L20 - 10-10b Borel-Cantelli

Friday, October 11, 2024 11:32 AM

Borel - Cantelli Lemma. $(\Omega, \mathcal{F}, P)$

① B.C.(I): Let $A_n \in \mathcal{F}$ $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} P(A_n) < \infty$

Then $P(A_n \text{ i.o.}) = 0$ (i.o. = infinitely often.)

count $\omega \in A_n$ if the count is infinite

then i.o.

$\{A_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$ Eg. ω appear in many A_n

$= \{\omega \in \Omega : \limsup_{n\to\infty} \{\mathbb{1}_{A_n}(\omega)\} = 1\}$ $\mathbb{1}_{A_n}(\omega) \begin{cases} 1 & \text{if } \omega \in A_n \\ 0 & \text{if } \omega \notin A_n \end{cases}$

Proof $N = \sum_{k=1}^{\infty} \mathbb{1}_{A_k}(\omega)$ $P(N < \infty) = 1$ or $P(N = \infty) = 0$

$$E(N) \overset{MCT}{=} E \sum_{k=1}^{\infty} \mathbb{1}_{A_k} = \sum_{k=1}^{\infty} P(A_k) < \infty$$

$S_n = \sum_{k=1}^{n} Y_k, \; 0 \le Y_k$

$\Rightarrow S_n \uparrow \sum_{k=1}^{\infty} Y_k$

By MCT

$E(S_n) \uparrow E\left(\sum_{k=1}^{\infty} Y_k\right)$

$= \sum_{k=1}^{\infty} E(Y_k)$.

$E\left(\sum_{k=1}^{\infty} Y_k\right) = \sum_{k=1}^{\infty} E(Y_k)$

if $P(N = \infty) > 0$ then $E(N) = \infty$

Conclude $P(N = \infty) = 0$

$P(N < 1) = 1$

<u>Why are we interested?</u>

Move from converge in Probablity to Converge Almost surely

Says $\xrightarrow{P}$ is for 1 dim $\xrightarrow{a.s.}$ is for many dim.

$X_n - X \xrightarrow[n\to\infty]{P} 0$

Corollary if $X_k \xrightarrow{P} X$ then $\exists \{n_k\}_{k \ge 1}$ $n_1 < n_2 < \ldots$

So that $X_{n_k} \xrightarrow[k\to\infty]{a.s.} X$.

Proof WLOG $X = 0$,

By B.C.(I) we can select $\{n_k\}_{k \ge 1}$ so that

$P(|X_{n_k}| > \frac{1}{k^2}) \le \frac{1}{k^2}$ $k = 1, 2, \ldots$

we know $\forall k$: $P(|X_n| > \frac{1}{k^2}) \xrightarrow[n\to\infty]{} 0$

$\exists\, n > n_{k-1}$ (increasing) so that $P(|X_n| > \frac{1}{k^2}) \le \frac{1}{k^2}$ *

$n_k = \min_{n > n_{k-1}} \{ n : * \text{ holds} \}$,

Since $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ we get $\underbrace{\sum P(|X_{n_k}| > \frac{1}{k^2}) < \infty}$

This event occurs Finitly often.

By B.C. (I) we get that a.s. $\exists L < \infty$ [L is Random!]

such that $|X_{n_k}| < \frac{1}{k^2}$, $k \geq L = L(\omega)$

if $\frac{1}{k^2}$, then $n \to \infty$ $\frac{1}{k^2} \to 0$. $\quad |a_k| \to 0 \Rightarrow a_k \to 0$

$\Rightarrow X_{n_k} \xrightarrow[k\to\infty]{a.s.} 0$

Application : DCT $X_n \xrightarrow[n\to\infty]{\cancel{a.s.}\ P} X$ and $|X_n| \leq Y$, $E(Y) < \infty$
then $E(X_n) \xrightarrow[n\to\infty]{} E(X)$

What if $X_n \xrightarrow{P} X$ and $E(Y) < \infty$, $\underset{n \geq 1}{|X_n|} \leq Y$

Does $E(X_n) \xrightarrow[n\to\infty]{} 0$? **YES** and $n_k \uparrow \infty$

Assume No. namely $E(X_n) \not\xrightarrow[n\to\infty]{} 0$ $\exists \varepsilon_0 > 0$ so that $\underbrace{|E(X_{n_k})| \geq \varepsilon_0}$ $k = 1, 2, \ldots$

As ε_0 goes to Zero.
There is Bigger.

However $X_{n_k} \xrightarrow[k\to\infty]{P} 0$

So: $\exists \{n_{k_\ell}\}_{\ell = 1, 2, \ldots}$ so that $X_{n_{k_\ell}} \xrightarrow[\ell\to\infty]{as} 0$
sub, sub seq

$|X_{n_{k_\ell}}| \leq Y$ $\quad E(Y) < \infty$

by DCT $E(X_{n_{k_\ell}}) \xrightarrow[\ell\to\infty]{} 0$

Recall $X_n \xrightarrow[n\to\infty]{a.s.} X$ then $X_n \xrightarrow[n\to\infty]{P} X$

upto now WLLN $xv\{X_n\}$ iid; $E|X| < \infty$ then $\frac{S_n}{n} \xrightarrow[n\to\infty]{P} E(X)$

$xP(|X| > x) \xrightarrow[n\to\infty]{} 0$

SLLN: $\{X, X_n\}_{n \geq 1}$ iid $E|X| < \infty$ then $\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s.} E(X)$

Assume Now $E(X^4) < \infty$

$$\left(\sum_{k=1}^{n} a_k\right)^4 = \sum_{\substack{\sum_{k=1}^{n} i_k = 4 \\ n \geq i_k \geq}} \frac{4!}{\prod_{k=1}^{n} i_k!} \prod_{k=1}^{n} a_k^{i_k}$$

$$= \sum_{k=1}^{n} a_k^4 + 6\sum a_k^2 a_\ell^2 \quad 4\sum a_k a_\ell^3 \quad +$$

$$= \sum_{k=1}^{n} a_k^4 + 6 \sum a_k^2 a_\ell^2 \quad 4 \sum_{k\ell} a_k a_\ell^3 \quad +$$

$\frac{4!}{2!2!}$