The Control of an Inverted Pendulum

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1 – Introduction

1.1 – Objectives, Goals, and Purposes

The main objective of the lab was to use a controller to control an inverted pendulum that is attached to a gantry/cart with the goal to keep the pendulum from swinging to the down position. This is done by getting the physical characteristics of the pendulum and the cart, using pole placement and LQR methods to obtain gains for the controller, the two methods of obtaining the gains for the controller make up the two parts of the lab. The performance of each method is compared to each other. This is done such that when small outer perturbations to the angle of the pendulum will be corrected by the movement of cart which is only able to move back and forth in one direction.

1.2 – Intended Methods

The intended methods in choosing the poles is to guess and check the poles for the system to obtain gains for the controller. Once the performance is recorded and analyzed, the poles can be adjusted accordingly to obtain even better performance. The performance will be based on the steady state error, rise time, overshoot, and the controller's ability to maintain a stable upright pendulum.

2 – Procedure

2.1 – Definition of Variables

Variable	Description	Units
xc	Position of the cart	M, cm, or mm
Alpha	Angle of the pendulum	
Fc	Cart force	kg/s
K	Controller gains	
ωn	Natural frequency	m
G	gravity	kg/s
B_{eq}	Damping of cart due to friction	kg/s
V	Voltage	V

Symbol	Description	Value	Unit
R_m	motor armature resistance	2.6	Ω
L_m	motor armature inductance	0.18	mH
K_t	motor torque constant	0.00767	$N.m/A$
η_m	motor efficiency	100%	%
K_m	back-electromotive-force(EMF) constant	0.00767	$V.s/rad$
J_m	rotor moment of inertia	3.9×10^{-7}	$kg.m^2$
K_g	planetary gearbox ratio	3.71	
η_g	planetary gearbox efficiency	100%	%
M_{c2}	cart mass	0.57	kg
M_w	cart weight mass	0.37	kg
M_c	total cart weight mass including motor inertia	1.0731	kg
B_{eq}	viscous damping at motor pinion	5.4000	$N.s/m$
L_t	track length	0.990	m
T_c	cart travel	0.814	m
P_r	rack pitch	1.664×10^{-3}	$m/tooth$
r_{mp}	motor pinion radius	6.35×10^{-3}	m
N_{mp}	motor pinion number of teeth	24	
r_{pp}	position pinion radius	0.01482975	m
N_{pp}	position pinion number of teeth	56	
K_{EP}	cart encoder resolution	2.275×10^{-5}	$m/count$
M_p	long pendulum mass with T-fitting	0.230	kg
M_{pm}	medium pendulum mass with T-fitting	0.127	kg
L_p	long pendulum length from pivot to tip	0.6413	m
L_{pm}	medium pendulum length from pivot to tip	0.3365	m
l_p	long pendulum length: pivot to center of mass	0.3302	m
l_{pm}	medium pendulum length: pivot to center of mass	0.1778	m
J_p	long pendulum moment of inertia $\circlearrowleft$ center of mass	7.88×10^{-3}	$kg.m^2$
J_{pm}	medium pendulum moment of inertia $\circlearrowleft$ center of mass	1.20×10^{-3}	$kg.m^2$
B_p	viscous damping at pendulum axis	0.0024	$N.m.s/rad$
g	gravitational constant	9.81	m/s^2

Figure 1 – Additional Variable Definitions

2.2 – Schematic and Description of Apparatus

Figure 2 – Experimental Apparatus

Similar to the previous lab, a cart is on a toothed track which interfaces with the gear on the cart to induce movement with a motor. The cart is connected, electronically, to the computer such that data is passed through and recorded to the computer which can be interacted with using MATLAB and Simulink. This time, instead of stabilizing the pendulum in the down position, the pendulum will be stabilized in the up position. The pendulum is 12 inches long and is allowed to freely rotate about the connection point.

