L08 09-12 lecture

Friday, September 13, 2024 11:32 AM

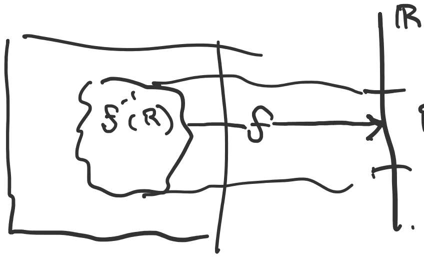
$(\Omega, \mathcal{F}, \mu)$, μ σ-finite measure

$$\|f\|_p = \int_\Omega \left[|f(\omega)|^p d\mu(\omega)\right]^{1/p} \equiv I^{1/p}(|f|^p) \qquad 1 \le p < \infty$$

$f: \Omega \to \mathbb{R}$

$f^{-1}(B) \in \mathcal{F} \quad \forall\ B$- borel sets $\mathcal{B}$

f is measurable $\mathcal{F}/\mathcal{B}$



Holder's Inequality.

$$I(|fg|) \le \|f\|_p \cdot \|g\|_q \qquad p \ge 1 \quad \frac{1}{p} + \frac{1}{q} = 1$$

Alternately $\|fg\|$ $\qquad p = 1 \Rightarrow q = \infty$

what $\|f\|_\infty$?

$$\int_\Omega |fg|\, d\mu \le \left(\int_\Omega |f|\, d\mu\right) \times$$

1st Attempt try " $\sup_\Omega \{|g|\}$ "

ESSSUP.

$\|g\|_\infty$ = essential sup of g.

ignore measure 0.

$A = \{a \ge 0 : \mu[g^{-1}([a, \infty])] = 0\}$

short hand $\mu(g \ge a)$

$\mu(\omega : g(\omega) \ge a) = 0$

$\|g\|_\infty = \inf\{A\}$

$$\le \left(\int |f|\, d\mu\right) \|g\|_\infty$$

esssup

how to find sup of g.

$A = \{a \ge 0 : g^{-1}[[a, \infty]] = \phi\}$

no measure, Not The Difference

$L_p = \{$ measurable $f : \|f\|_p < \infty\}$, $1 \le p$.

Full Notation.

$L^p(\Omega, \mathcal{F}, \mu)$

$L^p = \{ \ldots < \infty \ \|f\|_p < \infty \}, \quad 1 \le p$

Full Notation.

$L^p(\Omega, \mathcal{F}, \mu)$

triangle inequality.

$$\|f+g\|_p \le \|f\|_p + \|g\|_p \qquad 1 < p.$$

Trivial It's if true.

if ~~$p=\infty$~~ or $p=1$

$$\|f+g\|_\infty \le |f+g| \le |f| + |g|$$

<u>Proof</u>: $|f+g|^p = |f+g| \cdot |f+g|^{p-1} \le |f| \cdot |f+g|^{p-1} + |g| |f+g|^{p-1}$

$\Rightarrow$

$$I(|f+g|^p) \le I(|f||f+g|^{p-1}) + I(|g||f+g|^{p-1})$$

By holder.

$$\le \|f\|_p \cdot I^{1/q}(|f+g|^{\cancel{(p-1)q}\ p}) + \|g\|_p I^{1/q} (|f+g|^{\cancel{(p-1)q}\ p})$$

$(p-1)q = pq - q = p$

$$\|f+g\|_p^p = (\|f\|_p + \|g\|_p) \cdot \|f+g\|_p^{p/q}$$

$$\|f+g\|_p^q$$

$p - p/q$

$p + p(-\frac{1}{q})$

$p + p(1 - \frac{1}{p})$

$p + p - 1$

$2p - 1$

$\frac{1}{p} + \frac{1}{q} = 1$

$\frac{1}{p} - 1 = \frac{1}{q}$

$$d(f,g) = \|f-g\|_p$$

$$d(f,0) > 0, \quad f \ne 0$$

<u>Bounded convergence theorem.</u> (Book name)

Assume $\mu(\Omega) < \infty$, $|f_n| \le M < \infty$, $n = 1, 2, 3 \ldots$

$|f| \le M$

<u>Def</u> $f_n \xrightarrow[n\to\infty]{\mu} 0$ Converge in measure.

$$\mu\{|f_n| \ge \varepsilon\} \xrightarrow[n\to\infty]{} 0$$

$$f_n \xrightarrow[n\to\infty]{\mu} f \iff f_n - f \xrightarrow[n\to\infty]{\mu} 0$$

("Almost everywhere")

<u>Def</u> $f_n \xrightarrow[n\to\infty]{a.e.} f$.

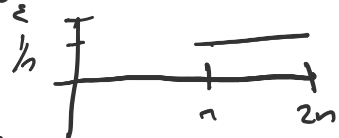
$$f_n \xrightarrow[n\to\infty]{a.e.} 0 : \mu\{\omega \in \Omega : f_n(\omega) \not\to_{n\to\infty} 0\} = 0$$

$$f_n \xrightarrow[n\to\infty]{a.e.} 0 \;:\; \mu\{\omega \in \Omega : f_n(\omega) \underset{n\to\infty}{\not\to} 0\} = 0$$

Lebesgue measure

$$\Omega = \mathbb{R}, \quad \mathcal{F} = \mathcal{B}, \quad \mu = \lambda \qquad \lambda([a,b]) = b - a$$

take $f_n(\omega) = \frac{1}{n} \cdot 1_{(n,2n]}, \quad n = 1, 2, \ldots$



$f_n \xrightarrow{a.e.} 0$

$$\mu\{|f_n| > \varepsilon\} = \begin{cases} n & \text{if } \varepsilon < \frac{1}{n} \\ 0 & \text{if } \varepsilon > \frac{1}{n} \end{cases}$$

$f_n \xrightarrow[n\to\infty]{\mu} 0$ ✓

think : tg.

$$\int_{\mathbb{R}} f_n \, d\lambda = 1 \not\to 0$$

Ex 2 $\quad f_n(\omega) = \mathbb{1}_{(n,2n]}(\omega), \quad n = 1, 2, \ldots$

$$\mu\{|f_n| > \varepsilon\} = \begin{cases} n & \text{if } \varepsilon \le 1 \\ 0 & \text{if } \varepsilon > 1 \end{cases}$$

$$n = \int_{\mathbb{R}} f_n \, d\lambda = 1 \xrightarrow[n\to\infty]{} \infty$$

if $\mu(\Omega) \le \infty$ then $f_n \xrightarrow[n\to\infty]{a.e.} f \Rightarrow f_n \xrightarrow[n\to\infty]{\mu} f$

if $f_n \xrightarrow[n\to\infty]{\mu} f$ (or $f_n \xrightarrow[n\to\infty]{a.e.} f$)

then $I(f_n) \xrightarrow[n\to\infty]{} I(f)$

Convergence in Almost surely $\to$ Convergence in Probablity

shrinking one inside another

$$f_n \xrightarrow[n\to\infty]{a.e.} 0 \Rightarrow \forall \varepsilon > 0 \; \mu\left\{\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{|f_k| > \varepsilon\}\right\} = 0$$

since $\mu(\Omega) < \infty$ we get: $\mu\{\bigcup_{k=n}^{\infty} |f_k| > \varepsilon\} \downarrow 0$ as $n \to \infty$

$$\vee|$$

$$\mu\{|f_n| > \varepsilon\} \xrightarrow[n\to\infty]{} 0.$$