The Big Book of Real Analysis - Johar

Notes

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Contents

Preface

Part I

Real Analysis 1: Numbers and Sequences

Chapter 1

Logic and Sets

Example 1.0.1. This is left empty

Definition 1.0.2. This is left empty

Remark 1.0.3. This is left empty

1.1 Introduction to Logic

Mathematical statements (often called propositions) require proof to be determined if true or false conditional to some definitions or axioms accepted to be true. Mathematical proofs require base level axioms as opposed to absolute truths.

Remark 1.1.1. This is left empty

1.1.1 And, Or, Not

Combinations of mathematical statements can be made or manipulated to create new ones. Negation is done by writing the opposite of a statement. With statements P and Q, they can be combined with “and” or “or”, “and” is called a logical conjunction and “or” is called a logical disjunction. The combination of statements is called a compound statement whose truth can be deduced as well.

Example 1.1.2. Consider

$$P : \text{A is a vowel}, \quad \text{and} \quad Q : \text{B is a vowel}$$

We know that P is true and Q is false.

1. Negating each statement results in

   $$\neg P : \text{A is not a vowel}, \quad \text{and} \quad \neg Q : \text{B is not a vowel}$$

   Which results in $\neg P$ being false and $\neg Q$ true. Negation switches the truth of a statement.

2. Looking at “and” and “or”:

   (a) The “and” connective is denoted with $\wedge$. $P \wedge Q$ says “A is a vowel and B is a vowel”, this is false because both statements need to be true in order for the compound statement to be true.

   (b) The “or” connective is denoted with $\vee$. $P \vee Q$ says “A is a vowel or B is a vowel (or both)”. This is true because either one of the statements needs to be true or both.

Remark 1.1.3. This is left empty.

Example 1.1.4. Consider

$$P : \text{Lucy likes coffee} \quad \text{and} \quad Q : \text{Lucy likes tea}$$

If $P \wedge Q$ is true, then Lucy likes both tea and coffee, if $P \vee Q$ is true then she likes either of them or both.

The order of connectives does not matter. For example $P \vee Q = Q \vee P$ and similar for $\wedge$ (this is symmetry of the connectives).

Definition 1.1.5. (Logically Equivalent Statements). We say two statements P and Q are logically equivalent if their truth or falseness are the same. In other words, if either one is true, the other must be true as well. Written as $P \equiv Q$.

With three statements P, Q, R, statements such as $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ likewise for $\vee$ because they are associative connectives, so the brackets/parentheses are not necessary.

Example 1.1.6. Consider

$$P : \text{Lucy likes coffee} \quad \text{and} \quad Q : \text{Lucy likes tea} \quad \text{and} \quad R : \text{Lucy likes juice}$$

$P \wedge Q \wedge R = T$ says that Lucy likes all three options, $P \vee Q \vee R = T$ says that Lucy likes at least one of them.

Example 1.1.7. This is left empty.

Example 1.1.8. This is left empty.

1.1.2 Conditional Statement

Remark 1.1.9. This is left empty.

Remark 1.1.10. This is left empty.

Example 1.1.11. This is left empty.

Example 1.1.12. This is left empty.

Example 1.1.13. This is left empty.

Remark 1.1.14. This is left empty.

1.1.3 Modus Ponens and Modus Tollens

Definition 1.1.15. This is left empty.

Example 1.1.16. This is left empty.

Remark 1.1.17. This is left empty.

Example 1.1.18.

$$\neg(P \implies Q) \equiv \neg(\neg P \vee Q) \equiv P \wedge (\neg Q) \tag{1.1}$$

Definition 1.1.19. This is left empty.

1.2 Proofs

Conjecture 1.2.1. This is left empty.

Proposition 1.2.2. *This is left empty.*

Remark 1.2.3. This is left empty.

Definition 1.2.4. This is left empty.

Remark 1.2.5. This is left empty.

Proposition 1.2.6. *This is left empty.*

Remark 1.2.7. This is left empty.

1.3 Sets

Definition 1.3.1. This is left empty.

Example 1.3.2. This is left empty.

Example 1.3.3. This is left empty.

Remark 1.3.4. This is left empty.

Definition 1.3.5. This is left empty.

Example 1.3.6. This is left empty.

Definition 1.3.7. This is left empty.

Example 1.3.8. This is left empty.

