03-19 OST

Wednesday, March 19, 2025 11:30 AM

$\{X_n, \mathcal{F}_n\}_{0 \le n \le N}$ sub MG. (Durrett § 5.7 P. 229)

T stopping time (S.T.) $0 \le T \le N$

we saw $E(X_0) \le E(X_T) \le E(X_N)$

Derived By Gambling System.

$D_k = X_k - X_{k-1}$ $\quad (H \cdot X)_n = \sum_{k=0}^{n} H_k \cdot D_k$ $\quad$ Predictable $H_n \in \mathcal{F}_{n-1}$

$H_k \ge 0 \Rightarrow ((H \cdot X)_n, \mathcal{F}_n)$ subMG.

Easy way take $H = 1$

$(H \cdot X)_0 = 0$ $\quad H_k = 1_{\{k \le T\}}$.

$(H \cdot X)_N = X_T - X_0$

$E(H \cdot X)_N \ge 0$

(2) Take S s.t. $S \le T$ a.s. $\quad (0 \le S \le T \le N)$

then $E(X_S) \le E(X_T)$

$Y_n = X_{T \wedge n}$ $\quad 0 \le n \le N$ $\quad \{Y_n, \mathcal{F}_n\}_{0 \le n \le N}$ subMG.

$E(Y_0) \le E(Y_S) \le E(Y_N)$

$E(X_0) \le E(X_S) \le E(X_T) \le E(X_N)$

(3) $S \le T$ both S.T. $\quad \{X_n, \mathcal{F}_n\}_{0 \le n \le N}$ sub MG.

$E_{\mathcal{F}_S}(X_T) \ge X_S$

$E(X_T; A) \ge E(X_S; A) \quad \forall A \in \mathcal{F}_S$.

Define $\tilde{S} = \begin{cases} S & \text{on } A \\ T & \text{on } A^c \end{cases}$ $\qquad \tilde{S} \le T$

$E(X_{\tilde{S}}) \le E(X_T)$

Dcp. $E(X_{\tilde{S}}; A^c) = E(X_T; A^c)$

we get

$E(X_S; A^c) \le E(X_T; A) = E(E_{\mathcal{F}_S}(X_T); A)$, $\quad A \in \mathcal{F}_S$

$\therefore \Rightarrow X_S \le E_{\mathcal{F}_S}(X_T)$ a.s.

Now $N = \infty$.

Two methods. to show $X_S \le / = / \ge E_{\mathcal{F}_S}(X_T)$ a.s.

1) UI method. works for MG, sub, sup.

2) Using Fatou Lemma works for positive sup MG.

theorem Let $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ U.I. sub MG.

then if T is S.T ($\{T=\infty\}$ may be non-trivial)

then $\{X_{T\wedge n}, \mathcal{F}_n\}_{n\geq 0}$ is UI as well

note: MGCT works

Proof: since $\{X_n\}_{n\geq 1}$ is UI then we get $X_n \xrightarrow[n\to\infty]{a.s. L^1} X_\infty$

convex, nondecreasing. then

$\{X_n^+, \mathcal{F}_n\}_{n\geq 0}$ subMG. so, $E(X_{T\wedge n}^+) \leq E(X_n^+)$ All positive.

$$\Rightarrow \sup_{n\geq 0}\{E(X^+_{T\wedge n})\} \leq \sup_{n\geq 0}\{E(X_n^+)\} < \infty$$

condition for convergence of MG. Eg. The theorem.

we get by MGCT

① $X_{T\wedge n} \xrightarrow[n\to\infty]{a.s.} X_T$

② $E(|X_T|) < \infty$

$|X_{T\wedge n}| \leq |X_T| + |X_n|$ a.s.

we have seen

$\{X_\alpha\}_{\alpha\in A}$ $\{Y_\beta\}_{\beta\in B}$ are both UI

then $\{X_\alpha + X_\beta\}_{\alpha\in A, \beta\in B}$ is UI as well.

