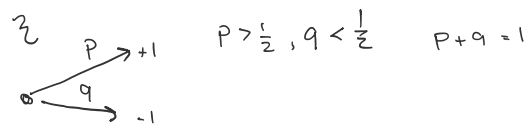


Example



$$\frac{S_n}{n} \xrightarrow[n \rightarrow \infty]{a.s.} p - q.$$

$$S_n = \sum_{k=1}^n Z_k, \quad n \geq 1, \quad S_0 = 0$$

$$\mathcal{F}_n = \sigma\{Z_1, \dots, Z_n\}$$

we use $\left(\psi(S_n) = \left(\frac{q}{p}\right)^{S_n}, \mathcal{F}_n\right)$ MG.

 $\{Z_n\}$

$$T = T_a \wedge T_b, \quad a, b \in \mathbb{Z}$$

$$\{\psi(S_{T \wedge n})\}_{n \geq 0} \text{ is Bounded.}$$

$$\boxed{T_b < \infty \text{ a.s.}}$$

From last time we got,

$$P(T_a < \infty) = \left(\frac{p}{q}\right)^a \text{ or } \left(\frac{q}{p}\right)^{-a} < 1, \quad P(T_a = \infty) > 0 \Rightarrow E(T_a) = \infty$$

$$P(T_b < \infty) = 1$$

we want to calculate $E(T_b)$

$$E(Z - (p - q)) = 0$$

$$\{S_n - n(p - q), \mathcal{F}_n\}_{n \geq 0} \text{ MG}$$

Since $E_{\mathcal{F}_n}(Z_{n+1} - (p - q)) \stackrel{\text{is}}{=} 0$

MG diff.

$$\Rightarrow E(S_{T_b \wedge n} - (p - q)T_b \wedge n) = 0$$

Not in the subscript

 \uparrow less than n

$$\Rightarrow E(S_{T_b \wedge n}) = (p - q)E(T_b \wedge n)$$

$$T_b \wedge n \uparrow T_b \text{ as } n \rightarrow \infty.$$

$$E(T_b \wedge n) \uparrow E(T_b) \quad \text{By MCT}$$

$$\begin{aligned} \text{then } E(S_{T_b \wedge n}) &= (p-q)E(T_b \wedge n) \\ b = E(S_{T_b}) &= (p-q)E(T_b) \end{aligned}$$

we want \uparrow

$$\Rightarrow b = (p-q)E(T_b)$$

$$E(T_b) = \frac{b}{p-q} < \infty$$

to prove we need $\lim E = E \lim$ either DCT or LI

why $\{S_{T_b \wedge n}\}_{n \geq 0}$ LI?

we know.

$$b \geq S_{T_b \wedge n} > \min_{n \geq 0} \{S_n\} \xrightarrow{a.s.} -\infty$$

$$\min_{n \geq 0} \{S_n\} \leq 0$$

$$\text{Claim } E(\min_{n \geq 0} \{S_n\}) > -\infty$$

says $(\frac{p}{q})^a < 1$ is going to save us.

$$Y \in \mathbb{Z}^- \Rightarrow E(Y) = -E(-Y) = -\sum_{k=1}^{\infty} P(-Y \geq k) \stackrel{\text{summation by parts.}}{=} -\sum_{k=1}^{\infty} P(Y \leq -k)$$

$$\{T_{-k} < \infty\} \quad T_{-k} \text{ first time RW hits } -k.$$

$$\text{related to } \{\min_n \{S_n\} \leq -k\} = \{T_{-k} < \infty\} \quad k=1,2,\dots$$

$$E(\min_{n \geq 0} \{S_n\}) = -\sum P(T_{-k} < \infty) = -\sum_{k=1}^{\infty} \left(\frac{q}{p}\right)^k > -\infty$$

since $0 < \frac{q}{p} < 1$
Geometric series.

$$\Rightarrow \{S_{T_b \wedge n}\}_{n \geq 0} \text{ LI}$$

\downarrow a.s. L_1

$$S_{T_b}$$

$$\text{then } E(T_b) = \frac{b}{p-q} < \infty$$

$$L(1,0) = p - q$$

positive supermartingale

Let $\{X_n, \mathcal{F}_n\}$ be positive SUP MG.

