

# Calculus

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Part I

Preliminaries





# Chapter 1

## Functions

### 1.1 Sets

Before defining what a function is or what it does, it is important to briefly discuss what goes into function and what comes out. Simply, *sets* are a collection of items and each one of those items are usually referred to as *elements*. Without getting into the weeds of set theory, sets can contain pretty much anything from numbers, functions, and other sets [1].

Some common sets that you may be familiar with are the *natural numbers*  $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$ , the *integers*  $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ , and the *real numbers*  $\mathbb{R}$ , which is usually represented via a number line. Sets can also be intervals on the real line (i.e.  $[a, b)$  is an interval on  $\mathbb{R}$  containing  $a$  but not  $b$ ) or even the possible results of flipping a coin  $C = \{H, T\}$ .

We will now define the basic notation when dealing with sets and the operations that can be performed on sets. We say that  $x$  is an element of a set  $A$  with the notation  $x \in A$  and when  $x$  is not in  $A$ , we say  $x \notin A$ . For example, given the set  $A = \{1, 2, 3, 4\}$ , we can say that  $1 \in A$  is true as well as  $5 \notin A$ .

The notion of combining sets comes with *unions* and *intersections*. Given  $A$  and  $B$  are sets, the union of  $A$  and  $B$  is denoted as  $A \cup B$  and is equal to the set that contains elements in either  $A$  or  $B$ . Similarly, the intersection between  $A$  and  $B$  is denoted as  $A \cap B$  and is the set that contains elements that are in both  $A$  and  $B$ . For example, given the sets  $A = \{1, 2, 3, 4\}$  and  $B = \{3, 4, 5, 6\}$ , the union and intersection between  $A$  and  $B$  is

$$\begin{aligned} A \cup B &= \{1, 2, 3, 4, 5, 6\} \\ A \cap B &= \{3, 4\} \end{aligned} \tag{1.1}$$

Subsets are important to relate different sets. A set  $A$  is said to be a subset of another set  $B$  if all of the elements of  $A$  are also within  $B$  and is denoted as  $A \subseteq B$  and a set  $A$  is equal to a set  $B$  if and only if  $A \subseteq B$  and  $B \subseteq A$ . An important subset is the empty set represented by the symbols  $\emptyset$ ,  $\emptyset$ , or simply  $\{\}$ . It is important to note that the empty set is also a subset of all sets.

Using sets by listing them out can become cumbersome and sometimes confusing, instead set builder notation is used to build a set based on a rule. For example, the set of all positive even integers can be written as

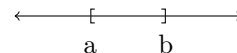
$$A = \{2, 4, 6, 8, \dots\} = \{z : z \text{ is an positive even integer}\} = \{z : z = 2n, n \in \mathbb{Z} \text{ and } n > 0\} \tag{1.2}$$

Here, the  $:$  stands for "such that" which indicates the rule (the words "such that" or the symbol  $|$  is also often used). The *rational numbers*  $\mathbb{Q}$  can also be constructed via the integers with

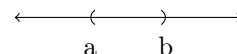
$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\} \tag{1.3}$$

As mentioned previously, intervals on the real number line can be represented as sets. Given two values  $a$  and  $b$  and assuming that  $a \leq b$ , intervals on the real line are represented as

- Closed interval:  $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

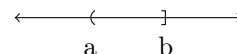


- Open interval:  $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

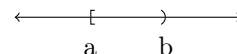


- Half-open interval:

$$- (a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

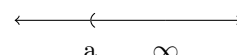


$$- [a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

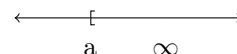


- Infinite interval:

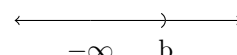
$$- (a, \infty) = \{x \in \mathbb{R} : a < x\}$$



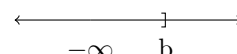
$$- [a, \infty) = \{x \in \mathbb{R} : a \leq x\}$$



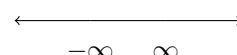
$$- (-\infty, b) = \{x \in \mathbb{R} : x < b\}$$



$$- (-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$$



$$- (-\infty, \infty) = \mathbb{R}$$



Here, the open brackets ( and ) indicate that the respective endpoint is not included, while the closed brackets [ and ] indicate that the respective endpoint is included in the interval.

## 1.2 What is a function?

*Functions* are objects in math that describe a relationship or mapping between two sets. Given two sets  $X$  and  $Y$ , a function  $f$  maps the elements of  $X$ , called the *domain* of the function, to elements in the set  $Y$  called the *codomain* of the function and the relationship is denoted as  $f : X \rightarrow Y$  (" $f$  maps from  $X$  to  $Y$ ") or  $y = f(x)$  (" $y$  equals  $f$  of  $x$ "), where  $y \in Y$  is known as the *dependent variable* and  $x \in X$  is known as the *independent variable* or the *argument* of the function. The *range* of a function is the set of all possible values that  $f(x)$  is able to output with  $X$  as its domain, note that the range is a subset of  $Y$  but is not always equal.

Some real-world examples of functions are

- The area of a circle: .....  $A(r) = \pi r^2$
- The height of a falling ball: .....  $h(t) = h_0 + v_0 t - (1/2)gt^2$
- Compound interest: .....  $A(t) = P(1 + \frac{r}{n})^{nt}$
- Temperature conversion: .....  $C(F) = \frac{5}{9}(F - 32)$
-

## **1.3 The Graph of a Function**

### **1.4 Types of Functions**

### **1.5 Common Functions**

### **1.6 Inverse Function**

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### **1.8 Function Transformations and Operations**

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Part II

Differential Calculus



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Part III

Integral Calculus



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# Multivariable Calculus



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17.4 Fluid Flows



Part V

Appendix



# Bibliography

- [1] Richard Hammack. *Book of Proof*. 3rd Edition. Richard Hammack, 2018.

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