Sec 39, Prob 1:

Deflection of a freely falling body from the vertical to leading order in Ω.

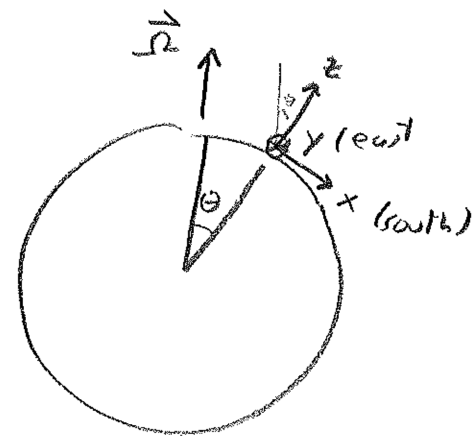
$$m\vec{a} = m\vec{g}_0 - m\underbrace{\vec{\Omega}\times(\vec{\Omega}\times\vec{R})}_{\vec{W}\equiv\ddot{\vec{R}}_f} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) - 2m\vec{\Omega}\times\vec{v}$$

linear acceleration of origin

$$\rightarrow \quad m\frac{d\vec{v}}{dt} = m\vec{g} - 2m\vec{\Omega}\times\vec{v}$$

[neglecting $\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$ relative to $\vec{\Omega}\times(\vec{\Omega}\times\vec{R})$]

where $\vec{g} \equiv \vec{g}_0 - \underbrace{\vec{\Omega}\times(\vec{\Omega}\times\vec{R})}_{O(\Omega^2)}$



Thus:

$$\frac{d\vec{v}}{dt} = \vec{g} - 2\vec{\Omega}\times\vec{v}$$

[1st order ODE for $\vec{v}$]

Solve iteratively: $\vec{v} = \vec{v}_1 + \vec{v}_2$

$\vec{v}_1$: 0th order; $\vec{v}_2$: 1st order in Ω

where $\dfrac{d\vec{v}_1}{dt} = \vec{g} \rightarrow \boxed{\vec{v}_1 = \vec{g}t + \vec{v}_0}$

$$\cancel{\frac{d\vec{v}_1}{dt}} + \frac{d\vec{v}_2}{dt} = \cancel{\vec{g}} - 2\vec{\Omega}\times(\vec{v}_1 + \vec{v}_2)$$

$\vec{v}_2$: ignore since multiplied by Ω

$$\frac{d\vec{v}_2}{dt} = -2\vec{\Omega}\times\vec{v}_1$$

$$= -2\vec{\Omega}\times(\vec{g}t + \vec{v}_0)$$

$$= -2(\vec{\Omega}\times\vec{g})t - 2\vec{\Omega}\times\vec{v}_0$$

$$\rightarrow \quad \vec{v}_2 = -(\vec{\Omega}\times\vec{g})t^2 - 2(\vec{\Omega}\times\vec{v}_0)t$$

$$\rightarrow \quad \vec{V} = \vec{g}t + \vec{v_0} - (\vec{\Omega}\times\vec{g})t^2 - 2(\vec{\Omega}\times\vec{v_0})t$$

$$= \vec{v_0} + \vec{g}t - 2(\vec{\Omega}\times\vec{v_0})t - (\vec{\Omega}\times\vec{g})t^2$$

Integrate.

$$\boxed{\vec{r} = \underbrace{\vec{r_0} + \vec{v_0}t + \tfrac{1}{2}\vec{g}t^2}_{\text{standard solution.}} - (\vec{\Omega}\times\vec{v_0})t^2 - \tfrac{1}{3}(\vec{\Omega}\times\vec{g})t^3}$$

Want to find deflection when body hits the ground.

<u>ICs</u>: $\vec{r_0} = h\hat{z}$

$\vec{v_0} = 0$ [dropped]

$$\rightarrow \quad \vec{r}(t) = h\hat{z} - \tfrac{1}{2}g\hat{z}t^2 - \tfrac{1}{3}(\vec{\Omega}\times\vec{g})t^3$$

0^{th} order:

$$0 = h - \tfrac{1}{2}gT^2$$

$$T = \sqrt{\frac{2h}{g}}$$

Thus,

$$\vec{r}(T) = (h - \cancelto{0}{\tfrac{1}{2}gT^2})\hat{z} - \tfrac{1}{3}(\vec{\Omega}\times\vec{g})T^3$$

$$= -\frac{1}{3}\left[-\Omega\sin\theta\,\hat{x} + \Omega\cos\theta\,\hat{z}\right]\times(-g\hat{z})\left(\frac{2h}{g}\right)^{3/2}$$

$$= \underbrace{+\frac{1}{3}\Omega g\sin\theta\left(\frac{2h}{g}\right)^{3/2}}_{\text{positive (deflected to the East)}}\hat{y}$$

