HLMA408: Traitement des données

Loi normale / gaussienne

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Sommaire

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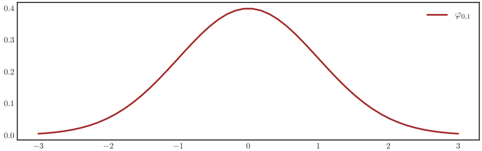
Loi normale standard (ou centrée-réduite)

- Une variable aléatoire (v.a.) réelle X suit une "**loi normale**" ou "**loi gaussienne**" ou "loi de Laplace-Gauss" si sa densité (: *probability density function, pdf*) vaut:

$$\forall x \in \mathbb{R}, \qquad \boxed{\varphi(x) = \varphi_{0,1}(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}$$



- Notation: $X \sim \mathcal{N}(0,1)$

- Propriétés:

$$\begin{cases} \mathbb{E}(Y) &= 0 & \text{(espérance nulle)} \\ \mathbb{V}\mathrm{ar}(X) &= \mathbb{E}(X - \mathbb{E}(X))^2 = 1 & \text{(variance unitaire)} \end{cases}$$

Loi normale

- Une v.a. Y suit une loi normale de paramètres μ et σ^2 si

$$Y = \mu + \sqrt{\sigma^2} X$$

où $X \sim \mathcal{N}(0,1)$, c'est-à-dire si sa densité vaut:

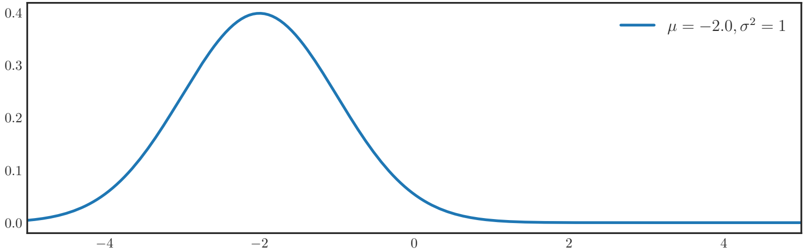
$$\boxed{\varphi_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

- Notation: $$Y \sim \mathcal{N}\Big(\underbrace{\mu}_{\textbf{Espérance}} , \underbrace{\sigma^2}_{\textbf{Variance}} \Big)$$

- Propriétés:

$$\begin{cases} \mathbb{E}(Y) & = & \mu & (\textbf{Espérance}) \\ \mathbb{V}\mathrm{ar}(Y) & = & \mathbb{E}(Y - \mathbb{E}(Y))^2 = \sigma^2 & (\textbf{Variance}) \end{cases}$$

Exemple: variation en μ (centrage)



TO DO: voir jupyter notebook `GaussianDistribution.ipynb`

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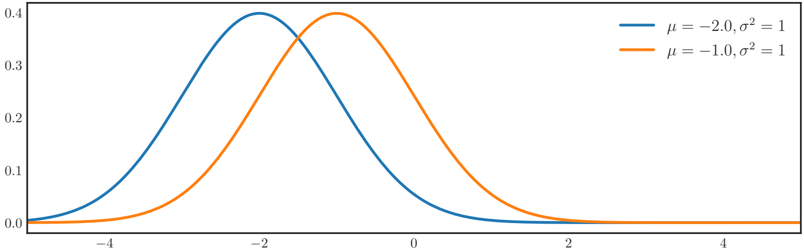

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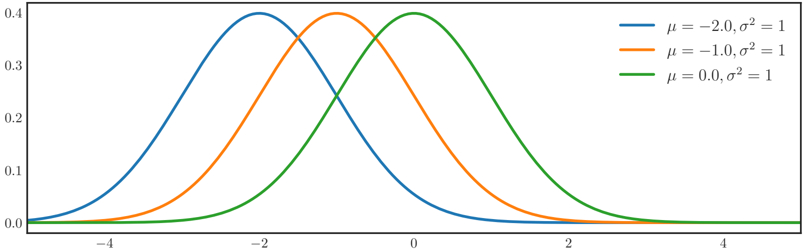

