3/8/17

GRAPH THEORY AND COMBINOTORICS

Introduction

In a hockey league there are 8 teams which we denote by S, T, U, V, W, X, Y, Z. After a few weeks, off the season, the following teams have played with each other.

S has played with → X and Z

T → W, X, Z

U → Y, Z

V → W, Y

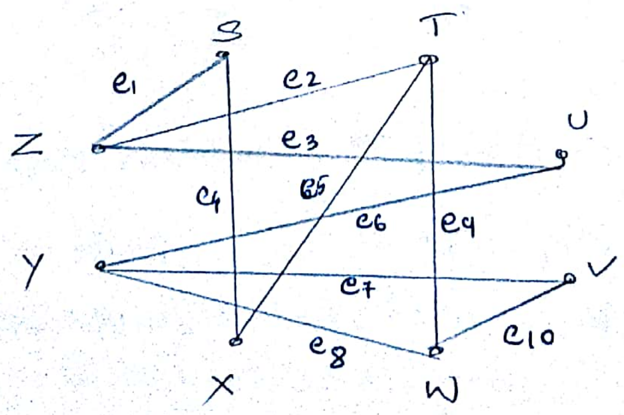
W → T, V, Y

X → S, T

Y → U, V, W

Z → S, T, U

The teams are represented by dots and the corresponding teams which have played with each other are joined by a line.



$V(G) = \{S, T, U, V, X, W, Y, Z\}$

$E(G) = \{e_1, e_2, e_3 \cdots, e_{10}\}$

$G = [V(G), E(G))$

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→ So many real life situations can conveniently be described by means of drawings which we call graphs.

→ Here, the vertex set $V(G) = \{S, T, U, V, W, X, Y, Z\}$ and the edge set $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$

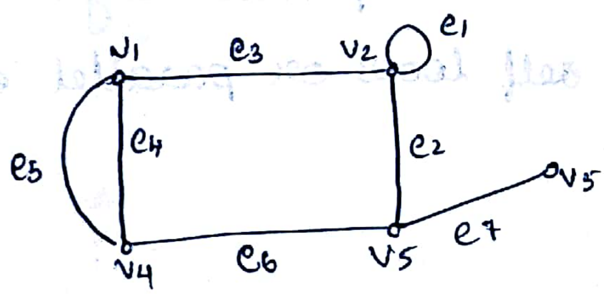
MODULE - I

Introductory Concepts of Graphs

GRAPH

A graph $G = (V(G), E(G))$ consists of two ~~non-empty~~ sets.

- the vertex set of the graph $V(G)$, which consist of ~~non~~ non empty set of elements $\{v_1, v_2, v_3 \ldots\}$ called the vertices.
- the edge set of a Graph $E(G)$ which consist of sets $\{e_1, e_2, e_3 \ldots\}$ called the edges, such that each edge e_k in E is assigned an unordered pair of vertices v_i, v_j are called end vertices



vertices $= V(G) = \{V_1, V_2, V_3, V_4, V_5\}$

edge $= E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

end vertices : of $e_1 = v_1, v_2$

$e_2 = v_2, v_4$ $\quad$ $e_3 = v_3, v_2$ $\quad$ $e_4 = v_1, v_3$

$e_5 = v_1, v_3$ $\quad$ $e_6 = v_3, v_4$ $\quad$ $e_7 = v_4, v_5$

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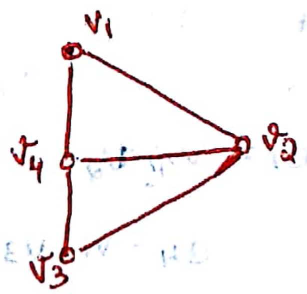
⇒ The vertices v_i, v_j associated with edge e_k are called end vertices of e_k.

⇒ The most common representation of a graph is by means of a diagram, in which end vertices are represented by points and each edge as a line segment joining its end vertices.

⇒ An edge having the same vertex as both its end vertices is called a self loop or simply a loop.
Here in this figure e_1 is a self loop

⇒ If more than one edge is associated with a given pair of vertices then such edges are called parallel edges or multiple edges.
In this figure, e_4 and e_5 are the parallel edges.

⇒ A graph that has neither self loops or parallel edges is called a simple graph.



NOTE

* A graph is also called a linear complex or a 1-complex or a 1-dimensional complex.
* A vertex is also referred to as node, a junction, a point 0-cell or 0-simplex.

* Other terms used for edges are branch, a line, an element, 1-cell, an arc and 1-simplex.

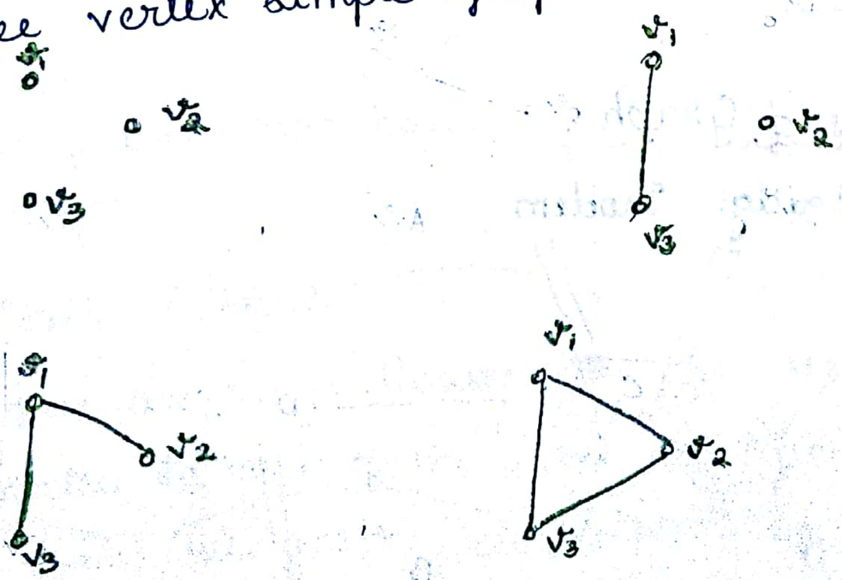
Problems

1. Draw all simple graphs of one, two, three and four vertices

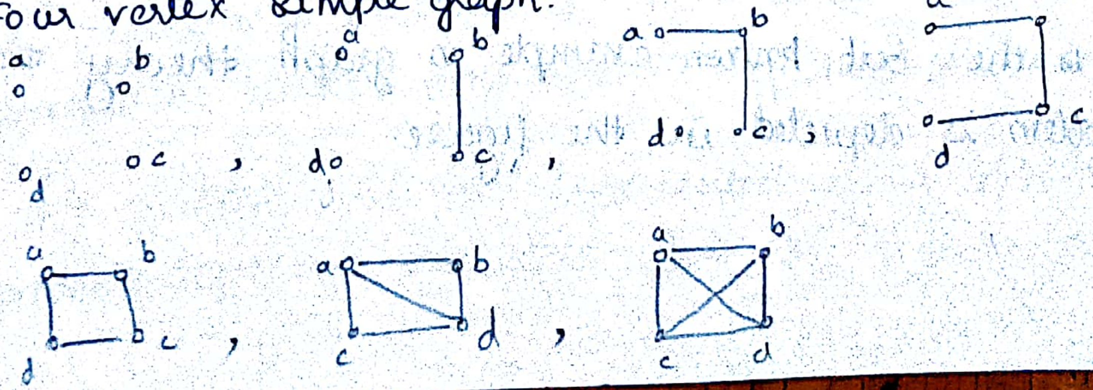
Ans one vertex simple graph: $\circ$ a

two vertex simple graph:

