

$(\Omega, \mathcal{A}, \mu)$, σ -algebra \mathcal{A} a measure in Ω

Expansion to $\mathcal{F} = \sigma(\mathcal{A})$

Define outer measure M^*

$$M^*(A) = \inf_{\substack{A \in \mathcal{A} \\ A_i \in A}} \sum_{i=1}^{\infty} \mu(A_i), \quad A \subset \Omega$$

this is F under partition E

Def " $E \subset \Omega$ is measurable" \leq
if $\forall F \subset \Omega$ we get $M^*(F) = M^*(F \cap E) + M^*(F \cap E^c)$

Always $M^*\left(\bigcup_{i=1}^{\infty} B_i\right) \leq \sum_{i=1}^{\infty} M(B_i)$ $\forall B_i \subset \mathbb{R}$ - subadditive.

$M^*(B) \leq M^*(D)$ if $B \subset D$ - monotonicity.

Claim: if $M^*(E) = 0$, then E is measurable and so is any $A \subset E$

$$M^*(F) \geq M^*(F \cap E) + M^*(F \cap E^c)$$

$$F \cap E \subset E$$

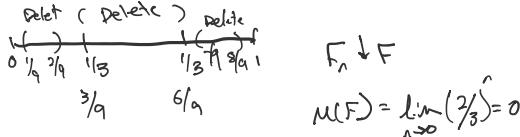
$$0 \leq M^*(F \cap E) \leq M^*(E) = 0$$

$$F \supset F^c$$

Example (cont'd) $\Omega = \mathbb{R}$, $\alpha = \left\{ \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}$

$$\sigma\{\alpha\} = \sigma\left\{ \text{open subsets of } \mathbb{R} \right\} \cup \left\{ (a_i, b_i] \right\}$$

Consider $[0, 1]$ Cantor subset of Ω



$\bar{F} = \bar{\mathbb{R}}$ the card of $[0, 1]$ equals card \mathbb{R} .

$$F = \left\{ \sum_{i=1}^{\infty} \frac{e_i}{3^i} \right\} \quad e_i = 0, 1, 2$$

In CANTOR SET we Delete 1
these have SAME CARD

$$[0, 1] = \left\{ \sum_{i=1}^{\infty} \frac{e_i}{2^i} \right\} \quad e_i = 0, 1$$

$\overline{\{ \text{all subsets of } F \}} > \bar{F} = \bar{\mathbb{R}}$ Cantor set are Bigger than \mathbb{R}

Conclusion: All subsets of Cantor set are measurable.

$$\sigma\{\alpha\} = \bar{\mathbb{R}}$$

cardiology - set is too large,

$$\sigma\{\alpha\} \neq \alpha^*, \quad \alpha^* = 2^{\mathbb{R}}, \quad \sigma(\alpha) = \bar{\mathbb{R}}$$

Borel σ -algebra is the minimal σ -algebra

Borel σ -algebra

$$\alpha = \left\{ \bigcup_{i=1}^{\infty} (a_i, b_i] \right\} \subset \sigma\{\alpha\} = \beta \subset \alpha^* \subset \text{Lebesgue } \sigma\text{-Algebra} \subset \{ \text{all subsets of } \mathbb{R} \}$$

Outer Measure

E is measurable if

$$\forall F \subset \Omega, \quad M^*(F) = M^*(F \cap E) + M^*(F \cap E^c)$$

Cantor Set

Borel σ -Algebra

Axiom of choice.

is there a choice of σ -Algebra that is not measurable? Yes!

