# Solutions to Part I of 17th PMO (Area Level)

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## Notations

| | |
|---|---|
| $AB$ | The segment $AB$ or the length of the segment $AB$. The usage is easily seen in context |
| $[ABC]$ | The area of the region (triangle) $ABC$. |
| $\angle BDA$ | The angle or the measure ($m\angle$) of the angle $BDA$. |

## 1 PART I.

1. What is the fourth smallest positive integer having exactly 4 positive integer divisors, including 1 and itself?

   **Solution:** The answer provided by PMO is 14. We can get this result if we follow the folloing solution:

$$1 \times 2 \times 2 = 4 \tag{1}$$
$$1 \times 2 \times 3 = 6 \tag{2}$$
$$1 \times 2 \times 5 = 10 \tag{3}$$
$$1 \times 2 \times 7 = 14 \tag{4}$$

   But I fail to see why 4 is included since 2 is a divisor of multiplicity 2. All the others have unique divisors. The four smallest positive integers having exactly 4 positive integer divisors should have been: 6, 10, 14, 15. That is, the answer should have been $15 = 1 \times 3 \times 5$. ■

2. Let $f(x) = a^x - 1$. Find the largest value of $a > 1$ so that if $0 \le x \le 3$, then $0 \le f(x) \le 3$.

   **Solution:** Since $a > 1$, $f(x) = a^x - 1$ is an increasing function, which means that $f(3) = 3$. That is $a^3 - 1 = 3 \implies a^3 = 4 \implies a = \sqrt[3]{4}$. ■

3. Simplify the expression $\left(1 + \frac{1}{i} + \frac{1}{i^2} + \cdots + \frac{1}{i^{2014}}\right)^2$.

   **Solution:** We note that

$$\frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = -i$$
$$\frac{1}{i^2} = \frac{1}{i^2} = \frac{1}{-1} = -1$$
$$\frac{1}{i^3} = \frac{1}{i^2} \cdot \frac{i}{i} = i$$
$$\frac{1}{i^4} = 1$$

and in general, it can be shown that

$$\begin{aligned}\frac{1}{i^{4n+1}} &= -i\\ \frac{1}{i^{4n+2}} &= -1\\ \frac{1}{i^{4n+3}} &= i\\ \frac{1}{i^{4n}} &= 1\end{aligned}$$

So that

$$\left(1+\frac{1}{i}+\frac{1}{i^2}+\cdots+\frac{1}{i^{2014}}\right)^2 = (1-i-1+i+\cdots 1-i-1)^2 = -1$$

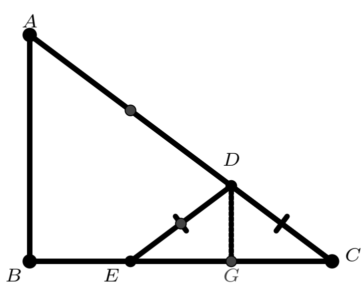
■

4. Find the numerical value of $(1-\cot 37°)(1-\cot 8°)/$

**Solution:** We know that $\cot 45° = 1$. We recall the identity $cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$. Letting $x = 37°$ and $y = 8°$ then $1 = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$ or $\cot x + \cot y = \cot x \cot y - 1$ from which we have $1 = \cot x \cot y - \cot x - \cot y$. Now, $(1-\cot x)(1-\cot y) = 1 \underbrace{-\cot x - \cot y + \cot x \cot y}_{1} = 1+1$. ■

5. Triangle $ABC$ has right angle at $B$, with $AB = 3$ and $BC = 4$. If $D$ and $E$ are points on $AC$ and $BC$, respectively, such that $CD = DE = \frac{5}{3}$, find the perimeter of quadrilateral $ABED$.

**Solution:** Consider the figure below.




$CD$ has length $\frac{5}{3}$ or $\frac{1}{3}$ the length of $CA$. Draw segment $DG \perp BC$. Then $DG$ is $\frac{1}{3}$ of $AB$, or $DG = 1$. Also, $CG = \frac{1}{3} \cdot BC = \frac{4}{3}$ so that $EC = \frac{8}{3}$ and $BE = 4 - \frac{8}{3} = \frac{4}{3}$. Therefore, the perimeter of $ABED$ is $\frac{4}{3} + \frac{5}{3} + \frac{10}{3} + 3 = 9\frac{1}{3}$.

