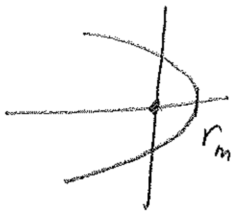
Sec 15, Prob 1

$U = -\frac{\alpha}{r}$, parabola : $E = 0,\ e = 1$



$r_{min} = \frac{p}{2}$

$$\frac{p}{r} = 1 + e\cos\phi$$

$$\boxed{\frac{p}{r} = 1 + \cos\phi}$$

$$t = \sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{E - U_{eff}(r)}} + \text{const} \quad , \quad U_{eff}(r) = \frac{l^2}{2mr^2} - \frac{\alpha}{r}$$

$$= \sqrt{\frac{m}{2}} \int_{r_{min}}^{r} \frac{dr}{\sqrt{0 - \frac{l^2}{2mr^2} + \frac{\alpha}{r}}}$$

$$= \sqrt{\frac{m}{2}} \int_{\frac{p}{2}}^{r} \frac{dr}{\sqrt{\frac{\alpha}{r} - \frac{l^2}{2mr^2}}}$$

$$= \sqrt{\frac{m}{2\alpha}} \int_{p/2}^{r} \frac{r\,dr}{\sqrt{r - \frac{l^2}{2m\alpha}}} \qquad \left[\text{recall: } p = \frac{l^2}{m\alpha}\right]$$

$$= \sqrt{\frac{m}{2\alpha}} \int_{p/2}^{r} \frac{r\,dr}{\sqrt{r - \frac{p}{2}}}$$

Define: $\cancel{r - \frac{p}{2} = \eta^2 \rightarrow dr = 2\eta\,d\eta \qquad r = \frac{p}{2} \rightarrow \eta = 0}$

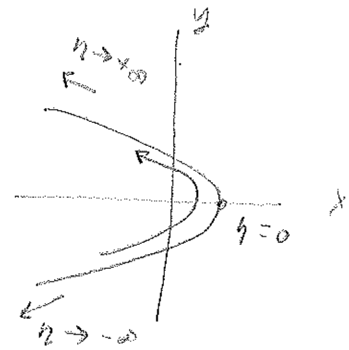
$\cancel{r = \eta^2 + \frac{p}{2}}$

~~Then~~

$\cancel{t = \sqrt{\frac{m}{2\alpha}} \int_0^{\eta} \frac{\left(\eta^2 + \frac{p}{2}\right) 2\eta\,d\eta}{\eta} = \sqrt{\frac{2m}{\alpha}}\left(\frac{1}{3}\eta^3 + \frac{p}{2}\eta\right) = \sqrt{\frac{2m}{\alpha}}\,\eta\left(\frac{\eta^2}{3} + \frac{p}{2}\right)}$

Thus, $\cancel{t = \frac{p}{2} + \eta^2}$

$$\cancel{t = \sqrt{\frac{2m}{\alpha}}\, \eta \left(\eta^2 + \frac{p}{2} \right)}$$



~~Define~~:

Change of variables:

$$\eta^2 = \frac{\left(r - \frac{p}{2}\right)}{\left(p/2\right)}$$

to eliminate square root

to normalize

Thus, $\frac{p}{2}\eta^2 = r - \frac{p}{2}$

$$\boxed{r = \frac{p}{2}\left(1 + \eta^2\right)}$$

parametric representation of r

$\eta = 0 \Longleftrightarrow r = \frac{p}{2} = r_{min}$

Then:

$$t = \sqrt{\frac{m}{2\alpha}} \int_0^{\eta} \frac{\frac{p}{2}(1+\eta^2)\, \frac{p}{\cancel{2}} \cdot \cancel{2}\cancel{\eta}\, d\eta}{\sqrt{\frac{p}{2}\cancel{\eta}^2}}$$

$$= \sqrt{\frac{m}{2\alpha}} \sqrt{\frac{p}{2}}\, p \int_0^{\eta} (1 + \eta^2)\, d\eta$$

$$= \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}} \left(\eta + \frac{1}{3}\eta^3\right)$$

$$= \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}}\, \eta\left(1 + \eta^2\right)$$

