L22 - 10-16 bc 2 ext

Wednesday, October 16, 2024 11:32 AM

Extension of B.C. II

Let $A_1, A_2, \dots$ be Pairwise ind. and $\sum_{k=1}^{\infty} (A_k) = \infty$ (theorem 2.3.8 Durrett)

Then $\sum_{m=1}^{n} \mathbb{1}_{A_m} \Big/ \sum_{m=1}^{n} P(A_m) \xrightarrow[n\to\infty]{a.s} 1$ Much stronger BC II.

Example $r, r_1, r_2, \dots$ Are iid r symmetric Bernoulli

$P(r=1) = \frac{1}{2} = P(r = -1)$

we look At $\left\{ \sum_{k=1}^{m} r_{n_k} \right\}_{n_1 < n_2 < \dots < n_m}$ ← countable subset $n_1 < n_2 < \dots$

Claim : Pairwise Independence, But not InD.

Consider

$r_1, r_2, r_3 = r_1 \cdot r_2$ not ind. for All 3.

$P(r_1 = 1, r_2 = 1, r_1 r_2 = -1) = 0.$

$P(r_1 = 1) = \frac{1}{2} = P(r_2 = 1) = P(r_1 r_2 = -1) = 1/8$ $0 \neq 1/8$ $\therefore$ not InD,

$\underset{1/4}{P(r_2 = 1, \overset{r_1 \cancel{r_2} = -1}{r_1 = -1})} = \underset{1/2}{P(r_1 = 1)}\, \underset{1/2.}{P(r_1 r_2 = -1)}$

Proof of BC II Ext.

$V\left(\sum_{m=1}^{n} 1_{A_m}\right) = \sum_{m=1}^{n} V(1_{A_m}) = \overset{\text{no } \cancel{\text{cov}}\ 0}{\sum} P(A_m)(1 - P(A_m)) \le \sum_{m=1}^{n} P(A_m) = \mathbb{E} \sum_{m=1}^{n} 1_{A_m}$

$S_n = \sum_{m=1}^{n} 1_{A_m}$. we get $V(S_n) \le E(S_n)$

chebychev.

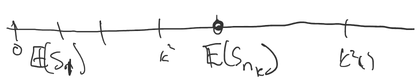
$\frac{S_n}{E(S_n)} \xrightarrow[n\to\infty]{P} 1$

Build subseq. ← $\{n_k\}_{1 \le k}$ $n_k \uparrow \infty$ with.

Let $k^2 + 1 \ge \mathbb{E}(S_{n_k}) \ge k^2$, $k = 1, 2,$

why does $\{n_k\}_{1 \le k}$ exist?

$E(S_n) = \sum_{m=1}^{n} P(A_m)$



$\mathbb{E}(S_n + 1) = \sum_{m=1}^{n+1} P(A_m)$

$\mathbb{E}(S_{n+1}) - \mathbb{E}(S_n) = P(A_{n+1}) < 1$

$\mathbb{E}(S_n) \uparrow \infty$ ... due. study chebychev.

$\mathbb{E}(S_{n+1}) - \mathbb{E}(S_n) = P(A_{n+1}) < 1$

$$\mathbb{E}(S_n) \uparrow \infty$$

study chebychev.

$$P(|S_{n_k} - \mathbb{E}(S_{n_k})| > \delta(k^2+1)) \overset{\text{chebychev.}}{\leq} \frac{V(S_{n_k})}{\delta^2(k^2+1)^2} \leq \frac{\cancel{\mathbb{E}(S_{n_k})}\ \cancel{k^2+1}}{\delta^2(k^2+1)^{\cancel{2}}} = \frac{1}{\delta^2(k^2+1)}$$

$$\sum_{k=1}^{\infty} P(|S_{n_k} - \mathbb{E}(S_{n_k})| > \delta(k^2+1)) \leq \frac{1}{\delta^2}\sum_{k=1}^{\infty}\frac{1}{k^2+1} \leq \infty$$

B.C. I. know

B.C. I says

$$P\left(\frac{|S_{n_k} - \mathbb{E}(S_{n_k})|}{k^2+1} > \delta \text{ i.o.}\right) = 0$$

i.o.

$$\Rightarrow \frac{\mathbb{E}(S_{n_k})}{k^2+1}\frac{[S_{n_k} - \mathbb{E}(S_{n_k})]}{\mathbb{E}(S_{n_k})} \xrightarrow[k\to\infty]{a.s.} 0$$

$$\frac{k^2}{k^2+1} \leq \frac{\mathbb{E}(S_{n_k})}{k^2+1} \leq 1 \quad \Rightarrow \quad \frac{S_{n_k}}{\mathbb{E}(S_{n_k})} \xrightarrow[k\to\infty]{a.s.} 1$$

$$\frac{k^2}{k^2+1} \xrightarrow{k\to\infty} 1, \qquad \frac{\mathbb{E}(S_{n_k})}{k^2+1} \xrightarrow{k\to\infty} 1$$

"how do we go from subseq to seq?"

