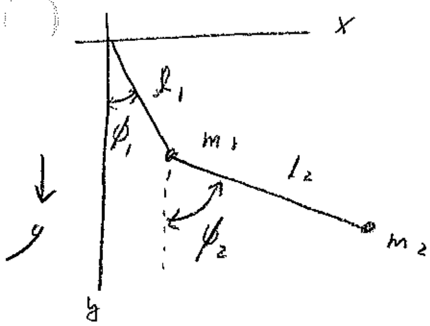
Section 5
Prob 1



$$U = -\vec{F}\cdot\vec{r}$$
$$= -mg\hat{y}\cdot[x\hat{x}+y\hat{y}]$$
$$= -mgy \quad \text{(single particle)}$$

$$\boxed{U = U_1 + U_2 = -m_1 g y_1 - m_2 g y_2}$$

$y_1 = l_1\cos\phi_1$, $x_1 = l_1\sin\phi_1$

$y_2 = l_1\cos\phi_1 + l_2\cos\phi_2$, $x_2 = l_1\sin\phi_1 + l_2\sin\phi_2$

$$T = \sum_a \frac{1}{2} m_a v_a^2$$
$$= \frac{1}{2} m_1(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2} m_2(\dot{x}_2^2+\dot{y}_2^2)$$

$\dot{x}_1 = l_1\cos\phi_1\,\dot{\phi}_1 \rightarrow \dot{x}_1^2 = l_1^2\cos^2\phi_1\,\dot{\phi}_1^2$

$\dot{y}_1 = -l_1\sin\phi_1\,\dot{\phi}_1 \rightarrow \dot{y}_1^2 = l_1^2\sin^2\phi_1\,\dot{\phi}_1^2$

Thus, $\dot{x}_1^2+\dot{y}_1^2 = l_1^2\dot{\phi}_1^2$

$\dot{x}_2 = l_1\cos\phi_1\,\dot{\phi}_1 + l_2\cos\phi_2\,\dot{\phi}_2$

$\dot{x}_2^2 = l_1^2\cos^2\phi_1\,\dot{\phi}_1^2 + l_2^2\cos^2\phi_2\,\dot{\phi}_2^2 + 2l_1l_2\cos\phi_1\cos\phi_2\,\dot{\phi}_1\dot{\phi}_2$

$\dot{y}_2 = -l_1\sin\phi_1\,\dot{\phi}_1 - l_2\sin\phi_2\,\dot{\phi}_2$

$\dot{y}_2^2 = l_1^2\sin^2\phi_1\,\dot{\phi}_1^2 + l_2^2\sin^2\phi_2\,\dot{\phi}_2^2 + 2l_1l_2\sin\phi_1\sin\phi_2\,\dot{\phi}_1\dot{\phi}_2$

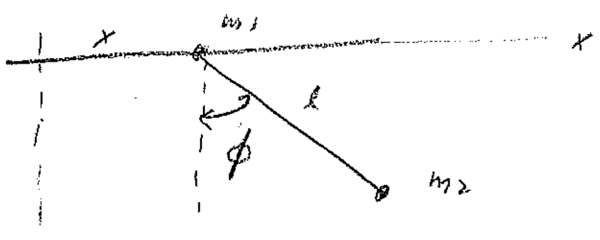
$\longrightarrow \dot{x}_2^2+\dot{y}_2^2 = l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2l_1l_2\cos(\phi_1-\phi_2)\,\dot{\phi}_1\dot{\phi}_2$

Thus,
$$T = \frac{1}{2}m_1 l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2l_1l_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2\right)$$
$$= \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2 + m_2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)$$

so $L = T - U$ with

$$\boxed{L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2 + m_2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2) + (m_1+m_2)gl_1\cos\phi_1 + m_2gl_2\cos\phi_2}$$

Sec 5, Prob 2:



$x_1 = x$
$y_1 = 0$

$x_2 = x + l\sin\phi$
$y_2 = l\cos\phi$

$$U = -mgy_2 = \boxed{-mgl\cos\phi} \quad \text{(reference to } y=0\text{)}$$

$$T = \frac{1}{2}\left(m_1 \dot{x}^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$\dot{x}_2 = \dot{x} + l\cos\phi\,\dot{\phi} \quad \rightarrow \quad \dot{x}_2^2 = \dot{x}^2 + l^2\cos^2\phi\,\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi}$$

$$\dot{y}_2 = -l\sin\phi\,\dot{\phi} \quad \rightarrow \quad \dot{y}_2^2 = l^2\sin^2\phi\,\dot{\phi}^2$$