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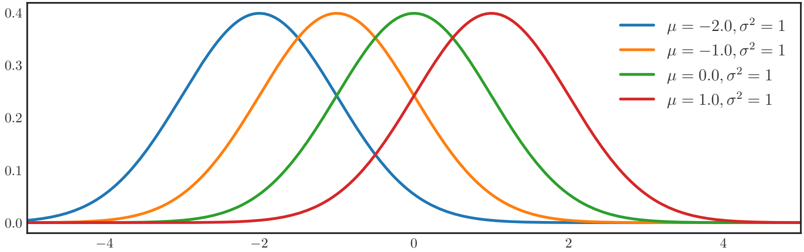

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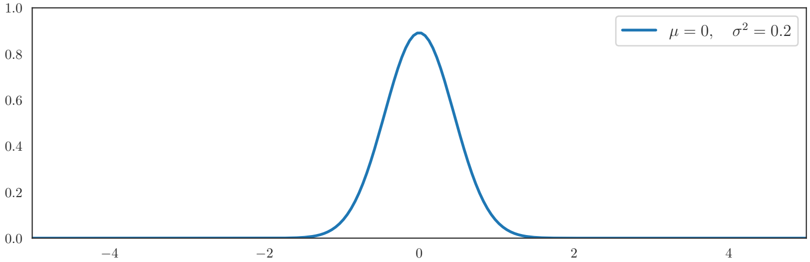

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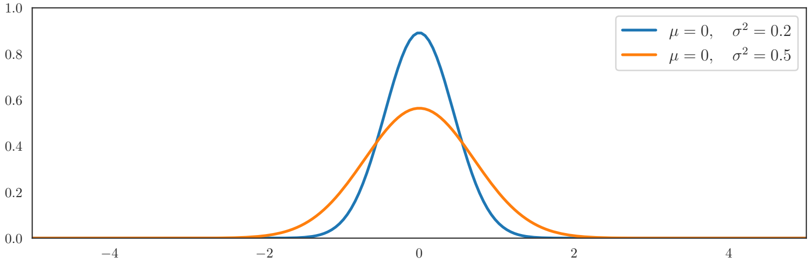

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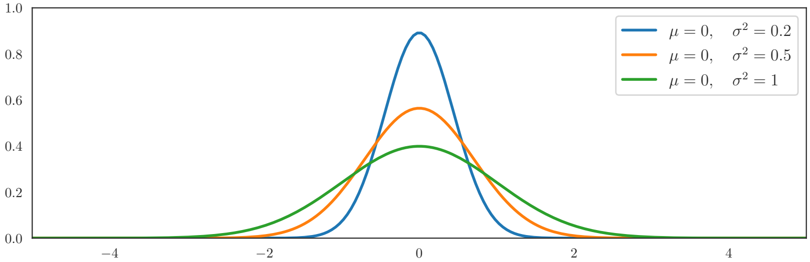

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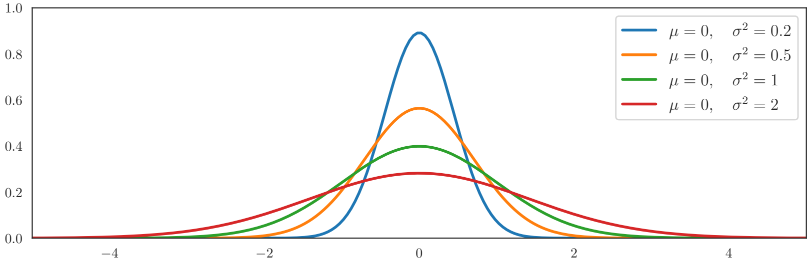

