§22, Probl:

Find $x(t)$ for different forces:

a) $F = F_0$, b) $F = \alpha t$, c) $F = F_0 \exp(-\alpha t)$

d) $F = F_0 \exp(-\alpha t) \cos\beta t$

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Use $\xi(t) = \exp(i\omega t)\left(\int_0^t \frac{1}{m} F(t) \exp(-i\omega t) dt + \xi_0 \right)$

$\xi = \dot{x} + i\omega x$

$= 0$ since
$\xi_0 = \dot{x}_0 + i\omega x_0$
with $x_0 = 0$, $\dot{x}_0 = 0$

Thus,

$$x(t) = \frac{1}{\omega} \mathrm{Im}\left[\exp(i\omega t) \int_0^t \frac{1}{m} F(t) \exp(-i\omega t) dt \right]$$

a) $F = F_0$

$$\int_0^t \frac{F_0}{m} \exp(-i\omega t) = \frac{F_0}{m}\left(\frac{1}{-i\omega}\right) e^{-i\omega t} \Big|_0^t$$

$$= \frac{F_0}{-im\omega}\left(e^{-i\omega t} - 1\right)$$

$$\rightarrow x(t) = -\frac{F_0}{m\omega^2} \mathrm{Im}\left[\frac{1}{i} e^{i\omega t}\left(e^{-i\omega t} - 1\right)\right]$$

$$= \frac{F_0}{m\omega^2} \mathrm{Im}\left[i\left(1 - e^{i\omega t}\right)\right]$$

$$= \frac{F_0}{m\omega^2} \mathrm{Im}\left[i\left(1 - \cos(\omega t) - i\sin\omega t\right)\right]$$

$$= \boxed{\frac{F_0}{m\omega^2}\left(1 - \cos(\omega t)\right)}$$

b) $F = at$

$$\int_0^t \frac{a}{m} t e^{-i\omega t} dt = \frac{a}{m} \int_0^t t e^{-i\omega t} dt$$

$$= \frac{a}{m}\left[-\frac{t}{i\omega} e^{-i\omega t} \Big|_0^t + \frac{1}{i\omega}\int_0^t e^{-i\omega t} dt \right]$$

Let $u = t$
$du = dt$
$dV = e^{-i\omega t} dt$
$V = \frac{-1}{i\omega} e^{-i\omega t}$

$$= \frac{a}{m}\left[-\frac{t}{i\omega} e^{-i\omega t} + \frac{1}{\omega^2}\left(e^{-i\omega t} - 1\right) \right]$$

Thus, $$x(t) = \frac{a}{m\omega} Im\left[e^{i\omega t}\left(-\frac{t}{i\omega} e^{-i\omega t} + \frac{1}{\omega^2}\left(e^{-i\omega t} - 1\right)\right)\right]$$

$$= \frac{a}{m\omega} Im\left[-\frac{t}{i\omega} + \frac{1}{\omega^2} - \frac{1}{\omega^2} e^{i\omega t} \right]$$

$$= \frac{a}{m\omega} Im\left[i\frac{t}{\omega} + \frac{1}{\omega^2}\left(1 - \cos(\omega t)\right) - \frac{i}{\omega^2} \sin \omega t \right]$$

$$= \frac{a}{m\omega} Im\left[\frac{1}{\omega^2}\left(1 - \cos(\omega t)\right) + i\left(\frac{t}{\omega} - \frac{1}{\omega^2} \sin \omega t\right)\right]$$

$$= \frac{a}{m\omega^2}\left(t - \frac{1}{\omega} \sin \omega t\right)$$

$$= \boxed{\frac{a}{m\omega^3}\left(\omega t - \sin \omega t\right)}$$

c) $F = F_0 \exp(-\alpha t)$

$$\int_0^t \frac{1}{m} F_0 e^{-\alpha t} e^{-i\omega t}\, dt = \frac{F_0}{m} \int_0^t e^{-(\alpha + i\omega)t}\, dt$$

$$= \frac{F_0}{m} \frac{1}{-(\alpha + i\omega)} e^{-(\alpha+i\omega)t} \Big|_0^t$$

$$= -\frac{F_0}{m} \frac{1}{(\alpha + i\omega)} \left(e^{-(\alpha+i\omega)t} - 1 \right)$$

$$\rightarrow x(t) = \frac{-F_0}{m\omega} \operatorname{Im}\left[e^{i\omega t} \frac{1}{(\alpha+i\omega)} \left(e^{-(\alpha+i\omega)t} - 1 \right) \right]$$

$$= \frac{-F_0}{m\omega} \operatorname{Im}\left[\frac{1}{(\alpha+i\omega)} \left(e^{-\alpha t} - e^{+i\omega t} \right) \right]$$

