Last time try to prove SLLN.

Reminder of steps.

$\{X, X_k\}_{k\geq 1}$ identically distributed & Pairwise $E|X|<\infty$ $E(X)=\mu$

then $\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s.} \mu$, $S_n = \sum_{k=1}^{n} X_k$.

Proof

Step 1 $Y_k = X_k 1_{\{|X_k|\leq k\}}$ $k\geq 1$

$P((X_k \neq Y_k \text{ i.o.})) = 0$

ETS $\frac{T_n}{n} \xrightarrow[n\to\infty]{a.s.} \mu$ $T_n = \sum_{k=1}^{n} Y_k$

Step 2 $\sum_{k=1}^{\infty} \frac{E(Y_k^2)}{k^2} < \infty$

Step 3 $U_n = \frac{\sum_{k=2^n+1}^{2^{n+1}} Y_k - E(Y_k)}{2^n}$

we got $\sum_{n=1}^{\infty} E(U_n^2) \overset{MCT}{=} E\sum_{n=1}^{\infty} U_n^2 < \infty \Rightarrow \sum_{n=1}^{\infty} U_n^2 < \infty \Rightarrow U_n^2 \xrightarrow[n\to\infty]{a.s.} 0$

$\Rightarrow U_n \xrightarrow[n\to\infty]{a.s.} 0$

$\frac{T_{2^n} - E(T_{2^n})}{2^n} = \sum_{m=1}^{n-1} \frac{2^m}{2^n} U_m \xrightarrow[n\to\infty]{a.s.} 0$

Lemma: $\underset{\text{seq of \#}}{a_k} \xrightarrow[k\to\infty]{} a$ $\underset{\text{triangular Array}}{\{P_{n,k}\}_{1\leq k\leq n}}$, $n=1,2,\ldots$ with

1) $P_{n,k}\geq 0$ 2) $\underbrace{\sum_{k=1}^{n} P_{n,k} \xrightarrow[n\to\infty]{} 1}_{\text{sum} \to 1}$ 3) $P_{n,k} \xrightarrow[n\to\infty]{} 0$ $\forall k=1,2,\ldots$

then $\sum_{k=1}^{n} P_{n,k} a_k \xrightarrow[n\to\infty]{} a$

Probability Explanation.

$P(Z_n = a_k) = P_{n,k}$.

Claim $E(Z_n) \xrightarrow[n\to\infty]{} a$ same

WLOG Assume $a=0$

Let $\varepsilon>0$ take N st. $k\geq N \Rightarrow |a_k|\leq\varepsilon$

"Just split"

$\sum_{k=1}^{n} P_{n,k} a_k = \sum_{n=1}^{N} P_{n,k} a_k + \sum_{k=N+1}^{n} P_{n,k} a_k$ $n\geq N$

$\downarrow n\to\infty$ weight Average.

0 $\left(\sum_{k=N+1}^{N} P_{n,k}\right)\left\{\sum_{k=N+1}^{n} \frac{P_{n,k}\cdot|a_k|}{\sum_{k=N+1}^{n} P_{n,k}}\right\}$ ε less than.

$\downarrow n\to\infty$ $\downarrow$

1 $\lim_{n\to\infty}(\cdot) \leq \varepsilon$ where ε is Arbitrary.

"Following the Lemma"

$\frac{T_{2^n} - E(T_{2^n})}{2^n} = \sum_{k=1}^{n-1} \frac{2^k}{2^n} U_k \xrightarrow[n\to\infty]{a.s.} 0$

Fix k. $\frac{2^k}{2^n} \xrightarrow[n\to\infty]{} 0$

$\sum_{k=1}^{n-1} \frac{2^k}{2^n} \longrightarrow 1$

"split $X = X^+ - X^-$"

Step 4: WLOG $X\geq 0$, a.s.

$E(T_{2^n}) = \frac{\sum_{k=1}^{2^n} E(\overbrace{X_k ; |X_k|\leq k}^{Y_k})}{2^n} \xrightarrow[n\to\infty]{} E(X)$

$\underset{\text{like } a_k}{E(X;|X|\leq k)} \xrightarrow[k\to\infty]{DCT} \underset{\text{like } a}{E(X)}$

we got:

$\frac{T_{2^n}}{2^n} \xrightarrow[n\to\infty]{a.s.} E(X) = \mu$. "write 2^n but we really mean α^n."

this is a subsequence.

we need:

$\frac{T_n}{\cancel{n}\, m} \xrightarrow[n\to\infty]{a.s.} \mu$

there are 2^n m's between.

take $2^n \leq m \leq 2^{n+1}$

By multiple by one.

