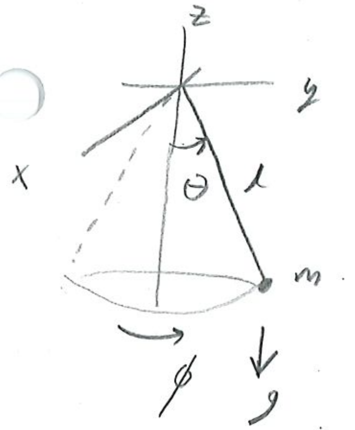
Sec 14, Prob 1 spherical pendulum



$$z = -l\cos\theta$$
$$x = l\sin\theta\cos\phi$$
$$y = l\sin\theta\sin\phi$$

$$U = mgz = -mgl\cos\theta$$
$$T = \frac{1}{2}mv^2$$
$$= \frac{1}{2}ml^2\left[\sin^2\theta\,\dot\phi^2 + \dot\theta^2\right]$$

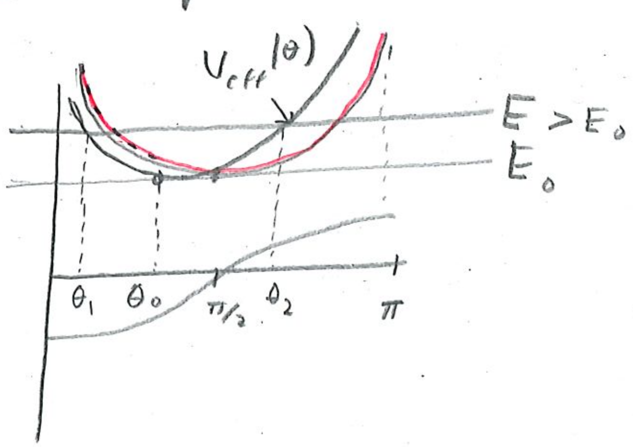
$$L = T - U = \frac{1}{2}ml^2\left[\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right] + mgl\cos\theta$$

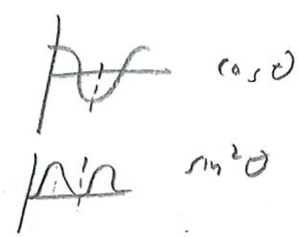
$$\rightarrow \quad p_\phi = \frac{\partial L}{\partial\dot\phi} = ml^2\sin^2\theta\,\dot\phi = \text{const}$$

$$E = T + U = \text{const}$$
$$= \frac{1}{2}ml^2\left[\dot\theta^2 + \sin^2\theta\,\frac{p_\phi^2}{m^2l^4\sin^4\theta}\right] - mgl\cos\theta$$

$$= \frac{1}{2}ml^2\dot\theta^2 + \underbrace{\frac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta}_{U_{eff}(\theta)}$$

Effective potential





E_0 : $\theta = \theta_0 = \text{const}$

Turning points:

$$E = U_{eff}(\theta)$$

$$E = U_{eff}(\theta_1) = U_{eff}(\theta_2)$$

(see page 3; cubic equation)

EOM's:

$$\sqrt{\frac{2}{m l^2}\left(E - U_{eff}(\theta)\right)} = \frac{d\theta}{dt}$$

$$\int dt = \int \frac{d\theta}{\sqrt{\frac{2}{ml^2}}\sqrt{E - U_{eff}(\theta)}}$$

$$\rightarrow \boxed{t = \sqrt{\frac{ml^2}{2}} \int \frac{d\theta}{\sqrt{E - U_{eff}(\theta)}} + \text{const}} \qquad (1)$$

$$\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)} = \left(\frac{d\theta}{dt}\frac{dt}{d\phi}\right)\frac{d\phi}{dt}$$

$$\sqrt{} = \frac{d\theta}{d\phi}\frac{p_\phi}{ml^2 \sin^2\theta}$$

$$\rightarrow d\phi = \frac{p_\phi}{ml^2 \sin^2\theta}\frac{d\theta}{\sqrt{\frac{2}{ml^2}\left(E - U_{eff}(\theta)\right)}}$$