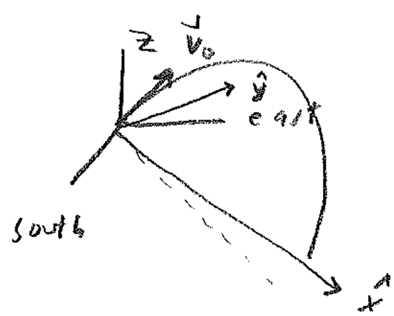
<u>NOTE</u>:

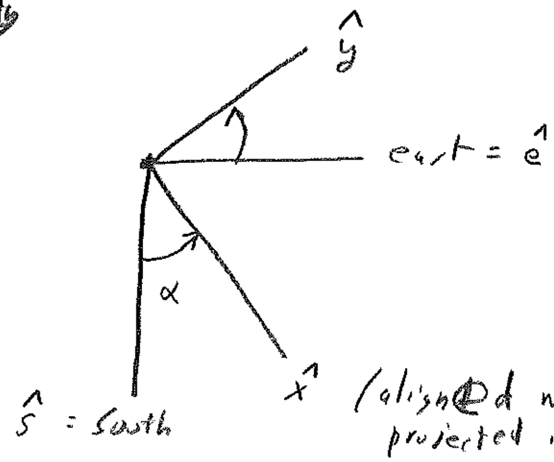
1) $h = 100\,m$, $\theta = \pi/2$ (equator) $\rightarrow$ $\Delta y = \frac{1}{3}\left(\frac{2\pi}{1\,day}\right)\left(9.8\,\frac{m}{s^2}\right)\left(\frac{2\cdot 100\,m}{9.8\,m/s^2}\right)^{3/2} = 0.02\,m = \boxed{2\,cm}$

2) $\underbrace{h = 100\,km}_{\text{edge of space}} = 10^5\,m$, $\theta = \pi/2$ $\rightarrow$ $\Delta y = 692\,m = \boxed{0.69\,km}$

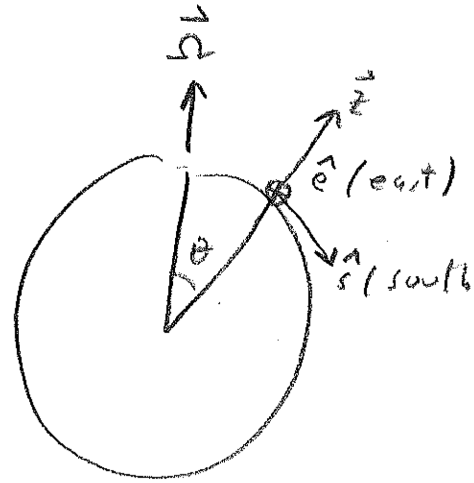
Sec. 39, Prob 2:

Deflection from coplanarity





(aligned with direction of $\vec{v}_0$ projected into $\hat{s}, \hat{e}$ plane)



So $\hat{s} = \cos\alpha\,\hat{x} - \sin\alpha\,\hat{y}$

$$\vec{\Omega} = -\Omega\sin\theta\,\hat{s} + \Omega\cos\theta\,\hat{z}$$

$$= -\Omega\sin\theta\cos\alpha\,\hat{x} + \Omega\sin\theta\sin\alpha\,\hat{y} + \Omega\cos\theta\,\hat{z}$$

$$\boxed{\vec{\Omega} = \Omega_x\hat{x} + \Omega_y\hat{y} + \Omega_z\hat{z}}$$

Using result of Sec 39, Prob 1:

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{g}t^2 - (\vec{\Omega}\times\vec{v}_0)t^2 - \tfrac{1}{3}(\vec{\Omega}\times\vec{g})t^3$$

I.C.'s: $\vec{r}_0 = 0$

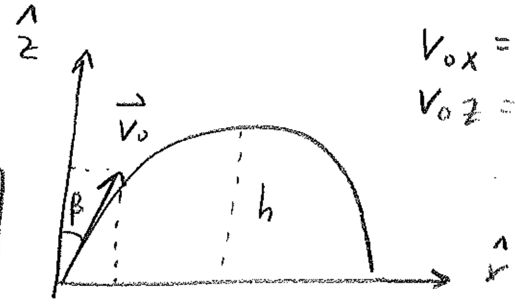
$\vec{V}_0 = V_{0x}\hat{x} + V_{0z}\hat{z}$

