L38 - 11-25 LF Truncation

Monday, November 25, 2024 11:33 AM

Recall: $|e^{ix}-1| \le |x| \wedge 2.$

$|x| = \int_0^x 1\,du \ge \left|\int_{u=0}^x e^{iu}\,du\right| = \left|\frac{e^{ix}-1}{i}\right|$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $|a| = |re^{i\theta}| = r$

$b = r_2 e^{i\theta}$

$\left|\int_{u=0}^x (e^{iu}-1)\,du\right| \le \int_{u=0}^x |e^{iu}-1|\,du \le \int_{u=0}^x |u|\,du \wedge \int_{u=0}^x 2\,du$

$\left|\left[\frac{e^{iu}-1}{i} - u\right]_0^x\right|$ $\qquad |e^{ix}-(1+ix)| \le \frac{x^2}{2} \wedge 2|x|$

$= |e^{ix}-1-ix|$

we need:

$$|e^{ix}-(1+ix-\frac{x^2}{2})| \le \frac{|x|^3}{3!} \wedge x^2$$

Last time:

if $\varphi_{x_n}(t) \xrightarrow[n\to\infty]{} g(t)$, $\forall t \in \mathbb{R}$, and g is Cont at $t=0$ then $\{x_n\}_{n\ge1}$ is tight Furthermore $g(t) = \varphi_Y(t)$, $t\in\mathbb{R}$ Y r.v. and $x_n \Rightarrow Y$

now look at characteristic functions

Replace x with xt and take Expectation

$|E(\quad)| \le E\,|e^{itx}-(1+itx+\frac{x^2}{2}t^2)| \le E\left[\frac{|t|^3|x|^3}{3!} \wedge t^2x^2\right]$

$$|\varphi_X(t) - (1 + itE(x) + \frac{t^2E(x^2)}{2})| \le E\left[\frac{|t|^3|x|^3}{3!} \wedge x^2t^2\right] \quad (*)$$

if $E(X)=0$. then $itE(x)=0$

Linberg-Feller. LF

$\{X_{n,m}\}_{1\le m\le n}$, $E(X_{m,n})=0$, $\{X_{n,m}\}$ rowwise ind. $\forall n\ge1$

if $\sum_{m=1}^n E(X_{m,n}^2) = 1$, $n\ge1$ and $\underbrace{L_n(\varepsilon) = \sum_{m=1}^n E(X_{n,m}^2; |X_{n,m}|>\varepsilon) \xrightarrow[n\to\infty]{} 0 \quad \forall\varepsilon>0}_{\text{Lindberg Condition.}}$

then $S_n = \sum_{m=1}^n X_{nm} \Rightarrow N(0,1)$

Recall: $\max_{1\le m\le n}\{\sigma_{n,m}^2\} \xrightarrow[n\to\infty]{} 0$

Proof $Z_{n,m} = \varphi_{X_{n,m}}(t)$, $E(X_{nm}^2) = \sigma_{n,m}^2$

Fix $t\in\mathbb{R}$.

$W_{n,m} = 1 - t^2\sigma_{n,m}^2/2$ take As positive since At the limit $t^2\sigma_{n,m}^2/2$ is small

then using $(*)$ $\forall\varepsilon>0$

if ε -

$$|Z_{n,m} - W_{n,m}| \le E\left[\frac{|t|^3|X_{nm}|^3}{3!} \wedge |X_{n,m}|^2t^2\right]$$

$$\le t^2E\left(\frac{|t||X_{n,m}|^3}{6}; |X_{n,m}|<\varepsilon\right) + t^2E(X_{nm}^2; |X_{nm}|>\varepsilon)$$

? where did this go?

since $|X_{n,m}|<\varepsilon$

$$\le \frac{\varepsilon t^2}{6}E[\,|t|\underbrace{X_{n,m}^2}_{E|X_{n,m}^2|=1 \text{ given.}}\,] + t^2E(X_{n,m}^2; |X_{n,m}|>\varepsilon)$$

$$\leq \frac{\varepsilon t^2}{6} E\left[|t| X_{n,m}^2 \right) + t^2 E(X_{n,m}^2 ; |X_{n,m}| > \varepsilon)$$

$\underline{E|X_{n,m}^2| = 1 \text{ given.}}$

$$\sum_{m=1}^{n} |Z_{n,m} - W_{n,m}| \leq \frac{\varepsilon |t|^3}{6} + t^2 c_n(\varepsilon) \xrightarrow[n\to\infty]{} \varepsilon \frac{|t|^3}{6}$$

$$= 0$$

Recall: $\prod_{m=1}^{n} (1 + a_{n,m}) \xrightarrow[n\to\infty]{} e$

(i) $\sum_{m=1}^{n} a_{n,m} \xrightarrow[n\to\infty]{} a.$

(ii) $\sup_n \sum_{m=1}^{n} |a_{n,m}| < \infty$

(iii) $\max_{1 \leq m \leq n} |a_{n,m}| \xrightarrow[n\to\infty]{} 0$

Another Result.

