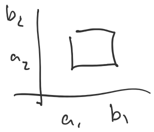
Last time
inversion formula.



$X \in \mathbb{R}^d$

$\varphi_X(t) = E e^{i(t\cdot x)}$

$t \in \mathbb{R}^d$

$tX = \sum_{i=1}^{d} t_i X_i$

$A = \prod_{i=1}^{A} [a_i, b_i]$

Inversion formula

$U \sim$ uniform(A).

move - From U to t. Because $\varphi_U(t) = E e^{\widehat{itU}}$

$$P(X \in A) = \lim_{T\to\infty} (2\pi)^{-d} \lambda(A) \cdot \int_{[-T,T]^d} \varphi_U(-t)\, \varphi_X(t)\, dt.$$

Lebesgue measure AKA Volume.
$\lambda($

Important trick.

$Y = X - U$, $U \sim$ uniform(A), (X, U) IND.

$$\varphi_Y(t) = E\left(e^{it\cdot(x-U)}\right) = E\left(e^{itx} \cdot e^{i(-t)U}\right)$$
$$= \varphi_X(t) \cdot \varphi_U(-t).$$

$$f_Y(0) = \frac{P(X \in A)}{\lambda(A)}$$

Let Y be Random vector. with "nice" density and

$\varphi_Y(t)$ is the ch. func.

then $\lim_{T\to\infty} (2\pi)^{-d} \cancel{\lambda(A)} \int_{[-T,T]^d} \varphi_Y(t)\, dt = f_Y(0)$

sq. Expanding

if we use Fubini
Mass Accumulates At 0.

$$f_U(u) = \frac{1}{\lambda(A)}, \quad u \in A$$

$u = \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix}$ $\quad U_k \sim$ Uniform(a_k, b_k)

$\perp\!\!\!\perp (U_k)_{1 \le k \le d}$

$$\varphi_U(t) = \prod_{k=1}^{d} \varphi_{U_k}(t_k)$$

$t = \begin{pmatrix} t_1 \\ \vdots \\ t_d \end{pmatrix}$ $\quad \searrow \int_{y=a_k}^{b_k} e^{it_k y} \cdot \frac{1}{b_k - a_k}\, dy = \frac{e^{it_k y}}{i t_k (b_k - a_k)} \Bigg|_{a_k}^{b_k}$

$$t = \begin{pmatrix} t_1 \\ \vdots \\ t_d \end{pmatrix} \qquad \int_{y=a_k} e^{i t_k y} \cdot \frac{1}{b_k - a_k}\, dy = \frac{e^{\cdots}}{i t_k (b_k - a_k)} \Big|_{a_k}$$

two R.V. have same c.f. have SAME distribution.

Application: $X_1, \ldots, X_d$ are IND.

iff $\varphi_X(t) = \prod_{k=1}^{d} \varphi_{X_k}(t_k)$

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_d \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_d \end{pmatrix}$$

how Do we know. $\varphi_X(t) = \prod_{k=1} \varphi_{X_k}(t_k) \to \perp\!\!\!\perp X_k$?

Because φ is c.f.

take $Y_1 \ldots Y_d$ and $Y_k \overset{D}{=} X_k$, $k = 1, \ldots, d$.
and $Y_1 \ldots Y_d$ Are IND.

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_d \end{pmatrix}$$

$$\underbrace{\varphi_Y(t) = \prod_{k=1}^{d} \varphi_{Y_k}(t_k) = \prod_{k=1}^{d} \varphi_{X_k}(t_k) = \varphi_X(t)}_{=}.$$

Recall X, Y are r.v. $\in \mathbb{R}$

$\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$. $\forall t \in \mathbb{R}$.

X, Y Are not inD.

Dist of X

x \ y	1	2	3	
1	1/9	0	2/9	1/3
2	2/9	1/9	0	1/3
3	0	2/9	1/9	1/3
Dist of Y	1/3	1/3	1/3	

not IND.

what does it mean To have Vectors To Converge in distribution?
eg. weak convergence $X_n \Rightarrow X$

Def $X_n \in \mathbb{R}^d$, $n = 1, 2, \ldots$

$$X_n \Rightarrow X_\infty \text{ if } E f(X_n) \to E f(X), \quad \forall f \in C_B(\mathbb{R}^d)$$

theorem The following Are Equivalent TFAE

(1) $X_n \Rightarrow X_\infty$

(2) $\overline{\lim_{n\to\infty}} P(X_n \in K) \leq P(X \in K)$, K is closed. $\subset \mathbb{R}^d$

(3) $\underline{\lim_{n\to\infty}} P(X_n \in O) \geq P(X \in O)$, O is open. $\subset \mathbb{R}^d$.
use for interior A.

Boundary ↓

(3) $\underset{n\to\infty}{\liminf} P(X_n \in O) \geq P(X \in O)$, O is open. $\subset \mathbb{R}$

use for interior A.

(4) $P(X_n \in A) \to P(X_\infty \in A)$ $\forall A \subset \mathcal{B}(\mathbb{R}^d)$ AND. $P(X_\infty \in \overset{\text{Boundry}}{\partial A}) = 0$

A° The interior set, , $\bar{A}$ closed set.

$A^\circ \subseteq A \subseteq \bar{A}$

$\partial A = \bar{A} \setminus A^\circ$

implies.

$\therefore P(X_\infty \in A^\circ) = P(X_\infty \in \bar{A})$

how to Prove 1 $\Rightarrow$ 2 & 3. Prob. space

in 1 dim we used let $X_n \Rightarrow X_\infty$ then $\exists$ Y_n, Y_∞ so that $Y_n \xrightarrow[n\to\infty]{a.s.} Y_\infty$ $Y_n \overset{D}{=} X_n$, $n \geq 1$

got Y from X used socall quantile function. $Y_\infty \overset{D}{=} X_\infty$

$Y_n = F^{-1}_{X_n}(U)$ where $U \sim$ uniform$(0,1)$

in one Dim.

$F_{X}(x) \to F_{X_\infty}(x)$, $P(X_\infty = x) = 0$.

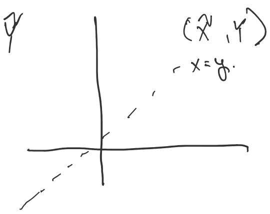
All most All are zero except countable

solved by skoro

Can't do it with $r = 2^+$

if $X \sim Y$ in dist then find $X \overset{\text{close}}{\sim} Y$ a.s.

$\tilde{X} \sim \tilde{Y}$ $(\tilde{X}, \tilde{Y})$ $\tilde{X} \overset{D}{=} X$, $\tilde{Y} \overset{D}{=} Y$



$\therefore$ point should be on the diagonal.