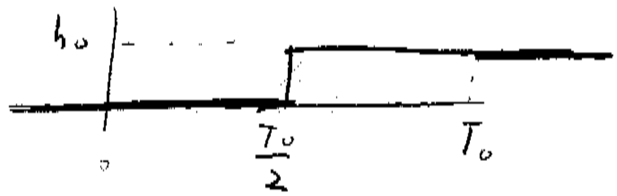
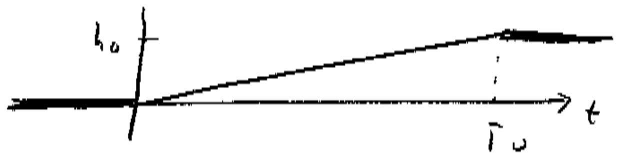
GW types

1. step

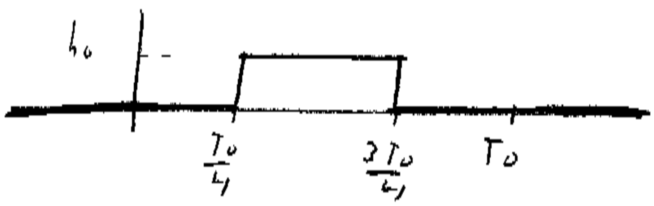


2. ramp



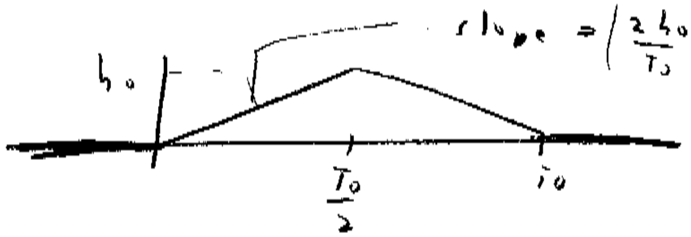
$h(t) = \left(\frac{h_0}{T_0}\right) t$

3. square

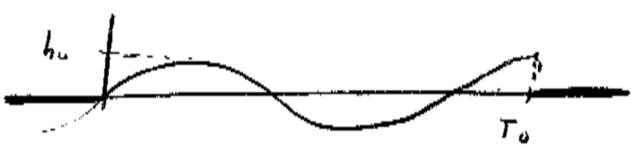


4. triangle

slope $= \left(\frac{2h_0}{T_0}\right)$



5. sine



Analytic expressions

$$T_{tot,x}(t) = 2L_* + \frac{1}{2}\int_0^{L_*}[h_+(t-x) + h_+(t-2L_*+x)]\,dx$$

Simplify notation:

$$L_* \to L$$
$$h_+(u) \to h(u)$$
$$T_0 \to T$$

Note: $T = NL = 8L$ (in program)
$= NL$

$T/2 = \left(\frac{N}{2}\right)L$

Integral:

$$I(t) = \int_0^L [h(t-x) + h(t-2L+x)]\,dx = I_1(t) + I_2(t)$$

where

$$I_1(t) = \int_0^L h(t-x)\,dx$$

$u = t - x$
$du = -dx$
$x = 0, L \to u = t, t-L$

$$= \int_t^{t-L} h(u)(-du)$$

$$= \int_{t-L}^{t} h(u)\,du$$

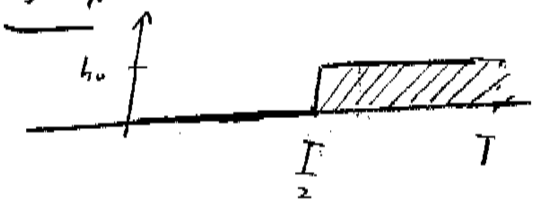
$$I_2(t) = \int_0^L h(t-2L+x)\,dx$$

$u = t - 2L + x$
$du = dx$
$x = 0, L \to$
$u = t-2L, t-L$

$$= \int_{t-2L}^{t-L} h(u)\,du$$

thus, $\boxed{I(t) = \int_{t-2L}^{t} h(u)\,du}$

step:



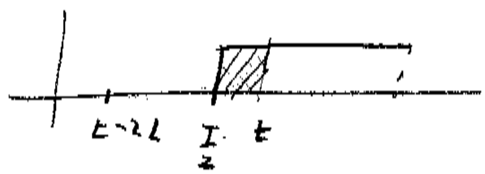
$$I(t) = \int_{t-2L}^{t} h(u)\,du$$

= area under curve between $t-2L$ and t

If $t < T/2$ then $I(t) = 0$

If $t - 2L > \frac{T}{2}$ then $I(t) = h_0 \cdot 2L$

If $t - 2L < \frac{T}{2}$ but $t > \frac{T}{2}$,

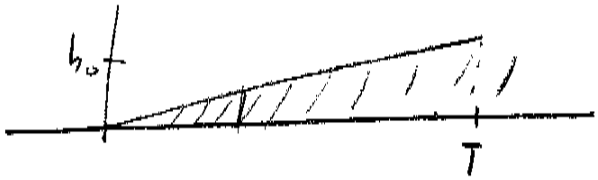


then $I(t) = h_0 (t - \frac{T}{2})$

$$\text{So} \quad I(t) = \begin{cases} 0 & t < \frac{T}{2} \\ h_0 (t - \frac{T}{2}) & \frac{T}{2} < t < \frac{T}{2} + 2L \\ h_0 \cdot 2L & t > \frac{T}{2} + 2L \end{cases}$$

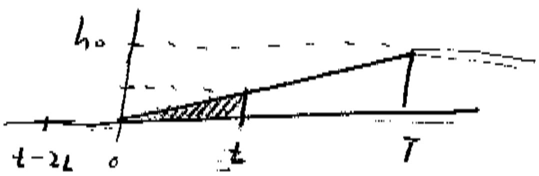
$$\rightarrow \boxed{T_{tot,x}(t) = \begin{cases} 2L & t < \frac{T}{2} \\ 2L + \frac{1}{2} h_0 \cdot (t - \frac{T}{2}) & \frac{T}{2} < t < \frac{T}{2} + 2L \\ 2L + h_0 L & \frac{T}{2} + 2L < t < T \end{cases}}$$

<u>ramp.</u>



$I(t)$ = area under curve between $t-2L$ and t

If $t > 0$ ~~and~~ and $t-2L < 0$

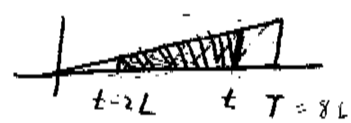


$$I(t) = \frac{1}{2} \cdot \underbrace{t}_{\text{base}} \cdot \underbrace{\left(\frac{h_0}{T}\right) t}_{\text{height}}$$

$$= \frac{1}{2}\left(\frac{h_0}{T}\right) t^2$$

If $t < 0$ then $I(t) = 0$

If $t-2L > 0$ and $t < T$ then



$$I(t) = \frac{1}{2}\, t \left(\frac{h_0}{T}\right) t - \frac{1}{2}(t-2L)\left(\frac{h_0}{T}\right)(t-2L)$$

$$= \frac{1}{2}\left(\frac{h_0}{T}\right)\left[t^2 - (t-2L)^2\right]$$

$$= \frac{1}{2}\left(\frac{h_0}{T}\right)\left[\cancel{t^2} - \left(\cancel{t^2} + 4L^2 - 4Lt\right)\right]$$

$$= \frac{1}{2}\left(\frac{h_0}{T}\right)\left[4Lt - 4L^2\right]$$

$$= \frac{1}{2}\left(\frac{h_0}{T}\right) 4L(t-L)$$

$$= 2L\left(\frac{h_0}{T}\right)(t-L)$$

Thus,

$$I(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ \frac{1}{2}\left(\frac{h_0}{T}\right) t^2 & 0 < t < 2L \\ \frac{1}{2}\left(\frac{h_0}{T}\right) 4L(t-L) & 2L < t < T \end{cases}$$

