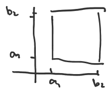
01-06

Monday, January 6, 2025 8:48 AM

Random vectors in $\mathbb{R}^d$, $d \geq 2$.

$$\underline{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_d \end{pmatrix}$$



$P(\underline{X} \in \prod_{i=1}^{d} [a_i, b_i]$ · $X_1 \in [a_1, b_1]$, $X_2 \in [a_2, b_2]$, ..., $X_d \in [a_d, b_2]$

examples

Binomial 2 outcomes
this have more than 2.

① $\underline{X} \sim$ multinomial $(n; p_1, \ldots, p_d)$ where $\sum_{i=1} p_i = 1$ $p_i \geq 0$

d outcomes : $O_1, \ldots, O_d$

$X = (X_i)_{1 \leq i \leq d}$ X_i = # of outcomes i $1 \leq i \leq d$.

$X_i \in (0, 1, \ldots, n)$

$\underline{X} = \sum_{i=1}^{n} Y_i$, $\{Y_i\}_{1 \leq k \leq d}$ IND.

$$Y_k \in \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \cancel{k} \atop i \right\}$$ if i occured in k experiments.

∴ X_1 is...

$X_1 \sim$ Binomial (n, p_1)

$$P\left(\underline{X} = \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix}\right) = \frac{n!}{\prod_{k=1}(n_k)!} \prod_{i=1}^{d} p_i^{n_i} \quad \Big\} \text{ CDF of Multinomial}$$

$\sum_{i=1}^{d} n_i = n$.

People are interested in covariance matrix of X.
have d coordinates ∴ $d \times d$ Matrix.

Covariance matrix $\Gamma(\underline{X}) = [\Gamma_{ij}]_{1 \leq i, j \leq d}$.

$$\Gamma_{i,j} = \mathrm{Cov}(X_i, X_j)$$
$$= E(X_i, X_j) - E(X_i)E(X_j)$$

$\Gamma_{i,i} = V(X_i) = np_i(1 - p_i)$ $1 \leq i \leq d$.

$\Gamma_{i,j} = E(X_i X_j) - E(X_i)E(X_j)$

$X_i = \sum_{k=1}^{n} \varepsilon_k$, $\varepsilon_k \sim \mathrm{Ber}(p_i)$, $\{\varepsilon_k\}_{1 \leq k \leq n}$ IND.

$X_j = \sum_{\ell=1}^{n} \delta_\ell$, $\delta_\ell \sim \mathrm{Ber}(p_j)$ $\{\delta_\ell\}_{1 \leq k \leq n}$ IND

$\mathrm{Cov}(X_i, X_j) = \sum_{1 \leq k, \ell \leq n} \mathrm{cov}(\varepsilon_k, \delta_\ell) = \sum_{k=1}^{n} \mathrm{cov}(\varepsilon_k, \delta_k)$
Bivariate linear so you can pull out.

Either $\varepsilon_1 = 1$ or $\delta_1 = 0$
$\delta_1 = 1$
$\varepsilon_1 = 0$
∴ $\varepsilon_1 \delta_1 = 0$.

ber

$= n\,\mathrm{cov}(\varepsilon_1, \delta_1) = n[E(\varepsilon_1 \delta_1) - E(\varepsilon_1)E(\delta_1)]$

$n[0 - p_i p_j$ Cov

The covariance is negative

∴ $\Gamma_{i,j} = -np_i p_j$.

