Product Measure

Definition 1

Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be any two measure spaces. In $\Omega_1 \times \Omega_2$ let $\mathcal{R}$ be the collection of all rectangles $B \times C$ with $B \in \mathcal{F}_1$ and $C \in \mathcal{F}_2$. The product σ-algebra $\mathcal{F}_1 \times \mathcal{F}_2$ is the smallest σ-algebra on $\Omega_1 \times \Omega_2$ containing $\mathcal{R}$.

Notation

Throughout, we will assume that μ_1 and μ_2 are finite measures. Also, we let

$$\Omega = \Omega_1 \times \Omega_2$$

$$\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$$

$$I_1^f(\omega_1) = \int f(\omega_1,\cdot)d\mu_2$$

$$I_2^f(\omega_2) = \int f(\cdot,\omega_2)d\mu_1 \ .$$

Lemma 1

Let f be a bounded $\mathcal{F}$-measurable function on Ω then

a) for each $\omega_1 \in \Omega_1$, the map $f(\omega_1,\cdot): \omega_2 \mapsto f(\omega_1, \omega_2)$ is $\mathcal{F}_2$-measurable on Ω_2 , and

b) for each $\omega_2 \in \Omega_2$, the map $f(\cdot, \omega_2): \omega_1 \longmapsto f(\omega_1, \omega_2)$ is $\mathcal{F}_1$- measurable on Ω_1 .

Proof

Let $g: \omega_2 \mapsto (\omega_1, \omega_2)$ and note that $f(\omega_1, \cdot) = f \circ g$. Now f is $\mathcal{F}$-measurable and g is $\mathcal{F}_2/\mathcal{F}$-measurable. Hence $f(\omega_1, \cdot)$ is $\mathcal{F}_2$-measurable. Similarly, we can show that $f(\cdot, \omega_2)$ is $\mathcal{F}_1$-measurable.

Lemma 2

Let f be a bounded $\mathcal{F}$-measurable function on Ω then

a) the function I_1^f is $\mathcal{F}_1$-measurable,

b) the function I_2^f is $\mathcal{F}_2$-measurable, and

c) $\int I_1^f \, d\mu_1 = \int I_2^f \, d\mu_2$.

Proof

Let $f = I_{A\times B}$. Then $I_1^f(\omega_1) = I_A(\omega_1)\mu_2(B)$ which is clearly $\mathcal{F}_1$-measurable. Define

$$\mathcal{G} = \left\{F \in \mathcal{F}: I_1^f \text{ is } \mathcal{F}_1\text{-measurable}, f = I_F\right\}.$$

Then $\mathcal{G}$ is a σ-algebra containing $\mathcal{R}$. If follows that I_1^f is $\mathcal{F}_1$-measurable for $f = I_F, F \in \mathcal{F}$. That I_1^f is $\mathcal{F}_1$-measurable for simple function f follows from linearity of the integral and that it is $\mathcal{F}_1$-measurable for nonnegative f follows from MCT. Finally that I_1^f is $\mathcal{F}_1$-measurable for bounded measurable f follows from measurability of $I_1^{f^+}$ and $I_1^{f^-}$. The measurability of I_2^f is proved similarly.

To prove (c), first assume that $f = I_{A\times B}$ with $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Then $I_1^f(\omega_1) = I_A(\omega_1) \cdot \mu_2(B)$ and $I_2^f(\omega_2) = \mu_1(A) \cdot I_B(\omega_2)$. Now,

$$\int I_1^f d\mu_1 = \int I_A d\mu_1 \cdot \mu_2(B) = \mu_1(A)\mu_2(B)$$

and

$$\int I_2^f d\mu_2 = \mu_1(A) \cdot \int I_B d\mu_2 = \mu_1(A)\mu_2(B).$$

Hence, the result is true for sets of the form $A \times B$ with $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Since these sets generate $\mathcal{F}$ then the result must also be true for I_F with $F \in \mathcal{F}$. By the linearity of integrals, the result is also true for simple *f*. The result can then be extended to nonnegative *f* by MCT. Finally, it can be extended to bounded functions by noting that $f = f^+ - f^-$.

