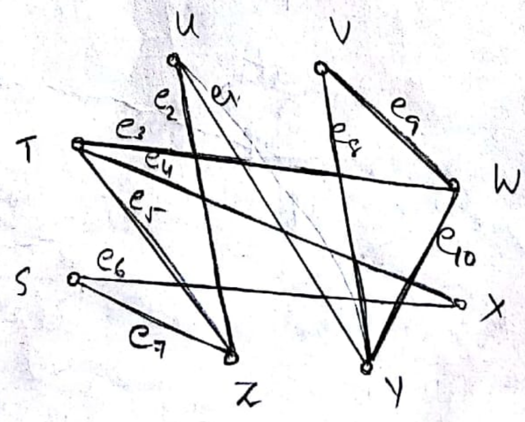
03/08/17

INTRODUCTION:

In a hockey league, there are eight teams which we denote by S, T, U, V, W, X, Y, Z. After a few weeks of season, the following teams have been played each other.

1. S has played with X and Z.
2. T has played with W, X, Z.
3. U has played with Y, Z.
4. V with W and Y.
5. W with T, V and Y.
6. X has played with S, T.
7. Y has played with U, V, W.
8. Z has played with S, T, U.

The teams are represented by dots and the corresponding team which have played each other are joined by a line.



So, many real life situations can conveniently be described by means of drawings which we call **graphs**.

Here, the vertex set

$$V(G) = \{S, T, U, V, W, X, Y, Z\}$$

and the edge set

$$E(G) = \{e_1, e_2, \ldots, e_{10}\}$$

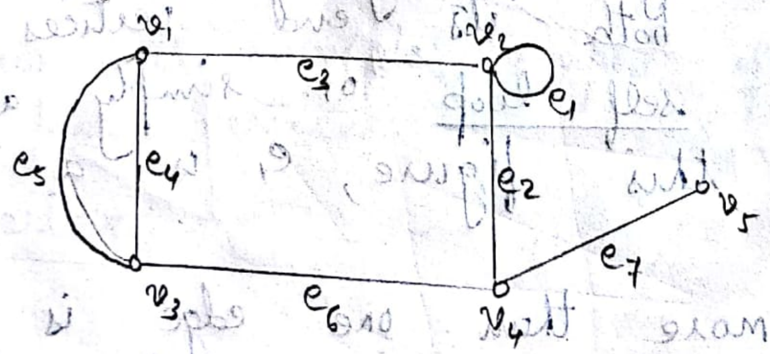
Module - I

Introductory Concepts of Graphs

Graph:

A graph $G = (V(G), E(G))$ consist of two non-empty sets,

1. The vertex set of the graph $V(G)$ (V) which consist of non-empty set of elements $\{v_1, v_2, v_3 \ldots\}$ called the vertices.
2. The edge set of a graph $E(G)$ (E) which consist of non-empty sets $\{e_1, e_2, e_3 \ldots\}$ called the edges such that each edge e_k in E ~~are~~ is assigned with an unordered pair of vertices (v_i, v_j), called the end vertices.



Vertex set, $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$

Edge set, $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

End vertices of

$e_1 \Rightarrow v_2$

$e_2 \Rightarrow v_2, v_4$

$e_3 \Rightarrow v_1, v_2$

$e_4 \Rightarrow v_1, v_3$

$e_5 \Rightarrow v_1, v_3$

$e_6 \Rightarrow v_3, v_4$

$e_7 \Rightarrow v_4, v_5$

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* The vertices v_i, v_j associated with edge e_k are called the end vertices of e_k. The most common representation of a graph is by means of a diagram in which end vertices are represented by points & each edge as a line segment joining its end vertices.

An edge having the same vertex as both its end vertices is called a self loop or simply a loop. Here, in this figure, e_1 is a self loop.

If more than one edge is associated with a given pair of

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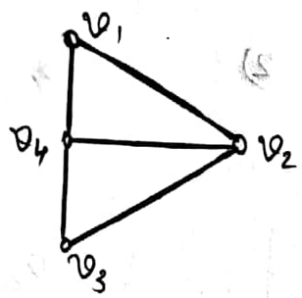
vertices, then such edges are called <u>parallel edges</u> or <u>multiple edges.</u> In this figure, e_4 & e_5 are the parallel edges.

A graph that has neither self loops nor parallel edges is called a <u>simple graph.</u>

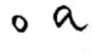


<u>Note:</u>

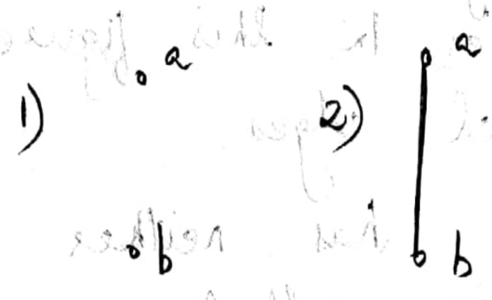
* A graph is also called a linear complex or a 1-complex or a 1 dimensional complex.
* A vertex is also refered to as node, a junction, a point, 0-cell or 0-Simplex.
* Other terms used for edges are branch, a line, an element, 1-cell, an arc and 1-Simplex.

Q1. Draw all simple graphs of one, two, three & four vertices.

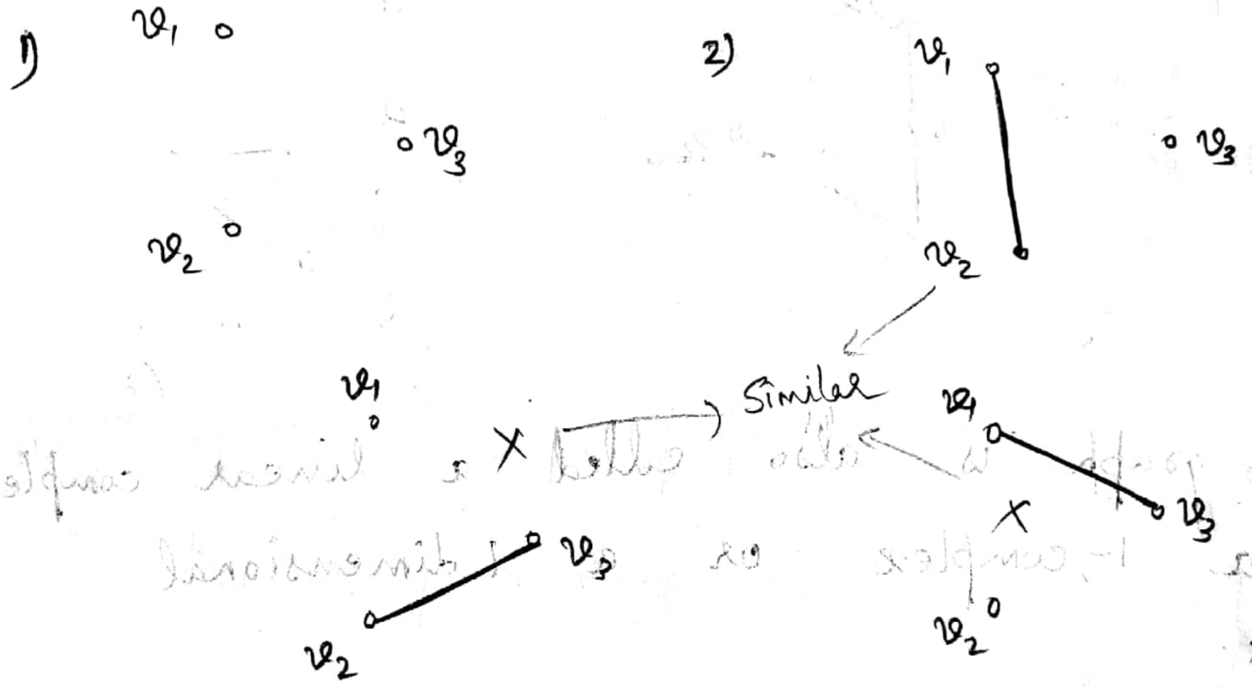
<u>1 vertex:</u>

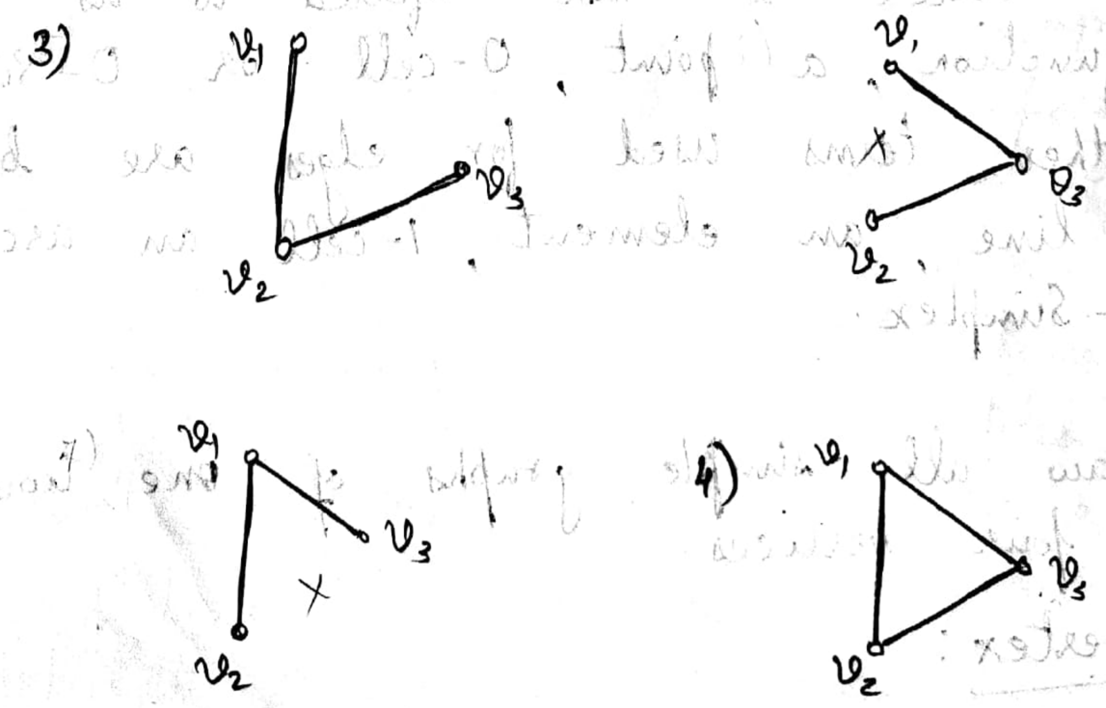


2 vertices:



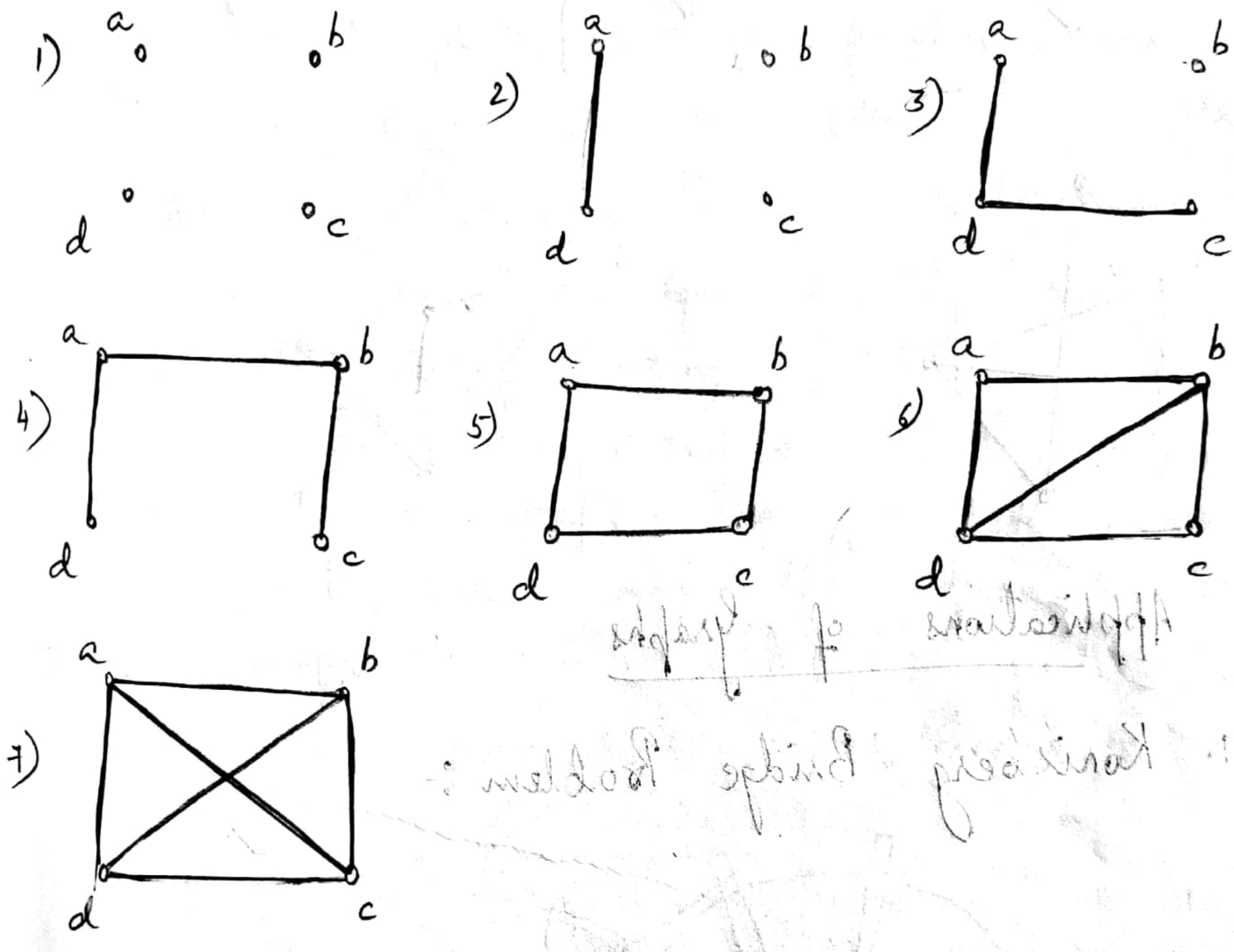
3 vertices:



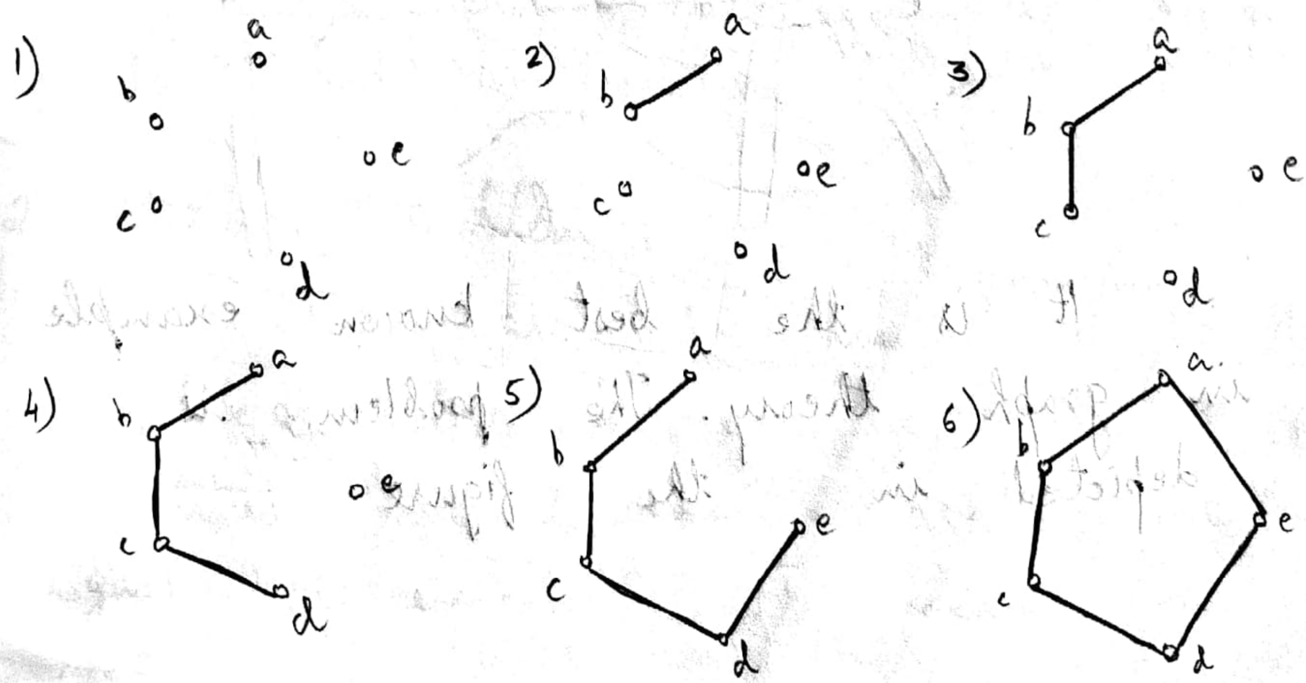


The naming of the vertices is not important.

4 vertices : X



5 vertices : X



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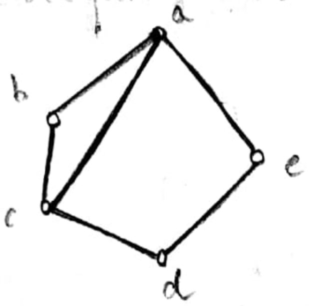

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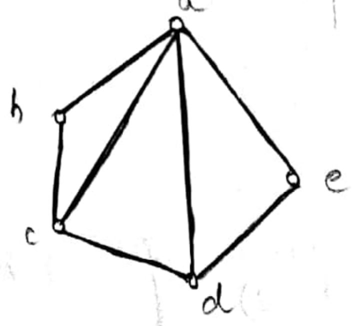

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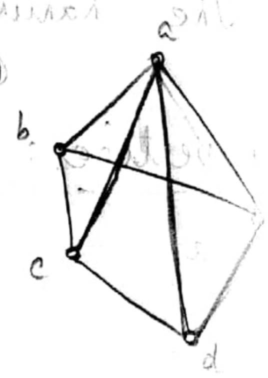

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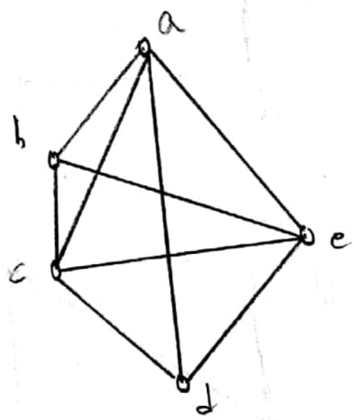

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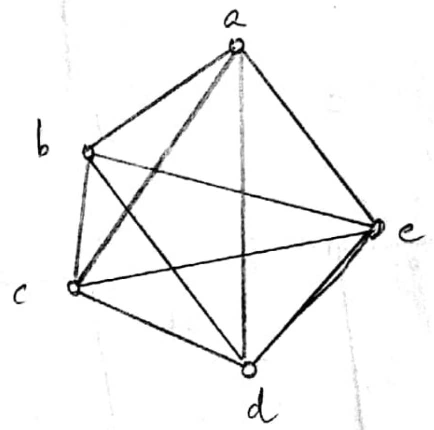
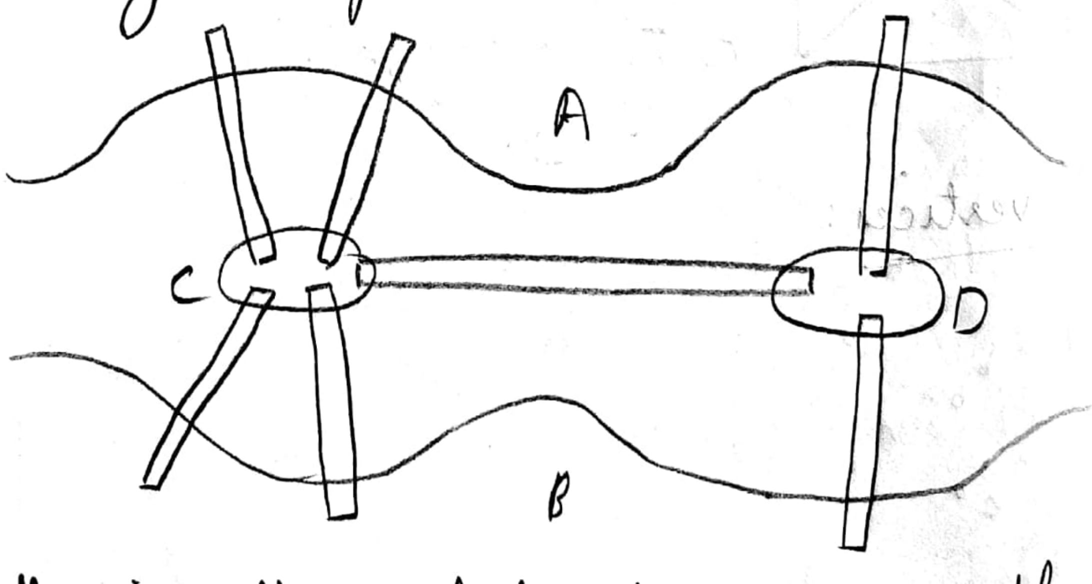


7/8/17 <u>Applications of Graphs</u>

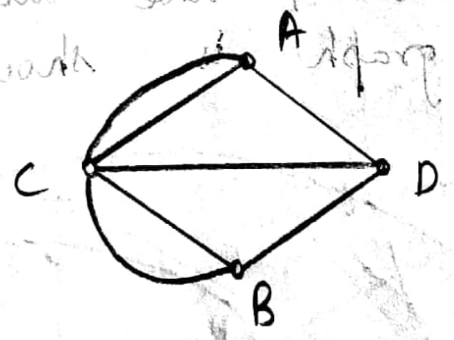
1. Konisberg Bridge Problem :-



It is the best known example in graph theory. The problem is depicted in the figure.