① **theorem 2** Let $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ UI sup. Sub MG Let $0\leq T\leq\infty$ be S.T.

then $E(X_0) \leq E(X_T) \leq E(X_\infty)$

Proof $\forall\ 0\leq n\leq\infty$

$$E(X_0) \leq E(X_{T\wedge n}) \leq E(X_n)$$

$n\to\infty$ $\downarrow$ $\downarrow$ $\downarrow$ if not UI, then may not be true.

$$E(X_0) \leq E(X_T) \leq E(X_\infty)$$

Corollary.

② if $S\leq T$ S. S.T. then $E(X_S) \leq E(X_T)$

③ $X_S \leq E_{\mathcal{F}_S}(X_T)$. a.s.

Simple Random walk not Symmetric.

Ex simple Random walk which is not symmetric

$\{\xi_k\}_{k\geq 1}$ iid ξ $\begin{cases} +1 & p \\ -1 & q\end{cases}$ where $p > q$ $p+q=1$ $1\geq p > 1/2$, $0\leq q < 1/2$

q^{-1} $\qquad$ $0 \le q < \frac{1}{2}$

$S_n = \sum_{k=1}^{n} \xi_k$, $n \le 0$ $\qquad$ $S_0 \equiv 0$

Let $x \in \mathbb{Z}$ $\quad T_x \equiv \inf\{ n ; S_n = x \}$.

if $b > 0$, then $T_b < \infty$ a.s.

$$\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s.} E(\xi) = p - q > 0.$$

$$\Rightarrow S_n \xrightarrow[n\to\infty]{a.s.} \infty$$

$\frac{q}{p} < 1 \Rightarrow \left(\frac{q}{p}\right)^x < 1$

Claim: $\quad a < 0 < b.$

then $P(T_a < T_b) = \dfrac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)}$ $\quad \varphi(x) = \left(\frac{q}{p}\right)^x, \; x \in \mathbb{Z}$

① $\{ \{ \varphi(S_n) \}, \mathcal{F}_n \}_{n \ge 0}$ is a MG?

$$E_{\mathcal{F}_n}(\varphi(S_{n+1})) = E_{\mathcal{F}_n}\left[\left(\frac{q}{p}\right)^x\right.$$

$$= E_{\mathcal{F}_n}\left[\left(\frac{q}{p}\right)^{S_n} \cdot E\left(\frac{q}{p}\right)^{\xi_{n+1}}\right.$$

$$= \left(\frac{q}{p}\right)^{S_n} \cdot E_{\cancel{\mathcal{F}_n}} \left(\frac{q}{p}\right)^{\xi_{n+1}}$$ which $\xi_{n+1} \perp\!\!\!\perp \mathcal{F}_n$

$$= \left(\frac{q}{\cancel{p}}\right)\cancel{p} + \left(\frac{q}{p}\right)^{-1} \cdot q$$

$$= q + p = 1$$ ✓

Define $T = T_a \wedge T_b < \infty$ a.s.

$\{ \varphi(S_{T\wedge n}), \mathcal{F}_n \}_{n \ge 0}$ is MG.

$0 \le \varphi(S_{T\wedge n}) \le \left(\frac{q}{p}\right)^a = \left(\frac{p}{q}\right)^{-a}$

(ii) $\varphi(S_{T\wedge n}) \le \left(\frac{p}{q}\right)^{-a}$, $n \ge 0$. $\qquad$ shows Bounded. ∴ UI

$\Rightarrow \{ \varphi(S_{T\wedge n}) \}_{n \ge 0}$ UI

we get by our results.

$$E(\varphi(S_T)) = E(\varphi(S_0)) = 1$$

$S_T \in \{a, b\}$

$$E\varphi(S_T) = P(T_a < T_b) \cdot \left(\frac{q}{p}\right)^a + P(T_b < T_a)\left(\frac{q}{p}\right)^b.$$

$$P(T_a < T_b) + P(T_b < T_a) = 1$$

Two equations two unknowns solve and get

$$P(T_a < T_b) = \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)} \xrightarrow[b \to \infty]{} \frac{0 - 1}{0 - \left(\frac{q}{p}\right)^a} = \left(\frac{p}{q}\right)^a.$$

$E(T_b)$