L21 - 10-14 Borel-Cantelli 2, ext

Monday, October 14, 2024 11:30 AM

Theorem Let $\{X, X_i\}_{i\ge1}$ be IID R.V. $E(X^4)<\infty$

$E(X)=\mu$, then $\frac{S_n}{n} \xrightarrow{a.s.}$

Collolary: Let $\{A_k\}_{A\ge1}$ be IND $P(A)=p>0$, $k=1,2,\dots$

then $\frac{\sum_{k=1}^{n} 1_{A_k}}{n} \xrightarrow[n\to\infty]{a.s.} p$ (observe : $\{1_{A_k}\}_{k\ge1}$ are iid r.v.)

$\mathbb{E}(1_{A_k}^4) = \mathbb{E}(1_{A_k}) = p<\infty$

"The power of an indicator is the indicator."

$\frac{\sum_{k=1}^{n} 1_{A_k}}{n} = \frac{\#\text{ of times } A_k \text{ occurs } k=1,\dots,n}{n} \xrightarrow[n\to\infty]{a.s} p$

"number of times A_k occurs" over n times.

Empirical Average.

By Markov

Proof

WLOG $\mu=0$, let $\varepsilon>0$ $P(|S_n|>n\varepsilon) = P(S_n^4>n^4\varepsilon^4) \le \frac{\mathbb{E}(S_n^4)}{\varepsilon^4 n^4}$

$\mathbb{E}(S_n^4) = \mathbb{E}\left(\sum_{k=1}^{n} X_k\right)^4 = E\left[\sum_{k=1}^{n} X_k^4 + \dots +\right]$

$= n\,\mathbb{E}X_k^4 + n(n-1)\cdot\cancel{\tfrac{4}{2}}^{3}[EX^2]^2$

Note $E(\cancel{24\,X_1X_2X_3X_4}) = 0$ $\quad \mathbb{E}(X_i)=0$

$E(\cancel{X_1X_2X_3^2}) = 0$

$\le n^2 C$

$\therefore \frac{E(S_n^4)}{\varepsilon^4 n^4} = \frac{C}{\varepsilon^4}\frac{1}{n^2}$

by BCI

$P\left(\frac{|S_n|}{n}>\varepsilon \text{ io}\right)=0$ (because $\frac{C}{\varepsilon^4}\sum_{n=1}^{\infty}\frac{1}{x^2}<\infty$)

$\Rightarrow P\left(\limsup_{n\to\infty}\left\{\frac{|S_n|}{n}\right\}\le\varepsilon\right)=1$ Almost surely

Equivalently

$\limsup_{n\to\infty}\frac{|S_n|}{n}=0$ A.s.

Consider $\varepsilon=\frac{1}{m}$ $\forall m\ge1$

$P\left(\limsup_{n\to\infty}\left\{\frac{|S_n|}{n}\right\}\le\frac{1}{m}\right)=1$ this event shrinks with $m\uparrow$

$\Rightarrow P\left(\bigcap_{n=1}^{\infty}\left\{\overline{\lim_{n\to\infty}}\left\{\frac{|S_n|}{n}\right\}\le\frac{1}{m}\right\}\right)=1$

$\Rightarrow \overline{\lim_{n\to\infty}}\frac{|S_n|}{n}=0$ a.s. $\qquad 0\le \underline{\lim_{n\to\infty}}\frac{|S_n|}{n}\le\overline{\lim_{n\to\infty}}\frac{|S_n|}{n}=0$, a.s.

$$\lim_{n\to\infty} \frac{|S_n|}{n} = 0 \quad \text{a.s.}$$

$$\Rightarrow \lim_{n\to\infty} \frac{S_n}{n} = 0, \text{ a.s.}$$

BC II (Assume) Events IID $\{A_n\}_{n\ge 1}$, $\sum_{n=1}^{\infty} P(A_n) = \infty$

then

$$P(A_n \text{ i.o.}) = 1$$

Proof. $P\left(\bigcup_{n=m}^{\infty} A_n\right)$ ← if Ω Belongs to one then Belongs to All.

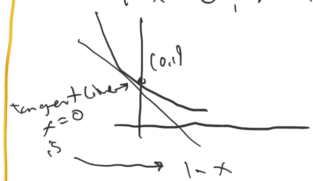
look at complement,

$$P\left(\left[\bigcup_{n=m}^{\infty} A_n\right]^c\right) = P\left(\bigcap_{n=m}^{\infty} A_n^c\right) = \prod_{n=m}^{\infty} P(A_n^c) = \prod_{n=m}^{\infty}\left[1 - P(A_n)\right]$$

$$\le \prod_{n=m}^{\infty} e^{-P(A_n)}$$

$$= e^{-\sum_{n=m}^{\infty} P(A_n)} = 0$$

Always Remember,
$1 - x \le e^{-x}$, $x \in \mathbb{R}$



$1 + x \le e^x$

thus.

$$P\left(\left[\bigcup_{n=m}^{\infty} A_n\right]^c\right) = 0$$

And

$$P\left(\bigcup_{n=m}^{\infty} A_n\right) = 1 \Rightarrow P\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} A_n\right) = \lim_{m\to\infty}\left(\bigcup_{n=m}^{\infty} A_n\right) = 1$$

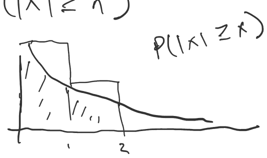
A Infinitely Often

$$\Rightarrow P(A_n \text{ i.o.})$$

Application. Let $\{X, X_k\}_{k\ge 1}$ iid $E|X| = \infty$

then $P(|X_n| \ge n, \text{ i.o.}) = 1$ and $P\Big(\underbrace{\lim_{n\to\infty} \frac{S_n}{n} \text{ Exist and is in } (-\infty,\infty)}_{A}\Big) = 0$

Proof: $E|X| = \int_{x=0}^{\infty} P(|X| \ge x)\,dx \le \sum_{n=0}^{\infty} P(|X| \ge n)$



$$\sum_{n=0}^{\infty} P(|X| \ge n) = \infty$$

ADD n to X

$$\sum_{n=0}^{\infty} P(|X_n| \ge n) = \infty \quad \text{then By BC II}$$

$$P(|X_n| \ge n, \text{ i.o.}) = 1$$

Part 2.

Consider

$$\frac{S_{n+1}}{n+1} - \frac{S_n}{n} = \frac{X_{n+1}}{n+1} + S_n\left(\frac{1}{n+1} - \frac{1}{n}\right) = \frac{X_{n+1}}{n+1} - S_n \frac{1}{n(n+1)}$$

a, a — gone to 0; → 0; → 0

if Left goes to Zero Then Right.

Given
$S_{n+1} = S_n + X_{n+1}$

Assume A $P(A) > 0$ then Above goes to Zero $\rightarrow\leftarrow$

wednesday. 10/16

Extension Borel Cantelli II Let A_1 A_2 Be Pairwise IND,

and $\sum P(A_n) = \infty$ then $\dfrac{\sum_{k=1}^{n} 1_{A_k}}{\sum_{k=1}^{n} P(A_k)} \xrightarrow[n\to\infty]{a.s.} 1$

It follows $P(A_n \text{ i.o.}) = 1$

Three other's Extensions. in the Book