Two islands C and D formed by Pregal River in Konisberg were connected to each other & to the banks A & B with 7 bridges. The problem was to start at any of the four land areas of the city A, B, C or D, walk over each of the seven bridges exactly once and return to the starting point.

Euler represented this situation by means of a graph. The vertices represent the land areas and the edges represent the bridges.

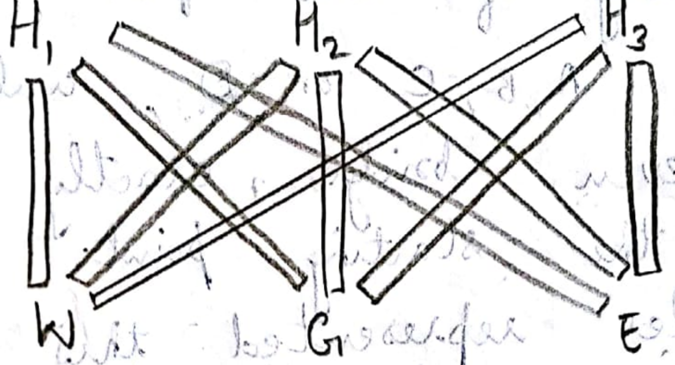


Euler proved that Konisberg bridge problem has no solution.

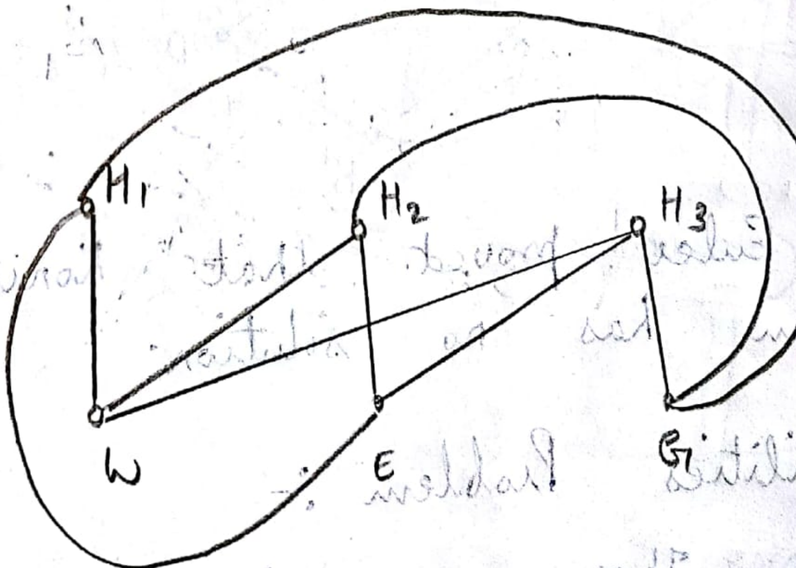
2. Utilities Problem :-

There are 3 houses H_1, H_2, H_3 each to be connected to each of the 3 utilities. The 3 utilities are water (W), gas (G), electricity (E) by means of conduits.

The problem is that 'is it possible to make such connections without any cross-over of the conduits.'



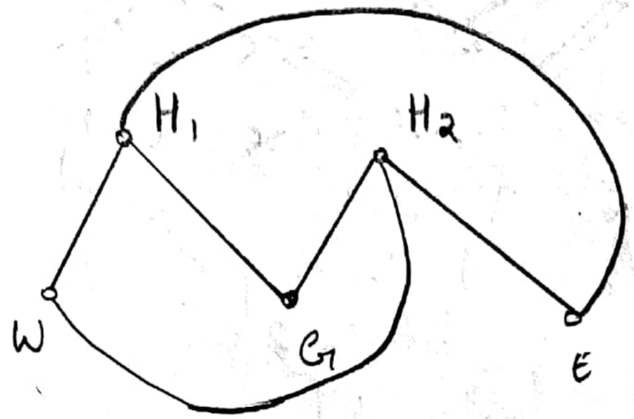
This problem can be represented by a graph. The conduits represent the edges while the houses & the utilities represent by vertices. The graph is shown below



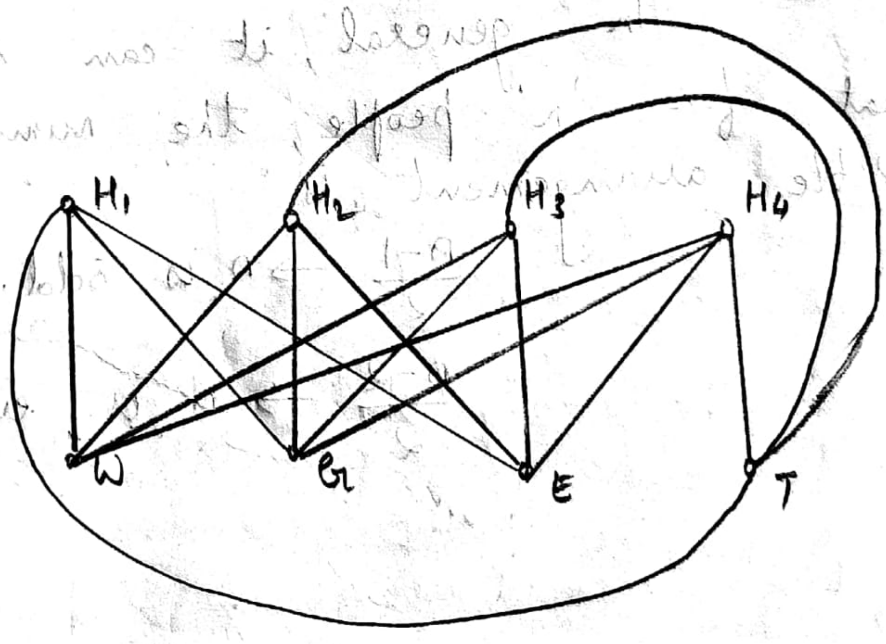
We can see that the graph can't be drawn in the plain without edges crossing each other. Thus, the answer to the problem is No.

Q. Draw graphs representing problems of
1. Two houses & 3 utilities
2. 4 houses & 4 utilities say, water, gas, electricity, telephone.

1.



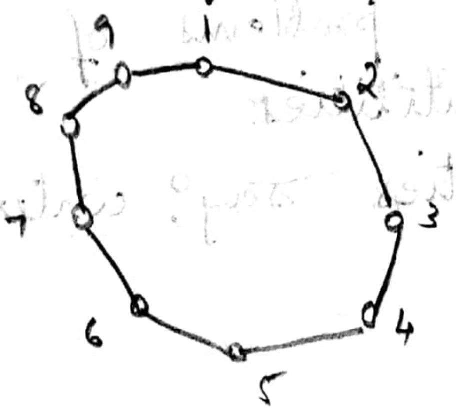
2.

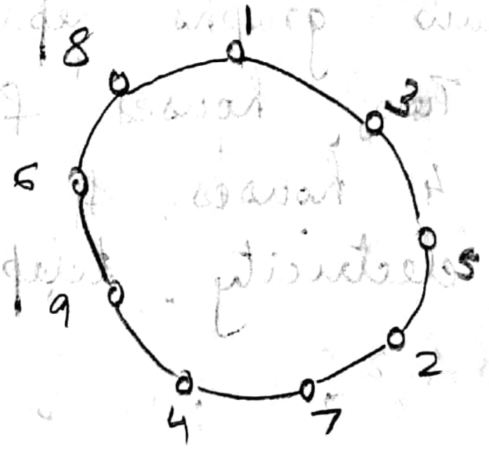


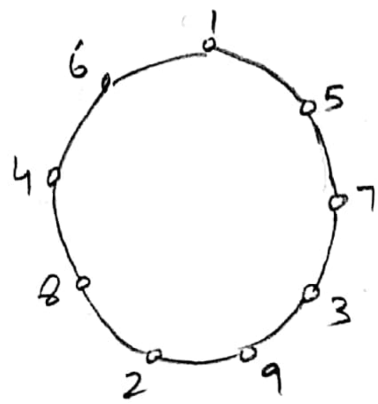
3. Seating Problem :-

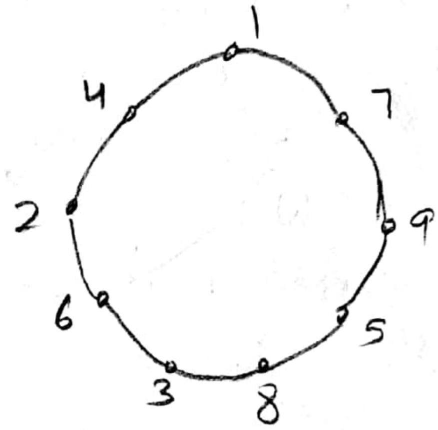
9 members of a new club meet each day for lunch at an round table. They decide to sit such that every member has differ neighbours at each lunch.

How many ways can this arrangement last?









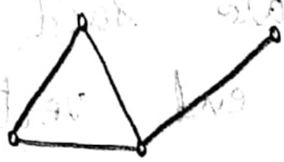
In general, it can be shown that for 'n' people, the number of such possible arrangement is

$$\frac{n-1}{2} \rightarrow n \text{ is odd.}$$

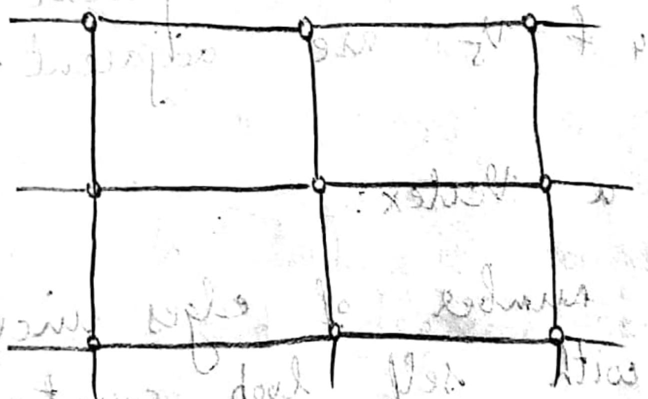
$$\frac{n-2}{2} \rightarrow n \text{ is even}$$

Finite & Infinite Graphs

A graph with finite number of vertices as well as finite number of edges is called a finite graph, otherwise it is an infinite graph.



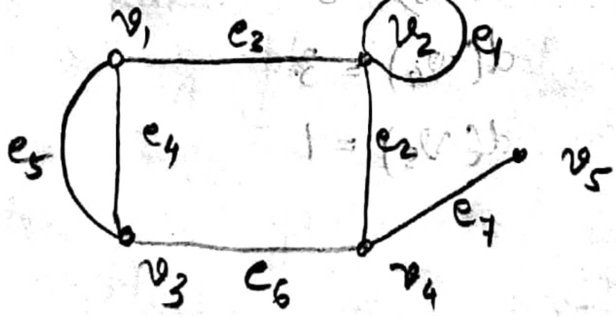
Finite



Infinite

Incidence & Degree

When a vertex v_i is an end vertex of some edges e_j then, v_i and e_j are said to be incident with each other.



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Two non-parallel edges are said to be adjacent, if they are incident on a common vertex.

Eg: e_2, e_6, e_7 are adjacent on a common vertex v_4.

Two vertices are said to be adjacent if they are the end vertices of the same edge.

Eg: v_1 & v_2 are adjacent.
v_4 & v_5 are adjacent.

Imp **Degree of a Vertex:**

The number of edges incident on a vertex with self loop counted twice is called the degree of the vertex and is denoted by

$$deg(v_i) \text{ or } d(v_i)$$

Eg: $d(v_1) = 3$
$d(v_2) = 4$
$d(v_3) = 3$
$d(v_4) = 3$
$d(v_5) = 1$

Note: The degree of a vertex is sometimes referred to as valency.

Fundamental Theorem of Graph Theory

Consider a graph G with 'e' edges and 'n' vertices say $v_1, v_2, \ldots v_n$. Then,

$$\sum_{i=1}^{n} d(v_i) = 2e$$

PROOF:

Each edge has two end vertices and hence contribute 2 to the sum of the degrees, i.e; when degrees of vertices are added, each edge is counted twice. Also, we know that a loop is counted twice while taking the degree. Therefore,

$$\sum_{i=1}^{n} d(v_i) = 2e.$$

Hence proved.

Theorem 2:

The number of vertices of odd degree in a graph is always even.

PROOF:

Consider a graph G with 'e' edges and 'n' vertices say $v_1, v_2, \ldots v_n$.

Then,
$$\sum_{i=1}^{n} d(v_i) = 2e \quad \text{——— (1)}$$

Consider the vertices with odd degree and even degree separately. Then (1) can be expressed as a sum of two sums each taken over the vertices of even & odd degrees.

i.e.,
$$\underbrace{\sum_{i=1}^{n} d(v_i)}_{=2e\ (\text{even})} = \underbrace{\sum_{\text{even}} d(v_j)}_{(\text{even})} + \sum_{\text{odd}} d(v_k) \quad \text{——— (2)}$$

Since the L.H.S of (2) is even, and the first expression on R.H.S is even, the second expression must be even.

$$\therefore \sum_{\text{odd}} d(v_k) = \text{even no.} \quad \text{——— (3)}$$

Since each $d(v_k)$ is odd, the total number of terms in the sum ~~to~~ must be even to make the sum ~~as~~ an even number. Hence the theorem.

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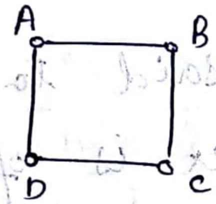
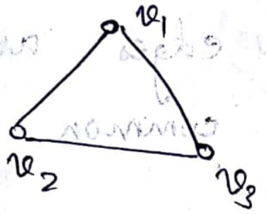
* A graph in which all the vertices are of equal degree is called a regular graph.

Eg:

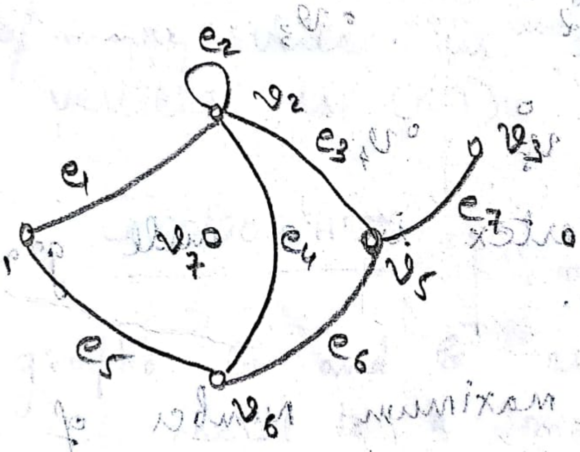


$d(A) = 2$
$d(B) = 2$
$d(C) = 2$
$d(D) = 2$



$d(v_1) = 2$
$d(v_2) = 2$
$d(v_3) = 2$

• Isolated Vertex, Pendant Vertex & Null Graph



* A vertex having no incident edge is called isolated vertex.

Eg: v_4 & v_7 are isolated vertex.

In other words, they are vertices with 0 degree.

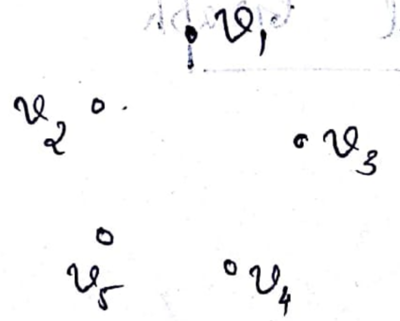
* A vertex of degree 1 is called a pendant vertex.

Eg: v_3 is a pendant vertex.

* Two adjacent edges are said to be in series if their common vertex is of degree 2.

Eg: Two edges e_1 and e_5 which are incident on v_1 are in series.

* A graph without any edges is called a null graph.



Every vertex in a null graph is an isolated vertex.

Q1. Show that maximum number of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$.

An edge is constructed by choosing any two vertices out of a set of 'n' vertices & joining them.

Therefore, the number of ways of choosing any two elements out of a set of 'n' elements is $^nC_2$

$$^nC_2 = \frac{n(n-1)}{2}$$

∴ The maximum number of edges is equal to $\boxed{\dfrac{n(n-1)}{2}}$

Q2. Show that maximum degree of any vertex in a simple graph with 'n' vertices is $(n-1)$.

Let G be a simple graph with 'n' vertices. Consider any vertex v of G. Since, the graph is simple, v can be adjacent to atmost $(n-1)$ vertices. Hence maximum degree of any vertex in a simple graph with 'n' vertices is $(n-1)$.

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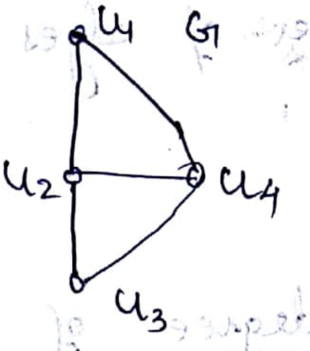
⊛ ISOMORPHISM

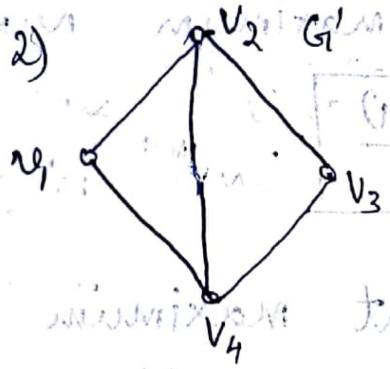
Two graphs G and G' are said to be isomorphic if there is a one-to-one correspondance between their ~~end~~ vertices & edges such that the incidence relation is preserved.

That is, if the edge e is incident on the vertices v_1 and v_2 in G, then the corresponding edge e' in G' must be incident on the vertices v_1' and v_2' that corresponds to v_1 & v_2 respectively.

Eg: 1)



2)



$u_1 \to v_1$
$u_2 \to v_2$
$u_3 \to v_3$
$u_4 \to v_4$

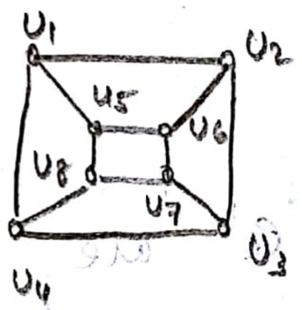
$u_1 \to v_3$
$u_2 \to v_2$
$u_3 \to v_1$
$u_4 \to v_4$

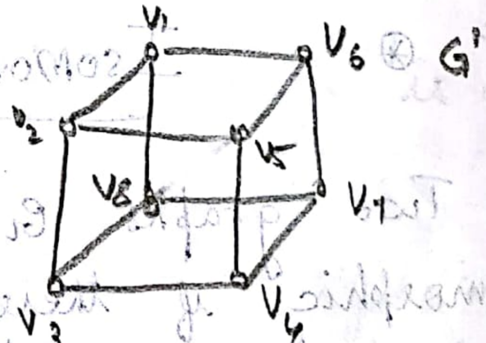
$\therefore G$ & G' are isomorphic.

Two isomorphic graphs must have

(1) Same number of vertices.
(2) Same number of edges.
(3) Equal no. of vertices with a given degree.

2)





$U_1 \to V_1$

$U_2 \to V_6$

$U_3 \to V_7$

$U_4 \to V_8$

$U_5 \to V_2$

$U_6 \to V_5$

$U_7 \to V_4$

$U_8 \to V_3$

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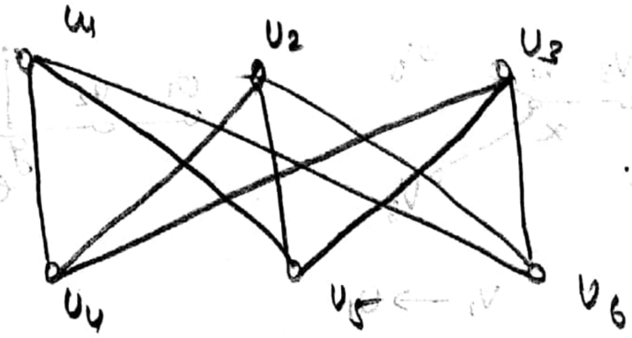
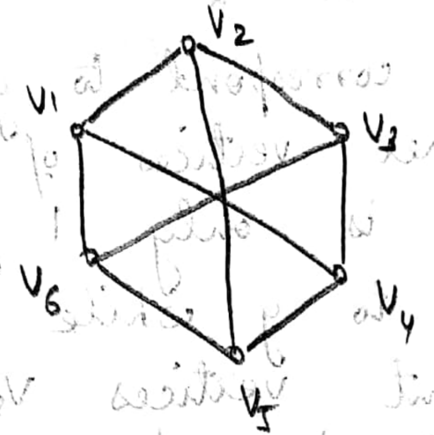
3)





$u_1 \to v_1$

$u_4 \to v_6$

$u_2 \to v_5$

$u_5 \to v_4$

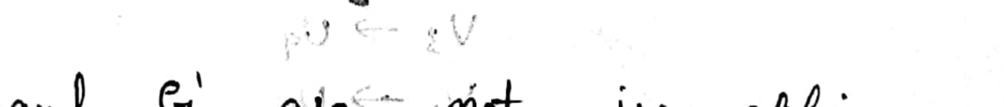
$u_3 \to v_3$

$u_6 \to v_2$

HW

Q- Check whether G & G' are isomorphic.

1)



G and G' are not isomorphic. as they have different no. of vertices & edges.