Let τ be a S.T.

$$\text{WTS } E(X_0) \geq E(X_\tau) \geq E(X_\infty)$$

condition MGCT
 $\sup_n \{X_n\} < \infty$

$$X_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} X_\infty$$

$$E|X_\infty| < \infty$$

$$\underline{Z} \equiv \{Z_k\}_{k \geq 1} \stackrel{w}{\sim} N(0,1)$$

$$X_n = e^{S_n - \frac{n}{2}}, \quad n \geq 0$$

$$S_n = \sum_{k=1}^n Z_k, \quad n \geq 1, \quad S_0 = 0$$

$$E(e^Z) = e^{1/2} \Rightarrow e^{Z - 1/2} = 1$$

$$E(e^{\frac{t^2}{2}}) = e^{\frac{t^2}{2}}, \quad t \in \mathbb{R}.$$

$$X_n = e^{\sum_{k=1}^n (Z_k - 1/2)} = \prod_{k=1}^n e^{Z_k - 1/2}$$

$$E(e^{S_{n+1} - \frac{n+1}{2}}) = e^{S_n - \frac{n}{2}}.$$

$$e^{S_n - \frac{n}{2}} E_{\mathcal{F}_n}(e^{Z_{n+1} - 1/2}) = e^{S_n - \frac{n}{2}} \quad \text{Example of MG.}$$

$\mathbb{E}ND: Z_{n+1} \perp \mathcal{F}_n$

the problem

$$\frac{\sum_{k=1}^n Z_k}{n} \xrightarrow{\text{a.s.}} 0$$

$$e^{S_n - \frac{n}{2}} = e^{n(\frac{S_n}{n} - 1/2)} \xrightarrow{\text{a.s.}} 0$$

$$e^{-\infty} \rightarrow 0.$$

a.s. $\exists N$

$$S_n < \frac{1}{n}, \quad n \geq N(\omega)$$

$$a_n \geq N$$

$$\frac{S_n}{n} < \frac{1}{4}, \quad n \geq N(\omega)$$

$$\frac{S_n}{n} - \frac{1}{2} < -\frac{1}{4}, \quad n \geq N(\omega)$$

$$E(e^{S_n - n/2}) = 1 \quad n \geq 1$$

it should go to 0, The MG. goes to 0.
therefore
not LI

Prove.

$$E(X_0) \geq E(X_T)$$

we have

$$E(X_0) \geq E(X_{T \wedge n})$$

given

$$X_{T \wedge n} \xrightarrow[n \rightarrow \infty]{a.s.} X_T$$

Because of Fatou Lemma

$$E(X_0) \geq \lim_{n \rightarrow \infty} E(X_{T \wedge n})$$

Fatou

$$\geq E(\lim_{n \rightarrow \infty} X_{T \wedge n}) = E(X_T)$$

used Gambling System enter at 0. end at T+1

$$(H \cdot X)_n$$

Profit if we start After this.

$$Y_n = \begin{cases} X_n - X_T & n > T \\ 0 & n \leq T \end{cases}$$

$$Y_n \geq -X_T$$

$$E|X_T| = E X_T < \infty$$

Fatou Condition \rightarrow Require Positive
can extend. if $Y_n \geq W$, $E|W| < \infty$.
then Fatou holds. for $\{Y_n\}_{n \geq 1}$

we can use Fatou $Y_n - W$ which is positive then
through at W .

\therefore we get X_∞ .

\therefore we get X_{∞} .

Can be Automatically extend to Two S.T.

$$E(X_s) \geq X_T \text{ a.s.}$$

\mathcal{F}_T