Sec 20, Prob 1

Start from (18.4)

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{\rho/r^2 \, dr}{\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2U}{mv_\infty^2}}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho/r^2 \, dr}{\sqrt{\left(1 - \frac{\rho^2}{r^2}\right) + \epsilon}} \qquad \left[\epsilon \equiv \left| \frac{2U}{mv_\infty^2} \right| << 1 \right]$$

$$= \int_{r_{min}}^{\infty} \frac{\rho/r^2 \, dr}{\sqrt{1 - \frac{\rho^2}{r^2}} \sqrt{1 + \frac{\epsilon}{(1 - \rho^2/r^2)}}}$$

$$\simeq \int_{r_{min}}^{\infty} \frac{\rho/r^2 \, dr}{\sqrt{1 - \frac{\rho^2}{r^2}}} \left[1 - \frac{1}{2} \frac{\epsilon}{(1 - \rho^2/r^2)} \right]$$

$$= \int_{r_{min}}^{\infty} \frac{\rho/r^2 \, dr}{\sqrt{1 - \rho^2/r^2}} + \int_{r_{min}}^{\infty} \frac{\rho/r^2 \, dr}{\left(1 - \frac{\rho^2}{r^2}\right)^{3/2}} \frac{U}{mv_\infty^2}$$

In absence of a potential $r_{min} = \rho$ and for $\left| \frac{U}{mv_\infty^2} \right| << 1$, $r_{min} \simeq \rho$

1st integral

$$= \int_{\rho}^{\infty} \frac{\rho/r^2 \, dr}{\sqrt{1 - \rho^2/r^2}}$$

let: $u = \frac{1}{r}$

$du = -\frac{1}{r^2} dr$

$$= \int_{0}^{\frac{1}{\rho}} \frac{\rho \, du}{\sqrt{1 - \rho^2 u^2}}$$

$$= \int_{0}^{\pi/2} \frac{\cancel{\rho} \frac{1}{\cancel{\rho}} \cancel{\cos\theta} \, d\theta}{\sqrt{1 - \sin^2\theta}}$$

let: $u = \frac{1}{\rho} \sin\theta$

$du = \frac{1}{\rho} \cos\theta \, d\theta$

$u = 0 \longleftrightarrow \theta = 0$

$u = \frac{1}{\rho} \longleftrightarrow \theta = \pi/2$

$$= \int_{0}^{\pi/2} d\theta$$

$$= \boxed{\pi/2}$$

2nd integral

$$= \int_{r_{min}}^{\infty} \frac{\rho/r^2 \, dr}{\left(1 - \frac{\rho^2}{r^2}\right)^{3/2}} \frac{U(r)}{m v_\infty^2}$$

$$= \frac{d}{d\rho} \left[\frac{1}{m v_\infty^2} \int_{r_{min}}^{\infty} dr \frac{U(r)}{\sqrt{1 - \frac{\rho^2}{r^2}}} \right]$$

$$= \frac{1}{m v_\infty^2} \frac{d}{d\rho} \left[\int_{\rho}^{\infty} dr \frac{U(r) \, r}{\sqrt{r^2 - \rho^2}} \right]$$

Now: we can integrate $\int_\rho^\infty dr \frac{U(r)\, r}{\sqrt{r^2-\rho^2}}$ by parts.

$$\int_\rho^\infty dr \frac{U(r)\, r}{\sqrt{r^2-\rho^2}} = uv\Big|_\rho^\infty - \int_\rho^\infty v\, du$$

let: $u = U(r) \rightarrow du = dU(r) = \frac{dU}{dr}dr$

$dv = \frac{r\, dr}{\sqrt{r^2-\rho^2}} \rightarrow v = \sqrt{r^2-\rho^2}$

$$= \underbrace{U(r)\sqrt{r^2-\rho^2}\Big|_\rho^\infty}_{0} - \int_\rho^\infty \sqrt{r^2-\rho^2}\left(\frac{dU}{dr}\right)dr$$

since $U(\infty) \rightarrow 0$
faster than $\sqrt{r^2-\rho^2} \rightarrow \infty$
(assume this)

$$= -\int_\rho^\infty dr \left(\frac{dU}{dr}\right)\sqrt{r^2-\rho^2}$$

Thus,

$$\phi_0 = \frac{\pi}{2} - \frac{1}{mv_\infty^2}\frac{\partial}{\partial\rho}\left[\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\sqrt{r^2-\rho^2}\right]$$

$$= \frac{\pi}{2} + \frac{\rho}{mv_\infty^2}\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}$$

$$\boxed{\chi} = |\pi - 2\phi_0|$$

$$= \pi - 2\phi_0 \qquad \text{(repulsive)}$$

$$= \cancel{\pi} - 2\left(\cancel{\frac{\pi}{2}} + \frac{\rho}{m v_\infty^2}\int_\rho^\infty dr \left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}\right)$$

$$= \boxed{-\frac{2\rho}{m v_\infty^2}\int_\rho^\infty dr \left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}}$$

<u>Now:</u>

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi} \qquad (17.4)$$

$$\rightarrow \qquad \theta_1 \simeq \frac{m_2\chi}{m_1+m_2}$$

so

$$\boxed{\theta_1} = -\frac{2\rho}{m v_\infty^2}\left(\frac{m_2}{m_1+m_2}\right)\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}$$

$$= \boxed{\frac{-2\rho}{m_1 v_\infty^2}\int_\rho^\infty dr\left(\frac{dU}{dr}\right)\frac{1}{\sqrt{r^2-\rho^2}}} \qquad (20.3)$$

