01-22

Wednesday, January 22, 2025 11:30 AM

Last week we Did Random vectors in $\mathbb{R}^d$

we Did Poisson Convergence.

→ Assign As Homework.

New topic: Chapter 4 section 1 Stopping times.

non-negative integer that is Random → Random time

Stopping time is a special case. (invented By Wald Popularize By Do)

Basis: $(\Omega, \mathcal{F}, P)$ Probablity space

Filtration: A non decreasing sequence of sigma Algebra, start At 0, or 1

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}$$

Natural filtration. (special case) Let $X_1, X_2, \ldots, X_n$

the sigma Algebra Generated By: $X_1 \cdots X_n$

$\mathcal{F}_n = \sigma(X_1, \ldots, X_n\}$, $n \geq 1$

the idea

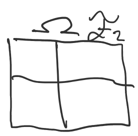
what is The information we have as time moves forward.

so The sigma-Algebra increase As n time increas.

At time 1

For $\omega \in \Omega$ is ω in set1 or set2?

....



Can define with out Random variables.

$$T: \Omega \longrightarrow \mathbb{Z}^+ = \{0, 1, 2, \ldots\}$$

is called stopping time. Relative to $\{\mathcal{F}_n\}_{n \geq 0}$ Filtration.

if $\{T = n\} \in \mathcal{F}_n$ $n \geq 0$ (Equivalently: $\{T \leq n\} \in \mathcal{F}_n$, $n \geq 0$)

what is $T \leq n$ $\qquad = \bigcup_{k \geq 0}^{n} \{t = k)$

$$\{T = n\} = \{T \leq n\} - \{T \leq n-1\},$$

$\{T \leq n\} \in \mathcal{F}_n$, $\{T \leq n-1\}$ Belongs to $\mathcal{F}_{n-1}$

we Don't have information About The Future.

$\{ t = n \} \in \mathcal{F}_n = \sigma\{ X_k \}_{0 \le k \le n.}$

Ex 1

Some Borel set on Real Line

$T = \inf_{k \ge 0} \{ X_k \in A \}$ what do it mean $\{T = n\}$.

$\{ T = n \}$

$= \{ X_0 \in A^c, X_1 \in A^c, \ldots X_{n-1} \in A^c, X_n \in A \} \in \mathcal{F}_n$

"Last Time X_k in A."

Ex 2 $T = \sup_{k \ge 0} \{ X_k \in A \}$.

is this a stopping time?

$\{ T = n \}$

No, is not stopping time Because we need info from The Future

if you Ask Directions? They say turn Left one street Before Big intersect? is a statement of the Future not a stopping time

if They say turn Left at 4th street, $n=4$ is stopping time.

Redefine $T = \sup_{k>0} \{ X_0 \in A_1, X_1 \in A_2, X \in A_k \}$.

$T+1$ is ST.

$T = c$ constant. is stopping time.

Let T be a st. relative to $\{\mathcal{F}_n\}_{n \ge 0}$.

$\mathcal{F}_T$ = "information up to T"

$= \{ A \in \mathcal{F} : A \cap \{T = n\} \in \mathcal{F}_n, \ 0 \le n < \infty \}$

Stopping time can be infinite stop when Level 10, which never happens.

Core Definition of Stopping time

Natural filteration: in The case of $\mathcal{F} = \sigma\{ X_1 \ldots, X_n \}, n \ge 0$.

SAme can be shown,

$$\boxed{\mathcal{F}_T = \sigma\{ X_{n \wedge T}, n \ge 0 \}}$$

Better Def.

Describes a seq from $n = 0, 1, \ldots, T$

Basics if S, T are stopping time (Relative to Filtration) omitted.

then $S \wedge T$ is s.t as well.

to Prove WTS. $\{ S \wedge T = n \} \in \mathcal{F}_n$. $n \ge 0$

One Possibility $\{ S = n, T \ge n \} \overset{\text{Said or}}{\cup} \{ S \ge n, T \ge n \}$

One Possibility $\{S = n, T \geq n\}$ (in $\mathcal{F}_n$, in $\mathcal{F}_n$ by …) $\cup$ $\{S \geq n, T \geq n\}$ Said or

$\{T \geq n\}^c = \{T \leq n-1\} \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$

$$X_T(\omega) = \left(X_{T(\omega)}\right)(\omega)$$

* First you show this is sigma Algebra :

(a) $\mathcal{F}_T$ is a σ-Algebra

Assume $A_k \in \mathcal{F}_T$, $k \geq 1 \Rightarrow A_k \cap \{T = n\} \in \mathcal{F}_n$, $k \geq 1$

$\Rightarrow \left(\bigcup_{k=1}^{\infty} A_k\right) \cap (T = n) \in \mathcal{F}_n$

is measurable w.r.t. σ-Alg

(b) $T \in \mathcal{F}_T$

But have to Prove it

$\{T = k\} \in \mathcal{F}_T$

$\{T = k\} \cap \{T = n\} \in \mathcal{F}_n$ $n \geq 1$

if $k \neq n$ The Empty set

if $k = n$ Then $= \{T = k\}$, trivial.

(c) if $S \leq T$ a.s., where S, T stopping times. then

want to show $\mathcal{F}_S \subset \mathcal{F}_T$

Because $S \leq T \Rightarrow \{T = n\} \subset \bigcup_{k=1}^{n} \{S = k\}$

if this happens, then this must of happen Because $S \leq T$.

$$A \cap \{T = n\} = \bigcup_{k=1}^{n} \Big[\overbrace{A \cap \{S = k\}}^{\text{Belongs } \mathcal{F}_k} \cap \underbrace{\{T = n\}}_{\text{in } \mathcal{F}_n} \Big] \Rightarrow A \in \mathcal{F}_T$$

Assume $A \in \mathcal{F}_S$ NTS $A \in \mathcal{F}_T$, $\mathcal{F}_k \subset \mathcal{F}_n$

Let $\{\mathcal{F}_n\}_{n \geq 1}$ be a filtration. "Don't Assume Natural Filtration

$\{X_n\}_{n \geq 1}$ is a sequence of Random Variable.

Assume $X_n \in \mathcal{F}_n$, $n \geq 1$ ("$\{X_n\}_{n \geq 1}$ is ADAPTED)

Assume $X_n \in \mathcal{F}_n$, $n \geq 1$ ("$\{X_n\}_{n\geq 1}$ is ADAPTED")

Let T be a s.t. w.r.t. $\{\mathcal{F}_n\}_{n\geq 1}$

then $X_T \in \mathcal{F}_T$.

Borel set.

Proof $\{X_T \in B\} \cap \{T = n\} \in \mathcal{F}_n$, $n \geq 1$

$$= \underbrace{\{X_n \in B\}}_{\text{in } \mathcal{F}_n} \cap \underbrace{\{T = n\}}_{\text{in } \mathcal{F}_n} \in \mathcal{F}_n$$