So $\boxed{t = \frac{1}{2}\sqrt{\frac{mp^3}{\alpha}}\, \eta\left(1+\eta^2\right)}$ parametric rep. of t

Sec 15, Prob 2: $U = -\frac{\alpha}{r^2}, \quad \alpha > 0$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{\alpha}{r}$$

$$\rightarrow \quad p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi} \equiv l = \text{const}$$

Thus, $\dot{\phi} = \frac{l}{m r^2}$

$$\rightarrow \quad E = \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} + \dot{r} \frac{\partial L}{\partial \dot{r}} - L = \text{const}$$

$$= T + U$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \left(\frac{l}{m r^2}\right)^2 - \frac{\alpha}{r^2}$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{l^2}{2 m r^2} - \frac{\alpha}{r^2}$$

$$= \frac{1}{2} m \dot{r}^2 + \left(\frac{l^2}{2m} - \alpha\right) \frac{1}{r^2}$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{\beta}{r^2}$$

where $\beta \equiv \frac{l^2}{2m} - \alpha$

NOTE: If $\beta = 0$ then $E = \frac{1}{2} m \dot{r}^2 \rightarrow \dot{r} = \sqrt{\frac{2E}{m}}$

$$\rightarrow \quad \boxed{r = \sqrt{\frac{2E}{m}}\, t + r_0}$$

Also: $\sqrt{\frac{2E}{m}} = \dot{r} = \frac{dr}{d\phi} \dot{\phi} = \frac{dr}{d\phi} \frac{l}{m r^2}$

$$\frac{\sqrt{2mE}}{l} = \frac{dr}{d\phi} \frac{1}{r^2}$$

$$\int \frac{\sqrt{2mE}}{l}\, d\phi = \int \frac{dr}{r^2} = \quad \Rightarrow \quad \frac{\sqrt{2mE}}{l} (\phi - \phi_0) = -\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

$\phi = 0$ when $r = r_0$ $\rightarrow$ $\boxed{\frac{\sqrt{2mE}}{l} \phi = \frac{1}{r} - \frac{1}{r_0}}$

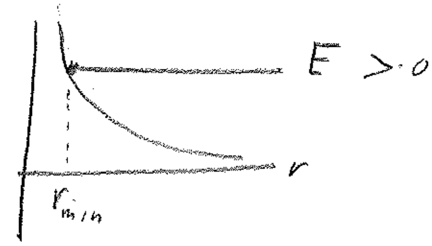
We will now just consider the cases where $\beta \gtrless 0$

Case (i): $\beta \equiv \frac{l^2}{2m} - \alpha > 0$

$E > 0$

$r \geq r_{min}$

$E = \frac{\beta}{r_{min}^2}$

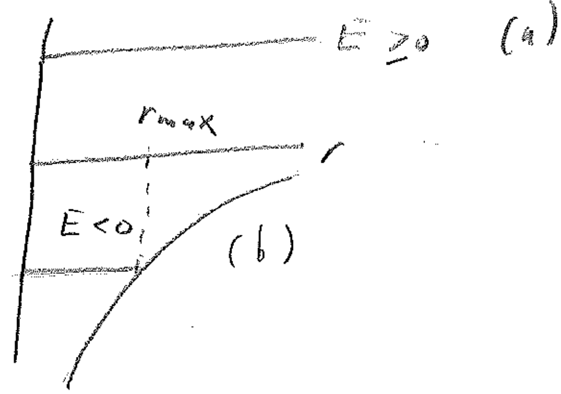


Case (ii): $\beta \equiv \frac{l^2}{2m} - \alpha < 0$

a) $E > 0$, $0 \leq r < \infty$

b) $E < 0$, $r \leq r_{max}$

$E = \frac{\beta}{r_{max}^2}$



Now: $E = \frac{1}{2} m \dot{r}^2 + \frac{\beta}{r^2}$

$$\sqrt{\frac{2}{m}\left(E - \frac{\beta}{r^2}\right)} = \dot{r} = \frac{dr}{d\phi}\dot{\phi} = \frac{dr}{d\phi}\frac{l}{mr^2}$$

