

$(\Omega, \mathcal{A}, \mu)$, \mathcal{A} - algebra μ a measure on \mathcal{A}

Extension $\mathbb{P}_0 \mathbb{F} = \sigma(\mathcal{A})$

Define outer measure μ^*

$$\mu^*(A) = \inf_{\substack{A \in \mathcal{A} \\ \bigcup_{i=1}^{\infty} A_i \supseteq A}} \sum_{i=1}^{\infty} \mu(A_i), A \subseteq \Omega$$

Def "E $\subseteq \Omega$ is measurable" \leq this is \mathbb{F} under partition E

$$\text{if } \forall F \subseteq \Omega \text{ we get: } \mu^*(F) = \mu^*(F \cap E) + \mu^*(F \cap E^c)$$

$$\text{Always } \mu^*\left(\bigcup_{i=1}^{\infty} B_i\right) \leq \sum_{i=1}^{\infty} \mu(B_i) \quad \forall B_i \subseteq \mathbb{R} - \text{subadditive.}$$

$$\mu^*(B) \leq \mu^*(D) \text{ if } B \subseteq D - \text{monotonicity.}$$

Claim: if $\mu^*(E) = 0$, then E is measurable and so is any ACE

$$\mu^*(F) \geq \mu^*(F \cap E) + \mu^*(F \cap E^c)$$

$$F \cap E \subseteq E$$

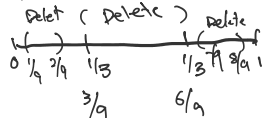
$$0 \leq \mu^*(F \cap E) \leq \mu^*(E) = 0$$

$$F \geq F \cap E$$

Example (Cantor) $\Omega = \mathbb{R}$, $\mathcal{A} = \left\{ \bigcup_{i=1}^{\infty} (a_i, b_i) \right\}$

$$\sigma\{\mathcal{A}\} = \sigma\{\text{open subset of } \mathbb{R}\} = \bigcup_{i=1}^{\infty} (a_i, b_i).$$

Consider $[0, 1]$ Cantor subset of Ω .



$$F_n \downarrow F$$

$$\mu(F) = \lim_{n \rightarrow \infty} (2/3)^n = 0$$

$$\bar{F} = \bar{\mathbb{R}} \text{ the cond } * [0, 1] \text{ equals cond } \mathbb{R}.$$

$$F = \left\{ \sum_{i=1}^{\infty} \frac{\delta_i}{3^i} \right\} \quad \delta_i = 0, 1, 2$$

in Cantor set we Delete 1

$$[0, 1] = \left\{ \sum_{i=1}^{\infty} \frac{\epsilon_i}{2^i} \right\} \quad \epsilon_i = 0, 1$$

these have same CARD

$$\overline{\left\{ \text{all subsets of } F \right\}} = \bar{F} = \bar{\mathbb{R}} \quad 2^{\bar{\mathbb{R}}} \text{ Cantor set are Bigger than } \bar{\mathbb{R}}$$

Conclusion: All subsets of Cantor set are measurable.

$$\sigma\{\mathcal{A}\} = \bar{\mathbb{R}} \quad \text{Cantor set} - \text{set is too large.}$$

$$\sigma\{\mathcal{A}\} \subsetneq \mathcal{A}^*, \quad \mathcal{A}^* = 2^{\bar{\mathbb{R}}}, \quad \sigma(\mathcal{A}) = \bar{\mathbb{R}}$$

Borel σ -algebra is the minimum σ -algebra

Borel σ -algebra

$$\mathcal{A} = \left\{ \bigcup_{i=1}^{\infty} (a_i, b_i) \right\} \subsetneq \sigma\{\mathcal{A}\} = \mathcal{B} \subsetneq \mathcal{A}^* \subsetneq \left\{ \text{all subset of } \mathbb{R} \right\}$$

Lebesgue σ -Algebra

Borel σ -Algebra

Axiom of choice.

is there a choice of σ -Algebra that is not measurable? Yes.

