

11. Musical Scales, Temperament, Elementary Music Theory

PURPOSE AND BACKGROUND

The arrangement of musical notes today is the result of centuries of changes in both musical style and taste. As music became more complex, the tuning of instruments, for instance the piano, took various forms. The older styles of tuning are primarily of historical interest, but it is still worthwhile to understand how music has arrived at the modern *equal temperament*. In tuning any instrument, a starting pitch must be chosen. For a violin, this pitch is concert A_4 ($f = 440$ Hz). The rest of the strings are tuned to this pitch. The manner in which they are tuned is called the *musical temperament*. A violin has only four strings and therefore tuning is not difficult. But how is a piano tuned with so many keys? What should be the starting pitch, and how do you tune the other keys? These questions arose from the need for standardization. Here we will discuss briefly some features of older temperaments, their benefits and drawbacks. Equal temperament will be treated in more detail.

I Pythagorean Temperament

When music started using multiple parts, chords became integral objects of melodic structure. Initially, *perfect fifths* were used to tune all notes on a keyboard. A perfect fifth is rather easy to discern, even by the musically challenged. The term *fifth* refers to the fifth note in a major scale. For example, in the C-major scale C, D, E, F, G, A, B, the note “G” is the fifth. The frequency ratio between the perfect fifth (G) and the *tonic* (C) is 3:2.

The perfect fifth was the first ratio to be used in tuning. Early piano tuners would first tune middle C_4 , then the notes G_4 , D_5 , A_5 , E_6 , B_6 , F_7^\sharp all in perfect fifths above C_4 . They also tuned F_3 , B_2^\flat , E_2^\flat , A_1^\flat , D_1^\flat in perfect fifths below C_4 . F_6^\sharp is then tuned by a perfect octave down from F_7^\sharp , and C_7^\sharp is tuned a perfect fifth above F_6^\sharp . In this way, the twelve different keys in the *chromatic scale* were determined. All other keys were tuned from these keys by octaves with frequencies in the ratio 2:1. The resulting temperament was known as the *Pythagorean temperament*.

1. Middle C is commonly taken as $C_4 = 261.63$ Hz. What are the frequencies of a perfect fifth above middle C (G_4) and a perfect fifth below middle C (F_3)?
2. Consider the notes D_1^\flat and C_8^\sharp tuned by perfect fifths according to the Pythagorean temperament. Since all 12 keys in the chromatic scale are determined in this manner, there are 12 jumps of $3/2$ in frequency, or twelve perfect fifths, from D_1^\flat to C_8^\sharp . Calculate the frequency ratio C_8^\sharp/D_1^\flat .
3. D^\flat and C^\sharp are the same key on the piano. We thus would expect D_8^\flat and C_8^\sharp to have the same pitch. (Note that C_8^\sharp is one half step above the highest key on the piano, but answer this question anyway.) Going from D_1^\flat to D_8^\flat in octaves spans 7 octaves. Each octave increases the frequency by a factor of 2. Calculate the frequency ratio D_8^\flat/D_1^\flat .
4. Questions 2 and 3 arrive at the same notes by two different means, one by 12 perfect fifths, the other by 7 octaves. How much do the frequency ratios in questions 2 and 3 differ?

The discrepancy is the *Pythagorean comma*. Octaves are retained as being perfect, with a ratio of 2:1. But the fifths must be re-tuned for 12 fifths to equal 7 octaves. In the Pythagorean temperament, all fifths are kept perfect but the last one. The final fifth sounds very bad and is known as the “wolf fifth” because of its growling sound and beats. This became unacceptable as music evolved.

II Just Temperament and Mean Tone Temperament

When music began to have more differentiated harmonies, the inclusion of a *perfect third* with a frequency ratio 5:4 became increasingly important. The Pythagorean temperament had particularly bad thirds and so its use was slowly discarded and replaced by other temperaments.

