02-14 Doob Decomp,

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Doob's Decomposition

$\{X_n, \mathcal{F}_n\}_{n\geq 0}$ is subMG $\exists \{A_n\}_{n\geq 0}$ Predictable ($A_n \in \mathcal{F}_{n-1}$)

And $A_0 = 0$, $A_n \leq A_{n+1}$ a.s. $n \geq 0$ $\{M_n, \mathcal{F}_n\}_{n\geq 0}$ MG.

so that $X_n = A_n + M_n$, $n \geq 0$ and this is unique

sub MG or MG Diff.

Ex $D_k = X_k - X_{k-1}$ $k \geq 1$

$$X_k = X_0 + \sum_{k=1}^{n} D_k \quad n \geq 0$$

$\{X_n\}_{n\geq 0}$ sub MG $\Rightarrow$ $E_{\mathcal{F}_{k-1}}(D_k) \geq 0$ a.s. $k \geq 1$

$0 \leq E(D_k) =$

$$= E_{\mathcal{F}_{k-1}}(X_k - X_{k-1}) = E_{\mathcal{F}_{k-1}}(X_k) - E_{\mathcal{F}_{k-1}}(X_{k-1}) = E_{\mathcal{F}_{k-1}}(X_k) - X_{k-1}$$

$$E_{\mathcal{F}_{k-1}}(X_k) \geq X_{k-1} \text{ a.s.}$$

Baby example

$\{D_k\}_{k\geq 1}$ IND $E(D_k) \geq 0$ $k \geq 1$

$$X_n = \underbrace{X_0 + \sum_{k=1}^{n}[D_k - E(D_k)]}_{M_n} + \underbrace{\sum_{k=1}^{n} E(D_k)}_{A_n}$$

M_n - The conditional Exp. is 0 ∴ MG.

Ex $D_k = X_k - X_{k-1}$ $k \geq 1$

$$X_n = X_0 + \sum_{k=1}^{n} D_k \quad n \geq 0, \quad E_{\mathcal{F}_{k-1}}(D_k) \geq 0 \quad k \geq 1$$

$$X_n = \underbrace{X_0 + \sum_{k=1}^{n}[D_k - E_{\mathcal{F}_{k-1}}(D_k)]}_{M_n - \text{MG.}} + \underbrace{\sum_{k=1}^{n} E_{\mathcal{F}_{k-1}}(D_k)}_{A_n \in \mathcal{F}_{n-1}}$$

Proof of uniqueness.

$X_n = M_n + A_n = \tilde{M}_n + \tilde{A}_n$

Proof of uniqueness.

$X_n = M_n + A_n = \tilde{M}_n + \tilde{A}_n$

$A_0 = 0 = \tilde{A}_0$

$\Rightarrow M_0 = \tilde{M}_0$

next look at consecutive Diff $M_n - M_{n-1}$

WTS $M_n - M_{n-1} = \tilde{M}_n - \tilde{M}_{n-1}$ a.s. $n \geq 1$

$\Rightarrow M_n = \tilde{M}_n$ $n \geq 1$ a.s.

Observe: $\{M_n - \tilde{M}_n, \mathcal{F}_n\}_{n \geq 0}$ is MG (since $E(M_n - \tilde{M}_n) + M_n - \tilde{M}_n$)

$\Rightarrow E_{\mathcal{F}_{n-1}}(M_n - \tilde{M}_n) = M_{n-1} - \tilde{M}_{n-1}$

is Also Predictable.

$M_n - \tilde{M}_n = \underbrace{\tilde{A}_n - A_n}_{\text{in } \mathcal{F}_{n-1}}$ "predictable" implies

$E_{\mathcal{F}_{n-1}}(M_n - \tilde{M}_n) = M_n - \tilde{M}_n$

Equate $M_n - M_{n-1} = \tilde{M}_n - \tilde{M}_{n-1}$ $n \geq 1$

$\Rightarrow M_n = M_{n-1}$ a.s. $\forall n \geq 0$.

