Analysis I - Amann & Escher

Notes

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Preface

Chapter 1

Foundations

1.1 Fundamentals of Logic

Symbolic logic is about **statements** which can be either true (T) or false (F), not both or in between. A statement A can have a **negation** $\neg A$ ('not A') which is defined as: $\neg A$ true if A false. A truth table can be used to show this:

A	T	F
$\neg A$	F	T

Two statements A and B can be combined using **conjunction** or **disjunction** to make new ones. The statement $A \wedge B$ ('A and B') is true when both A and B are true. The statement $A \vee B$ ('A or B') is true when either A or B or both are true (inclusive or) and is false only when both are false. Refer to the following truth table:

A	B	$A \wedge B$	$A \vee B$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

If $E(x)$ is a statement where x is replaced by an object (member, thing) of a specified class (collection, universe), then E is a **property**. 'x has property E' is equivalent to '$E(x)$ is true'. If x is a member (an **element**) of a class X we write $x \in X$ otherwise $x \notin X$. Then,

$$\{x \in X : E(x)\}$$

is the class of all elements x of the collection X that have the property E.

The **quantifier** $\exists$ denotes existence and is read 'there exists'. The expression

$$\exists x \in X : E(x)$$

says 'There is (at least) one object x in (the class) X which has the property E'. The unique quantifier $\exists!$ says that there is only one (a unique) object.

The quantifier $\forall$ denotes 'for all' or 'for each' object in a collection. The expression

$$\forall x \in X : E(x) \tag{1.1}$$

says 'For each $x \in X$ the statement $E(x)$ is true'. This can also be written as

$$E(x), \quad \forall x \in X \tag{1.2}$$

which says 'The property $E(x)$ is true for all x in X'. Sometimes the quantifier is left out

$$E(x), \quad x \in X \tag{1.3}$$

The symbol := means ‘is defined by’. Thus,

$$a := b$$

says ‘the object (or symbol) a is defined by the object (or expression) b’. Of course $a = b$ means a and b are equal.

Example 1.1.1. Let A and B be statements, X and Y be classes of objects, and E is a property. Truth tables can verify the following statements:

(a) $\neg\neg A := \neg(\neg A) = A$

A	$\neg A$	$\neg\neg(A)$
T	F	T
F	T	F

(b) $\neg(A \wedge B) = (\neg A) \vee (\neg B)$ (De Morgan’s Law 1)

A	B	$A \wedge B$	$\neg(A \wedge B)$	$(\neg A) \vee (\neg B)$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

(c) $\neg(A \vee B) = (\neg A) \wedge (\neg B)$ (De Morgan’s Law 2)

A	B	$A \vee B$	$\neg(A \vee B)$	$(\neg A) \wedge (\neg B)$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

(d) $\neg(\forall x \in X : E(x)) = (\exists x \in X : \neg E(x))$ (for example: the negation of ‘everyone wears glasses’ is ‘at least one person does not wear glasses’)

(e) $\neg(\exists x \in X : E(x)) = (\forall x \in X : \neg E(x))$ (for example: the negation of ‘there is at least one car on the road that is red’ is ‘there are no cars on the road that is read’)

(f) $\neg(\forall x \in X : (\exists y \in Y : E(x, y))) = (\exists x \in X : (\forall y \in Y : \neg E(x, y)))$ (for example: the negation of ‘every person has something in at least one pocket’ is ‘there is at least one person that has whose pockets are all empty’. Here X is the collection of people, Y is the collection of pockets on a person, and E is the property that the pocket is occupied and depends on the person and the pocket.)

(g) $\neg(\exists x \in X : (\forall y \in Y : E(x, y))) = (\forall x \in X : (\exists y \in Y : \neg E(x, y)))$ (for example: the negation of ‘there is at least one car with every window open’ is ‘in the collection of all cars there is at least one window that is closed’)

Remark 1.1.2. (a) Parenthesis keep the statements exact but aren’t always used; similarly, the membership symbol is not always used. For example: $\forall x \exists y : E(x, y)$ is still valid and says ‘For all x there is at least one y such that $E(x, y)$ is true’, thus y depends on x. Another example: $\exists x \forall y : E(x, y)$ which says ‘there is at least one x with every y such that $E(x, y)$ is true’ it is sufficient to find one y which is true for all x. For example, if $E(x, y)$ is the statement ‘reader x of this book find the concept of y to be trivial’ then the first statement is: ‘Each reader of this book finds at least one concept that is trivial’ and the second statement is ‘there is at least one statement that every reader finds trivial.’

(b) The quantifiers $\exists$ and $\forall$ as well as the logical ‘and’ and ‘or’ are mechanically interchanged in negation without changing order while the statements are negated.