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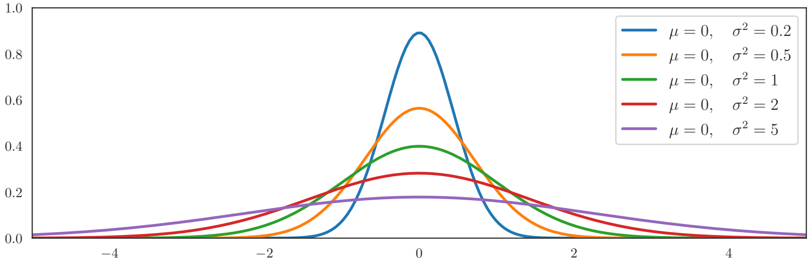

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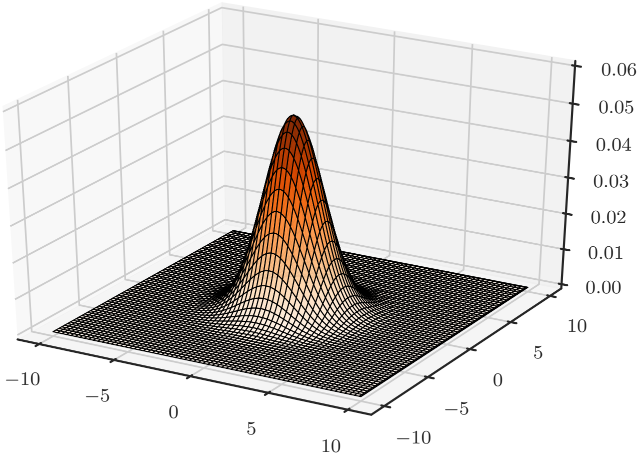
Exemple 2D ($p = 2$)



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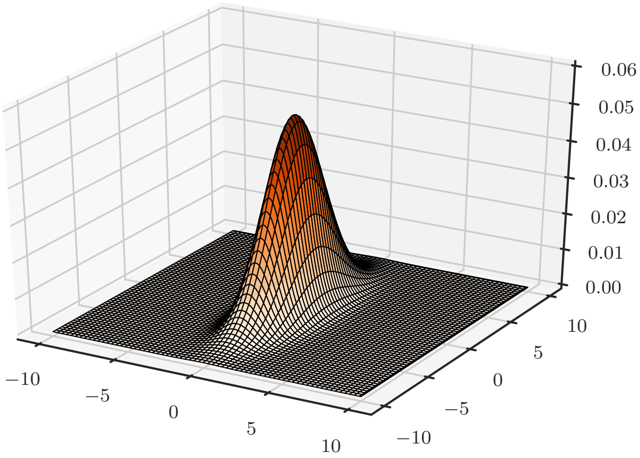

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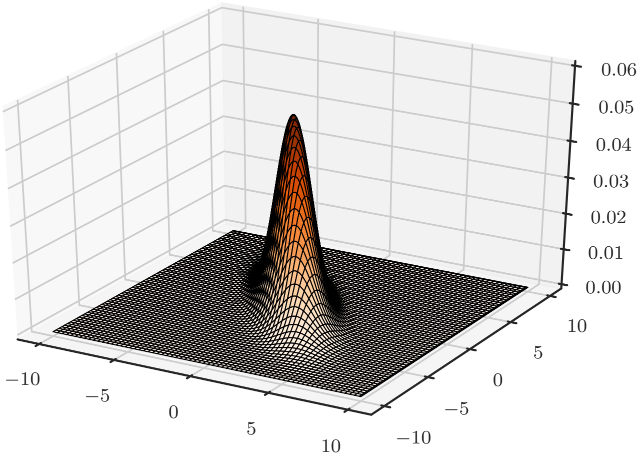

