Construction of the Integral

Definition 1

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A nonnegative measurable function f is called simple if f may be written as a finite sum $f = \sum_{k=1}^{m} a_k I_{A_k}$ where $a_k \in [0, \infty]$ and $A_k \in \mathcal{F}$. We then define the integral of f with respect to μ by $\int f \, d\mu = \sum_{k=1}^{m} a_k \mu(A_k)$.

Theorem 1

a) The integral $\int f\, d\mu$ is well-defined

b) If f, g are simple functions and $c \geq 0$ then $f+g$ and cf are simple functions and $\int (f+g) d\mu = \int f\, d\mu + \int g\, d\mu$, $\int cf\, d\mu = c \int f\, d\mu$.

c) If f and g are simple functions with $f \leq g$ then $\int f\, d\mu \leq \int g\, d\mu$.

d) If f and g are simple functions then $f \wedge g$ and $f \vee g$ are simple functions

Proof

a) For distinct nonnegative values b_j of f define $B_j = \{\omega \in \Omega : f(\omega) = b_j\}$. Then

$$\begin{aligned}\sum b_j \mu(B_j) &= \sum \mu(B_j) \sum \{a_k : A_k \supseteq B_j\} \\ &= \sum a_k \sum \{\mu(B_j) : B_j \subseteq A_k\} \\ &= \sum a_k \mu(A_k).\end{aligned}$$

This proves that f is well-defined.

b) Note that $f+g=\sum\left(a_i+b_j\right)I_{A_i\cap B_j}$ and $cf=\sum ca_iI_{A_i}$. So that $f+g$ and cf are both simple. Now

$$\int(f+g)d\mu=\sum_i a_i\sum_j\mu\left(A_i\cap B_j\right)+\sum_j b_j\sum_i\mu\left(A_i\cap B_j\right)$$

$$=\sum_i a_i\mu\left(A_i\right)+\sum_j b_j\mu\left(B_j\right)$$

$$=\int fd\mu+\int gd\mu$$

and

$$\int cf\,d\mu=c\sum a_iI_{A_i}=c\int f\,d\mu\ .$$

c) Note that for fixed i, j we have $a_i \mu(A_i \cap B_j) \leq b_j \mu(A_i \cap B_j)$. It follows from this that

$$\int f d\mu = \sum a_i \mu(A_i) = \sum_i \sum_j a_i \mu(A_i \cap B_j)$$

$$\leq \sum_i \sum_j b_j \mu(A_i \cap B_j) \leq \sum_j b_j \mu(B_j) = \int g d\mu.$$

d) Note that $f \wedge g = \sum \min(a_i, b_j) I_{A_i \cap B_j}$ and $f \vee g = \sum \max(a_i, b_j) I_{A_i \cap B_j}$. Hence, both $f \wedge g$ and $f \vee g$ are simple.

Definition 2

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and f be a nonnegative measurable function, we define

$\int f \, d\mu = \sup\left\{\int g \, d\mu : 0 \leq g \leq f, \; g \text{ simple}\right\}$.

Theorem 2

For any measurable $f \geq 0$, there exists an increasing sequence of simple functions f_n with $f = \lim_{n \to \infty} f_n$.

Moreover for any such sequence $\{f_n\}$, $\int f \, d\mu = \lim_{n \to \infty} \int f_n d\mu$.

Proof

Let $A_{nj} = f^{-1}((j/2^n, (j+1)/2^n]), j = 0, \dots, 2^n n - 1$ and $A_n = f^{-1}((n, \infty])$. Define the simple functions $f_n = \sum_{j=0}^{2^n n-1} j 2^{-n} I_{A_{nj}} + n I_{A_n}$. Then f_n are increasing with $f = \lim_{n\to\infty} f_n$. Since f_n are increasing then so are $\int f_n d\mu$ and thus $\lim_{n\to\infty} \int f_n d\mu = c$ for some $c \in [0, \infty]$. Now, $f \geq f_n$ for all n so that $\int f d\mu \geq \int f_n d\mu$ for all n. This implies that $\int f d\mu \geq \lim_{n\to\infty} \int f_n d\mu = c$.

To show that $\int f d\mu \leq c$ let g be a simple function with $0 \leq g \leq f$. Assume, without loss of generality, that $g = \sum a_i I_{A_i}$ where A_i are disjoint and their union is Ω. Note that $\int f_n d\mu = \sum \int f_n I_{A_i} d\mu$ and $\int g d\mu = \sum a_i \mu(A_i)$. This implies that to prove that $c = \lim_{n\to\infty} \int f_n d\mu \geq \int g d\mu$ it is enough to show that for each i, $\lim_{n\to\infty} \int f_n I_{A_i}\, d\mu \geq a_i \mu(A_i)$. If $a_i = 0$, this is clear. Let $A_{ni} = \left\{\omega \in A_i : f_n(\omega) > (1-\varepsilon) a_i\right\}$. Note that $\{A_{ni}\}_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{F}$.

