L37 - 11-22 Continuity Theorem

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$$\frac{b-a}{2\pi}\int_{t=-T}^{T}\varphi_X(t)\,\varphi_{-U}(t)\,dt \xrightarrow[T\to\infty]{} P(a<X<b)+\frac{P(X=a)+P(X=b)}{2}$$

theorem 3.3.4 Inversion Formula.

$U \sim$ uniform (

① $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-ita}\varphi_X(t)\,dt = P(X=a)$

$\int_{x=-\infty}^{\infty}e^{itx}\,dF_X(x)$ $\quad E(F_X(x))$ $\quad \forall a\in\mathbb{R}$

$\int_{-T}^{T}\int_{x=-\infty}^{\infty}e^{it(x-a)}\,dF_X(x)\,dt$

$\int_{x=-\infty}^{\infty}\int_{t=-T}^{T}e^{it(x-a)}\,dt\,dF_X(x)$

$\int_{x=-\infty}^{\infty}\int_{t=-T}^{T}\cos(t(x-a))\,dt\,dF_X(x)$,

$\lim_{T\to\infty}\int_{x=-\infty}^{\infty}\frac{2\cdot\sin(T(x-a))}{T(x-a)}\,dF_X(x) = P(X=a)$ fix.

$\left\{ \frac{S(y)}{y} = \begin{array}{l} 1 \text{ if } y=0 \\ \le 1/y \text{ if } y\neq 0 \end{array}\right.$

"hope $x-a=0$."

② if $\int_{t=-\infty}^{\infty}|\varphi_X(t)|<\infty$ then X has $f_X(x)$ PDF

theorem 3.3.5

and $f_X(x) = \frac{1}{2\pi}\int_{t=-\infty}^{\infty}e^{-ity}\cdot\varphi_X(t)\,dt$

$\lim_{T\to\infty}\frac{1}{2\pi}\left|\int_{-T}^{T}e^{-iTa}\varphi_X(t)\,dt\right| \le \lim_{T\to\infty}\frac{1}{2T}\int_{t=-T}^{T}|\varphi_X(t)|\,dt. = 0$

Proof ① $\Rightarrow P(X=a)=0\ \forall a\in\mathbb{R}$

use formula:

$P(\overset{a}{x}\le X\le \overset{b}{x+h}) = \frac{h}{2\pi}\int_{t=-\infty}^{\infty}\varphi_X(t)\int_{y=x}^{x+R}e^{-ity}\,dy\,dt$

$\int_{y=x}^{x+h}\underbrace{\frac{1}{2\pi}\int_{t=-\infty}^{\infty}e^{-ity}\varphi_X(t)}_{f_X(y)\text{ Density.}}\,dy$

$X\sim N(0,1)$, $\varphi_X(t)=e^{-\frac{t^2}{2}}$

$f_X(y)=\frac{1}{\sqrt{2\pi}}e^{-y^2/2}.$

ch. 3.3.2 Weak Convergence

Continuity theorem

theorem 3.3.6 Durrett. Continuity theorem

① Let $\{X_n\}_{n\ge1}$ be R.V. and $X_n \xrightarrow[n\to\infty]{} X$

then $\varphi_{X_n}(t)\xrightarrow[n\to\infty]{}\varphi_X(t)$, $t\in\mathbb{R}$.

② Let $\{\varphi_n(t)\}_{n\ge1}$ be C.F. of $\{X_n\}_{n\ge1}$ respectively

② Let $\{\varphi_n^{(t)}\}_{n\geq 1}$ be C.F. of $\{X_n\}_{n\geq 1}$ respectively

Assume $\varphi_n(t) \xrightarrow[n\to\infty]{} g(t)$, $t \in \mathbb{R}$, and g is continuous at $t=0$

Then $\{X_n\}_{n\geq 1}$ is tight, g is CF of X

and $X_n \Rightarrow X$

Part of 3.3.6 Proof

Lemma $\forall u > 0$ $\frac{1}{u}\int_{t=-u}^{u} (1-\varphi(t))\, dt = 2 - \frac{1}{u}\int_{t=-u}^{u} \cancel{\varphi(t)\, dt}$. $P(|X| \geq \frac{2}{u})$

$= 2 - \frac{1}{u}\int_{t=-u}^{u} \frac{\sin(tx)}{x}\, dF_x(x)$

$$2\left(\int_{x=-\infty}^{\infty} \underbrace{1 - \frac{\sin(ux)}{ux}}_{\text{Always positive.}}\, dF_x(x) \geq 2\int_{\substack{u|x|\geq 2 \\ |x|u \geq 2}} 1 - \frac{\sin(ux)}{ux}\, dF_x(x)\right.$$

$$\geq \int_{|x|>\frac{2}{u}} 1 - \frac{1}{u|x|}\, dF_x(x)$$

$\Big\{ |x| > \frac{2}{u} \geq P(|x| > \frac{2}{u})$

tightness

$\lim_{n\to 0} \frac{1}{u}\int_{-u}^{u} 1 - P\varphi A$ $\quad \underbrace{\frac{1}{u}\int_{-u}^{u} (1-g(t))\, dt}_{<\epsilon} \leq \epsilon$

$0 \leq u \leq \delta$

By DCT on integral converge to

$\forall a > 0$ $\exists N$ s.t. $n \geq N$
$\left|\frac{1}{u}\int_{-u}^{u}(1-\varphi_n(t))\, dt\right| < 2\epsilon$

for all epsilon $P(|X_n| \geq M) < \epsilon$