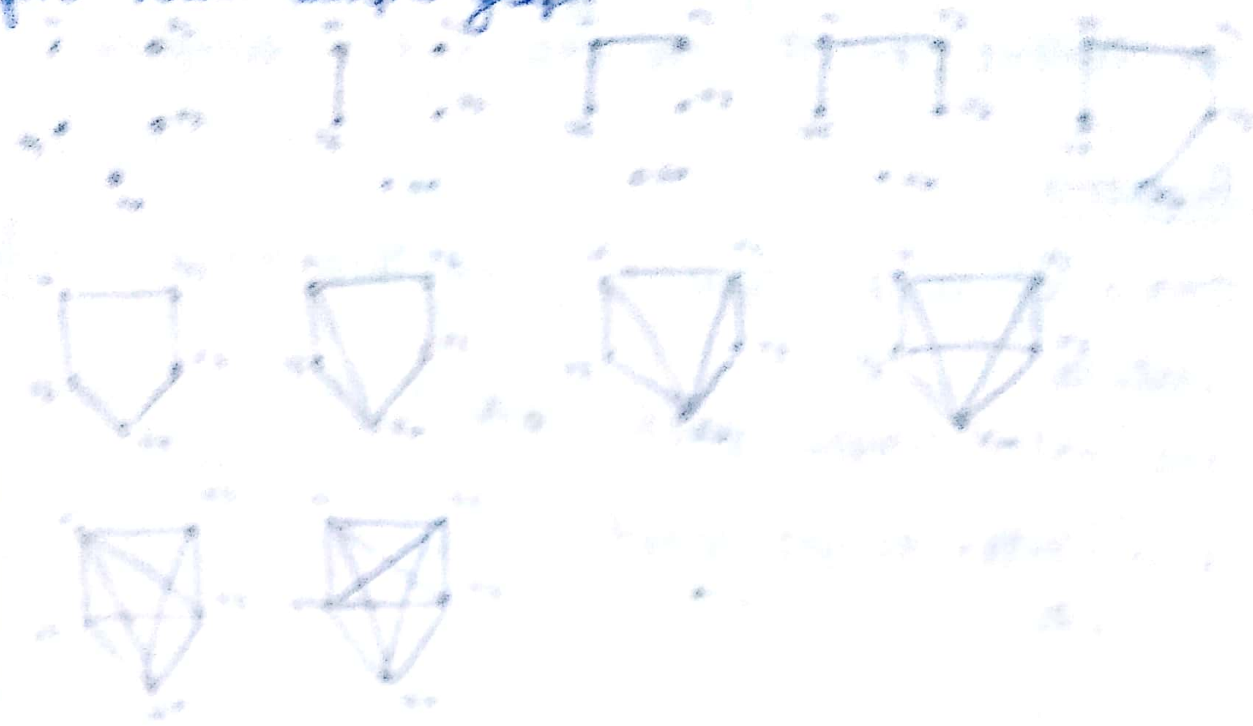
three vertex simple graph:



Four vertex simple graph:



+ five vertex simple graph



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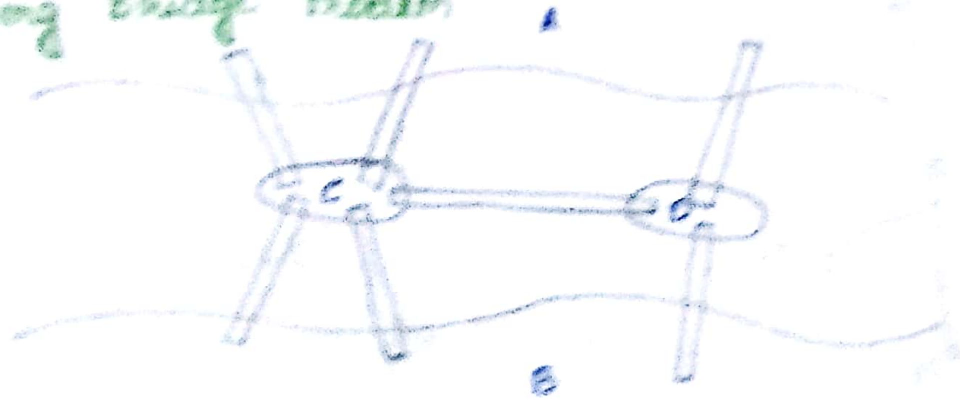
Applications of Graph

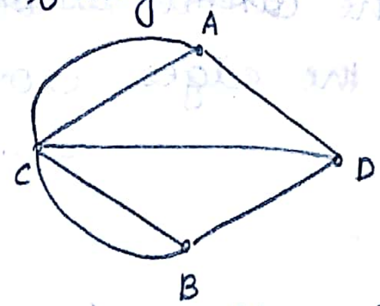
I Konigsberg Bridge Problem



It is the best known example in graph theory the problem is depicted in the figure

→ Two islands C and D formed by Pregel river in Konisberg were connected to each other and to the banks A and B with seven bridges. The problem was to start at any of the four land area of the city A, B, C or D, walkover each of the seven bridges, exactly once and return to the starting point.

→ Euler represented this situation by means of a graph. the vertices represent the land areas and the edges represent the bridges.

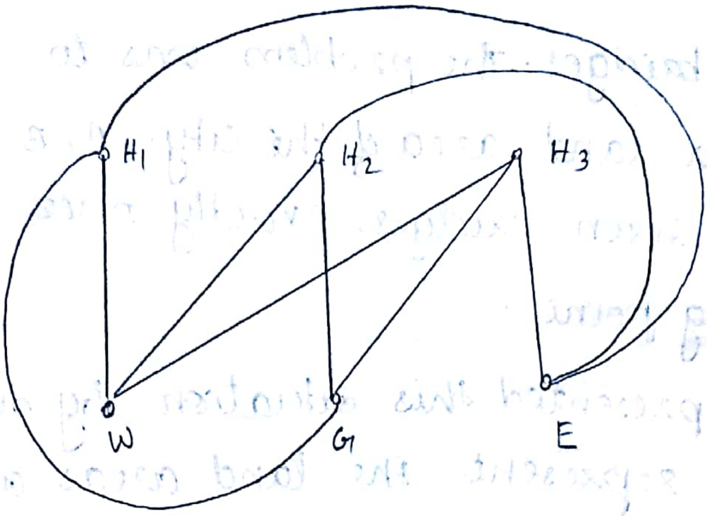


→ Euler proved that Konisberg bridge problem has no solution.

