01-24

Friday, January 24, 2025 11:32 AM

$(\Omega, \mathcal{F}, P)$

Filtration $\{\mathcal{F}_n\}_{n\geq 1}$ $\mathcal{F}_n \subset \mathcal{F}_{n+1}$

Stopping time $T: \Omega \to \mathbb{Z}^+$ $\{T = n\} \in \mathcal{F}_n$, $n \geq 1$

$\mathcal{F}_T = \{A \in \mathcal{F};\ A \cap \{T = n\} \in \mathcal{F}_n,\ n \geq 1\}$.

Natural Filtration : $\mathcal{F}_n = \sigma\{X_1, \dots, X_n\}$, $n \geq 1$

in the $\mathcal{F}_T = \sigma\{X_{n \wedge T}\}$

if $T_1 < T_2$ a.s. and T_1, T_2 are s.t.

The $\mathcal{F}_{T_1} \subset \mathcal{F}_{T_2}$ create filtration from s.t.

Corallary.

Let $T_1 < T_2 < \dots < T_n < \dots$ be increasing seq of s.t.

then $\{\mathcal{F}_{T_n}\}_{n\geq 1}$ is a filtration.

Pur. 4.1.3

theorem Let $\{X_n\}_{n\geq 1}$ be iid let T be a s.t.

w.r.t. the natural filltration. $P(T < \infty) = 1$

← from the stopping time onward.

then ① $\{X_{T+n}\}_{n\geq 1} \overset{\mathcal{D}}{=} \{X_n\}_{n\geq 1}$

② $\exists\ T$, $\{X_{N+n}\}_{n\geq 1}$ are IND

Proof.

intersection

T ← stopping time not N.

Let $A \in \mathcal{F}_T$ wts. $P(A, X_{\cancel{N}+n} \in B_n,\ n = 1, \dots, k) = P(A) \cdot \prod_{n=1}^{k} P(X_n \in B_n)$

if $A = \Omega$ then $P(A) = 1$

$= P(A, T=\ell, X_{T+n} \in B_n, n=1,\dots,k)$

$= P(\underbrace{A, T=\ell}_{\in \mathcal{F}_\ell}, \underbrace{X_{\ell+n} \in B_n}_{\in \sigma\{X_m\}_{m \geq \ell+1}}, n=1,\dots,k)$

therefore independent and Identically dist.

$= \prod_{n=1}^{k} P(X_n \in B_n)$

$= \sum_{\ell=1}^{\infty} P(A \cap \{T=\ell\}) =$

Type equation here.

EX symple symmetric Random walk. (S.S.R.W.)

$S_0=0$, $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$, $\{X_k\}_{k \geq 1}$ iid $P(X_k = \pm 1) = 1/2$.

Start At Zero.

now Define Stopping times

$T_1 = \min\{n>0 ; S_n = 0\}$. use CLT & 0-1 scenario

$T_2 = \min\{n > T_1 ; S_n = 0\}$.

$\{T_k\}_{k \geq 1}$.

$\{T_1, T_2 - T_1\}$ iid. go to zero we start over.

$T_0=0$ $\{T_k - T_{k-1}\}_{k \geq 1}$ iid.

Celebrate Wald Equations

Wald 1st Equations. (Generalized in the next chapter 5)

Let $\{X_k\}_{k \geq 1}$ be iid, $S_n = \sum_{k=1}^{n} X_k$

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Let T be a s.t. w.r.t. natural filltration.

Assume $E(T) < \infty$ and $E(|X_1|) < \infty$

then $E(S_T) = E(X_1) \cdot E(T)$

$S_T = \sum_{k=1}^{T} X_k$, $S_T(\omega) \overset{??}{=} S_{T(\omega)}^{(\omega)}$

infinite But After time we have All zeros

$S_T = \sum_{n=0}^{\infty} \left(S_{T\wedge(n+1)} - S_{T\wedge n} \right)$

$S_{T\wedge(n+1)} - S_{T\wedge n} = X_{n+1} \cdot \mathbb{1}_{\{T \geq n+1\}}$. see Recordings for

Observe: $\{T \geq n+1\} \in \mathcal{F}_n$ $\in \sigma\{X_1, \dots X_n\}$

The complement $\{T < n\} \in \mathcal{F}_n$, Since σ-Alg $\therefore$ $\{T \geq n+1\} \in \mathcal{F}_n$. $\mathcal{F}_n \therefore \perp\!\!\!\perp$ ind.

$E|S_{T\wedge(n+1)} - S_{T\wedge n}| = E|X_1| \cdot P(T \geq n+1)$

$E(S_{T\wedge(n+1)} - S_{T\wedge n}) = E(X_1) \cdot P(T \geq n+1)$

since positive.

$\sum_{n=1}^{\infty} E|S_{T\wedge(n+1)} - S_{T\wedge n}| = E|X_1| \sum_{n=1}^{\infty} P(T \geq n+1) = E|X| \, E(T) < \infty$

using DCT $E(S_T) = E(X_1) E(T)$

slight Generalization of WalD 1st Eq.

can't have smaller filtration than natural

Adds $\{\mathcal{F}_n\}_{n\geq 1}$ filltration

$X_n \in \mathcal{F}_n$, $n \geq 1$

and X_{n+1}, $\mathcal{F}_n$ IND. $n \geq 1$

if T, $\{X_k\}_{k\geq 1}$ IND $\mathcal{F}_n = \sigma\{T, X_1, \dots, X_n\}$ $n \geq 1$

T is a S.T. w.r.t. to $\{\mathcal{F}_n\}_{n\geq 1}$ Say " T is known from $t=1$

· UPComing
WalD 2rD Eq is same type of Proof.
3RD 3
0-1 Law.