L15 - 09-30 Radon-nikodym, prob

Monday, September 30, 2024 11:31 AM

Lebesgue decomposition

$(\Omega, \mathcal{F}, \{\mu, \nu\})$ μ, ν are σ-finite

then $\exists$ 2 measures ν_r ν_s

Regular. Singular

A	A^c
ν_s	μ

The measure $\nu_s \perp \mu$ (i.e. $\exists A \in \mathcal{F}$ s.t. $\nu_s(A^c) = 0 = \mu(A)$)

function of omega

and $\frac{\partial \nu_r}{\partial \mu} = g$ satisfy $\nu_r(B) = \int_B g \, d\mu$, $\forall B \in \mathcal{F}$

So $\nu_r << \mu$

Absoluty continuous with Respect to μ.

if $\mu(B) = 0$ them Then $\nu_r(B) = 0$

special case

RADON - NYKODyn theorem.

if ν_s is 0, the ν

Radon - Nikodym theorem

Appendix A.4. durrett.

if $\nu << \mu$ then $\exists g$ so that $\frac{\partial \nu}{\partial \mu} = g$

i.e. $\nu(B) = \int_B g \, d\mu$ $\forall B \in \mathcal{F}$

Example Are of Probablity mesures.

Ex: 1 Example not true.

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, μ is lebesgue

$\nu = \frac{1}{2}\mu + \frac{1}{2}\delta_{\{\frac{1}{2}\}}$

dirac

$\left(\delta_{\{1/2\}}(A) = \begin{cases} 1 & \text{if } \frac{1}{2} \in A \\ 0 & \text{if } \frac{1}{2} \notin A \end{cases} \right.$, $A \subset [0,1]$

$\nu_r = \frac{1}{2}\mu$ $\nu_s = \frac{1}{2}\delta_{\{\frac{1}{2}\}}$

Not cointius to lebesgue measure

if $\mu(B) = 0$

$\frac{d\nu_r}{\partial \mu} = \frac{1}{2}$

EX 2. $X \sim \{X_k\}_{k \geq 1}$ IND. & IDentical Distributed. (IID)

$P(X=0) = \frac{1}{2} = P(X=2)$

converges this geometric.

$Y = \sum_{k=1}^{\infty} \frac{X_k}{3^k} = \frac{X_1}{3^1} + \frac{X_2}{3^2} + \dots$

$0 \leq Y \leq 2\sum_{k=1}^{\infty} \frac{1}{3^k} = 2 \cdot \frac{1/3}{1 - 1/3} = 2/2 = 1$

ν will come from CDF Y.

$F(y) = P(Y \leq y)$, $0 < y < 1$, $\nu((a,b]) = F(b) - F(a) = P(a < Y \leq b)$

$P(Y \in C) = 1$ C is cantor set.

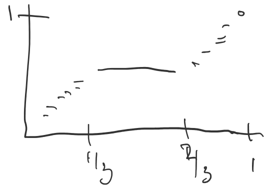
$\mu(C) = 0$

$V_s = V \perp M.$
$V_r = 0$

cantor

ADD TO force increasing after

MODIFICATION: $V_2 = \frac{1}{2}V + \frac{1}{2}\mu$; $V_n = \frac{1}{2}\mu$, $g = \frac{1}{2}$.
$V_s = \frac{1}{2}V$



continuous function
moves only on cantor set

$$F_{V_2}(y) = P(V_2 \le y)$$

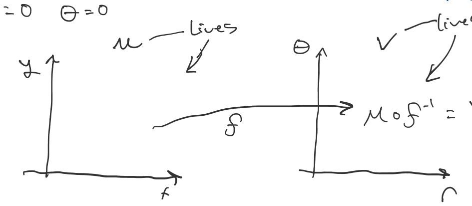
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,\partial x\,\partial y$$

But make change of variable,

$x = r\cos(\theta)$
$y = r\sin(\theta)$

$$= \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} g(r\cos\theta, r\sin\theta)\, r\, \partial r\, \partial\theta$$

Says r is RADON - Nykodym Derivative,



f takes $(x,y) \to \left(\sqrt{x^2+y^2}, \tan^{-1}\left(\frac{y}{x}\right)\right)$

$$r = \frac{\partial(\mu \circ f^{-1})}{\partial r\, \partial\theta}$$

Jacobian.

$$J = \begin{bmatrix} \frac{\partial x}{\partial\theta}^{A} & \frac{\partial x}{\partial r}^{B} \\ \frac{\partial y}{\partial\theta}^{C} & \frac{\partial y}{\partial r}^{D} \end{bmatrix}$$ then $\det J = AD - BC = r$.

Says Jacobian & Radon - Nykodym Are very Related.

Probability.

$(\Omega, \mathcal{F}, P)$ $\quad P(\Omega) = 1$

$X: \Omega \to \mathbb{R}$ is called Random Variable (r.v.) if X is $\mathcal{F}/\mathcal{B}(\mathbb{R})$ Measurable

dist of X ("the law of X") : $\mu(A) = P(X \in A)$, $A \in$ Borel $(\mathbb{R})$

density of X : $\frac{\partial\mu}{\partial x}$ using Radon-N. $\quad (\mathbb{R}, \text{Borel})$

Examples. Uniform $[0,1]$, Exponential (λ) $\lambda > 0$, $N(0,1)$, $N(\mu, \sigma^2)$

$X = (X_1, \ldots, X_d) : \Omega \to \mathbb{R}^d.$

what is that

X is $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$ Measureable if $X^{-1}(B) \in \mathcal{F}$ $\forall B \in \mathcal{B}(\mathbb{R}^d)$

$\underline{X} = (X_1, \dots, X_d) : \Omega \to \mathbb{R}^d$

what that

$\underline{X}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$ measureabe if $X^{-1}(B) \in \mathcal{F}$, $\forall B \in \mathcal{B}(\mathbb{R}^d)$

take too measureable things, is a measurable thing,

Let $f : \mathbb{R}^d \to \mathbb{R}$ which is $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$ measurable,

and $\underline{X} = (X_1, \dots, X_d)$ $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$ measurable.

Then $f(\underline{X}) : \Omega \to \mathbb{R}$ is measureable r.v.

E.X: $\sum_{k=1}^{d} X_k$ $\quad f(x_1, \dots, x_d) = \sum_{k=1}^{d} x_k$.

" in theory, check whole Borel set, But Really check. Rectangles"

" if $\underline{X}$ vector is made from Random vectors $(X_1, \dots X_n)$ then $\underline{X}$ is Random vector that is measureable,

$\inf_{k \geq 1} \{\underline{X}_k\}$ is a R.V. same for sup.

Also true from $\liminf_{\infty \geq k \geq 1}$ & $\limsup_{\infty \geq k \geq 1}$.