$$= \frac{p_\phi}{\sqrt{ml^2 \cdot 2}}\frac{d\theta/\sin^2\theta}{\sqrt{E - U_{eff}(\theta)}}$$

$$\rightarrow \boxed{\phi = \frac{p_\phi}{\sqrt{2ml^2}}\int \frac{d\theta/\sin^2\theta}{\sqrt{E - U_{eff}(\theta)}} + \text{const}} \qquad (2)$$

Now: $$t = \sqrt{\frac{m l^2}{2}} \int \frac{d\theta}{\sqrt{E - U_{eff}(\theta)}} + \text{const}$$

$$E - U_{eff}(\theta) = E - \frac{p_\phi^2}{2ml^2 \sin^2\theta} + mgl\cos\theta$$

$$= \frac{1}{2ml^2\sin^2\theta}\left[2mEl^2\sin^2\theta - p_\phi^2 + 2m^2gl^3\sin^2\theta\cos\theta\right]$$

$$= \frac{1}{2ml^2\sin^2\theta}\left[2mEl^2(1-\cos^2\theta) - p_\phi^2 + 2m^2gl^3(1-\cos^2\theta)\cos\theta\right]$$

$$= \frac{1}{2ml^2\sin^2\theta}\left[(2mEl^2 - p_\phi^2) + 2m^2gl^3\cos\theta - 2mEl^2\cos^2\theta - 2m^2gl^3\cos^3\theta\right]$$

$$= \frac{1}{2ml^2\sin^2\theta}\left[a + b\cos\theta + c\cos^2\theta + d\cos^3\theta\right]$$

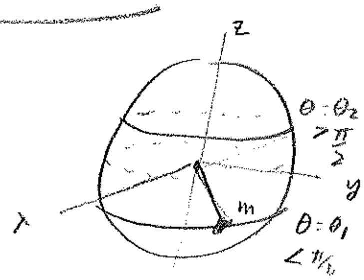
Turning points: $0 = a + b\cos\theta + c\cos^2\theta + d\cos^3\theta$

Solve cubic: Two roots between -1 and $+1$

$0 \le \theta_{1,2} \le \pi$

Also: $$\phi = \frac{p_\phi}{\sqrt{2ml^2}} \int \frac{d\theta}{\sqrt{\sin^4\theta\,(E - U_{eff}(\theta))}}$$

$$\sin^4\theta\left(E - U_{eff}(\theta)\right) = \frac{\sin^2\theta}{2ml^2}\underbrace{\left[a + b\cos\theta + \cos^2\theta + d\cos^3\theta\right]}_{\text{Same cubic to get turning points.}}$$



Form of integrals:

$$I_1 \equiv \int \frac{d\theta}{\sqrt{E - U_{eff}(\theta)}} \qquad \text{and} \qquad I_2 \equiv \int \frac{d\theta}{\sin^2\theta \sqrt{E - U_{eff}(\theta)}}$$

Let $x = \cos\theta \rightarrow dx = -\sin\theta \, d\theta = -\sqrt{1-x^2}\, d\theta$

$$I_1 = \int \frac{d\theta \sqrt{1-x^2}}{\sqrt{a + bx + cx^2 + dx^3}} \sqrt{2ml^2}$$

$$= -\sqrt{2ml^2} \int \frac{dx}{\sqrt{a+bx+cx^2+dx^3}} \xrightarrow{\text{GBR}} \text{Elliptic function of 1st Kind } F(\ ,\ )$$

$$I_2 = \int \frac{d\theta \sqrt{2ml^2}}{\sqrt{(1-x^2)(a+bx+cx^2+dx^3)}}$$

$x = \cos\theta$

$dx = -\sin\theta \, d\theta = -\sqrt{1-x^2}\, d\theta$

$d\theta = -\dfrac{dx}{\sqrt{1-x^2}}$

$$= -\sqrt{2ml^2} \int \frac{dx}{(1-x^2)\sqrt{a+bx+cx^2+dx^3}}$$

$\longrightarrow$ Combination of elliptic functions of 1st and 3rd Kind

Elliptic Integrals

1st kind:

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \qquad 0<k^2<1$$

$$= \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-k^2t^2)(1-t^2)}}$$

2nd kind:

$$E(\phi, k) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\; d\theta$$

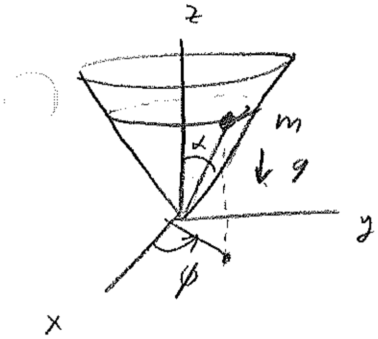
$$= \int_0^{\sin\phi} \sqrt{\frac{1-k^2t^2}{1-t^2}}\; dt$$

3rd kind:

$$\Pi(n, \phi, k) = \int_0^{\pi} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}$$

$$= \int_0^{\sin\phi} \frac{dt}{(1-nt^2)\sqrt{(1-k^2t^2)(1-t^2)}}$$

Sec 14, Prob 2



spherical coords: $(r, \theta = \alpha, \phi)$

$$T = \tfrac{1}{2} m\left(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2\right)\Big|_{\theta=\alpha}$$

$$= \tfrac{1}{2} m\left(\dot r^2 + r^2\sin^2\alpha\,\dot\phi^2\right)$$

$$U = mgz$$

$$= mg\,r\cos\alpha$$

$$L = T - U$$

$$= \tfrac{1}{2} m\left(\dot r^2 + r^2\sin^2\alpha\,\dot\phi^2\right) - mg\,r\cos\alpha$$

No explicit t or ϕ dependence

$$\rightarrow \quad p_\phi = \frac{\partial L}{\partial\dot\phi} = \text{const}$$

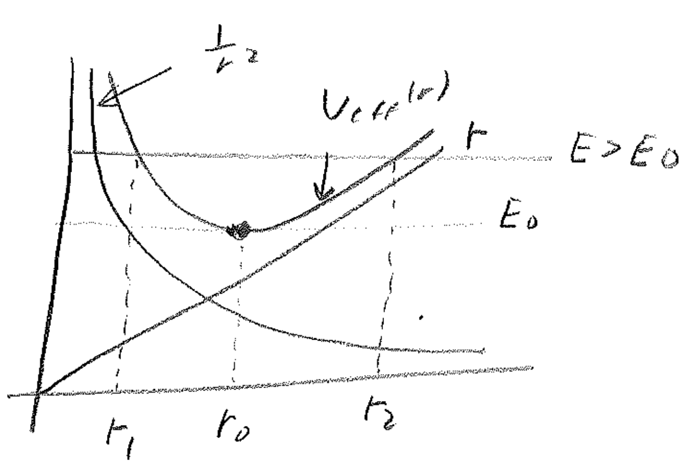
$$\& \quad E = T + U = \text{const}$$

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = m r^2\sin^2\alpha\,\dot\phi \quad \rightarrow \quad \dot\phi = \frac{p_\phi}{m r^2\sin^2\alpha}$$

$$E = \tfrac{1}{2} m\left(\dot r^2 + r^2\sin^2\alpha\left(\frac{p_\phi}{m r^2\sin^2\alpha}\right)^2\right) + mg\,r\cos\alpha$$

$$= \tfrac{1}{2} m\dot r^2 + \frac{p_\phi^2}{2 m r^2\sin^2\alpha} + mg\,r\cos\alpha$$

$$= \tfrac{1}{2} m\dot r^2 + U_{eff}(r)$$



$$U_{eff}(r) = \frac{p_\phi^2}{2 m r^2\sin^2\alpha} + mg\,r\cos\alpha$$

Turning points:

$$E = U_{eff}(r_1) = U_{eff}(r_2)$$

Find minimum of U_{eff}:

