L36 - 11-20 Characteristic Functions

Wednesday, November 20, 2024 11:30 AM

Friday. On Zoom.

"characteristic functions"

last time Examples

Example of $\{\varphi_X(t)\}_{-\infty < t < \infty}$

Example 3.3.4

① $X \sim$ uniform(a,b)

$$\varphi(t) = \int_a^b \frac{e^{itx}}{b-a} = \frac{e^{itb} - e^{ita}}{it(b-a)}$$

$f_X(x) = \frac{1}{b-a}$, $a < x < b$.

$$\int e^{ix} dx = \frac{e^{ix}}{i} = -ie^{ix}$$

$$\varphi_{-X}(t) = \frac{e^{-ita} - e^{-itb}}{it(b-a)}$$

$-b - -a$

$-b + a$

$a - b$, ?

$-X \sim$ Uniform$(-b, -a)$

Example 3.3.2.

② $X \sim$ Poisson(λ) $\lambda > 0$

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$\varphi(t) = \sum_{k=0}^{\infty} e^{itk} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} (\lambda e^{it})^k / k! = e^{-\lambda} e^{\lambda e^{it}} = e^{\lambda(e^{it} - 1)} \quad \forall \in$$

③ $\varphi_X(t) = e^{itc}$

$X \equiv c$ constant.

Example 3.3.3

(4) $X \sim N(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty \qquad f_X(-x) = f_X(x), \quad x \in \mathbb{R}$$

$$E e^{itx} = \int_{x=-\infty}^{\infty} \cos(tx) + i\sin(tx)\, f_X(x)\, dx$$

seems $e^{itx} = \cos(tx) + i\sin(tx)$

sin is Anti symmetric. $f_x(x) + f_x(-x) = 0.$

$$\therefore \quad = \int_{x=-\infty}^{\infty} \cos(tx)\, f_x(x)\, dx. \qquad \frac{d}{dx} e^{-\frac{x^2}{2}} = (-x)e^{-\frac{x^2}{2}}$$

$$\varphi'(t) = \int_{-\infty}^{\infty} x \sin(tx)\, f_X(x)\, dx$$

"use integration by parts" $uv - \int v\,du.$

$$\cancel{\sin(tx)\, e^{-x^2/2}}\Big|_{-\infty}^{\infty} - t\int_{-\infty}^{\infty} \cos(tx) e^{-x^2/2} dx = -t\,\varphi(t)$$

$\varphi'(t) = (-t)\varphi(t), \quad t \in \mathbb{R}$

$\varphi(0) = 1$

$\varphi(t) = e^{-t^2/2} \quad t \in \mathbb{R}.$

"A Better way" Not in The Book.

Alternate method for solving N c.f.

Requires Complex Analysis' $\quad \varphi(t) = \psi(it)$

$$\psi_X(t) = E e^{tx} = \frac{1}{\sqrt{2\pi}}\int_{x=-\infty}^{\infty} e^{tx} e^{-x^2/2}\, dx = \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2 + tx - \frac{x^2}{2}}\, dx$$

MGF.

"complete the square"

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{x=\infty}^{\infty} e^{-\frac{(x-t)^2}{2}}\, dx$$

then change of variable

$x - t = y, \quad -\infty < y < \infty$

$dx = dy$

Sideshow Proof of gaussian integral.

Sideshow: Solve gaussian with polar.

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx$$

$$2\pi = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \int_{-\infty}^{\infty} e^{-y^2/2}\, dy$$

$$2\pi = \iint e^{-\frac{x^2+y^2}{2}}\, dx\,dy$$

$x = r\cos(\theta)$

$y = r\sin(\theta).$

$dx\,dy = r\,dr\,d\theta$

complex Analysis:

"Analytic function on the Real line"

can Be Extended to Realline

$$\psi_X(t) = \sum_{k=0}^{\infty} \frac{\left(\frac{t^2}{2}\right)^k}{k!}$$

$$\psi_X(it) = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{2^k k!}}_{(-1)^{k+1} t^2} = e^{-t^2/2}.$$

Is the characteristic continuous in t?

$$|\varphi(t+h) - \varphi(t)| = |E e^{i(t+h)X} - e^{itX}|$$

$$\le E|e^{i(t+h)X} - e^{itX}|$$

$$= E|e^{itX}(e^{ihX} - 1)|$$

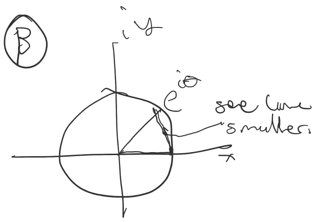
$$= E|e^{ith} - 1|$$

$$= E(|e^{iXh} - 1|; |X| \ge M)$$

$$+ E(|e^{iXh} - 1|; |X| < M) \quad (B)$$

$$E[e^{itX}] = 1$$

on normal circle with norm of 1



$$\boxed{|e^{i\theta} - 1| \le |\theta|^2}$$

$$\left|\int_{u=0}^{\theta} e^{iu}\, du\right| = |e^{i\theta} - 1|$$

$$\le \left|\int_0^{\theta} du\right| = |\theta|$$

$E(|e^{ihX} - 1|; |X| \le M)$

$$(B) = 2P(|X| \ge M)) + |h| \cdot M$$

ϵ like δ.

$$\therefore \quad |\varphi(t+h) - \varphi(t)| \le 2P(|X| \ge M) + |h| \cdot M, \quad \forall M > 0$$

$\Rightarrow \{\varphi(t)\}_{-\infty \le t \le \infty}$ is uniformly cont.

if $\varphi_X(t) = \varphi_Y(t)$, $\forall t \iff X \overset{D}{=} Y$

Theorem 3.3.4

Inversion Formula.

Assume that $a < b$. $P(X = a) = P(X = b) = 0$

$$\frac{b-a}{2\pi} \int_{-T}^{T} \varphi_X(t) \cdot \varphi_{-U}(t)\, dt \xrightarrow[T \to \infty]{} P(a < X < b)$$

$U \sim \text{Uniform}(a,b)$

$-U \sim \text{Uniform}(-b,-a)$

$$\varphi_{-U}(t) = \frac{e^{-ita} - e^{-itb}}{it(b-a)}$$

Assume $X \perp\!\!\!\perp U$

$$Y = X - U \quad \leftarrow \text{like smoothing}$$

$$\varphi_Y(t) = \varphi_X(t)\,\varphi_{-U}(t)$$

$$\frac{b-a}{2\pi}\int_{-T}^{T} \varphi_Y(t)\,dt \xrightarrow[T\to\infty]{} P(a < X < b)$$

$$f_Y(0) \xleftarrow{\text{has Density}} = \frac{P(a<X<b)}{b-a}$$

used to recover the density at zero

$$f_Y(y) = \frac{P(a+y < X < b+y)}{b-a}$$

has Bounded Variation, Diff. of 2 increasing functions

$$\frac{1}{2\pi}\int_{-T}^{T}\varphi_Y(t)\,dt = \frac{1}{\pi}\int_{y=-\infty}^{\infty}\frac{\sin(Ty)}{y} f_Y(y)\,dy \xrightarrow[T\to\infty]{} f_Y(0) = P(a<X<b)$$

What is going on here.

$$\forall T: \int_{-\infty}^{\infty}\frac{\sin Ty}{y}\,dy = \int \frac{\sin(y)}{y}\,dy.$$