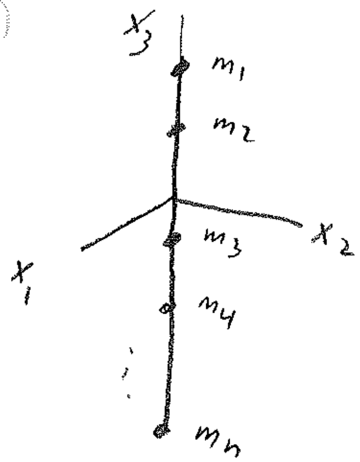
§32, Prob 1:

a) molecule of collinear atoms:



Take origin at COM:

$$I_3 = 0$$

$$I_1 = I_2 = \sum_a m_a x_{3a}^2$$

For simplicity, let $z_a \equiv x_{3a}$

Then $I_1 = I_2 = \sum_a m_a z_a^2$

where $\sum_a m_a z_a = 0$ (COM)

It turns out that we can rewrite $I_1 = I_2$ in terms of the distances $l_{ab} \equiv z_a - z_b$ between the masses.

$\Rightarrow$ That is, given:

$$\left.\begin{aligned} m_1 z_1 + m_2 z_2 + \cdots + m_n z_n &= 0 \\ z_1 - z_2 &= l_{12} \\ z_2 - z_3 &= l_{23} \\ z_3 - z_4 &= l_{34} \\ &\vdots \\ z_{n-1} - z_n &= l_{n-1,n} \end{aligned}\right\} \text{n-equations}$$

we can solve for $z_1, z_2, \ldots, z_n$ in terms of l_{12}, l_{23}, etc. so that

$$I_1 = I_2 = \sum_a m_a z_a^2 = f(l_{ab}^2)$$

NOTE: $l_{ab} + l_{bc} = (z_a - z_b) + (z_b - z_c) = l_{ac}$

also $l_{ab} = -l_{ab}$, $l_{ab}^2 = l_{ba}^2$

Examples:

$\boxed{n=2}$:

$$m_1 z_1 + m_2 z_2 = 0$$
$$z_1 - z_2 = l_{12} \equiv l$$

Thus, $z_2 = z_1 - l \rightarrow$

$$m_1 z_1 + m_2 (z_1 - l) = 0$$
$$(m_1 + m_2) z_1 - m_2 l = 0$$
$$z_1 = \left(\frac{m_2}{m_1 + m_2}\right) l$$
$$z_2 = \left(\frac{-m_1}{m_1 + m_2}\right) l$$

$\rightarrow$

$$I_1 = I_2$$
$$= m_1 z_1^2 + m_2 z_2^2$$
$$= \frac{m_1 m_2^2}{(m_1+m_2)^2} l^2 + \frac{m_2 m_1^2}{(m_1+m_2)^2} l^2$$
$$= \frac{m_1 m_2}{(m_1+m_2)^2} l^2 \left[\cancel{m_2 + m_1}\right]$$
$$= \frac{m_1 m_2 l^2}{m_1 + m_2} = \boxed{\mu l^2}$$

reduced mass

$\boxed{n=3}$:

$$m_1 z_1 + m_2 z_2 + m_3 z_3 = 0$$
$$z_1 - z_2 = l_{12}$$
$$z_2 - z_3 = l_{23}$$
$$z_3 - z_1 = l_{31} = -l_{23} - l_{12}$$
$$= l_{32} + l_{21}$$

$\cancel{I_1 \sim I_2}$

$$\cancel{= m_1 z_1^2 + m_2 z_2^2 + m_3 z_3^2}$$

$$Z_2 = z_1 - l_{12}$$

$$Z_3 = z_2 - l_{23}$$

$$= z_1 - l_{12} - l_{23}$$

$$= z_1 - l_{13}$$

$$0 = m_1 z_1 + m_2 z_2 + m_3 z_3$$

$$= m_1 z_1 + m_2 (z_1 - l_{12}) + m_3 (z_1 - l_{13})$$

$$= (m_1 + m_2 + m_3) z_1 - (m_2 l_{12} + m_3 l_{13})$$

$$\boxed{z_1} = \frac{m_2 l_{12} + m_3 l_{13}}{m_1 + m_2 + m_3}$$

$$= \boxed{\frac{m_2 l_{12} + m_3 l_{13}}{M}}$$

$$\boxed{Z_2} = z_1 - l_{12}$$

$$= \frac{\cancel{m_2 l_{12}} + m_3 l_{13} - (m_1 + \cancel{m_2} + m_3) l_{12}}{M}$$

$$= \frac{-m_1 l_{12} + m_3 (l_{13} - l_{12})}{M}$$

$$= \frac{-m_1 l_{12} + m_3 l_{23}}{M}$$

$$= \boxed{\frac{m_1 l_{21} + m_3 l_{23}}{M}}$$

$$\boxed{z_3 = z_1 - l_{13} = \frac{m_2 l_{12} + \cancel{m_3 l_{13}} - (m_1 + m_2 + \cancel{m_3})\, l_{13}}{M} = \frac{m_2 (l_{12} - l_{13}) - m_1 l_{13}}{M} = \frac{m_2 l_{32} + m_1 l_{31}}{M}}$$

Thus,

$$I_1 = I_2$$

$$= m_1 z_1^2 + m_2 z_2^2 + m_3 z_3^2$$

$$= \frac{1}{M^2} \Big[m_1 (m_2 l_{12} + m_3 l_{13})^2 + m_2 (m_1 l_{21} + m_3 l_{23})^2 + m_3 (m_2 l_{32} + m_1 l_{31})^2 \Big]$$

$$= \frac{1}{M^2} \Big[m_1 (m_2^2 l_{12}^2 + m_3^2 l_{13}^2 + 2 m_2 m_3 l_{12} l_{13}) + m_2 (m_1^2 l_{21}^2 + m_3^2 l_{23}^2 + 2 m_1 m_3 l_{21} l_{23}) + m_3 (m_2^2 l_{32}^2 + m_1^2 l_{31}^2 + 2 m_2 m_1 l_{32} l_{31}) \Big]$$

$$= \frac{1}{M^2} \Big[m_1 m_2 (m_1 + m_2) l_{12}^2 + m_1 m_3 (m_1 + m_3) l_{31}^2 + m_2 m_3 (m_2 + m_3) l_{23}^2 + 2 m_1 m_2 m_3 (l_{12} l_{13} + l_{21} l_{23} + l_{32} l_{31}) \Big]$$

Now:

$$2(l_{12}l_{13} + l_{21}l_{23} + l_{32}l_{31})$$

$$= l_{12}l_{13} + l_{21}l_{23} + l_{32}l_{31} + l_{12}l_{13} + l_{21}l_{23} + l_{32}l_{31}$$

$$= l_{12}l_{13} + l_{12}l_{32} + l_{32}l_{31} + l_{21}l_{31} + l_{21}l_{23} + l_{23}l_{13}$$

$$= l_{12}(l_{13}+l_{32}) + l_{31}(l_{32}+l_{21}) + l_{23}(l_{21}+l_{13})$$

$$= l_{12}^2 + l_{31}^2 + l_{23}^2$$

Thus,

$$I_1 = I_2$$

$$= \frac{1}{M^2}\Big[m_1 m_2 (m_1+m_2) l_{12}^2 + m_1 m_3 (m_1+m_3) l_{31}^2 + m_2 m_3 (m_2+m_3) l_{23}^2 + m_1 m_2 m_3 (l_{12}^2 + l_{31}^2 + l_{23}^2)\Big]$$

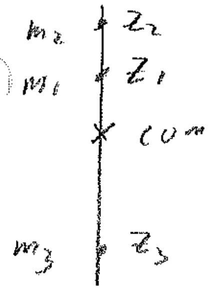
$$= \frac{1}{M^2}\Big[m_1 m_2 (m_1+m_2+m_3) l_{12}^2 + m_1 m_3 (m_1+m_2+m_3) l_{31}^2 + m_2 m_3 (m_1+m_2+m_3) l_{23}^2 \Big]$$

$$= \frac{1}{M}\left[m_1 m_2 l_{12}^2 + m_2 m_3 l_{23}^2 + m_3 m_1 l_{31}^2 \right]$$

$$= \boxed{\frac{1}{M}\sum_{a\neq b} m_a m_b l_{ab}^2}$$

this is actually the general expression for any n.

