Calculus

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Contents

Part I

Preliminaries

Chapter 1

Functions

1.1 Sets

Before defining what a function is or what it does, it is important to briefly discuss what goes into function and what comes out. Simply, **sets** are a collection of items and each one of those items are usually referred to as **elements**. Without getting into the weeds of set theory, sets can contain pretty much anything from numbers, functions, and other sets [3].

Example 1.1. Common sets of numbers

Some common sets that you may be familiar with are the **natural numbers** $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$, the **integers** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, and the **real numbers** $\mathbb{R}$, which is usually represented via a number line. Sets can also be intervals on the real line (i.e. [a,b) is an interval on $\mathbb{R}$ containing a but not b) or even the possible results of flipping a coin $C = \{H, T\}$.

We will now define the basic notation when dealing with sets and the operations that can be performed on sets. We say that x is an element of a set A with the notation $x \in A$ and when x is not in A, we say $x \notin A$. For example, given the set $A = \{1, 2, 3, 4\}$, we can say that $1 \in A$ is true as well as $5 \notin A$.

The notion of combining sets comes with **unions** and **intersections**. Given A and B are sets, the union of A and B is denoted as $A \cup B$ and is equal to the set the contains elements in either A or B. Similarly, the intersection between A and B is denoted as $A \cap B$ and is the set that contains elements that are in both A and B.

Example 1.2. Union and Intersection of two finite sets.

Given the sets $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, the union and intersection between A and B is

$$A \cup B = \{1, 2, 3, 4, 5, 6\}$$
$$A \cap B = \{3, 4\}$$

Example 1.3. Constructing the integers from the natural numbers.

Given $A = \mathbb{N}$, $B = \{-n : n \in N\}$, and $C = \{0\}$, then

$$(A \cup B) \cup C = \mathbb{Z}$$

Another elementary set operation is the **set difference**. Suppose A and B are sets, then the set difference $A - B$ (or $A \setminus B$) is the set A with the elements that both A and B share removed, using set notation, $A - B = \{a \in A : a \notin B\}$. Note that $A - B \neq B - A$. Similarly, the **symmetric difference** of two sets A and B, denoted $A \Delta B$ is the set of elements in *either* A or B but *both*. In set notation, $A \Delta B = A \cup B - A \cap B = \{x \in A \cup B : x \notin A \cap B\} = (A - B) \cup (B - A)$.

Example 1.4. Set difference and symmetric difference.

Suppose $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{3, 4, 5, 6, 8\}$, then

$$A - B = \{1, 2, 7\}$$
$$B - A = \{8\}$$
$$A \Delta B = B \Delta A = \{1, 2, 7, 8\}$$

Subsets are often used to relate different sets and their elements. A set A is said to be a subset of another set B if all of the elements of A are also within B and is denoted as $A \subseteq B$ and a set A is equal to a set B if and only if $A \subseteq B$ and $B \subseteq A$. An important subset is the empty set represented by the symbols $\varnothing$, $\emptyset$, or simply $\{\}$. It is important to note that the empty set is also a subset of all sets.

Using sets by listing them out can become cumbersome and sometimes confusing, instead set builder notation is used to build a set based on a rule. For example, the set of all positive even integers can be written as

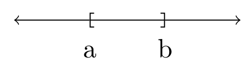
$$A = \{2, 4, 6, 8, \dots\} = \{z : z \text{ is an positive even integer}\} = \{z : z = 2n, n \in \mathbb{Z} \text{ and } n > 0\} \tag{1.1}$$

Here, the ":" stands for "such that" which indicates the rule (the words "such that" or the symbol | is also often used). The **rational numbers** $\mathbb{Q}$ can also be constructed via the integers with

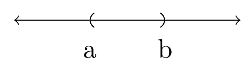
$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\} \tag{1.2}$$

As mentioned previously, intervals on the real number line can be represented as sets. Given two values a and b and assuming that $a \leq b$, intervals on the real line are represented as

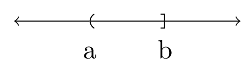
- Closed interval: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$



- Open interval: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

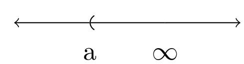


- Half-open interval:
  - $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

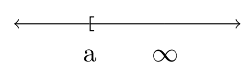


  - $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$

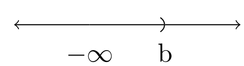
- Infinite interval:
  - $(a, \infty) = \{x \in \mathbb{R} : a < x\}$



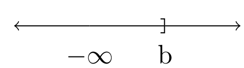
  - $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$



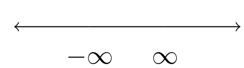
  - $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$



  - $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$



  - $(-\infty, \infty) = \mathbb{R}$



Here, the open brackets (and) indicate that the respective endpoint is not included in the interval, while the closed brackets [and] indicate that the respective endpoint is included in the interval.

