# 02-17 BCII ext,

Monday, February 17, 2025 11:30 AM

Last time $\underline{X}_0 = 0$, $\{\underline{X}_n, \mathcal{F}_n\}_{k \geq 0}$ MG $|D_k| \leq M < \infty$ a.s. $k \geq 1$

then $\Omega = C \cup D$, $C \cap D = \phi$

$$C = \{ \lim_{n\to\infty} X_n = x,\ |\underline{x}| < \infty \}$$

$$D = \{ \overline{\lim_{n\to\infty}}\, X_n = \infty,\ \underline{\lim_{n\to\infty}}\, X_n = -\infty \}$$

B.C. II Extended.

Let $A_n \in \underset{\sim}{\mathcal{F}_n}$ $n \geq 0$ $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ $n \geq 0$. ← filtration

Claim $\{A_n \text{ i.o}\} = \{ \sum_{n=0}^{\infty} \mathbb{1}_{A_n} = +\infty \} = \{ \sum_{n=1}^{\infty} P_{\mathcal{F}_{n-1}}(A_n) = +\infty \}$

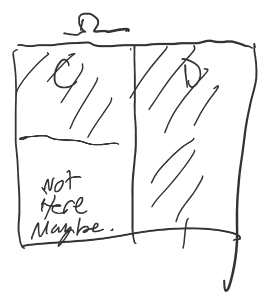
Proof Look at MG.

$$\underline{X}_n = \left\{ \sum_{k=1}^{n} \left[ \mathbb{1}_{A_k} - P_{\mathcal{F}_{k-1}}(A_n) \right],\ \mathcal{F}_n \right\}_{n \geq 1} \quad \text{is MG.}$$

sum of increasing predictable process.

$Y_n = \sum_{k=1}^{n} 1_{A_k}$ increasing $\therefore$ sub MG.

$\{X_n, \mathcal{F}_n\}_{n \geq 1}$ MG, $|D_n| \leq 1 < \infty$ satisfying Req.



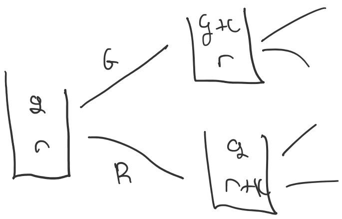

on $C$ we have: $\sum_{n=1}^{\infty} 1_{A_n} = \infty$ iff. $\sum P_{\mathcal{F}_{n-1}}(A_n) = \infty$.

on $D$ we have Both $\sum_{n=1}^{\infty} 1_{A_n} = \infty$ ; $\sum_{n=1}^{\infty} P_{\mathcal{F}_{n-1}}(A_n) = \infty$

Polya scheme (Durrett § 5.3.2 p. 205)

$g$ - green marbles
$r$ = red marbles
$g + r$ = total marble.

Pick Marble at Random
Replace
& ADD 1 of The same color.




Let $G_n$ = # Greens After $n^{th}$ selections
$R_n$ = # Reds " " "

$G_n + R_n = g + r + n \cdot c$     $c$ # of marbles ADD.
$n$ # of Times played

$X_0 = \frac{g}{r+g}$ , $X_n = \frac{G_n}{G_n + R_n}$    $0 \le X_n \le 1$

$P_{\mathcal{F}_n}($ $n+1$ selection is Green $)$

Claim $\{X_n, \mathcal{F}_n\}_{n \ge 0}$ is MG.

wts $E_{\mathcal{F}_n}(X_{n+1}) = X_n$

$$X_{n+1} = \begin{cases} \dfrac{G_n + c}{g + r + (n+1)c} & \text{w.p. } \dfrac{G_n}{g+r+nc} \\[2ex] \dfrac{G_n}{g+r+(n+1)c} & \text{w.p. } \dfrac{R_n}{g+r+nc} \end{cases}$$

$$E_{\mathcal{F}_n}(X_{n+1}) = \left(\frac{G_n + c}{g+r+(n+1)c}\right) \cdot \left(\frac{G_n}{g+r+nc}\right) + \left(\frac{G_n}{g+r+(n+1)c}\right) \cdot \left(\frac{R_n}{g+r+nc}\right)$$

$$= \frac{G_n}{g+r+nc} = 1$$     Because is Bound.

