01-27

Monday, January 27, 2025 11:30 AM

1st Wald Equation

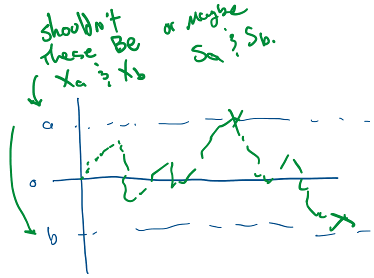
$\{\mathcal{F}_n\}_{n\geq 1}$ T is a stopping time w.r.t. $\{\mathcal{F}_n\}_{n\geq 1}$ and $X_{n+1} \perp\!\!\!\perp \mathcal{F}_n$ $\forall n \geq 1$

$\{X_n\}_{n\geq 1}$ iid Adapted to $\{\mathcal{F}_n\}_{n\geq 1}$ Adapted $\equiv$ ($X_n \in \mathcal{F}_n$, $n \geq 1$)

if $E|X_1| < \infty$, $E(T) < \infty$ then $E(S_T) = E(X_1)\cdot E(T)$

$S_0 = 0$, $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$ $S_T = \sum_{k=1}^{T} X_k$

$\mathcal{F}_n = \sigma\{X_1, \dots, X_n\}$.



<u>Example</u> SSRW $P(X_k = \pm 1) = 1/2$.

$T_a = \min_{n\geq 1}\{S_n = a\}$, $a \in \mathbb{Z}$

T_a, $a > 0$, T_b, $b < 0$

$T_{a,b} = T_a \wedge T_b$

Book calls

if this → if $E(a,b) < \infty$ then $E(S_{T_{a,b}}) = E(X_1)E(T_{a,b}) = 0$

then this → $= T_{a,b} < \infty$ a.s.

$$S_{T_{a,b}} = \begin{cases} a & \text{if } T_a < T_b \\ b & \text{if } T_b < T_a \end{cases}$$

$0 = a\,P(T_a < T_b) + b\,P(T_b < T_a)$

$P(T_a < T_b) + P(T_b < T_a) = 1$ disjoint. two Eq. w/ two variables

$P(T_b < T_a) = \frac{a}{a-b}$ plug in to verify

$P(T_a < T_b) = \frac{-b}{a-b}$.

Claim $T_a < \infty$ a.s. $\not\Rightarrow$ $T_b < \infty$ a.s.

$P(T_a < T_b) \xrightarrow[b\to-\infty]{} P(T_a < \infty)$

$P(T_a < T_b) = \frac{-b}{a-b} \xrightarrow[b\to-\infty]{} 1$ $\Rightarrow$ $P(T_a < \infty) = 1$

Question $E(T_a) = \infty$ or $< \infty$?

if $E(T_a) < \infty$
then $E(S_{T_a}) = E(T_a)E(X_1) = 0$

Just a constant. which is a.
$\therefore E(a) = a$. $\Rightarrow$ $a = 0$ but $a > 0$ contradiction

$\therefore$ $E(T_a) = \infty$.

in Prior years we had $E(T_{a,b}) < \infty$ As prelim Q.

Recall: $Y \sim \text{Geom}(p)$, $0<p<1$ then $E(Y) = \frac{1}{p} < \infty$

fail - fail - fail we succeed. $(1-p)^k p$

Blocks of $a+|b|$ length.

$[1, \dots, a+|b|], [a+|b|+1, \dots, 2[a+|b|], \dots$

Success

$X_i = +1, +1, \dots, +1$

with Prob. $\left(\frac{1}{2}\right)^{a+|b|}$

$$E(\underbrace{1_s \text{ Block with } S}_{T}) = \frac{1}{\left(\frac{1}{2}\right)^{a+|b|}} < \infty$$

$\therefore E(T)$

$$E(T_{a,b}) \le E(T) \cdot (a+|b|) < \infty$$

$$T_{a,b} < (a+|b|)T$$

<u>Wald 2nd Equation</u>

$\{\mathcal{F}_n\}_{n\ge1}$ T is a stopping time w.r.t. $\{\mathcal{F}_n\}_{n\ge1}$ and $X_{n+1} \perp\!\!\!\perp \mathcal{F}_n$ $\forall n\ge1$

$\{X_n\}_{n\ge1}$ iid Adapted to $\{\mathcal{F}_n\}_{n\ge1}$ Adapted $\equiv$ ($X_n \in \mathcal{F}_n$, $n\ge1$)

Assume $E(X_1)=0$

if $E|X_1| < \infty$, $E(T)<\infty$ then $E(S_T^2) = \cancel{E(X_1)}\,\sigma^2 \cdot E(T)$

$S_0 = 0$, $S_n = \sum_{k=1}^{n} X_k$, $n\ge1$ $\quad S_T = \sum_{k=1}^{T} X_k$ $\quad E(X^2) = \sigma^2$

$\mathcal{F}_n = \sigma\{X_1, \dots, X_n\}$.

