# Analysis I - Amann & Escher

Notes

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# Preface

# Chapter 1

# Foundations

## 1.1 Fundamentals of Logic

Symbolic logic is about **statements** which can be either true (T) or false (F), not both or in between. A statement $A$ can have a **negation** $\neg A$ ('not $A$') which is defined as: $\neg A$ true if $A$ false. A truth table can be used to show this:

| $A$ | $T$ | $F$ |
|---|---|---|
| $\neg A$ | $F$ | $T$ |

Two statements $A$ and $B$ can be combined using **conjunction** or **disjunction** to make new ones. The statement $A \wedge B$ ('$A$ and $B$') is true when both $A$ and $B$ are true. The statement $A \vee B$ ('$A$ or $B$') is true when either $A$ or $B$ or both are true (inclusive or) and is false only when both are false. Refer to the following truth table:

| $A$ | $B$ | $A \wedge B$ | $A \vee B$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $F$ |

If $E(x)$ is a statement where $x$ is replaced by an object (member, thing) of a specified class (collection, universe), then $E$ is a **property**. '$x$ has property $E$' is equivalent to '$E(x)$ is true'. If $x$ is a member (an **element**) of a class $X$ we write $x \in X$ otherwise $x \notin X$. Then,

$$\{x \in X : E(x)\}$$

is the class of all elements $x$ of the collection $X$ that have the property $E$.

The **quantifier** $\exists$ denotes existence and is read 'there exists'. The expression

$$\exists x \in X : E(x)$$

says 'There is (at least) one object $x$ in (the class) $X$ which has the property $E$'. The unique quantifier $\exists!$ says that there is only one (a unique) object.

The quantifier $\forall$ denotes 'for all' or 'for each' object in a collection. The expression

$$\forall x \in X : E(x) \tag{1.1.1}$$

says 'For each $x \in X$ the statement $E(x)$ is true'. This can also be written as

$$E(x), \quad \forall x \in X \tag{1.1.2}$$

which says 'The property $E(x)$ is true for all $x$ in $X$'. Sometimes the quantifier is left out

$$E(x), \quad x \in X \tag{1.1.3}$$

The symbol $:=$ means 'is defined by'. Thus,

$$a := b$$

says 'the object (or symbol) $a$ is defined by the object (or expression) $b$'. Of course $a = b$ means $a$ and $b$ are equal.

**Example 1.1.1.** Let $A$ and $B$ be statements, $X$ and $Y$ be classes of objects, and $E$ is a property. Truth tables can verify the following statements:

(a) $\neg\neg A := \neg(\neg A) = A$

| $A$ | $\neg A$ | $\neg\neg(A)$ |
|---|---|---|
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ |

(b) $\neg(A \wedge B) = (\neg A) \vee (\neg B)$ (de Morgan's Law 1)

| $A$ | $B$ | $A \wedge B$ | $\neg(A \wedge B)$ | $(\neg A) \vee (\neg B)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ |

(c) $\neg(A \vee B) = (\neg A) \wedge (\neg B)$ (De Morgan's Law 2)

| $A$ | $B$ | $A \vee B$ | $\neg(A \vee B)$ | $(\neg A) \wedge (\neg B)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $T$ |

(d) $\neg(\forall x \in X : E(x)) = (\exists x \in X : \neg E(x))$ (for example: the negation of 'everyone wears glasses' is 'at least one person does not wear glasses')

(e) $\neg(\exists x \in X : E(x)) = (\forall x \in X : \neg E(x))$ (for example: the negation of 'there is at least one car on the road that is red' is 'there are no cars on the road that is read')

(f) $\neg(\forall x \in X : (\exists y \in Y : E(x, y))) = (\exists x \in X : (\forall y \in Y : \neg E(x, y)))$ (for example: the negation of 'every person has something in at least one pocket' is 'there is at least one person that has whose pockets are all empty'. Here X is the collection of people, Y is the collection of pockets on a person, and E is the property that the pocket is occupied and depends on the person and the pocket.)

