L02 - 08-28 lecture

Wednesday, August 28, 2024 11:30 AM

$(\Omega, \mathcal{A}, \mu)$ — measure; μ is the measure; "Algebra" collection

$\mathcal{A} = \{A : A \subseteq \Omega\}$ that satisfy.

Algebra $\mathcal{A}$

① $\Omega \in \mathcal{A}$ — contains whole set

② $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ — closed under complement.

③ $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$ — closed under finite intersection

Main Example $A \in \mathcal{A}$ as $A = \bigcup_{i=1}^{n} (a_i, b_i]$, $a_i < b_i$ — semi open.

Assume to be disjoint. if not we can merge non-disjoint overlap

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Main Sample

Set of semi-open sets
$A = \bigcup_{i=1} (a_i, b_i]$, $a_i < b_i$
= disjoint =

Measure μ

① $\mu(A) \geq 0$, $A \in \mathcal{A}$

② $\mu(\emptyset) = 0$

③ $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ if $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, $A_i \in \mathcal{A}$ — Sigma Additivity

Why is this true?

$A_i \cap A_j = \emptyset$, $1 \leq i \neq j < \infty$

Measure μ

- Nonnegative
- Measure of Emptyset is 0
- Sigma additivity.

Ex. of measure

$\mu((a,b]) = b - a$ — Lebesgue measure. The $\mu(\text{Interval } A)$ is length(A)

$\mu(A) = \sum_{i=1}^{n} b_i - a_i$

"want Reals because completeness"

② Let $F : \mathbb{R} \to \mathbb{R}$, $F(x) \leq F(y)$ $x \leq y$. Non Decreasing.

Right continuous. $\lim_{\substack{y > x \\ y \to x}} F(y) = F(y)$

Generalize measure
$\mu((a,b]) = F(b) - F(a)$

General Form of Measure

$\mu((a,b]) = F(b) - F(a)$

Let Assume Ω not $\mathbb{R}$.

we want to verify $\mu\left(\bigcup_{i=1}^{\infty} A_i\right)$ from Req 3.

work for Main Example.

Assume: $\mathcal{A} = \{A \subset (0,1]\}$, $\mu((0,1]) = 1 - 0 = 1 \therefore \mu(\Omega) < \infty$ — Lebesgue; is finite.

$\mathcal{A} = \{A \subset (0,1]\}$ that is All the semi open sets

if $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ Then $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$ and $\bigcup_{i=n+1}^{\infty} A_i \in \mathcal{A}$

$\bigcup_{i=n+1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{n} A_i \in \mathcal{A}$

$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{n} A_i + \bigcup_{i=n+1}^{\infty} A_i$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \left[\sum_{i=1}^{n} \mu(A_i)\right] + \mu\Big(\underbrace{\bigcup_{i=n+1}^{\infty} A_i}_{B_{n+1}}\Big)$$

$\downarrow n \to \infty$ $\sum_{i=1}^{\infty} \mu(A_i)$; $\downarrow n \to \infty$ 0

total intersection.

we see $B_n \supset B_{n+1}$, $\bigcap_{n}^{\infty} B_n = \emptyset$ $(B_n \downarrow \emptyset)$

$\sum_{i=1} \mu(A_i)$

$\leftarrow$ total intercection.

we see $B_n > B_{n+1}$, $\bigcap_{n=1}^{\infty} B_n = \emptyset$ $\quad (B_n \downarrow \emptyset)$

Changed signs (, [

this works.

what About $D_n = (0, \frac{1}{n}]$, $n = 1, 2, \ldots$

SIDE Example.

$D_n \downarrow \emptyset$

$\mu(D_n) = \frac{1}{n} \downarrow 0$

Does not go down to?

if $\mu(B_n) \not\downarrow 0$ then $\exists\, a > 0$ s.t. $\mu(B_n) \geq a$, $n \geq 1$

By contra Dict.

[,] closed interval.

Compact subset of $\mathbb{R}$ is closed and bounded subset

IF B_n are all compact.

$B_n > B_{n+1}$, $n \geq 1$ then $\bigcap_{n=1}^{\infty} B_n \neq \emptyset$

$b_n \in B_n$, $n \geq 1$ {

subsequence

$b_{n_k} \xrightarrow[k \to \infty]{} b^* \in B_1$

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ADD pt. Then Becomes compact.

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Let $C_n \subset B_n$ $\forall n$, $\bar{C}_n \subset B_n$

$\mu(B_n \setminus C_n) \leq \frac{a}{2^n}$, $n \geq 1$

what is the Distance Between B_n C_i

$\mu\left(B_n \setminus \bigcap_{i=1}^{n} C_i\right)$ $\quad \mu(A \cup B) \leq \mu(A) + \mu(B)$

$= \mu\left(B_n \cap \left(\bigcap_{i=1}^{n} C_i\right)^c\right) =$

$= \mu\left(B_n \cap \bigcup_{i=1}^{n} (C_i)^c\right)$

$\leq \mu\left(\bigcup_{i=1}^{n} B_i \cap (C_i)^c\right)$

$\leq \sum_{i=1}^{n} \mu(B_i \setminus C_i)$

simplest geometric series.

$\leq \sum_{i=1}^{n} \frac{a}{2^i} < a$

what happens if $\bigcap_{i=1}^{n} C_i = \emptyset$? Does not happening.

$\Rightarrow \bigcap_{i=1}^{n} C_i \neq \emptyset$ $\quad \therefore C_i$ is not compact.

$\bar{C}_i \leftarrow$ ADD A Point.

$\bigcap_{i=1}^{n} \bar{C}_i > \bigcap_{i=1}^{n} C_i \neq 0.$

$\bigcap_{n=1}^{\infty} B_n > \bigcap_{i=1}^{\infty} \bar{C}_i \neq \emptyset \quad \rightarrow\leftarrow$

$$\bigcap_{n=1}^{\infty} B_n \supset \bigcap_{i=1}^{\infty} \bar{C}_i \neq \phi \quad \rightarrow\leftarrow$$

$$\Rightarrow \mu(B_n) \downarrow 0$$

σ - Algebra. $\mathcal{A}^*$

① $\Omega \in \mathcal{A}^*$

② $A \in \mathcal{A}^* \Rightarrow A^c \in \mathcal{A}^*$ countable union.

③ $A_i \in \mathcal{A}^* \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}^*$

then measure μ:

① $\mu(A) \geq 0$, $A \in \mathcal{A}$

② $\mu(\phi) = 0$

③ $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$, $A_i \in \mathcal{A}^*$

$\sigma(\mathcal{A})$ - "minimal" σ-Algebra $\supset \mathcal{A}$

$\sigma(\mathcal{A})$ in our example is called "Borel σ-algebra."

Caratheodory algebra is bigger.

Friday class online.