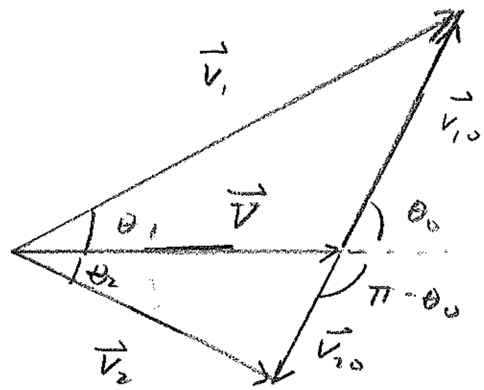
Sec 16, Prob 1



$$p_0 = m_1 v_{10} = m_2 v_{20}$$

$$v_{10} = \frac{p_0}{m_1}, \quad v_{20} = \frac{p_0}{m_2}$$

$$\frac{p_0^2}{2m} = \epsilon, \quad m = \frac{m_1 m_2}{m_1 + m_2}$$

$$\vec{v}_1 = \vec{V} + \vec{v}_{10}$$

$$\vec{v}_2 = \vec{V} + \vec{v}_{20}$$

$$\boxed{\tan\theta_1 = \frac{v_{10}\sin\theta_0}{v_{10}\cos\theta_0 + V}} \qquad (16.5)$$

$$\tan\theta_2 = \frac{v_{20}\sin(\pi - \theta_0)}{v_{20}\cos(\pi - \theta_0) + V}$$

Now: $\sin(\pi - \theta_0) = \cancel{\sin\pi}^{\,0}\cos\theta_0 - \cos\pi\sin\theta_0$

$$= \sin\theta_0$$

$$\cos(\pi - \theta_0) = \cos\pi\cos\theta_0 + \cancel{\sin\pi}^{\,0}\sin\theta_0$$

$$= -\cos\theta_0$$

$$\rightarrow \boxed{\tan\theta_2 = \frac{v_{20}\sin\theta_0}{-v_{20}\cos\theta_0 + V}}$$

To find a relationship between θ_1 and θ_2, solve above equations for $\sin\theta_0$, $\cos\theta_0$ then use $\sin^2\theta_0 + \cos^2\theta_0 = 1$ to eliminate θ_0.

$$\tan\theta_1 \left(\cos\theta_0 + \frac{V}{v_{10}}\right) = \sin\theta_0$$

$$\rightarrow \boxed{\sin\theta_0 - \tan\theta_1 \cos\theta_0 = \frac{V}{v_{10}} \tan\theta_1}$$

$$\tan\theta_2 \left(-\cos\theta_0 + \frac{V}{v_{20}}\right) = \sin\theta_0$$

$$\rightarrow \boxed{\sin\theta_0 + \tan\theta_2 \cos\theta_0 = \frac{V}{v_{20}} \tan\theta_2}$$

Matrix equation

$$\underbrace{\begin{bmatrix} 1 & -\tan\theta_1 \\ 1 & \tan\theta_2 \end{bmatrix}}_{A} \begin{bmatrix} \sin\theta_0 \\ \cos\theta_0 \end{bmatrix} = \begin{bmatrix} \frac{V}{v_{10}} \tan\theta_1 \\ \frac{V}{v_{20}} \tan\theta_2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\tan\theta_1 + \tan\theta_2} \begin{bmatrix} \tan\theta_2 & \tan\theta_1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sin\theta_0 \\ \cos\theta_0 \end{bmatrix} = \frac{1}{\tan\theta_1 + \tan\theta_2} \begin{bmatrix} \tan\theta_2 & \tan\theta_1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{V}{v_{10}} \tan\theta_1 \\ \frac{V}{v_{20}} \tan\theta_2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus,

$$\sin\theta_0 = \frac{1}{(\tan\theta_1 + \tan\theta_2)} \tan\theta_1 \tan\theta_2 \left(\frac{V}{v_{10}} + \frac{V}{v_{20}} \right)$$

$$\cos\theta_0 = \frac{1}{(\tan\theta_1 + \tan\theta_2)} \left(-\frac{V}{v_{10}} \tan\theta_1 + \frac{V}{v_{20}} \tan\theta_2 \right)$$

