L34 - 11-15 CLT

Friday, November 15, 2024 11:31 AM

CLT without c.f. ← characteristic function

Version: let $X, X_1, X_2, \dots$ be iid, assume $E(X)=0$, $E(X^2)=1$, $E(|X|^3) \le \infty$

then $\dfrac{\sum_{k=1}^{n} X_k}{\sqrt{n}} \Rightarrow Z \sim N(0,1)$

Proof
WTS $E\left[f\left(\sum_{k=1}^{n} \frac{X_k}{\sqrt{n}}\right)\right] \longrightarrow E[f(Z)]$ $\quad f \in C_B^\infty(\mathbb{R})$ $\quad$ we need only $f \in C_B^3(\mathbb{R})$ ← 3RD Derivative

$\forall f \in C_B^3(\mathbb{R})$

$E f(Z) = E f\left(\sum_{k=1}^{n} \frac{Z_k}{\sqrt{n}}\right)$, $\quad (Z_k)_{k\ge 1}$ iid $N(0,1)$.

$\{\{Z_k\}_{k\ge1}, \{X_k\}_{k\ge1}\}$ ind

original

$$\left| E\left(f\left(\frac{\sum_{k=1}^{n} X_k}{\sqrt{n}}\right)\right) - E f\left(\frac{\sum_{k=1}^{n-1} X_k + Z_n}{\sqrt{n}}\right)\right| + \left| E f\left(\sum_{k=1}^{n-1}\frac{X_k}{\sqrt{n}} + \frac{Z_n}{\sqrt{n}}\right) - f\left(\frac{\sum_{k=1}^{n-2} X_k}{\sqrt{n}} + \frac{Z_{n-1}+Z_n}{\sqrt{n}}\right)\right|$$

small

continue Replace small 1 everytime.

(last one)

$+ \dots +$ $\quad + \left| E\left(f\left(\frac{X_1}{\sqrt{n}} + \frac{\sum_{k=2}^{n} Z_k}{\sqrt{n}}\right) - f\left(\frac{\sum_{k=1}^{n} Z_k}{\sqrt{n}}\right)\right)\right|$ $\quad \Bigg\} = \cancel{A}\ I$

change, X_k to Z_k,

$$\left|E\left(f\left(\frac{\sum_{k=1}^{n} X_k}{\sqrt{n}}\right) - f(Z)\right)\right| \le$$

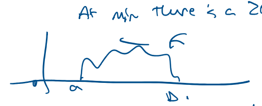
taylor theorem: Assume $f \in C^1(\mathbb{R})$

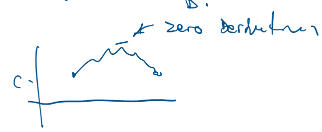
$f(x+h) = f(x) + \overbrace{f'(\xi)}^{c} h$, $\quad |\xi - x| < |h| \quad x \in \mathbb{R}, h \in \mathbb{R}$

$\vdots$

$f(x+h) = f(x) + f'(x)h + \dfrac{f^{(2)}(x)h^2}{2!} + \dfrac{f^{3}(\xi)h^3}{3!}$, $\quad |\xi - x| < |h|$ ← psi

Idea Behind taylor series:
Between two zeros there is a Derivative
At min there is a zero derivative
$f \in C^1(\mathbb{R})$





Lemma: let B, Y, W be R.V. with $E(Y) = E(W)$

$E(Y^2) = E(W^2)$, $B \perp\!\!\!\perp Y \perp\!\!\!\perp W$.

then $C = \sup_{x \in \mathbb{R}} \{|f^{(3)}(x)|/3\} < \infty$. $\quad f \in C_B^3(\mathbb{R})$

then $|E f(B+Y) - E f(B+W)| \le C\, E(|Y|^3 + |W|^3)$

we know $E\left[f(B)+f^{(1)}(B)Y+f^{(2)}(B)Y^2\right]=E\left[f(B)+f^{(1)}(B)W+\frac{f^{(2)}(B)W^2}{2}\right]$

The key is ind.

$$E[f'(B)Y]=Ef'(B)\cdot EY=Ef'(B)EW=E[f^{(1)}(B)W]$$

then Do for second dirivative.