Remark 1.3.9. This is left empty.

1.3.1 Set Algebra

Example 1.3.10. This is left empty.

Remark 1.3.11. This is left empty.

Definition 1.3.12. This is left empty.

Proposition 1.3.13. *This is left empty.*

Definition 1.3.14. This is left empty.

Proposition 1.3.15. *This is left empty.*

Definition 1.3.16. This is left empty.

Proposition 1.3.17. *This is left empty.*

Remark 1.3.18. This is left empty.

Theorem 1.3.19 (De Morgan's Laws)**.** *Let X and Y be sets contained in the universe U. Then:*

1. $(X \cup Y)^c = X^c \cap Y^c$
2. $(X \cap Y)^c = X^c \cup Y^c$

Proof. This is left empty. □

Definition 1.3.20 (Difference)**.** This is left empty.

Proposition 1.3.21. *This is left empty.*

Proof. This is left empty. □

Definition 1.3.22. This is left empty.

1.3.2 Power Sets and Cartesian Product

Definition 1.3.23 (Power Set)**.** This is left empty.

Example 1.3.24. This is left empty.

Definition 1.3.25 (Cartesian Product)**.** This is left empty.

Proposition 1.3.26. *This is left empty.*

Proof. This is left empty. □

Proposition 1.3.27. *This is left empty.*

Proof. This is left empty. □

Definition 1.3.28 (n-fold Cartesian Product)**.** This is left empty.

1.4 Quantifiers

Remark 1.4.1. This is left empty.

Example 1.4.2. This is left empty.

Definition 1.4.3 (Universal, Existential Quantifiers)**.** This is left empty.

Remark 1.4.4. We make several remarks here:

1. Universal Quantifier:
   (a) Similar to the "and" connective, require all statements involved to be true for the compound statement to be true.
   (b) $\forall$ read as "for all" or "for every" or "for each" or "for any" or "for arbitrary".
   (c) $(\forall x \in X), P(x)$ read as "For all $x \in X$, $P(x)$ is true" or "$P(x)$ is true for all $x \in X$".
2. Existential Quantifier:
   (a) Similar to the "or" connective, require at least one statement involved to be true for the compound statement to be true.
   (b) $\exists$ read as "there exists" or "there are some" or "there is at least one" or "for some" or "for at least one".
   (c) $(\exists x \in X) : P(x)$ read as "There exists an $x \in X$ such that $P(x)$ is true" or "$P(x)$ is true for some $x \in X$".
3. The colon : used in similar manner to set builder notation. Read as "such that". Not necessary in the universal quantifier example, same for the comma in the existential quantifier example.
4. Most of the time, quantifier parentheses not used.

Example 1.4.5.

1. Suppose X and Y are sets.
   (a) $X \subseteq Y \iff (\forall x \in X), P(x)$
   (b) $X \cap Y \neq \varnothing \iff (\exists x \in X) : P(x)$
   (c) $X \cap Y = \varnothing \iff (\forall x \in X), \neg P(x)$
2. Let X be the set of months $X = \{\text{Jan, Feb, ..., Dec}\}$ and $P(x)$ is "The month x has 30 days in it"
   (a) $(\forall x \in X), P(x)$ reads as "For every month x in X, the month x has 30 days". This compound statement is false, because February has 28 or 29 days and some months have 31 days. Logically, and used in many proofs, there can be found at least one $x \in X$ that does not satisfy $P(x)$.

   (b) $(\exists x \in X) : P(x)$ reads as "There exists a month x in X such that the month x has 30 days in it". This is true because January has 30 days, thus the compound statement is true.

3. Let Γ be the set of all polygons. For each $\gamma \in \Gamma$, we define:

$$P(\gamma) : \gamma \text{ is a square,} \quad \text{and} \quad Q(\gamma) : \gamma \text{ is a rectangle}$$

   (a) $(\forall \gamma \in \Gamma), P(\gamma)$ is false because it reads as "For any polygon $\gamma \in \Gamma$, it is a square"; but a pentagon is a polygon and it is not a square.
   (b) $(\exists \gamma \in \Gamma) : P(\gamma)$ reads as "There exists a polygon $\gamma \in \Gamma$, such that γ is square" which is true because squares are polygons.
   (c) $(\forall \gamma \in \Gamma), (P(\gamma) \implies Q(\gamma))$ is read as "For all polygons $\gamma \in \Gamma$, if γ is a square then it is a rectangle" which is true because all squares are rectangles.

