02-05

Wednesday, February 5, 2025 11:31 AM

Durrett § 5.1.3 p. 193

Regular Conditional Probability / Distribution.

$(\Omega, \mathcal{F}_0, P)$, $\mathcal{F} \in \mathcal{F}_0$

$\mu : \mathbb{R} \times \tilde{\mathcal{F}}_0 \longrightarrow [0,1]$.

we want ① $\underbrace{\mu(\cdot, A) = P_{\mathcal{F}}(A)(\cdot) \text{ a.s.}}_{\forall A \in \mathcal{F}_0}$

② $\underset{A \in \tilde{\mathcal{F}}_0}{\mu(\omega, \cdot)}\Big\}$ Prob. measure on to

$\Omega_0 \subset \Omega$

Application $E_{\mathcal{F}}(f(x))(\omega) = \int_{x=-\infty}^{\infty} f(x)\, d\mu(\omega, x)$

> in theory μ doesn't always exist.

simpliest case in which it exist for sure..

Proof in the Following case $(\mathbb{R}, \mathcal{B}, P)$, $\tilde{\mathcal{F}} \subseteq \mathcal{B}$ [Realline, Borel sets]

$\mathbb{Q} = \{ q \text{ rationals} \}$.

<u>step 1</u> Define $M_q(\omega) = P_{\mathcal{F}}((-\infty, q])(\omega)$ where $q \in \mathbb{Q}$, $\omega \in \mathbb{R}$

<u>step 2</u> Reduce Ω to Ω_0 so that $P(\Omega_0^c) = 0$ and $\forall \omega \in \Omega_0$

we have (a) $M_{q_2}(\omega) \geq M_{q_1}(\omega)$. $\forall q_2 > q_1$, $\omega \in \Omega_0$ [Non Decreasing.]

right continuity.

(b) $M_q(\omega) \downarrow M_{q_0}(\omega)$

"Expectation of $\uparrow$ lead to Expectation"

(c) $M_{q_n}(\omega) \xrightarrow[q_n \to \infty]{} 1$, $M_{q_n}(\omega) \xrightarrow[q_n \to -\infty]{} 0$, $\omega \in \Omega_0$

<u>step 3</u>

Define $M_x(\omega) = \inf M_q(\omega)$, $\omega \in \Omega_0$

Step 3

Define $M_x(\omega) = \inf_{q \geq x} M_q(\omega)$, $\omega \in \Omega_0$

WTS. $P_{\mathcal{F}}((-\infty, x])(\omega) = M_x(\omega)$ a.e.

Step 4 μ_ω is a CDF

maps 1-1

$$\mu_\omega \longleftrightarrow \mu(\omega, \cdot) \quad , \omega \in \Omega_0$$

$$\mu(\omega, (a,b]) = \mu_\omega((-\infty, b]) - \mu_\omega((-\infty, a])$$

$$= M_b(\omega) - M_a(\omega)$$

wts: $P_A((A)(\omega) = \mu(\omega, A), \ \forall A \in \mathbb{B}, \ \omega \in \Omega_0$

$(\Omega, \mathcal{F}_0, P)$ $\quad X$ is an R.V. $\quad X: \Omega \to \mathbb{R}$, X is measurable wrt $\mathcal{F}_0 \backslash \mathcal{B}(\mathbb{R})$

$\mathcal{F} \subset \mathcal{F}_0$ $\quad \{X \in B\} \in \mathcal{F}_0, \ \forall B \in \mathcal{B}(\mathbb{R})$

$\sigma\{X\} = \{X \in B : B \in \mathcal{B}(\mathbb{R})\}$

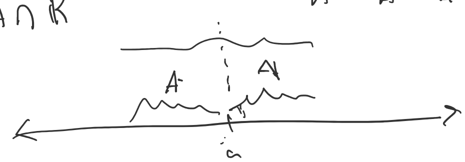
$\{X \in B\}$ .... some cont.... $\boxed{29:00}$

we have r.c.p with $(\Omega, \sigma(X), P)$, $\mathcal{F} \subset \mathcal{F}_0$

$$\underset{\substack{\parallel \\ \{X \in B\}}}{\mu(\omega, A)} = P_{\mathcal{F}}(X \in B)(\omega) \quad \text{a.s.}, \quad B \in \mathcal{B}(\mathbb{R})$$

