The Big Book of Real Analysis - Johar

Notes

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Contents

Preface

Part I

Real Analysis 1: Numbers and Sequences

Chapter 1

Logic and Sets

Example 1.0.1. This is left empty

Definition 1.0.2. This is left empty

Remark 1.0.3. This is left empty

1.1 Introduction to Logic

Mathematical statements (often called propositions) require proof to be determined if true or false conditional to some definitions or axioms accepted to be true. Mathematical proofs require base level axioms as opposed to absolute truths.

Remark 1.1.1. This is left empty

1.1.1 And, Or, Not

Combinations of mathematical statements can be made or manipulated to create new ones. Negation is done by writing the opposite of a statement. With statements P and Q, they can be combined with "and" or "or", "and" is called a logical conjunction and "or" is called a logical disjunction. The combination of statements is called a compound statement whose truth can be deduced as well.

Example 1.1.2. Consider

$$P : \text{A is a vowel,} \quad \text{and} \quad Q : \text{B is a vowel}$$

We know that P is true and Q is false.

1. Negating each statement results in

$$\neg P : \text{A is not a vowel,} \quad \text{and} \quad \neg Q : \text{B is not a vowel}$$

   Which results in $\neg P$ being false and $\neg Q$ true. Negation switches the truth of a statement.

2. Looking at "and" and "or":

   (a) The "and" connective is denoted with $\wedge$. $P \wedge Q$ says "A is a vowel and B is a vowel", this is false because both statements need to be true in order for the compound statement to be true.

   (b) The "or" connective is denoted with $\vee$. $P \vee Q$ says "A is a vowel or B is a vowel (or both)". This is true because either one of the statements needs to be true or both.

Remark 1.1.3. This is left empty.

Example 1.1.4. Consider

$$P : \text{Lucy likes coffee} \quad \text{and} \quad Q : \text{Lucy likes tea}$$

If $P \wedge Q$ is true, then Lucy likes both tea and coffee, if $P \vee Q$ is true then she likes either of them or both.

The order of connectives does not matter. For example $P \vee Q = Q \vee P$ and similar for $\wedge$ (this is symmetry of the connectives).

Definition 1.1.5. (Logically Equivalent Statements). We say two statements P and Q are logically equivalent if their truth or falseness are the same. In other words, if either one is true, the other must be true as well. Written as $P \equiv Q$.

With three statements P, Q, R, statements such as $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ likewise for $\vee$ because they are associative connectives, so the brackets/parentheses are not necessary.

Example 1.1.6. Consider

$$P : \text{Lucy likes coffee} \quad \text{and} \quad Q : \text{Lucy likes tea} \quad \text{and} \quad R : \text{Lucy likes juice}$$

$P \wedge Q \wedge R = T$ says that Lucy likes all three options, $P \vee Q \vee R = T$ says that Lucy likes at least one of them.

Example 1.1.7. This is left empty.

Example 1.1.8. This is left empty.

1.1.2 Conditional Statement

Remark 1.1.9. This is left empty.

Remark 1.1.10. This is left empty.

Example 1.1.11. This is left empty.

Example 1.1.12. This is left empty.

Example 1.1.13. This is left empty.

Remark 1.1.14. This is left empty.

1.1.3 Modus Ponens and Modus Tollens

Definition 1.1.15. This is left empty.

Example 1.1.16. This is left empty.

Remark 1.1.17. This is left empty.

Example 1.1.18. This is left empty.

Definition 1.1.19. This is left empty.

1.2 Proofs

Conjecture 1.2.1. This is left empty.

Proposition 1.2.2. *This is left empty.*

Remark 1.2.3. This is left empty.

Definition 1.2.4. This is left empty.

Remark 1.2.5. This is left empty.

Proposition 1.2.6. *This is left empty.*

Remark 1.2.7. This is left empty.

1.3 Sets

Definition 1.3.1. This is left empty.

Example 1.3.2. This is left empty.

