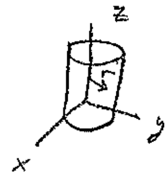
Section 9: (Prob 1)

$$\vec{M} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v}$$
$$M_x = m(y\dot{z} - z\dot{y})$$
$$M_y = m(z\dot{x} - x\dot{z})$$
$$M_z = m(x\dot{y} - y\dot{x})$$

Cylindrical coords (r, ϕ, z)

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$



$$r^2 = x^2 + y^2$$

$$\dot{x} = \dot{r}c\phi - rs\phi\,\dot{\phi}$$
$$\dot{y} = \dot{r}s\phi + rc\phi\,\dot{\phi}$$
$$\dot{z} = \dot{z}$$

$$M_x = m(\,rs\phi\,\dot{z} - z(\dot{r}s\phi + rc\phi\,\dot{\phi}))$$
$$= m(\,(r\dot{z} - z\dot{r})s\phi - rzc\phi\,\dot{\phi})$$
$$M_y = m(\,z(\dot{r}c\phi - rs\phi\,\dot{\phi}) - rc\phi\,\dot{z})$$
$$= m(\,-(r\dot{z} - z\dot{r})c\phi - rzs\phi\,\dot{\phi})$$
$$M_z = m[\,rc\phi(\cancel{\dot{r}s\phi} + rc\phi\,\dot{\phi}) - rs\phi(\cancel{\dot{r}c\phi} - rs\phi\,\dot{\phi})]$$
$$= mr^2[\,(c^2\phi + s^2\phi)\dot{\phi}\,]$$
$$= mr^2\dot{\phi}$$

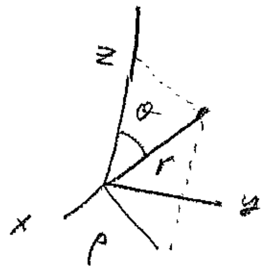
$$M^2 = M_x^2 + M_y^2 + M_z^2$$
$$= m^2[\,(\;)^2 s^2\phi + r^2z^2c^2\phi\,\dot{\phi}^2 \cancel{- 2rz\,s\phi\,c\phi\,(\;)\dot{\phi}}$$
$$+ (\;)^2c^2\phi + r^2z^2s^2\phi\,\dot{\phi}^2 \cancel{+ 2rz\,s\phi\,c\phi\,(\;)\dot{\phi}}$$
$$+ r^4\dot{\phi}^2\,]$$
$$= m^2[\,(\;)^2 + r^2z^2\dot{\phi}^2 + r^4\dot{\phi}^2\,]$$
$$= m^2[\,(r\dot{z} - z\dot{r})^2 + r^2(r^2+z^2)\dot{\phi}^2\,]$$

NOTE:

(☆) convert to sph. polar coordinate

Notation: cylindrical (ρ, ϕ, z)

spherical (r, θ, z)



$$M^2 = m^2\left[(\rho\dot z - z\dot\rho)^2 + \rho^2(\rho^2+z^2)\dot\phi^2\right]$$

$\rho = r\sin\theta$

$z = r\cos\theta$

$$\dot\rho = \dot r\, s\theta + r\, c\theta\,\dot\theta$$
$$\dot z = \dot r\, c\theta - r\, s\theta\,\dot\theta$$

Thus
$$\rho\dot z - z\dot\rho = r s\theta(\cancel{\dot r c\theta} - r s\theta\,\dot\theta) - r c\theta(\cancel{\dot r s\theta} + r c\theta\,\dot\theta)$$
$$= -r^2 s^2\theta\,\dot\theta - r^2 c^2\theta\,\dot\theta$$
$$= -r^2\dot\theta$$

$$\to M^2 = m^2\left[r^4\dot\theta^2 + r^2 s^2\theta(r^2 s^2\theta + r^2 c^2\theta)\dot\phi^2\right]$$
$$= m^2\left[r^4\dot\theta^2 + r^4 s^2\theta\,\dot\phi^2\right]$$
$$= m^2 r^4(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)$$

