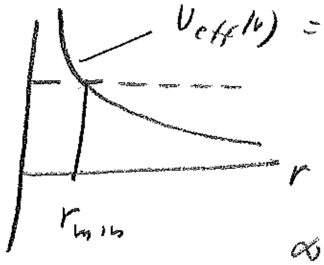
Sec 19, Prob 1: Find $\frac{d\sigma}{d\Omega}$ for $U(r) = \frac{\alpha}{r^2}$, $(\alpha > 0)$

repulsive

$$U_{eff}(r) = \frac{l^2}{2mr^2} + U(r)$$



$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho/r^2 \, dr}{\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2\alpha}{m v_\infty^2 r^2}}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho/r^2 \, dr}{\sqrt{1 - \frac{1}{r^2}\underbrace{\left(\rho^2 + \frac{2\alpha}{m v_\infty^2}\right)}_{A^2}}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho/r^2 \, dr}{\sqrt{1 - \frac{A^2}{r^2}}}$$

Let: $u = \frac{1}{r} \rightarrow du = -\frac{1}{r^2} dr$

$r = r_{min}, \infty \rightarrow u = \frac{1}{r_{min}}, 0$

$$\phi_0 = \int_0^{\frac{1}{r_{min}}} \frac{\rho \, du}{\sqrt{1 - A^2 u^2}}$$

Let: $u = \frac{1}{A} \sin\theta$

$du = \frac{1}{A} \cos\theta \, d\theta$, $\sqrt{1 - A^2u^2} = \cos\theta$

$u = 0 \Leftrightarrow \theta = 0$, $u = \frac{1}{r_{min}} = \frac{1}{A}\sin\theta$

$\frac{1}{A} = \frac{1}{A}\sin\theta \rightarrow \theta = \pi/2$

Thus,

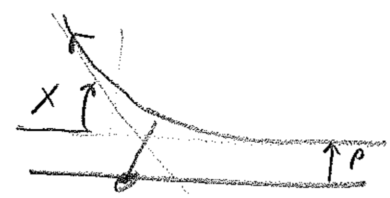
$$\phi_0 = \int_0^{\pi/2} \frac{\rho \frac{1}{A} \cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \frac{\rho}{A}\frac{\pi}{2}$$

so

$$2\phi_0 = \frac{\pi\rho}{A}$$

$$= \frac{\pi \rho}{\sqrt{\rho^2 + \frac{2\alpha}{m v_\infty^2}}}$$

$$= \frac{\pi}{\sqrt{1 + \frac{2\alpha}{\rho^2 m v_\infty^2}}}$$



<u>Now:</u> $\chi = \pi - 2\phi_0$ (repulsive)

$$d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega \quad , \quad d\Omega = 2\pi \sin\chi\, d\chi$$

Thus

$$\chi = \pi - \frac{\pi}{\sqrt{1 + \frac{2\alpha}{\rho^2 m v_\infty^2}}}$$

$$\frac{\pi}{\sqrt{}} = \pi - \chi$$

$$\frac{\pi^2}{1 + \frac{2\alpha}{\rho^2 m v_\infty^2}} = (\pi - \chi)^2$$

$$\frac{\pi^2}{(\pi - \chi)^2} = 1 + \frac{2\alpha}{\rho^2 m v_\infty^2}$$

$$\frac{2\alpha}{\rho^2 m V_\infty^2} = \frac{\pi^2}{(\pi-\chi)^2} - 1$$

$$= \frac{\pi^2 - (\pi-\chi)^2}{(\pi-\chi)^2}$$

$$= \frac{\cancel{\pi^2} - (\cancel{\pi^2}+\chi^2 - 2\pi\chi)}{(\pi-\chi)^2}$$

$$= \frac{2\pi\chi - \chi^2}{(\pi-\chi)^2}$$

$$= \frac{\chi(2\pi-\chi)}{(\pi-\chi)^2}$$

Thus, $$\rho^2 = \frac{2\alpha(\pi-\chi)^2}{mV_\infty^2\,\chi(2\pi-\chi)}$$

$$\rightarrow \boxed{\rho = \sqrt{\frac{2\alpha}{mV_\infty^2}}\;\frac{(\pi-\chi)}{\sqrt{\chi(2\pi-\chi)}}}$$

