03-17 backwards MG

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Backwards Martingale BMG.

$\{X_n, \mathcal{F}_n\}_{n\geq 0}$, $\mathcal{F}_n \downarrow \mathcal{F}_\infty$ $\left(\mathcal{F}_\infty = \bigcap_{n=0}^{\infty} \mathcal{F}_n\right)$

$E|X_n| < \infty$ $n \geq 0$ $\quad E_{\mathcal{F}_{n+1}}(X_n) = X_{n+1}$ $n \geq 0$

L^1 means.

therefore $X_n \xrightarrow[n\to\infty]{a.s.\ L^1} X_\infty$ $\quad E(X_n) \xrightarrow[n\to\infty]{} E(X_\infty)$

$X_n = E_{\mathcal{F}_n}(X_0)$, $\quad X_\infty = E_{\mathcal{F}_\infty}(X_0)$

Application

1.) Recall Dominated Covergence (DC) for Conditional Expectations

$Y_n \xrightarrow[n\to\infty]{a.s.} Y$ $\quad |Y_n| \leq Z$ a.s. $n \geq 1$, $\quad \mathcal{F}_n \uparrow \mathcal{F}_\infty$

now we ADD $\boxed{\text{or } \mathcal{F}_n \downarrow \mathcal{F}_\infty}$

then $E_{\mathcal{F}_n}(Y_n) \xrightarrow[n\to\infty]{a.s.} E_{\mathcal{F}_\infty}(Y)$

$E_{\mathcal{F}_n}(Y) \xrightarrow[n\to\infty]{a.s.} E_{\mathcal{F}_\infty}(Y)$

$\Rightarrow$ NTS $E_{\mathcal{F}_n}(Y_n - Y) \xrightarrow[n\to\infty]{a.s.} 0$.

good Exercise

2) SLLN:

$\{\xi_k, \xi\}_{k\geq 1}$ iid $E(|\xi|) < \infty$ $\quad S_n = \sum_{k=1}^{n} \xi_k$, $n \geq 1$

then $\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s.} E(\xi)$ $\quad$ want prove with BMG

$\mathcal{F}_n = \sigma\{S_n, S_{n+1}, \ldots\}$ $\quad$ says its Decreasing.

$\mathcal{F}_n \downarrow \mathcal{F}_\infty$

By Hewitt-Savage 0-1 law for iid.

if we take symmetric Event under finite permutation
All the Prob. of permutation is either 0 or 1
So switch the order of S_n Nothing happens.
For first n coordinations.
we look at $n \to \infty$. For every n we have invariate
we get $\mathcal{F}_\infty$ is trivial.

$A \in \mathcal{F}_\infty \Rightarrow P(A) \in \{0,1\}$.

claim $E_{\mathcal{F}_n}(\xi_1) = \frac{S_n}{n}$

$\frac{S_n}{n} = E_{\mathcal{F}_n}(\xi_1) \longrightarrow E\{\xi_1\} = E\{\xi_1\}$

$E_{\sigma\{S_n\}}(\xi_1)$

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$E(\xi_1 ; A)$

$A \in \sigma\{S_n\}$ what happens if we permute ξ_i under S_n
Eg. $1 \longrightarrow 2$

π is a permut. on $\{1, \dots, n\}$.
$\pi : 1 \to 2$

$$E(\xi_1 ; A) = E(\xi_{\pi_1} ; \pi A) = E(\xi_2 ; A)$$

$$A \in \sigma\{S_n\} \Rightarrow E(\xi_k ; A) = E(\xi_1 ; A) \quad 1 \le k \le n.$$

$$\Rightarrow \sum_{k=1}^{n} E(\xi_k ; A) = E(\underbrace{\textstyle\sum_i \xi_k}_{S_n} ; A)$$

$$n\, E(\xi_1 ; A) = E(S_n ; A)$$

$$\Rightarrow E(\xi_1 ; A) = E\left(\frac{S_n}{n} ; A\right) \quad A \in \sigma\{S_n\}.$$

So $\dfrac{S_n}{n} = E_{\mathcal{F}_n}(\xi_1)$ if we Replace $\mathcal{F}_n$ w/ $\mathcal{F}_{n+1}$

$\dfrac{S_{n+1}}{n+1} = E_{\mathcal{F}_{n+1}}(\xi_1)$ $\therefore$ $\dfrac{S_n}{n}$ is A BMG.

From iid to exchangeable R.V. (leads to ~~Finitte~~ Foundational Definati theorem)

exchangeable R.V.S.

