Sec 40 : prob 1

a) single particle, Cartesian (x, y, z)

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

$$H = \sum_i p_i \dot{q}_i - L \equiv H(p, q, t)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \ , \ p_y = m\dot{y} \ , \ p_z = m\dot{z}$$

so $\dot{x} = \frac{p_x}{m}$, $\dot{y} = \frac{p_y}{m}$, $\dot{z} = \frac{p_z}{m}$

Thus,

$$\boxed{H} = p_x \left(\frac{p_x}{m}\right) + p_y \left(\frac{p_y}{m}\right) + p_z \left(\frac{p_z}{m}\right) - \frac{1}{2} m \left(\frac{p_x^2}{m^2} + \frac{p_y^2}{m^2} + \frac{p_z^2}{m^2} \right) + U(x, y, z)$$

$$= \boxed{\frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + U(x, y, z)}$$

b) single particle, cylindrical (r, ϕ, z)

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2)$$

$$p_r = m\dot{r} , \ p_\phi = m r^2 \dot{\phi} \ , \ p_z = m\dot{z}$$

$$\rightarrow \dot{r} = \frac{p_r}{m} \ , \ \dot{\phi} = \frac{p_\phi}{m r^2} \ , \ \dot{z} = \frac{p_z}{m}$$

Thus,

$$\boxed{H} = p_r \left(\frac{p_r}{m}\right) + p_\phi \left(\frac{p_\phi}{m r^2}\right) + p_z \left(\frac{p_z}{m}\right) - \frac{1}{2} m \left(\frac{p_r^2}{m^2} + r^2 \frac{p_\phi^2}{m^2 r^4} + \frac{p_z^2}{m^2} \right) + U(r, \phi, z)$$

$$= \boxed{\frac{1}{2m} \left(p_r^2 + \frac{p_\phi^2}{r^2} + p_z^2 \right) + U(r, \phi, z)}$$

c) single particle, spherical (r, θ, ϕ)

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right)$$

$$p_r = m\dot{r}, \quad p_\theta = m r^2 \dot{\theta}, \quad p_\phi = m r^2 \sin^2\theta \, \dot{\phi}$$

$$\rightarrow \quad \dot{r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{p_\theta}{m r^2}, \quad \dot{\phi} = \frac{p_\phi}{m r^2 \sin^2\theta}$$

Thus,

$$\boxed{\begin{aligned} \boxed{H} &= p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{m r^2}\right) + p_\phi\left(\frac{p_\phi}{m r^2 \sin^2\theta}\right) \\ &\quad - \frac{1}{2} m \left(\frac{p_r^2}{m^2} + r^2 \frac{p_\theta^2}{m^2 r^4} + r^2 \sin^2\theta \, \frac{p_\phi^2}{m^2 r^4 \sin^4\theta} \right) \\ &\quad + U(r, \theta, \phi) \\ &= \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta} \right) + U(r, \theta, \phi) \end{aligned}}$$

Sec 40:
Prob 2:

In a uniformly rotating frame of reference (no translation)

$$L = \frac{1}{2}mv^2 + m\vec{v}\cdot(\vec{\Omega}\times\vec{r}) + \frac{1}{2}m(\vec{\Omega}\times\vec{r})^2 - U$$

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + m(\vec{\Omega}\times\vec{r}) = m(\underbrace{\vec{v}+\vec{\Omega}\times\vec{r}}_{\vec{v}_0})$$

$\vec{v}_0 \equiv$ velocity wrt inertial frame

Invert:

$$\vec{v}+\vec{\Omega}\times\vec{r} = \frac{\vec{p}}{m}$$

$$\vec{v} = \frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}$$

Thus,

$$H = \vec{p}\cdot\vec{v} - L$$

$$= \vec{p}\cdot\left(\frac{\vec{p}}{m} - \vec{\Omega}\times\vec{r}\right) - \frac{1}{2}m\left(\frac{\vec{p}}{m}-\vec{\Omega}\times\vec{r}\right)\cdot\left(\frac{\vec{p}}{m}-\vec{\Omega}\times\vec{r}\right)$$

$$- m\left(\frac{\vec{p}}{m}-\vec{\Omega}\times\vec{r}\right)\cdot(\vec{\Omega}\times\vec{r}) - \frac{1}{2}m(\vec{\Omega}\times\vec{r})^2 + U$$

$$= \frac{p^2}{m} - \cancel{\vec{p}\cdot(\vec{\Omega}\times\vec{r})} - \frac{1}{2}m\left(\frac{p^2}{m^2} - \cancel{2\frac{\vec{p}}{m}\cdot(\vec{\Omega}\times\vec{r})} + \cancel{(\vec{\Omega}\times\vec{r})^2}\right)$$

$$- \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + \cancel{m(\vec{\Omega}\times\vec{r})^2} - \cancel{\frac{1}{2}m(\vec{\Omega}\times\vec{r})^2} + U$$

$$= \frac{1}{2}\frac{p^2}{m} - \vec{p}\cdot(\vec{\Omega}\times\vec{r}) + U$$

$$= \frac{1}{2}\frac{p^2}{m} - \vec{\Omega}\cdot(\vec{r}\times\vec{p}) + U$$

$$= \frac{1}{2}\frac{p^2}{m} - \vec{\Omega}\cdot\vec{M} + U \qquad \text{where } \vec{M} = \vec{r}\times\vec{p} = \vec{r}\times\vec{p}_0 = \vec{M}_0$$