The **Cartesian product** (also known as the **direct product**) is used often to describe ordered pairs or even higher-dimension coordinates. Some examples include the xy-plane also known as $\mathbb{R}^2$, 3D space with $\mathbb{R}^3$. The Cartesian product for two sets A and B is defined as

$$A \times B = \{(a,b) : a \in A, b \in B\} \tag{1.3}$$

Note that, in general, $A \times B \neq B \times A$ as the operation is order dependent. Higher-order products are defined with sets $A_1, A_2, \ldots, A_n$ as

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i, i \in \{1, 2, \ldots, n\}\} \tag{1.4}$$

As previously mentioned, the 2D plane as well as the 3D plane can be constructed using Cartesian products as well and is done as such with the set of real numbers $\mathbb{R}$

$$\begin{aligned} \mathbb{R}^2 &= \mathbb{R} \times \mathbb{R} = \{(x,y) : x, y \in \mathbb{R}\} \\ \mathbb{R}^3 &= \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) : x, y, z \in \mathbb{R}\} \\ &\vdots \\ \mathbb{R}^n &= \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i \in \{1, 2, \ldots, n\}\} \end{aligned} \tag{1.5}$$

For further and more formal discussion on sets and naive set theory as well as formal logic and proof writing, refer to [2] or [3].

Exercises

1.2 What is a function?

Functions are objects in math that describe a relationship or mapping between two sets. Given two sets X and Y, a function f maps the unique elements of X, called the **domain** of the function, to elements in the set Y called the **codomain** of the function and the relationship is denoted as $f : X \to Y$ ("f maps from X to Y") or $y = f(x)$ ("y equals f of x" [1]), where $y \in R \subseteq Y$ is known as the **dependent variable** with R as the range of the function and $x \in X$ is known as the **independent variable** or the **argument** of the function. The **range** of a function is the set of all possible values that $f(x)$ is able to output with X as its domain, note that the range is a subset of the codomain Y but is not always equal. Formally, the definition is as follows

Definition 1.5. A **function** is a mapping or rule that assigns each element from a set called the domain X of the function to a unique element in the range $R = f(X)$ of the function which is a subset of the codomain $R \subseteq Y$ and the element $x \in X$ is mapped to an element in $y \in R \subseteq Y$ with $x \mapsto y$.

Further emphasis must be made on the elements $y = f(x)$. The function $f(x)$ maps the element $x \in X$ to exactly one element $y \in Y$ [4]. For single variable functions, this can be tested using the vertical line test on a function's graph (covered shortly). If a formula results in two answers with one input (i.e. the equation for a circle: $x^2 + y^2 = r$ gives two points of y for each x) it is no longer considered a function. Figure 1.1 shows how a function maps elements of the domain to the range.

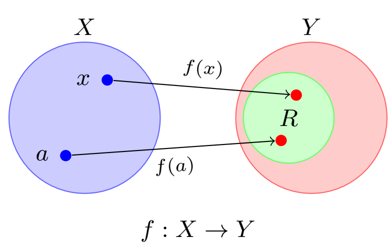


Figure 1.1: Function mapping diagram showing the domain X, codomain Y, and range $f(X)$

Example 1.6. Real world examples of functions.

- The area of a circle: .... $A(r) = \pi r^2$
- The height of a falling ball: .... $h(t) = h_0 + v_0 t - (1/2)gt^2$
- Compound interest: .... $A(t) = P(1 + \frac{r}{n})^{nt}$
- Temperature conversion: .... $C(F) = \frac{5}{9}(F - 32)$
-

Intuitively, it can be useful to think of functions as machines that take in an input and produces an output. The examples in Example 1.6 show such cases using mathematical formulas that give a set of instructions on how the input is transformed from the element in the domain to the element in the range.

1.2.1 Graphs of Functions

Another way to represent functions is with its **graph** which is a set of points consisting of the input and its corresponding output and is represented with the set of ordered pairs $\{(x, y) : y = f(x), x \in \mathbb{R}\}$ for functions that accept real numbers.

Example 1.7. Graph of a linear function.

A familiar example is the linear function $f(x) = mx + b$ where m is the slope of the line and b is the vertical offset. Suppose $m = 1$ and $b = 0$, we get the following graph for the function $f(x) = x$ in Figure 1.7

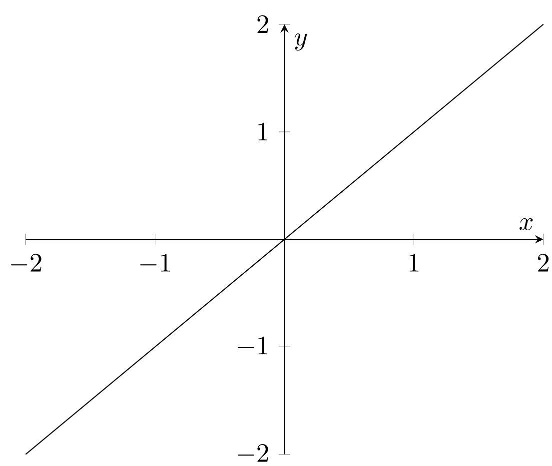


Figure 1.2: Graph of the function $f(x) = x$

Example 1.8. Graph of a quadratic function.