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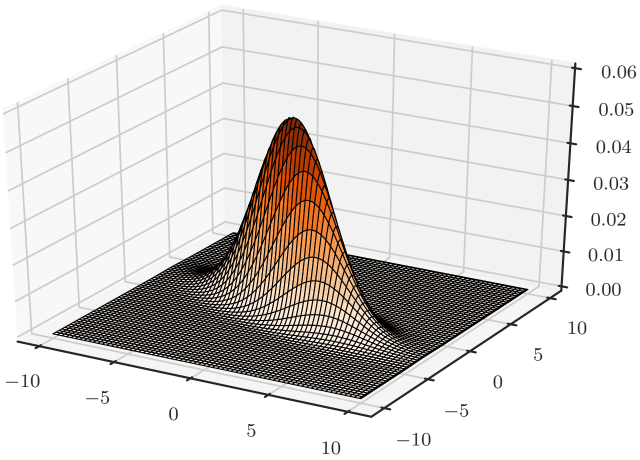

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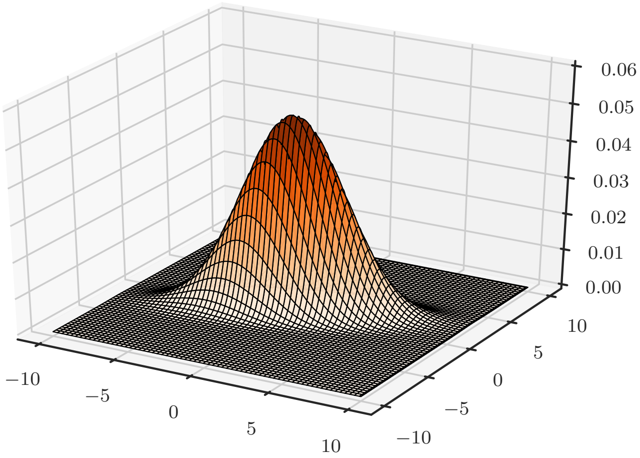

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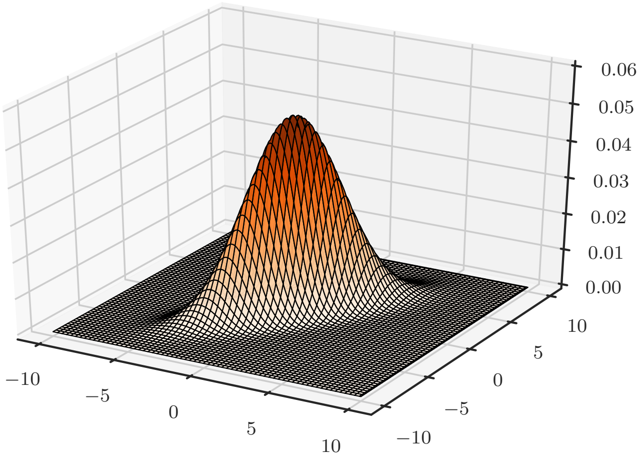


Vecteurs gaussiens (hors programme)

Notation: $\boxed{X \sim \mathcal{N}(\mu, \Sigma)} \iff$ le vecteur aléatoire $X \in \mathbb{R}^p$ suit une loi gaussienne d'espérance μ et de covariance Σ.

Densité à deux paramètres: $\varphi_{\mu,\Sigma} : \mathbb{R}^p \mapsto \mathbb{R}$

- le vecteur d'**espérance**: $\mu \in \mathbb{R}^p$
- la matrice de **covariance** $\Sigma \in \mathbb{R}^{p \times p}$ est symétrique

$$\boxed{\varphi_{\mu,\Sigma}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{p}{2}} \sqrt{\det(\Sigma)}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)^\top \Sigma^{-1}(\mathbf{x}-\mu)\right\}}\ .$$

Rem: $\det(\Sigma)$ est le déterminant de Σ, *i.e.*, le produit des valeurs propres de Σ. On parle de cas dégénéré quand $\det(\Sigma) = 0$

