# L06 - 09-09 lecture

Monday, September 9, 2024 11:30 AM

we want to Extend to step 3.

setup: $f \geq 0$, $\mu$ - is $\sigma$-finite.

$(\Omega, \mathcal{F}, \mu)$, $f \in F$ ( $f: \Omega \to \mathbb{R}$ is Borel Measureable" i.e. $f^{-1}(B) \in \mathcal{F}$, $\forall B$ - Borel subset

$\int_\Omega f \, d\mu = I(f)$

$\int_E f \, d\mu = I(f)$

$I(f) \equiv \sup \{ I(h) : f \geq h \geq 0,\ h \text{ bdd},\ \mu(x: h(x) > 0) < \infty \}$

$\begin{cases} E_{n+1} \supset E_n, \forall n \geq 1 \\ \bigcup_{n=1}^{\infty} E_n = \Omega \end{cases}$

Lemma Let $E_n \uparrow \Omega$, $E_n \in F$, $\mu(E_n) < \infty$, $n \geq 1$ then

$I(f \wedge n; E_n) \uparrow I(f)$ (Min.)

seq. of integrals

$\lim_{n \to \infty} I(f \wedge n; E_n) \xrightarrow[n \to \infty]{} I\left[ \lim_{n \to \infty} [(f \wedge n) \cdot \mathbb{1}_{E_n}] \right] = I(f)$

can switch Between Lim & Int.
sometimes can't switch.

## Monotone convergence theory.

if $f_n \geq 0$, $f_n \uparrow f$ as $n \to \infty$

almost everywhere a.e. then

in Probably almost surely a.s.
when measure is 1

$I(f_n) \uparrow I(f)$

Satisfies.

Does not mean
$\forall \omega \in \Omega: f_n(\omega) \uparrow f(\omega)$

$\mu(f_n \not\uparrow f) = 0$

Measures of zero Don't go into Integral so Ignore.

Can use for 1.4.1. HW 2.
Don't use ★

Prove: N.T.S. $\forall h \geq 0$, bounded, $\mu(h > 0) < \infty$

$\lim_{n \to \infty} I(f \wedge n; E_n) \geq I(h)$.

(min.)
obviously smaller than $f$

By Bounded.

Assume: $0 \leq h \leq f \wedge M$

$\Rightarrow \forall n \geq M$ we get $0 \leq h \leq f \wedge n$

Because of Additivity.

$I(h) = I(h; E_n) + I(h; E_n^c)$

$\int_\Omega h \wedge E_n \, d\mu + \int_\Omega h \wedge E_n^c \, d\mu$

zero does not contribute.

$$\int_\Omega h \wedge E_n \, d\mu + \int_\Omega h \wedge E_n^c \, d\mu$$

zero does not contribute.

$$\leq I(f \wedge n; E_n) + M \cdot \mu(E_n^c \cap \{h>0\})$$

$E_n^c \cap \{\omega : h(\omega) > 0\}$.

$E_n^c \downarrow \phi$

$\bigcap_{n=1}^{\infty} E_n^c = \phi$

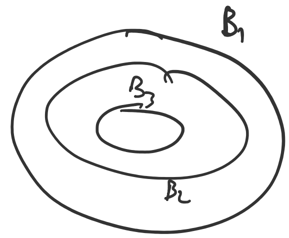
$E_n \supset E_{n+1}^c \quad C \; n \geq 1$

$$E_n^c \cap \{h>0\} \downarrow \phi$$

NTS $M\mu(\underbrace{E_n^c \cap \{h>0\}}_{B_n}) \xrightarrow[n\to\infty]{} 0$

we have: $B_n \downarrow \phi$, $\mu(B_1) < \infty$ $\qquad \mu(B_n) \downarrow 0$

Compact means closed and open Bounded.




Disjoint

$$B_1 = \bigcup_{k=1}^{\infty} B_k \setminus B_{k+1} \cup \bigcap_{k=1}^{\infty} B_k$$

$B_n \downarrow \phi$ means this is $\phi$

This is not True.

$$\mu(B_1) = \sum_{k=1}^{\infty} \left[\mu(B_k) - \mu(B_{k+1})\right]$$

telescoping sums. lots of cancels.

if $\sum_{k=1}^{\infty} \mu(B_k) - \mu(B_{k+1})$

$$= \mu(B_1) - \mu(B_{n+1}) \xrightarrow{n\to\infty} \mu(B_1)$$

$$= \mu(B_{n+1}) \xrightarrow[n\to\infty]{} 0$$

Can Delete $\mu(B_1)$ from Both sides - Because $\mu(B_1) < \infty$ finite.

$f, g \geq 0$ "$f, g \in F$"

$$I(f+g) = I(f) + I(g)$$

Proof: First Direction.

can use $h$ as $h_1 + h_2$ but maybe more $h$'s if so sup is Bigger.

$$I(f) + I(g) = \sup_{\substack{0 \leq h_1 \leq f \\ 0 \leq h_2 \leq g \\ \mu(h_1>0)<\infty \\ \mu(h_2>0)<\infty \\ h_1 \,\&\, h_2 \text{ Bdd.}}} I(h_1 + h_2) \leq \sup_{\substack{0 \leq h \leq f+g \\ h \text{ bdd} \\ \mu(h>0)<\infty}} I(h) = I(f+g)$$

$$(f+g) \wedge n \leq (f \wedge n) + (g \wedge n)$$

Case 1: $f > n$ or $g > n$

Case 2: $f < n$ and $g < n$

$$I((f+g) \wedge n \,; E_n)$$

$$I((f+g)\wedge n ; E_n)$$

Multiple indicator of $E_n$

$$\Big\downarrow n \to \infty \qquad \le I((f\wedge n) ; E_n) + I((g \wedge n) ; E_n)$$

$$I(f+g) \le I(f) + I(g)$$

Define $I(f)$ when $I(|f|) < \infty$

$f = f^+ - f^-$ $\quad \underbrace{f^+ = f \vee 0 \quad f^- = (f \wedge 0)(-1)}_{\text{positive}}$

we know how to integrate each . .

$$|f| = f^+ + f^- \Rightarrow I(f^+) < \infty$$

$$I(f) = I(f^+) - I(f^-)$$

Proving Additivity for $f, g$ with $I(|f|) < \infty$, $I(|g|) < \infty$

when $I(|f|) < \infty \to$ is integrable.

if $f > 0$ then $I(f)$ is always defined. (we can have $I(f) = \infty$)

$$I(f+g) = I(f^+ - f^- + g^+ - g^-)$$

$$= I(f^+ + g^+ - (f^- + g^-)) \leftarrow \text{use additivity because all positive}$$

$$= I(f^+ + g^+) - (I(f^-) + I(g^-))$$

$$= I(f) + I(g)$$

$f, g \ge 0 \quad I(f-g) = I(f) - I(g)$