Thus,

$$\vec{r} = \vec{V}_0 t - \tfrac{1}{2}g\hat{z}t^2 - (\vec{\Omega}\times\vec{V}_0)t^2 - \tfrac{1}{3}(\vec{\Omega}\times\vec{g})t^3$$

0th order:

$$\vec{r} = \vec{v}_0 t - \tfrac{1}{2}gt^2\hat{z}$$



$V_{0x} = V_0\sin\beta$

$V_{0z} = V_0\cos\beta$

$$mgh = \tfrac{1}{2}mV_{0z}^2 \rightarrow \boxed{h = \frac{V_{0z}^2}{2g}}$$

~~$h = \frac{V_0^2}{2g}$~~

Time to height h:

$$v_z = 0 = v_{0z} - gt$$

$$t = \frac{v_{0z}}{g}$$

Time to landing:

$$T = 2t = \boxed{\frac{2v_{0z}}{g}}$$

Range:

$$R = v_{0x} T$$

$$= v_0 \sin\beta \, \frac{2 v_0 \cos\beta}{g}$$

$$= \frac{2v_0^2}{g} \sin\beta \cos\beta$$

$$= \frac{v_0^2}{g} \sin(2\beta)$$

max range for $\beta = \pi/4$

$$\vec{r} = (v_{0x}\hat{x} + v_{0z}\hat{z})t - \tfrac{1}{2} g t^2 \hat{z} - (\vec{\Omega} \times \vec{v}_0) t^2 - \tfrac{1}{3}(\vec{\Omega} \times \vec{g}) t^3$$

$$\rightarrow \delta\vec{r} = -(\vec{\Omega} \times \vec{v}_0) T^2 - \tfrac{1}{3}(\vec{\Omega} \times \vec{g}) T^3$$

$$= -T^2 \left[(\vec{\Omega} \times \vec{v}_0) + \tfrac{1}{3}(\vec{\Omega} \times \vec{g}) T \right]$$

$$= \frac{-4 v_{0z}^2}{g^2} \left[(\vec{\Omega} \times \vec{v}_0) + \frac{2}{3}\frac{v_{0z}}{g}(\vec{\Omega} \times \vec{g}) \right]$$

Now, $\vec{\Omega} \times \vec{v}_0 = (\Omega_x \hat{x} + \Omega_y \hat{y} + \Omega_z \hat{z}) \times (v_{0x}\hat{x} + v_{0z}\hat{z})$

$$= -\Omega_x v_{0z} \hat{y} - \Omega_y v_{0x} \hat{z} + \Omega_y v_{0z} \hat{x} + \Omega_z v_{0x} \hat{y}$$

$$= (-\Omega_x v_{0z} + \Omega_z v_{0x})\hat{y} + \underbrace{\Omega_y v_{0z}\hat{x} - \Omega_y v_{0x}\hat{z}}_{\text{ignore}}$$

$$\vec{\Omega} \times \vec{g} = (\Omega_x \hat{x} + \Omega_y \hat{y} + \Omega_z \hat{z}) \times (-g\hat{z})$$

$$= +\Omega_x g \hat{y} \underbrace{- g\Omega_y \hat{x}}_{\text{ignore}}$$

Thus $\delta\vec{r}_\perp = \frac{-4v_{0z}^2}{g^2}\left[-\Omega_x v_{0z} + \Omega_z v_{0x} + \frac{2}{3}\frac{v_{0z}}{g}\Omega_x g \right]\hat{y}$

$\perp$ to plane

$$\boxed{\delta \vec{r}_\perp = \frac{4 v_{0z}^2}{g^2}\left[\frac{1}{3}\Omega_x v_{0z} - \Omega_z v_{0x}\right]\hat{y}}$$

In terms of α, β, θ: ($\theta = 0$, NP; $\theta = \frac{\pi}{2}$, equator)

($\beta = 0$, vertical)

$$v_{0z} = v_0 \cos\beta$$

$$v_{0x} = v_0 \sin\beta$$

$$\Omega_x = -\Omega \sin\theta \cos\alpha \qquad \left(\begin{array}{l}\alpha = 0, \text{ south}\\ \alpha = \frac{\pi}{2}, \text{ east}\end{array}\right)$$

$$\Omega_z = \Omega \cos\theta$$

$$\delta \vec{r}_\perp = \frac{4 v_0^2 \Omega}{g^2}\cos^2\beta\left[-\frac{1}{3}\sin\theta\cos\alpha\, v_0\cos\beta - \cos\theta\, v_0\sin\beta\right]\hat{y}$$

$$= \frac{4 v_0^3 \Omega}{g^2}\cos^2\beta\left[-\frac{1}{3}\cos\alpha\sin\theta\cos\beta - \cos\theta\sin\beta\right]\hat{y}$$

