Limit Theorems on Integrals

Theorem 1

Monotone Convergence Theorem

If $\{f_n\}$ is an increasing sequence of nonnegative measurable functions such that $\lim_{n\to\infty} f_n = f$, then $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$.

Proof

For each n, take increasing simple functions $\{f_{mn}\}$ with $\lim_{m\to\infty} f_{mn} = f_n$. Define $g_n = \max(f_{1n}, f_{2n}, \dots, f_{nn})$. Then $\{g_n\}$ is an increasing sequence of simple functions with $\lim_{n\to\infty} g_n = f$. It follows that $\lim_{n\to\infty} \int g_n d\mu = \int f d\mu$. Since $g_n \leq f_n \leq f$, we get $\int f d\mu = \lim_{n\to\infty} \int g_n \leq \lim_{n\to\infty} \int f_n d\mu \leq \int f d\mu$. This implies that $\lim_{n\to\infty} \int f_n \, d\mu = \int f d\mu$.

Lemma 1

Let f be a nonnegative measurable function and $N \in \mathcal{F}$ with $\mu(N) = 0$. Then $\int f I_N \, d\mu = 0$.

Proof

Let $\{f_n\}$ be a sequence of simple functions with $\lim_{n\to\infty} f_n = f$. Then $\{f_n I_N\}$ is an increasing sequence of simple functions with $\lim_{n\to\infty} f_n I_N = f I_N$. If $f_n = \sum a_{ni} I_{A_{ni}}$ then $f_n I_N = \sum a_{ni} I_{A_{ni}\cap N}$. This implies that $\int f_n I_N \, d\mu = \sum a_{ni}\mu(A_{ni} \cap N) = 0$ since $\mu(A_{ni} \cap N) \le \mu(N) = 0$. Therefore $\int f I_N d\mu = \lim_{n\to\infty} \int f_n I_N d\mu = 0$.

Theorem 2

a) If f and g are nonnegative measurable functions and $f = g$ a.e., then $\int f \, d\mu = \int g \, d\mu$.

b) If $\{f_n\}$ and f are nonnegative measurable functions with $\{f_n\}$ increasing and $\lim_{n\to\infty} f_n = f$ a.e. . Then $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$.

Proof

a) Let

$f_n = \sum_{j=0}^{n2^n-1} j2^{-n} I_{f^{-1}((j2^{-n},(j+1)2^{-n}])} + nI_{f^{-1}((n,\infty])}$

and

$g_n = \sum_{j=0}^{n2^n-1} j2^{-n} I_{g^{-1}((j2^{-n},(j+1)2^{-n}])} + nI_{g^{-1}((n,\infty])}$.

Then $f_n = g_n$ a.e. and so $\int f_n \, d\mu = \int g_n \, d\mu$. The result now follows by monotone convergence.

b) Let $N = \left\{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) \neq f(\omega)\right\}$. Then $\lim_{n\to\infty} f_n I_{N^c} = f I_{N^c}$. From this it follows that

$$\lim_{n\to\infty} \int f_n d\mu = \lim_{n\to\infty} \int f_n I_{N^c} d\mu$$

$$= \int f I_{N^c}\, d\mu = \int f\, d\mu.$$

Lemma 2 (Fatou Lemma)

For a sequence $\{f_n\}$ nonnegative measurable functions

$$\int \liminf f_n d\mu \leq \liminf \int f_n d\mu \ .$$

Proof

Let $g_n = \inf_{k \geq n} f_k$. Then $\{g_n\}$ is an increasing sequence with $\lim\limits_{n\to\infty} g_n = \liminf f_n$. For $k \geq n$, we have $f_k \geq g_n$, so that $\int f_k \, d\mu \geq \int g_n \, d\mu$. This implies that $\inf_{k \geq n} \int f_k \, d\mu \geq \int g_n \, d\mu$. Thus

$\int \liminf f_n \, d\mu \leq \lim\limits_{n} \inf\limits_{k \geq n} \int f_k \, d\mu = \liminf \int f_n \, d\mu$.

Lemma 3 (Reverse Fatou)

If $\{f_n\}$ is a sequence of nonnegative measurable functions such that for some nonnegative measurable function g, we have $f_n \leq g$ for all n and $\int g\, d\mu$, then

$$\int \limsup f_n d\mu \geq \limsup \int f_n d\mu \ .$$

Proof

For each n, let $h_n = g - f_n$. Then $\{h_n\}$ is a sequence of nonnegative measurable functions and by Fatou lemma we have

$$\begin{aligned}\int g\,d\mu - \limsup \int f_n\,d\mu &= \liminf \int h_n\,d\mu \\ &\geq \int \liminf h_n\,d\mu \\ &= \int g\,d\mu - \int \limsup f_n\,d\mu\,.\end{aligned}$$

Subtracting $\int g d\mu$ on both sides of the inequality gives the result.

Theorem 3

Dominated Convergence Theorem

Suppose that $\{f_n\}, f$ are measurable functions, that $\lim_{n\to\infty} f_n = f$ on Ω and that $|f_n| \leq g$ for some integrable function g. Then

$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0$$

and, hence, $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$.

Proof

Note that $|f_n - f| \leq 2g$ and so by reverse Fatou lemma

$$\limsup \int |f_n - f| d\mu \leq \int \limsup |f_n - f| = 0.$$

Consequently,

$$\lim_{n\to\infty} \left| \int f_n d\mu - \int f d\mu \right| = \lim_{n\to\infty} \left| \int (f_n - f) d\mu \right|$$

$$\leq \lim_{n\to\infty} \int |f_n - f| d\mu = 0\,.$$

Corollary

Bounded Convergence Theorem

Suppose that $\{f_n\}, f$ are measurable functions, that $\lim_{n\to\infty} f_n = f$ on Ω and that $|f_n| \le M$ for some real number $M < \infty$. Then $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$.

Theorem 4

a) If $f_n \geq 0$ for all n, then $\int \sum_n f_n \, d\mu = \sum_n \int f_n d\mu$.

b) If $\sum_n f_n$ converges a.e. and $|\sum_{k=1}^{n} f_k| \leq g$ a.e., where g is integrable, then $\sum_n f_n$ and the f_n are integrable and $\int \sum_n f_n \, d\mu = \sum_n \int f_n \, d\mu$.

Proof

a) Let $g_k = \sum_{n=1}^{k} f_n$. Then $\{g_k\}$ is an increasing sequence of nonnegative functions with $\lim_{k\to\infty} g_k = \sum f_n$. It follows from the Monotone Convergence theorem that

$$\int \sum f_n d\mu = \lim \int g_k d\mu = \lim_{k\to\infty} \sum_{n=1}^{k} \int f_n d\mu = \sum \int f_n d\mu .$$

b) Same proof as above except that it follows from Dominated instead of Monotone Convergence.