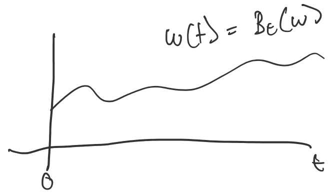
04-09

Wednesday, April 9, 2025 11:32 AM

Markov Property.

$\Omega = C[0,\infty]$



$\mathcal{F}_t^0 = \sigma\{B_s : 0 \le s \le t\}$ $\qquad \mathcal{F}_t^0 \subset \mathcal{F}_t^{+} \subset \mathcal{F}_t$

$\{P_x\}_{x \in \mathbb{R}}$ $\qquad P_x(B_0 = x) = 1$

$P_0(B_0 = 0) = 1 \implies$ Dis of SBM ← Standard Brownian motion

$P_x(A) = P_0(A - x)$

Properties are same just Different starting point.

~~Examples of r.v.~~

2

(A) Claim $\quad E^x_{\mathcal{F}_t^0}(Y \circ \theta)_t = E^{B_t^*}(Y), \quad \forall Y : ([0,\infty] \to \mathbb{R},$ bdd. measureable,

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Example

$Y = \int_0^1 B_s \, ds.$ $\qquad$ this integral is Random

$Y = \prod_{k=1}^{d} f_k(B_{h_k})$

$0 < h_1 < h_2 < \dots < h_d$

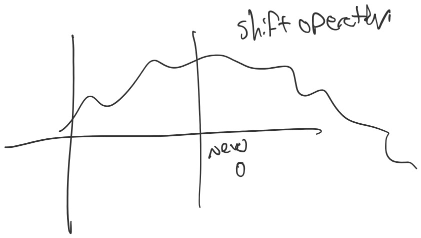
$f_k : \mathbb{R} \to \mathbb{R}$, bdd. $\quad k = 1, \dots, d.$

$$Y = \max_{0 \le t \le 7} \{B_t\}$$

$$Y \circ \Theta_t = \prod_{k=1}^{d} f_k(B_{t+h_k})$$

$$\Theta_t(\omega)(s) = \omega(t+s), \quad 0 \le s$$

shift operator



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$E_{\mathcal{F}_t^0}$

↑ conditions on up to time t.

$$g(y) = E^{y}(Y), \quad y \in \mathbb{R}$$

$$g(B_t^x) = E^{B_t^x}(Y)$$

Leventhal's B

Lemma: $X \in \mathcal{F}$, $\vec{Y} = (Y_1, \dots, Y_d)$, $\vec{Y} \perp\!\!\!\perp \mathcal{F}$

then: $E_{\mathcal{F}} f(X, \vec{Y}) = g(X)$ $\quad g(x) = E[f(x, \vec{Y})]$ $\quad x \in \mathbb{R}$.

$$f : \mathbb{R}^{d+1} \to \mathbb{R}$$

Says X & Y are ind.

Easy to prove w/ Fubini

WTS: $E(f(X,Y)\cancel{1_A}\; G) = E(g(X) \cdot \cancel{1_A}\; G)$ $\quad G \in \mathcal{F}$, G is nn Bound.

$A \in \mathcal{F}$

Fix X & G. integrate over Y then.

$X, G \in \mathcal{F}$ $\quad$ calc this first.

$Y \perp\!\!\!\perp \mathcal{F}$ $\quad E[f(x, \vec{Y}) g]$ $\quad x, y \in \mathbb{R}$.