From: $$\frac{\pi^2}{(\pi-\chi)^2} = 1 + \frac{2\alpha}{\rho^2 m V_\infty^2}$$

$$+\frac{2\pi^2}{(\pi-\chi)^3}\,d\chi = \frac{-4\alpha}{\rho^3 m V_\infty^2}\,d\rho$$

Thus, $$\left|\frac{d\rho}{d\chi}\right|^{*} = \frac{\rho^3 m V_\infty^2}{4\alpha} \frac{2\pi^2}{(\pi-\chi)^3}$$

$$= \rho^3 \pi^2 \left(\frac{m V_\infty^2}{2\alpha}\right) \frac{1}{(\pi-\chi)^3}$$

$$= \left(\frac{2\alpha}{m V_\infty^2}\right)^{3/2} \frac{\cancel{(\pi-\chi)^3}}{(\chi(2\pi-\chi))^{3/2}} \pi^2 \left(\frac{m V_\infty^2}{2\alpha}\right) \frac{1}{\cancel{(\pi-\chi)^3}}$$

$$= \sqrt{\frac{2\alpha}{m V_\infty^2}} \frac{\pi^2}{(\chi(2\pi-\chi))^{3/2}}$$

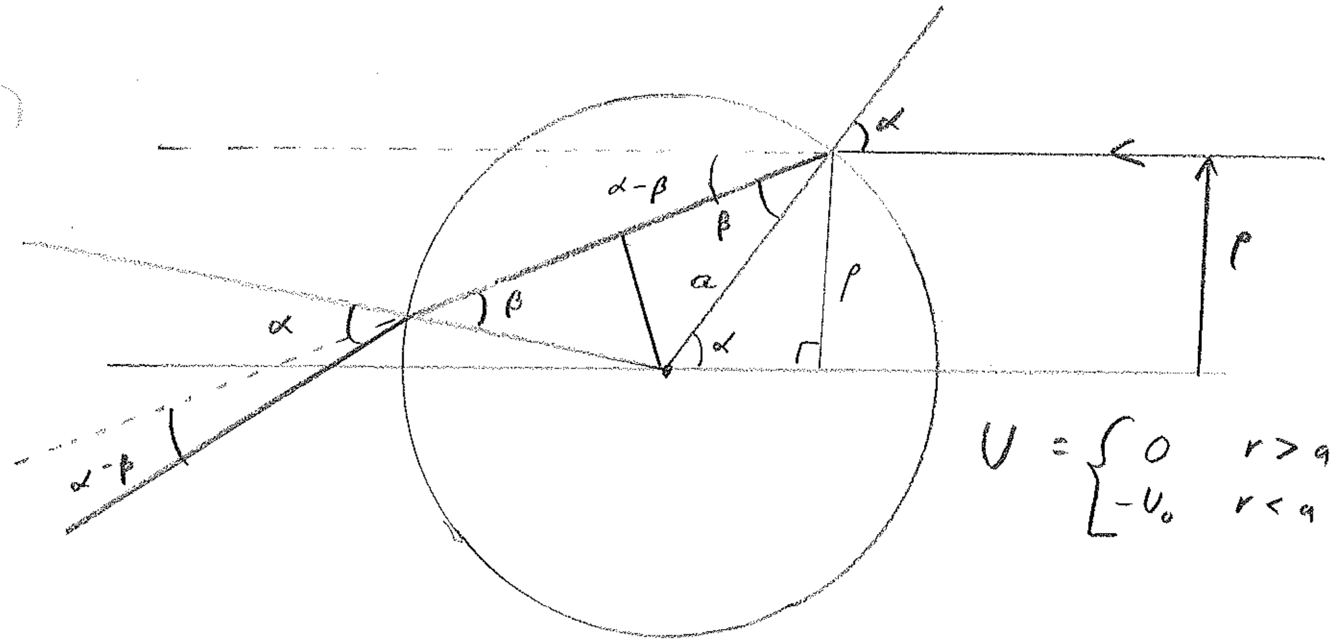
So

$$\frac{d\sigma}{d\Omega} = \frac{\rho(\chi)}{\sin\chi} \left|\frac{d\rho}{d\chi}\right|$$

$$= \frac{1}{\sin\chi} \sqrt{\frac{2\alpha}{m V_\infty^2}} \frac{\pi-\chi}{\sqrt{\chi(2\pi-\chi)}} \sqrt{\frac{2\alpha}{m V_\infty^2}} \frac{\pi^2}{(\chi(2\pi-\chi))^{3/2}}$$

$$= \boxed{\frac{1}{\sin\chi} \left(\frac{2\alpha}{m V_\infty^2}\right) \frac{\pi^2(\pi-\chi)}{\chi^2(2\pi-\chi)^2}}$$

Sec 18, prob 2



$$U = \begin{cases} 0 & r > a \\ -U_0 & r < a \end{cases}$$

$$\chi = 2(\alpha - \beta)$$

$$\sin\alpha = \frac{\rho}{a}$$

$$v_1 \sin\theta_1 = v_2 \sin\theta_2$$

$$\rightarrow \frac{\sin\theta_1}{\sin\theta_2} = \sqrt{1 + \frac{2}{m v_1^2}(U_1 - U_2)}$$