Consider $\Omega = [-1, 2]$

Show a not measurable

Look at pair $x, y \in \mathbb{R}$, $x-y \notin \mathbb{Q}$

$$B \subset [0, 1]$$

If we look at

$$B_+ ? \{x + Q \cap [-1, 1]\}, x \in B.$$

$$\begin{aligned} [0, 1] &\subseteq \bigcup \{B + Q\} \subset [-1, 2] \\ q \in B \cap [-1, 1] \quad M(B_{q, 1}) &= 3 \\ M(B + q) &= M(q) \quad \text{if } q \text{ translation invariant} \\ \text{thus not measurable} \end{aligned}$$

translation invariant

$$M(B + q) = M(B)$$

$$\text{if } A \in \mathcal{A}^* \rightarrow \exists B \in \mathcal{B} \text{ and } N \in \mathcal{B}$$

so that $B \cup N > A$

$$A = B \cup D, D \in N$$

$$\begin{aligned} (\Omega, \mathcal{F}, \mu) & \downarrow \quad \text{completion of sigma algebra,} \\ (\Omega, \bar{\mathcal{F}}, \bar{\mu}) & \quad \text{ADD ALL SUBSETS OF } [0, 1] \end{aligned}$$

will not really use it as we use lebesgue measure.

$$\begin{array}{c} \text{Integration } (\Omega, \mathcal{F}, \mu) \quad \text{Chapter 1.4} \\ \sigma\text{-Algebra} \quad \nearrow \quad \nwarrow \sigma\text{-finite measure} \\ \exists E_n \uparrow \Omega, E_n \in \mathcal{F}, M(E_n) < \infty \end{array}$$

$$\text{Example: } \lambda([-n, n]) = 2n < \infty \quad \forall n.$$

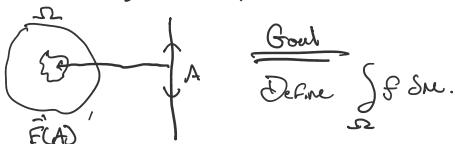
$$\begin{array}{c} \bigcup_{n=1}^{\infty} (-n, n) = \mathbb{R} \\ \lambda(\mathbb{R}) = \infty \end{array}$$

λ is σ -finite measure

$$(\Omega, \mathcal{F}) \quad (\mathbb{R}, \mathcal{B})$$

Def: $f: \Omega \rightarrow \mathbb{R}$ is called \mathcal{F} -B measurable

$$\text{if } f^{-1}(A) \in \mathcal{F}, \forall A \in \mathcal{B}.$$



σ -finite measure

$$\exists E_n \uparrow \Omega, E_n \in \mathcal{F}, M(E_n) < \infty$$

there exist a sequence of sets E_n that goes Ω s.t. The measure of the sequence is finite.

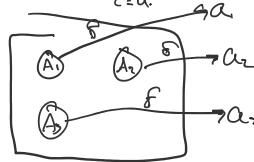
Note: Even if $M(\Omega) = \infty$

Step 1 $f \rightarrow$ simple function.

$$f = \sum_{i=0}^n a_i \mathbf{1}_{A_i}$$

$$M(A_i) < \infty, A_i \cap A_j = \emptyset, \text{ if } i \neq j$$

$$\int f d\mu = \sum_{i=0}^n a_i M(A_i)$$



Lemma A: Let φ and ψ be simple functions then

- 1) $\psi \geq 0 \Rightarrow I(\psi) \geq 0$
- 2) $I(a\psi) = aI(\psi), \forall a \in \mathbb{R}$
- 3) $I(\varphi + \psi) = I(\varphi) + I(\psi)$

$$\varphi \quad A_1 \quad A_2 \quad \dots$$

$$\left. \begin{array}{l} \text{where} \\ I(f) = \int f d\mu \end{array} \right\} \Omega$$

Properties of Integration

- 1) if $\varphi \geq 0, I(\varphi) \geq 0$
- 2) $I(a\varphi) = aI(\varphi) \quad \forall a \in \mathbb{R}$
- 3) $I(\varphi + \psi) = I(\varphi) + I(\psi)$

$$\text{3) } I(\varphi + \psi) = I(\varphi) + I(\psi) \quad | \quad \text{Ex. } \sum_{i=1}^n a_i \mu(A_i \cap B_i) = \sum_{i=1}^n a_i \mu(A_i) + \sum_{i=1}^n b_i \mu(B_i)$$

Envy Extension