~~Consider~~ $\cancel{\beta = \pi/4}$

In terms of range: $R = \frac{2v_0^2}{g}\sin\beta\cos\beta$ (0^{th} order)

$$\rightarrow v_0 = \sqrt{\frac{gR}{2\sin\beta\cos\beta}}$$

$$\cancel{v_0^3 = \frac{gR}{2\sin\beta\cos\beta}\sqrt{\frac{gR}{2\sin\beta\cos\beta}}}$$

$$\delta \vec{r}_\perp = \frac{4\Omega\cos^2\beta}{g^2}\left(\frac{gR}{2\sin\beta\cos\beta}\right)^{3/2}\left[-\frac{1}{3}\cos\alpha\sin\theta\cos\beta - \cos\theta\sin\beta\right]\hat{y}$$

$$= \cancel{4}\,\frac{4}{2\sqrt{2}}\,\Omega\sqrt{\frac{\cos\beta}{g}}\left(\frac{R}{\sin\beta}\right)^{3/2}\left[\quad '' \quad\right]\hat{y}$$

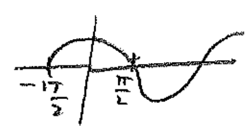
$$= \Omega\sqrt{\frac{2\cos\beta}{g}}\left(\frac{R}{\sin\beta}\right)^{3/2}\left[-\frac{1}{3}\cos\alpha\sin\theta\cos\beta - \cos\theta\sin\beta\right]\hat{y}$$

<u>Take:</u> $\beta = \pi/4$ (which gives maximum zeroth order range)

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\delta\vec{r}_\perp = \Omega\sqrt{\frac{2}{g}}\;R^{3/2}\;\frac{1}{\cancel{(\sqrt{2}/2)}}\left[-\frac{1}{3}\cos\alpha\sin\theta - \cos\theta\right]\cancel{\left(\frac{\sqrt{2}}{2}\right)}\hat{y}$$

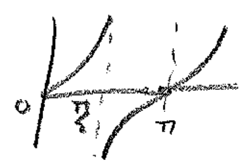
$$= \Omega\sqrt{\frac{2}{g}}\;R^{3/2}\left[-\frac{1}{3}\cos\alpha\sin\theta - \cos\theta\right]\hat{y}$$



<u>NOTE:</u>

$$-\frac{1}{3}\cos\alpha\sin\theta - \cos\theta = 0$$

$$-\frac{1}{3}\cos\alpha\sin\theta = \cos\theta$$



$$\Rightarrow\quad \boxed{\cos\alpha = -\frac{3}{\tan\theta}}$$

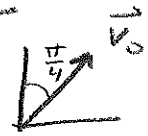
Thus, for a given θ can choose α so that trajectory stays in a plane.

$\rightarrow$ $\theta = \frac{\pi}{2}$ (equator)

$\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$ $\rightarrow$ $-\frac{1}{3}\cos\alpha = 0$ $\rightarrow$ $\alpha = \pm\frac{\pi}{2}$ (aim East or west)

Numerical values:

$\beta = \pi/4$



$, \quad R = 100\,m, \quad g = 9.8\,\frac{m}{s^2}, \quad \Omega = \frac{2\pi}{24 \cdot 3600\,s}$

$$\delta \vec{r}_+ = \sqrt{\frac{2\Omega^2 R}{g}}\, R \left[-\tfrac{1}{3} \cos\alpha \sin\theta - \cos\theta \right] \hat{y}$$

$\theta = 0$: (North pole)

$$\delta \vec{r}_+ = -\sqrt{2}\, R \sqrt{\frac{\Omega^2 R}{g}}\, \hat{y} \qquad \forall \alpha$$

$$= -0.033\, m\, \hat{y}$$

$$= \boxed{-3.3\,cm\ \hat{y}}$$

$\theta = \pi/4$:

$$\delta \vec{r}_\perp = \sqrt{\frac{2\Omega^2 R}{g}}\, R\, \frac{\sqrt{2}}{2} \left[-\tfrac{1}{3}\cos\alpha - 1 \right] \hat{y}$$

$$= \sqrt{\frac{\Omega^2 R}{g}}\, R \left(-\frac{1}{3}\right)(3 + \cos\alpha)\, \hat{y} \longleftarrow \text{always} < 0$$

$$= \begin{cases} -\frac{4}{3} R \sqrt{\frac{\Omega^2 R}{g}}\, \hat{y} = \boxed{-3.1\,cm\ \hat{y}} & (\alpha = 0) \quad \text{South} \\ -R \sqrt{\frac{\Omega^2 R}{g}}\, \hat{y} = \boxed{-2.3\,cm\ \hat{y}} & (\alpha = \pi/2) \quad \text{East} \\ -\frac{2}{3} R \sqrt{\frac{\Omega^2 R}{g}}\, \hat{y} = \boxed{-1.55\,cm\ \hat{y}} & (\alpha = \pi) \quad \text{North} \end{cases}$$

