Markov Property: start At 0

$\mathcal{F}_t^0 = \sigma\{B_s ; s \le t\}, \ t \ge 0$, $\Omega = C[0,\infty) = \{\omega : [0,\infty) \to \mathbb{R}$, ω is continuous function$\}$

$$E^x_{\mathcal{F}_t^0}(Y \circ \Theta_t) = E_{B_t^x}(Y) = \varphi(B_t^x)$$

shift operator.

we calculate using

$$E^x(Y \circ \Theta_t) = E^x[E_{B_t^x}(Y)]$$

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Good idea

$\mathcal{F}_t^0$ $\mathcal{F}_t^+ = \bigcap_{n=1}^{\infty} \mathcal{F}^0_{t+\frac{1}{n}}$, $\mathcal{F}_t^0 \subset \mathcal{F}^0_{t+\frac{1}{n}}$, $\forall n$

$\bigcup \mathcal{F}_t^0$??? ...

$\{\mathcal{F}_t^+\}_{t \ge 0}$ is Right Continuous. At each $t > 0$

$$(\mathcal{F}_t^+)^+ = \bigcap_{n=1}^{\infty} \mathcal{F}^+_{t-\frac{1}{n}}$$

says they Are the SAME.

we have to Prove $\mathcal{F}_t^0 = \tilde{\mathcal{F}}_t^+$ up to $\underbrace{\text{null sets}}_{P(A) \in \{0,1\}}$. w.r.t P_x $\forall x \in \mathbb{R}$

in particular $\mathcal{F}_0^0 = \mathcal{F}_0^+$ Blumenthal 0-1 law. ← hardest Book to Read.

trivial σ-algebra

$P^x(B(0) = x) = 1$

$y \ne x$ $P^y(B(0) = x) = 0$

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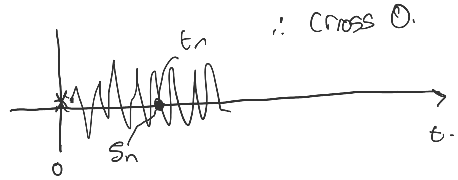
theorem. Let $\{B_t\}_{t \ge 0}$ be SBM.

$T = \inf\{t > 0 : B_t > 0\}$



claim: $P_0(T = 0) = 1$

$t_n \downarrow 0$ with $B_{t_n} > 0$, $n \ge 1$ $t_n \downarrow 0$

$t_n \downarrow 0$ with $B_{t_n} > 0$, $n \geq 1$

$t_n \downarrow 0$
$B_{t_n} > 0$, $n > 1$

$S = \inf\{T > 0 : B_t < 0\}$

$s_n \downarrow 0$.
$B_{t_n} < 0$, $n \geq 1$

claim: $P_0(S = 0) = 1$

$\exists\, m < n$ $\qquad w_n$ a seq Between B_{t_n} & B_{s_n}.

$P_0(\exists\, w_n \downarrow 0 : B_{w_n} = 0) =$

2

theorem: $\mathcal{F}_t^0 = \mathcal{F}_t^+$ up to null sets w.r.t. P_x $\forall x \in \mathbb{R}$.

1

$E^x_{\mathcal{F}_t^+}(Y) = E^x_{\mathcal{F}_t^0}(Y)$ as where Y bdd r.v. on Ω.

$= 1_A$ But not in $\mathcal{F}_t^0$

Proof: $Y = X \cdot (Z \circ \theta_t)$, $X \in \mathcal{F}_t^0$

$$E^x_{\mathcal{F}_t^+}(X \cdot (Z \circ \theta_t)) = X\, E^x_{\mathcal{F}_t^+}(Z \circ \theta_t)$$
$$= E^x_{\mathcal{F}_t^0}(Z \circ \theta_t)$$
$$= E_{\mathcal{F}_t^0}(X \cdot Z \circ \theta_t)$$

why $E^x_{\mathcal{F}_t^+}(Y) = E^x_{\mathcal{F}_t^0}(Y)$ Assume $Y = 1_A$ where $0 < P(A) < 1$ $A \in \mathcal{F}_t^+$, $A \notin \mathcal{F}_t^0$

monotone class theorem works for 1_A - then works for $\forall Y$

Ets (Enough to show) $\quad E^x_{\mathcal{F}_t^+}(Y \circ \theta_t) = E_{B_t}(Y) = E^Y_{\mathcal{F}_t^0}(Y \circ \theta_t)$

we will Do for specific Y.

