L19 - 10-09

Wednesday, October 9, 2024 11:29 AM

$\{X_{n,k}\}_{1 \le k \le n}$ $n = 1, 2, \ldots$ IND for each n

$$P\left(\frac{S_n - a_n}{b_n}\right) \le \underbrace{\sum_{k=1}^{n} P(|X_{k,n}| > b_n)}_{\mathrm{I}} + \underbrace{\sum_{k=1}^{n} \frac{E(X_{n,k}^2 ; |X_{n,k}| \le b_n)}{\varepsilon^2 b_n^2}}_{\mathrm{II}}$$

$$a_n = \sum_{k=1}^{n} E(X_{k,n} ; |X_{k,n}| \le b_n)$$

if $(\mathrm{I}) \xrightarrow[n\to\infty]{} 0$ and $(\mathrm{II}) \xrightarrow[n\to\infty]{} 0$ then $\frac{S_n - a_n}{b_n} \xrightarrow[n\to\infty]{P} 0$

in case $\{X, X_k\}_{k \ge 1}$ Are iid. where $S_n = \sum_{k=1}^{n} X_k$

$$P\left(\frac{S_n - a_n}{b_n}\right) \le \underbrace{n \cdot P(|X| \ge b_n)}_{\mathrm{I}} + \underbrace{\frac{n\, E(X^2 ; |X| \le b_n)}{\varepsilon^2 b_n^2}}_{\mathrm{II}}$$

$$a_n = n\, E(X ; |X| \le b_n)$$

if $|\mathrm{I}| \xrightarrow[n\to\infty]{} 0$ and $\mathrm{II} \xrightarrow[n\to\infty]{} 0$ then

St. Petersburg Paradox (Example 2.2.7 Durrett)

Ex

$$P(X = 2^k) = 2^{-k} \quad k = 1, 2 \ldots$$

$$E(X) = \sum_{k=1}^{\infty} 2^k 2^{-k} = \sum_{n=1}^{\infty} 1 = \infty$$

we know: $\frac{S_n}{n} \xrightarrow[n\to\infty]{P} \infty$

ie $P\left(\frac{S_n}{n} > M\right) \xrightarrow[n\to\infty]{} 1, \ \forall M > 0$

"how do I Prove it"

"Truncated At Level M"

Look At $X_M = X \wedge M$ $M > 0$

$\{X_M, X_{k,M}\}_{k \ge 1} = \{X_M, X_k \wedge M\}_{k \ge 1}$ "still IID"

$E(X \wedge M) < M < \infty$ By WLLN

$$\frac{S_n^{(M)}}{n} \xrightarrow[n\to\infty]{P} E(X_M) \Rightarrow P\left(\frac{S_n}{n} > E(X_M) - \varepsilon\right) \xrightarrow[n\to\infty]{} 1$$

$S_k^{(M)} = \sum_{k=1}^{n} X_k \wedge M$ $\quad P(X_k \ge \underline{X_k \wedge M}) = 1 \ \forall k.$

Truncated.

$P(S_n \ge S^{(M)}) = 1$

$$S_k^{(M)} = \sum_{k=1} X_k \wedge M \qquad P(X_k \geq \underbrace{X_k \wedge M}_{\text{Truncated.}}) = 1 \quad \forall k.$$

$$P(S_n \geq \underbrace{S_n^{(M)}}_{\text{Truncated}}) = 1$$

$$P\left(\frac{S_n}{n} > \mathbb{E}(X_M) - \varepsilon\right) \xrightarrow[n\to\infty]{} 1$$

Since M is Arbitrary.

$$\mathbb{E}(X_M) = \mathbb{E}(X \wedge M) \xrightarrow[M\to\infty]{}$$

$$0 \leq X \wedge M \uparrow X \quad \text{As} \quad M \to \infty$$

By MCT $\quad E(X \wedge M) \uparrow E(X) = \infty$

"if Expectation is Positive have to Divide By something Bigger if we want it to to Converge."

