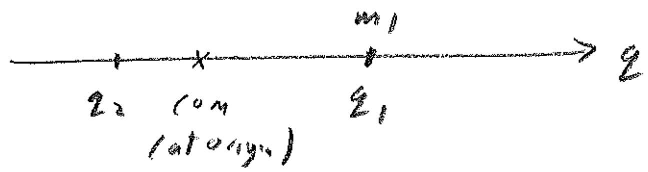
Sec 21, prob 2

Diatomic molecule: masses m_1, m_2



COM frame

$$m_1 q_1 + m_2 q_2 = 0$$

$q = q_1 - q_2$ (relative position vector)

$$\longrightarrow \quad q_1 = \left(\frac{m_2}{m_1+m_2}\right) q$$

$$q_2 = \left(\frac{-m_1}{m_1+m_2}\right) q$$

$$T = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2$$

$$= \frac{1}{2}\left[m_1 \frac{m_2^2}{(m_1+m_2)^2} \dot{q}^2 + m_2 \frac{m_1^2}{(m_1+m_2)^2} \dot{q}^2\right]$$

$$= \frac{1}{2}\left[\frac{m_2}{(m_1+m_2)} m\dot{q}^2 + \frac{m_1}{(m_1+m_2)} m\dot{q}^2\right]$$

$$= \frac{1}{2} m \dot{q}^2$$

$$U \simeq \frac{1}{2} k (q-q_0)^2$$

where q_0 = separation for equilibrium configuration.

Define: $x = q - q_0 \longrightarrow L = T - U$

$$= \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{k(m_1+m_2)}{m_1 m_2}}$$

For another diatomic molecule with masses m_1', m_2'

$$\omega' = \sqrt{\frac{k}{m'}} = \sqrt{\frac{k\,(m_1' + m_2')}{m_1' m_2'}}$$

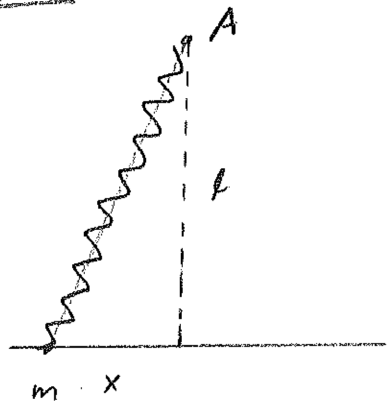
$$\rightarrow \quad \frac{\omega'}{\omega} = \sqrt{\frac{\cancel{k}}{m'}}\sqrt{\frac{m}{\cancel{k}}}$$

$$= \sqrt{\frac{m}{m'}}$$

$$= \sqrt{\frac{m_1 m_2}{m_1' m_2'} \frac{(m_1' + m_2')}{(m_1 + m_2)}}$$

Section 21

Prob 3



A force F is required to stretch the spring to length ℓ.

Potential energy:

$$U = F\,\delta\ell$$

$$= F\left(\sqrt{\ell^2 + x^2} - \ell\right)$$

$$= F\ell\left(\sqrt{1 + \left(\frac{x}{\ell}\right)^2} - 1\right)$$

$$\simeq F\ell\,\frac{1}{2}\left(\frac{x}{\ell}\right)^2 \quad \left(\text{for small oscillations}\right)$$

$$= \frac{1}{2}\left(\frac{F}{\ell}\right)x^2$$

$$= \frac{1}{2}kx^2$$

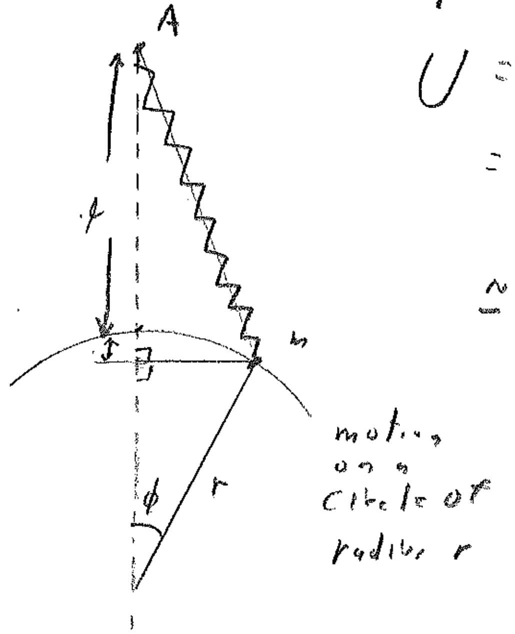
where $k = \frac{F}{\ell}$

$\rightarrow$ $T = \frac{1}{2}m\dot{x}^2$

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{F}{m\ell}}$$

Sec 21, Prob 41



$$T = \frac{1}{2} m r^2 \dot{\phi}^2$$

$$U = F\,\delta l$$

$$= F\left(\sqrt{\left(l + r(1-\cos\phi)\right)^2 + (r\sin\phi)^2} - l\right)$$

$$\simeq F\left(\sqrt{\left(l + \frac{r}{2}\phi^2\right)^2 + (r\phi)^2} - l\right)$$