■

6. Rationalize the denominator of $\dfrac{6}{\sqrt[3]{4}+\sqrt[3]{16}+\sqrt[3]{64}}$ and simplify.

**Solution:**

$$\frac{6}{\sqrt[3]{4}+\sqrt[3]{16}+\sqrt[3]{64}} = \frac{6}{\sqrt[3]{4}+2\sqrt[3]{2}+4} \tag{5}$$

Letting $x = 2$ and $y = \sqrt[3]{2}$, we can see that $4 + 2\sqrt[3]{2} + \sqrt[3]{4} = x^2 + xy + y^2$, which we multiply by $(x-y) = 2 - \sqrt[3]{2}$ to get $x^3 - y^3 = 8 - 2 = 6$. Therefore, multiplying (5) by $\dfrac{2-\sqrt[3]{2}}{2-\sqrt[3]{2}}$, we get $\dfrac{6(2-\sqrt[3]{2})}{6} = 2 - \sqrt[3]{2}$. ■

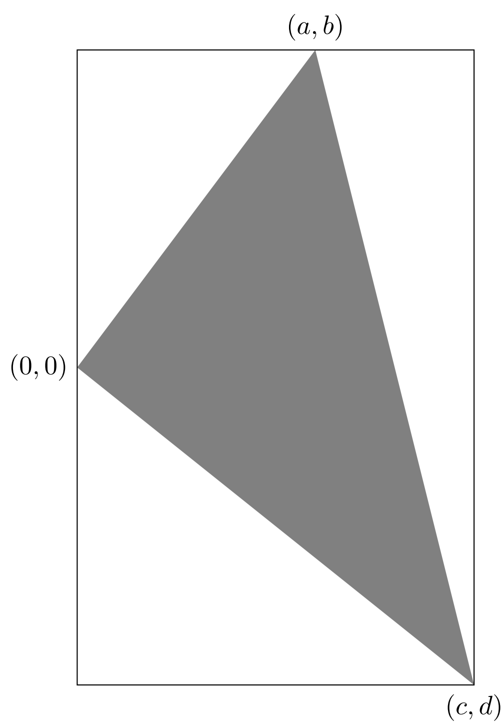
7. Find the area of the triangle having vertices $A(10, -9)$, $B(19, 3)$, and $C(25, -21)$.

**Solution:** You can use the shoelace formula which states that if $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ are the coordinates of the vertices of the triangle, then the area of the triangle is

$$[ABC] = \frac{1}{2}\left|\det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix}\right|. \tag{6}$$

But this is not very intuitive. We can derive a special case of the shoelace formula. The special case states that if one of the vertices is at the origin, and the coordinates of the other two vertices are $(a, b)$ and $(c, d)$, then the area of the triangle is $\frac{1}{2}|ad - bc|$.

To see this, consider the case below. Here, we fixed one of the vertices at $(0, 0)$. The area of the shaded triangle is simply the area of the rectangle minus the areas of the unshaded triangles. It is easy to show that it is equal to $\frac{1}{2}(ad - bc)$ in this case.




To apply this to our current problem, we simply translate the origin to $A(10, -9)$ so that the new coordinates become $A'(0, 0)$, $B'(9, 12)$, and $C'(15, -12)$. Then using the special case of the shoestring formula, we have

$$[ABC] = [A'B'C'] = \frac{1}{2}|(9 \times -12) - (12 \times 15)| = 144.$$

■

8. How many ways can 6 boys and 6 girls be seated in a circle so that no two boys sit next to each other?

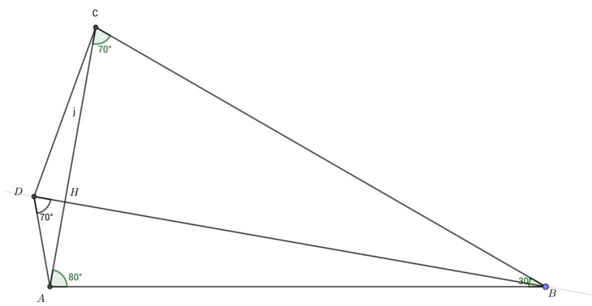
**Solution:** The 6 boys can pair up with the 6 girls in 6! ways in such a way that no two boys (hence no two girls) will sit next to each other. However, each pair can only be arranged 5! ways in a circle. Therefore, there are $6!5! = 86400$ ways that the desired seating arrangement can be done. ■

