# 03-10

<u>Doob $L^p$ inequality.</u>   ($p > 1$)

$\{X_n, \mathcal{F}_n\}_n$    $L^p$ MG.  $p > 1$   ($E|X_n|^p < \infty$)

then ,

$$E(|X_n|^p) \leq E\left(\max_{1 \leq k \leq n}\{|X_k|^p\}\right) \leq C_p \, E(|X_n|^p)$$

} Equivalent

now <u>B. D. G inequality.</u> :    $p \geq 1$

Euclidean norm

$$\tilde{C}_p \, E\left(\left(\sum_{k=1}^n D_k^2\right)^{p/2}\right) \leq E\left(\max_{1 \leq k \leq n} |X_k|^p\right) \leq C_p \, E\left(\left(\sum D_k^2\right)^{1/2^p}\right)$$

$$D_k = X_k - X_{k-1}$$

these can be the SAME   when  $P = 2$.

if  $P = 2$                        called  Quadratic Variation.

$$E\left(\sum_{k=1}^n D_k^2\right) = E\left(\left(\sum_k D_k\right)^2\right) = E(X_n^2)$$

$$E(D_k D_j) = 0 \quad \text{if } k \neq j$$

$$j < k \qquad \to 0 \quad \text{By MG-diff.}$$

$$E(D_j \, E_{\mathcal{F}_j}(D_k))$$

For  $P = 1$   we   use   log.

<u>Section 5</u>   $P = 1$   AKA  Uniform Integrability.  $L^1$  MG.  (Durrett §5.5 p. 220)

Uniform Integrability. (UI)  and   $L^1$  Convergence.

<u>Recall:</u>   $\{X_n\}_{n \geq 1}$  is   UI  if  $\lim_{M \to \infty}\left(\sup_n E(|X_n|; |X_n| > M)\right) = 0$

Alternatively:    UI  $\Rightarrow$  1) $\sup_n E|X_n| < \infty$
                  2) $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $P(A) < \delta \Rightarrow E(|X_n|; A) < \varepsilon, \forall n \geq 1$

Relationship to martingale
Answers Question " How to create MG that is UI ? "
<u>Theorem</u> (B)

Let $X$ be a r.v. on $(\Omega, \mathcal{F}_0, P)$

$E|X| < \infty$   then $\{E_{\mathcal{G}}(X) : \mathcal{G} \subset \mathcal{F}_0\}$ is UI   $\leftarrow$ creates many r.v.

$\mathcal{F}_n \uparrow \mathcal{F}_\infty \subset \mathcal{F}_0$

$\{E_{\mathcal{F}_n}(X), \mathcal{F}_n\}_{n \geq 1}$ UI. M.G.

Examples of UI (1) $\sup_{n \geq 1} E|X_n|^p < \infty \Rightarrow \{X_n\}_{n \geq 1}$ UI

(2) $|X_n| \leq Z$ $n \geq 1$ $E(Z) < \infty$

Examples of UI (1) $\sup_{n\geq1} E|X_n|^p < \infty \Rightarrow \{X_n\}_{n\geq1}$ UI

(2) $|X_n| \leq Z$, $n\geq1$, $E(Z) < \infty$

$\Rightarrow \{X_n\}_{n\geq1}$ UI

Proof   Fix $\varepsilon > 0$, $\exists \delta < 0$

st. $E(|X|; A) < \varepsilon$ if $P(A) < \delta$

Assume $E|X| \leq M\cdot\delta$,        Note $|E(X)| \leq E|X|$

then $E|E_{\mathscr{G}}(X)| : |E_{\mathscr{G}}(X)| > M)$

$\underset{\nearrow}{\text{measurable}}$ wrt $\mathscr{G}$

$\leq E(E_{\mathscr{G}}|X| : E_{\mathscr{G}}(|X|) > M$

$= E(|X| ; E_{\mathscr{G}}(|X|) > M)$     $\overset{By}{\text{Def}}$ of Conditional Expectation.

