The Big Book of Real Analysis - Johar
Notes

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Preface

Part I

Real Analysis 1: Numbers and Sequences

Chapter 1

Logic and Sets

1.1 Introduction to Logic

Mathematical statements (often called propositions) require proof to be determined if true or false conditional to some definitions or axioms accepted to be true. Mathematical proofs require base level axioms as opposed to absolute truths.

Remark 1.1.1. This is left empty

1.1.1 And, Or, Not

Combinations of mathematical statements can be made or manipulated to create new ones. Negation is done by writing the opposite of a statement. With statements P and Q, they can be combined with "and" or "or", "and" is called a logical conjunction and "or" is called a logical disjunction. The combination of statements is called a compound statement whose truth can be deduced as well.

Example 1.1.2. Consider

$$P: \text{A is a vowel,} \quad \text{and} \quad Q: \text{B is a vowel}$$

We know that P is true and Q is false.

1. Negating each statement results in

$$\neg P: \text{A is not a vowel,} \quad \text{and} \quad \neg Q: \text{B is not a vowel}$$

   Which results in $\neg P$ being false and $\neg Q$ true. Negation switches the truth of a statement.

2. Looking at "and" and "or":

   (a) The "and" connective is denoted with $\wedge$. $P \wedge Q$ says "A is a vowel and B is a vowel", this is false because both statements need to be true in order for the compound statement to be true.

   (b) The "or" connective is denoted with $\vee$. $P \vee Q$ says "A is a vowel or B is a vowel (or both)". This is true because either one of the statements needs to be true or both.

Remark 1.1.3. This is left empty

Example 1.1.4. Consider

$$P: \text{Lucy likes coffee} \quad \text{and} \quad Q: \text{Lucy likes tea}$$

If $P \wedge Q$ is true, then Lucy likes both tea and coffee, if $P \vee Q$ is true then she likes either of them or both.

The order of connectives does not matter. For example $P \vee Q = Q \vee P$ and similar for $\wedge$ (this is symmetry of the connectives).

Definition 1.1.5. (Logically Equivalent Statements). We say two statements P and Q are logically equivalent if their truth or falseness are the same. In other words, if either one is true, the other must be true as well. Written as $P \equiv Q$.

With three statements P, Q, R, statements such as $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ likewise for $\vee$ because they are associative connectives, so the brackets/parentheses are not necessary.

Example 1.1.6. Consider

$$P : \text{Lucy likes coffee} \quad \text{and} \quad Q : \text{Lucy likes tea} \quad \text{and} \quad R : \text{Lucy likes juice}$$

$P \wedge Q \wedge R = T$ says that Lucy likes all three options, $P \vee Q \vee R = T$ says that Lucy likes at least one of them.

1.2 Proofs

1.3 Sets

1.4 Quantifiers

Remark 1.4.1. We make several remarks here:

1. Universal Quantifier:
   (a) Similar to the "and" connective, require all statements involved to be true for the compound statement to be true.
   (b) $\forall$ read as "for all" or "for every" or "for each" or "for any" or "for arbitrary".
   (c) $(\forall x \in X)$, $P(x)$ read as "For all $x \in X$, $P(x)$ is true" or "$P(x)$ is true for all $x \in X$".
2. Existential Quantifier:
   (a) Similar to the "or" connective, require at least one statement involved to be true for the compound statement to be true.
   (b) $\exists$ read as "there exists" or "there are some" or "there is at least one" or "for some" or "for at least one".
   (c) $(\exists x \in X) : P(x)$ read as "There exists an $x \in X$ such that $P(x)$ is true" or "$P(x)$ is true for some $x \in X$".
3. The colon : used in similar manner to set builder notation. Read as "such that". Not necessary in the universal quantifier example, same for the comma in the existential quantifier example.
4. Most of the time, quantifier parentheses not used.

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2.4 Integers $\mathbb{Z}$

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