03-12

Wednesday, March 12, 2025 11:30 AM

Last time: monday

① $(\Omega, \mathcal{F}_n, P)$ $E|X| < \infty$

$\{E_{\mathcal{F}}(X) ; \mathcal{F} \subset \mathcal{F}_0\}$ is UI

② $\{X, \mathcal{F}_n\}_{n \geq 1}$ U.I. and M.G. then

$X_n \xrightarrow[n\to\infty]{L^1 \text{ a.s.}} X$ $\quad X_n \xrightarrow{L^1} X \iff E|X_n - X| \xrightarrow[n\to\infty]{} 0$

③ $X_n = E_{\mathcal{F}_n}(X)$ $n \geq 1$

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theorem $(\Omega, \mathcal{F}_0, P)$ is Algebra Not σ-Alg.

Assume $E|X| < \infty$ $\mathcal{F}_n \uparrow \mathcal{F} \subset \mathcal{F}_0$ $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, $n \geq 1$ and $\sigma\{\bigcup_{n=1}^{\infty} \mathcal{F}_n\} = \mathcal{F}$ is

then $E_{\mathcal{F}_n}(X) \xrightarrow{\text{a.s. } L^1} E_{\mathcal{F}}(X)$ here convergence of filtration which is given.

Proof. 1st observation $\{E_{\mathcal{F}_n}(X), \mathcal{F}_n\}_{n \geq 1}$ MG. and it is UI

By Result 2, we get.

$E_{\mathcal{F}_n}(X) \xrightarrow{\text{a.s. } L^1} X_\infty \in \mathcal{F}$ what is say integrable measurable w.r.t $\mathcal{F}$

what is the relationship between X_n & X

Claim: $X_\infty = E_{\mathcal{F}}(X) \iff E(X_\infty ; A) = E(X ; A), \forall A \in \mathcal{F}$

we know. $X_n = E_{\mathcal{F}_n}(X_\infty) \iff E(X_n ; A) = E(X_\infty ; A), \forall A \in \mathcal{F}_n$

$n+1 > n, \therefore$ $\quad = E(X_{n+1} ; A) = E(X_n ; A) = E(X_\infty ; A)$

$X_n \xrightarrow{\text{a.s. } L^1} X_\infty$ $\quad E|X_n - X_\infty| \xrightarrow[n\to\infty]{} 0$

$E(X_n ; A) \xrightarrow[n\to\infty]{} E(X_\infty ; A)$ $\quad$ the L^1 Distance $E(|X_n - X_\infty| ; A)$ that is bigger than

$A \in \mathcal{F}_N$ $\quad N$ - fixed. (Arbitrary)

$\Rightarrow E(X_n ; A) = E(X_\infty ; A)$, $A \in \mathcal{F}_n$, $A \in \cancel{\bigcup_{n=1}^{\infty} \mathcal{F}_n}$ $\mathcal{F}$ By Dynkin.

$= E(X ; A)$

we need for All $\mathcal{F}$. "Dynkin" π-λ system.

¤ Application (a) $\{\mathcal{F}_n\}_{n \geq 1}$ Filtration $\mathcal{F} = \sigma\{\mathcal{F}_n, n \geq 1\}$

Let $A \in \mathcal{F}$ then $P_{\mathcal{F}_n}(A) \xrightarrow[n\to\infty]{\text{a.s. } L^1} \mathbb{1}_A$ $\quad$ $\mathcal{F}_n$ is more & more Refined information about omega

(b) New Proof of Kolgomorov 0-1 Law. $\quad$ then Either 0 or 1 Eventually

Kologorov 0-1 if seq $X_1 \ldots X_n$ Turn As $n \to \infty$
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Eventually

Kologorov 0-1 if seq $X_1, \ldots X_n$ Then As $n \to \infty$

Then tail event is either 0 or 1.

Let $\{X_n\}_{n\geq1}$ ind., $\mathcal{F}_n = \sigma\{X_1, \ldots X_n\}$, $T = \{\text{Tail Events}\}$

$T = \bigcap_{n=1}^{\infty} \sigma\{X_n, X_{n+1} \ldots\}$.

ecreasing set of σ-Alg.
which is σ-Alg.

