03-28

Friday, March 28, 2025 11:30 AM

Brownian motion. $(\Omega, \mathcal{F}, P)$ $B(t)$ is R.V.

$\{B(t)\}_{t\ge 0}$ is called standard Brownian motion. if

① $B_0 = 0$ ② stationary & Ind increments.
a) $B(t+h) - B(t) \overset{d}{=} B(h)$, $h,t \ge 0$.
b) $B(t+h) - B(t) \perp\!\!\!\perp \sigma\{B_s\}_{0\le s\le t}$ $h,t \ge 0$

~~③ B(t) has continuous sample paths.~~

③ $B(t+h) - B(t) \sim N(0,h)$
$B_t(\omega)$, $0 \le t < \infty$, $\omega \in \Omega$

④ $\{B_t\}_{t\ge 0}$ has continuous sample
fixed ω. then $B_t(\omega)$ is just a function t.

Fixed $\omega \in \Omega$, then $\{B_t(\omega)\}_{t\ge 0}$ is continuous function of $t \in \mathbb{R}^+$

$0 < t_1 < t_2 < \dots t_k$.

Practices
Find $f_{(B_{t_1}, \dots, B_{t_k})}(x_1, \dots, x_k)$

$$\left.\begin{aligned} U_1 &= B_{t_1} \\ U_2 &= B_{t_2} - B_{t_1} \\ U_3 &= B_{t_3} - B_{t_2} \\ &\vdots \\ U_k &= B_{t_k} - B_{t_{k-1}} \end{aligned}\right\} \text{IND Because IND. INC.}$$

$$P_t(\omega) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{u^2}{2t}}, \quad -\infty < u < \infty, \quad t \ge 0$$

$$f_{U_1, \dots, U_k}(u_1, \dots, u_k) = P_{t_1}(\omega) \cdot P_{t_2 - t_1}(\omega) \cdot \dots \cdot P_{t_k - t_{k-1}}(U_k)$$

$$f_{\vec{U}}(\vec{u}) \left| \det\left[\frac{\partial \vec{U}}{\partial \vec{x}}\right] \right| = f_{\vec{X}}(\vec{x})$$

change of variable
Jacobian.

$$f_{\vec{U}}(x_1, x_2 - x_1, \dots x_k - x_{k-1})$$

Need to calc $\frac{\partial \vec{U}}{\partial \vec{x}}$ $k \times k$ matrix.

Main Diagonal

$$\det \begin{bmatrix} 1 & 0 & \dots & & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ & & & \ddots & \end{bmatrix} = 1 \cdot 1 \cdot \dots \cdot 1 = 1$$

therefore Jacobian is 1

$$\begin{bmatrix} \ddots \end{bmatrix}_{K\times K}$$

$$f_{B_{t_1}\cdots B_{t_K}}(x_1 \cdots x_K) = \prod_{i=1}^{K} p_{t_i - t_{i-1}}(x_i - x_{i-1})$$

Another Def.

$\{B_t\}_{t\geq 0}$ is SBM if

① $\{B_t\}_{t\geq 0}$: is a Gaussian Process

namely $\forall$ $t_1 < t_2 < \cdots < t_n$ $n \geq 1$

we have $(B_{t_1}, \cdots, B_{t_n}) \sim$ Multivariate Normal

Gauss: 1778 - 1850.

3re

$B_{t_2} - B_{t_1} \overset{\mathcal{D}}{=} \sqrt{t_2 - t_1}\, Z$: $Z \sim N(0,1)$.

What is Important About MVN? μ & Σ - covariance matrix.

$Cov(B_{t_1}, B_{t_2}) = E(B_{t_1}, B_{t_2})$ where $0 < t_1 < t_2$

$E[B_t(\overbrace{B_{t_1} + B_{t_2} - B_{t_1}}^{B_{t_2}})$

$E[B_{t_1}^2] + \underbrace{E[B_{t_1}\cdot(B_{t_2} - B_{t_1})]}_{0\cdot 0 = 0.}$

$= t_1$

② $E(B_s \cdot B_t) = s \wedge t$ $E(B_t) = 0$ $t, s \geq 0$.