(g) $\neg(\exists x \in X : (\forall y \in Y : E(x, y))) = (\forall x \in X : (\exists y \in Y : \neg E(x, y)))$ (for example: the negation of 'there is at least one car with every window open' is 'in the collection of all cars there is at least one window that is closed')

**Remark 1.1.2.** (a) Parenthesis keep the statements exact but aren't always used; similarly, the membership symbol is not always used. For example: $\forall x \exists y : E(x, y)$ is still valid and says 'For all $x$ there is at least one $y$ such that $E(x, y)$ is true', thus $y$ depends on $x$. Another example: $\exists x \forall y : E(x, y)$ which says 'there is at least one $x$ with every $y$ such that $E(x, y)$ is true' it is sufficient to find one $y$ which is true for all $x$. For example, if $E(x, y)$ is the statement 'reader $x$ of this book find the concept of $y$ to be trivial' then the first statement is: 'Each reader of this book finds at least one concept that is trivial' and the second statement is 'there is at least one statement that every reader finds trivial.'

(b) The quantifiers $\exists$ and $\forall$ as well as the logical 'and' and 'or' are mechanically interchanged in negation without changing order while the statements are negated.

Let $A$ and $B$ be statements, the **implication** $A \implies B$, ('$A$ implies $B$') is defined as:

$$(A \implies B) := (\neg A) \vee B \tag{1.1.4}$$

Thus $A \implies B$ is false if $A$ is true and $B$ is false, and true in all other cases. In other words, it is true when $A$ and $B$ are both true, or when $A$ is false (independent of whether $B$ is true or false) meaning a true statement cannot imply a false statement, also a false statement implies any statement. Common to say 'To prove $B$ it **suffices** to prove $A$' or '$B$ is **necessary** for $A$ to be true', in other words, $A$ is a **sufficient condition** for $B$ and $B$ is a **necessary condition** for $A$.

The **equivalence** $A \iff B$ ('$A$ and $B$ are equivalent') of the statements is defined by:

$$(A \iff B) := (A \implies B) \wedge (B \implies A)$$

| $A$ | $B$ | $A \implies B$ | $B \implies A$ | $A \iff B$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |

$B \implies A$ is the **converse** of $A \implies B$, and $A$ is a **necessary and sufficient** condition for B (or vice versa), another way to express this is to say '$A$ is true **if and only if** (iff) $B$ is true'.

A fundamental observation:

$$(A \implies B) \iff (\neg B \implies \neg A) \tag{1.1.5}$$

| $A$ | $B$ | $\neg A$ | $\neg B$ | $A \implies B$ | $\neg B \implies \neg A$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

This follows directly from Equation 1.1.4 and is called the **contrapositive** of the statement $A \implies B$.

For example, if $A$ is 'There are clouds in the sky' and $B$ is 'It is raining', then $B \implies A$ is the statement 'If it is raining, then there are clouds in the sky'. Its contrapositive is 'If there are no clouds in the sky, then it is not raining'.

If $B \implies A$ is true it does not, in general, follow that $\neg B \implies \neg A$ is true, for example, even if 'It is not raining' it is possible that 'There are clouds in the sky'.

The following defines $A$ is true whenever $B$ is true and is read as '$A$ is true, by definition, if $B$ is true':

$$A :\iff B$$

In math, a true statement is often called a **proposition**, **theorem**, **lemma**, or **corollary**. Typically, propositions are of the form $A \implies B$ and since the statement is automatically true if $A$ is false (see last two lines of the truth table), to prove the statement true, suppose $A$ is true then show that $B$ is also true.

Proofs can be done directly or 'by contradiction'. Directly, one can use the fact that

$$(A \implies C) \wedge (C \implies B) \implies (A \implies B) \tag{1.1.6}$$

If $A \implies C$ and $C \implies B$ are each separately already known to be true, then by Equation 1.1.6 $A \implies B$ is also true.

A proof by **contradiction** supposes that $B$ is false. The one proves, using the assumption that $A$ is true, a statement $C$ which is already known to be false. It follows from this 'contradiction' that $\neg B$ cannot be true, and hence $B$ is true.

Sometimes easier to prove contrapositive.

