

03-21

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Example

$$\begin{array}{c} \mathbb{Z} \\ \nearrow p \rightarrow +1 \\ \bullet \quad \searrow q \rightarrow -1 \\ S_n = \sum_{k=1}^n \mathbb{Z}_k, \quad n \geq 1 \end{array} \quad P > \frac{1}{2}, \quad q < \frac{1}{2} \quad P+q=1$$

$$\frac{S_n}{n} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} P-q.$$

$$\mathcal{F}_n = \sigma \{ \mathbb{Z}_1, \dots, \mathbb{Z}_n \}, \quad \text{we use } (\varphi(S_n) = \left(\frac{q}{P}\right)^{S_n}, \mathcal{F}_n) \text{ MG.}$$

$\{\mathbb{Z}_n\}$

$$\{\varphi(S_{T \wedge n})\}_{n \geq 0} \text{ is Bounded.}$$

$$T = T_a \wedge T_b \quad a, b \in \mathbb{Z}$$

$$\boxed{\begin{aligned} a < 0, b > 0. \\ T_b < \infty \text{ a.s.} \end{aligned}}$$

From last time we got,

$$P(T_a < \infty) = \left(\frac{P}{q}\right)^a \text{ or } \left(\frac{q}{P}\right)^{-a} < 1, \quad P(T_a = \infty) > 0 \Rightarrow E(T_a) = \infty$$

$$P(T_b < \infty) = 1$$

we want to calculate $E(T_b)$

$$E(\mathbb{Z} - (P-q)) = 0$$

$$\left\{ S_n - n(P-q), \mathcal{F}_n \right\}_{n \geq 0} \text{ MG}$$

$$\text{since } E_{\mathcal{F}_n}(\mathbb{Z}_{n+1} - (P-q)) = 0$$

MG D.R.F.

$$\Rightarrow E(S_{T_b \wedge n} - (P-q) T_b \wedge n) = 0 \quad \text{Not in the subscript}$$

$$\Rightarrow E(S_{T_b \wedge n}) = (P-q) E(T_b \wedge n)$$

$$T_b \wedge n \uparrow T_b \text{ as } n \rightarrow \infty.$$

$$E(T_b \wedge n) \uparrow E(T_b) \quad \text{By NCT}$$

then

$$\begin{aligned} E(S_{T_b \wedge n}) &= (p-q) E(T_b \wedge n) \\ b = E(S_{T_b}) &= (p-q) E(T_b) \end{aligned}$$

we want \downarrow

$$\Rightarrow b = (p-q) E(T_b)$$

$$E(T_b) = \frac{b}{p-q} < \infty$$

to prove we need $\lim E = E \text{ in either DCT or LI}$

why $\{S_{T_b \wedge n}\}_{n \geq 0}$ LI?

we know.

$$b \geq S_{T_b \wedge n} > \min_{n \geq 0} \{S_n\} \xrightarrow{a.s.} -\infty$$

$$\min_{n \geq 0} \{S_n\} \leq 0$$

Claim $E(\min_{n \geq 0} \{S_n\}) > -\infty$

says $(\frac{p}{q})^a < 1$ is going to save us.

$$Y \in \mathbb{Z}^- \Rightarrow E(Y) = -E(-Y) = -\sum_{k=1}^{\infty} P(-Y \geq k) \stackrel{\text{summation by parts.}}{=} -\sum_{k=1}^{\infty} P(Y \leq -k)$$

$\{T_{-k} < \infty\}$ T_{-k} first time RW hits $-k$.

related to $\{\min_n \{S_n\} \leq -k\} = \{T_{-k} < \infty\} \quad k=1, 2, \dots$

$$E(\min_{n \geq 0} \{S_n\}) = -\sum_{k=1}^{\infty} P(T_{-k} < \infty) = -\sum_{k=1}^{\infty} \left(\frac{q}{p}\right)^k > -\infty$$

since $0 < \frac{q}{p} < 1$

Geometric series.