$\left(\frac{2^n}{2^{n+1}} \frac{T_{2^n}}{2^n}\right) = \frac{T_{2^n}}{2^{n+1}} \leq \frac{T_m}{m} \leq \frac{T_{2^{n+1}}}{m} \leq \frac{T_{2^{n+1}}}{2^n} = \frac{T_{2^{n+1}}}{2^{n+1}} \cdot \frac{2^{n+1}}{2}$

$\frac{1}{2}\mu$ $\qquad\qquad\qquad\qquad \downarrow \mu \quad 2$

we got $\limsup_{m\to\infty} \frac{T_m}{m} \leq 2\mu$ Put in α. For 2.

$\alpha > 1$

$\leq \limsup_{m\to\infty} \frac{T_m}{m} \leq \alpha\cdot\mu$

$\mu \leq \liminf_{m\to\infty} \frac{T_m}{m} \leq \limsup_{m\to\infty} \frac{T_m}{m} \leq \mu$ a.s.

which is what we we Are After.

weighted Convergence of triangular Array.

Given $a_k \xrightarrow[k\to\infty]{Pw} a$, $\{P_{n,k}\}_{1\leq k\leq n}$ triangular Array.

(i) $P_{n,k}\geq 0$ (ii) $\sum_{k=1}^{n} P_{n,k} \xrightarrow[n\to\infty]{} 1$ (iii) $P_{n,k} \xrightarrow[n\to\infty]{} 0$ $\forall k$.

then $\sum_{k=1}^{n} P_{n,k} a_k \xrightarrow[n\to\infty]{} a$

$P_{n,k}$	$k=1$	$k=2$	$\cdots$	$k=\infty$
$n=1$	1	–	–	–
$n=2$	1/2	1/2	–	–
$\downarrow$				
$n=\infty$	$\lim_{n\to\infty} 1/n$,	$\lim_{n\to\infty} 1/n$,	$\cdots$	$\lim_{n\to\infty} 1/n$

EX $P(X=k) = P_k \qquad k=1,\dots,K$

called entropy.

then $\dfrac{\log \prod_{k=1}^{n} P_{X_k}}{n} \xrightarrow[n\to\infty]{A.S.} -H \equiv \sum P_k \log(P_k)$

Log Likilihood.

$$\| \qquad\qquad\qquad\qquad /\!/$$

$$\frac{\sum_{k=1}^{n} \log P_{X_k}}{n} \xrightarrow[n\to\infty]{a.s.} E(\log P_X)$$

$E(\log P_X)$

Convergence of Random Series.

Def Let $\{X_k\}_{k\geq 1}$ be sequence of r.v.s.

sigma Algebra Generated By

$$\mathcal{F}_{[n,\infty)} \equiv \sigma\{X_1, X_2, \dots\}.$$

$$\mathcal{F}_{[n,\infty]} \supset \mathcal{F}_{[n+1,\infty]} \qquad \mathcal{F} \equiv \bigcap_{m=1}^{\infty} \mathcal{F}_{[m,\infty)} \subset \mathcal{F}_{[n,\infty]} \qquad n = 1, 2, \dots$$

tail Events

EX 1 $\{A_n \text{ i.o.}\}. = \{\sum_{k\geq n}^{\infty} 1_{A_k} = \infty,\ n \geq 1\}.$ all ω

$\qquad\qquad\qquad\qquad \cap$

$\qquad\qquad\qquad \mathcal{F}_{[n,\infty)} \qquad$ where $A_k \in \mathcal{F}_{[k,\infty)}$

EX 2 $\{\sum_{k=1}^{\infty} X_k \text{ Converge}\}$

SAME AS $\sum_{k=n}^{\infty} X_k$ coverge $\qquad$ "convergent partial sum"

$\mathcal{F}_{[n,\infty)} \ni$

Tail Events.

cauchy criterian

$$\{S_n \text{ Converge}\} = \bigcap_{k=1}^{\infty} \bigcup_{N=N_0}^{\infty} \bigcap_{N\leq m<l} \{|\sum_{m}^{l} X_k| < \tfrac{1}{k}\} \quad \forall N_0$$

$\mathcal{F}_{[N,\infty)}$