### Exercises

1. TODO: do this
2. TODO: do this

## 1.2 Sets

### 1.2.1 Elementary Sets

If $X$ and $Y$ are **sets**, then $X \subseteq Y$ ('$X$ is a **subset** of $Y$') means that each element in $X$ is contained in $Y$: $\forall x \in X : x \in Y$. Sometimes written as $Y \supseteq X$ as well. Equality of sets defined as

$$X = Y :\iff (X \subseteq Y) \wedge (Y \subseteq X)$$

The statements

$$\begin{aligned} X &\subseteq X && \textbf{(reflexivity)} \\ (X \subseteq Y) \wedge (Y \subseteq Z) &\implies (X \subseteq Z) && \textbf{(transitivity)} \end{aligned}$$

are true. If $X \subseteq Y$ and $X \nsubseteq Y$ then $X$ is a **proper subset** of $Y$ denoted with $X \subset Y$. If $X$ is a set and $E$ a property then $\{x \in X : E(x)\}$ is a subset of $X$ with elements that satisfy $E(x)$. The empty set of $X$ is defined as

$$\varnothing_X := \{x \in X : x \neq x\}.$$

**Remark 1.2.1.** (a) Let $E$ be a property, then $x \in \varnothing_X \implies E(x)$ is true for each $x \in X$ ('The empty set possesses every property').

*Proof.* From Equation 1.1.4 we have

$$(x \in \varnothing_X \implies E(x)) = \neg(x \in \varnothing_X) \vee E(x)$$

The negation $\neg(x \in \varnothing_X)$ is true for each $x \in X$. □

(b) If $X$ and $Y$ are sets, then $\varnothing_X = \varnothing_Y$, i.e., **empty set** is unique and denoted as $\varnothing$ and is a subset of any (every) set.

*Proof.* From the first remark, $x \in \varnothing_X \implies x \in \varnothing_Y$, hence $\varnothing_X \subseteq \varnothing_Y$, by symmetry $\varnothing_Y \subseteq \varnothing_X$, thus $\varnothing_X = \varnothing_Y$. □

The set containing the single element $x$ is denoted $\{x\}$ (a singleton). The set containing $a, b, ..., *, \odot$ is written $\{a, b, ..., *, \odot\}$.

### 1.2.2 The Power Set

$X$ is a set, its **power set** $\mathcal{P}(X)$ whose elements are the subsets of $X$, sometimes it is written as $2^X$. The following are true:

$$\varnothing \in \mathcal{P}(X), X \in \mathcal{P}(X). \tag{1.2.1}$$
$$x \in X \iff \{x\} \in \mathcal{P}(X). \tag{1.2.2}$$
$$Y \subseteq X \iff Y \in \mathcal{P}(X). \tag{1.2.3}$$

$\mathcal{P}$ is never empty.

**Example 1.2.2.** (a) $\mathcal{P}(\varnothing) = \{\varnothing\}$, $\mathcal{P}(\{\varnothing\}) = \{\varnothing, \{vno\}\}$.

(b) $\mathcal{P}(\{*, \odot\}) = \{\varnothing, \{*\}, \{\odot\}, \{*, \odot\}\}$

### 1.2.3 Complement, Intersection, and Union

Let $A$ and $B$ be subsets of $X$. Then

$$A \setminus B := \{x \in X : (x \in A) \wedge (x \notin B)\}$$

is the **relative complement** of $B$ in $A$. When $X$ is clear from context, the **complement** of $A$ in $X$ is

$$A^c := X \setminus A$$

The set

$$A \cap B := \{x \in X : (x \in A) \wedge (x \in B)\}$$

is called the **intersection** of $A$ and $B$. If they have no elements in common $A \cap B = \varnothing$ and are **disjoint**. $A \setminus B = A \cap B^c$. The set

$$A \cup B := \{x \in X : (x \in A) \vee (x \in B)\}$$

is called the **union** of $A$ and $B$.

**Remark 1.2.3.** Useful to use **Venn Diagrams** set!Venn diagrams. They cannot be used as proofs but help build intuition and hints at proofs.

