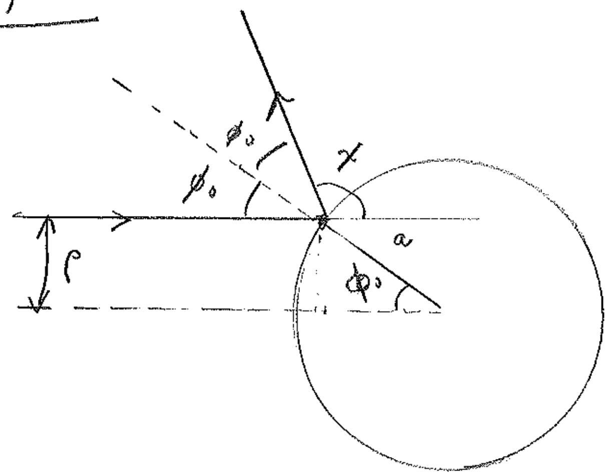
Sec 18, Prob 1



$$2\phi_0 + \chi = \pi$$
$$\chi = \pi - 2\phi_0$$
$$\phi_0 = \frac{\pi}{2} - \frac{\chi}{2}$$

$\sin\phi_0 = \frac{\rho}{a} \longrightarrow$

$$\begin{aligned}\rho &= a \sin\phi_0 \\ &= a\sin\left(\frac{\pi}{2} - \frac{\chi}{2}\right) \\ &= a\left(\sin\frac{\pi}{2}\cos\frac{\chi}{2} - \cancelto{0}{\cos\frac{\pi}{2}}\sin\frac{\chi}{2}\right) \\ &= a\cos\frac{\chi}{2}\end{aligned}$$

$$\begin{aligned}d\sigma &= 2\pi\rho(\chi)\left|\frac{d\rho}{d\chi}\right| d\chi \\ &= 2\pi\, a\cos\left(\frac{\chi}{2}\right)\frac{a}{2}\sin\left(\frac{\chi}{2}\right)d\chi \\ &= \pi a^2 \sin\left(\frac{\chi}{2}\right)\cos\left(\frac{\chi}{2}\right)d\chi \\ &= \boxed{\frac{\pi a^2}{2}\sin\chi\, d\chi} \leftarrow d\sigma\end{aligned}$$

Total cross-section

$$\begin{aligned}\sigma = \int d\sigma &= \int_0^{\pi} \frac{\pi a^2}{2}\sin\chi\, d\chi \\ &= -\frac{\pi a^2}{2}\cos\chi\Big|_0^{\pi} \\ &= -\frac{\pi a^2}{2}(-1-1) \\ &= \pi a^2 \qquad \text{(as expected)}\end{aligned}$$

NOTE: Using $d\Omega = 2\pi\sin\chi\, d\chi$

$$\rightarrow \boxed{d\sigma = \frac{a^2}{4}d\Omega}$$

Convert to L system: (sphere at rest)

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}, \qquad \theta_2 = \frac{1}{2}(\pi - \chi)$$

For all cases m_1, m_2

$$\boxed{\chi = \pi - 2\theta_2}$$

$$\boxed{\cos\chi = -\frac{m_1}{m_2}\sin^2\theta_1 \pm \cos\theta_1\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1}}$$

where for $m_1 < m_2$, $+$ sign must be taken

and for $m_1 > m_2$, then $\pm$ values are both needed

and for $m_1 = m_2$, $\theta_1 = \frac{\chi}{2} \to \chi = 2\theta_1$

$$d\sigma_2 = \frac{\pi a^2}{2}|\sin\chi|\,d\chi$$

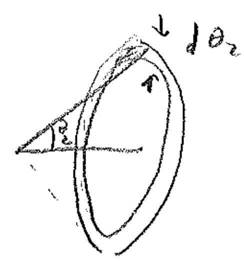
$$= \frac{\pi a^2}{2}\left|\sin(\pi - 2\theta_2)\,d(\pi - 2\theta_2)\right|$$

$$= \frac{\pi a^2}{\cancel{2}}\left|\left[\overset{0}{\sin\pi}\cos(2\theta_2) - \underset{-1}{\cos\pi}\sin(2\theta_2)\right]\cancel{-2}\,d\theta_2\right|$$