The *just temperament* tuned a perfect fifth above and below middle C, resulting in G_4 and F_3 , respectively. It also tuned perfect thirds above F_3 and above middle C. In this way, *major triads*, composed of the tonic, third, and fifth notes on a scale, sounded perfectly in tune. Tuning the other eight keys in just temperament is slightly more complicated than in the Pythagorean temperament.

1. Calculate the frequencies of a perfect third above middle C (E_4) and a perfect third below middle C (A_3^b).

As music expanded into *minor keys*, one of the serious weaknesses of just temperament became painfully apparent. Minor triads sound especially bad in this temperament. The error again is all concentrated in one area. In this case, above E^b , the pair F^\sharp and D^b should be a perfect fifth, but each note is reached by two different routes. Therefore this fifth is off by a significant amount. The *mean tone temperament* was created to compensate for the error by spreading it over all the fifths, not just one. This too, however, caused problems when musical keys farther from the key of C were used.

III Equal Temperament

All these problems eventually led to a resolution with the introduction of *equal temperament* or the *equal-tempered scale*. Here the error is spread *equally across all twelve notes* in the chromatic sequence. All keys then sound the same. However, there are no perfect fifths or thirds or any other chords, except the octave. The twelve keys in equal temperament are spaced by equal frequency ratios. Since an octave must have a 2:1 ratio, the interval between keys must be multiplied 12 times in order to give a value of two. This interval therefore is the 12th root of two, namely $2^{1/12}$. The frequency of each note is multiplied by this number to give the next note one half step or semitone higher.

1. Use a calculator to compute the value of $2^{1/12}$. Calculate the frequencies of the remaining 11 notes of the chromatic scale, starting with middle C. Insert the values in Table 1 below.

Note	Frequency	Note	Frequency
C_4	261.63Hz	F^\sharp_4/G^b_4	
C^\sharp_4/D^b_4		G_4	
D_4		G^\sharp_4/A^b_4	
D^\sharp_4/E^b_4		A_4	
E_4		A^\sharp_4/B^b_4	
F_4		B_4	

Table 1: Frequencies of the notes of the chromatic scale in equal temperament, starting with C_4 .

- Compare your calculated values in Table 1 with the frequencies shown in Figure 1 for the piano keyboard. Equal temperament is the way all pianos are tuned today, including modern electronic keyboards.
- Compare the frequencies of the C-major triad in just temperament and equal temperament. For just temperament, take the answers for the perfect fifth G_4 from Part I, Question 1 and perfect third E_4 from Part II, Question 1. For equal temperament, use values from Table 1. Insert all values in Table 2 below.

Just		Equal	
Note	Frequency	Note	Frequency
C_4	261.63 Hz	C_4	261.63 Hz
E_4		E_4	
G_4		G_4	

Table 2: Frequencies of the major triad based on C_4 , in just and equal temperament.

- Which triad would you think sounds “better”, the one tuned in just temperament or the one tuned in equal temperament? Why?

