03-14 ,BMG

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things we cover(ed).

① $\{E_{\mathcal{F}_\alpha}(X)\}_{\alpha\in I}$ is UI if $E(|X|)<\infty$

② $\{X_n,\mathcal{F}_n\}_{n\ge1}$ MG and UI then $\exists X,\ E|X|<\infty$

and $X_n = E_{\mathcal{F}_n}(X)$ $\quad$ ($X = \lim_{n\to\infty} X_n$ a.s.)

③ $\mathcal{F}_n \uparrow \mathcal{F}$ and $E|X|<\infty$ then

calls this MGCT Extension.

$$E_{\mathcal{F}_n}(X) \xrightarrow[n\to\infty]{a.s.,\ L^1} E_{\mathcal{F}}(X)$$

④ Buried in Proof.

if $X_n \xrightarrow{a.s.} 0$ $\quad |X_n|\le Y$, $E(Y)<\infty$ $\quad n\ge1$

then $E_{\mathcal{F}}(X_n) \xrightarrow[n\to\infty]{a.s.} \cancel{E_{\mathcal{F}}(X)}\ 0$

⑤ Ex $\{Y_n, Z_n\}_{n\ge1}$ IND.

$Y_n \sim$ Bernoulli$(\frac{1}{n})$ $\quad n\ge1$ $\qquad X_n \equiv Z_n Y_n$.

$Z_n \sim n\cdot$ Bernoulli$(\frac{1}{n})$ $\quad n\ge$.

$P(X_n>0) =$

$\mathcal{F}=\sigma\{Y_n\}_{n\ge1}$

$P(Y_n=1, Z_n=n) = \frac{1}{n}\frac{1}{n} = \frac{1}{n^2}$.

Given $\mathcal{F}$, Y_n Are constant.

$\sum_{n=1}^{\infty} P(X_n>0) = \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$ BCI $\Rightarrow X_n \xrightarrow[n\to\infty]{a.s.} 0$

$$E_{\mathcal{F}}(X_n) = E(Z_n\cdot Y_n)$$
$$= Y_n E(Z_n)$$
$$= Y_n \not\xrightarrow{a.s.} 0$$

$\sum_{n=1}^{\infty} P(Y=1) = \sum_{n=1}^{\infty}\frac{1}{n} = \infty$ $\quad$ BCII says $P(Y_n=1 \text{ i.o.})=1$

Also $\quad P(Y_n=0 \text{ i.o.})=1$

Say $\quad Y_n \xrightarrow[n\to\infty]{P} 0$

Observation $\{X_n\}_{n\ge1}$ UI

$E(X_n) = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$ $\qquad$ Def. of UI

check UI Look At $\sup_n E(X_n; X_n>M) \xrightarrow[M\to\infty]{} 0$

The moral of the story is that UI $\not\Rightarrow$ DCT.

in this Example: how does $\{X_n\}_{n\ge1}$ dist. After conditioning on $\mathcal{F}$

look like at $\omega\in\Omega$? $\quad$ y Becomes Fixed.

$\exists n_k\uparrow\infty$ so that $\quad Y_n(\omega) = \begin{cases}1 & \text{if } n\in\{n_k\}_{k\ge1}\\ 0 & \text{if } n\notin\{n_k\}_{k\ge1}\end{cases}$

Z is IND. therefore Dne ~~not~~ care about conditioning $\mathcal{F}=\sigma\{Y_n\}$

$\{ 0 \quad \text{if} \quad n \notin \{n_k\}_{k \geq 1}$

Z is IND. therefore Dare not care about conditioning $\mathcal{F} = \sigma\{Y_n\}$

$$X_n = \begin{cases} Z_{n_k} & \text{if } \cancel{n \in \{n_k, k \geq 1\}} \quad n = n_k \\ 0 & \text{if } n \notin \{n_k, k \geq 1\} \end{cases}$$

$\{Z_{n_k}, k \geq 1\}$ not UI, same since truncate at M.

$Z_n \sim n \, \text{Bern}(1/n)$

$E(Z_n) = n/n = 1$

So After Conditioning we lose UI But not DCT.

