Stat235
Problem Set 2

1. Show that if f and g are simple functions then $f + g$ is a simple function and $\int (f + g)\, d\mu = \int f\, d\mu + \int g\, d\mu$.

2. Suppose that $\mu(A) < \infty$ and $\alpha \leq f \leq \beta$ on A. Show that $\alpha\mu(A) \leq \int_A f\, d\mu \leq \beta\mu(A)$.

3. Prove that if $f_n \geq 0, f_n \to f$ and $\int f_n\, d\mu \leq A < \infty$ then $\int f\, d\mu \leq A$.

4. Let $f_n \leq g_n \leq h_n$ with $f_n \to f, g_n \to g$, and $h_n \to h$. Suppose further that $\int f_n\, d\mu \to \int f\, d\mu$ and $\int h_n\, d\mu \to \int h\, d\mu$. Show that $\int g_n\, d\mu \to \int g\, d\mu$.

5. Suppose that f_n are integrable and $\sup_n \int f_n\, d\mu < \infty$. Show that if $f_n \uparrow f$ then $\int f_n\, d\mu \to \int f\, d\mu$.

6. Show that if f is integrable, then for each ε there is a δ such that $\mu(A) < \delta$ implies that $\int_A |f| d\mu < \varepsilon$.

7. Let $S = [0,1], \Sigma = \mathcal{B}([0,1])$ and μ Lebesgue measure. Define $f_n = nI_{(0,1/n)}$. Prove that $f_n(s) \to 0$ for every $s \in S$, but that $\lim_{n\to\infty} \int f_n\, d\mu \neq 0$.

8. Suppose that $\nu(\{y : (x, y) \in E\}) = \nu(\{y : (x, y) \in F\})$ for all x. Show that $(\mu \times \nu)(E) = (\mu \times \nu)(F)$.

9. Let $f, g \in \mathcal{L}^2$. Prove that $\|f + g\|_2^2 + \|f - g\|_2^2 = 2\|f\|_2^2 + 2\|g\|_2^2$.

10. Let λ, ν, and μ be finite measures such that $\lambda \prec \nu$ and $\nu \prec \mu$. Show that $\lambda \prec \mu$ and that $\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \cdot \frac{d\nu}{d\mu}$. (Note that for any measures λ and μ, $\frac{d\lambda}{d\mu}$ denotes the Radon Nikodym derivative of λ with respect to μ.)