Another example is the quadratic function of the form $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ are the coefficients of the function and determine its shape. Suppose that $a = 1, b = 0, c = -1$ resulting in the function $f(x) = x^2 - 1$. Using knowledge of the roots a quadratic function, it can be deduced that this will result in a parabola with x-intersections at $x = -1$ and $x = 1$ which can be confirmed in the graph in Figure 1.8

1.2.2 Table Functions

The last representation of a function that will be discussed are functions that are in the form of tables. Consider two finite sets $X = \{1, 2, 3, 4, 5\}$ and $Y = \{2, 4, 6, 8, 10\}$ and a function that maps the elements of X to the elements of Y. A table can be created that shows the assignment of the elements (Table 1.1) via the function $f(x)$.

The function shown in Table 1.1 is constructed from a mathematical formula ($y = 2x$), this is not necessary, as long as there is only one output for any given input, then, the table is considered a function.

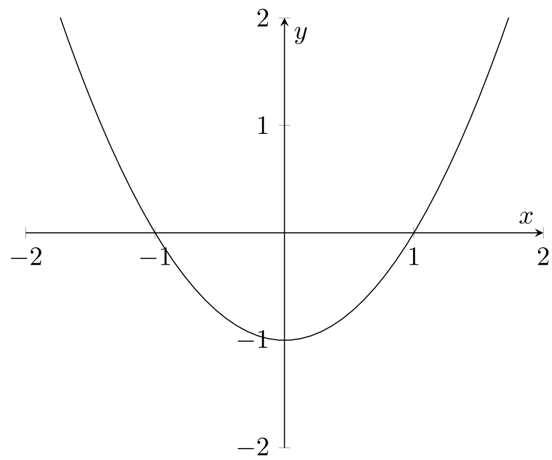


Figure 1.3: Graph of the function $f(x) = x^2 - 1$

$x \in X$	1	2	3	4	5
$y \in Y$	2	4	6	8	10

Table 1.1: Function Mapping from X to Y

Example 1.9. A table function.

For example, consider the following table that shows the function that maps $X = \{1, 3, 5, 8, 4\}$ to $Y = \{3, 5, 2, 1, 1\}$ in Table 1.2. Though a value in Y repeats once, it still passes the vertical line test as the

$x \in X$	1	3	5	8	4
$y \in Y$	3	5	2	1	1

Table 1.2: Function Mapping from X to Y

function maps only one output y from any input x.

This conceptual foundation of what a function is prepares us for the introduction and review of commonly used functions and function families (functions with similar forms and properties).

Exercises

1.3 Common Functions

Functions, namely those defined via mathematical formulas, can be classified into function families whose members share common features, forms, and properties. Some such families often differ only in coefficients and others may change the base in which they operate. The following types of functions are used frequently in the study of calculus.

1.3.1 Linear Functions

Beginning with the most elementary form of functions created using mathematical formulas is the **linear function** or the equation for a line.

Definition 1.10. Functions of the form

$$f(x) = mx + b \tag{1.6}$$

where m is the slope and b is the y-intercept, are **linear functions**. The domain and range of linear functions is all the real numbers, $(-\infty, \infty)$.

The graphs of linear functions are straight lines, if the domain of the function is $\mathbb{R}$, then the graph of f extends from $-\infty$ to $+\infty$. It's slope determines the angle of the line and can be reconstructed with any two points on the line with the following formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.7}$$

where y_i corresponds to x_i and $x_2 > x_1$. The y-intercept the value of the y-coordinate of the function when $x = 0$ and denotes where the line crosses the y-axis on the graph. An example for the graph of a linear function is shown in Figure 1.7 where $f(x) = x$.

Example 1.11. Linear functions can model the conversion between temperatures in degrees Celsius and Fahrenheit with the function $C(F) = \frac{5}{9}(F - 32) = \frac{5}{9}F - \frac{160}{9}$. Here the independent variable is F, the slope is $\frac{5}{9}$, and the y-intercept is $\frac{160}{9}$ and the graph is shown in Figure 1.11.