$P(W_n \xrightarrow[n \to \infty]{a.s.} 0) = 1$

$P_{\mathcal{F}}(W_n \xrightarrow[n \to \infty]{} 0) = 1$ a.s.

converges goes through The conditioning.

if $Y \geq |W_n|$, $E|Y| < \infty$

then $E_{\mathcal{F}}(Y) \geq E_{\mathcal{F}}(|W_n|) \geq |E_{\mathcal{F}} W_n|$ a.s.

D.C.T. under Conditioning.

For conditional Expectation.

$Y_n \xrightarrow[n \to \infty]{a.s.} Y$, $|Y_n| \leq Z$, $|Y| \leq Z$, $E(Z) < \infty$, $n \geq 1$

Assume $\mathcal{F}_n \uparrow \mathcal{F}$ then $E_{\mathcal{F}_n}(Y_n) \xrightarrow[n \to \infty]{a.s.} E_{\mathcal{F}}(Y)$

Proof:

by $E_{\mathcal{F}_n}(Y) \xrightarrow[n \to \infty]{a.s.} E_{\mathcal{F}}(Y)$ if we ADD proves

Enough To show: ETS $E_{\mathcal{F}_n}(Y_n - Y) \xrightarrow[n \to \infty]{a.s.} 0$

$W_n \equiv Y_n - Y$, $W_n \xrightarrow[n \to \infty]{} 0$ $\Rightarrow |W_n| \to 0$ $|W_n| \leq 2Z$ $E(2Z) < \infty$

$|E_{\mathcal{F}_n}(W_n)| \leq E_{\mathcal{F}_n}(|W_n|) \leq \overline{\lim_{n \to \infty}} E_{\mathcal{F}_n}(|W_n|) \leq \overline{\lim_{n \to \infty}} E_{\mathcal{F}_n}(\sup_{k=0,1,2,\ldots} \{|W_{N+k}|\})$

Fix N

$= E_{\mathcal{F}}(\sup_{k \geq 0} \{|W_{N+k}|\}) \xrightarrow[N \to \infty]{a.s.} 0$

Do we have Bounded? yes $2Z$ we know Because? $W_n \to 0$

Done with section 5 onto 6.

(Durrett § 5.6 p.225)

Backwards (Reverse) Martingale (MG).

Imagine MG. is index By Negative Numbers.

$\mathcal{F}_{-n} \uparrow$ as $-n \uparrow$ ($\Leftrightarrow n \downarrow$)

$\{X_{-n}, \mathcal{F}_{-n}\}_{n=0,1,2\ldots}$ is MG. such that X_{-2}, X_{-1}, X_0

Therefore $\mathcal{F}_0 \supset \mathcal{F}_{-n}$.

$E_{\mathcal{F}_{-(n+1)}}(X_{-n}) = X_{-(n+1)} \quad n > 0$

upcrossing Lemma is still in Effect. is just index By
MGCT still holds.

σ-(n+1)

Upcoarsing Lemma is still in Effect. is Just index By negative numbers.

$\therefore$ MGCT still holds.

Observe: $E_{\mathcal{F}_{-n}}(X_0) = X_{-n}$

$\Rightarrow \{X_{-n}\}_{n \geq 1}$ UI   Automatically.

MGCT still holds.

$X_{-n} \xrightarrow{L^1, a.s.} X_{-\infty}$

Claim: $X_{-\infty} = E_{\bigcap_{n=1}^{\infty} \mathcal{F}_{-n}}(X_0)$ used notation $\mathcal{F}_{-\infty} = \bigcap_{n=1}^{\infty} \mathcal{F}_{-n}$

$E(X_0; A) = E(X_{-n}; A)$ $A \in \mathcal{F}_{-\infty} \subset \mathcal{F}_{-n}$

$= E(X_{-\infty}; A)$

Says As we go to -infinity X_{-n} becomes more Constant since The $\mathcal{F}_{n-1}$ Becomes. more Finite.

to talk About we Show X_n for The filtration is shrink $\mathcal{F}_n \downarrow$

Application: Prove SLLN.