$$1 = \sin^2\theta_0 + \cos^2\theta_0$$

$$= \frac{1}{(\tan\theta_1 + \tan\theta_2)^2} \left[\tan^2\theta_1 \tan^2\theta_2 \left(\left(\frac{V}{v_{10}}\right)^2 + \left(\frac{V}{v_{20}}\right)^2 + 2\frac{V^2}{v_{10}v_{20}} \right) + \left(\frac{V}{v_{10}}\right)^2 \tan^2\theta_1 + \left(\frac{V}{v_{20}}\right)^2 \tan^2\theta_2 - 2\frac{V^2}{v_{10}v_{20}} \tan\theta_1 \tan\theta_2 \right]$$

$$= \frac{1}{(\tan\theta_1 + \tan\theta_2)^2} \left(\frac{V}{p_0}\right)^2 \left[\tan^2\theta_1 \tan^2\theta_2 \left(m_1^2 + m_2^2 + 2m_1 m_2 \right) + m_1^2 \tan^2\theta_1 + m_2^2 \tan^2\theta_2 - 2 m_1 m_2 \tan\theta_1 \tan\theta_2 \right]$$

Cross multiply

$$(\tan\theta_1 + \tan\theta_2)^2 \left(\frac{p_0}{V}\right)^2 = \tan^2\theta_1 \tan^2\theta_2 \left(m_1^2 + m_2^2 + 2m_1 m_2 \right) + m_1^2 \tan^2\theta_1 + m_2^2 \tan^2\theta_2 - 2m_1 m_2 \tan\theta_1 \tan\theta_2$$

Now:

$$\tan\theta_1 + \tan\theta_2 = \frac{\sin\theta_1}{\cos\theta_1} + \frac{\sin\theta_2}{\cos\theta_2}$$

$$= \frac{\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1}{\cos\theta_1\cos\theta_2}$$

$$= \frac{\sin(\theta_1+\theta_2)}{\cos\theta_1\cos\theta_2}$$

$$\rightarrow (\tan\theta_1 + \tan\theta_2)^2 = \frac{\sin^2(\theta_1+\theta_2)}{\cos^2\theta_1\cos^2\theta_2}$$

$$LHS = \frac{\sin^2(\theta_1+\theta_2)}{\cos^2\theta_1\cos^2\theta_2}\ \frac{2mE}{V^2}$$

$$= \frac{\sin^2(\theta_1+\theta_2)}{\cos^2\theta_1\cos^2\theta_2}\ \frac{2E\,m_1m_2}{(m_1+m_2)V^2}$$

$$RHS = \frac{\sin^2\theta_1\sin^2\theta_2}{\cos^2\theta_1\cos^2\theta_2}\ m_1m_2\left(\frac{m_1}{m_2}+\frac{m_2}{m_1}+2\right)$$

$$+ \frac{m_1m_2}{\cos^2\theta_1\cos^2\theta_2}\left(\frac{m_1}{m_2}\sin^2\theta_1\cos^2\theta_2 + \frac{m_2}{m_1}\sin^2\theta_2\cos^2\theta_1 - 2\sin\theta_1\sin\theta_2\cos\theta_1\cos\theta_2\right)$$

$$= \frac{m_1 m_2}{\cos^2\theta_1 \cos^2\theta_2} \Big[\frac{m_1}{m_2} \sin^2\theta_1 \sin^2\theta_2 + \frac{m_2}{m_1} \sin^2\theta_1 \sin^2\theta_2 + 2\sin^2\theta_1 \sin^2\theta_2$$

$$+ \frac{m_1}{m_2} \sin^2\theta_1 \cos^2\theta_2 + \frac{m_2}{m_1} \sin^2\theta_2 \cos^2\theta_1$$

$$- 2\sin\theta_1 \sin\theta_2 \cos\theta_1 \cos\theta_2 \Big]$$

$$= \frac{m_1 m_2}{\cos^2\theta_1 \cos^2\theta_2} \Big[\frac{m_1}{m_2} \sin^2\theta_1 + \frac{m_2}{m_1} \sin^2\theta_2$$

$$- 2\sin\theta_1 \sin\theta_2 \big(\underbrace{\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2}_{\cos(\theta_1+\theta_2)} \big) \Big]$$

