04-14

Monday, April 14, 2025 11:30 AM

Example (From Book)

$$\overline{\lim_{n\to\infty}} \frac{B_t}{\sqrt{t}} = +\infty \quad a.s. \qquad \left(\Rightarrow \overline{\lim_{t\to\infty}} B_t = +\infty \text{ , a.s} \right)$$

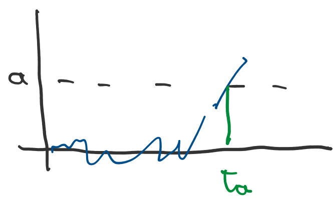
$$\Rightarrow T_a < \infty \text{ a.s}$$

By continuity of sample pass.

$$\underline{\lim_{n\to\infty}} \frac{B_t}{\sqrt{t}} = -\infty \quad a.s.$$

2

$$T_a = \inf \{ t > 0 : B_t = a \}$$ is Almost surely Finite for all a



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Proof of $\odot$

we can Prove this without this chapter.

$$\overline{\lim_{n\to\infty}} \frac{B_n}{\sqrt{n}} = +\infty \quad a.s.$$

$$B_n = \sum_{k=1}^{n} \underbrace{(B_k - B_{k-1})}_{\{Z_n\} \overset{iid}{\sim} N(0,1)}$$

2

$$S_n = \sum_{k=1}^{n} Z_k \text{ , } \{Z_k\}_{k\geq 1} \text{ iid } N(0,1)$$

ADDED Let (1)

$$P\left(\overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{n}} > 1 \right) \geq P(Z > 1) > 0$$

$$\frac{S_n}{\sqrt{n}} \sim N(0,1)$$

$$P\left(\frac{S_n}{\sqrt{n}} > 1 \right) = P(Z > 1) > 0$$

(1)

[illegible]

(1)

$$P\left\{\overline{\lim_{n\to\infty}}\left\{\frac{S_n}{\sqrt{n}}\right\} > 1\right\} = P\left\{\lim_{n\to\infty} \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left\{\frac{S_k}{\sqrt{n}} > 1\right\}\right\}$$

Decreasing intersection.

$$\geq \underbrace{\overline{\lim_{n\to\infty}}\, P\left(\frac{S_n}{\sqrt{n}} > 1\right)}_{\text{Permutable Event.}} = P(Z > 1)$$

we have two Rules for tails

1) Kolmogorov 0-1
2) Heavy savage 0-1 Req iid.

then H.S. 0-1 law $\Rightarrow$ $P\left(\overline{\lim}\left\{\frac{S_n}{\sqrt{n}}\right\} > M\right)$, $M < \infty$

1

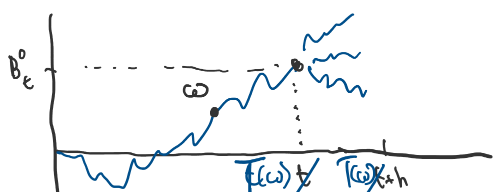
For Proof of (2)

$$\overline{\lim_{t\to\infty}} \frac{B_t}{\sqrt{n}} \geq \overline{\lim_{n\to\infty}} \frac{B_n}{\sqrt{n}}$$

Next section: Strong Markov Property.

Strong Markov Property For BM.

Markov Property of BM.



new Brownian motion from the ω.

Replace t with stopping time T(ω)

$$\mathcal{F}^0_t = \sigma\{B_s ;\ 0 \leq s \leq t\},\ t \geq 0.$$

$$\hat{\mathcal{F}}^+_t = \bigcap_{n=1}^{\infty} \mathcal{F}^0_{t+\frac{1}{n}}$$

$\mathcal{F}^+_t = \mathcal{F}_t$ up to null set.

So where here we have the usual conditions w.r.t. filtration.

canonical filtration.

script N.

$$\hat{\mathcal{F}}_t = \mathcal{F}^+_t \vee \mathcal{N} \supset \mathcal{F}^+_t \supset \mathcal{F}^0_t$$

these are all same upto Null set

What Does this Imply? see prev lecture.

$\mathcal{N} = \{A \in \mathcal{F}^0 : P(A) = 0 \ \forall \ldots\}$

$\mathcal{F}_t = \mathcal{F}_t \vee \mathcal{N} \supset \mathcal{F}_t \supset \mathcal{F}_t$

these Are All same · upto Null set

$\mathcal{N} = \{A \in \mathcal{F}^0_\infty ; P_x(A) = 0, \forall x \in \mathbb{R}\}.$

$P_x(B_{1000} = 7) = 0 \ \forall x \in \mathbb{R}.$

2

$\Omega = C[0,\infty]$

Def $T: \Omega \to [0,\infty]$ is stopping Time (S.T.)

if $\{T \leq t\} \in \mathcal{F}_t, \quad 0 \leq t < \infty$

observe if $\{T < t\} \in \mathcal{F}_t, \quad 0 < t < \infty$ then T is a S.T.

