

MG For BM. (Ω, \mathcal{F}, P)

Def. of MG in continuous time

$\{X_t, \mathcal{F}_t\}_{t \geq 0}$ is MG if

- ① $E|X_t| < \infty$
- ② $X_t \in \mathcal{F}_t$
- ③ $E_{\mathcal{F}_t}(X_{t+h}) = X_t \quad \forall t \geq 0, h > 0$

we want to study MG associated with BM.

$M(x, t), x \in \mathbb{R}, t \geq 0$ w/ natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ or canonical filtration.

$M(x, t) \quad x \in \mathbb{R}, t \geq 0.$

Question when is $\{M(B_t^x, t), \mathcal{F}_t\}_{t \geq 0} \propto$ MG?

$$E_x M(B_{t+h}, t+h | \mathcal{F}_t) = M(B_t^x, t) \text{ a.s. } \forall x \in \mathbb{R}, t \geq 0, h > 0$$

By Markov Property.

$$\begin{aligned} \psi(B_t^x) &= E_{B_t^x} [M(B_{t+h}, t+h)] \\ &= M(B_t^x, t) \end{aligned} \quad \begin{array}{l} \text{this is the} \\ \text{formula.} \end{array} \quad \forall x \in \mathbb{R}, t \geq 0, h > 0$$

$$\psi(y) = E_y [M(B_{t+h}, t+h)]$$

Example Famous MG Exponential

$$M(x, t) = e^{\theta x - \theta^2 t / 2} \quad \theta \in \mathbb{R}, \theta \text{ constant.}$$

$$E_x (M(B_{t+h}, t+h)) = E_x (e^{\theta B_{t+h} - \theta^2 (t+h)/2})$$

$$= e^{-\theta^2 (t+h)/2} \cdot E_x (e^{\theta B_{t+h}})$$

$$= e^{-\theta^2 (t+h)/2} \cdot E(e^{\theta(x + \sqrt{t+h} Z)})$$

$$B_h \sim N(x, h)$$

$$B_h \stackrel{d}{=} x + \sqrt{h} Z \sim N(x, h)$$

$$= e^{-\theta^2(t+h)/2} \frac{E(e^{\theta(x+\sqrt{t+h}Z)})}{e^{\theta x} e^{\theta^2 t/2}} \\ e^{-\frac{\theta^2 t}{2}} e^{-\frac{\theta^2 h}{2}} = e^{-\frac{\theta^2 t}{2} + \theta x} = u(x, t)$$

$$\left(e^{\theta B_t^x - \frac{\theta^2 t}{2}}, \tilde{\mathcal{F}}_t \right)_{t \geq 0} \text{ is mg with } P_x, x \in \mathbb{R}$$

$$\frac{\partial}{\partial \theta} E_x[u(B_n, t+h; \theta)] = u(x, t; \theta)$$

$$E_x \left[\frac{\partial}{\partial \theta} u(B_n + t+h; \theta) \right] = \frac{\partial}{\partial \theta} u(x, t; \theta) \quad \forall \theta$$

$$\{u(B_t^x, t, \theta), \tilde{\mathcal{F}}_t\} \text{ is mg.}$$

$$\frac{\partial}{\partial \theta} (B_t^x, t, \theta), \tilde{\mathcal{F}}_t \text{ is mb.}$$

$$\text{Continue to take Derivatives } \frac{\partial^2}{\partial \theta^2} u(\cdot) \neq 0$$

$$u(\cdot) = e^{\theta x - \theta^2 t/2}$$

$$\frac{\partial}{\partial t} u(\cdot) = u(\cdot)(x - \theta t)$$

$$\Rightarrow \frac{\partial}{\partial \theta} u(0) \Big|_{\theta=0} = x$$

$$\Rightarrow \{B_t^x, \tilde{\mathcal{F}}_t\}_{t \geq 0}$$

implies Brownian motion is mg.

$$\frac{\partial^2}{\partial \theta^2} = M(1) \cdot (x - \theta t)^2 \Big|_{\theta=0} = x^2$$

$$\Rightarrow (B_t^2 - t F_t)_{t \geq 0}$$

$$\frac{\partial^3}{\partial \theta^3} = M(1) (x - \theta t)^3 + M(1) 2(x - \theta t)(x - t) - t M(0)(x - \theta t)$$

$$\Rightarrow_{\theta=0} \{B_t^3 - 3t B_t, F_t, t \geq 0\} M_6.$$

M_6 . What we get: $1, B_t, B_t^2 - t, B_t^3 - 3t B_t, B_t^4 - 6t B_t^2 + 3t^2, \dots$

In the book let $M(x, t)$ satisfy $\frac{\partial M}{\partial t} + \frac{1}{2} \frac{\partial^2 M}{\partial x^2} = 0 \quad \forall x \in \mathbb{R}, t \geq 0$

then $M(B_t, t)$ is M_6 .

The book proved $\frac{\partial}{\partial t} E_x(M(B_t, t)) = 0$

$$\Leftrightarrow E_x M(B_t, t) = M(x, 0) \quad \forall t \geq 0, x \in \mathbb{R}$$

Start with:

$$P_t(x, y) = \frac{P(F_t)}{N(x, t)} \quad , \text{ it turns out } \frac{\partial P_t}{\partial t} = \frac{1}{2} \frac{\partial^2 P_t}{\partial y^2} \quad \forall x \in \mathbb{R}.$$

$$\tau \text{ s.t. } \tau \leq t \text{ a.s.} \quad E_{\mathcal{F}_\tau} (M(B_t, t)) = M(B_\tau, \tau)$$