$\theta_1 = \alpha \qquad U_1 = 0 \qquad v_1 = v_\infty$

$\theta_2 = \beta \qquad U_2 = -U_0$

$$\frac{\sin\alpha}{\sin\beta} = \sqrt{1 + \frac{2}{m v_\infty^2} U_0} \equiv n > 1$$

Now: $\chi = 2(\alpha - \beta) \rightarrow \beta = \alpha - \frac{\chi}{2}$

Thus,

$$\begin{aligned} \sin\beta &= \sin\left(\alpha - \frac{\chi}{2}\right) \\ &= \sin\alpha \cos\frac{\chi}{2} - \cos\alpha \sin\frac{\chi}{2} \\ &= \frac{\rho}{a}\cos\frac{\chi}{2} - \sqrt{1 - \left(\frac{\rho}{a}\right)^2}\,\sin\frac{\chi}{2} \end{aligned}$$

Also,

$$\frac{\sin\alpha}{\sin\beta} = n \longrightarrow \sin\beta = \frac{\sin\alpha}{n} = \frac{\rho}{an}$$

Thus,

$$\frac{\rho}{a}\frac{1}{n} = \frac{\rho}{a}\cos\frac{\chi}{2} - \sqrt{1-\left(\frac{\rho}{a}\right)^2}\,\sin\frac{\chi}{2}$$

$$\sqrt{1-\left(\frac{\rho}{a}\right)^2}\,\sin\frac{\chi}{2} = \frac{\rho}{a}\left(\cos\frac{\chi}{2} - \frac{1}{n}\right)$$

$$\rightarrow \left(1-\left(\frac{\rho}{a}\right)^2\right)\sin^2\frac{\chi}{2} = \left(\frac{\rho}{a}\right)^2\left(\cos^2\frac{\chi}{2} + \frac{1}{n^2} - \frac{2}{n}\cos\frac{\chi}{2}\right)$$

$$\sin^2\frac{\chi}{2} = \left(\frac{\rho}{a}\right)^2\left[\underbrace{\sin^2\frac{\chi}{2} + \cos^2\frac{\chi}{2}}_{1} + \frac{1}{n^2} - \frac{2}{n}\cos\frac{\chi}{2}\right]$$

$$\sin^2\frac{\chi}{2} = \left(\frac{\rho}{a}\right)^2\left[1 + \frac{1}{n^2} - \frac{2}{n}\cos\frac{\chi}{2}\right]$$

