01-31 Cond. Ex.

Friday, January 31, 2025 11:32 AM

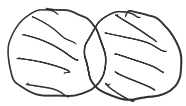
Basics

① $P(A \Delta B) \leq P(A \Delta C) + P(C \Delta B)$

② if $P(A_n \Delta A) \xrightarrow[n\to\infty]{} 0$ then $P(A_n) \to P(A)$ and $P(A_n \cap A) \to P(A)$

③ if $P(A_n \Delta A) \xrightarrow[n\to\infty]{} 0$ and $\underbrace{P(B_n \Delta A) \xrightarrow[n\to\infty]{} 0}_{P(B_n \cap A) \to P(A)}$

then $P(A_n \cap B_n) \xrightarrow[n\to\infty]{} P(A)$



$A_n \Delta A$

$P(A_n \Delta A)$

$= P(A_n) - P(A_n \cap A)$

$+ P(A) - P(A_n \cap A)$

$\vee$ $P(A_n \cap A) \to P(A)$

$P(A_n) \to P(A)$

① $\Rightarrow$ $P(A_n \Delta B_n) \xrightarrow[n\to\infty]{} 0$

$P(A_n) - P(A_n \cap B_n) \to 0$

$\downarrow$ $P(A)$ $\qquad$ $\downarrow$ $P(A)$

Hewitt-savage 0-1

Let $\{X_k\}_{k\geq 1}$ i.i.d.

Let $A \in \sigma\{X_k\}_{k\geq 1}$ be permutatable $(\pi A = A,\ \forall \pi$ - Finite permutation$)$

$P(A) = P(\pi A)$ $\forall A \in \sigma\{X_k\}_{k\geq 1}$

Example

Consider $A = \{\overline{\lim_{n\to\infty}} S_n > 55\}$

Then $P(A) \in \{0, 1\}$.

Proof take $A \in \sigma\{X_k\}_{k\geq 1}$

$\leftarrow$ Approximate

$\exists A_n \in \sigma\{X_1, \dots, X_n\}$, $n \geq 1$ so that $P(A_n \Delta A) \xrightarrow[n\to\infty]{} 0$

So we know $P(A_n) \to P(A)$

Assume A is permutable. (from now on)

take a seq of $\{\pi\}_{n\geq 1}$ of Finite permutation.

$P(A_n \Delta A) = P(\pi_n(A_n \Delta A)) = P(\pi_n(A_n) \Delta A) \xrightarrow[n\to\infty]{} 0$

$\downarrow$ $\pi_n(A) = A$ set eq.

$= P(\pi_n(A_n) \cap A_n) \xrightarrow[n\to\infty]{} P(A)$

if $\pi_n(A_n)$, A_n are IND,

$= P(\pi_n(A_n))P(A_n) = P^2(A_n) \to P^2(A)$

Say $n=3$

$(1,2,3,4,5,6) \xrightarrow{\pi_3} (4,5,6,1,2,3)$

$A_3 \in \sigma(X_1, X_2, X_3)$

$\pi_3(A_3) \in \sigma\{X_4, X_5, X_6\}$

New chapter: Conditioning. (Durret ch. 5)

$(\Omega, \mathcal{F}_0, P)$ $\quad \mathcal{F} \subset \mathcal{F}_0$

X r.v., $E(|X|) < \infty$

$E(X|\mathcal{F}) \in E_{\mathcal{F}}(X)$

$E_{\mathcal{F}}(X) = Y \iff$ 1. $Y \in \mathcal{F}$

2. $E(X;A) = E(Y;A)$, $A \in \mathcal{F}$.

(2) $\Rightarrow$ (3) $E(X \cdot Z) = E(Y \cdot Z)$ $Z \in \mathcal{F}$ and $|Z| < C$ a.s.

Examples

Assume $E_{\mathcal{F}}(X) = Y$ $\qquad E_{\mathcal{F}}(X) = Y'$ then $Y = Y'$ a.s.

if not then $P(Y - Y' > 0) + P(Y - Y' < 0)$

create contradiction.

Assume wlog $P(\underbrace{Y - Y'}_{A} > 0) > 0$

$A \in \mathcal{F}$

$E[Y';A] = E(X;A) = E(Y;A)$

$\Rightarrow E(Y - Y';A) = 0$

this can't happen $P(A) > 0$ $\rightarrow\leftarrow$

Example (1) if $X \in \mathcal{F}$ then $E_{\mathcal{F}}(X) = X$

[illegible]

Example ① if $X \in \mathcal{F}$ then $E_{\mathcal{F}}(X) = X$

② if $\{X, \mathcal{F}\}$ ind then $E_{\mathcal{F}}(X) = E(X)$

Prove 2 $\quad E(X \cdot Z) = E(E(X)\, Z)$, $\forall Z$ bdd $Z \in \mathcal{F}$

$E(X)\,E(Z)$

$\Omega \qquad \mathcal{F} = \sigma\{A_k\}_{1 \le k \le n}$

A_1	A_2	A_3
A_4	A_5	A_6

$E_{\mathcal{F}}(X)(\omega) = \dfrac{E(X ; A_k)}{P(A_k)}$ if $\omega \in A_k$ $k = 1, \dots, 6$

The purpose of transformation

For

Idea of Conditional Expectation.

Radon - Nikodym.

$(\Omega, \mathcal{F}_0)$ Measure space.

$\mu \ll P$ (μ is Absolutely continuous w.r.t. P)

$P \sim$ sigma-finite measure

if $A \in \mathcal{F}_0$ and $P(A) = 0$ then $\mu(A) = 0$

Ex Let $f_{X,Y}(x, y)$ is joint density of $(X, Y) \in \mathbb{R}^2$

then $\displaystyle E_{\sigma\{X\}}(Y) = \int_{-\infty}^{\infty} y\, f_{Y|X=x_0}(x_0, y)\, dy$

$$f_{Y|X=x_0}(x_0, y) = \frac{f_{XY}(x_0, y)}{f_X(x_0)} \qquad f_X(x_0) = \int_{-\infty}^{\infty} f(x_0, y)\, dy$$