04-04

Friday, April 4, 2025 11:30 AM

Brownian motion Construction using Hilbert Spaces.

Last step

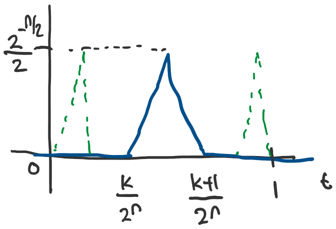
$$B(t) = \sum_{k=0}^{\infty} S_k(t) Z_k$$

$Z_k \overset{iid}{\sim} N(0,1)$

$2^n \leq k \leq 2^{n+1}$

$k' = k - 2^n$

$k' = 0, 1, \dots, 2^n - 1$



$$\frac{2^{-n/2}}{2} \geq \sum_{k=2^n}^{2^{n+1}-1} S_k(t) \geq 0 \quad \forall\, 0 \leq t \leq 1$$

$$\sum_{k=0}^{\infty} S_k(t_1) \cdot S_k(t_2) = t_1 \wedge t_2 = E(B_{t_1} \cdot B_{t_2}) = Cov(B_{t_1}, B_{t_2})$$

$0 \leq t_1, t_2 \leq 1$

$E(B_{t_1} \cdot B_{t_2}) = t_1 \wedge t_2$

$E[B_t) = 0$

Need to Prove Formal is Continuous.

$$B(t) = \sum_{k=0}^{\infty} S_k(t) Z_k$$

Basic Result

if $f_k(x) \in C[0,1]$ $k \geq 1$

and $\max_{0 \leq x \leq 1} |f_k(x)| \leq a_k$, $k = 1, 2, \dots$ when $\sum_{k=1}^{\infty} a_k < \infty$

then $\sum_{k=1}^{\infty} f_k(x) \in C[0,1]$

$$\sum^{\infty} S_k(t) Z_k$$

$$\sum_{k=0}^{\infty} S_k(t) Z_k$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=2^n}^{2^{n+1}-1} S_k(t) Z_k\right]$$

$$= \sum_{k=2^n}^{2^{n+1}-1} S_k(t)|Z_k| \le \left(\frac{C(\omega)}{2}\right)\sqrt{n+1}\cdot 2^{-n/2} < \infty$$

here we took maximum of Z_k

which converges By ratio test.

Last time $\omega \in \Omega$

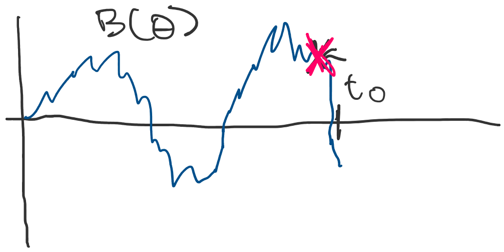
$\exists\ C(\omega) < \infty$ so that $|Z_k(\omega)| \le C(\omega)\sqrt{\operatorname{Log}(k)}$, $k \ge 2$.

if $2^n \le k < 2^{n+1}$

then $|Z_k(\omega)| \le C(\omega)\sqrt{n+1}$, $k \ge 2$.

next section.: Markov processes.

Markov Property. For Brownian motion



where will it go from Here?

$$\text{Dist of } \{B(t_0+h)\}_{h\ge 0} \mid \mathcal{F}^0_{t_0} = \sigma\{B_s\}_{0\le s\le t_0}$$

$$B(t_0+h) - B(t_0) \perp\!\!\!\perp \mathcal{F}^0_{t_0}$$

$$B(t+h_0) = B(t_0) + [B(t_0+h) - B(t_0)]$$

$$B(t_0) \in \mathcal{F}_{t_0}$$

$X(h) \equiv B(t_0+h) - B(t_0)$

what can we say About this?

$X(h) \sim N(0,h)$

$E(X(h)) = 0, \quad h \geq 0$

$Cov(X(h_1), X(h_2)) = h_1 \wedge h_2$

$(X(h))_{h\geq 0}$ has continuous sAmple Path.

$\{X(h)\}_{h\geq 0}$ is standard Brownian motion.

<u>SETUP</u> $\Omega = C[0,\infty)$, $B_t(\omega) = \omega(t)$, $t \geq 0$, $\omega \in \Omega$

$\mathcal{F} = \sigma\{B_t ; t > 0\}$

P = weiner measure Denoted By W

$\{P_x\}_{x\in\mathbb{R}}$, P_x is the probablity measure on Ω, $x \in \mathbb{R}$

$P_x(B_0 = x) = 1$

$P_x(A) = P_0(A - x)$, $A \in \mathcal{F}$

Dist of $\{B(t_0+h)\}_{h\geq 0}$

$= P_{B(t_0)}$

$\Updownarrow$

$P_x(x+A) = P_0$

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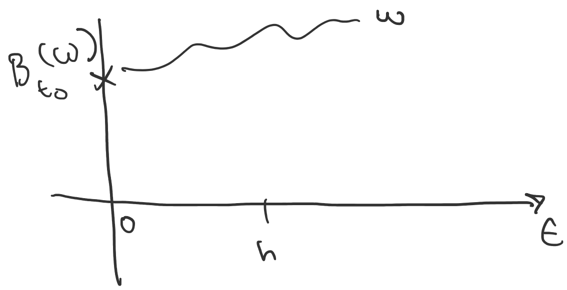
shift operator.

Let $s \geq 0$

$\theta_s(\omega) \in \Omega$

$\theta_s(\omega)(t) = \omega(s+t)$, $t \geq 0$



$\Theta_{t_0}(\omega)(h) = w(t_0+h), \ h \geq 0$

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$\mathcal{F}_t^0 = \sigma\{B_n, n \leq t\}, \ \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}, \ t_2 \geq t_1$

$\mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_s^0 \supset \mathcal{F}_t^0$

we will se $\mathcal{F}_t^0 = \mathcal{F}_t$

$\mathcal{F}_t^+$ is Right Continuous in $t \geq 0$

$\bigcap_{s>t} \mathcal{F}_s^+ = \mathcal{F}_t^+$

π - λ system. equility $\Rightarrow$

monotone class theroem (MCT) uses π-λ to Prove

Let $\mathcal{Q}$ be a collecton of events in Ω.
Assume $\mathcal{Q}$ is a π-system.

script H z

Let $\mathcal{H}$ be a collection of R.V.

assumption; ① $A \in \mathcal{Q} \Rightarrow \mathbb{I}_A \in \mathcal{H}$

② $f, g \in \mathcal{H} \Rightarrow x+y < f \in \mathcal{H}$

③ $f_n \geq 0, \ f_n \uparrow f, \ f$ is bdd $\Rightarrow f \in \mathcal{H}$
$c \in \mathbb{R}$

④ $1 \in \mathcal{H}$

then $\mathcal{H} > \{f \text{ bdd}, \ f \in \sigma\sqrt{\mathcal{Q}})$