

02-28 L2 MG, Polya

Friday, February 28, 2025 11:32 AM

L2 MG

Setup: $X_0 = 0$ $\{X_n, \mathcal{F}_n\}_{n \geq 0}$ MG. $E(X_n^2) < \infty \forall n$
 $\{X_n^2, \mathcal{F}_n\}_{n \geq 0}$ submg. $D_k = X_k - X_{k-1}$ $A_n = \sum_{k=1}^n D_k^2$ $n \geq 1$
 $X_n^2 = M_n + A_n$ $\{M_n, \mathcal{F}_n\}$ MG $A_0 = 0$, $A_n \in \mathcal{F}_{n-1}$ $A_n \uparrow$ a.s.
 Prob. $E(A_\infty) \leq E(\sup_{n \geq 0} |X_n|^2) \leq 4E(A_\infty)$

Convergence of series

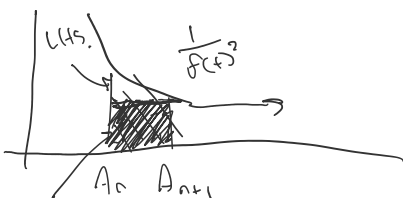
$$X_n \xrightarrow[n \rightarrow \infty]{a.s.} 0 \text{ on } \{A_\infty < \infty\}$$

SLLN: let $f(t)$, $t \geq 1$ be increasing. $\int_1^\infty \frac{dt}{f(t)^2} < \infty$

Then $\frac{X_n}{f(A_n)} \xrightarrow[n \rightarrow \infty]{a.s.} 0$ on $\{A_\infty = \infty\}$ Proved test t.h.w

upper bound.

should draw picture.



$$\frac{A_{n+1} - A_n}{f^2(A_{n+1})} \leq \int_{t=A_n}^{A_{n+1}} \frac{dt}{(f(t))^2}$$

right side.

B.C. II, $B_n \in \mathcal{F}_n$ $n \geq 1$ then $\{B_n \text{ i.o.}\} = \{\sum_{n=1}^\infty P_n(B_n) = \infty\}$

B.C. II⁺ $\frac{\sum_{k=1}^n \mathbb{1}_{B_k}}{\sum_{k=1}^n P_k} \xrightarrow[n \rightarrow \infty]{a.s.} 1$ on $\{\sum_{k=1}^\infty P_k = \infty\}$ $\uparrow P_n$

this goes to infinite.

Rewrite as $\sum_{n=1}^\infty \mathbb{1}_{B_n} = \infty$

omega by omega it is the same.

Proof Wts A_n $X_n = \sum_{k=1}^n \underbrace{\mathbb{1}_{B_k} - P_k}_{D_k \text{ MG. diff.}}$ $\leftarrow P_k \text{ is measurable in } \mathcal{F}_{k-1}$
 thus $P_k - P_k = 0$.

$$\mathbb{E}_{\mathcal{F}_{k-1}}(D_k^2)$$

what is the variance of Bernoulli $P(1-P)$.

$$= P_k(1-P_k) \therefore A_n = \sum_{k=1}^n P_k(1-P_k)$$

$$\left\{ \sum_{k=1}^\infty P_k = \infty \right\} \stackrel{?}{=} \left\{ A_\infty = \sum_{k=1}^\infty P_k(1-P_k) = \infty \right\}$$

something smaller.

\Rightarrow we think that.

Take $P_k = 1$

$$\{\infty\} \supset \{0\}$$

$$\{ \infty \} > \{ 0 \}$$

$\{ A_{\infty} < \infty \} \cap \{ \sum_{k=1}^{\infty} p_k = \infty \}$ what happens here? need to use X_n MB

$$\frac{X_n}{\sum_{k=1}^n p_k} = \frac{\sum_{k=1}^n 1_{B_k} - p_k}{\sum_{k=1}^n p_k} \rightarrow 0$$

$$\frac{X_n}{\sum_{k=1}^n p_k} \xrightarrow[n \rightarrow \infty]{\text{Finite A.S.}} 0. \quad (\text{B.C. II} +)$$

What happens is $\{ A_{\infty} = \infty \}$

use SLLN:

$$F(t) = t \quad \int_1^{\infty} \frac{dt}{t^2} < \infty. \quad \text{need finite time finite.} \quad F(A_n) = A_n.$$

$$\frac{X_n}{\sum_{k=1}^n p_k(1-p_k)} \xrightarrow[n \rightarrow \infty]{\text{A.S.}} 0 \text{ on } \sum_{k=1}^{\infty} p_k(1-p_k) = \infty.$$

so Divide by something bigger.

$$\frac{X_n}{\sum_{k=1}^n p_k} \xrightarrow[n \rightarrow \infty]{\text{A.S.}} 0.$$

where do you use it? Polya scheme.