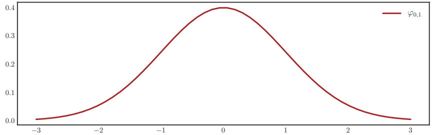
⚠: Σ doit être supposée définie positive (*i.e.*, toutes ses valeurs propres ≥ 0) pour être une matrice de covariance

Loi normale

- Rôle central en statistique
- De nombreuses données suivent (approx.) cette loi
- Le théorème central limite (TCL) assure que certaines variables aléatoires suivent (approx.) cette loi si n est grand

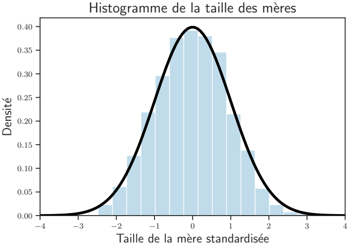
$$\text{TCL :} \qquad \frac{\bar{x}_n - \mu}{\sigma/\sqrt{n}} = \sqrt{n}\left(\frac{\bar{x}_n - \mu}{\sigma}\right) \to \mathcal{N}(0,1)$$

si $x_1, \ldots, x_n$ *i.i.d.* d'espérance μ et de variance σ^2



Lien histogramme–densité et TCL

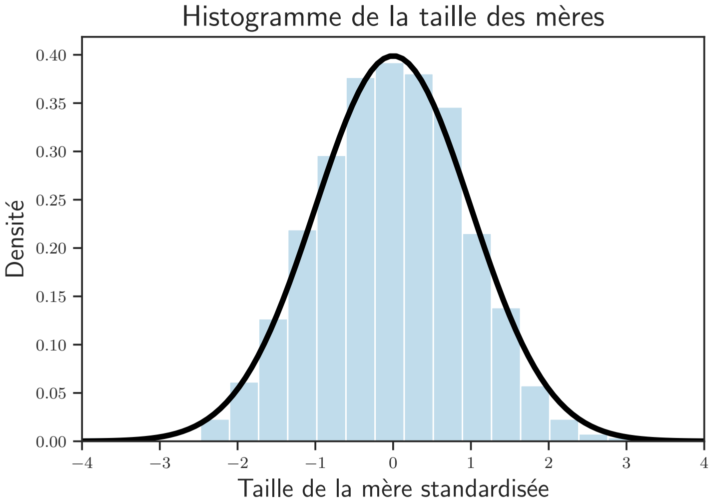
- Si des données suivent approximativement une loi normale, alors l'histogramme des données standardisées doit ressembler à la courbe ci-dessous
- Standardiser les données $x_1, \ldots, x_n$: $\dfrac{x_i - \bar{x}_n}{s_n}, \quad i = 1, \ldots, n$ (retrancher la moyenne, diviser par l'écart-type)



Comparaison:
histogramme / loi normale

- Les données semblent être bien représentées par une loi normale
- On peut alors utiliser cette loi pour répondre à des questions statistiques

Zoom



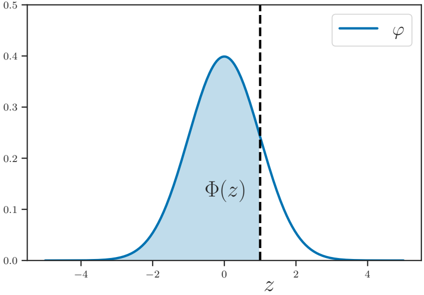
Rem: noter que l'abscisse est sans unité et varie de -4 à 4

Calcul des probabilités

- La probabilité d'être plus petit qu'un nombre z correspond à l'aire sous la courbe φ entre $-\infty$ et z

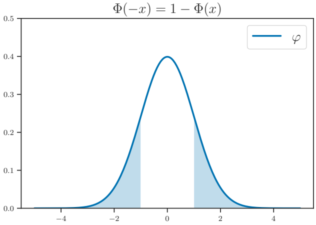
$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