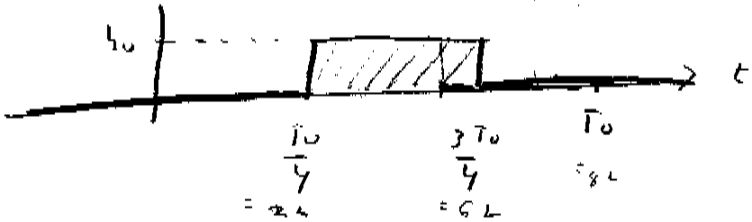
so

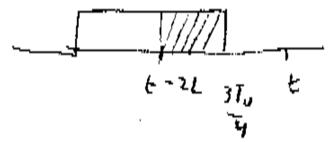
$$\boxed{T_{tur,x}(t) = \begin{cases} 2L & t < 0 \\ 2L + \frac{1}{4}\left(\frac{h_0}{T}\right) t^2 & 0 < t < 2L \\ 2L + \frac{1}{4}\left(\frac{h_0}{T}\right) 4L(t-L) & 2L < t < T \end{cases}}$$

$t = 2L$

Square:

$\frac{3T_0}{4} - t + 2L$





$$I(t) = \int_{t-2L}^{t} h(u)\,du$$

$\frac{3T_0}{4} \cdot \frac{T_0}{4}$

If $t < \frac{T_0}{4}$ then $I(t) = 0$

If $t > \frac{T_0}{4}$ and $t - 2L < \frac{T_0}{4}$ then $I(t) = h_0\left(t - \frac{T_0}{4}\right)$

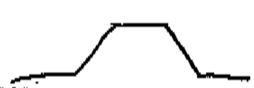
If $t - 2L > \frac{T_0}{4}$ and $t < \frac{3T_0}{4}$ then $I(t) = h_0 \cdot 2L$

If $t > \frac{3T_0}{4}$ and $t - 2L < \frac{3T_0}{4}$ then $I(t) = h_0\left(\frac{3T_0}{4} - t + 2L\right)$

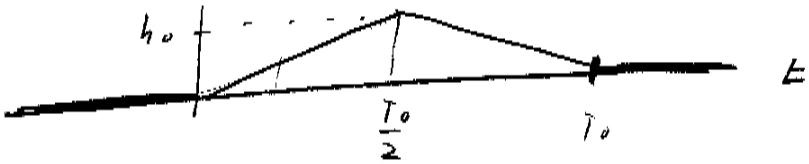
If $t - 2L > \frac{3T_0}{4}$, then $I(t) = 0$ $\left(\frac{3T_0}{4} - (t-2L)\right)$

so
$$I(t) = \begin{cases} 0 & \text{if } t < \frac{T_0}{4} \\ h_0\left(t - \frac{T_0}{4}\right) & \text{if } \frac{T_0}{4} < t < \frac{T_0}{4} + 2L \\ h_0 \cdot 2L & \text{if } \frac{T_0}{4} + 2L < t < \frac{3T_0}{4} \\ h_0\left(\frac{3T_0}{4} - t + 2L\right) & \text{if } \frac{3T_0}{4} < t < \frac{3T_0}{4} + 2L \\ 0 & \text{if } t > \frac{3T_0}{4} + 2L \end{cases}$$

$\rightarrow$ $T_{tot,x}(t) = 2L_x + \frac{1}{2} \cdot I(t)$

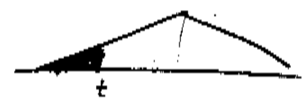


Triangle:



$$I(t) = \int_{t-2L}^{t} h(u)\,du$$

If $t < 0 \rightarrow I(t) = 0$



If $t > 0$ and $t - 2L < 0 \rightarrow I(t) = \underset{\text{base}}{\frac{1}{2}\, t}\, \underbrace{\left(\frac{2h_0}{T_0}\right) t}_{\text{height}}$