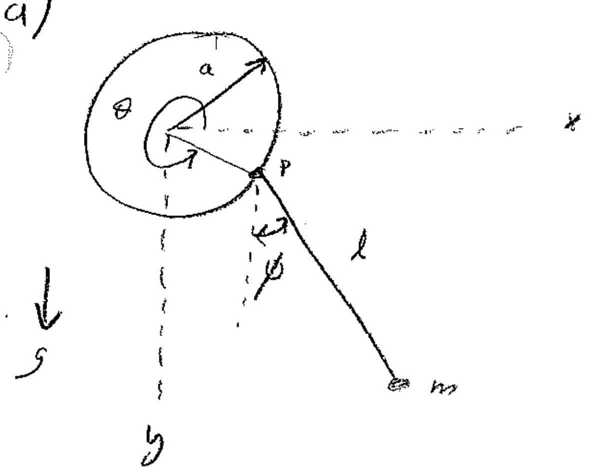
$$\rightarrow \quad \boxed{T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi}\right) = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2\left(l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi}\right)}$$

Thus,

$$\boxed{L = T - U = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2\left(l^2\dot{\phi}^2 + 2l\cos\phi\,\dot{x}\dot{\phi}\right) + m_2 g l\cos\phi}$$

Sec 5, Prob 3.

(a)



$\theta = \gamma t$

γ : angular freq

Thus, $x_p = a\cos\gamma t$
$y_p = -a\sin\gamma t$

$U = -mgy$

so $x = x_p + l\sin\phi$
$= a\cos\gamma t + l\sin\phi$

$y = y_p + l\cos\phi$
$= -a\sin\gamma t + l\cos\phi$

$$\boxed{U = -mg\left[-a\sin\gamma t + l\cos\phi\right]}$$

$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$

$\dot{x} = -a\gamma\sin\gamma t + l\cos\phi\,\dot{\phi}$

$\dot{y} = -a\gamma\cos\gamma t - l\sin\phi\,\dot{\phi}$

$\longrightarrow \dot{x}^2 = a^2\gamma^2\sin^2\gamma t + l^2\cos^2\phi\,\dot{\phi}^2 - 2al\gamma\sin\gamma t\cos\phi\,\dot{\phi}$

$\dot{y}^2 = a^2\gamma^2\cos^2\gamma t + l^2\sin^2\phi\,\dot{\phi}^2 + 2al\gamma\cos\gamma t\sin\phi\,\dot{\phi}$

so $\boxed{T} = \frac{1}{2}m\left(a^2\gamma^2 + l^2\dot{\phi}^2 + 2al\gamma\underbrace{(\sin\phi\cos\gamma t - \sin\gamma t\cos\phi)}_{\sin(\phi-\gamma t)}\dot{\phi}\right)$

$$= \frac{1}{2}ma^2\gamma^2 + \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\sin(\phi - \gamma t)$$

$\boxed{L} = T - U$

$$= \frac{1}{2}ma^2\gamma^2 + \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma\dot{\phi}\sin(\phi-\gamma t) + mg\left[-a\sin\gamma t + l\cos\phi\right]$$

Let's ignore functions of time, constants, and total time derivatives:

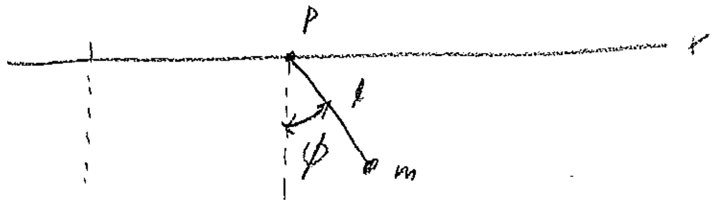
1) $\frac{1}{2} m a^2 \gamma^2 = \text{const}$