Thus, $$dt = \frac{dr}{\sqrt{\frac{2}{m}}\sqrt{E - \frac{\beta}{r^2}}} \rightarrow \boxed{t = \sqrt{\frac{m}{2}}\int \frac{dr}{\sqrt{E - \frac{\beta}{r^2}}} + const}$$

also $$d\phi = \frac{l\, dr/r^2}{m\sqrt{\frac{2}{m}}\sqrt{E - \frac{\beta}{r^2}}} \rightarrow \boxed{\phi = \frac{l}{\sqrt{2m}}\int \frac{dr/r^2}{\sqrt{E - \frac{\beta}{r^2}}} + const}$$

t - equation:

$$t = \sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{E - \frac{\beta}{r^2}}} + \text{const} = \sqrt{\frac{m}{2}} \int \frac{r\,dr}{\sqrt{\underbrace{Er^2 - \beta}_{>0}}} + \text{const}$$

Define: $\eta^2 = Er^2 - \beta$ $\qquad -\infty < \eta < \infty$

$\qquad \eta = 0 \Leftrightarrow t = 0$

$$\cancel{2}\eta\, d\eta = \cancel{2}E\, r\,dr$$

$$r\,dr = \frac{1}{E}\eta\, d\eta$$

Thus,

$$\boxed{t} = \sqrt{\frac{m}{2}} \int \frac{1}{E} \frac{\cancel{\eta}\, d\eta}{\cancel{\sqrt{\eta^2}}}$$

$$= \sqrt{\frac{m}{2}}\, \frac{1}{E}\, \eta$$

$$= \sqrt{\frac{m}{2}}\, \frac{1}{E} \sqrt{Er^2 - \beta}$$

$$\boxed{= \frac{1}{E}\sqrt{\frac{m}{2}} \sqrt{Er^2 - \left(\frac{l^2}{2m} - \alpha\right)}}$$

Valid for all cases $E<0$ or $E \geq 0$ and $\beta > 0$ or $\beta < 0$

ϕ - equation:

$$\phi = \frac{l}{\sqrt{2m}} \int \frac{dr/r^2}{\sqrt{E - \beta/r^2}} + \text{const}$$

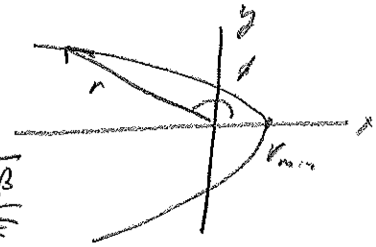
$$= \frac{l}{\sqrt{2m}} \int \frac{-du}{\sqrt{E - \beta u^2}}$$

Let: $u = \frac{1}{r}$

$du = -\frac{1}{r^2}\,dr$

Case (i): $\beta = \frac{\ell^2}{2m} - \alpha > 0$

$E > 0, \quad E = \frac{\beta}{r_{min}^2}, \; r_{min} = \sqrt{\frac{\beta}{E}}$



$$\phi = \frac{-\ell}{\sqrt{2m}} \int_{\frac{1}{r_{min}}} \frac{du}{\sqrt{\frac{\beta}{r_{min}^2} - \beta u^2}}$$

$$= \frac{\ell}{\sqrt{2m\beta}} \int_{\frac{1}{r_{min}}} \frac{-du}{\sqrt{\frac{1}{r_{min}^2} - u^2}}$$

$$= \frac{\ell}{\sqrt{2m\beta}} \int_0^\theta \frac{\frac{\cancel{\sin\theta}}{\cancel{r_{min}}} \, d\theta}{\frac{1}{\cancel{r_{min}}} \cancel{\sin\theta}}$$

$$= \frac{\ell}{\sqrt{2m\beta}} \cos^{-1}(r_{min} u)$$

$$= \frac{\ell}{\sqrt{2m\beta}} \cos^{-1}\left(\frac{r_{min}}{r}\right)$$

$1 = \cos^2\theta + \sin^2\theta$

$u = \frac{1}{r_{min}} \cos\theta$

$du = \left(\frac{1}{r_{min}}\right) - \sin\theta \, d\theta$

$\sqrt{\frac{1}{r_{min}^2} - u^2} = \frac{1}{r_{min}} \sqrt{1 - \cos^2\theta}$

$= \frac{\sin\theta}{r_{min}}$

$u = \frac{1}{r_{min}} \Leftrightarrow \theta = 0$

Thus, $\frac{\sqrt{2m\beta}}{\ell} \phi = \cos^{-1}\left(\frac{r_{min}}{r}\right)$

$$\rightarrow \quad \frac{r_{min}}{r} = \cos\left(\frac{\sqrt{2m\beta}}{\ell} \phi\right)$$

