02-21

Friday, February 21, 2025 11:31 AM

$(\Omega, \mathcal{F}_0, P)$, $\mathcal{F} \subset \mathcal{F}_0$ $X \geq 0$, $E(X) < \infty$

$\mu(A) = E(X; A)$, $A \in \mathcal{F}$

μ: is a finite measure on $(\Omega, \mathcal{F})$

$\mu \ll P$ $(P(A) = 0 \Rightarrow \mu(A) = 0, A \in \mathcal{F})$

By R.N. $\frac{\partial \mu}{\partial P} = Y$, $Y \in \mathcal{F}$, $\mu(A) = E(Y; A) = \int_A Y \partial P$. $A \in \mathcal{F}$

$E_{\mathcal{F}}(X) = Y$

if X is not Positive.

$X = X^+ - X^-$

$E_{\mathcal{F}}(X) = E_{\mathcal{F}}(X^+) - E_{\mathcal{F}}(X^-)$

Doob.

Section 4 : L^p convergence, $p > 1$ (Durrett § 5.4 P. 212)

constant.

Theorem Let $\{X_n, \mathcal{F}_n\}_{n \geq 0}$ subMG. , T is S.t., $T \leq M < \infty$

then

$$E(X_0) \overset{\text{I}}{\leq} E(X_T) \overset{\text{II}}{\leq} E(X_M)$$

Proof "use Gambling system." if invest positive units

if H positive $H \cdot X$ subMG.

$(H \cdot X)_n = \sum_{k=1}^{n} H_k D_k$ $D_k = X_k - X_{k-1} \in \mathcal{F}_k$ $H_k \in \mathcal{F}_{k-1}$ $(H \cdot X)_0 = 0$

"Decide at $k-1$ Before. thus need to choose Appropriate H.

for I we select $H_k = \mathbb{1}_{\{k \leq T\}}$ $H_1 = 1_{\{1 \leq T\}}$ $H_2 = 1_{\{2 \leq T\}}$. ...

$(H \cdot X)_n = X_{T \wedge n} - X_0$

subMG.

$(H \cdot X)_M = X_T - X_0$

$E(X_T - X_M) \geq 0$

II $H_n = 1_{\{T < n\}}$.

$(H \cdot X)_n = X_n - X_{T \wedge n}$ $0 \leq n \leq M$.

$$(H\cdot X)_M = X_M - X_T$$
$$E(X_M - X_T) \geq 0$$

Corollary: Under the same conditions & setup.

$$E_{\mathcal{F}_T}(X_M) \geq X_T \quad \text{a.s.}$$

Proof: take $A \in \mathcal{F}_T$, Def $\tilde{T} = \begin{cases} T & \text{on } A \\ M & \text{on } A^c \end{cases}$ if I_M not in A then M .:. St. $\tilde{T} \leq M$.

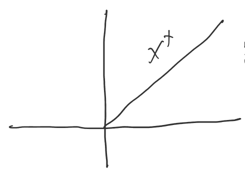
$$E(X_M) \geq E(X_{\tilde{T}})$$

$$E(X_M; A) \geq E(\cancel{X_M; A^c}) \geq E(X_{\tilde{T}}; A) + E(\cancel{X_T; A^c}) = E(X_T; A) \quad , \; A \in \mathcal{F}$$

SAME.

"says Bonding is Important."

Doob inequality. "Kolgomorov Maximality." - Had. IND. .:. special case of Doob



nondecreasing & convex. subMG.

Let $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ be Sub MG. Let $\lambda > 0$

Let $A = \{\max_{0\leq m\leq n}\{X_m\} \geq \lambda\}$ then

obviously. $\leq$ The E over whole set is Bigger than E given.

$$\lambda \cdot P(A) \leq E(X_n^+; A) \leq E(X_n^+)$$

Proof WLOG $X_n > 0$ Define $T = \inf_{0\leq m\leq n}\{m; X_m \geq \lambda\} \wedge n$, $T \leq n$

$$X_T \cdot \mathbb{1}_A \geq \lambda \cdot \mathbb{1}_A$$

$$\Rightarrow \quad E(X_n; A) \geq E(X_T : A) \geq \lambda P(A)$$

Similar, Because MG.

Kolmogorov' Inequality.

$\{Z_k\}$ are IND, $E(Z_k) = 0$, $E(Z_k^2) < \infty$ $k = 1,2,3.$

$$P\left(\max_{1\leq m\leq n} \cancel{|} S_n^2 \cancel{|} \geq x^2\right) \leq x^{-2} E(S_n^2).$$

$$S_m = \sum_{k=1}^{m} Z_k \quad , \; m \geq 1$$

$\{S_m, \mathcal{F}_m\}_{m\geq 0}$ is MG

where $\mathcal{F}_m = \sigma\{Z_1, \ldots, Z_m\}$

$\Rightarrow \{S_m^2, \mathcal{F}_m\}_{m\geq 0}$ is Sub MG

L^p maximal inequality. $p>1$ $\frac{1}{p}+\frac{1}{q}=1$, $p+q=pq$ p-norm.

Let $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ be L^p subMG $E(|X_m|^p)<\infty$ $\|X_n\|_p = E(|X_n|^p)^{1/p}$

then $\|\max_{1\leq m\leq n}\{X_m^+\}\|_p \leq q\cdot\|X_n\|_p$

Proof. WLOG $X_n>0$ can use the tail to integrate

$$E\left[\left(\max_{0\leq m\leq n} X_m\right)^p\right] = \int_{\lambda=0}^{\infty} p\lambda^{p-1} P(\max_{m\leq n} X_m > \lambda)\, d\lambda$$

By Doob inequality.

Everything is positive ∴ switch order

$$\leq \int_{\lambda=0}^{\infty} p\lambda^{p-1}\left[\int_\Omega \lambda^{-1}\cdot X_n\cdot \mathbb{1}_{\{\max_{m\leq n} X_n\geq\lambda\}}\right] d\lambda$$

$$= \int_\Omega X_n\left[\int_{\lambda=0}^{\max_{m\leq n}\{X_n\}} p\lambda^{p-2}\, d\lambda\right] dP$$

$\frac{p}{p-1}=q$

By integration.

$$= \frac{p}{p-1}\int_\Omega X_n\cdot\left(\max_{m\leq n} X_m\right)^{p-1} dP.$$

By holder.

$$\leq q\,\|X_n\|_p\left[E\left(\max_{m\leq n} X_m\right)^p\right]^{1/q}$$

$$E\left[\left(\max_{m\leq n} X_n\right)^p\right] \leq q\|X_n\|\left[E\left(\max_{n\leq n} X_m^p\right)\right]^{1/q}$$

Belongs to L^p — Belongs to L^p

$$\max_{1\leq m\leq n} X_m \leq \boxed{\sum_{m+1}^{n} X_m} \Rightarrow \left\|\max_{1\leq m\leq n}\{X_m^+\}\right\|_p \leq q\|X_n\|_p$$

" $p=1$ Does not work.

L^1 case

$\{X_n, \mathcal{F}_n\}$ subMG

$$E\max_{m\leq n} X_m^+ \leq \frac{\ell}{\ell-1}\left(1+E(X_n^+[\log(X_n^+)]^+\right)$$

is Related Dominated convergence theorem $p=2$.