§ Sec 32 - Prob 1(a) (General solution)



Rotator:

$$I_3 = 0, \quad I_1 = I_2 = \sum_a m_a x_{3a}^2 = \sum_a m_a z_a^2$$

where $z_a \equiv x_{3a}$ (For simplicity of notation)

COM condition: $0 = \sum_a m_a z_a$

Define: $l_{ab} \equiv z_a - z_b$

NOTE: $l_{aa} = 0$, $l_{ba} = -l_{ab}$, $l_{ab} + l_{bc} = (z_a - z_b) + (z_b - z_c)$
$= z_a - z_c$
$= l_{ac}$

N-equations:

$$0 = \sum_a m_a z_a$$

$$l_{12} = z_1 - z_2$$

$$l_{13} = z_1 - z_3$$

$$\vdots$$

$$l_{1N} = z_1 - z_N$$

Can solve for z_a in terms of l_{ab}.

$$\sum_b m_b l_{ab} = \sum_b (z_a - z_b) m_b$$

$$= z_a \sum_b m_b - \underbrace{\cancel{\sum_b z_b m_b}}_{\to 0 \text{ com}}$$

$$= \mu z_a$$

Thus, $\boxed{z_a = \frac{1}{\mu} \sum_b m_b l_{ab}}$ — can sum over all b, noting that when $b = a$, $l_{ab} = 0$, so $z_a = \frac{1}{\mu} \sum_{a \neq b} m_b l_{ab}$ also works

Thus,

$$I_1 = I_2 = \sum_a m_a z_a^2$$

$$= \sum_a m_a \frac{1}{M} \sum_b m_b l_{ab} \frac{1}{M} \sum_c m_c l_{ac}$$

$$= \frac{1}{M^2} \sum_a \sum_b \sum_c m_a m_b m_c l_{ab} l_{ac}$$

$$= \frac{1}{M^2} \sum_a \sum_b m_a m_b^2 l_{ab}^2 + \frac{1}{M^2} \sum_a \sum_b \sum_{c\,(c \neq b)} m_a m_b m_c l_{ab} l_{ac}$$

$$= \frac{1}{2M^2} \sum_a \sum_b m_a m_b^2 l_{ab}^2 + \frac{1}{2M^2} \sum_b \sum_a m_b m_a^2 l_{ba}^2$$

$$+ \frac{1}{2M^2} \sum_a \sum_b \sum_{c\,(c \neq b)} m_a m_b m_c l_{ab} l_{ac} + \frac{1}{2M^2} \sum_b \sum_a \sum_{c\,(c \neq a)} m_b m_a m_c l_{ba} l_{bc}$$

$$= \frac{1}{2M^2} \sum_a \sum_b m_a m_b l_{ab}^2 (m_a + m_b)$$

$$+ \frac{1}{2M^2} \sum_a \sum_b \sum_{c\,\binom{c \neq b}{c \neq a}} m_a m_b m_c \underbrace{(l_{ab} l_{ac} + l_{ba} l_{bc})}$$

$$= l_{ab} l_{ac} + l_{ab} l_{cb}$$
$$= l_{ab} [l_{ac} + l_{cb}]$$
$$= l_{ab}^2$$

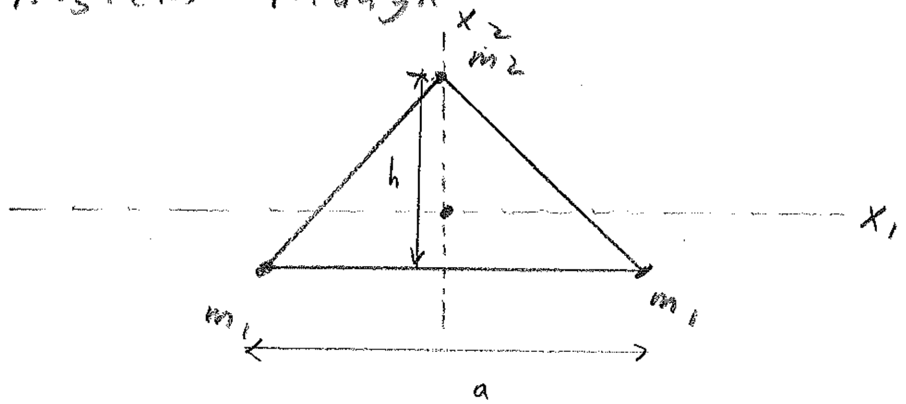
$$= \frac{1}{2M^2} \sum_a \sum_b m_a m_b l_{ab}^2 (m_a + m_b) + \frac{1}{2M^2} \sum_a \sum_b \sum_{c\,\binom{c \neq b}{c \neq a}} m_a m_b l_{ab}^2 m_c$$

$$= \frac{1}{2M^2} \sum_a \sum_b m_a m_b l_{ab}^2 \left[m_a + m_b + \sum_{\substack{c \\ c \neq b \\ c \neq a}} m_c \right]$$

$$= \frac{1}{2M} \sum_a \sum_b m_a m_b l_{ab}^2$$

$$= \frac{1}{M} \sum_a \sum_{b > a} m_a m_b l_{ab}^2 \quad \text{(so no repeat since } l_{12}^2 = l_{21}^2 \text{)}$$

(b) coplanar triatomic molecule shaped like an isosceles triangle



CoM at origin $(x_1, x_2, x_3) = (0,0,0)$

All masses have ~~m~~ $x_3 = 0$ so

$$I_1 = \sum m(x_2^2 + x_3^2) = \sum m x_2^2$$
$$I_2 = \sum m (x_1^2 + x_3^2) = \sum m x_1^2$$
$$I_3 = \sum m(x_1^2 + x_2^2) = I_1 + I_2$$

~~m_1 : (~~

<u>CoM</u>: $m_2 x_2 - m_1(h - x_2) - m_1(h - x_2) = 0$

$$(m_2 + 2m_1) x_2 - 2m_1 h = 0$$

$$x_2 = \frac{2m_1 h}{M}, \quad h - x_2 = \frac{m_2 h}{M}$$

Thus,
m_1 : $\left(-\frac{a}{2}, -\frac{m_2 h}{M}, 0\right)$

m_1 : $\left(\frac{a}{2}, -\frac{m_2 h}{M}, 0\right)$

m_2 : $\left(0, \frac{2m_1 h}{M}, 0\right)$

$$\rightarrow I_1 = m_1\left(\frac{-m_2 h}{M}\right)^2 + m_1\left(\frac{-m_2 h}{M}\right)^2 + m_2\left(\frac{2m_1 h}{M}\right)^2$$

$$= \frac{m_1 m_2 h^2}{M^2}(2m_2 + 4m_1)$$

$$= \boxed{2\frac{m_1 m_2}{M} h^2}$$

$$I_2 = m_1\left(\frac{-a}{2}\right)^2 + m_1\left(\frac{a}{2}\right)^2 + m_2 \cdot 0^2$$

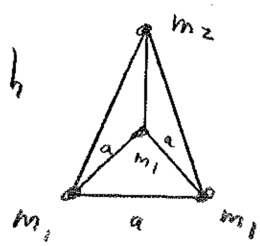
$$= 2 m_1 \frac{a^2}{4}$$

$$= \boxed{\frac{1}{2} m_1 a^2}$$

$$I_3 = I_1 + I_2$$

$$= \boxed{\frac{1}{2} m_1 a^2 + \frac{2 m_1 m_2}{M} h^2}$$

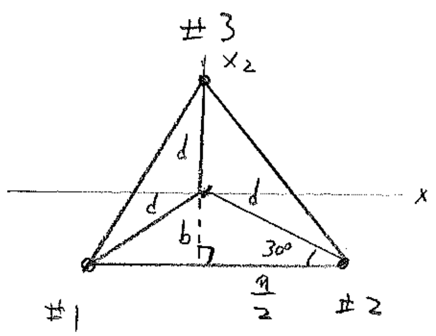
(c) tetratomic molecule which is an equilateral-based tetrahedron



put equilateral triangle in x_3 = constant plane.
x_3 axis passes through m_2 and com of equilateral triangle.

$$COM_3 : (3m_1 + m_2) R = m_2 h$$

$$\boxed{R = \frac{m_2 h}{M}}$$ (height of com above plane of equilateral triangle)



$$\cos 30° = \frac{a/2}{d} = \frac{a}{2d}$$

$$\cos 30° = \frac{\sqrt{3}}{2}$$

$$\rightarrow \boxed{d = \frac{a}{\sqrt{3}}}$$

$$\sin 30° = \frac{b}{d} \rightarrow \boxed{b = d \sin 30° = d/2 = \frac{a}{2\sqrt{3}}}$$