4. Consider the set of birds B and family of statements $\{P(b) : b \in B\}$ where $P(b)$ is "The bird b can fly". This statement is false because there are birds that cannot fly. Therefore $(\forall b \in B), P(b)$ which says "For each bird b in B it can fly" is false. The negation $\neg((\forall b \in B), P(b))$ must be true. The negation says "There is at least one bird b such that the bird b cannot fly" or $(\exists b \in B) : \neg P(b)$. So we have the equivalence:

$$\neg((\forall b \in B), P(b)) \equiv (\exists b \in B) : \neg P(b)$$

5. Let X and Y be sets. For each $x \in X$, define $P(x)$ to be the statement $x \in Y$ and Q be the statement "$X \cap Y \neq \varnothing$".

   (a) Q says that there is at least one element in both X and Y and $Q \equiv (\exists x \in X) : P(x)$.
   (b) The negation of Q namely $\neg Q$ is $X \cap Y = \varnothing$ and $\neg Q \equiv (\forall x \in X), \neg P(x)$.

   Thus, we have the equivalence:

$$(\forall x \in X), \neg P(x) \equiv \neg Q \equiv\neq ((\exists x \in X) : P(x))$$

In examples 1.4.5(4) and (5) the following rules hold:

1. $\neg((\exists x \in X) : P(x)) \equiv (\forall x \in X), \neg P(x)$
2. $\neg((\forall x \in X), P(x)) \equiv (\exists x \in X) : \neg P(x)$

where the negation of a compound quantifier statement results in the flipping of the quantifier and negation of the specified statement. These are called De Morgan's laws in formal logic.

Example 1.4.6. Define two sets Γ and Δ where Γ is the set of letters in Latin and Δ is the set of all words in *The Oxford English Dictionary*. Define a mathematical statement that depends on two variables $(\gamma, \delta) \in \Gamma \times \Delta$ which says "The word δ begins with the letter γ" as $P(\gamma, \delta)$

1. For any fixed $\gamma \in \Gamma$, we can create $Q \equiv (\delta \in \Delta) : P(\gamma, \delta)$. Varying γ, a family of mathematical statements $\{Q(\gamma) : \gamma \in \Gamma\}$ which is parametrized by $\gamma \in \Gamma$. Therefore, we can append with a quantifier for γ to create a mathematical statement:

   (a) The statement:
$$(\forall \gamma \in \Gamma), Q(\gamma) \equiv (\forall \gamma \in \Gamma), (\exists \delta \in \Delta) : P(\gamma, \delta)$$
   which reads as "For every letter $\gamma \in \Gamma$, there is a word $\delta \in \Delta$ such that the word δ starts with the letter γ." This is true, because there are words starting with any letter in the alphabet in the dictionary.
   (b) Another statement:
$$(\forall \gamma \in \Gamma) : Q(\gamma) \equiv (\exists \gamma \in \Gamma) : (\exists \delta \in \Delta) : P(\gamma, \delta)$$
   which reads as "There exists a letter $\gamma \in \Gamma$ such that there exists a word $\delta \in \Delta$ such that the word δ starts with the letter γ." This is also true, because for every letter there is a word that starts with that letter.

2. Other combinations of quantifiers:

   (a) $(\forall \delta \in \Delta), (\exists \gamma \in \Gamma) : P(\gamma, \delta)$ reads as "For every word $\delta \in \Delta$, there exists a letter $\gamma \in \Gamma$ such that the word δ starts with the letter γ." Which means every word starts with some letter which is true.

   (b) $(\exists \delta \in \Delta) : (\forall \gamma \in \Gamma), P(\gamma, \delta)$ reads as "There is a word $\delta \in \Delta$ such that for every letter $\gamma \in \Gamma$ the word starts with that letter". This is saying that there is a word that starts with every letter in the alphabet which is false.

   (c) $(\exists \delta \in \Delta) : (\exists \gamma \in \Gamma) : P(\gamma, \delta)$ reads as "There exists a word such that there is a letter that the word starts with." This is true because every word starts with a letter.

Remark 1.4.7. Some remarks of Example 1.4.6.

