Continuity and Uniqueness

Theorem 1

Let μ be a measure on a σ-algebra $\mathcal{F}$.

i. *If* $\{A_n\}$ *is an increasing sequence in* $\mathcal{F}$ *with* $A = \bigcup A_n$ *then* $\lim_{n\to\infty} \mu(A_n) = \mu(A)$.

ii. *If* $\{A_n\}$ *is a decreasing sequence in* $\mathcal{F}$ *with* $A = \bigcap A_n$ *and* $\mu(A_k) < \infty$ *for some* k *then* $\lim_{n\to\infty} \mu(A_n) = \mu(A)$.

iii. *If* $\{A_n\}$ *is a sequence in* $\mathcal{F}$, *then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n) \ .$$

Proof

Let $B_1 = A_1, B_n = A_n - A_{n-1}, n > 1$. Then the B_n's are disjoint with $\bigcup B_n = \bigcup A_n$. Hence

$$\mu(A) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu(A_n).$$

For (*ii*), let $B_n = A_k \backslash A_n$, $n > k$. Then $\{B_n\}_{n>k}$ is an increasing sequence and

$$\mu(A_k) - \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(B_n) = \mu(A_k) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right).$$

For (iii), let $B_n = \bigcup_{i=1}^{n} A_i$ then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(B_n) \le \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) = \sum_{n=1}^{\infty} \mu(A_n).$$

Theorem 2

Let Ω be a set. Let $\mathcal{I}$ be a π-system on Ω. Let $\mathcal{F} = \sigma(\mathcal{I})$. Suppose that μ_1 and μ_2 are measures on $(\Omega, \mathcal{F})$ such that $\mu_1(\Omega) = \mu_2(\Omega) < \infty$ and $\mu_1 = \mu_2$ on $\mathcal{I}$. Then $\mu_1 = \mu_2$ on $\mathcal{F}$.

Proof

Let

$$\mathcal{G} = \{A \in \mathcal{F} : \; \mu_1(A) = \mu_2(A)\}\,.$$

Then $\mathcal{G}$ is a *d*-system containing the π-system containing $\mathcal{J}$. Hence, $\mathcal{F} = \sigma(\mathcal{J}) \subset \mathcal{G}$.

Corollary

If two probability measures agree on a π-system, then they agree on the σ-algebra generated by that π-system.