If $t - 2L > 0$ and $t < \frac{T_0}{2}$

$$\rightarrow I(t) = \frac{1}{2}\, t \left(\frac{2h_0}{T}\right) t - \frac{1}{2}(t-2L)\left(\frac{2h_0}{T}\right)(t-2L)$$

$$= \frac{1}{2}\left(\frac{2h_0}{T_0}\right)\left[t^2 - (t-2L)^2\right]$$

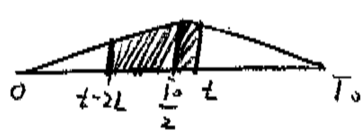
$$= \frac{1}{2}\left(\frac{2h_0}{T_0}\right)\left[\cancel{t^2} - (\cancel{t^2} - 4Lt + 4L^2)\right]$$

$$= \frac{1}{2}\left(\frac{2h_0}{T_0}\right)\left[4Lt - 4L^2\right]$$

$$= \frac{1}{2}\left(\frac{2h_0}{T_0}\right) 4L(t-L)$$

$=$

If $t - 2L < \frac{T_0}{2}$ and $t > \frac{T_0}{2}$



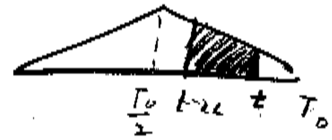
$$\rightarrow I(t) = \frac{1}{2}\left(\frac{T_0}{2}\right) h_0 - \frac{1}{2}(t-2L)\left(\frac{2h_0}{T_0}\right)(t-2L)$$

$$+ \frac{1}{2}\left(\frac{T_0}{2}\right) h_0 - \frac{1}{2}(T_0 - t)\left(\frac{2h_0}{T_0}\right)(T_0 - t)$$

$$= \left(\frac{T_0}{2}\right) h_0 - \frac{1}{2}\left(\frac{2h_0}{T_0}\right)\left[t^2 + 4L^2 - 4Lt + T_0^2 + t^2 - 2T_0 t\right]$$

$$= \left(\frac{T_0}{2}\right) h_0 - \frac{1}{2}\left(\frac{2h_0}{T_0}\right)\left[2t^2 - 2t(2L + T_0) + T_0^2 + 4L^2\right]$$

If $t - 2L > \frac{T_0}{2}$ and $t < T_0$



$$\rightarrow I(t) = \frac{1}{2}\left(T_0 - (t-2L)\right)\left(\frac{2h_0}{T_0}\right)\left(T_0 - (t-2L)\right)$$
$$- \frac{1}{2}\left(T_0 - t\right)\left(\frac{2h_0}{T_0}\right)\left(T_0 - t\right)$$

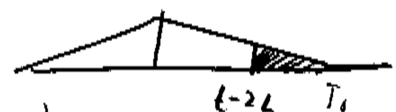
$$= \frac{1}{2}\left(\frac{2h_0}{T_0}\right)\Big[\; T_0^2 + (t-2L)^2 - 2T_0(t-2L)$$
$$- \left(T_0^2 + t^2 - 2T_0 t\right) \Big]$$

$$= \frac{1}{2}\left(\frac{2h_0}{T_0}\right)\Big[\; \cancel{T_0^2} + \cancel{t^2} + 4L^2 - 4Lt - \cancel{2T_0 t} + 4LT_0$$
$$- \cancel{T_0^2} - \cancel{t^2} + \cancel{2T_0 t} \Big]$$

$$= \frac{1}{2}\left(\frac{2h_0}{T_0}\right)\left[4L^2 - 4Lt + 4LT_0\right]$$

$$= \frac{1}{2}\left(\frac{2h_0}{T_0}\right) 4L \left[L + (T_0 - t)\right]$$

If $t > T_0$ and $t - 2L < T_0$



$$\rightarrow I(t) = \frac{1}{2}\left(T_0 - (t-2L)\right)\left(\frac{2h_0}{T_0}\right)\left(T_0 - (t-2L)\right)$$

$$= \frac{1}{2}\left(\frac{2h_0}{T_0}\right)\left[T_0^2 + (t-2L)^2 - 2T_0(t-2L)\right]$$

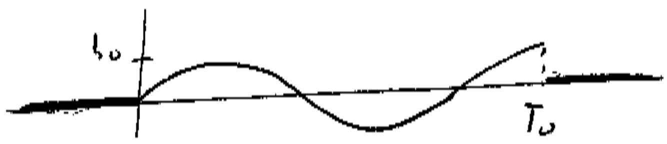
If $t > T_0$ and $t - 2L > T_0$ then $I(t) = 0$

