Theorem let $\{X_{n,k}\}_{1 \leq k \leq n, n \geq 1}$ get values $\mathbb{Z}^+ = \{0,1,2,\dots\}$

if ① $\sum_{k=1}^{n} P(X_{n,k} = 1) \xrightarrow[n \to \infty]{} \lambda$

② $\max_{1 \leq k \leq n} P(X_{n,k} = 1) \xrightarrow[n \to \infty]{} 0$

③ $\sum_{k=1}^{n} P(X_{n,k} \geq 2) \xrightarrow[n \to \infty]{} 0$

then

$\sum_{k=1}^{n} X_{n,k} \implies Poisson(\lambda)$

$Y_{n,k} = X_{n,k} \mathbb{1}_{X_{n,k} \leq 1}$, $T_n = \sum_{k=1}^{n} Y_{k,n}$, $S_n = \sum_{k=1}^{n} X_{n,k}$

From 3 we conclude

$P(S_n - T_n \neq 0) \xrightarrow[n \to \infty]{} 0$ APP P here!

$\{S_n - T_n \neq 0\} \subseteq \bigcup_{k=1}^{n} \{X_{n,k} \neq Y_{n,k}\}$

$= P\left(\bigcup_{k=1}^{n} \{X_{n,k} \geq 2\}\right)$

$\leq \sum_{k=1}^{n} P(X_{n,k} \geq 2) \xrightarrow[n \to \infty]{③} 0$

Bernoulli. not independent.
    such that limit the dependence disappears.

Poisson Approximation of Dependent Events.

<u>Ex</u>   random Permutations $(1, \dots, n)$ has $n!$ Perms.

Given   1, 2, 3   numbers.
Perm   2, 3, 1   Reorder the numbers

how many Perm 3!

$P(a\ perm) = \dfrac{1}{n!}$ e.g. Uniform

$S_n = \#$ of fixed Points of $(1, \dots, n)$

1,2,3   0 Fixed Points   1 2 3   1 fixed Point.
2,3,1   1 3 2

$$1\,2\,3$$
$$2\,3\,1$$
0 fixed points
$$1\,2\,3$$
$$1\,3\,2$$
1 fixed point.

$$P(S_n = 0) \xrightarrow[n \to \infty]{} e^{-1}$$

Done By Complement

$$1 - P(S_n \geq 1).$$

$A_k = \{k \text{ is fixed point.}\}$ $1 \leq k \leq n$

Exclusion-Inclusion n choose 2 $\binom{n}{3}$

$$P(S_n \geq 1) = P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k) - \sum_{1 \leq k < \ell \leq n} P(A_k \cap A_\ell) + \sum_{1 \leq k < \ell < m \leq n} P(A_k \cap A_\ell \cap A_n) \ldots \pm P\left(\bigcap_{k=1}^{m} A_k\right)$$

$$= n \cdot \frac{1}{n} - \binom{n}{2} \frac{(n-2)!}{n!} \qquad \binom{n}{3} \frac{(n-3)!}{n!} \pm \ldots \pm \frac{1}{n!}$$

$$P(A_k) = \frac{(n-1)!}{n!} = \frac{1}{n} \qquad = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots \pm \frac{1}{n!}$$

$$P(A_k \cap A_\ell) = \frac{(n-2)!}{n!} \qquad P(S_n = 0) = \frac{1}{0!} - \left(\sum_{k=1}^{n} \frac{(-1)^{k+1}}{n!}\right) = -$$
$$k < \ell$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!} = e^{-1} \qquad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

First step To Prove,

$$S_n \Rightarrow \text{Poisson}(\lambda = 1)$$

NTs $P(S_n = k) \longrightarrow \dfrac{e^{-1} 1^k}{k!} = \dfrac{e^{-1}}{k!}$, $k = 0, 1, \ldots$

Preceding showed $k=0$

"Convergence in total Variation $\Rightarrow$ Convergence Distribution"

$$P(S_n = k) \longrightarrow \frac{e^{-1}}{k!}$$

How many left
$n-k$ has fixed pnt

$$P(S_n = k) = \frac{\binom{n}{k} \cdot P(S_{n-k} = 0)(n-k)!}{n!}$$

$$= \frac{P(S_{n-k} = 0)}{k!} \xrightarrow[n \to \infty]{} \frac{e^{-1}}{k!}$$

↳ Depends on n.

Boxes.

Ex 2 r_n balls. inserted Randomly in n boxes. ⊔ ⊔ ... ⊔
                                                    1 2   n

$S_n = \#$ of Empty Boxes, $\xrightarrow{n\to\infty}$?

$n \sim n \log(\frac{\lambda}{n})$

<u>IDEA</u>

if $E(S_n) \xrightarrow{n\to\infty} \lambda$ then $S_n \Rightarrow$ Poisson(λ)

Condition $Y_k = \begin{cases} 1 & \text{if box } k \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$

$S_n = \sum_{k=1}^{n} Y_k$

$E S_n = \sum_{k=1}^{n} E Y_k = n E(Y_1) = n P(\text{box 1 Empty})$

we want

$E S_n = n \cdot (1 - \frac{1}{n})^n \xrightarrow{n\to\infty} \lambda$

Replace $n \cdot S_n / n$ then $(1 - \frac{1}{n})^n = e^{-1}$

Condition $n e^{-n/n} \xrightarrow{n\to\infty} \lambda$

Important Extension! of Poisson Convergence
if we Remove ③ from theorem & Assume $X_{n,k} \sim Ber(P_{n,k})$

$P_{n,k} = P(X_{n,k} = 1)$

<<Copy from Above>>

Remove Due to Party school

$P(X_{n,k} = a_i) = P_{n,i}^{(k)} = P_{n,i}$ $i \leq k \leq n$, $\{X_{n,k}\}_{1 \leq k \leq n}$ iid.

$_{1 \leq i \leq I}$

if this condition, Asymptotically Independent.

if $n \cdot P_{n,i} \xrightarrow{n\to\infty} \lambda_i$: $1 \leq i \leq I$

Must go to Zero. or not converge.

then $\sum_{k=1}^{n} X_{n,k} \Longrightarrow \sum_{i=1}^{I} a_i Y_i$ where $Y_i \Rightarrow$ Poisson(λ_i) $1 \leq i \leq n$

Y_i counts how many there we got a_i

"Finite Dimension R.V. & cfi for multidimensio."
Last chapter of Durrent 3.

$\overset{\mathcal{D}}{=} \sum_{k=1}^{T} Z_k$ $\{Z_k\}_{i, k \geq 1}$ iid.

compound Poisson. $P(Z_k = a_i) = \frac{\lambda_i}{\lambda}$ where $\lambda = \sum_{i=1}^{I} \lambda_i$ $T \sim$ Poisson(λ)

compound Poisson.

$$P(Z_k = a_i) = \frac{\lambda_i}{\lambda} \qquad \text{where} \quad \lambda = \sum_{i=1}^{I} \lambda_i \quad T \sim \text{Poisson}(\lambda)$$

$$T \perp\!\!\!\perp \{Z_k\} \quad \text{in } D.$$

Exam, 6.

* Some state theorems.

Prob. 2.   Prove something.   All in class
        Convergence in Distribution
        Slutsky.

Prob 3.   convergen to standard Normal
        truncation.   step A.B.C.   see class

Prob 4   Triangle Array,   check Lindberg condition -

Prob 5   Converge in Dist   to Poisson.
        May need truncation

Prob 6   Calc. Directly charactistic function
        $S_n \longrightarrow$ target
        some Approx.
    not theorem