Φ est la **fonction de répartition** d'une loi normale
(🇬🇧 : *Cumulative distribution function, cdf*)



Quelques propriétés de Φ

- $\Phi(-x) = 1 - \Phi(x)$ (symétrie):



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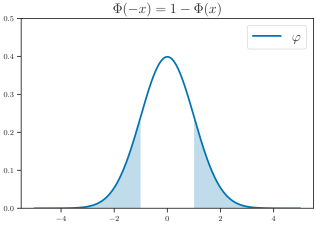
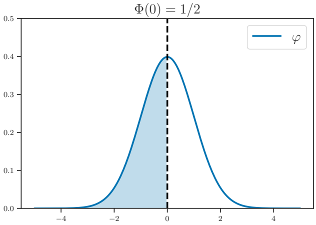
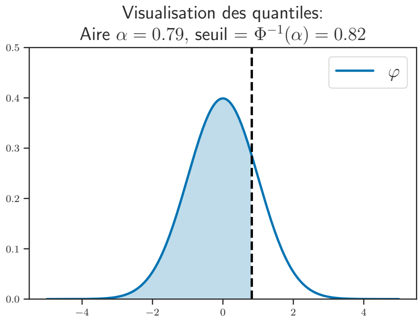


- $\Phi(0) = \frac{1}{2}$ (0 est la médiane):



Visualisation de la fonction de répartition



Exemple de la taille de la mère:

$$\mathbb{P}[\text{Taille} \leq 168] = \mathbb{P}\left[\frac{\text{Taille} - \bar{x}_n}{s_n} \leq \frac{168 - \bar{x}_n}{s_n}\right] \approx \Phi(0.82) = 0.79$$

on calcule la moyenne ($\bar{x}_n = 162.7$) et l'écart-type ($s_n = 6.428$) de l'échantillon pour obtenir ce nombre

<u>Rem</u>: *cf.* notebook `GaussianDistribution.ipynb`

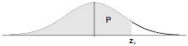
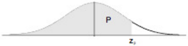


TABLE C.1. Cumulative normal distribution—values of P corresponding to z_p for the standard normal curve.

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.719	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8707	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998



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2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Quantiles gaussiens

Utiliser plutôt:

En Python:

```
>>> from scipy.stats import norm
>>> norm.ppf(0.95, 0, 1)
1.6448536269514722
```

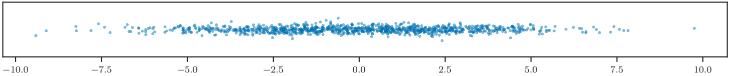
En R:

```
>>> qnorm(.95,mean=0,sd=1)
1.6448536269514722
```

Tirage / échantillon gaussiens: cas 1D

- Cas 1D

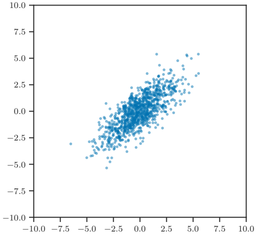
```
n_samples = 1000
X = np.random.normal(loc=0,
                     scale=3,
                     size=n_samples)
```