II Utilities Problem

There are three houses H_1, H_2, H_3 each to be connected to each of the three utilities. The three utilities are water (W), gas (G), electricity (E) by means of conduit. The problem is that is it possible to make such connections without any crossover of the conduits.

This problem can be represented by a graph. the conduits represent the edges while the houses and the utilities represented by the vertices. the graph is shown below.



we can see that the ~~cannot can be also~~ graph cannot be drawn without the edges crossing ones.

∴ the answer is no.

Problem

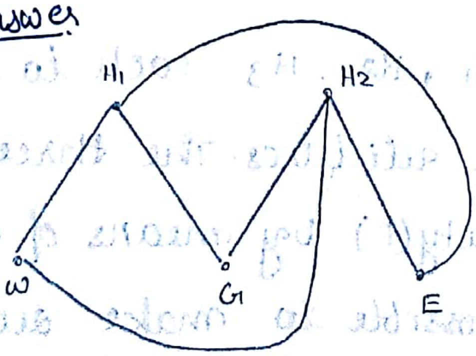
① Draw graph representing problems of

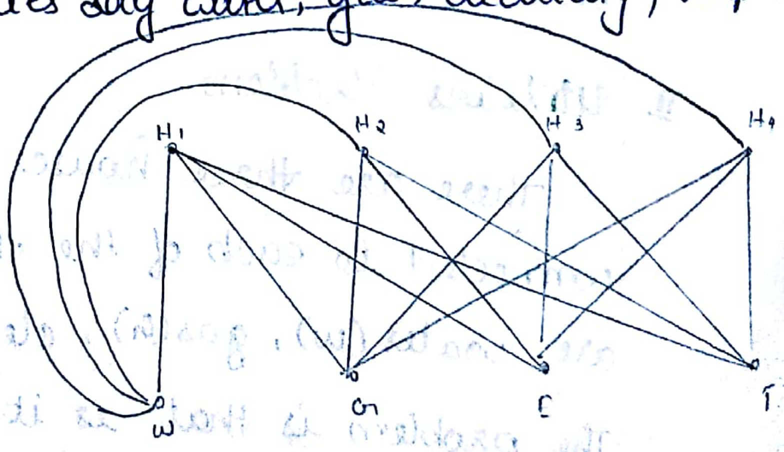
i) Two houses and three utilities

ii) Four houses and 4 utilities say water, gas, electricity & telephone

Answer

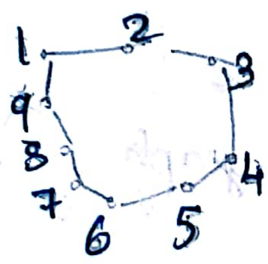
i)

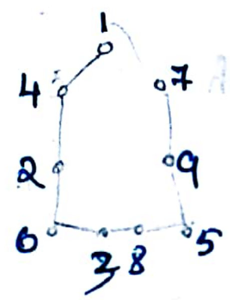


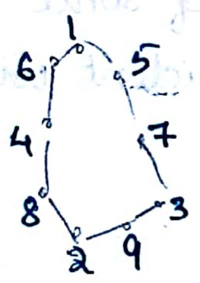


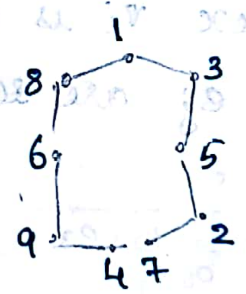
III Seating Problem

Nine members of a new club meet each day for lunch at a round table. They decide to sit such that every member has different neighbours at each lunch. How many days can this arrangement last









In general, it can be shown that for 'n' people the number of such possible arrangement is

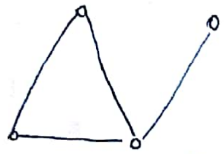
$\frac{n-1}{2} \rightarrow n$ is odd

$\frac{n-2}{2} \rightarrow n$ is even

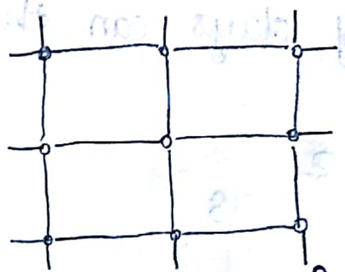
1	2	3	4	5	6	7	8	9	1
1	3	5	2	7	4	9	6	8	1
1	5	7	3	9	2	8	4	6	1
1	7	9	5	8	3	6	2	4	1

Finite And Infinite Graph

A Graph with finite number of vertices as well as finite number of edges is called a finite graph ; otherwise it is an infinite graph



finite graph



infinite graph

Incidence And Degree

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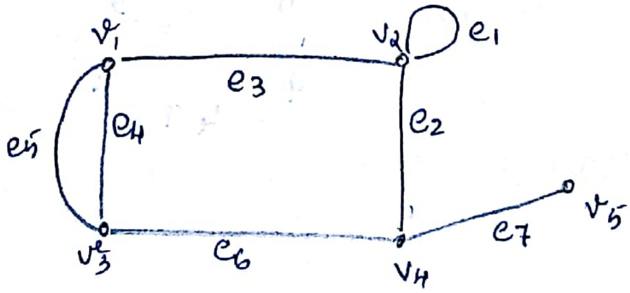
* When a vertex v_i is an end vertex of some edges e_j then v_i and e_j are said to be incident with each other.



* Two non-parallel edges are said to be adjacent if they are incident to a common vertex
  eg:- e_2, e_6, e_7 are adjacent on a common vertex v_4

* Two vertices are said to be adjacent if they are the end vertices of the same edge.
  eg:- v_1 and v_2 are adjacent.

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Degree of a vertex

→ The no: of edges incident on a vertex ~~in~~ with self loop counted twice is called the degree of the vertex and is denoted by $deg(v_i)$ or $d(v_i)$.

$d(v_1) = 3$ $\quad$ $d(v_4) = 3$

$d(v_2) = 4$ $\quad$ $d(v_5) = 1$

$d(v_3) = 3$

→ The degree of a vertex is sometimes referred to as valency.

Fundamental theorem of Graph Theory

Theorem I

consider a graph G with 'e' edges and 'n' vertices say $v_1, v_2, \ldots v_n$ then

$$\boxed{\sum_{i=1}^{n} d(v_i) = 2e}$$

Proof

Each edge has two end vertices and hence contribute two to the sum of degrees. i.e., when degrees of vertices are added each edge is counted twice. also we know that a loop is counted twice while taking the degree.