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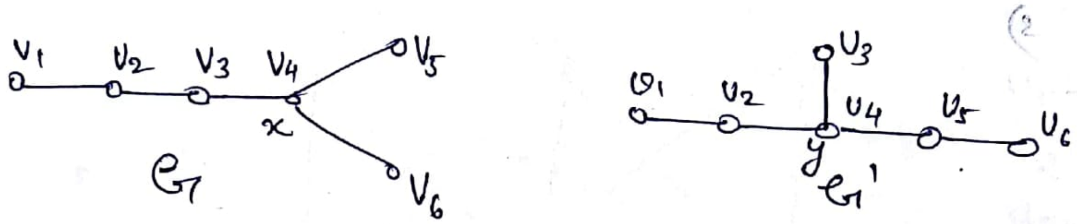
2)

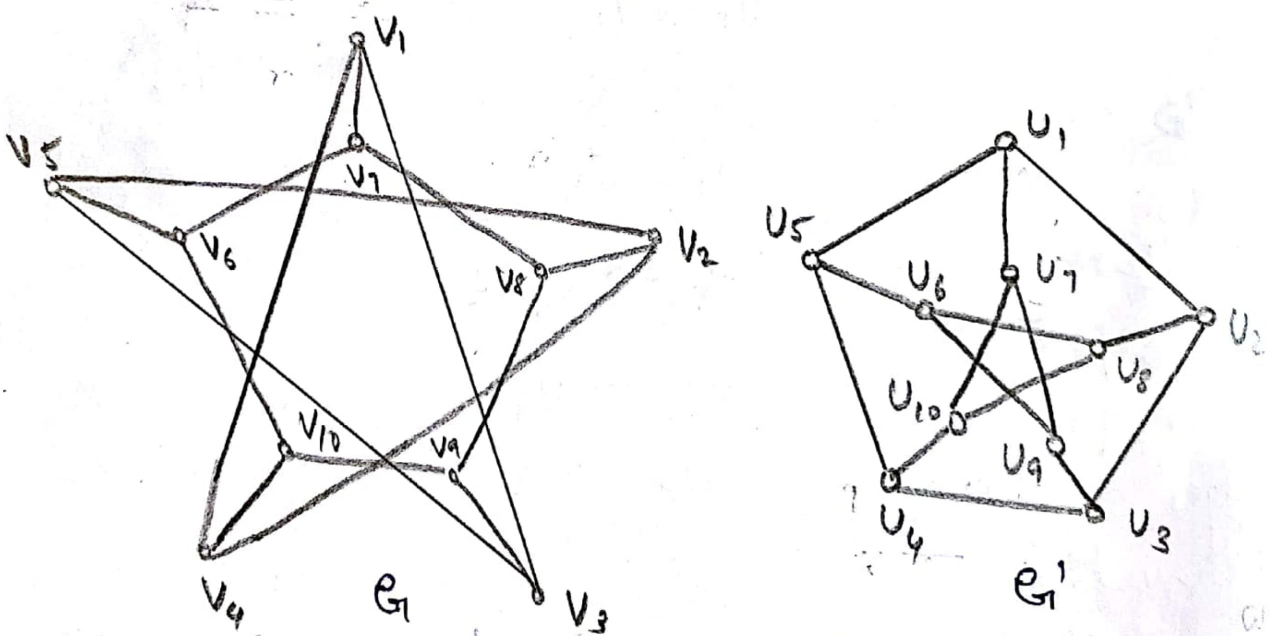


$V_1 \rightarrow U_1$

$V_2 \rightarrow U_2$

x must correspond to y because there are no other vertices of degree 3. In G', there is only 1 pendant vertex U_3 adjacent to y while in G, there are 2 pendant vertices V_5 & V_6 adjacent to x. ∴ G is not isomorphic to G'.

3)



$V_1 \rightarrow U_7$

$V_2 \rightarrow U_8$

$V_3 \rightarrow U_9$

$V_4 \rightarrow U_{10}$

$V_5 \rightarrow U_6$

$V_7 \rightarrow U_1$

$V_8 \rightarrow U_2$

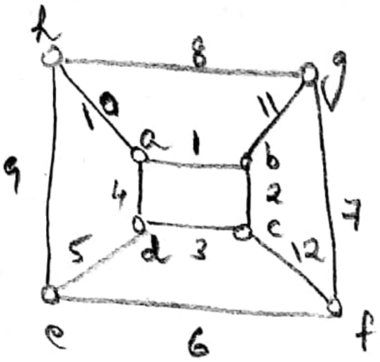
$V_9 \rightarrow U_3$

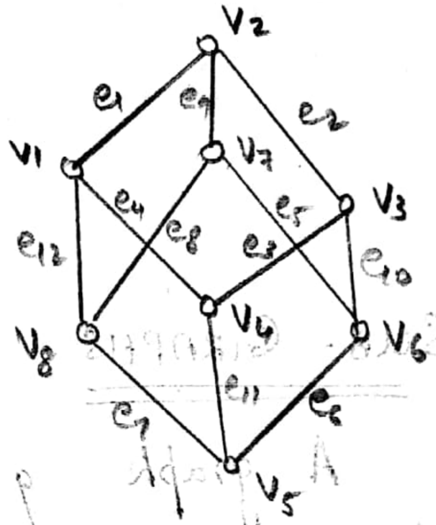
$V_{10} \rightarrow U_4$

$V_6 \longrightarrow U_5$

They are isomorphic.

4)





$a \rightarrow V_1$ $\quad$ $1 \rightarrow e_1$

$b \rightarrow V_2$ $\quad$ $2 \rightarrow e_2$

$c \rightarrow V_3$ $\quad$ $3 \rightarrow e_3$

$d \rightarrow V_4$ $\quad$ $4 \rightarrow e_4$

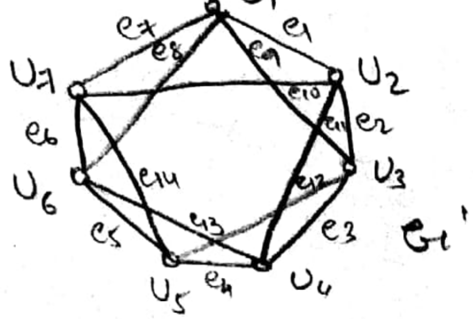
$e \rightarrow V_5$

$f \rightarrow V_6$

$g \rightarrow V_7$

$h \rightarrow V_8$

5)



Verify that the two graphs are isomorphic. Label the corresponding vertices & edges. Show their correspondance.

$V_1 \to U_1$

$V_2 \to U_2$

$V_3 \to U_3$

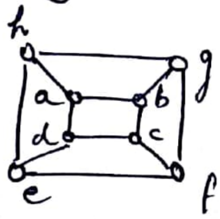
Since we can't find a 1-1 correspondance between the vertices & edges of G & G', they are not isomorphic.

SUB-GRAPHS

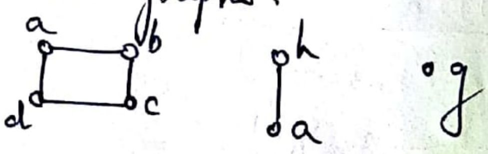
A graph g is said to be a sub-graph of G if all vertices and all edges of g are in G and each edge of g has the same end vertices in g as in G.

We denote it by $\boxed{g \subset G}$. stating as g is a sub-graph of G.

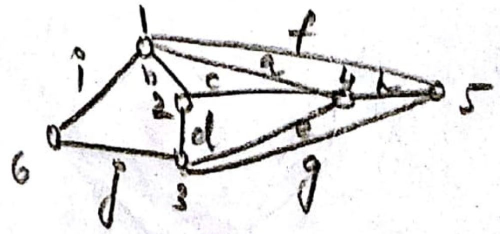
Eg: 1)



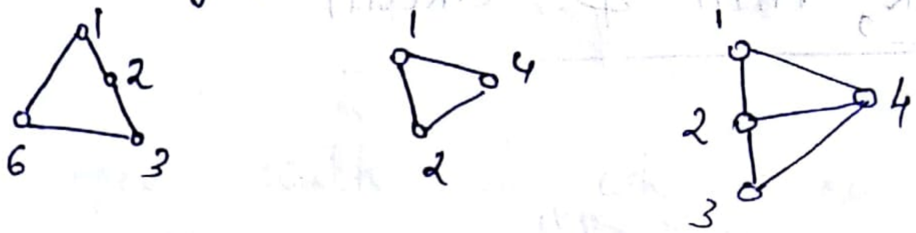
Sub-graphs:



2)



Sub-graphs :



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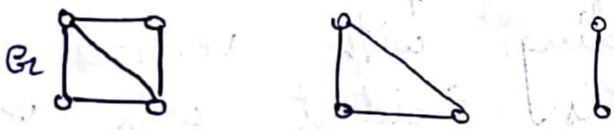
Note :

1) Every graph is its own subgraph.
2) A sub graph of a sub graph of G is a sub graph of G.



3) A single vertex in a graph G is a sub graph of G.
4) A single edge in G together with its end vertices is also a sub graph of G.

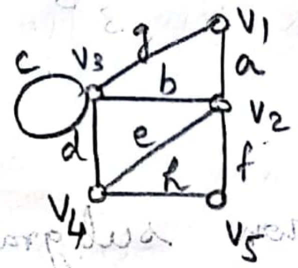
EDGE DISJOINT SUBGRAPH

Two or more sub graphs g_1 and g_2 of a graph G are said to be edge disjoint if g_1 & g_2 do not have edges in common.

VERTEX DISJOINT SUBGRAPH

Subgraphs that do not have vertices in common are said to be vertex disjoint subgraph.

WALK, PATH & CIRCUIT



A walk is defined as a finite alternating sequence of vertices & edges beginning and ending with vertices such that each edge is incident with vertices preceding and following it. No edge will appear more than once in a walk. A vertex however may appear more than once.

Eg: $v_1\ a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2\ f\ v_5$ is a walk.

A walk is also referred to as edge train or chain.

Vertices with which a walk begins and ends are called its terminal vertices.

Eg: v_1 & v_5 are the terminal vertices of a walk.

A walk which begins and ends at the same vertex is called a closed walk.

A walk that is not closed is called an <u>open walk</u>.

An open walk in which no vertex appear more than once is called a <u>path</u> or simple path or elementary path.

Eg: $V_1\ a\ V_2\ b\ V_3\ d\ V_4$ is a path.

$V_1\ a\ V_2\ b\ V_3\ c\ V_3\ d\ V_4\ e\ V_2\ f\ V_5$ is not a path.

The number of edges ~~of~~ in a path is called the <u>length</u> of a path.

An edge which is not a self loop is a path of length 1.

A self loop can be included in a walk but not in a path.

The terminal vertices of a path are of degree 1 & the rest of the vertices called intermediate vertices are of degree 2.

A closed walk in which no vertex except the initial and the final vertex appears more than once is called a <u>circuit</u> or a cycle or elementary cycle or circular path or polygon.

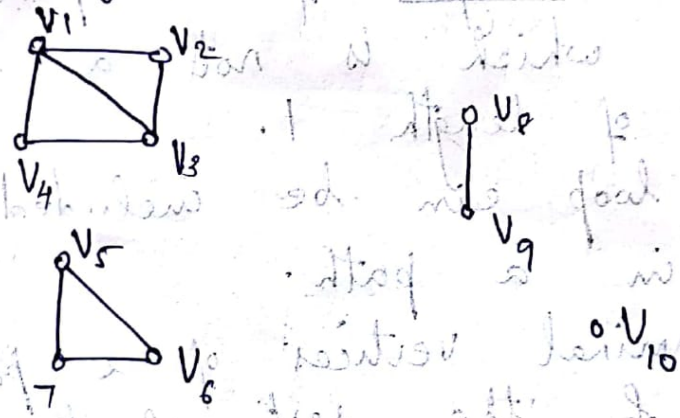
Connected Graphs, Disconnected Graphs & Components

A graph is said to be connected if there is atleast one path between every pair of vertices in G.

Otherwise, it is a disconnected graph.

Disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component.

Eg: 1)



This is a disconnected graph with 4 components.

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2)

G_2

G is a disconnected graph with 2 components.

THEOREM

A graph G is disconnected if and only if its vertex set V can be partitioned into 2 non-empty disjoint ~~subjects~~ set, V_1 and V_2 such that there exist no edge in G whose one end vertex is in subset V_1 & the other in subset V_2.

Proof: Suppose that such a partition exists. Consider 2 arbitrary vertices a and b of G such that $a \in V_1$ and $b \in V_2$. No path can exist between vertices a and b. Otherwise, there would be atleast one edge whose one end vertex would be in V_1 & other in V_2.

Hence, if a partition exist, G is not connected.

Conversely, let G be a disconnected graph. Consider a vertex a in G. Let V_1 be the set of all vertices that are joined by the paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a set V_2. which is non-empty. No vertex in V_1 is joined to any in V_2 by an edge. Hence, the partition.

Hence the theorem.

THEOREM: If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Proof: 

let G be a graph with all even vertices (ie, vertex with even degree), except vertices v_1 & v_2 which are odd. The number of vertices of odd degree is even. This holds for every graph and therefore, for every component of a disconnected graph. No graph can have an odd no. of odd vertices.

$\therefore$ In graph G, v_1 and v_2 must belong to the same component and hence must have a path between them.

Hence the theorem.

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THEOREM: A simple graph with n vertices & k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof: Let the number of vertices in each of the k components of a graph G be $n_1, n_2, \ldots, n_k$.

Thus, we have

$$n_1+n_2+\cdots+n_k = n$$

where $n_i \geq 1$

We have $\sum_{i=1}^{k} n_i = n$

$\therefore \sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$

$= n - k$

i.e., $\sum_{i=1}^{k} (n_i - 1) = n - k$

Squaring on both the sides, we have

$$\left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2$$

$$\Rightarrow \left[(n_1 - 1) + (n_2 - 1) + \dots + (n_k - 1)\right]^2 = (n-k)^2$$

$$\Rightarrow (n_1-1)^2 + (n_2-1)^2 + \dots + (n_k-1)^2 + 2(n_1-1)(n_2-1) + 2(n_2-1)(n_3-1) + \dots + 2(n_{k-1}-1)(n_k-1)$$

$$= n^2 + k^2 - 2nk$$

$$\Rightarrow n_1^2 - 2n_1 + 1 + n_2^2 + 1 - 2n_2 + \dots + n_k^2 - 2n_k + 1 + \text{non negative cross terms} = n^2 + k^2 - 2nk$$

$$\Rightarrow n_1^2 + n_2^2 + \dots + n_k^2 - 2(n_1 + n_2 + \dots + n_k) + k \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + k \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2n + k \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk + 2n - k$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 + k(k-2n) - (k-2n)$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 + (k-2n)(k-1)$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k) \quad \text{——— (1)}$$

Since, the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$,

the maximum number of edges in the i^{th} component of G is $\frac{n_i(n_i-1)}{2}$.

$\therefore$ The maximum no. of edges in G is

$$\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$

$$\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \sum_{i=1}^{k} \left(\frac{n_i^2 - n_i}{2}\right)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$$

$$\leq \frac{1}{2}\left[n^2 - (k-1)(2n-k) - n\right]$$

From (1)

$$= \frac{1}{2}\left[n^2 - 2nk + k^2 + 2n - k - n\right]$$

$$= \frac{1}{2}\left[n^2 - nk - nk + k^2 + n - k\right]$$

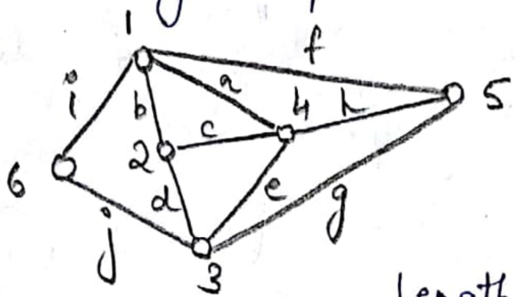
$$= \frac{1}{2}\left[n(n-k) - k(n-k) + (n-k)\right]$$

$$= \frac{1}{2}\left[(n-k)(n-k+1)\right]$$

∴ Maximum number of edges in G

$$\le \frac{(n-k)(n-k+1)}{2}.$$

Q. List all different paths between the vertices 5 and 6 in the figure given below. Give the length of each of these path.



		Length
1)	5 g 3 j 6	2
2)	5 f 1 i 6	2
3)	5 h 4 a 1 i 6	3
4)	5 h 4 e 3 j 6	3
5)	5 h 4 c 2 b 1 i 6	4
6)	5 h 4 c 2 d 3 j 6	4
7)	5 f 1 b 2 d 3 j 6	4
8)	5 g 3 d 2 b 1 i 6	4
9)	5 f 1 a 4 e 3 j 6	4
10)	5 g 3 e 4 a 1 i 6	4
11)	5 h 4 a 1 b 2 d 3 j 6	5
12)	5 h 4 e 3 d 2 b 1 i 6	5
13)	5 f 1 a 4 c 2 d 3 j 6	5
14)	5 g 3 e 4 c 2 b 1 i 6	5

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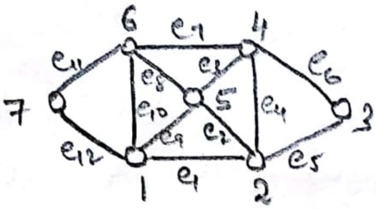
MODULE - II

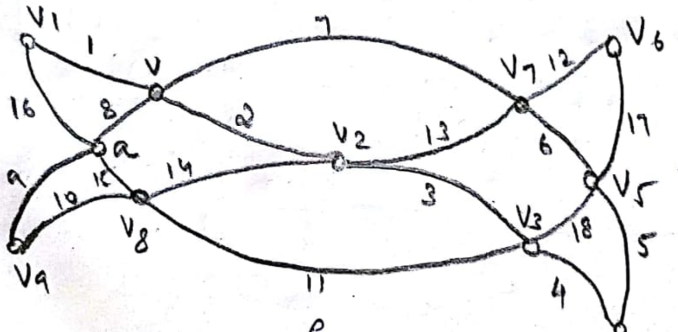
EULER GRAPHS

If some closed walk in a graph contains all the edges of the graph, then the walk is called an Euler line & the graph is called an Euler graph.

Eg:



$1\, e_1\, 2\, e_5\, 3\, e_6\, 4\, e_7\, 6\, e_{11}\, 7\, e_{12}\, 1\, e_9\, 5\, e_2\, 2\, e_4\, 4$
$e_3\, 5\, e_8\, 6\, e_{10}\, 1$



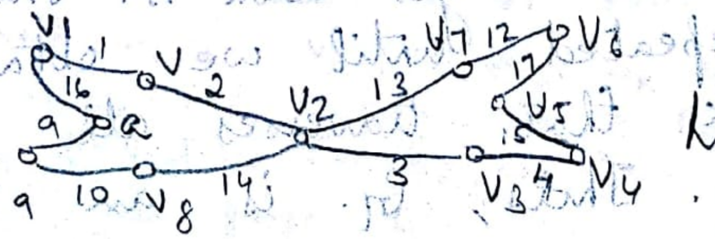
G

THEOREM: A given connected graph G is an Euler graph if and only if all vertices of G are of even degree.

Proof: Suppose that G is an Euler graph. It therefore contains an Euler line which is a closed walk. In tracing this walk, we observe that every time the walk meets a vertex v, it goes through 2 new edges incident on v with one entered to v & with the other exited.

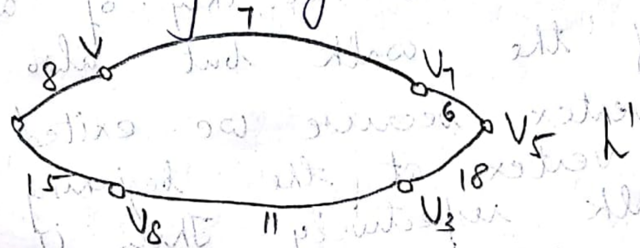
This is true not only of all intermediate vertices of the walk but also of the terminal vertex because we exited & entered the same vertex at the beginning and end of the walk respectively. Thus, if G is an Euler graph, the degree of every vertex is even.

29/8/'17 To prove the sufficiency of the condition, assume that all vertices of G are of even degree. Now, we construct a walk starting at an arbitrary vertex v and going through the edges of G such that no edge is traced more than once. We continue tracing as far as possible. Since every vertex is of even degree, we can exit from every vertex we enter. The tracing cannot stop at any vertex but v.



Since v is of even degree, we shall eventually reach v, when the tracing comes to an end.

If the closed walk h includes all the edges of G, then G is an Euler graph. If not, remove from G all the edges in h and obtain a subgraph h' of G formed by the remaining edges.

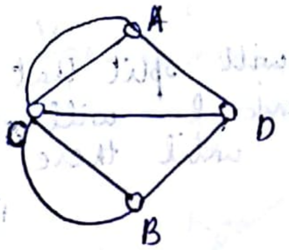


Since, both G & h have all their vertices of even degree, the degree of h' are also even.

Moreover h' must touch h atleast at one vertex a because G is connected. Starting from a we can again construct a new walk in graph h'. Since all the vertices of h' are of even degree, this walk in h' must terminate at a. But this walk in h' can be combined with h to form a new walk which starts & ends at vertex v and has more edges than h. This process can be repeated until we obtain a closed walk that traverses all the edges of G. Thus, G is an Euler Graph.

Hence, the theorem.

Note: 1) By looking at the graph of Konisberg bridge problem, we find that not all of its vertices are of even degree. Hence, it is not an Euler graph. Thus, it is not possible to walk over each of the seven bridges exactly once & return to the starting point.



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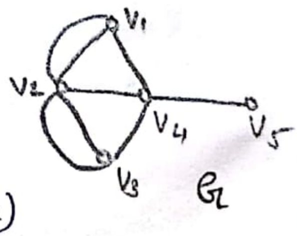
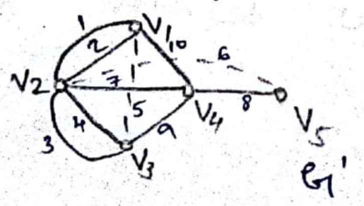
2) An open walk that includes all the edges of a graph without retracing any edge is a unicursal line or an open Euler line.

A connected graph that has a unicursal line is called a Unicursal graph.

THEOREM: In a connected graph G with exactly $2k$ odd vertices, there exists k edge disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

PROOF: Let the odd vertices of the given graph be named $v_1, v_2, \dots v_k$; $w_1, w_2, \dots w_k$ in any arbitrary order. Add k edges to G between the pair of vertices $(v_1, w_1), (v_2, w_2), \dots (v_k, w_k)$ to form a new graph G'.