By taylor.

$$Ef(B+Y)=E\left[\cancel{f(B)}+\cancel{f^{(1)}(B)Y}+\cancel{f^2(B)\frac{Y^2}{2}}+f^3(\eta)\frac{Y^3}{3!}\right]$$

$$Ef(B+W)=E\left[\cancel{f(B)}+\cancel{f'(B)W}+\cancel{f^2(B)\frac{W^2}{2}}+f^3(\eta)\frac{W^3}{3!}\right]$$

$$|Ef(B+Y)-Ef(B+W)|\le c\,E(|Y|^3+|W|^3) \quad B$$

then use Lemma on ~~A~~ I

$$T_m=\sum_{k=1}^{m-1}Z_k+\sum_{k=n}^{n}X_k$$

$$I=\sum_{m=1}^{n}\left|Ef\left(\frac{T_m}{\sqrt{n}}+\frac{X_m}{\sqrt{n}}\right)-f\left(\frac{T_m}{\sqrt{n}}+\frac{Z_m}{\sqrt{n}}\right)\right|$$

Becomes,

$$B \leftrightarrow \frac{T_m}{\sqrt{n}},\quad Y \leftrightarrow \frac{X_m}{\sqrt{n}},\quad W \leftrightarrow \frac{Z_m}{\sqrt{n}}$$

$$\le C\sum_{m=1}^{n}E\left[\frac{|X_m|^3}{n^{1.5}}+\frac{|Z_m|^3}{n^{1.5}}\right]=C\cdot\cancel{n}\,E\left[\frac{|X|^3}{\cancel{n^{1.5}}\ \sqrt{n}}+\frac{|Z|^3}{\cancel{n^{1.5}}\ \sqrt{n}}\right]\xrightarrow[n\to\infty]{}0 \quad C$$

Main Idea!

we like h^3 when h is small.

h^3 if $h\sim 0$

instead we want h^2

h^2 if h is large

taylor

$$f(x+h)=f(x)+f^{(1)}(x)h+f^{(2)}(\eta)\frac{h^2}{2}$$

we want

$$f(x+h)=f(x)+f^{(1)}(x)h+f^{(2)}(x)\frac{h^2}{2}+\underbrace{\left[f^{(2)}(\eta)-f^{(2)}(x)\right]}_{\le c}\frac{h^2}{2}$$

Now Equal, The Difference is less than $2/c$.

$$\left|\,|f(x+h)-(f(x)+f'(x)h+f''(x)\frac{h^2}{2})\right|\le c(h^2\wedge h^3)$$

$$B\quad \le CE(\{Y^2\wedge|Y|^3\}+|W|^3)$$

$$C\quad \le C\,n\,E\left(X^2\cdot\mathbb{1}_{|X|>\varepsilon\sqrt{n}}\right)+E\frac{|X|^3\cdot\mathbb{1}_{|X|<\varepsilon\sqrt{n}}}{\sqrt{n}}+E\frac{|Z|^3}{\sqrt{n}}=\varepsilon$$

$\frac{1}{n}$

$\downarrow n\to\infty \qquad \downarrow n\to\infty$

$$\le n E\left(\frac{X^2}{n} \cdot \mathbb{1}_{|X| > \varepsilon\sqrt{n}}\right) + E\,\frac{|X|^3}{\sqrt{n}} \cdot \mathbb{1}_{|X| < \varepsilon\sqrt{n}} + E\,\frac{|X|^3}{\sqrt{n}}$$

$\downarrow n \to \infty$ 0 By DCT

$\le \dfrac{E(\varepsilon\sqrt{n} \cdot X^2)}{\sqrt{n}} = \varepsilon$ since $E|X^2| = 1$

$\downarrow n \to \infty$ 0

$\overline{\lim_{n\to\infty}}\,(I) \le \varepsilon$ is Arbitrary.

$\therefore \lim_{n \to \infty} (I) = 0.$

Next time Lindberg Feller CLT.