04-25

Friday, April 25, 2025 11:31 AM

Thursday. C506 Midterm.

if we have $P(X = x_k) = p_k \quad k = 1, \dots, n \quad E(X) = 0$

then $\exists$ s.t. τ w.r.t $\{\mathcal{F}_t\}_{t\geq 0}$

so that $B^0(\tau) \overset{\mathcal{D}}{=} X$, and $E(\tau) = E(B_\tau^2)$

Implies why?

$\{B_t^2 - t\}_{t>0}$ is MG.

$|B^0_{t\wedge\tau}| \leq C < \infty$ therefore UI MG.

$$E_0(B^2_{\tau\wedge\tau} - (t\wedge\tau)) = 0$$

$\therefore$

$$E_0(B^2_{t\wedge\tau}) = E(\tau\wedge t)$$

$t < \infty$ a.s.

$t\wedge\tau \uparrow \tau$ at $t\uparrow$

$$\downarrow t\to\infty \qquad \downarrow t\to\infty$$

$$E_0(B_\tau^2) = E(\tau)$$

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Case $X \equiv 0$

$\tau_0 = \inf\{t > 0 : B^0_t = 0\} = 0$

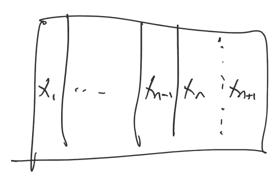
$B^0(0) = 0$

induction: if X gets n-values, then $\exists\, \tau$ s.t. $X \overset{\mathcal{D}}{=} B(\tau)$

Prove that if X get $(n+1)$ values then ..... $X \overset{\mathcal{D}}{=} B(\tau)$ 3

$E(X) = 0 \qquad x_1 < x_2 < \dots < x_n < x_{n+1}$

$P(X = x_k) = p_k \qquad k = 1, \dots, n+1$



$\mathcal{F} = \sigma\{X = x_k,\ k = 1, \dots, n-1,\ \{X = x_n\} \sqcup \{X = x_{n+1}\}\}$

$X \in \{x_n, x_{n+1}\}$

Wts

Calculate: $E_{\mathcal{F}}(X) = \begin{cases} x_k & \text{wp } p_k \text{ if } 1 \leq k \leq n-1 \\ \dfrac{x_n p_n + x_{n+1} p_{n+1}}{p_n + p_{n+1}} & \text{w.p. } p_n + p_{n+1} \end{cases}$

$$Y \equiv E_{\mathcal{F}}(X)$$

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Y gets n values $E[Y] = 0$ By assumption $\exists$ s.t. $B(\eta) \overset{\mathcal{D}}{=} Y$

$\tau = \cancel{\inf\{t > \eta\}}$ $\begin{cases} \eta \text{ if } X = x_k, \quad 1 \leq k \leq n-1 \\ \inf\{t \geq \eta : B_t \in \{x_n, x_{n+1}\}\} \quad \text{if } B(\eta) \notin \{x \quad x\} \end{cases}$

η some constant?

$$\tau = \begin{cases} \ldots \ \text{if } x = x_k, \ 1 \le k \le n-1 \\ \inf\{t > n : B_t \in \{x_n, x_{n+1}\}\} \quad \text{if } B(\tau) \notin \{x_1, \ldots, x_{n+1}\} \end{cases}$$ some constant?

Problem: Let X a r.v., $E(X) = 0$, $E(X^2) < \infty$

Find τ st. so that $B^0(\tau) \overset{\mathcal{D}}{=} X$, $E(\tau) = E_0(B_\tau^2)$

Assume $\Omega = \mathbb{R}$, $\mathcal{F}$ = Borel set, P = dist of X.

Assume $X(\underline{x}) = \underline{x}$, $\underline{x} = \mathbb{R}$.