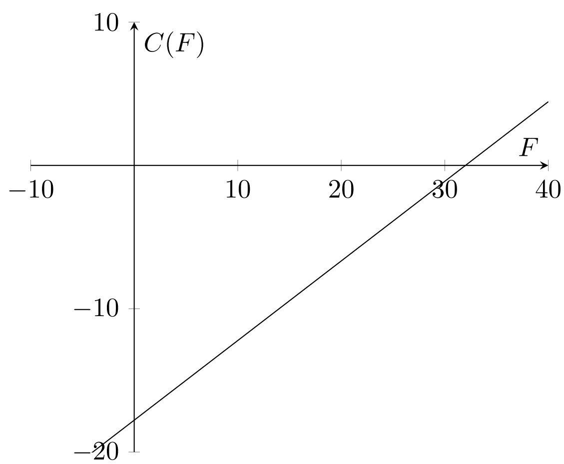


Figure 1.4: Graph of the Fahrenheit to Celsius Conversion, $C(F) = \frac{5}{9}(F - 32)$

1.3.2 Quadratic Functions

Quadratic functions are the next natural progression after linear functions. The simplest form of a quadratic function $f(x) = x^2$ consists of x multiplied by itself. The graph of a quadratic equation is a **parabola** and can be seen in Figure 1.8.

Definition 1.12. Functions of the form

$$f(x) = ax^2 + bx + c \tag{1.8}$$

are **quadratic functions**, this is the standard form for a quadratic function. Alternative forms to the standard quadratic function are the root form $f(x) = (x - a)(x - b)$ where a and b are the x-intercepts of the function and the vertex form $f(x) = a(x - h)^2 + k$ where the point h, k is the vertex of the parabola. The domain and range of quadratic functions span the entirety of the real numbers.

Quadratic functions have some useful properties such as the root(s) of the parabola (the x-intercept(s)) as well as having an explicit formula for computing the root(s) of the function known as the **quadratic formula**. The quadratic formula solves for the x-coordinate(s) of the roots for a given quadratic function (where $y = 0$)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.9}$$

where a, b, and c are the coefficients in the standard form of a quadratic function. This formula gives real roots when the **discriminant** $b^2 - 4ac > 0$, a single real root (with a multiplicity of two) when $b^2 - 4ac = 0$, complex roots when $b^2 - 4ac < 0$, and purely imaginary roots when $b = 0$ and $4ac > 0$.

Example 1.13. An example of a quadratic function is **vertical projectile motion** under gravity:
Given an object with initial height h_0 and initial vertical velocity v_0, its height at time t is modeled by

$$h(t) = -\frac{1}{2}gt^2 + v_0t + h_0$$

where $h(t)$ is the height at time t, g is the the acceleration due to gravity ($g \approx 9.81\text{m/s}^2$) at Earth's surface, and air resistance is neglected. The graph showing the height of a particle thrown from a height of 20 meters and at an initial vertical velocity of 2 m/s^2 is shown in Figure 1.13.

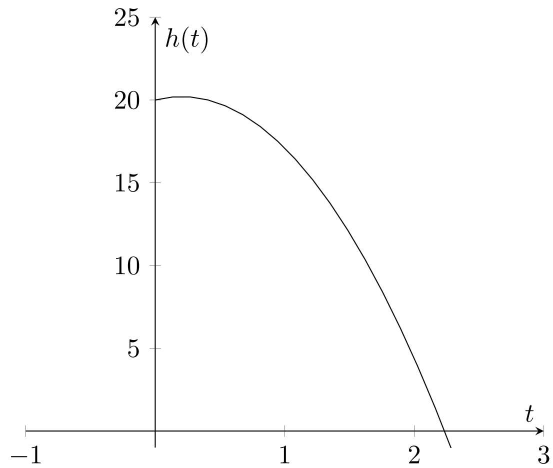


Figure 1.5: Graph of a Particle Falling Due to Gravity over time, $h(t) = -\frac{1}{2}gt^2 + v_0t + h_0$

1.3.3 Polynomial and Rational Functions

Generalizing upon the linear and quadratic functions further, polynomial functions are the next step.

Definition 1.14. Functions of the form

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \tag{1.10}$$

are **polynomial functions**. Where n is the **degree** of the polynomial (when $a_n \neq 0$) and a_i for $i \in \{0, 1, \ldots, n-1, n\}$ are the coefficients. The domain and range of polynomials, like linear and quadratic functions, span the entirety of the real numbers.

Similar to quadratic functions, polynomials have up to n roots but do not have an explicit formula to compute those roots for polynomials with degree $n > 3$. Linear functions and quadratic functions can be considered special cases of polynomial functions with degree $n = 1$ and $n = 2$ respectively. Common higher-order polynomials are **cubic functions** $n = 3$, **quartic functions** $n = 4$, and **quintic functions** $n = 5$. A useful property of polynomials is that there are at most n roots for a polynomial of degree n, note that when a polynomial has less (real and complex) roots than its degree, one of its roots may have a multiplicity greater than one meaning that it is repeated. For example, the polynomial (in root form) $f(x) = (x+1)(x+1)(x+1) = (x+1)^3 = x^3 + 3x^2 + 3x + 1$ has all three roots at $x = -1$ which means that the root $(x = -1, y = 0)$ has multiplicity of 3.