By MGCT : $X_n \xrightarrow[n\to\infty]{a.s.} X$ , $E|X_n - X| \xrightarrow[n\to\infty]{} 0$

" $X_n$ is Random "

Desity of $X_n$

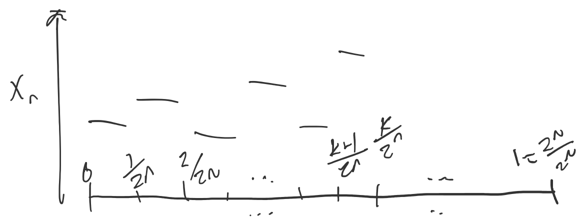
$$f_X(x) = C_{g,r,c} \cdot x^{\frac{g}{c}-1} \cdot (1-x)^{\frac{r}{c}-1} \qquad 0 \le x \le 1$$

$X \sim \text{Beta}\left(\frac{g}{c}, \frac{r}{c}\right)$

$\ast$ Can use MGCT to prove Radon-Nikodym — maybe seem like circular logic

<u>Ex</u> Let $\mu, \gamma$ be 2 p.m on $([0,1], \mathcal{B})$

Let $I_{n,k} = \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right), \quad k = 1, \dots, 2^n.$




$$\rho = \frac{\mu + \gamma}{2}$$

Define $X_n(t) = \dfrac{\mu(I_{n,k})}{\rho(I_{n,k})}, \quad t \in I_{n,k}$

$X_n \equiv \mathcal{F}_n \equiv \sigma\{I_{n,k}\}_{1 \le k \le 2^n}$ contains $2^{2^n}$ values.

(a) $\{X_n, \mathcal{F}_n\}$ is MG in $([0,1], \mathcal{B}, \rho)$ $\quad \rho(I_{n,k}) > 0 \ \forall I_{n,k}$

where $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ increasing $\sigma$-Alg,

and $0 \le X_n \le 2$

The Rest is less than 1.

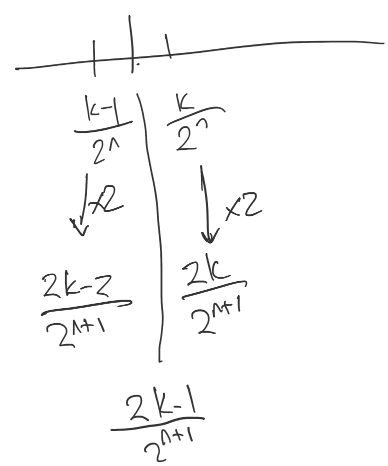
$$X_n(t) = \frac{\mu(I_{n,k})}{\frac{\mu(I_{n,k}) + \gamma(I_{n,k})}{2}}, \quad t \in I_{n,k}.$$

$$\int_{I_{n,k}} X_{n+1}\, d\rho = \int_{I_{n,k}} X_n\, d\rho$$

$$\int_{I_{n,k}} \frac{\mu(I_{n,k})}{\rho(I_{n,k})}\, d\rho$$

$$= \mu(I_{n,k})$$

split.



$$= \mu(I_{n,k})$$

$$\frac{2k-1}{2^{n+1}}$$

$$\downarrow$$

$$\frac{\mu(I_{n+1,2k-1})}{\rho(\cancel{I_{n+1,2k-1}})} \cancel{\rho('')} + \frac{\mu(I_{n+1,2k})}{\rho(\cancel{''})} \cdot \cancel{\rho('')} = \mu(I_{n,k})$$

by MGCT,

(b) $X = \lim_{n\to\infty} X_n$, $\rho$ a.s., $0 \le x \le 2$

Also $\mu(A) = \int_A X \, d\rho$, $A \in \mathcal{B}$,

1st step: $A \in \mathcal{F}_n$

we have: $\mu(A) = \int X_m \, d\rho$ $m \ge n$, is MG Let $n\to\infty$ says,

$$\Big\downarrow m \to \infty$$

$$\int_A X \, d\rho$$

we get it for each $\mathcal{F}_n$ we get it for $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ ← Algebra not $\sigma$-Alg.

$\pi$-$\lambda$ system Extends By Dynkin.