Properties of integral

$(\Omega, \mathcal{F}, \mu)$ μ - σ - finite.

$\int_\Omega f \, d\mu$ is integrable if $\int_\Omega |f| \, d\mu < \infty$

$|f| = f^+ + f^-$, $f^+ = f \vee 0$, $f^- = -(f \wedge 0) \geq 0$

$I(f) \equiv I(f^+) - I(f^-)$

f, g integrable

① $f \geq g \Rightarrow I(f) \geq I(g)$ $(I(f) \geq 0, f \geq 0)$

② $a \in \mathbb{R} \Rightarrow I(af) = a I(f)$

③ $I(f+g) = I(f) + I(g)$

④ $|I(f)| \leq I(|f|)$

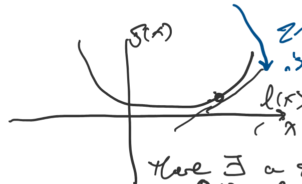
$|I(f)| = |I(f^+) - I(f^-)| = \max\{I(f^+) - I(f^-), I(f^-) - I(f^+)\}$

$I(|f|) = I(f^+ + f^-) \underset{\text{by 3.}}{=} I(f^+) + I(f^-)$

Jensen inequality.

Assumption $\mu(\Omega) = 1$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$, φ is convex $(\sim f'' \geq 0)$



$l(x) = a + bx$

2nd derivative is positive.

There $\exists$ a straight line ① $\varphi \geq l(x)$ $\forall x$

$\forall x \in \mathbb{R}, \exists x_0 : \varphi(x_0) = l(x_0)$

definition of Convex

① $\varphi \geq l(x)$ $\forall x$

② $\forall x \in \mathbb{R}$ $\exists x_0 : \varphi(x_0) = l(x_0)$

Assume: $f, \varphi(f)$ are integrable. for

Eg $\varphi = x^2$ or e^x.

$= (f)^2$ or $e^{f(x)}$

then $\varphi(I(f)) \leq I(\varphi(f))$

Proof: $X_0 = I(f)$

$I[\varphi(f)] \geq I(l(f))$ where $l(x) = a + bx$.

$\int_\Omega l(f) \, d\mu = \int_\Omega a + bf \, d\mu$ since μ is 1

$= a + b I(f) = l(I(f))$

$= l(I(f))$

$= \varphi(I(f))$ since $l(f) = \varphi(f)$ At x_0.

Prove let $x_1, \ldots, x_n \in \mathbb{R}^+$

$\bar{x} = \frac{x_1 + \ldots + x_n}{n} \geq (x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{\frac{1}{n}}$

$\frac{\sum_{k=1}^{n}}{n} = \bar{x} \geq \left(\prod_{k=1}^{n} x_k\right)^{1/n} \geq \prod_{k=1}^{n} [(x_k)^{1/n}]$

Connection

$\Omega = \{1, 2, \ldots n\}$

$\mu(\{k\}) = \cancel{1/n}$, $1 \leq k \leq n$, $\mu(\Omega) = 1$ $\log(\bar{x})$. $\log(\frac{1}{n} \sum \log x_k)$

Connection

$\Omega = \{1, 2, \dots n\}$

$\mu(\{k\}) = \cancel{1/n}\ \omega_k$, $1 \le k \le n$, $\mu(\Omega) = 1$ $\log(\bar{x})$. $\log(\frac{1}{n}\Sigma \log x_k)$

$\omega_k > 0$, $\sum_{k=1}^{n} \omega_k = 0$

$f : k \to k_k$

$I(f) = \bar{x}$ select $f(x) = \ln(x)$, $x > 0$

$\log(I(f)) \ge I[\log(f)]$

$\log(\bar{x}) \ge \sum_{k=1}^{n} \frac{1}{n} \log(x_k) = \log \prod_{k=1}^{n} (x_k^{1/n})$

$\sum_{k=1}^{n} \omega_k x_k \ge \prod_{k=1}^{n} x_k^{\omega_k}$ generalization of proof.

Used in...

Let $p \ge 1$, $\frac{1}{p} + \frac{1}{q} = 1$ $x > 0$, $y > 0$.

$q \ge 1$?

then: $xy \le \frac{x^p}{p} + \frac{y^q}{q}$ can use Jensen Inequality.

Proof $x_1 = x^p$, $x_2 = y^q$ $\omega_1 = \frac{1}{p}$, $\omega_2 = \frac{1}{q}$.

$xy = (x^p)^{1/p} \cdot (y^q)^{1/q} \le \frac{1}{p} x^p + \frac{1}{q} \cdot y^q$

By Jensen.

Holder inequality. let f, g be functions. $\Omega \to \mathbb{R}$.

$|f|^p, |g|^q$ Are integrable.

then $I(|f \cdot g|) \le (\int_\Omega |f|^p d\mu)^{1/p} (\int |g|^q d\mu)^{1/q}$

holder inequality

$\|f \cdot g\|_{L_1} \le \|f\|_{L_p} \cdot \|g\|_{L_q}$

$\underbrace{\|f \cdot g\|_{L_1}}_{\text{L1 norm}} \le \underbrace{\|f\|_{L_p}}_{\text{L-p Norm}} \cdot \|g\|_{L_q}$ — Lq Norm

Replace f By $\frac{f}{\|f\|_{L_p}}$ & g by $\frac{g}{\|f\|_{L_q}}$

then

$\Rightarrow \left\| \frac{f}{\|f\|_{L_p}} \right\|_{L_p} = 1 = \left\| \frac{g}{\|g\|_{L_q}} \right\|_{L_q}$ this is a common trick the norm of norm equals 1

WLOG we assume $\|f\|_p = \|g\|_q$ WTS $I(|fg|) \le 1$

$\int_\Omega (fg)\, d\mu \le \int_\Omega \frac{|f|^p}{p} + \frac{|g|^q}{q}\, d\mu.$

int of is one. since $\frac{|f|^p}{p} = 1$

Triangle inequality in $L^p(\Omega, F, \mu)$, $p \ge 1$

$L^p(\Omega, F, \mu) = \{f : \Omega \to \mathbb{R} : \int_\Omega |f|^p d\mu < \infty\}$

Let $\|f + g\|_p \le \|f\|_p + \|g\|_q$