$\theta = \pi/2$: (equator)

$$\delta \vec{r}_+ = -\frac{1}{3}\sqrt{\frac{2\Omega^2 R}{g}}\, R \cos\alpha$$

$$= \begin{cases} -\frac{\sqrt{2}}{3} R \sqrt{\frac{\Omega^2 R}{g}}\, \hat{y} = \boxed{-1.1\,cm} & (\alpha = 0) \quad \text{South} \\ \boxed{0} & (\alpha = \frac{\pi}{2}) \quad \text{East} \\ +\frac{\sqrt{2}}{3} R \sqrt{\frac{\Omega^2 R}{g}}\, \hat{y} = \boxed{+1.1\,cm} & (\alpha = \pi) \quad \text{North} \end{cases}$$

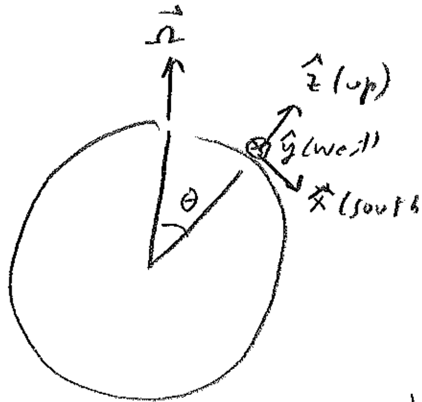
Foucault's pendulum

Sec 39, Prob 3:

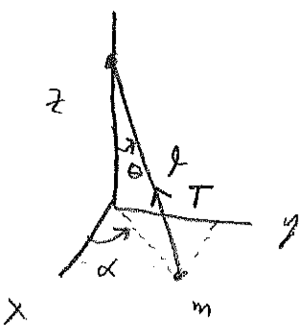
$$m\frac{d\vec v}{dt} = -\frac{\partial U}{\partial r} - m\vec W - m\cancel{\dot{\vec\Omega}}^{\,0}\times\vec r - 2m\vec\Omega\times\vec v - m\vec\Omega\times(\vec\Omega\times\vec r)$$

$$= m\vec g_0 + \vec T - m\vec\Omega\times(\vec\Omega\times\vec R) - 2m\vec\Omega\times\vec v - m\vec\Omega\times(\vec\Omega\times\vec r)$$

$$= \underbrace{m\vec g_0 - m\vec\Omega\times(\vec\Omega\times\underbrace{(\vec R+\vec r)}_{\approx\vec R})}_{m\vec g} - 2m\vec\Omega\times\vec v + \vec T$$

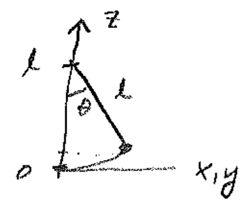


thus, $$\boxed{m\frac{d\vec v}{dt} = m\vec g + \vec T - 2m\vec\Omega\times\vec v}$$



Small oscillations in xy plane:

$$z = l(1-\cos\theta) \approx 0$$
$$x = l\sin\theta\cos\alpha \approx l\theta\cos\alpha$$
$$y = l\sin\theta\sin\alpha \approx l\theta\sin\alpha$$



Tension: $\vec T$:

$$T_z = T\cos\theta \approx T$$
$$T_x = -T\sin\theta\cos\alpha = -T\frac{x}{l}$$
$$T_y = -T\sin\theta\sin\alpha = -T\frac{y}{l}$$

thus, $$\boxed{\vec T \approx -T\frac{x}{l}\hat x - T\frac{y}{l}\hat y + T\hat z}$$

so $$\boxed{m\frac{d\vec v}{dt} = -mg\hat z + \left(-T\frac{x}{l}\hat x - T\frac{y}{l}\hat y + T\hat z\right) - 2m\vec\Omega\times\vec v}$$

$$\vec\Omega = -\Omega\sin\theta\,\hat x + \Omega\cos\theta\,\hat z = \Omega_x\hat x + \Omega_z\hat z$$
$$\vec v = v_x\hat x + v_y\hat y + v_z\hat z \approx v_x\hat x + v_y\hat y \quad (v_z\approx 0)$$

$$\to \vec\Omega\times\vec v = (\Omega_x\hat x + \Omega_z\hat z)\times(v_x\hat x + v_y\hat y) = \Omega_x v_y\hat z + \Omega_z v_x\hat y - \Omega_z v_y\hat x$$

