L12 - 09-23 up, fubini

Monday, September 23, 2024   11:30 AM

① Def: $\{X_n\}_{n\geq 1}$ UI if $\Psi(M) \xrightarrow[M\to\infty]{} 0$, $\Psi(M) \equiv \sup_{n\geq 1} \mathbb{E}\left[|X_n| \cdot \mathbb{1}_{\{|X_n| > M\}}\right]$

② Alternative: $\{X_n\}_{n\geq 1}$ UI iff (i) $\sup_n \mathbb{E}|X_n| \leq \infty$

(ii) $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. if $P(A) < \delta$
then $\sup_{n\geq 1} \mathbb{E}(|X_n| \cdot \mathbb{1}_A) < \varepsilon$

③ Application: $\{X_n\}_{n\geq 1}$ $\{Y_m\}_{m\geq 1}$ are both UI, then so is $\{X_n + Y_m\}_{\substack{n = 1,2,\dots \\ m = 1,2,\dots}}$

④ $\mathbb{E}|X_n| < \infty$, $n = 1,2,\dots$ $\mathbb{E}|X| < \infty$, and $\mathbb{E}|X_n - X| \to 0$ then $\{X_n\}_{n\geq 1}$ in UI

Step 1: $X = 0$, $\mathbb{E}|X_n| \xrightarrow[n\to\infty]{} 0 \Rightarrow \{X_n\}_{n\geq 1}$ UI

Step 2: $\{X_n - X\}_{n\geq 1}$ is UI

$\{X\}$ is UI

$\Rightarrow \{X_n\}$ UI   by ③

⑤ $X, \{X_n\}_{n\geq 1}$ so that $\{X_n\}_{n\geq 1}$ UI$_n$ and $X_n \xrightarrow[n\to\infty]{a.s} X$

then $\mathbb{E}|X_n - X| \xrightarrow[n\to\infty]{} 0$ implies $\mathbb{E}(X_n) \xrightarrow[n\to\infty]{} \mathbb{E}(X)$

Step 1: $\mathbb{E}|X| < \infty$ is finite.
is integrable.

Fatou Lemma

$\underset{n\geq 1}{\sup \mathbb{E}|X_n|}$ $\underset{n\to\infty}{\underline{\lim}} \mathbb{E}|X_n| \geq \mathbb{E}\left[\underset{n\to\infty}{\lim}|X_n|\right] = \mathbb{E}|X|$

$\overset{\wedge}{\underset{\infty}{}}$

we reduce the problem: $\{X_n\}_{n\geq 1}$ UI and $X_n \xrightarrow{a.s.} 0$

then $\mathbb{E}|X_n| \xrightarrow[n\to\infty]{} 0$

Proof $\overset{\underset{\overline{\lim}}{n\to\infty}}{}$ $\overset{\overline{\lim}}{\underset{n\to\infty}{}}$

$\underset{n\to\infty}{\overline{\lim}} \mathbb{E}|X_n| \leq \mathbb{E}\left[|X_n| \cdot \mathbb{1}_{\{|X| > M\}}\right] + \mathbb{E}\left[|X_n| \cdot \mathbb{1}_{\{|X_n| \leq M\}}\right]$ $\Big)$ By.

$\leq \Psi(M)$ Bound converges theorem.

where M is Arbitrary.

conclusion $0 = \underline{\lim} \mathbb{E}|X_n| \leq \overline{\lim}_{n\to\infty} \mathbb{E}|X_n| = 0$

$\Rightarrow \mathbb{E}(|X_n|) \xrightarrow[n\to\infty]{} 0$

Fubini theorem:
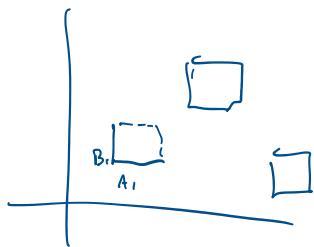
Product measures and Fubini theorem.