Example 1.15. To show the differences between polynomials with varying order, Figure 1.6 graphs polynomials of the form x^n up to the 6th degree.

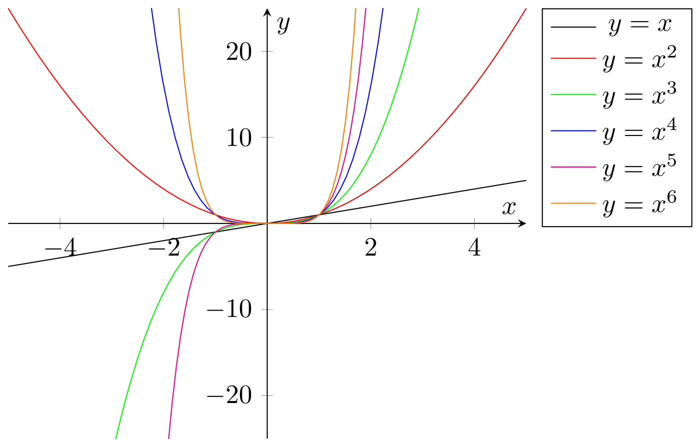


Figure 1.6: Polynomials of varying degrees $n \in \{1, 2, 3, 4, 5, 6\}$

Rational functions branch from polynomials in a similar way to how rational numbers branch from the integers as they are constructed by dividing one polynomial by another.

Definition 1.16. Functions of the form

$$f(x) = \frac{p(x)}{q(x)} \tag{1.11}$$

where $p(x)$ and $q(x) \neq 0$ are polynomial functions are known as **rational functions**. Their domains are $(-\infty, \infty)$ (not including asymptotes and holes) and their ranges are $(-\infty, \infty)$.

Like polynomials, rational functions have **zeros** or roots when the numerator is equal to zero, but also zeros when the denominator of the function is equal to zero called **poles**. These describe discontinuous points as the the denominators reach zero resulting in holes (when a zero and a pole cancel) and an **asymptote** when the pole is not cancelled.

Example 1.17. An example of a rational function that has both holes and asymptotes is

$$f(x) = \frac{(x-1)(x+1)(x+2)}{(x-1)(x+1)(x-2)}$$

This function has holes at $x = \pm 1$ where the poles and zeros cancel out asymptotes at $x = 2$ where the denominator becomes zero and $y = 1$ for the horizontal asymptote. The graph for this function is shown in Figure 1.7.

1.3.4 Trigonometric Functions

While the progression from linear to polynomial functions follows an algebraic development (any functions formed using algebraic operations: $\{+, -, \times, \div, x^a, \sqrt[a]{(\cdot)}, \dots\}$), trigonometric functions like $f(x) = \sin(x)$ and $f(x) = \cos(x)$ come from the fundamental geometric concepts of triangles and **angle** and are considered **transcendental**. Transcendental functions are any functions that are not algebraic, including trigonometric, exponential, and logarithmic functions (discussed in Section 1.3.5).

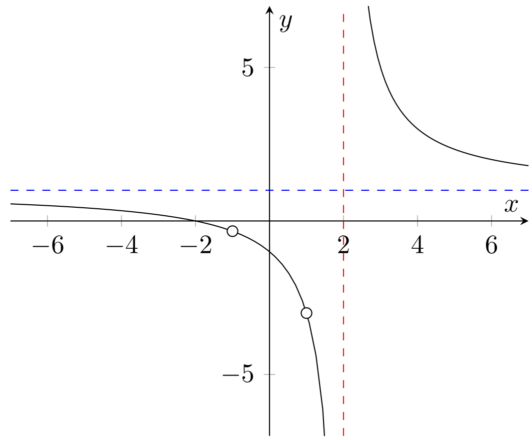


Figure 1.7: Rational function $f(x) = \frac{(x-1)(x+1)(x+2)}{(x-1)(x+1)(x-2)}$ with holes at $(1, -3)$ and $(-1, -\frac{1}{3})$, vertical asymptote at $x = 2$, and horizontal asymptote at $y = 1$

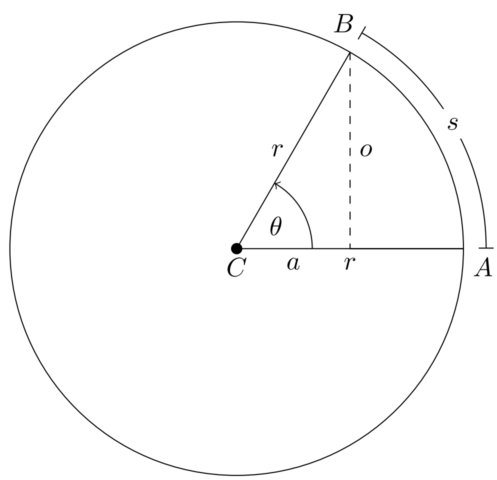


Figure 1.8: Unit circle showing angle θ, radius r, arclength s, and a right triangle.