Write $v_x = \dot{x}, \; v_y = \dot{y}, \; v_z = \dot{z}$

$$\frac{d\vec{v}}{dt} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$$

<u>then:</u>

$$m\ddot{x} = -T\frac{x}{l} + 2m\Omega_z \dot{y} \qquad (1)$$

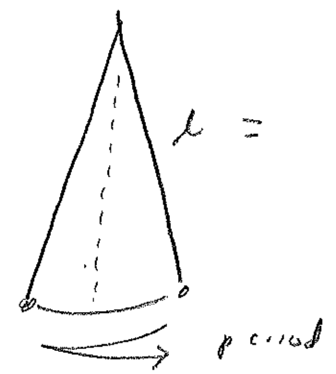
$$m\ddot{y} = -T\frac{y}{l} - 2m\Omega_z \dot{x} \qquad (2)$$

$$m\ddot{z} = -mg + T - 2m\Omega_x \dot{y} \qquad (3)$$

For small oscillation in (x,y) plane $\ddot{z} \approx 0$, thus

$$0 \approx -mg + T - 2m\Omega_x \dot{y} \qquad (3)$$

Compare g and $\Omega_x \dot{y}$:



$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$

$\Delta y = D << l$

max displacement

$$\left|\frac{\Omega_x \dot{y}}{g}\right| = \left|\frac{\Omega \sin\theta}{g} \frac{\Delta y}{\Delta t}\right|$$

$$\lesssim \frac{\Omega}{g} \frac{D}{2\pi\sqrt{\frac{l}{g}}}$$

$$\lesssim \Omega \left(\frac{D}{l}\right) \sqrt{\frac{l}{g}}$$

$$\sim \frac{\Omega}{\sqrt{g/l}} \left(\frac{D}{l}\right)$$

<u>Now:</u> $D << l$ so $\frac{D}{l} << 1$

$$\sqrt{\frac{g}{l}} = \text{angular freq} = \frac{2\pi}{10\text{'s of second}}$$

$$\Omega = \frac{2\pi}{24\,\text{hrs}} = \frac{2\pi}{24 \cdot 3600} \approx \frac{2\pi}{100{,}000\ \text{sec}}$$

$$\begin{array}{r} 3600 \\ \times\ 24 \\ \hline 14400 \\ 7200\ \\ \hline 86400 \end{array}$$

so $\frac{\Omega}{\sqrt{g/l}} << 1, \; \frac{D}{l} << 1 \longrightarrow \left|\frac{\Omega_x \dot{y}}{g}\right| << 1$

$\longrightarrow$ can ignore last term relative to mg

So:

$$0 \approx -mg + T \qquad (3)$$

$$\to \boxed{T = mg}$$

Thus,

$$\cancel{m}\ddot{x} \approx -\cancel{m}g\frac{x}{\ell} + 2\cancel{m}\Omega_z \dot{y}$$

$$\boxed{\ddot{x} \approx -\frac{g}{\ell}x + 2\Omega_z \dot{y} \qquad (1)}$$

$$\boxed{\ddot{y} \approx -\frac{g}{\ell}y - 2\Omega_z \dot{x} \qquad (2)}$$

Define:

$$\zeta = x + iy \qquad [\text{complex-valued}]$$

$$\dot{\zeta} = \dot{x} + i\dot{y}$$

$$\ddot{\zeta} = \ddot{x} + i\ddot{y}$$

Add: $(1) + i(2)$

$$\ddot{x} + i\ddot{y} \approx -\frac{g}{\ell}(x+iy) + 2\Omega_z(\dot{y} - i\dot{x})$$

$$\ddot{\zeta} = -\frac{g}{\ell}\zeta - i\,2\Omega_z\dot{\zeta}$$

$$\boxed{\ddot{\zeta} + 2i\Omega_z\dot{\zeta} + \frac{g}{\ell}\zeta = 0}$$

Ansatz:

$$\zeta = Ae^{i\lambda t}$$

$$\dot{\zeta} = Ae^{i\lambda t}\, i\lambda$$

$$\ddot{\zeta} = -Ae^{i\lambda t}\lambda^2$$

$$\to \quad Ae^{i\lambda t}\left[-\lambda^2 + 2i\Omega_z i\lambda + \frac{g}{\ell}\right] = 0$$

$$\lambda^2 + 2\lambda\Omega_z - \frac{g}{\ell} = 0$$

quadratic:

