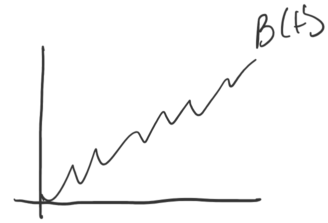
03-31

Monday, March 31, 2025 11:30 AM

Theorem with Prob 1 $\{B(t)\}_{0 \le t \le 1}$ is not holder continuous.

Eventually Becomes 5/6

with $1 \ge \gamma > 1/2$ at any point



For $\gamma > 1/2$

$\nexists\ 0 < s < t$ s.t. $|B(t) - B(s)| \le C|t-s|^\gamma$, $|t-s| \le \delta(\omega)$

$\therefore$ There is no $\partial/\partial x$ for any point.

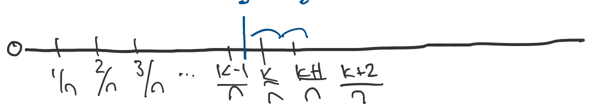
continuous But no Derivatives.

vice.

Proof $A_n = \{\omega \in \Omega : \exists\ 0 < s < 1;\ |B_t - B_s| < C|t-s|^\gamma,\ |t-s| < \frac{3}{n}\}$

$A_n \uparrow$ as $n \uparrow$

these pt. satisfy $\frac{5C}{n^\gamma}$



the Answer will be $\frac{k}{n}$

Define $Y_{k,n} = \max\{|B(\frac{k}{n}) - B(\frac{k-1}{n})|,\ |B(\frac{k+1}{n}) - B(\frac{k}{n})|,\ |B(\frac{k+2}{n}) - B(\frac{k+1}{n})|\}$

then. Observe,

$$B_n = \bigcup_{k=1}^{n} \{Y_{k,n} < \frac{5C}{n^\gamma}\}$$

$$A_n \subseteq B_n$$

$$P(A_n) \le P(B_n)$$

$$|B(\tfrac{k}{n}) - B(\tfrac{k+1}{n})| \le |B(s) - B(\tfrac{k-1}{n})| + |B(\tfrac{k}{n}) - B(s)|$$

$$\le C|s - \tfrac{k-1}{n}|^\gamma$$

$$\le C|\tfrac{k}{n} - \tfrac{k-1}{n}|^\gamma + C|\tfrac{k}{n} - \tfrac{k-1}{n}|^\gamma)$$

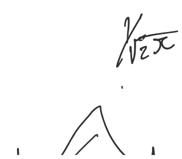
$$= \frac{2C}{n^\gamma}$$

max $\{|B\cdot - B\cdot|, \ldots$

so consider $B_n = \bigcup \{Y_{k,n} \le \frac{5C}{\cdot}\}$

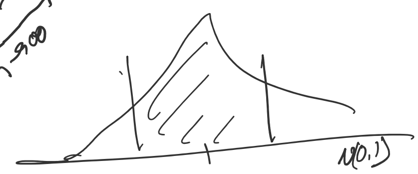
$\gamma > 5/6$

900



so consider $B_n = \bigcup_{k=1}^{n} \{ Y_{k,n} \le \frac{5C}{n^\gamma} \}$

$\gamma > 5/6$, $n \to \infty$



$$\overline{\lim_{n\to\infty}} P(A_n) \le \overline{\lim_{n\to\infty}} P(B_n) \le \overline{\lim_{n\to\infty}} n \cdot P\left(|B(\tfrac{1}{n})| < \frac{5C}{n^\gamma}\right)^3$$

STD. Normal

$$= n \cdot P\left(|Z| \le \frac{5C}{n^{\gamma - 1/2}}\right)^3$$

$B(\frac{1}{n}) \sim N(0, \frac{1}{n})$

$B(\frac{1}{n}) \stackrel{d}{=} \sqrt{1/n} \cdot Z$

$Z \sim N(0,1)$

$$\sim \frac{n}{(n^{\gamma - 1/2})^3} = \frac{n}{n^{3\gamma - 1.5}}$$

$3\gamma - 1.5 > 1$

$3\gamma > 5/2$

$\gamma > 5/6$.