Thus, coordinate

$$m_1 : \left(-\frac{a}{2},\ \frac{-a}{2\sqrt{3}},\ \frac{-m_2 h}{m} \right)$$

$$m_1 : \left(\frac{a}{2},\ \frac{-a}{2\sqrt{3}},\ -\frac{m_2 h}{m} \right)$$

$$m_1 : \left(0,\ \frac{a}{\sqrt{3}},\ -\frac{m_2 h}{m} \right)$$

$$m_2 : \left(0, 0, \frac{3 m_1 h}{m} \right)$$

$$I_1 = \sum m (x_2^2 + x_3^2)$$

$$= m_1\left(\left(\frac{-a}{2\sqrt{3}}\right)^2 + \left(\frac{-m_2 h}{m}\right)^2\right) + m_1\left(\left(\frac{-a}{2\sqrt{3}}\right)^2 + \left(\frac{-m_2 h}{m}\right)^2\right)$$

$$+ m_1\left(\left(\frac{a}{\sqrt{3}}\right)^2 + \left(\frac{-m_2 h}{m}\right)^2\right) + m_2\left(\frac{3 m_1 h}{m}\right)^2$$

$$= 3\frac{m_1 m_2^2 h^2}{m^2} + 9\frac{m_2 m_1^2 h^2}{m^2} + m_1 \frac{a^2}{3}\left(\frac{1}{4} + \frac{1}{4} + 1\right)$$

$$= \frac{3 m_1 m_2 h^2}{m^{\cancel{2}}}\left(\cancel{m_2 + 3 m_1}\right) + \frac{1}{2} m_1 a^2$$

$$= \boxed{\frac{1}{2} m_1 a^2 + 3\frac{m_1 m_2}{m} h^2}$$

$$I_2 = \sum m (x_1^2 + x_3^2)$$

$$= m_1\left(\left(\frac{-a}{2}\right)^2 + \left(\frac{-m_2 h}{m}\right)^2\right) + m_1\left(\left(\frac{a}{2}\right)^2 + \left(\frac{-m_2 h}{m}\right)^2\right)$$

$$+ m_1\left(\frac{-m_2 h}{m}\right)^2 + m_2\left(\frac{3 m_1 h}{m}\right)^2$$

$$= 3\frac{m_1 m_2^2 h^2}{m^2} + 9\frac{m_2 m_1^2 h^2}{m^2} + 2 m_1 \frac{a^2}{4}$$

$$= \frac{3 m_1 m_2}{M^{\cancel{2}}} h^2 \cancel{(m_2 + 3m_1)} + \frac{1}{2} m_1 a^2$$

$$= \boxed{\frac{1}{2} m_1 a^2 + \frac{3 m_1 m_2}{M} h^2}$$ — same as I_1

$$I_3 = \sum m (x_1^2 + x_2^2)$$

$$= m_1 \left(\left(\frac{-a}{2}\right)^2 + \left(\frac{-a}{2\sqrt{3}}\right)^2 \right) + m_1 \left(\left(\frac{a}{2}\right)^2 + \left(\frac{-a}{2\sqrt{3}}\right)^2 \right)$$

$$+ m_1 \left(\frac{a}{\sqrt{3}}\right)^2 + \cancel{m_2 \cdot 0}$$

$$= m_1 \left[\frac{a^2}{4} + \frac{a^2}{4 \cdot 3} + \frac{a^2}{4} + \frac{a^2}{4 \cdot 3} + \frac{a^2}{3} \right]$$

$$= m_1 a^2 \left[\frac{1}{2} + \frac{1}{3} + \frac{2}{12} \right]$$

$$\frac{3}{6} + \frac{2}{6} \quad \frac{1}{6}$$

$$= \boxed{m_1 a^2}$$

__Note:__ If $m_1 = m_2$, $h = \sqrt{\frac{2}{3}}\, a$ (regular tetrahedron)

then

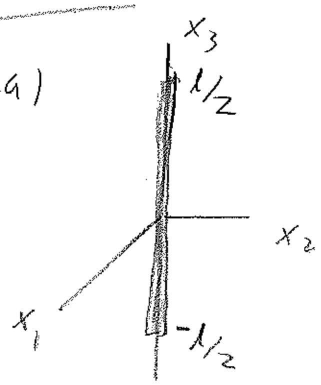
$$I_1 = I_2 = \frac{1}{2} m_1 a^2 + \frac{\cancel{3}\, m_1^{\cancel{2}}}{(4 \cancel{m_1})} \frac{2}{\cancel{3}} a^2$$

$$= \boxed{m_1 a^2}$$

$I_3 = \boxed{m_1 a^2}$ — all equal

§32, Prob 2:

(a)



$$\rho = \frac{M}{l}\,\delta(x)\,\delta(y)$$

$$I_3 = 0$$

$$I_1 = I_2 = \int \rho\, dV\, z^2$$

$$= \int_{-l/2}^{l/2} \frac{M}{l}\, dz\, z^2$$

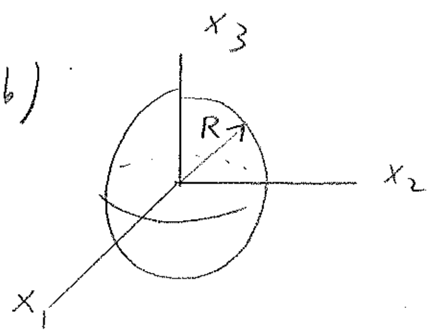
$$= \frac{M}{l}\,\frac{z^3}{3}\Big|_{-l/2}^{l/2}$$

$$= \frac{M}{l}\,\frac{1}{3}\left(\frac{l}{2}\right)^3 \cdot 2$$

$$= \frac{M}{l}\,\frac{1}{3}\,\frac{l^3}{8}\cdot 2$$

$$= \boxed{\frac{1}{12} M l^2}$$

(b)



$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

$$I_1 = I_2 = I_3 \equiv I$$

$$I_1 = \int \rho\, dV\,(y^2 + z^2)$$

$$I_2 = \int \rho\, dV\,(z^2 + x^2)$$

$$I_3 = \int \rho\, dV\,(x^2 + y^2)$$

$$3I = I_1 + I_2 + I_3$$

$$= \int \rho\, dV\,[\,y^2 + z^2 + z^2 + x^2 + x^2 + y^2\,]$$

$$= 2\int \rho\, dV\,(x^2 + y^2 + z^2)$$

$$= 2\int \rho\, dV\, r^2$$

$$= 2\rho \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \int_0^R r^2 dr\, r^2$$

$$3I = 2\rho \cdot 2\pi \cdot 2 \left.\frac{r^5}{5}\right|_0^R$$

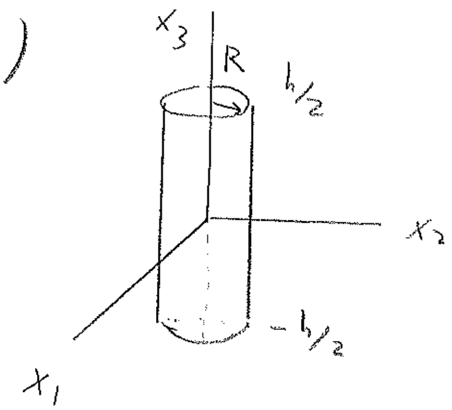
$$= \frac{8\pi\rho}{5} R^5$$

$$= \frac{8\cancel{\pi}}{5} \frac{M}{\frac{4}{3}\cancel{\pi} R^3} R^5$$

$$= \frac{2\cdot 3}{5} MR^2$$

$$\rightarrow \boxed{I = \frac{2}{5} MR^2}$$

(c)



$$\rho = \frac{M}{\text{volume}} = \frac{M}{\pi R^2 h}$$

$$I_3 = \int \rho\, dV (x^2+y^2)$$

$$= \rho \int_{-h/2}^{h/2} dz \int_0^{2\pi} d\phi \int_0^R s\, ds\, s^2$$

$$= \rho h 2\pi \left.\frac{s^4}{4}\right|_0^R$$

$$= \rho h \frac{\pi}{2} R^4$$

$$= \frac{M}{\cancel{\pi} R^2 \cancel{h}} \cancel{h} \frac{\pi}{2} R^4$$

$$= \boxed{\frac{1}{2} MR^2} \quad \text{(indep. of } h)$$

$$I_1 = I_2$$

$$2I_1 = I_1 + I_2$$

$$= \int \rho\, dV (y^2+z^2) + \int \rho\, dV (z^2+x^2)$$

$$= \int \rho\, dV \left[2z^2 + x^2+y^2\right]$$

$$= 2\int \rho\, dV z^2 + I_3$$

$$\int \rho\, dV \cdot z^2 = \rho \int_{-h/2}^{h/2} dz \int_0^{2\pi} d\phi \int_0^R s\, ds \cdot z^2$$

$$= \rho \frac{z^3}{3} \Big|_{-h/2}^{h/2} 2\pi \frac{s^2}{2} \Big|_0^R$$

$$= \rho \frac{2}{3} \left(\frac{h}{2}\right)^3 \cancel{2}\pi \frac{1}{\cancel{2}} R^2$$

$$= \frac{M}{\cancel{\pi}\cancel{R^2} h} \cdot \frac{2}{3} \frac{h^3}{8} \cancel{\pi} \cancel{R^2}$$

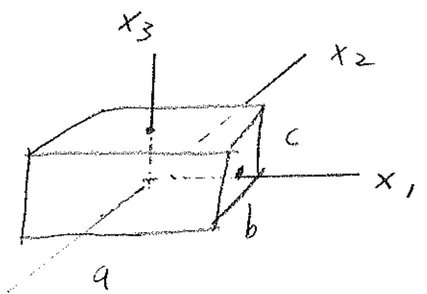
$$= \frac{1}{12} M h^2$$

Thus, $2 I_1 = 2\left(\frac{1}{12} M h^2\right) + \frac{1}{2} M R^2$

$$\rightarrow \quad I_1 = I_2 = \frac{1}{12} M h^2 + \frac{1}{4} M R^2$$

$$= \boxed{\frac{1}{4} M \left(R^2 + \frac{1}{3} h^2\right)}$$

(d)



