Caratheodory's Extension Theorem

Theorem

Caratheodory's Extension Theorem

Let Ω be a set, let $\mathcal{A}$ be an algebra on Ω, and let $\mathcal{F} = \sigma(\mathcal{A})$. If μ_0 is a countably additive map $\mu_0 : \mathcal{A} \to [0, \infty]$, then there exists a measure μ on $(\Omega, \mathcal{F})$ such that $\mu = \mu_0$ on $\mathcal{A}$. If $\mu_0(S) < \infty$ then this extension is unique.

Definition 1

For any set $A \subseteq \Omega$ define the outer measure $\mu^*(A)$ of A by

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu_0(A_n) : A_n \in \mathcal{A}, A \subseteq \bigcup_{n=1}^{\infty} A_n\right\}.$$

Lemma 1

For any sets A and A_n in Ω, if $A \subseteq \bigcup_{n=1}^{\infty} A_n$ then

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n) \ .$$

Proof

If the latter sum is $+\infty$, there is no problem. Otherwise, given $\varepsilon > 0$, let $A_n \subseteq \bigcup_m A_{nm}$ and $\sum_m \mu_0(A_{nm}) < \mu^*(A_n) + \varepsilon/2^n$ with $A_{nm} \in \mathcal{A}$ for all m,n. Then $A \subseteq \bigcup_{m,n} E_{nm}$ and

$$\mu^*(A) \leq \sum_n \sum_m \mu_0(A_{nm}) \leq \sum_n \mu^*(E_n) + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ proves the result.

Lemma 2

For any $A \in \mathcal{A}$, $\mu^*(A) = \mu_0(E)$.

Proof

Clearly, $\mu^*(A) \leq \mu_0(A)$. Let $A \subseteq \bigcup_n A_n$ and $B_n = A_n \setminus \bigcup_{j<n} A_j$ with $A_n \in \mathcal{A}$. Then

$$\mu_0(A) \leq \mu_0(\bigcup B_n) = \sum \mu(B_n) \leq \sum \mu(A_n).$$

Since the A_n's are arbitrary then $\mu_0(A) \leq \mu^*(A)$.

Definition 2

A set $E \subseteq \Omega$ is called μ^*-measurable if for every set $A \subseteq \Omega$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

We will denote the collection of μ^*-measurable sets by $\mathcal{M}(\mu^*)$.

Lemma 3

All sets in $\mathcal{A}$ are μ^*-measurable, i.e., $\mathcal{A} \subseteq \mathcal{M}(\mu^*)$.

Proof

Let $A \in \mathcal{A}$ and $B \subseteq \Omega$. Given $\varepsilon > 0$, take $B \subseteq \bigcup B_n$ with $B_n \in \mathcal{A}$ and $\sum \mu_0(B_n) \leq \mu^*(B) + \varepsilon$. Then $B \cap A \subseteq \bigcup (B_n \cap A), B \cap A^c \subseteq \bigcup (B_n \cap A^c)$ with $B_n \cap A \in \mathcal{A}$ and $B_n \cap A^c \in \mathcal{A}$. So that

$$
\begin{aligned}
&\mu^*(B \cap A) + \mu^*\left(B \cap A^c\right) \\
&\qquad \leq \sum \left[\mu^*(B_n \cap A) + \mu^*\left(B_n \cap A^c\right)\right] \\
&\qquad \leq \sum \mu_0(B_n) \leq \mu^*(B) + \varepsilon .
\end{aligned}
$$

Letting $\varepsilon \downarrow 0$ gives the result.

Lemma 4

$\mathcal{M}(\mu^*)$ is a σ-algebra and μ^* is a measure on it.

Proof

Clearly $A \in \mathcal{M}(\mu^*)$ then $A^c \in \mathcal{M}(\mu^*)$. If $A, B \in \mathcal{M}(\mu^*)$ then for $C \subseteq \Omega$,

$$\begin{aligned}\mu^*(C) &= \mu^*(C \cap A) + \mu^*\left(C \cap A^c\right)\\ &= \mu^*(C \cap A \cap B) + \mu^*\left(C \cap A \cap B^c\right) + \mu^*\left(C \cap A^c\right)\\ &= \mu^*(C \cap (A \cap B)) + \mu^*\left(C \cap (A \cap B)^c\right).\end{aligned}$$

Thus $\mathcal{M}(\mu^*)$ is an algebra. Let $A_n \in \mathcal{M}(\mu^*)$, $A = \bigcup_n A_n$, $B_n = \bigcup_{j<n} A_j$, and assume, without loss of generality, that the A_n's are disjoint.

Note that for $E \subseteq \Omega$

$$\mu^*(E) = \mu^*\left(E \cap B_n^c\right) + \mu^*\left(E \cap B_n\right)$$

$$= \mu^*\left(E \cap B_n^c\right) + \mu^*\left(E \cap A_n\right) + \mu^*\left(\bigcup_{i<n}\left(E \cap A_i\right)\right).$$

From this it follows by induction on n that

$$\mu^*(E) \geq \mu^*\left(E \cap A^c\right) + \sum_{j=1}^{n} \mu^*\left(E \cap A_j\right).$$

Letting $n \to \infty$

$$\mu^*(E) \geq \mu^*\left(E \cap A^c\right) + \mu^*(E \cap A).$$

Thus $A \in \mathcal{M}(\mu^*)$ and

$$\mu^*(E) = \mu^*\left(E \cap A^c\right) + \sum_{j=1}^{\infty} \mu^*\left(E \cap A_j\right)$$

Letting $E = A$ shows μ^* is countably additive on $\mathcal{M}(\mu^*)$.

Proof of Caratheodory's

Note that by Lemmas 3 and 4, $\mathcal{M}(\mu^*)$ is a σ-algebra containing $\mathcal{A}$ and, hence, $\mathcal{F} \subseteq \mathcal{M}(\mu^*)$. Take $\mu = \mu^*$ on $\mathcal{F}$. Then μ is a measure on $\mathcal{F}$ and for $A \in \mathcal{A}$,

$$\mu(A) = \mu^*(A) = \mu_0(A)\,.$$

Uniqueness follows from Theorem 2 of Lecture 3.