**Proposition 1.2.4.** *Let $X$, $Y$, and $Z$ be subsets of a set.*

*(i)* $X \cup Y = Y \cup X$, $X \cap Y = Y \cap X$. *(commutativity)*

*(ii)* $X \cup (Y \cup Z) = (X \cup Y) \cup Z$, $X \cap (Y \cap Z) = (X \cap Y) \cap Z$. *(associativity)*

*(iii)* $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$,
$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$. *(distributivity)*

*(iv)* $X \subseteq Y \iff X \cup Y = Y \iff X \cap Y = X$

### 1.2.4 Products

An **ordered pair** is made of two elements $a$ and $b$ to make $(a, b)$. Equality is defined as

$$(a, b) = (c, d) :\iff (a = c) \wedge (b = d)$$

The objects $a$ and $b$ are the first and second **components**. For $x = (a, b)$, we define

$$\mathrm{pr}_1(x) := a, \quad \mathrm{pr}_2 := b$$

generally for any sized ordered sets, $\mathrm{pr}_j(x)$ is the $j^{\text{th}}$ **projection** of $x$.

If $X$ and $Y$ are sets, then the **Cartesian product** $X \times Y$ is the set of all ordered pairs $(x, y)$ with $x \in X$ and $y \in Y$.

**Example 1.2.5** (also a remark). (a) For $X := \{a, b\}$ and $Y := \{c, d, e\}$ we have

$$X \times Y = \{(a,c), (a,d), (a,e), (b,c), (b,d), (b,e)\}$$

(b) Useful to have a graphical representation. If $X$ and $Y$ are number lines, their Cartesian product can be seen as a rectangle. Or using the previous example, a table can be made with $X$ and $Y$ as below

| $X \backslash Y$ | $c$ | $d$ | $e$ |
|---|---|---|---|
| $a$ | $(a,c)$ | $(a,d)$ | $(a,e)$ |
| $b$ | $(b,c)$ | $(b,d)$ | $(b,e)$ |

**Proposition 1.2.6.** *Let $X$ and $Y$ be sets.*

*(i)* $X \times Y = \varnothing \iff (X = \varnothing) \vee (Y = \varnothing)$.

*(ii) In general:* $X \times Y \neq Y \times X$.

*Proof.* (i) Need to prove forward statement ( $\implies$ ) and backwards (converse, $\impliedby$).

'$\implies$': Using contradiction. Suppose $X \times Y = \varnothing$ is true and $(X = \varnothing) \vee (Y = \varnothing)$ is false. By Example 1.1.1(c) (de Morgan's Law 2), the negation of the second statement is $\neg((X = \varnothing) \vee (Y = \varnothing)) = (X \neq \varnothing) \wedge (Y \neq \varnothing)$ and is considered true (taking the non-negated statement as true). So there are $x \in X$ and $y \in Y$. But then $(x, y) \in X \times Y$, contradicting $X \times Y = \varnothing$. Thus $X \times Y \implies (X = \varnothing) \vee (Y = \varnothing)$.

'$\impliedby$' Prove using the contrapositive of

$$(X = \varnothing) \vee (Y = \varnothing) \implies X \times Y = \varnothing.$$

Which is

$$\begin{aligned} \neg(X \times Y = \varnothing) &\implies \neg((X = \varnothing) \vee (Y = \varnothing)) \\ (X \times Y \neq \varnothing) &\implies (X \neq \varnothing) \wedge (Y \neq \varnothing) \end{aligned}$$

If $X \times Y \neq \varnothing$ then there is some $(x, y) \in X \times Y$ which implies that $X \neq \varnothing$ and $Y \neq \varnothing$.

Thus, $(X = \varnothing) \vee (Y = \varnothing) \implies X \times Y = \varnothing$.

(ii) See Exercise 4.

□

The product of three sets is defined by

$$X \times Y \times Z := (X \times Y) \times Z$$

Repeating this to define the product of $n$ sets:

$$X_1 \times \cdots X_n := (X_1 \times \cdots \times X_{n-1}) \times X_n$$

For $x \in X_1 \times \cdots X_n$ we write $(x_1, \ldots, x_n)$ instead of $(\ldots((x_1, x_2), x_3), \ldots x_n)$ and call $x_j$ the $j^{\text{th}}$ component of $x$ for $1 \leq j \leq n$ and is equal to $\mathrm{pr}_j(x)$. We can also write

$$X_1 \times \cdots X_n = \prod_{j=1}^{n} X_j \quad .$$

If all the factors in the product are the same, $X_j = X$ for $j = 1, ..., n$ then the product is written $X^n$.