(x_1, x_2, x_3)
(a, b, c)

$$\rho = \frac{M}{abc}$$

$$I_1 = \int \rho\, dV\, (y^2 + z^2)$$

$$= \rho \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz\, (y^2 + z^2)$$

$$= \rho a \left(c \frac{y^3}{3} \Big|_{-b/2}^{b/2} + b \frac{z^3}{3} \Big|_{-c/2}^{c/2} \right.$$

$$= \rho a \left(c \frac{2}{3} \left(\frac{b}{2}\right)^3 + b \frac{2}{3} \left(\frac{c}{2}\right)^3 \right)$$

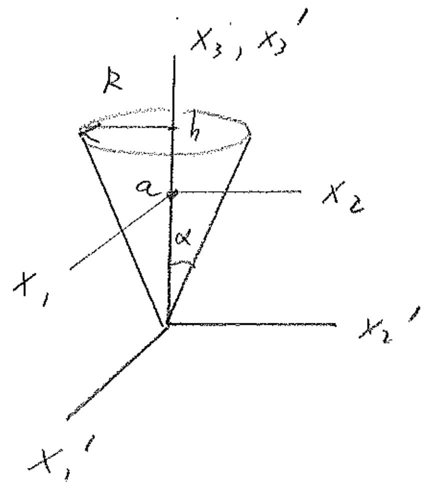
$$= \rho a \, \frac{2}{3} \cdot \frac{1}{8} \left(c b^3 + b c^3 \right)$$

$$= \frac{m}{\cancel{abc}} \cancel{abc} \, \frac{1}{12} \left(b^2 + c^2 \right)$$

$$= \boxed{\frac{1}{12} m \left(b^2 + c^2 \right)} = I_1$$

similarly, $\boxed{\begin{aligned} I_2 &= \frac{1}{12} m \left(c^2 + a^2 \right) \\ I_3 &= \frac{1}{12} m \left(a^2 + b^2 \right) \end{aligned}}$

(e)



a = height of com

$$vol = \int_h dV$$

$$= \int_0^h dz \int_0^{\frac{R}{h} z} s \, ds \int_0^{2\pi} d\phi$$

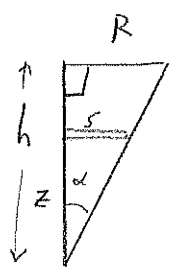
$$= \cancel{2}\pi \int_0^h dz \, \frac{s^2}{\cancel{2}} \Big|_0^{\frac{R}{h} z}$$

$$= \pi \left(\frac{R}{h} \right)^2 \int_0^h dz \, z^2$$

$$= \pi \left(\frac{R}{h} \right)^2 \frac{z^3}{3} \Big|_0^h$$

$$= \pi \frac{R^2}{h^2} \frac{1}{3} h^3$$

$$= \boxed{\frac{1}{3} \pi R^2 h}$$



$$\tan \alpha = \frac{R}{h}$$

$$s(z) = \frac{R}{h} z$$

$$\rho = \frac{m}{vol} = \frac{m}{\frac{1}{3} \pi R^2 h}$$

$$
\begin{aligned}
a &= z_{com} \\
&= \frac{1}{M}\int \rho\, dV \cdot z \\
&= \frac{\rho}{M}\int_0^h dz \int_0^{\frac{R}{h}z} s\, ds \int_0^{2\pi} d\phi \;\; z \\
&= \frac{\cancel{2}\pi\rho}{M}\int_0^h dz \cdot z \;\; \frac{s^2}{\cancel{2}}\Bigg|_0^{\frac{R}{h}z} \\
&= \frac{\pi\rho}{M}\int_0^h dz \;\; z^3 \left(\frac{R}{h}\right)^2 \\
&= \frac{\pi}{M}\rho \;\; \frac{R^2}{h^2}\;\; \frac{z^4}{4}\Bigg|_0^h \\
&= \frac{\cancel{\pi}}{\cancel{M}} \;\; \frac{3\cancel{M}}{\cancel{\pi}\cancel{R^2}h} \;\; \frac{\cancel{R^2}}{h^2} \;\; \frac{h^4}{4} \\
&= \boxed{\frac{3}{4}h}
\end{aligned}
$$

$$
\begin{aligned}
I_3' &= \int \rho\, dV\, (x^2+y^2) \\
&= \rho\int_0^h dz \int_0^{\frac{R}{h}z} s\, ds \int_0^{2\pi} d\phi \;\; s^2 \\
&= 2\pi\rho \int_0^h dz \;\; \frac{s^4}{4}\Bigg|_0^{\frac{R}{h}z} \\
&= \frac{\pi}{2}\rho\left(\frac{R}{h}\right)^4 \int_0^h dz\; z^4 \\
&= \frac{\pi}{2}\rho\left(\frac{R}{h}\right)^4 \frac{h^5}{5}
\end{aligned}
$$

$$= \frac{\pi}{2} \frac{3m}{\cancel{\pi} R^2 \cancel{h}} \frac{R^4}{\cancel{h^4}} \frac{\cancel{h^5}}{5}$$

$$= \boxed{\frac{3}{10} m R^2}$$

$I_1' = I_2'$

$$2 I_1' = I_1' + I_2'$$

$$= \int \rho \, dV (y^2 + z^2) + \int \rho \, dV (z^2 + x^2)$$

$$= \int \rho \, dV (x^2 + y^2) + 2 \int \rho \, dV \, z^2$$

$$= I_3 + 2\rho \int dV z^2$$

<u>Now</u>: $\rho \int dV z^2 = \rho \int_0^h dz \int_0^{\frac{R}{h} z} s\,ds \int_0^{2\pi} d\phi \; z^2$

$$= \cancel{2}\pi \rho \int_0^h dz \, z^2 \left. \frac{s^2}{\cancel{2}} \right|_0^{\frac{R}{h} z}$$

$$= \pi \rho \left(\frac{R}{h}\right)^2 \int_0^h dz \; z^4$$

$$= \pi \rho \left(\frac{R}{h}\right)^2 \frac{h^5}{5}$$

$$= \pi \frac{3m}{\pi \cancel{R^2} h} \frac{\cancel{R^2}}{h^2} \frac{h^5}{5}$$

$$= \frac{3}{5} m h^2$$

So $\quad 2 I_1' = 2 \cdot \frac{3}{5} m h^2 + \frac{3}{10} m R^2$

$$I_1' = \frac{3}{5} m h^2 + \frac{3}{20} m R^2$$

$$= \boxed{\frac{3}{5} m\left(h^2 + \frac{1}{4} R^2\right)}$$

<u>Use</u>: $I_{ik} = I_{ik}' - m\left(a^2 \delta_{ik} - a_i a_k\right)$

$$a_i = \frac{3}{4} h \delta_{i3}$$

$$I_3 = I_3' - m\left(\cancel{a^2 \cdot 1 - a \cdot a}\right)$$

$$= I_3' = \boxed{\frac{3}{10} m R^2}$$

$$I_1 = I_1' - m\left(a^2 - 0 \cdot 0\right)$$

$$= I_1' - m a^2$$

$$= \frac{3}{5} m\left(h^2 + \frac{1}{4} R^2\right) - m\left(\frac{3}{4} h\right)^2$$

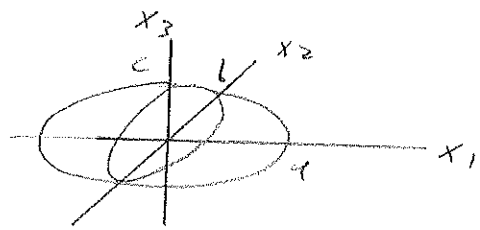
$$= \frac{3}{5} m\left(h^2 + \frac{1}{4} R^2\right) - m \frac{9}{16} h^2$$

$$= m\left(\underbrace{\left(\frac{3}{5} - \frac{9}{16}\right)}_{= \frac{48-45}{80} = \frac{3}{80}} h^2 + \frac{3}{20} R^2\right)$$

$$= \boxed{\frac{3}{20} m\left(R^2 + \frac{1}{4} h^2\right)}$$

$$I_2 = I_1$$

(f) Ellipsoid



$(x_1, x_2, x_3) \leftrightarrow (a, b, c)$

$Vol = \frac{4}{3}\pi abc$

$\rho = \frac{M}{\frac{4}{3}\pi abc}$

surface:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \qquad \begin{pmatrix} x_1 = x \\ x_2 = y \\ x_3 = z \end{pmatrix}$$

$$dV = dx\,dy\,dz$$

Transformation

$$u = \frac{x}{a}, \quad v = \frac{y}{b}, \quad w = \frac{z}{c}$$

$$u^2 + v^2 + w^2 = 1 \quad \text{(boundary)}$$

$$dx = a\,du, \quad dy = b\,dv, \quad dz = c\,dw$$

$$dV = abc\,du\,dv\,dw$$

$$I_1 = \int \rho\,dV\,(y^2 + z^2)$$

$$I_2 = \int \rho\,dV\,(z^2 + x^2)$$

$$I_3 = \int \rho\,dV\,(x^2 + y^2)$$

Let's calculate I_3:

