L13 - 09-26 Kolmogorov ext thm.

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Kolmogorov Extension theorem

KolmoGorov. 1930 - 1940
- Few Papers - Died 1986.
- very influencial

$\Omega = \{ \omega = (x_1, x_2, \ldots) \} : x_k \in \mathbb{R}, k = 1, 2, \ldots$

Define sequence of sigma Algebras.

Gives a Prob space when goes multidemision.

$$\mathcal{F}_1 = \{ \overset{x_1}{A} \times \overset{x_2}{\mathbb{R}} \overset{\ldots}{}$$

$A \in \mathbb{B}(\mathbb{R}^1)$

$$\mathcal{F}_1 = \{ \omega : x_1 \in A , A \in \mathbb{B}(\mathbb{R}) \}$$

$$\mathcal{F}_2 = \{ \omega : (x_1, x_2) \in A , A \in \mathbb{B}(\mathbb{R}^2) \}$$

$Ex : A = \{ x_1^2 + x_2^2 < 2 \}$

$$\mathcal{F}_n = A = \{ \omega : (x_1, x_2, \ldots , x_n) \in A , A \in \mathbb{B}(\mathbb{R}^n) \}$$

Another Example 2.

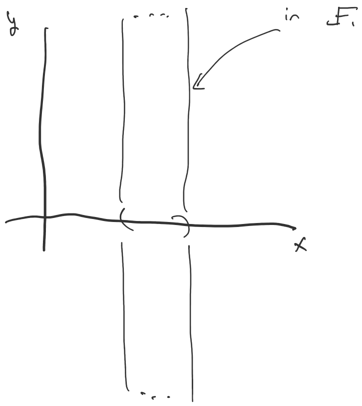
$$A = \{ \overset{\omega :}{} \max_{1 \le k \le n} \{ |\overset{\omega}{x_k}| \} < 2 \}$$

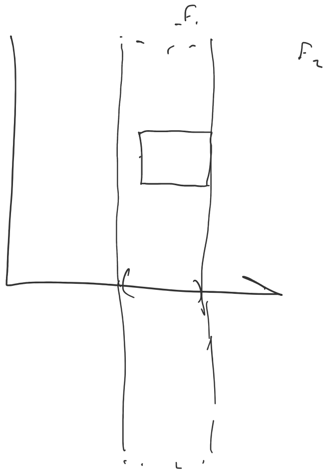
$n = 1, 2, \ldots$

$\mathcal{F}_n$ - σ algebra, $n \geq 1$

$\mathcal{F}_n \subset \mathcal{F}_{n+1}$

$\therefore \mathcal{F}_n \uparrow \bigcup_{n=1}^{\infty} \mathcal{F}_n$ an Algebra.





$A = \{ \omega : \sum_{k=1}^{\infty} x_k^2 \le 1 \}$ not in $\mathcal{F}_n$

$x_1^2 + x_2^2 \le 1$

Hyper sphereas.

$A \in \sigma \{ \bigcup_{n=1}^{\infty} \mathcal{F}_n \}$

$\mu(\Omega) = 1$

$\mu : \bigcup_{n=1}^{\infty} \mathcal{F}_n \longrightarrow \mathbb{R}^+$

μ is a p.m. on Each $\mathcal{F}_n$ $n \ge 1$

Kologoumerov

μ_n is competitaly.

we sum

μ over All $\mathcal{F}_n$

Goal Extend μ To a probablity measure on $\mathcal{F}$

we want to establish lebesgue measure on $\mathbb{R}$, we want to establish lebesgue measure on $\mathbb{R}$,

Carathéodory - countable members. | converted it instead.

and U. Disjoint -

were sigma Additivity.

They are in The Alg.

NTS : $B_n \in \mathcal{F}_n$, $B_n \downarrow \emptyset$ $\left(B_n \supset B_{n+1} \ n \geq \text{ and } \bigcap_{n=1}^{\infty} B_n = \emptyset \right)$

Then $\mu(B_n) \downarrow 0$ μ is a measure on alg.

By Caratheodory Result we can Extend μ to $\mathcal{F}$

Suppose not $\exists \delta > 0 : \mu(B_n) \geq \delta$, $\forall n \geq 1$

$\overset{\text{compact.}}{C_n} \subset B_n$, $\mu(B_n \setminus C_n) \leq \frac{\delta}{2^n}$, $B_n \in \mathcal{F}_n$

$\mu\left(B_n \setminus \bigcap_{k=1}^{n} C_k\right) \leq \frac{\delta}{2^n}$, $n \geq 1$ $\longrightarrow \longleftarrow$

$\Rightarrow \bigcap_{k=1}^{\infty} C_k \neq \emptyset \quad \subset \bigcap_{n=1}^{\infty} B_n = \emptyset$

what did that mean again?

if $D_k \subseteq \mathbb{R}$, D_k compact, $D_k \neq \emptyset$, $D_{k+1} \subset D_k$.

$a_k \in D_k$. $\exists \{a_{k_\ell}\}_{\ell \geq 1}$ st. $a_{k_\ell} \xrightarrow[\ell \to \infty]{} a \in \bigcap_{k=1}^{\infty} D_k$

need to take infinite subset of infinite sequence

diagonalization.

Example

Say 1, 2, 3 ....

subseq. $a_{k,n} = a_{k \cdot 2^n}$ $k = 1, 2, \ldots$, $n = 1, 2 \ldots$

$a_{k,1} = 2k.$, $a_{k,2.} = \overset{?}{a_{4k}}$

$\vdots$

$\therefore$ 2, 4, 6, 8 10

2	4	8	16
4	8	16	32
6	12	24	48
8	16	32	64
10	20	40	80

$\{a_{nn}\}_{n \geq m} \subset \{a_{k,m};\, k \geq 1\}. \forall m$

so ok Because Throw away Finite number, the subseq converge to same #,

a sub sequence of All $\overset{a_{k,1}}{\text{}}$ is $\{a_{n,n}\}$

- why can you construct a lebesgue measure,

Say on $[0, 1]$?

$X, X_1, X_2, \ldots$ iid, $P(X=0) = \frac{1}{2} = P(X=1)$

$Y = \sum_{k=1}^{\infty} \frac{X_k}{2^k}$ — The Distribution is Lebesgue

Eg. $\mu(0 \le Y < 1) = 1/k$

What is P.S. use Kolmov.

$\mu((0,1,1)) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 1/8$

By independence.

$X_k(\omega) = x_k$ $k=1,2,\ldots$

$\omega = \{x_1, x_2, \ldots)$

Dominate Convergence theorem.