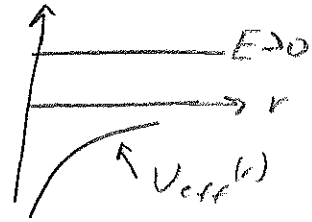
$$= \cos\left(\sqrt{\frac{2m}{\ell^2}\left(\frac{\ell^2}{2m} - \alpha\right)} \; \phi\right)$$

$$= \cos\left(\sqrt{1 - \frac{2m\alpha}{\ell^2}} \; \phi\right)$$

$$\rightarrow \quad \frac{1}{r} = \sqrt{\frac{E}{\left(\frac{\ell^2}{2m} - \alpha\right)}} \cos\left(\sqrt{1 - \frac{2m\alpha}{\ell^2}} \; \phi\right) = \boxed{\sqrt{\frac{2mE}{\ell^2 - 2m\alpha}} \cos\left(\sqrt{1 - \frac{2m\alpha}{\ell^2}} \; \phi\right) = \frac{1}{r}}$$

Case (ii), (a): $\quad \beta = \frac{\ell^2}{2m} - \alpha < 0$

$E > 0, \quad 0 \le r < \infty$



$$\phi = \frac{\ell}{\sqrt{2m}} \int_0^{\frac{1}{r}} \frac{+\,du}{\sqrt{E - \beta u^2}}$$

Integrate inward from $r = \infty$ $\left(u = \frac{1}{r} = 0\right)$

$$= \frac{\ell}{\sqrt{2m|\beta|}} \int_0^{\frac{1}{r}} \frac{+\,du}{\sqrt{u^2 - \frac{E}{\beta}}}$$

$$= \frac{\ell}{\sqrt{2m|\beta|}} \int_0^{\frac{1}{r}} \frac{+\,du}{\sqrt{u^2 + \frac{E}{|\beta|}}}$$

Now:

$\cosh^2 - \sinh^2 = 1$

$\cosh^2 = 1 + \sinh^2$

Let: $u = \sqrt{\frac{E}{|\beta|}} \sinh\chi$

$du = \sqrt{\frac{E}{|\beta|}} \cosh\chi \, d\chi$

$\sqrt{} = \sqrt{\frac{E}{|\beta|}} \sqrt{1 + \sinh^2} = \sqrt{\frac{E}{|\beta|}} \cosh\chi$

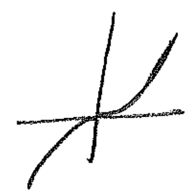
$$= \frac{\ell}{\sqrt{2m|\beta|}} \int_0^{\sinh^{-1}\left(\sqrt{\frac{|\beta|}{E}} \frac{1}{r}\right)} d\chi$$

$$= \frac{\ell}{\sqrt{2m|\beta|}} \sinh^{-1}\left(\sqrt{\frac{|\beta|}{E}} \frac{1}{r}\right)$$

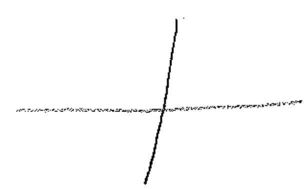
Thus,

~~choose such that ... when~~ $\phi = 0$

$$\sinh\left(\frac{\sqrt{2m|\beta|}}{\ell} \phi\right) = \sqrt{\frac{|\beta|}{E}} \frac{1}{r}$$



$$\rightarrow \boxed{\frac{1}{r}} = \sqrt{\frac{E}{|\beta|}} \sinh\left(\frac{\sqrt{2m|\beta|}}{\ell} \phi\right)$$

