L14 - 09-25 fubini

Wednesday, September 25, 2024 11:32 AM

two measure spaces $(\Omega_1, \mathcal{F}_1, \mu_1), (\Omega_2, \mathcal{F}_2, \mu_2)$

$\Omega = \Omega_1 \times \Omega_2 \quad \mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2, \quad \mu = \mu_1 \times \mu_2$

Not $\sigma(\mathcal{F}_1 \times \mathcal{F}_2)$

$f : \Omega \to \mathbb{R} \quad f(x,y), \quad x \in \Omega_1, \; y \in \Omega_2$

Fubini Let f be a measureable $(\mathcal{F} \mid \mathcal{B}(\mathbb{R}))$

if $f \geq 0$ or $\int |f| \, d\mu < \infty$ then

$$\int_{\Omega_1} \int_{\Omega_2} f(x,y) \, d\mu_2(y) \, d\mu_1(x) = \int_{\Omega} f \, d\mu = \int_{\Omega_2} \int_{\Omega_1} f(x,y) \, d\mu_1(x) \, d\mu_2(y)$$

we started with an algebra $\mathcal{L} = \{ \bigcup_{i=1}^{n} A_i \times B_i, \; A_i \in \Omega_1, \; B_i \in \Omega_2 \}$

$$\mu(A \times B) = \mu_1(A)\mu_2(B), \quad A \in \Omega_1, \; B \in \Sigma_2$$

By caratheodory μ is Extended to the σ-algebra $\mathcal{F}$.

if μ_1 & μ_2 σ-finite then μ is σ-finite.

to Prove FUBINI

step $f = \mathbb{1}_D, \quad D \in \mathcal{F}$

"suppose we know ~~how~~ to do it."

$c \in \mathbb{R} \quad D_i \in \mathcal{F}$

Step 2 Assume $f > 0$, $h : \Omega \to \mathbb{R}$ is a simple function $h = \sum_{i}^{n} c_i \mathbb{1}_{D_i}$

"By Linearity it is true"

step 3. $f > 0 \quad \exists \, 0 \leq h_n \uparrow f \quad h_n$ simple function.

Example:

if $\frac{k}{2^n} < f \leq \frac{k+1}{2^n}$ then $h_n = \frac{k}{2^n} \quad k = 0, 1, 2, \ldots$

Restrict k to $k \leq N$ or $2^n n$.

$$\int h_n \, d\mu \uparrow \int_{\Omega} f \, d\mu \quad \text{By MCT.}$$

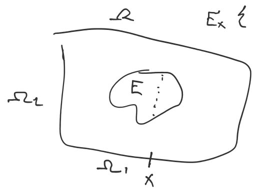
step 4 $\int |f_n| \, d\mu < \infty$ Ingreble (Integrable) Look at $f = f^+ - f^-$

$|f| = f^+ + f^-$

Recall f^+ & f^- are positive

$$\int_{\Omega} f \, d\mu \quad \int_{\Omega} f^+ \, d\mu - \int_{\Omega} f^- \, d\mu$$

Back to setup one.



$E_x \{ y \in \Omega_2 ; (x,y) \in E \}$

$E_y = \{ x \in \Omega_1 : (x,y) \in E \}$

$E_x \subseteq \Omega_1$, $E_y \subseteq \Omega_2$

Lemma 1: if $E \in \mathcal{F}$ then $E_x \in \mathcal{F}$, $\forall x \in \Omega_1$ similar for E_y

if E $A \times B$, $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$

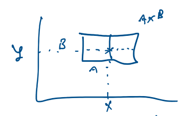
$$E_x = \begin{cases} B & x \in A \\ \phi & x \in A^c \end{cases}$$

says E_x will be sigma Algebra.
need to verify

Lemma 2: $E \in \mathcal{F}$ then $g(x) = \mu_2(E_x)$ $\forall x \in \Omega_1$ is $\mathcal{F}_1$ measureable.

and $\int_{\Omega_1} g(x)\, d\mu_1 = \mu(E)$

Dinkin: say if we have a π-system with a λ-system we have a σ-Algebra.
show contains
π system - Rectangles



All the E's Are a λ-system.

integrate x then get constant y.

$\underline{\lambda\text{-system}}$
$E_n \uparrow E$ $\quad E = \bigcup_{n=1}^{\infty} E_n \ni E_n \subset E_{n+1}$

Requirement 1

2 $E, F \in \mathcal{F}$ then the result holds for $E \setminus F$
$F \subset E$

Application of Fubini: "used thought course."
"why is Fubini better than integration by parts"

Let $X \geq 0$, $P > 0$

then $E[X^P] = \int_{\Omega} [X(\omega)]^P \, dP(\omega)$ "Lebesgue measure means dx → calculus."

" have to posh measure P to $\mathbb{R}$.

$= \int_{x=0}^{\infty} x^P \, dF_X(x)$ where $\underline{\text{CDF}}$ $F_X(x) = P(X < x)$

↑ Lebesgue-stieltjes measure

$\mu_X((b,a]) = F_X(a) - F_X(b)$, $b<a$

$= P(b < X \leq a)$

not Fubini.

Fubini.

$$\int_{x=0}^{\infty} p x^{p-1} P(X \geq x)\, dx$$

similar | same as integration by parts.

Constant w/ respect to ∂P

$$= \int_{x=0}^{\infty} \int_{\Omega} \left[p x^{p-1} \mathbb{1}_{\{X \geq x\}} \, \partial P \right] \partial x$$

$$= \int_{\Omega} \int_{x=0}^{\cancel{\infty}\, X} p x^{p-1} \cancel{\mathbb{1}_{\{X \geq x\}}} \, \partial x \; \partial P$$

$$\int_{x=0}^{\infty} p x^{p-1} P(X \geq x) \partial x = \mathbb{E}[X^p]$$

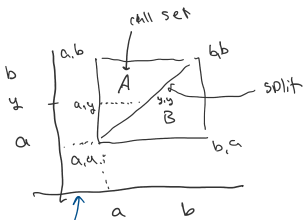
$$= \int_{\Omega} X^p \, \partial P = \mathbb{E}(X^p)$$

Probability measures on $\mathbb{R}$

$\Omega = \mathbb{R}^2$, $\mu = \mu_1 \times \mu_2$,

$F(x) = \mu_1((-\infty, x])$, $x \in \mathbb{R}$

$G(y) = \mu_2((-\infty, y])$, $y \in \mathbb{R}$



Counted twice.

$$\mu(A) = \int_{a \leq y \leq b} F(y) - F(a) \, \partial G(y)$$

$$\mu(B) = \int G(b) - G(a) \, \partial F(x)$$

2.

$$\mu(A) + \mu(B) = (F(b) - F(a))(G(b) - G(a)) + \mu(\{(x,x) : a < x \leq b\})$$

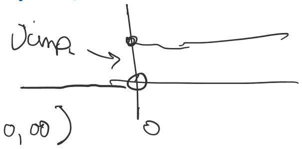
$$\int_{a \leq b \leq b} F(y) \partial G(y) + \int_{a \leq y \leq b} G(y) \partial F(y) = [F(b)G(b) - F(a)G(a)] + \mu\{(x,x), a \leq x \leq b\}$$

$$\sum_{a < x \leq b} \mu_1(x) \mu(x)$$

We say finite # of jumps with measure

$$\sum_{a < x \leq b} \Delta F(x) \cdot \Delta G(x)$$

$G = F = \mathbb{1}_{[0,\infty)}$ integrate $\mathbb{1}[0,\infty)$



$$\int_{(a,b]} \partial F(x) \partial G(x)$$

$a < 0 \leq b$, then constant.