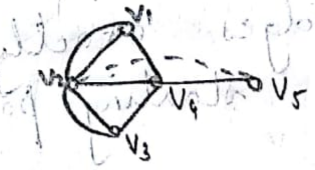


(In G, v_1, v_2, v_3, v_5 are the vertices of odd degree. We call v_3 as w_1 & v_5 as w_2).

Since every vertex of G' is of even degree, G' consist of an Euler line S.

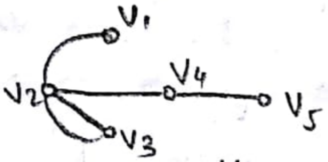
$S = V_1 1 V_2 3 V_3 4 V_2 7 V_4 8 V_5 6 V_2 2 V_1 10 V_4 9 V_3 5 V_1$

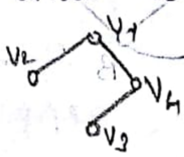
Now if we remove from S the k edges we just added S will split into k walks each of which is a unicursal line.



$S = V_1 1 V_2 3 V_3 4 V_2 7 V_4 8 V_5 6 V_2 2 V_1 10 V_4 9 V_3$

The second removal will split that into 2 unicursal line. And each successive removal will split the unicursal line into 2 unicursal lines until there are k of them.





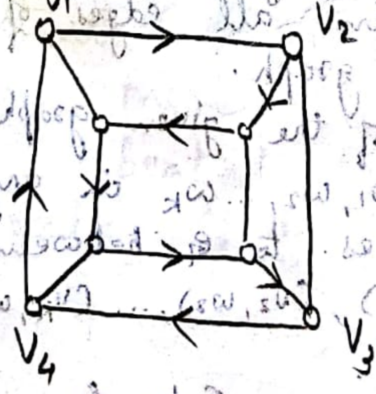
1) $V_1 1 V_2 3 V_3 4 V_2 7 V_4 8 V_5$

2) $V_2 2 V_1 10 V_4 9 V_3$

Hence the theorem.

HAMILTONIAN PATH & CIRCUITS

A Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once except the initial and the final vertex.



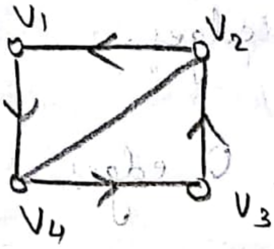
G

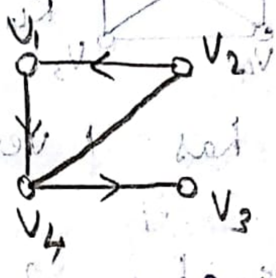
Eg:

Not every connected graph has a Hamiltonian circuit.

⇒ If we remove any one edge from a Hamiltonian circuit, we are left with a path. This path is called Hamiltonian Path.

Eg:





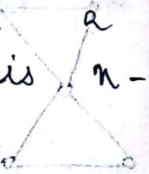
Clearly, a Hamiltonian Path in a graph G traverses every vertex of G. Since, a Hamiltonian path is a subgraph of a Hamiltonian circuit, every graph that has a Hamiltonian circuit also has a Hamiltonian path.

But, there are graphs with Hamiltonian path that have no Hamiltonian circuit.

<u>Note</u>:

1. A self loop or parallel edges cannot be in a Hamiltonian circuit.
2. A Hamiltonian circuit in a graph of n vertices consist of exactly n edges.

3. Length of a Hamiltonian path in a connected graph of n vertices is $n-1$.



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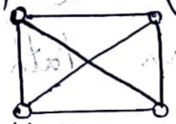
Complete Graph

A simple graph in which there exits an edge between every pair of vertices is called a complete graph.



K_4 4 vertices
6 edges.

K_1 K_1 has 1 vertex, 0 edges

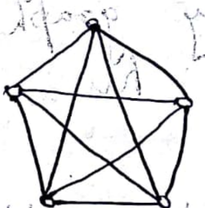
K_2

K_2 has 2 vertices, 1 edge

K_3

K_3 - vertices - 3, 3 edges

K_5



5 vertices
10 edges.

$\therefore$ Total number of edges in a complete graph with n vertices is given by

$${}^nC_2 = \frac{n(n-1)}{2}$$

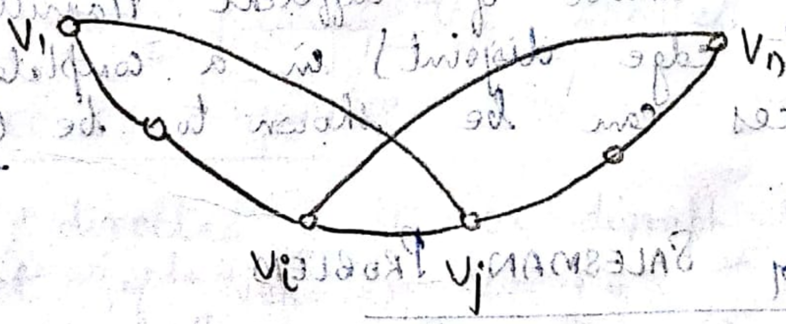
DIRAC'S THEOREM:

A sufficient condition for a simple graph G to have a Hamiltonian circuit is that the degree of every vertex in G be atleast $n/2$ where 'n' is the number of vertices.

PROOF: We assume the theorem false and derive a contradiction. So let G be a simple non Hamiltonian graph with n vertices satisfying the given condition on the vertex degrees.

By adding extra edges if necessary, we may assume that G is non Hamiltonian & simple in the sense that addition of any further edges gives a Hamiltonian graph.

But, since G is non Hamiltonian, vertices v_1 and v_n are not adjacent. Given that $\deg(V_i) \geq n/2$ for each i then it follows that there must be some vertex v_j adjacent to v_1 and some vertex v_i adjacent to v_n.



But, this gives us the required contradiction since

$$V_1 \rightarrow V_2 \rightarrow \cdots V_i \rightarrow V_n \rightarrow \cdots \rightarrow V_j \rightarrow \cdots V_1$$

which is a Hamiltonian circuit.

Hence the proof.

DIFFERENT TYPES OF WALK:

- WALK
  - Open Walk
    - Path
      - Hamiltonian Path
    - Unicursal line
  - Closed Walk
    - Circuit
      - Hamiltonian Circuit
    - Euler graph

Weighted Graph

In a graph, every edge e_i, there is associated a real number (the distance in miles say) and is denoted by $w(e_i)$. Such a graph is known as weighted graph. $w(e_i)$ being the weight of edge e_i.

Note: The total number of different Hamiltonian circuits (not edge disjoint) in a complete graph of n vertices can be shown to be $\frac{(n-1)!}{2}$.

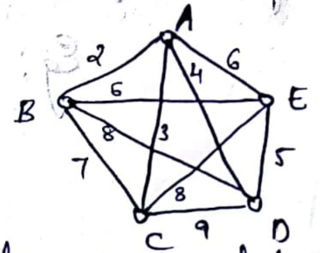
18/09/17 TRAVELLING SALESMAN PROBLEM

In this problem, a travelling salesman wishes to visit several given cities and return to his starting point covering the least possible total distance.

For eg: If there are 5 cities A, B, C, D, E and if the distance are given below, then the shortest possible route is

$A \to B \to D \to E \to C \to A$

giving a total distance of 26 as seen by inspection.



This problem can also be reformulated in terms of weighted graphs. In this case, the requirement is to find a Hamiltonian circuit of least possible total weight in a weighted complete graph. Note that, as in the shortest problem, the numbers can also refer to times taken to travel between their cities or costs involved in doing so.

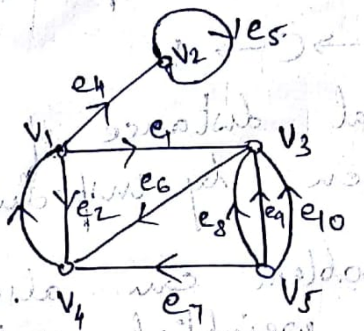
Theoretically, the problem of travelling salesman can always be solved by enumerating all $\frac{(n-1)!}{2}$ Hamiltonian circuits, calculating the distance travelled in each and then picking the shortest one.

DIRECTED GRAPHS

A directed graph or digraph G consist of a set of vertices $V = \{v_1, v_2, \dots\}$ a set of edges $E = \{e_1, e_2, \dots\}$ and a mapping ψ that maps every edge and some ordered pair of vertices (v_i, v_j).

In case of directed graph, a vertex is represented by a point and an edge by a line segment between the vertices v_i & v_j with an arrow directed from v_i to v_j.

Eg:



This is a directed graph with 5 vertices & 10 edges.

* The vertex v_i which the edge e_k is incident out of is called the initial vertex of e_k.
The vertex v_j which the edge e_k is incident into is called the terminal vertex of e_k.
In the above figure, v_5 is the initial vertex & v_4 is the terminal vertex of the edge e_7.

* An edge for which the initial & the terminal vertex are the same forms a self loop such as e_5.

* The number of edges incident out of a vertex v_i is called the out-degree or out-valence or outward demidegree of v_i and is written as

$$d^+(v_i).$$

The number of edges incident into v_i is called the in-degree or in-valence or inward demidegree of v_i and is written as

$$d^-(v_i)$$

In the above figure,

$v_1 \longrightarrow d^+(v_1) = 3$

$d^-(v_1) = 1$

$v_2 \longrightarrow d^+(v_2) = 1$

$d^-(v_2) = 2$

$v_5 \longrightarrow d^+(v_5) = 4$

$d^-(v_5) = 0$

In any digraph G the sum of all out-degrees is equal to sum of all in-degrees, each sum being equal to number of edges in G.

i.e; $\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i)$.

An isolated vertex is a vertex in which the in-degree & out-degree are both equal to 0.

A vertex in a digraph is called a pendant vertex if it is of degree 1 i.e; $d^+(v) + d^-(v) = 1$.

Two directed edges are said to be parallel if they are mapped onto the same ordered pair of vertices.

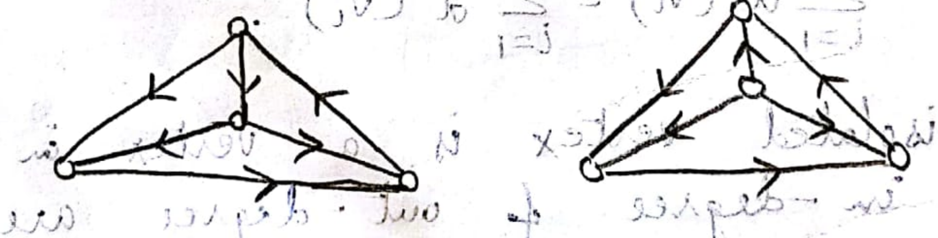
Eg: e_8, e_9, e_{10} are parallel edges whereas e_2, e_3 are not parallel.

25/09/17

* Given an undirected graph H, we can assign each edge of H some arbitrary direction. The resulting digraph designated by $\vec{H}$ is called an Orientation of H.

Isomorphic Digraphs

~~Iso~~ Two digraphs are said to be isomorphic not only their corresponding undirected graphs be isomorphic but the directions of the corresponding edges must also agree.



Two non-isomorphic graphs.

TYPES OF DIGRAPHS

1. Simple Digraph.

A digraph that has no loop or parallel edges is called a simple digraph.

2. Asymmetric Digraph.

Digraphs that have atmost one directed edge between every pair of vertices but are allowed to have self loop are called asymmetric digraph or anti-symmetric digraph.

3. Symmetric Digraph

Digraphs in which for every edge (a, b) (from vertex a to vertex b), there is also an edge (b, a)

A digraph that is both simple & symmetric is called simple symmetric digraph.

A digraph that is both simple & asymmetric is called simple asymmetric digraph.

4. Complete Digraph

A complete undirected graph is defined as a simple graph in which every vertex is joined to every other vertex exactly by one edge.

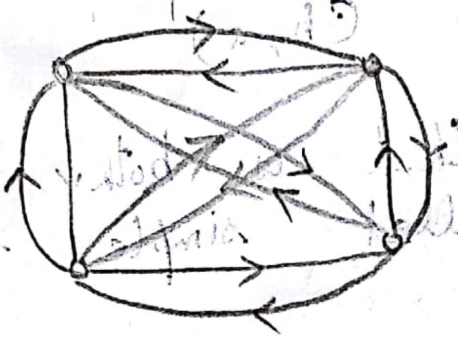
A complete symmetric digraph is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex.

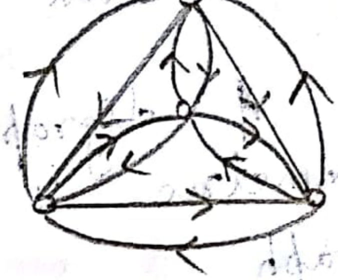
Scanned by CamScanner

A complete asymmetric digraph is an asymmetric digraph in which there is exactly one edge between every pair of vertices.

* A complete symmetric digraph contains $n(n-1)$ edges.

Q. Draw a complete symmetric digraph with 4 vertices.





A complete asymmetric digraph is also called a tournament or complete tournament.

A digraph is said to be balanced if for every vertex v_i, the in-degrees equals the out-degrees.

i.e., $d^-(v_i) = d^+(v_i)$

It is also referred to as pseudo-symmetric digraph or iso graph.

A balanced digraph is said to be regular if every vertex has the same in-degree & out-degree as every other vertex.

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Digraphs & Binary Relation

In a set of objects X where $X = \{x_1, x_2, \ldots\}$, a binary relation R between (x_i, x_j) may exist.

We write it as

$x_i R x_j$

and say that x_i has relation R to x_j.

Relation R may be (R→) 'is greater than', 'is equal to' and so on in the case where X consists of numbers.

A digraph is the most natural way of representing a binary relation on a set X. Each $x_i \in X$ is represented a vertex x_i. If x_i has a specified relation R to x_j, a directed edge is drawn from x_i to x_j for every pair (x_i, x_j).

For eg: the digraph given below represents the relation 'greater than' on a set consisting of 5 numbers (say 3, 4, 5, 7, 8).

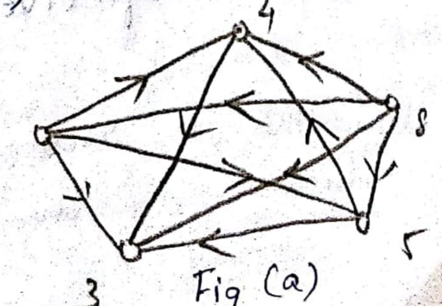


Fig (a)

Clearly, every binary relation on a finite set can be represented by a digraph without parallel edges.

Conversely, every digraph without parallel edges defines a binary relation on a set of its vertices.

Reflexive Relation

For some relation R it may happen that every element is in relation R to itself. For eg: A number is always equal to itself or a line is always parallel to itself.

Such a relation R on a set X that satisfies $x_i R x_i$ for every $x_i \in X$ is called reflexive relation.

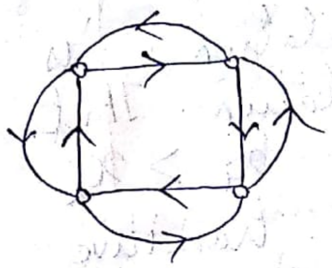
The digraph of reflexive relation will have a self loop at every vertex. Such a digraph representing a reflexive binary relation on its vertex set may be called a reflexive digraph.

A digraph in which no vertex has a self loop is called irreflexive relation.

SYMMETRIC RELATION

For some relation R, it may happen that for all x_i & x_j if $x_i R x_j$ holds then $x_j R x_i$ must hold. Such a relation is called a symmetric relation.

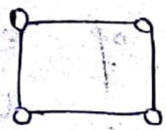
Eg: 'Is spouse of' is a symmetric relation but irreflexive.
'is equal to' is a symmetric & reflexive relation.



The above figure is irreflexive, symmetric digraph.

The digraph of a symmetric relation is a symmetric digraph because for every directed edge from vertex x_i to x_j, there is a directed edge from x_j to x_i.

The same relation can be represented by drawing just one undirected edge between every pair of vertices that are related.



Thus, every undirected graph is a representation of some Symmetric binary relation.

Furthermore, every undirected graph with e edges can be thought of as a symmetric digraph with $2e$ directed edges.

TRANSITIVE RELATION

A relation R is said to be transitive if for any three elements x_i, x_j, x_k in the set X, $x_i R x_j$ and $x_j R x_k$ always implies $x_i R x_k$.

The binary relation 'is greater than' is a transitive relation. That is, if $x_i > x_j$ and $x_j > x_k$ implies $x_i > x_k$.

Eg: Fig (a) is a transitive relation but irreflexive & asymmetric.

A digraph representing a transitive relation on its vertex set is called a transitive directed graph.

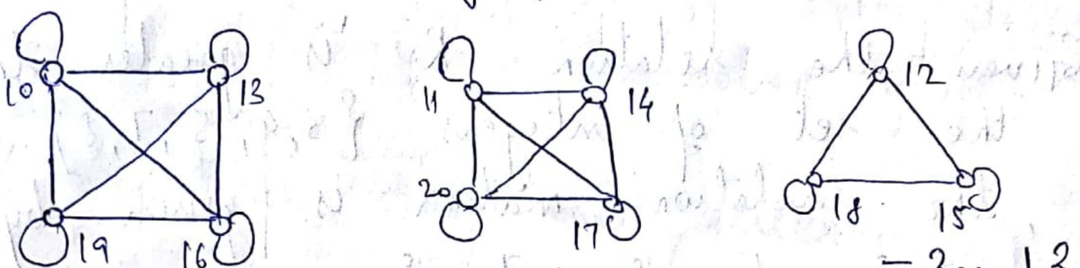
EQUIVALENCE RELATION

A binary relation is called an equivalence relation if it is reflexive, symmetric & transitive.

Some examples of equivalence relation are 'is parallel to', 'is equal to', 'is congruent to' & 'is isomorphic to'.

The graph representing equivalence relation may be called an equivalence directed graph.

Consider a relation 'is congruent to modulo 3' defined on the set of 11 integers from 10 to 20. The graph is shown below.



$\equiv 1 \bmod 3$ $\equiv 2 \bmod 3$ $\equiv 3 \bmod 3$

In the above figure we see that the vertex set of the graph is divided into three disjoint classes each in a separate component. Each component is an undirected graph due to symmetry with a self loop at each vertex, due to reflexivity.

Furthermore, in each component, every vertex is related to every other vertex.

$\therefore$ The given relation is an equivalence relation.

An equivalence relation on a set partitions the elements of the set into classes called the equivalence classes such that two elements are in the same class iff they are related.

Relation Matrices

A binary relation R on a set can also be represented by a matrix called the relation matrix. It is is (0, 1) $n \times n$ matrix where n is the no. of elements in the set. The (i,j)th entry in the matrix is 1 if $x_i R x_j$ is true & is 0 otherwise.

Eg: Given the relation R 'is greater than' on the set of integers $\{3,4,5,7,8\}$. Then, the relation matrix is given by

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 3 & 4 & 5 & 7 & 8 \end{array} \\
\begin{array}{c} 3 \\ 4 \\ 5 \\ 7 \\ 8 \end{array} &
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$

03/10/17

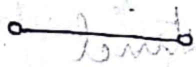
MODULE – III

TREES

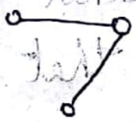
A tree is a connected graph without any circuits.

Eg: Tree with one vertex.

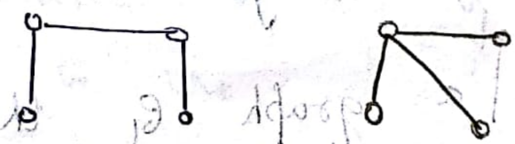
Tree with two vertices.



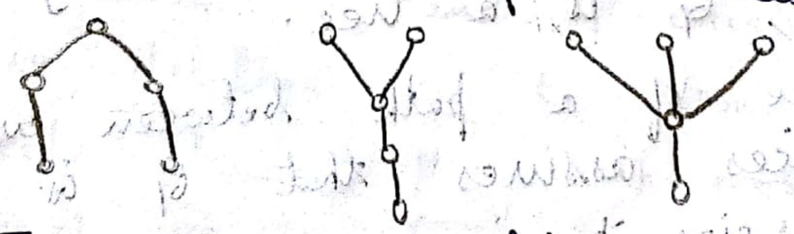
Tree with three vertices



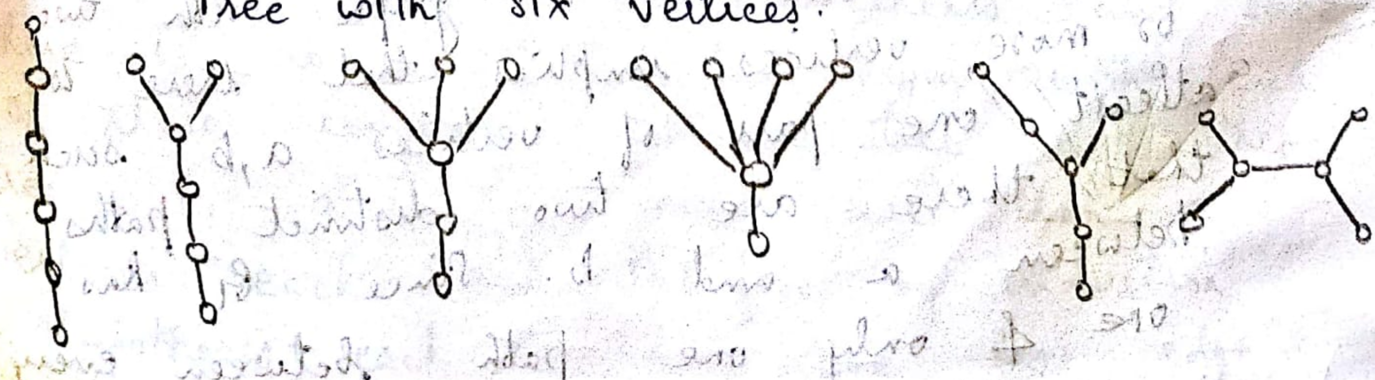
Tree with four vertices.



