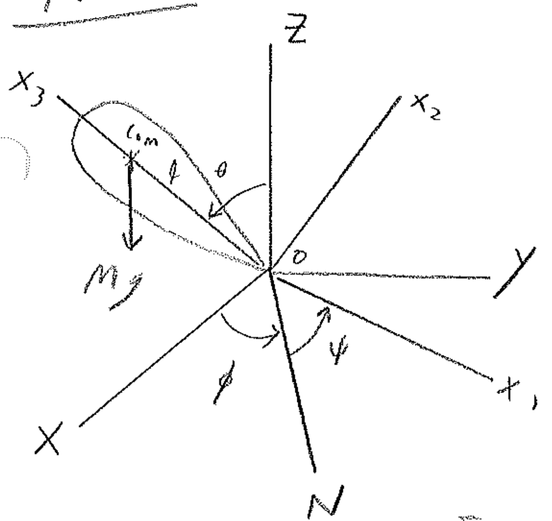


symmetric top: $I_1 = I_2$

$$U = Mgl\cos\theta$$
$$T = T_{com} + T_{rot}$$
$$T_{com} = \frac{1}{2}MV^2$$
$$\vec{V} = \frac{d\vec{R}}{dt}$$
$$\vec{R} = l\sin\theta\cos(\phi - \tfrac{\pi}{2})\,\hat{x} + l\sin\theta\sin(\phi - \tfrac{\pi}{2})\,\hat{y} + l\cos\theta\,\hat{z}$$
$$= l\sin\theta\sin\phi\,\hat{x} - l\sin\theta\cos\phi\,\hat{y} + l\cos\theta\,\hat{z}$$

$$V^2 = |\dot{\vec{R}}|^2 = l^2(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)$$

so $\boxed{T_{com} = \frac{1}{2}Ml^2(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)}$

$$T_{rot} = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2),$$

I_1, I_2, I_3: rotational inertia w.r.t. principal axes (translated version of x_1, x_2, x_3 passing through origin)

$$\Omega_1 = \dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi$$
$$\Omega_2 = \dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi$$
$$\Omega_3 = \dot\phi\cos\theta + \dot\psi$$

$$\rightarrow \boxed{T_{rot}} = \frac{1}{2}I_1\big(\dot\phi^2\sin^2\theta\sin^2\psi + \dot\theta^2\cos^2\psi + \cancel{2\dot\phi\dot\theta\sin\theta\sin\psi\cos\psi} + \dot\phi^2\sin^2\theta\cos^2\psi + \dot\theta^2\sin^2\psi - \cancel{2\dot\phi\dot\theta\sin\theta\cos\psi\sin\psi}\big) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2$$

$$= \frac{1}{2}I_1(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2$$

so $\boxed{T} = \frac{1}{2}(I_1 + Ml^2)(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2$

$$\boxed{= \frac{1}{2}I_1'(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2}$$

where $I_1' = I_1 + Ml^2$ (rotational inertia wrt x_1)

$$L = T - U$$

$$= \frac{1}{2} I_1' \left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) + \frac{1}{2} I_3 \left(\dot\phi\cos\theta + \dot\psi\right)^2 - Mgl\cos\theta$$

No t, ϕ, ψ dependence:

1) $$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3\left(\dot\phi\cos\theta + \dot\psi\right) = \text{const} \quad (\equiv M_3)$$

2) $$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1'\sin^2\theta\,\dot\phi + I_3\left(\dot\phi\cos\theta + \dot\psi\right)\cos\theta = \text{const} \quad (\equiv M_z)$$

$$= I_1'\sin^2\theta\,\dot\phi + \cos\theta\, p_\psi$$

<u>NOTE:</u> $$I_1'\sin^2\theta\,\dot\phi + \cos\theta\, p_\psi = p_\phi$$

$$\boxed{\dot\phi = \frac{p_\phi - \cos\theta\, p_\psi}{I_1'\sin^2\theta}}$$

and $$p_\psi = I_3\dot\phi\cos\theta + I_3\dot\psi$$

$$\boxed{\dot\psi = \frac{p_\psi - I_3\dot\phi\cos\theta}{I_3} = \frac{p_\psi}{I_3} - \cos\theta\left(\frac{p_\phi - \cos\theta\, p_\psi}{I_1'\sin^2\theta}\right)}$$

