# L40 - 12-04

Wednesday, December 4, 2024 11:30 AM

$$P(X_n = k) \xrightarrow[n\to\infty]{} P(X=k), \quad k \in \mathbb{Z}$$

Why this Implies

$T_{n,k}$ $\qquad$ $p_k$ $\qquad$ "Genuine Probabilities!"

$\sum_{k\in\mathbb{Z}} p_{n,k} = 1 \qquad \sum_{k\in\mathbb{Z}} p_k = 1$

$p_n = (p_{n,k} \geq 0)_{k\in\mathbb{Z}} \qquad p = (p_k \geq 0)_{k\in\mathbb{Z}}$

then $\|p_n - p\| = \sum_{k\in\mathbb{Z}} |p_{n,k} - p_k| \xrightarrow[n\to\infty]{} 0$

$P(X_n \leq k) - P(X \leq k) \xrightarrow[n\to\infty]{} 0, \quad k\in\mathbb{Z}$

wts. $\left|\sum_{\ell\leq k} p_{n,\ell} - \sum_{\ell \leq k} p_\ell\right| \xrightarrow[n\to\infty]{} 0 \qquad k \in \mathbb{Z}$

$$\|p_n - p\| \geq \sum_{\ell\leq k} |p_{n,\ell} - p_\ell| \xrightarrow[n\to\infty]{} 0$$

$$|p_{n,k} - p_k| = (p_k - p_{n,k})^+ + (p_k - p_{n,k})^-$$

$$p_k - p_{n,k} = (p_k - p_{n,k})^+ - (p_k - p_{n,k})^-$$

$$\sum_k (p_k - p_{n,k}) = \sum_k (p_k - p_{n,k})^+ - \sum_k (p_k - p_{n,k})^-$$

$$\sum_k p_k = 1$$

$$\sum_k p_{k,n} = 1$$

$p_k - p_{n,k} < 1$ & $p_k + p_{n,k} < 2$.

since Dist $=1$

$$\sum_k |p_{n,k} - p_k| = \sum_k (p_k - p_{n,k})^+ + \sum_k (p_k - p_{n,k})^- = 2\sum_k (p_k - p_{n,k})^+ \leq 2\sum_k p_k = 2$$

$\therefore \quad p_{n,k} \xrightarrow[n\to\infty]{} p_k, \quad k\in\mathbb{Z}.$

Insert Dominance Converge theorem, DCT.

$f_n, f : (\Omega, \mathcal{F}, \mu) \to \mathbb{R}.$

$\mu$ - $\sigma$ finite measure

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$\mu$ - $\sigma$ finite measure

$\Omega_n \subset \Omega$, $\bigcup_n \Omega_n = \Omega$, $\mu(\Omega_n) \uparrow \mu(\Omega)$

$\mu(\Omega_n) < \infty$

if $f_n(\omega) \xrightarrow[n\to\infty]{} f(\omega)$, $\mu$-a.e.

and $|f_n(\omega)| \le g(\omega)$ with $\int_\Omega g\, d\mu < \infty$

then $\int_\Omega f_n d\mu \to \int_\Omega f d\mu$

$\Omega = \mathbb{Z}$, $\{\mathcal{F}$ all subsets of $\mathbb{Z}\}$, $\mu(k) = 1$, $k \in \mathbb{Z}$
$\mu(A) =$ cardinality of $A$.

$a = (a_n)_{n \in \mathbb{Z}}$

+18 $\int_\Omega a\, d\mu = \sum_{n \in \mathbb{Z}} a_n$

$\forall k \in \mathbb{R}$.

$P_{n,k} \xrightarrow[n\to\infty]{} P_k$

$P_k \ge (P_k - P_{n,k})^+ \xrightarrow[k\to\infty]{} 0$

then $\|P_n - P\| = \sum |P_{n,k} - P_k| \xrightarrow[n\to\infty]{} 0$ ✓ By DCT.

$f_{X_n}(x) - f_n(x)$, $f_X(x) = f_n(x)$,

statement

if $f_n(x) \xrightarrow[n\to\infty]{} f(x), x \in \mathbb{R}$,

then $\int_{x\in\mathbb{R}} |f_n(x) - f(x)|\, dx \xrightarrow[n\to\infty]{} 0$

$\|\mu_{X_n} - \mu_X\| \xrightarrow[n\to\infty]{} 0$

+25 $\prod_{k=1}^{n} (1 + a_{n,k}) \xrightarrow[n\to\infty]{} e^a$ Triangle Array of Bernoullis.

+25 $\prod_{k=1}^{n}(1+a_{n,k}) \xrightarrow[n\to\infty]{} e^{a}$ Triangle Array of Bernoullis.