$\underline{X} \sim$ multivariate normal (μ, Γ) $\vec{\mu} \in \vec{\mathbb{R}}^d$

$E(X) = \mu$

$\Gamma = [\mathrm{cov}(X_i, X_j)]$

$\vec{X} \sim$ Multivariate normal (μ, Γ) $\quad \vec{\mu} \in \vec{\mathbb{R}}$

$E(X) = \mu$

$E(X) = \begin{pmatrix} E X_1 \\ \vdots \\ E X_n \end{pmatrix}$

$\Gamma = \left[\text{cov}(X_i, X_j) \right]_{1 \le i,j \le d}$

if $X^d = \vec{M} + A\vec{Z}$

where $A \in M_{d \times d}$ $\leftarrow$ matrix

$A = [a_{ij}]_{1 \le i,j \le d}$

$\vec{Z} = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_d \end{pmatrix}$ $\quad \{Z_i\}_{1 \le i \le d}$ IND $N(0,1)$.

what happens to covariance matrix

calc covariance Γ of Multivariate normal(μ, Γ)

$\vec{Y} \in \mathbb{R}^d$, $E(Y) = 0$

$\Gamma(Y) =$

$\underbrace{\begin{pmatrix} Y_1 \\ \vdots \\ Y_d \end{pmatrix} (Y_1 \ \cdots \ Y_d)}_{Y Y^t} = \left[Y_i Y_j \right]^{d \times d}_{1 \le i,j \le d}$

$\Gamma(Y) = E(Y \cdot Y^t)$

$\Gamma = E\left((X - \mu)(X - \mu)^t\right)$

transpose

$(AB)^t = B^t A^t$

$= AZ(AZ)^t$

$= A Z Z^t A^t$

$A \underbrace{E(Z Z^t)}_{I} A^t$

$= AA^t$

AA^t - is symmetric matrix.

X is R.V. is Multi Normal

iff $t \cdot X = \langle t, X \rangle$ is $N(\mu, \sigma^2)$. $\forall \vec{t} \in \mathbb{R}^d$.

$t X = \sum_{i=1}^{d} t_i x_i$

Multivariate & Multi normals are 2 important examples.

How to characterize the distribution

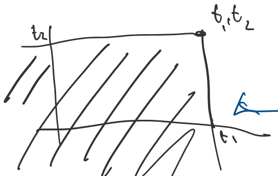
Let $X \in \mathbb{R}^d$

$F_X(t_1, \ldots, t_d) = P(X_1 \le t_1, \ldots, X_d \le t_d)$, $Z \in \mathbb{R}^d$

you characterize the distribution w/ Rect

t_1 $\quad t_1 t_2$

you characterize the distribution w/ Rect

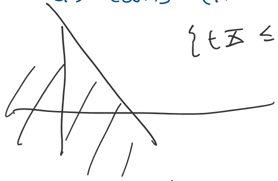


The Important char is not That Even though Book says,

Distribution of $\mathbb{X}$ is characterized by Dis $\{t \cdot \mathbb{X}\}_{t \in \mathbb{R}^d}$

Namely, $P(t \cdot \mathbb{X} \le u)$ $u \in \mathbb{R}$

this Looks like half planes



$\{t \mathbb{X} \le u\}$ in 2D.

if $\|t\|_2 = 1$ Norm $\sum_{i=1}^{d} t_i^2 = 1$

$\varphi_{t \cdot \mathbb{X}}(1) = E e^{i(t \mathbb{X})\cancel{u}}$ it like I multiplied t by u.

Since for all t we can consider u part of t.

$\varphi_X(t) = \varphi_{t \cdot \mathbb{X}}(1) = E e^{i(t \cdot \mathbb{X})}$, $t \in \mathbb{R}^d$.

Claim this is the characteristic function of X.

$X \in \mathbb{R}^d$. $t \in \mathbb{R}^d$.

inversion formula

Proved similar to 1D

no char. func. in infinite Dim.

Problem use $\varphi_X(t)$ to calculate $P(X \in A)$ where
$A = \prod_{i=1}^{d} (a_i, b_i)$?, can't Reach the Boundary.

$P(\mathbb{X} \in \partial A)$

X subtract ind. uniform.

$Y = \mathbb{X} - U$. $U \sim \text{uniform}(A)$

$f_U(x) = \frac{1}{\text{Volume}(A)}$. $x \in A$.

Where U & X ind. if $x \notin A$ Then 0.

Y is Bounded integrable, Even if X is not,

$f_{\vec{Y}}(\vec{0}) = P(X \in A)$.