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r=r_0}$$

$$= \frac{-p_\phi^2}{m\sin^2\alpha\, r_0^3} + mg\cos\alpha$$

$$\rightarrow \quad \frac{p_\phi^2}{m\sin^2\alpha\, r_0^3} = mg\cos\alpha$$

$$r_0^3 = \frac{p_\phi^2}{m^2 g\sin^2\alpha\cos\alpha}$$

$$r_0 = \left(\frac{p_\phi^2}{m^2 g\sin^2\alpha\cos\alpha}\right)^{\frac{1}{3}}$$ — for a circular orbit


$r = r_0$

EOM:

$$\frac{2}{m}\left(E - U_{eff}(r)\right) = \dot{r}^2 = \left(\frac{dr}{dt}\right)^2$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}$$

$$\int dt = \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}}$$

$$\rightarrow \quad \boxed{t = \sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{E - \frac{p_\phi^2}{2mr^2\sin^2\alpha} - mgr\cos\alpha}} + \text{const}}$$

Also:

$$\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)} = \frac{dr}{dt} = \left(\frac{dr}{d\phi}\right)\frac{d\phi}{dt}$$

$$\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)} = \left(\frac{dr}{d\phi}\right)\frac{p_\phi}{mr^2\sin^2\alpha}$$

$$d\phi = \left(\frac{p_\phi}{m\sin^2\alpha}\right)\frac{dr}{r^2\sqrt{\frac{2}{m}\left(E - U_{eff}(r)\right)}}$$

$$\rightarrow \boxed{\phi = \frac{p_\phi}{\sqrt{2m}\,\sin^2\alpha}\int\frac{dr}{r^2\sqrt{E - \frac{p_\phi^2}{2mr^2\sin^2\alpha} - mgr\cos\alpha}} + \text{const}}$$

Now:

$$E - U_{eff}(r) = E - \frac{p_\phi^2}{2mr^2\sin^2\alpha} - mgr\cos\alpha$$

$$= \frac{p_\phi^2}{2mr^2\sin^2\alpha}\left(\frac{2mE\sin^2\alpha}{p_\phi^2}r^2 - 1 - \frac{2m^2g\sin^2\alpha\cos\alpha}{p_\phi^2}r^3\right)$$

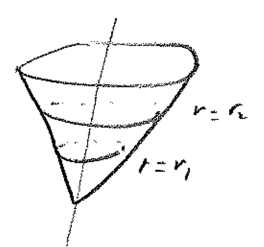
$$= \frac{p_\phi^2}{2mr^2\sin^2\alpha}\left[a + br + cr^2 + dr^3\right]$$

$a = -1$

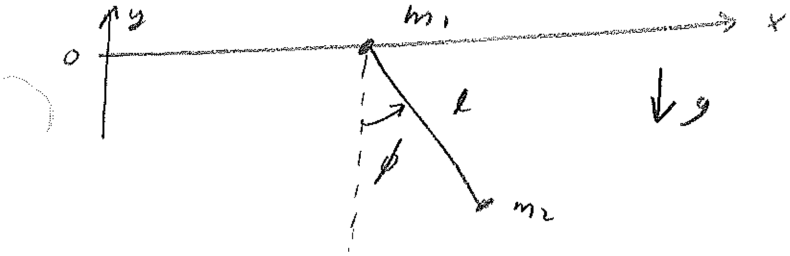
$b = 0$

$$= (r - r_1)(r - r_2)(r - r_3) \quad \text{—— two roots } r_1, r_2 > 0 \text{ with } r_1 < r < r_2$$

$$c = \frac{2mE\sin^2\alpha}{p_\phi^2}, \quad d = -\frac{2m^2g\sin^2\alpha\cos\alpha}{p_\phi^2}$$