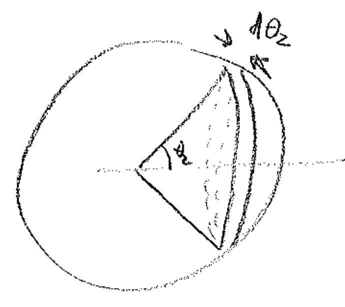
$$= +\pi a^2|\sin(2\theta_2)|\,d\theta_2$$

$$= +2\pi a^2|\sin\theta_2\cos\theta_2|\,d\theta_2$$

<u>Now:</u> $d\Omega_2 = 2\pi\sin\theta_2\,d\theta_2$



$$\rightarrow \quad \boxed{d\sigma_2 = a^2|\cos\theta_2|\,d\Omega_2}$$



For $m_1 = m_2$: $\qquad \chi = 2\theta_1$

$$d\sigma_1 = \frac{\pi a^2}{\cancel{2}} \sin(2\theta_1)\, d(\cancel{2}\theta_1)$$

$$= 2\pi a^2 \sin\theta_1 \cos\theta_1 \, d\theta_1$$

$$= a^2 \cos\theta_1 \; 2\pi \sin\theta_1 \, d\theta_1$$

$$= a^2 \cos\theta_1 \, d\Omega_1$$

$$\rightarrow \boxed{d\sigma_1 = a^2 |\cos\theta_1| \, d\Omega_1}$$

For $m_1 < m_2$:

$$\cos\chi = -\frac{m_1}{m_2}\sin^2\theta_1 + \cos\theta_1 \sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1}$$

$$\rightarrow \quad d\sigma_1 = \frac{\pi a^2}{2} |\sin\chi| \, d\chi$$

$$= \frac{\pi a^2}{2} d(\cos\chi)$$

$$= \frac{\pi a^2}{2} d\left[-\frac{m_1}{m_2}\left(1 - \cos^2\theta_1\right) + \cos\theta_1 \sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \left(1 - \cos^2\theta_1\right)} \right]$$

$$= \frac{\pi a^2}{2} \Bigg[\frac{m_1}{m_2} 2\cos\theta_1 \, d(\cos\theta_1) + d(\cos\theta_1)\sqrt{}$$

$$+ \frac{1}{\cancel{2}\sqrt{}} \cos\theta_1 \cdot \cancel{2} \cos\theta_1 \left(\frac{m_1}{m_2}\right)^2 d(\cos\theta_1) \Bigg]$$

$$= \frac{\pi a^2}{2} d(\cos\theta_1) \left[2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \sqrt{} + \left(\frac{m_1}{m_2}\right)^2 \frac{\cos^2\theta_1}{\sqrt{}} \right]$$

$$d\sigma_1 = \frac{\pi a^2}{2} \, d(\cos\theta_1) \left[2\left(\frac{m_1}{m_2}\right) + \frac{1-\left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1 + \left(\frac{m_1}{m_2}\right)^2 \cos^2\theta_1}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1}} \right]$$

$$\cancel{\frac{2\pi a^2}{4} \sin\theta_1 \, d\theta_1}$$

$$= \frac{a^2}{4} \underbrace{2\pi \sin\theta_1 \, d\theta_1}_{d\Omega_1} \left[2\left(\frac{m_1}{m_2}\right) + \frac{1+\left(\frac{m_1}{m_2}\right)^2 (\cos^2\theta_1 - \sin^2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1}} \right]$$

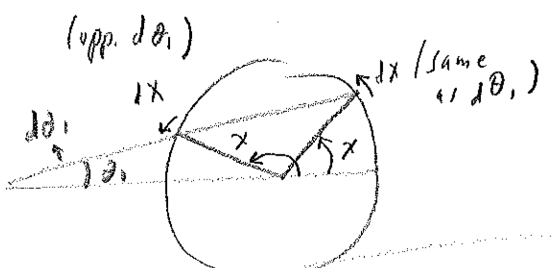
$$= \frac{1}{4} a^2 \, d\Omega_1 \left[2\left(\frac{m_1}{m_2}\right) + \frac{1+\left(\frac{m_1}{m_2}\right)^2 \cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1}} \right] \quad \checkmark$$

<u>For $m_1 > m_2$</u>

Two contributions from $\pm\sqrt{}$

$$d\sigma_1^{(+)} = \frac{1}{4} a^2 \, d\Omega_1 \left[2\left(\frac{m_1}{m_2}\right) + \frac{1+\left(\frac{m_1}{m_2}\right)^2 \cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1}} \right]$$

$$d\sigma_1^{(-)} = \frac{1}{4} a^2 \, d\Omega_1 \left[2\left(\frac{m_1}{m_2}\right) - \frac{1+\left(\frac{m_1}{m_2}\right)^2 \cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1}} \right]$$



$$\rightarrow \boxed{d\sigma_1 = d\sigma_1^{(+)} - d\sigma_1^{(-)} = \frac{1}{2} a^2 \, d\Omega_1 \, \frac{1+\left(\frac{m_1}{m_2}\right)^2 \cos(2\theta_1)}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1}}}$$

start with: $(m_1 < m_2, \quad 0 \le \theta_1 \le \pi)$

$$d\sigma_1 = \frac{1}{4}a^2\left[2\left(\frac{m_1}{m_2}\right)\cos\theta_1 + \frac{1+\left(\frac{m_1}{m_2}\right)^2\cos 2\theta_1}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\theta_1}}\right]d\Omega_1$$

$$d\Omega_1 = 2\pi\sin\theta_1\, d\theta_1 = -2\pi\, d(\cos\theta_1)$$

Total cross section: $\sigma_1 = \int d\sigma_1 = \int_0^{\pi}\left(\frac{d\sigma_1}{d\theta_1}\right)d\theta_1$

Not necessary to calculate since total cross-section should be the same in lab and CoM Frames!