Q = {all rationals},

$Q = \{q_n\}_{n \ge 1}$

step 1

take q_1, $\mathcal{F}_1 = \sigma\{\overset{A_1}{(-\infty, q]}, \overset{A_2}{(q, \infty)}\}$.

$$E[X \mid \mathcal{F}_1] = \begin{cases} \dfrac{E[X; X \in A_1]}{P(A_1)} \\ \dfrac{E(X; X \in A_2]}{P(A_1)} \end{cases}$$

$\therefore$ Find τ_1 s.t. $B^0(\tau_1) \overset{\mathcal{D}}{=} X_1$

step 2: look at $\{q_1, q_2\}$, $q_{(1)} < q_{(2)}$

q_1

$q_{(1)}$ $q_{(2)}$

$$F_2 = \sigma\{(-\infty, q_{(1)}], (q_{(1)}, q_{(2)}], (q_{(2)}, \infty)\}$$

$X_2 = E[X \mid \mathcal{F}_2]$, $E[X_2] = 0$

$B^0(\tau_2) \overset{\mathcal{D}}{=} X_2$ $\quad \tau_2 > \tau_2$

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Every time we add one, then get.

$\{q_1, \ldots q_n\} = \{q_{(1)}, q_{(2)}, \ldots q_{(n)}\}$

$X_n = E(X \mid \mathcal{F}_n)$, $\quad \tau_n \ge \tau_{n-1} \ldots$

$X_n = E(X \mid \mathcal{F}_n)$, $\tau_n \geq \tau_{n-1} \dots$

$|X_n(x) - \cancel{X(x)}| \leq q_{(k+1)} - q_{(k)}$ if

then Expectation is some point q_1 & q_2.

$q_{(k)} < x < q_{(k+1)}$

$|q_1 - x|$ is smaller $|q_1 - q_2|$

$\tau_n \uparrow \tau$ a.s. τ is st.

$X_n \xrightarrow[n\to\infty]{a.s} X$ (so $X_n \Rightarrow X$)

$B^\circ(\tau_n) \overset{\mathcal{D}}{=} X_n$

Conditional Expectation in L^2 is contraction

$\infty > E(X^2) \geq E(X_n^2) = E[\tau_n]$ $\forall n \geq 0$ Implies $E[\tau_n] < \infty \therefore$

$E(\tau) < \infty$.

$E(X_n^2) \xrightarrow[n\to\infty]{} E(X^2)$

$E(X_n - X)^2 \xrightarrow[n\to\infty]{} 0$ $\Rightarrow$ $E(X^2) = E(\tau)$

How to get CLT w/o issue?

Application $\{X_k\}_{k\geq 1}$ iid $E(X)=0$ $E(X^2)=1$

$S_n = \sum_1^n X_k$ Prove $\frac{S_n}{\sqrt{n}} \Rightarrow N(0,1)$

$\tau_0 := 0$

Let τ_1 be st. so that $X_1 \overset{\mathcal{D}}{=} B^\circ(\tau_1)$

we hit τ_1 now we Find X_2 @ τ_2

Find $\tau_2 > \tau_1$ $\tau_2 - \tau_1 \perp\!\!\!\perp \tau_1$ $B(\tau_2) - B(\tau_1) \overset{\mathcal{D}}{=} X_2$

$= S_2 = B^\circ(\tau_2)$

want to continue to n.

$\{\tau_{k+1} - \tau_k\}_{0 \le k \le n-1}$ i.i.D.

$S_n \stackrel{\mathcal{D}}{=} B^0(\tau_n)$

$\frac{S_n}{\sqrt{n}} \stackrel{\mathcal{D}}{=} \frac{B^0(\tau_n)}{\sqrt{n}}$ A

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Fix n

$W_n(t) = \frac{B(nt)}{\sqrt{n}}, \ t \ge 0 \stackrel{\mathcal{D}}{=} \{B^0(t)\}_{t \ge 0}$

$\frac{nt \wedge ns}{n} = t \wedge s$

A $= W_n\left(\frac{\tau_n}{n}\right) \Rightarrow W_n(1)$ which is $N(0,1)$

by SLLN.

$\frac{\tau_n}{n} = \sum_{k=0}^{n-1} \frac{\overset{\text{i.i.D.}}{\tau_{k+1} - \tau_k}}{n} \xrightarrow[n\to\infty]{a.s.} E(\tau_1)$

Donsker invariance principle