2.3 – Procedure of Experiments

Part (i): The Inverted Pendulum

a) Enter K gains that were calculated using pole placement
b) Place cart in the middle of the track, the downward position is equal to alpha = pi
c) Slowly the pendulum to the upward position, set the initial angle for the pendulum and let go of the cart and pendulum, record the data
d) Tap the pendulum and record the data, this is to test to see if the cart returns to the middle
e) Hold down the cart and move the pendulum to 0.2rad then let go of both the cart and the pendulum, record data on how the cart reacts
f) Record best K gains

Part (ii): Moving the Balanced Pendulum

a) Set maximum and minimum position limits
b) Enter K gains calculated with LQR
c) Place cart in the middle of the track
d) Test response with a pulse generator with an amplitude of 0.2, period of 10 seconds, PWM of 50% and phase delay of 10 seconds, record the data
e) Record bets K gains

3 – Results

3.1 – The Inverted Pendulum

Pole Placement -

Poles	-15.0000	-8.0000 + 3.0000i	-8.0000 - 3.0000i	-1.0000
Gains (K)	-16.4551	63.5987	-28.9504	9.5700

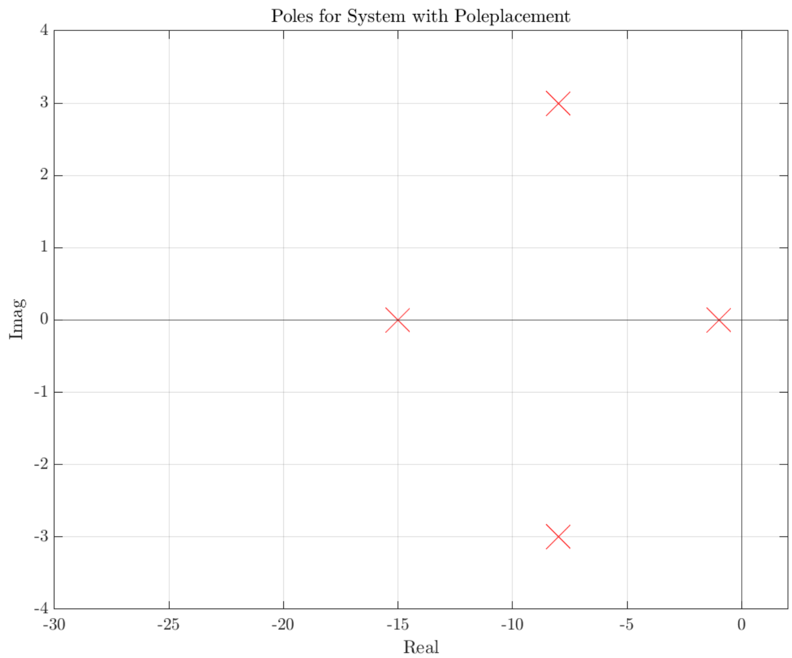

Figure 3 – System Poles with Pole Placement

LQR -

Poles	-28.7837	-4.7610 +1.8401i	-4.7610 -1.8401i	-0.8419
Gains (K)	-9.4868	59.6207	-22.7821	9.1995

Q Matrix:

