PHYS 4023
Version 1

Homework Assignment 3: Linear Algebra
Due Oct. 10th

1. **(NB)** Consider the following matrices $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}$ and F.

$$\mathsf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \qquad \mathsf{B} = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \qquad \mathsf{C} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & -1 \end{bmatrix} \qquad E = \begin{bmatrix} 3 & 1 \\ 0 & 3 \\ -2 & 2 \end{bmatrix} \qquad F = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 3 & -2 \end{bmatrix}$$

A) What is the product of $\mathsf{AB}, \mathsf{CD}, \mathsf{EF}$? B) Consider the products $\mathsf{A}^\mathsf{T}\mathsf{B}, \mathsf{B}\mathsf{D}^\mathsf{T}, \mathsf{D}^\mathsf{T}\mathsf{F}^\mathsf{T}$. Is the matrix product defined for any of these? If so calculate the result or explain why it does not work.

2. **(NB)** Find the inverse of the matrix M below. Demonstrate that the inverse satisfies the expected relation ($MM^-1 = I$).

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

3. **(NB)** An orthogonal matrix has the special property where it's transpose is also it's inverse. Consider the matrix P below. Find P^T, and demonstrate P is indeed an orthogonal matrix by showing $PP^T = I$ and $P^TP = I$.

$$P = \frac{1}{7}\begin{bmatrix} 3 & -2 & -6 \\ -6 & -3 & -2 \\ -2 & 6 & -3 \end{bmatrix}$$

4. **(NB)** Solve the following system of equations for x_1, x_2, x_3, x_4. Additionally, show that the determinate of the system of equations in matrix form is nonzero.

$$-x_1 + 3x_2 + 5x_3 + 2x_4 = 10 \tag{1}$$
$$x_1 + 9x_2 + 8x_3 + 4x_4 = 15 \tag{2}$$
$$x_2 + x_4 = 2 \tag{3}$$
$$2x_1 + x_2 + x_3 - x_4 = -3 \tag{4}$$

5. **(NB)** Following the example of the Gauss-Seidel method we discussed in class. Find the solution vector $\vec{x}$ for $A\vec{x} = b$, where:

$$A = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

How many iterations does it take for the solution to converge to to the point where the error terms are of order 10^{-5} or better?[1] Repeat the process for the vector $b = [4, 2, 10, -2]$.

[1] While this problem demonstrate the power of the Gauss-Seidel method, we are only guaranteed a convergent solution if the matrix A is diagonal dominate.

6. **(Problem 4.12)** Use LU composition to calculate the inverse of a matrix A as per equation 4.99, and the determinate as per equation 4.102. Test your answers by comparing the output of 'np.linalg.inv()' and np.linalg.det().

7. **(NB)** In this problem we will solve a matrix equation of the form $A\vec{x} = \vec{b}$, however, we will examine the possible pitfalls with 'black box' solvers to be aware of. Consider the matrices below:

$$A = \begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{bmatrix} \qquad b = \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix}$$

   The exact solution is $\vec{x} = (1, -1)$. Consider two approximate solutions $\vec{x}_\alpha = (0.999, -1.001)$ and $\vec{x}_\beta = (0.341, -0.087)$.

   - Compute residuals of the form $r = \vec{b} - A\vec{x}$ for the approximate solutions $\vec{x}_\alpha$ and $\vec{x}_\beta$. Does the more accurate solution have a smaller residual?
   - Calculate the determinate of the Matrix A, does this help us to understand the result of the residual analysis?
   - Last, use the 'solve' function in np.linalg() to find the solution vector. What does this result tell us?

8. **(Problem 4.26)** We will now use QR decomposition to solve a linear system of equations $\mathsf{A}\vec{x} = \vec{b}$. This equation can be re-written as $\mathsf{QR}\vec{x} = \vec{b}$. We can take advantage of the orthogonality of Q to write this as: $\mathsf{R}\vec{x} = \mathsf{Q}^{\mathsf{T}}\vec{b}$. But now the right-hand side equation contains only known quantities and the left-hand side has the upper-triangular matrix R, so a back substitution isn't all that needed. Implement this approach in python using classical Gram-Schmidt. (*Hint*: It will likely help to reference the code for qr_dec() function in the course notes and/or textbook).

9. **(NB)** Matrices need not be composed of real numbers, in fact the fact that complex matrices exist is an important feature in Quantum Mechanics. The description of spin 1 particles uses the matrix operators:

$$M_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \qquad M_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

   show the following:

   - Test the commutation relation (that is $[M_x, M_y] = M_xM_y - M_yM_x$) to show $[M_x, M_y] = iM_z$ and so on for cyclic permutations of indecies (i.e., $[M_i, M_j] = iM_k$ for $i \neq j \neq k$, which are the commutation relations of angular momentum
   - $M^2 = M_x^2 + M_y^2 + M_z^2 = 2I$ where I is the identity matrix
   - $[L^+, L^-] = 2M_z$ where $L^+ \equiv M_x + iM_y$ and $L^- \equiv M_x - iM_y$

10. **(NB)** The L^+ and L^- operators from the last problem are the 'ladder' operators from Quantum Mechanics. The L^+ operating on a system of spin projection m will raise it to the state $m+1$ if m is below m_{max}. Note that applying L^+ on m_{max} results in zero. The L^- operator reduces the spin state from m to $m-1$ in a similar fashion. Dividing out the expressions by $\sqrt{2}$ we have:

$$L^+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad L^- = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

   Show that $L^+|-1\rangle = |0\rangle$, $L^-|-1\rangle$ = Null vector, $L^+|0\rangle = |1\rangle$, $L^-|0\rangle = |-1\rangle$, $L^+|1\rangle$ = Null vector, $L^-|1\rangle = |0\rangle$
   where vectors are represented as:

$$|-1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad |0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$