Angles, measured in **radians** or **degrees** where a full circle is measured have an angle of 2π radians or $360°$ (360 degrees). The measure of angles and inputs/outputs for trigonometric functions will always be assumed to be in radians unless specified. One radian is equal to the radius of the unit circle and wraps around the circumference of that circle 2π times and is defined using θ as the angle measure $\theta = \frac{s}{r}$ where s is the arclength (see Figure 1.8).

The definition of the **sine** and **cosine** functions come from right triangle trigonometry, where θ is an acute angle (not the right angle). In the context of the unit circle, draw a radius from the origin forming an angle θ with the positive x-axis. From the point where the radius intersects the circle, drop a perpendicular to the x-axis to form a right triangle. The horizontal leg, labeled x, is the adjacent side to θ, and the vertical leg, labeled y, is the opposite side. The hypotenuse is the radius r of the circle. By definition, $\sin(\theta) = \frac{y}{r}$ and $\cos(\theta) = \frac{x}{r}$. On the unit circle, where $r = 1$, this simplifies to $\sin(\theta) = y$ and $\cos(\theta) = x$. Figure 1.8 shows this triangle embedded in the circle.

The main two trigonometric functions $f(x) = \sin(x)$ and $f(x) = \cos(x)$, unlike the previously mentioned functions, have restricted ranges as their values vary within the interval $[-1, 1]$. Their graphs can be seen in Figure 1.9.

From the basic trigonometric functions, an additional function $f(x) = \tan(x)$ can be constructed with the following formula

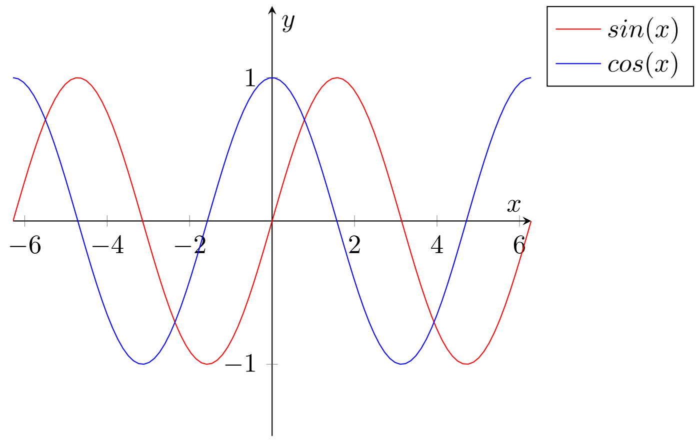


Figure 1.9: Graphs of $\sin(x)$ and $\cos(x)$

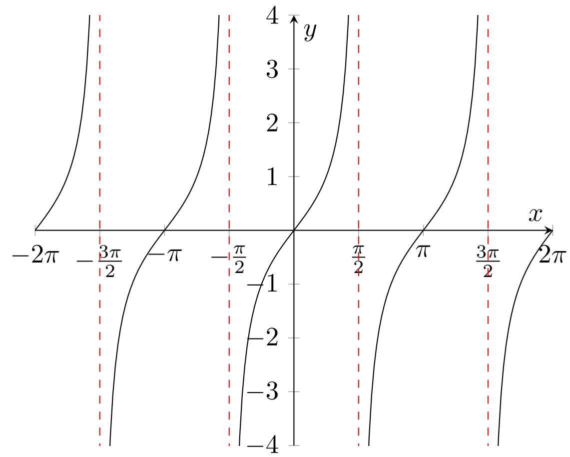


Figure 1.10: Graph of $\tan(x)$ with vertical asymptotes at $x = \frac{\pi}{2} + k\pi$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \tag{1.12}$$

The domain of the trigonometric **tangent** function is determined by the $\cos(x)$ in the denominator, where $\cos(x) = 0$ there is a vertical asymptote though it is still periodic. These asymptotes occur at $x = \frac{\pi}{2} + k\pi$ where $n \in \mathbb{Z}$. The graph of $\tan(x)$ can be seen in Figure 1.10.

As noted with the sine, cosine, and tangent functions, these are periodic, meaning they identically repeat after a set period, this is further discussed in Section 1.4.4.