$\therefore \sum_{i=1}^{n} d(v_i) = 2e$

Hence Proved

Theorem II

~~con~~ The number of vertices of odd degree, in a graph is always even.

Proof

Consider a graph G with 'e' edges and 'n' vertices say $v_1, v_2, v_3 \cdots v_n$ then

$$\sum_{i=1}^{n} d(v_i) = 2e \quad \text{——} \ (1)$$

Consider the vertices with odd degree and even degree seperately. Then equation (1) can be expressed as the a sum of two sums each taken over the vertices of even and odd degree

$$\underbrace{\sum_{i=1}^{n} d(v_i)}_{=2e \text{ (even no)}} = \underbrace{\sum_{\text{even}} d(v_j)}_{\text{even no}} + \sum_{\text{odd}} d(v_k) \quad \text{——} \ (2)$$

Since the L.H.S of equation number (2) is even and the first expression on R.H.S is even, the second expression must be even

$$\therefore \sum_{\text{odd}} d(v_K) = \text{even no.} \quad \text{——} \ (3)$$

~~Since each $d(v_K)$ is odd, the total number of terms in the sum to be even, to make the sum an even number.~~

~~Hence the Theorem~~

Since each $d(v_K)$ is odd, the total number of terms in the sum must be even, to make the sum an even number

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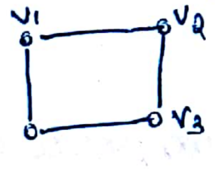
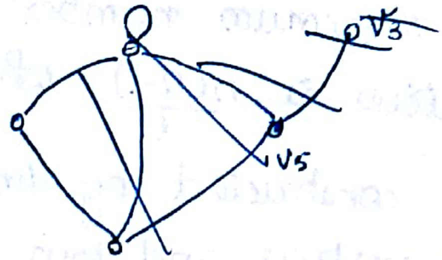
Regular Graph

A graph in which all the vertices are of equal degree is called regular graph

eg:-





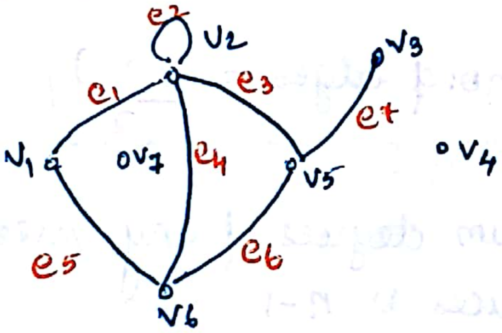
$d(v_1) = 2$

$d(v_2) = 2$

$d(v_3) = 2$

$d(v_4) = 2$

Isolated vertex, Pendant vertex & Null Graph



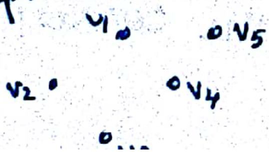
* A vertex having no incident edge is called isolated vertex
  eg:- v_7, v_4 are isolated vertex
* In other words, they are vertices with zero degree.

* A vertex of degree one is called a pendant vertex
  eg:- v_3

* Two adjacent edges are said to be in series, if their common vertex is of degree two.
  eg:- two edges e_1 and e_5 which are incident on v_1 are in series

* A graph without any edges is called a null Graph

eg:-



Every vertex in a null graph is an isolated vertex.

Problems

Q) Show that maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$. why?

Ans An edge is constructed by choosing any two vertices out a set of n vertices and join them. Therefore, the number of ways choosing any two elements out of a set of n elements is nC_2

$$nC_2 = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$$

$$\boxed{\therefore \text{The maximum no. of edges} = \frac{n(n-1)}{2}}$$

2) Show that maximum degrees of any vertex in a simple graph with 'n' vertices is $n-1$

Ans Let G be a simple graph, with n vertices.
Consider any vertex ~~v~~ v of G. Since the graph is simple, v can be adjacent to atmost $n-1$ vertices. Hence, maximum degree of any vertex in a simple graph with n vertices is $n-1$

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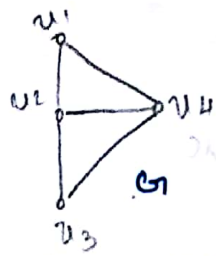
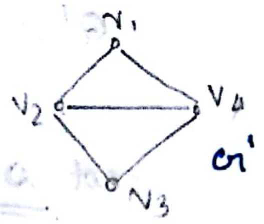
Isomorphism

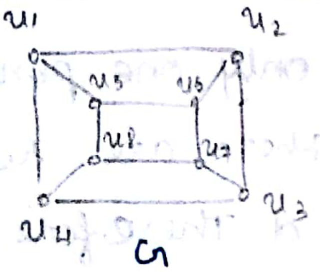
Two graphs G and G' are said to be isomorphic if there is a one to one correspondance between their vertices and edges such that the incidence relation is preserved. i.e., if the edge e is incident on the vertices v_1 and v_2 in G then the corresponding edge e' in G' must be incident on the vertices v_1' and v_2' that corresponds to v_1 and v_2

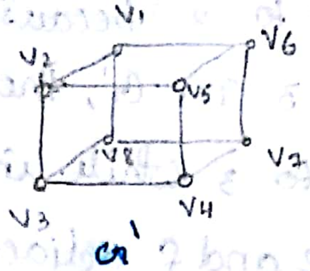
1) 



$u_1 \to v_1$
$u_2 \to v_2$
$u_3 \to v_3$
$u_4 \to v_4$

$u_1 \to v_3$
$u_2 \to v_2$
$u_3 \to v_1$
$u_4 \to v_4$

2.) 



$u_1 \to v_1$
$u_2 \to v_6$
$u_3 \to v_2$
$u_4 \to v_5$
$u_5 \to v_8$
$u_6 \to v_7$
$u_7 \to v_4$
$u_8 \to v_3$

G and G' are isomorphic in both cases.

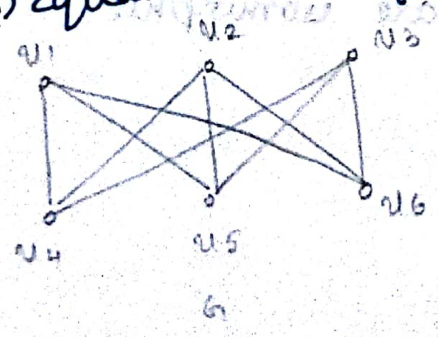
NOTE

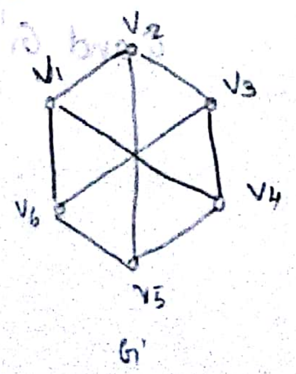
Two isomorphic graphs must have

i) same number of vertices

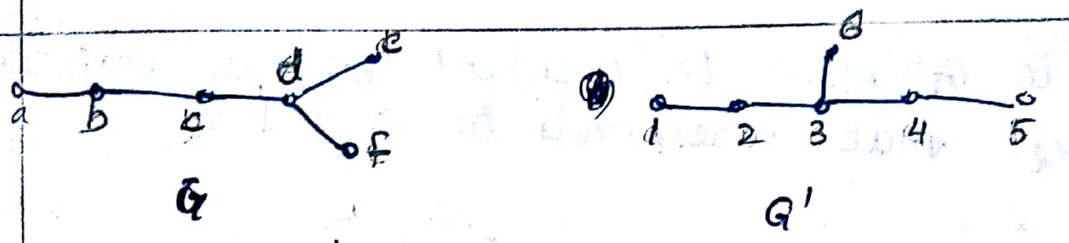
ii) same number of edges

iii) Equal number of vertices with a given degree





$u_1 \to$
$u_2 \to$
$u_3 \to$
$u_4 \to$
$u_5 \to$
$u_6 \to$

i) 

