L11 - 09-20 uniform integration

Friday, September 20, 2024 11:31 AM

Uniform Integrability $Y \geq 0$

Claim if $I(|Y|) < \infty$ then $E(|Y| \cdot 1_{\{|Y|<M\}}) \to 0$

1st $Y \cdot 1_{\{Y>M\}} \xrightarrow[M\to\infty]{a.e.} 0$ the set $Y > m$ decreasing.

more and more Become zero.

in fact $Y \, 1_{\{Y>M\}} \downarrow 0$ a.e, as $M \to \infty$

2nd $\leq Y \cdot 1_{\{Y>M\}} \leq Y$

we can use D.C.T. : $E(Y \cdot 1_{\{Y>M\}}) \xrightarrow[M\to\infty]{\infty}$

$\{X_n\}_{n\geq 1}$ is Uniform Ingreble (U.I.) if

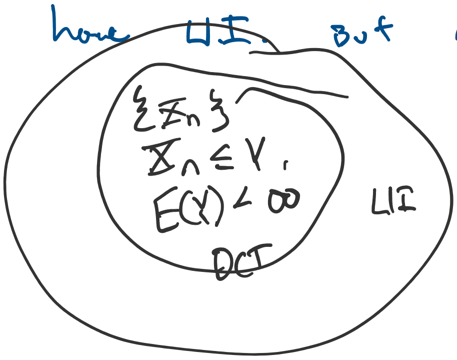
$\varphi(M) \xrightarrow[M\to\infty]{} 0$ where $\varphi(M) = \sup_{n\geq 1} E(|X_n| \cdot 1_{\{|X_n|>M\}})$. bigger than Y

this one.

observation 1 if $X_n \leq Y$, $\forall n \geq 1$, $E(Y) < \infty$ then

$\{X_n\}_{n\geq 1}$ is U.I.

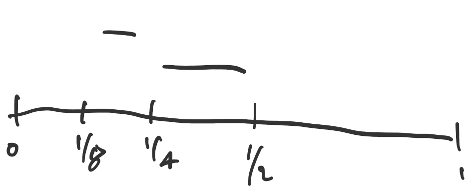
have U.I. but not DCT



Ex. $\Omega = [0,1]$ Lebesgue measure

The Length

AKA dx.



$\left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$

$\left(\frac{1}{2^{n+2}}, \frac{1}{2^{n+1}}\right] \leftarrow$ closer to Zero. Disjoint.

$$X_n = \frac{2^n}{n} \mathbb{1}_{(2^{n+1}, \frac{1}{2^n}]}(x)$$

$$X_n \xrightarrow[n\to\infty]{a.s.} 0$$

Q. is there Y so that $X_n \le Y$, $\forall n \ge 1$? $E(Y) \le \infty$

A. No

minimal Y with $Y \ge X_n$, $\forall n$

is $Y(x) = \sum_{n=1}^{\infty} \frac{2^n}{n} \mathbb{1}_{(\frac{1}{2^{n+1}}, \frac{1}{2^n}]}(x)$ $\quad 0 < x < 1$

$$E(Y) = \sum_{n=1}^{\infty} \frac{2^n}{n} \mathbb{1}_{(\frac{1}{2^{n+1}}, \frac{1}{2^n}]}(x)$$

The Expection of $\mathbb{1}_{(a,b)}$ = b-a.

$$= \sum_{n=1}^{\infty} \frac{\cancel{2^n}}{n} \frac{1}{\cancel{2^n} 2} = \frac{1}{2} \sum$$

$$E(X_n) = \frac{2^n}{n} \cdot \frac{1}{2^{n+1}} = \frac{1}{2n} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow \{X_n\}$. U.I. But not DCT.

main Result of U.I.

Switch to Prob space

theorem if $\{X_n\}_{n \ge 1}$ is UI and $X_n \xrightarrow[n\to\infty]{a.s.} X$ Then $E(X_n) \xrightarrow[n\to\infty]{} E(X)$ } goal

$(E|X_n - X| \xrightarrow[n\to\infty]{} 0)$

Stronger which is true.

Example U.I. seq

$\{X_n\}$ which is UI.

if $\underbrace{\sup_n \{E|X_n|^p\}}_{C} < \infty$ $\quad p > 1$

Bounded.

then $\{X_n\}_{n \ge 1}$ is UI.