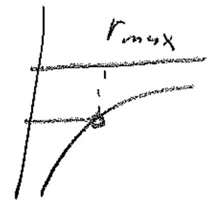


$$= \sqrt{\frac{E}{\alpha - \frac{\ell^2}{2m}}} \sinh\left(\sqrt{\frac{2m}{\ell^2}\left(\alpha - \frac{\ell^2}{2m}\right)} \, \phi\right)$$

$$= \boxed{\sqrt{\frac{2mE}{2m\alpha - \ell^2}} \sinh\left(\sqrt{\frac{2m\alpha}{\ell^2} - 1} \, \phi\right)}$$

Case (ii), (b): $\beta = \frac{l^2}{2m} - \alpha < 0$

$E = \frac{\beta}{r_{max}^2}$ $E < 0, \quad 0 \le r \le r_{max}$



$$\phi = \frac{l}{\sqrt{2m}} \int_{\frac{1}{r_{max}}}^{\frac{1}{r}} \frac{\mp du}{\sqrt{E - \beta u^2}} + const$$

Integrate inward from $r = r_{max}$ $\left(u = \frac{1}{r_{max}}\right)$

$$= \frac{l}{\sqrt{2m|\beta|}} \int_{\frac{1}{r_{max}}}^{\frac{1}{r}} \frac{du}{\sqrt{u^2 - \frac{E}{\beta}}} + const$$

$$= \frac{l}{\sqrt{2m|\beta|}} \int_{\frac{1}{r_{max}}}^{\frac{1}{r}} \frac{du}{\sqrt{u^2 - \frac{|E|}{|\beta|}}}$$

Now:
$ch^2 - sh^2 = 1$
$ch^2 - 1 = sh^2$
Let $u = \sqrt{\frac{|E|}{|\beta|}} \cosh\chi$
$du = \sqrt{\frac{|E|}{|\beta|}} \sinh\chi$

$$= \frac{l}{\sqrt{2m|\beta|}} \int_{\cosh^{-1}(1) = 0}^{\cosh^{-1}\left(\frac{r_{max}}{r}\right)} d\chi$$

using $\sqrt{\frac{|\beta|}{|E|}} = r_{max}$

$$= \frac{l}{\sqrt{2m|\beta|}} \cosh^{-1}\left(\sqrt{\frac{|\beta|}{|E|}} \frac{1}{r}\right)$$

Thus,

$$\cosh\left(\frac{\sqrt{2m|\beta|}}{l} \phi\right) = \sqrt{\frac{|\beta|}{|E|}} \frac{1}{r}$$

$$\rightarrow \boxed{\frac{1}{r} = \sqrt{\frac{|E|}{|\beta|}} \cosh\left(\frac{\sqrt{2m|\beta|}}{l} \phi\right) = \sqrt{\frac{2m|E|}{2m\alpha - l^2}} \cosh\left(\sqrt{\frac{2m\alpha}{l^2} - 1} \; \phi\right)}$$

$$\downarrow$$

$$\sqrt{} = \frac{1}{r_{max}}$$

For cases (ii) (a), (b), the particle eventually falls to the origin $r = 0$.

The time taken to go from $r = r_0$ to $r = 0$ is given by (see page 3):

$$\Delta t = \frac{1}{E}\sqrt{\frac{m}{2}}\left(\sqrt{E r_0^2 - \left(\frac{l^2}{2m} - \alpha\right)} - \sqrt{-\left(\frac{l^2}{2m} - \alpha\right)}\right)$$

$$= \frac{1}{E}\sqrt{\frac{m}{2}}\left(\sqrt{E r_0^2 + \alpha - \frac{l^2}{2m}} - \sqrt{\alpha - \frac{l^2}{2m}}\right)$$

Sec 15, Prob 3 Find $\delta\phi$ For (a) $\delta U = \frac{\beta}{r^2}$, (b) $\delta U = \frac{\gamma}{r^3}$

$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{l\, dr/r^2}{\sqrt{2m(E-U) - \frac{l^2}{r^2}}}$$ (for closed orbit $\Delta\phi = 2\pi\left(\frac{m}{n}\right)$)

For $U = -\frac{\alpha}{r}$, $\Delta\phi = 2\pi$

NOTE: In general, we can write

$$\frac{l/r^2}{\sqrt{2m(E-U) - \frac{l^2}{r^2}}} = -\frac{\partial}{\partial l}\left(\sqrt{2m(E-U) - \frac{l^2}{r^2}}\right)$$