$$\lambda = \frac{-2\Omega_z \pm \sqrt{4\Omega_z^2 + 4\frac{g}{l}}}{2}$$

$$= -\Omega_z \pm \sqrt{\Omega_z^2 + \frac{g}{l}}$$

$$= -\Omega_z \pm \sqrt{\frac{g}{l}}\sqrt{1 + \frac{\Omega_z^2}{(g/l)}}$$

$$\epsilon = \frac{\Omega_z^2}{(g/l)} << 1$$

$$\approx -\Omega_z \pm \sqrt{\frac{g}{l}}\left(1 + \frac{1}{2}\frac{\Omega_z^2}{(\frac{g}{l})}\right)$$

$$= \pm\sqrt{\frac{g}{l}} - \Omega_z \underbrace{\pm \frac{1}{2}\frac{\Omega_z^2}{\sqrt{\frac{g}{l}}}}_{2^{nd}\text{ order in }\Omega}$$

$$\approx \pm\sqrt{\frac{g}{l}} - \Omega_z$$

Thus,

$$\xi(t) = A\, e^{i\left(\sqrt{\frac{g}{l}} - \Omega_z\right)t} + B\, e^{i\left(-\sqrt{\frac{g}{l}} - \Omega_z\right)t}$$

$$= e^{-i\Omega_z t}\left[\cancel{A e^{i\sqrt{\frac{g}{l}}t} + B e^{-i\sqrt{\frac{g}{l}}t}}\right]$$

$$\cancel{x + iy = \left[\cos(\Omega_z t) - i\sin(\Omega_z t)\right]\left\{A\left(\cos\left(\sqrt{\frac{g}{l}}t\right) + i\sin\left(\sqrt{\frac{g}{l}}t\right)\right) + B\left(\cos\left(\sqrt{\frac{g}{l}}t\right) - i\sin\left(\sqrt{\frac{g}{l}}t\right)\right)\right\}}$$

$$\cancel{\xi(t) = a\, e^{i\left(\left(\sqrt{\frac{g}{l}} - \Omega_z\right)t + \alpha\right)} + b\, e^{i\left(\left(-\sqrt{\frac{g}{l}} - \Omega_z\right)t + \beta\right)}}$$

$$\cancel{= a\left(\cos\left[\left(\sqrt{\frac{g}{l}} - \Omega_z\right)t + \alpha\right] + i\sin\left[\left(\sqrt{\frac{g}{l}} - \Omega_z\right)t + \alpha\right]\right) + b\left(\cos\left[\left(-\sqrt{\frac{g}{l}} - \Omega_z\right)t + \beta\right] + i\sin\left[\left(-\sqrt{\frac{g}{l}} - \Omega_z\right)t + \beta\right]\right)}$$

$\zeta = x + iy$ where

$$\cancel{x(t) = a\cos\left[\left(\sqrt{\tfrac{g}{l}} - \Omega_z\right)t + \alpha\right] + b\cos\left[\left(-\sqrt{\tfrac{g}{l}} - \Omega_z\right)t + \beta\right]}$$

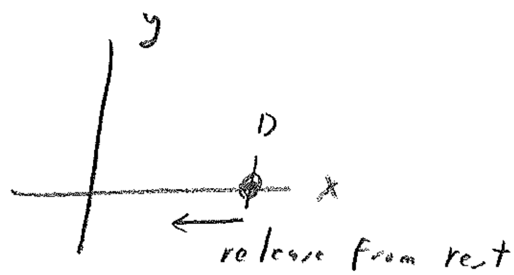
$$\cancel{y(t) = a\sin\left[\left(\quad\right)t + \alpha\right] + b\sin\left[\left(\quad\right)t + \beta\right]}$$

~~where a, b, α, β determined by I.C's~~ $x(0), y(0), \dot{x}(0), \dot{y}(0)$:

$$\zeta(t) = e^{-i\Omega_z t}\left[A e^{i\sqrt{\frac{g}{l}}\,t} + B e^{-i\sqrt{\frac{g}{l}}\,t}\right]$$

I.C's

$x(0) = D$
$y(0) = 0$
$\dot{x}(0) = 0$
$\dot{y}(0) = 0$



$\zeta(0) = x(0) + i y(0) = D$
$\dot{\zeta}(0) = \dot{x}(0) + i\dot{y}(0) = 0$

Thus,

$$D = A + B$$

$$0 = -i\Omega_z(A+B) + i\sqrt{\tfrac{g}{l}}\,A - i\sqrt{\tfrac{g}{l}}\,B$$

$$= i\left[A\left(-\Omega_z + \sqrt{\tfrac{g}{l}}\right) + B\left(-\Omega_z - \sqrt{\tfrac{g}{l}}\right)\right]$$

$$\rightarrow \quad B = A\,\frac{\left(-\Omega_z + \sqrt{\frac{g}{l}}\right)}{\left(\Omega_z + \sqrt{\frac{g}{l}}\right)}$$

$$\rightarrow \quad D = A\left(1 + \frac{(-\Omega_z + \sqrt{\ })}{(\Omega_z + \sqrt{\ })}\right) = A\,\frac{2\sqrt{\ }}{\Omega_z + \sqrt{\ }}$$

