$\mathcal{L}^p$ Spaces

Definition 1

For any measurable space $(\Omega, \mathcal{F}, \mu)$ and $0 < p < \infty$, $\mathcal{L}^p(\Omega, \mathcal{F}, \mu)$ denotes the set of all measurable functions *f* on Ω such that $\int |f|^p \, d\mu < \infty$ and the values of *f* are real numbers except possibly on a set of measure 0, where *f* may be undefined or infinite.

For $1 \leq p < \infty$, let $\|f\|_p = \left(\int |f|^p \, d\mu\right)^{1/p}$, called the L^p norm of *f*.

Theorem 1 (Hölder Inequality)

If $1 < p < \infty$, $p^{-1} + q^{-1} = 1, f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$, then $fg \in \mathcal{L}^1$ and

$$\left|\int fg d\mu\right| \leq \int |fg| d\mu \leq \|f\|_p \|g\|_q.$$

Proof

If $\|f\|_p = 0$, then $f = 0$ a.e., so $fg = 0$ a.e., $\int |fg| d\mu = 0$, and the inequality holds, and likewise if $\|g\|_q = 0$. Assume that the norms are not 0. Now, for any constant $c > 0$, $\|cf\|_p = c\|f\|_p$, dividing out by the norms, we can assume $\|f\|_p = \|g\|_q = 1$. We will use the fact that for any numbers u,v and $0 < \alpha < 1, u^\alpha v^{1-\alpha} \leq \alpha u + (1-\alpha)v$. This implies that with $\alpha = 1/p$, $u = |f|^p$ and $v = |g|^q$,

$$|fg| \leq \alpha|f|^p + (1-\alpha)|g|^q.$$

Integrating gives

$$\int |fg|\,d\mu \leq \alpha \int |f|^p d\mu + (1-\alpha) \int |g|^q d\mu$$

$$= \alpha \|f\|_p^p + (1-\alpha)\|g\|_q^q = 1\,.$$

Corollary (Cauchy-Schwarz)

For any f and g in $\mathcal{L}^2$, we have $fg \in \mathcal{L}^1$, and

$$\left| \int fg d\mu \right| \leq \|f\|_2 \|g\|_2 .$$

Theorem 2 (Minkowski's)

For $1 \leq p < \infty$, if f and g are in $\mathcal{L}^p(S, \mathcal{M}, \mu)$, then $f + g \in \mathcal{L}^p(S, \mathcal{M}, \mu)$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \, .$$

Proof

Since $|f + g| \leq |f| + |g|$, we can replace f and g by their absolute values and so assume $f \geq 0, g \geq 0$. If $f = 0$ a.e. or $g = 0$ a.e., the inequality is clear. If $p = 1$ or ∞ the inequality is straightforward. For $1 < p < \infty$ we have

$$(f + g)^p \leq 2^p \max(f^p, g^p) \leq 2^p(f^p + g^p).$$

Applying Holder inequality gives

$$\begin{aligned}\|f+g\|_p^p &= \int (f+g)^p \, d\mu \\ &= \int f(f+g)^{p-1} d\mu + \int g(f+g)^{p-1} d\mu \\ &\le \|f\|_p \left\|(f+g)^{p-1}\right\|_q + \|g\|_p \left\|(f+g)^{p-1}\right\|_q .\end{aligned}$$

Now $(p-1)q = p$, so

$$\|f+g\|_p^p \le \left(\|f\|_p + \|g\|_p\right)\|f+g\|_p^{p/q}$$

which leads to the result.

Definition 2

A seminorm on a real vector space X is a function $\|\cdot\|$ from X into $[0,\infty)$ such that

(i) $\|cx\| = |c|\,\|x\|$ for all $c \in \mathbb{R}$ and $x \in X$, and

(ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

A seminorm $\|\cdot\|$ is called a norm if and only if $\|x\| = 0$ only for $x = 0$. A normed space $(X, \|\cdot\|)$ is complete if for any sequence $\{x_n\}$ in X with $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$ there exists an $x \in X$ with $\lim_{n\to\infty} \|x_n - x\| = 0$.

Remark

The function $\|\cdot\|_p$ is a seminorm on $\mathcal{L}^p$ but not a norm. For each *f*, we define the equivalence class $[f]$by $[f] = \{g : f = g$ a.e.$\}$. For each $\mathcal{L}^p$, we define the factor space of equivalence classes L^p by $L^p = \{[f] : f \in \mathcal{L}^p\}$. On L^p we define the real valued function $\|\cdot\|_p$ by $\|[f]\|_p = \|f\|_p$. Then $\left(L^p, \|\cdot\|_p\right)$ is a complete normed linear space.