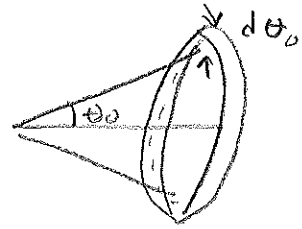
$$= \frac{m_1 m_2}{\cos^2\theta_1 \cos^2\theta_2} \Big[\frac{m_1}{m_2} \sin^2\theta_1 + \frac{m_2}{m_1} \sin^2\theta_2$$

$$- 2\sin\theta_1 \sin\theta_2 \cos(\theta_1+\theta_2) \Big]$$

Thus, LHS = RHS iff

$$\boxed{\frac{\sin^2(\theta_1+\theta_2)\, 2\epsilon}{(m_1+m_2)\, V} = \frac{m_1}{m_2} \sin^2\theta_1 + \frac{m_2}{m_1} \sin^2\theta_2 - 2\sin\theta_1 \sin\theta_2 \cos(\theta_1+\theta_2)}$$

Sec 16, Prob 2: Find $\frac{dN}{N}$ in L frame



$$\frac{dN}{N} = \frac{1}{4\pi}\, d\Omega$$

$$= \frac{1}{4\pi}\, 2\pi \sin\theta_0\, d\theta_0$$

$$= \frac{1}{2} \sin\theta_0\, d\theta_0$$

$$= -\frac{1}{2}\, d(\cos\theta_0)$$

Now:

$$\cos\theta_0 = -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{1-\left(\frac{V}{v_0}\right)^2\sin^2\theta} \qquad (16.6)$$

For $v_0 > V$ take $+\sqrt{\ }$

For $v_0 < V$ take $\pm\sqrt{\ }$

$$-d(\cos\theta_0) = d\left[\frac{V}{v_0}\sin^2\theta \mp \cos\theta\sqrt{\ }\ \right]$$

$$= \frac{V}{v_0}\, 2\sin\theta\cos\theta\, d\theta \pm \sin\theta\, d\theta\sqrt{\ }$$

$$\mp \cos\theta\,\frac{1}{\cancel{2}\sqrt{\ }}\left(-\left(\frac{V}{v_0}\right)^2\right)\cancel{2}\sin\theta\cos\theta\, d\theta$$

$$= \sin\theta\, d\theta\left[2\left(\frac{V}{v_0}\right)\cos\theta \pm \sqrt{\ } \pm \left(\frac{V}{v_0}\right)^2\frac{1}{\sqrt{\ }}\cos^2\theta\right]$$

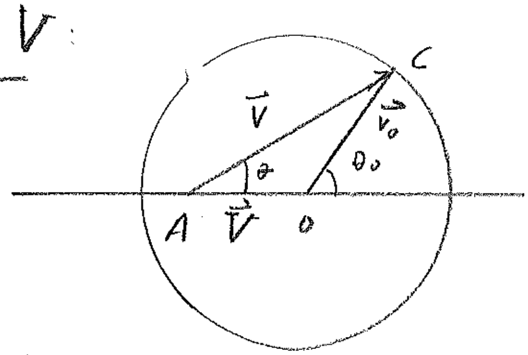
$$= \frac{\sin\theta\, d\theta}{\sqrt{\ }}\left[2\left(\frac{V}{v_0}\right)\cos\theta\sqrt{\ } \pm \left(1-\left(\frac{V}{v_0}\right)^2\sin^2\theta\right) \pm \underbrace{\left(\frac{V}{v_0}\right)^2\cos^2\theta}_{}\right]$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$-d(\cos\theta_0) = \frac{\sin\theta \, d\theta}{\sqrt{}} \left[2\left(\frac{V}{v_0}\right)\cos\theta \sqrt{} \pm \left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right) \right]$$

$$= \sin\theta \, d\theta \left[2\left(\frac{V}{v_0}\right)\cos\theta \pm \frac{1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