$$M_x = m\left((\rho\dot z - z\dot\rho)s\phi - \rho z\, c\phi\,\dot\phi\right)$$
$$= m(-r^2\dot\theta\, s\phi - r^2 s\theta\, c\theta\, c\phi\,\dot\phi)$$
$$= m r^2(-s\phi\,\dot\theta - s\theta\, c\theta\, c\phi\,\dot\phi)$$

$$M_y = m\left(-(\rho\dot z - z\dot\rho)c\phi - \rho z\, s\phi\,\dot\phi\right)$$
$$= m(r^2\dot\theta\, c\phi - r^2 s\theta\, c\theta\, s\phi\,\dot\phi)$$
$$= m r^2\left[c\phi\,\dot\theta - s\theta\, c\theta\, s\phi\,\dot\phi\right]$$

$$M_z = m\rho^2\dot\phi$$
$$= m r^2\sin^2\theta\,\dot\phi$$

Section 9 (Prob 2)

$$\vec{M} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v}$$
$$M_x = m(y\dot{z} - z\dot{y})$$
$$M_y = m(z\dot{x} - x\dot{z})$$
$$M_z = m(x\dot{y} - y\dot{x})$$

sph. polar

$$x = r s\theta c\phi$$
$$y = r s\theta s\phi$$
$$z = r c\theta$$

$$\rightarrow \dot{x} = \dot{r} s\theta c\phi + r c\theta c\phi\,\dot{\theta} - r s\theta s\phi\,\dot{\phi}$$
$$\dot{y} = \dot{r} s\theta s\phi + r c\theta s\phi\,\dot{\theta} + r s\theta c\phi\,\dot{\phi}$$
$$\dot{z} = \dot{r} c\theta - r s\theta\,\dot{\theta}$$

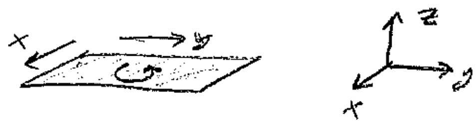
$$M_z = m\left[r s\theta c\phi\left(\cancel{\dot{r} s\theta s\phi} + r c\theta s\phi\,\dot{\theta} + r s\theta c\phi\,\dot{\phi}\right) - r s\theta s\phi\left(\cancel{\dot{r} s\theta c\phi} + r c\theta c\phi\,\dot{\theta} - r s\theta s\phi\,\dot{\phi}\right)\right]$$
$$= mr^2\left[\cancel{s\theta c\theta s\phi c\phi\,\dot{\theta}} + s^2\theta c^2\phi\,\dot{\phi} - \cancel{s\theta c\theta s\phi c\phi\,\dot{\theta}} + s^2\theta s^2\phi\,\dot{\phi}\right]$$
$$= mr^2\sin^2\theta\,\dot{\phi}$$

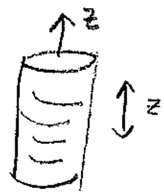
$$M_x = m\left[r s\theta s\phi\left(\cancel{\dot{r} c\theta} - r s\theta\,\dot{\theta}\right) - r c\theta\left(\cancel{\dot{r} s\theta s\phi} + r c\theta s\phi\,\dot{\theta} + r s\theta c\phi\,\dot{\phi}\right)\right]$$
$$= mr^2\left[-s^2\theta s\phi\,\dot{\theta} - c^2\theta s\phi\,\dot{\theta} - s\theta c\theta c\phi\,\dot{\phi}\right]$$
$$= mr^2\left[-s\phi\,\dot{\theta} - s\theta c\theta c\phi\,\dot{\phi}\right] \checkmark$$