Consider $\Omega = [-1, 2]$

show a not measurable

look at pair $x, y < 1$, $x - y \notin \mathbb{Q}$

Outer Measure

E is measurable if

$$\forall F \subseteq \Omega, \quad \mu^*(F) = \mu^*(F \cap E) + \mu^*(F \cap E^c)$$

Cantor Set

$$B \subset [0,1]$$

if we look at

$$B + ? \quad \{x + \mathbb{Q} \cap [-1,1]\}, \quad x \in B.$$

$$[0,1] \subseteq \bigcup_{q \in \mathbb{Q} \cap [-1,1]} \{B + q\} \subset [-1,2]$$

$$M(\mathbb{R}_{1,2}) = 3$$

$$M(B+q) = M(B) \quad \forall q. \quad \text{translation invariant}$$

translation invariant

$$M(B+q) = M(B)$$

thus not measurable

$$\text{if } A \in \mathcal{Q}^* \rightarrow \exists B \in \mathcal{B} \text{ and } N \in \mathcal{B}$$

$$\text{so that } B \cup N \supset A$$

$$A = B \cup D, \quad D \subset N$$

$$(\Omega, \mathcal{F}, \mu)$$

completion of sigma algebra.

$$(\Omega, \tilde{\mathcal{F}}, \mu)$$

ADD all subsets of $[0,1]$

will not really use it as we use Lebesgue measure.

Integration on measure spaces.

chapter 1.4

σ -Algebra

σ -finite measure

$$\exists E_n \uparrow \Omega, E_n \in \mathcal{F}, \mu(E_n) < \infty$$

$$\text{Example: } \lambda([-n,n]) = 2n < \infty \quad \forall n.$$

$$\bigcup_{n=1}^{\infty} [-n,n] = \mathbb{R}$$

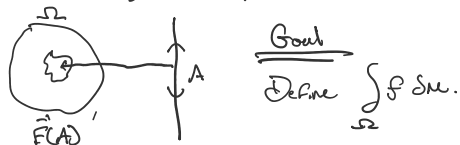
$$\lambda(\mathbb{R}) = \infty$$

λ is σ -finite measure

$$(\Omega, \mathcal{F}) \quad (\mathbb{R}, \mathcal{B})$$

Def: $f: \Omega \rightarrow \mathbb{R}$ is called \mathcal{F} - \mathcal{B} measurable,

$$\text{if } f^{-1}(A) \in \mathcal{F}, \quad \forall A \in \mathcal{B}.$$

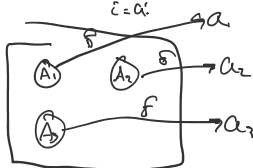


Step 1 $f \mapsto$ simple function.

$$1_{A_i} = \begin{cases} 1 & \text{on } A_i \\ 0 & \text{on } A_i^c \end{cases}$$

$$f = \sum_{i=1}^n \alpha_i 1_{A_i} \quad \mu(A_i) < \infty, \quad A_i \cap A_j = \emptyset, \quad \text{if } i \neq j$$

$$\int f d\mu = \sum_{i=1}^n \alpha_i \mu(A_i)$$



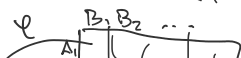
Lemma: Let ψ and φ be simple functions then

- 1) $\psi \geq 0 \Rightarrow I(\psi) \geq 0$
- 2) $I(a\psi) = aI(\psi), \quad a \in \mathbb{R}$
- 3) $I(\psi + \varphi) = I(\psi) + I(\varphi)$

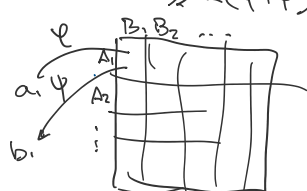
$$\text{where } I(\psi) = \int \psi d\mu$$

Properties of Integration

- (1) if $\psi \geq 0, \quad I(\psi) \geq 0$
- (2) $I(a\psi) = aI(\psi) \quad \forall a \in \mathbb{R}$
- (3) $I(\psi + \varphi) = I(\psi) + I(\varphi)$



$$\sum_i I(\varphi + \psi) = I(\varphi) + I(\psi) \quad \Bigg| \quad \sum_i I(\varphi + \psi) = I(\varphi) + I(\psi)$$



$\sum_i \sum_j a_i + b_j u(A_i \cap B_j)$

Every Extension