3) $$E = \dot\psi\frac{\partial L}{\partial\dot\psi} + \dot\phi\frac{\partial L}{\partial\dot\phi} + \dot\theta\frac{\partial L}{\partial\dot\theta} - L \quad (= \text{const})$$

$$= \dot\psi\, I_3\left(\dot\phi\cos\theta + \dot\psi\right) + \dot\phi\left(I_1'\sin^2\theta\,\dot\phi + I_3\left(\dot\phi\cos\theta + \dot\psi\right)\cos\theta\right)$$

$$+ \dot\theta\, I_1'\dot\theta - \frac{1}{2}I_1'\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) - \frac{1}{2}I_3\left(\dot\phi\cos\theta + \dot\psi\right)^2 + Mgl\cos\theta$$

$$= I_3\left[\dot\psi\dot\phi\cos\theta + \dot\psi^2 + \dot\phi^2\cos^2\theta + \dot\phi\dot\psi\cos\theta - \frac{1}{2}\left(\dot\phi\cos\theta + \dot\psi\right)^2\right]$$

$$+ I_1'\left[\sin^2\theta\,\dot\phi^2 + \dot\theta^2 - \frac{1}{2}\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right)\right] + Mgl\cos\theta$$

$$= \frac{1}{2}I_3\left(\dot\phi\cos\theta + \dot\psi\right)^2 + \frac{1}{2}I_1'\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) + Mgl\cos\theta$$

$$= T + U$$

$$E = \frac{1}{2} I_3 \left(\frac{p_\psi^2}{I_3^2}\right) + \frac{1}{2} I_1' \left(\dot\theta^2 + \sin^2\theta \frac{(p_\phi - \cos\theta\, p_\psi)^2}{(I_1' \sin^2\theta)^2}\right) + Mgl\cos\theta$$

$$= \frac{1}{2}\frac{p_\psi^2}{I_3} + \frac{1}{2} I_1' \dot\theta^2 + \frac{1}{2}\frac{1}{I_1' \sin^2\theta}(p_\phi - \cos\theta\, p_\psi)^2 + Mgl\cos\theta$$

$$E - \frac{1}{2}\frac{p_\psi^2}{I_3} = \frac{1}{2} I_1' \dot\theta^2 + \frac{1}{2 I_1' \sin^2\theta}(p_\phi - \cos\theta\, p_\psi)^2 + Mgl\cos\theta$$

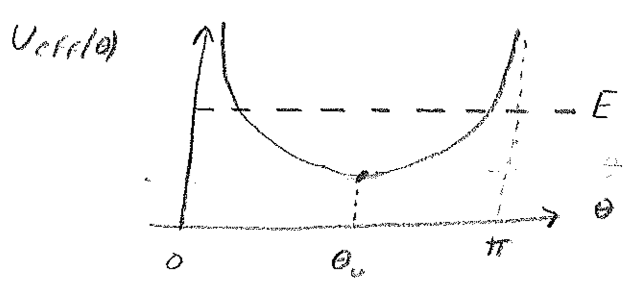
To normalize the energy we will subtract off Mgl = const so $U(0) = 0$, instead of $U(0) = Mgl$.

$$\underbrace{E - \frac{1}{2}\frac{p_\psi^2}{I_3} - Mgl}_{E'} = \frac{1}{2} I_1' \dot\theta^2 + \underbrace{\frac{1}{2 I_1' \sin^2\theta}(p_\phi - \cos\theta\, p_\psi)^2 - Mgl(1-\cos\theta)}_{U_{eff}(\theta)}$$

$$\boxed{\begin{aligned} E' &= \frac{1}{2} I_1' \dot\theta^2 + U_{eff}(\theta) \\ U_{eff}(\theta) &= \frac{1}{2 I_1' \sin^2\theta}(p_\phi - \cos\theta\, p_\psi)^2 - Mgl(1-\cos\theta) \end{aligned}}$$