### 1.2.5 Families of Sets

Let $\mathsf{A}$ be a nonempty set, for each $\alpha \in \mathsf{A}$, let $A_\alpha$ be a set. Then $\{A_\alpha : \alpha \in \mathsf{A}\}$ is called a **family of sets** and $\mathsf{A}$ is an **index set** for this family. (Notation here is a little weird, typically $\{A_i : i \in \mathsf{I}\}$ is used) $A_\alpha$ does not need to be nonempty for each index but a family of sets is never empty.

Let $X$ be a set and $\mathcal{A} := \{A_\alpha : a \in \mathsf{A}\}$ a family of subsets of $X$. Generalizing the above concepts we define the intersection and union of this family by (Note: Germans use ';' as such that as well. Americans just use one ':' and use commas to separate the 'such that' quantifier and conditions)

$$\bigcap_\alpha A_\alpha := \{x \in X : \forall \alpha \in \mathsf{A}, x \in A_\alpha\} = \bigcap \mathcal{A},$$

and

$$\bigcup_\alpha A_\alpha := \{x \in X : \exists \alpha \in \mathsf{A}, x \in A_\alpha\} = \bigcup \mathcal{A}.$$

These are both subsets of $X$. If $\mathcal{A}$ is a finite family of sets, then it can be indexed by finitely many natural numbers $\{0, 1, ..., n\}$: $\mathcal{A}\{A_j : j = 0, .., n\}$ and the union

**Proposition 1.2.7.** *Let $\{A_\alpha : \alpha \in \mathsf{A}\}$ and $\{B_\beta : \beta \in \mathsf{B}\}$ be families of subsets of $X$.*

*(i)* $(\bigcap_\alpha A_\alpha) \cap \left(\bigcap_\beta B_\beta\right) = \bigcap_{(\alpha,\beta)} A_\alpha \cap B_\beta,$
$(\bigcup_\alpha A_\alpha) \cup \left(\bigcup_\beta B_\beta\right) = \bigcup_{(\alpha,\beta)} A_\alpha \cup B_\beta.$ *(associativity)*

*(ii)* $(\bigcap_\alpha A_\alpha) \cup \left(\bigcap_\beta B_\beta\right) = \bigcap_{(\alpha,\beta)} A_\alpha \cup B_\beta,$
$(\bigcup_\alpha A_\alpha) \cap \left(\bigcup_\beta B_\beta\right) = \bigcup_{(\alpha,\beta)} A_\alpha \cap B_\beta.$ *(distributivity)*

*(iii)* $(\bigcap_\alpha A_\alpha)^c = \bigcup_\alpha A_\alpha^c,$
$(\bigcup_\alpha A_\alpha)^c = \bigcap_\alpha A_\alpha^c.$ *(de Morgan's laws)*

*Here, $(\alpha, \beta)$ runs through the index set $\mathsf{A} \times \mathsf{B}$.*

*Proof.* Follows easily from definitions. For (iii), see Examples 1.1.1. □

**Remark 1.2.8.** Insert 'philosophy of sets' and 'what is a set' here. **Axioms** are rules that assumed to be fundamentally true and need no/cannot be proved.

## Exercises

1. TODO: do this
2. TODO: do this
3. TODO: do this
4. TODO: do this
5. TODO: do this
6. TODO: do this
7. TODO: do this
8. TODO: do this

## 1.3 Functions

In this section $X$, $Y$, $U$, and $V$ are arbitrary sets.

A **function** or **map** $f$ from $X$ to $Y$ is a rule which, $\forall x \in X$, specifies *exactly* one element of $Y$ written as

$$f : X \to Y \qquad \text{to} \qquad X \to Y, \quad x \mapsto f(x),$$

Here $f(x) \in Y$ is the **value** of $f$ at $x$. The set $X$ is called the **domain** of $f$ and is denoted $\mathrm{dom}(f)$ and $Y$ is the **codomain** of $f$. Finally

$$\mathrm{im}(f) := \{y \in Y : \exists x \in X : y = f(x)\}$$

is called the **image** or **range** of $f$, it is the subset of the codomain that is 'reachable' by the function.