$\Omega = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots$
$\mathcal{F} = \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \dots$

$X_n(\omega) = \omega_n \quad n \ge 1, \ \omega \in \Omega.$

$\{X_n\}_{n \ge 1}$ Are called Exchangeable if $\forall \pi$ Finite Permutation on $\{1, 2, \dots\}$

$$(X_{\pi_1}, X_{\pi_2}, \dots) \stackrel{\mathcal{D}}{=} (X_1, X_2, \dots)$$

A Place At which π Does not Do Any thing.
Eg. iid. scenario

Example
Consider $\mathbb{R}^d$, $d < \infty$ how many permutations $d!$
where $X = (X_1, X_2, \dots, X_d)$ have

$f_X(x_1, \dots, x_d)$ density.

$$\hat{f}_X(x_1, \dots, x_d) = \sum_{\pi \in \Pi_d} f_X(x_{\pi_1}, \dots, x_{\pi_d}) \Big/ d!$$

$$f_x(x_1, \dots, x_0) = \sum_{\pi \in \pi_0} f_{\bar{x}}(x_{\pi_1}, \dots, x_{\pi_0}) / \partial!$$

DeFonetti theorem.

$$\mathcal{E}_n = \{A \in \mathcal{F} : \pi(A) = A,\ \pi \in \Pi_n\},\ n \geq 1$$

$$\mathcal{E}_n \downarrow \mathcal{E}_\infty = \{A \in \mathcal{F} : \pi(A) = A,\ \pi \in \Pi_n,\ n \geq 1\}$$

Definati theorem

if $\{x_n\}_{n\geq 1}$ Exchangeable then

"Given $\mathcal{E}_\infty$ $\{x_n\}_{n \geq 1}$ in i.i.D." Follows From BMG + calc

Book Example:

Ex. $\{x_i \in \{0,1\}\}$ $x_1 \sim \text{Bernoulli}(\cdot)$

$\{x_i\}_{i \geq 1}$ exchangable.

$$P(x_1 = \dots = x_k = 1,\ x_{k+1} = \dots = x_n = 0)$$

$$= \int \theta^k (1-\theta)^{n-k} \cdot dF(\theta)$$ CDF of measures on 0-1

$\theta = 0$ some Distribution on 0-1 which will create.

Say we have Function $\varphi : \mathbb{R}^k \to \mathbb{R}$ $k \leq n$

$$E_{\mathcal{E}_n}(\varphi(x_1, \dots, x_k))$$ $\mathcal{E}_n$ symetric under n Coordinates

$$= \sum_{\pi \in \Pi_n} \varphi(x_{\pi_1}, \dots, x_{\pi_k}) / \cancel{n!}\ k! \equiv A_n(\varphi)$$

$$\text{BMG}\quad A_n(\varphi) \xrightarrow[n\to\infty]{a.s.} E_{\mathcal{E}_\infty}(\varphi(x_1, \dots, x_k))$$

Given $\mathcal{E}_F$ we have independence

$$\varphi(x_1, \dots, x_k) = f(x_1, \dots, x_{k-1}) g(x_k)$$ one coordinate At a time, that's the Rule

$$A_n(\varphi) = \left(\frac{n}{n-k+1}\right) A_n(f) \cdot A_n(g) - \left(\frac{1}{n-k+1}\right) \sum_{j=1}^{k-1} A_n(\varphi_j)$$

$n \to \infty$: $\frac{n}{n-k+1} \to 1$; $n \to \infty$: $\left(\frac{1}{n-k+1}\right)\sum_{j=1}^{k-1} A_n(\varphi_j) \to 0$

$$\left[\varphi_j(x_1, \dots, x_{k-1}) = f(x_1, \dots, x_k) g(x_j)\right]$$

$$E_{\mathcal{E}_\infty} f(x_1, \dots, x_{k-1}) E_{\mathcal{E}_\infty} g(x_k) = E_{\mathcal{E}_\infty}[f(x_1, \dots, x_{k-1}) g(x_k)]$$

Then Prove By induction, we will see

$$E_{\mathcal{E}_\infty}\left(\prod_{k=1}^{n} f_i(x_i)\right) = \prod_{k=1}^{n} E_{\mathcal{E}_\infty}(f_i(x_i)) \quad \forall f_i \text{ bounded.}$$

$$E_{\xi_\infty}\left(\prod_{k=1}^{n} f_i(x_i)\right) = \prod_{k=1}^{n} E_{\xi_\infty}\left(f_i(x_i)\right) \quad \forall f_i \text{ bounded.}$$

with this we get first n coords are independent.