For $v_0 > V$:

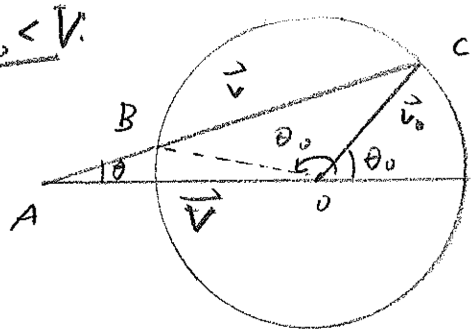


$0 \le \theta \le \pi$
single-valued
Take $+\sqrt{}$

$$\frac{dN}{N} = -\frac{1}{2} d(\cos\theta_0)$$

$$= \frac{1}{2} \sin\theta \, d\theta \left[2\left(\frac{V}{v_0}\right)\cos\theta + \frac{\left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}} \right]$$

For $v_0 < V$:



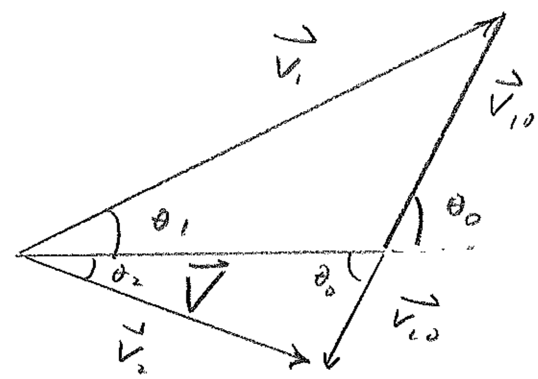
Two different values of θ_0 correspond to the same θ. $(0 \le \theta \le \theta_{max})$

For increasing θ, one value of θ_0 increases, the other decreases

Thus, $$\frac{dN}{N} = \left.\frac{dN}{N}\right|_+ - \left.\frac{dN}{N}\right|_-$$ ← 1st terms cancel, while $\frac{1}{\sqrt{}}$ terms add.

$$= \frac{\sin\theta \, d\theta \left(1 + \left(\frac{V}{v_0}\right)^2 \cos 2\theta\right)}{\sqrt{1 - \left(\frac{V}{v_0}\right)^2 \sin^2\theta}}$$

Sec 16, Prob 3:



$$V_1 \sin\theta_1 = V_{10} \sin\theta_0$$

$$V_1 \cos\theta_1 = V + V_{10} \cos\theta_0$$

$$\rightarrow \boxed{\tan\theta_1 = \frac{V_{10} \sin\theta_0}{V + V_{10}\cos\theta_0}} \quad (16.5)$$

$$V_2 \sin\theta_2 = V_{20} \sin\theta_0$$

$$V_2 \cos\theta_2 = V - V_{20} \cos\theta_0$$

$$\rightarrow \boxed{\tan\theta_2 = \frac{V_{20} \sin\theta_0}{V - V_{20}\cos\theta_0}}$$

Now:

$$\tan(\theta_1+\theta_2) = \frac{\sin(\theta_1+\theta_2)}{\cos(\theta_1+\theta_2)}$$

$$= \frac{\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2}{\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2}$$

$$= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

$$\tan(\theta_1+\theta_2) = \frac{\dfrac{V_{10}\sin\theta_0}{V+V_{10}\cos\theta_0} + \dfrac{V_{20}\sin\theta_0}{V-V_{20}\cos\theta_0}}{1 - \dfrac{V_{10}\sin\theta_0}{V+V_{10}\cos\theta_0}\,\dfrac{V_{20}\sin\theta_0}{V-V_{20}\cos\theta_0}}$$

$$= \frac{V_{10}\sin\theta_0\,(V-V_{20}\cos\theta_0) + V_{20}\sin\theta_0\,(V+V_{10}\cos\theta_0)}{(V+V_{10}\cos\theta_0)(V-V_{20}\cos\theta_0) - V_{10}V_{20}\sin^2\theta_0}$$