Thus,

$$I(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2}\left(\frac{2h_0}{T_0}\right)t^2 & \text{if } 0 < t < 2L \\ \frac{1}{2}\left(\frac{2h_0}{T_0}\right)4L(t-L) & \text{if } 2L < t < \frac{T_0}{2} \\ \left(\frac{T_0}{2}\right)h_0 - \frac{1}{2}\left(\frac{2h_0}{T_0}\right)\left[2t^2 - 2t(2L+T_0) + T_0^2 + 4L^2\right] & \text{if } \frac{T_0}{2} < t < \frac{T_0}{2} + 2L \\ \frac{1}{2}\left(\frac{2h_0}{T_0}\right)4L(L+T_0-t) & \text{if } \frac{T_0}{2}+2L < t < T_0 \\ \frac{1}{2}\left(\frac{2h_0}{T}\right)\left(T_0 - (t-2L)\right)^2 & \text{if } T_0 < t < T_0 + 2L \\ 0 & \text{if } t > T_0 + 2L \end{cases}$$

$$\Gamma_{tot,x}(t) = 2L_* + \frac{1}{2} I(t)$$

Sine wave:

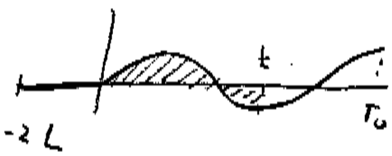


$$h(t) = \begin{cases} h_0 \sin(2\pi f_{gw} t) & 0 < t < T_0 \\ 0 & \text{otherwise} \end{cases}$$

$$I(t) = \int_{t-2L}^{t} h(u)\, du$$

If $t < 0 \rightarrow I(t) = 0$

If $t > 0$ and $t - 2L < 0$



$$I(t) = \int_0^t h_0 \sin(2\pi f_{gw} u)\, du$$

$$= \left(\frac{-h_0}{2\pi f_{gw}}\right) \cos(2\pi f_{gw} u) \Big|_0^t$$

$$= \left(\frac{-h_0}{2\pi f_{gw}}\right)\left(\cos(2\pi f_{gw} t) - 1\right)$$

$$= \left(\frac{h_0}{2\pi f_{gw}}\right)\left(1 - \cos(2\pi f_{gw} t)\right)$$

If $t - 2L > 0$ and $t < T_0$

$$I(t) = \int_{t-2L}^{t} h_0 \sin(2\pi f_{gw} u)\, du$$

$$= \frac{-h_0}{2\pi f_{gw}} \cos(2\pi f_{gw} u) \Big|_{t-2L}^{t}$$

$$= \left(\frac{h_0}{2\pi f_{gw}}\right)\left(\cos[2\pi f_{gw}(t-2L)] - \cos(2\pi f_{gw} t)\right)$$

If $t > T_0$ and $t - 2L < T_0$,

$$I(t) = \int_{t-2L}^{T_0} h_0 \sin(2\pi f_{gw} u)\, du$$

$$= \frac{-h_0}{2\pi f_{gw}} \cos(2\pi f_{gw} u) \Big|_{t-2L}^{T_0}$$

$$= \left(\frac{h_0}{2\pi f_{gw}}\right)\left(\cos[2\pi f_{gw}(t-2L)] - \cos(2\pi f_{gw} T_0)\right)$$

If $t - 2L > T_0$, $I(t) = 0$

Thus,

$$I(t) = \begin{cases} 0 & \text{if } t < 0 \\ \left(\frac{h_0}{2\pi f_{gw}}\right)\left(1 - \cos(2\pi f_{gw} t)\right) & \text{if } 0 < t < 2L \\ \left(\frac{h_0}{2\pi f_{gw}}\right)\left(\cos[2\pi f_{gw}(t-2L)] - \cos(2\pi f_{gw} t)\right) & \text{if } 2L < t < T_0 \\ \left(\frac{h_0}{2\pi f_{gw}}\right)\left(\cos(2\pi f_{gw}(t-2L)] - \cos(2\pi f_{gw} T_0)\right) & \text{if } T_0 < t < T_0 + 2L \\ 0 & \text{if } T_0 + 2L < t \end{cases}$$

$$T_{tot,+}(t) = 2L_* + \frac{1}{2} I(t)$$