Let $A \in T$

$\mathcal{F} = \sigma\{X_1, X_2, \ldots$

Ind of $\mathcal{F}_n$. ✓

$A \in \sigma\{\underline{X_{n+1}, \ldots}\}$

says this is #. (I guess constant)

Probability. $P_{\mathcal{F}_n}(A) = P(A) \xrightarrow[n\to\infty]{a.s. L'} 1_A$

$\Rightarrow P(A) \in \{0,1\}$.

Dominated convergence theorem for Condition Expectation

before general Result.

Lemma $\mathcal{F} \subset \mathcal{F}_0$ Let $X_n \downarrow 0$ a.s. ($\Rightarrow X_n \geq 0$, Decreasing), $E(X_1) < \infty$ ($\overset{LDL}{\Rightarrow}$ $E(X_n) \xrightarrow[n\to\infty]{} 0$) (Lebesgue Dominated convergence)

then $E_{\mathcal{F}}(X_n) \xrightarrow[n\to\infty]{a.s.} 0$

Proof $0 < E_{\mathcal{F}}(X_n) \downarrow X \geq 0 \overset{LDL}{\Rightarrow} E(E_{\mathcal{F}}(X_n)) \downarrow E(X)$ (wts $X = 0$

$= E(X_n) \xrightarrow[n\to\infty]{} 0$

$E_{\mathcal{F}}(X_1) \geq E_{\mathcal{F}}(X_n) \geq 0$

$\therefore E(X) = 0$

$0 < EX = E[E_{\mathcal{F}}(X_1)] \geq E[E_{\mathcal{F}}(X_n)]$. Since $X \geq 0$ we get a.s. $X = 0$.

Extension

$X_n \xrightarrow{a.s.} 0$ $|X_n| \leq Y$, $n \geq 1$, $E(Y) < \infty$

then $E_{\mathcal{F}}(X_n) \xrightarrow[n\to\infty]{a.s.} 0$

Proof $|X_n| \xrightarrow[n\to\infty]{a.s.} 0$ Then $\overline{\lim_{n\to\infty}}\{|X_n|\} = 0$ a.s.

Monotone sequence Decreasing

$\lim\sup_{n\to\infty} \{|X_n|, |X_{n+1}|, \ldots$

Monotone goes to 0.

$Y \geq Y_n = \sup\{|X_n|, |X_{n+1}|, \ldots\} \downarrow 0$ a.s.

$\Rightarrow E_{\mathcal{F}}(Y_n) \xrightarrow{a.s.} 0$

$0 \leq E_{\mathcal{F}}(|X_n|) \xrightarrow{a.s.} 0$

thus a Dominated convergence.

Lemma let $X_n \xrightarrow[n\to\infty]{a.s.} 0$, $|Y| \geq X_n$, $n \geq 1$ $E(Y) < \infty$

then $E_{\mathcal{F}}(X_n) \xrightarrow[n\to\infty]{a.s.} 0$

Why not uniformly in? is not working

Example $\{Y_n \perp\!\!\!\perp Z_n\}_{n\geq1}$ all IND

Example $\{Y_n \perp\!\!\!\perp Z_n\}_{n\geq 1}$ all IND

$P_1 \; \frac{1/n}{1-1/n} \; \begin{matrix}1\\0\end{matrix} \qquad Z_n \; \frac{1/n}{1-\frac{1}{n}} \; \begin{matrix}0\\1\end{matrix}$

$X_n = Y_n Z_n$, $E(X_n) = E(Y_n) = E(Z_k) = \frac{1}{K} = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

$\mathcal{F} = \sigma\{Y_1, Y_2$

$$E_{\mathcal{F}}(X_n) = E_{\mathcal{F}}[Y_n Z_n]$$

$= Y_n \cancel{E_{\mathcal{F}}(Z_n)}$ since IND,

$= Y_n$

$\sum P(Y = 1) = \sum \frac{1}{n} = \infty$ by BCII

when you go to integrable, we lose