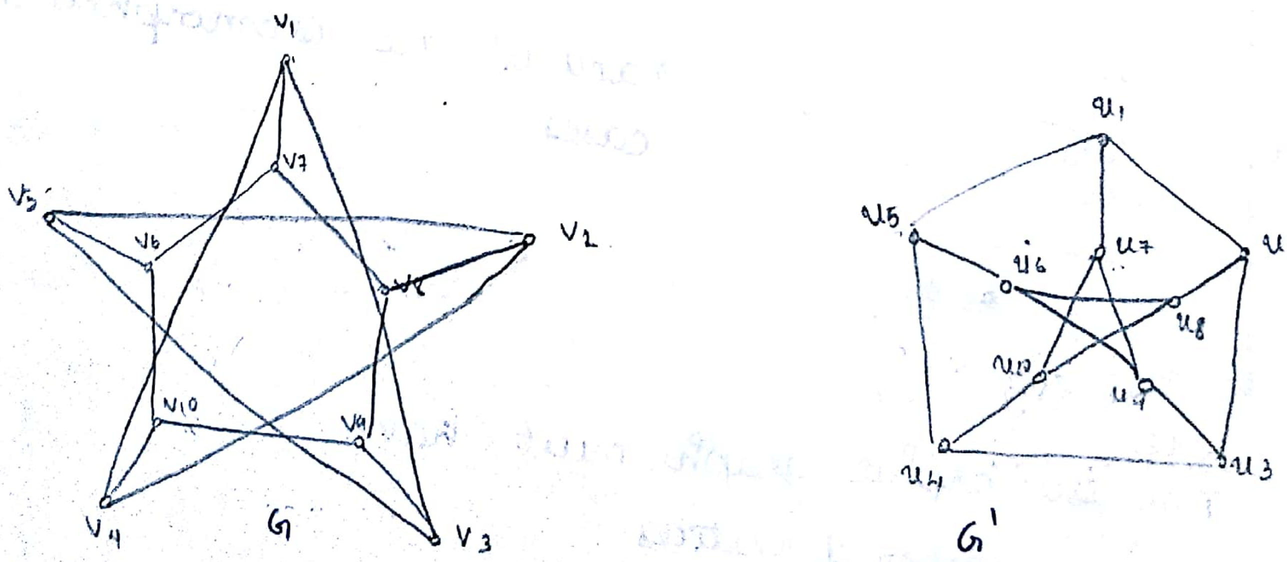
a → 1　　d →

b → 2　　e →

c →　　f →

not isomorphic

d must correspond to 3 because there are no other vertices of degree 3. In G', there is only one pendant vertex 6 adjacent to 3. While in G, there are two pendant vertices e and f adjacent to d. Therefore, G is not isomorphic to G'

ii) 

$v_1 \rightarrow u_7$

$v_2 \rightarrow u_8$

$v_3 \rightarrow u_9$

$v_4 \rightarrow u_{10}$

$v_5 \rightarrow u_6$

$v_6 \rightarrow u_5$

$v_7 \rightarrow u_1$

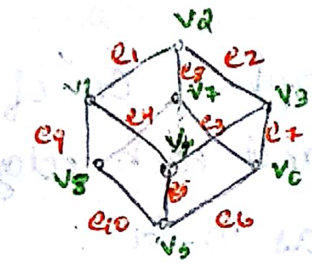
$v_8 \rightarrow u_2$

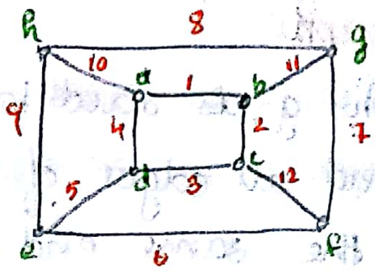
$v_9 \rightarrow u_3$

$v_{10} \rightarrow u_4$

G and G' are isomorphic

iii)





$v_1 \to a$

$v_2 \to b$

$v_3 \to c$

$v_4 \to d$

$v_5 \to e$

$v_6 \to f$

$v_7 \to g$

$v_8 \to h$

$e_1 \to 1$

$e_2 \to$

$e_3 \to$

$e_4 \to$

$e_5 \to$

$e_6 \to$

$e_7 \to$

$e_8 \to$

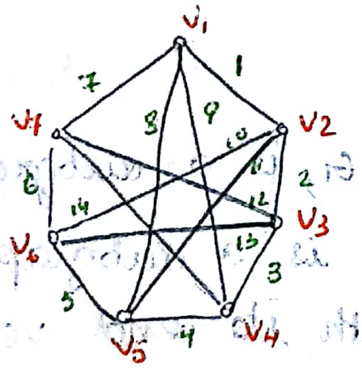
$e_9 \to$

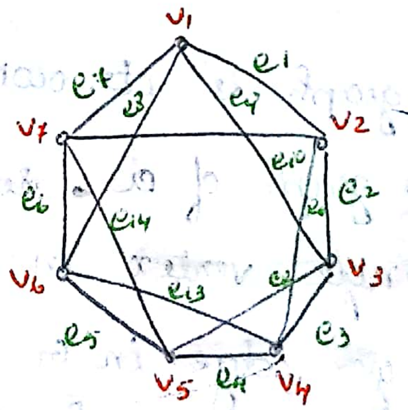
$e_{10} \to$

$e_{11} \to$

$e_{12} \to$

iv) Check whether G and G' are isomorphic
(verify that the two graphs are isomorphic. Label the corresponding edges and vertices and write their correspondance





G

G'

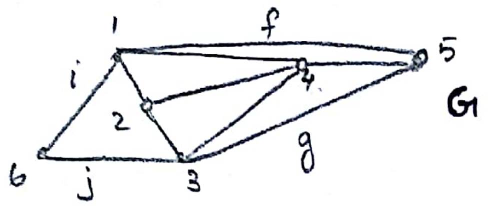
Since we can't find a correspondance between the vertices & edges of G & G', they are not isomorphic

Sub-Graph

A Graph 'g' is said to be a sub-graph of G if all vertices and all edges of 'g' are in G and each edge of 'g' has the same end vertices in 'g' as in G.

we denote it by $\boxed{g \subset G.}$ stating as g is a subgraph of G.

eg:-



sub-graphs

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NOTE

1. Every graph is its own sub-graph
2. A sub graph of a sub graph of G, is a subgraph of G
3. A single vertex in a graph G is a subgraph of G.
4. A single edge in G together with its end vertices is also a subgraph of G.

