Measurable Functions and Mappings

Definition 1

Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two measurable spaces. For a mapping $T: \Omega \to \Omega'$ and $A' \subseteq \Omega'$, define

$$T^{-1}(A') = \{\omega \in \Omega : T(\omega) \in A'\}.$$

Then the mapping T is called $\mathcal{F}/\mathcal{F}'$-measurable if $T^{-1}(A') \in \mathcal{F}$ for every $A' \in \mathcal{F}'$.

Theorem 1

The map T^{-1} preserves set operations:

a) $T^{-1}(\bigcup_\alpha A_\alpha) = \bigcup_\alpha T^{-1}(A_\alpha)$

b) $T^{-1}(\bigcap_\alpha A_\alpha) = \bigcap_\alpha T^{-1}(A_\alpha)$

c) $T^{-1}(A^c) = (T^{-1}(A))^c$

Proof

a) $\omega \in T^{-1}(\bigcup_\alpha A_\alpha)$ if and only if $T(\omega) \in A_\alpha$ for some α. This is equivalent to $\omega \in T^{-1}(A_\alpha)$ for some α, i.e., $\omega \in \bigcup_\alpha T^{-1}(A_\alpha)$.

b) Similar to (a) with union replaced by intersection.

c) $\omega \in T^{-1}(A^c)$ if and only if $T(\omega) \notin A$. This is equivalent to $\omega \notin T^{-1}(A)$, i.e., $\omega \in (T^{-1}(A))^c$.

Theorem 2

Let T be a mapping from the measurable space $(\Omega, \mathcal{F})$ into the measurable space $(\Omega', \mathcal{F}')$.
Then $T^{-1}(\mathcal{F}') = \{T^{-1}(A') : A' \in \mathcal{F}'\}$ is a σ-algebra on Ω, whereas $\{A' \subseteq \Omega' : T^{-1}(A') \in \mathcal{F}\}$ is a σ-algebra on Ω'.

Proof

$\Omega = T^{-1}(\Omega')$ so that $\Omega \in T^{-1}(\mathcal{F}')$. Since $(T^{-1}(A'))^c = T^{-1}(A'^c)$ then $T^{-1}(\mathcal{F}')$ is closed under complementation. Lastly, $\bigcup_{n=1}^{\infty} T^{-1}(A'_n) = T^{-1}(\bigcup_{n=1}^{\infty} A'_n)$. Hence, $T^{-1}(\mathcal{F}')$ is a σ-algebra. Now, let $\mathcal{G} = \{A' \subseteq \Omega' : T^{-1}(A') \in \mathcal{F}\}$. $T^{-1}(\Omega') = \Omega \in \mathcal{F}$ so that $\Omega' \in \mathcal{G}$. $T^{-1}(A'^c) = (T^{-1}(A'))^c \in \mathcal{F}$ whenever $A' \in \mathcal{G}$. Hence, $\mathcal{G}$ is closed under complementation. Finally, $T^{-1}(\bigcup A'_n) = \bigcup T^{-1}(A'_n) \in \mathcal{F}$ whenever $A'_n \in \mathcal{G}$. Therefore $\mathcal{G}$ is a σ-algebra.

Theorem 3

If $\mathcal{C} \subseteq \mathcal{F}', \sigma(\mathcal{C}) = \mathcal{F}'$ and $T^{-1}(A') \in \mathcal{F}$ for every $A' \in \mathcal{C}$, then T is $\mathcal{F}/\mathcal{F}'$-measurable.

Proof

Let $\mathcal{G} = \{A' \in \mathcal{F}' : T^{-1}(A') \in \mathcal{F}\}$. Then $\mathcal{G}$ is a σ-algebra containing $\mathcal{C}$ and, hence, $\mathcal{G} = \mathcal{F}'$.

