02-03

Monday, February 3, 2025 11:31 AM

$\cancel{\mathbb{E}(X \mid \mathcal{F}) = E_{\mathcal{F}}(X)}$ $(\Omega, \mathcal{F}, P)$ $\mathcal{F} \subset \mathcal{F}_0$

$\mathbb{E}(X ; A)$

$E(X \cdot \mathbb{1}_A) = \mathbb{E}(\mathbb{E}(X) ; A) \quad \forall A \in \mathcal{F}$

if satisfies then ...

we need ① $E_{\mathcal{F}}(X) \in \mathcal{F}$

② $E(X; A) = E(E_{\mathcal{F}}(X; A)) \quad \forall A \in \mathcal{F}$

then we say $E_{\mathcal{F}}(X)$ in "Conditional Expectation of X Given $\mathcal{F}$)

example Assume X, Y are ind $\varphi(X;Y) : \mathbb{R}^2 \to \mathbb{R}$ $\mathcal{F} = \sigma\{X\} = \{X \in A\}_{A \in \mathcal{B}(\mathbb{R})}$

claim $E_{\mathcal{F}}\, \varphi(X, Y) = g(X)$

where $g(x) = \mathbb{E}(\varphi(x, Y)]$, $x \in \mathbb{R}$

$$E[\varphi(X, Y) ; A] = \int_{x \in A} \underbrace{\int_{y \in \mathbb{R}} \varphi(x, y)\, dF_Y(y)}_{g(x)}\, dF_X(x)$$ use Fubini

$$g(x) = \int_{y \in \mathbb{R}} \varphi(x, y)\, dF_Y(y)$$

$$= \int_{x \in A} g(x)\, dF_X(x) = \mathbb{E}(g(X) ; A)$$

now All sorts of Properties of independent Expectation

Properties of $E_{\mathcal{F}}(\cdot)$

① linearity $E_{\mathcal{F}}(aX + bY) = aE_{\mathcal{F}}(X) + bE_{\mathcal{F}}(Y)$

② monotonicity if $Y \geq X$ then $E_{\mathcal{F}}(Y) \geq E_{\mathcal{F}}(X)$

③ M.C.T. $X_n \geq 0$, $X_n \uparrow X$, $E(X) < \infty$, then $E_{\mathcal{F}}(X_n) \xrightarrow[n \to \infty]{a.s.} E_{\mathcal{F}}(X)$

④ Jensen inequality. $\varphi(x)$ is convex function, $E|X|<\infty$, $E|\varphi(x)|<\infty$

Then $\varphi(E_{\mathcal{F}}(x)) \leq E_{\mathcal{F}}\varphi(x)$ a.s.

$\Rightarrow E\varphi(E_{\mathcal{F}}(x)) \leq (\varphi(X))$

Application of 4. Let $X \in L_p$, $p \geq 1$ $(E(|X|^p) < \infty)$, $\|X\|_p = (E|X|^p)^{1/p}$

then $\|E_{\mathcal{F}}(X)\|_p \leq \|X\|_p$ ← say if true then $\|E_{\mathcal{F}}(X)\|_p^p \leq \|X\|_p^p$

The conditional Expectation is a contraction in L^p.
That is the L^p norm shrinks.

$\varphi(x) = |x|^p$, $x \in \mathbb{R}$, $p \geq 1$

(5) if $\mathcal{F}_1 \subset \mathcal{F}_2$ Then $E_{\mathcal{F}_1}(E_{\mathcal{F}_2}(X)) = E_{\mathcal{F}_1}(X) = E_{\mathcal{F}_2}(E_{\mathcal{F}_1}(X))$

Becomes the smaller of the two.

(6) if $X \in \mathcal{F}$, then $E_{\mathcal{F}}(XY) = XE_{\mathcal{F}}(Y)$ we Assume $E|XY|<\infty$
$E(Y) < \infty$

Proof we need to show $E(E_{\mathcal{F}}(XY) \cdot Z) = E(XE_{\mathcal{F}}(Y)Z)$

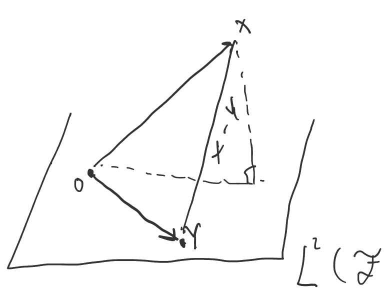
$\forall Z \in \mathcal{F}$
Z is bdd

$E(E_{\mathcal{F}}(XY)Z) = E(XYZ)$

$E(E_{\mathcal{F}}(Y)\underbrace{XZ}_{\text{call this my new } Z}) = E(XYZ)$ [if X is Bounded use Proceedure from 881]

⑦ we Assume $E|X|^2 < \infty$ $(X \in L_2)$ then

$\|X - E_{\mathcal{F}}(X)\|_{L_2} \leq \|X - Y\|_{L_2}$, $Y \in L_2(\mathcal{F})$ — Y is measurable with respect to $\mathcal{F}$



$$\|X-Y\|_{L^2}^2 = \|X-E_{\mathcal{F}}(X)\|_2^2 + \|E_{\mathcal{F}}(X)-Y\|_2^2$$

$$E(X-Y)^2 = E(X-E_{\mathcal{F}}(X))^2 + E(E_{\mathcal{F}}(X)-Y)^2 + E\,2(\overbrace{X-E_{\mathcal{F}}(X)}^{a})(\overbrace{E_{\mathcal{F}}(X)-Y}^{b})$$

then take the Expectation of every term $\quad 2E(\cancel{b\,E_{\mathcal{F}}(a)})\ 0$

J.L. Doob. inverted martingales

$$E_{\mathcal{F}}\,g(x)(\omega) = \int_{\mathbb{R}} g(x)\,d\mu_\omega(x)$$

$$\omega \longleftrightarrow \underbrace{\mu_\omega}_{\text{something}} \quad \text{PM on } \mathbb{R}$$

something Important.

$$P_{\mathcal{F}}(A)(\omega) \equiv E_{\mathcal{F}}(1_A)\,\omega \quad A \in \mathcal{F}_0$$

$$\text{if } P_{\mathcal{F}}(A\cup B) = P_{\mathcal{F}}(A) + P_{\mathcal{F}}(B)$$

$$P_{\mathcal{F}}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P_{\mathcal{F}}(A_i)$$

$$A_i \cap A_j = \emptyset \quad i \neq j$$

what was the trick of doob?

Focus on the Realline.