04-16

Wednesday, April 16, 2025 11:31 AM

Formal Def.

$(\Omega, \mathcal{F}, P_x)_{x \in \mathbb{R}}$

Strong Markov Property for BM.

$Y : (\mathbb{R}^+ \times \Omega) \longrightarrow \mathbb{R}$, Y measurable

Let $S : \Omega \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ be a s.t. then

$$E_x(Y_s \circ \theta_s \mid \mathcal{F}_s) = E_{B^x(s)}(Y_s) \quad \text{on } (s < \infty)$$

$$E_x[(Y_s \circ \theta_s) \mathbb{1}_{\{s<\infty\}} \mid \mathcal{F}_s] = E^x_{B(s)}(Y_s)\, \mathbb{1}_{\{s<\infty\}}$$

this is Random

(B) $$E_x[(Y_s \circ \theta_s) \mathbb{1}_{\{s<\infty\}}] = (A)\ E_x(E_{B(s)}(Y_s) \mathbb{1}_{\{s<\infty\}}$$

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More intuitive Formulation

Let S be a s.t. then

Note
$(P(S<\infty) = 1$

① $\{B(s+t)\}_{t\geq 0} \overset{\mathcal{D}}{=} \{B_t^{B(s)}\}_{t\geq 0}$

$\cancel{\{B(s+t) - B(t)\}_{t\geq 0} = \{B_t^0\}_{t\geq 0}}$

② $\cancel{\{B(s+t)\}_{t>0} \perp\!\!\!\perp \mathcal{F}_s}$

$\{B(s+t) - B(s)\}_{t\geq 0} \perp\!\!\!\perp \mathcal{F}_s$

2.5

Example

$S = T_a \quad a > 0$

$P_x(T_a < \infty) = 1$

$T_a = \inf\{t > 0 : B_t = a\}$

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Last Time $\overline{\lim_{t\to\infty}} \dfrac{B_t}{\sqrt{t}} = +\infty$ a.s.

$\Rightarrow \overline{\lim_{t\to\infty}}\ B_t = +\infty$ a.s.

∴ we will cross every a as we go to inf.

2.5

① $\{B(T_a + t)\}_{t\geq 0} = \{B^a(t)\}_{t\geq 0}$ ← Brown start at a.

② $\{B(T_a+t)\}_{t\geq 0} \perp\!\!\!\perp \mathcal{F}_{T_a}$.

notice this is different. we should subtract a
But doesn't matter since a is constant.

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<u>sketch of Proof</u>

① Assume $\exists \{t_n\}_{n\geq 1}$, $t_n \in \mathbb{R}^+$ so that $\sum_{n=1}^{\infty} P(S=t_n) = 1$

so S only gets a countable # of Points

on the event $\{S=t_n\}$ we apply the M.P.

then For each t_n we Divide and conquer.
we use the Regular Markov Property.

② Recall $\exists$ S_n s.t. so that $S_n \downarrow S$, a.s.

$S_n \in Q_{2,n} = \left\{\frac{k}{2^n}\right\}_{k\geq 0}$ $\qquad S_n = \frac{k+1}{2^n}$ if $\frac{k}{2^n} \leq S < \frac{k+1}{2^n}$

Then $n\to\infty$, $S_n \to S$.

we will use the monotone class theorem
But choose simple Y

$Y_s(\omega) = \prod_{k=1}^{\ell} f_k(B_{t_k}) \cdot f_0(s)$, $\quad \{f_k\}_{k\geq 0}$ Bounded & Continuous.

where $t_0 < t_1 < t_2 < \ldots$ $\qquad f_k : \mathbb{R} \to \mathbb{R}$ $\quad k>0$

theorem works for S_n, Ref. Board 1

we will get $B_{t_k+S_n}$ From Formula.

By The Conditional DCT.

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$\mathcal{F}_{S_n} \downarrow \mathcal{F}_S$.
want to Prove
Reflection Priciple

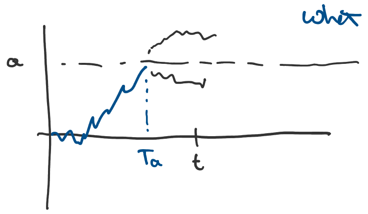
2.5

Reflect Pricipal for BM $\quad$ its like Levy. maximal inequality.

