Chapter 1 - Sets

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1 Introduction to Sets

A **set** is a collection of **elements**, elements can be anything (numbers, points, functions, etc.). An **infinite** has an infinite number of elements, otherwise it is a **finite** set. Two sets are **equal** if they contain the same elements.

An element a is represented as being **in** a set A by $a \in A$ and an element b is **not in** A is shown as $b \notin A$

The set of **natural numbers**, the set of **integers**, and the **rational numbers** have reserved symbols

$$\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$$

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$$

$$\mathbb{Q} = \{x : x = \frac{m}{n}, \text{ where } m, n \in \mathbb{Z} \text{ and } n \neq 0\}$$

as well as the set of **real numbers**, $\mathbb{R}$.

For finite sets, the **cardinality** or **size** is the number of elements and is denoted as $|A|$, not to be confused with the absolute value of a number, this notation is used only for sets.

The **empty set** is unique and is a set with no elements usually denoted as $\emptyset$, $\varnothing$, or $\{\}$ and $|\emptyset| = 0$. A set containing the empty set is not empty as $M = \emptyset$ contains the empty set, and thus has a cardinality of 1. An analogy often used is thinking of sets as boxes containing things, these things can be other boxes, whether empty or not.

Set-builder notation is a special type of notation used to describe sets by giving its elements rules. Consider the even integers, it can be written as $E = \{0, \pm 2, \pm 4, \pm 6, ...\} = \{2n : n \in \mathbb{Z}\} = \{n : n \text{ is an even integer}\} = \{n : n = 2k, k \in \mathbb{Z}\}$. The general format is $X = \{\text{expression} : \text{rule}\}$

Example 1. Examples of set-builder notation

1. $\{n : n \text{ is a prime number}\}$
2. $\{n^2 : n \in \mathbb{Z}\} = \{0, 1, 4, 9, 16, \dots\}$
3. $\{x \; in \mathbb{R} : x^2 - 2 = 0\} = \{\pm\sqrt{2}\}$

Example 2. Describe the set $A = \{7a + 3b : a, b \in \mathbb{Z}\}$
Solution: A contains all numbers of the form $7a + 3b$ where a and b are integers, for example $(a, b) = (0, 0) \to 0$, etc. It can be determined that A only contains integers, but which ones? Suppose $n \in \mathbb{Z}$, and $n = 7n + 3(-2n)$. Here $a = n$ and $b = -2n$, therefore $n \in A$, thus every integer is in A and $A = \mathbb{Z}$.

Intervals on the number line are sets as well, for the real number line, these sets are infinite in size. These sets are given as any two numbers $a, b \in \mathbb{R}$ with $a < b$.

Example. These are examples of intervals

- Closed interval: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- Open interval: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
- Half-open interval:
  - $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
  - $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- Infinite interval:
  - $(a, \infty) = \{x \in \mathbb{R} : a < x\}$
  - $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$
  - $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$
  - $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$

Exercises

A. Write each of the following sets by listing their elements between braces

1. $\{5x - 1 : x \in \mathbb{Z}\} = \{-16, -11, -6, -1, 4, 9, 15\}$
3. $\{x \in \mathbb{Z} : -2 \leq x < 7\} = \{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$
5. $\{x \in \mathbb{R} : x^2 = 3\} = \{\pm\sqrt{3}\}$
7. $\{x \in \mathbb{R} : x^2 + 5x = -6\} = \{2, 3\}$
9. $\{x \in \mathbb{R} : \sin \pi x = 0\} = \{0, \pm 2, \pm 3, \pm 4, \dots\} = \mathbb{Z}$
11. $\{x \in \mathbb{Z} : |x| < 5\} = \{0, \pm 1, \pm 2, \pm 3, \pm 4\}$
13. $\{x \in \mathbb{Z} : |6x| < 5\} = \{0\}$
15. $\{5a + 2b : a, b \in \mathbb{Z}\} = \mathbb{Z}$

    Proof. If a and b are integers, then $5a + 2b$ is also an integer since it is simple multiplication and addition. Then if $n = 5a + 2b = 5n + 2(-2n) = 5n - 4n = n$ where $a = n$ and $b = -2n$, then $5a + 2b$ can create any integer, thus the set is equal to the set of all integers, $\mathbb{Z}$. (not sure if this is even a valid proof, just copied the proof from example 1.2) ■

