L17 - 10-04 WLLN, Bernstein

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Weak Law of Large Numbers, WLLN — chapter 2.2 Durrett.

talks About converges in Probability.
a.e. → Prob
more Applicatable than SLLN.

WLLN
Def $X_n \xrightarrow[n\to\infty]{p} 0$ if $\forall \varepsilon > 0$ $P(|X_n| > \varepsilon) \xrightarrow[n\to\infty]{} 0$

SLLN
Def $X_n \xrightarrow[n\to\infty]{a.s.} 0$ if $P(X_n \xrightarrow[n\to\infty]{} 0) = 1$

SLLN → WLLN $P(\omega \in \Omega : X_n(\omega) \xrightarrow[n\to\infty]{} 0) = 1$

✱ SLLN implies WLLN

Result: from Khinchin
$\{Y_n\}_{n\geq 1}$ be R.V. if $\frac{V(Y_n)}{b_n^2} \xrightarrow[n\to\infty]{} 0$

then $\mathbb{E}\left[\left(\frac{Y_n - \mathbb{E}(Y_n)}{b_n}\right)^2\right] \xrightarrow[n=\infty]{} 0$ $\left(\frac{Y_n - E(Y_n)}{b_n} \xrightarrow[n\to\infty]{L^2} 0\right)$

$\Rightarrow \frac{Y_n - \mathbb{E}(Y_n)}{b_n} \xrightarrow[n\to\infty]{p} 0$

"Converge L^2 implies Converge P"

Assume $X_n \xrightarrow[n\to\infty]{L_2} 0$ then $P(|X_n| > \varepsilon) \leq \frac{E[X_n^2]}{\varepsilon^2}$ Erased.

chebychev.

Khinchin $\left\{\begin{array}{l}\text{Result: Let } \{Y_n\}_{n\geq 1} \text{ be R.V. if } \frac{V(Y_n)}{b_n^2} \xrightarrow[n\to\infty]{} 0 \\ \text{then } \frac{Y_n - E(Y_n)}{b_n} \xrightarrow[n\to\infty]{P} 0\end{array}\right.$

Khinchin

Let $\{Y_n\}_{n\geq 1}$ be seq. of R.V.

if $\frac{Var(Y_n)}{b_n^2} \xrightarrow[n\to\infty]{pw.} 0$ for scaling seq b_n

then $\frac{Y_n - E(Y_n)}{b_n} \xrightarrow[n\to\infty]{P} 0$

Ex 1 coupon Problem (Example 2.2.3 Durrett)

Let $X, X_1, X_2, \ldots$ be iid, $X \sim \text{uniform}(1, 2, \ldots, n)$ $(P(X=k) = \frac{1}{n},\ k = 1, \ldots, n)$

Let $\tau_k^{(n)} =$ minimal # of observations to get k different coupons

$\tau_1^{(n)} = 1$, $X_{n,k} = \tau_k^{(n)} - \tau_{k-1}^{(n)}$ $\tau_0^{(n)} = 0$.

$\{X_{n,k}\}_{k=1}^{n}$ $\quad X_{n,k+1} \sim \text{Geometric}\left(p = \frac{n-k}{n}\right)$ Implies.

$T_n = \tau_n^{(n)} = \sum_{k=1}^{n} X_{n,k}$ $\quad E(X) = \frac{1}{p},\ V(X) \frac{q}{p^2},\ q = 1-p$

$\mathbb{E}(T_n) = \sum_{k=1}^{n} \mathbb{E}(X_{n,k}) = n\sum_{k=1}^{n}\frac{1}{n-k} = n\sum_{k=1}^{n}\frac{1}{k} \sim n\log_2(n)$, when n is Large.

$V(T_n) = \sum_{k=1}^{n} V(X_{n,k}) \leq n^2 \sum_{k=1}^{n}\frac{1}{(n-k)^2} = n^2\sum_{k=1}^{n}\frac{1}{k^2} \leq Cn^2$

for $\frac{q}{p^2} \leq \frac{1}{p} \cdot n$

$q^2 =$

$\frac{1}{p^2} - \frac{p}{p^2}$ $\quad \frac{q-p}{q^2} = x$

$\frac{1}{q^2}$

$\frac{T_n - E(T_n)}{n\log(n)} \xrightarrow[n\to\infty]{L^2} 0$ which leads to what,

Div. to make work.