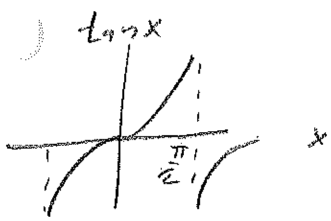
$$= \frac{V_{10}V\sin\theta_0 - \cancel{V_{10}V_{20}\,s\theta_0\,c\theta_0} + V_{20}V\sin\theta_0 + \cancel{V_{20}V_{10}\,s\theta_0\,c\theta_0}}{V^2 - V_{10}V_{20}\cos^2\theta_0 + V\cos\theta_0(V_{10}-V_{20}) - V_{10}V_{20}\sin^2\theta_0}$$

$$= \frac{(V_{10}+V_{20})\,V\sin\theta_0}{V^2 - V_{10}V_{20} + V(V_{10}-V_{20})\cos\theta_0}$$

$$\tan(\theta_1+\theta_2) = f(\theta_0) = \frac{(V_{10}+V_{20})\,V\sin\theta_0}{V^2 - V_{10}V_{20} + V(V_{10}-V_{20})\cos\theta_0}$$

when $\theta_0 = 0$, RHS $= 0$ $\rightarrow$ $\tan(\theta_1+\theta_2) = 0$

$\rightarrow$ $\Theta \equiv \theta_1+\theta_2 = 0$



$$\tan(\theta_1+\theta_2) = \frac{(v_{10}+v_{20})\, V \sin\theta_0}{V^2 - v_{10}v_{20} + V(v_{10}-v_{20})\cos\theta_0} \equiv F(\theta_0)$$

Extremize the RHS w.r.t. θ_0:

$$0 = F'(\theta_0)$$

$$= \frac{\left[(v_{10}+v_{20})V\cos\theta_0\left(V^2 - v_{10}v_{20} + V(v_{10}-v_{20})\cos\theta_0\right) + V(v_{10}-v_{20})\sin\theta_0\,(v_{10}+v_{20})V\sin\theta_0\right]}{\text{bottom}^2}$$

$$0 = \text{numerator}$$

$$= V(v_{10}+v_{20})(V^2 - v_{10}v_{20})\cos\theta_0 + V^2(v_{10}^2 - v_{20}^2)\cos^2\theta_0 + V^2(v_{10}^2 - v_{20}^2)\sin^2\theta_0$$

$$\cancel{= V\left[\quad v_{10}\right.}$$

$$= V(v_{10}+v_{20})(V^2 - v_{10}v_{20})\cos\theta_0 + V^2(v_{10}^2 - v_{20}^2)$$

$$= V(v_{10}+v_{20})\left[(V^2 - v_{10}v_{20})\cos\theta_0 + V(v_{10}-v_{20})\right]$$

so
$$0 = (V^2 - v_{10}v_{20})\cos\theta_0 + (v_{10}-v_{20})V$$

$$\boxed{\cos\bar{\theta}_0 = \frac{(v_{10}-v_{20})V}{v_{10}v_{20} - V^2}}$$

Define $\theta \equiv \theta_1 + \theta_2$

$$\tan\theta = \frac{(v_{10}+v_{20})\, V\sin\theta_0}{V^2 - v_{10}v_{20} + V(v_{10}-v_{20})\cos\theta_0}$$

$$\rightarrow \qquad \tan\bar{\theta} = \frac{(v_{10}+v_{20})\,V\sin\bar{\theta}_0}{V^2 - v_{10}v_{20} + V(v_{10}-v_{20})\cos\bar{\theta}_0}$$

Now:

$$\sin\bar{\theta}_0 = \sqrt{1-\cos^2\bar{\theta}_0}$$

$$= \sqrt{1 - V^2\left(\frac{v_{10}-v_{20}}{v_{10}v_{20}-V^2}\right)^2}$$

$$= \frac{\sqrt{(v_{10}v_{20}-V^2)^2 - V^2(v_{10}-v_{20})^2}}{|v_{10}v_{20}-V^2|}$$

$$= \frac{\sqrt{v_{10}^2v_{20}^2 + V^4 - \cancel{2v_{10}v_{20}}V^2 - V^2(v_{10}^2+v_{20}^2-\cancel{2v_{10}v_{20}})}}{|v_{10}v_{20}-V^2|}$$

