L05 - 09-06 lecture

Friday, September 6, 2024 11:30 AM

K = kantor set

$K \subset [0,1]$, K closed, $\bar{\bar{K}} = \bar{\bar{\mathbb{R}}} = 2^{\bar{\bar{\mathbb{N}}}} > \bar{\bar{\mathbb{N}}}$ — not countable, Every seq. has a limit in the set.

$K := \bigcap_{n=1}^{\infty} K_n$, $K_n \downarrow$ $(K_n > K_{n+1})$ $\forall n \geq 1$, K_n-closed set

$\Rightarrow$ intersection K closed.

The question: is kantor set Borel.

$\mathcal{F}$ - Borel σ-Algebra = smallest σ-Algebra that $\supset$ closed set

therefore Borel. $\bar{\bar{\mathcal{F}}} = \bar{\bar{\mathbb{R}}}$ (no proof was Borel)

why is the Lebesgue σ-Algebra $\mathcal{F}^* \supsetneq \mathcal{F}$ Borel?

is the completion of $\mathcal{F}$

smallest sigma Algebra that contains $\mathcal{F}$ and All subset of Null sets in $\mathcal{F}$

$A \subset N$, $\mu(N) = 0$, $N \in \mathcal{F} \Rightarrow A \in \mathcal{F}^*$

Kantor theorem: $\forall$ set A we get. $\overline{\overline{\{\text{all subset of } A\}}} > \bar{\bar{A}}$

therefore $\bar{\bar{\mathcal{F}}}^* > \bar{\bar{\mathcal{F}}} = \bar{\bar{\mathbb{R}}}$

Last time: integral of simple function.

LAST TIME

$(\Omega, \mathcal{F}, \mu)$ μ is σ-finite $(\exists E_n \in \mathcal{F}, E_n \uparrow \Omega, \mu(E_n) < \infty, n \geq 1)$

$I(f) \equiv \int_\Omega f \, d\mu$

Integral

Step 1 Define $I(\varphi)$ φ-simple function $\varphi = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$, $a_i \in \mathbb{R}$

$I(\varphi) = \sum_{i=1}^{n} a_i \mu(A_i)$ Define for simple functions $A_i \in \mathcal{F}$, $\mu(A_i) < \infty$

$A_i \cap A_j = \emptyset$, $i \neq j$

$\bigcup_{i=1}^{\infty} A_i = \Omega$

Proved: ① $\varphi \geq 0 \Rightarrow I(\varphi) \geq 0$ "Positivity"

② $I(a\varphi) = aI(\varphi)$ "homogeneity"

③ $I(\varphi + \psi) = I(\varphi) + I(\psi)$

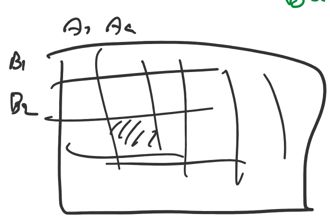
Based on 1.4.1

Proof ③

Let $\psi = \sum_{j=1}^{m} b_j \mathbb{1}_{B_j}$

then

$\sum_i \sum_j (a_i + b_j) \mathbb{1}_{A_i \cap B_j}$



$I(\varphi + \psi) = \sum_i \sum_j a_i \mathbb{1}_{A_i \cap B_j} + \sum_i \sum_j b_j \mathbb{1}_{A_i \cap B_j}$

$$I(\varphi+\psi) = \sum_i\sum_j a_i \mathbb{1}_{A_i\cap B_j} + \sum_i\sum_j b_j \mathbb{1}_{A_j\cap B_j}$$

$$= I\left(\sum_{i=1}^{n} a_i \sum_{j=1}^{} \mathbb{1}_{A_i\cup B_j}\right) + \dots$$ likewise

$$I\left(\sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mu(A_i\cap B_j)\right) + \dots$$

$$I\left(\sum_{i=1}^{n} \mu(A_i\cap B)\right) + \dots$$

$$= I(\varphi) + I(\psi)$$

step 2 $\mu(\Omega<\infty)$, $f\geq 0$, f is Bounded, ($\exists M<\infty$ s.t. $f\leq M$)

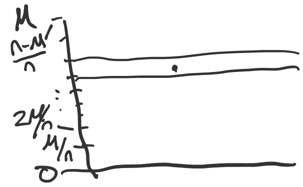
$$I(f) \equiv \sup_{0\leq\varphi\leq f} I(\varphi)$$

Claim. $$\tilde{I}(f) \equiv \inf_{\psi\geq f} I(\psi)$$

need to prove. show are the same.

WTS $I(f) = \tilde{I}(f)$

The trick Divide Equal n-pieces



Define $E_k^{(n)} = \{x\in\Omega : \frac{(k-1)M}{n} < f(x) \leq \frac{kM}{n}\}$, $1\leq k\leq n$

$$\psi_n(x) = \sum_{k=1}^{n} \frac{kM}{n} \mathbb{1}_{E_k^{(n)}} \geq f$$

$$\varphi_n(x) = \sum_{k=1}^{n} \frac{(k-1)M}{n} \mathbb{1}_{E_k^{(n)}} < f$$

$$\psi_n(x) - \varphi(x) = \sum_{k=1}^{n} \frac{M}{n} \mathbb{1}_{E_k^{(n)}}$$

$$I(\psi_n) \geq \inf_{\psi\geq f} I(\psi) \geq \sup_{\varphi\leq f} I(\varphi) \geq I(\varphi_n)$$

$$\Rightarrow \inf_{\psi\geq f} I(\psi) - \sup_{\varphi\leq f} I(\varphi) \leq I(\psi_n) - I(\varphi_n)$$

$$= I(\psi_n - \varphi_n)$$

$$= \frac{M}{n}\mu(n) \overset{n\to\infty}{=} 0$$

By additivity of simple measures.

$$\sum_{k=1}^{n} \mathbb{1}_{E_k^{(n)}} = \mathbb{1}_{\bigcup_{(\epsilon)}}$$

$f, g\geq 0$. $f\leq M$. $g\leq M$, $\mu(\Omega)<\infty$

① $I(f)\geq 0$

② $I(\alpha f) = \alpha I(f) \quad \alpha \in \mathbb{R}$

③ $I(f+g) = I(f) + I(g)$

Proof ③

$$I(f+g) = \inf_{\varphi \geq f+g} I(\varphi) \leq \inf_{\substack{\varphi_1 \geq f \\ \varphi_2 \geq g}} (\varphi_1 + \varphi) = \inf_{\varphi_1 \geq f} I(\varphi_1) + \inf_{\varphi_2 \geq g} (\varphi_2) = I(f) + I(g)$$

Can find $\varphi = \varphi_1 + \varphi_2$, But there are more things

$$I(f+g) = \sup_{\varphi \leq f+g} I(f+g) \geq \sup_{\substack{\varphi_1 \leq f \\ \varphi_2 \leq g}} \quad \dots \quad I(f) + I(g)$$

Because is more & less equal.

Define the integral of f - positive.