02-19 Radon, kaketuri

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trying to Prove Radon - nikodym (Durrett §5.3.3 P. 206)

RN for $((0,1], \mathcal{B})$ with 2 P.m μ δ, $\rho = \frac{\mu+\delta}{2}$

$I_{n,k} = \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]$, $k = 1, \ldots, 2^n$, $n = 1, 2, \ldots$

$\mathcal{F}_n = \sigma\{I_{n,k}\}_{1 \le k \le 2^n}$, $\mathcal{F}_n \subset \mathcal{F}_{n+1}$

$$X_n(t) = \frac{\mu(I_{n,k})}{\rho(I_{n,k})}, \quad t \in I_{n,k}.$$

① $\{X_n, \mathcal{F}_n\}_{n \ge 1}$ on $((0,1], \mathcal{B}, \rho) \underset{MGCT}{\Longrightarrow} X_n \xrightarrow[n\to\infty]{a.s.} X$.

② $0 \le X_n \le 2$.

sometimes people write $X = \frac{d\mu}{d\rho}$

③ $\mu(A) = \int_A X \, d\rho$, $A \in \mathcal{B}$. Borel.

④ $\int_A 2 - X \, d\mu = \int_A X \, d\gamma$, $A \in \mathcal{B}$. Also, $\gamma\{x = 2\} = 0$. Gamma

opps!

$$\int_A \cancel{x} \, d\rho = \int_A X \, \frac{d\mu + d\gamma}{2} = \frac{1}{2}\left[\int_A x \, d\mu + \int_A x \, d\gamma\right]$$

why is $d\rho = \frac{d\mu + d\gamma}{2}$

$$\int_A 2 - x \, d\mu = \int_A x \, d\gamma$$

Goal (intuition).

$(2 - X) \, d\mu = X \, d\gamma$

$d\mu = \frac{X}{2 - X} d\gamma$ we want to get

can Be 0.

⑤ $\int_{(0,1]} \cancel{\mathbb{1}_A}^{\,Y} \cdot (2 \cdot X) \, d\mu = \int_{(0,1]} \cancel{\mathbb{1}_A}^{\,Y} X \, d\gamma$, $A \in \mathcal{B}$.

Replace By simple function $\sum_{i\ge1}^{n} a_i \cdot \mathbb{1}_{A_i} \to Y$. By DCT.

select Y in smart way.

if x is close to $2-\epsilon$ then zero. otherwise. x?

Take $Y = \dfrac{\mathbb{1}\{\{x<2-\epsilon\} \cap A\}}{2-x} \leq \dfrac{1}{\epsilon}$

(6) $\mu(A \cap \{x \leq 2-\epsilon\}) = \displaystyle\int_{A\cap\{x\leq 2-\epsilon\}} \frac{x}{2-x}\, d\gamma$

since $\int_{(0,1]}$

$\epsilon \to 0$

$\mu(A \cap \{x<2\}) = \displaystyle\int_{A\,\cancel{\cap\{x<2\}}} \frac{x\, 1\{x<2\}}{2-x}\, d\gamma$

$\mu_r(A) =$

← Regular. we are spliting into 2 measure the singular and the regular.

not nessesarily prob. measure, Depends on x 2

$\dfrac{d\mu_r}{d\gamma} = \dfrac{x\, 1\{x\leq 2\}}{2-x}$

Relation Between r & γ

observe:

$\gamma(A) = 0 \Rightarrow \mu_r(A) = 0.$

we say μ_r is Absolutely continous w.r.t. γ Denoted $\mu \ll \gamma$

Randon nikodym says. if $\mu_r \ll \gamma$ then $\exists\ \dfrac{d\mu_r}{d\gamma}$

if $\mu_r((0,1]) < 1$ μ - singular.

we Define $\mu_s(A) = \mu(A) - \mu_r(A)$.

$\mu_r + \mu_s = \mu.$

← Alive when $x = 2.$

in the Measure space > 0.

$\mu_r(A) = \mu(A \cap \{x \neq 2\})$ live in seperate worlds

$\mu_s(A) = \mu(A \cap \{x = 2\})$

$\gamma(x=2) = 0.$

trivial Example. $\gamma = \lambda$ on $[0,1]$.

$\mu(A) = \begin{cases} 1 & \text{if } 1/2 \in A \\ 0 & \text{if } 1/2 \notin A \end{cases}$

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$$\mu = \delta_{\{1/2\}}.$$

$\mu_s = \mu,$ we started with $\frac{\partial \mu}{\partial \rho}$

$\mu_r = 0$

μ, γ

$Y_n(t) = \frac{\mu(I_{nk})}{\gamma(I_{nk})}, \quad t \in I_{nk}.$ Assume positive

$\{Y_n, \mathcal{F}_n\}_{n \geq 1}$ MG. w.r.t. γ

think super MG converge Almost surely..

$Y_n \xrightarrow[n \to \infty]{a.s.} Y, \quad E_\gamma(Y) \cancel{< \infty} \leq 1$ by Fatou Lemma

$Y_n = \frac{X_n}{2 - X_n}$ it strictly less than 1

this is Lebesgue Decomposition

$E(Y) = 0 \Rightarrow \mu_s = \mu$ w.r.t γ

γ is not in $x = 2$.

— in modern writing tensor

<u>Kakutani</u> $\mu = \mu_1 \times \mu_2 \times \mu_3 \times \dots$ a P.n on $\mathbb{R}^\infty$

$P = P_1 \times P_2 \times P_3$ on $\mathbb{R}^\infty$

$\mu = \prod_{k=1}^{\infty} \mu_k \qquad \mu_k << P_k \quad \forall k = 1, 2, \dots$

$P = \prod_{k=1}^{\infty} P_k$

$\mathcal{F}_n = \sigma\{X_1, \dots X_n\} \qquad X_k(\omega) = \omega_k.$

$$Y_n = \left.\frac{\partial \mu}{\partial P}\right|_{\mathcal{F}_n} = \prod_{k=1}^{n} \frac{\partial \mu_k}{\partial P_k}.$$

What do we get with Point mass on μ_i

Kakutani:

Let $Y_n \xrightarrow[n\to\infty]{a.s.} Y$

Either $E_p(Y) = 0$ or $E_p(Y) = 1$

$$\mu = \mu_s \qquad\qquad \mu = M_r$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\prod_{k=1}^{\infty} E_p\left(\sqrt{\frac{\partial \mu}{\partial P_k}}\right) = 0 \qquad\qquad \prod_{k=1}^{\infty} E_p\left(\sqrt{\frac{\partial \mu_k}{\partial P_k}}\right) > 0$$