Preliminary Exam: Probability.

Time: 9:00am - 2:00pm, Friday, August 28, 2015

Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.
The exam consists of six main problems, each with several steps designed to help you in the overall solution.

Important: If you cannot solve a certain part of a problem, you still may use its conclusion in a later part!

Please make sure to apply the following guidelines:

1. On each page you turn in, write your assigned code number. Don't write your name on any page.

2. Start each problem on a new page.

3. Write only on one side of a page.

1. Let $\{X, X_k\colon\ k = 1,2,\dots\}$ be random variables and let $S_n = \sum_{k=1}^{n} X_k$.
   a. Prove: $E|X| < \infty$ if and only if $\sum_{n=1}^{\infty} P(|X| \geq n) < \infty$
   b. Prove: If $\frac{S_n}{n} \to c$, a.s. where $-\infty < c < \infty$, then $\frac{X_n}{n} \to 0$, a.s.
   c. Assume now that $\{X, X_k\colon\ k = 1,2,\dots\}$ are **i.i.d** random variables. **Show by using the two earlier parts** that if $\frac{S_n}{n} \to c$, a.s. where $-\infty < c < \infty$, then $E|X| < \infty$.

2. Let $\{X, X_k\colon\ k = 1,2,\dots\}$ be random variables. Assume that $X_k \to X$ in probability.
   a. Assume that $\sup_{k\geq 1} E|\,X_k| < \infty$.
      (i) Prove that $E|X| < \infty$.
      (ii) Is it true that $E|\,X_k| \to E|X|$? Prove or provide a counter example.
   b. Now assume: $\sup_{k\geq 1} E|\,X_k|^{1+\alpha} < \infty$, where $\alpha > 0$. Prove: $E|\,X_k - X| \to 0$.

3. Let $\{G_k, X_k\colon\ k = 1,2,\dots\}$ be random variables. Assume that $E\left(X_k^2\right) = 1, G_k \sim N\left(0, \sigma_k^2\right), k = 1,2,\dots$. Assume also that $G_1, G_2, \dots$ are independent and the two families of random variables $\{X_k\}$ and $\{G_k\}$ are independent as well.
   a. Let $S_n = \sum_{k=1}^{n} G_k X_k$, $n \geq 1$. Calculate $E[(G_1X_1)\cdot(G_2X_2)]$ and $E(S_n^2)$.
   b. Prove that $\{S_n\colon\ n = 1,2,\dots\}$ is a martingale sequence with respect to the sequence of σ-algebras $\sigma\{G_k, X_k\colon\ k = 1,\dots,n\}$, $n = 1,2,\dots$
      Hint: If $\mathcal{F} \subset \mathcal{G}$ are σ-algebras and Y is a random variable then $E(E(Y|\mathcal{G})|\mathcal{F}) = E(Y|\mathcal{F})$.
   c. Assume that $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$. Decide if each of the following series converges a.s. Explain your reasoning.
      (i). $\sum_{k=1}^{\infty} G_k X_k$
      (ii). $\sum_{k=1}^{\infty} G_k$
      (iii). $\sum_{k=1}^{\infty} G_k^2$

4. Let $\{Y_k\colon\ k = 1,2,\dots\}$ be independent random variables and let $X_k = \begin{cases} Y_k & \text{if } |Y_k| < 1 \\ 0 & \text{if } |Y_k| \geq 1 \end{cases}$

Assume that $\sum_{k=1}^{\infty} \mathrm{Var}(X_k) = \infty$.

a. Prove that $\frac{\sum_{k=1}^{n} X_k - E(X_k)}{\sqrt{\sum_{k=1}^{n} \mathrm{Var}(X_k)}} \Rightarrow Z$, as $n \to \infty$, where $Z \sim N(0,1)$ and $\Rightarrow$ stands for convergence in distribution.

b. Prove that if $\sum_{k=1}^{\infty} Y_k$ converges a.s. then

(i) $\sum_{k=1}^{\infty} P(X_k \neq Y_k) < \infty$, and

(ii) $\sum_{k=1}^{\infty} X_k$ converges a.s.

c. Prove that if $\sum_{k=1}^{\infty} Y_k$ converges a.s. then $\frac{\sum_{k=1}^{n} E(X_k)}{\sqrt{\sum_{k=1}^{n} \mathrm{Var}(X_k)}} \Rightarrow Z$. Do you think that this result makes sense?

5. There are 3 locations on a circle denoted by 1, 2, 3. The movement in the direction $1 \to 2 \to 3$ is clockwise. A particle moves between the locations according to the following rule: It moves one step clockwise with probability p where $0 < p < 1$ and it moves one step counterclockwise with probability $1 - p$.

a. What is the transition matrix, P, of the Markov chain that represents the particle movement?

b. Prove that P^n converges as $n \to \infty$. Write down the limit matrix.

6. Let $\{W(t), t \geq 0\}$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$. It is known that with probability 1, the function $t \mapsto W(t)$ is nowhere differentiable. The problem is designed to prove by an elementary step-by-step method a stronger result: Let $\gamma > \frac{5}{6}$ be a fixed constant throughout. Then for any constant $C > 0$

(*) $\quad P\{\exists\, \mathrm{s} \in [0,1] \text{ and } \delta > 0 \text{ so that } |W(t) - W(s)| \leq C|t-s|^{\gamma}, \text{ when } |t-s| \leq \delta\} = 0.$

a. In (i), (ii) and (iii) below the constant $C > 0$ is given and you have to find, in each case, constants $K > 0, \beta > 0$ that are as small as possible so that the inequality holds (K, β may be different in each of the inequalities!)

(i). $P\left(|W\left(\frac{1}{n}\right)| \leq Cn^{-\gamma}\right) \leq K \cdot n^{-\gamma+\beta}, \; n \geq 1$. [Hint: reduce it to inequality about $Z \sim$N(0, 1).]

(ii). $P(\max\{|W\left(\frac{1}{n}\right)|, |W\left(\frac{2}{n}\right) - W\left(\frac{1}{n}\right)|, |W\left(\frac{3}{n}\right) - W\left(\frac{2}{n}\right)|\} \leq Cn^{-\gamma}) \leq K \cdot n^{-3\gamma+\beta}, \; n \geq 1.$

(iii). $P(B_n) \leq K \cdot n^{-3\gamma+\beta}, \; n \geq 1$, where B_n is defined by

$$B_n = \left\{\exists\; k \in \{1, \dots, n\} \text{ so that } \max_{j=0,1,2} \{|W\left(\frac{k+j}{n}\right) - W\left(\frac{k-1+j}{n}\right)| \leq Cn^{-\gamma}\}\right\}$$

b. For a constant $C > 0$ we denote for $n \geq 3$

$$A_n = \left\{\; \exists\, \mathrm{s} \in (0,1) \text{ so that } |W(t) - W(s)| \leq C|t-s|^{\gamma}, \text{ when } |t-s| \leq \frac{3}{n}\right\}$$

(i). Prove that $A_n \subset A_{n+1}, \; n \geq 3$.

(ii). Prove that for each $C > 0$ there is $C' > 0$ so that if C, C' are used in the definitions of A_n, B_n, respectively, we have: $A_n \subset B_n, \; n \geq 3$. [Hint: look at k that satisfies $\frac{k-1}{n} < s < \frac{k}{n} < \frac{k+1}{n} < \frac{k+2}{n}$.]

c. Prove $P(A_n) = 0, \; n \geq 3$. How do you prove (*) using this result?