Let (Ω_1, F_1, M_1), (Ω_2, F_2, M_2) be σ-finite measure spaces

$\Omega = \Omega_1 \times \Omega_2 = \{w = (w_1, w_2) : w_1 \in \Omega_1, w_2 \in \Omega_2\}$.

$$\mathcal{L} = \left\{\bigcup_{i=1}^{n} A_i \times B_i : \text{union is Disjoint}, A_i \in F_1, B_i \in F_2, i = 1, \ldots, n, n \geq 1\right\}$$
$(A_i \times B_i) \cap (A_j \times B_j) = \phi$ if $i \neq j$

$\mathcal{L}$ is Algebra. 1) closed under finite union. 2) Complement

Theorem: there exists UNIQUE measure $F = \sigma(\mathcal{L})$ so that

$$M(A \times B) = M_1(A) M_2(B) \quad \forall A \in \Omega_1, B \in \Omega_2$$

Step 1: show M is a measure on $\mathcal{L}$ $[D_1, D_2, \ldots,$ with $D_n \in \mathcal{L}$ $n \geq 1$, $D_i \cap D_j \neq \phi$ $\forall i \neq j$

one Reet split in many   and $\bigcup_{n=1}^{\infty} D_n \in \mathcal{L}$ then $M\left(\bigcup_{n=1}^{\infty} D_n\right) = \sum_{n=1}^{\infty} M(D_n)$

Case: $A \times B = \bigcup_{n=1}^{\infty} A_n \times B_n$, $\{A_n \times B_n\}_{n \geq 1}$ Disjoint.

WTS: $M(A \times B) = \sum_{n=1}^{\infty} M(A_n \times B_n) = \sum_{n=1}^{\infty} M_1(A_n) M_2(B_n)$

$\mathbb{1}_A^{(w_1)} \mathbb{1}_B^{(w_2)} = \mathbb{1}_{A \times B}^{(w_1, w_2)} = \sum_{n=1}^{\infty} \mathbb{1}_{A_n}^{(w_1)} \mathbb{1}_{B_n}^{(w_2)}$ Integrate w/ Respect to M_2.

$= \mathbb{1}_A^{(w_1)} M_2 \quad = \sum_{n \leq 1}^{\infty} \mathbb{1}_A^{(w_1)} M_2(B_n)$ nonnegative $\therefore$ MCT   integrate again + MCT.

$= M_1(A) M_2(B) = \sum_{n \leq 1}^{\infty} M_1(A_n) M_2(B_n)$ ✓

Fobini

                                        2 variables   Measure by corollary.

Let $f \geq 0$ on Ω or $\int_\Omega |f| \, dM < \infty$

then $\int_{\Omega_1} \int_{\Omega_2} \left[f(x,y) \, dM_2(y) \right] dM_1(x) = \int_\Omega f \, dM = \cdots = \int_{\Omega_2} \int_\Omega \left[f(x,y) \, dM_1(x) \right] dM_2(y)$

Can't Integrate unless this is measurable, on (Ω_1, F_1)

Example: $\Omega_1 = (0, 1)$, $\Omega_2 = (1, \infty)$

M_1, M_2 are lebesgue measure. M.

$$f(x,y) = e^{-xy} - 2e^{-2xy}, \quad 0 < x < 1, \quad 1 < y < \infty$$

$\int_0^1 \int_1^\infty f(x,y) \, dy \, dx > 0$

$\int_1^\infty \int_0^1 f(x,y) \, dx \, dy < 0$ Fobini. Fails Because $\int_\Omega |f| \, dM$

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$\Omega_2 = \Omega_1 = \mathbb{N} = \{1, 2, \ldots\}$
$F_2 = F_1 = 2^{\mathbb{N}} =$
$M_2 = M_1 = $ counting measure (still σ-finite.)

$M_1(A) = \bar{\bar{A}}$, $M(\{n\}) = 1$ for $n \in \mathbb{N}$

$\mu_2 = \mu_1 =$ counting measure (still σ finite.)

$\mu_1(A) = \bar{\bar{A}}$, $\mu(\{n\}) = 1$ for $n \in \mathbb{N}$

$\Omega = \{(n,m) ; n,m \in \mathbb{N}\}$

$f((n,m)) = a_{n,m}$ $n,m = 1,2,\dots$

$a_{i,i} = 1$, $i \geq 0$

$a_{i+1,i} = -1$

$0 \neq 1$