Thus, $$\boxed{\left(\frac{\rho}{a}\right)^2 = \frac{\sin^2\frac{\chi}{2}}{1+\frac{1}{n^2}-\frac{2}{n}\cos\frac{\chi}{2}}} = \frac{n^2\sin^2\frac{\chi}{2}}{n^2+1-2n\cos\frac{\chi}{2}}$$

<u>Now</u>. $$d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$

Need to differentiate (1).

$$2\left(\frac{\rho}{a}\right)\frac{1}{a}\,d\rho = \frac{\left[\cancel{2}\sin\frac{\chi}{2}\cos\frac{\chi}{2}\,\frac{d\chi}{\cancel{2}}\left(1+\frac{1}{n^2}-\frac{2}{n}\cos\frac{\chi}{2}\right) - \frac{\cancel{2}}{n}\sin\frac{\chi}{2}\,\frac{d\chi}{\cancel{2}}\sin^2\frac{\chi}{2}\right]}{\left(1+\frac{1}{n^2}-\frac{2}{n}\cos\frac{\chi}{2}\right)^2}$$

$$= \frac{\sin\frac{\chi}{2}\,d\chi\left(\cos\frac{\chi}{2}+\cos\frac{\chi}{2}\frac{1}{n^2}-\frac{2}{n}\cos^2\frac{\chi}{2}-\frac{1}{n}\sin^2\frac{\chi}{2}\right)}{\left(1+\frac{1}{n^2}-\frac{2}{n}\cos\frac{\chi}{2}\right)^2}$$

$$= \frac{\sin\frac{\chi}{2}\,d\chi\left(\cos\frac{\chi}{2}+\cos\frac{\chi}{2}\frac{1}{n^2}-\frac{1}{n}-\frac{1}{n}\cos^2\frac{\chi}{2}\right)}{\left(1+\frac{1}{n^2}-\frac{2}{n}\cos\frac{\chi}{2}\right)^2}$$

$$= \frac{n^2\sin\frac{\chi}{2}\,d\chi\left(n^2\cos\frac{\chi}{2}+\cos\frac{\chi}{2}-n-n\cos^2\frac{\chi}{2}\right)}{\left(n^2+1-2n\cos\frac{\chi}{2}\right)^2}$$

$$= \frac{n^2\sin\frac{\chi}{2}\,d\chi\left(n-\cos\frac{\chi}{2}\right)\left(n\cos\frac{\chi}{2}-1\right)}{\left(n^2+1-2n\cos\frac{\chi}{2}\right)^2}$$

Thus,

$$\left|\frac{d\rho}{d\chi}\right| = \left|\frac{1}{2}\,\frac{a\left(n^2+1-2n\cos\frac{\chi}{2}\right)^{\frac{1}{2}}}{\cancel{n}\,\cancel{\sin\frac{\chi}{2}}}\;\frac{n^{\cancel{2}}\,\cancel{\sin\frac{\chi}{2}}\left(n-\cos\frac{\chi}{2}\right)\left(n\cos\frac{\chi}{2}-1\right)}{\left(n^2+1-2n\cos\frac{\chi}{2}\right)^2}\right|$$

$$\left|\frac{d\rho}{d\chi}\right| = \frac{a n}{2}\left|\frac{\left(n-\cos\frac{\chi}{2}\right)\left(n\cos\frac{\chi}{2}-1\right)}{\left(n^2+1-2n\cos\frac{\chi}{2}\right)^{3/2}}\right|$$