Example 1.3.3. This is left empty.

Remark 1.3.4. This is left empty.

Definition 1.3.5. This is left empty.

Example 1.3.6. This is left empty.

Definition 1.3.7. This is left empty.

Example 1.3.8. This is left empty.

Remark 1.3.9. This is left empty.

1.3.1 Set Algebra

Example 1.3.10. This is left empty.

Remark 1.3.11. This is left empty.

Definition 1.3.12. This is left empty.

Proposition 1.3.13. *This is left empty.*

Definition 1.3.14. This is left empty.

Proposition 1.3.15. *This is left empty.*

Definition 1.3.16. This is left empty.

Proposition 1.3.17. *This is left empty.*

Remark 1.3.18. This is left empty.

Theorem 1.3.19 (De Morgan's Laws)**.** *Let X and Y be sets contained in the universe U. Then:*

1. $(X \cup Y)^c = X^c \cap Y^c$
2. $(X \cap Y)^c = X^c \cup Y^c$

Proof. This is left empty. □

Definition 1.3.20 (Difference)**.** This is left empty.

Proposition 1.3.21. *This is left empty.*

Proof. This is left empty. □

Definition 1.3.22. This is left empty.

1.3.2 Power Sets and Cartesian Product

Definition 1.3.23 (Power Set)**.** This is left empty.

Example 1.3.24. This is left empty.

Definition 1.3.25 (Cartesian Product)**.** This is left empty.

Proposition 1.3.26. *This is left empty.*

Proof. This is left empty. □

Proposition 1.3.27. *This is left empty.*

Proof. This is left empty. □

Definition 1.3.28 (n-fold Cartesian Product)**.** This is left empty.

1.4 Quantifiers

Remark 1.4.1. This is left empty.

Example 1.4.2. This is left empty.

Definition 1.4.3 (Universal, Existential Quantifiers)**.** This is left empty.

Remark 1.4.4. We make several remarks here:

1. Universal Quantifier:
   (a) Similar to the "and" connective, require all statements involved to be true for the compound statement to be true.
   (b) $\forall$ read as "for all" or "for every" or "for each" or "for any" or "for arbitrary".
   (c) $(\forall x \in X)$, $P(x)$ read as "For all $x \in X$, $P(x)$ is true" or "$P(x)$ is true for all $x \in X$".
2. Existential Quantifier:
   (a) Similar to the "or" connective, require at least one statement involved to be true for the compound statement to be true.
   (b) $\exists$ read as "there exists" or "there are some" or "there is at least one" or "for some" or "for at least one".
   (c) $(\exists x \in X) : P(x)$ read as "There exists an $x \in X$ such that $P(x)$ is true" or "$P(x)$ is true for some $x \in X$".
3. The colon : used in similar manner to set builder notation. Read as "such that". Not necessary in the universal quantifier example, same for the comma in the existential quantifier example.
4. Most of the time, quantifier parentheses not used.

Example 1.4.5. 1. Suppose X and Y are sets.
   (a) $X \subseteq Y \iff (\forall x \in X), P(x)$
   (b) $X \cap Y \neq \varnothing \iff (\exists x \in X) : P(x)$
   (c) $X \cap Y = \varnothing \iff (\forall x \in X), \neg P(x)$

2. Let X be the set of months $X = \{\text{Jan, Feb, ..., Dec}\}$ and $P(x)$ is "The month x has 30 days in it"
   (a) $(\forall x \in X), P(x)$ reads as "For every month x in X, the month x has 30 days". This compound statement is false, because February has 28 or 29 days and some months have 31 days. Logically, and used in many proofs, there can be found at least one $x \in X$ that does not satisfy $P(x)$.
   (b) $(\exists x \in X) : P(x)$ reads as "There exists a month x in X such that the month x has 30 days in it". This is true because January has 30 days, thus the compound statement is true.