2) $-mga \sin\gamma t = \text{fcn of time}$

3) $mal\gamma\dot\phi \sin(\phi-\gamma t) = \frac{d}{dt}\left[-mal\gamma\cos(\phi-\gamma t)\right]$
$+ mal\gamma^2 \sin(\phi-\gamma t)$

Thus,

$$\boxed{L \overset{\text{equiv}}{=} \frac{1}{2} m l^2 \dot\phi^2 + mal\gamma^2 \sin(\phi-\gamma t) + mgl\cos\phi}$$

1) 

$x_p = a\cos\gamma t$ (oscillates)

$x = x_p + l\sin\phi = a\cos\gamma t + l\sin\phi$

$y = l\cos\phi$

$U = -mgy = -mgl\cos\phi$

$T = \frac{1}{2} m(\dot x^2 + \dot y^2) = \frac{1}{2} m\left((-a\gamma\sin\gamma t + l\cos\phi\,\dot\phi)^2 + (-l\sin\phi\,\dot\phi)^2\right)$

$= \frac{1}{2} m\left[a^2\gamma^2\sin^2\gamma t + l^2\cos^2\phi\,\dot\phi^2 - 2al\gamma\sin\gamma t\cos\phi\,\dot\phi\right.$
$\left. + l^2\sin^2\phi\,\dot\phi^2\right]$

$= \frac{1}{2} m\left[a^2\gamma^2\sin^2\gamma t + l^2\dot\phi^2 - 2al\gamma\sin\gamma t\cos\phi\,\dot\phi\right]$

$$\rightarrow \boxed{L = \frac{1}{2} m a^2\gamma^2\sin^2\gamma t + \frac{1}{2} m l^2\dot\phi^2 - mal\gamma\sin\gamma t\cos\phi\,\dot\phi + mgl\cos\phi}$$

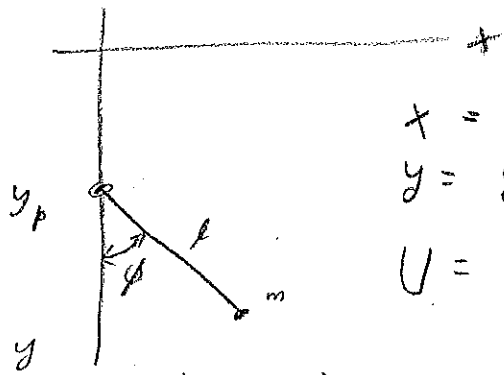
As before, ignore consts, functions of time, and total time derivatives:

$$-mal\gamma \sin\gamma t \cos\phi\,\dot\phi = \frac{d}{dt}\left[-mal\gamma\sin\gamma t\sin\phi\right] + mal\gamma^2\cos\gamma t\sin\phi$$

Thus,

$$\boxed{L \overset{\text{equiv}}{=} \frac{1}{2}ml^2\dot\phi^2 + mal\gamma^2\cos\gamma t\sin\phi + mgl\cos\phi}$$

c)



$y_p = a\cos\gamma t$ (oscillator)

$x = l\sin\phi$

$y = y_p + l\cos\phi = a\cos\gamma t + l\cos\phi$

$U = -mgy = -mga\cos\gamma t - mgl\cos\phi$

$$T = \frac{1}{2}m(\dot x^2+\dot y^2) = \frac{1}{2}m\left(l^2\cos^2\phi\,\dot\phi^2 + (-a\gamma\sin\gamma t - l\sin\phi\,\dot\phi)^2\right)$$

$$= \frac{1}{2}m\left[l^2\cos^2\phi\,\dot\phi^2 + a^2\gamma^2\sin^2\gamma t + l^2\sin^2\phi\,\dot\phi^2 + 2al\gamma\sin\phi\sin\gamma t\,\dot\phi\right]$$

$$= \frac{1}{2}ma^2\gamma^2\sin^2\gamma t + \frac{1}{2}ml^2\dot\phi^2 + mal\gamma\sin\phi\sin\gamma t\,\dot\phi$$

Ignore consts, functions of time, total time derivatives:

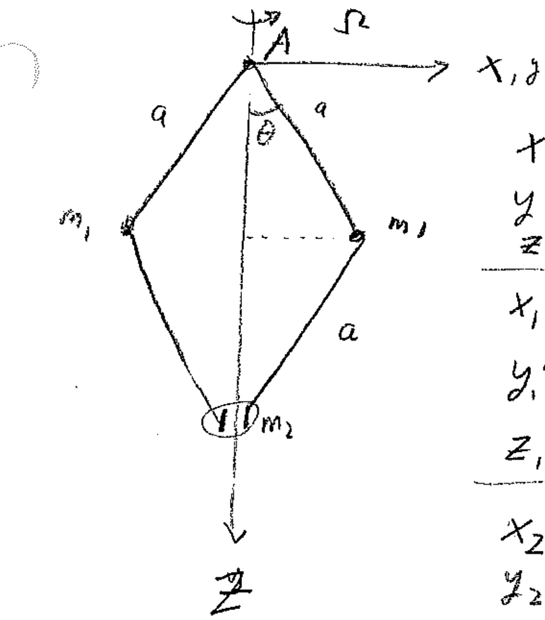
$$mal\gamma\sin\phi\sin\gamma t\,\dot\phi = \frac{d}{dt}\left[-mal\gamma\cos\phi\sin\gamma t\right] + mal\gamma^2\cos\phi\cos\gamma t$$

Thus,

$$\boxed{L \overset{\text{equiv}}{=} \frac{1}{2}ml^2\dot\phi^2 + mal\gamma^2\cos\gamma t\cos\phi + mgl\cos\phi}$$

Sec 5, prob 4: LI

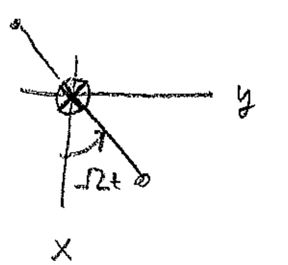
$$U = -mgz$$



$$x_1 = a\sin\theta\cos\Omega t$$
$$y_1 = a\sin\theta\sin\Omega t \qquad \text{(right)}$$
$$z_1 = a\cos\theta$$

$$x_1' = -a\sin\theta\cos\Omega t$$
$$y_1' = -a\sin\theta\sin\Omega t \qquad \text{(left)}$$
$$z_1' = a\cos\theta$$

$$x_2 = 0$$
$$y_2 = 0$$
$$z_2 = 2a\cos\theta$$



$$\boxed{U} = -m_1 g z_1 - m_1 g z_1' - m_2 g$$
$$= -2m_1 g a\cos\theta - 2m_2 g a\cos\theta$$
$$= -2(m_1+m_2)ga\cos\theta$$

$$\boxed{T} = \left(\frac{1}{2}\right)m_1\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2\right)\cdot 2 + \frac{1}{2}m_2\dot{z}_2^2$$
$$= m_1\Big[\left(a\cos\theta\,\dot\theta\cos\Omega t - a\Omega\sin\theta\sin\Omega t\right)^2$$
$$+ \left(a\cos\theta\,\dot\theta\sin\Omega t + a\Omega\sin\theta\cos\Omega t\right)^2$$
$$+ a^2\sin^2\theta\,\dot\theta^2\Big]$$
$$+ \frac{1}{2}m_2\left(4a^2\sin^2\theta\,\dot\theta^2\right)$$
$$= m_1 a^2\Big[\dot\theta^2\big(\overbrace{c^2c^2\Omega + c^2s^2\Omega}^{=c^2} + s^2\big) + \Omega^2\left(s^2s^2\Omega + s^2c^2\Omega\right)$$
$$\cancel{-2\dot\theta\Omega\, sc\, c\Omega\, s\Omega} \cancel{+2\dot\theta\Omega\, cs\, s\Omega\, c\Omega}\Big]$$
$$+ 2m_2 a^2\sin^2\theta\,\dot\theta^2$$
$$= m_1 a^2\left[\dot\theta^2 + \sin^2\theta\,\Omega^2\right] + 2m_2a^2\sin^2\theta\,\dot\theta^2$$

$$\boxed{L = T - U = m_1a^2\left(\dot\theta^2 + \sin^2\theta\,\Omega^2\right) + 2m_2a^2\sin^2\theta\,\dot\theta^2 + 2(m_1+m_2)ga\cos\theta}$$