Similiar to Discrete. But in Discrete $\geq \Rightarrow >$ gives Predictable. here No.

1

$$\bigcap_{n=1}^{\infty} \mathcal{F}_{t+\frac{1}{n}} = \mathcal{F}_t$$

2

Proof

$$\{T \leq t\} \in \mathcal{F}_t \quad t \geq 0 \quad \Rightarrow \quad \underbrace{\bigcup_{n=1}^{\infty} \{T \leq t - \tfrac{1}{n}\}}_{\in \mathcal{F}_t} \overset{\text{By eq.}}{=} \underbrace{\{T < t\}}_{\in \mathcal{F}_t}$$

now the other way.

$\{T < t\} \in \mathcal{F}_t, \ \forall t > 0$

$$\underbrace{\bigcap_{n=1}^{\infty} \underbrace{\{T < t + \tfrac{1}{n}\}}_{\in \mathcal{F}_{t+\frac{1}{n}}}}_{\mathcal{F}_t} = \underbrace{\{T \leq t\}}_{\in \mathcal{F}_t}$$

By intersection $\mathcal{F}_t$ in $\in \mathcal{F}_t$

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Some Properties.

① $T_n \uparrow T$ a.s. T_n is S.T., $n \geq 1$
Then T is S.T. as well.

What is the Relationship Between t & T?

Proof

$$\{T \leq t\} = \underbrace{\bigcap_{n=1}^{\infty} \{T_n \leq t\}}_{\mathcal{F}_t} \in \mathcal{F}_t, \ \forall t > 0$$

why eq?

$T_n \uparrow T$; $T_n \leq T$, $\forall n.$ since $T_n \uparrow T$

② $T_n \downarrow T$, T_n S.T. $\forall n$

then T is a S.T.

$$\underbrace{\bigcup_{n=1}^{\infty} \{T_n < t\}}_{\in \mathcal{F}_t} = \underbrace{\{T < t\}}_{\mathcal{F}_t} \quad \forall t \geq 0$$

There must some n: t_n Does True job.

③ if T is a S.T.

Then $\exists T_n \downarrow T$ where T_n is S.t $\forall n \geq 1$

where $T_n \in Q_{2,n} = \{\frac{k}{2^n} ; k \geq 1\}$

Proof.

$T_n = \frac{k+1}{2^n}$ if $\frac{k}{2^n} \leq T < \frac{k+1}{2^n}$

of course this converges to T

show T_n is S.T. $\forall n$

1

④ Let S, T be S.T.

then $S \wedge T$, $S \vee T$, $S+T$ are All S.T.

Proof that $S+T$ is a S.T.

$\exists S_n \downarrow S$, S_n S.t. and $\exists T_n \downarrow T$ T_n S.T

then $S_n + T_n \downarrow S + T$

and $S_n + T_n$ is S.T. $\forall n$.

So we use ②

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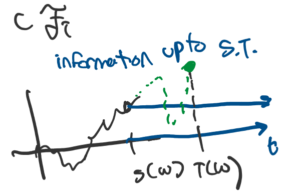
⑤ $S \leq T$ a.s. S, T are S.T. then $\mathcal{F}_S \subset \mathcal{F}_T$



Def Let S be S.T. then

$\mathcal{F}_S = \sigma\{B_{t \wedge S},\ t \geq 0\}$

Alternatively: $\mathcal{F}_S = \{A : A \cap \{S \leq t\} \in \mathcal{F}_t \ \forall 0 \leq t < \infty\}$

But the same.

$= \{A : A \cap \{S < t\} \in \mathcal{F}_t \ \forall 0 \leq t < \infty\}$.

2.5

⑥ $T \in \mathcal{F}_T$.

NTS: $\{T \leq t_0\} \cap \{T \leq t\} \in \mathcal{F}_t$, $\forall t$

if $t_0 < t$ then $\{T \leq t_0\} \in \mathcal{F}_t$.

$t_0 \geq t$ then intersection $\in \mathcal{F}_t$.

if $t_0 < t$ then $\{T \le t_0\} \in \mathcal{F}_t$.
$t_0 \ge t$ then intersection $\in \mathcal{F}_t$.

⑦ Let S, T be s.t. then

$\{S < T\}, \{S > T\}, \{T = S\} \in \mathcal{F}_T \cap \mathcal{F}_S$.

show $\{S < T\} \in \mathcal{F}_T \cap \mathcal{F}_S$.

$\{S < T\} \cap \{S < t\} \in \mathcal{F}_t \quad \forall t \ge 0$.

$= \bigcup_{\substack{q < t \\ q \in \mathbb{Q}}} \{S < q\} \cap \{T > q\}$

Next Formal Markov Property.