Tree with five vertices (Assg. Q1)



Tree with six vertices.



Tree has no self loop nor parallel edges.
∴ Trees have to be a simple graph.

Properties of Trees

Theorem 1:

There is one and only one path between every pair of vertices in a tree, T.

Proof: Since T is a connected graph, there must exist atleast one path between every pair of vertices in T. Now suppose that, between two vertices a and b of T, there are two distinct paths. The union of these two distinct paths will contain a circuit and T cannot be a tree.

∴ Our assumption that there are two distinct path between every pair of vertices is wrong. Hence the theorem.

Theorem 2:

If in a graph G there is one and only one path between every pair of vertices, G is a tree.

Proof: Existance of a path between every pair of vertices assures that G is connected.

A circuit in a graph with two or more vertices implies that there is atleast one pair of vertices a, b such that there are two distinct paths between a and b. Since G has one & only one path between every

pair of vertices, G can have no circuits.
Thus, G is connected & G has no circuits
This shows that G is a tree.

Hence the theorem.

Theorem 3: A tree with n vertices has $(n-1)$ edges.

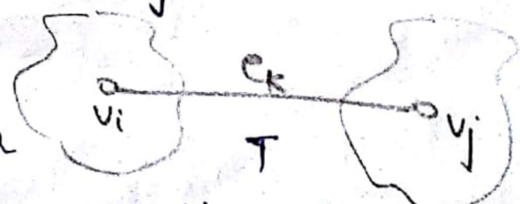
Proof: The theorem will be proved by induction on the number of vertices.

Suppose that $n=1$. Then T has no edges which implies, no. of edges $=0=(n-1)$

It is easy to see that the theorem is true for $n=1,2,3$.

Assume that the theorem holds for all trees with fewer than 'n' vertices.

Let us now consider a tree T with n vertices. In T, let e_k be an edge with end vertices v_i and v_j.



According to the theorem, there is no other path between v_i and v_j except e_k.
Therefore, deletion of e_k from T will disconnect the graph. Furthermore, $T-e_k$ consists of exactly two components and since there is no circuit to begin with, each of these component is a tree. Both these trees t_1 and t_2 have fewer than n vertices each and therefore by induction hypothesis, each contains one less edge

than the number of vertices in it.
Thus, $T - e_k$ consist of $(n-2)$ edges.
Hence T has exactly $(n-1)$ edges.

Assg. Q2.
Q.
Theorem 4: Any connected graph with n vertices and $(n-1)$ edges is a tree.

Let G be a connected graph with n vertices and $(n-1)$ edges. We show that G contains no cycle. Assume to the contrary that G contains cycle.

Remove an edge from a cycle so that the resulting graph is again connected. Continue this process of removing one edge from the cycle one at a time till the resulting graph H is a tree. As H has n vertices so number of edges in G is greater than the number of edges in H, so $n-1 > n-1$, which is not possible. Hence G has no cycles and therefore is a tree.

A connected graph is said to be minimally connected if removal of any one edge from it disconnects the graph.

A minimally connected graph cannot have a circuit and hence, a minimally connected graph is a tree.

HW: Theorem 5: A graph is a tree iff it is minimally connected.

Let G be minimally connected. It is therefore a connected graph. Also the removal of any one edge from it will disconnect G. Therefore there is only one path between every pair of vertices. Hence it is a tree.

Conversely, let G be a tree. Then G has no circuits and deletion of any one edge will disconnect it. Hence, it is minimally connected.

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THEOREM 6: A graph G with n vertices, $(n-1)$ edges and no circuit is connected.

Proof: Suppose that there exist a circuitless graph G with n vertices & $(n-1)$ edges which is disconnected. In that case, G will consist of two or more circuitless components. Without loss of generality, let G consist of two components g_1 and g_2. Add an edge e between a vertex v_1 in g_1 and v_2 in g_2. Since there was no path between v_1 and v_2 in G, adding e did not create a circuit.

Thus $G \cup e$ is a circuitless connected graph that is a tree of n vertices and n edges which is not possible since a tree with n vertices has $(n-1)$ edges.

$\therefore$ G is a connected graph.

Hence the theorem.

NOTE: 1) A graph G with n vertices is called a tree if

i) G is connected and G is circuitless

OR

ii) G is connected and has $(n-1)$ edges

OR

iii) G is circuitless and has $(n-1)$ edges

OR

iv) There is exactly one path between every pair of vertices in G

OR

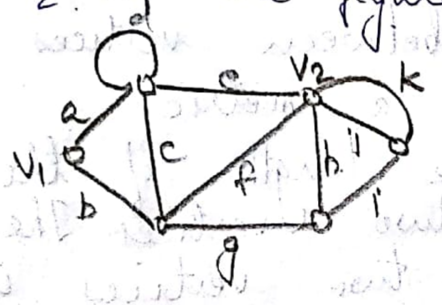
v) G is minimally connected.

Distance & Centres in a Tree

In a connected graph G, the distance between two of its vertices v_i & v_j denoted by $d(v_i, v_j)$ is the length of the shortest path that is, the no. of edges in the shortest path.

The definition of distance between any two vertices is valid for any connected graph not necessarily a tree.

Eg: Q. Find the distance between the vertices v_1 and v_2 in the figure given below

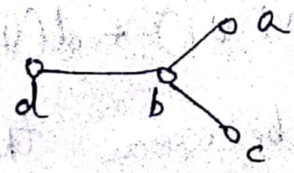


(a, e)
(b, c, e)
(b, f)
(b, g, h)

The distance is 2.

In a tree since there is exactly one path between any two vertices, the determination of distance is much easier.

Eg:



$d(a, b) = 1$
$d(a, c) = 2$
$d(b, c) = 1$
$d(a, d) = 2$

METRIC

A function which satisfies the following three conditions is called a metric.

1> Non-negativity ie; $f(x,y) \geq 0$ and $f(x,y) = 0$ iff $x = y$

2> Symmetry
$$f(x, y) = f(y, x)$$

3> Triangle inequality
$$f(x, y) \leq f(x, z) + f(z, y) \text{ for any } z.$$

06/10/17

Theorem 8:

The distance between vertices of a connected graph is a metric.

Proof: 1> Distance is the length of the shortest path between two vertices. Therefore, the distance between two vertices is always greater than or equal to 0.

i.e; $d(v_i, v_j) \geq 0$ or $d(v_i, v_j) = 0$ iff $v_i = v_j$

2> $d(v_i, v_j) = d(v_j, v_i)$

3> Since, $d(v_i, v_j)$ is the length of the shortest path between the vertices v_i and v_j, this path cannot be longer than another path between v_i & v_j which goes through the specified vertex v_k.

Hence $d(v_i, v_j) \leq d(v_i, v_k) + d(v_k, v_j)$

Hence, the distance between the vertices is a metric.

Hence the theorem.

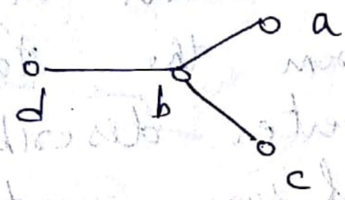
ECCENTRICITY

The eccentricity $E(v)$ of a vertex v in a graph G is the distance from v to the vertex farthest from v in G.

i.e; $E(v) = \max\limits_{v_i \in G} d(v, v_i)$

A vertex with minimum eccentricity in a graph G is called the 'center' of G.

Eg:

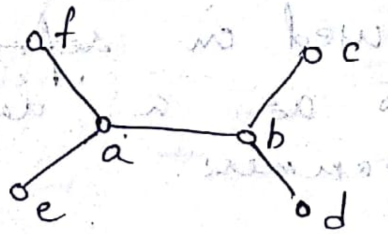


$E(a) = 2$
$E(b) = 1$
$E(c) = 2$
$E(d) = 2$

Center : b

Here, the vertex b is the center of the tree.

Q.



$E(a) = 2$
$E(b) = 2$
$E(c) = 3$
$E(d) = 3$
$E(e) = 3$
$E(f) = 3$

Center of the tree is a, b.

Theorem 7: In any tree with two or more vertices, there are atleast two pendant vertices.

Eg. Given a sequence of integers no two which are the same. Find the largest monotonically increasing subsequence in it. Suppose that the given sequences to us is

4 1 13 7 0 2 8 11 3

It can be represented by a tree in which the vertices except the start vertex represent the individual number in the sequence & the path from the start vertex to a particular vertex describes the monotonically increasing sub sequence terminating in v. The figure is shown below.

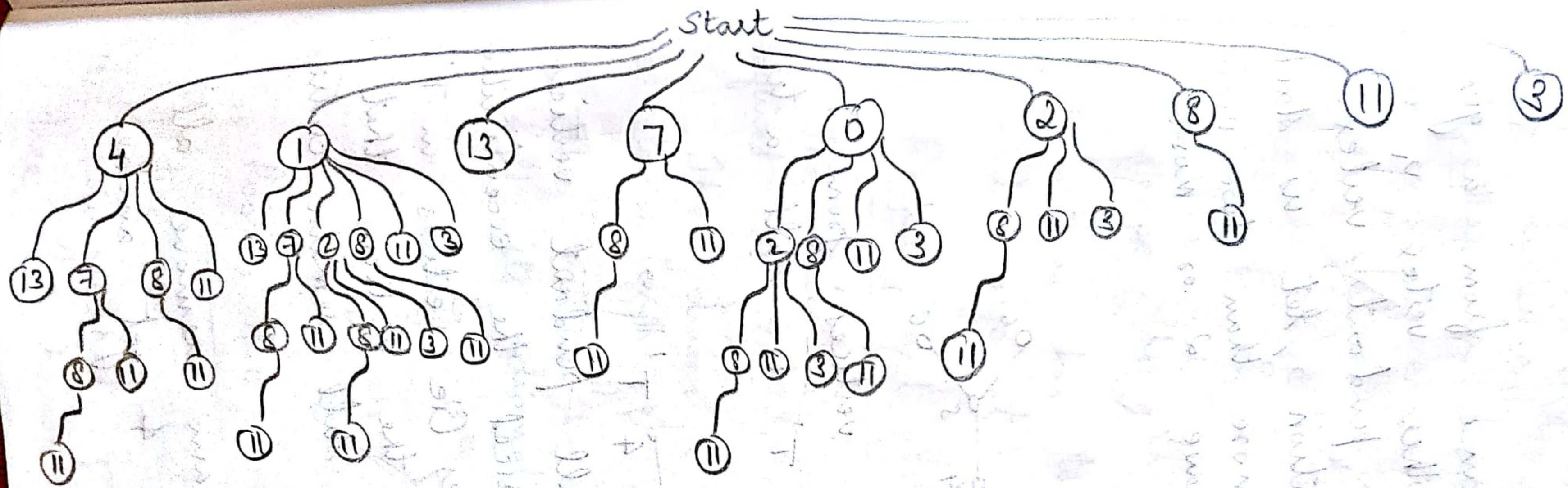
Figure →

The sequence contains 4 longest monotonically increasing subsequences

(4, 7, 8, 11), (1, 7, 8, 11), (1, 2, 8, 11), (0, 2, 8, 11).

Each is of length 4.

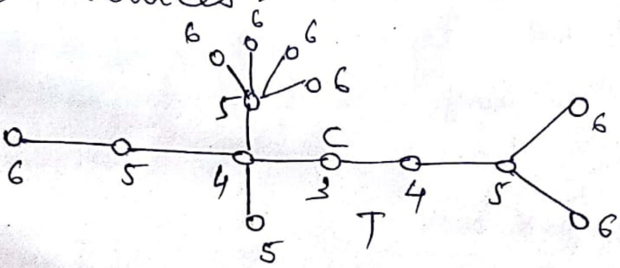
Such a tree used in representing data is referred to as a 'data tree' by computer programmers.


Start

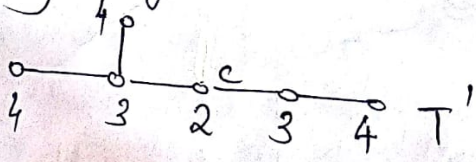
Theorem 9: Every tree has either one or two centers.

Proof: The maximum distance from a given vertex v to any other vertex v_i occurs only when v_i is a pendant vertex.

With this observation, let us start with a tree having more than two vertices. The tree T must have 2 or more pendant vertices.

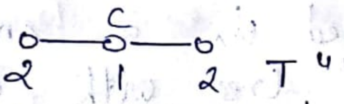


Delete all pendant vertices from T. The resulting graph T' is still a tree.



The removal of all pendant vertices from T uniformly reduced the eccentricities of the remaining vertices (ie vertices in T') by one. Therefore all the vertices that T had as centres will still remain centres in T'.

From T' we again remove all the pendant vertices & get another tree T''.



We continue this process until there is left either a vertex which is the center of T or an edge whose end vertices are the two centres of T.

Hence the theorem.

<u>Note</u>: If a tree has two centres, the two centres must be adjacent.

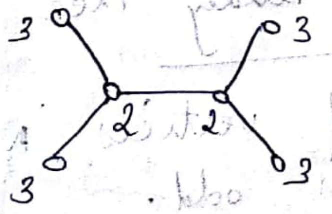
07/10/17 <u>Radius and Diameter</u>

The eccentricity of a center in a tree is defined as the radius of the tree.

The diameter of a tree T is defined as the length of the longest path in T.

<u>Note</u>: The radius in a tree is not necessarily half its diameter.

Eg:



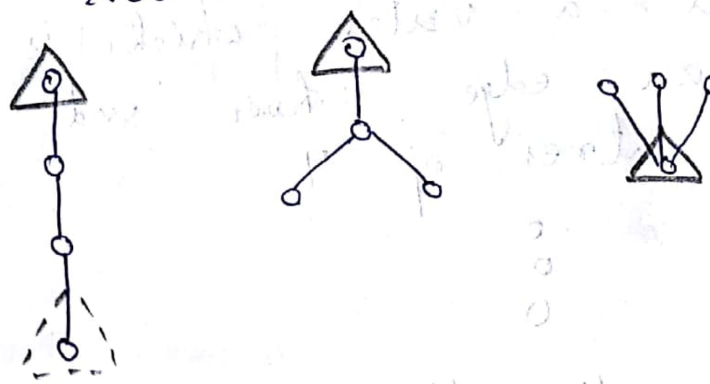
R = 2
D = 3

<u>ROOTED AND BINARY TREE</u>

A tree in which one vertex (called the root) is distinguished from all other vertices. Such tree is known as a rooted tree.

The root is enclosed in a small triangle.

Eg: All the rooted trees with 4 vertices are shown below



Tree without any root are called free trees or non-rooted trees.

Binary Trees

A binary tree is defined as a tree in which there is exactly one vertex of degree 2 and each of the remaining vertices is of degree 1 or 3.

Since the vertex of degree 2 is distinct from all other vertices, this vertex serves as the root.

Properties of Binary Tree

1) The number of vertices 'n' in a binary tree is always odd.

$1+3+2+1$

This is because there is exactly one vertex of even degree and the remaining $(n-1)$ vertices are of odd degree. Since from a theorem, we know that the number of vertices of odd degree is always an even number.

i.e; $(n-1)$ is an even number

which shows that 'n' is an odd number.

2) Let p be the number of pendant vertices in a tree. Then $(n-p-1)$ is the number of vertices of degree 3. Then prove that $p = \frac{n+1}{2}$

Total number of vertices $= n$

No. of pendant vertices $= p$

No. of vertex with even degree $= 1$

No. of vertices with degree $3 = (n-p-1)$

We know that a tree with n vertices has $(n-1)$ edges.

By a theorem $\sum d(v) = 2 \times$ no. of edges

i.e; $1 \times 2 + p \times 1 + (n-p-1) \times 3 = 2 \times (n-1)$

$\Longrightarrow p = \frac{n+1}{2}$

A non pendant vertex in a tree is called an internal vertex.

Q. Show that the no. of internal vertices in a binary tree is one less than the number of pendant vertices.

We know that number of pendant vertices equal to $\frac{n+1}{2}$

Total no. of vertices $= n$

i.e; No. of pendant vertices + No. of internal vertices $= n$

$\Rightarrow$ No. of internal vertices $= n - \frac{n+1}{2}$

$= \frac{n-1}{2}$

i.e; No. of pendant vertices $-1 = \frac{n+1}{2} - 1$

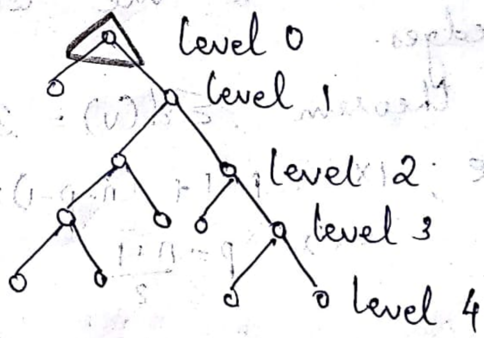
$= \frac{n-1}{2}$

= No. of internal vertices

09/10/'17

* In a binary tree, a vertex v_i is said to be at a level l_i if v_i is at a distance of l_i from the root. Thus, the root is at level 0.

Eg:



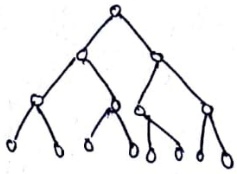
A 13 vertex with 4 level binary tree is shown above.

The number of vertices at level 1, 2, 3, 4 are 2, 2, 4, 4.

Clearly, there can only be one vertex at level 0, atmost 2 vertices at level 1, atmost 4 vertices at level 2 and so on.

$\therefore$ Maximum number of vertices possible in a k level binary tree is $2^0+2^1+2^2+\dots+2^k$.

The maximum level l_{max} of any vertex in a binary tree is called the height of the tree.



Minimum possible height of n vertices in a binary tree is

$$\min l_{max} = \lceil \log_2 (n+1) - 1 \rceil$$

Proof: $2^0+2^1+2^2+\dots+2^k = n$

$$\frac{1(2^{k+1}-1)}{2-1} = n \qquad S = \frac{a(r^n-1)}{r-1}$$

$$2^{k+1} = n+1 \qquad \log a^b = b\log a$$

$$\log_2 2^{(k+1)} = \log_2 (n+1)$$

$$(k+1)\log_2 2 = \log_2 (n+1)$$

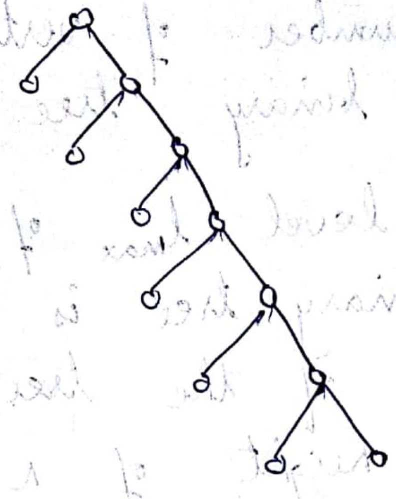
$$(k+1) = \log_2 (n+1)$$

$$\therefore k = \lceil \log_2 (n+1) - 1 \rceil \to \text{greatest integer}$$

On the other hand, to construct a binary tree for a given n vertices such that the farthest vertex is as far as possible from the root, we must have exactly two vertices at each level except at level 0.

i.e; $\max l_{max} = \frac{n-1}{2}$

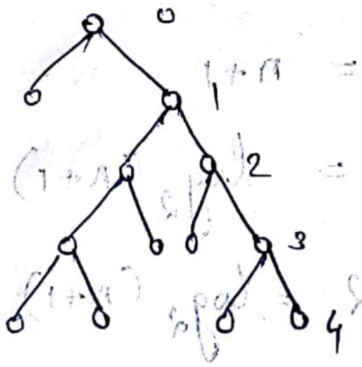
i.e; $1+2k = n \quad \therefore k = \frac{n-1}{2}$



Path length or External Path length

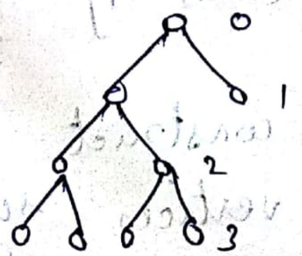
The path length of a tree can be defined as the sum of path lengths from the root to all pendant vertices.

1)



1 + 4 + 3 + 4 + 3 + 4 + 4 = 23

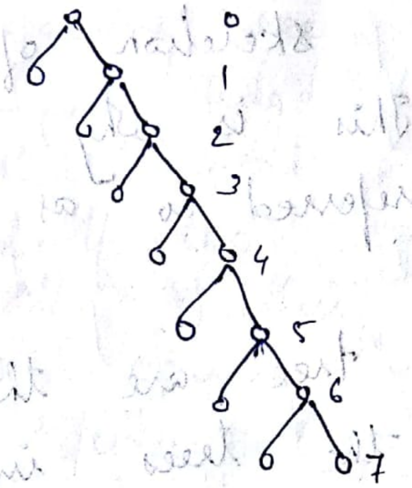
2)



~~1+4+4+4+4~~ +

1 + 3 + 3 + 3 + 3 = 13

3)



$1 + 2 + 3 + 4 + 5 + 6 + 7 + 7 = 35$

Spanning Tree

~~A spanning tree is a connected graph G~~

A tree is said to be a spanning tree of a connected graph G if T is a subgraph of G & T contains all the vertices of G.

(vertices can repeat)

Hamiltonian path $\Rightarrow$ spanning Tree

But spanning Tree $\not\Rightarrow$ Hamiltonian Path

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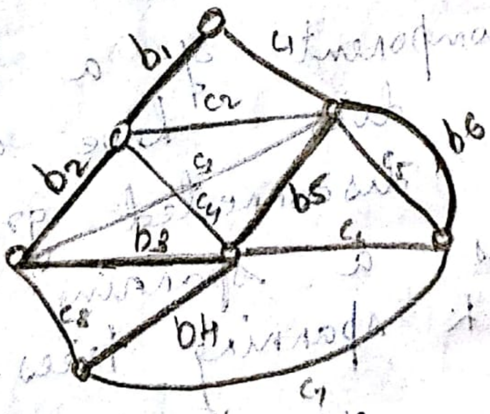

The subgraph in heavy lines in the above figure is a spanning tree of the graph.

