L24 - 10-23 SLLN proof

Wednesday, October 23, 2024 11:30 AM

SLLN: $\{X, X_k\}_{k\geq 1}$ ~~iid~~ Pairwise $E|X| > \infty$, $E(X) = \mu$

then $\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s.} \mu$ $S_n = \sum_{k=1}^{n} X_k$, $n = 1, 2, \dots$

First observed WLOG $X \geq 0$ a.s.

$X_k = X_k^+ - X_k^-$, $X_k^+ = \max(X, 0)$, $E(X^+) < \infty$

$\{X_k^+, X^+\}$ iid.

$S_n^+ \equiv \sum_{k=1}^{n} X_k^+$

$\frac{S_n^+}{n} - \frac{S_n^-}{n} = \frac{S_n}{n}$

Step 1 $Y_k = X_k \mathbb{1}_{\{|X_k| \leq k\}}$, $k = 1, 2, \dots$

$$\sum_{k=1}^{\infty} P(X_k \neq Y_k) = \sum_{k=1}^{\infty} P(|X_k| > k) = \sum_{k=1}^{\infty} P(|X| > k) \approx \int_{x=0}^{\infty} P(|X| \geq x)\, dx = E|X| < \infty$$

By BC I: $P(X_k \neq Y_k \text{ i.o}) = 0$

$\underset{\text{if}}{\Rightarrow}$ $\frac{\sum_{k=1}^{n} Y_k}{n} \xrightarrow[n\to\infty]{a.s.} E(X)$ then $\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s.} \mu$.

$P(X_k \neq Y_k \text{ i.o}) \rightarrow X_k = Y_k$ a.s.

ASIDE.

Def. Let $a_n \uparrow \infty$ we say $\{a_n\}_{n\geq 1}$ is "Good" (Leventhal named)

if $\sum_{n=m}^{\infty} a_n^{-2} \leq C m \cdot a_m^{-2}$, $m = 1, 2, \dots$

Ex: $a_n = \frac{1}{n^{1/p}}$, $0 < p < 2$ more generally:
if $\frac{a_n}{n^{1/p}}$ is non-decreasing. then $\{a_n\}$ "is Good"
"note $\frac{1}{n^{1/p}}$ is Decreasing."

<u>Lemma</u>

Let $Y_n = X_n \cdot \mathbb{1}_{\{|X| \leq a_n\}}, n \geq 1$ in this case $a_n = n$.

Assume $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$

Then $\sum_{n=1}^{\infty} \frac{E(Y_n^2)}{a_n^2} < \infty$

Step 2. $\sum_{k=1}^{\infty} \frac{E(Y_k^2)}{k^2} < \infty$ "in the Book he uses Diff Proof, not the following"

$a_0 = 0$

$$\sum_{m=1}^{\infty} \frac{E(Y_m^2)}{a_n^2} = \sum_{n=1}^{\infty} a_n^{-2} \sum_{m=1}^{n} E(X^2 ; a_{m-1} < |X| \le a_m)$$

note $1 \le m \le n < \infty$ $\quad 1 \le n < \infty$

$$\sum_{m=1}^{\infty} \left[\sum_{n=m}^{\infty} a_n^{-2} \right] E(X^2 ; a_{m-1} < |X| \le a_m)$$

$$\le \sum_{m=1}^{\infty} c \cdot m \cdot \underbrace{\cancel{a_m^{-2}}}_{\text{constant}} E\left(\frac{X^2}{a_m^2} ; a_{m-1} < |X| \le a_m\right)$$

$$\le c \sum_{m=1}^{\infty} m \, P(a_{m-1} < |X| \le a_m) = c \sum_{m=0}^{\infty} P(|X| > a_m) < \infty$$

Fubini ?

note $\sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} \right) \left(P(\quad " \quad) \right) = c \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} P(\quad " \quad)$

$$= c \sum_{k=1}^{\infty} P(|X| > a_{k-1}) < \infty$$

"similar to find $P(X > x) = \sum_{x}^{\infty} x \, P(X = x)$"

Step 3 let $T_n \equiv \sum_{k=1}^{n} Y_k$, $n \ge 1$

$\forall \alpha > 1$ we get $\frac{T_{\alpha^n} - E(T_{\alpha^n})}{\alpha^n} \xrightarrow[n\to\infty]{a.s.} 0$.

if α is $\mathbb{Q} \setminus \mathbb{Z}$ get messy

Proof ($\alpha = 2$) "chow-bon-yan" Proof.

Claim $U_n \equiv \sum_{k=2^n+1}^{2^{n+1}} \frac{Y_k - E(Y_k)}{2^n} \xrightarrow[n\to\infty]{a.s.} 0$

we will show: $\sum_{n=1}^{\infty} E(U_n^2) < \infty$ $\left(\Rightarrow \sum_{n=1}^{\infty} Y_n^2 < \infty \text{ a.s.} \right)$

implies $U_n^2 \xrightarrow[n\to\infty]{a.s.} 0$

$= U_n \xrightarrow[n\to\infty]{a.s.} 0$

" $E(U_n^2) = V\left(\sum^{2^{n+1}} \frac{Y_k - E(Y_k)}{} \right) \le 4 \sum^{2^{n+1}} \frac{Var(Y_k)}{}$

$$= U_n \xrightarrow[n\to\infty]{a.s.} 0$$

then $E(U_n^2) = Var\left(\sum_{k=2^n+1}^{2^{n+1}} \frac{Y_k - E(Y_k)}{2^n}\right) \le 4\sum_{k=2^n+1}^{2^{n+1}} \frac{Var(Y_k)}{k^2}$

$2^n < 2^n+1 \le k \le 2^{n+1}$

$(2^n)^2 \le k^2 \le (2^{n+1})^2 = (2^n)^2 \cdot 4$

$T_{2^n} = \sum_{k=1}^{2^{n+1}} \frac{Y_k - E(Y_k)}{2^n}$

$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$

$$\le 4 \cdot \sum_{2^n+1}^{2^{n+1}} \frac{Var(Y_k)}{k^2}$$

$$\sum_{n=1}^{\infty} E(U_n^2) \le 4 \sum_{n=1}^{\infty} \sum_{k=} \frac{E(Y_k^2)}{k^2} = 4 \sum_{k\ge 1}^{\infty} \frac{E(Y_k^2)}{k^2}$$

we know $U_n \to 0$ A.S.

$$\frac{T_{2^n} - E(T_{2^n})}{2^n} = \sum_{m=1}^{n-1} \frac{2^m}{2^n} U_m \xrightarrow[n\to\infty]{a.s.} 0$$

so for small n this is 0,
Then for large m $U_m \to 0$.
$\therefore$ 0

$$\frac{E(T_{2^n})}{2^n} = \sum_{k=1}^{2^n} \frac{E(Y_k)}{2^n} \xrightarrow[n\to\infty]{} E(X)$$

$$E(Y_k) = E(X; |X| \le k) \xrightarrow[k\to\infty]{\text{By DCT}} E(X)$$

$$\frac{T_{\alpha^n} - E(T_{\alpha^n})}{\alpha^n} \xrightarrow[n\to\infty]{a.s.} 0$$

$$\Rightarrow \frac{T_{\alpha^n}}{\alpha^n} \xrightarrow[n\to\infty]{a.s.} \mu$$