

Optimal Control  
with Aerospace Applications - Longuski  
Notes

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# Chapter 1

## Parameter Optimization

### 1.1 Introduction

Two main branches of optimization: parameter optimization and optimal control theory. Parameter optimization is finite dimensional, and a function is minimized via a set of optimal parameters. Optimal control is infinite dimensional find a trajectory  $\mathbf{x}(t)$ , generally an  $n$ -vector, that minimizes a **functional**.

Parameter optimization seeks to find maxima and minima using calculus. The problem is stated as:  
Find:

$$\mathbf{x}$$

to minimize:

$$J = f(\mathbf{x}) \tag{1.1}$$

where  $J$  is a scalar cost function or index of performance and  $\mathbf{x}$  is a constant  $n$ -vector. The solution,  $\mathbf{x}^*$ , can be found by (if  $f$  has continuous partial derivatives and  $x_i$  are independent)

$$\frac{\partial f}{\partial x_i} = 0, \quad i = 1, 2, 3, \dots, n \tag{1.2}$$

Equation 1.2 is a necessary condition for an extreumum (maximum or minimum, note that  $f$  can be maximized by minimizing  $-f$ ). The stationary point  $\mathbf{x}^*$  is the local minimum if the matrix  $\frac{\partial^2 f}{\partial x_i \partial x_j}$  evaluated at  $\mathbf{x}^*$  is a **positive-definite** matrix, which is a sufficient condition for a local minimum. The second partial derivatives of  $f$  must also be continuous for the matrix to be well defined.

### 1.2 Parameter Optimization with Constraints

#### 1.2.1 Lagrange Multipliers

#### 1.2.2 Parameter Optimization: The Hohmann Transfer (1925)

#### 1.2.3 Extensions of the Hohmann Transfer (1959)

#### 1.2.4 The Bi-parabolic Transfer

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