Denote $\sqrt{\frac{g}{l}} = \omega$:

$$A = D\frac{(\Omega_z + \omega)}{2\omega}$$

$$B = A\frac{(-\Omega_z+\omega)}{(\Omega_z+\omega)} = D\frac{\cancel{(\Omega_z+\omega)}}{2\omega}\frac{(-\Omega_z+\omega)}{\cancel{(\Omega_z+\omega)}} = \frac{D(-\Omega_z+\omega)}{2\omega}$$

Thus

$$s(t) = e^{-i\Omega_z t}\left(\frac{D}{2\omega}\right)\left[(\Omega_z+\omega)e^{i\omega t} + (-\Omega_z+\omega)e^{-i\omega t}\right]$$

$$= \left(\frac{D}{2\omega}\right)e^{-i\Omega_z t}\left[\Omega_z\underbrace{(e^{i\omega t}-e^{-i\omega t})}_{2i\sin\omega t} + \omega\underbrace{(e^{i\omega t}+e^{-i\omega t})}_{2\cos\omega t}\right]$$

$$= \left(\frac{D}{\omega}\right)e^{-i\Omega_z t}\left[\omega\cos\omega t + i\Omega_z\sin\omega t\right]$$

$$= \left(\frac{D}{\omega}\right)\left[\cos(\Omega_z t) - i\sin(\Omega_z t)\right]\left[\omega\cos(\omega t) + i\Omega_z\sin\omega t\right]$$

$$= \left(\frac{D}{\omega}\right)\Big\{\left[\omega\cos(\,)\cos(\,) + \Omega_z\sin(\,)\sin(\,)\right]$$

$$- i\omega\sin(\Omega_z t)\cos(\omega t) + i\Omega_z\cos(\Omega_z t)\sin\omega t\Big]$$

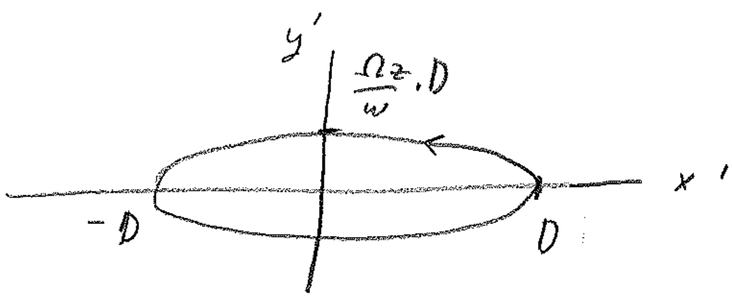
$$= x + iy$$

$$\boxed{\begin{aligned} x &= D\left[\cos(\,)\cos(\,) + \frac{-\Omega_z}{\omega}\sin(\,)\sin(\,)\right] \\ y &= D\left[-\sin(\Omega_z t)\cos(\omega t) + \frac{\Omega_z}{\omega}\cos(\Omega_z t)\sin(\omega t)\right] \end{aligned}}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\Omega_z t) & \sin(\Omega_z t) \\ -\sin(\Omega_z t) & \cos(\Omega_z t) \end{bmatrix}}_{\text{Rotation by } \Omega_z t} \begin{bmatrix} D\cos(\omega t) \\ D\frac{\Omega_z}{\omega}\sin(\omega t) \end{bmatrix}$$

In corotating frame:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} D\cos(\omega t) \\ D\frac{\Omega_z}{\omega}\sin(\omega t) \end{bmatrix} \qquad \frac{\Omega_z}{\omega} = \frac{\Omega\cos\theta}{\sqrt{\frac{g}{l}}} \ll 1$$



ellipse

Precession period:

$$T_{precession} = \frac{2\pi}{\Omega_z} = \frac{2\pi}{\Omega\cos\theta}$$

$$= \frac{\cancel{2\pi}}{\left(\frac{\cancel{2\pi}}{1\,\text{day}}\right)\cos\theta}$$

$$= \frac{1\,\text{day}}{\cos\theta}$$

NOTE: $\theta = 0$ (NP) $\rightarrow T_{precession} = 1$ day

$\theta = \frac{\pi}{2}$ (equator) $\rightarrow T_{precession} = \infty$