Since: $RHS = -\frac{\partial}{\partial l}\sqrt{}$

$$= -\frac{1}{2}\frac{1}{\sqrt{}}\left(-\frac{2l}{r^2}\right)$$

$$= +\frac{l/r^2}{\sqrt{2m(E-U) - \frac{l^2}{r^2}}}$$

Thus, $$\boxed{\Delta\phi = -2\frac{\partial}{\partial l}\left[\int_{r_{min}}^{r_{max}} dr\sqrt{2m(E-U) - \frac{l^2}{r^2}}\right]}$$

(we will derive something similar for $\delta\phi$ shortly)

$$\delta\phi = \Delta\phi - 2\pi$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{\ell\, dr/r^2}{\sqrt{2m\left(E+\frac{\alpha}{r}-\delta U\right)-\frac{\ell^2}{r^2}}} - 2\int_{r_{min}}^{r_{max}} \frac{\ell\, dr/r^2}{\sqrt{\underbrace{2m\left(E+\frac{\alpha}{r}\right)-\frac{\ell^2}{r^2}}_{A}}}$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{\ell\, dr}{r^2}\left[\frac{1}{\sqrt{A-2m\delta U}} - \frac{1}{\sqrt{A}}\right]$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{\ell\, dr}{r^2}\left[\frac{1}{\sqrt{A}\sqrt{1-\frac{2m\delta U}{A}}} - \frac{1}{\sqrt{A}}\right]$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{\ell\, dr/r^2}{\sqrt{A}}\underbrace{\left(\frac{1}{\sqrt{1-\frac{2m\delta U}{A}}} - 1\right)}_{\cancel{1}+\frac{1}{2}\left(\frac{2m\delta U}{A}\right)-\cancel{1}}$$

$$= \int_{r_{min}}^{r_{max}} \frac{\ell\, dr/r^2}{A^{3/2}}\, 2m\delta U$$

$$= \int_{r_{min}}^{r_{max}} \frac{2m\ell\, dr/r^2\, \delta U}{\left(2m\left(E+\frac{\alpha}{r}\right)-\frac{\ell^2}{r^2}\right)^{3/2}}$$

Now:

$$\frac{\partial}{\partial \ell}\left(\frac{1}{\sqrt{2m\left(E+\frac{\alpha}{r}\right)-\frac{\ell^2}{r^2}}}\right) = -\frac{1}{\cancel{2}(\ )^{3/2}}\ \frac{-\cancel{2}\ell}{r^2}$$

$$= \frac{\ell/r^2}{(\ \ )^{3/2}}$$

Thus,

$$\boxed{\delta\phi = 2m\frac{\partial}{\partial \ell}\left[\int_{r_{min}}^{r_{max}} \frac{\delta U(r)\, dr}{\sqrt{2m\left(E+\frac{\alpha}{r}\right)-\frac{\ell^2}{r^2}}}\right]}$$

Since $\delta U(r)$ is small, we can substitute for r inside integrand using equation for ellipse

$$\frac{p}{r} = 1+e\cos\phi$$

$$\frac{1}{r} = \frac{1+e\cos\phi}{p} \qquad \text{where } p = \frac{\ell^2}{m\alpha}$$

$$\text{and } e = \sqrt{1+\frac{2E\ell^2}{m\alpha^2}}$$

Thus,
$$-\frac{1}{r^2}dr = \frac{-e}{p}\sin\phi\, d\phi$$

$$dr = \left(\frac{e}{p}\right) r^2 \sin\phi\, d\phi$$

and:
$$2m\left(E+\frac{\alpha}{r}\right) - \frac{\ell^2}{r^2} = 2mE + \frac{2m\alpha}{p}(1+e\cos\phi) - \frac{\ell^2}{p^2}(1+e\cos\phi)^2$$

$$= 2mE + \frac{2m\alpha}{p} + \frac{2m\alpha}{p}e\cos\phi - \frac{\ell^2}{p^2}\left(1+e^2\cos^2\phi + 2e\cos\phi\right)$$

$$= 2mE + 2\frac{m^2\alpha^2}{l^2} + \cancel{\frac{2m^2\alpha^2 e\cos\phi}{l^2}}$$