③ $\{B_t\}_{t\geq 0}$ has Continuous sample paths.

$$\Omega = \{\omega : \omega : \mathbb{R}^+ \to \mathbb{R}\}$$

$B_t(\omega) = \omega(t)$, $t \geq 0$

$(t_1, t_2, \cdots, t_n)$

$$(B_{t_1}, \cdots B_{t_n}) \sim MVN\Big(\vec{0}, \underbrace{[t_i \wedge t_j]}_{\Sigma}{}_{1\leq i,j\leq n}\Big)$$

Kolgomorov. if Measures are compatiable, then Algebra can be Extended

Not Good. Idea fails. Kolgomorow Extension theorem.

Continuous are uncountable. $\Rightarrow$ Idea of chaining.

Kolgomorov continuity criterion not necessarily Brownian.

Continuous are uncountable => Idea of chaining.

Kolgomorov continuity criterion

we have a process $\{X_t\}_{0 \le t \le 1}$ not necessarily Brownian. was $t \ge 0$

if $E|X_t - X_s|^\beta < C \cdot |t-s|^{1+\alpha}$ say this Ineq is weak.

then $\{X(t)\}_{t \ge 0}$ is holder continuous.

with $\gamma < \frac{\alpha}{\beta}$ gamma

smaller γ then Better Ineq. Holds.

which is smaller? $\left(\frac{1}{4}\right)^{.8} < \left(\frac{1}{4}\right)^{.7}$

Holder continuous.

$|X_t + X_s| \le C_2 |t-s|^\gamma \quad \exists\ |t-s| < \delta > 1$

For $0 \le s < t$

$B_t - B_s \sim N(0, t-s) \overset{\mathcal{D}}{=} \sqrt{t-s}\, Z \qquad Z \sim N(0,1)$

$\beta = 2m, \quad \alpha = m-1 \quad \Rightarrow \quad 2+1 = m$

$E(|B_t - B_s|)^{2m} = |t-s|^m \cdot \underbrace{E[Z^{2m}]}_{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2m-1)}$

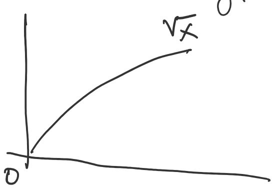
$\gamma < \frac{m-1}{2m} \longrightarrow \frac{1}{2}$

$\therefore \quad \gamma < \frac{1}{2}$

$\left(0, \frac{1}{2^n}, \frac{2}{2^n}, \ldots, 1 = \frac{2^n}{2^n}\right)$

we get For SBM Holder Cont. with $\gamma < 1/2$

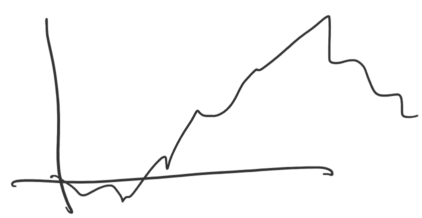
Holder Continuity Ex.



it will have a Holder Continuity. $C = 1/2$

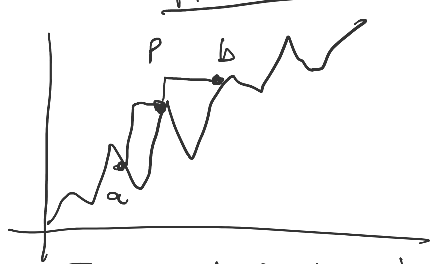
Because of 0

Look a Brownian motion.



Dvoretzky - CLt for Mg. As A warm up.
EROOS Beyond $C = 1/2$ it fails
KAKutani Do on monday.

Not one point of Increase



$\exists$ a point p st. a & b Exist.

Says simple Random walk converges to Brownian motion & $C = 1/2$.

is It Really $1/2$? No