^2} = \frac{1}{a^2} + \cos^2\phi\left(\frac{1}{b^2} - \frac{1}{a^2}\right)$$