Since the vertices of G are barely hanging together in a spanning tree, it is a sort of skeleton of the original graph G. This is why spanning tree is sometimes referred to as a <u>skeleton</u> or <u>scaffolding</u>.

Since spanning tree are the largest tree among all the trees in G, it is quite appropriate to call a spanning tree a Maximal tree subgraph or <u>Maximal tree</u> of G.

It is to be noted that the spanning tree is defined only for a connected graph because a tree is always connected & in a disconnected graph of n vertices we cannot find a connected subgraph of n vertices.

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Each component of a disconnected graph however does have a spanning tree. Thus, a disconnected graph with k components has a spanning forest consisting of k spanning trees.

<u>Theorem 10:</u> Every connected graph has atleast one spanning tree.

Proof: Let G be a connected graph. If G has no circuit, it is own spanning tree. If G has a circuit, delete an edge from the circuit. This will still lead the graph to be connected. If there are more circuits, repeat the operation till an edge from the last circuit is deleted leaving a connected circuitless graph that contains all the vertices of G.

Thus we have a connected graph that has atleast one spanning tree.

Hence, the theorem.

* An edge in a spanning tree T is called a <u>branch</u> of T.

An edge of G that is not in a given spanning tree T is called <u>chord</u>.

In the above figure, edges b1, b2, b3, b4, b5, b6 are the branches of spanning tree while the edges $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ are the chords.

It must be kept in mind that the branches & chords are defined only with respect to a given spanning tree.

An edge that is a branch of spanning tree T_1 may be a chord with w.r.to another spanning tree T_2.

13/10/17 A connected graph G can be considered as a union of two subgraphs T and $\overline{T}$.

i.e, $G = T \cup \overline{T}$

where T is the spanning tree and $\overline{T}$ is the complement of T in G. Since the subgraph $\overline{T}$ is the collection of chords it is referred to as the chord set or ~~T~~ Tie set or Co-tree.

Theorem: With respect to any of its spanning tree, a connected graph of 'n' vertices and 'e' edges has $(n-1)$ tree branches and $e-n+1$ chords.

Proof: Given that G is connected and contains a spanning tree. From the definition of spanning tree, it is clear that this tree contains all the n vertices. Therefore by a theorem, if a tree contains 'n' vertices, it has $(n-1)$ edge
$\therefore$ A spanning tree contains $(n-1)$ tree branches.

Let e be the number of edges in a graph G. Then the number of chords is equal to total no. of edges - no. of branches.

$$= e-(n-1) = e-n+1$$

∴ No. of chords = $e - n + 1$.

Hence the theorem.

Note: In the above figure, the graph has 6 branches & 8 chords. Any other spanning will yield the same number.

Q1. If we have an electrical network with e elements (edges) and n nodes (vertices) what is the minimum no. of elements we must remove to eliminate all circuits in a network.

It's spanning tree will have $(n-1)$ edges.

∴ Minimum no. of elements we must remove to eliminate all circuits in the network $= e - (n-1) = e - n + 1$

Q2. If we have a farm consisting of 6 walled plots as shown in the figure and the plots are full of water how many walls will have to be broken so that all water can be drained out.

No. of edges in a spanning tree of vertices = 7

∴ No. of walls to be removed $= 13 - 7 = 6$

Rank and Nullity

When someone specifies a graph G, the first thing we mostly like to mention is the number of vertices 'n', in G. Immediately following comes the number of edges in G. Then, the number of components k in G. If $k = 1$, then, the graph is connected.

How are these 3 numbers of a graph related?

Since every component of a graph must have atleast one vertex, the n is greater than or equal to k. Moreover, the number of edges in a component can be no less that the number of vertices in that component -1.

$\therefore e \geq n - k$

Apart from the constraints $(n-k) \geq 0$ and $(e - n + k) \geq 0$, these 3 numbers n, e, k are independent & they are the fundamental numbers in the graph. From these 3 numbers are derived two other important numbers called rank & nullity which is defined as

$$\boxed{\begin{aligned}\text{Rank}(\gamma) &= n-k \\ \text{Nullity}(\mu) &= e-n+k\end{aligned}}$$

$\frac{n(n-1)}{2} - n + 1$

* Rank of G = no. of branches in any spanning tree of G. $\frac{n^2 - 2n+2}{2}$

Nullity of G = no. of chords in G $\frac{n^2-3n+2}{2}$

$\therefore$ Rank + Nullity = No. of edges in G.

* The nullity of a graph is also referred to as cyclomatic number or petti number.

Weighted Path Length

It is denoted by a positive real number ω_j which is associated with every pendant vertex v_j of a binary tree.

On- Counting Trees

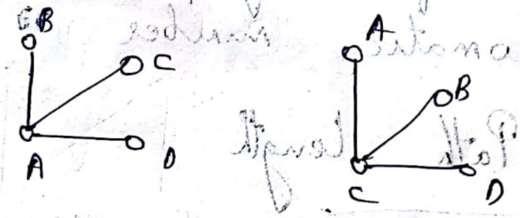
Problem of counting trees is the problem of counting structural isomers of a given hydrocarbon. For counting the number of structural isomers of the saturated hydrocarbon C_kH_{2k+2}, a connected graph is used to represent a C_kH_{2k+2} molecule. Corresponding to their chemical valencies, a carbon atom was represented by a vertex of degree 4 & hydrogen atom by a vertex of degree 1.

This graph is connected & the number of edges is 1 less than the number of vertices, i.e; it is a tree.

LABELLED GRAPH

A graph in which each vertex is assigned a unique name or label. That is no two vertices have the same label. Such graphs are called labelled graphs.

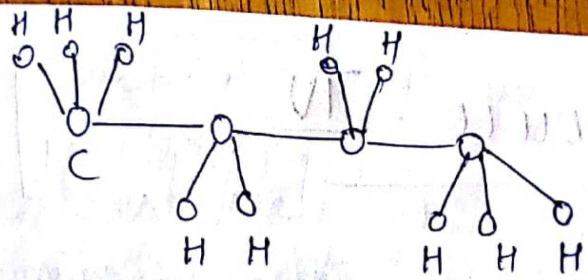
Eg:

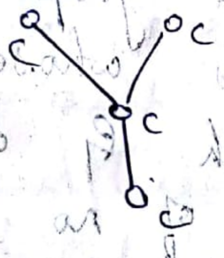


Note: In actual counting of isomers of C_kH_{2k+2}, since the vertices representing hydrogen are pendant they go with carbon atoms only one way & hence make no contribution to isomerism.

∴ We need not show any hydrogen vertices; Thus the tree representing C_kH_{2k+2} reduces to one with k vertices, each representing a carbon atom.

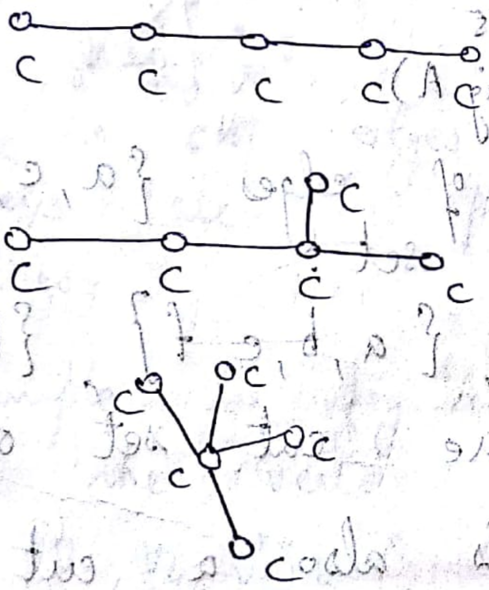
Q. How many isomers does C_4H_{10} have?



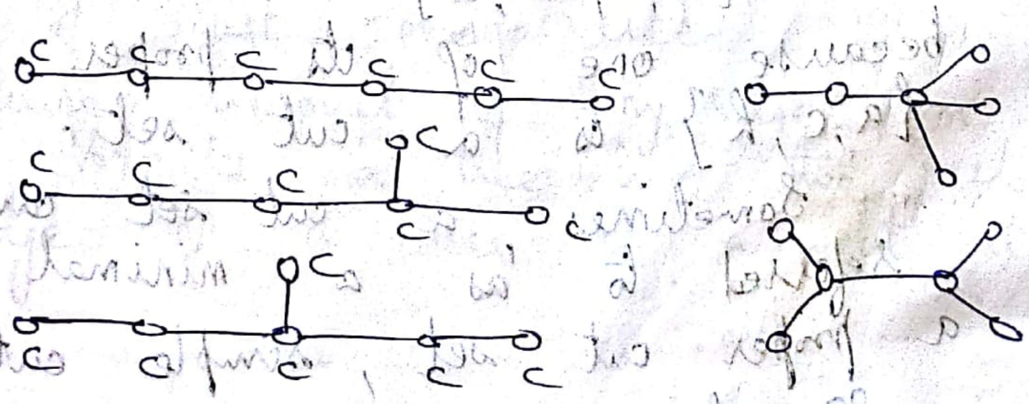


∴ 2 isomers.

2) C_5H_{12}



3) C_6H_{14}



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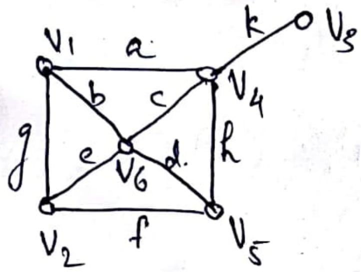
MODULE - IV

Cut-Set

In a connected graph G, a cut set is a set of edges whose removal from G leaves the graph G disconnected provided the removal of no proper subset of these edges disconnects the graph G.

Eg:



G (fig A)

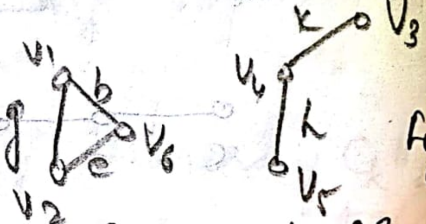


Fig. B

The set of edges {a, c, d, f} is a cut set.

{a, b, g}, {a, b, e, f}, {k} and {d, h, f} are cut set of G.

{a, c, h} is also a cut set.

But {a, c, h, d} is not a cut set because one of its proper subset {a, c, h} is a cut set.

Sometimes a cut set can be referred to as a minimal cut set, a proper cut set, simple cut set or co-cycle.

A cut set always cuts a graph into two.

∴ A cut set can also be defined as a minimal set of edges in a connected graph whose removal reduces the rank of the graph by one.

∴ The rank of the graph in fig. B is 4, one less than the rank of the graph in fig. A.

Another way of looking at a cut set is that if we partition all the vertices of a connected graph G into two mutually exclusive subsets, a cut set is a minimal no. of edges whose removal from G destroys all the paths between these two vertices.

For example in fig. A, cut set {a, c, d, f} connects the vertex set {v_1, v_2, v_6}, with {v_3, v_4, v_5}. Sometimes, one or both of these two subsets of vertices may consist of just one vertex.

Since removal of any edge from a tree breaks a tree into two parts every edge of a tree is a cut set.

Application

Suppose that the six vertices in Fig (A) represent 6 cities connected ~~in~~ by telephone lines. We wish to find out if there are any weak spots in the network that need strengthening by means of ~~an~~ additional telephone lines. We look at all cut sets of the graph and the one with smallest number of edges is most ~~vult~~ vulnerable.

In fig (A) the city represented by vertex v_3 can be saved from the rest of the network by the destruction of just one edge.

PROPERTIES:

1) ~~Consider~~ Every cut set in a connected graph G must contain atleast one branch of every spanning tree of G.

Q. Consider a spanning tree T in a connected graph G & an arbitrary cut set S in G. Is it possible for S not to have any edge common with T? Justify.

No. Otherwise the removal of the cut set from G would not disconnect

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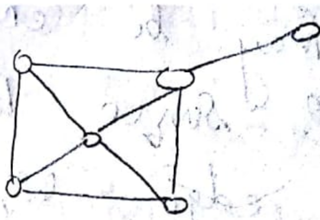
MOD

the graph.



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CAYLEY'S THEOREM:

The number of labelled trees with 'n' vertices where $n \geq 2$ is n^{n-2}.

Proof: Let $T(n, k)$ be the number of labelled trees on 'n' vertices in which a given vertex 'v' has degree k. We shall derive an expression for $T(n, k)$ and the result follows on summing from $k=1$ to $k=(n-1)$.

Let A be any labelled tree in which $\deg(v)=k$. The removal from A of any edge w, z that is not incident with v leaves 2 sub-trees, one containing v & either w or z (say w) & the other containing z.

If we now join the vertices v and z, we obtain a labelled tree B in which $\deg(v)=k$.

Call a pair (A, B) of labelled trees a linkage if B can be obtained from A by the above construction.

Our aim is to count the possible linkages (A, B).

Since A may be chosen in $T(n, k-1)$ ways & since B is uniquely defined by the edge w, z which may be chosen in $(n-1) - (k-1) = (n-k)$ ways, the total number of linkages (A, B) is given by $(n-k) \cdot T(n, k-1)$ ——— ①.

On the other hand, let B be a labelled tree in which $\deg(v) = k$ and let $T_1, T_2, \dots T_k$ be the subtrees obtained from B by removing the vertex v & each edge incident with v.

Then we obtain a labelled tree A with $\deg(v) = (k-1)$ by removing from B just one of these edges (vw_i, where w_i lies in T_i) & joining w_i to any vertex u of any other subtree T. Note that the corresponding pair (A, B) of labelled trees is a linkage & that all linkages can be obtained in this way.

Since B can be chosen in $T(n, k)$ ways and the number of ways joining w_i to vertices in any other T_j is $(n-1) - n_i$ where n_i is the number of vertices of T_i, the total number of linkages is

$$T(n,k)\left[(n-1-n_1)+(n-1-n_2)+\dots+(n-1-n_k)\right]$$
$$= T(n,k)\left[kn-k-(n_1+n_2+\dots+n_k)\right]$$
$$= T(n,k)\left[k(n-1)-(n-1)\right] \quad \because n_1+n_2+\dots+n_k=(n-1)$$
$$= T(n,k).(n-1).(k-1)$$

That is,

the total no. of linkages $= (n-1)(k-1).T(n,k)$ —— ②

From ① & ② we get,

$$(n-k).T(n,k-1) = (n-1)(k-1)T(n,k) \quad —— ③$$

~~Next~~ $\cancel{T(n,k)} = \dfrac{\cancel{(n-k).T(n,k-1)}}{\cancel{(n-1).(k-1)}}$

Putting $k=k+1$ in ③ we get,

$$(n-k-1).T(n,k) = (n-1)(k)T(n,k+1)$$

i.e; $T(n,k) = \dfrac{(n-1).k.T(n,k+1)}{(n-k-1)}$ —— ④

Putting $k=k+2$ in ③ we get

$$(n-k-2).T(n,k+1) = (n-1)(k+1)T(n,k+2)$$

i.e; $T(n,k+1) = \dfrac{(n-1).(k+1).T(n,k+2)}{(n-k-2)}$ —— ⑤

Substituting ⑤ in ④ we get

$$T(n,k) = \frac{(n-1)^2.k.(k+1).T(n,k+2)}{(n-k-1)(n-k-2)}$$

Continuing this iteration we get,

$$T(n,k) = \frac{(n-1)^{n-k-1} \cdot k \cdot (k+1) \cdots (k+n-2-k) \cdot T(n,n-1)}{(n-k-1)(n-k-2) \cdots (n-(n-1))}$$

We know that $T(n,n-1) = 1$

$$\therefore T(n,k) = \frac{(n-1)^{n-k-1} \cdot k \cdot (k+1) \cdots (n-2)}{(n-k-1)(n-k-2) \cdots 1}$$

$$= \frac{(n-1)^{n-k-1} \cdot 1 \cdot 2 \cdots (k-1) \cdot k(k+1) \cdots (n-2)}{1 \cdot 2 \cdots (k-1) \cdot 1 \cdots (n-k-2)(n-k-1)}$$

$$= \frac{(n-2)!}{(k-1)!(n-k-1)!} \cdot (n-1)^{n-k-1}$$

$$= \left({}^{n-2}C_{k-1}\right) \cdot (n-1)^{n-k-1}$$

i.e; $T(n,k) = \left({}^{n-2}C_{k-1}\right) \cdot (n-1)^{n-k-1}$

On summing over all possible values of k, we deduce that the number $T(n)$ of labelled trees on n vertices is given by

$$T(n) = \sum_{k=1}^{n-1} T(n,k)$$

$$= \sum_{k=1}^{n-1} \left({}^{n-2}C_{k-1}\right)(n-1)^{n-k-1}$$

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$$= {}^{(n-2)}C_0 \cdot (n-1)^{n-2} + {}^{(n-2)}C_1 \cdot (n-1)^{n-3} + {}^{(n-2)}C_2 \cdot (n-1)^{n-4} + \dots + {}^{n-2}C_{n-2} (n-1)^0$$

$$= (n-1)^{n-2} + {}^{(n-2)}C_1 (n-1)^{n-3} + {}^{(n-2)}C_2 (n-1)^{n-4} + \dots + 1$$

$$= 1 + \dots + {}^{(n-2)}C_1 (n-1)^{n-3} + (n-1)^{n-2}$$

$$= [1 + (n-1)]^{n-2}$$

$$= n^{n-2}$$

Hence the theorem.

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Properties :

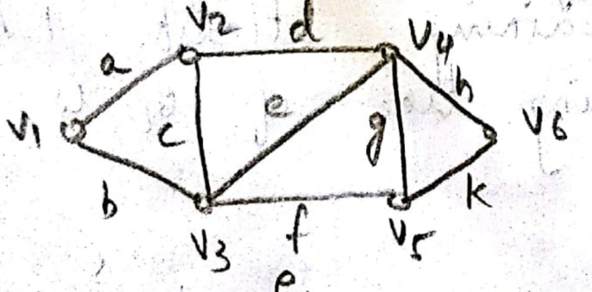
2. In a connected graph G, any minimal set of edges containing atleast 1 branch of every spanning tree of G is a cut set.

3. Every circuit has an even no. edges in common with any cut set.

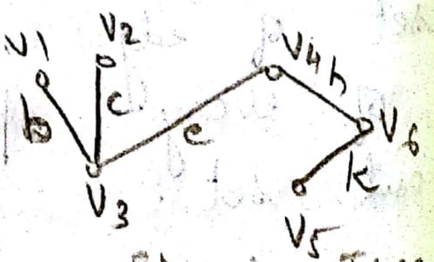
Fundamental Cut-Set

Consider a spanning tree T of a connected graph G. Take any branch in T. Since set B is a cut-set in T, set B partitions all vertices of T into two disjoint sets one at each end of B.

Consider the same partition of vertices in G and the cut set S in G that corresponds to this partition. Cut set S will contain only 1 branch B of T, and the rest (if any) of the edges in S are chords with respect to T. Such a cut set S containing exactly one branch of a tree T is called fundamental cut set or basic cut set.

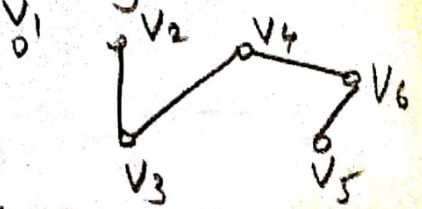


G

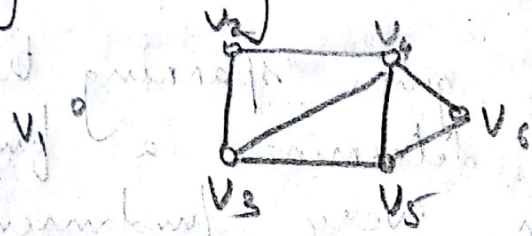


Spanning Tree

Removing edge h



by removing the necessary edges from G,



Here we remove $\{a, b\}$

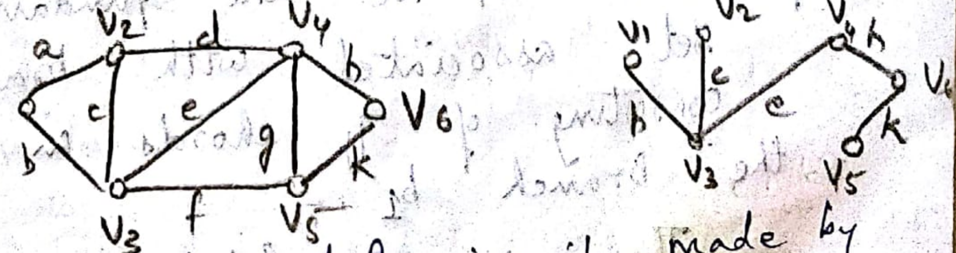
$\therefore$ we see that there is only one branch 'b' in the set of edges we remove.

Fundamental Circuit

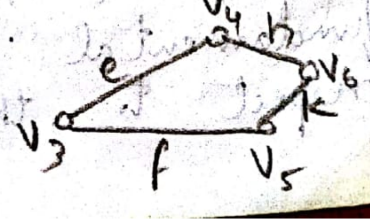
Consider a spanning tree T in a given connected graph G. Let c_i be a chord with respect to T. And let the fundamental circuit made by c_i be called 'Γ' consisting of k branches $b_1, b_2, \ldots b_k$ in addition to the chord c_i,

i.e,

$$\Gamma = \{ \underline{c_i}, b_1, b_2, \ldots b_k \}$$ is a fundamental circuit with respect to T.

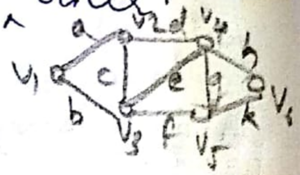


Q. Find a fundamental circuit made by the chord f.



Theorem 1:

With respect to a given spanning tree T, a chord c_i that determines a fundamental circuit occurs in every fundamental cut set associated with the branches in fundamental circuit & in no other.