9	0	0	0
0	9	0	0
0	0	7	0
0	0	0	1

R = $\boxed{0.1}$

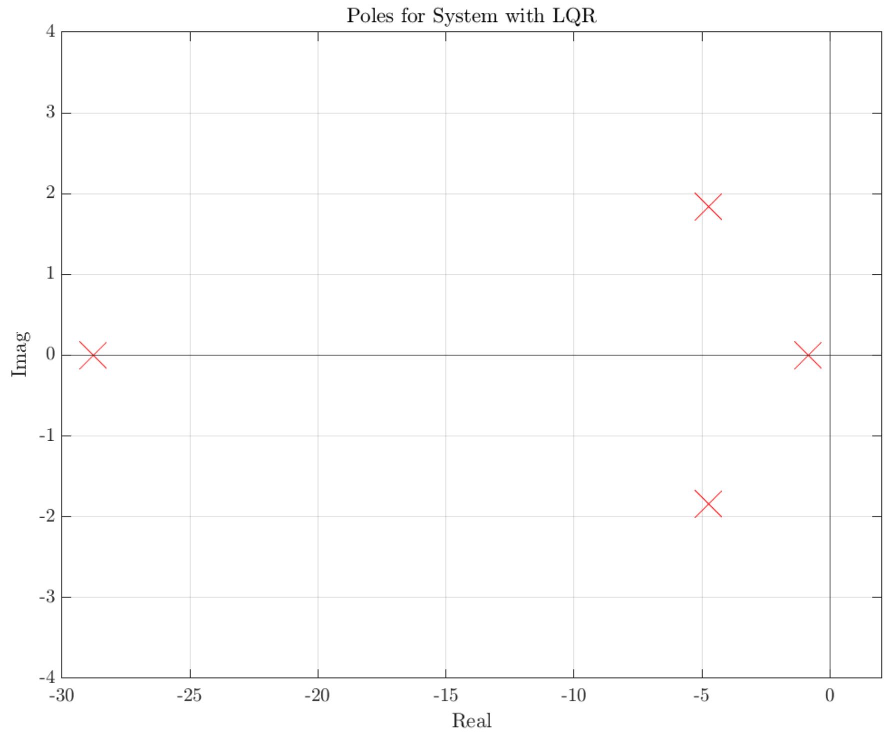

Figure 4 – System Poles with LQR

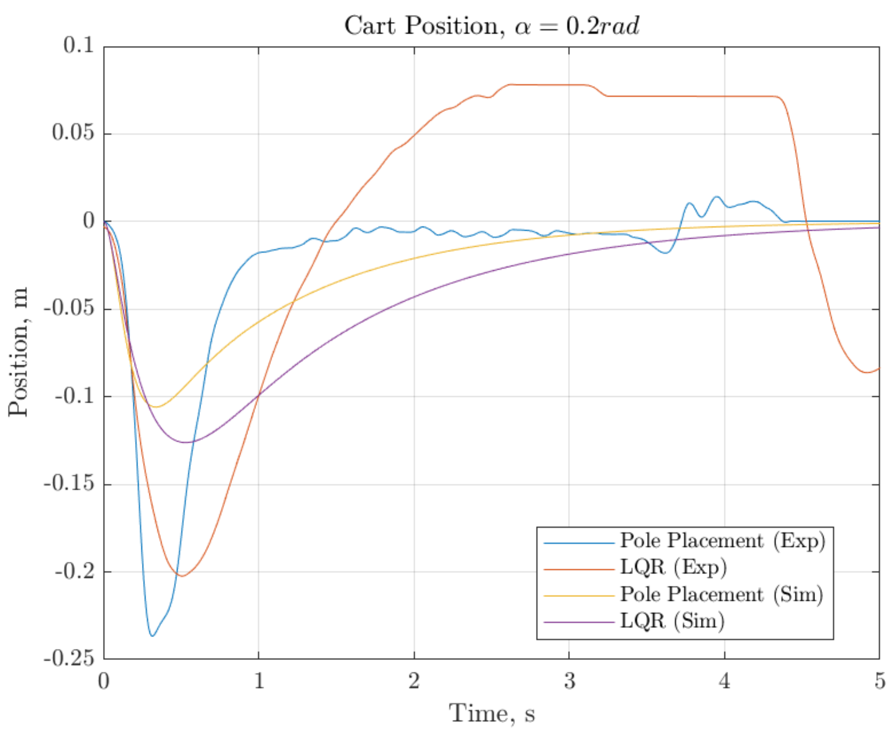

Figure 5 – Cart Position Response to Initial Angle

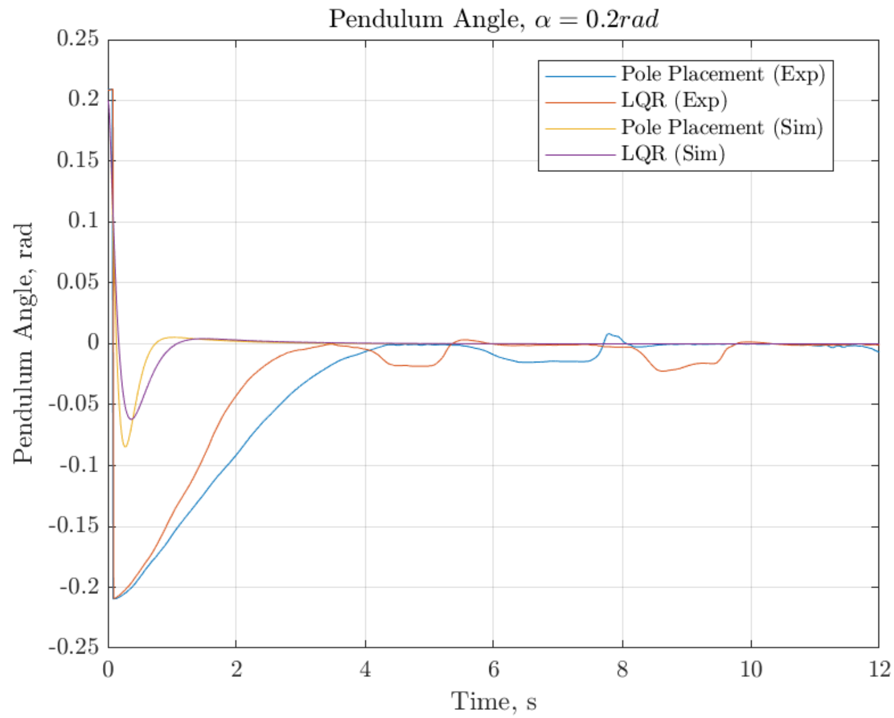

Figure 6 – Pendulum Angle Response to Initial Angle

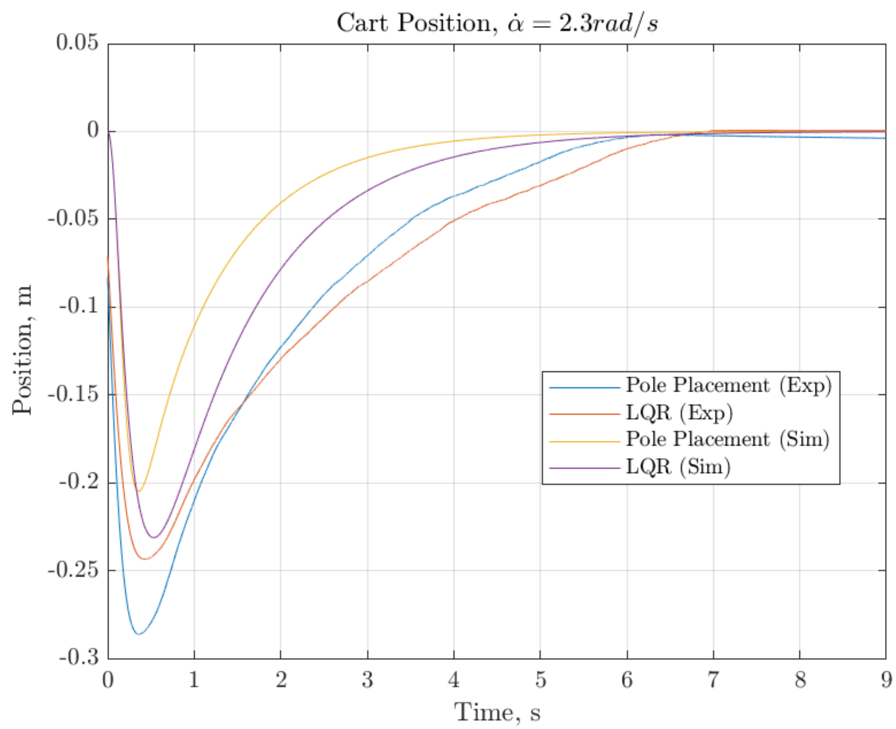