Definition 3

A semi-inner product on a real vector space H is a function $(\cdot,\cdot)$from $H \times H$ into $\mathbb{R}$ such that

(i) $(cf + g, h) = c(f, h) + (g, h)$ for all $c \in \mathbb{R}$ and $f, g \in H$.

(ii) $(f, g) = (g, f)$ for all $f, g \in H$.

(iii) $(f, f) \geq 0$ for $f \in H$.

A semi-inner product $(\cdot,\cdot)$ is called an inner product if and only if $(f, f) = 0$ implies $f = 0$.

Definition 4

Let $\big(H, (\cdot,\cdot)\big)$ be an inner product space. Then H is called a Hilbert space if it is complete for the norm $\|x\| = (x, x)^{1/2}$.

Theorem 3

$L^2(\Omega, \mathcal{F}, \mu)$ is a Hilbert space with inner product

$$([f],[g]) = (f,g) = \int fg d\mu \ .$$

Theorem 4

A function f from a Hilbert space H into $\mathbb{R}$ is linear and continuous if and only if for some $h \in H$, $f(x) = (x, h)$ for all $x \in H$. If so, then h is unique.

Theorem 5

Let $(\Omega, \mathcal{F}, \mu)$ be any finite measure space. Then for $1 \leq r < s < \infty, \mathcal{L}^s(\Omega, \mathcal{F}, \mu) \subseteq \mathcal{L}^r(\Omega, \mathcal{F}, \mu)$, and the identity function from L^s into L^r is continuous.

Proof

By Holder inequality with $p = s/r$,

$$\int |f|^r d\mu = \int |f|^r \cdot 1 d\mu \leq \left(\int |f|^{r(s/r)}\right)^{\frac{r}{s}} \mu(\Omega)^{\frac{1}{q}} < \infty.$$

Thus $\|f\|_r \leq \|f\|_s \mu(\Omega)^{\frac{1}{qr}}$ for all $f \in \mathcal{L}^s$. This implies continuity.

Definition 5

Let ν and μ be two measures on the same measurable space $(\Omega, \mathcal{F})$. We say that ν is absolutely continuous with respect to μ, denoted $\nu \prec \mu$, if and only if $\nu(A) = 0$ whenever $\mu(A) = 0$. We say that μ and ν are singular, denoted $\nu \perp \mu$, if and only if there is an $A \in \mathcal{F}$ with $\mu(A) = \nu(A^c) = 0$.

Theorem 6

Lebesgue Decomposition

Let $(\Omega, \mathcal{F})$ be a measurable space and μ and ν two σ-finite measures on it. Then there are unique measures ν_{ac} and ν_s such that $\nu = \nu_{ac} + \nu_s, \nu_{ac} \prec \mu$ and $\nu_s \perp \mu$.

Theorem 7 (Radon-Nikodym)

On the measurable space $(\Omega, \mathcal{F})$ let μ be a σ-finite measure. Let ν be a finite measure, absolutely continuous with respect to μ. Then there exists a nonnegative f such that $\nu(A) = \int_A f\, d\mu$ for all A in $\mathcal{F}$. Any two such f are equal a.e. (μ).

Proof

Form the Hilbert space $H = L^2(\Omega, \mathcal{F}, \mu + \nu)$ then $L^2 \subseteq L^1$ and the identity function from L^2 into L^1 is continuous . The linear function $h \mapsto \int h d\nu$ is continuous from H to the set of real numbers and, hence, there is a $g \in H$, such that $\int h d\nu = \int hg d(\mu + \nu)$ for all $h \in H$. Note that the above can be written as

$$\int h(1-g)d(\mu+\nu) = \int h d\mu$$

for all $h \in H$.

Let $A = \{\omega : g(\omega) = 1\}$. For all $E \in \mathcal{F}$, let $\nu_s(E) = \nu(E \cap A)$ and $\nu_{ac}(E) = \nu(E \cap A^c)$. Then ν_s and ν_{ac} are measures with $\nu = \nu_s + \nu_{ac}$, $\nu_{ac} \prec \mu$ and $\nu_s \perp \mu$. If $\nu \prec \mu$ then $\nu = \nu_{ac}$ and $\nu_s = 0$. Let $f = g/(1-g)$ on A^c and $f = 0$ on A. Then $\int_E f d\mu = \int_{E \cap A^c} g d(\mu + \nu) = \nu(E \cap A^c) = \nu(E)$.