$$= \frac{\sqrt{V^4 - V^2(v_{10}^2+v_{20}^2) + v_{10}^2v_{20}^2}}{|v_{10}v_{20}-V^2|}$$

$$= \frac{\sqrt{(V^2-v_{10}^2)(V^2-v_{20}^2)}}{|v_{10}v_{20}-V^2|}$$

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thus,

$$\tan\bar{\theta} = \frac{(v_{10}+v_{20})\,V\,\dfrac{\sqrt{(V^2-v_{10}^2)(V^2-v_{20}^2)}}{|v_{10}v_{20}-V^2|}}{V^2 - v_{10}v_{20} + V(v_{10}-v_{20})\dfrac{(v_{10}-v_{20})V}{(v_{10}v_{20}-V^2)}}$$

$$\text{denom} = \frac{-(V^2 - v_{10}v_{20})^2 + V^2(v_{10} - v_{20})^2}{v_{10}v_{20} - V^2}$$

$$\text{num} = \frac{V(v_{10}+v_{20})\sqrt{(V^2 - v_{20}^2)(V^2 - v_{10}^2)}}{|v_{10}v_{20} - V^2|}$$

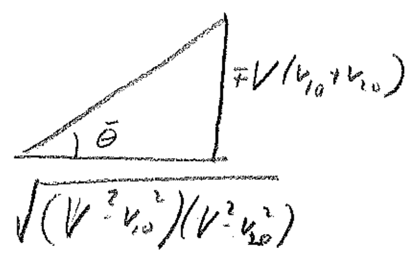
$$\rightarrow \boxed{\tan\bar{\theta} = \pm\frac{V(v_{10}+v_{20})\sqrt{(V^2 - v_{10}^2)(V^2 - v_{10}^2)}}{V^2(v_{10}-v_{20})^2 - (V^2 - v_{10}v_{20})^2}}$$

<u>NOTE</u>:

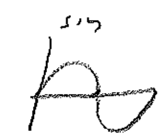
$$\begin{aligned}
\text{denom} &= V^2(v_{10}^2 + v_{20}^2 - \cancel{2v_{10}v_{20}}) - (V^4 + v_{10}^2 v_{20}^2 - \cancel{2V^2 v_{10}v_{20}}) \\
&= -V^4 + V^2(v_{10}^2 + v_{20}^2) - v_{10}^2 v_{20}^2 \\
&= -\left(V^4 - V^2(v_{10}^2+v_{20}^2) + v_{10}^2 v_{20}^2\right) \\
&= -(V^2 - v_{10}^2)(V^2 - v_{20}^2)
\end{aligned}$$

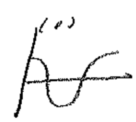
$-$: if $v_{10}v_{20} - V^2 > 0$
$+$: if $v_{10}v_{20} - V^2 < 0$

$$\text{so} \quad \boxed{\tan\bar{\theta} = \frac{\mp V(v_{10}+v_{20})}{\sqrt{(V^2 - v_{10}^2)(V^2 - v_{20}^2)}}}$$