$\Rightarrow \{S_{T_b \wedge n}\}_{n \geq 0}$ LI

\downarrow a.s. L

S_{T_b}

then $E(T_b) = \frac{b}{p-q} < \infty$

$$L(\cdot) = P^{\alpha}$$

positive super martingale

Let $\{X_n, \mathcal{F}_n\}$ be positive SUP MG.

Let T be a ST.

$$\text{WTS } E(X_0) \geq E(X_T) \geq E(X_\infty)$$

Condition MGCI
 $\sup \{X_n\} < \infty$.
 \cap
 $X_n \xrightarrow[n \rightarrow \infty]{\text{as.s.}} X_\infty$
 $E|X_\infty| < \infty$

$$\underline{\text{Ex }} \left\{ Z_k \right\}_{k \geq 1} \sim N(0, 1)$$

$$X_n = e^{S_n - \frac{\gamma}{2}}, \quad n \geq 0.$$

$$S_n = \sum_{k=1}^n Z_k, \quad n \geq 1, \quad S_0 = 0$$

$$E(e^z) = e^{\gamma z} \Rightarrow e^{z - \frac{\gamma}{2}} = 1$$

$$E(e^{\frac{t^2}{2}}) = e^{\frac{t^2}{2}}, \quad t \in \mathbb{R}.$$

$$X_n = e^{\sum_{k=1}^n (Z_k - \gamma_k)} = \prod_{k=1}^n e^{Z_k - \gamma_k}$$

$$E\left(e^{S_{n+1} - \frac{\gamma_{n+1}}{2}}\right) = e^{S_n - \gamma_n}.$$

$$e^{S_n - \gamma_n} E\left(e^{Z_{n+1} - \gamma_{n+1}}\right) = e^{S_n - \gamma_n} \quad \text{Example of MG.}$$

END. $Z_{n+1} \perp \mathcal{F}_n$

the Problem

$$\underbrace{\sum_{k=1}^n Z_k}_{n} \xrightarrow{\text{a.s.}} 0$$

$$e^{S_n - \gamma_n} = e^{\sum_{k=1}^n (\frac{Z_k}{n} - \gamma_k)} \xrightarrow{\text{a.s.}} e^{-\infty \rightarrow 0}$$

a.s. $\exists N$

$$S_n < \gamma_n \quad n \geq N(\omega)$$

a.s. $\exists N$

$$\frac{s_n}{n} < \gamma_A, n \geq N(\omega)$$

$$\frac{s_n - \gamma_A}{n} < -\gamma_A, n \geq N(\omega)$$

$$E(e^{s_n - \gamma_A}) = 1, n \geq 1$$

it should go to 0, the M.G. goes to 0.

therefore

not LI

Prove.

$$E(x_0) \geq E(x_T)$$

we have

$$E(x_0) \geq E(x_{T \wedge n}) \quad X_{T \wedge n} \xrightarrow[n \rightarrow \infty]{a.s.} x_T$$

given

because of Fatou Lemma

$$E(x_0) \geq \lim_{n \rightarrow \infty} E(x_{T \wedge n})$$

$$\stackrel{\text{Fatou}}{\geq} E\left(\lim_{n \rightarrow \infty} x_{T \wedge n}\right) = E(x_T)$$

used Gambling System enter at 0. end at T+1

$$(H \cdot X)_n$$

Profit if we start after this.

$$Y_n = \begin{cases} x_n - x_T & n > T \\ 0 & n \leq T \end{cases}$$

$$E|Y_n| = E|x_T| < \infty$$

Fatou Condition \rightarrow require positive

can extend. if $Y_n \geq W, E|W| < \infty$.

thus Fatou holds. for $\{Y_n\}_{n \geq 1}$

We can use Fatou $Y_n - W$ which is positive then through at W.

\therefore we get X_∞ .

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Can be automatically extend to two ST.

$$E(X_s) \geq X_T \text{ a.s.}$$

$\tilde{\mathcal{F}}_t$