Homework Assignment #6: ODES
Due Dec. 5th

Instructions: Find solutions to each of the listed problems below.

1. (NB) Use the Euler method to solve the initial value problem $y'' + 2y' + 4y = 0$ over the range $x = [0, 5]$. Use initial conditions $y(0) = 2$ and $y'(0) = 0$. Repeat the calculation using step sizes $h = 0.1, 0.01$, and 0.001. Plot the results and compare with the known analytic solution $y(t) = 2e^{-x}\left[\cos(\sqrt{3}x) + \frac{1}{\sqrt{3}}\sin(\sqrt{3}x)\right]$.

2. (NB) Use the 4th order Runge-Kutta method to solve the initial value problem $y'' = (1 + x^2)y$ over the range $x = [0, 1]$. Use initial conditions $y(0) = 1$ and $y'(0) = 0$. Use step size $h = 0.1$ and plot the results.

3. (Problem 8.3) Reproduce Fig. 8.4, augmented to also include results for explicit midpoint and explicit trapezoid methods (forms of predictor-corrector method), given by equations 8.54 and 8.75.

4. (NB) Solve the initial value problem $y'' + 2by' + y = k$, where b and 2 are constants. Create a plot of the solution $y(t)$, with values $b = 0.5$ and $k = 3$. Use initial values $y(0) = 1$ and $y'(0) = 0$.

5. (NB) Solve the boundary value problem $y'' = \frac{1}{8}(32 + 2x^3 - 4y')$ over the range $[1, 3]$ with initial values $y(1) = 17$ and $y(3) = 43/3$. Plot the solution and compare with the exactly known solution $y(x) = x^2 + 16/x$.

6. (Problem 8.18) Apply the shooting method to the following 4th-order BVP:

$$w''''(x) = -13w''(x) - 36w(x)$$

$$w(0) = 0; \quad w'(0) = -3; \quad w(\pi) = 2; \quad w'(\pi) = -9$$

   Note that we are missing two pieces of information starting at the point $w''(0)$ and $w'''(0)$, so you will have to use multi-dimensional Newton's method. Specifically, make sure to define one function for the four right-hand sides of our system of ODEs (after you rewrite the fourth-order ODE) and another function for the two conditions we wish to satisfy ($w(\pi) - 2 = 0$ and $w'(\pi) + 9 = 0$).

7. (NB) Solve the boundary value problem $y'' + \frac{1}{4}y = 8$ over the range $[0, 10]$ with initial values $y(0) = y(10) = 0$. Use the finite difference approach we discussed in class. As a final step, plot the solution.

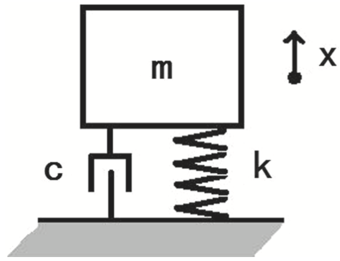
8. (NB) A mass m falls from an airplane with initial velocity $v_0 = 0$. The mass drops vertically with a starting position $y_0 = 3000$ m. Determine the position and velocity of the mass as a function of time until it hits the ground. Solve this as a two part problem where first you assume the only force acting on the mass is gravity (basically the physics I version of this problem). Then, consider an air drag force acting on the person which is of the form αv, where α a constant (for this problem adopt $\alpha = 0.5$). Create plots of the position and velocity of the object in both scenarios and compare the results.

9. (Problem 8.32) Use the shooting method for the quantum-mechanical eigenvalue problem of the infinite square well. Specifically, take the potential to be zero over the range $-a < x < a$ and infinite at the boundaries; compute the first six eigenvalues and compare with:

$$E_n = \frac{\hbar^2\pi^2}{8ma^2}n^2$$

   which you learned in a course on quantum mechanics. Take $\hbar/m = 1$ and $a = 0.5$

10. (NB) Consider the following model of a mass-spring-damper (MSD) system in one dimension. In this figure m denotes the mass of the block, c is called the damping coefficient, and k is the spring stiffness. The equation of motion for the MSD is given by: $mx'' + cx' + kx = 0$.



Start this problem by re-writting the MSD equation as set of first order ODEs like we discussed in class. Then use *solve_ivp* to find solutions to the problem. Use values $m = 1$, $k = 10$, and evaluate times in the range $t = [0, 20]$ with $\Delta t = 0.1$. Use initial values $x(0) = 1$ and $v(0) = 0$. Evaluate scenarios with different values of the damping term $c = 0$, $c = 1$, and $c = 10$. Plot the solutions and compare the results.