$$I_3 = \rho \iiint dx\,dy\,dz\,(x^2 + y^2)$$

$$= \rho\,abc \iiint du\,dv\,dw\,(a^2u^2 + b^2v^2)$$

Use spherical polar:

$$u = r\sin\theta\cos\phi$$
$$v = r\sin\theta\sin\phi$$
$$w = r\cos\theta$$

$$du\,dv\,dw = r^2 dr \sin\theta\, d\theta\, d\phi$$
$$= r^2 dr\, d(\cos\theta)\, d\phi$$

$$I_3 = \rho abc \int_0^1 r^2 dr \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\phi \left(a^2 r^2 \sin^2\theta \cos^2\phi + b^2 r^2 \sin^2\theta \sin^2\phi \right)$$

$$= ① + ②$$

$$① = \rho abc\, a^2 \underbrace{\int_0^1 r^4 dr}_{\frac{r^5}{5}\Big|_0^1 = \boxed{\frac{1}{5}}} \underbrace{\int_{-1}^1 d(\cos\theta)(1-\cos^2\theta)}_{\int_{-1}^1 dx(1-x^2)} \underbrace{\int_0^{2\pi} d\phi \cos^2\phi}_{2\pi \cdot \frac{1}{2} = \boxed{\pi}}$$

$$\int_{-1}^1 dx(1-x^2) = \left(x - \frac{x^3}{3}\right)\Big|_{-1}^1 = 2 - \frac{2}{3} = \boxed{\frac{4}{3}}$$

$$= \rho abc\, a^2 \frac{4\pi}{15}$$

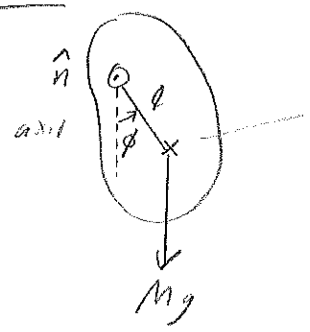
$$= \frac{M}{\cancel{\frac{4}{3}\pi abc}} \cancel{abc}\, a^2 \frac{4}{\cancel{15}_5} \cancel{\pi}$$

$$= \frac{1}{5} M a^2$$

$$② = \frac{1}{5} M b^2$$

$$\rightarrow \boxed{\begin{aligned} I_3 &= \frac{1}{5} M (a^2 + b^2) \\ \text{so} \quad I_1 &= \frac{1}{5} M (b^2 + c^2) \\ I_2 &= \frac{1}{5} M (c^2 + a^2) \end{aligned}}$$

§32, Prob 3



$$\tau = I\alpha$$

$$\tau = -Mgl\sin\phi$$

$$\simeq -Mgl\phi \quad \text{(small oscillations)}$$

$$\alpha = \ddot{\phi}$$

$$I = I_{com}(\hat{n}) + Ml^2 \quad \text{(parallel axis theorem)}$$

$$= I_{ik}\, n_i\, n_k + Ml^2$$

$$= \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} + Ml^2$$

$$= \left(I_1 n_1^2 + I_2 n_2^2 + I_3 n_3^2\right) + Ml^2$$

where
$$n_1 \equiv \hat{n}\cdot\hat{x}_1 \equiv \cos\alpha$$
$$n_2 \equiv \hat{n}\cdot\hat{x}_2 = \cos\beta$$
$$n_3 = \hat{n}\cdot\hat{x}_3 = \cos\gamma$$

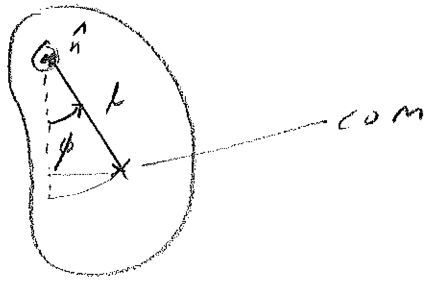
Thus,
$$I = \left(I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right) + Ml^2$$

$$I\ddot{\phi} = -Mgl\,\phi$$

$$\rightarrow \quad \omega = \sqrt{\frac{Mgl}{I}}$$

$$= \sqrt{\frac{Mgl}{Ml^2 + \left(I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)}}$$

Alternate solution: (Prob 3)



l : $\perp$ distance between com and axis of rotation $\hat{n} = \hat{\Omega}$

$V = l\dot{\phi}$ (com velocity)

$\vec{\Omega} = \Omega\hat{n} = \dot{\phi}\hat{n}$

$$T_{com} = \frac{1}{2} M l^2 \dot{\phi}^2$$

$$T_{rot} = \frac{1}{2} I_{ij}\Omega_i\Omega_j$$

$$= \frac{1}{2}\left[I_1(\vec{\Omega}\cdot\hat{x}_1)^2 + I_2(\vec{\Omega}\cdot\hat{x}_2)^2 + I_3(\vec{\Omega}\cdot\hat{x}_3)^2\right]$$

$$= \frac{1}{2}\dot{\phi}^2\left[I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right]$$

$$\rightarrow \quad T = \frac{1}{2}Ml^2\dot{\phi}^2 + \frac{1}{2}\left[I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right]\dot{\phi}^2$$

$$U = Mgl(1-\cos\phi), \quad \phi = 0 \text{ equilibrium}$$

$$\simeq \frac{1}{2}Mgl\phi^2$$

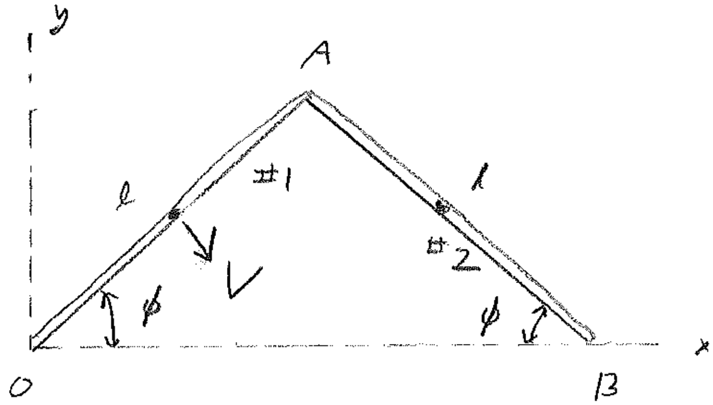
so

$$L = \frac{1}{2}\left[Ml^2 + (I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma)\right]\dot{\phi}^2 - \frac{1}{2}Mgl\phi^2$$

SHM with

$$\omega = \sqrt{\frac{Mgl}{Ml^2 + (I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma)}}$$

§ 32, Prob 4



$$T_1 = \frac{1}{2} M V^2 + \frac{1}{2} I_{ik} \Omega_i \Omega_k$$

$$V = \left(\frac{l}{2}\right)\dot\phi \ , \quad \vec\Omega = -\dot\phi \hat z \ , \quad I_{zz} = \frac{1}{12} M l^2$$

Thus,
$$T_1 = \frac{1}{2} M \left(\frac{l}{2}\right)^2 \dot\phi^2 + \frac{1}{2}\left(\frac{1}{12} M l^2\right) \dot\phi^2$$

$$= M l^2 \dot\phi^2 \left[\frac{1}{8} + \frac{1}{24}\right]$$

$$= \boxed{\frac{1}{6} M l^2 \dot\phi^2}$$

COM of 2nd rod:

$$x_2 = l\cos\phi + \frac{l}{2}\cos\phi = \frac{3}{2} l \cos\phi$$

$$y_2 = \frac{l}{2}\sin\phi$$

$$V^2 = \dot x_2^2 + \dot y_2^2$$

$$= \left(\frac{3}{2} l\right)^2 \sin^2\phi \, \dot\phi^2 + \left(\frac{l}{2}\right)^2 \cos^2\phi \, \dot\phi^2$$

