02-07

Friday, February 7, 2025 11:31 AM

(Durret ch 5.2 p. 198)

Martingale (MG). Given $(\Omega, \mathcal{F}, P)$

$\{X_n, \mathcal{F}_n\}_{n\ge 0}$ is MG if $\{\mathcal{F}_n\}_{n \ge 0}$ is a filtration.

and ① $X_n \in \mathcal{F}_n$ $n \ge 0$

② $E|X_n| < \infty$, $n \ge 0$

③ $E_{\mathcal{F}_n}(X_{n+1}) = X_n$ a.s.

Notation Question

$E_{\mathcal{F}_n}(X) = E(X|\mathcal{F})$?

(B) $\{X_n, \mathcal{F}_n\}_{n \ge 0}$ is SUB MG. if

①② hold and ③ $E_{\mathcal{F}_n}(X_{n+1}) \ge X_n$ a.s.

(C) $\{X_n, \mathcal{F}_n\}_{n\ge 0}$ is super MG. if

①② hold and ③ $E_{\mathcal{F}_n}(X_{n+1}) \le X_n$ a.s.

"think of X_n as S_n that is a sum of ...

Random walk.

Assume $\{X_n, \mathcal{F}_n\}_{n \ge 0}$ is MG.

$X_n - X_{n-1} = D_n$ $n \ge 1$ $\Rightarrow E_{\mathcal{F}_n}(D_{n+1}) = E_{\mathcal{F}_n}(X_{n+1}) - E_{\mathcal{F}_n}(X_n) = X_n - X_n = 0$

$\{D_n, \mathcal{F}_n\}_{n\ge 1}$ is called Martingale Difference

$X_n = \sum_{k=1}^{n} D_k$, $n \ge 1$, $E_{\mathcal{F}_n}(D_{n+1}) = 0$, a.s. $n \ge 0$.

sub MG. $E(D_n) \ge 0$

super MG $E(D_n) \le 0$.

Ex: $\{D_k\}_{k \ge 1}$ IND. $\overset{E|D_n|<\infty \ n\ge 1}{\wedge}$ $\mathcal{F}_n = \sigma\{D_1, \dots D_n\}$ $n \ge 1$ AND $E(D_n) = 0$ $n \ge 1$

then $\{X_n, \mathcal{F}_n\}$ is MG. The key is to check

Ex 2 $X_n = \prod_{k=1}^{n} \xi_k$, $\{\xi_k\}_{k\ge 1}$ IND. $E|\xi_k| < \infty$, $E(\xi_k) = 1$ $k \ge 1$

$\mathcal{F}_n = \sigma\{\xi_1, \dots, \xi_n\}$ $n \ge 1$.

if you have something measurable you go out.

$E(X_{n+1}) = E(\prod_{k=1}^{n+1} \xi_k) = E_{\mathcal{F}_n}((\prod_{k=1}^{n} \xi_k)\xi_{n+1}) = (\prod_{k=1}^{n} \xi_k) E_{\mathcal{F}_n}(\xi_{n+1}) = \prod_{k=1}^{n} \xi_k = X_n$

$\prod_{1}^{n} \xi_k \in \mathcal{F}_n$.

$\prod_{1} \xi_k \in \mathcal{F}_n$.

$$\prod_1^{n+1} \xi_k = \left(\prod_1^{n} \xi_k\right)\xi_{n+1}$$

From 1 MG to many MG By Transformation.

Assume $\{X_n, \mathcal{F}_n\}_{n\ge 0}$ is MG.

$X_n - X_{n-1} = D_n \quad n \ge 1 \quad \Rightarrow \quad E_{\mathcal{F}_n}(D_{n+1}) = 0 \quad n \ge 0.$

$\{D_n, \mathcal{F}_n\}_{n\ge 1}$ is called MD. $\quad X_n = \sum_{k=1}^{n} D_k, \ n \ge 1$

we take $H_n \in \mathcal{F}_{n-1}, \ n \ge 1$ $\left(\{H_n\}_{n\ge 1} \text{ is called "Predictable" w.r.t. } \{\mathcal{F}_n\}_{n\ge 0}. \right)$

if $H_{n-1} \in \mathcal{F}_{n-1}$ Then we call it ADAPTED

Look at sequence

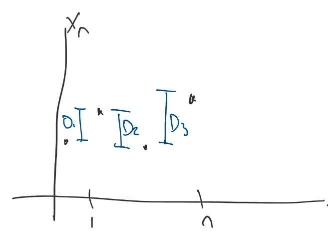
$$(H \cdot X)_0 = 0,$$

not Dot Product

$$(H \cdot X)_n = \sum_{m=1}^{n} H_m D_m, \quad n \ge 1$$

if $E|H_m \cdot D_m| < \infty, \ n \ge$ $\quad$ (or $|H_m| \le C_m < \infty \ \forall m$)

Claim $\{(H \cdot X)_n, \mathcal{F}_n\}_{n\ge 0}$ is MG.



Book calls it Gambling System.

want subMG. H_n Positive.

$n-1$ To n

Profit D_n for 1 unit.

play with H_n units

total profit

$n-1 \xrightarrow{H_n D_n} n$

total profit from time 0. To time n.

$(H \cdot X)_n$

Summary

Assume that $\{X_n, \mathcal{F}_n\}_{n\ge 0}$ is MG {Super MG, sub MG}

and $\{H_n, \mathcal{F}_n\}_{n\ge 1}$ is Predictable $\quad 0 \le H_n \le C_n, \ n \ge 1$

then $\{(H \cdot X)\}_{n\ge 0}$ is Also MG {super, sub} Respectively.

gambling system

EX Let T be a stopping time w.r.t. $\{\mathcal{F}_n\}_{n\geq 0}$.

take $H_n = \mathbb{1}_{\{T \geq n\}}$, $n \geq 0$. play 1 unit until stop time, so beyond stopping is 0.

$$(H \cdot X)_n = \begin{cases} X_n - X_0 & \text{if } n \leq T \\ X_T - X_0 & \text{if } n > T \end{cases} = X_{T\wedge n} - X_0$$

← known at $\mathcal{F}_0$

if $\{X_n, \mathcal{F}_n\}_{n\geq 1}$ subMG (supMG).

and T is a S.T. then

$\{X_{T\wedge n}, \mathcal{F}_n\}_{n\geq 0}$ is subMG (superMG)

key to prove martingale convergence theorem.