Fundamental cut set formed by

1) e

$$S = \{e, d, f\} \qquad \{v_1, v_2, v_3\} \quad \{v_4, v_5, v_6\}$$

2) h

$$S = \{h, g, f\} \qquad \{v_1, v_2, v_3, v_4\} \quad \{v_5, v_6\}$$

3) k

$$S = \{k, g, f\} \qquad \{v_1, v_2, v_3, v_4, v_6\} \quad \{v_5\}$$

Fundamental circuit made by f

$$\Gamma = \{f, e, h, k\}$$

Proof: Let S_1 be the fundamental cut set associated with branch b_1 consisting of q chords in addition to the Branch b_1.

That is $S_1 = \{b_1, c_1, c_2, \ldots c_q\}$ is a fundamental cut set associated with respect to the spanning tree T.

Since, every circuit has even number of edges common with any cut set there must be even number of edges common to Γ and S_1. Edge b_1 is in both Γ and S_1 and there is only one other edge in Γ which is c_i that can be possibly also be in S_1. Therefore, we must have two edges b_1 and c_i common to both S_1 & Γ. Thus, the chord c_i is one of the chords $c_1, c_2, \ldots, c_q$.

On the other hand, suppose that only one edge c_i occurs in the fundamental cut set S' made by a branch other than $b_1, b_2, \ldots, b_k$. Since none of the branches $b_1, b_2, \ldots b_k$ are in S' there is only one edge c_i common to S' & Γ. which is a contradiction to our theorem that every circuit has even number of edges common with any cut-set.

Hence, the theorem.

As an example, consider the spanning tree $\{b, c, e, h, k\}$ in the above figure.

The fundamental circuit made by the chord f is

$$\Gamma = \{f, e, h, k\}$$

The 3 fundamental cut sets determined by the 3 branches e, h, k are

Determined by branch $e = \{e, d, f\}$
Determined by branch $h = \{h, f, g\}$
Determined by branch $k = \{k, f, g\}$

Chord f occurs in each of these 3 fundamental cut sets & there is no other fundamental cut set that contain f.

Theorem 2: With respect to a given spanning tree T, a branch b_i that determines a fundamental cut-set is contained in every fundamental circuit associated with chords in the cut set and in no others.

For eg: Consider the branch e of the spanking tree $\{b, c, e, h, k\}$.
The fundamental cut-set determined by the branch $e = \{e, d, f\}$.
The two fundamental ckts determined by the chord d and f are
Determined by chord $f = \{f, e, h, k\}$
Determined by chord $d = \{d, c, e\}$
Branch e is contained in both these fundamental ckts and there is no other fundamental ckt that contains the branch e.

Proof: Let the fundamental cut set S determined by the branch b_i be $S = \{b_i, c_1, c_2, \dots c_p\}$ and let Γ be the fundamental circuit determined by the chord c_1, that is,

$$\Gamma = \{c_1, b_1, b_2, \dots b_q\}$$

Since the no. of edges common to S and Γ must be even, b_i must be in Γ. The same is true for the fundamental ckt. made by the chords $c_1, c_2, \dots c_p$.

On the other hand, suppose that, b_i occurs in a fundamental circuit Γ' made by a chord other than $c_1, c_2, \dots c_p$. Since none of the chords $c_1, c_2, \dots c_p$ are in Γ', there is only one edge b_i common to a ckt. Γ' & the cut set S which is a contradiction.

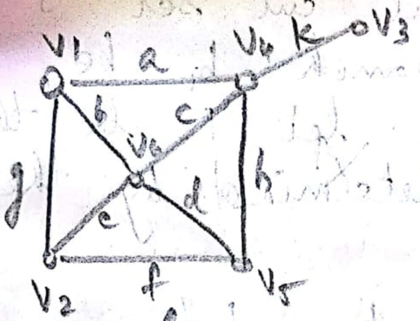
Hence the theorem.

Edge Connectivity

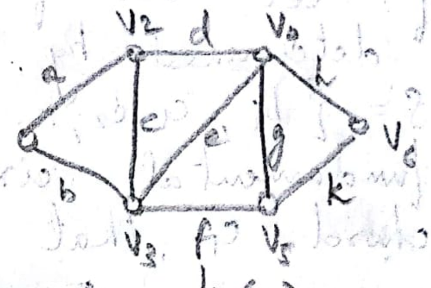
The number of edges in the smallest cut-set (i.e; cut set with fewest number of edges) is defined as the edge connectivity of G.

OR

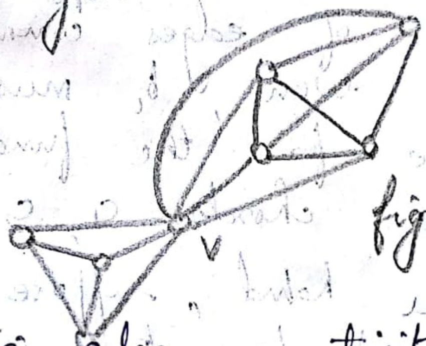
Edge connectivity of a connected graph G can be defined as the minimum number of edges whose removal reduces the rank by 1.



fig(1)



fig(2)



fig(3)

The edge connectivity of a tree is 1.

The edge connectivity of fig 1, 2, 3 are 1, 2, 3 respectively.

07/11/17 **Vertex Connectivity**

Vertex Connectivity or connectivity of a connected graph G is defined as the minimum no. of vertices whose removal from G leaves the remaining graph to be disconnected.

The vertex connectivity of fig 1, 2, 3 are 1, 2, 1 respectively.

Note: Vertex connectivity is meaningful only for the graphs that have 3 or more vertices & are not complete.

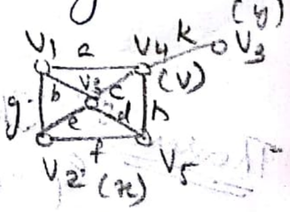
Separable Graphs

A connected graph is said to be separable if its vertex connectivity is 1. All other connected graphs are called non-separable.

In a separable graph a vertex whose removal disconnects the graph is called a cut-vertex or cut-node or articulation point.

In fig (1), v_4 is the cut-vertex and in fig (3), v is the cut-vertex.

Theorem 3: A vertex v in a connected graph G is a cut vertex iff there exists two vertices x and y in G such that every path between x and y passes through v.



Proof: Let v be a cut vertex of a connected graph G which implies $G-v$ is disconnected.
Let x and y belong to 2 components of $G-v$.
Then, $x, y \neq v$.

Since G is connected, there exists a xy path in G. As they are not contained in $G-v$, all these path contain v which implies v is in every xy path in G.

Conversely, suppose that v is in every xy path in G (which implies) $\Longrightarrow$ there exists no path between x & y

in $G - v$. $\Rightarrow$

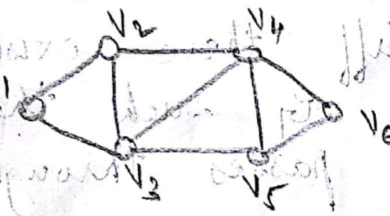
$G - v$ is disconnected

$\Rightarrow$ v is a cut-vertex.

Theorem 4: The edge connectivity of a graph G cannot exceed the degree of the vertex with the smallest degree in G.

Proof: Let v_i be the vertex with smallest degree in G. Let $d(v_i)$ be the degree of v_i. Vertex v_i can be separated from G by removing the $d(v_i)$ of edges incident on a vertex v_i.

Hence the theorem.



Edge connectivity = 2

Degree of vertex with smallest degree = 2

Theorem 5: The vertex connectivity of any graph G can never exceed the edge connectivity of G.

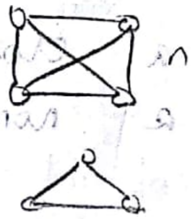
Proof: Let α denote the edge connectivity of G. Therefore, there exists a cut-set S in G with α edges. Let S partition the vertices in G in two subsets V_1 and V_2. By removing atmost α vertices from V_1 or V_2 on which the edges in S are incident, we can affect the removal of S together with all other edges incident on these vertices from G.

Hence the theorem.

Note: vertex connectivity ≤ edge connectivity ≤ degree of smallest vertex.

Problems

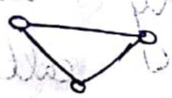
Q. What is the edge connectivity of a complete graph with n vertices?



For n = 2

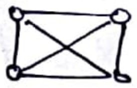
Edge connectivity = 1

For n = 3



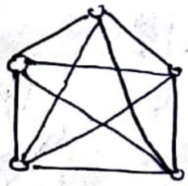
Edge connectivity = 2

For n = 4



Edge connectivity = 3

For n = 5



Edge connectivity = 4

∴ Edge connectivity of a complete graph with n vertices = (n−1).

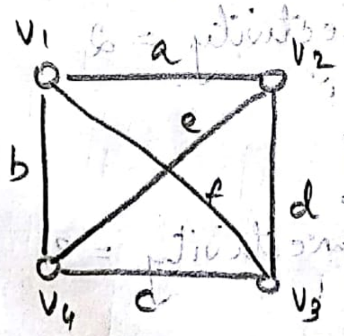
Planar Graphs

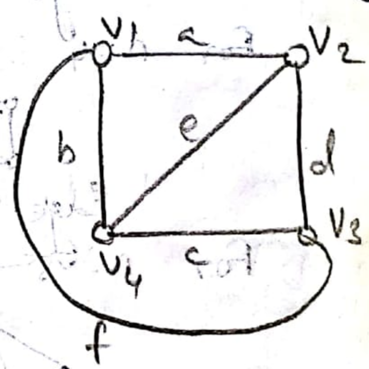
A graph G is said to be planar if there exist some geometrical representation of G which can be drawn on a plane such than no two of its edges intersect.

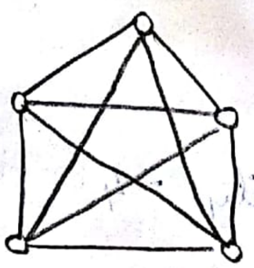
A graph that cannot be drawn without a cross-over between its edges is called a non-planar graph.

A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding.

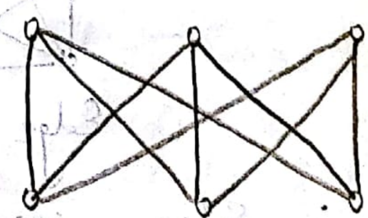
Eg:







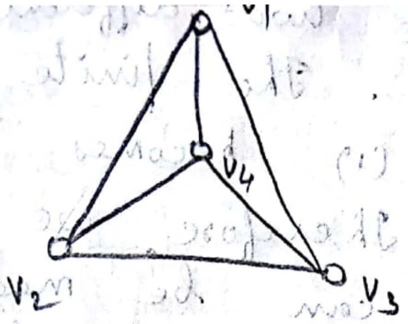
K_5



$K_{3,3}$

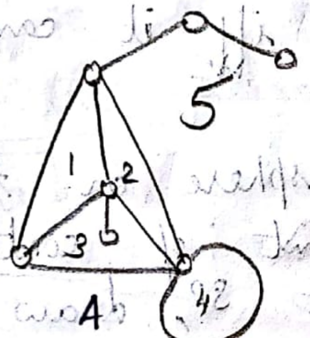
Different Representation of a Planar graph:

* Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.

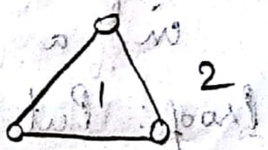


* Region

A planar representation of a graph divides the plane into regions (faces/windows). A region is characterized by the set of edges or the set of vertices forming its boundary. Note that a region is not defined in a non-planar graph.



A



Infinite Region: The portion of the plane lying outside a graph in a plane such as region 5 in fig A is infinite. Such a region is called infinite; unbounded, outer or exterior region for that particular plane representation.

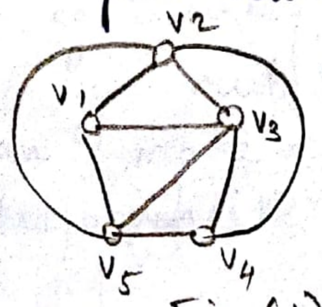


Fig. (1)

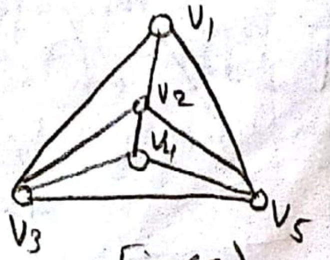


Fig. (2)

Scanned by CamScanner

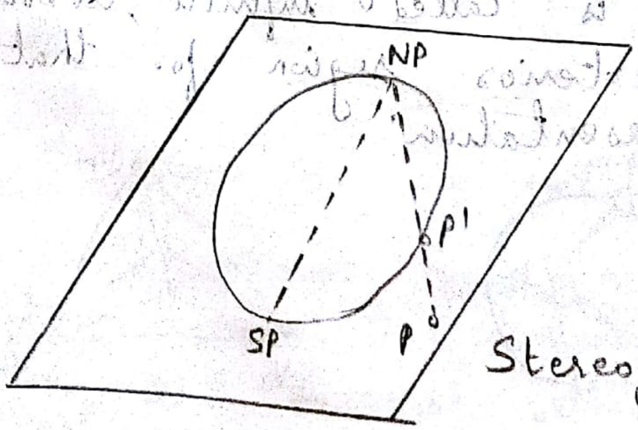
Fig (1) and (2) are two different embedding of the same graph. The finite region v_1, v_3, v_5 in fig (1) becomes the infinite region in fig (2). Therefore we can show that any region can be made the infinite region by proper embedding.

* Embedding on a sphere

To eliminate the distinction between the finite and the infinite region, a planar graph is often embedded on the surface of the sphere.

Theorem: A graph can be embedded on the surface of a sphere iff it can be embedded in a plane.

Proof: Put the sphere on the plane and call the point of contact SP (South Pole). At point SP, draw a straight line ⊥r to the plane and let the point where this line intersects the surface of the sphere be called NP (North Pole).



Stereographic projection

Now, corresponding to any point P on the plane, there exist a unique point P' on the sphere & vice-versa, where P' is the point at which the straight line from the point P to the point NP intersects the surface of the sphere. Thus, there is a one-to-one correspondance between the points on the sphere & the finite pts. on the plane, and points at infinity in the plane corresponding to the point NP on the sphere. Hence, any graph that can be embedded in a plane (i.e, drawn on a plane such that its edges do not intersect) can also be embedded in the surface of the sphere & vice-versa.

Hence the theorem.

Theorem: A planar graph maybe embedded in a plane such that any specified region can be made infinite.

Proof: A planar graph embedded on the surface of a sphere divides the surface into different regions. Each region of the sphere is finite, infinite region on the plane having being mapped onto the region containing the pt. NP.

By suitably rotating the sphere, we can make any specified region mapped onto the infinite region on the plane.

10/11/17

Euler's Formula / Euler's Theorem

A connected planar graph with 'n' vertices and 'e' edges has $e-n+2$ regions.

Proof: The proof is by induction on number of edges of G. If $e=0$ then $n=1$ since G is connected and $f=1$ i.e, the infinite region. Then, $e-n+2 = 0-1+2 = 1 = f$. That is, the theorem is true in this case.

Now suppose that the theorem holds for all graphs with atmost $(e-1)$ edges and let G be a graph with e edges.

If G is a tree, then $e=(n-1)$ and $f=1$. so that $e-n+2 = (n-1)-n+2 = 1 = f$

If G is not a tree, let e_1 be an edge in some cycle of G. Then $G-e_1$ is a connected planar graph with n vertices, $(e-1)$ edges and $(f-1)$ faces so that $n-(e-1)+(f-1) = 2$. By induction hypothesis it follows that $n-e+f=2$ ie, $f = e-n+2$.

$\therefore$ By induction hypothesis, the theorem is proved.

Hence the theorem.

In any simple connected planar graph with f regions, n vertices & e edges where $e > 2$, the following inequalities must hold.

1) $e \geq \frac{3}{2} f$

2) $e \leq 3n - 6$

$e - n + 2 = f$

$e - n + 2 = \frac{2e}{3}$

$3e - 2e - n + 6 = 0$

$e = n - 6$

Proof: Since each region is bounded by atleast 3 edges and each edge belongs to exactly 2 regions. Then $2e$ is greater than or equal to $3f$. That is

$$e \geq \frac{3}{2} f$$

Substituting for f from euler's formula we have $e \geq \frac{3}{2} f$

$$\implies e \geq \frac{3}{2} \times e - n + 2$$

$$\Rightarrow e \geq \frac{3}{2} e - n + 2$$

$$\Rightarrow \frac{3}{2} e - e \leq n - 2$$

$$\Rightarrow e \leq 3n - 6$$

Hence, the theorem.

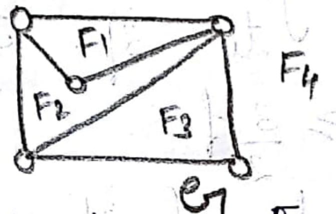
Note: $e \leq 3n - 6$ is only a necessary condition but not a sufficient condition for planarity of a graph.

For eg: In case of K_5, $n=5$, $e=10$. Then $3n-6=9<e$. Hence, the graph is not planar. But

In $K_{3,3}$, $e=9$ and $n=6$, then $3n-6=12$. Still yet the graph is not planar.

Geometric Dual

Consider the plane representation of a graph in fig (A)



$n=5$
$e=7$
$f=4$

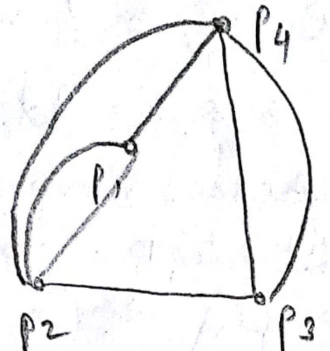
with 4 faces F_1, F_2, F_3, F_4. Let us place the 4 points P_1, P_2, P_3, P_4 one in each region. Next, let us join these 4 points according to the following procedure.

If two regions F_i and F_j are adjacent (i.e, they have a common edge), draw a line joining the points P_i and P_j that intersects the common edge between F_i & F_j exactly once.

If there is more than one edge common between F_i and F_j, draw one line b/w P_i and P_j for each of the common edges.

For an edge e lying entirely

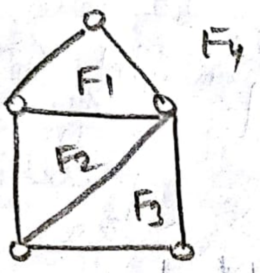
in one region say F_k draw a self loop at point P_k intersecting e exactly once. By this procedure, we obtain a new graph G^*. Such a graph G^* is called a dual of G.



G^*

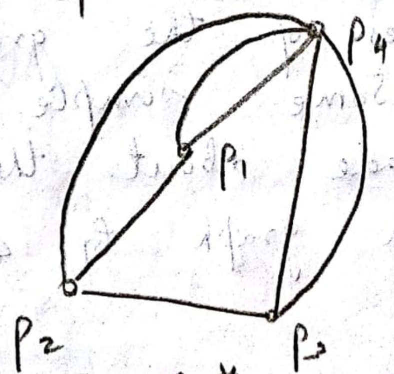
$n^* = 4$
$e^* = 7$
$f^* = 5$

Q.



G

$n = 5$
$e = 7$
$f = 4$



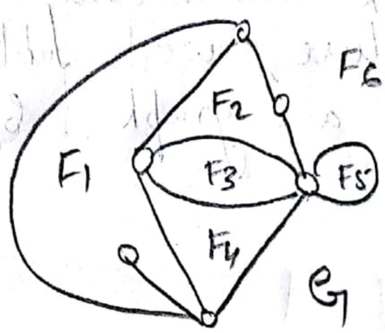
G^*

$n^* = 4$
$e^* = 7$
$f^* = 5$

$n^* = f$
$e^* = e$
$f^* = n$

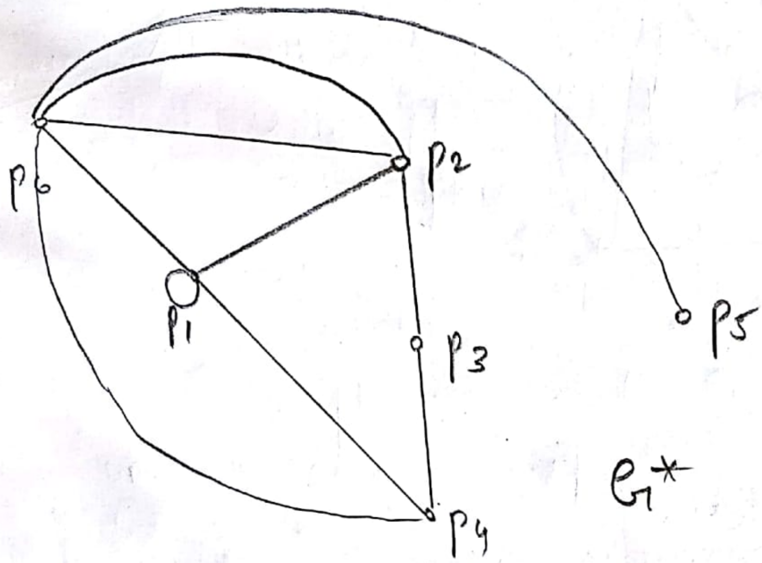
/11/17

Q. Find geometric dual of



G

$n = 6$
$e = 10$
$f = 6$



$n^* = 6$
$e^* = 10$
$f^* = 6$

G^*

~~Theorem~~: ~~All~~ ~~duals~~

Clearly, there is a 1 to 1 correspondance between the edges of the graph G & its dual G^*. Some simple observations that can be made about the relationship between a planar graph G & its dual G^* are

1) An edge forming a self loop in G yields a pendant edge in G^*. That is, an edge cincident on a pendant vertex is called a pendant edge.

2> A pendant edge in G yields a self loop in G^*.

3> Edges that are in series in G produce parallel edges in G^*.

4> Parallel edges in G produce edges in series in G^*.