$$= \frac{l^2}{4} \dot\phi^2 \left[9 \sin^2\phi + \cos^2\phi\right]$$

$$= \frac{l^2}{4} \dot\phi^2 \left[1 + 8\sin^2\phi\right]$$

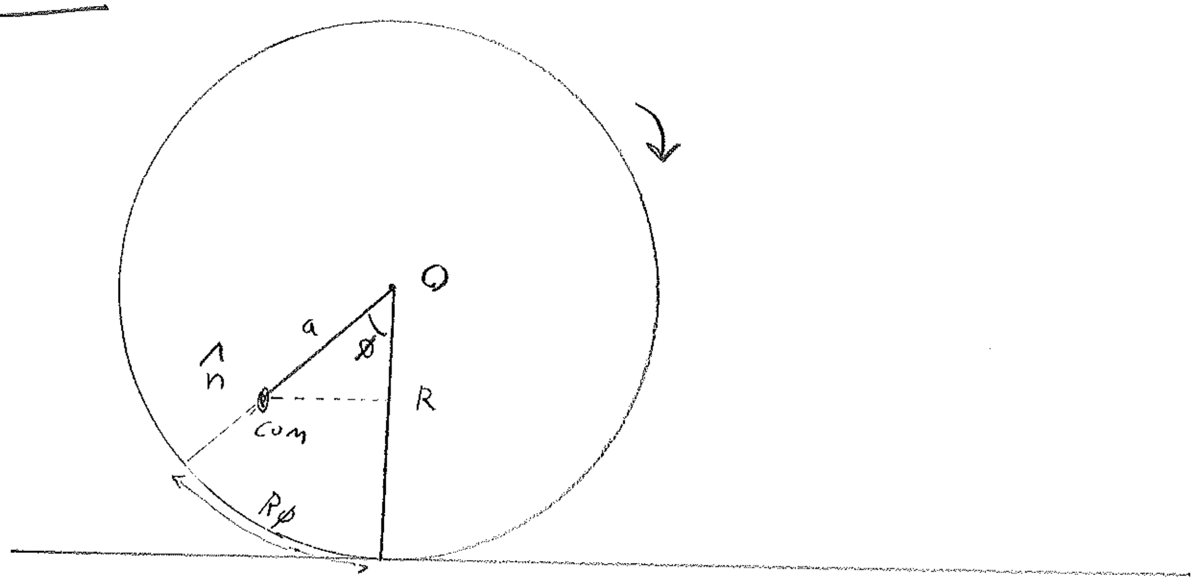
$$T_{2,rot} = \frac{1}{2} I_{ik} \Omega_i \Omega_k$$
$$= \frac{1}{24} m l^2 \dot{\phi}^2 \quad \text{(as before)}$$

$$T_2 = \frac{1}{2} M V^2 + \frac{1}{2} I_{ik} \Omega_i \Omega_k$$
$$= \frac{1}{2} \frac{m l^2}{4} \dot{\phi}^2 \left[1 + 8 \sin^2\phi \right] + \frac{1}{24} m l^2 \dot{\phi}^2$$
$$= m l^2 \dot{\phi}^2 \left[\frac{1}{8} + \frac{1}{24} + \sin^2\phi \right]$$
$$= m l^2 \dot{\phi}^2 \left[\frac{1}{6} + \sin^2\phi \right]$$
$$= \boxed{\frac{1}{6} m l^2 \dot{\phi}^2 + m l^2 \dot{\phi}^2 \sin^2\phi}$$

Thus,

$$T = T_1 + T_2$$
$$= \frac{1}{6} m l^2 \dot{\phi}^2 + \frac{1}{6} m l^2 \dot{\phi}^2 + m l^2 \dot{\phi}^2 \sin^2\phi$$
$$= \frac{1}{3} m l^2 \dot{\phi}^2 + m l^2 \dot{\phi}^2 \sin^2\phi$$
$$= \boxed{\frac{1}{3} m l^2 \dot{\phi}^2 \left[1 + 3 \sin^2\phi \right]}$$

§32, Prob 5



As the cylinder rolls without slipping to the right, the center of the cylinder O moves a distance

$$s = R\phi$$

The axis $\hat{n}$, through which the com passes, therefore moves a distance

$$x_{com} = s - a\sin\phi = R\phi - a\sin\phi$$

$$y_{com} = R - a\cos\phi$$

Thus,

$$T_{com} = \frac{1}{2} M V^2$$

$$= \frac{1}{2} M\left(\dot{x}_{com}^2 + \dot{y}_{com}^2\right)$$

$$= \frac{1}{2} M\left(\left(R\dot{\phi} - a\cos\phi\,\dot{\phi}\right)^2 + \left(a\sin\phi\,\dot{\phi}\right)^2\right)$$

$$= \frac{1}{2} M\left[R^2\dot{\phi}^2 + \underline{a^2\cos^2\phi\,\dot{\phi}^2} - 2aR\cos\phi\,\dot{\phi}^2 + \underline{a^2\sin^2\phi\,\dot{\phi}^2}\right]$$

$$= \frac{1}{2} M\dot{\phi}^2\left[R^2 + a^2 - 2aR\cos\phi\right]$$

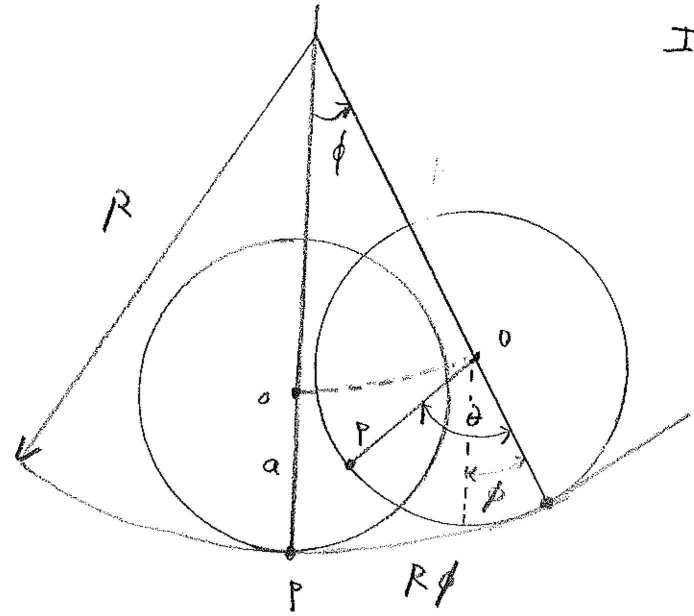
Also, $T_{rot} = \frac{1}{2} I\Omega^2 = \frac{1}{2} I\dot{\phi}^2$

$$\rightarrow \boxed{T = \frac{1}{2}\dot{\phi}^2\left[I + M\left(R^2 + a^2 - 2aR\cos\phi\right)\right]}$$

§ 32, Prob 6

uniform cylinder:

$I_3 = \frac{1}{2} M a^2$



COM moves a distance $s = (R-a)\phi$:

$\rightarrow \quad T_{com} = \frac{1}{2} M V^2 = \frac{1}{2} M \dot{s}^2 = \frac{1}{2} M (R-a)^2 \dot{\phi}^2$

Note: $R\phi = a\theta$ for rolling without slipping

The line OP has turned through an angle (wrt vertical)

$$\alpha = \theta - \phi$$

So angular velocity is $\Omega = \dot{\alpha} = \dot{\theta} - \dot{\phi}$

$= \left(\frac{R}{a} - 1\right)\dot{\phi}$

$= \left(\frac{R-a}{a}\right)\dot{\phi}$

$\rightarrow \quad T_{rot} = \frac{1}{2} I \Omega^2$

$= \frac{1}{2}\left(\frac{1}{2} M a^2\right)(\dot{\theta} - \dot{\phi})^2$

$= \frac{1}{2}\left(\frac{1}{2} M \cancel{a^2}\right)\frac{(R-a)^2}{\cancel{a^2}} \dot{\phi}^2$

$= \frac{1}{4} M (R-a)^2 \dot{\phi}^2$

Thus $T = T_{com} + T_{rot}$

$= \frac{1}{2} M (R-a)^2 \dot{\phi}^2 + \frac{1}{4} M (R-a)^2 \dot{\phi}^2$

$= \boxed{\frac{3}{4} M (R-a)^2 \dot{\phi}^2}$

Alternate solution: (Prob 6)

com velocity: $V = (R-a)\dot{\phi}$

$\vec{\Omega} = \Omega_3 \hat{x}_3$ (contact between rolling cylinder and cylindrical surface)

a : $\perp$ distance between com and $\vec{\Omega}$

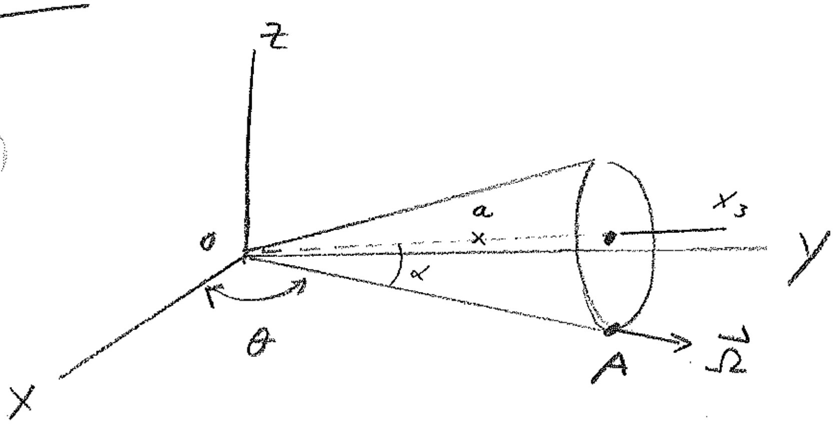
$\rightarrow V = a\Omega$

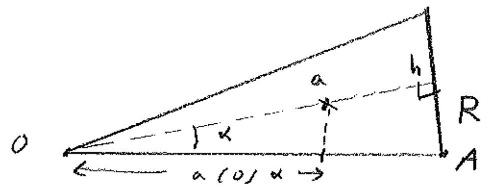
Thus, $a\Omega = (R-a)\dot{\phi}$

$\Omega_3 = \Omega = \left(\frac{R-a}{a}\right)\dot{\phi}$

Thus,
$$T = \frac{1}{2}MV^2 + \frac{1}{2}I_3\Omega_3^2$$
$$= \frac{1}{2}M(R-a)^2\dot{\phi}^2 + \frac{1}{2}I_3\left(\frac{R-a}{a}\right)^2\dot{\phi}^2$$
$$= \frac{1}{2}M(R-a)^2\dot{\phi}^2 + \frac{1}{2}\left(\frac{1}{2}M\cancel{a^2}\right)\frac{(R-a)^2}{\cancel{a^2}}\dot{\phi}^2$$
$$= \frac{3}{4}M(R-a)^2\dot{\phi}^2$$