$$= \frac{-F_0}{m\omega} \operatorname{Im}\left[\frac{\alpha - i\omega}{\alpha^2+\omega^2} \left(e^{-\alpha t} - \cos(\omega t) - i \sin \omega t \right) \right]$$

$$= \frac{-F_0}{m\omega} \left(\frac{1}{\alpha^2+\omega^2} \right) \operatorname{Im}\Big[\alpha \left(e^{-\alpha t} - \cos(\omega t) \right) - \omega \sin \omega t$$
$$- i\omega \left(e^{-\alpha t} + \cos(\omega t) \right) - i\alpha \sin \omega t \Big]$$

$$= \frac{-F_0}{m\omega} \left(\frac{1}{\alpha^2+\omega^2} \right) \left[-\omega \left(e^{-\alpha t} - \cos(\omega t) \right) - \alpha \sin \omega t \right]$$

$$= \frac{F_0}{m\omega} \left(\frac{1}{\alpha^2+\omega^2} \right) \left[\omega e^{-\alpha t} - \omega \cos \omega t + \alpha \sin \omega t \right]$$

$$= \boxed{\frac{F_0}{m} \left(\frac{1}{\alpha^2+\omega^2} \right) \left[e^{-\alpha t} - \cos \omega t + \frac{\alpha}{\omega} \sin \omega t \right]}$$

d) $F = F_0 e^{-\alpha t} \cos \beta t$

$$= F_0 e^{-\alpha t} \frac{1}{2}\left(e^{i\beta t} + e^{-i\beta t}\right)$$

$$= \frac{F_0}{2}\left[e^{(-\alpha + i\beta)t} + e^{(-\alpha - i\beta)t}\right]$$

$$\int_0^t \frac{1}{m}\frac{F_0}{2}\left(e^{(-\alpha + i\beta - i\omega)t} + e^{(-\alpha - i\beta - i\omega)t}\right) dt$$

$$= \frac{F_0}{2m}\int_0^t dt \left(e^{(-\alpha + i(\beta-\omega))t} + e^{(-\alpha - i(\beta+\omega))t}\right)$$

$$= \frac{F_0}{2m}\left[\frac{1}{-\alpha + i(\beta-\omega)}\left(e^{(-\alpha + i(\beta-\omega))t} - 1\right) + \frac{1}{-\alpha - i(\beta+\omega)}\left(e^{(-\alpha - i(\beta+\omega))t} - 1\right)\right]$$

$$= \frac{F_0}{2m}\left[\frac{-\alpha - i(\beta-\omega)}{\alpha^2 + (\beta-\omega)^2}\left(e^{(-\alpha + i(\beta-\omega))t} - 1\right) + \frac{-\alpha + i(\beta+\omega)}{\alpha^2 + (\beta+\omega)^2}\left(e^{(-\alpha - i(\beta+\omega))t} - 1\right)\right]$$

$$= \frac{-F_0}{2m}\left[\frac{\alpha + i(\beta-\omega)}{\alpha^2 + (\beta-\omega)^2}\left(e^{(-\alpha + i(\beta-\omega))t} - 1\right) + \frac{\alpha - i(\beta+\omega)}{\alpha^2 + (\beta+\omega)^2}\left(e^{(-\alpha - i(\beta+\omega))t} - 1\right)\right]$$

$$\rightarrow x(t) = \frac{-F_0}{2m\omega} \mathrm{Im}\left[e^{i\omega t} [\,/\!/\!/\!/\,] \right]$$

$$= \frac{-F_0}{2m\omega} \mathrm{Im}\left[\frac{\alpha + i(\beta-\omega)}{\alpha^2+(\beta-\omega)^2}\left(e^{(-\alpha+i\beta)t} - e^{i\omega t} \right) \right.$$

$$\left. + \frac{\alpha - i(\beta+\omega)}{\alpha^2+(\beta+\omega)^2}\left(e^{(-\alpha-i\beta)t} - e^{i\omega t} \right) \right]$$

$$= \frac{-F_0}{2m\omega} \mathrm{Im}\left[\frac{\alpha + i(\beta-\omega)}{\alpha^2+(\beta-\omega)^2}\left(e^{-\alpha t}(\cos\beta t + i\sin\beta t) - (\cos\omega t + i\sin\omega t) \right) \right.$$

$$\left. + \frac{\alpha - i(\beta+\omega)}{\alpha^2+(\beta+\omega)^2}\left(e^{-\alpha t}(\cos\beta t - i\sin\beta t) - (\cos\omega t + i\sin\omega t) \right) \right]$$

$$= \frac{-F_0}{2m\omega} \left[\frac{1}{\alpha^2+(\beta-\omega)^2}\left((\beta-\omega)\left(e^{-\alpha t}\cos\beta t - \cos\omega t\right) + \alpha\left(e^{-\alpha t}\sin\beta t - \sin\omega t\right) \right) \right.$$

$$\left. + \frac{1}{\alpha^2+(\beta+\omega)^2}\left(-(\beta+\omega)\left(e^{-\alpha t}\cos\beta t - \cos\omega t\right) - \alpha\left(e^{-\alpha t}\sin\beta t + \sin\omega t\right) \right) \right]$$