Edge-disjoint subgraph

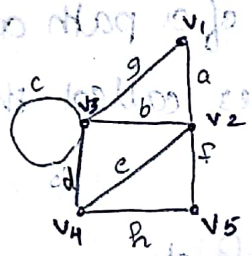
Two or more subgraphs g_1 and g_2 of a graph G are said to be edge disjoint if g_1 and g_2 do not have edges in common.

Vertex disjoint subgraphs

subgraphs that do not have vertices in common, are said to be 'vertex disjoint subgraph'.

walk, path and circuit

* A **walk** is defined as a finite alternating sequence of vertices and edges begininning and ending with vertices such that each edge is incident with vertices preceeding and following it. No edge will appear more than once in a walk. A vertex however may appear more than once.



examples for walk

⇒ v_1 a v_2 b v_3 c v_3 d v_4 e v_2 f v_5

⇒ A walk is also referred to as edge-train or chain

* Vertices with which a walk begins and ends are called its terminal vertices.

eg:- v_1 and v_5 are the terminal vertices of a walk.

* A walk which begins and ends at the same vertex is called a closed walk

eg:- v_3 g v_1 a v_2 b v_3

* A walk that is not closed is called an open walk

eg:- v_1 a v_2 f v_5 h v_4 d v_3

* An open walk, in which no vertex appear more than once is called a path or a simple path or elementary path

eg:- v_1 a v_2 b v_3 d v_4 - is a path

v_1 a v_2 b v_3 c v_3 d v_4 e v_2 f v_5 - is not a path.

⇒ The no. of edges in a path is called the length of a path.

⇒ An edge which is not a self loop is a path of length 1.

⇒ A self loop can be included in walk, but not in a path.

⇒ The terminal vertices of a path are of degree 1 and the rest of the vertices called the intermediate vertices are of degree 2.

* A closed walk in which no vertex except the initial and the final vertex appears more than once is called a circuit or cycle or elementary cycle or circular path or polygon.

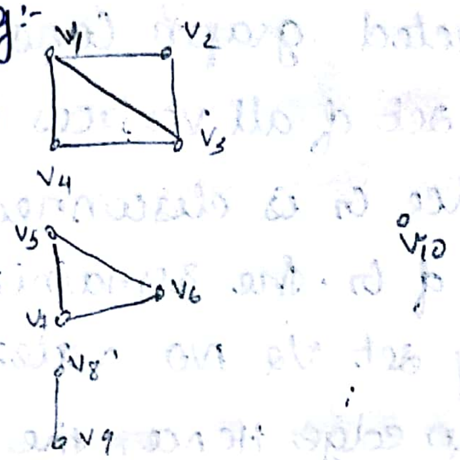
Connected Graph, disconnected Graph and components

⇒ A graph is said to be connected if there is atleast one path between every pair of vertices in graph G.

⇒ otherwise it is a disconnected graph

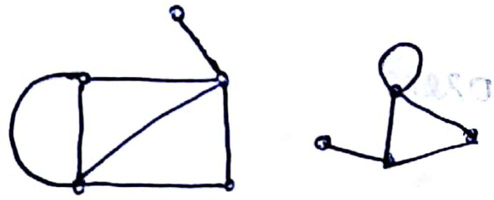
⇒ Disconnected graph consist of two or more connected graph. Each of these connected graphs is called a component.

eg:-



24/8/17 Thursday This is a disconnected graph with four components

eg:-



G

This a graph with two components.

Theorem-I

A graph G is disconnected if and only if its vertex set V can be partioned into two non-empty disjoint subsets V_1 and V_2 such that there exists no edge in G, whose one end vertex is in subset V_1 and the other end in subset V_2.

Suppose that such a partition exists. consider, two arbitary vertices 'a' and 'b' of G such that a belongs to V_1 and b belongs to V_2. No path can exist between a and b otherwise, there would be atleast one edge whose one end vertex would be in V_1 and other in V_2. Hence, if a partition exists, G is not connected.

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Conversely, let G be a disconnected graph. Consider a vertex 'a' in G. Let V_1 be the set of all vertices that are joined by paths to 'a'. Since G is disconnected, V_1 doesnot include all vertices of G. The remaining vertices will form a non-empty set V_2. No vertex in V_1 is joined to any in V_2 by an edge. Hence, the partition.

Hence the theorem

Theorem-II

If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Let G be a graph with all even vertices (vertex with even degree) except vertices v_1 and v_2 which are odd. The number of vertices of odd degree is even. This holds for every graph and therefore for every component of a disconnected graph. No graph can have an odd number of odd vertices. Therefore, in graph G, v_1 and v_2 must belong to the same component and hence must have a path between them.

Hence the theorem

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Theorem - III

A simple graph with 'n' vertices and and 'k' components can have atmost $\boxed{\frac{(n-k)(n-k+1)}{2}}$ edges.

PROOF

Let the number of vertices in each of the k components of a graph G be $n_1, n_2 \ldots n_k$

Thus we have, $n_1 + n_2 \ldots + n_k = n$ where $n_i \geq 1$

we have $\sum_{i=1}^{k} n_i = n$

$$\therefore \sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$$

$$= n - k$$

i.e, $\sum_{i=1}^{k} (n_i - 1) = n - k$

squaring on both the sides we have,

$$\left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2$$

$$\Rightarrow [(n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1)]^2 = (n-k)^2$$

$$\Rightarrow (n_1 - 1)^2 + (n_2 - 1)^2 \ldots + (n_k - 1)^2 + 2(n_1 - 1)(n_2 - 1) + 2(n_2 - 1)(n_3 - 1) + \ldots$$

$$\ldots + 2(n_{k-1} - 1)(n_k - 1) = n^2 + k^2 - 2nk$$

$\Rightarrow n_1^2 - 2n_1 + 1 + n_2^2 - 2n_2 + 1 \cdots + n_k^2 - 2n_k + 1 +$ non-negative cross terms
$$= n^2 + k^2 - 2nk$$

$$\Rightarrow n_1^2 + n_2^2 + \cdots + n_k^2 - 2(n_1 + n_2 + \cdots n_k) + k \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + k \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2n + k \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk + 2n - k$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 + k(k-2n) - (k-2n)$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 + (k-2n)(k-1)$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k) \quad \text{———} \quad (1)$$