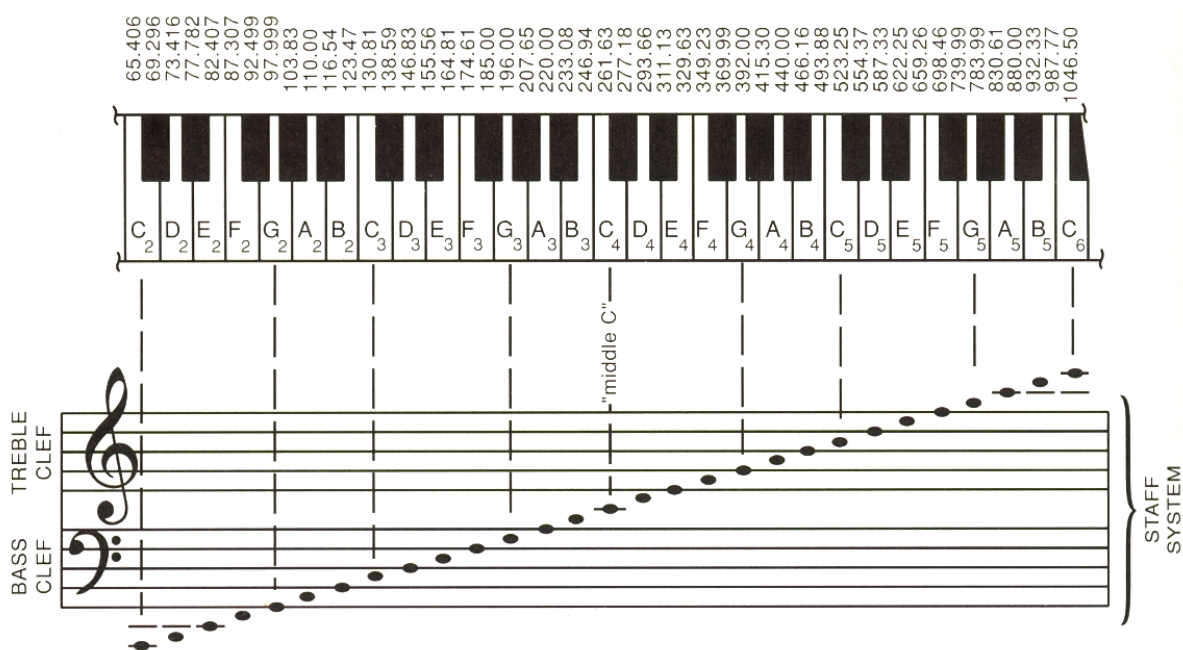


Figure 1: Frequencies of the equal temperament scale. (From “Physics of Sound,” by R.E. Berg and D.G. Stork.)

IV Tuning a Violin and Beats

String instruments in chamber music often are tuned to Pythagorean temperament. The A string of a violin is first tuned *beatless* to concert $A_4 = 440$ Hz by comparing, for instance, with a tuning fork. The term *beatless* refers to the absence of “beats” that would otherwise be heard if a note were slightly out of tune with another. (Recall that if two notes with frequencies $f_1 \approx f_2$ are played simultaneously, you can hear two things: A tone with a frequency equal to the average of f_1 and f_2 , and a slow amplitude variation “beating” with the difference frequency $\Delta f = |f_2 - f_1|$. This difference frequency is the *beat frequency*.)

Once the A_4 string on the violin is tuned to 440 Hz, the E_5 string, a fifth higher than A_4 , is tuned by beats as follows: The 3rd harmonic of A_4 , i.e., $E_6 = 1320$ Hz, is tuned beatless with the 2nd harmonic of the E_5 string, which again is $E_6 = 1320$ Hz. In the same manner, the D_4 string, which is a fifth down from the A_4 string, is tuned beatless with the A_4 string. Finally, the G_3 string, which is a fifth down from the D_4 string, is tuned beatless with the D_4 string. An experienced string-instrument player can easily hear the beats between two strings that are out of tune and thus tune them to be beatless. All strings are tuned by *perfect fifths* in this way according to the Pythagorean temperament. The resulting sound from a string ensemble can be very clean and pleasing. But slight dissonances may arise when string instruments tuned to Pythagorean temperament and an equal-tempered piano play together.

1. What are the frequencies of the four strings on the violin tuned to Pythagorean temperament, starting with $A_4 = 440$ Hz? Put your entries in the first column of Table 3 below. For the second column, use the relevant frequency values from Figure 1.

String	Violin Tuned f	Equal Temp. f
G_3		
D_4		
A_4		
E_5		

Table 3: Frequencies of the four open violin strings in Pythagorean temperament and comparison with equal temperament.

2. Start with middle $C_4 = 261.63$ Hz and calculate the frequency of E_4 in Pythagorean temperament. Compare this frequency with the value of E_4 in equal temperament (you can get this value from Figure 1). If both of these notes were played simultaneously, would you expect to hear beats? If so, what would you hear for the beat frequency?