(for $\phi << 1$) ~~with~~

Thus, $$U \simeq F\left(\sqrt{l^2 + r l \phi^2 + \underbrace{\frac{1}{4} r^2 \phi^4}_{\text{ignore}} + r^2\phi^2} - l\right)$$

$$\simeq F\left(\sqrt{l^2 + r(l+r)\phi^2} - l\right)$$

$$= F\left(l\sqrt{1 + \frac{r(l+r)}{l^2}\phi^2} - l\right)$$

$$\simeq F\left(l\left(\cancel{1} + \frac{1}{2}\frac{r(l+r)}{l^2}\phi^2\right) \cancel{-l}\right)$$

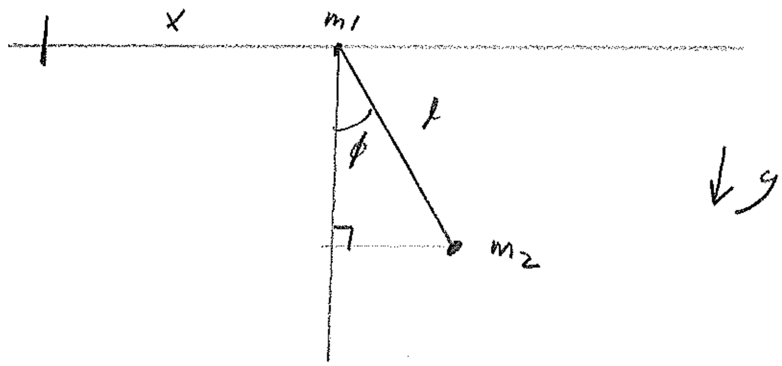
$$= \frac{1}{2}\frac{F}{l} r(l+r)\phi^2$$

$$= \frac{1}{2} k r^2 \phi^2 \quad \text{where } k = \frac{F}{lr}(l+r)$$

$$L = T - U = \frac{1}{2} m r^2 \dot{\phi}^2 + \frac{1}{2} k r^2 \phi^2$$

$$\rightarrow \quad \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{F}{m l r}(l+r)}$$

Sec 21, Prob 5



$x_1 = x$
$y_1 = 0$

$x_2 = x + l\sin\phi$
$y_2 = -l\cos\phi$

$U = m_2 g y_2 = -m_2 g l\cos\phi$

$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$

$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left((\dot{x} + l\cos\phi\,\dot\phi)^2 + l^2\sin^2\phi\,\dot\phi^2 \right)$

$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left(\dot{x}^2 + 2l\cos\phi\,\dot{x}\dot\phi + l^2\cos^2\phi\,\dot\phi^2 + l^2\sin^2\phi\,\dot\phi^2 \right)$

$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + m_2 l\cos\phi\,\dot{x}\dot\phi + \frac{1}{2} m_2 l^2 \dot\phi^2$

$L = T - U$

$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2\dot\phi^2 + m_2 l\cos\phi\,\dot{x}\dot\phi + m_2 g l\cos\phi$

Now: L does not depend explicitly on x, $\rightarrow$

$P_x = \frac{\partial L}{\partial \dot{x}} = \text{const}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} = 0$$