§20, Prob 2:

$$U = \frac{\alpha}{r^n} \quad (n > 0) \qquad \text{small angle scattering}$$

$$\theta_1 = \frac{-2\rho}{m_1 v_\infty^2} \int_\rho^\infty \frac{dU}{dr} \frac{dr}{\sqrt{r^2 - \rho^2}}$$

$$= \frac{+2\rho n \alpha}{m_1 v_\infty^2} \int_\rho^\infty \frac{1}{r^{n+1}} \frac{dr}{\sqrt{r^2 - \rho^2}}$$

$$= \frac{2\rho n \alpha}{m_1 v_\infty^2} \int_\rho^\infty \frac{1}{r^{n+2}} \frac{dr}{\sqrt{1 - \left(\frac{\rho}{r}\right)^2}}$$

Let: $t = \left(\frac{\rho}{r}\right)^2 \qquad r = \rho \rightarrow t = 1$

$\qquad\qquad\qquad\qquad r = \infty \rightarrow t = 0$

$$dt = \rho^2 \left(\frac{-2}{r^3}\right) dr$$

$$= \frac{-2\rho^2}{r^3} dr$$

$$\rightarrow dr = -\frac{1}{2\rho^2} r^3 \, dt$$

$$= -\frac{1}{2\rho^2} \frac{\rho^3}{t^{3/2}} dt$$

$$= -\frac{1}{2} \frac{\rho}{t^{3/2}} dt$$

$$t = \frac{\rho^2}{r^2}$$

$$r^2 = \frac{\rho^2}{t}$$

$$r = \frac{\rho}{\sqrt{t}}$$

$$\theta_1 = \frac{\cancel{2}\rho n \alpha}{m_1 v_\infty^2} \int_0^1 dt \, \frac{\rho}{\cancel{\rho} t^{3/2}} \, \frac{1}{\dfrac{\rho^{n+2}}{t^{(n+2)/2}}} \, \frac{1}{\sqrt{1-t}}$$

$$= \frac{\rho n \alpha}{m_1 v_\infty^2} \int_0^1 dt \, \rho^{-(n+1)} \, \frac{t^{\frac{n}{2}+1}}{t^{3/2}} \, \frac{1}{\sqrt{1-t}}$$

$$= \frac{\rho^{-n} n \alpha}{m_1 v_\infty^2} \int_0^1 dt \, t^{\frac{n}{2}-\frac{1}{2}} \, (1-t)^{-1/2}$$

$$= \frac{\rho^{-n} n \alpha}{m_1 v_\infty^2} \int_0^1 dt \, t^{\left(\frac{n+1}{2}\right)-1} \, (1-t)^{\frac{1}{2}-1}$$

$$= \frac{\rho^{-n} n \alpha}{m_1 v_\infty^2} \, B\left(\frac{n+1}{2}, \frac{1}{2}\right)$$

Using $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$

$$= \frac{\rho^{-n} n \alpha}{m_1 v_\infty^2} \, \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

Using $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

$$= \frac{\rho^{-n} \cancel{n} \alpha}{m_1 v_\infty^2} \, \frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{\frac{\cancel{n}}{2}\Gamma\left(\frac{n}{2}\right)}$$

Using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma(z+1) = z\Gamma(z)$

$$= \boxed{\frac{2\alpha\sqrt{\pi}}{\rho^n m_1 v_\infty^2} \, \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}}$$

Differential cross-section:

$$d\sigma = \left|\frac{d\rho}{d\theta_1}\right| \frac{\rho(\theta_1)}{\theta_1} \, d\Omega_1 ,$$

Now:

$$\rho^n = \frac{2\alpha\sqrt{\pi}}{\theta_1 m_1 v_\infty^2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

$$\rightarrow \rho = \frac{(2\alpha_1\sqrt{\pi})^{\frac{1}{n}}}{(\theta_1 m_1 v_\infty^2)^{\frac{1}{n}}} \left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{1}{n}}$$

$$\frac{d\rho}{d\theta_1} = -\frac{1}{n} \theta_1^{-\frac{1}{n}-1} \frac{(2\alpha\sqrt{\pi})^{\frac{1}{n}}}{(m_1 v_\infty^2)^{\frac{1}{n}}} \left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{1}{n}}$$

$$\rightarrow d\sigma = \frac{1}{n} \frac{\theta_1^{-\frac{1}{n}-1}}{\theta_1} \frac{(2\alpha\sqrt{\pi})^{\frac{1}{n}}}{(m_1 v_\infty^2)^{\frac{1}{n}}} \left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{1}{n}} \frac{(2\alpha)^{\frac{1}{n}}}{\theta_1^{\frac{1}{n}} (m_1 v_\infty^2)^{\frac{1}{n}}} \left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{1}{n}} d\Omega_1$$

$$= \frac{1}{n} \theta^{-\frac{2}{n}-2} \left[\frac{2\alpha\sqrt{\pi}}{m_1 v_\infty^2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]^{\frac{2}{n}} d\Omega_1 ,$$