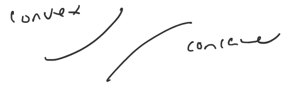
02-10

Monday, February 10, 2025 11:31 AM



Theorem ⓐ Let $\{X_n, \mathcal{F}_n\}_{n \geq 0}$ be a M.G. let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex (concave) function and $E|\varphi(X_n)| < \infty$, $n \geq 1$

then $\{\varphi(X_n), \mathcal{F}_n\}_{n \geq 1}$ is sub MG. (super MG)

$y(x) = x^2$ convex

$y(x) = \sqrt{x}$, $x > 0$ concave.

Proof $E_{\mathcal{F}_n}(\varphi(X_{n+1})) \underset{\leq \text{ concave}}{\overset{\text{convex }\geq}{}} \varphi(E_{\mathcal{F}_n}(X_{n+1})) = \varphi(X_n)$ a.s. $\forall n \geq 1$

concave " $= \varphi(X_n)$ ∴ super MG.

EX ① $\varphi(x) = x^+$

$\{X_n^+, \mathcal{F}_n\}$ sub MG.

~~$\{|X_n|^p, \mathcal{F}_n\}$ " $p \geq 1$~~ wait for part B. of theorem.

ⓑ $\{X_n, \mathcal{F}_n\}$ sub MG (super MG), φ convex (concave) and increasing.

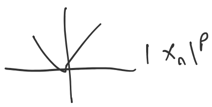
Then $\{\varphi(X_n), \mathcal{F}_n\}_{n \geq 0}$ is sub MG (super MG) as well.

Proof: only for sub MG.

$$E_{\mathcal{F}_n}(\varphi(X_{n+1})) \geq \varphi(E_{\mathcal{F}_n}(X_{n+1})) \geq \varphi(X_n)$$

convex.

② $|X_n|^p$, $p \geq 1$ → sub MG. $p \geq 1$

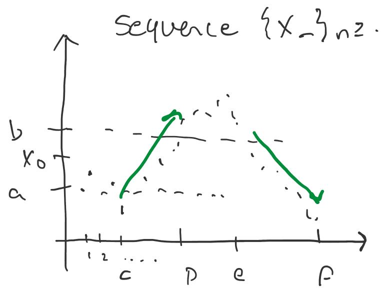


$H_n = \mathbb{1}_{\{T \geq n\}}$

Last Time: if $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ is MG (Sub MG, Super MG)

and T is a S.T. then $\{X_{n\wedge T}, \mathcal{F}_n\}$ is MG (super, sub) as well

"so we can stop and still MG"



on interval (c, D)
we up-courses. Go from Below a to Above b.

likewise Down course e, f

Doob - we want $X_n \xrightarrow[n\to\infty]{} X$ Then. $\underset{\text{time from 0 to } n}{U_n^{ab}}$ (# of up coursing) $\uparrow U_\infty^{ab}$ with $U_\infty^{ab} < \infty$ Finite.

As n increases $U_n^{a,b}$ increasing

if $U_\infty^{ab.}$ is not finite it never Converges.

Doob trick: try to get $E(U_\infty^{ab}) < \infty$
which Implies $U_\infty^{ab} < \infty$, a.s.

Look at only Rationals.

The up coursing Lemma

Let $a < b$, $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ is sub MG.

then $(b-a) E(U_n^{a,b}) \leq E[(X_n - a)^+] \underbrace{- E[(X_0 - a)^+]}_{\text{we can Ignore using this As Bound}} \leq \underset{n\geq 0}{E(X_n^+)} + |a|$

$(c+d)^+ < c^+ + d^+$

Martingale Convergence Theorem.

Let $\{X_n, \mathcal{F}_n\}_{n\ge 0}$ is subMG. (super MG)

Assume $\sup_{n\ge 0} E(X_n^+) < \infty$ $\quad \left(\sup_{n\ge 0} E(X_n^-) < \infty\right)$

Then,

① $X_n \xrightarrow[n\to\infty]{a.s.} X$

② $E|X| < \infty$ (The limit is in L1)

(subMG) $\quad$ (sup $E(X_n^-)$) minus.

Remark. For $\{X_n\}$ subMG: $\sup E(X_n^+) < \infty$ iff $\sup_{n\ge 0} E|X_n| < \infty$

$$E|X_n| = E(X_n^+) + E(X_n^-)$$

$$E(X_n^-) = E(X_n^+) - E(X_n) \le E(X_n^+) - E(X_0)$$

subMG ↑

$E(X_0) \le E(X_1) \le \dots \le E(X_n) \le \dots$

sup MG.

$E(X_0) \ge E(X_1) \ge \dots \ge E(X_n)$

$$\sup_n E(X_n^-) \le \sup_{n\ge 0}\{E(X_n^+)\} - \underbrace{E(X_0)}_{\text{constant.}} < \infty.$$

$$\sup_n E|X_n| \le \sup_{n\ge 0} E(X_n^+) + \sup_n E(X_n^-) < \infty$$

Proof of ② given ①: ① $\Rightarrow |X_n| \xrightarrow[n\to\infty]{a.s.} |X|$ and we use Fatou Lemma.

Fatou: $E|X| = E \underline{\lim} |X_n| \le \underline{\lim} E|X_n| < \infty$ $\quad$ Prove 2.

Proof ① From upcrossing

Application of super MG.

EX $\{X_n, \mathcal{F}_n\}_{n\ge 1}$ SuperMG, $X_n \ge 0$ a.s., $n \ge 0$.

need to show $\sup_{n\ge 1} E(X_n^-) < \infty$. $\quad X_n^- \equiv 0$.

$$\sup_{n\ge 1} E(X_n^-) = 0. < \infty.$$

then $X_n \xrightarrow{a.s.} X \ge$ $\quad E(X) < \infty$

SSRW. ind.

martingale diff is IND.