If $f : X \to Y$ then

$$\mathrm{graph}(f) := \{(x, y) \in X \times Y : y = f(x)\} = \{(x, f(x) \in X \times Y : x \in X)\}$$

is called the **graph** of $f$.

**Remark 1.3.1.** Let $G \subseteq X \times Y$ with the property $\forall x \in X, \exists! y \in Y$ with $(x, y) \in G$. The function $f : X \to Y$ with the rule that $\forall x \in X, f(x) := y, y \in Y$ where $y$ is the unique element such that $(x, y) \in G$. Clearly, $\mathrm{graph}(f) = G$. So a function, $f : X \to Y$ is defined as the ordered triple $f = (X, G, Y), G \subseteq X \times Y$ such that $\forall x \in X, \exists! y \in Y, (x, y) \in G$.

### 1.3.1 Simple Examples

$X = \varnothing$ and $Y = \varnothing$ are not excluded. If $X = \varnothing$ then there is only one function called the **empty function**, $\varnothing : \varnothing \to Y$. If $Y = \varnothing$ but $X \neq \varnothing$ then there are no functions from $X$ to $Y$. Two functions $f : X \to Y$ and $g : U \to V$ are **equal**, $f = g$, if

$$X = U, \quad Y = V \quad \text{and} \quad f(x) = g(x), \quad \forall x \in X$$

**Example 1.3.2.** (a) $\mathrm{id}_X : X \to X, x \mapsto x$ is the **identity function** (of $X$). Sometimes written id if $X$ is clear from context.

(b) If $X \subseteq Y$, then $i : X \to Y, x \mapsto x$ is called the **inclusion (embedding, injection)** of $X$ into $Y$. (sends each element of $X$ to itself while the image is viewed as an element of $Y$). Note that $i = \mathrm{id}_X \iff X = Y$.

(c) If $X$ and $Y$ nonempty and $b \in Y$, then $f : X \to Y, x \mapsto b$ is a **constant function**.

(d) if $f : X \to Y$ and $A \subseteq X$, then $f|A : A \to Y, x \mapsto f(x)$ is the **restriction of $f$ to** $A$. Also, $f|A = f \iff A = X$.

(e) Let $A \subseteq X$ and $g : A \to Y$. Then any function $f : X \to Y$ with $f|A = g$ is called an **extension of** $g$, $f \supseteq g$.

(f) Let $f : X \to Y$ with $\mathrm{im}(f) \subseteq U \subseteq Y \subseteq V$, then there are '**induced**' functions $f_1 : X \to U$ and $f_2 : X \to V$ defined by $f_j(x) := f(x)$ for $x \in X$ and $j = 1, 2$ using the same symbol for the function as needed because they are the same function just different codomains.

(g) Let $X \neq \varnothing$ and $A \subseteq X$, then the **characteristic function** of $A$ is

$$\chi_A : X \to \{0, 1\}, \quad x \mapsto \begin{cases} 1, & x \in A, \\ 0, & x \in A^c. \end{cases}$$

(h) If $X_1, \ldots, X_n$ are nonempty, then projections

$$\mathrm{pr}_k : \left(\prod_{j=1}^{n} X_j\right) \to X_k, \quad x = (x_1, \ldots, x_n) \mapsto x_k, \quad k = 1, \ldots, n$$

are functions.

### 1.3.2 Composition of Functions

Let $f : X \to Y$ and $g : Y \to V$ be two functions. Then $g \circ f$ is the **composition** of $f$ and $g$ ('$f$ followed by $g$') and is defined as

$$g \circ f : X \to V, \quad x \mapsto g(f(x))$$

**Proposition 1.3.3.** *Let $f : X \to Y$ and $g : Y \to$ and $h : U \to V$. Then $(h \circ g) \circ f$ and $h \circ (g \circ f) : X \to V$ are well defined and*

$$(h \circ g) \circ f = h \circ (g \circ f) \tag{1.3.1}$$

*(associativity of composition).*

*Proof.* Follows directly from the definition.

$$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x))),$$
$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x))).$$

□

*Thus parenthesis are not required.*

### 1.3.3 Commutative Diagrams

Useful to represent compositions of functions in diagrams where $X \xrightarrow{f} Y$ instead of $f : X \to Y$. The **diagram**

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & V \end{tikzcd}$$

is **commutative** if $h = g \circ f$.