3. Let Γ be the set of all polygons. For each $\gamma \in \Gamma$, we define:

$$P(\gamma) : \gamma \text{ is a square}, \quad \text{and} \quad Q(\gamma) : \gamma \text{ is a rectangle}$$

   (a) $(\forall \gamma \in \Gamma), P(\gamma)$ is false because it reads as "For any polygon $\gamma \in \Gamma$, it is a square"; but a pentagon is a polygon and it is not a square.
   (b) $(\exists \gamma \in \Gamma) : P(\gamma)$ reads as "There exists a polygon $\gamma \in \Gamma$, such that γ is square" which is true because squares are polygons.
   (c) $(\forall \gamma \in \Gamma), (P(\gamma) \implies Q(\gamma))$ is read as "For all polygons $\gamma \in \Gamma$, if γ is a square then it is a rectangle" which is true because all squares are rectangles.

4. Consider the set of birds B and family of statements $\{P(b) : b \in B\}$ where $P(b)$ is "The bird b can fly". This statement is false because there are birds that cannot fly. Therefore $(\forall b \in B), P(b)$ which says "For each bird b in B it can fly" is false. The negation $\neg((\forall b \in B), P(b))$ must be true. The negation says "There is at least one bird b such that the bird b cannot fly" or $(\exists b \in B) : \neg P(b)$. So we have the equivalence:
$$\neg((\forall b \in B), P(b)) \equiv (\exists b \in B) : \neg P(b)$$

5. Let X and Y be sets. For each $x \in X$, define $P(x)$ to be the statement $x \in Y$ and Q be the statement "$X \cap Y \neq \varnothing$".
   (a) Q says that there is at least one element in both X and Y and $Q \equiv (\exists x \in X) : P(x)$.
   (b) The negation of Q namely $\neg Q$ is $X \cap Y = \varnothing$ and $\neg Q \equiv (\forall x \in X), \neg P(x)$.

   Thus, we have the equivalence:
$$(\forall x \in X), \neg P(x) \equiv \neg Q \equiv\neq ((\exists x \in X) : P(x))$$

In examples 1.4.5(4) and (5) the following rules hold:

1. $\neg((\exists x \in X) : P(x)) \equiv (\forall x \in X), \neg P(x)$
2. $\neg((\forall x \in X), P(x)) \equiv (\exists x \in X) : \neg P(x)$

where the negation of a compound quantifier statement results in the flipping of the quantifier and negation of the specified statement. These are called De Morgan's laws in formal logic.

Example 1.4.6. Define two sets Γ and Δ where Γ is the set of letters in Latin and Δ is the set of all words in *The Oxford English Dictionary*. Define a mathematical statement that depends on two variables $(\gamma, \delta) \in \Gamma \times \Delta$ which says "The word δ begins with the letter γ" as $P(\gamma, \delta)$

1.

1.5 Functions

Chapter 2

Integers

2.1 Relations

2.2 Natural Numbers $\mathbb{N}$

2.3 Ordering on $\mathbb{N}$

2.4 Integers $\mathbb{Z}$

2.5 Algebra on $\mathbb{Z}$

2.6 Ordering on $\mathbb{Z}$

Chapter 3

Construction of Real Numbers

3.1 Rational Numbers $\mathbb{Q}$

3.2 Algebra on $\mathbb{Q}$

3.3 Ordering on $\mathbb{Q}$

3.4 Cardinality

3.5 Irrational Numbers $\bar{\mathbb{Q}}$

3.6 Bounds, Supremum, and Infimum

3.7 Dedekind Cuts

3.8 Algebra and Ordering of Dedekind Cuts

Chapter 4

Real Numbers

4.1 Properties of Real Numbers $\mathbb{R}$

4.2 Exponentiation

4.3 Logarithm

4.4 Decimal Representation of the Real Numbers

4.5 Topology on $\mathbb{R}$

4.6 Real n-Space and Complex Numbers

Part II

Real Analysis 2: Series, Continuity, and Differentiability

Part III

Real Analysis 3: Integration