Figure 7 – Cart Position Response to Tap

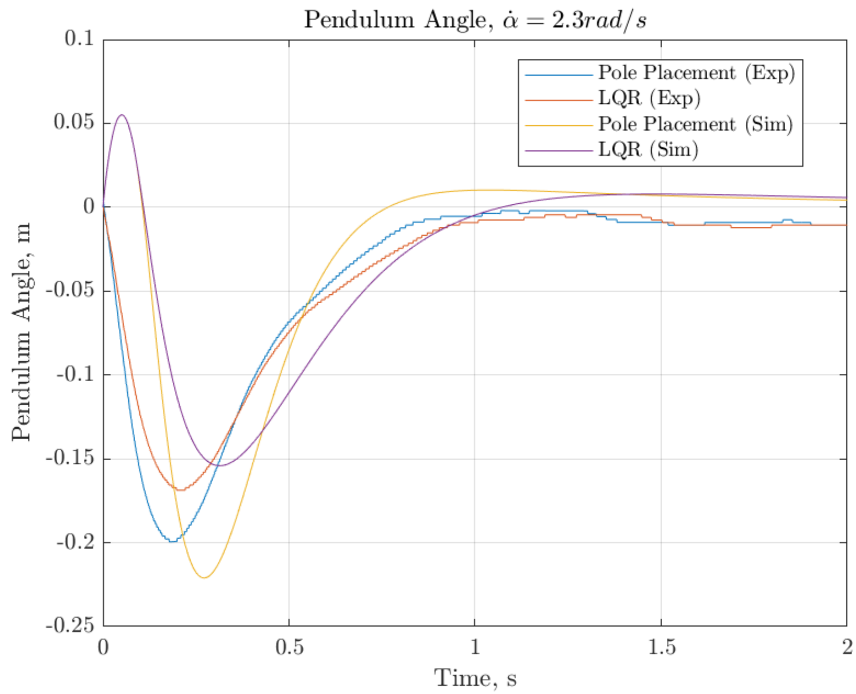

Figure 8 – Pendulum Angle Response to Tap

3.2 – Moving the Balanced Pendulum

LQR –

Poles	-40.8724	-3.3671 + 1.7481i	-3.3671 - 1.7481i	-1.5493 + 0.9965i	-1.5493 - 0.9965i
Gains (K)	-42.4888	84.3299	-33.5350	13.3860	30.0000

Q Matrix:

25	0	0	0	0
0	25	0	0	0
0	0	3	0	0
0	0	0	3	0
0	0	0	0	90

R =

0.1

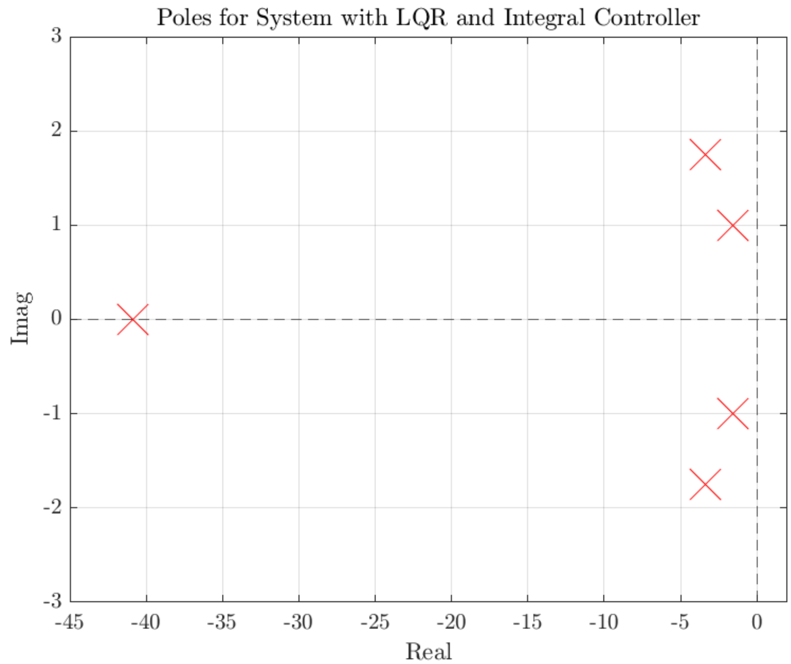

Figure 9 – System Poles with LQR and Integral Controller

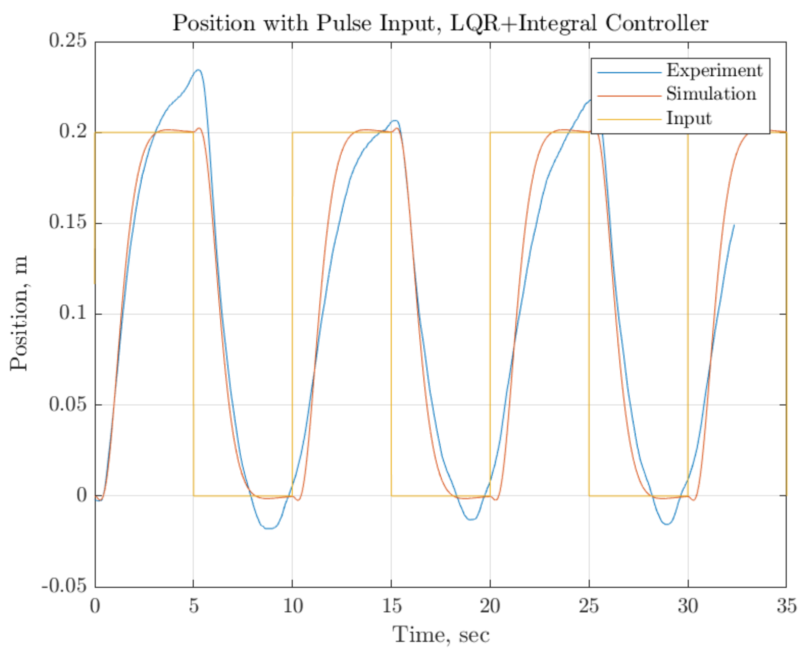

Figure 10 – Cart Position Response to Square Wave Input

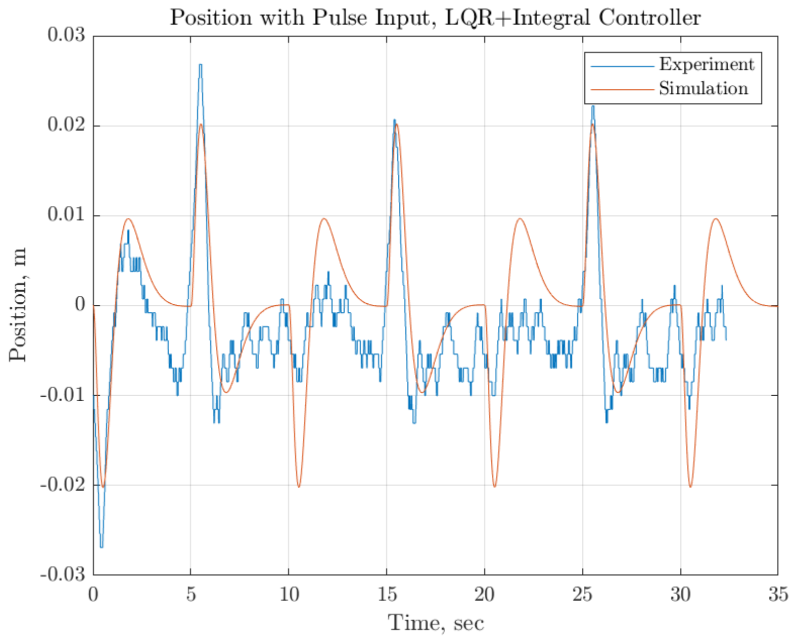

Figure 11 – Pendulum Angle Response top Square Wave Input

4 – Analysis and Discussion

Equations of Motion (Nonlinear):

$$
\begin{aligned}
&\left((M_c+M_p)I_p + M_cM_pl_p^2 + M_p^2l_p^2\sin(\alpha)^2\right)\ddot{x}_c + \left(I_p+M_pl_p^2\right)B_{eq}\dot{x}_c \\
= \quad & -M_pl_pB_p\cos(\alpha)\dot{\alpha} - \left(M_p^2l_p^3 + I_pM_pl_p\right)\sin(\alpha)\dot{\alpha}^2 + \left(I_p+M_pl_p^2\right)F_c + M_p^2l_p^2g\cos(\alpha)\sin(\alpha);
\end{aligned}
$$

$$
\begin{aligned}
&\left((M_c+M_p)I_p + M_cM_pl_p^2 + M_p^2l_p^2\sin(\alpha)^2\right)\ddot{\alpha} + (M_c+M_p)B_p\dot{\alpha} \\
= \quad & (M_c+M_p)M_pgl_p\sin(\alpha) - M_pl_p\cos(\alpha)B_{eq}\dot{x}_c - M_p^2l_p^2\sin(\alpha)\cos(\alpha)\dot{\alpha}^2 + M_pl_p\cos(\alpha)F_c;
\end{aligned}
$$

$$
F_c = \frac{\eta_gK_g\eta_mK_t\left(vr_{mp} - K_gK_m\dot{x}_c\right)}{R_mr_{mp}^2}.