Also, since $g \le f$ and $f = \lim_{n\to\infty} f_n$ it follows that for each $\omega \in A_i$ there exists an n such that $f_n(\omega) > (1-\varepsilon)g(s) = (1-\varepsilon)a_i$. This implies that $A_i = \bigcup_n A_{ni}$. From continuity of measures it follows that $\lim_{n\to\infty} \mu(A_{ni}) = \mu(A_i)$. Thus

$$(1-\varepsilon)a_i\mu(A_i) = \lim_{n\to\infty}\int(1-\varepsilon)a_i I_{A_{ni}}d\mu \le \lim_{n\to\infty}\int f_n I_{A_i}d\mu .$$

Therefore, $\lim_{n\to\infty}\int f_n d\mu \ge \int g d\mu$ for any simple function $0 \le g \le f$. This implies that $\lim_{n\to\infty}\int f_n d\mu \ge \int f d\mu$. This completes the proof.

Theorem 3

If f is a nonnegative measurable function and $\int f d\mu = 0$, then $\mu(\{f > 0\}) = 0$.

Proof

Note $\{f > n^{-1}\}$ is an increasing sequence of elements in $\mathcal{F}$ and $\bigcup\{f_n > n^{-1}\} = \{f > 0\}$. Hence, if $\mu(\{f > 0\}) > 0$ then for some n, $\mu\left(\{f > n^{-1}\}\right) > 0$. This implies that $\int f\, d\mu \geq \int n^{-1} I_{\{f > n^{-1}\}}\, d\mu > 0$, a contradiction.

Remark 1

A statement $\mathcal{S}$ about points ω of Ω is said to hold almost everywhere (a.e.) if $F = \{\omega: \mathcal{S}(\omega) \text{ is false}\} \in \mathcal{F}$ and $\mu(F) = 0$. The previous theorem states that if a nonnegative measurable function has zero integral then it must be equal to 0 a.e.

Remark 2

For a measurable function f, let

$$f^+(s) = \max(f(s), 0)\ , f^-(s) = \max(-f(s), 0).$$

Then f^+, f^- are measurable, $f = f^+ - f^-$, and $|f| = f^+ + f^-$.

Definition 3

For a measurable function f, we say that f is integrable, and write $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ if

$$\int |f|\, d\mu = \int f^+\, d\mu + \int f^-\, d\mu < \infty,$$

and then we define

$$\int f\, d\mu = \int f^+\, d\mu - \int f^-\, d\mu\ .$$

Theorem 4

a) If f and g are integrable and $f \leq g$, then $\int f \, d\mu \leq \int g \, d\mu$.

b) For nonnegative real numbers α, β and nonnegative measurable functions f, g, we have $\int (\alpha f + \beta g) d\mu = \alpha \int f d\mu + \beta \int g d\mu$.

c) If f and g are integrable and α, β are finite real numbers, then $\alpha f + \beta g$ is integrable and $\int (\alpha f + \beta g) \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu$.

Proof

a) Note that $f^{+} \leq g^{+}$ and $-f^{-} \leq -g^{-}$. Hence, $\int f^{+}\,d\mu \leq \int g^{+}\,d\mu$ and $\int f^{-}\,d\mu \geq \int g^{-}\,d\mu$. Thus

$\int f\,d\mu = \int f^{+}\,d\mu - \int f^{-}\,d\mu \leq \int g^{+}\,d\mu - \int g^{-}\,d\mu = \int g\,d\mu$.

b) Let $\{f_n\}, \{g_n\}$ be sequences of simple functions with $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$. Then

$$\int(\alpha f + \beta g)d\mu = \alpha \cdot \lim \int f_n d\mu + \beta \cdot \lim \int g_n d\mu$$

$$= \alpha \int f\,d\mu + \beta \int g\,d\mu\,.$$

c) First, $\alpha f + \beta g$ is integrable since $\int |\alpha f + \beta g| d\mu \leq |\alpha| \int |f| d\mu + |\beta| \int |g| d\mu < \infty$. For $\alpha \geq 0$, clearly $\int \alpha f \, d\mu = \alpha \int f \, d\mu$. By definition $\int -f \, d\mu = -\int f \, d\mu$. Combining this yields $\int \alpha f \, d\mu = \alpha \int f \, d\mu$ for all $\alpha \in \mathbb{R}$. Hence, we can assume without loss of generality, $\alpha = \beta = 1$. Now $(f+g)^+ - (f+g)^- = f + g = f^+ - f^- + g^+ - g^-$.

Hence

$$
\begin{aligned}
\int (f+g)\,d\mu &= \int (f+g)^{+}\,d\mu - \int (f+g)^{-}\,d\mu \\
&= \int f^{+}d\mu - \int f^{-}d\mu + \int g^{+}d\mu - \int g^{-}d\mu \\
&= \int f\,d\mu + \int g\,d\mu\,.
\end{aligned}
$$

Corollary

If f is integrable then

$$\left|\int f \, d\mu\right| \leq \int |f| \, d\mu \, .$$

Proof

$$\begin{aligned}\left|\int f\,d\mu\right| &= \left|\int f^{+}d\mu + \int - f^{-}d\mu\right| \\ &\leq \left|\int f^{+}d\mu\right| + \left|\int - f^{-}d\mu\right| \\ &= \int f^{+}d\mu + \int f^{-}d\mu = \int |f|\,d\mu\end{aligned}$$