$\sum_{k=1}^{n} a_{n,k} \xrightarrow[n\to\infty]{} a$

<u>theorem</u> if $\{X_{n,m}\}_{1\le m\le n}$, $n\ge 1$, $\{X_{n,m}\}_{1\le m\le n}$ $X_{n,m} \sim Ber(P_{n,m})$

if ① $\sum_{m=1}^{n} P_{n,m} \xrightarrow[n\to\infty]{} \lambda$

② $\max_{1\le m\le n} P_{n,m} \to 0$

there $S_n = \sum_{m=1}^{n} X_{n,m} \Rightarrow$ Poisson. "so we know it converge in Total Variation.

+58 "we want a Proof for converge in uniform"

<u>Goal</u>

$P_n = (P_{n,m})_{1\le m\le n}$, $n\ge 1$

$P = (P_m)_{1\le m}$

$\|P_n - P\| = \sum_{m=1}^{\infty} |P(X_n = m) - P(X = m)|$

instead, we will look At total variation.

$Q_n = (q_{n,m})_{1\le m}$

$q_n = P(\text{Poisson}(\sum_{m=1}^{n} P_{n,m}))$

Goal: $\|P_n - q_n\| \le 2\sum_{m=1}^{n} P_{n,m}^2$

Example $\|B(n, \frac{\lambda}{n}) - \text{Poisson}(\lambda)\|$ "only Rint number of del

$\le n\cdot\left(\frac{\lambda}{n}\right)^2 \le 2\frac{\lambda^2}{n}$

$2\cdot\sum_{m=1}^{n} P_{n,m}^2 \le 2\max_{1\le m\le n}\{P_{n,m}\} \to 0$ by Assumption.

Redefine

$q_n = P($ Poisson

Step 1 $\|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\| \le \|\mu_1 - \nu_1\| + \|\mu_2 - \nu_2\|$

$\mu_1$ = dist of $X$

$\mu_2$ = dist of $Y$

$X \perp Y$.

$\mu_1 \times \mu_2(k, l) = P(X = k)(Y = l)$

$\mu_1 \times \mu_2(k, l) - \nu_1 \times \nu_2(k, l)$

$= \mu_1(k) \cdot \mu_2(l) - \nu_1(k)\nu_2(l)$

$= \mu_1(k)\mu_2(l) - \mu_1(k)\nu_2(l) + \mu_1(k)\nu_2(l) - \nu_1(k)\nu_2(l)|$

with $\sum_l \sum_k \nu_2(l)|$

$= \sum \mu_1(k) = | \mu_2(l) - \nu_2(l) + \nu_2(\mu_1(k) - \nu_1(k))$

$\sum \mu_1(k) \, \|\mu_2 - \nu_2 \qquad \|\mu_1 - \nu_1\|$

$\|\mu_2 - \nu_2\|$

~~tensor product~~ ? convolution

Step 2 $\|\mu_1 * \mu_2 - \nu_1 * \nu_2\| \le \|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\|$

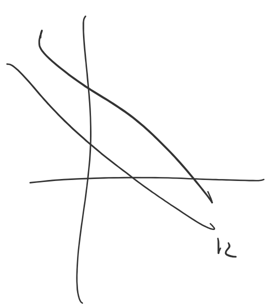
$\mu_1 * \mu_2(k) = P(X + Y = k) = \sum_l \mu_1(l)\mu_2(k-l)$

Back to comparison.

$\|\mu_1 * \mu_2 - \nu_1 * \nu_2\| = \sum_k \sum_l \mu_1(l)\mu_2(k-l) - \sum_k \sum_l \nu_1(l)\nu_2(k-l)$

$\le \sum_k \sum_l |\mu_1(l)\mu_2(k-l) - \nu_1(l)\nu_2(k-l)|$

$= \|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\|$




"theory by Sauer"

"note: $\|\mu_1 \times \mu_2 - \nu_1\nu_2\| \le \|\mu_1 - \nu_1\| + \|\mu_2 - \nu_2\| \le \|\mu_1 * \mu_2 - \nu_1 * \nu_2\|$

$\|p_n - q_n\| \le$

$\sum_k |p_{n,k} - q_{n,k}|$

$k$

$$P_{n,k} = P\left(\sum_{k=1}^{1} Ber(p_{n,k}) = m\right)$$

$Ber(p_{n,1})$ $\uparrow$ $M_1$ $\qquad$ $Ber(p_{n,m})$ $\uparrow$

$M_1 * M_2 * M_3 \cdots * M_n$

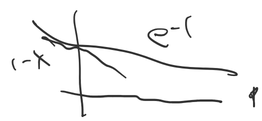
$U_1 * \cdots$

$* \, Poisson(p_{n,1}) \qquad * \, Poisson(p_{n,m})$

What is the total variation between

$$\leq |1-p-e^{-p}| + |p-pe^{-p}| + P(Poisson(p) \geq 2)\; 1-e^{-p}-pe^{-p}$$

using $e^{-p}-(1-p)$




$$\leq e^{-p}-(1-p) + p - pe^{-p} + 1 - e^{-p} - pe^{-p}$$

$$= 2p(1-e^{-p}) \leq 2p^2$$