~~Solution~~

$$p_x = \frac{\partial L}{\partial \dot{x}} \quad (= \text{const})$$

$$= (m_1 + m_2)\dot{x} + m_2 l \cos\phi \, \dot{\phi}$$

$$= \frac{d}{dt}\left[(m_1+m_2) x + m_2 l \sin\phi \right]$$

$$= \frac{d}{dt}\left[X_{com} \right]$$

☆ Let us choose a reference frame where $x_{com} = 0$

Then,

$$(m_1 + m_2) x + m_2 l \sin\phi = 0$$

$$\boxed{m_1 x + m_2 (x + l \sin\phi) = 0}$$

In this frame

$$m_1 \dot{x} + m_2(\dot{x} + l \cos\phi\, \dot{\phi}) = 0$$

$$\boxed{\dot{x} = \frac{-m_2 l \cos\phi\, \dot{\phi}}{m_1 + m_2}}$$

$$\rightarrow \quad T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 l \cos\phi\, \dot{x}\dot{\phi}$$

$$= \frac{1}{2}(m_1 + m_2)\frac{m_2^2 l^2 \cos^2\phi\, \dot{\phi}^2}{(m_1+m_2)^2} + \frac{1}{2} m_2 l^2 \dot{\phi}^2 - \frac{m_2^2 l^2 \cos^2\phi\, \dot{\phi}^2}{m_1 + m_2}$$

$$= \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left[1 - \frac{m_2 \cos^2\phi}{m_1 + m_2} \right]$$

so

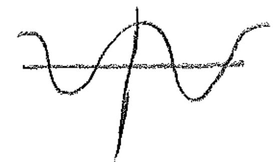
$$\boxed{L = \frac{1}{2} m_2 l^2 \dot{\phi}^2 \left[1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2\phi \right] + m_2 g l \cos\phi}$$

↑

only 1 DOF

Now restrict to small oscillations around a stable equilibrium position.

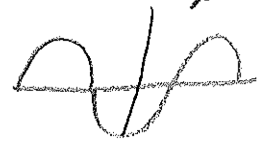
$\cos\phi$



$$U = -m_2 g l \cos\phi$$

$\phi = 0$, stable equilibrium

$-\cos\phi$



$$U(\phi) = -m_2 g l \cos\phi$$

$$\simeq -m_2 g l \left[1 - \frac{\phi^2}{2} \right]$$

$$= \underbrace{-m_2 g l}_{\text{const}} + \frac{1}{2} m_2 g l \phi^2$$

$$T(\phi) = \frac{1}{2} m_2 l^2 \underset{\substack{\uparrow \\ 2^{nd}\text{ order}}}{\dot\phi^2} \left[1 - \left(\frac{m_2}{m_1+m_2}\right) \underset{\substack{\uparrow \\ \text{replace by } 1}}{\cos^2\phi} \right]$$

$$\simeq \frac{1}{2} m_2 l^2 \dot\phi^2 \left[1 - \frac{m_2}{m_1+m_2} \right]$$

$$= \frac{1}{2} m_2 l^2 \dot\phi^2 \left(\frac{m_1}{m_1+m_2} \right)$$

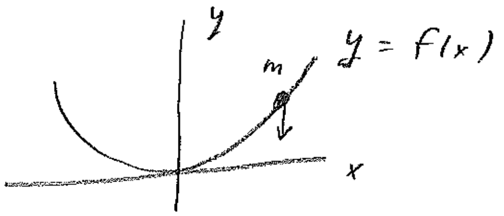
$$= \frac{1}{2} m l^2 \dot\phi^2 \qquad \text{where} \qquad m \equiv \frac{m_1 m_2}{m_1+m_2}$$

$$\rightarrow L = \frac{1}{2} m l^2 \dot\phi^2 - \frac{1}{2} m_2 g l \phi^2$$

$$\omega = \sqrt{\frac{m_2 g l}{m l^2}} = \sqrt{\frac{g}{l} \frac{\cancel{m_2}}{m_1 \cancel{m_2}} (m_1+m_2)}$$

$$= \sqrt{\frac{g}{l} \left(\frac{m_1+m_2}{m_1} \right)}$$

Sec 21, Prob 6:



Particle constrained to move along the curve $y = f(x)$.

$U = mgy = mg f(x)$

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y})^2$

$= \frac{1}{2} m(\dot{x}^2 + (f'(x))^2 \dot{x}^2)$

$= \frac{1}{2} m \dot{x}^2 [1 + (f'(x))^2]$

$L = \frac{1}{2} m\dot{x}^2 [1 + (f'(x))^2] - mg f(x)$

No explicit t dependence

$E = \dot{x} \frac{\partial L}{\partial \dot{x}} - L = \text{const}$

$= \frac{1}{2} m \dot{x}^2 [1 + (f'(x))^2] + mg f(x)$

$$\sqrt{\frac{\frac{2}{m}(E - mgy)}{1 + y'^2}} = \frac{dx}{dt}$$

$$\int dt = \int dx \sqrt{\frac{1+y'^2}{\frac{2}{m}(E - mgy)}} + \text{const}$$

$$T(E) = 4\sqrt{\frac{m}{2}} \int_0^{x_0} dx \sqrt{\frac{1+y'^2}{E - mgy}} = \frac{4}{\sqrt{2g}} \int_0^{x_0} dx \sqrt{\frac{1+y'^2}{y_0 - y}}$$

where $E = mgy_0$, $y_0 \equiv f(x_0)$

☆ want period to be independent of ~~independent of~~ starting position (x_0, y_0)