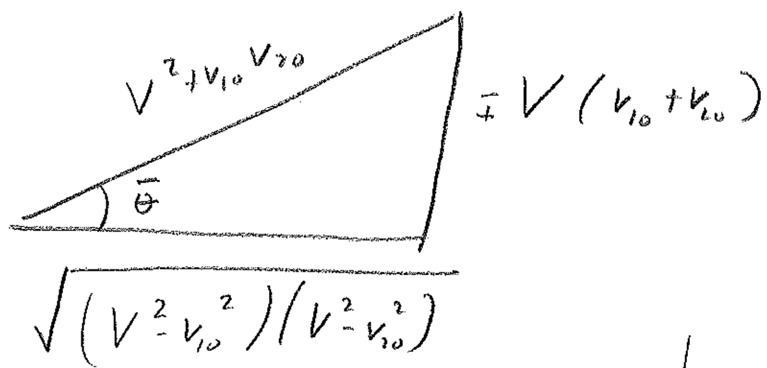


$$\begin{aligned}
\text{hypotenuse} &= \sqrt{(V^2 - v_{10}^2)(V^2 - v_{20}^2) + V^2(v_{10}+v_{20})^2} \\
&= \sqrt{V^4 \cancel{- V^2(v_{10}^2+v_{20}^2)} + v_{10}^2 v_{20}^2 + V^2(\cancel{v_{10}^2 + v_{20}^2} + 2v_{10}v_{20})} \\
&= \sqrt{V^4 + 2V^2 v_{10}v_{20} + v_{10}^2 v_{20}^2} \\
&= \sqrt{(V^2 + v_{10}v_{20})^2} \\
&= V^2 + v_{10}v_{20}
\end{aligned}$$





Thus,



$$\longrightarrow \boxed{\sin\bar{\theta} = \frac{\mp V(v_{10}+v_{20})}{V^2 + v_{10}v_{20}}}$$

~~Recall~~

~~(or) $\bar{\theta} = \frac{(v_{10} - v_{20})V}{V^2 - v_{10}v_{20}}$~~

Lets consider relative values of V, v_{10}, v_{20}

Assume without loss of generality that $v_{20} > v_{10}$

(i) $v_{10} < V < v_{20}$ $\longrightarrow$ $v_{10}v_{20} - V^2 \gtreqless 0$, $(V^2 - v_{10}^2)(V^2 - v_{20}^2) < 0$

(ii) $V < v_{10} < v_{20}$ $\longrightarrow$ $v_{10}v_{20} - V^2 > 0$, $(V^2 - v_{10}^2)(V^2 - v_{20}^2) > 0$

(iii) $v_{10} < v_{20} < V$ $\longrightarrow$ $v_{10}v_{20} - V^2 < 0$, $(V^2 - v_{10}^2)(V^2 - v_{20}^2) > 0$

~~Case (i): $v_{10} < V < v_{20}$; $\bar{\theta} = \frac{(v_{10} - v_{20})V}{V^2 - v_{10}v_{20}}$ $+, -, 0$~~

~~$v_{10}v_{20} - V^2 < (v_{20} - v_{10})V < V^2 - v_{10}v_{20}$~~

__Case (i):__ since $(V^2 - v_{10}^2)(V^2 - v_{20}^2) < 0$, $\tan\bar{\theta} = \frac{\mp V(v_{10}+v_{20})}{\sqrt{(V^2 - v_{10}^2)(V^2 - v_{20}^2)}}$

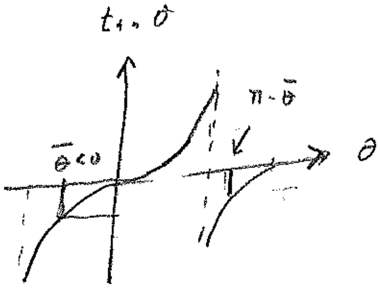
is ill defined.

So no extremum $\bar{\theta}$

$$\rightarrow \boxed{0 < \theta < \pi}$$

Case (ii'): since $(V^2-v_{10}^2)(V^2-v_{20}^2)>0$ and $v_{10}v_{20}-V^2>0$

we have $\tan\bar{\theta} = \dfrac{-V(v_{10}+v_{20})}{\sqrt{(V^2-v_{10}^2)(V^2-v_{20}^2)}} < 0$

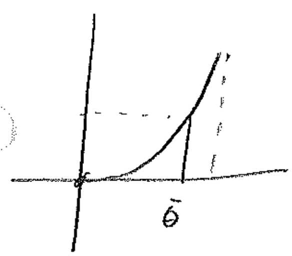


thus, $\boxed{\pi-|\bar{\theta}|<\theta<\pi}$

$\tan(\pi)=0$

Case (iii) since $(V^2-v_{10}^2)(V^2-v_{20}^2)>0$ and $v_{10}v_{20}-V^2<0$

we have $\tan\bar{\theta} = \dfrac{+V(v_{10}+v_{20})}{\sqrt{(V^2-v_{10}^2)(V^2-v_{20}^2)}} > 0$



thus $\boxed{0<\theta<\bar{\theta}}$