Sec 14, Prob 3:



$$x_1 = x \,, \quad y_1 = 0$$
$$x_2 = x + l \sin\phi$$
$$y_2 = -l\cos\phi$$

$$U = m_2 g y_2 = -m_2 g l \cos\phi$$

$$T = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 \left(\dot{x}^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2\dot{x}\dot{\phi}\, l\cos\phi + l^2\sin^2\phi\,\dot{\phi}^2 \right)$$

$$= \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 (\dot{x}^2 + l^2\dot{\phi}^2) + m_2 l\cos\phi\,\dot{x}\dot{\phi}$$

$$= \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + \tfrac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l\cos\phi\,\dot{x}\dot{\phi}$$

$$L = T - U$$

$$= \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + \tfrac{1}{2} m_2 l^2\dot{\phi}^2 + m_2 l\cos\phi\,\dot{x}\dot{\phi} + m_2 g l\cos\phi$$

Since L does not depend explicitly on t or on x

$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2 l\cos\phi\,\dot{\phi} = \text{const}$$

$$E = \dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L = \text{const}$$

where

$$E = \dot{x}\left[(m_1+m_2)\dot{x} + \cancel{m_2 l\cos\phi\,\dot{\phi}} \right]$$
$$+ \dot{\phi}\left[m_2 l^2\dot{\phi} + m_2 l\cos\phi\,\dot{x} \right] - \tfrac{1}{2}(m_1+m_2)\dot{x}^2$$
$$- \tfrac{1}{2} m_2 l^2\dot{\phi}^2 - \cancel{m_2 l\cos\phi\,\dot{x}\dot{\phi}} - m_2 g l\cos\phi$$

$$E = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2 l\cos\phi\,\dot{x}\dot{\phi} - m_2 g l\cos\phi$$
$$= T+U$$

Now: $p_x = (m_1+m_2)\dot{x} + m_2 l\cos\phi\,\dot{\phi} = \text{const}$

$$\rightarrow \dot{x} = \frac{p_x - m_2 l\cos\phi\,\dot{\phi}}{m_1+m_2}$$

Thus,

$$E = \frac{1}{2}(m_1+m_2)\frac{1}{(m_1+m_2)^2}\left(p_x^2 + m_2^2 l^2\cos^2\phi\,\dot{\phi}^2 - 2\cancel{p_x m_2 l\cos\phi\,\dot{\phi}}\right)$$
$$+\frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2 l\cos\phi\,\dot{\phi}\,\cancel{\frac{1}{(m_1+m_2)}}\left(\frac{\cancel{p_x} - m_2 l\cos\phi\,\dot{\phi}}{m_1+m_2}\right)$$
$$- m_2 g l\cos\phi$$

$$= \frac{1}{2(m_1+m_2)}p_x^2 - \frac{1}{2}\frac{1}{(m_1+m_2)}m_2^2 l^2\cos^2\phi\,\dot{\phi}^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 - m_2 g l\cos\phi$$

$$= \frac{1}{2(m_1+m_2)}p_x^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2\left[1-\left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right] - m_2 g l\cos\phi$$

$$= \underbrace{\frac{1}{2(m_1+m_2)}p_x^2}_{\text{constant}} + \frac{1}{2}m_2 l^2\dot{\phi}^2\left[1-\left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right] - m_2 g l\cos\phi$$

Thus,
$$\Big(\underbrace{E - \frac{1}{2(m_1+m_2)}p_x^2}_{E' = \text{const}} + m_2 g l\cos\phi\Big) = \frac{1}{2}m_2 l^2\left[1-\left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right]\dot{\phi}^2$$

$$\sqrt{\frac{2}{m_2 l^2\left[1-\left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi\right]}\left(E'+m_2 g l\cos\phi\right)} = \frac{d\phi}{dt}$$