1. We cannot generally move around quantifiers if there are many in a statement.

2. However, switching two quantifiers of the same type that are adjacent is allowed. For example, $(\exists \gamma \in \Gamma) : (\exists \delta \in \Delta) : P(\gamma, \delta)$ and $(\exists \delta \in \Delta) : (\exists \gamma \in \Gamma) : P(\gamma, \delta)$ are the same and can be read as $\exists (\gamma \in \Gamma) \wedge (\delta \in \Delta) : P(\gamma, \delta) \equiv (\exists (\gamma, \delta) \in \Gamma \times \Delta) : P(\gamma, \delta)$. So we have:

$$\begin{aligned} (\exists \gamma \in \Gamma) : (\exists \delta \in \Delta) : P(\gamma, \delta) &\equiv (\exists (\gamma, \delta) \in \Gamma \times \Delta) : P(\gamma, \delta) \\ &\equiv (\exists \delta \in \Delta) : (\exists \gamma \in \Gamma) : P(\gamma, \delta) \end{aligned}$$

   Similarly,

$$\begin{aligned} (\forall \gamma \in \Gamma), (\forall \delta \in \Delta), P(\gamma, \delta) &\equiv (\forall (\gamma, \delta) \in \Gamma \times \Delta), P(\gamma, \delta) \\ &\equiv (\forall \delta \in \Delta), (\forall \gamma \in \Gamma), P(\gamma, \delta) \end{aligned}$$

3. If there is more than one quantifier in a statement, they can be treated as nested statements. Allows to define negations more systematically. Recall that negation of a quantifier statement flips the quantifier and negates its statement. The following are equivalent and is the negation is done by moving inwards one quantifier at at time:

$$\begin{aligned} \neg((\exists \gamma \in \Gamma) : (\exists \delta \in \Delta) : P(\gamma, \delta)) &\equiv (\forall \gamma \in \Gamma), \neg((\exists \delta \in \Delta) : P(\gamma, \delta)) \\ &\equiv (\forall \gamma \in \Gamma), (\forall \delta \in \Delta), \neg P(\gamma, \delta) \end{aligned}$$

Example 1.4.8. X and Y are sets, consider two families of statements: $\{P(x) : x \in X\}$ and $\{Q(y) : y \in Y\}$. Suppose we have the statement $(\forall x \in X), (\exists y \in Y) : P(x) \implies Q(y)$ and want to find negation. Using Remark 1.4.7(3):

$$\begin{aligned} \neg((\forall x \in X), (\exists y \in Y) : P(x) \implies Q(y)) &\equiv (\exists x \in X) : \neg((\exists y \in Y) : P(x) \implies Q(y)) \\ &\equiv (\exists x \in X) : (\forall y \in Y), \neg(P(x) \implies Q(y)) \end{aligned}$$

Using the equivalence in Equation 1.1:

$$\begin{aligned} \neg((\forall x \in X), (\exists y \in Y) : P(x) \implies Q(y)) &\equiv (\exists x \in X) : (\forall y \in Y), \neg(P(x) \implies Q(y)) \\ &\equiv (\exists x \in X) : (\forall y \in Y), P(x) \wedge (\neg Q(y)) \end{aligned}$$

The existential quantifier denotes that for at least one element $x \in X, P(x)$ is true, the following clarifies when a unique element holds the statement true.

Definition 1.4.9 (Unique, Non-existential Quantifier)**.** Let X be a non-empty set and $\{P(x) : x \in X\}$ be a set of mathematical statements with domain X.

1. Unique Existential Quanifier: A unique existential quantifier is a symbol "$(\exists! x \in X) :$" where the statement $(\exists! x \in X) : P(x)$ is true when $P(x)$ is true for exactly one $x \in X$.

2. Non-existential Quantifier: A non-existential quantifier is a symbol "$(\nexists x \in X)$:" where the statement $(\nexists x \in X)$: $P(x)$ is true when $P(x)$ is true for none of $x \in X$ (or, in other words, $P(x)$ is false $\forall x$).

Remark 1.4.10. Some remarks:

1. The symbol $\exists!$ reads as "there exists a unique" or "there exists exactly one".
2. The symbol $\nexists$ reads as "there does not exist" or "there are no".
3. Note that the non-existential quantifier is the negation of the existence quantifier. Therefore, the following equivalence holds

$$(\nexists x \in X) : P(x) \equiv (\forall x \in X), \neg P(x) \equiv \neg((\exists x \in X) : P(x))$$

Example 1.4.11. Let X be the set of planets in the solar system and $P(x)$ be the sentence "Humans can live on planet x" for $x \in X$.