B. Write each of the following sets in set-builder notation

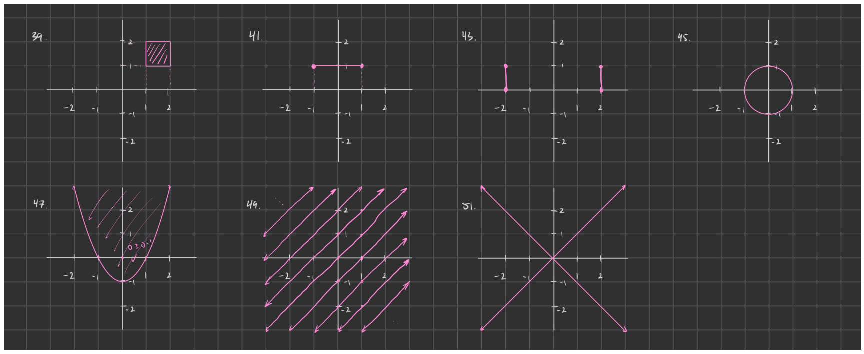

Figure 1: Exercises 1.1D

17. $\{2, 4, 8, 16, 13, 64, \dots\} = \{2^x : x \in \mathbb{N}\}$
19. $\{\dots, -6, -3, 0, 3, 6, 9, 12, \dots\} = \{3x : x \in \mathbb{Z}\}$
21. $\{0, 1, 4, 9, 16, 25, 36, \dots\} = \{x^2 : x \in \mathbb{Z}\}$
23. $\{3, 4, 5, 6, 7, 8\} = \{x \in N : 2 < x < 9\}$
25. $\{\dots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \dots\} = \{2^a : a \in \mathbb{Z}\}$
27. $\{\dots, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \dots\} = \{\frac{a\pi}{2} : a \in \mathbb{Z}\}$

C. Find the following cardinalities

29. $|\{\{1\}, \{2, \{3, 4\}\}, \emptyset\}| = 3$
31. $|\{\{\{1\}, \{2, \{3, 4\}\}\emptyset\}\}| = 1$
33. $|\{x \in \mathbb{Z} : |x| < 10\}| = 19$
35. $|\{x \in \mathbb{Z} : x^2 < 10\}| = 7$
37. $|\{x \in \mathbb{N} : x^2 < 0\}| = 0$

D. Sketch the following sets of points in the $x - y$ plane

39. $\{(x, y) : x \in [1, 2], y \in [1, 2]\}$
41. $\{(x, y) : x \in [-1, 1], y = 1\}$
43. $\{(x, y) : |x| = 2, y \in [0, 1]\}$
45. $\{(x, y) : x, y \in \mathbb{R}, x^2 + y^2 = 1\}$
47. $\{(x, y) : x, y \in \mathbb{R}, y \geq x^2 - 1\}$
49. $\{(x, x+k) : x \in \mathbb{R}, k \in \mathbb{Z}\}$, this generates a set of parallel lines with slope $m = 1$, because $(x, y) = (x, x + k) \to y = x + k$.
51. $\{(x, y) \in \mathbb{R}^2 : (y - x)(y + x) = 0\}$, here, either $(y - x) = 0$ or $(y + x) = 0$ or both. The first case gives $y = x$ a line through the origin with slope $m = 0$, the second case gives $y = -x$ a line through the origin with slope $m = -1$, and when both are zero, this gives the intersection which is the origin.

2 The Cartesian Product

Given two sets A and B, "multiplying" them results a new set denoted as $A \times B$ and is called the **Cartesian product** of A and B.

Definition 1. An **ordered pair** is a list (x, y) of two elements x and y, enclosed in parentheses and separated by a comma.

For example, $(2, 4) \neq (4, 2)$ because the order is different even though they contain the same elements. Like sets, the elements don't have to be just numbers.

Definition 2. The **Cartesian product** of two sets A and B is another set, denoted as $A \times B$ defined as $A \times B = \{(a, b) : a \in A, b \in B\}$.

The elements of a Cartesian product are ordered pairs of elements from both sets. For example, if $A = \{k, l, m\}$ and $B = \{q, r\}$, then $A \times B = \{(k, q), (k, r), (l, q), (l, r), (m, q), (m, r)\}$.

Fact 1. If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$.

Example 3.