Stat 235 Exercise 1

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February 19, 2016

Theorem ($\star$)**.** *If $\mathcal{J}$ is a π-system and $\mathcal{D}$ is a d-system with $\mathcal{J} \subset \mathcal{D}$, then $\sigma(\mathcal{J}) \subset \mathcal{D}$.*

Note(s) 1. *$\sigma(\mathcal{J}) \subset \mathcal{D}$ means that $\mathcal{D}$ contains the smallest σ-algebra contained in $\mathcal{J}$.*

Proof. To show $\sigma(\mathcal{J}) \subset \mathcal{D}$, we need only show that $\mathcal{D}$ is a π-system. Why? If $\mathcal{D}$ is a π-system, then, since it is also a d-system, then it is a σ-algebra. $\sigma(\mathcal{J})$ and $\mathcal{D}$ both being σ-algebras and $\sigma(\mathcal{J})$ being the smallest σ-algebra containing $\mathcal{J}$ means $\sigma(\mathcal{J}) \subset \mathcal{D}$.

Now without loss of generality, we can also let $\mathcal{D}$ be the smallest such d-system that contains $\mathcal{J}$. If it holds that $\sigma(\mathcal{J}) \subset \mathcal{D}$ for this $\mathcal{D}$ then it will hold for other bigger $\mathcal{D}$'s.

Consider $\mathcal{F} = \{A \in \mathcal{D} \mid A \cap B \in \mathcal{D} \text{ for all } B \in \mathcal{J}\}$. We need to show that $\mathcal{J} \subset \mathcal{F}$ and $\mathcal{F}$ is a d-system.

Note than since $\mathcal{J}$ is a π-system, for $A \in \mathcal{J}$ then $A \cap B \in \mathcal{D}$ for all $B \in \mathcal{J}$. Hence, $A \in \mathcal{F}$ and $\mathcal{J} \subset \mathcal{F}$.

We now proceed to show that $\mathcal{F}$ is a d-system.

(i) Now $\Omega \cap B = B \in \mathcal{J}$ for all $B \in \mathcal{J}$ and hence $\Omega \in \mathcal{F}$.

(ii) Moreover, if $A, C \in \mathcal{F}$ and $A \subset C$ then for any $B \in \mathcal{J}$, $(C \setminus A) \cap B = (C \cap B) \setminus (A \cap B)$. Now $C \cap B \in \mathcal{D}$ and $A \cap B\mathcal{D}$ by definition of $\mathcal{F}$. So $(C \setminus A) \cap B \in \mathcal{D}$ since $\mathcal{D}$ is a d-system. Thus $C \setminus A$ satisfies the condition of $\mathcal{F}$ so $C \setminus A \in \mathcal{F}$.

(iii) $A_1, A_2, \ldots \in \mathcal{F}$ and $A_1 \subset A_2 \subset \cdots$. Then for any $B \in \mathcal{J}$, $(\bigcup_{i=1}^{\infty} A_i) \cap B = \bigcup_{i=1}^{\infty}(A_i \cap B)$. Now $A_i \cap B \in \mathcal{D}$ for all i since $A_i \in \mathcal{F}$ and $(A_1 \cap B) \subset (A_2 \cap B) \subset (A_3 \cap B) \subset \cdots$. This means that $(\bigcup_{i=1}^{\infty} A_i) \cap B = \bigcup_{i=1}^{\infty}(A_i \cap B) \in \mathcal{D}$, since $\mathcal{D}$ is a d-system. Therefore, $(\bigcup_{i=1}^{\infty} A_i) \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a d-system.

Now since $\mathcal{D}$ is the smallest d-system containing $\mathcal{J}$, and $\mathcal{J} \subset \mathcal{D}$ it follows that $\mathcal{F} = \mathcal{D}$. Therefore, if $A \in \mathcal{D}$ and $B \in \mathcal{J}$, then $A \cap B \in \mathcal{D}$.

Now define $\mathcal{G} = \{B \in \mathcal{D} \mid A \cap B \in \mathcal{D} \text{ for all } A \in \mathcal{D}\}$. Note that $\mathcal{J} \subset \mathcal{G}$. We can show in a similar fashion that $\mathcal{G}$ is a d-system (see Exercise 1). Hence, $\mathcal{G} = \mathcal{D}$. Therefore, if $A, B \in \mathcal{D}$ then $A \cap B \in \mathcal{D}$ and hence $\mathcal{D}$ is a π-system. Now $\mathcal{D}$ is then both a π-system and a d-system, and, whence, a σ-algebra. Therefore, $\sigma(\mathcal{J}) \subset \mathcal{D}$. $\square$

Exercises

1. Show that $\mathcal{G} = \{B \in \mathcal{D} \mid A \cap B \in \mathcal{D} \text{ for all } A \in \mathcal{D}\}$ in the proof of Theorem ($\star$) is a d-system.

   Proof. (i) Now $\Omega \in \mathcal{D}$ since $\mathcal{D}$ is a d-system. Let $A \in \mathcal{D}$. Now, $\Omega \cap A = A \in \mathcal{D}$. Therefore, $\Omega \in \mathcal{G}$.

   (ii) Moreover, if $B, C \in \mathcal{G}$ and $B \subset C$ then for any $A \in \mathcal{D}$, $(C \setminus B) \cap A = (A \cap C) \setminus (A \cap B)$. Now $A \cap C \in \mathcal{D}$ and $A \cap B \in \mathcal{D}$ by definition of $\mathcal{G}$. So $(C \setminus A) \cap B \in \mathcal{D}$ since $\mathcal{D}$ is a d-system. Thus $C \setminus A$ satisfies the condition of $\mathcal{G}$ so $C \setminus A \in \mathcal{G}$.

(*iii*) Suppose $B_1, B_2, \ldots \in \mathcal{G}$ and $B_1 \subset B_2 \subset \cdots$. Then for any arbitrary $A \in \mathcal{D}$, $(\bigcup_{i=1}^{\infty} B_i) \cap A = \bigcup_{i=1}^{\infty}(B_i \cap A)$. Now $B_i \cap A \in \mathcal{D}$ for all i since $B_i \in \mathcal{G}$ and $(B_1 \cap A) \subset (B_2 \cap A) \subset (B_3 \cap A) \subset \cdots$. This means that $(\bigcup_{i=1}^{\infty} B_i) \cap A = \bigcup_{i=1}^{\infty}(B_i \cap A) \in \mathcal{D}$, since $\mathcal{D}$ is a d-system. Therefore, $(\bigcup_{i=1}^{\infty} B_i) \in \mathcal{G}$. Therefore, $\mathcal{G}$ is a d-system.

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