Denote: $\frac{m_1}{m_2} \equiv \eta < 1$

$x_1 = \cos\theta_1, \quad \theta_1 = 0 \leftrightarrow x = 1$
$\theta_1 = \pi \leftrightarrow x = -1$

$[\cos 2\theta_1 = \cos^2\theta_1 - \sin^2\theta_1 = 2\cos^2\theta_1 - 1]$

Thus,

$$d\sigma_1 = \frac{1}{4}a^2\left[2\eta x + \frac{1+\eta^2(2x^2-1)}{\sqrt{1-\eta^2(1-x^2)}}\right](-2\pi)dx$$

$$= -\frac{1}{4}a^2(2\pi)\left[2\eta x + \frac{(1-\eta^2)+2\eta^2x^2}{\sqrt{(1-\eta^2)+\eta^2x^2}}\right]dx$$

$$= -\frac{\pi a^2}{2}\left[2\eta x + \frac{A^2+2\eta^2x^2}{\sqrt{A^2+\eta^2x^2}}\right]dx \qquad (A^2 \equiv 1-\eta^2) < 1$$

Thus,

$$\sigma_1 = \int_0^{\pi}\left(\frac{d\sigma_1}{d\theta_1}\right)d\theta_1 = \int_1^{-1} -\frac{\pi a^2}{2}\left[2\eta x + \frac{A^2}{\sqrt{A^2+\eta^2x^2}} + 2\eta^2\frac{x^2}{\sqrt{A^2+\eta^2x^2}}\right]dx$$

$$= \frac{\pi a^2}{2}\int_{-1}^{1} dx\left[\cancelto{0}{2\eta x} + \frac{A^2}{\sqrt{A^2+\eta^2x^2}} + 2\eta^2\frac{x^2}{\sqrt{A^2+\eta^2x^2}}\right]$$

Wolfram α:

$$\int_{-1}^{1} \frac{a\,dx}{\sqrt{a+(1-a)x^2}} = \frac{2a\sinh^{-1}\left(\sqrt{\frac{1}{a}-1}\right)}{\sqrt{1-a}}$$

$$\int_{-1}^{1} \frac{2(1-a)x^2}{\sqrt{a+(1-a)x^2}}\,dx = \frac{2-2a\sinh^{-1}\left(\sqrt{\frac{1}{a}-1}\right)}{\sqrt{1-a}}$$

Same as integrals we want to do with

$$a \Leftrightarrow A^2$$
$$1-a \Leftrightarrow 1-A^2 = \eta^2$$

Thus,

$$\sigma_1 = \frac{\pi a^2}{2}\left[\cancel{\frac{2a\sinh^{-1}(\ )}{\sqrt{\ }}} + \frac{2 \cancel{-2a\sinh^{-1}(\ )}}{\cancel{\sqrt{\ }}}\right]$$

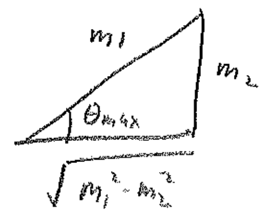
$$= \boxed{\pi a^2} \quad \text{(just like we found in C Frame)}$$

Start with: $\left(m_1 > m_2 , \quad 0 < \theta_1 < \theta_{max} = \sin^{-1}\left(\frac{m_2}{m_1}\right) \right)$

$$d\sigma_1 = \frac{1}{2} a^2 \frac{1 + \left(\frac{m_1}{m_2}\right)^2 \cos 2\theta_1}{\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1}} d\Omega_1$$