5> Remarks 1, 2, 3, 4 are the result of the general observation that the no. of edges constituting the boundary of a region f_i in G is equal to the the degree of the corresponding vertex v_i in G^* & vice-versa.

6> Graph G^* is also embedded in the plane and is therefore planar.

7> Considering the process of drawing a dual G^* from G, it is evident that G is a dual of G^*.

Therefore, instead of calling G^* a dual of G we usually say that G & G^* are dual graphs.

8> If n, e, f, r, μ denote the no. of vertices, no. of edges, regions, rank and nullity of a connected planar graph G and if $n^*, e^*, f^*, r^*, \mu^*$ are the corresponding numbers in the dual graph G^* then,

$$n^* = f$$

$$e^* = e$$
$$f^* = n$$

Using the above relationship, we can immediately get that

$$r^* = \mu$$
$$\mu^* = r$$

<u>Theorem:</u> All duals of a planar graph G are two-isomorphic and every graph two-isomorphic to a dual of G is also a dual of G.

<u>Note:</u> * Two graphs G_1 and G_2 are said to be one-isomorphic if they become isomorphic to each other under repeated application of the following operation.

1> Split a cut vertex into 2 vertices to produce 2 disjoint subgraphs.

* Two-isomorphism

Two graphs are said to be two-isomorphic if they become isomorphic after undergoing operation 1 or operation 2, or both operations any number of times.

2> Split the vertex x into x_1 & x_2 and the vertex y into y_1 & y_2 such that the graph G is split into G_1 and $\bar{G_1}$. Let the vertices x_1 and y_1 go with G_1 and x_2 & y_2 in $\overline{G_1}$. Now, rejoin the graphs G_1 & $\overline{G_1}$ by merging x_1 with y_2 & x_2 with y_1. Clearly, edges whose end vertices were x and y in G_1 could have gone with G_1 or $\overline{G_1}$ without affecting the final graph.

From the definition, it follows that isomorphic graphs are always one-isomorphic, and one-isomorphic graphs are two-isomorphic. But two-isomorphic graphs are not necessarily one-isomorphic, one-isomorphic graphs are not necessarily isomorphic.

However, for graphs with connectivity 3 or more, isomorphism, one-iso., two-iso. are synonymous.

<u>Result 1</u>: Two graphs are two-isomorphic iff they have circuit correspondance.

Module - V

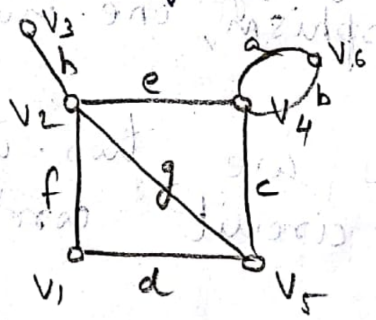
Matrix Representations of Graphs

1) Incidence Matrix :

Let G be a graph with n vertices, e edges & no self loops.
Define an $n \times e$ matrix $A = [a_{ij}]$ whose n rows correspond to n vertices and the e columns correspond to e edges as follows:

The matrix element $a_{ij} = 1$ if the jth edge e_j is incident on the ith vertex v_i, equal to 0 otherwise.

Eg:



$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

Such a matrix A is called the vertex-edge incidence matrix or simply incidence matrix.

Matrix A for a graph G is also written as A(G).

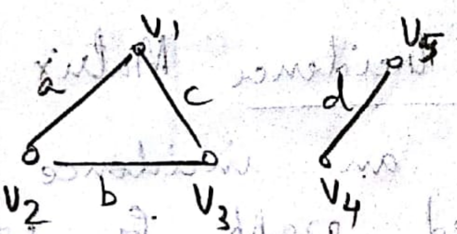
The incidence matrix contains only 2 elements 0 & 1. Such a matrix is called a binary matrix or a (0, 1) matrix.

The following observations about the incidence matrix A are as follows:

1) Since every edge is incident on exactly two vertices, each column of A has exactly two 1's.
2) The number of 1's in each row equals the degree of the corresponding vertex.
3) A row with all zeroes represents an isolated vertex.
4) Parallel edges in a graph produce identical columns in its incidence matrix.
5) If a graph G is a disconnected graph, and consists of 2 components g_1 & g_2, the incidence matrix A(G) of graph G can be written in a block diagonal form as

$$A(G) = \left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right]$$

Eg:



$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

where $A(g_1)$ and $A(g_2)$ are incident matrices of components g_1 & g_2. This observation results from the fact that no edge in g_1 is incident on vertices of g_2 & vice-versa. Obviously, this remark is also true for a disconnected graph with any no. of components.

6) Permutation of any 2 rows or columns in a incidence matrix simply corresponds to relabelling the vertices & edges of the same graph. This observation lead to us saying that

'two graphs G_1 & G_2 are isomorphic iff their incident matrices $A(G_1)$ and $A(G_2)$ differ only by permutation of rows & columns.

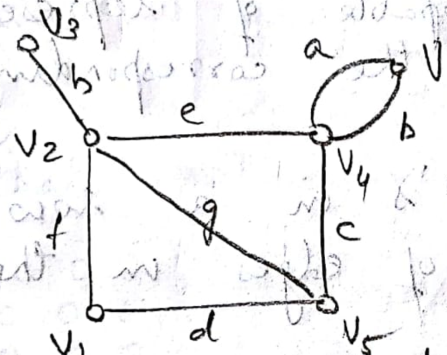
Rank of an Incidence Matrix :

Theorem: If $A(G)$ is an incidence matrix of a connected graph G with n vertices, then the rank of $A(G)$ is $(n-1)$.

Circuit Matrix :

Let the number of different circuits in graph G be q and the no. of edges in G be e. Then, a circuit matrix $B = [b_{ij}]$ of G is a $q \times e$ (0, 1) matrix defined as follows :

$$b_{ij} = 1 \text{ if } i^{th} \text{ circuit includes } j^{th} \text{ edge}$$
$$= 0 \text{ otherwise}$$



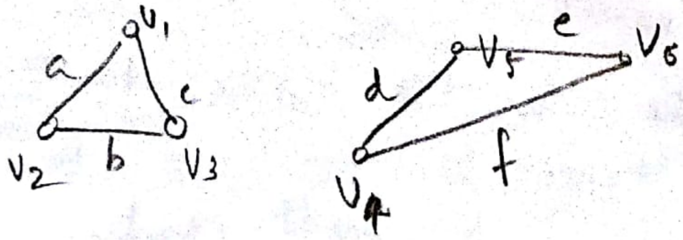
The graph has 4 different circuits that is $\{a, b\}$, $\{c, e, g\}$, $\{d, f, g\}$, $\{c, d, f, e\}$.
Therefore its circuit matrix is a 4 it is written as 4×8 (0, 1) matrix

$$B(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The following observations can be made about a circuit matrix $B(G)$ of a graph G as shown below:

1) A column of all zeroes corresponds to a non-circuit edge (i.e; an edge that does not belong to any circuit).
2) Each row of $B(G)$ is a circuit vector.
3) Unlike the incidence matrix, a circuit matrix is capable of representing a self loop, i.e, the corresponding will have a single 1. 7,
4) The no. of 1's in a row is equal to the no. of edges in the corresponding circuit. 15|
5) If graph G is separable or disconnected and consists of 2 components g_1 & g_2, a circuit matrix $B(G)$ can be written as a block diagonal form as

$$B(G) = \left[\begin{array}{c|c} b(g_1) & 0 \\ \hline 0 & b(g_2) \end{array}\right]$$



$$B(G_1) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

6) Permutation of any 2 rows or columns in a circuit matrix simply corresponds to relabelling the circuits & edges.

7) Two graphs G_1 & G_2 will have the same circuit matrix iff G_1 & G_2 are isomorphic.

15/11/17 Show that $AB^T = BA^T = 0$ under modulo 2.

$AB^T = BA^T = 0$ modulo 2.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A \cdot B^T = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix} \bmod 2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B \cdot A^T = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 & 2 & 0 \\ 2 & 2 & 0 & 0 & 2 & 0 \\ 2 & 2 & 0 & 2 & 2 & 0 \end{bmatrix} \bmod 2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ We've shown that

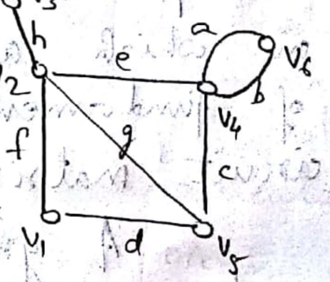
$A \cdot B^T = B \cdot A^T = 0$ under modulo 2.

Theorem: Let B and A be the circuit and incidence matrix respectively of a self loop free graph, whose columns are arranged using the same order of edges. Then every row of B is orthogonal to every row of A. That is,

$$A \cdot B^T = B \cdot A^T = 0$$

under modulo 2 where superscript T denotes the transposed matrix.

Proof: Consider a vertex v and a circuit Γ in the graph G. Either v is in Γ or it is not. If v is not in Γ, there is no edge in the circuit Γ that is incident on v. On the other hand, if v is in Γ, the no. of those edges in the circuit Γ that are incident on v is exactly two.



With this remark in mind, consider the i^{th} row in A and j^{th} row in matrix B. Since the edges are arranged in the same order, the non-zero entries in the corresponding positions occur only if a particular edge is incident on the i^{th} vertex & is also in the j^{th} circuit.

If the i^{th} vertex is not in the j^{th} circuit, there is no such non-zero entry & the dot product of these two rows is zero. If the i^{th} vertex is in the j^{th} circuit there will be exactly two 1s in the sum of products in the individual entries. Since $1+1=0$ under modulo 2, the dot product of the two arbitrary rows one from A & the other from B is zero.

Hence the theorem.

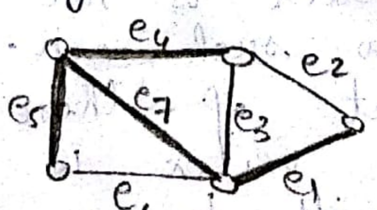
$v = v_2$
$CKT' = \{c, e, g\}$

Fundamental Circuit Matrix & Rank of B_f

A sub matrix of a circuit matrix in which all rows correspond to a set of fundamental circuits is called a fundamental circuit matrix B_f.

A graph and its fundamental circuit matrix with respect to a spanning tree is given below.

As in the matrices A and B, permutation of rows or columns do not affect B_f. If n is the no. of vertices & e is the no. of edges in a connected graph, then B_f is an $e-n+1 \times e$ matrix because the no. of fundamental circuits is $e-n+1$, each fundamental circuit being produced by one chord.



$\Gamma_{e_2} = \{e_2, e_4, e_7, e_1\}$

$\Gamma_{e_3} = \{e_3, e_4, e_7\}$

$\Gamma_{e_6} = \{e_6, e_5, e_7\}$

$$B_f = \begin{array}{c} \begin{array}{ccccccc} e_2 & e_3 & e_6 & e_1 & e_4 & e_5 & e_7 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

Let us arrange the columns in B_f such that all the $e-n+1$ chords correspond to the first $e-n+1$ columns. Furthermore, let us rearrange the rows such that the first row correspond to the fundamental circuit made by the chord in the first column. The second row to the fundamental ckt. made by the 2nd and so on. This indeed is how the fundamental ckt. matrix is arranged.

A matrix B_f thus arranged can be written as

$$B_f = [I_\mu \mid B_t]$$

where identity matrix I_μ is of order $\mu = e-n+1$ and B_T is the remaining $\mu \times (n-1)$ matrix corresponding to the branches of the spanning tree.

$\therefore$ Rank of $B_f = e-n+1 = \mu$

16/11/17 Since B_f is a sub matrix of the circuit matrix B, the rank of B is greater than or equal to rank of B_f.

i.e, Rank $B \geq e-n+1$

Theorem: If B is a circuit matrix of a connected graph G with e edges and n vertices, rank of $B = e-n+1$.

We know that if A is an incidence matrix of G, then $A \cdot B^T = 0 \mod 2$.

$\therefore$ According to Sylvester's Theorem, we have rank of A + rank of $B \leq e$

i.e. Rank of $B \leq e -$ Rank of A

Since rank of $A = (n-1)$,

rank of $B \leq e-n+1$

But we have,

rank of $B \geq e-n+1$.

$\therefore$ We have

rank of $B = e-n+1$

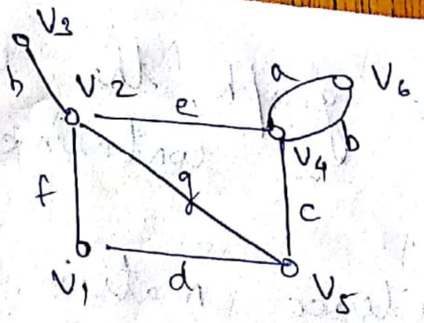
Cut-Set Matrix

A cut set matrix C which is equal to $[c_{ij}]$ in which the row correspond to the cut-set & the columns to the edges of the graph.

We can define a cut set matrix as follows:

$$c_{ij} = 1 \quad \text{if the } i^{th} \text{ cut-set contains the } j^{th} \text{ edge}$$

$$= 0 \quad \text{otherwise}$$



Cut-sets:

$\{h\}^{1}$, $\{e, g, f\}^{4}$, $\{f, d\}^{6}$, $\{d, c, g\}^{7}$,
$\{a, b\}^{2}$, $\{e, c\}^{3}$, $\{c, f, g\}^{5}$, $\{d, e, g\}^{8}$

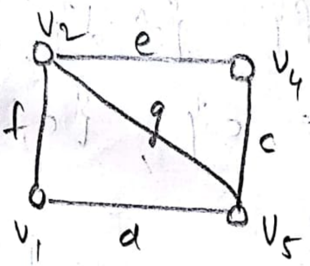
$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

The following remarks can be made about a cut-set matrix

1) As in the case of incidence matrix, a permutation of rows or columns in a cut set matrix corresponds simply to a renaming of cut-sets & edges respectively.
2) Each row in $C(G)$ is a cut-set vector.
3) A column with all zeroes corresponds to an edge forming a self loop.
4) Parallel edges produces identical columns in a cut-set.

5) For a separable graph, the incidence matrix of each block is contained in the cut-set matrix.

For eg: the incidence matrix of the block {c, d, e, f, g



$$A(G_1) = \begin{array}{c|ccccc} & c & d & e & f & g \\ \hline v_1 & 0 & 1 & 0 & 1 & 0 \\ v_2 & 0 & 0 & 1 & 1 & 1 \\ v_4 & 1 & 0 & 1 & 0 & 0 \\ v_5 & 1 & 1 & 0 & 0 & 1 \end{array}$$

6) According to the above observation, $C(G_1)$ contains $A(G_1)$.

$\therefore$ Rank of $C(G_1) \geq$ Rank of $A(G_1)$

Hence for a connected graph of n vertices, rank of $C(G_1) \geq n-1$.

7) $B \cdot C^T = C \cdot B^T = 0$ under modulo 2.

__Theorem:__ If C is a cut-set matrix of a connected graph G with n vertices, rank of $C = (n-1)$.

We know if B is a circuit matrix of a graph G,

$$B \cdot C^T = 0 \text{ under modulo } 2.$$

$\therefore$ By Sylvester Theorem,

$$\text{Rank}(B) + \text{Rank}(C) \leq e$$

and since for a connected graph G,

$$\text{rank}(B) = e - n + 1.$$

$$\therefore \text{Rank}(C) \leq e - \text{rank}(B)$$

$$\text{i.e., } \text{Rank}(C) \leq n - 1$$

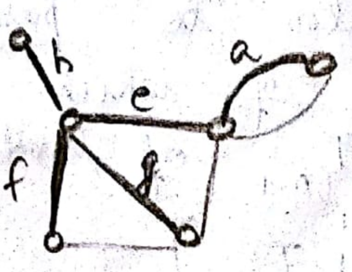
But we have $\text{Rank}(C) \geq n-1$.

$\therefore$ Rank of $C = n-1$

Result: The rank of a cut-set matrix $C(G)$ = rank of incidence matrix $A(G)$ = Rank of the graph G.

Fundamental Cut-Set Matrix

A fundamental cut-set matrix C_f is an $(n-1) \times e$ sub matrix of C such that the rows correspond to the fundamental cut-set with respect to some spanning tree.



Fundamental cut-set determined by

1) a

$\{a, b\}$

2) e

$\{e, c\}$

3) g 4

$\{c, g, d\}$

4) f 3

$\{f, d\}$

5) h 5

$\{h\}$

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} b & c & d & a & e & f & g & h \end{array} \\
\left[\begin{array}{ccc|ccccc}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

As in the case of a fundamental ckt matrix, a fundamental cut-set matrix C_f can also be partitioned into 2 sub-matrices one of which is an identity matrix I_{n-1} of order $(n-1)$.

i.e, $C_f = \left[C_c \,\middle|\, I_{n-1} \right]$.

where the last $(n-1)$ columns forming the identity matrix correspond to the $(n-1)$ branches of spanning tree & the first $e-(n+1)$ columns C_c correspond to the chords.

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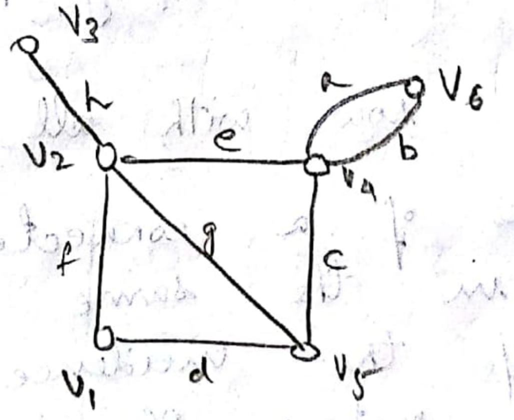
Path Matrix

A path matrix is a $(0,1)$ matrix and is defined for a specific pair of vertices in a graph, say, (x,y) and is written as $P(x,y)$. The rows in $P(x,y)$ corresponds to different paths between the vertices x and y, and the columns correspond to edges in G. That is, the path matrix for (x,y) is $P(x,y) = [p_{ij}]$ where $p_{ij} = 1$ if the j^{th} edge lies in the i^{th} path.
$= 0$ otherwise



Consider all paths between vertices V_3 and V_4. They are

$\{h, e\}$ (1), $\{h, f, d, c\}$ (3), ~~{h, f, d, g, e}~~ $\{h, g, c\}$ (2)

$$
\begin{array}{c|cccccccc}
 & a & b & c & d & e & f & g & h \\
\hline
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
3 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{array}
$$

Some of the observations are listed below:

1) A column of all 0s corresponds to an edge that does not lie in any path between x and y.
2) A column of all 1s correspond to an edge that lies in every path between x & y.
3) There is no row with all zeroes.

Theorem: If the edges of a connected graph are arranged in the same order for the columns of the incidence matrix A and the path matrix $P(x, y)$, then the product under modulo 2

$$A \cdot P^T(x, y) = M$$

where the matrix M has 1's in two rows x, y and the rest of the $(n-2)$ rows are all 0s.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

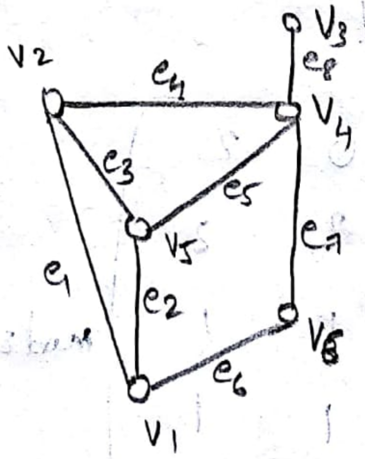
$$A \cdot P^T = \begin{bmatrix} 0 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \bmod 2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Adjacency Matrix

As an alternative to incidence matrix it is sometimes more convenient to represent a graph by its adjacency matrix or connection matrix.

The adjacency matrix of a graph G with n vertices & no parallel edges is an $n \times n$ symmetric binary matrix $X = [x_{ij}]$ and is defined as

$$x_{ij} = \begin{cases} 1 & \text{if there is an edge between the } i^{th} \text{ \& } j^{th} \text{ vertices} \\ 0 & \text{(otherwise) if there is no edge between them} \end{cases}$$



Ob..
about

1>

2>

3>

$$\begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 0 & 0 & 1 & 1 \\ v_2 & 1 & 0 & 0 & 1 & 1 & 0 \\ v_3 & 0 & 0 & 0 & 1 & 0 & 0 \\ v_4 & 0 & 1 & 1 & 0 & 1 & 1 \\ v_5 & 1 & 1 & 0 & 1 & 0 & 0 \\ v_6 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Observations that can be made immediately about the adjacency matrix are

1> The entries along the principle diagonal of X are all 0s iff the graph has no self loops.
A self loop at the i^{th} vertex correspond to $x_{ii} = 1$

2> The definition of adjacency matrix makes no provision for parallel edges. That is why, adjacency matrix X was defined for graphs without parallel edges.

3> If the graph has no self loop, the degree of the vertex equals the no. of 1s in the corresponding row or column of X.

4> Permutation of rows & of the corresponding columns imply reordering the vertices. It must be noted however that the rows & columns must be arranged in the same order. Thus, if two rows are interchanged in X, the corresponding columns must also be interchanged. Hence, 2 graphs G_1 & G_2 with no parallel edges are isomorphic iff their adjacency matrices $X(G_1)$ & $X(G_2)$ are related. That is,

$$X(G_2) = R^{-1} X(G_1) \cdot R$$

where R is the permutation matrix.

5> A graph G is disconnected & is in two components g_1 and g_2 iff its adjacency matrix $X(G)$ can be partitioned into block diagonal form, i.e,

$$X(G) = \left[\begin{array}{c|c} X(g_1) & 0 \\ \hline 0 & X(g_2) \end{array}\right]$$

where $X(g_1)$ is the adjacency matrix of component g_1 & $X(g_2)$ is that of the component g_2.

This partition clearly implies that there exists no edge joining any vertex in subgraph g_1 to any vertex in subgraph g_2.