§32, Prob 7





$a = \frac{3}{4} h$

$\tan\alpha = \frac{R}{h}$

$x_{com} = a \cos\alpha \cos\theta \rightarrow \dot{x}_{com} = -a\cos\alpha \sin\theta\, \dot{\theta}$

$y_{com} = a\cos\alpha \sin\theta \rightarrow \dot{y}_{com} = a\cos\alpha\cos\theta\,\dot{\theta}$

$z_{com} = a\sin\alpha \rightarrow \dot{z}_{com} = 0$

$$\rightarrow T_{com} = \frac{1}{2} M \left[a^2 \cos^2\alpha \underline{\sin^2\theta}\, \dot{\theta}^2 + a^2\cos^2\alpha \underline{\cos^2\theta}\, \dot{\theta}^2 \right]$$

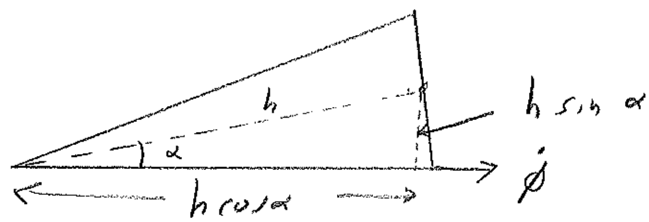
$$= \frac{1}{2} M a^2 \cos^2\alpha\, \dot{\theta}^2$$

$$= \frac{1}{2} M \left(\frac{3}{4} h\right)^2 \cos^2\alpha\, \dot{\theta}^2$$

$$= \boxed{\frac{9}{32} M h^2 \cos^2\alpha\, \dot{\theta}^2}$$

Angular Velocity vector:

$$\vec{\Omega} = \Omega\cos\alpha\, \hat{x}_3 + \Omega\sin\alpha\, \hat{x}_1$$



$$I_1 = I_2 = \frac{3}{20} M\left(R^2 + \frac{1}{4}h^2\right)$$

$$I_3 = \frac{3}{10} M R^2$$

$$\cancel{h} \sin\alpha\, \phi = \cancel{h} \cos\alpha\, \theta$$

$$\Omega \equiv \dot\phi = \cot\alpha\, \dot\theta$$

Thus,

$$\begin{aligned} T_{rot} &= \frac{1}{2} I_{ik} \Omega_i \Omega_k \\ &= \frac{1}{2}\left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2\right) \\ &= \frac{1}{2}\left(I_1 \Omega^2 \sin^2\alpha + I_3 \Omega^2 \cos^2\alpha\right) \\ &= \frac{1}{2} \dot\theta^2 \cot^2\alpha \left[I_1 \sin^2\alpha + I_3 \cos^2\alpha\right] \\ &= \boxed{\left(\frac{1}{2} I_1 \cos^2\alpha + \frac{1}{2} I_3 \frac{\cos^4\alpha}{\sin^2\alpha}\right)\dot\theta^2} \end{aligned}$$

$$\begin{aligned} \rightarrow T &= T_{cm} + T_{rot} \\ &= \frac{9}{32} M h^2 \cos^2\alpha\, \dot\theta^2 + \frac{1}{2}\left(\frac{3}{20}\right) M\left(R^2 + \frac{1}{4}h^2\right)\cos^2\alpha\, \dot\theta^2 \\ &\quad + \frac{1}{2}\left(\frac{3}{10}\right) M R^2 \frac{\cos^4\alpha}{\sin^2\alpha}\dot\theta^2 \\ &= \frac{9}{32} M h^2 \cos^2\alpha\, \dot\theta^2 + \frac{3}{40} M\left(\tan^2\alpha + \frac{1}{4}\right) h^2 \cos^2\alpha\, \dot\theta^2 \\ &\quad + \frac{3}{20} M h^2 \underbrace{\tan^2\alpha \frac{\cos^4\alpha}{\sin^2\alpha}}_{\cos^2\alpha} \dot\theta^2 \end{aligned}$$

$$= Mh^2\dot{\theta}^2\Big[\frac{9}{32}\cos^2\alpha + \frac{3}{40}\Big(\sin^2\alpha + \frac{1}{4}\cos^2\alpha\Big)$$
$$+ \frac{3}{20}\cos^2\alpha\Big]$$

$$= Mh^2\dot{\theta}^2\Big[\frac{9}{32}\cos^2\alpha + \frac{3}{40}\Big(1-\cos^2\alpha + \frac{1}{4}\cos^2\alpha\Big) + \frac{3}{20}\cos^2\alpha\Big]$$

$$= Mh^2\dot{\theta}^2\Big[\frac{3}{40} + \cos^2\alpha\underbrace{\Big(\frac{9}{32} - \frac{3}{40} + \frac{3}{160} + \frac{3}{20}\Big)}\Big]$$

$$= \frac{45 - 12 + 3 + 24}{160}$$

$$= \frac{36+24}{160}$$

$$= \frac{60}{160}$$

$$= \frac{15}{40}$$

$$= Mh^2\dot{\theta}^2\Big[\frac{3}{40} + \frac{15}{40}\cos^2\alpha\Big]$$

$$= \boxed{\frac{3}{40}Mh^2\dot{\theta}^2\big[1 + 5\cos^2\alpha\big]}$$

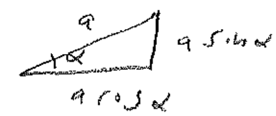
Alternative calculation:

velocity of com:

$$V = a\cos\alpha\,\dot\theta$$

~~Dot~~ $\vec\Omega$: along OA with magnitude Ω

$\perp$ distance from $\vec\Omega$ to com: $a\sin\alpha$



Rolling without slipping:

$$\Omega\, a\sin\alpha = V$$

$$\rightarrow \Omega = \frac{V}{a\sin\alpha} = \frac{\cancel{a}\cos\alpha\,\dot\theta}{\cancel{a}\sin\alpha} = \cot\alpha\,\dot\theta$$

$$\vec\Omega = \Omega\cos\alpha\,\hat{x}_3 + \Omega\sin\alpha\,\hat{x}_1$$

Thus,

$$T = \frac{1}{2}MV^2 + \frac{1}{2}\left(I_1\Omega_1^2 + I_3\Omega_3^2\right)$$

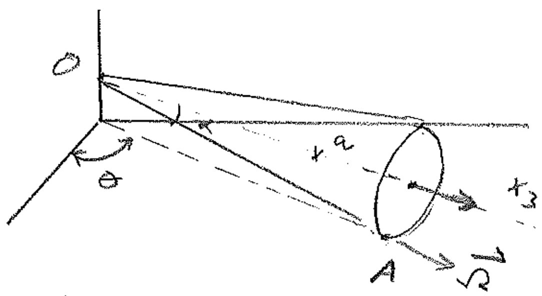
$$= \frac{1}{2}M a^2\cos^2\alpha\,\dot\theta^2 + \frac{1}{2}\left(I_1\sin^2\alpha\cot^2\alpha\,\dot\theta^2 + I_3\cos^2\alpha\cot^2\alpha\,\dot\theta^2\right)$$

$$= \frac{1}{2}M a^2\cos^2\alpha\,\dot\theta^2 + \frac{1}{2}\left(I_1\cos^2\alpha + I_3\frac{\cos^4\alpha}{\sin^2\alpha}\right)\dot\theta^2$$

...