$\overset{By \text{ Given}}{}$

$P(E_Z(|X|) > M) \leq \underbrace{\dfrac{E(E_{\mathscr{G}}|X|)}{M}} = \dfrac{E|X|}{M} \leq \dfrac{M\delta}{M} = \delta < \varepsilon \cdot \forall \mathscr{G} \subset \mathscr{F}_0$

we Are Dealing with

$\sup_{\mathscr{G}\subset\mathscr{F}_0} \{E|E_{\mathscr{G}}(X)|\} \leq \sup_{\mathscr{G}\subset\mathscr{F}_0} \{E(E_{\mathscr{G}}(|X|))\} = E|X| < \infty$

Theorem  Assume $X_n \overset{a.s.}{\underset{n\to\infty}{\to}} X$  $\left(X_n \overset{p}{\underset{n\to\infty}{\to}} X\right)$ ← Really Just need. But...

and $E|X_n| < \infty$   The following are Equivalent.   TFAE

(1) $\{X_n\}$ UI

(2) $\overset{\text{converge in } L^1}{E|X_n - X| \underset{n\to\infty}{\to} 0}$, and $E|X| < \infty$

(3) $E|X_n| \to E|X|$, $E|X| < \infty$

then to Prove   $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$.

Prove $(1) \Rightarrow (2)$ is in STT 881

Fatou's, $\varliminf E|X_n| \geq E \varliminf |X_n| = EX$.   there for $E|X| < \infty$
$\infty$        $n\to\infty$

To Prove $(2) \Rightarrow (3)$ By $\triangle$ineq. $E|X_n - X| \geq |E|X_n| - E|X||$

(3) $\Rightarrow$ (1)

Consider $X_n \cdot \mathbb{1}_{\{|X_n|\leq M\}} \overset{a.s}{\underset{n\to\infty}{\to}} X \mathbb{1}_{\{|X|<M\}}$ if $P(|X| = M) = 0$

$|X_n| \mathbb{1}_{\{|X_n| < M\}} < M$.   $\Rightarrow$ Bounded then DCT

look At complement.

$E|X|\mathbb{1}$ _____

$\cdots < |x_n| < m$

look At complement.

$$E|x_n| \mathbb{1}_{\{|x_n| \geq m\}} \xrightarrow[n \to \infty]{} E\left(|x| \mathbb{1}_{\{|x| < m\}}\right)$$

$$E\left(|x_n| \cdot \mathbb{1}_{\{|x_n| \geq m\}}\right) \to E\left(|x| \cdot \mathbb{1}_{\{|x| > m\}}\right)$$

$$\therefore \quad \bigg| \quad E(x_n) \to E(x)$$

As you increase $m$ the event $\{|x_n| \geq m\}$ Becomes smaller.

#Enter in the MG world#

$\forall n \geq 1$ <u>Theorem</u> : Let $\{x_n, \mathcal{F}_n\}_{n=1}$ be $\nearrow^{\text{or}}$ Sub MG TFAE

(1) $\{x_n\}_{n \geq 1}$ UI

(2) $x_n \xrightarrow[n \to \infty]{a.s.} x$, $E|x_n - x| \xrightarrow[n \to \infty]{} 0$ $E|x| < \infty$

(1)$\Rightarrow$(2)

(3) if MG $\exists x$, $E|x| < \infty$ s.t. $x_n = E(x)|_{\mathcal{F}_n}$, $n \geq 1$

(2)$\Rightarrow$(3) Proof.

$$E(x_n ; A) = E(x_m ; A) \xrightarrow[n \to \infty]{} E(x ; A) \text{ by (2)}$$
$$, \quad m \geq 0$$

$A \in \mathcal{F}_n$, $E_{\mathcal{F}_n}(x_m) = x_m$ for All $m$.

$$\therefore E(x_n ; A) = E(x ; A)$$

(3)$\Rightarrow$(1) from. By Ⓑ