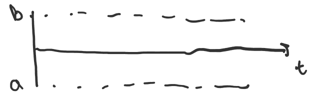
04-23

Wednesday, April 23, 2025 11:34 AM



$T = T_b \wedge T_a.$

Question $P_0(T = T_a) = P_0(T_a < T_b)$

$|B_{T\wedge t}| \overset{\text{Bound.}}{\leq} b \vee |a| \quad \forall t \geq 0 \Rightarrow$ UI

Bounded

$B_{T\wedge t} \xrightarrow[t\to\infty]{a.s.} B_T$

2

$E_0[B_{T\wedge t}] = E_0(B_0) = 0$

DCT $\downarrow t \to \infty$

~~$P(T_a < T_b)\ a \cdot P$~~

$E_0(B_t) = 0 = P(T_a < T_b)\cdot a + P(T_b < T_a) b.$

$P(T_a < T_b) + P(T_a > T_b) = 1$

$B_{T_a} = a$

$B_{T_b} = b$

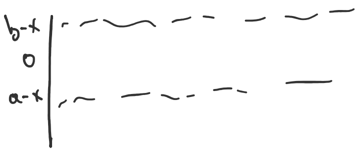
2.5

$\Rightarrow P(T_a < T_b) = \frac{b}{b-a} = \frac{b}{b+|a|}$

$P(T_b < T_a) = \frac{|a|}{b+|a|}$

3

translate to the zero case



$P_x(T_a < T_b) = \frac{b-x}{b-a}$

$P_x(T_b < T_a) = \frac{x-a}{b-a}$

new question.

what is $E_0(T_a \wedge T_b)$?

start with MG.

$$E_0(T_a \wedge T_b) = E_0(T)$$

MG: $B_t^2 - t$

$\{B^2_{T\wedge t} - t \wedge T\}_{t \geq 0}$ is MG.

t BoundeD → UI

$$E(B^2_{T\wedge t} - (t\wedge T)) = 0$$

Since time increases, & Positive.

$t \wedge T$ is:

$$E_0[B^2_{T\wedge t}] = E_0(t \wedge T)$$

DCT $\downarrow$ $\qquad$ $\downarrow$ by MCT

$$E_0(B_T^2) = E_0(T)$$

Remember this is finite.

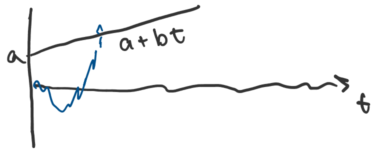
$$E(B_T^2) = a^2 \cdot \frac{b}{b-a} + b^2 \frac{a}{b-a}$$

$$E_0[T) = |a|b.$$

now in The starting At x Case.

$a < x < b$

$$E_x(T) = (b-x)(x-a)$$



Find $P_0(T<\infty)$ where $T = \inf\{t \geq 0;\ B_t^0 = a+bt$

not Gaurunteed. since B move $\sqrt{t}$ & line moves bT

2

$$M_t = e^{\theta B_t - \theta^2 t/2}$$

$$E_0(M_0) \Rightarrow E_0(M_t) = 1$$

Question: Is $\{M_t\}_{0 \leq t \cancel{\leq T} < \infty}$ UI? No.

$$M_t \quad E_t[M_T] \quad 0 \leq t \leq T.$$

$$M_t = e^{\theta B_t - \theta^2 t/2}$$

let say $\theta = 1$

$$M_t = e^{B_t - t/2} = e^{t\left(\frac{B_t}{t} - 1/2\right)}$$

$$\lim_{t\to\infty} M(t) \overset{a.s.}{=} 0$$

$UI \Rightarrow E(M_T) \underset{t\to\infty}{\not\longrightarrow} 0$

$\longrightarrow 1$

3

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$\frac{B_t}{t} \xrightarrow[t\to\infty]{a.s.} 0$: $e^{-t/2} \to 0$ Ⓐ

SBM start At 0.

$\{B_t\}_{t\ge 0} \overset{D}{=} \{t\cdot B(\frac{1}{t})\}_{t>0}$ under P_0

if $\frac{B_t}{t} \xrightarrow[t\to\infty]{a.s.} 0$ then $B(\frac{1}{t}) \xrightarrow[t\to\infty]{} Y \equiv 0$

$B(0) = 0$

1

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$E[e^{\theta\cdot B_{t\wedge T} - \frac{\theta^2}{2}(T\wedge t)}] = 1 \quad \forall t>0$

$B_{t\wedge T} \le a+b(t\wedge T), \quad t\ge 0$

$\theta = 2b$

2

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$E[e^{2bB_{t\wedge T} - 2b^2(T\wedge t)}]$

$\le e^{2b[a+b(t\wedge T)] - 2b^2(T\wedge t)}$ how these cancel?

$\le e^{2ab}$

2.5

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$\theta = 2b$

$M(t\wedge T) \le e^{2ab}, \quad t\ge 0$

$a+bT$

$E_0[M(t\wedge T)] \xrightarrow[t\to\infty]{a.s} \begin{cases} M_T = e^{2bB_T - 2b^2T} \text{ if } \{T<\infty\} \\ \quad = e^{2ab}. \\ M_T = 0 \quad \text{on } \{T=\infty\} \end{cases}$

we show since SLLN argument from Ⓐ

$E_0(M_T) = E_0[e^{2ab}; T<\infty] = 1$

$\underline{0\cdot P(T=\infty)}$

$P_0(T<\infty) = e^{-2ab}$

Ex R.V. X

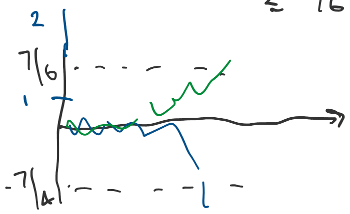
x	$-7/4$	1	2	
$P(x)$	$.4$	$.5$	$.1$	$=1$

Define a stopping time τ such that $B^0_\tau \overset{\mathcal{D}}{=} X$

$$\mathcal{F}_1 = \sigma\{X = -7/4 \ , \ X \in \{1,2\}\}$$

wts. $X_1 = E_{\mathcal{F}_1}(X) = \begin{cases} -7/4 & \text{if } X = -7/4 \\ \dfrac{1\cdot.5+2\cdot.1}{.6} & X \in \{1,2\} \\ & \quad .6 \end{cases}$

$\leq 7/6$



$E(X_1) = 0$

$B_{\tau_1} \overset{\mathcal{D}}{=} X_1$

$\tau_1 = T_{7/6} \wedge T_{-7/4}$