Common denominator

$$\frac{1}{\alpha^2+(\beta-\omega)^2}\;\frac{1}{\alpha^2+(\beta+\omega)^2} = \frac{1}{\alpha^4+(\beta-\omega)^2(\beta+\omega)^2+\alpha^2\left((\beta+\omega)^2+(\beta-\omega)^2\right)}$$

$$= \frac{1}{\alpha^4+(\beta^2-\omega^2)^2+\alpha^2\left(\beta^2+\omega^2+\cancel{2\beta^2\omega^2}+\beta^2+\omega^2-\cancel{2\beta^2\omega^2}\right)}$$

$$= \frac{1}{\alpha^4+(\beta^2-\omega^2)^2+2(\beta^2+\omega^2)\alpha^2}$$

$$= \frac{1}{\alpha^4+\beta^4+\omega^4-2\beta^2\omega^2+2\alpha^2\beta^2+2\alpha^2\omega^2}$$

$$= \frac{1}{\omega^4+(\alpha^2-\beta^2)^2+4\alpha^2\beta^2+2(\alpha^2-\beta^2)\omega^2}$$

$$= \frac{1}{\left(\omega^2+(\alpha^2-\beta^2)\right)^2+4\alpha^2\beta^2}$$

$$= \frac{1}{\text{denom}}$$

Thus

$$x(t) = \frac{-F_0}{2m\omega}\left(\frac{1}{\text{denom}}\right)\Big[\left(\alpha^2+(\beta+\omega)^2\right)\Big(e^{-\alpha t}\left((\beta-\omega)\cos\beta t+\alpha\sin\beta t\right) - \left((\beta-\omega)\cos\omega t+\alpha\sin\omega t\right)\Big)$$

$$+\left(\alpha^2+(\beta-\omega)^2\right)\Big(-e^{-\alpha t}\left((\beta+\omega)\cos\beta t+\alpha\sin\beta t\right) + \left((\beta+\omega)\cos\omega t-\alpha\sin\omega t\right)\Big)\Big]$$

Terms multiplying:

i) $\cos \omega t$: $\left(\alpha^2+(\beta+\omega)^2\right)\left(-(\beta-\omega)\right) + \left(\alpha^2+(\beta-\omega)^2\right)(\beta+\omega)$

$$= \underline{-\alpha^2(\beta-\omega)} - (\beta+\omega)(\beta^2-\omega^2) \underline{+ \alpha^2(\beta+\omega)} + (\beta-\omega)(\beta^2-\omega^2)$$

$$= \alpha^2 \underbrace{\left[-(\cancel{\beta}-\omega)+(\cancel{\beta}+\omega)\right]}_{2\omega} + (\beta^2-\omega^2)\underbrace{\left[-(\cancel{\beta}+\omega)+(\cancel{\beta}-\omega)\right]}_{-2\omega}$$

$$= 2\omega\left(\alpha^2-(\beta^2-\omega^2)\right)$$

$$= \boxed{+2\omega\left(\omega^2+\alpha^2-\beta^2\right)}$$

ii) $\sin \omega t$: $\left(\alpha^2+(\beta+\omega)^2\right)(-\alpha) + \left(\alpha^2+(\beta-\omega)^2\right)(-\alpha)$

$$= -\alpha\left(\alpha^2+\beta^2+\omega^2+\cancel{2\beta\omega^2}+\alpha^2+\beta^2+\omega^2-\cancel{2\beta\omega}\right)$$

$$= \boxed{-2\alpha\left(\omega^2+\alpha^2+\beta^2\right)}$$

iii) $e^{-\alpha t}\cos\beta t$: $\left(\alpha^2+(\beta+\omega)^2\right)(\beta-\omega) + \left(\alpha^2+(\beta-\omega)^2\right)\left(-(\beta+\omega)\right)$

