L32 - 11-11 Scheffe's, Total Variation

Monday, November 11, 2024 11:33 AM

Next week thursday.

Scheffe's theorem

Let f_n be PDF of X_n, f is PDF X

assume $f_n(x) \longrightarrow f(x)$ $x \in \mathbb{R}$. "Pointwise"

(*) $|P(X_n \in B) - P(X \in B)| = |\int_B f_n(x)\,dx - \int_B f(x)\,dx|$

$= |\int_B f_n(x) - f(x)\,dx| \leq \int_B |f_n(x) - f(x)|\,dx$

(*) $\leq \int_{-\infty}^{\infty} |f(x) - f_n(x)|\,dx$

$= \int_{-\infty}^{\infty} (f(x) - f_n(x))^+ + \int_{-\infty}^{\infty} (f(x) - f_n(x))^- dx$

$= 2\int_{-\infty}^{\infty} (f(x) - f_n(x))^+ dx \xrightarrow[n\to\infty]{} 0$

Since $a = a^+ - a^-$
$|a| = a^+ + a^-$

Consider $\int_{\mathbb{R}} [f(x) - f_n(x)]^+ dx - \int_{\mathbb{R}} [f(x) - f_n(x)]^- dx$

$= \int_{\mathbb{R}} (f(x) - f_n(x))\,dx$

therefore same.

Can't just $\lim \int$ need $\int \lim$.

Consider DCT

$[f(x) - f_n(x)]^+ \leq$

case ① $f_n(x) > f(x) \Rightarrow [f(x) - f_n(x)]^+ \leq f(x)$.

case ② $f_n(x) < f(x) \Rightarrow \overset{''}{f(x) - f_n(x)} \leq f(x)$

then By DCT with $g = f(x)$ true

B disappeared ≥ 0 and $\sup_B$ (*)

$\sup_{B \in \mathcal{B}(\mathbb{R})} \{|P(X \in B) - P(X \in B)|\} \to 0$ as $n \to \infty$

In our case $P(X = x) = 0$; $x \in \mathbb{R}$.

increasing, continuous function
$\Rightarrow$ uniform convergence.

$X_n \Rightarrow X$ means that $F_{X_n}(x) \longrightarrow F_X(x)$ As $n \to \infty$, $x \in \mathbb{R}$.

$P(X_n \leq x) \longrightarrow P(X \leq x)$ As $n \to \infty$, $x \in \mathbb{R}$
$B = (-\infty, x]$

"convergence should be uniformly."

$\mu_n(B) = P(X_n \in B)$, $B \in \mathcal{B}(\mathbb{R})$

$\mu(B) = P(X \in B)$

total variation norm.

$\| \mu_n - \mu \| \equiv \sup_B | \mu_n(B) - \mu(B) | \to 0$ As $n \to \infty$

Example (Durrett 3.2.6)

$\{U_k\}$ i.i.d. $U[0,1]$

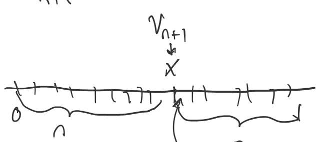
$1 \le k \le 2n+1$

order statistics,

$V_1 \le V_2 < \dots < V_{2n+1}$

Empirical median $= V_{n+1}$

$f_{V_{n+1}}(x) =$



Probability be close to x is $x\,dx$.

density of uniform = 1

$f_{V_{n+1}}(x)\,dx = x^n(1-x)^n \, 1 \cdot \cancel{dx}$

says disappears,

Every element can be here.
so times $\left(\frac{2n+1!}{(n!)^2} \right)$ $\binom{2n+1}{n,n,1}$

$E(V_{n+1}) = \frac{1}{2}$

$\int_0^1 x^k (1-x)^l \, dx = \frac{k!\,l!}{(k+l+1)!}$, $k = 0,1,\dots$ $l = 0,1,\dots$

induction, then use integration by parts.

$Var(V_{n+1}) = E(V_{n+1}^2) - \left(\frac{1}{2}\right)^2$

$Y_n = 2\sqrt{2n}\left(V_{n+1} - \frac{1}{2}\right)$ "the book"

$Var(Y_n) = \frac{n}{n+1.5}$ "I got"

$f_{Y_n}(y) = \varphi(n)\left(1 - \frac{y^2}{2n}\right)^n \longrightarrow f_Z(y)$, $y \in \mathbb{R}$, $Z \sim N(0,1)$

theorem 32.5

TFAE (the following are equivalent).

(1) $X_n \Rightarrow X$

(2) $\underline{\lim}_{n\to\infty} P(X_n \in O) \ge P(X \in O)$, $\forall O$ - openset

(3) $\overline{\lim}_{n\to\infty} P(X_n \in C) \le P(X \in C)$, $\forall C$ - closed set

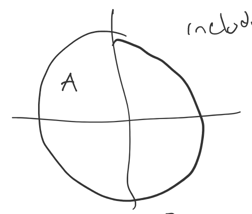
(4) if $P(X \in \delta A) = 0$ then $P(X_n \in A) \longrightarrow P(X \in A)$ as $n \to \infty$.

Prove (4) → (1).

Each A contains the interior.

$\delta A = \bar{A} \cap \bar{A^c}$

$\bar{A} \supset A \supset A^o$



including circle.

(1) $X_n \Rightarrow X$ means $P(X_n \le x) \to P(X \le x)$ if $P(X = x) = 0$

$P(X_n \in (-\infty, x]) \to P(X \in (-\infty, x])$ if $P(X = x) = 0$.

Boundary

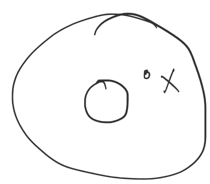
(1) Implies (2)

Because (1) Assum $X_n \xrightarrow{a.s.} X$, $n \to \infty$

$\underline{\lim}_{n\to\infty} P(X_n \in O) \ge P(X \in O)$.

$\lim_{n\to\infty} 1_O(X_n) \ge 1_O(x)$ if

Either 1 or 0



$\lim_{n\to\infty} E[1(X_n)] \overset{FATO}{\le} E[\lim 1(X_n)] \ge E(I_O(X))$

$\ge P(X \in O)$

(2) to (3) The complement is becomes 1 minus.

(2 & 3) to (4). use $\overset{A}{\text{interior}}$ & ~~Exterior~~ and closure of A.

$A^o \subset A \subset \bar{A}$ A is in the middle