L23 - 10-18 SLLN

Friday, October 18, 2024 11:32 AM

Head Runs "talk about symmetric Bernoulli" Example 2.3.3 Head runs.

$P(X=1) = \frac{1}{2} = P(X=-1)$

$\{X, X_k\}_{k \in \mathbb{Z}}$ i.i.d.

$\ell_n = \max\{m : X_{n-m-1} = X_{n-m-2} = \dots = X_n = 1\}$

$-1, 1, 1, 1, \underset{n=2}{\underset{\uparrow}{1}}$ $\{\ell_n\}_{n \geq 1}$ identically distrib,

$\ell_2 = 4.$

What is?

$P(\ell_n = 0) = 1/2$ $P(L_n = k) = \frac{1}{2^{k+1}}$

Prove that $\lim_{n\to\infty} \frac{\ell_n}{\log_2(n)} = 0$ a.s. $P(\ell_n \geq k) = \sum_{m=k}^{\infty} \frac{1}{2^{m+1}} = \frac{1}{2^k}$, $k = 0, 1 \dots$

$P(\ell \leq k-1) = 1 - \frac{1}{2^k} \leq e^{-2^{-k}}$

$\sum_{k=1}^{\infty} P(X_k = -1) = \infty$ $\{X_k\}_{k \geq 1}$ IND

$P(X_k = 0 \text{ i.o}) = 1$

$P(\ell_n = 0 \text{ i.o}) = 1$

Longest chain is of "-1" or "1" is.

Goal: $\lim_{n\to\infty} \frac{L_n}{\log_2(n)} = 1$, a.s. $L_n \equiv \max_{m=1,2,\dots,n} \ell_m$, $n = 1, 2, 3, \dots$

$\varepsilon > 0$

$P(L_n > (1+\epsilon)\log_2(n)) = n^{-(1+\varepsilon)}$

$\sum$ " $\leq \sum n^{-(1+\varepsilon)} < \infty$

BCI

$\Rightarrow$, $P\left(\frac{\ell_n}{\log_2(n)} > 1+\varepsilon, \text{ i.o.}\right) = 0$

Arbitrary.

$\overline{\lim_{n\to\infty}} \frac{\ell_n}{\log_2(n)} \leq 1 \cancel{+\varepsilon}$ a.s.

$\Rightarrow \overline{\lim_{n\to\infty}} \frac{L_n}{\log_2(n)} \overset{?}{\leq} 1$ a.s.

$\frac{L_n}{\log_2(n)} = \frac{\ell_{k_n}}{\log_2(n)}$ $k_n \in \{1, \dots, n\}$

$$\le \frac{\ell_{k_n}}{\log_2(k_n)}$$

$$\underline{\lim}_{n\to\infty} \frac{\ell_n}{\log_2(n)} \le \overline{\lim}_{n\to\infty} \frac{\ell_n}{\log_2(n)} \le \overline{\lim}_{n\to\infty} \frac{\ell_n}{\log_2(n)} \le 1 \quad \text{a.s.}$$

$$\Rightarrow \overline{\lim}_{n\to\infty} \frac{\ell_n}{\log_2(n)} = \lim_{n\to\infty} \frac{L_n}{\log_2(n)}$$

<u>trick</u>
look at max.
As Bounds of Seq.

Goal $\underline{\lim} \frac{L_n}{\log_2(n)} \ge 1$ a.s.

$1, \dots, n$

$[1, \dots, (1-\epsilon)\log_2(n)]$,

$((1-\epsilon)\log_2(n), \dots 2(1-\epsilon)\log_2(n)]$

$(2(1-\epsilon)\log_2(n), \dots, 3(1-\epsilon)\log_2(n)$

$\vdots$

$n_k = k\cdot(1-\epsilon)\log_2(n)$

$$P(L_n < (1-\epsilon)\log_2(n))$$

$$\le P\left(\bigcap_{k=1}^{n/\log(n)} \{\ell_{n_k} \le (1-\epsilon)\log_2(n)\}\right)$$

$$\prod_{k=1}^{n/\log_2(n)} (1 - n^{-(1-\epsilon)})^{n/\log_2(n)}$$

$$\le \left[e^{-n^{-(1-\epsilon)}}\right]^{n/\log_2(n)} = e^{-n^{\epsilon}/\log_2(n)} \le \frac{1}{n^2} \quad \text{for } n \ge N$$

"Prove: take log of each side"

$$\sum_{n=1}^{\infty} P(L_n < (1-\epsilon)\log_2(n)) \le \sum_{n=1}^{\infty} e^{-n^{\epsilon}/\log_2(n)} < \infty$$

$$\lim_{n\to\infty} \frac{L_n}{\log_2(n)} > 1-\epsilon, \quad \text{a.s.}$$

$$\Rightarrow \lim_{n\to\infty} \frac{L_n}{\log_2(n)} \ge 1, \quad \text{a.s.}$$

Strong law of large numbers

Strong law of large numbers

SLLN $\{X_k\}_{k\ge1}$ are iid are Pairwise ind. Also $E|X| < \infty$

then $\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s.} \mu = E(X)$

Proof

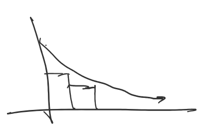
<u>Step 1</u> $Y_k = X_k \cdot \mathbb{1}_{\{X_k \le k\}}$

Claim: $P(Y_k \ne X_k \text{ i.o.}) = 0$

BCI $P(Y_k \ne X_k) = P(|X_k| > k)$

$$\sum_{k=1}^{\infty} P(Y_k \ne X_k) \le \sum_{k=1}^{\infty} P(|X_k| \ge k) \le E(|X|) < \infty$$

$$\text{''} \int_{x=0}^{\infty} P(|X| \ge x)\,dx$$



use BC I to finish proof

$\{X_k^+\}_{k\geq 1}$, $\{X_k^-\}_{k\geq 1}$

$X_k = X_k^+ - X_k^-$

$\mathbb{E}(X_k^-) \vee \mathbb{E}(X_k^+) < \mathbb{E}|X| = \mathbb{E}[X_k]$

$\mu^+ = \mathbb{E}(X^+)$

$\mu^- = \mathbb{E}(X^-)$

Ets: $T_n = \sum_{k\geq 1}^{n} X_k^+$ $\quad \frac{T_n}{n} \xrightarrow[n\to\infty]{a.s.} \mu^+$

$U_n = \sum_{k=1}^{n} X_k^-$

$\frac{U_n}{n} \xrightarrow[n\to\infty]{a.s.} \mu_-$

$\mu^+ - \mu^- = \mathbb{E}(X) = \mu.$

Ets

$\Rightarrow \frac{\sum_{k=1}^{n} Y_k}{n} \xrightarrow[n\to\infty]{a.s.} \mu$