~~Then~~

$$dt = \frac{d\phi}{\sqrt{}}$$

$$\rightarrow \quad t = \int \frac{d\phi}{\sqrt{\frac{2}{m_2 l^2}}\sqrt{\frac{E'+m_2 g l\cos\phi}{1-\left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi}}} + \text{const}$$

$$= \sqrt{\frac{m_2 l^2}{2}}\int d\phi\sqrt{\frac{1-\left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi}{E'+m_2 g l\cos\phi}} + \text{const}$$

$$= l\sqrt{\frac{m_2}{2(m_1+m_2)}}\int d\phi\sqrt{\frac{(m_1+m_2)-m_2\cos^2\phi}{E'+m_2 g l\cos\phi}} + \text{const}$$

$$= l\sqrt{\frac{m_2}{2(m_1+m_2)}}\int d\phi\sqrt{\frac{m_1+m_2\sin^2\phi}{E'+m_2 g l\cos\phi}} + \text{const}$$

(using $1-\cos^2\phi = \sin^2\phi$)

If we work in reference frame where $p_x = 0$,

then
$$0 = (m_1 + m_2)\dot{x} + m_2 l \cos\phi \, \dot{\phi}$$
$$= (m_1 + m_2)\dot{x} + m_2 l \frac{d}{dt}[\sin\phi]$$
$$= \frac{d}{dt}\left[(m_1 + m_2)x + m_2 l \sin\phi\right]$$

$$\rightarrow \boxed{(m_1 + m_2)x + m_2 l \sin\phi = \text{const}} \Longleftrightarrow \cancel{x = \left(\frac{c - m_2 l \sin\phi}{m_1 + m_2}\right)}$$

Recall:

Choose const so that $\phi = 0 \Longleftrightarrow x = x_0$

then, $\text{const} = x_0(m_1 + m_2)$

$$\rightarrow \quad x + \left(\frac{m_2}{m_1 + m_2}\right) l \sin\phi = x_0$$

$$x = x_0 - \left(\frac{m_2}{m_1 + m_2}\right) l \sin\phi$$

Recall: m_2 located at

$$x_2 = x + l\sin\phi = x_0 - \left(\frac{m_2}{m_1 + m_2}\right) l \sin\phi + l \sin\phi$$
$$= x_0 + l \sin\phi \left(\frac{m_1}{m_1 + m_2}\right)$$

$$y_2 = -l\cos\phi$$

Thus,

$$x_2 - x_0 = l \sin\phi \left(\frac{m_1}{m_1 + m_2} \right)$$

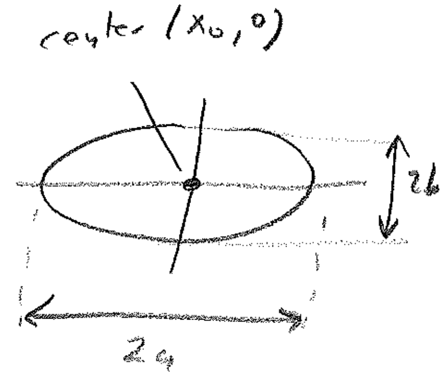
$$y_2 = -l \cos\phi$$

$$\rightarrow \quad \frac{x_2 - x_0}{l\left(\frac{m_1}{m_1+m_2}\right)} = \sin\phi \quad , \quad \frac{y_2}{l} = -\cos\phi$$

$$\frac{(x_2 - x_0)^2}{l^2\left(\frac{m_1}{m_1+m_2}\right)^2} + \frac{y_2^2}{l^2} = 1$$

$$: \quad \frac{(x_2 - x_0)^2}{a^2} + \frac{(y_2 - y_0)^2}{b^2} = 0$$

$$\longrightarrow \quad \boxed{a = l\left(\frac{m_1}{m_1 + m_2}\right) , \quad b = l}$$



NOTE:

limit $m_1 \gg m_2 \rightarrow a \simeq l , \; b = l$

$\rightarrow$ circle.