Since, the maximum no. of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$, the maximum no. of edges in the i^{th} component of G is $\frac{n_i(n_i-1)}{2}$

$\therefore$ the maximum no. of edges in G is

$$\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$

$$\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \sum_{i=1}^{k} \left(\frac{n_i^2 - n_i}{2}\right)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$$

$$\leq \frac{1}{2}\left[n^2 - (k-1)(2n-k) - n\right] \quad \text{(from eq ①)}$$

$$\leq \frac{1}{2}\left[n^2 - (2nk - k^2 - 2n + k) - n\right]$$

$$\leq \frac{1}{2}\left[n^2 - 2nk + k^2 + 2n - k - n\right]$$

$$\leq \frac{1}{2}\left[n^2 - nk - nk + k^2 + n - k\right]$$

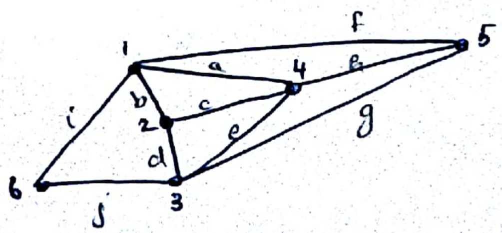
$$\leq \frac{1}{2}\left[n(n-k) - k(n-k) + (n-k)\right]$$

$$\leq \frac{1}{2}\left[(n-k)(n-k+1)\right]$$

$$\boxed{\text{max no: of edges in } G \leq \frac{(n-k)(n-k+1)}{2}}$$

Problems

Q) List all different paths between the vertices 5 and 6 in the figure given below. Give the length of each of these path.



i) 5 g 3 j 6   length: 2

ii) 5 h 4 e 3 j 6   length: 3

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iii) 5 h 4 c 2 d 3 j 6 : length : 4

iv) 5 h 4 a 1 b 2 d 3 j 6 : length : 5

v) 5 f 1 b 2 d 3 j 6 : length : 4

vi) 5 f 1 i 6 : length : 2

vii) 5 h 4 a 1 i 6 : length : 3

viii) 5 h 4 c 2 b 1 i 6 : length : 4

ix) 5 h 4 e 3 d 2 b 1 i 6 length : 5

x) 5 g 3 d 2 b 1 i 6 length : 4

xi) 5 f 1 a 4 e 3 j 6 length : 4

xii) 5 g 3 e 4 a 1 i 6 length : 5

xiii) 5 g 3 e 4 c 2 b 1 i 6 length : 5

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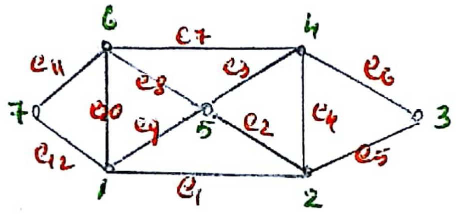
Saturday

MODULE - II

Euler Graph

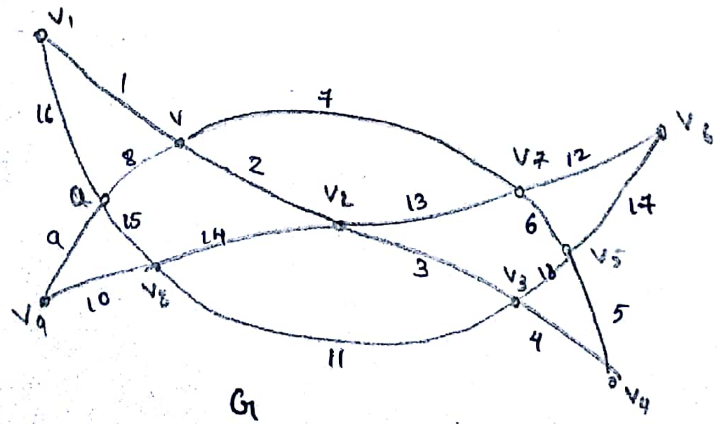
If some closed walk in a graph contains all the edges of the graph, then the walk is called an Euler line and the graph is called an euler graph

eg:-
i)



4 e_3 5 e_9 1 e_1 2 e_2 5 e_8 6 e_{10} 1 e_{12} 7 e_{11} 6 e_7 4 e_4 2 e_5 3 e_6 4

ii)



Theorem

A given connected graph G is an euler graph, if and only if all vertices of G are of even degree.

Suppose that G is an euler graph, it therefore contains an euler line, which is a closed walk. In tracing this walk, we observe that, everytime the walk meets a vertex v, it goes through two new edges incident on v with one entered to v and with other exited.

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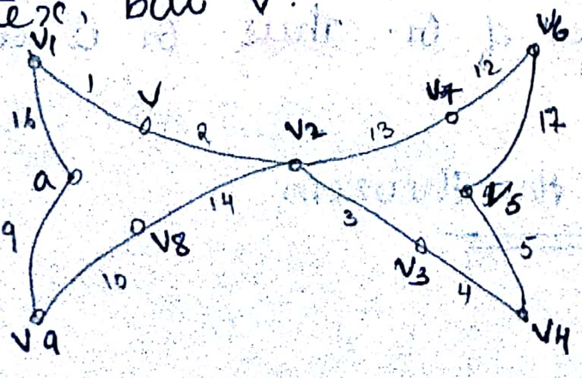
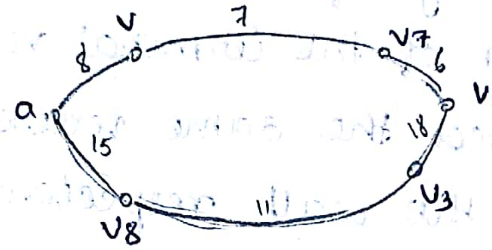
This is true not only of all intermediate vertices of the walk, but also of the terminal vertex, because we exited and entered the same vertex at the begining and end of the walk respectively.
Thus if G is an euler graph, the degree of every vertex is even.

To prove the sufficiency of the condition, assume that all vertices of G are of even degree.
Now, we construct a walk, starting at an arbitrary vertex 'v' and going through the edges of G such that no edge is traced more than once. We continue tracing as far as possible. Since, every vertex is of even degree, we can exit from every vertex we entered. The tracing cannot stop any vertex, but 'v'.



since v is of even degree, we shall eventually reach v when tracing comes to an end. If the closed walk H includes all edges of G, then G is an Euler graph. If not, remove from G, all the edges in H and obtain a subgraph H' of G formed by the remaining edges.



H'

since, both G and H have all their vertices of even degree the degree of H' are also even. more over, H' must touch H atleast at one vertex 'a' because G is connected. starting from 'a', we can again construct a new walk in graph H, since all the vertices of H' are of even degree, this walk in H' must terminate at 'a'. But, this walk in H' can be combined with H to form a new walk, which starts and ends at vertex v and has more edges than H. This process can be repeated until we obtain a closed walk that traverses all the edges of G. Thus G is an Euler Graph.