Appears to Let $b_n = n\log(n)$.

$\frac{T_n}{n\log n} \xrightarrow[n\to\infty]{P} 1$

$\frac{V(Y_n)}{b_n^2} = \frac{V(T_n)\ \cancel{E(T_n))}}{n^2\log_2^2(n)} \leq \frac{C\cancel{n^2}}{\cancel{n^2}\log^2(n)}$ $\quad \frac{C}{\log^2(n)} \xrightarrow[n\to\infty]{} 0$

Box Ball Example (Example 2.2.2 Durrett)

Application: Let $f : [0,1] \to \mathbb{R}$ be continuous. function.

Problem: Prove that $\forall \varepsilon > 0$ there is a polynomial

$P_n(x) = \sum_{k=0}^{n} a_k x^k$ $\quad 0 \leq x \leq 1$, $a_k \in \mathbb{R}$ so that

$\max_{0\leq x\leq 1} |f(x) - f_n(x)| < \varepsilon$ $\quad$ Bernstein Polynomial. 1920's.

Berstein Polynomial.
Example 2.2.1

Solution, Berenstein's Polynomial

$0 \leq x \leq 1$, $Y_{x,n} \sim Bin(n, x)$

$P_n(x) = \mathbb{E}\left(f\left(\frac{Y_{x,n}}{n}\right)\right) = \sum f\left(\frac{k}{n}\right)\cdot\binom{n}{k} x^k (1-x)^{n-k}$ Polynomial

$\to [0,1]$ $\quad$ Rank of n. math.

Claim

$\max_{0\leq x\leq 1} |f_n(x) - P_n(x)| \xrightarrow[n\to\infty]{} 0$

$\mathbb{E}\left|f\left(\frac{Y_{x,n}}{n}\right) - f(x)\right|$ $\quad$ split into two

Use Prob.

$= \underbrace{\mathbb{E}\left|f\left(\frac{Y_{x,n}}{n}\right) - f(x)\right| \cdot \mathbb{1}\left\{\left|\frac{Y_{x,n}}{n} - x\right| \leq \delta\right\}}_{\text{less than } \varepsilon.\ \text{therefore Bound.}} + \mathbb{E}\left| \quad \right| \mathbb{1}\left\{\left|\frac{Y_{x,n}}{n} - x\right| \geq \delta\right\}$

$\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $|x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon$, $x, y \in [0,1]$

"uniform continuity gives δ"

f is bdd by $\|f\|_\infty < \infty$

Split Expectation trick.

$E[|f(x) - f(y)|] = \mathbb{E}[|f(x) - f(y)|\,\mathbb{1}_{(x-y<\delta)}]$
$+ \mathbb{E}[|f(x) - f(y)|\,\mathbb{1}_{(x-y\geq\delta)}]$

Uniform Continuity.

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.
$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

"uniform continuity gives δ"

f is bdd by $\|f\|_\infty < \infty$

$\max_{0 \le x \le 1} |f| = \|f\|_\infty$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

$+ 2 \cdot \|f\|_\infty P\left(\left|\frac{Y_{x,n}}{n} - x\right| \geq \delta\right)$

$\leq \underset{\text{for b}}{2\|f\|_\infty} \frac{V(Y_{x,n}/n)}{\delta^2}$

$= 2\|b\| \cdot \frac{n x (1-x)}{n^2 \cdot \delta^2} \xrightarrow[n\to\infty]{} 0$

$\limsup_{n\to\infty} E|f(Y_{x,n}/n) - f(x)| \leq \varepsilon$ since Epsilon is Arbitrary

$\xrightarrow[n\to\infty]{} 0$

$\limsup_{n\to\infty} \sup_{0 \le x \le 1}$