Let $n_k \leq n \leq n_{k+1}$

$$\frac{S_{n_k}}{\mathbb{E}(S_{n_{k+1}})} \leq \frac{S_n}{\mathbb{E}(S_n)} \leq \frac{S_{n_{k+1}}}{\mathbb{E}(S_{n_k})}$$

S_n is always smaller than S_{n_k}

$$= \frac{S_{k+1}}{\mathbb{E}(S_{n_{k+1}})} \cdot \frac{\mathbb{E}(S_{n_{k+1}})}{\mathbb{E}(S_{n_k})}$$

$$\frac{S_{k+1}}{\mathbb{E}(S_{n_{k+1}})} \xrightarrow[k\to\infty]{a.s.} 1, \qquad \frac{\mathbb{E}(S_{n_{k+1}})}{\mathbb{E}(S_{n_k})} \xrightarrow{k\to\infty} 1$$

What is ration.

$$S_{n_{k+1}} \geq S_{n_k}$$

$$\therefore 1 \leq \frac{\mathbb{E}(S_{n_{k+1}})}{\mathbb{E}(S_{n_k})} \leq \frac{(k+1)^2+1}{k^2} \xrightarrow{k\to\infty} 1$$

trick: we have Ratio Then shows it is Between Two Ratios That converge to 1

$$\left[\frac{\mathbb{E}(S_{n_k})}{\mathbb{E}(S_{n_{k+1}})}\right] \cdot \frac{S_{n_k}}{\mathbb{E}(S_{n_k})} =$$

$$\frac{\mathbb{E}(S_{n_k})}{\mathbb{E}(S_{n_{k+1}})} \xrightarrow{k\to\infty} 1, \qquad \frac{S_{n_k}}{\mathbb{E}(S_{n_k})} \xrightarrow[k\to\infty]{a.s.} 1$$

which is proves BC II Ext.

Record Wave:

Let $\{x, X_i\}_{i\geq 1}$ i.i.d. $P(X = x) = 0 \quad x \in \mathbb{R}$.

$$A_k = \{X_k > \max_{1\leq i\leq k-1}\{X_i\}\}$$

given some k is The highest.

$$\text{Prob}(A_k) = P(\pi_k = k)$$
$$= \frac{(k-1)!}{k!} = \frac{1}{k}$$

Random Permutation on $\{1, 2, \ldots, k\}$.

$\pi : \pi_1, \pi_2, \ldots, \pi_k$

Permutations is $k!$

they want pairwise independence.

$P(\pi) = \frac{1}{k!}$

Let $j < k$

The Probablity of Events A

$P(A_k \cap A_j) = P(\pi_k = k, \pi_j > \max\{\pi_1, \dots, \pi_{j-1}\})$

$P(A \cap B) = P(A)P(B)$

$= \frac{(j-1)! \; \cancel{(k-1)}! \binom{k-1}{j}}{k!} = \frac{1}{kj}$

how many times Before.

uncorrelated

from $\frac{1}{k}$

IND $\to$ Pairwise

$\frac{(k-1)!}{j!(k-1-j)!}$

$\frac{(k-1)!}{(k-1-j)!(\cancel{k-1-k+1}+j)!}$

A_k, A_j Are IND.

Now use The Resent (Extension of BC II)

$$\frac{R_n}{\sum_{k=1}^{n} 1/k} \xrightarrow[n \to \infty]{a.s.} 1$$

$$\frac{R_n}{\log_e(n)} \xrightarrow[n \to \infty]{a.s.} 1$$

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trick:

Replace $Y_i = \frac{X_i}{M}$, $0 \le Y_i \le 1$

$0 \le X_i \le M$ $\quad \{X_i\}_{i \ge 1}$ uncorrelated.

$$\sum_{k=1}^{\infty} E(X_k) = \infty$$

then:

$$\frac{S_n}{E(S_n)} \xrightarrow[n \to \infty]{a.s.} 1$$