L09 - 09-16

Monday, September 16, 2024 11:30 AM

Last Time

Bounded Convergence theorem. (in space $(\Omega, \mathcal{F}, \mu)$)

$\mu(\Omega) < \infty$ $f_n : \Omega \to \mathbb{R}$. measureable $n = 1, 2 \dots$

and $|f_n| \leq M < \infty$, $\forall n$

if $f_n \xrightarrow[n\to\infty]{\text{measure}} f$ Then $I(f_n) \xrightarrow[n\to\infty]{\mu} I(f)$

Proof

WLOG $f = 0$ Assume $f = 0$ (otherwise look at $f_n \leftrightarrow f_n - f$

Let $\varepsilon > 0$, $|I(f_n)| \leq I(|f_n|) = I(|f_n| \cdot 1_{G_n}) + I(|f_n| \cdot 1_{G_n^c})$

Define set $G_n = \{x \in \Omega, |f_n| \cancel{>} \leq \varepsilon\}$. know $\mu(G_n^c) \xrightarrow[n\to\infty]{} 0$

meaning of converging ~~measure~~.

Denote $\overline{\text{Lim}}$ $\leq \varepsilon \mu(\Omega) + M \cdot \mu(G_n^c)$

$\text{Lim sup } |I(f_n)| \leq \varepsilon \mu(\Omega) + M \overline{\text{Lim}}\, \mu(G_n^c) \to 0$

Since ε is Arbitrary.

$\overline{\text{Lim}}\, |I(f_n)| \xrightarrow[n\to 0]{} 0 \Rightarrow I(f_n) \xrightarrow[n\to 0]{} 0$

Last Time

Given $\mu(\Omega) < \infty$. If $f_n \xrightarrow[n\to\infty]{a.e.} f$ then $f_n \xrightarrow[n\to\infty]{\mu} f$.

So $f_n \xrightarrow{a.} f$ and $|f_n| \leq M$ then $I(f_n) \to I(f)$

Recall: $f_n \xrightarrow[n\to\infty]{a.e} f$ means that $\mu\{x \in \Omega : f_n(x) \not\xrightarrow[n\to\infty]{} f(x)\} = 0$

Fatou's Lemma

μ is sigma finite.

if $f_n \geq 0$ then $\underset{n\to\infty}{\underline{\text{Lim}}}\, I(f_n) \geq I(\underset{n\to\infty}{\underline{\text{Lim}}}\, f_n)$ (liminf)

When Def. $I(f)$: $f \geq 0$ ...

First figure out liminf.

sequence of $\mathbb{R}$: $a_1, a_2, \dots$

$\underline{\text{lim}}\, a_n$ of all the sequences what is the GLB.

$\underline{\lim}_{n\to\infty} a_n$ of all the sequences what is the GLB.

$\lim_{n\to\infty}\left[\inf_{k\geq n}\{a_k\}\right]$ Eg $a_n, a_{n+1}, a_{n+2}, \dots$

$a_{n+1}, a_{n+2}, \dots$ which has a bigger inf.

By def of inf.

$a_n \geq A_n = \inf_{k\geq n}\{a_k\}$, we get $A_n \uparrow A$ as $n\to\infty$

By Def of inf.

$g_n(x) \equiv \inf_{m\geq n} f_n(x)$, $f_n(x) \geq g_n(x)$, $x\in\Omega$

$g_n(x) \uparrow g(x) \equiv \underline{\lim}_{n\to\infty} f_n(x)$

since $I(f_n) \geq I(G_n)$ By Prop.   Implicitly.

Enough to show E.T.S.

then $\lim_{n\to\infty} I(g_n) \geq I(g)$

because $\underline{\lim}\, I(f_n) \geq \underline{\lim}\, I(g_n) = \lim_{n\to\infty} I(g_n) \geq I(g) = I(\underline{\lim}\, f_n)$

Now show $\lim_{n\to\infty} I(g_n) \geq I(g)$.

trivially $\lim_{n\to\infty} I(g_n) \leq I(g)$.