Conclusion

$P(A_n) = 0 \quad \forall n \ge 1$

if we Repeat with the max of m intervals

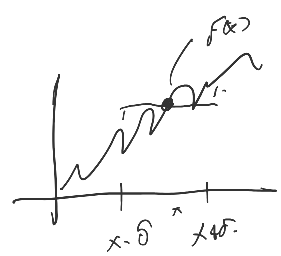
if $\gamma > \frac{1}{2} + \frac{1}{m}$ then $\nexists$ s st.

As $m \uparrow$ $\searrow 1/2$

Prove not 1.

Point of Increase. $(x, f(x))$ is Point of Increase

if $\exists \delta > 0$ s.t.



▶ Levy models of Continuity

step 1 Let $\{X_k\}_{k \ge 1}$ be inD. symmetric R.V. (Not in Book)

$$S_n = \sum_{k=1}^{n} X_k \quad n \ge 1 \quad S_0 = 0.$$

$$\forall t > 0 : \quad P(\max_{0 \le k \le n} |S_k| \ge t) = 2P(|S_n| > t) = 4P(S_n > t)$$

symmetric $\Rightarrow$ characteristic function is Real valued.

Step 2 $\quad t \in Q_2 = \left\{ \frac{m}{2^n},\ m=0,\dots,2^n,\quad n=1,2,3,\dots \right\}$

Diadic Rotionals.

$$Q_{2,n} = \left\{ \frac{m}{2^n},\ m=0,\dots,2^n \right\} \qquad Q_{2,n} \subset Q_{2,n+1}$$

$$I_{m,n} \equiv \left\{ t \in Q_2 ;\ \frac{m}{2^n} < t \le \frac{m+1}{2^n} \right\}$$

$$\Delta_{m,n} = \sup_{t \in I_{n,m}} \left\{ \left| B_t - B_{\frac{m_0}{2^{n_0}}} \right| \right\}$$

transposed?

the number of terms Does not matter.

then

Because Levy inEq.

$$P(\Delta_{m,n} \ge a 2^{-n/2}) < 4P\left(B_{\frac{1}{2^n}} > a \cdot 2^{-\frac{n_0}{2}}\right)$$

translate to std Normal

$$= 4P(Z > a) \le 4e^{-a^2/2}, \quad a > 1$$

$$P\left(\bigcup_{m=0}^{2^{n-1}} \{ \Delta_{m,n} > a 2^{-m/2} \} \right) \le 2^n \cdot 4 \cdot e^{-a^2/2} \le 4 \cdot 2^{-\varepsilon \cdot n}$$

fix $\varepsilon > 0$.

$$a_n = (b_n)^{1/2}, \quad b = 2(1+\varepsilon)\ln(2)$$

then Borel Cantelli kicks in.

$$\sum_{n=1}^{\infty} 2^{-\varepsilon n} < \infty \quad \forall \varepsilon < \infty$$

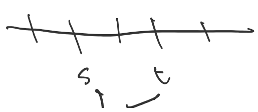
By BCI a.s. $\forall \omega \in \Omega\ \exists N(\omega)$ s.t. for $n \ge N(\omega)$

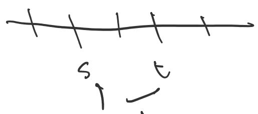
$$\sup_{0 \le m \le 2^n} \{ \Delta_{m,n} \} \le (b_n)^{1/2} \cdot 2^{-n/2}.$$

Conclusion if $n \ge N(\omega)$ then $|s-t| \le 2^{-n}$, $s,t \in Q_2$

$$\Rightarrow |B_t - B_s| \le 3(b_n)^{1/2} 2^{-n/2}.$$

Why 3?, we work with $\frac{k}{2^n}, \frac{k+1}{2^n}, \frac{k+2}{2^n}$





to control we need Δ ineq. $\times$ 3.

We are try to Build Brownian motion Here?

Thm Kolgomorov Extension theorem

Dyadic Rationals $\Rightarrow \mathbb{R}$.

Last step tricky.

$2^{-(n+1)} < \delta < 2^{-n}$. this give "modulus of continuity" will be $\sqrt{\delta \log(\frac{1}{\delta})}$

Says gamma = 1/2 needs help By