L29 - 11-04 Feller Theorem

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Theorem (M.Z.) $\{X_k, X\}_{i \geq 1}$ iid $E|X|^p \cancel{<} = \infty$ $1 \leq p < 2$, $E(X) = 0$

then $\frac{S_n}{n^{1/p}} \xrightarrow[n\to\infty]{a.s.} 0$ where $S_n = \sum_{k=1}^{n} X_k$.

Proof $Y_k = X_k \mathbb{1}_{\{|X| \leq k^{1/p}\}}$ $k = 1, 2 \ldots$ A

$T_n = \sum_{1}^{n} Y_k$ $n \geq 1$

$$\sum_{k=1}^{\infty} P(Y_k \neq X_k) = \sum_{k=1}^{\infty} P(|X| > k^{1/p}) = \sum_{k=1}^{\infty} P(|X|^p > k) \sim E|X^p| < \infty$$

By BCI $\Rightarrow$ $P(X_k \neq Y_k \text{ i.o.}) = 0$

$\frac{S_n}{n^{1/p}} \xrightarrow[n\to\infty]{a.s.} 0$ will follow from $\frac{T_n}{n^{1/p}} \xrightarrow[n\to\infty]{a.s.} 0$

Because $\frac{S_n - T_n}{n^{1/p}} \xrightarrow[n\to\infty]{a.s.} 0$

Lemma: "a is Good"

if $\{a_n\}_{n\geq 1}$ is "Good" and $\sum P(|X| > a_n) < \infty$. Let $Y_n = X_n \mathbb{1}_{\{|X| \leq a_n\}}, n \geq 1$

then $\sum_{n=1}^{\infty} \frac{E(Y_n^2)}{a_n^2} < \infty$

if $\left\{\frac{a_n}{n^{1/p}}\right\}_{n\geq 1}$ with $0 < p < 2$ is non-Decreasing, then $\{a_n\}_{n\geq 1}$ is Good.

in paritculary we can take $A_n = n^{1/p}$, $1 < p < 2$.

So we get $\frac{\sum E(Y_k^2)}{k^{2/p}} < \infty \Rightarrow \sum_{k=1}^{\infty} Var\left(\frac{Y_k}{k^{1/p}}\right) < \infty$

Since Y_k are inD. we get: $\frac{\sum Y_k - E(Y_k)}{k^{1/p}} \xrightarrow{a.s.} < \infty$

Kronecker: $\sum_{k=1}^{n} \frac{Y_k - E(Y_k)}{n^{1/p}} \xrightarrow[n\to\infty]{a.s.} 0$

$$\frac{T_n - \sum_{k=1}^{n} E(Y_k)}{n^{1/p}} \xrightarrow[n\to\infty]{a.s.} 0 \text{ so } \frac{S_n - \sum_{k=1}^{n} E(Y_k)}{n^{1/p}} \xrightarrow[n\to\infty]{a.s.} 0$$

we need $\frac{\sum_{k=1}^{n} E(Y_k)}{n^{1/p}} \xrightarrow[n\to\infty]{} 0$

A

$\Rightarrow E(Y_k) = -E(X_k : |X| > k^{1/p})$

(4)

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$|E(Y_k)| \le E(|X_k| \cdot \mathbb{1}_{\{|X_k| > k^{1/p}\}})$

By identical Distributed.

$\le \cancel{k^{-(p-1)/p}} \dfrac{E[|X|^p \mathbb{1}\{|X| > k^{1/p}\}]}{k^{(p-1)/p}}$

Note $E|X|^p < \infty \Rightarrow E|X| < \infty$

if $x > k^{1/p}$ then $\dfrac{x^p}{k^{(p-1)/p}} \ge x$, $x > 0$

Because $x^{(p-1)} > k^{(p-1)/p} \rightarrow \dfrac{x^{p-1}}{k^{(p-1)/p}} \ge 1$

$\sum_{k=1}^{n} \dfrac{|E(Y_k)|}{n^{1/p}} \le \dfrac{\underline{\sum k^{1/p-1} E(|X|^p \mathbb{1}_{\{|X| > k^{1/p}\}})}}{n^{1/p}}$ $\quad E(|X|^p ; |X| > k^{1/p}) \xrightarrow{k \to \infty} 0$

what about $\sum_{k=1}^{n} k^{1/p-1} \le C \cdot n^{1/p}$

$\int x^{1/p-1} \, dx = [x^{1/p}]_{x=1}^{n} (1/p)$

is decreasing.

$\dfrac{\sum_{k=1}^{n} k^{1/p-1}}{n^{1/p}} \le C$ $\quad \therefore$ It Becomes 0.

William Feller.

Theorem (W. Feller) Let $\{X_i, X\}_{i \ge 1}$ be iid. $E|X| < \infty$

Let $a_n > 0$, $n = 1, 2, \ldots$ we Assume $\{\frac{a_n}{n}\}_{n \ge 1}$ is non-decreasing.

then: (a) if $\sum_{n=1}^{\infty} P(|X| \ge a_n) < \infty$ then $\overline{\lim_{n \to \infty}} \dfrac{|S_n|}{a_n} = 0$ a.s. $\Rightarrow \left(\lim_{n \to \infty} \dfrac{S_n}{a_n} = 0 \text{ a.s.} \right)$

(b) if $\sum_{n=1}^{\infty} P(|X| \ge a_n) = \infty$ then $\overline{\lim_{n \to \infty}} \dfrac{|S_n|}{a_n} = \infty$ a.s.

Proof (b) $\frac{a_n}{n} \uparrow \Rightarrow a_{k_n} \ge k \cdot a_n$ Fix $k > 0$

Because, $\dfrac{a_{kn}}{kn} \ge a_n$

$\sum_{n=1}^{\infty} P(|X| > k a_n) \ge \sum_{n=1}^{\infty} P(|X| \ge a_{kn}) \overset{?}{\ge} \frac{1}{k} \sum_{n=k}^{\infty} P(|X| > a_n) = \infty$

$$\sum_{n=1}^{\infty} P(|X| > k a_n) \geq \sum_{n=1}^{\infty} P(|X| \geq a_{kn}) \overset{?}{\geq} \frac{1}{k} \sum_{n=k}^{\infty} P(|X| > a_n) = \infty$$

Consider $P\left(\frac{|X|}{a_n} > k\right)$ Then By BC II $P\left(\frac{|X_n|}{a_n} > k, \text{ i.o.}\right) = 1$

what happens with limsup.

$$\overline{\lim_{n\to\infty}} \left\{ \frac{|X_n|}{a_n} \right\} \geq k \text{ a.s.} \Rightarrow \overline{\lim_{n\to\infty}} \left|\frac{|X_n|}{a_n}\right|$$

$$|X_n| = |S_n - S_{n-1}| \leq |S_n| + |S_{n-1}| \leq 2 \max\{|S_n|, |S_{n-1}|\}$$

$$\frac{|X_n|}{2 a_n} \leq \max\left\{ \frac{|S_n|}{a_n}, \frac{|S_{n-1}|}{a_n} \right\}.$$

Since the $\overline{\lim}$ is infinity

So is

Proof (a). "then BC I, sum of variances the same as MZ proof

Then use Kronecker Lemma

$$\sum_{k=1}^{n} \frac{X_k - E(Y_k)}{a_n} \xrightarrow[n\to\infty]{\text{as}} 0$$

need $\dfrac{\sum E(Y_k)}{a_n} \xrightarrow[n\to\infty]{} 0$

need to show $\left\{\frac{a_n}{n}\right\}_{n\geq 1}$ goto inf

By $E|X_i| = \infty$

see Book for proof.

two Examples next time.