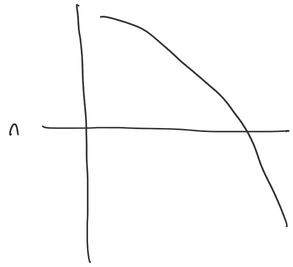
03-24 CLT for MD

Monday, March 24, 2025 11:31 AM

Central Limit theorem (For Martingales)

Prove By — Devoretsky - used taylor
— Hiddie used characteristic function.

Relates to Autoregressive Proff



Lindberg - Feller CLT for MD.

$\{D_{n,j}, \mathcal{F}_{n,j}\}_{i \le j \le n} \quad n = 1, 2, \dots$

$E_{n,j-1}[D_{n,j}] = 0 \quad , \quad E[D^2_{n,j}] < \infty$

theorem: if i) $\sum_{j=1}^{n} E_{n,j-1}(D^2_{n,j}) \xrightarrow[n\to\infty]{P} 1$ and

ii) $\sum_{j=1}^{n} E_{n,j-1}(D^2_{n,j} |D_{n,j}| > \varepsilon) \xrightarrow[n\to\infty]{P} 0$ (Lindeburg Condition)

then

$$S_n = \sum_{j=1}^{n} D_{n,j} \Rightarrow N(0,1)$$

WLOG we can add (iii)

(iii) $\sum E_{n,j-1}(D^2_{n,j}) \le 2$ a.s. $\forall n \ge 1$

says this is predictable, therefore we can use stopping time

we will replace $D_{n,j}$ by $D_{n,j} \mathbb{1}_{\{j \le T_n\}}$
where $T_n = \mathrm{Max}\{ j \, ; \, \sum_{k=1}^{j} E_{n,k-1}(D^2_{n,k}) \le 2 \}$
condition on J+1

$P(T_n = n) \xrightarrow[n\to\infty]{P} 1$

Proof: NTS $E[e^{itS_n}] \xrightarrow[n\to\infty]{} e^{-t^2/2} \quad \forall t \in \mathbb{R}$

$$E\left[E_{n-1}\left[\frac{e^{itS_n}}{\prod_{k=1}^{n} E_{n,k-1}[e^{itD_{n,k}}]}\right]\right] = E\left[\frac{e^{it\sum_{k=1}^{n-1} D_{n,k}}}{\prod_{k=1}^{n-1} E_{n,k-1}(e^{itD_{n,k}})}\right] E_{n,n-1}\left[\cancel{\frac{e^{itD_{n,n}}}{E_{n,n-1}(e^{itD_{n,n}})}}\right]$$

$S_n = \sum_{k=1}^{n} D_{n,k}$

$\xrightarrow{} 1$

now take expectation of Cond. Exp.

therefore

$\left[\; e^{itS_n} \; \right] \qquad e^{itS_n}$

therefore

$$E\left[\frac{e^{itS_n}}{\prod_{k=1}^{n} E[e^{itD_{n,k}}]_{n,k+1}}\right] = 1 = E\frac{e^{itS_n}}{\prod_{k=1}^{n}(1+r_{n,k})}$$

Notice $\frac{1}{1+x} \approx 1-x$ when x is small

$$\sim E\left[e^{itS_n}\prod_{k=1}^{n}(1-r_{n,k})\right] \quad (i)$$

$$E_{n,k-1}[e^{itD_{n,k}}] = 1 + r_{n,k}$$

NtS:

1) $(i) \xrightarrow[n\to\infty]{} 1$

2) $E\left(\prod_{k=1}^{n}(1-r_{n,k})\right) \xrightarrow[n\to\infty]{} e^{t^2/2}$ in L^1

$\therefore E\left|\prod_{k=1}^{n}(1-r_{n,k}) - e^{t^2/2}\right| \to 0$

$$r_{n,k} = E_{n,k-1}[e^{itD_{n,k}} - 1 - itD_{n,k}]$$

From 881 To carry the Problem.

step 1: show

(1) $\sum_{j=1}^{n} r_{n,j} \xrightarrow[n\to\infty]{P} -\frac{t^2}{2}$

(2) $\sum_{j=1}^{n}|r_{n,j}| \le 2t^2$

(3) $\max_{1\le j\le n}|r_{n,j}| \xrightarrow[n\to\infty]{P} 0$

step 2: (1),(2),(3) $\Rightarrow \prod_{j=1}^{n}(1-r_{n,j}) \xrightarrow[n\to\infty]{P} e^{t^2/2}$

881 reminder:

Let $\{a_{n,m}\}_{1\le m\le n}$, $n\ge 1$ complex numbers

if (1) $\sum_{m=1}^{n} a_{n,m} \xrightarrow[n\to\infty]{} a$

(2) $\sup_n\{\sum_{m=1}^{n}|a_{n,m}|\} < \infty$

(3) $\max_{1\le m\le n}|a_{n,m}| \xrightarrow[n\to\infty]{} 0$

then

$$\prod_{m=1}^{n}(1+a_{n,m}) \xrightarrow[n\to\infty]{} e^{a}$$

Auto Regressive sequence.

Ex $X_n = \theta X_{n-1}$ then, $n\ge 1$, $|\theta|<1$

$\{U_n\}_{n\ge 1}$ are iid. $E[U_n]=0$, $E(U_n^2) = \sigma^2 < \infty$

$X_0, \{W_n\}_{n\ge 1}$ are InD

if $\hat{\theta}_n = \arg\min_{\theta}\left\{\sum_{k=1}^{n}(X_k - \theta X_{k-1})^2\right\}$

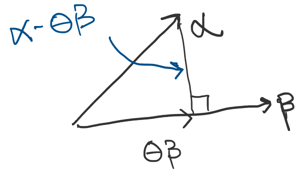
estimator

then $\sqrt{n}(\hat{\theta}_n - \theta)$

$$= \frac{\cancel{\sqrt{n}}\sum_{k=1}^{n}U_k X_{k-1}\cancel{/\sqrt{n}} \cdot \frac{1}{n}}{\sum_{k=1}^{n}X_{k-1}^2 \cancel{/n} \cdot \frac{1}{n}}$$

Proof $\alpha = (X_1, \dots, X_n)$, $\beta = (X_0, \dots, X_{n-1})$

<u>Proof</u> $\alpha = (X_1, \ldots, X_n)$, $\beta = (X_0, \ldots, X_{n-1})$



$$\|\alpha - \theta\beta\|^2$$

$$(\alpha - \hat{\theta}_n\beta, \beta) = 0$$

$$(\alpha, \beta) - \hat{\theta}_n(\beta, \beta) = 0$$

$$\hat{\theta}_n = \frac{(\alpha, \beta)}{(\beta \beta)}$$

$$\hat{\theta}_n - \theta = \frac{(\alpha - \theta\beta, \beta)}{(\beta, \beta)}$$

<u>Notation</u>

(A, B)

is Dot Product $A \cdot B$

[Finish on WED]