Probability Spaces, Random Variables and Distribution Functions

Definition 1

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A measurable function $X\colon \Omega \to \mathbb{R}$ is called a random variable. The law of the random variable is the probability measure μ on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu(A) = \mathbb{P}(X \in A),\ \ A \in \mathcal{B}.$$

The distribution function of X is defined by

$$F(x) = \mu((-\infty, x]) = \mathbb{P}(X \leq x)$$

for real x.

Theorem 1

Let F be a distribution function of some random variable X. Then

a) $F(x) \leq F(y)$ whenever $x \leq y$,

b) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$,

c) F is right-continuous

Proof

a) If $x \leq y$ then $(-\infty, x] \subseteq (-\infty, y]$ and it follows that $F(x) = \mu((-\infty, x]) \leq \mu\big((-\infty, y]\big) = F(y)$.

b) Note that $\bigcap_{n=1}^{\infty}(-\infty, -n] = \emptyset$ so that

$$\lim_{x \to -\infty} F(x) = \mu\left(\bigcap_{n=1}^{\infty}(-\infty, -n]\right) = 0.$$

Similarly, $\bigcup_{n=1}^{\infty}(-\infty, n] = \mathbb{R}$ and

$$\lim_{x \to +\infty} F(x) = \mu\left(\bigcup_{n=1}^{\infty}(-\infty, n]\right) = 1.$$

c) For right-continuity, note that

$$\lim_{n \to \infty} F\left(x + \frac{1}{n}\right) = \mu\left(\bigcap_{n=1}^{\infty}\left(-\infty, x + \frac{1}{n}\right]\right) = F(x).$$

Theorem 2

If F is a nondecreasing, right-continuous function with $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$, then there exists on some probability space a random variable X for which $F(x) = P(X \leq x)$.

Proof

Define the random variable X on $([0,1], \mathcal{B}[0,1], \lambda)$ by $X(\omega) = \inf\{z: F(z) \geq \omega\}$. Now, if $z > X(\omega)$ then $F(z) \geq \omega$, so by right-continuity, $F\big(X(\omega)\big) \geq \omega$. If, in addition $X(\omega) \leq c$ then $\omega \leq F\big(X(\omega)\big) \leq F(c)$. Thus $\omega \leq F(c)$ if and only if $X(\omega) \leq c$, so that $P(X \leq c) = \lambda([0, F(c)]) = F(c)$.