At $\theta = 0$: $\quad U_{eff}(0) \rightarrow \frac{1}{0^2}(p_\phi - p_\psi)^2 - 0 \rightarrow \infty$ for $p_\phi \neq p_\psi$

At $\theta = \pi$ $\quad U_{eff}(0) \rightarrow \frac{1}{0^2}(p_\phi + p_\psi)^2 - 2Mgl \rightarrow \infty$ for $p_\phi \neq -p_\psi$



$$0 = \frac{dU_{eff}}{d\theta}$$

$$= -\frac{1}{I_1' \sin^3\theta} \cos\theta \left(p_\phi - \cos\theta p_\psi\right)^2 + \frac{1}{I_1' \sin^2\theta}\left(p_\phi - \cos\theta p_\psi\right)\sin\theta\, p_\psi$$
$$- Mgl\sin\theta$$

$$= -\frac{\cos\theta}{I_1' \sin^3\theta}\left(p_\phi - \cos\theta p_\psi\right)^2 + \frac{p_\psi}{I_1'\sin\theta}\left(p_\phi - \cos\theta p_\psi\right) - Mgl\sin\theta$$

$$= -\frac{1}{I_1'\sin^3\theta}\Big[\cos\theta\left(p_\phi - \cos\theta p_\psi\right)^2 - \sin^2\theta\, p_\psi\left(p_\phi - \cos\theta p_\psi\right)$$
$$+ I_1' Mgl \sin^4\theta\Big]$$

So: $$\boxed{0 = \cos\theta_0\, \beta^2 - \sin^2\theta_0\, p_\psi\, \beta + Mgl I_1' \sin^4\theta_0}$$

Quadratic equation for $\beta \equiv p_\phi - \cos\theta_0 p_\psi$:

$$\rightarrow \beta = \frac{\sin^2\theta_0\, p_\psi \pm \sqrt{\sin^4\theta_0\, p_\psi^2 - 4\cos\theta_0\, Mgl I_1' \sin^4\theta_0}}{2\cos\theta_0}$$

$$= \frac{\sin^2\theta_0\, p_\psi}{2\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4MglI_1'\cos\theta_0}{p_\psi^2}}\right)$$

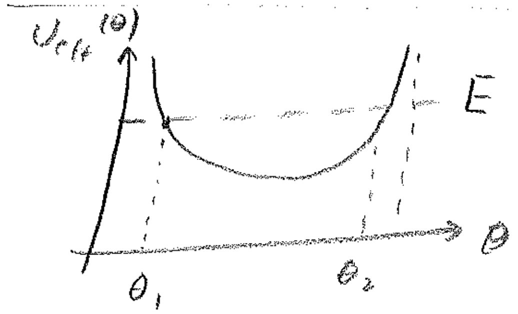
Transcendental equation for θ_0:

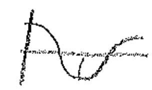
$$p_\phi - \cos\theta_0\, p_\psi = \frac{\sin^2\theta_0\, p_\psi}{2\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4MglI_1'\cos\theta_0}{p_\psi^2}}\right)$$

<u>NOTE</u>: real solution for θ_0 provided

$$1 - \frac{4 M g l I_1' \cos\theta_0}{p_\psi^2} \geq 0$$

$$p_\psi^2 > 4 M g l I_1' \cos\theta_0$$



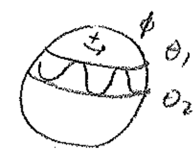


Turning point at θ_1, θ_2 where $E' = U_{eff}(\theta_{1,2})$

Since $\dot\phi = \dfrac{p_\phi - \cos\theta\, p_\psi}{I_1' \sin^2\theta}$, the sign of $\dot\phi$

depends on the sign of $p_\phi - \cos\theta\, p_\psi$ for $\theta_1 \leq \theta \leq \theta_2$

i) $p_\phi - \cos\theta\, p_\psi > 0 \quad \forall\, \theta \in [\theta_1, \theta_2]$



ii) $p_\phi - \cos\theta_1\, p_\psi = 0$, $p_\phi - \cos\theta\, p_\psi > 0$ for $\theta_1 < \theta \leq \theta_2$