Exercise.

(in Fact $\{X_n^{p'}\}$ is UI for $1 < p' < p$)

$$E[|X_n|^{\cancel{p}} : |X_n| > M] \le E\left[\frac{|X|^p}{M^{p-1}} ; |X_n| > M\right] \le \frac{\sup_n \{E|X_n|^p ; |X_n| > M\}}{M^{p-1}} \le \frac{C}{M^{p-1}} \xrightarrow[M\to\infty]{} 0$$

if $|X_n| > m$

Regardless of n

To Justify.

take one $|X|$

$$\frac{|X|^p}{M^{p-1}} = |X_n| \cdot \underbrace{\frac{|X_n|^{p-1}}{M}}$$

Result

$\{X_n\}$ is UI iff { (i) $\sup_n P(E|X_n|) < \infty$ more than one.

(ii) $\forall \varepsilon > 0\ \exists \delta > 0$ so that ... Roh Slit.

Result

$\{X_n\}$ is UI iff $\begin{cases} \text{(i) } \sup_n E|X_n| < \infty & \text{more than one.} \\ \text{(ii) } \forall \varepsilon < 0 \; \exists \delta > 0 \text{ so that} & \text{in probability.} \end{cases}$

if $P(A) < \delta$ then

$\sup E[|X_n| : A] \le \varepsilon$

say

$\delta = 1/k$. large x_n on zero.

such as n on zero.

Proof. (i) (wlog $x_n > 0$)

$\Rightarrow \quad E(X_n) = E(X_n; X_n > M) + E(X_n; X_n \le M)$

$\sup_{n \ge 1}\{E(X_n)\} \cdot \varphi(M) + M < \infty$

(ii) $\sup_n E[X_n; A] \le \sup_n E(X_n; A \cap \{X_n > M\})$

$+ \sup_n E(X_n; A \cap \{X_n \le M\})$

$\sup_n(a_n + b_n) \le \sup_n(a_n) + \sup_n(b_n)$

Describe a_n as $(1, 0, 0, 0 \ldots$

b_n as $(0, 1, 0, 0 \ldots$

Then the sum is $a_n + b_n = (1+0, 0+1, 0+0, \ldots)$

$\le \varphi(M) + P(A) M$

$\le \varphi(M)\, \delta M \le \varepsilon$

"$\Leftarrow$" Assume $x_n \ge 0$

Denote $C = \sup_n E(X_n)$ so $P(X_n > M) \le \frac{E(X_n)}{M} \le \frac{C}{M}$

$\sup_n P(X_n > M) \le \frac{C}{M} \xrightarrow[M \to \infty]{} 0$

Now we get $\varphi(M) \xrightarrow[M \to \infty]{} 0$ by using (ii)

Corollary: If $\{X_n\}_{n \ge 1}$ and $\{Y_n\}_{n \ge 1}$ are UI

then. $\{X_n + Y_n\}_{n \ge 1}$ is also UI.

Result $E[X_n - X] \xrightarrow[x \to \infty]{} 0$, $E|X_n| < \infty, \forall n > 1$, $E|X| < \infty$

then

$\{X_n\}_{n \ge 1}$ is UI.

Proof step 1. Assume $X = 0$

$E|X_n| \xrightarrow[n \to \infty]{} 0$ have to show $\sup E(|X_n| \mathbb{1}_{|X_n| > n})$

Let $\varepsilon > 0$

$\exists N \ldots \sup E(|X_n| : |X_n| > n) \le \varepsilon$

$\mathbb{E}|X_n| \xrightarrow[n\to\infty]{} 0$ have to show $\sup \mathbb{E}(|X_n| \mathbb{1}_{|X_n|>M})$

Let $\varepsilon > 0$

$\exists$ N so that $\sup_{n \geq N} \mathbb{E}(|X_n| ; |X_n| > M) \leq \varepsilon$

$$\varphi(M) \leq \sup_{n \leq N} \mathbb{E}|X_n| ; |X_n| > M + \varepsilon$$

$$\leq \sum_{n=1}^{N} \underbrace{\mathbb{E}(|X_n| ; |X_n| > M)}_{\downarrow \, M\to\infty \atop 0} + \varepsilon$$

$$\xrightarrow[M\to\infty]{} \varepsilon$$

$$\limsup_{M\to\infty} \varphi(M) \leq \varepsilon$$