9. Two numbers $p$ and $q$ are both chosen randomly (and independently of each other) form the interval $[-2, 2]$. Find the probability that $4x^2 + 4px + 1 - q^2 = 0$ has imaginary roots.

**Solution:** The sample space is the square region defined by the diagonal $(-2,-2)$ and $(2,2)$, that is the region $[-2,2] \times [-2,2] = \{(p,q) : p \in [-2,2], q \in [-2,2]\}$, which has an area of 16. The event that $4x^2 + 4px + 1 - q^2 = 0$ will have imaginary roots occurs when the discriminant $b^2 - 4ac = 16p^2 - 16(1-q^2) < 0$, that is when $p^2 + q^2 < 1$, whih is a circular region centered at $(0,0)$, radius 1, and area $\pi$. Therefore, the probablity that $4x^2 + 4px + 1 - q^2 = 0$ will have imaginary roots is equal to the area of the circle over the area of the square or $\frac{\pi}{16}$. ■

10. In $\triangle ABC$, $\angle A = 80°$, $\angle B = 30°$, and $\angle C = 70°$. Let $BH$ be an altitude of the triangle. Extend $BH$ to a point $D$ on the other side of $AC$ so that $BD = BC$. Find $\angle BDA$.

**Solution:** Consider the figure below.




By the isosceles triangle theorem, we have $BD = BC$ and $\angle BDC = \angle BCD$. Since $\angle CHB$ is a right angle, we have $m\angle CBH = 20$. Therefore, $m\angle BDC = m\angle BCD = 80$. Consequently, we have $m\angle HCD = m\angle HBA = 10$. Therefore, by $AA$ theorem, we have $\triangle DHC \sim \triangle AHB$. We will now show that $\triangle DHA \sim \triangle CHB$ and therefore conclude that $m\angle BDA = m\angle HDA = 70$. To show this, we only need to show that corresponding adjacent sides are proportional since the included angles $\angle DHA$ and $\angle CHB$ are already congruent. That is,

$$\frac{DH}{HC} = \frac{HA}{HB} \tag{7}$$

But since $\triangle DHC \sim \triangle AHB$, we have their corresponding sides proportional, in particular

$$\frac{DH}{HA} = \frac{HC}{HB} \tag{8}$$

from which (7) is easily derived. ■

11. Find all integer values of $n$ that will make $\dfrac{6n^3 - n^2 + 2n + 32}{3n+1}$ an integer.

**Solution:** By long division, we have

$$\frac{6n^3 - n^2 + 2n + 32}{3n+1} = 2n^2 - n + 1 + \frac{31}{3n+1} \tag{9}$$

For (9) to be an integer, we must have $(3n+1) \mid 31$. But since 31 is prime, the only divisors of 31 are 1 and itself. That is, either $3n + 1 = 1 \implies n = 0$ or $3n + 1 = 31 \implies n = 10$. Therefore, the desired values are 0 and 10. ■

12. Suppose that the function $y = f(x)$ satisfies $1 - y = \dfrac{9e^x + 2}{12e^x + 3}$. If $m$ and $n$ are consecutive integers so that $m < \dfrac{1}{y} < n$ for all real $x$, find the value of $mn$.

