Calculus

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Part I

Preliminaries

Chapter 1

Functions

1.1 Sets

Before defining what a function is or what it does, it is important to briefly discuss what goes into function and what comes out. Simply, *sets* are a collection of items and each one of those items are usually referred to as *elements*. Without getting into the weeds of set theory, sets can contain pretty much anything from numbers, functions, and other sets [3].

Some common sets that you may be familiar with are the *natural numbers* $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$, the *integers* $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, and the *real numbers* $\mathbb{R}$, which is usually represented via a number line. Sets can also be intervals on the real line (i.e. [a,b) is an interval on $\mathbb{R}$ containing a but not b) or even the possible results of flipping a coin $C = \{H, T\}$.

We will now define the basic notation when dealing with sets and the operations that can be performed on sets. We say that x is an element of a set A with the notation $x \in A$ and when x is not in A, we say $x \notin A$. For example, given the set $A = \{1, 2, 3, 4\}$, we can say that $1 \in A$ is true as well as $5 \notin A$.

The notion of combining sets comes with *unions* and *intersections*. Given A and B are sets, the union of A and B is denoted as $A \cup B$ and is equal to the set the contains elements in either A or B. Similarly, the intersection between A and B is denoted as $A \cap B$ and is the set that contains elements that are in both A and B. For example, given the sets $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, the union and intersection between A and B is

$$\begin{aligned} A \cup B &= \{1, 2, 3, 4, 5, 6\} \\ A \cap B &= \{3, 4\} \end{aligned} \tag{1.1}$$

Subsets are often used to relate different sets and their elements. A set A is said to be a subset of another set B if all of the elements of A are also within B and is denoted as $A \subseteq B$ and a set A is equal to a set B if and only if $A \subseteq B$ and $B \subseteq A$. An important subset is the empty set represented by the symbols $\varnothing$, $\emptyset$, or simply $\{\}$. It is important to note that the empty set is also a subset of all sets.

Using sets by listing them out can become cumbersome and sometimes confusing, instead set builder notation is used to build a set based on a rule. For example, the set of all positive even integers can be written as

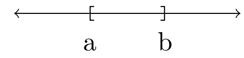
$$A = \{2, 4, 6, 8, \dots\} = \{z : z \text{ is an positive even integer}\} = \{z : z = 2n, n \in \mathbb{Z} \text{ and } n > 0\} \tag{1.2}$$

Here, the : stands for "such that" which indicates the rule (the words "such that" or the symbol | is also often used). The *rational numbers* $\mathbb{Q}$ can also be constructed via the integers with

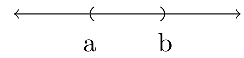
$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0\} \tag{1.3}$$

As mentioned previously, intervals on the real number line can be represented as sets. Given two values a and b and assuming that $a \leq b$, intervals on the real line are represented as

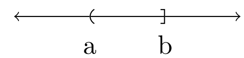
- Closed interval: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$



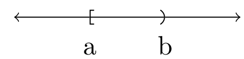
- Open interval: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$



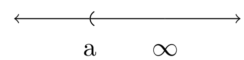
- Half-open interval:
  - $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$



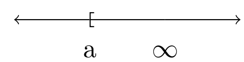
  - $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$



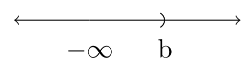
- Infinite interval:
  - $(a, \infty) = \{x \in \mathbb{R} : a < x\}$



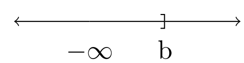
  - $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$



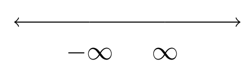
  - $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$



  - $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$



  - $(-\infty, \infty) = \mathbb{R}$



Here, the open brackets (and) indicate that the respective endpoint is not included, while the closed brackets [and] indicate that the respective endpoint is included in the interval.

The *Cartesian product* is used often to describe ordered pairs or even higher-dimension coordinates. Some examples include the xy-plane also known as $\mathbb{R}^2$, 3D space with $\mathbb{R}^3$. The Cartesian product for two sets A and B is defined as

$$A \times B = \{(a, b) : a \in A, b \in B\} \tag{1.4}$$

Note that $A \times B \neq B \times A$ as the operation is order dependent. Higher-order products are defined with sets $A_1, A_2, \ldots, A_n$ as

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i, i \in \{1, 2, \ldots, n\}\} \tag{1.5}$$

As previously mentioned, the 2D plane as well as the 3D plane can be constructed using Cartesian products as well and is done as such with the set of real numbers $\mathbb{R}$

$$\begin{aligned}
\mathbb{R}^2 &= \mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\} \\
\mathbb{R}^3 &= \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) : x, y, z \in \mathbb{R}\} \\
&\vdots \\
\mathbb{R}^n &= \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i \in \{1, 2, \ldots, n\}\}
\end{aligned} \tag{1.6}$$

For more discussion on sets and set theory, refer to [2] or [3].

1.2 What is a function?

Functions are objects in math that describe a relationship or mapping between two sets. Given two sets X and Y, a function f maps the elements of X, called the *domain* of the function, to elements in the set Y called the *codomain* of the function and the relationship is denoted as $f : X \to Y$ ("f maps from X to Y") or $y = f(x)$ ("y equals f of x" [1]), where $y \in Y$ is known as the *dependent variable* and $x \in X$ is known as the *independent variable* or the *argument* of the function. The *range* of a function is the set of all possible values that $f(x)$ is able to output with X as its domain, note that the range is a subset of Y but is not always equal.

Example 1.1. Some real-world examples of functions are

- The area of a circle: .... $A(r) = \pi r^2$
- The height of a falling ball: .... $h(t) = h_0 + v_0 t - (1/2)gt^2$
- Compound interest: .... $A(t) = P(1 + \frac{r}{n})^{nt}$
- Temperature conversion: .... $C(F) = \frac{5}{9}(F - 32)$
-

These examples help model and compute

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Bibliography

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