Hence the theorem

NOTE :

1) By looking at the graph of Konisberg bridge problem, we find that not all of its vertices are of even degree. Hence, it is not an Euler graph. Thus, it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

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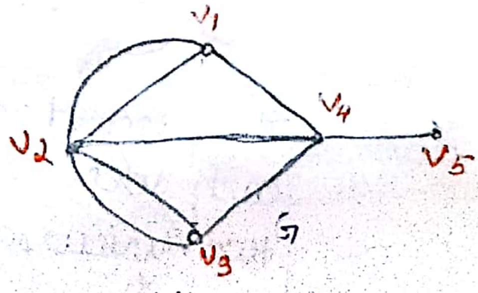
2) An open walk that includes all the edges of a graph without retracing any edge is a unicursal line or an open euler line. A connected graph that has a unicursal line is called a unicursal graph.

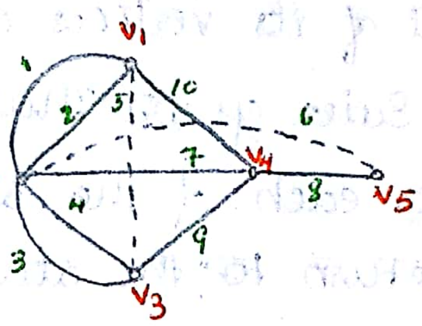
Theorem

In a connected graph G, with exactly 2k odd vertices, there exists k edge-disjoint subgraphs such that they together contain all edges of G and such that each is a unicursal graph.



Let the odd vertices of the given graph G be named $v_1, v_2, \ldots v_k$; $w_1, w_2 \ldots w_k$ in any arbitrary order. Add k edges to G between the pair of ~~ver~~ vertices $v_1 ; v_3$ and $v_2 - v_5$ to form a new graph G'.

(In G, v_1, v_2, v_3, v_5 are the vertices of odd degree. we call v_3 as w_1 and v_5 as w_2)

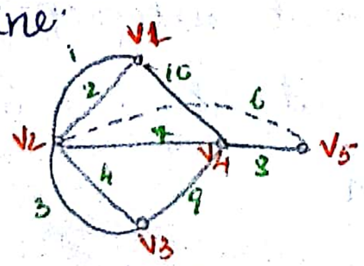


G'

Since, every vertex of G' is of even degree, G' consists of an euler line ρ

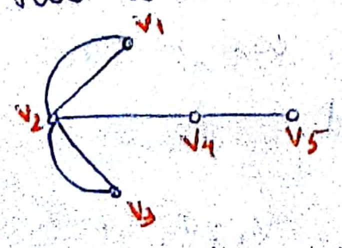
$\rho = v_1\ 1\ v_2\ 3\ v_3\ 4\ v_2\ 7\ v_4\ 8\ v_5\ 6\ v_2\ 2\ v_1\ 10\ v_4\ 9\ v_3\ 5\ v_1$

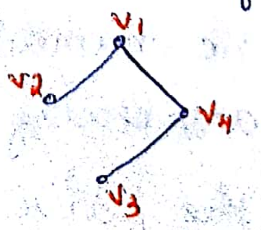
Now, if we remove from ρ, the k edges we just added, ρ will split into k walks each of which is a unicursal line.



$\rho = v_1\ 1\ v_2\ 3\ v_3\ 4\ v_2\ 7\ v_4\ 8\ v_5\ v_6\ v_2\ 2\ v_1\ 10\ v_4\ 9\ v_3$

The second removal will split that into two unicursal line and each successive removal will split a unicursal line into two unicursal lines until there are k of them





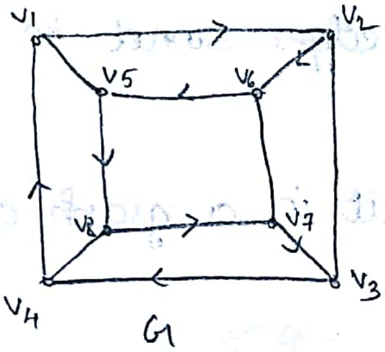
$v_1\ 1\ v_2\ 3\ v_3\ 4\ v_2\ 7\ v_4\ 8\ v_5$

$v_2\ 2\ v_1\ 10\ v_4\ 9\ v_3$

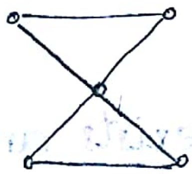
Hence the theorem

Hamiltonian Paths & Circuit

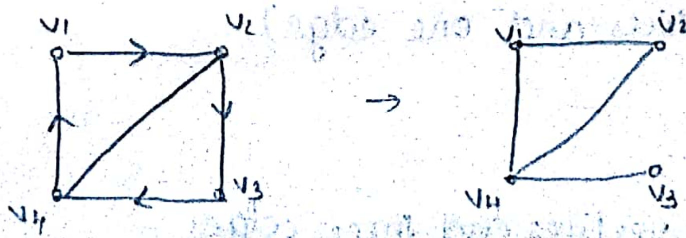
⇒ A hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once except the initial and the final vertex.



⇒ Every connected graph doesnot contain a have a hameltonian circuit



* If we remove any one edge from a hameltonian circuit we are left with a path. This path is called hameltonian path.



* Clearly, a hameltonian path in a graph G traverses every vertex of G. Since, a hameltonian path is a subgraph of a hameltonian circuit, every graph that has a

hamiltonian circuit also has a hamiltonian path.

But there are graphs with hamiltonian path that has no hamiltonian circuit.

NOTE:

1* A self loop or parallel edges cannot be included in a hamiltonian circuit.

2* A hamiltonian circuit in a graph of 'n' vertices consists of exactly 'n' edges.

3* Length of a hamiltonian path in a connected graph of 'n' vertices is $(n-1)$

complete Graph

14/9/17

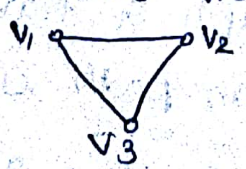
A simple graph in which there exists an edge between every pair of vertices is called a complete graph.

⇒ ov_1

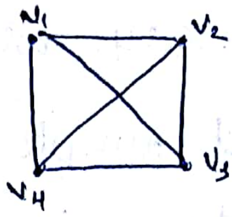
K_1 (has one vertex and zero edges)

⇒ K_2 (has two vertices and one edge)

⇒ K_3 has three vertices and three edges

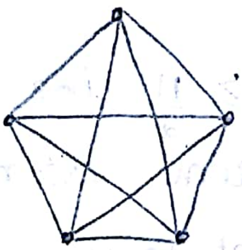


$\Rightarrow K_4$



Four vertices and six edges

$\Rightarrow K_5$



5 vertices and 10 edges

$\therefore$ Total number of edges in a complete graph with 'n' vertices is given by $\;{}^{n}C_2 = \frac{n(n-1)}{2}$

DIRAC'S THEOREM