Now: $d\Omega_1 = 2\pi \sin\theta_1 \, d\theta_1 = -2\pi \, d(\cos\theta_1)$

Let: $\eta \equiv \frac{m_1}{m_2} > 1$



$x = \cos\theta_1 \rightarrow d\Omega_1 = -2\pi dx$

$\theta_1 = 0, \theta_{max} \rightarrow x = 1, \cos(\theta_{max})$

$$\cos(\theta_{max}) = \frac{\sqrt{m_1^2 - m_2^2}}{m_1} = \sqrt{1 - \left(\frac{m_2}{m_1}\right)^2} = \sqrt{1 - \frac{1}{\eta^2}}$$

$$\cos 2\theta_1 = \cos^2\theta_1 - \sin^2\theta_1 = 2\cos^2\theta_1 - 1 = 2x^2 - 1$$

$$\sin^2\theta_1 = 1 - \cos^2\theta_1 = 1 - x^2$$

$$\sqrt{1 - \left(\frac{m_1}{m_2}\right)^2 \sin^2\theta_1} = \sqrt{1 - \eta^2(1 - x^2)} = \sqrt{1 - \eta^2 + \eta^2 x^2} = \sqrt{\eta^2 x^2 - (\eta^2 - 1)} = \sqrt{\eta^2 x^2 - A^2}$$

$A^2 = \eta^2 - 1$

substitution

$\eta x = A \cosh u$

$\eta^2 x^2 - A^2 = A^2(\cosh^2 u - 1) = A^2 \sinh^2 u$

etc.

~~Thus,~~ $\cancel{\sqrt{1 - \frac{1}{\eta^2}}}$

$$\sigma_1 = \cancel{\int_1^{\sqrt{1-\frac{1}{\eta^2}}} -\frac{1}{2} a^2 (2\pi) \frac{1 + \eta^2(2x^2 - 1)}{\sqrt{\eta^2 x^2 - A^2}} dx}$$

$$= \cancel{\pi a^2 \int_{\sqrt{1-\frac{1}{\eta^2}}}^{1} dx \left[\frac{(1 - \eta^2)}{\sqrt{\eta^2 x^2 - A^2}} + \frac{2\eta^2 x^2}{\sqrt{\eta^2 x^2 - A^2}} \right]}$$

Thus,

$$\sigma_1 = \int_{\sqrt{1-\frac{1}{\eta^2}}}^{1} \frac{1}{\cancel{2}} a^2 (\cancel{2}\pi\, dx) \quad \frac{1 + \eta^2(2x^2-1)}{\sqrt{\eta^2 x^2 - A^2}}$$

$$= \pi a^2 \int_{\sqrt{1-\frac{1}{\eta^2}}}^{1} dx \quad \frac{-A^2 + 2\eta^2 x^2}{\sqrt{\eta^2 x^2 - A^2}}$$

$$= \pi a^2 \int_{\sqrt{1-\frac{1}{\eta^2}}}^{1} dx \quad \frac{-A^2 + 2\eta^2 x^2}{\eta\sqrt{x^2 - \frac{A^2}{\eta^2}}}$$

$$A^2 = \eta^2 - 1$$
$$\frac{A^2}{\eta^2} = 1 - \frac{1}{\eta^2}$$

$$= \pi a^2 \int_{\sqrt{1-\frac{1}{\eta^2}}}^{1} dx \quad \frac{-\eta\left(1-\frac{1}{\eta^2}\right) + 2\eta x^2}{\sqrt{x^2 - \left(1-\frac{1}{\eta^2}\right)}}$$

$$= \pi a^2 \int_{B}^{1} dx \quad \frac{-\eta B^2 + 2\eta x^2}{\sqrt{x^2 - B^2}}$$

$$B^2 = 1 - \frac{1}{\eta^2} < 1$$

$$= \pi a^2 \int_{B}^{1} dx\, (-\eta) \left(\frac{B^2}{\sqrt{x^2-B^2}} - \frac{2x^2}{\sqrt{x^2-B^2}} \right)$$

<u>wolfram α</u>:

$$\int_B^1 \frac{dx}{\sqrt{x^2-B^2}} = \log\left(\frac{\sqrt{1-B^2}+1}{B}\right) \qquad 0<B<1$$

$$\int_B^1 \frac{dx(2x^2)}{\sqrt{x^2-B^2}} = \left(x\sqrt{x^2-B^2} + B^2 \log\left(\sqrt{x^2-B^2} + x\right)\right)\Big|_B^1$$

$$= \sqrt{1-B^2} + B^2 \log\left(\sqrt{1-B^2}+1\right) - B^2 \log(B)$$

$$= \sqrt{1-B^2} + B^2 \log\left(\frac{\sqrt{1-B^2}+1}{B}\right)$$

Thus,

$$\sigma_1 = \pi a^2 \left[-\eta \left(\cancel{B^2 \log\left(\frac{\sqrt{1-B^2}+1}{B}\right)} \right)\Big| + \sqrt{1-B^2} - \cancel{B^2 \log\left(\frac{\sqrt{1-B^2}+1}{B}\right)} \right]$$

