Measures

Definition 1

A set function on an algebra $\mathcal{F}$ in Ω is a measure if

i. $\mu(A) \in [0, \infty]$ *for* $A \in \mathcal{F}$,

ii. $\mu(\emptyset) = 0$,

iii. *if* $\{A_n\}$ *is a disjoint collection of sets in* $\mathcal{F}$ *with* $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ *then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \ .$$

If μ is a measure on a σ-algebra $\mathcal{F}$ then the triple $(\Omega, \mathcal{F}, \mu)$ is called a measure space.

Definition 2

Let μ be a measure on $\mathcal{F}$. Then μ is called

i. *finite if* $\mu(\Omega) < \infty$,

ii. *σ-finite if there is a sequence* $\{A_n\}$ *of elements of* $\mathcal{F}$ *such that* $\mu(A_n) < \infty$ *for all n and* $\bigcup A_n = \Omega$,

iii. *a probability measure if* $\mu(\Omega) = 1$.

Example 1

A measure μ on $(\Omega, \mathcal{F})$ is discrete if there are countably many points ω_i in Ω and numbers m_i in $[0, \infty]$ such that $\mu(A) = \sum_{\omega_i \in A} m_i$ for A in $\mathcal{F}$. If $\mathcal{F}$ contains each singleton $\{\omega_i\}$, then μ is σ-finite if and only if $m_i < \infty$ for all i.

Example 2

Let $\mathcal{F}$ be the σ-algebra of all subsets of Ω, and let $\mu(A)$ be the number of elements in A, where $\mu(A) = \infty$ if A is not finite. This μ is counting measure; it is finite if and only if Ω is finite and is σ-finite if and only if Ω is countable.

Example 3

Let $\Omega = (0,1]$. For $A \subset \Omega$, say that $A \in \mathcal{F}$ if it may be written as a finite union

$$A = (a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n]$$

where $0 \le a_1 \le b_1 \le \cdots \le a_n \le b_n \le 1$. Then $\mathcal{F}$ is an algebra on $(0,1]$. For $A \subseteq \mathcal{F}$ define

$$\mu(A) = \sum_{k=1}^{n} (b_k - a_k)\ .$$

Then μ is a measure on $\mathcal{F}$.

Theorem

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Then

i. $\mu(A) \leq \mu(B)$ *if* $A \subseteq B$

ii. $\mu(\bigcup_{i \leq n} A_i) \leq \sum_{i \leq n} \mu(A_i) \quad (A_1, \ldots, A_n \in \mathcal{F})$.

Furthermore, if $\mu(\Omega) < \infty$, then

iii. *for* $A_1, A_2, \ldots, A_n$ *in* $\mathcal{F}$,

$$\mu\left(\bigcup_{i \leq n} A_i\right) = \sum_{i \leq n} \mu(A_i) - \sum\sum_{i<j \leq n} \mu\left(A_i \cap A_j\right)$$
$$+ \sum\sum\sum_{i<j<k \leq n} \mu\left(A_i \cap A_j \cap A_k\right) -$$
$$\cdots + (-1)^{n-1} \mu\left(A_1 \cap A_2 \cap \cdots \cap A_n\right)$$