1. The statement $(\exists! x \in X) : P(x)$ reads as "There is only one planet in the solar system where humans can live", which is unknown because humans have not tried to live on any other planets, yet.
2. The statement $(\nexists x \in X) : P(x)$ reads as "There are no planets in the solar system which humans can live on" which is false because we live on the Earth which is in the solar system.

1.5 Functions

Functions describe how sets transform from one to another. An output is produced when a single input is "fed" into it.

Definition 1.5.1 (Function)**.** A function $f : X \to Y$ is a correspondence between two sets X and Y which assigns each element $x \in X$ a single element $f(x) \in Y$. In symbols, this is

$$(\forall x \in X), (\exists! y \in Y) : f(x) = y$$

Remark 1.5.2. Some remarks:

1. X and Y are called the domain and codomain of the function respectively.
2. f is sometimes called a map, mapping, assignment, or simply as a function. Note that this is an abuse of terminology as, technically, a function is the triple (X, Y, f) which each component specified.
3. The element $f(x) \in Y$ is called the image of the element x under f. $f(x)$ is an element while f is the mapping, they are not the same.
4. Usually when a function is specified it is written as $f : X \to Y$ such that $x \mapsto f(x)$ and is read as "the element x (in X) is mapped to the element $f(x)$ (in Y)".

Example 1.5.3. Let $X = \{a, b, c\}$ and $Y = \{\clubsuit, \diamondsuit, \spadesuit, \heartsuit\}$.

- Consider the following functions, $f : X \to Y$ via $a \mapsto \clubsuit, b \mapsto \diamondsuit, c \mapsto diamondsuit$ and $g : X \to Y$ via $a \mapsto \clubsuit, b \mapsto \diamondsuit, c \mapsto \spadesuit$. Both f and g are functions because each element in X is assigned to only one element in Y. Even for g where multiple elements from X map to a single element in Y, g is still a function because there is no requirement for unique/distinct mappings.
- Consider the following "functions", $p : X \to Y$ via $a \mapsto \clubsuit, b \mapsto \diamondsuit, c \mapsto \spadesuit$ and $\heartsuit$ and $q : X \to Y$ via $a \mapsto \clubsuit, b \mapsto \diamondsuit$. Here, neither p nor q are functions. p assigns two outputs for a single input, while q does not map its entire domain, thus both violate the definitions of being a function.

Functions can be represented by pictures or by listing the pairs (input, output) in the Cartesian product $X \times Y$ (or various other representations). This is called a graph:

Definition 1.5.4 (Graph)**.** Let $f : X \to Y$ be a function between X and Y. The graph of function f is given by the collection of pairs $G_f = \{(x, f(x)) : x \in X\} \subseteq X \times Y$.

1.5.1 Image and Preimage

For a function $f : X \to Y$, the domain and codomain are X and Y and the domain can be written as $X = \mathrm{Dom}(f)$. The image of a function is defined as the set $f(X) = \{f(x) : x \in X\} \subseteq Y$ (Note: codomain = target set, range/image = actual output).

Remark 1.5.5. This is left blank.

If $Z \subseteq X$, then $f(Z)$ is the image of the subset under the mapping f:

$$f(Z) = \{f(x) : x \in Z\} \subseteq f(X).$$

The image of a function may or may not be equal to the codomain, $f(X) \subseteq Y$. In Example 1.5.3, $f(X)$ and $g(X)$ are proper subsets of Y, if the image coincides with the codomain, $f(X) = Y$, then the function is called a surjective function or a surjection.

