01-29 HS0-1 Law

Wednesday, January 29, 2025 11:29 AM

Theorem Let $\{X_1, X_2, \ldots\}$ be iid, $E(X_1) = 0$, $E(X^2) = 1$

$S_n = \sum_1^n X_k \quad n \geq 1$ Define $T_c = \inf\{n \geq 1 ; S_n > c\sqrt{n}\}$, $c > 0$

then $E[T_c] = \begin{cases} \infty & \text{if } c \geq 1 \\ < \infty & \text{if } c < 1 \end{cases}$

Prove First: $P(T_c < \infty) = 1 \quad \forall c > 0$

Proof: using CLT: $\frac{S_n}{\sqrt{n}} \Rightarrow N(0,1)$

we get $P\left(\frac{|S_n|}{\sqrt{n}} > c\right) = \underbrace{P(|Z| > c)}_{Z}$ if n is Large
if $n > N(c)$

Claim: $P\left(\frac{|S_n|}{\sqrt{n}} > c, \text{ i.o}\right) > 0$

$$\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} \left\{\frac{S_k}{\sqrt{k}} > c\right\}}_{\text{Decreasing.}} = \left\{\frac{|S_n|}{\sqrt{n}} > c, \text{ i.o.}\right\}$$

$$P\left(\bigcup_{k=n}^{\infty} \frac{|S_k|}{\sqrt{k}}\right) \xrightarrow[n \to \infty]{} P\left(\frac{|S_n|}{\sqrt{n}} > c \text{ i.o.}\right)$$

$\quad \geq$ Bigger than

$$P\left(\frac{|S_n|}{\sqrt{n}} > c\right) > 0$$

Remember: Kolgomorov 0-1 Law.

$P\left(\frac{|S_n|}{\sqrt{n}} > c \text{ i.o.}\right) > 0$ is this a tail event?

Lets look At first coordinate X_1 Does it matter? no.

$X_1 \to 0$. $\quad P\left(\frac{|X_1|}{\cancel{\sqrt{1}}} > c \text{ i.o}\right) \to 0$ yes?

$$P\left(\overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{n}} > c\right) = 1$$

$\Rightarrow P(T_c < \infty) = 1$

Prove " ∞ if $c \geq 1$ "

By useing WALD 2nd Equation and get answer Immediately

Proof of $E(T_c) = \infty$, $c \geq 1$

Assue by contradiction $E(T_c) < \infty$

By Wald 2nd. Equation

$$E(S_T^2) = E(X_1^2)\, E(T_c) = E(T_c)$$

$$E(T_c) = E(S_T^2) > E(c^2 T_c) = c^2 E(T_c) \geq E(T_c)$$

$$E(T_c) > E(T_c) \quad \rightarrow\leftarrow \quad \therefore \quad E(T_c) < \infty$$

Prove other is Difficult. Move-on.

Hewitt - Savage 0-1 Law

Let $\{X_k\}_{k\geq 1}$ be iid Let A be a "permutable Event"

permutable Event — $\pi(A) = A$, $\forall\ \pi$ - finite permutation

says every A that is permutable then P(A) is 0 or 1

then $P(A) \in \{0, 1\}$

Example: ① A is a tail Event

Notes S_n Does not change if you permute

Notes First 100 Events can't change tail Event.

Example ② $\{S_n \in B \text{ i.o.}\}$.

Said something About tail Events Belonging or Not.

Applications of HS 0-1 Law.

Example From Book. Assume $X_1 \dots X_n$ iid.

Ex: there are 4 possibilities for $\{S_n\}_{n\geq 1}$ $\left(S_n = \sum_{n=1}^{\infty} X_n, \ \{X_i\} \text{ iid} \right)$

describes Random walk.

① $X_1 = 0$, a.s. $\Rightarrow$ $S_n = 0$ a.s. $\forall n \geq 1$

② $S_n \xrightarrow[n\to\infty]{a.s.} \infty$

③ $S_n \xrightarrow[n\to\infty]{a.s.} -\infty$

④ $-\infty \overset{a.s.}{=} \underline{\lim}_{n\to\infty} \{S_n\} < \overline{\lim}_{n\to\infty} \{S_n\} \overset{a.s.}{=} \infty$

no Example for (1), its just a.s.

Ex for (2): $E(X_1) > 0$ By SLLN $\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s.} E(X_1) > 0$

$S_n \xrightarrow[n\to\infty]{a.s.} \infty$

Ex for (3) $E(X_1) < 0$ $S_n \xrightarrow[n\to\infty]{a.s.} -\infty$

Ex for (4) (a) $X \overset{D}{=} -X$ $\Rightarrow$ $S_n \overset{D}{=} -S_n$

(b) if $E(X) = 0$, $E(X^2) < \infty$, $X \not\equiv 0$

then $P(S_n > \sqrt{n} \text{ i.o.}) > 0$

$\Rightarrow$ $P(S_n > \sqrt{n} \text{ i.o.}) = 1$

by Kolgomorv 0-1 or HS 0-1 since S_n is permutable.

What happens if I Drop $E(X^2) < \infty$. then variance would be infinite. See Problem 4.1.8 4.1.11.

Proof By H-S 0-1 Law.

$$\overline{\lim_{n\to\infty}}\ S_n = C \quad a.s. \qquad -\infty \le c \le \infty \qquad C \text{ is a constant.}$$

Exmple

by H-S. 0-1

$$P\{\overline{\lim_{n\to\infty}}\ S_n > 17\} \in \{0,1\}$$

say $\overline{\lim_{n\to\infty}}\ S_n$ is trivial.

Let $\{S_n'\} = \{S_{n+1} - X_1\} \left(= \sum_{k=2}^{n+1} X_k \right) \overset{D}{=} \{S_n\}_{n\ge1}$

$$\limsup_{n\ge1} \{S_{n+1}\} = \limsup_{n\to\infty} \{S_n\}$$

$C - X_1 = C \Rightarrow$ if $|C|$ is finite, then $X_1 = 0$

$\liminf_{n\to\infty} S_n = \partial$ $\quad \partial$ is constant, $\partial \le c$

if $X_1 \not\equiv 0$ then $c \in \{-\infty, \infty\}$

$\partial \in \{-\infty, \infty\}$

Proof of H-S 0-1

Basics $\quad P(A \Delta C) \le P(A \Delta B) + P(B \Delta C) \quad \forall A, B, C$ events

metric which means

1) symmetric

2) triangle ineq. ✓

$$A \Delta C \subseteq \{A \Delta B\} \cup \{B \Delta C\}$$

Symmetric Different

$A \Delta C := (A \cap C^c) \cup (C \cap A^c)$

② if $P(A_n \Delta A) \xrightarrow[n\to\infty]{} 0$ then (i) $P(A_n) \to P(A)$

(ii) $P(A_n \cap A) \longrightarrow P(A)$

③ if $P(A_n \Delta A) \to 0$ and $P(B_n \Delta A) \to 0$

then $P(A_n \cap B_n) \xrightarrow[n\to\infty]{} P(A)$

then $P(A_n \cap B_n) \xrightarrow[n\to\infty]{} P(A)$