Tirage / échantillon gaussiens: cas 2D

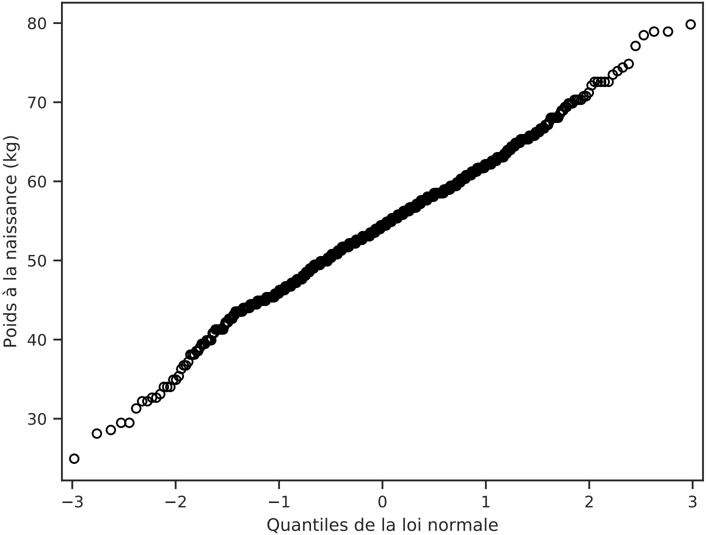
- Cas 2D (et plus):

```
mu = [0, 0]
Sigma = [[3, 2], [2, 2.5]]
n_samples = 1000
X = np.random.multivariate_normal(mu,
                                  Sigma,
                                  n_samples=100)
```



Sommaire

Diagramme quantile-quantile[1]: exemple



[1] M. B. Wilk and R. Gnanadesikan. "Probability plotting methods for the analysis for the analysis of data". In: *Biometrika* 55.1 (1968), pp. 1–17.

Diagrammes quantiles-quantiles (qq-plots)

- Représentation graphique comparant des distributions de type:
  - observées vs observées
  - observées vs théoriques
  - théoriques vs théoriques
- Utilité des qq-plots:
  - Vérifier si les données suivent une loi particulière
  - Vérifier si deux jeux de données ont la même loi
- Construction pour le cas gaussien:
  on ordonne l'échantillon $x_1, \ldots, x_n$ en $x_{(1)} \leq \cdots \leq x_{(n)}$ et on affiche les points de coordonnées

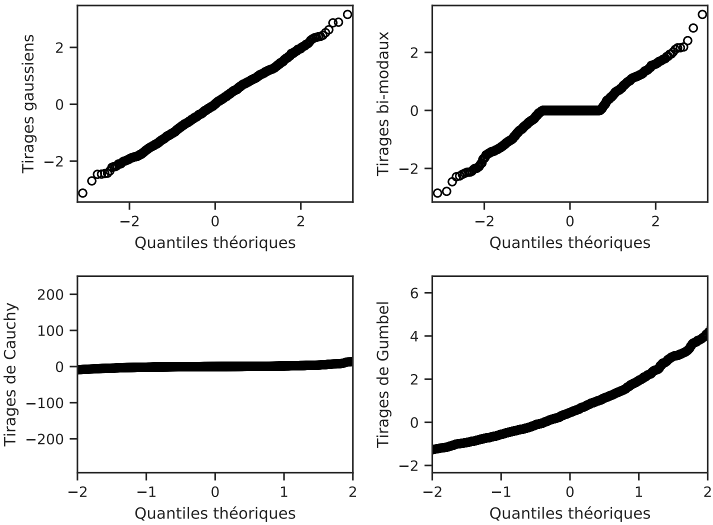
$$\left(\underbrace{\Phi^{-1}\left(\frac{i}{n+1}\right)}_{\text{quantile théorique}}, \quad \underbrace{x_{(i)}}_{\text{quantile empirique}} \right), \text{ pour } i = 1, \ldots, n$$

<u>Rem</u>: détails en TD / TP

Interprétation de qq-plot: poids à la naissance vs loi normale

- Si les observations étaient $\mathcal{N}(0,1)$ alors le nuage de points se concentrerait autour de la droite $y = x$
- Si le nuage de points se concentre autour d'une droite mais pas $y = x$, disons $y = ax + b$
  - Si $b \neq 0 \implies$ Translation
  - Si $a \neq 1 \implies$ Changement d'échelle

Quelques qq-plots pathologiques (vs. loi normale)



Bibliographie I

- Wilk, M. B. and R. Gnanadesikan. "Probability plotting methods for the analysis for the analysis of data". In: *Biometrika* 55.1 (1968), pp. 1–17.