$$= \pi a^2 \eta \sqrt{1-B^2}$$

$$= \pi a^2 \cancel{\eta} \frac{1}{\cancel{\eta}}$$

$$= \boxed{\pi a^2}$$ just like we found in the C frame

~~Just~~

For $m_1 = m_2$,

$$(2\cos^2\theta_1 - 1)$$

$$d\sigma_1 = \frac{1}{2} a^2 \frac{1 + \cos 2\theta_1}{\sqrt{1-\sin^2\theta_1}} \, d\Omega_1$$

$$= \frac{1}{2} a^2 \frac{2\cos^2\theta_1}{|\cos\theta_1|} \, d\Omega_1$$

$$= a^2 |\cos\theta_1| \, d\Omega_1$$

$$= a^2 |\cos\theta_1| \, 2\pi \sin\theta_1 \, d\theta_1$$

$$= -2\pi a^2 |\cos\theta_1| \, d(\cos\theta_1)$$

$$= -2\pi a^2 |x| \, dx$$

$0 \le \theta_1 \le \pi/2$
$1 \le x \le 0$

$$\rightarrow \sigma_1 = -\int_1^0 2\pi a^2 |x| dx$$

$$= \int_0^1 \pi a^2 \, d(x^2)$$

$$= \pi a^2 x^2 \Big|_0^1$$

$$= \boxed{\pi a^2}$$

For all cases;

$\chi : 0 \to \pi$

$\theta_2 : \frac{\pi}{2} \to 0$

$$\theta_2 = \frac{1}{2}(\pi - \chi) \longleftrightarrow 2\theta_2 = \pi - \chi$$

$$\cancel{d\theta_2 = -\frac{1}{2} d\chi} \qquad \chi = \pi - 2\theta_2$$

$$d\sigma_2 = \frac{1}{2} \pi a^2 \sin\chi \, d\chi$$

$$= \frac{1}{\cancel{2}} \pi a^2 \sin(\pi - 2\theta_2)(-\cancel{2} d\theta_2)$$

$$= -\pi a^2 [\sin\pi \cos(2\theta_2) - \underbrace{\cos\pi}_{=-1} \sin(2\theta_2)] d\theta_2$$

$$= -\pi a^2 \sin 2\theta_2 \, d\theta_2$$

$$= -2\pi a^2 \sin\theta_2 \cos\theta_2 \, d\theta_2$$

$$= 2\pi a^2 \cos\theta_2 \, d(\cos\theta_2)$$

$$= 2\pi a^2 x \, dx \quad \text{where } x = \cos\theta_2$$

$\theta_2 : \pi/2 \to 0$

$\cos\frac{\pi}{2} = 0$

$\cos 0 = 1$

$$\rightarrow \sigma_2 = \int_0^1 2\pi a^2 x \, dx$$

$$= \cancel{2}\pi a^2 \frac{x^2}{\cancel{2}} \Big|_0^1$$

$$= \boxed{\pi a^2}$$

Sec 18, Prob 2

Scattering off of a hard sphere of radius a (Problem 1)

$$d\sigma = \frac{1}{4} a^2 \, d\Omega = \frac{1}{2}\pi a^2 \sin\chi \, d\chi$$

In terms of energy lost

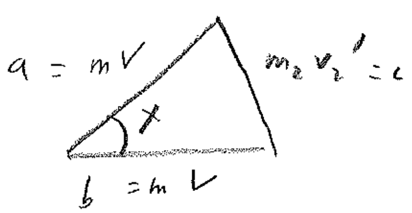
ϵ = energy lost by m_1

= energy gained by m_2 (hard sphere)

$= \frac{1}{2} m_2 v_2'^2$

Use: $$v_2' = \left(\frac{2m_1}{m_1+m_2}\right) v_\infty \sin\left(\frac{1}{2}\chi\right) \qquad (17.5)$$

Proof of this equation (from Fig 16)



$$m_2^2 v_2'^2 = 2m^2v^2 - 2m^2v^2\cos\chi$$
$$= 2m^2v^2(1-\cos\chi)$$
$$= 4m^2v^2\sin^2\left(\frac{\chi}{2}\right)$$

$$\rightarrow \quad v_2' = 2\frac{m}{m_2} v \sin\left(\frac{\chi}{2}\right)$$
$$= \left(\frac{2m_1}{m_1+m_2}\right) v \sin\left(\frac{\chi}{2}\right)$$

Thus, $$\epsilon = \frac{1}{2} m_2 \frac{4m_1^2}{(m_1+m_2)^2} v_\infty^2 \sin^2\left(\frac{1}{2}\chi\right)$$
$$= 2\frac{m^2}{m_2} v_\infty^2 \sin^2\left(\frac{1}{2}\chi\right)$$
$$= \epsilon_{max} \sin^2\left(\frac{1}{2}\chi\right)$$