$$\left| \prod_{i=1}^{n} z_i - \prod_{i=1}^{n} w_i \right| \leq \sum_{i=1}^{n} |z_i - w_i| \text{, if } |z_i| \leq 1, \ |w_i| \leq 1 \quad 1 \leq i \leq n.$$

$$|z_{n,m}| \leq |w_{n,m}| \leq 1 \ , \quad n \geq N$$

$$\lim_{n\to\infty} \left| \prod_{m=1}^{n} z_{n,m} - \prod_{m=1}^{n} \left(1 - \frac{t^2 \sigma_{n,m}^2}{2}\right) \right| \leq \lim_{n\to\infty} \sum_{m=1}^{n} |z_{n,m} - w_{n,m}| = 0$$

$$\lim_{n\to\infty} \left| \varphi_{S_n}(t) - \prod_{m=1}^{n} \left(1 - \frac{t^2 \sigma_{n,m}^2}{2}\right) \right| = 0$$

$$\left\{ \prod_{n=1} \left(1 - \frac{\sigma_{n,m}^2 t^2}{2}\right) \to e^{-\frac{t^2}{2}} = \varphi_{N(0,1)}(t) \right.$$

$$\Rightarrow \varphi_{S_n}(t) \to e^{-t^2/2} \quad \forall t \in \mathbb{R}.$$

gives Result $S_n \Rightarrow N(0,1)$.

truncation of CLT.

Example 1

using truncation.

$\{X, X_k\}_{k \geq 1}$ iid., $X \overset{\mathcal{D}}{=} -X$ symmetric.

$$P(|X| \geq x) = x^{-2} \ , \quad x \geq 1 \qquad P(|X| < 1) = 0$$

$E(X) = 0$ (from symmetric).

$$E(X^2) \geq 2\int_{x=1}^{\infty} P(|X| \geq x) \cdot x \, dx = \int_1^{\infty} 2 \cdot x^{-2} x \, dx = 2 \int_1^{\infty} x^{-1} dx = \infty$$

Want trucation.

if $n = 7$
$\log\log 7 = e^{e^7}$

$$Y_{n,m} = X_m \mathbb{1}_{|X_m| \leq \sqrt{n \cdot \log\log n}} \ , \ 1 \leq m \leq n \qquad c_n = \sqrt{n \log\log n}$$

$\log\log Z = e^{t}$

$$Y_{n,m} = X_m \mathbb{1}_{|X_m| \le \sqrt{n \cdot \log\log n}} , \quad 1 \le m \le n \qquad C_n = \sqrt{n \log\log n}$$

$$S_n = \sum_{m=1}^{n} X_{n,m} , \quad T_n \sum_{m=1}^{n} Y_{n,m}$$

(1) $P(S_n \ne T_n) \le \sum_{m=1}^{n} P(Y_{n,m} \ne X_m) = n \cdot P(|X| > C_n) = \frac{n}{C_n^2} = \frac{1}{\log\log n} \xrightarrow[n\to\infty]{} 0$

$$E(Y_{n,m}^2) = \int_1^{C_n} y^2 \cdot f_{Y_{n,m}}(y)\, dy = \int_1^{C_n} \cancel{y^2} \frac{2}{y^{\cancel{3}}}\, dy = \log(C_n) = \frac{1}{2}\left(\log(n) + \log\log\log(n)\right)$$

note $\frac{\partial}{\partial x} x^{-2} = \frac{2}{y^3}$

$E(Y_{n,m}^2) \sim \log(n)$, $\quad E(Y_{n,m}) = 0.$

$Var(T_n) \equiv n\log(n)$

L-F theron.

$$\frac{T_n}{\sqrt{n\log n}} \Rightarrow N(0,1).$$

$$\Rightarrow \frac{S_n}{\sqrt{n\log(n)}} \Rightarrow N(0,1)$$

$S_n = T_n + (S_n - T_n)$

Divide By $\sqrt{n\log n}$

Paul Levy. Result Domain of Attraction of $N(0,1)$

$\{X_n\}$ iid. $\exists a_n, b_n$ s.t. $\frac{S_n - a_n}{b_n} \Rightarrow N(0,1)$

iff $\frac{y^2 \cdot P(|X| > y)}{E(X^2; |X| > y)} \xrightarrow[y\to\infty]{} 0$