Theorem 4

If $(\Omega, \mathcal{F}), (\Omega', \mathcal{F}')$ and $(\Omega'', \mathcal{F}'')$ are measurable spaces, and if T is $\mathcal{F}/\mathcal{F}'$-measurable and T' is $\mathcal{F}'/\mathcal{F}''$-measurable, then $T' \circ T$ is $\mathcal{F}/\mathcal{F}''$-measurable.

Proof

Let $A'' \in \mathcal{F}''$. It follows that

$$(T' \circ T)^{-1}(A'') = T^{-1}(T'^{-1}(A'')) \in \mathcal{F}$$

since $T'^{-1}(A'') \in \mathcal{F}'$.

Definition 2

Let $(\Omega, \mathcal{F})$ be a measurable space. A function $f: \Omega \to \mathbb{R}$ is called measurable if $f^{-1}(B) \in \mathcal{F}$ for every Borel set B.

Theorem 5

The function $f: \Omega \to \mathbb{R}$ is measurable if and only if

$$\{f \leq c\} \coloneqq \{\omega \in \Omega : f(\omega) \leq c\} \in \mathcal{F}$$

for all $c \in \mathbb{R}$.

Proof

The class $\{(-\infty, c] : c \in \mathbb{R}\}$ generates the Borel σ-algebra. Also, note that $\{f \leq c\} = f^{-1}((-\infty, c])$. The result then follows from Theorem 3.

Theorem 6

Let $\lambda \in \mathbb{R}$ and f, f_1, f_2 be measurable. Then

$$f_1 + f_2 \,,\, f_1 f_2 \text{ and } \lambda f$$

are measurable.

Proof

Note that

$$\{f_1 + f_2 > c\} = \bigcup_{r \in \mathbb{Q}} (\{f_1 > r\} \cap \{f_2 > c - r\}) \ .$$

Hence, $f_1 + f_2$ is measurable. If $\lambda = 0$, then $\{\lambda f \leq c\}$ is either $\emptyset$ (if $c < 0$) or Ω (if $c \geq 0$). Hence λf is measurable. If $\lambda \neq 0$, note that

$$\{\lambda f > c\} = \{f > c / \lambda\} \text{ if } \lambda > 0$$

and

$$\{\lambda f > c\} = \{f < c / \lambda\} \text{ if } \lambda < 0 \, .$$

This shows λf is measurable.

To show that $f_1 f_2$ is measurable, first note that

$$\{f^2 > c\} = \{f > \sqrt{c}\} \cup \{f < -\sqrt{c}\}.$$

This shows that f^2 is measurable whenever f is measurable. Observe that

$$f_1 f_2 = \frac{1}{2}[(f_1 + f_2)^2 - f_1^2 - f_2^2]$$

and, hence, $f_1 f_2$ is measurable.

Theorem 7

Let $\{f_n : n \in \mathbb{N}\}$ be a sequence of measurable functions. Then

$$\inf f_n, \liminf f_n \text{ and } \limsup f_n$$

are measurable (into $([-\infty, +\infty], \mathcal{B}[-\infty, +\infty])$.

Further $\{\omega \in \Omega : \lim f_n(\omega) \text{ exists}\} \in \mathcal{F}$.

Proof

Note that $\{\inf f_n > c\} = \bigcap_n \{f_n > c\} \in \mathcal{F}$. Hence, $\inf f_n$ is measurable. Let $g_n = \inf_{k \geq n} f_k$. Then

$$\{\liminf f_n \leq c\} = \{\sup g_n \leq c\} = \bigcap \{g_n \leq c\} \in \mathcal{F}.$$

Hence, $\liminf f_n$ is measurable. Now, $\limsup f_n = -\liminf -f_n$ and so $\limsup f_n$ is measurable. Finally,

$$\begin{aligned}\{\lim f_n \text{ exists}\} &= \{\limsup f_n < \infty\} \cap \{\liminf f_n > -\infty\} \\ &\cap g^{-1}(\{0\}),\end{aligned}$$

where $g = \limsup f_n - \liminf f_n$. This is in $\mathcal{F}$.