~~where~~ $\cancel{\epsilon_{max} = 2\frac{m^2}{m_2} v_\infty^2}$

where $$\epsilon_{max} \equiv 2\frac{m^2}{m_2} v_\infty^2$$

Thus,

$$d\epsilon = \epsilon_{max}\,\cancel{2}\,\sin\left(\tfrac{1}{2}\chi\right)\cos\left(\tfrac{\chi}{2}\right)\cancel{\tfrac{1}{2}}\,d\chi$$

$$= \epsilon_{max}\,\sin\left(\tfrac{\chi}{2}\right)\cos\left(\tfrac{\chi}{2}\right)d\chi$$

$$= \frac{1}{2}\epsilon_{max}\,\sin\chi\,d\chi$$

$$\rightarrow \quad \sin\chi\,d\chi = \frac{2}{\epsilon_{max}}\,d\epsilon$$

$$\rightarrow \quad d\sigma = \frac{1}{\cancel{2}}\pi a^2\,\frac{\cancel{2}}{\epsilon_{max}}\,d\epsilon$$

$$= \frac{\pi a^2}{\epsilon_{max}}\,d\epsilon$$

or $\boxed{\dfrac{d\sigma}{d\epsilon} = \dfrac{\pi a^2}{\epsilon_{max}}}$ uniform for $0 \leq \epsilon \leq \epsilon_{max}$

Sec 18, Prob 3 Find $d\sigma$ as a function of v_∞ for $U \sim r^{-n}$

solution:

$U(r) \sim r^{-n}$ means that U is homog. of degree $k = -n$ in positions.

From section 10, we have $\frac{v'}{v} \sim \left(\frac{l'}{l}\right)^{\frac{k}{2}} = \left(\frac{l'}{l}\right)^{-n/2}$

Now: $\underbrace{d\sigma}_{\text{area}} = 2\pi\rho\, d\rho \longrightarrow \frac{d\sigma'}{d\sigma} \sim \left(\frac{l'}{l}\right)^2 \sim \left(\frac{v'}{v}\right)^{-4/n}$

so: $\boxed{d\sigma \sim v_\infty^{-4/n}\, d\Omega}$

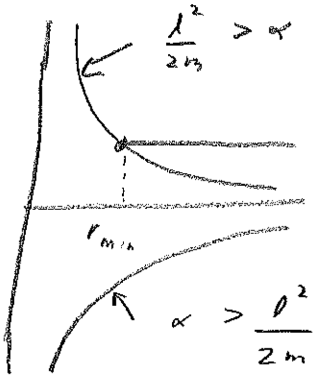
as angles are not affected by a similarity transformation $d\Omega = d\Omega'$

Section 18
Prob 4 Determine effective cross-section to "fall" to center of a field $U = -\alpha/r^2$ $(\alpha > 0)$

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{l\, dr/r^2}{\sqrt{2m(E-U(r)) - \frac{l^2}{r^2}}}$$

$$= \int_{r_{min}}^{\infty} \frac{\rho\, dr/r^2}{\sqrt{1-\rho^2/r^2 - 2U/mv_\infty^2}}$$

making the substitutions $E = \frac{1}{2} m v_\infty^2 > 0$
$l = m\rho v_\infty$

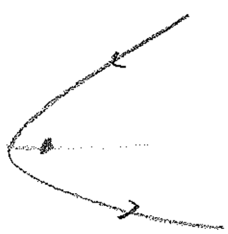


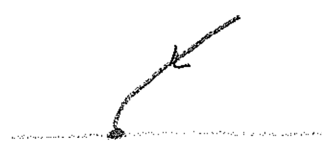
For $U(r) = -\alpha/r^2$

$$U_{eff}(r) = \frac{l^2}{2mr^2} - \frac{\alpha}{r^2}$$

$$= \frac{1}{r^2}\left(\frac{l^2}{2m} - \alpha\right)$$

< 0 if $\alpha > \frac{l^2}{2m}$





So to fall to center, need

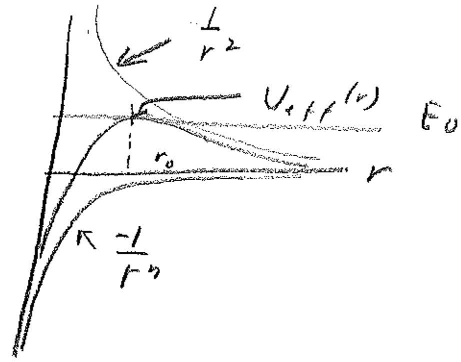
$$\alpha > \frac{l^2}{2m} = \frac{m^2\rho^2 v_\infty^2}{2m} = \frac{1}{2} m\rho^2 v_\infty^2$$

$$\rightarrow \quad \rho^2 < \frac{2\alpha}{m v_\infty^2} = \rho_{max}^2$$

Thus, $\sigma = \pi\rho_{max}^2 = \boxed{\frac{2\pi\alpha}{m v_\infty^2}}$

Sec 18
Prob 5

Same as Prob 4 but for $U = -\alpha / r^n$
$(\alpha > 0, \quad n > 2)$



$$U_{eff}(r) = \frac{\ell^2}{2mr^2} - \frac{\alpha}{r^n}$$

Fall to the center if
$E > U_{eff}(r_0) \equiv E_0$

r_0 corresponds to max $U_{eff}(r)$:

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r_0}$$

$$= \frac{-\ell^2}{mr_0^3} + \frac{n\alpha}{r_0^{n+1}}$$

$$\frac{\ell^2}{mr_0^3} = \frac{n\alpha}{r_0^{n+1}}$$

$$\frac{\ell^2}{m} = \frac{n\alpha}{r_0^{n-2}}$$

So $\boxed{r_0^{n-2} = \frac{n\alpha m}{\ell^2}} \iff \boxed{r_0 = \left(\frac{n\alpha m}{\ell^2}\right)^{\frac{1}{n-2}}}$

Thus, $E_0 = U_{eff}(r_0)$

$$= \frac{\ell^2}{2mr_0^2} - \frac{\alpha}{r_0^n}$$

$$= \frac{\ell^2}{2mr_0^2} - \frac{\ell^2}{nmr_0^2}$$

$$= \frac{\ell^2}{2mr_0^2}\left(1 - \frac{2}{n}\right)$$

$\cancel{E > E_0 \quad E = \tfrac{1}{2}mv_\infty^2}$

so $$E_0 = \frac{l^2}{2 m r_0^2}\left(\frac{n-2}{n}\right)$$

$$= \frac{l^2}{2m\left(\frac{n \alpha m}{l^2}\right)^{\frac{2}{n-2}}}\left(\frac{n-2}{n}\right)$$

$$= \frac{l^2 \, l^{\frac{4}{n-2}}}{2m\,(n\alpha m)^{\frac{2}{n-2}}}\left(\frac{n-2}{n}\right)$$

<u>Now:</u> $$l^2 \, l^{\frac{4}{n-2}} = l^{\left(2 + \frac{4}{n-2}\right)}$$

$$= l^{\frac{2(n-2)+4}{n-2}}$$

$$= l^{\frac{2n}{n-2}}$$

$$m \cdot m^{\frac{2}{n-2}} = m^{\left(1 + \left(\frac{2}{n-2}\right)\right)}$$

$$= m^{\left(\frac{n-\cancel{2}+\cancel{2}}{n-2}\right)}$$

$$= m^{\frac{n}{n-2}}$$

$$\rightarrow \quad (n\alpha)^{\frac{2}{n-2}} = \frac{1}{n\alpha}\,(n\alpha)^{\left(1+\left(\frac{2}{n-2}\right)\right)}$$

$$= \frac{1}{n\alpha}\,(n\alpha)^{\frac{n}{n-2}}$$

Thus, $$E_0 = \frac{\left(l^2\right)^{\frac{n}{n-2}}}{2\left(\frac{1}{\cancel{n}\alpha}\right)(n\alpha m)^{\frac{n}{n-2}}}\left(\frac{n-2}{\cancel{n}}\right)$$

$$= \frac{(n-2)}{2}\,\alpha\left(\frac{\rho^2 m^2 v_\infty^2}{n\alpha m}\right)^{\frac{n}{n-2}}$$

$$\boxed{E_0 = \left(\frac{n-2}{2}\right) \alpha \left(\frac{\rho^2 m V_\infty^2}{n \alpha}\right)^{\frac{n}{n-2}}}$$

Fall to center when $E > E_0$

Thus,

$$\cancel{\tfrac{1}{2}} m V_\infty^2 > \left(\frac{n-2}{\cancel{2}}\right) \alpha \left(\frac{\rho^2 m V_\infty^2}{n\alpha}\right)^{\frac{n}{n-2}}$$

$$\left(\frac{m V_\infty^2}{(n-2)\alpha}\right)^{\frac{n-2}{n}} > \frac{\rho^2 m V_\infty^2}{n \alpha}$$

$$\rho^2 < \frac{n \alpha}{m V_\infty^2} \left(\frac{m V_\infty^2}{(n-2)\alpha}\right)^{\frac{n-2}{n}}$$

$$= \frac{n}{(n-2)^{\frac{n-2}{n}}} \left(\frac{m V_\infty^2}{\alpha}\right)^{-2/n}$$