$Y = \prod_{k=1} f_k(B_{h_k})$ $\quad f_k : \mathbb{R} \to \mathbb{R}$, f_n bdd, continuous functions $k = 1, \dots, d$.

1st step: $F^t(Y)$ is continuous in u

$k = 1, \ldots, d$

1st step: $E^y(Y)$ is continuous in y.

2

trick

continuous.

Step 2 in Blue.

$$E^{y_n}(Y) = E^0(\cancel{y+Y}) = E^0 \prod_{k=1}^{d} f_k(y_n + B_{n_k}) \xrightarrow[n\to\infty]{} E^{y_\infty}(Y)$$

Fit. $E^0[Y(y_n + \omega)]$

$f_k(y_n + B_{n_k})$

$y_n \xrightarrow[n\to\infty]{} y_\infty$

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By Markov Prop.

Step cont. Brownian.

$B^x_{t_n} \xrightarrow[n\to\infty]{} B^x_t$

$$E^x_{\mathcal{F}^0_{t_n}}(Y \circ \theta_{t_n}) = E_{B^x_{t_n}}(Y)$$

$t_n \downarrow t$ $\quad t_n > t$

By Cond'l. D.C.T. $\quad n\to\infty$ $\quad = E_{B^x_t}(Y)$

$$E_{\mathcal{F}^+_t}(Y \circ \theta_t)$$

$Y \circ \theta_n \xrightarrow{a.s} Y \circ \theta_t$

$Y \circ \theta_n$ is bdd

$\mathcal{F}_{t^0_n} \downarrow \mathcal{F}_{t^+_n}$

1

Theorem Let $\{B_t\}_{t\geq 0}$ be SBM

then $B(t) = \begin{cases} t \cdot B(\frac{1}{t}) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$

$\{\tilde{B}$

$E(t \cdot B(\frac{1}{T})) = 0.$

$$\mathrm{Cov}\left[t\cdot B\left(\tfrac{1}{t}\right),\ sB\left(\tfrac{1}{s}\right)\right] = ts\cdot\left(\tfrac{1}{t}\wedge\tfrac{1}{s}\right)$$

$s>0,\ t>0$ $$= t\wedge s$$

$\{t\cdot B(\tfrac{1}{t})\}_{t>0}$ has cont. sample.

NTS. $\lim_{t\to 0}\tilde{B}(t) = 0$ a.s.

if $\{B_t\}_{t\geq 0}$ is same as $\{t\cdot B(\tfrac{1}{t})\}_{t>0}$

then $=0$ $\qquad \approx 0$

theorem if A is a tail event of Bm $\{B_t^x\}_{t>0}$. way to Denote non-standard Brownian motion

then $P(A)\in\{0,1\}$. Also $P^0(A) = P^x(A)$, $\forall x$

work w/ SBM.

first step, let $\{B_t\}$ be SBM.

then $P(A)\in\{0,1\}$

$A\in\mathcal{F}_0^+$

null event

$A = \sigma\{B_{t_1},\ t\geq 1\}$

$B\circ\Theta_1 = A$

$B = A\circ\Theta_{-1}$

$A = \{B_2 > 1/2\}$ $\Rightarrow$ $B = \{B_1 > 1/2\}$.

we can use formulas for Markov Properties.

$$P_0(A) = E^0(1_B\circ\Theta_1) = E_0[E_{B_1^0}[1_B]]$$

$$= \int_{y=-\infty}^{\infty}(2\pi)^{-1/2}e^{\frac{(y-0)^2}{2}}P_y(B)\,dy = 0$$

$$\Rightarrow P_y(B) = 0.$$

next time strong markov property.
section 3.