For every element in the image of f, $y \in f(X)$, it must be mapped from at least one element in the domain. This collection is called the preimage of the element y written as:

$$f^{-1}(\{y\}) = \{x \in X : f(x) = y\}.$$

If y is not in the image of the f ($y \notin f(X)$), then $f^{-1}(\{y\}) = \varnothing$ since there are no elements in X that are mapped to y by f. If $W \subseteq Y$, $f^{-1}(W)$ is the preimage of a subset of the codomain of f and is all the elements in X which are mapped to any element in W:

$$f^{-1}(W) = \{x \in X : f(x) \in W\} \subseteq X.$$

Example 1.5.6. Recall the function $f : X \to Y$ in Example 1.5.3, we have:

1. $f^{-1}(\{\diamond\}) = \{b, c\}$
2. $f^{-1}(\{\spadesuit\}) = \varnothing$
3. $f^{-1}(\{\diamond, \clubsuit\}) = \{a, b, c\} = X$

Proposition 1.5.7. *Let $f : X \to Y$ be a function with $A \subseteq X$ and $B, C \subseteq Y$.*

1. *If $B \subseteq C$ then $f^{-1}(B) \subseteq f^{-1}(C)$.*
2. *$f^{-1}(f(X)) = X$.*
3. *$f(f^{-1}(Y)) = f(X) \subseteq Y$.*
4. *$f(X \setminus A) \supseteq f(X) \setminus f(A)$.*
5. *$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ or in other words $f^{-1}(B^c) = (f^{-1}(B))^c$.*

Proof. See Exercise 1.25. □

Proposition 1.5.7(5) says that the preimage operation preserves complements. This is not true for image operations as strict inclusion may still occur for some functions and sets in Proposition 1.5.7(4), this proposition says that given X and $A \subseteq X$ points from $X \setminus A$ can still map to points in the subset $f(A) \subseteq f(X)$ while the subset $f(X) \setminus f(A)$ do not contain any points in $f(A)$. So $f(X \setminus A) \supseteq f(X) \setminus f(A)$ the former completely contains all of the latter, but also points that may be in $f(A)$ (this is only a partial proof).

Proposition 1.5.8. *Let $f : X \to Y$ be a function, $V_i \subseteq X$ be a collection of subsets of X for reach $i \in I$, and $W_j \subseteq Y$ be a collection of subsets of Y for each $j \in J$ where I and J are some indexing sets. Then:*

1. *$f(\bigcap_{i \in I} V_i) \subseteq \bigcap_{i \in I} f(V_i)$*
2. *$f(\bigcup_{i \in I} V_i) = \bigcup_{i \in I} f(V_i)$*
3. *$f^{-1}(\bigcap_{j \in J} W_j) \subseteq \bigcap_{j \in J} f^{-1}(W_j)$*

4. $f^{-1}(\bigcup_{j\in J} W_j) \subseteq \bigcup_{j\in J} f^{-1}(W_j)$

Proof. Only prove the first two.

1. Pick $y \in f(\bigcap_{i\in I})$. Then, by definition:

$$\begin{aligned}\exists x \in \bigcap_{i\in I} V_i \text{ such that } f(x) = y &\implies \exists x \in V_i \text{ for all } i \in I \text{ such that } f(x) = y\\ &\implies y \in f(V_i) \text{ for all } i \in I\\ &\implies y \in \bigcap_{i\in I} f(V_i),\end{aligned}$$

and since y is arbitrary, we obtain the inclusion $f(\bigcap_{i\in I} V_i) \subseteq \bigcap_{i\in I} f(V_i)$. (Proof method: morph LHS into the RHS, this only proves inclusion, not the reverse inclusion in order to get equality.)

2. Use double inclusion to prove this equality.

($\subseteq$): Pick an element $y \in f(\bigcup_{i\in I} V_i)$. Then, by definition:

$$\begin{aligned}\exists x \in \bigcup_{i\in I} V_i \text{ such that } f(x) = y &\implies \exists i \in I \text{ such that } x \in V_i \text{ with } f(x) = y\\ &\implies y \in f(V_i) \text{ for some } i \in I\\ &\implies y \in \bigcup_{i\in I} f(V_i),\end{aligned}$$

which proves the first inclusion $f(\bigcup_{i\in I} V_i) \subseteq \bigcup_{i\in I} f(V_i)$.

($\supseteq$): Pick an arbitrary $y \in \bigcup_{i\in I} f(V_i)$. Then, by definition:

$$\begin{aligned}\exists i \in I \text{ s.t. } y = f(V_i) &\implies \exists i \in I \text{ s.t. } \exists x \in V_i \text{ with } f(x) = y\\ &\implies \exists x \in \bigcup_{i\in I} V_i \text{ s.t. } f(x) = y\\ &\implies y \in f\left(\bigcup_{i\in I} V_i\right),\end{aligned}$$

which shows the reverse inclusion $f(\bigcup_{i\in I} V_i) \supseteq \bigcup_{i\in I} f(V_i)$.