$$- \frac{m^2\alpha^2}{l^2}\left[1 + e^2\cos^2\phi + \cancel{2e\cos\phi}\right]$$

$$= \frac{m^2\alpha^2}{l^2} + 2mE - \frac{m^2\alpha^2 e^2\cos^2\phi}{l^2}$$

$$= \cancel{\frac{m^2\alpha^2}{l^2} + 2mE - \frac{m^2\alpha^2}{l^2}\left(1 + \frac{2El^2}{m\alpha^2}\right)\cos^2\phi}$$

$$= \cancel{2mE(1-\cos^2\phi)}$$

$$= \cancel{2mE\sin^2\phi}$$

Thus, $\cancel{\sqrt{2m\left(E + \frac{\alpha}{r}\right) - \frac{l^2}{r^2}}} = \sqrt{2mE}\sin\phi$

$$\rightarrow \delta\phi = \cancel{2m\frac{\partial}{\partial l}\left[2m\int_0^\pi \frac{\delta U(r)\left(\frac{p}{p}\right)r^2\sin\phi\, d\phi}{\sqrt{2mE}\,\sin\phi}\right]}$$

$$= \frac{m^2\alpha^2}{l^2} + 2m\left(\frac{m\alpha^2}{2l^2}\right)\left(e^2-1\right) - \frac{m^2\alpha^2}{l^2}e^2\cos^2\phi$$

$$= \cancel{\frac{m^2\alpha^2}{l^2}} + \frac{m^2\alpha^2}{l^2}\left(e^2 - \cancel{1}\right) - \frac{m^2\alpha^2}{l^2}e^2\cos^2\phi$$

$$= \frac{m^2\alpha^2}{l^2}e^2\left(1-\cos^2\phi\right)$$

$$= \frac{m^2\alpha^2}{l^2}e^2\sin^2\phi$$

Thus,

$$\delta\phi = \frac{\partial}{\partial l}\left[2m \int_0^{\pi} \frac{\delta U(r)\left(\frac{\cancel{e m\alpha}}{l^2}\right) r^2 \cancel{\sin\phi}\, d\phi}{\cancel{\frac{m\alpha}{l}}\,\cancel{e}\,\cancel{\sin\phi}} \right]$$

$$= \frac{\partial}{\partial l}\left[\frac{2m}{l} \int_0^{\pi} \delta U(r)\, r^2\, d\phi \right]$$

(a) $\delta U(r) = \dfrac{\beta}{r^2}, \quad \beta > 0$

$$\delta\phi = \frac{\partial}{\partial l}\left[\frac{2m}{l} \int_0^{\pi} \frac{\beta}{\cancel{r^2}} \cancel{r^2}\, d\phi \right]$$

$$= \frac{\partial}{\partial l}\left[\frac{2m\beta\pi}{l} \right]$$

$$= \boxed{-\frac{2\pi m\beta}{l^2}}$$

(b.) $\delta U(r) = \dfrac{\gamma}{r^3}, \quad \gamma > 0$

$$\delta\phi = \frac{\partial}{\partial l}\left[\frac{2m}{l} \int_0^{\pi} \frac{\gamma}{r^3} r^2\, d\phi \right]$$

$$= \frac{\partial}{\partial l}\left[\frac{2m\gamma}{l} \int_0^{\pi} \frac{1}{r}\, d\phi \right]$$

$$= \frac{\partial}{\partial l}\left[\frac{2m\gamma}{l} \int_0^{\pi} \frac{1 + e\cos\phi}{p}\, d\phi \right]$$

$$= \frac{\partial}{\partial l}\left[\frac{2m\gamma}{l}\left(\frac{m\alpha}{l^2}\right)\left(\underbrace{\int_0^{\pi} d\phi}_{= \pi} + \underbrace{e\int_0^{\pi} \cos\phi \, d\phi}_{0}\right)\right]$$

since $\sin\phi\Big|_0^{\pi} = 0$

$$= 2\pi m^2 \alpha\gamma \frac{\partial}{\partial l}\left(\frac{1}{l^3}\right)$$

$$= \boxed{-6\pi \frac{m^2 \alpha\gamma}{l^4}}$$