Proof

$$(S_{T\wedge(n+1)} - S_{T\wedge n})^2 = X_{n+1}^2 \cdot \mathbb{1}_{\overbrace{\{T \ge n+1\}}^{\in \mathcal{F}_n}}$$

$$E(\mathbb{1}_a) = a.$$

$$E(S_{T\wedge n+1} - S_{T\wedge n})^2 = E(X_{n+1}^2)P(T \ge n+1)$$

$$S_T = \left(\sum_{n=0}^{\infty} S_{T\wedge n+1} - S_{T\wedge n}\right)^2$$

$S_0 = 0$

$$\cancel{S_{T\wedge n}^2 = \left[\sum_{k=0}^{N} \left(S_{T\wedge(k+1)} - S_{T\wedge k}\right)\right]^2}$$

Aside

$$\left(\sum_{k=1}^{n} a_k\right)^2 = \sum_{k=1}^{n} a_k^2 + 2\sum a_k a_\ell$$

$$E\Big[\underbrace{(S_{T\wedge(n+1)} - S_{T\wedge n})}_{X_{n+1}\,\mathbb{1}_{\{T\ge n+1\}}}(S_{T\wedge(n+k)} - S_{T\wedge(n+k-1)})\Big] \quad k\ge1$$

$X_{n+k} \cdot \mathbb{1}_{\{T \ge n+k\}}$

$$X_{n+1} \mathbb{1}\{T \geq n+1\} \qquad X_{n+k} \quad - \quad \mathbb{1}\{T \geq n+k\}$$

$$\cap \qquad\qquad \cap$$

$$\mathcal{F}_{n+k-1} \qquad \mathcal{F}_{n+k} \qquad \mathcal{F}_{n+k-1}$$

thus indep

$$\therefore \quad E[X_{n+k}] \, E[\underbrace{X_{n+1} \mathbb{1}_{T \geq n+1} \mathbb{1}_{T \geq n+k})}_{\text{say this is } 0.}]$$

So no Crossproduct.

$$E(S_{T\wedge(n+k)} - S_{T\wedge n})^2 = \sum_{l=n}^{n+k+1} E(S_{T\wedge(l+1)} - S_{T\wedge l})^2$$

$$= \sigma^2 \sum_{l=n}^{n+k-1} P(T \geq l+1)$$

$$\leq \sum_{l=1}^{\infty} P(T \geq l) = E(T) < \infty$$

$\{S_{T\wedge(n+1)} - S_{T\wedge n}\}_{n\geq 1}$ is a Cauchy sequence as $L^2(\Omega)$ is complete.

omega

$\therefore$ $S_{T\wedge n}$ is Cauchy.

$$S_{T\wedge n} \xrightarrow{L^2} Y$$

$$E(S_{T\wedge n} - Y)^2 \xrightarrow[n\to\infty]{} 0$$

$$S_{T\wedge n} \xrightarrow[n\to\infty]{a.s} S_T$$

$$\Rightarrow Y = S_T \text{ a.s.}$$

$$E(S_{T\wedge n}^2) \longrightarrow E(S_T^2) = E(T)\cdot\sigma^2$$

this is only for $E(X_1) = 0$, what happen when $E(X_1) = \mu \neq 0$

we can $E(X_1) = \mu$. Then $Y_k = X_k - \mu$ then $E(Y_k) = 0$.

$$E(S_T - T\mu)^2 = Var(X)\, E(T)$$

$$E(S_T^2) = E(S_T - T\mu + T\mu)^2$$

$$E(S_T^2) = E(S_T - T\mu + T\mu)^2$$
$$= V(X)\,E(T) + E(T^2)\cdot\mu^2.$$
$$+ 2E[(S_T - T\mu)\,T\mu]$$

it is indeed 0

in STT 886,
if T ind X_k
$(S_T - T\mu)T\mu) = 0.$