**Solution:** Solving $1 - y = \dfrac{9e^x + 2}{12e^x + 3}$ for $y$, we have

$$y = \frac{3e^x + 1}{12e^x + 3} = \frac{3 + \dfrac{1}{e^x}}{12 + \dfrac{3}{e^x}}. \tag{10}$$

Note that $y \to \frac{1}{4}$ as $x \to +\infty$ or as $x \to -\infty$ and that $y \to \dfrac{4}{15}$ as $x \to 0$. This means that $\dfrac{1}{4} < y < \dfrac{4}{15}$ or $\dfrac{15}{4} < \dfrac{1}{6} < 4$. But $3 < \dfrac{15}{4}$ so that $m = 3$ and $n = 4$. Therefore, $mn = 12$. ∎

13. The product of the two roots of $\sqrt{2014}\, x^{\log_{2014} x} = x^{2014}$ is an integer. Find its units digit.

**Solution:** We have the following derivations

$$\begin{aligned} \sqrt{2014}\, x^{\log_{2014} x} &= x^{2014} \\ \Longleftrightarrow \quad 2014^{1/2} x^{\log_{2014} x} &= x^{2014} \\ \Longleftrightarrow \quad \log_{2014} 2014^{1/2} + \log_{2014} x^{\log_{2014} x} &= \log_{2014} x^{2014} \\ \Longleftrightarrow \quad \frac{1}{2} + (\log_{2014} x)^2 &= 2014 \log_{2014} x \end{aligned}$$

Letting $y = \log_{2014} x$, and completing squares, we easily get

$$\log_{2014} x_{1,2} = 1007 \pm \sqrt{\frac{1007^2 - 2}{2}} \tag{11}$$

Solving (11) for $x$, we get

$$x_{1,2} = 2014^{1007 \pm \sqrt{\frac{1007^2-2}{2}}}. \tag{12}$$

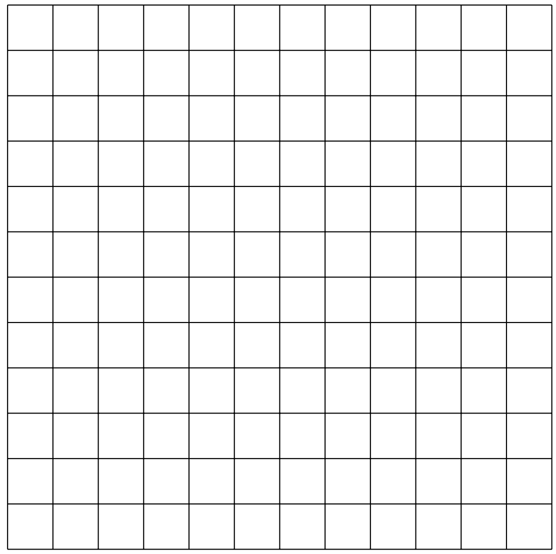
Therefore, the product of the two roots is simply $2014^{2014}$. Since 2014 ends in 4, the odd powers will end in 4 and the even powers will end in 6. Therefore, $2014^{2014}$ will have 6 as its digits place. ∎

14. In how many ways can Alex, Billy, and Charles split 7 identical marbles among themselves so that no two have the same number of marbles? It is possible for someone not to get any marbles.

**Solution:** The only possible combinations of numbers of marbles are: 6, 1, 0; 5, 2, 0; 4, 3, 0; and 4, 2, 1. Each of these combinations are permuted 3! times. Therefore, there are $4 \times 3! = 24$ ways that Alex, Billy and Charles can split the 7 identical marbles. ∎

15. In a Word Finding game, a player tries to find a word in a $12 \times 12$ array of letteres by looking at blocks of adjacent letters that are arranged horizontally, arranged vertically, or arranged diagonally. How many such 3-letter blocks are there in a given $12 \times 12$ array of letters?

**Solution:** There is a missing condition that only left-right, up-down, and top-down diagonal blocks can be considered. Counting the number of blocks for a $12 \times 12$ grid is manageable.