Thus,

$$d\sigma = \frac{a\,n\sin\frac{\chi}{2}}{\left(n^2+1-2n\cos\frac{\chi}{2}\right)^{\frac{1}{2}}}\,\frac{1}{\sin\chi}\cdot\frac{a n}{2}\left|\frac{\left(n-\cos\frac{\chi}{2}\right)\left(n\cos\frac{\chi}{2}-1\right)}{\left(n^2+1-2n\cos\frac{\chi}{2}\right)^{3/2}}\right| d\Omega$$

$$= \frac{a^2 n^2}{2}\left(\frac{\cancel{\sin\frac{\chi}{2}}}{2\cancel{\sin\frac{\chi}{2}}\cos\frac{\chi}{2}}\right)\frac{\left(n-\cos\frac{\chi}{2}\right)\left|n\cos\frac{\chi}{2}-1\right|}{\left(n^2+1-2n\cos\frac{\chi}{2}\right)^2}\,d\Omega$$

$$= \boxed{\frac{1}{4}\,\frac{a^2 n^2}{\cos\frac{\chi}{2}}\;\frac{\left(n-\cos\frac{\chi}{2}\right)\left|n\cos\frac{\chi}{2}-1\right|}{\left(n^2+1-2n\cos\frac{\chi}{2}\right)^2}\,d\Omega}$$

NOTE: $0 < \chi < \chi_{max}$ where χ_{max} occurs when $\rho = \rho_{max} = a$.

$$\rightarrow \quad 1 = \frac{n^2\sin^2\left(\frac{\chi_{max}}{2}\right)}{n^2+1-2n\cos\frac{\chi_{max}}{2}}$$

$$n^2+1-2n\cos\left(\frac{\chi_{max}}{2}\right) = n^2\sin^2\left(\frac{\chi_{max}}{2}\right)$$

$$n^2\left(1-\sin^2\left(\frac{\chi_{max}}{2}\right)\right)+1-2n\cos\left(\frac{\chi_{max}}{2}\right)=0$$

$$n^2 \cos^2\left(\frac{X_{max}}{2}\right) - 2n \cos\left(\frac{X_{max}}{2}\right) + 1 = 0$$

$$\left(n \cos\left(\frac{X_{max}}{2}\right) - 1\right)^2 = 0$$

$$n \cos\left(\frac{X_{max}}{2}\right) - 1 = 0$$

$$\rightarrow \boxed{\cos\left(\frac{X_{max}}{2}\right) = \frac{1}{n}}$$

Thus, $n \cos\left(\frac{X}{2}\right) - 1 \geq 0$ so I can remove the absolute value signs.

<u>Total cross-section:</u>

$$\sigma = \int d\sigma = \int_0^{X_{max}} \left(\frac{d\sigma}{dX}\right) dX$$

<u>Now:</u> $$d\sigma = \left(\frac{d\sigma}{d\Omega}\right) d\Omega$$

$$= \left(\frac{d\sigma}{d\Omega}\right) 2\pi \sin X \, dX$$

$$= \frac{d\sigma}{dX} \, dX$$

$$\rightarrow \sigma = \int_0^{X_{max}} dX \; 2\pi \underbrace{\sin X}_{2\sin\left(\frac{X}{2}\right)\cos\left(\frac{X}{2}\right)} \; \frac{1}{4} \frac{a^2 n^2}{\cos\frac{X}{2}} \frac{\left(n - \cos\left(\frac{X}{2}\right)\right)\left(n \cos\left(\frac{X}{2}\right) - 1\right)}{\left(n^2 + 1 - 2n \cos\frac{X}{2}\right)^2}$$

$$= \pi a^2 n^2 \int_0^{X_{max}} dX \; \sin\left(\frac{X}{2}\right) \frac{\left(n - \cos\left(\frac{X}{2}\right)\right)\left(n \cos\left(\frac{X}{2}\right) - 1\right)}{\left(n^2 + 1 - 2n \cos\frac{X}{2}\right)^2}$$