Let $E_n \uparrow \Omega$ with $\mu(E_n) < \infty$. ($\{E_n\}_{n\geq 1}$ exists as μ-σ-finite)

Fix $m > 0$ $\in \mathbb{N}$ cut above by m.

$$(g_n \wedge m)\mathbb{1}_{E_m} \xrightarrow[n\to\infty]{a.e.} (g\wedge m)\mathbb{1}_{E_m}$$

use Bound convergence theorem

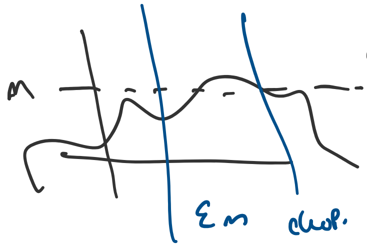
use bdd converge theorem

$\lim_{n\to\infty} I(g_n) \geq \lim_{n\to\infty} I(g_n\wedge m)\mathbb{1}_{E_m} = I(g\wedge m)\mathbb{1}_{E_m}$.

we conclude

$$\underline{\lim}_{n\to\infty} I(g_n) \geq \underbrace{I((g\wedge m)\mathbb{1}_{E_m})}_{\downarrow\, m\to\infty \;\; I(g).} \quad \forall\, m\geq 1$$

$$(g \wedge m) 1_{E_m}(x) = \begin{cases} 0 & \text{if } x \in E_m^c \\ g \wedge m & \text{if } x \in E_m \end{cases}$$



Wei Y:

Boddine
Suman
Forruk
Ganeshki
Qi Ping.
Jiaxin.
Naifan.
Lei Alice
Siling.
William.
Y: Xiao

Jorge Martínez Ortiz
Chia-Yu Wei
Zilong Zhang

monotone convergence theorem. (M.C.T)

μ-sigma-finite.

if $f_n \geq 0$, $f_n \uparrow f$ then $I(f_n) \uparrow I(f)$, $n \to \infty$

Proof Fatou Implies

$$\underline{\lim}\, I(f_n) \geq I(\underline{\lim}\, f_n) = I(f)$$

obs.

$$\overline{\lim_{n\to\infty}}\, I(f_n) \leq I(f)$$

$$I(f) \leq \underline{\lim}\, I(f_n) \overset{=}{\leq} \overline{\lim}\, I(f_n) \leq I(f).$$

Equal lim of f_n & $\lim I(f_n) \to \lim I(f)$.

Concerning $f_n > 0$.

if $f_n \geq g$, $I(|g|)$ $f_n \uparrow f$ then $I(f_n) \uparrow I(f)$

use. $f_n - g_n$

Dominated convergence theorem. DCT.

μ-sigma finite.

$$f_n \xrightarrow[n\to\infty]{a.e} f, \quad \underbrace{|f_n| \leq g}, \quad I(g) < \infty$$

g is positive.. by ABS.

$$\text{then } I(f_n) \xrightarrow[n\to\infty]{} I(f).$$

observe $f_n + g \geq 0$, $g - f_n \geq 0$

$\Downarrow$ By Fatou $\qquad\qquad \Downarrow$ Fatou.

$\lim I(f_n + g) \geq I(f + g) \qquad \lim I(g - f_n) \geq I(g - f)$

Observe $f_n + g \ge 0$, $g - f_n \ge 0$

$\Downarrow$ By Fatou

$$\underline{\lim}\, I(f_n + \cancel{g}) \ge I(f + \cancel{g})$$

$\Downarrow$ Fatou.

$$\underline{\lim}\, I(\cancel{g} - f_n) \ge I(\cancel{g} - f)$$

Finite. $\Rightarrow$ Drop from Both sides.

$$\underline{\lim}\, I(f) \ge I(f). \quad \blacksquare$$

$$\underline{\lim}\, -I(f_n) \ge -I(f)$$

$$\Rightarrow -\overline{\lim}\, I(f_n) \ge -I(f) \quad \text{mult } -1$$

$$\Rightarrow \overline{\lim}\, I(f_n) \le I(f)$$