$$

Figure 12 – Nonlinear Equations of Motion

$$
\begin{aligned}
\ddot{x}_c &= \frac{M_p^2l_p^2g\alpha - (I_p+M_pl_p^2)B_{eq}\dot{x}_c - M_pl_pB_p\dot{\alpha} + (I_p+M_pl_p^2)F_c}{(M_c+M_p)I_p + M_cM_pl_p^2} \\
\ddot{\alpha} &= \frac{(M_c+M_p)M_pgl_p\alpha - (M_c+M_p)B_p\dot{\alpha} - M_pl_pB_{eq}\dot{x}_c + M_pl_pF_c}{(M_c+M_p)I_p + M_cM_pl_p^2} \\
F_c &= \frac{\eta_gK_g\eta_mK_t\left(vr_{mp} - K_gK_m\dot{x}_c\right)}{R_mr_{mp}^2}.
\end{aligned}
$$

Figure 13 – Linearized Equations of Motion (about alpha = 0)

To calculate the equilibrium points, all f the higher order terms (xdot, xddot, alphadot, alphaddot) are set to 0 and solve for alpha which will be the possible equilibrium points either alpha = 0 or alpha = pi. The equilibrium point of interest for this lab is 0 radians which is the upright position as the lab is meant to stabilize an upright (inverted) pendulum on a cart. The state variables are x and xdot which are the cart position and velocity respectively and alpha and alphadot which are the angle of the pendulum and the angular rate of the pendulum respectively. The input for the system is the voltage.