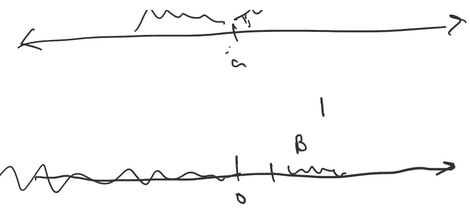
Question $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^- = (-\infty, 0)$

$A = A^+ \cup A^-$ $\quad A^+ = A \cap \mathbb{R}^+$, $\quad A^- = A \cap \mathbb{R}^-$ $\quad A = A^- + A^+$.



$\mathcal{F} = \{B, B \cup \mathbb{R}^- ; B \in \mathcal{B}(\mathbb{R}^+)\}$

$\tilde{\mathcal{F}} = \{B,\ B \cup \mathbb{R}^- \ ;\ B \in \mathcal{B}(\mathbb{R}^+)\}$



$P_{\tilde{\mathcal{F}}}(A)(\omega) = P_{\tilde{\mathcal{F}}}(A^+)(\omega) + P_{\tilde{\mathcal{F}}}(A^-)(\omega).$

$\mathbb{1}_{A^+}(\omega) \quad +$

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$\mu(\omega, A) = \mathbb{1}_A(\omega), \quad A \in \tilde{\mathcal{F}}$

if $X \in \tilde{\mathcal{F}}$ and $X(\omega) = x_0$ or $\neq x_0$

then $\mu(\omega, \{X = x_0\}) = 1$ $\quad = 0$

On the term $P_{\tilde{\mathcal{F}}}$

$\therefore = \mathbb{1}_{A^+}(\omega) + \frac{P(A^-)}{P(\mathbb{R}^-)} \mathbb{1}_{\mathbb{R}^-}(\omega)$

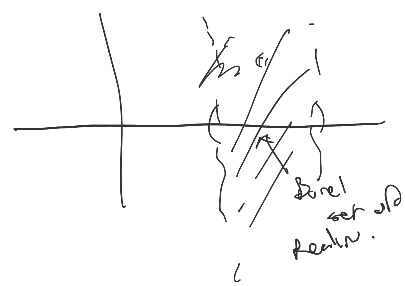
Ex. (X,Y) have a joint density $f(x,y)$, $(x,y) \in \mathbb{R}$

r.c.d. of (x,y) give $\sigma(x)$

Simplify

$(\Omega = \mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), P)$

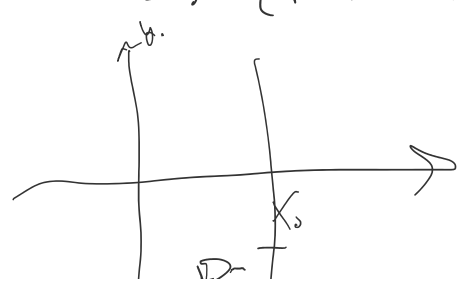
$X(x,y) = x, \quad Y(x,y) = y, \ (x,y) \in \mathbb{R}^2$



$\sigma(X) = \{B = A \times \mathbb{R},\ A \in \mathcal{B}(\mathbb{R})\}$

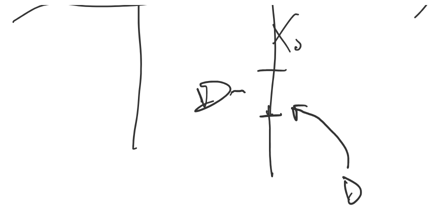
$P_{\tilde{\mathcal{F}}}((x,y) \in D)(\omega) = \quad X(\omega)$

consider $\omega = \{x_0, y_0\}$



$$\int_D \frac{f(x_0, y)\, dy}{f_X(x_0)}$$ conditional Density

$(x_0, y_0) \in D$



$$f_X(x_0) = \int^{\infty} f(x_0, y)\, dy$$

Note we