$\Rightarrow A_n = \tilde{A}_n$, $n \geq 0$.

Examples. §5.3. (Durrett §5.3 p. 204

$D_k = X_k - X_{k-1}$ $k \geq 1$

$\{X_k, \mathcal{F}_k\}_{k \geq 0}$ MG. $\exists$ $M < \infty$ so that $|D_k| \leq M_k$ a.s. $k \geq 1$

then $\Omega = C \cup D$, $C \cap D = \emptyset$ a.s.

where on $C = \{\lim_{n\to\infty} X_n$ Exists & finite $\}$

$D = \{\overline{\lim_{n\to\infty}} X_n = +\infty, \underline{\lim_{n\to\infty}} X_n = -\infty\}$

EX $\{D_k\}_{k \geq 1}$ iid $E(D_k) = 0$ $\mathcal{F}_n = \sigma\{D_1, \ldots, D_n\}$ $n \geq 1$

$|D_k| \leq M_1$

we saw that 4 cases. Pg 187.

1) $X_n = \cancel{0,}$ $X_0 = 0$ not happening.
2) $\overline{\lim_{n\to\infty}} \cancel{X_n} = +\infty$ a.s.
3) $\underline{\lim_{n\to\infty}} \cancel{X_n} = -\infty$ a.s.

$\overline{\lim_{k\to\infty}} X_k = \infty$, $\underline{\lim_{k\to\infty}} X_k = -\infty$ a.s.

$\therefore$ $P(D) = 1$, $P(C) = 0$

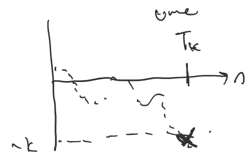
Example

= take $\{A_n\}_{n\ge1}$ independent Events. = Dm.

Proof.

← negative part of D_k

Assume $0 \le D_k^- \le M$ $\quad k\ge 0$ and $D_k^+ \le M$ (1)



Define: $T_k = \inf\{n : X_n \le -k\}$ $\quad k=1,2,\dots$

observe $\{T_k=\infty\}$ can be w.p. >0.

T_k is a S.t. $k\ge1$

How far can I Drop Below k at T_k? $D_k\le M$ therefore only Below M

we learned that $\{X_{n\wedge T_k}\}_{n\ge0}$ is a MG

Also: $X_{n\wedge T_k} \ge -k-M$, $\quad n\ge0$ a.s.

By MGCT: on $\{T_k=\infty\}$ we get $X_{n\wedge T_k}\xrightarrow[n\to\infty]{a.s.} X$ a.s.

MGCT
if $\{X_n,\mathcal{F}_n\}$ supMG and $\sup_n E(X_n^-)<\infty$
then $\lim_{n\to\infty} X_n = X$ a.s. and $E|X|<\infty$

C =

$$\left\{\varliminf_{n\to\infty} X_n > -\infty\right\} = \{\exists k\ge1 : X_n \ge -k,\ n\ge1\}.$$

$$= \bigcup_{k=1}^{\infty}\{T_k=\infty\}$$

$$= \{\lim_{n\to\infty} X_n \text{ Exist \& finite.}\}$$

$D = \{\varliminf_{n\to\infty} X_n = -\infty\}$ with Both we get $\overline{\lim}_{n\to\infty}$

At (1)

B.C. II if $\{A_n\}_{n\ge1}$ IND then $\{A_n \text{ i.o.}\} \overset{a.s.}{\Leftrightarrow} \sum_{n=1}^{\infty} P(A_n)=\infty$ (iff)

in this chapter:

$A_n\in\mathcal{F}_n$, $n\ge0$ $\qquad \{A_n \text{ i.o.}\} = \left\{\sum_{n=1}^{\infty} P(A_n \mid \mathcal{F}_{n-1}) = \infty\right\}$.