A =

$$\begin{pmatrix} 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \\ 0 & 0.8703 & -11.92 & -0.00943 \\ 0 & 46.26 & -51.64 & -0.5012 \end{pmatrix}$$

B =

$$\begin{pmatrix} 0 \\ 0 \\ 1.566 \\ 6.783 \end{pmatrix}$$

C =

$$\begin{pmatrix} 1.0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{pmatrix}$$

D =

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Figure 14 – State Space Matrices for the Short Pendulum

4.1 - The Inverted Pendulum

In this lab, the objective of the controller is to maintain the equilibrium point for the pendulum while resisting outside interference such as initial conditions or bumps to the pendulum. Performance of the system is also one of the goals for the controller as the test and repeat process during the lab consisted of changing the poles which in turn changed the gains to the controller. This allowed for better performance as the gains got closer to the best gains possible. Similar to the pole placements, the Q and R matrices were chosen in the lab such that the performance and speed of the system were optimized. The plots for this section are in figures 5 through 8. The data that was recorded in the lab did not match exactly to the simulated results. This may be an error in the way that we set up the lab and outside factors such as the speed of moving the pendulum to the upright position, bumping the cart longer than an initial impulse, etc. In the experiment where the pendulum was tapped, the angular rate was determined using the simulation and changing the angular rate in the model until it closely matches the experimental data. The pole placement gains ended up giving “better” results when compared to the LQR gains. This actually depends on the definition of “better” as the LQR gains tended to yield results that had a lot less overshoot while being slightly slower. On the other hand, the pole placement gains tended to perform faster, but had a much higher overshoot, so the “better” set of gains depends on the system requirements.

4.2 – Moving the Inverted Pendulum

The controller that was applied in this case was a state space integral controller whose goal was to move the cart without letting the pendulum fall. The design of the controller consisted of using LQR to find the gains. The integral controller implements a new state variable which is the integral of the difference between the cart position and the input variable. While performing the lab, the Q and R matrices were determined through trial and error until the performance of the system response was quick enough and had the least amount of error compared to the input into the system the penalizations of the state or control variables were decided when looking at the output plots and deciding which state needed better response times. The plots for this section of the experiment are in figures 10 and 11. These plots were generated in MATLAB using the experimental data and the simulated data used the same K gains as the experimental data. The system response was acceptable as the response followed close enough to the input signal without too much overshoot.

5 – Conclusions

5.1 – Main Points

The controllers were able to achieve design objectives despite having troubles in the first section of the lab. The performance of the system was acceptable in the second section of the lab, the second section of the lab had fair performance but was riddled with corrupt data and/or improper testing. Pole placement was much easier to use than LQR as the poles can be chosen on the specific requirements, but LQR may be more robust and better in some situations. For this, I prefer to use pole placement to design the controller as it is much simpler and intuitive.

5.2 – Limitations

The biggest assumption or limitation for this lab was that the equations of motion were linearized in order to design the controller. There is no easy way to design a controller for such a nonlinear system so that makes sense but there are limitations to linearizing a system such as the linearizations are only valid very close to the equilibrium points. Another limitation was the maximum gains that could be used as they were limited to 200 for all gains, these gains were able to be used in the simulations but were not allowed when performing the experiment physically in concern of damaging the equipment even further. The last limitation, similar to the last one, is the limitation on the voltage of the motor for the cart, this limits the maximum speed that the cart can move which limits the response of the system.

5.3 – Personal Lessons and Suggestions

I personally have no other suggestions for improving the lab. The lessons I learned for the lab was to make sure that the data was recorded properly and organized well so that the data can be used for the post lab. Another lesson I learned is to watch the instruction more carefully on how to manipulate the physical apparatus such that the data that is collected is valid.