

Last time: Monday

$$\textcircled{1} \quad (\Omega, \mathcal{F}_n, P) \quad E|X| < \infty$$

$\{E_{\mathcal{F}_n}(x); n \geq 1\}$ is UI

$$\textcircled{2} \quad \{\bar{X}_n\}_{n \geq 1} \text{ UI and MG. Then}$$

$$X_n \xrightarrow[n \rightarrow \infty]{L^1 \text{ a.s.}} X \quad X_n \xrightarrow[L^1]{\text{a.s.}} X \Leftrightarrow E|X_n - X| \xrightarrow{n \rightarrow \infty} 0$$

$$\textcircled{3} \quad X_n = E_{\mathcal{F}_n}(x) \quad n \geq 1$$

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$$\underline{\text{Theorem}} \quad (\Omega, \mathcal{F}_n, P)$$

is Algebra Not  $\sigma$ -Alg.

Assume  $E|X| < \infty$   $\mathcal{F}_n \uparrow \mathcal{F} \subset \mathcal{F}$   $\bar{X}_n \in \mathcal{F}_n, n \geq 1$  and  $\sigma\{\bar{X}_n\}_{n \geq 1} \subset \mathcal{F}$  is

then  $E_{\mathcal{F}_n}(x) \xrightarrow{\text{a.s. } L^1} E_{\mathcal{F}}(x)$  here converges of  
filtration which is  
given.

Proof. 1st observation  $\{E_{\mathcal{F}_n}(x), \mathcal{F}_n\}_{n \geq 1}$  MG. and it is UI

By Result 2, we get.

$$E_{\mathcal{F}_n}(x) \xrightarrow{\text{a.s. } L^1} X_\infty \in \mathcal{F}$$

what is sup  
integrable  
measurable wrt  $\mathcal{F}$

What is the relationship between  $X_n \rightarrow X$

$$\text{Claim: } X_\infty = E_{\mathcal{F}}(x) \Leftrightarrow E(X_\infty; A) = E(X; A), \forall A \in \mathcal{F}$$

$$\text{we know, } \bar{X}_n = E_{\mathcal{F}_n}(\bar{X}_\infty) \Leftrightarrow E(\bar{X}_n; A) = E(\bar{X}_\infty; A), \forall A \in \mathcal{F}_n$$

$$\stackrel{n+1 > n, \dots}{=} E(\bar{X}_{n+1}; A) = E(\bar{X}_n; A) = E(\bar{X}_\infty; A)$$

$$X_n \xrightarrow{\text{a.s. } L^1} X_\infty$$

$E|X_n - X_\infty| \xrightarrow{n \rightarrow \infty} 0$

$$E(X_n; A) \xrightarrow{n \rightarrow \infty} E(X_\infty; A)$$

A  $\in \mathcal{F}_N$  N - fixed. (Arbitrary)

the L<sup>1</sup> Distance  
 $E(|X_n - X_\infty|; A)$  that is bigger than

$$\Rightarrow E(X_n; A) = E(X_\infty; A), A \in \mathcal{F}_n, A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n \quad \text{By Dinkin's}$$

we need for all  $\mathcal{F}$ . "Dinkin's"  $x$ - $\lambda$  system.

Application (a)  $\{\mathcal{F}_n\}_{n \geq 1}$  filtration,  $\mathcal{F} = \sigma\{\bar{X}_n, n \geq 1\}$

Let  $A \in \mathcal{F}$  then  $P_{\mathcal{F}_n}(A) \xrightarrow{n \rightarrow \infty} 1_A$

$\mathcal{F}_n$  is more & more refined information about omega

(b) New Proof of Kolmogorov 0-1 Law. then Either 0 or 1  
Eventually

Kolmogorov 0-1 if seq  $x_1, \dots, x_n$  Thus  $n \rightarrow \infty$

Eventually

Kolmogorov 0-1 if seq  $x_1, \dots, x_n$  Tails  $\xrightarrow{n \rightarrow \infty}$   
 Then tail event is either 0 or 1.

Let  $\{x_n\}_{n=1}^{\infty}$  i.i.d.,  $\mathcal{F}_n = \sigma\{x_1, \dots, x_n\}$ ,  $T = \{\text{tail events}\}$

$T = \bigcap_{n=1}^{\infty} \sigma\{x_n, x_{n+1}, \dots\}$ .       $\text{eventually set of } \sigma\text{-A.s.}$   
 which is  $\sigma\text{-A.s.}$ .