Polya scheme.

$\begin{bmatrix} r \text{ red} \\ g \text{ green} \end{bmatrix} \xrightarrow{R} \begin{bmatrix} r+1 \\ g \end{bmatrix}$ we add c
 $G_n = \{ \# \text{ of Greens After } n \text{ times} \}.$

$$\left\{ \frac{G_n}{G_n + R_n}, \mathcal{F}_n \right\}_{n=1}^{\infty} \text{ MB} \quad \frac{G_n}{G_n + R_n} \xrightarrow[n \rightarrow \infty]{\text{A.S.}} X \sim \text{Beta}\left(\frac{g}{c}, \frac{r}{c}\right)$$

$$\text{FRIEDMAN} \quad \begin{bmatrix} r \text{ red} \\ g \text{ green} \end{bmatrix} \xrightarrow{R} \begin{bmatrix} r+1 \\ g+1 \end{bmatrix} \quad \frac{G_n}{G_n + R_n} \xrightarrow[n \rightarrow \infty]{\text{A.S.}} \frac{1}{2}$$

Even if we add 10 to other
add 1 to logics.

$G_n = \# \text{ green After } n \text{ times.}$

$R_n = \# \text{ Red after } n \text{ times}$

$$g_n = \frac{G_n}{R_n + G_n} \quad - R_n = n^{\text{th}} \text{ select is green}$$

$$g_n = \frac{w_n}{R_n + G_n} \quad - R_n = n^{\pi} \text{ select is green}$$

$$r_n = \frac{R_n}{R_n + G_n} \quad D_n = \sum_{k=1}^n \mathbb{1}_{B_k}$$

what is condition probability $P_{\tilde{g}_{n-1}}(B_n) = g_{n-1}$

what can say about g_n $g_n \geq \frac{g}{g+c+n(c+1)} \sim \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$$\therefore \sum_{n=1}^{\infty} g_n = \infty \text{ a.s.}$$

\uparrow Games played
 \uparrow winner
 \uparrow losses

$$\frac{D_n}{\sum_{k=1}^n g_{k-1}} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 1 \quad \text{since BC II +} \quad \frac{\sum \mathbb{1}_{B_k}}{\sum p_k} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 1$$

Doob inequality L^p for $p=2$.

Another version with same setup 1

$$E\left(\sup_{n \geq 0} |x_n|\right) \leq 3 \cdot E(\sqrt{A_{\infty}}) \quad \text{Proof is in the book.}$$

Application...

Corollary Let $\{x_n, \tilde{x}_n\}_{n \geq 0} \quad L^2 \text{ Mb.}$

If $E(\sqrt{A_{\infty}}) < \infty$ then $0 \leq x_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} x$ by MGCT

we only need $\sup_n E|x_n| < E\left(\sup_{n \geq 0} |x_n|\right) < \infty$.

$$2. E|x_n - x| \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{By DCT.}$$

Extension of Wald First Equation

Let $\{z_k\}$ be iid $E(z_k) = 0$. then T is st. wrt \mathcal{F}_n .

$$E(T) < \infty \Rightarrow T < \infty \text{ a.s.}$$

then $E(S_T) = 0$. wld first eq.

Version: we also assume $E(z^2) < \infty$, but

now $E(\sqrt{T}) < \infty$, then $E(S_T) = 0$
 $\Rightarrow T < \infty$.

Always true. $S = \sum_{i=1}^T x_i$

$\{S_{T \wedge n}\}$ Mg. $E(S_{T \wedge n}) = 0$.

$$S_{T \wedge n} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} S_T$$

solve using A_{∞}

$$A_n = \sum_{k=1}^n E_{\mathcal{F}_{n-1}} \{z_k^2\} = \sigma^2(n \wedge T)$$

$$A_{\infty \wedge T} = A_T = \sigma^2 T \Rightarrow E(\sqrt{A_{\infty}}) < \infty$$

$$A_{\infty} \wedge T = A_T = \sigma^2 T \Rightarrow E(\sqrt{A_{\infty}}) < \infty$$

$$\Rightarrow E(S_T) = \lim_{\lambda \rightarrow \infty} E(S_{T \wedge \lambda}) = 0$$

Ben-Gad's good:
Ben-Gad's? - chris's.