Let T_a, $a>0$ be $T_a = \inf\{t>0, B_t = a\}$.

Then $P_0(T_a \leq t) = 2P_0(B_t \geq a)$, $\forall t=0$.

what happens At time T_a?
50% chance Above a
" " below a



Given $\{T_a < t\}$ we get $P(B_t > a \mid \mathcal{F}_{T_a}) = 1/2$

$T_a \quad t$

Given $\{T_a < t\}$ we get $P(B_t > a \mid \mathcal{F}_{T_a}) = 1/2$

$$P(B_t < a \mid \mathcal{F}_{T_a}) = 1/2$$

$$P_0(T_a < t, B_t > a) = P_0(T_a < t, B_t < a) = P(T_a < t)$$

$$P_0(B_t > a \mid T_a < t) P(T_a < t) + P_0(\cdots \qquad = P(T_a < t)$$

$$\left(\tfrac{1}{2}\right) P(T_a < t) + \tfrac{1}{2} P(T_a < t)$$

since Prob. of term 1 & 2 is same. ∴

$$2 P_0(T_a < t, B_t > a) = P_0(T_a < t)$$

$$2 P_0(B_t > a) = P_0(T_a < t)$$

Formal Proof.

Fix $t > 0$, $a > 0$

Step 1 Build second term. A

$$Y_s(\omega) = \mathbb{1}_{\{\omega(t-s) > a\}} \cdot \mathbb{1}_{\{s < t\}}$$

$$E_{\cancel{x}\,a}(Y_s) = P_{\cancel{x}\,a}(B_{t-s} > a)\, \mathbb{1}_{\{s < t\}}$$

$$= \tfrac{1}{2} \mathbb{1}_{\{s < t\}}$$

$B(T_a) = a. \qquad s = T_a$

∴

$$E_{B(T_a)}(Y_{T_a}) = 1/2\, \mathbb{1}_{\{T_a < t\}}.$$

Random ∴ take Expectation,

$$E_0\left[E_{B(T_a)}(Y_{T_a})\right] = \tfrac{1}{2} P_0(T_a < t)$$

Step 2 For 1st term in SMP. B

$$Y_s \circ \Theta_s = \mathbb{1}_{\{\omega(t) > a;\, s < t\}}.$$

$$E_0(Y_{T_a} \circ \Theta_{T_a}) = P_0(B_t > a, \overset{\text{Implies.}}{T_a < t})$$

$$= P_0(B_t > a)$$

$$2 P_0(B_t > a) = P(T_a < t) \longrightarrow F_{T_a}(t)$$

Erased.

$$P_0(|B_t| > a) = P_0(T_a < t) = P_0\left(\max_{0 \le s \le t}\{B_s\} > a\right)$$

$$\Rightarrow |B_t^0| \overset{D}{=} \max_{0 \le s \le t}\{B_s\}, \quad \forall t \ge 0 \text{ from } \{B_t\}_{t \ge 0} \text{ SBM}$$

calc. Density from

$$P_0(B_t > a) = P(Z > \sqrt{t}\, a).$$

therefore need $\partial/\partial t\; P_0(B_t > a)$.

there should be Negative,

therefore need $\partial/\partial t \; P_0(B_t > a)$.

there should be Negative,

$$f_{T_a}(\quad = 2 \cdot \frac{d}{dt} P(Z > \sqrt{t}\, a) = -2 f_Z(\sqrt{t}\, a) \cdot a \frac{1}{2\sqrt{t}}$$

$$f_{T_a}(t) = (2\pi t^3)^{-1/2} e^{-1/2t}, \quad t > 0.$$

why t^3 why t not t^2

$E[T_1] = \infty$

$T_a \overset{\mathcal{D}}{=} a^2 T_1 \qquad T_2 - T_1 \overset{\mathcal{D}}{=} T_1$

$T_2 = 4T_1$