Note:

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m \dot{s}^2 \qquad \text{where} \quad ds^2 = dx^2 + dy^2$$
$$ds = \text{arc length}$$

$$U = mg\, y(s)$$

$$L = T - U = \frac{1}{2} m \dot{s}^2 - mgy$$

For period to be independent of s_0, need

$$U = mgy = \frac{1}{2} k s^2$$

$$\rightarrow \boxed{y = \frac{1}{2}\left(\frac{k}{mg}\right) s^2 = \frac{1}{2} \frac{\omega^2}{g} s^2} \rightarrow dy = \frac{\omega^2}{g} s\, ds$$

---

what is $x = x(s)$?

$$ds^2 = dx^2 + dy^2$$
$$= dx^2 + \frac{\omega^4}{g^2} s^2 ds^2$$

$$\rightarrow dx^2 = \left(1 - \frac{\omega^4}{g^2} s^2\right) ds^2$$

$$dx = \sqrt{1 - \frac{\omega^4}{g^2} s^2}\, ds$$

Thus,

$$x = \int \sqrt{1 - \left(\frac{\omega^4}{g^2}\right) s^2}\, ds \quad \underbrace{+ \text{const}}_{= 0 \text{ for } x(0) = 0}$$

$$= \int_0^s \sqrt{1 - \left(\frac{\omega^4}{g^2}\right) s^2}\, ds$$

Let $s = \frac{g}{\omega^2} \sin\theta \rightarrow 1 - \frac{\omega^4}{g^2} s^2 = 1 - \sin^2\theta = \cos^2\theta$

$$ds = \frac{g}{\omega^2} \cos\theta\, d\theta$$

$s = 0 \rightarrow \theta = 0$, $\theta = \sin^{-1}\left(\frac{\omega^2}{g} s\right)$

$$x = \int_0^{\sin^{-1}\left(\frac{w^2}{g}s\right)} \cos\theta \, \frac{g}{w^2} \cos\theta \, d\theta$$

$$= \frac{g}{w^2} \int_0^{\sin^{-1}\left(\frac{w^2}{g}s\right)} \cos^2\theta \, d\theta$$

Now:

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$\cos^2 x = \frac{1}{2}\left(1 + \cos 2x\right)$$

Thus,

$$x = \frac{g}{2w^2} \int_0^{\sin^{-1}\left(\frac{w^2}{g}s\right)} \left(1 + \cos 2\theta\right) d\theta$$

$$= \frac{g}{2w^2}\left[\theta + \frac{1}{2}\sin 2\theta\right]\Bigg|_0^{\sin^{-1}\left(\frac{w^2}{g}s\right)}$$

$$\boxed{x = \frac{g}{2w^2}\left[\sin^{-1}\left(\frac{w^2}{g}s\right) + \frac{1}{2}\sin\left[2\sin^{-1}\left(\frac{w^2}{g}s\right)\right]\right]}$$

see page 6

Find instead a relationship between x and y

$$ds^2 = dx^2 + dy^2 \quad , \quad dy = \frac{w^2}{g} s \, ds$$

$$\frac{g^2}{w^4 s^2} dy^2 = dx^2 + dy^2$$

$$\frac{g^2}{w^4} \frac{1}{2} \frac{w^2}{g} \frac{1}{y} dy^2 = dx^2 + dy^2$$

$$\frac{g}{2w^2} \frac{1}{y} dy^2 = dx^2 + dy^2$$

$$dx = dy\sqrt{\frac{g}{2w^2y} - 1} \rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{\frac{g}{2w^2y} - 1}}$$

$$= \sqrt{\frac{2w^2y}{g - 2w^2y}}$$

$$= \sqrt{\frac{\left(\frac{2w^2}{g}\right)y}{1 - \left(\frac{2w^2}{g}\right)y}}$$