= minus 1st term

$$= \boxed{-2\omega\left(\omega^2+\alpha^2-\beta^2\right)}$$

iv) $e^{-\alpha t}\sin\beta t$: $\left(\alpha^2+(\beta+\omega)^2\right)\alpha + \left(\alpha^2+(\beta-\omega)^2\right)(-\alpha)$

$$= \alpha\left[\cancel{\alpha^2}+\cancel{\beta^2}+\cancel{\omega^2}+2\beta\omega-\cancel{\alpha^2}-\cancel{\beta^2}-\cancel{\omega^2}+2\beta\omega\right]$$

$$= \boxed{4\alpha\beta\omega}$$

Thus,

$$x(t) = \frac{-F_0}{m}\left(\frac{1}{\text{denom}}\right)\Big[(\omega^2+\alpha^2-\beta^2)\cos\omega t - \frac{\alpha}{\omega}(\omega^2+\alpha^2+\beta^2)\sin\omega t$$

$$- (\omega^2+\alpha^2-\beta^2)e^{-\alpha t}\cos\beta t + 2\alpha\beta e^{-\alpha t}\sin\beta t\Big]$$

$$= \boxed{\frac{F_0}{m}\left(\frac{1}{\text{denom}}\right)\Big[-(\omega^2+\alpha^2-\beta^2)\cos\omega t + \left(\frac{\alpha}{\omega}\right)(\omega^2+\alpha^2+\beta^2)\sin\omega t + e^{-\alpha t}\left[(\omega^2+\alpha^2-\beta^2)\cos\beta t - 2\alpha\beta\sin\beta t\right]\Big]}$$

where $\text{denom} = \left(\omega^2+(\alpha^2-\beta^2)\right)^2 + 4\alpha^2\beta^2$

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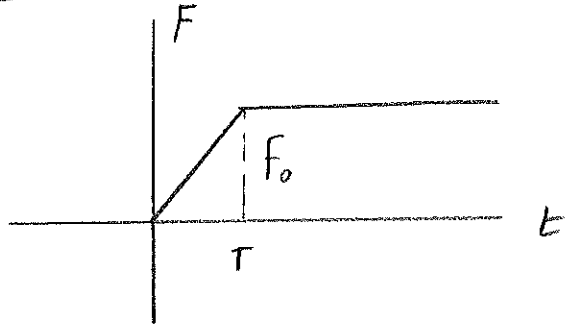
§22, Prob 2



where $\begin{cases} 0 < t < T \\ F(t) = F_0\left(\frac{t}{T}\right) \\ \\ T < t \\ F(t) = F_0 \end{cases}$

$$x(t) = \frac{1}{\omega}\, Im\left[e^{i\omega t}\left(\int_0^t \frac{1}{m} F(t)\, e^{-i\omega t}\, dt + \underbrace{\xi_0}_{=0}\right)\right]$$

since $x_0 = 0,\ \dot{x}_0 = 0$

$$= \frac{F_0}{m\omega}\, Im\left[e^{i\omega t}\left(\int_0^T \frac{t}{T} e^{-i\omega t}\, dt + \int_T^t e^{-i\omega t}\, dt\right)\right]$$

$$(A) = \frac{F_0}{m\omega T}\, Im\left[e^{i\omega t}\int_0^T t e^{-i\omega t}\, dt\right]$$

$u = t,\ du = dt$
$dv = e^{-i\omega t}\, dt$
$v = \frac{1}{-i\omega} e^{-i\omega t}$

$$= \frac{F_0}{m\omega T}\, Im\left[e^{i\omega t}\left(\frac{t}{-i\omega} e^{-i\omega t}\Big|_0^T + \frac{1}{i\omega}\int_0^T e^{-i\omega t}\, dt\right)\right]$$

$$= \frac{F_0}{m\omega T}\, Im\left[e^{i\omega t}\left(\frac{iT}{\omega} e^{-i\omega T} + \frac{1}{\omega^2} e^{-i\omega t}\Big|_0^T\right)\right]$$

$$= \frac{F_0}{m\omega T}\, Im\left[e^{i\omega t}\left(\frac{iT}{\omega} e^{-i\omega T} + \frac{1}{\omega^2} e^{-i\omega T} - \frac{1}{\omega^2}\right)\right]$$

$$= \frac{F_0}{m\omega T}\, Im\left[e^{i\omega(t-T)}\left(\frac{iT}{\omega} + \frac{1}{\omega^2}\right) - \frac{1}{\omega^2} e^{i\omega t}\right]$$

$$= \frac{F_0}{m\omega T} \operatorname{Im}\left[\left(\cos(\omega(t-T)) + i\sin(\omega(t-T))\right)\left(\frac{iT}{\omega} + \frac{1}{\omega^2}\right) - \frac{1}{\omega^2}\left(\cos\omega t + i\sin\omega t\right)\right]$$

$$= \frac{F_0}{m\omega T}\left[\frac{T}{\omega}\cos(\omega(t-T)) + \frac{1}{\omega^2}\sin(\omega(t-T)) - \frac{1}{\omega^2}\sin\omega t\right]$$