1.3.5 Exponential and Logarithmic Functions

The last family of functions we are going to present in this section are the **exponential** and **logarithmic**functions.

Definition 1.18. Functions of the form

$$f(x) = a^x \tag{1.13}$$

where a is the base, are called **exponential** functions. Their domain is $(-\infty, \infty)$ and range is $(0, \infty)$.

Logarithmic are inverses (discussed in Section 1.4.4) of exponents such that

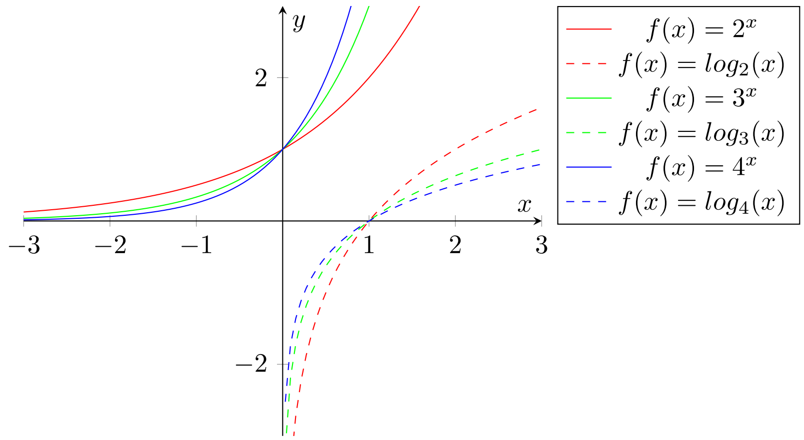


Figure 1.11: Graphs of Exponential and Logarithmic Functions with various bases, $f(x) = a^x$ and $f(x) = log_a(x)$ for $a \in \{2, 3, 4\}$.

$$\log_a(a^x) = x \tag{1.14}$$

where a is the base of the logarithm. Logarithms essentially compute the power required to bring the base to a specified number. For example $log_2(4) = 2$ and $log_3(81) = 4$.

Definition 1.19. Functions of the form

$$f(x) = \log_a(x) \tag{1.15}$$

where a is the base, are called **logarithmic** functions. The domain for the logarithm is $(0, \infty)$ and the range is $(-\infty, \infty)$.

The graphs of exponents and logarithms with various bases are shown in Figure 1.11.

Special exponential and logarithmic functions have the base of $e \approx 2.71828...$ which is Euler's number. This results in the following functions $f(x) = e^x$ for the exponential and using special notation $f(x) = \log_e(x) = \ln(x)$ and of course $\ln(e^x) = x$. These functions show up in numerous real world situations such as exponential growth/decay and sound decibels.

Example 1.20. Exponential growth and decay take the form of $f(x) = a(1 + r)^x$, where a is the initial amount, r is the growth rate, and x is the time interval. Growth is modeled with a positive $r > 0$ and decay with a negative $r < 0$. Choosing $a = 100$ and $r = \pm 0.1$, results in the following growth and decay graphs in Figure 1.12.

Example 1.21. An example of logarithms being used for modeling real world phenomena is the pH scale in chemistry $pH(H^+) = log_{10}(H^+)$ where H^+ is the number of hydrogen ions are in a solution. This essentially determines the order of magnitude (exponent of 10) of the number of hydrogen ions there are. The graph of this function is shown in Figure 1.13.

1.4 Types of Functions

It is important to be able to further distinguish functions and classify their behaviors into distinct characteristics in order to properly analyze them. Characteristics such as symmetry (evenness and oddness of a function) and periodicity (repeating functions) are defined in such ways that may give insight into solving certain problems or giving us the opportunity to exploit their definitions.

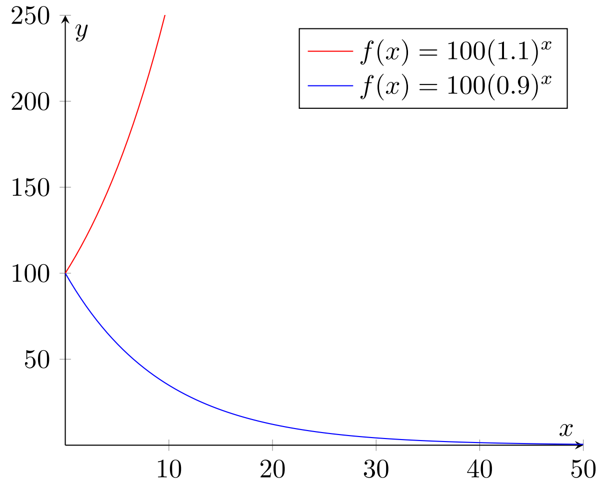


Figure 1.12: Graphs Showing Exponential Growth and Decay, $f(x) = 100(1.1)^x$ and $f(x) = 100(0.9)^x$