Let  $A \in T$   
 $\text{IND of } \mathcal{F}_n \checkmark$   
 $A \in \sigma\{x_{n+1}, \dots\}$

$$\mathcal{F} = \sigma\{x_1, x_2, \dots\}$$

says this is #. (I guess constant)

Probability.  $P_{\mathcal{F}_n}(A) = P(A) \xrightarrow{n \rightarrow \infty} 1_A$

$$\Rightarrow P(A) \in \{0, 1\}.$$

Dominated Convergence theorem for Condition Expectation

before general result.

Lemma  $\forall C > 0$   $\exists n > 0$ ,  $\text{Derivative}$   $\left( \text{absolute Dominated convergence} \right)$   
 Let  $X_n \downarrow 0$  a.s.,  $E(X_n) < \infty \left( \xrightarrow{\text{LCL}} E(X_n) \xrightarrow{n \rightarrow \infty} 0 \right)$

then  $E_{\mathcal{F}}(X_n) \xrightarrow{n \rightarrow \infty} 0$

Proof  $0 < E_{\mathcal{F}}(X_n) \downarrow X \geq 0 \xrightarrow{\text{a.s.}} E(E_{\mathcal{F}}(X_n)) + E(X)$   
 $= E(X_n) \xrightarrow{n \rightarrow \infty} 0$

$E_{\mathcal{F}}(X_n) \geq E_{\mathcal{F}}(X) \geq 0$

$\therefore E_{\mathcal{F}} = E[E_{\mathcal{F}}(X_n)] \geq E[E_{\mathcal{F}}(X_n)]$ .      since  $X \geq 0$  we get  
 a.s.  $X = 0$

Extension

$$X_n \xrightarrow{n \rightarrow \infty} 0, |X_n| \leq Y, n \geq 1, E(Y) < \infty$$

then  $E_{\mathcal{F}}(X_n) \xrightarrow{n \rightarrow \infty} 0$

Proof  $|X_n| \xrightarrow{n \rightarrow \infty} 0$  then  $\overline{\lim}_{n \rightarrow \infty} \{|X_n|\} = 0$  a.s.  
 monotone sequences decreasing

$\liminf_{n \rightarrow \infty} \{|X_n|, |Y_n|\} \downarrow 0$

$Y \geq Y_n = \sup \{|X_n|, |Y_n|\} \downarrow 0$  a.s.

$\Rightarrow E_{\mathcal{F}}(Y_n) \xrightarrow{n \rightarrow \infty} 0$

$0 < E_{\mathcal{F}}(|X_n|) \xrightarrow{n \rightarrow \infty} 0$

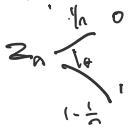
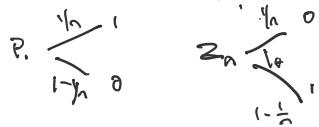
corner let  $X_n \xrightarrow{n \rightarrow \infty} 0$ ,  $|Y| = f_n$ ,  $n \geq 1$   $E(Y) < \infty$

then  $E_{\mathcal{F}}(f_n) \xrightarrow{n \rightarrow \infty} 0$

why not uniformly? is not uniform

Example  $\{Y_n + Z_n\}_{n=1}^{\infty}$  all i.i.d.

Example  $\{Y_n + Z_n\}_{n \geq 1}$  all ind



$$X_n = Y_n + Z_n, \quad E(X_n) = E(Y_n) + E(Z_n) = \frac{1}{p_n} - \frac{1}{1-p_n} \xrightarrow{n \rightarrow \infty} \infty$$

$\mathcal{F} = \sigma \{Y_n, Z_n\}$

$$E_{\mathcal{F}}(X_n) = E_{\mathcal{F}}[Y_n + Z_n]$$

$$= y_n E_{\mathcal{F}}(Z_n) \quad \text{since ind}$$

$$= y_n$$

$$\sum p_n y_n > 1 \Rightarrow \sum \frac{1}{p_n} = \infty \quad \text{by BCT}$$

when you go to investigate, we have,