(B) $$= \frac{F_0}{m\omega} \operatorname{Im}\left[e^{i\omega t}\int_T^t e^{-i\omega t}\,dt\right]$$

$$= \frac{F_0}{m\omega} \operatorname{Im}\left[e^{i\omega t}\frac{1}{-i\omega} e^{-i\omega t}\Big|_T^t\right]$$

$$= \frac{F_0}{m\omega^2} \operatorname{Im}\left[i\, e^{i\omega t}\left(e^{-i\omega t} - e^{-i\omega T}\right)\right]$$

$$= \frac{F_0}{m\omega^2} \operatorname{Im}\left[i\left(1 - e^{i\omega(t-T)}\right)\right]$$

$$= \frac{F_0}{m\omega^2} \operatorname{Im}\left[i\left(1 - \cos(\omega(t-T)) - i\sin(\omega(t-T))\right)\right]$$

$$= \frac{F_0}{m\omega^2}\left(1 - \cos(\omega(t-T))\right)$$

Thus,

$$x(t) = \cancel{\frac{F_0}{m\omega^2}\cos(\omega(t-T))} + \frac{F_0}{m\omega^3 T}\sin(\omega(t-T)) - \frac{F_0}{m\omega^3 T}\sin\omega t$$

$$+ \frac{F_0}{m\omega^2} - \cancel{\frac{F_0}{m\omega^2}\cos(\omega(t-T))}$$

$$\boxed{x(t) = \frac{F_0}{m\omega^2}\left(1 + \frac{\sin(\omega(t-T))}{\omega T} - \frac{1}{\omega T}\sin\omega t\right)}$$

Now: $\sin\omega t = \sin[\omega(t-T) + \omega T]$

$$= \sin(\omega(t-T))\cos\omega T + \cos(\omega(t-T))\sin\omega T$$

$$\rightarrow \quad x(t) = \frac{F_0}{m\omega^2}\left[1 + \frac{\sin(\omega(t-T))}{\omega T} - \frac{1}{\omega T}\sin(\omega(t-T))\cos\omega T - \frac{1}{\omega T}\cos(\omega(t-T))\sin\omega T\right]$$

$$= \frac{F_0}{m\omega^2} + \frac{F_0}{m\omega^3 T}\left[\sin(\omega(t-T))(1-\cos\omega T) - \cos(\omega(t-T))\sin\omega T\right]$$

Now: $1-\cos\omega T = 2\sin^2\left(\frac{\omega T}{2}\right)$

$$\boxed{x(t) = \frac{F_0}{m\omega^2} - \frac{F_0}{m\omega^2}\frac{\sin(\omega T)}{\omega T}\cos(\omega(t-T)) + \frac{2F_0}{m\omega^2}\frac{1}{\omega T}\sin^2\left(\frac{\omega T}{2}\right)\sin(\omega(t-T))}$$

NOTE:

$$x(t) = \frac{F_0}{m\omega^2} + c_1 \cos(\omega(t-T)) + c_2 \sin(\omega(t-T))$$

where $$c_1 = -\frac{F_0}{m\omega^2} \frac{\sin(\omega T)}{\omega T}$$

$$c_2 = \frac{2F_0}{m\omega^2} \frac{\sin^2(\frac{\omega T}{2})}{\omega T} = \frac{F_0}{m\omega^2} \frac{(1-\cos(\omega T))}{\omega T}$$

Now:

$$c_1 \cos\theta + c_2 \sin\theta = c_1 \operatorname{Re} e^{i\theta} + c_2 \operatorname{Im} e^{i\theta}$$
$$= c_1 \operatorname{Re} e^{i\theta} + c_2 \operatorname{Re}(-ie^{i\theta})$$
$$= \operatorname{Re}[c_1 e^{i\theta} - ic_2 e^{i\theta}]$$
$$= \operatorname{Re}[(c_1 - ic_2) e^{i\theta}]$$

where $$c_1 - ic_2 = \sqrt{c_1^2 + c_2^2}\; e^{i\alpha}$$

with $$\tan\alpha = -\frac{c_2}{c_1}$$

Thus, $$c_1 \cos\theta + c_2 \sin\theta = \operatorname{Re}[\sqrt{c_1^2+c_2^2}\; e^{i(\theta+\alpha)}]$$
$$= \sqrt{c_1^2+c_2^2} \cos[\theta+\alpha]$$