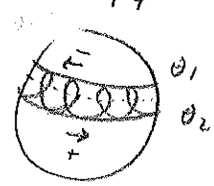
or $p_\phi - \cos\theta_2\, p_\psi = 0$, $p_\phi - \cos\theta\, p_\psi < 0$ for $\theta_1 \leq \theta < \theta_2$

also

iii) $p_\phi - \cos\theta_1\, p_\psi < 0$

$p_\phi - \cos\theta_2\, p_\psi > 0$

$p_\phi - \cos\theta\, p_\psi = 0$ for $\theta_1 < \theta < \theta_2$,



<u>NOTE</u>: $E' = \frac{1}{2} I_1' \dot\theta^2 + U_{eff}(\theta) \rightarrow \dot\theta = \sqrt{\frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)} = \frac{d\theta}{dt}$

$$\boxed{t = \int dt = \int \frac{d\theta}{\sqrt{\frac{2}{I_1'}\left(E' - U_{eff}(\theta)\right)}}}$$

§35, Prob 2 :

Rotation of symmetric top around vertical axis ($\theta=0$)

From Prob 1:

$$p_\psi = I_3(\dot\psi + \dot\phi\cos\theta)$$

$$p_\phi = I_1' \sin^2\theta\, \dot\phi + I_3(\dot\psi + \dot\phi\cos\theta)\cos\theta$$

Note: $\theta \to 0$: $p_\psi \to I_3(\dot\psi + \dot\phi)$, $p_\phi \to I_3(\dot\psi + \dot\phi)$ equal

Effective potential

$$U_{eff}(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1' \sin^2\theta} - Mgl(1-\cos\theta)$$

Limit $\theta \to 0$:

$$\lim_{\theta\to 0} U_{eff}(\theta) = \lim_{\theta \to 0} \frac{p_\psi^2 (1-\cos\theta)^2}{2 I_1' \sin^2\theta} - 0$$

$$= \lim_{\theta\to 0} \frac{p_\psi^2}{2I_1'} \frac{\left(\frac{\theta^2}{2}\right)^2}{\theta^2}$$

$$= \lim_{\theta\to 0} \frac{1}{8}\frac{p_\psi^2}{I_1'}\theta^2 \qquad \cos\theta \cong 1 - \frac{\theta^2}{2}$$

$$= 0$$

So $U_{eff}(\theta) \simeq \frac{1}{8}\frac{p_\psi^2}{I_1'}\theta^2 - \frac{Mgl\,\theta^2}{2}$

$$= \left(\frac{1}{8}\frac{p_\psi^2}{I_1'} - \frac{Mgl}{2}\right)\theta^2 \qquad \text{for } \theta \ll 1$$

$$\left.\frac{d^2 V_{eff}}{d\theta^2}\right|_{\theta=0} = 2\left(\frac{1}{8}\frac{p_\psi^2}{I_1'} - \frac{Mgl}{2}\right)$$

stable equilibrium requires that $\left.\frac{d^2 V_{eff}}{d\theta^2}\right|_{\theta=0} \equiv k > 0$

$$\frac{1}{4}\frac{p_\psi^2}{I_1'} - Mgl > 0$$

$$\boxed{p_\psi^2 > 4 I_1' Mgl}$$

In terms of $\Omega_3 = \dot\phi \cos\theta + \dot\psi = \frac{p_\psi}{I_3}$:

$$I_3^2 \Omega_3^2 > 4 I_1' Mgl$$

$$\boxed{\Omega_3^2 > \frac{4 I_1' Mgl}{I_3^2}}$$