First, we count the horizontal blocks. There are 10 blocks per row, for a total of 120 blocks for 12 rows. Consequently, there are 120 blocks for 12 columns. There are 20 blocks for the two main diagonals. For the smaller diagonals, we have $4(1+2+3+\cdots+9) = 4 \times 9 \times 10 \div 2 = 180$ blocks. Therefore, there are 440 such blocks, all in all. ■

16. Find the largest possible value of

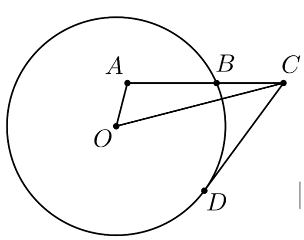
$$(\sin\theta_1)(\cos\theta_2) + (\sin\theta_2)(\cos\theta_3) + \cdots + (\sin\theta_{2013})(\cos\theta_{2014}) + (\sin\theta_{2014})(\cos\theta_1). \tag{13}$$

**Solution:** The largest value of $\sin x \cos y$ is attained when $x = y = \frac{\pi}{4}$. Therefore, the largest possible value of (13) is 1007. ■

17. What is the remainder when $16^{15} - 8^{15} - 4^{15} - 2^{15} - 1^{15}$ is divided by 96.

**Solution:** I don't know if there are some more elegant solutions. I just performed arithmetic (mod 96) to get 31. ■

18. Segment $CD$ is tangent to the circle with center $O$, at $D$. Point $A$ is in the interior of the circle, and segment $AC$ intersects the circle at $B$. If $OA = 2$, and $AB = 4$, $BC = 3$, and $CD = 6$, find the length of segment $OC$.




**Solution:** Wala pa. ■

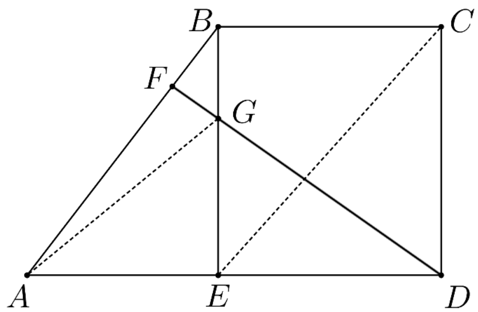
19. Find the maximum value of

$$(1-x)(2-y)(3-z)\left(x + \frac{y}{2} + \frac{z}{3}\right)$$

where $x < 1$, $y < 2$, $z < 3$, and $x + \frac{y}{2} + \frac{z}{3} > 0$.

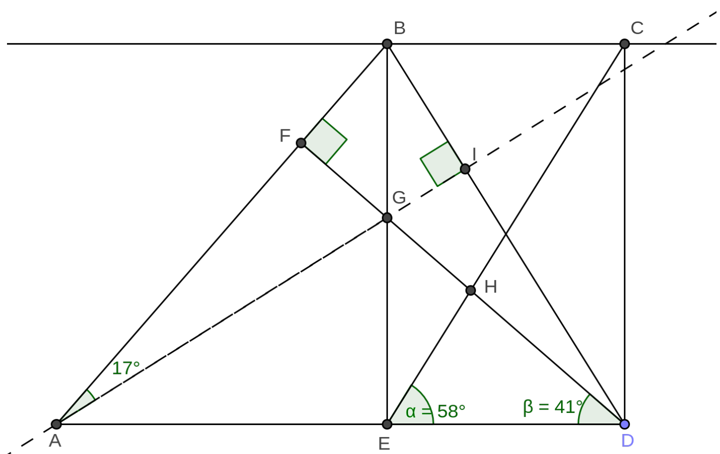
**Solution:** Wala pa. ■

20. Trapezoid $ABCD$ has right angles at $C$ and $D$, and $AD > BC$. Let $E$ and $F$ be the points on $AD$ and $AB$, respectively, such that $\angle BED$ and $\angle DFA$ are right angles. Let $G$ be the point of intersection of segments $BE$ and $DF$. If $\angle CED = 58°$ and $\angle FDE = 41°$, what is $\angle GAB$?




**Solution:** Note that $G$ is the orthocenter. If we draw line segment $BD$, then $AI \perp BD$. $\triangle ECD \simeq DBE$ so $m\angle EBD = m\angle DCE = 90 - 58 = 32$.

$m\angle ABG = m\angle FGB = m\angle EGD = 90 - 49 = 41$. $m\angle FBG = 90 - 32 = 58$. $m\angle ABI = 32 + 41 = 73$. $m\angle GAB = m\angle IAB = 90 - 73 = 17$.

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