Let: $u = \cos\left(\frac{X}{2}\right)$

$$du = -\frac{dX}{2}\sin\left(\frac{X}{2}\right)$$

$$X = 0 \rightarrow u = 1$$

$$X = X_{max} \rightarrow u = \frac{1}{n}$$

$$\rightarrow \sigma = \pi a^2 n^2 \int_{\frac{1}{n}}^{1} 2\,du \; \frac{(n-u)(nu-1)}{(n^2+1-2nu)^2}$$

$$= 2\pi a^2 n^2 \int_{\frac{1}{n}}^{1} du \; \frac{(n-u)(nu-1)}{(n^2+1-2nu)^2}$$

$$= 2\pi a^2 \cancel{n^2} \int_{\frac{1}{n}}^{1} du \; \frac{(n-u)(nu-1)}{4\cancel{n^2}\left(\left(\frac{n^2+1}{2n}\right)-u\right)^2}$$

$$= \frac{1}{2}\pi a^2 \int_{\frac{1}{n}}^{1} du \; \frac{(n-u)(nu-1)}{(A-u)^2}, \quad A = \frac{n^2+1}{2n}$$

~~Thus~~

Let, $v = A - u$, $dv = -du$

$$u = \frac{1}{n} \rightarrow v = A - \frac{1}{n} = \frac{n^2+1}{2n} - \frac{1}{n}$$

$$= \frac{n^2+1-2}{2n} = \frac{n^2-1}{2n}$$

$$= \frac{(n-1)(n+1)}{2n}$$

$$u = 1 \rightarrow v = A - 1 = \frac{n^2+1}{2n} - 1$$

$$= \frac{1}{2n}(n^2+1-2n) = \frac{(n-1)^2}{2n}$$

Thus,

$$\sigma = \frac{1}{2}\pi a^2 \int_{\frac{(n-1)^2}{2n}}^{\frac{(n-1)(n+1)}{2n}} dv \; \frac{(n-A+v)(n(A-v)-1)}{v^2}$$

<u>Now</u>

$$n - A = n - \left(\frac{n^2+1}{2n}\right)$$
$$= \frac{2n^2 - n^2 - 1}{2n}$$
$$= \frac{n^2-1}{2n}$$
$$= \frac{(n-1)(n+1)}{2n}$$

$$nA - 1 = n\left(\frac{n^2+1}{2n}\right) - 1$$
$$= \frac{n^2+1}{2} - 1$$
$$= \frac{n^2+1-2}{2}$$
$$= \frac{(n-1)(n+1)}{2}$$

$$\sigma = \frac{1}{2}\pi a^2 \int_{\frac{(n-1)^2}{2n}}^{\frac{(n-1)(n+1)}{2n}} dv \; \frac{\left(\frac{(n-1)(n+1)}{2n} + v\right)\left(\frac{(n-1)(n+1)}{2n} - v\right)n}{v^2}$$

$$= \frac{1}{2}\pi a^2 \, n \int_{\frac{(n-1)^2}{2n}}^{\frac{(n-1)(n+1)}{2n}} dv \; \frac{1}{v^2}\left(\left(\frac{(n-1)(n+1)}{2n}\right)^2 - v^2\right)$$

$$\sigma = \frac{1}{2}\pi a^2 n \left[\left(\frac{(n-1)(n+1)}{2n}\right)^2 \left(\frac{2n}{(n-1)^2} - \frac{2n}{(n-1)(n+1)}\right) - \frac{(n-1)(n+1)}{2n} + \frac{(n-1)^2}{2n} \right]$$

$$= \frac{1}{2}\pi a^2 \cancel{n} \left(\frac{1}{2\cancel{n}}\right) \left[(n+1)^2 - \underline{(n-1)(n+1)} - \underline{(n-1)(n+1)} + (n-1)^2 \right]$$

$$= \frac{1}{4}\pi a^2 \left[\cancel{n^2} + 2n + \underline{1} - \cancel{2n^2} \underline{+2} + \cancel{n^2} - 2n + \underline{1} \right]$$

$$= \frac{1}{4}\pi a^2 \cdot 4$$

$$= \boxed{\pi a^2}$$