A $\Rightarrow$ $\quad E^{x}(Y \circ \Theta_t) = E^{x}[E^{B_t^x}(Y)]$

By Expectation

Renaming

$$E^{\hat{x}}_{\mathcal{F}}(Y \circ \Theta_t) \overset{a.s}{=} E^{B_t^{\hat{x}}}(Y) \quad \forall Y : (C[0,\infty)) \to \mathbb{R} \text{ bdd measure}$$

Renaming

$$E_{\mathcal{F}_t^0}^{\tilde{x}}(Y \circ \theta_t) \overset{a.s}{=} E^{B_t^{\tilde{x}}}(Y), \quad \forall Y : ([0,\infty)) \to \mathbb{R}, \text{ bdd, measure}$$

MCT.

on Friday we mention Monotone class theorem · w/ π-system.

will take $Y = \prod_{k=1}^{d} f_k(B_{h_k})$ to prove Levantel Lemma

Nts.: $$E_{\hat{\mathcal{F}}_t^0}^{\tilde{x}} \prod_{k=1}^{d} f_k(B_t^{\tilde{x}} + h_k) = E^{B_t^{\tilde{x}}}\left[\prod_{k=1}^{d} f_k(B_{h_k})\right]$$

Notation: E^x Does this Denote Another given? a Starting Point.

to Prove

using the lemma. $X = B_t^{\tilde{x}}$, $Y_k = B_{t+h_k}^{\tilde{x}} - B_t^{\tilde{x}}$ $k = 1, \dots, d$.

$$\vec{Y} = (Y_1, \dots, Y_d) \perp\!\!\!\perp \mathcal{F}_t^0$$

Each is increments, therefore by ind. incre. From Brownian Motions

therefore we can use lemma

$$f(X, \vec{Y}) = \prod_{k=1}^{d} f_k\Big(\underset{X}{B_t^{\tilde{x}}} + \underset{Y_k}{(B_{t+h_k}^{\tilde{x}} - B_t^{\tilde{x}})}\Big)$$

By (B)

$$g(x) = E^{\tilde{x}} \prod_{k=1}^{d} f_k\Big(x + \underbrace{B_{t+h_k}^{\tilde{x}} - B_t^{\tilde{x}}}_{\overset{D}{=} B^0_{h_k}}\Big)$$

$$= E^{x} \prod_{k=1}^{d} f_k(x + B_{h_k})$$

$$= E^{x}\left[\prod_{k=1}^{d} f_k(B_{h_k})\right]$$

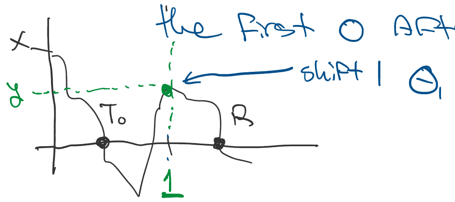
the Answer is $g(B_t^{\tilde{x}})$.

$$g(B_t^{\tilde{x}}) = E^{B_t^{\tilde{x}}}\left[\prod_{k=1}^{d} f_k(B_{h_k})\right]$$

* intuitive. given ind. inc. Then we will Do it right

EX: ① $R = \inf\{t > 1 ;\ B_t = 0\}$.

the First 0 After $t = 1$



$$T_0 = \inf\{t \geq 0 ;\ B_t = 0\}$$

shift By 1 eg. Ignore Before 1.

$$Y = \mathbb{1}_{\{T_0 > 1\}} \circ \theta_1 = \mathbb{1}_{\{R > 1+t\}}$$

$$P_x(R > 1+t) = \int_{y=-\infty}^{\infty} P_y(T_0 > t)\, P_1(x,y)\, dy$$

Density.

Probability if we start Brownian motion At y

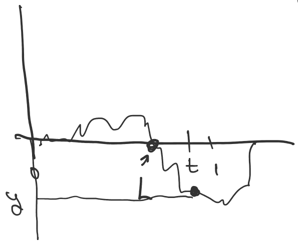
now At Point Y.
what is $P_t(x,y) = f_{N(x,t)}(y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(y-x)^2}$, $-\infty < y < \infty$

Example.

$$L = \sup\{ t \le 1 : B_t = 0 \}$$

then $P_0(L \le t) = \int_{-\infty}^{\infty} P_t(0,y)\, P_y(T_0 > 1-t)\, dy.$ $t \le 1$

$L < t < 1$



$$\mathbb{1}_{\{T_0 > 1-t\}} \circ \theta_t = \mathbb{1}_{\{L \le t\}}$$