Stat 235
Exercise 1

1. If $\mathcal{F}$ is a σ-algebra generated by a countable collection of disjoint sets $\{\Lambda_n\}$, such that $\bigcup_n \Lambda_n = \Omega$, show that each member of $\mathcal{F}$ is just the union of countable subcollection of these Λ_n's.

2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and A an element of $\mathcal{F}$. Define $\mathcal{G}$ to be the collection of sets in $\mathcal{F}$ that are subsets of A and $\mu_0: \mathcal{G} \to [0, \infty]$ with $\mu_0(E) = \mu(E)$ for $E \in \mathcal{G}$. Show that $(A, \mathcal{G}, \mu_0)$ is a measure space.

3. Suppose that μ_1 and μ_2 are finite measures on a σ-algebra $\mathcal{F}$ generated by the algebra $\mathcal{A}$. Show that, if $\mu_1(A) \leq \mu_2(A)$ holds for all A in $\mathcal{A}$, then it holds for all A in $\mathcal{F}$.

4. Let $(\Omega_n, \mathcal{F}_n, \mu_n)$ be measure spaces, $n = 1,2, \ldots$, and suppose that the Ω_n are disjoint. Let $\Omega = \bigcup_n \Omega_n$, let $\mathcal{F}$ consist of the sets of the form $A = \bigcup_n A_n$ with $A_n \in \mathcal{F}_n$ and for such an A let $\mu(A) = \sum_n \mu_n(A_n)$. Show that $(\Omega, \mathcal{F}, \mu)$ is a measure space.

5. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space with $\mu(\Omega) = \infty$. Show that for any $M < \infty$ there is some $A \in \mathcal{F}$ with $M < \mu(A) < \infty$.

6. Define for a sequence of numbers $\{x_n\}$ the quantity $\liminf x_n$ by $\liminf x_n = \sup_n \inf_{k \geq n} x_k$ and for a sequence of sets $\{A_n\}$ the set $\liminf A_n$ by $\liminf A_n = \bigcup_n \bigcap_{k \geq n} A_k$. Show that $\mu\left(\liminf A_n\right) \leq \liminf \mu(A_n)$.

7. If $\mu^*$is an outer measure and $\nu^*(E) = \mu^*(E \cap A)$, show that ν^* is monotone ($E \subset F$ implies $\nu^*(E) \leq \nu^*(F)$) and is countably subadditive.