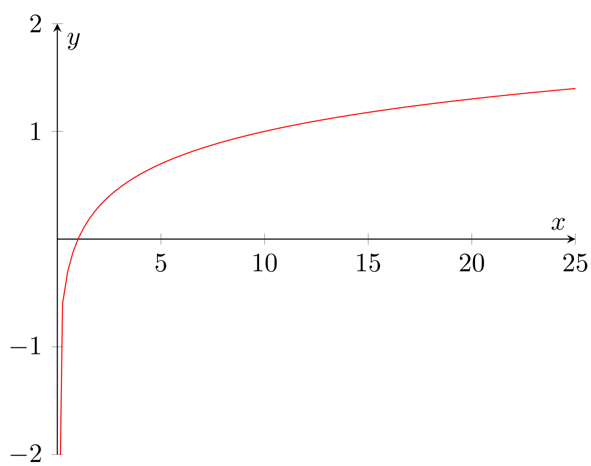


Figure 1.13: Graph Showing the pH Scale, $pH(H^+) = log_{10}(H^+)$

1.4.1 Piecewise Functions and Absolute Values

Piecewise functions are formed with multiple functions on separate intervals for the domain. For example, in a rational function with holes, $g(x) = \frac{(x+1)(x-1)}{(x+1)}$ the hole at $x = -1$ can be "plugged" with a piecewise function

$$f(x) = \begin{cases} g(x) = \frac{(x+1)(x-1)}{(x+1)} & \text{if } x \neq -1 \\ -2 & \text{if } x = -1 \end{cases}$$

An important type of piecewise function is the **absolute value** defined as

$$|x| = \begin{cases} x & \text{if } x >= 0 \\ -x & \text{if } x < 0 \end{cases} \tag{1.16}$$

The absolute value returns a positive value for all inputs so the range is $[0, \infty)$. The absolute value is important as it allows us to define a notion of **distance** between numbers. For example take $a = 10$ and $b = 3$, the distance between a and b is

$$d = |b - a| = |3 - 10| = |-7| = 7$$

so a is 7 units away from b. The absolute value has the property $|ab| = |a||b|$ and the **triangle inequality** with the following definition

Definition 1.22. The **Triangle Inequality**

$$\begin{gathered} |a + b| \leq |a| + |b| \\ |(a - c) + (c - b)| = |a - b| \leq |a - c| + |c - b| \end{gathered} \tag{1.17}$$

This states that the distance from point a to point b is less or equal to that of the distance from a to a point c and then from the point c to the point b. This may make more sense in higher dimensions where vector addition forms triangles, see Chapter 11.

1.4.2 Even and Odd Functions

Function symmetry is a useful classification for functions especially once integration is introduced in Part III. Function symmetry is divided into two classes, even functions and odd functions with their definitions following.

Definition 1.23. A function $f(x)$ is said to be an **even function** of the independent variable x if and only if

$$f(-x) = f(x)$$

for all x in the function's domain.

Definition 1.24. A function $f(x)$ is said to be an **odd function** of the independent variable x if and only if

$$f(-x) = -f(x)$$

for all x in the function's domain.

A simple example of an even function is the cosine function $f(x) = \cos(x)$ and an example of an odd function is the sine function $f(x) = \sin(x)$ whose graphs can be seen in Figure 1.9. Graphically, even functions symmetric about the vertical axis while odd functions hold rotational symmetry when rotating the function 180° about the origin. To further reinforce the symmetry of odd functions Example1.25.

Example 1.25. Consider the function $f(x) = x^3$. To rotate the function 180° about the origin, we set $x = -x$ and $y = -y$ such that every point (x, y) is mapped to $(-x, -y)$. This gives us $g(x) = -f(-x) = -(-(x)^3) = x^3$, which matches the original function.

1.4.3 Increasing and Decreasing Functions

Definition 1.26. A function f is **increasing** on an interval I if for all $x_1, x_2 \in I$ with $x_1 < x_2$,

$$f(x_1) \leq f(x_2).$$

If the inequality is strict ($f(x_1) < f(x_2)$), we say f is **strictly increasing**.

Definition 1.27. A function f is **decreasing** on an interval I if for all $x_1, x_2 \in I$ with $x_1 < x_2$,

$$f(x_1) \geq f(x_2).$$

If the inequality is strict ($f(x_1) > f(x_2)$), we say f is **strictly decreasing**.

1.4.4 Periodic Functions

Definition 1.28. A function is said to be a **period function** if and only if for some value a,

$$f(x) = f(x + a)$$

where a is the period of the function. The period defines the magnitude of the independent variable required for the function to repeat.

1.5 Function Transformations and Operations

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Appendix

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