Magnitude of oscillation

$$= \sqrt{c_1^2+c_2^2}$$
$$= \frac{F_0}{m\omega^3 T} \sqrt{\sin^2 \omega T + (1-\cos\omega T)^2}$$
$$= \frac{F_0}{m\omega^3 T} \sqrt{\sin^2\omega T + 1 + \cos^2\omega T - 2\cos\omega T}$$

$$\sqrt{c_1^2 + c_2^2} = \frac{F_0}{m\omega^3 T}\sqrt{2}\sqrt{(1-\cos\omega T)}$$

$$= \frac{F_0}{m\omega^3 T}\sqrt{2}\sqrt{2\sin^2\left(\frac{\omega T}{2}\right)}$$

$$= \boxed{\frac{2F_0}{m\omega^3 T}\sin\left(\frac{\omega T}{2}\right)}$$

Alternative method:

For $0 < t < T$:

$$x(t) = \frac{F_0}{m\omega^3 T}\left(\omega t - \sin(\omega t)\right)$$

using the result of Sec 22, Prob 1

For $T < t$, assume that

$$x(t) = \underbrace{\frac{F_0}{m\omega^2}}_{\text{particular soln}} + c_1\cos[\omega(t-T)] + c_2\sin[\omega(t-T)]$$

Determine c_1 and c_2 by matching $x(t)$, $\dot{x}(t)$ at $t=T$

(1) $$\frac{F_0}{m\omega^3 T}\left(\cancel{\omega T} - \sin(\omega T)\right) = \cancel{\frac{F_0}{m\omega^2}} + c_1$$

$$\rightarrow \boxed{c_1 = \frac{-F_0}{m\omega^3 T}\sin(\omega T)}$$

(2) $$\frac{F_0}{m\omega^3 T}\left(\omega - \omega\cos(\omega T)\right) = -\omega C_1 \sin\left[\cancel{\omega(T-T)}\right]^{\,0} + \omega C_2 \cos\left[\omega(T-T)\right]$$

$$= \omega C_2$$

$$\rightarrow \boxed{C_2 = \frac{F_0}{m\omega^3 T}\left(1-\cos(\omega T)\right) = \frac{2F_0}{m\omega^3 T}\sin^2\left(\frac{\omega T}{2}\right)}$$

which agree with previous calculation for C_1, C_2.

§22, Prob 3

$$F(t) = \begin{cases} F_0 & 0<t<T \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \frac{1}{\omega} \, \mathrm{Im}\left[e^{i\omega t} \left(\int_0^t \frac{1}{m} F(t) e^{-i\omega t} dt + \underbrace{\xi_0}_{\substack{=0 \text{ since} \\ x_0=0,\ \dot{x}_0=0}} \right) \right]$$

$$= \frac{F_0}{m\omega} \, \mathrm{Im}\left[e^{i\omega t} \int_0^T e^{-i\omega t} dt \right]$$

$$= \frac{F_0}{m\omega} \, \mathrm{Im}\left[e^{i\omega t} \left(\left(\frac{1}{-i\omega} \right) e^{-i\omega t} \right) \Big|_0^T \right]$$

$$= \frac{F_0}{m\omega^2} \, \mathrm{Im}\left[i \, e^{i\omega t} \left(e^{-i\omega T} - 1 \right) \right]$$

$$= \frac{F_0}{m\omega^2} \, \mathrm{Im}\left[i \left(e^{i\omega(t-T)} - e^{i\omega t} \right) \right]$$

$$= \frac{F_0}{m\omega^2} \, \mathrm{Im}\left[i \left(\left(\cos(\omega(t-T)) + i\sin(\omega(t-T)) \right) - \left(\cos(\omega t) + i\sin(\omega t) \right) \right) \right]$$

$$= \frac{F_0}{m\omega^2} \left(\cos(\omega(t-T)) - \cos(\omega t) \right)$$

Now: $\cos(\omega t) = \cos(\omega(t-T) + \omega T)$

$$= \cos(\omega(t-T)) \cos \omega T - \sin(\omega(t-T)) \sin \omega T$$

$$x(t) = \frac{F_0}{m\omega^2}\left[\left(1-\cos(\omega T)\right)\cos(\omega(t-T)) + \sin\omega T \; \sin(\omega(t-T))\right]$$

$$= c_1 \cos(\omega(t-T)) + c_2 \sin(\omega(t-T))$$

where $c_1 = \frac{F_0}{m\omega^2}\left(1-\cos(\omega T)\right) = \frac{2F_0}{m\omega^2}\sin^2\left(\frac{\omega T}{2}\right)$