$$= \sqrt{\frac{Ay}{1 - Ay}}$$

$$\boxed{A = \frac{2w^2}{g}}$$

$$\rightarrow \quad x = \int dx = \int dy\sqrt{\frac{g}{2w^2y} - 1}$$

$$= \int dy\sqrt{\frac{1}{Ay} - 1}$$

$$= \int_0^y dy\sqrt{\frac{1 - Ay}{Ay}}$$

To evaluate this integral, take

$$\boxed{y = \frac{1}{2A}(1 - \cos\theta)} \quad y = 0,\ \theta = 0$$

$$\rightarrow \quad dy = \frac{\sin\theta}{2A}\,d\theta = \frac{\sqrt{1 - \cos^2\theta}}{2A}\,d\theta$$

$$1 - Ay = 1 - \frac{1}{2}(1 - \cos\theta)$$

$$= \frac{1}{2}(1 + \cos\theta)$$

$$\rightarrow \quad x = \int \frac{\sqrt{1 - \cos^2\theta}}{2A}\,d\theta\sqrt{\frac{\cancel{\tfrac{1}{2}}(1 + \cos\theta)}{\cancel{\tfrac{1}{2}}(1 - \cos\theta)}}$$

$$= \frac{1}{2A}\int (1 + \cos\theta)\,d\theta$$

$$= \frac{1}{2A}(\theta + \sin\theta)$$

These two equations $y = y(\theta)$, $x = x(\theta)$ are the parametric equations for a cycloid

Relationship between s and θ:

$$x = \frac{1}{2A}(\theta + \sin\theta)$$
$$y = \frac{1}{2A}(1 - \cos\theta)$$
$$dx = \frac{1}{2A}(1 + \cos\theta)\, d\theta$$
$$dy = \frac{1}{2A}\sin\theta\, d\theta$$

$$ds^2 = dx^2 + dy^2$$

$$\rightarrow ds = \sqrt{dx^2 + dy^2}$$

$$= \frac{d\theta}{2A}\sqrt{(1+\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{d\theta}{2A}\sqrt{1 + \cos^2\theta + 2\cos\theta + \sin^2\theta}$$

$$= \frac{d\theta}{2A}\sqrt{2(1+\cos\theta)}$$

Now: $1 + \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)$

$$= \frac{d\theta}{2A}\sqrt{2 \cdot 2\cos^2\left(\frac{\theta}{2}\right)}$$

$$= \frac{d\theta}{A}\cos\left(\frac{\theta}{2}\right)$$

$$s = \frac{1}{A}\int d\theta \cos\left(\frac{\theta}{2}\right)$$

$$= \frac{1}{A}\, 2\sin\left(\frac{\theta}{2}\right)$$

$$= \frac{g}{\cancel{2}\omega^2} \cdot \cancel{2}\sin\left(\frac{\theta}{2}\right)$$

$$\boxed{= \frac{g}{\omega^2}\sin\left(\frac{\theta}{2}\right)}$$

Check:

$$y = \frac{1}{2}\frac{w^2}{g}s^2$$
$$= \frac{1}{2}\frac{w^2}{g}\frac{g^2}{w^4}\sin^2\left(\frac{\theta}{2}\right)$$
$$= \frac{1}{2}\frac{g}{w^2}\sin^2\left(\frac{\theta}{2}\right)$$
$$= \frac{1}{A}\sin^2\left(\frac{\theta}{2}\right)$$
$$= \frac{1}{A}\frac{1}{2}(1-\cos\theta)$$
$$= \frac{1}{2A}(1-\cos\theta) \quad \checkmark$$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$
$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$x = \frac{g}{2w^2}\left[\sin^{-1}\left(\frac{w^2}{g}s\right) + \frac{1}{2}\sin\left[2\sin^{-1}\left(\frac{w^2 s}{g}\right)\right]\right]$$

Now: $\frac{w^2}{g}s = \sin\left(\frac{\theta}{2}\right) \rightarrow \sin^{-1}\left(\frac{w^2}{g}s\right) = \sin^{-1}\left(\sin\left(\frac{\theta}{2}\right)\right)$

$$= \frac{\theta}{2}$$

and $\sin\left[2\sin^{-1}\left(\frac{w^2 s}{g}\right)\right] = \sin\left[\cancel{2}\cdot\frac{\theta}{\cancel{2}}\right] = \sin\theta$

Thus, $$x = \frac{1}{A}\left[\frac{\theta}{2} + \frac{1}{2}\sin\theta\right]$$
$$= \frac{1}{2A}(\theta + \sin\theta) \quad \checkmark$$