The two inclusions results in the equality of the sets. □

Remark 1.5.9. Note that in Proposition 1.5.8, the preimage operation f^{-1} preserves union and intersection. However, the image operation f only preserves unions. Intersections may not be preserved under f.

An example is the function $f : X \to Y$ from Example 1.5.3. If we set $U = \{b\}$ and $V = \{c\}$, we immediately get $U \cap V = \varnothing$ and thus $f(U \cap V) = f(\varnothing) = \varnothing$. However, $f(U) = f(V) = \{\diamond\}$ so $f(U) \cap f(V) = \{\diamond\} \neq \varnothing$. So this is an example for which $f(U \cap V) \subsetneq f(U) \cap f(V)$.

In fact, the preimage operations also satisfy the following:

Proposition 1.5.10. *Let $f : X \to Y$ be a function and $A, B \subseteq Y$. Then:*

1. $f^{-1}(B \setminus A) = f^{-1}(B) \setminus f^{-1}(A)$
2. $f^{-1}(B\Delta A) = f^{-1}(B)\Delta f^{-1}(A)$

Thus, the preimage operations preserves (finite and arbitrary) union, (finite and arbitrary) intersection, complements, set difference, and symmetric set difference. On the other hand, the image operations do not necessarily satisfy this.

1.5.2 Injection, Surjection, Bijection

If each element in the image of a function has exactly one preimage, then we call the function an injective function or an injection. In other words, an injective function maps distinct elements in the domain to distinct elements in the codomain.

Example 1.5.11. $f : X \to Y$ in Example 1.5.3 is not injective because the diamond suit has two possible preimages b and c. On the other hand, g is injective as each element in the image has exactly one preimage.

Definition 1.5.12 (Injection, Surjection, Bijection)**.** Let $f : X \to Y$ be a function. (Note: if a $y \in Y$ does not have a preimage it is not in the image $f(X)$, only in the codomain.)

1. The function f is called an injective function or an injection if for each element $y \in f(X)$, there exists exactly one element $x \in X$ such that $f(x) = y$. In other words, whenever $f(x) = f(z)$, necessarily $x = z$. In symbols:

$$(\forall y \in f(X)), (\exists! x \in X) : f(x) = y$$

2. The function f is called a surjective function or a surjection if for every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$. In other words, the image of the function coincides with the codomain, namely $f(X) = Y$. In symbols:

$$(\forall y \in Y), (\exists x \in X) : f(x) = y$$

3. The function f is called a bijective function or a bijection if it is both injective and surjective. In other words, every element in the codomain is mapped from exactly one element in the domain via f. In symbols:

$$(\forall y \in Y), (\exists! x \in X) : f(x) = y$$

Remark 1.5.13. Let $f : X \to Y$ be a function.

1. If f is an injection, we say f injects into Y. f is also called a one-to-one function.
2. If f is a surjection, we say f surjects onto Y. f is also called an onto function.
3. If f is a bijection, we call f a one-to-one correspondence between X and Y.

Note: the one-to-one terminology is usually not used as remark 1 and 3 can be confused.

1.5.3 Composite, Inverse, Restriction Functions

Chapter 2

Integers

2.1 Relations

2.2 Natural Numbers $\mathbb{N}$

2.3 Ordering on $\mathbb{N}$

2.4 Integers $\mathbb{Z}$

2.5 Algebra on $\mathbb{Z}$

2.6 Ordering on $\mathbb{Z}$

Chapter 3

Construction of Real Numbers

Chapter 4

Real Numbers

4.1 Properties of Real Numbers $\mathbb{R}$

4.2 Exponentiation

4.3 Logarithm

4.4 Decimal Representation of the Real Numbers

4.5 Topology on $\mathbb{R}$

4.6 Real n-Space and Complex Numbers

Chapter 5

Real Sequences

Chapter 6

Some Applications of real Sequences

6.1 Circular Arclength

6.2 Limit Points and Topology

6.3 Sequences in $\mathbb{C}$ and $\mathbb{R}^n$

6.4 Introduction to Metric Spaces

Part II

Real Analysis 2: Series, Continuity, and Differentiability

Part III

Real Analysis 3: Integration