Similarly the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"] & Y \arrow[d, "g"] \\ U \arrow[r, "\psi"] & V \end{tikzcd}$$

is commutative if $g \circ f = \psi \circ \varphi$. Such diagrams are commutative if $X$ and $Y$ are sets in the diagram and on can get from $X$ to $Y$ via two different paths following the arrows, for example

$$X \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} Y \quad \text{and} \quad X \xrightarrow{g_1} B_1 \xrightarrow{g_2} B_2 \xrightarrow{g_3} \cdots \xrightarrow{g_n} Y,$$

then $f_n \circ f_{n-1} \circ \cdots \circ f_1$ and $g_m \circ g_{m-1} \circ \cdots \circ g_1$ are equal. The diagram

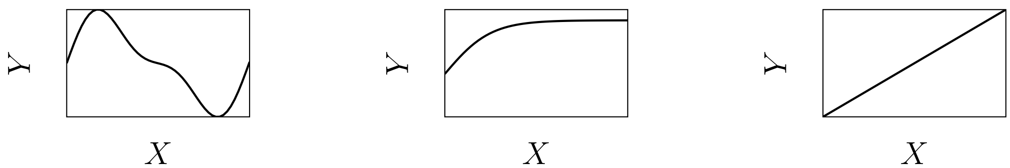
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, "\varphi"] & Y \arrow[d, "g"] \arrow[dl, "\psi"'] \\ V & U \arrow[l, "h"] \end{tikzcd}$$

is commutative if $\varphi = g \circ f$, $\psi = h \circ g$, $j = h \circ g \circ f = h \circ \varphi = \psi \circ f$ via associativity.

### 1.3.4 Injections, Surjections, and Bijections

Let $f : X \to Y$ be a function. Then $f$ is **surjective** if $\mathrm{im}(f) = Y$, **injective** if $f(x) = f(y) \implies x = y \forall x, y \in X$, and **bijective** if $f$ is both injective and surjective. The expressions 'onto' and 'one-to-one' often used to mean 'surjective' and 'injective' respectively. (Note: surjective intuitively means the image and the codomain are the same (the function maps to all points in the codomain), and injective intuitively means that each input maps to different outputs, i.e. each output comes from at most one input.)

**Example 1.3.4.** (a) These images illustrate these function properties:




Surjective, not injective — Injective, not surjective — Bijective

(b) Let $X_1, \ldots, X_n$ be nonempty. then $\forall k \in \{1, \ldots, n\}$ the $k^{\text{kth}}$ projection $\mathrm{pr}_k : \prod_{j=1}^{n} X_j \to X_k$ is surjective, but not injective generally.

**Proposition 1.3.5.** *Let $f : X \to Y$ be a function, then $f$ is bijective if and only if there is a function $g : Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$. In this case, $g$ is uniquely determined by $f$, i.e. there is only one function that satisfies these properties.*

*Proof.* Statement $A$ is '$f$ is bijective'. Statement $B$ is 'there is a function $g$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$'. The compound statement is $A \iff B$.

(i) '$\implies$': suppose that $f : X \to Y$ is bijective then $\forall y \in Y$, $\exists x \in X$ s.t. $y = f(x)$ (surjective) and $x$ is uniquely determined by $y$, there is, at most, one $y$ for each $x$ (injective), together there is exactly one $x \in X$ for every $y \in Y$ (bijective). This defines a function $g : Y \to X$ with the desired properties: We define a function $g : Y \to X$ such that there is exactly one $x \in X$ with $f(x) = y$, so for each $y \in Y$ define $g(y) = x$.

   Now we verify that $g \circ f = \mathrm{id}_X$. Let $x \in X$ be arbitrary, we know that $g \circ f = g(f(x))$ by definition of a composite function. By the definition of $g$, $g(f(x)) = x$ and is unique thus $g \circ f = \mathrm{id}_X$.

(ii) '$\Longleftarrow$':

(iii)

$\square$

### 1.3.5 Inverse Functions

### 1.3.6 Set Valued Functions

### Exercises