Theorem 1 (Fubini's Theorem)

For $A \in \mathcal{F}$ define

$$\mu(A) = \int I_1^f \, d\mu_1 = \int I_2^f \, d\mu_2$$

with $f = I_A$. Then μ is a measure on $\mathcal{F}$. Moreover, μ is the unique measure for which

$$\mu(A \times B) = \mu_1(A)\mu_2(B)$$

for $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. We write this as $\mu = \mu_1 \times \mu_2$.

For any nonnegative $\mathcal{F}$-measurable function f, we have

$$\int f d\mu = \int I_1^f d\mu_1 = \int I_2^f d\mu_2 \ .$$

Proof

The previous lemma guarantees that μ as given in the theorem is well-defined. Now that μ is a measure follows from the linearity of the integral and the MCT. That the measure μ is unique follows from the fact that the collection $\mathcal{R}$ of rectangles is a π-system. Now let $f = I_{A\times B}$ with $A \in \mathcal{F}_1$and $B \in \mathcal{F}_2$. Then

$$\int f d\mu = \mu(A \times B) = \mu_1(A)\mu_2(B)$$

$$= \int I_1^f d\mu_1 = \int I_2^f d\mu_2 \ .$$

Since sets of the form $A \times B$ with $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ generate $\mathcal{F}$ then the result must also be true for I_F with $F \in \mathcal{F}$. By the linearity of integrals, the result is also true for simple f and this can then be extended to nonnegative f by MCT.

Definition 2

Let X and Y be two random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then X and Y are said to be independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

for $A \in \sigma(X)$ and $B \in \sigma(Y)$.

Definition 3

Let X and Y be two random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The joint law $\mu_{X,Y}$ of the pair (X,Y) is defined by

$$\mu_{X,Y}(A) = \mathbb{P}\{(X,Y) \in A\}$$

for $A \in \mathcal{B} \times \mathcal{B}$. The joint distribution $F_{X,Y}$ of X and Y is defined by

$$F_{X,Y}(x,y) = \mathbb{P}\{X \le x, Y \le y\}.$$

Theorem 2

Let X and Y be two random variables with laws μ_X and μ_Y, respectively and distribution functions F_X and F_Y, respectively. The following three statements are equivalent:

i. X and Y are independent

ii. $\mu_{X,Y} = \mu_X \times \mu_Y$

iii. $F_{X,Y}(x, y) = F_X(x) F_Y(y)$.

Proof

Assume that X and Y are independent. Then

$$\begin{aligned}\mu_{X,Y}(A \times B) &= \mathbb{P}(\{X \in A\} \cap \{Y \in B\}) \\ &= \mathbb{P}(X \in A)\mathbb{P}(Y \in B) = \mu_X(A)\mu_Y(B).\end{aligned}$$

It follows that $\mu_{X,Y} = \mu_X \times \mu_Y$ by Fubini's theorem. Hence, (i) implies (ii). Now suppose that (ii) is true. Then

$$\begin{aligned}F_{X,Y}(x, y) &= \mu_{X,Y}((-\infty, x] \times (-\infty, y]) \\ &= \mu_X((-\infty, x])\mu_Y((-\infty, y]) = F_X(x)F_Y(y).\end{aligned}$$

This shows that (ii) implies (iii).

Finally to show that (iii) implies (i) note that

$$\mathbb{P}(X \leq x, Y \leq y) = F_{X,Y}(x, y) = F_X(x)F_Y(y)$$
$$= \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y).$$

Hence $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for A of the form $\{X \leq x\}$ for some $x \in \mathbb{R}$ and B of the form $\{Y \leq y\}$ for some $y \in \mathbb{R}$. But sets of the form $\{X \leq x\}$ generate $\sigma(X)$ and sets of the form $\{Y \leq y\}$ generate $\sigma(Y)$ and so the result extends to $A \in \sigma(X)$ and $B \in \sigma(Y)$. This shows that X and Y are independent. Therefore, all three statements are equivalent.