Classes of Sets

Set Operations

- Union

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

- Intersection

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

- Difference

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

- Subset

$$A \subset B \text{ if } x \in A \text{ implies } x \in B$$

De Morgan's Identities

$$\left[\bigcup_{i=1}^{\infty} A_i\right]^c = \bigcap_{i=1}^{\infty} A_i^c \text{ and } \left[\bigcap_{i=1}^{\infty} A_i\right]^c = \bigcup_{i=1}^{\infty} A_i^c$$

Distributive Law

$$A \cap \left(\bigcup_{i=1}^{\infty} B_i \right) = \bigcup_{i=1}^{\infty} \left(A \cap B_i \right)$$

$$A \cup \left(\bigcap_{i=1}^{\infty} B_i \right) = \bigcap_{i=1}^{\infty} \left(A \cup B_i \right)$$

Definition 1

Let Ω be a nonempty set. A class $\mathcal{F}$ of subsets of Ω is called an algebra if

i. $\Omega \in \mathcal{A}$,

ii. $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$,

iii. $A, B \in \mathcal{F}$ implies $A \cup B \in \mathcal{F}$.

Definition 2

A class $\mathcal{F}$ of subsets of Ω is called a σ-algebra if $\mathcal{F}$ is an algebra and $A_1, A_2, \ldots \in \mathcal{F}$ implies $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$. The pair $(\Omega, \mathcal{F})$ is called a measurable space. The elements of a σ-algebra $\mathcal{F}$ are called measurable sets.

Example 1

Let $A \subset \Omega$. Then $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$ is a σ-algebra.

Example 2

Let $\mathcal{F}$ consists of the finite and cofinite (A being cofinite if A^c is finite) subsets of a set Ω. Then $\mathcal{F}$ is an algebra. If Ω is finite then $\mathcal{F}$ is also a σ-algebra. If Ω is infinite, however, then $\mathcal{F}$ is not a σ-algebra.

Example 3

Let $\mathcal{F}$ consists of the countable and co-countable (A being co-countable if A^c is countable) subsets of a set Ω. Then $\mathcal{F}$ is a σ-algebra.

Definition 3

Let $\mathcal{C}$ be a class of subsets of Ω. Then $\sigma(\mathcal{C})$, the σ-algebra generated by $\mathcal{C}$, is the smallest σ-algebra $\mathcal{F}$ on Ω such that $\mathcal{C} \subseteq \mathcal{F}$.

Theorem 1

The σ-algebra generated by $\mathcal{C}$ is the intersection of all σ-algebras on Ω containing $\mathcal{C}$.

Example 4

The Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ is defined as the smallest σ-algebra containing the open sets in $\mathbb{R}$. An open set in $\mathbb{R}$ is either empty or a disjoint union of a countable collection of open intervals. This σ-algebra is generated by the collection $\pi(\mathbb{R})$ defined by

$$\pi(\mathbb{R}) = \{(-\infty, x]: x \in \mathbb{R}\}.$$

The elements of $\mathcal{B}$ are called Borel sets.

Definition 4

Let Ω be a nonempty set. A collection $\mathcal{J}$ of subsets of Ω is called a π-system if whenever $A, B \in \mathcal{J}$ we have $A \cap B \in \mathcal{J}$. A collection $\mathcal{D}$ of subsets of Ω is called a d-system if

i. $\Omega \in \mathcal{D}$,

ii. $A, B \in \mathcal{D}$ and $A \subseteq B$ implies $B \setminus A \in \mathcal{D}$,

iii. $A_1, A_2, \ldots \in \mathcal{D}$ and $A_1 \subset A_2 \subset \cdots$ implies $\bigcup_{i=1}^{\infty} A_i \in \mathcal{D}$.

Theorem 2

A collection $\mathcal{F}$ of subsets of Ω is a σ-algebra if and only if $\mathcal{F}$ is both a π-system and a d-system.

Theorem 3

If $\mathcal{I}$ is a π-system and $\mathcal{D}$ is a d-system with $\mathcal{I} \subset \mathcal{D}$, then $\sigma(\mathcal{I}) \subset \mathcal{D}$.