$c_2 = \frac{F_0}{m\omega^2}\sin\omega T$

~~Amplitude~~: Thus, $x(t) = a \cos(\omega(t-T)+\alpha)$

amplitude (a); phase (α): $\tan\alpha = -\frac{c_2}{c_1}$

where $a = \sqrt{c_1^2 + c_2^2}$

$$= \frac{F_0}{m\omega^2}\sqrt{(1-\cos\omega T)^2 + \sin^2\omega T}$$

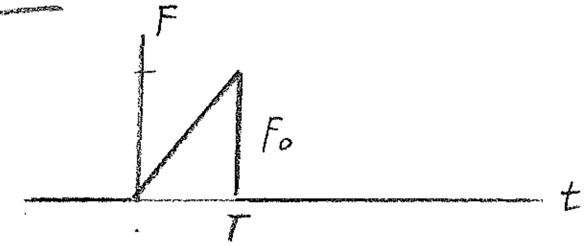
$$= \frac{F_0}{m\omega^2}\sqrt{1+\cos^2(\omega T) - 2\cos\omega T + \sin^2(\omega T)}$$

$$= \frac{F_0}{m\omega^2}\sqrt{2(1-\cos\omega T)}$$

$$= \frac{F_0}{m\omega^2}\sqrt{2\cdot 2\sin^2\left(\frac{\omega T}{2}\right)}$$

$$= \boxed{\frac{2F_0}{m\omega^2}\sin\left(\frac{\omega T}{2}\right)}$$

§22, prob 4:



$$F(t) = \begin{cases} F_0\left(\frac{t}{T}\right) & 0<t<T \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \frac{1}{\omega}\, Im\left[e^{i\omega t}\left(\int_0^t \frac{1}{m} F(t)\, e^{-i\omega t}\, dt + \xi_0 \right)\right]$$

$= 0$

Since $x_0 = 0$, $\dot{x}_0 = 0$

$$= \frac{F_0}{m\omega T}\, Im\left[e^{i\omega t} \int_0^T dt\; t\, e^{-i\omega t} \right]$$

Let: $u = t$, $du = dt$

$dv = e^{-i\omega t}\, dt$, $v = \frac{1}{-i\omega} e^{-i\omega t}$

$$x(t) = \frac{F_0}{m\omega T}\, Im\left[e^{i\omega t}\left(\frac{t e^{-i\omega t}}{-i\omega}\Big|_0^T + \frac{1}{i\omega}\int_0^T dt\, e^{-i\omega t} \right)\right]$$

$$= \frac{F_0}{m\omega T}\, Im\left[e^{i\omega t}\left(\frac{T e^{-i\omega T}}{-i\omega} + \frac{1}{\omega^2} e^{-i\omega t}\Big|_0^T \right)\right]$$

$$= \frac{F_0}{m\omega T}\, Im\left[i\frac{T}{\omega} e^{i\omega(t-T)} + \frac{1}{\omega^2}\left(e^{i\omega(t-T)} - e^{i\omega t} \right)\right]$$

$$= \frac{F_0}{m\omega T}\, Im\left[\left(i\frac{T}{\omega} + \frac{1}{\omega^2} \right) e^{i\omega(t-T)} - \frac{1}{\omega^2} e^{i\omega t} \right]$$

$$= \frac{F_0}{m\omega T}\, Im\left[\left(i\frac{T}{\omega} + \frac{1}{\omega^2} \right)\left(\cos(\omega(t-T)) + i\sin(\omega(t-T)) \right) - \frac{1}{\omega^2}\left(\cos\omega t + i\sin\omega t \right) \right]$$

$$x(t) = \frac{F_0}{m\omega T}\left[\frac{T}{\omega}\cos(\omega(t-T)) + \frac{1}{\omega^2}\sin(\omega(t-T)) - \frac{1}{\omega^2}\sin\omega t\right]$$

$$= \frac{F_0}{m\omega^2}\cos(\omega(t-T)) + \frac{F_0}{m\omega^3 T}\sin(\omega(t-T)) - \frac{F_0}{m\omega^3 T}\sin(\omega t)$$