$$M_y = m\left[r c\theta\left(\cancel{\dot{r} s\theta c\phi} + r c\theta c\phi\,\dot{\theta} - r s\theta s\phi\,\dot{\phi}\right) - r s\theta c\phi\left(\cancel{\dot{r} c\theta} - r s\theta\,\dot{\theta}\right)\right]$$
$$= mr^2\left[c\phi\,\dot{\theta} - s\theta c\theta s\phi\,\dot{\phi}\right] \checkmark$$

$$M^2 = M_x^2 + M_y^2 + M_z^2$$
$$= m^2r^4\left[\sin^4\theta\,\dot{\phi}^2 + s^2\phi\,\dot{\theta}^2 + s^2\theta c^2\theta c^2\phi\,\dot{\phi}^2 + \cancel{2 c\theta s\theta s\phi c\phi\,\dot{\theta}\dot{\phi}} + c^2\phi\,\dot{\theta}^2 + s^2\theta c^2\theta s^2\phi\,\dot{\phi}^2 - \cancel{2 s\theta c\theta s\phi c\phi\,\dot{\theta}\dot{\phi}}\right]$$
$$= m^2r^4\left[s^4\theta\,\dot{\phi}^2 + s^2\theta c^2\theta\,\dot{\phi}^2 + \dot{\theta}^2\right]$$
$$= m^2r^4\left[\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right]$$

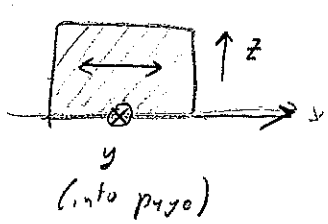
Section 9: (Prob 3)

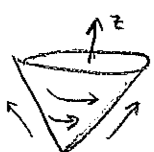
(a)  $\boxed{P_x, P_y, M_z}$

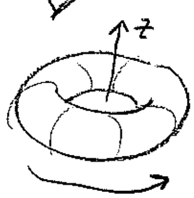
(b)  $\boxed{P_z, M_z}$

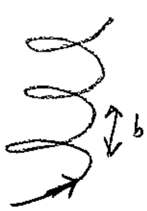
(c)  $\boxed{P_z}$

(d)  $\boxed{M_z}$ about line connecting the two points

(e)  $\boxed{P_x}$

(f)  only $\boxed{M_z}$

(g)  only $\boxed{M_z}$

(h) 

$$\vec{V} = 2\pi a\,\hat{\phi} + b\,\hat{z}$$

$$\vec{P}\cdot\vec{V} = 2\pi a\,P_\phi + b\,P_z$$

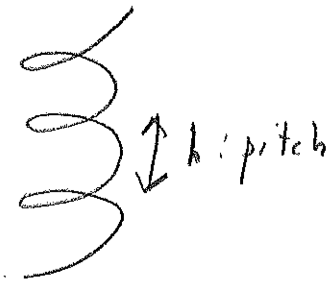
$$= 2\pi a\,M_z + b\,P_z$$

$$= \boxed{2\pi M_z + h\,P_z}$$ where $h \equiv \frac{b}{a}$

$= $ pitch

top view

(b)



$\delta L = 0$ for transformation that take $\phi \to \phi + \delta\phi$

$z \to z + \delta z$

$$\delta z = \left(\frac{\delta\phi}{2\pi}\right) h$$

Thus, $$\delta L = 0 = \left(\frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial z}\delta z\right)$$

$$= \delta\phi\left[\frac{\partial L}{\partial \phi} + \frac{\partial L}{\partial z}\frac{h}{2\pi}\right]$$

$$\rightarrow \quad 0 = \frac{\partial L}{\partial \phi} + \frac{\partial L}{\partial z}\frac{h}{2\pi}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot z}\right)\frac{h}{2\pi}$$

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot\phi} + \frac{h}{2\pi}\frac{\partial L}{\partial \dot z}\right]$$

$$= \frac{d}{dt}\Big[\underbrace{M_z + \frac{h}{2\pi}P_z}_{\text{conserved.}}\Big]$$