Integrals over Subsets

Lemma 1

Let T be a measurable map from $(\Omega_1, \mathcal{F}_1, \mu)$ to $(\Omega_2, \mathcal{F}_2)$. For $A \in \mathcal{F}_2$ define

$$(\mu \circ T^{-1})(A) = \mu(T^{-1}(A)) \; .$$

Then $\mu \circ T^{-1}$ is a measure on $(\Omega_2, \mathcal{F}_2)$.

Proof

For disjoint elements $A_1, A_2, \dots$ of $\mathcal{F}_2$ we have

$$\begin{aligned}(\mu \circ T^{-1})\left(\bigcup_n A_n\right) &= \mu\left(T^{-1}\left(\bigcup_n A_n\right)\right) \\ &= \mu\left(\bigcup_n T^{-1}(A_n)\right) \\ &= \sum_n \mu\left(T^{-1}(A_n)\right) = \sum_n (\mu \circ T^{-1})(A_n).\end{aligned}$$

Theorem 1 (Change of Variable)

Let T be a measurable map from $(\Omega_1, \mathcal{F}_1, \mu)$ to $(\Omega_2, \mathcal{F}_2)$. Let f be a measurable function on Ω_2. Then

$$\int f d(\mu \circ T^{-1}) = \int f \circ T \, d\mu$$

if either integral is defined (possibly infinite).

Proof

For $f = I_A$, note that $f \circ T = I_{T^{-1}(A)}$ and, hence,

$$\int fd\left(\mu \circ T^{-1}\right) = \left(\mu \circ T^{-1}\right)(A) = \mu\left(T^{-1}(A)\right) = \int f \circ T d\mu .$$

By linearity of the integral, the result is then true for simple function f. Let f be a nonnegative measurable function and $g_1, g_2, \ldots$ be an increasing sequence of simple functions with limit f. Then by the Monotone Convergence Theorem

$$\begin{aligned}\int f d\left(\mu \circ T^{-1}\right) &= \lim_{n\to\infty} \int g_n d\left(\mu \circ T^{-1}\right) \\ &= \lim_{n\to\infty} \int g_n \circ T d\mu = \int f \circ T d\mu .\end{aligned}$$

Finally, if f is an integrable function

$$\begin{aligned}\int f d\left(\mu \circ T^{-1}\right) &= \int f^{+} d\left(\mu \circ T^{-1}\right) - \int f^{-} d\left(\mu \circ T^{-1}\right) \\ &= \int f^{+} \circ T d\mu - \int f^{-} \circ T d\mu \\ &= \int f \circ T d\mu .\end{aligned}$$

Definition 1

The integral of f over a set A in $\mathcal{F}$ is defined by

$$\int_A f \, d\mu = \int f \cdot I_A \, d\mu \, .$$

If λ is Lebesgue measure then the integral $\int_{[a,b]} f d\lambda$ is called the Lebesgue integral of f and is usually denoted by $\int_a^b f(x) \, dx$.

Theorem 2

Let *f* be a bounded function defined on the closed, bounded interval [a,b]. If *f* is Riemann integrable over [a,b], then it is Lebesgue integrable over [a,b] and the two integrals are equal.

Definition 2

Let F be a nondecreasing and right-continuous function on $\mathbb{R}$. Define $\mu_F((x,y]) = F(y) - F(x)$ for $x \leq y$. Then the integral $\int_{[a,b]} f \, d\mu_F$ is called the Lebesgue-Stieltjes integral of f with respect to F and is usually denoted by $\int_a^b f(x) \, dF(x)$.

Lemma

If $A_1, A_2, \ldots$ are disjoint, and if f is either nonnegative or integrable, then

$$\int_{\bigcup_n A_n} f \, d\mu = \sum_n \int_{A_n} f \, d\mu \, .$$

Proof

$$\int_{\bigcup_n A_n} f \, d\mu = \int f \, I_{\bigcup_n A_n} \, d\mu$$

$$= \lim_{N \to \infty} \int f \, I_{\bigcup_{n=1}^{N} A_n} \, d\mu$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} \int f \, I_{A_n} d\mu$$

$$= \sum_n \int_{A_n} f \, d\mu \, .$$

Theorem 3

Let f be a nonnegative measurable function. Define the set function ν on $\mathcal{F}$ by

$$\nu(A) = \int_A f \, d\mu .$$

Then ν is a measure.

Proof

From the lemma

$$\nu\left(\bigcup A_n\right) = \sum \int_{A_n} f d\mu = \sum \nu(A_n)$$

for disjoint $A_1, A_2, \ldots$ in $\mathcal{F}$.

Remark

We say that the measure ν has density f relative to μ, and write $d\nu/d\mu = f$. Moreover, note that for E in $\mathcal{F}$, $\mu(E) = 0$ implies that $\nu(E) = 0$.

Theorem 4 (Chain Rule)

Let f be a nonnegative measurable function and suppose that $f = d\nu/d\mu$. If g is a measurable function. Then

$$\int g d\nu = \int g \cdot f \, d\mu \ .$$

whenever either side exists.

Proof

For $g = I_A$, we have

$$\int g d\nu = \nu(A) = \int_A f d\mu = \int g \cdot f d\mu \ .$$

By linearity of integrals, it follows that the result is true for simple functions. That the result follows for nonnegative functions now follows from MCT and the fact that any nonnegative function is an increasing limit of simple functions. Finally, the result follows for integrable f by applying the equation to f^+ and f^-, separately.