01-17

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CLT in $\mathbb{R}^d$

Convergence theorem

"881 continuity convergence theorem"

$\underline{X}_n \Rightarrow \underline{X}_\infty$ iff $\varphi_{\underline{X}_n}(t) \xrightarrow[n\to\infty]{} \varphi_{\underline{X}_\infty}(t) \quad \forall t \in \mathbb{R}^d$.

CRAMER-WOID Device.

sweden

Hungary Prf at columbia.

Invent seq. Stats.

$\underline{X}_n \Longrightarrow X_\infty$ if $t\cdot \underline{X} \Rightarrow t\underline{X}_\infty \quad \forall t \in \mathbb{R}$.

$\varphi_{t\cdot X_n}(1) \longrightarrow \varphi_{t\cdot X_\infty}(1)$ SAME.

Reduce Problem to 1 Dimension.

BASIC CLT

covariance matrix.

$\{X_n\}$ are iid in $\mathbb{R}^d$, $E(\underline{X})=0$, $\Gamma(\underline{X}) = [\Gamma_{i,j}]_{1\le i,j\le d}$ $\quad \Gamma_{i,j} = E(\xi_i\cdot\xi_j)$

$$\underline{X} = \begin{pmatrix}\xi_1\\ \vdots \\ \xi_d\end{pmatrix}$$

$$E\|\underline{X}\|^2 = E\sum_{k=1}^{d}\xi_k^2 < \infty$$

then $\frac{S_n}{\sqrt{n}} \Rightarrow G$ where $G \sim N(\vec{0}, \Gamma(\underline{X}))$

where $S_n = \sum_{k=1}^{n} \underline{X}_k$

$G = \underset{d\times d}{A}\,\vec{Z}$ $\qquad \vec{Z} = \begin{pmatrix}z_1\\ \vdots\\ z_d\end{pmatrix}$ $\{Z_k\}$ iid $N(0,1)$

cov. mat of G.

$$\Gamma(G) = E(A\cdot Z\cdot Z^t A^t) = A\cdot E(ZZ^t)A^t$$

$$= A \cdot \Gamma(z) A^t$$

$$= AA^t$$

$$\Gamma(\mathbb{Z}) = A \cdot A^t$$

why is there on A like this.

① $\Gamma(\mathbb{Z})$ is symmetric. $\Gamma(\mathbb{Z}) = \Gamma(\mathbb{Z})^t$

$a_{i,j} = a_{j,i}$

Also ② Positive semidefinite. $\Gamma(x) \succeq 0$

Def $\vec{t}^T \cdot \Gamma t \geq 0$

$$t^t \Gamma t = Var(t\mathbb{Z}) = E(t \cdot \mathbb{Z})^2 = E(t' \mathbb{Z}\mathbb{Z} t')$$

Also we can find Eigenvalues

O - orthogonal matrix if $O \cdot O^t = I$

Eigendecomposition.

$$\Gamma(X) = O D O^t$$

where $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_i \end{pmatrix}$ Diagonal matrix,

$E = \begin{pmatrix} \sqrt{\lambda_i} & & & \\ & \sqrt{\lambda_i} & & \\ & & \ddots & \\ & & & \sqrt{N_i} \end{pmatrix}$ $\quad E \cdot E^t = D$

$$OE(OE)^t = AA^t$$

why $G \sim N(0, \Gamma(z))$ & $G = A\vec{Z}$ have same Distrib.

wts: $\varphi_{S_n}(t) \longrightarrow \varphi_G(t)$, $t \in \mathbb{R}^d$

$\varphi_{t \cdot S_n}(1) \longrightarrow \varphi_{tG}(1)$

So what is $t \cdot S_n$

$$t \cdot S_n = \sum_{k=1}^{n} \underbrace{t \cdot X_k}$$

iid.

$t \in \mathbb{R}^d$.

$$\frac{\sum_{k=1}^{n} t \cdot X_k}{\sqrt{n}} \Rightarrow N(0, \sigma^2_{t \cdot X})$$

where $\sigma^2_{t \cdot X}$ $t' \Gamma t$

we Reduce to 1 dim through Kramer-Wold device thereby use 881 tech

$$\varphi^{(1)}_{t \cdot G} = e^{-\frac{t' \Gamma t}{2} \cdot \cancel{1^2}}$$ From Normal c.f.

CLT $\{\vec{X}_{n,k}\}_{1 \le k \le n}$ $\in \mathbb{R}^d$ IND. $E(X_{n,k}) = \vec{0}$, $Cov(X_{n,k}) = \Gamma_{n,k}$

like 881 But with vectors,

$S_n = \sum_{k=1}^{n} X_{n,k}$, $\Gamma(S_n) = \sum_{k=1}^{n} \Gamma_{n,k}$ if (1) $\sum_{k=1}^{n} \Gamma_{n,k} \longrightarrow \Gamma$

convergence of matrix Def As All Element converge

(2) $L_n(\varepsilon) = \sum_{k=1}^{n} E(\|X_{n,k}\|^2 ; \|X_{n,k}\| > \varepsilon) \xrightarrow[n \to \infty]{} 0$ $\forall \varepsilon > 0$

"lindeburg condition"

$E\|X_{n,k}\|^2 < \infty$ $1 \le k \le n$,

then $S_n \Rightarrow N(\vec{0}, \Gamma)$

uses cauchy-swartz inequality.

if $G \sim N(\mu, \Gamma)$ then $\varphi_X(t) = e^{it \cdot \vec{\mu}} \cdot e^{-t' \cdot \Gamma t/2}$ $\forall t \in \mathbb{R}^d$

if integrate get density.

$$f_G(\vec{x}) = (2\pi)^{-d/2} \exp\left\{-\frac{1}{2} \cdot (\vec{x} - \vec{\mu})^t \cdot \Gamma^{-1} (\vec{x} - \vec{\mu})\right\} \quad \forall x \in \mathbb{R}^d$$

$\det(\Gamma)^{-1/2}$

Multinomial with $d+1$ outcomes

vector $X_n \sim MN(n, p_{n,1}, \ldots p_{n,d}, q_n)$

Assume $n \cdot p_{n,k} \xrightarrow[n \to \infty]{} \lambda_k$ $k = 1, \ldots, d$.

vector

Assume $n \cdot p_{n,k} \xrightarrow[n\to\infty]{} \lambda_k$ $k = 1, \dots, d.$

this implies $p_{n,k} \to 0$ & $q_n \to 1$

want x coord with out q_n.

$$\begin{pmatrix} X_{n,1} \\ X_{n,2} \\ \vdots \\ X_{n,d} \\ X_{n,d+1} \end{pmatrix} \qquad Y_n = \begin{pmatrix} X_{n,1} \\ \vdots \\ X_{n,d} \end{pmatrix}$$

then $Y \Rightarrow (Y^{(1)}, \dots, Y^{(d)})$

$k = 1, \dots, d,$

$\{Y^{(k)}\}_{1 \le k \le d}$ IND

$Y^{(k)} \sim \text{Poisson}(\lambda k)$

use this theorem All the time,

$$\prod_{k=1}^{n} (1 + a_{n,k}) \to e^{a}$$

$$\sum_{k=1}^{n} a_{n,k} \xrightarrow[n\to\infty]{} a.$$

"the C.F. of the limit is the Product." "k" "just come out"

"becomes Asymptotically independent.