02-26

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L^2 MG.

$\{X_n, \mathcal{F}_n\}_{n\geq 0}$ MG, $X_0 = 0$ $E(X_n^2) < \infty$ $\forall n \geq 1$

$\hookrightarrow$ sub MG. is convex — convex MG is sub MG.

Doob Decomp.

$X_n^2 = M_n + A_n$, $A_0 \equiv 0$ $A_n \uparrow$ $A_n \in \mathcal{F}_{n-1}$ therefore increasing

$M_0 \equiv 0$, $\{M_n, \mathcal{F}_n\}_{n\geq 0}$ MG. $E[X_n^2] \uparrow$ as $n \uparrow$

Let $D_k^2 = (X_k - X_{k-1})^2 \Rightarrow E_{\mathcal{F}_{k-1}}(D_k^2) = E_{\mathcal{F}_{k-1}}[X_k^2] - (X_{k-1})^2$ ... $E(X_n)^2$

NAME the increasing process associated with it $\rightarrow$ $A_n = \sum_{k=1}^{n} E_{\mathcal{F}_{k-1}}[D_k^2] = \sum_{k=1}^{n} [E_{\mathcal{F}_{k-1}} X_k^2 - X_{k-1}^2]$.

Predictable MG.

there is only one Representation
$\therefore$ unique,

$A_\infty = \lim_{n\to\infty} A_n$ a.s. $E(A_n) = E(X_n^2)$ (as $E(M_n) = E(M_0) = 0$)

$E(A_\infty^2)$ $\lim_{n\to\infty} E(A_n) = \sup_{n\to\infty}\{E(X_n^2)\}$

$= E(A_\infty)$ By MCT.

Doob L^p, $p>1$ Inequality for $p=2$. eg. L^2

$$E(X_n^2) \leq E(\sup_{1\leq k\leq n}\{X_k^2\}) \leq 4\, E(X_n^2) = 4E(A_n$$

called Doob's L^2 Inequality.

used in Proof

$$E(A_\infty) \leq \sup_{n\geq 1} E(X_n^2) \leq E(\sup_{n\geq 1}\{X_k^2\}) \leq 4E(X_\infty)$$

Analogous Kolgomorov 3 series.

Theorem

no Assumption of independence.

$X_n = \sum_{n=1}^{k} D_k$ converge to finite R.V. on $\{A_\infty < \infty\}$ so increasing $A_n \xrightarrow[n\to\infty]{} A_\infty < \infty$

when Dealing with Mg use Stopping times.

Proof.

Let $a > 0$ $T_a = \inf\{n : A_{n+1} > a\}$ (can be ∞; Future? still a s.t. Because A_n is Predictable $A_n \in \mathcal{F}_{n-1}$)

leads to $A_{T_a} \leq a$ if $T_a < \infty$

$$E(\sup_n X_{m\wedge T_a}^2) \leq 4E(A_{n\wedge T_a}) \leq 4a.$$

MG.

$\Rightarrow X_{n\wedge T_a}$ Converge A.S. As $n\to\infty$

$\Rightarrow X_n$ converge A.S. On $\{T_a = \infty\}$

Because we never cross a $\therefore$ Bounded.

observe $\{A_\infty < \infty\} = \bigcup_{m=1}^{\infty} \underbrace{\{A_\infty < m\}}_{\{T_m = \infty\}}$, countable # of Events

$\bigstar$ Kronecker Lemma.

SLLN For MD MARTinGale Difference,

Let $f(x) \geq 1$, $f(x)\uparrow$ and $\int_1^\infty \frac{dx}{f^2(x)} < \infty$ square it and ... like

Let $f(x) \geq 1$, $f(x)\uparrow$ and $\int_1^\infty \frac{dx}{f^2(x)} < \infty$

square it and.. something like $u = \log x$.

ex: $f(x) = x$, $f(x) = \sqrt{x}\,[\log(x)]^{1/2+\epsilon}$

then $\frac{X_n}{f(A_n)} \xrightarrow[n\to\infty]{} 0$ on $\{A_\infty = \infty\}$

Proof. $\{Y_n = \sum_{k=1}^n \frac{D_k}{f(A_k)}, \mathcal{F}_n\}_{n\geq 1}$ is MG. Y_n is measurable w.r.t. $\mathcal{F}_n$

$X_n - X_{n-1}$

if $E_{\mathcal{F}_{n-1}}(Y_n - Y_{n-1}) = E_{\mathcal{F}_{n-1}}\left(\frac{D_n}{f(A_n)}\right) = \frac{1}{f(A_n)} \cancel{E_{\mathcal{F}_{n-1}} D_n} = 0$ $\therefore$ MG.

what is $\{A^Y_n\}$?

two MG competeing. A^Y_n is the increasing process of Y_n not X_n

$$E_{\mathcal{F}_{n-1}}[(Y_n - Y_{n-1})^2] = E_{\mathcal{F}_{n-1}} \frac{D_n^2}{[f(A_n)]^2} = \frac{1}{(f(A_n))^2} E_{\mathcal{F}_{n-1}}(D_n^2) = \frac{A_n - A_{n-1}}{f(A_n^2)}$$

$$A^Y_\infty = \sum_{n=1}^\infty \frac{A_n - A_{n-1}}{[f(A_n)]^2} < \infty \text{ A.S.}$$

Reiman integral..

$$\leq \sum_{n=1}^\infty \int_{A_{n-1}}^{A_n} \frac{dt}{f(t)^2} < \infty \quad A.S.$$

$\Rightarrow Y_n$ converge A.S. as $n \uparrow \infty$

on $\{A_\infty = \infty\}$ use Kronecker Lemma.

from 881 $\sum_{k=1}^\infty \frac{a_k}{b_k} < \infty$ and $b_k \uparrow \infty$

then $\frac{\sum a_k}{b_n} \xrightarrow[n\to\infty]{} 0$

Example:

Application let $X_n = \sum_{k=1}^n \xi_k$ $E(\xi_k) = 0$ $E(\xi_k^2) < \infty$,

$\{\xi_k\}_{k\geq 1}$ IND. $\sum_{k=1}^\infty E(\xi^2) = \infty$

$A_\infty = \infty$

then $\frac{\sum_{k=1}^n \xi_k}{\sum_{k=1}^n E(\xi_k^2)} \xrightarrow[n\to\infty]{AS} 0$

Borel Cantelli II $\{B_n\}_{n\geq 1}$ $\mathcal{F}_n = \sigma\{B_m, m \leq n\}$

$P_n = P_{\mathcal{F}_{n-1}}(B_n)$ then $\left\{\sum_{n=1}^\infty P_n = \infty\right\} = \{B_n \text{ i.o.}\}$

Stronger BC II

$$\frac{\sum_{m=1}^{n} \mathbb{1}_{B_m}}{\sum_{m=1}^{n} P_m} \to 1 \quad \text{on } \{B_n \text{ i.o.}\}$$

Both goes to inf. But At Equal Rates.