Now: $\sin\omega t = \sin(\omega(t-T) + \omega T)$

$$= \sin(\omega(t-T))\cos(\omega T) + \cos(\omega(t-T))\sin(\omega T)$$

$$x(t) = \frac{F_0}{m\omega^2}\left(1 - \frac{1}{\omega T}\sin(\omega T)\right)\cos(\omega(t-T))$$

$$+ \frac{F_0}{m\omega^3 T}\left(1 - \cos(\omega T)\right)\sin(\omega(t-T))$$

$$= \frac{F_0}{m\omega^3 T}\left[(\omega T - \sin(\omega T))\cos(\omega(t-T)) + (1-\cos(\omega T))\sin(\omega(t-T))\right]$$

$$= c_1\cos(\omega(t-T)) + c_2\sin(\omega(t-T))$$

$$= a\cos(\omega(t-T) + \alpha)$$

$$a = \sqrt{c_1^2 + c_2^2}, \quad \tan\alpha = -\frac{c_2}{c_1}$$

$$c_1 = \frac{F_0}{m\omega^3 T}\left(\omega T - \sin(\omega T)\right)$$

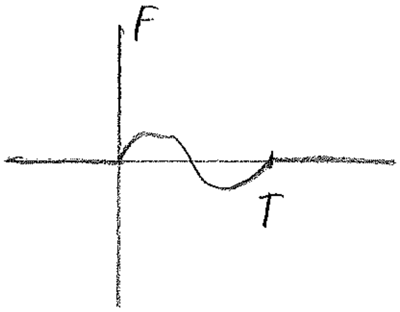
$$c_2 = \frac{F_0}{m\omega^3 T}\left(1 - \cos(\omega T)\right)$$

$$\rightarrow \quad a = \sqrt{c_1^2 + c_2^2}$$

$$= \frac{F_0}{m\omega^3 T}\sqrt{\omega^2 T^2 + \sin^2(\omega T) - 2\omega T \sin(\omega T) + 1 + \cos^2(\omega T) - 2\cos(\omega T)}$$

$$= \frac{F_0}{m\omega^3 T}\sqrt{\omega^2 T^2 - 2\omega T \sin(\omega T) + 2\left(1 - \cos(\omega T)\right)}$$

§22, Prob 5:



$$T = \frac{2\pi}{\omega}$$

$$F(t) = \begin{cases} F_0 \sin(\omega t) & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \frac{1}{\omega}\, \mathrm{Im}\left[e^{i\omega t}\left(\int_0^t \frac{F(t)}{m} e^{-i\omega t}\, dt + \underbrace{\xi_0}_{\substack{=0 \text{ since} \\ x_0 = 0,\ \dot{x}_0 = 0}} \right)\right]$$

$$= \frac{F_0}{m\omega}\, \mathrm{Im}\left[e^{i\omega t} \int_0^T \sin(\omega t)\, e^{-i\omega t}\, dt \right]$$

$$= \frac{F_0}{m\omega}\, \mathrm{Im}\left[e^{i\omega t} \int_0^T dt\, \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right) e^{-i\omega t} \right]$$

$$= \frac{F_0}{2m\omega}\, \mathrm{Im}\left[-i\, e^{i\omega t} \int_0^T dt \left(1 - e^{-i2\omega t}\right) \right]$$

$$= \frac{F_0}{2m\omega}\, \mathrm{Im}\left[-i\, e^{i\omega t}\left(T + \frac{1}{i2\omega}\underbrace{\left(e^{-i2\omega T} - 1\right)}_{} \right)\right]$$

$$= \frac{F_0 \cdot \cancel{2}\pi}{\cancel{2} m\omega^2}\, \mathrm{Im}\left[-i\left(\cos\omega t + i \sin\omega t\right)\right]$$

$$= \frac{-F_0 \pi}{m\omega^2} \cos\omega t$$

now $T = \frac{2\pi}{\omega}$

$\rightarrow$ $e^{-i2\omega T} = e^{-i2\cancel{\omega}\frac{2\pi}{\cancel{\omega}}} = e^{-i4\pi} = 1$

so $\left(e^{-i2\omega T} - 1\right) = 0$

②

Thus,

$$x(t) = \frac{-F_0 \pi}{m\omega^2} \cos(\omega(t-T) + \omega T)$$

$$= \frac{-F_0 \pi}{m\omega^2} \left[\cos(\omega(t-T))\cos\omega T - \sin(\omega(t-T))\sin\omega T\right]$$

$$= -\frac{F_0 \pi}{m\omega^2} \cos(\omega T) \cos(\omega(t-T))$$

$$+ \frac{F_0 \pi}{m\omega^2} \sin(\omega T) \sin(\omega(t-T))$$

$$= C_1 \cos(\omega(t-T)) + C_2 \sin(\omega(t-T))$$

$$= a \cos(\omega(t-T) + \alpha)$$

$$a = \sqrt{C_1^2 + C_2^2}, \quad \tan\alpha = -\frac{C_2}{C_1}$$

$$a = \sqrt{C_1^2 + C_2^2}$$

$$= \frac{F_0 \pi}{m\omega^2} \sqrt{\cos^2(\omega T) + \sin^2(\omega T)}$$

$$= \boxed{\frac{F_0 \pi}{m\omega^2}}$$