Sec 32, Prob 8

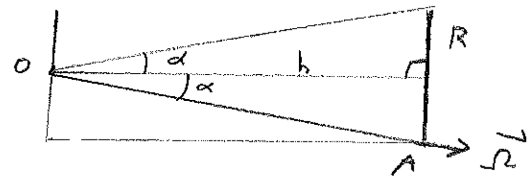
①



$$I_1 = I_2 = \frac{3}{20} M \left(R^2 + \frac{h^2}{4}\right)$$

$$I_3 = \frac{3}{10} M R^2$$

$$R = h \tan\alpha$$



$$\boxed{T_{com} = \frac{1}{2} M a^2 \dot{\theta}^2}, \qquad a = \frac{3}{4} h$$

$$\vec{\Omega} = \Omega \cos\alpha\, \hat{x}_3 + \Omega \sin\alpha\, \hat{x}_1, \quad \Omega = \dot{\phi} \text{ so } \Omega_1 = \dot{\phi}\sin\alpha$$

$$\Omega_3 = \dot{\phi}\cos\alpha$$

Now: $\dot{\phi}\cos\alpha\, R = h\dot{\theta}$

$$\dot{\phi} = \frac{h}{R\cos\alpha}\dot{\theta} = \frac{\cancel{h}}{\cancel{h}\tan\alpha\cos\alpha}\dot{\theta} = \frac{\dot{\theta}}{\sin\alpha}$$

Thus, $T_{rot} = \frac{1}{2}\left(I_1\Omega_1^2 + I_3\Omega_3^2\right)$

$$= \frac{1}{2}\left(I_1 \sin^2\alpha\,\dot{\phi}^2 + I_3\cos^2\alpha\,\dot{\phi}^2\right)$$

$$= \frac{1}{2}\left(I_1 \cancel{\sin^2\alpha}\frac{\dot{\theta}^2}{\cancel{\sin^2\alpha}} + I_3\cos^2\alpha\frac{\dot{\theta}^2}{\sin^2\alpha}\right)$$

$$= \boxed{\frac{1}{2}\left(I_1 + I_3\cot^2\alpha\right)\dot{\theta}^2}$$

~~NOTE:~~

$$\rightarrow T = \frac{1}{2} M a^2\dot{\theta}^2 + \frac{1}{2}\left(I_1 + I_3\cot^2\alpha\right)\dot{\theta}^2$$

$$= \frac{1}{2} M\left(\frac{3}{4}h\right)^2\dot{\theta}^2 + \frac{1}{2}\left[\frac{3}{20}M\left(h^2\tan^2\alpha + \frac{h^2}{4}\right) + \frac{3}{10}M h^2\tan^2\alpha\cot^2\alpha\right]\dot{\theta}^2$$

$$= M h^2\dot{\theta}^2\left[\frac{9}{32} + \frac{3}{40}\tan^2\alpha + \frac{3}{160} + \frac{3}{20}\right]$$

$$\frac{45 + 3 + 24}{160} = \frac{72}{160} = \frac{9}{20}$$

$$= Mh^2\dot\theta^2\left[\frac{9}{20} + \frac{3}{40}\tan^2\alpha\right]$$

$$= \frac{3}{40}Mh^2\dot\theta^2\left[\tan^2\alpha + 6\right]$$

$$= \frac{3}{40}Mh^2\dot\theta^2\left[\underbrace{\tan^2\alpha+1}_{\sec^2\alpha} + 5\right]$$

$$= \boxed{\frac{3}{40}Mh^2\dot\theta^2\left[\sec^2\alpha + 5\right]}$$

NOTE: This is just $\sec^2\alpha$ times the result from Prob 7

Alternative Calculation:

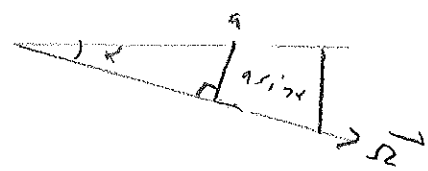
Velocity of com: $V = a\dot\theta$

$$\rightarrow T_{com} = \frac{1}{2}MV^2 = \frac{1}{2}Ma^2\dot\theta^2$$

$\vec\Omega$: directed along OA with magnitude Ω

$$\vec\Omega = \Omega\sin\alpha\,\hat{x}_1 + \Omega\cos\alpha\,\hat{x}_3$$

Rolling without slipping:



$$V = a\sin\alpha\,\Omega$$

$$\rightarrow \cancel{a}\dot\theta = \cancel{a}\sin\alpha\,\Omega$$

$$\Omega = \dot\theta/\sin\alpha$$

Thus, $T_{rot} = \frac{1}{2}\left(I_1\Omega_1^2 + I_3\Omega_3^2\right)$

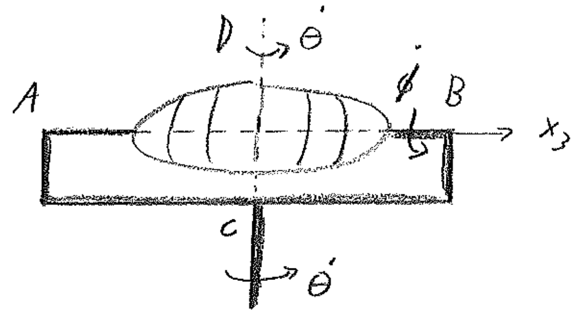
$$= \frac{1}{2}\left(I_1\cancel{\sin^2\alpha}\frac{\dot\theta^2}{\cancel{\sin^2\alpha}} + I_3\cos^2\alpha\frac{\dot\theta^2}{\sin^2\alpha}\right)$$

$$= \frac{1}{2}\left(I_1\dot\theta^2 + I_3\cot^2\alpha\,\dot\theta^2\right)$$

So $T = \frac{1}{2}Ma^2\dot\theta^2 + \frac{1}{2}\left(I_1 + I_3\cot^2\alpha\right)\dot\theta^2$

...

§32, Prob 9



x_1, x_2 rotating w.r.t CD

$\vec{\Omega} = \vec{\dot\theta} + \vec{\dot\phi}$ where $\vec{\dot\phi} = \dot\phi\,\hat{x}_3$

$\vec{\dot\theta} = \cos\phi\,\dot\theta\,\hat{x}_1 + \sin\phi\,\dot\theta\,\hat{x}_2$

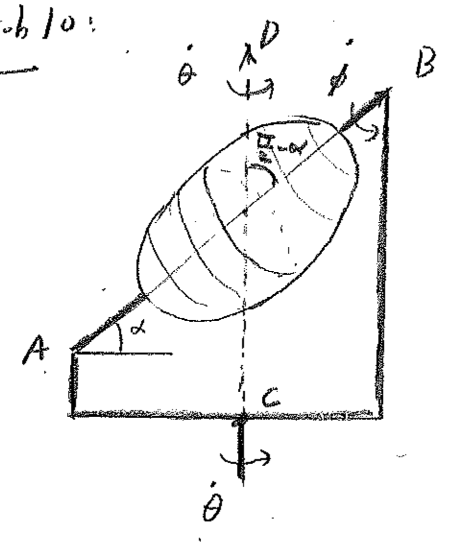
Thus, $\vec{\Omega} = \cos\phi\,\dot\theta\,\hat{x}_1 + \sin\phi\,\dot\theta\,\hat{x}_2 + \dot\phi\,\hat{x}_3$

$$\rightarrow T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$= \frac{1}{2}\left(I_1\cos^2\phi\,\dot\theta^2 + I_2\sin^2\phi\,\dot\theta^2 + I_3\dot\phi^2\right)$$

$$= \boxed{\frac{1}{2}\left(I_1\cos^2\phi + I_2\sin^2\phi\right)\dot\theta^2 + \frac{1}{2}I_3\dot\phi^2}$$

§32, Prob 10:



CD: $\vec{\dot\theta}$ $\qquad \vec{\Omega} = \vec{\dot\theta} + \vec{\dot\phi}$

AB: $\vec{\dot\phi}$

$$\vec{\dot\phi} = \dot\phi\,\hat{x}_3$$

$$\vec{\dot\theta} = \Big[\cos\left(\tfrac{\pi}{2}-\alpha\right)\hat{x}_3 + \sin\left(\tfrac{\pi}{2}-\alpha\right)\cos\phi\,\hat{x}_1 + \sin\left(\tfrac{\pi}{2}-\alpha\right)\sin\phi\,\hat{x}_1\Big]\dot\theta$$

Now: $\cos\left(\frac{\pi}{2}-\alpha\right) = \cancelto{0}{\cos\frac{\pi}{2}}\cos\alpha + \sin\frac{\pi}{2}\sin\alpha = \sin\alpha$

$\sin\left(\frac{\pi}{2}-\alpha\right) = \sin\frac{\pi}{2}\cos\alpha - \cancelto{0}{\cos\frac{\pi}{2}}\sin\alpha = \cos\alpha$

thus, $\vec{\Omega} = \vec{\dot\theta} + \vec{\dot\phi}$

$$= \dot\phi\,\hat{x}_3 + \sin\alpha\,\hat{x}_3 + \cos\alpha\cos\phi\,\hat{x}_1 + \cos\alpha\sin\phi\,\hat{x}_2$$

$$= \dot\theta\cos\alpha\cos\phi\,\hat{x}_1 + \dot\theta\cos\alpha\sin\phi\,\hat{x}_2 + (\dot\phi + \dot\theta\sin\alpha)\,\hat{x}_3$$

$I_1 = I_2$ (symmetric top)

I_3 (symmetry axis)

$$T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$= \frac{1}{2}\left(I_1\dot\theta^2\cos^2\alpha\cos^2\phi + I_1\dot\theta^2\cos^2\alpha\sin^2\alpha + I_3(\dot\phi + \dot\theta\sin\alpha)^2\right)$$

$$= \boxed{\frac{1}{2}\left[I_1\cos^2\alpha\,\dot\theta^2 + I_3(\dot\phi + \dot\theta\sin\alpha)^2\right]}$$