L10 - 09-18

Wednesday, September 18, 2024 11:31 AM

Example Monotone convergence theorem

$0 \le f_n$, $f_n \uparrow f$ then $I(f_n) \xrightarrow[n\to\infty]{} I(f)$

Take $g_n \ge 0$

$$0 \le S_n(x) = \sum_{k=1}^{n} g_k(x).$$

$$S_{n+1}(x) = \left[\sum_{k=1}^{n} g_k(x)\right] + g_{n+1}(x) \ge S_n(x)$$

Apply MCT.

$$\sum_{k=1}^{n} I(g_k) = I(S_n) \xrightarrow[n\to\infty]{} I\left(\sum_{k=1}^{\infty} g_k\right)$$

$\downarrow n\to\infty$

$$\sum_{k=1}^{\infty} I(g_k)$$

Conclusions.

$$I\left(\sum_{k=1}^{\infty} g_k\right) = \sum_{k=1}^{\infty} I(g_k) \quad g_k \ge 0,\ k = 1, 2, \dots$$

Fatou's. Example.

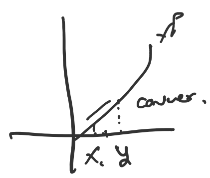
$$\|f\|_p = \left[\int_\Omega |f|^p \, d\mu\right]^{1/p}, \quad p \ge 1 \to \text{convex}$$

Δ - ineq.

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

Problem if $f_n \xrightarrow[n\to\infty]{a.e.} f$ and $\|f_n\|_p \xrightarrow{n\to\infty} \|f\|_p < \infty$.

observe: $\left(\frac{|x|+|y|}{2}\right)^p \le \frac{|x|^p + |y|^p}{2}$, $p \ge 1$



Says line connecting 2 Points in Convex Function is greater.

$$2^{p-1}(|x|^p + |y|^p) - |x-y| \ge 0$$

$x, y \in \mathbb{R}$

Fatou if $f_n \ge 0$ then $I(\underline{\lim} f_n) \le \underline{\lim} I(f_n)$.

take lim inf $2^{p-1}(|f_n|^p + |f|^p) - |f - f_n|^p$

$$= I\ 2^p |f|^p + \cancel{\underline{\lim}_{n\to\infty} |f - f_n|^p} = 2^p I(|f|^p)$$

$\|f\|_p^p$

this side

$$\underline{\lim}\left[2^{p-1}(\cancel{|f|^p + |f_n|^p})\right] + I(-|f - f_n|)$$

$2|f|^p$

$$= 2^{p-1} \cdot 2. \ + \underline{\lim} - I(|f - f_n|) \ge 2^p \|f\|^p$$

$$-\overline{\lim}\ I(|f-f_n|^p) \geq 0$$

$$0 \leq \underline{\lim}\ I(|f-f_n|^p) \leq \overline{\lim}\ I(|f-f_n|^p) \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow I(|f-f_n|^p) \xrightarrow[n\to\infty]{} 0$$

$$= \|f-f_n\|_p^p \xrightarrow[n\to\infty]{} 0$$

" in General liminf Does not have

Change of Variable formula

$\Omega_1 \to \Omega_2$

$(\Omega_1, \mathcal{F}_1, \mu_1)$ $\quad$ $(\Omega_2, \mathcal{F}_2)$

$\varphi : \Omega_1 \to \Omega_2$ $\quad$ φ is $\mathcal{F}_1 / \mathcal{F}_2$ measurable

$\varphi^{-1}(B) \in \mathcal{F}_1$, $\quad B \in \mathcal{F}_2$

$$\Omega_1 \xrightarrow{\varphi} \Omega_2 \xrightarrow{f} \mathbb{R}.$$

DNE

$$\int_{\Omega_1} f \circ \varphi \, d\mu_1 = \int_{\Omega_2} f \, d\mu_2, \quad \mu_2 = ?$$

μ_2 will be a Push of μ_1

Push Formula

$$\mu_2(B) = \mu_1(\varphi^{-1}(B)), \quad B \in \mathcal{F}_2$$

use it if $f \geq 0$ or $f \circ \varphi$ is integrable on Ω.

Definition of integrable

if $\int_\Omega |f \circ \varphi| \, d\mu < \infty$

μ_1, $P(\Omega_1) = 1$

EX: $\Omega_2 = \mathbb{R}$, $\varphi \to X$ is r.v.

$$\Omega_1 \xrightarrow{\varphi} \mathbb{R} \xrightarrow{f} \mathbb{R}$$

CDF: $F_X(x) = P(X \leq x)$, $-\infty < x < \infty$.

$$\mu((a,b]) = F_X(b) - F_X(a), \ldots$$

PDF

$$\mathbb{E}[f(X)] = \int_{\mathbb{R}} f(x) \, dF_X(x) = \int_{\mathbb{R}} f(x) \cdot f_X(x) \, dx$$

we don't integrate on Prob space instead Real Line'

Dominate convergence theorem.

σ-finite measure

If $|f_n| \leq g$, $I(|g|) < \infty$, $f_n \xrightarrow[n\to\infty]{a.e.} f$

Then $I(f_n) \xrightarrow[n\to\infty]{} I(f)$

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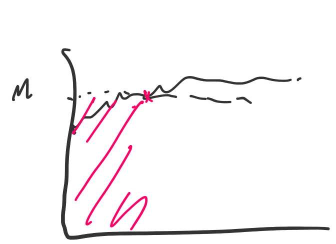
Uniform Integrability. $(\Omega, \mathcal{F}, P)$ $\quad P(\Omega) = 1$

Def $\{X_n\}_{n\geq 1}$ is U.I. if $\varphi(M) \xrightarrow[M\to\infty]{} 0$

where $\varphi(M) = \sup_{n\geq} E\left(|X| \cdot \mathbb{1}_{\{|X_n| \geq M\}}\right)$



integrate only where above M.

<u>Result</u>: if $\{X_n\}_{n\geq 1}$ U.I. and $X_n \xrightarrow[n\to\infty]{a.s.} X$

then $\mathbb{E}|X_n - X| \xrightarrow[n\to\infty]{} 0$

$\mathbb{E}(X_n) \xrightarrow[n\to\infty]{} \mathbb{E}(X)$

if $|X_n| \leq Y$, $\forall$, and $\mathbb{E}(Y) > \infty$

then $\{X_n\}_{n\geq 1}$ U.I.

Proof: $\mathbb{E}(Y : Y \geq M) \xrightarrow[M\to\infty]{} 0$

$\quad\quad\quad\quad \vee\!|$

$\sup_{n\geq 1} \mathbb{E}(|X_n| : |X_n| > M)$