<u>Claim</u> $\quad \dfrac{S_n}{n\log_2 n} \xrightarrow[n\to\infty]{P} 1 \quad$ where $S_n = \sum_{k=1}^{\infty} X_k$, X_k iid $\quad X_k \overset{D}{=} X$

$$b_n = n \log_2(n)$$

$$a_n = n \cdot \mathbb{E}(X ; X \leq n \log(n)) = n \sum_{2^k < n\log(n)} 2^k 2^{-k} = [\operatorname{Log}(n) + \log\operatorname{Log} \; n] n$$

$$\Rightarrow k \leq [\operatorname{Log}(n) + \operatorname{Log}\operatorname{Log}(n)] n$$

For $x > 0$ $\qquad\qquad k \log 2^k \qquad \log n + \log\log n$

$$\underbrace{\sum_{k=1}^{\infty} 2^{-k} \mathbb{1}_{\{2^k > x\}}}_{\text{simple series, is infinite.}} \leq \frac{2}{x} \; ; \quad \sum_{k=1}^{\infty} 2^k \mathbb{1}_{\{2^k < x\}} \leq 2X$$

Exercise check if true

Apparently used here

$$(\mathrm{I}) \quad nP(X \geq b_n) = n \cdot \sum_{2^k > b_n} 2^{-k} \leq \frac{2n}{b_n} = \frac{2\cancel{n}}{\cancel{n}\log(n)} \xrightarrow[n\to\infty]{} 0$$

$$(\mathrm{II}) \quad \frac{n E(X^2 \leq b_n)}{\varepsilon^2 b_n^2} = \frac{n}{\varepsilon^2 b_n^2} \sum_{2^k \leq b_n} \cancel{2^{2k}}\, \cancel{2^{-k}}\; 2^k = \frac{n \cdot 2b_n}{\varepsilon^2 b_n^2} = \frac{2n}{\varepsilon^2 b_n} = \frac{2\cancel{n}}{\varepsilon^2 \cancel{n} \operatorname{Log}(n)} = \frac{2}{\varepsilon^2 \operatorname{Log}(n)} \xrightarrow[n\to\infty]{} 0$$

WLLN for Positive iid R.V. with $E(X) = \infty$

$\mu(s) = E(X ; X \le s)$, $s \ge 0$ "as $s \to \infty$; $EX \to \infty$"

" $\mu(s)$ is increasing"

if $\dfrac{\mu(s)}{s(1-F_X(s))} \xrightarrow[s\to\infty]{} \infty$ then we can find $(b_n)_{n\ge1}$, $b_n \xrightarrow[n\to\infty]{} \infty$ so that $\dfrac{S_n}{b_n} \xrightarrow[n\to\infty]{P} 1$

Observe $\dfrac{\mu(s)}{s} = \dfrac{E[X ; X \le s]}{s} \xrightarrow[s\to\infty]{} 0$

$\dfrac{X(\omega)}{s} \xrightarrow[s\to\infty]{} 0$ $\quad = E\left[\dfrac{X}{s} ; X < s\right] \xrightarrow[n\to\infty]{DCT}$

$1 \ge \dfrac{X}{s} \cdot \mathbb{1}_{\{X \le s\}}$

$E(1) = 1 < \infty$

$\lim_{s\to\infty} \dfrac{X}{s} \mathbb{1}_{\{X \le s\}} = 0$

this is Justification of

Def of b_n $n \ge 1$

b_n satisfies $\dfrac{\mu(b_n)}{b_n} = \dfrac{1}{n}$, $n \ge 1$

$b_n = \min_{s\ge0} \left\{ \dfrac{\mu(s)}{s} \right\} \approx \dfrac{1}{n}$

Calls it a "Jump"

if $P(X = s_0) > 0$ then $\mu(s_0) > \lim_{\substack{s < s_0 \\ s \to s_0}} \mu(s)$

use this formula to check condition

$$\frac{\mu(b_n)}{b_n} = \frac{1}{n}, \quad n \ge N_0$$

$$\begin{aligned} a_n &= n\, E(X ; X \le b_n) \\ &= n\, \mu(b_n) \\ &= n \frac{b_n}{n} \approx b_n \end{aligned}$$

Enough to Demand $\dfrac{\mu(b_n)n}{b_n} \longrightarrow ?$

"want to talk about unfair-fair game." William Feller.

$E(X) = 0$

$P(X = 2^k - 1) = \dfrac{1}{2^k k(k+1)}$, $k \ge 1$

$P(X=-1) = 1 - P(X \geq 1)$

$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ get telescoping sum.

Result:

$$\frac{S_n}{n/\log_2 n} \xrightarrow[n\to\infty]{P} -1 \qquad \left(\frac{S_n}{n} \xrightarrow[n\to\infty]{P} 0 \right)$$

" the truncation leads to ϕ "

" After truncation, $E = 0$ unbalanced.