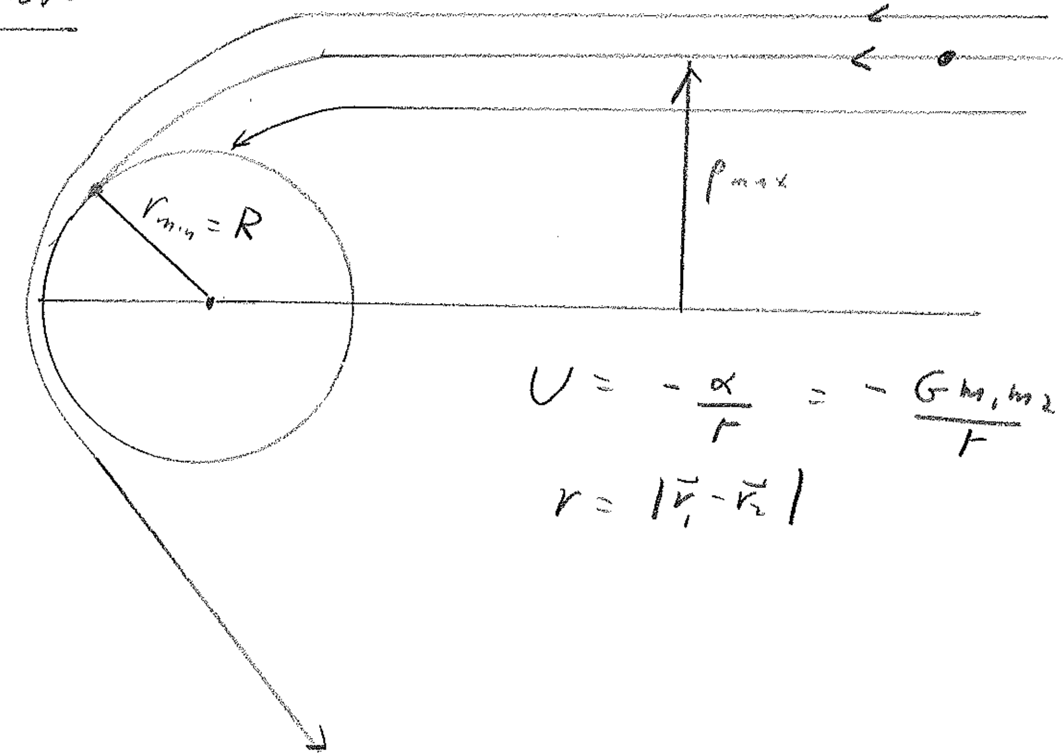
$$= n\,(n-2)^{\frac{2-n}{n}} \left(\frac{\alpha}{m V_\infty^2}\right)^{2/n} \equiv \rho_{max}^2$$

Thus,

$$\boxed{\sigma} = \pi \rho_{max}^2$$

$$\boxed{= n \pi\, (n-2)^{\frac{2-n}{n}} \left(\frac{\alpha}{m V_\infty^2}\right)^{2/n}}$$

Sec 18, Prob 6



$$U = -\frac{\alpha}{r} = -\frac{G m_1 m_2}{r}$$
$$r = |\vec{r}_1 - \vec{r}_2|$$

$E = \frac{1}{2} m v_\infty^2$

$l = m \rho v_\infty$

Effective cross-section:

$$\sigma = \pi \rho_{max}^2$$

where ρ_{max} corresponds to $r_{min} = R$.

Now:

r_{min} is a root of $E - U_{eff}(r) = 0$ (since $\dot{r} = 0$ at $r = r_{min}$)

Thus,

$$0 = E - U_{eff}(R)$$
$$= \frac{1}{2} m v_\infty^2 - \frac{l^2}{2 m R^2} + \frac{G m M}{R}$$
$$= \frac{1}{2} m v_\infty^2 - \frac{m^2 \rho_{max}^2 v_\infty^2}{2 m R^2} + \frac{G m M}{R}$$
$$= \frac{1}{2} m v_\infty^2 \left(1 - \frac{\rho_{max}^2}{R^2}\right) + \frac{G m M}{R}$$

Thus,

$$\frac{1}{2} v_\infty^2 \left(\frac{\rho_{max}^2}{R^2} - 1 \right) = \frac{GM}{R}$$

$$\frac{\rho_{max}^2}{R^2} - 1 = \frac{2GM}{R v_\infty^2}$$

$$\rho_{max}^2 = R^2 \left[1 + \frac{2GM}{R v_\infty^2} \right]$$

$$\rightarrow \quad \sigma = \pi \rho_{max}^2$$

$$= \pi R^2 \left[1 + \frac{2GM}{R v_\infty^2} \right]$$

<u>NOTE</u>:

$\sigma \rightarrow \pi R^2$ in the limit $v_\infty \rightarrow \infty$.