L28 - 11-01 Kronecker

Friday, November 1, 2024 11:30 AM

Let $X_1, X_2 \ldots$ inD.

① Let $\{X_n\}_{n\geq 1}$ InD. if $\sum_{n=1}^{\infty} V(X_n) < \infty$ then $\sum_{n=1}^{\infty}(X_n - E(X_n))$ conv. a.s.

② Kol. 3-series. theorem.

$\sum_{n=1}^{\infty} X_n$ con a.s. $\iff$ ① $\forall A>0$ $\sum^{\infty} P(|X_n| > A) < \infty$
② $\sum_{n=1}^{\infty} Var(Y_n) < \infty$
③ $\sum_{n=1}^{\infty} E(Y_n)$ conv. a.s.

③ Lemma that Follow H-J inequality. $S_n = \sum X_n$.

if $\sup_{n\geq 1}\{|S_n|\} < \infty$, a.s. and $E(\sup_{n\geq 1}\{X_n^2\}) < \infty$.

then. $\sum_{n=1}^{\infty} Var(X_n) < \infty$

$\overset{①}{\Rightarrow} \sum (X_n - E(X_n))$ conv. a.s.

It Follows From 3: $\{X_n\}_{n\geq 1}$ InD, $E(X_n) = 0$, $n \geq 1$

$\sup_{n\geq 1} |S_n| < \infty$, $|X_n| \leq A$, $n \geq 1$

then $\sum_{n=1}^{\infty} X_n$ Conv. As. if Partial sum is Bounded. then They converge,

$\lim_{n\to\infty} S_n$ conv. a.s.

used To prove $\Rightarrow$ on Kol. 3-series.

Kronecker Lemma.

Let $x_n \in \mathbb{R}$, $n > 1$ $0 < a_n \uparrow \infty$. if $\sum_{n=1}^{\infty} \frac{x_n}{a_n}$ converges.

then. $\frac{\sum_{k=1}^{n} x_k}{a_n} = \frac{S_n}{a_n} \xrightarrow[n\to\infty]{} 0$

Lebesgue stieltjes.

Recall: $G, F : (a,b] \to \mathbb{R}$.

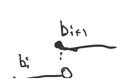
$$\int_{(a,b]} F(x)\,dG(x) + \int_{(a,b]} G(x)\,dF(x) = [F\cdot G]_a^b + \sum_{a<x<b} \Delta F(x)\,\Delta G(x)$$

$F(b)G(b) - F(a)G(a)$

of the Jumps. $\Delta F(x) = F(x) - F(x^-)$.

$a_0 = 0 = b_0$

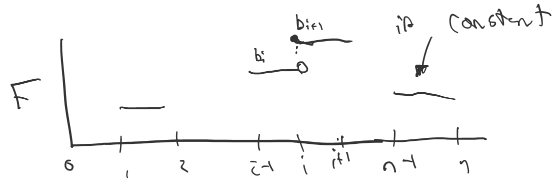
$b_n = \sum \frac{x_k}{a_k}$, $n \geq 1$



is constant

$a_0 = 0 = b_0$

$b_n = \sum \frac{X_k}{a_k}, \quad n \geq 1$



right continuous

$\overset{n}{\mathbb{Z}}$ $\quad F((c,d]) = F(d) - F(c)$

if $\mu(k, k+1) = 0$.

$$F(x) = \sum_{i=1}^{n} b_i \cdot \mathbb{1}_{\{i < x < i+1\}},$$

$$G(x) = \sum_{i=1}^{n} a_i \mathbb{1}_{\{i < x < i+1\}}$$

$$\sum_{i=1}^{n} b_i(a_i - a_{i-1}) + \sum_{i=1}^{n} a_i(b_i - b_{i-1})$$

minus that.

$$= b_n \cdot a_n + \sum_{n=1}^{n} (a_i - a_{i-1})(b_i - b_{i-1})$$

$\sum_{n=1}^{\infty} b_n = b_\infty$

$$\sum_{i=1}^{n} \frac{a_i(b_i - b_{i-1})}{a_n} = b_n - \sum_{i=1}^{n} b_{i-1} \left(\frac{a_i - a_{i-1}}{a_n} \right) \xrightarrow[n\to\infty]{} b_\infty - b_\infty = 0.$$

$b_i - b_{i-1} = \frac{X_i}{a_i} \qquad \sum_{i=1}^{n} \frac{a_i(b_i - b_{i-1})}{a_n}$

$$\sum_{i=1}^{n} \frac{X_i}{a_n} \xrightarrow[n\to\infty]{} 0$$

Theorem

$\text{Var}(X_n) \therefore \text{Var}(X_n) < \infty$

$\{X_n\}$ IND, $E(X_n) = 0$, $n \geq 1$, Assume $a_n \uparrow \infty$, $\sum_{n=1}^{\infty} \frac{E(X_n^2)}{a_n^2} < \infty$

then $\frac{S_n}{a_n} \xrightarrow[n\to\infty]{a.s.} 0$

$\Rightarrow \sum_{n=1}^{\infty} \frac{X_n}{a_n}$ conv a.s.

$\Rightarrow \frac{\sum_{k=1}^{n} X_k}{a_n} \xrightarrow[n\to\infty]{a.s.} 0$

Exactly kronecker lemma.

Ex $\{X_n\}_{n \geq 1}$, $E(X_n) = 0$, $\sup_n \{E(X_n^2)\} < c < \infty$

if second moment exists then. First moment exist $E|X| \leq E^{1/2}(X^2)$ By holder. says $L^1 \leq L^2$.

Since $\sum_{n=1}^{\infty} E\left[\frac{X_n^2}{n[\log_2 n]^{1+\varepsilon}}\right] \leq c \cdot \sum \frac{1}{n[\log(n)]^{1+\varepsilon}} < \infty \qquad \varepsilon > 0$

$$\int_{x=2}^{\infty} \frac{dx}{x (\log(x))^{1+\varepsilon}} \qquad \int_{y=\log(2)}^{\infty} \frac{1}{y^{1+\varepsilon}} dy < \infty$$

then $\dfrac{S_n}{\sqrt{n}\,[\log(n)]^{1/2+\epsilon}} \xrightarrow[n\to\infty]{a.s.} 0$

comes from $E[X_i^2]$. e.g. $n^{1/2}$.

if we Assume iid.

SLLN of Marcinkiewicz - Zygmond

$X_1, X_2, \ldots$ iid $\quad E(X_i) = 0 \quad E|X|^p < \infty \quad 1 \le p < 2$

then $\dfrac{S_n}{n^{1/p}} \xrightarrow[n\to\infty]{a.s.} 0.$

Proof $\quad Y_k = X_k \cdot \mathbb{1}_{\{|X_k| \le k^{1/p}\}}, \quad T_n = \sum_{k=1}^{n} Y_k.$

$\sum_{k=1}^{\infty} P(X_k \neq Y_k) = \sum_{k=1}^{\infty} P(|X|^p > k) \sim E|X|^p < \infty.$

By BCI. $\quad P(X_k \neq Y_k \text{ i.o.}) = 0$

Enough to show.

ETS $\quad \dfrac{T_n}{n^{1/p}} \xrightarrow[n\to\infty]{a.s.} 0$

Also: $\sum \operatorname{Var}\left(\dfrac{Y_k}{k^{1/p}}\right)$ $\quad$ use Lemma with good sequences