L30 - 11-06 ch.3 CLT, Weak Convergence

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Feller: let $\{X, X_i\}_{i\geq 1}$ be iid and $\{a_n\}_{n\geq 1}$, $a_n > 0$

(a) if $\frac{a_n}{n} \uparrow \infty$ and $\sum_{n=1}^{\infty} P(|X| > a_n) < \infty$, then $\frac{S_n}{a_n} \xrightarrow[n\to\infty]{as} 0$

(b) if $\{\frac{a_n}{n}\}_{n\geq 1}$ non-decrasing and $\sum_{n=1}^{\infty} P(|X| > a_n) = \infty$ then $\overline{\lim_{n\to\infty}} \frac{|S_n|}{a_n} = \infty$ a.s.

Remark: if assumption in (b) holds then $E|X| = \infty$

Hint: $\frac{a_n}{n} \geq a_1$ if not Decreasing.

$\Rightarrow a_n \geq n a_1$

$\sum_{n=1}^{\infty} P(X > n \cdot a_1) = \infty$

$E \frac{|X|}{a_1} = \infty$

in Part (a) $\frac{S_n - \sum_{k\leq 1}^{n} EY_k}{a_n} \xrightarrow[n\to\infty]{} 0$ then we show $\frac{\sum_{k=1}^{n} EY_k}{a_n} \to 0$.

$\frac{n}{a_n} \to 0$

Example 1. (has to do with (b))

St. Petersberg Paradox.

$P(X = 2^k) = 2^{-k}$, $k = 1, 2 \ldots$

$E(X) = \infty$

When we study WLLN

$\frac{S_n}{n \log_2(n)} \xrightarrow[n\to\infty]{P} 1$

$a_n = n \log_2(n)$.

this Implies $\frac{S_n}{a_n} \not\to 0$ a.s. therefore (b)

Gave formula to check fail that slows. $\sum P(|X| > a_n) > \infty$

Example 2. (uses (A)) $0 < p < 1$.

$E|X|^p < \infty$

$\sum_{n=1}^{\infty} P(|X| > n^{1/p}) < \infty$

$\sum_{n=1}^{\infty} P(|X|^p > n) < \infty$

$a_n = n^{1/p}$, $0 < p < 1$

$\frac{a_n}{n} \uparrow \infty$. as $n \to \infty$

$\overset{a}{\Longrightarrow} \quad \frac{|S_n|}{n^{1/p}} \xrightarrow[n\to\infty]{a.s.} 0$

Chapter 3 Central Limit theorem.

theorem 1

Let z_i, w_i, $i \le i \le n$ complex numbers $(a+bi)$: $i = \sqrt{-1}$

AKA $re^{i\phi}$ polar.

$$\left|\prod_{i=1}^{n} z_i - \prod_{i=1}^{n} w_i\right| \le \left|\sum_{i=1}^{n} z_i - \sum_{i=1}^{n} w_i\right| \cdot \prod_{k=1}^{n} \theta_i \quad (A) \qquad 0 \le \theta_i = \underset{1\le i\le n}{\max}\{1, |z_i|, |w_i|\}$$

$z_1 \cdots z_{n-1}\, z_n \to z_1 \cdots z_{n-2}\, z_{n-1} w_n \;(+) \to z_1 \cdots z_{n-2}\, w_{n-1} w_n \;(\#)$

$\to w_1 \cdots w_n$

$|c_1 + c_2| \le |c_1| + |c_2|$ Δ iny of complex #.

what is the Diff?

$$\left|\left(\prod_{k=1}^{n-1} z_k\right) z_n - \left(\prod_{k=1}^{n-1} z_k\right) w_n\right| = \left|\prod_{k=1}^{n-1} |z_k|\right| |z_n - w_n| \le$$

$$\le |z_n - w_n| \prod_{i=1}^{n} \theta_i$$

$$|(+) - (\#)| = |z_{n-1} - w_{n-1}| \left(\prod_{k=1}^{n} |z_k|\right) |w_n| \le |z_{n-1} - w_{n-1}| \left(\prod_{i=1}^{n} \theta_i\right)$$

(A) Better than $\sum_{i=1}^{n} |z_i - w_i| \cdot \prod_{k=1}^{n} \theta_i$

Important Result:

$\{a_{n,m}\}_{1 \le m \le n,\; n = 1,2,\dots}$ $a_{n,m} \in \mathbb{C}$

if the following 3 conditions hold.

(i) $\sum_{m=1}^{n} a_{n,m} \xrightarrow[n\to\infty]{} a$

(ii) $\sup_n \left\{\sum_{m=1} |a_{n,m}|\right\} < \infty$ Each Row take sup

(iii) $\underset{1\le m\le n}{\max}\{|a_{n,m}|\} \xrightarrow[n\to\infty]{} 0$

then $\prod_{m=1}^{n}(1 + a_{n,m}) \to e^a$

Example $\left(1 + \frac{x}{n}\right)^n \longrightarrow e^x$ (i) $\sum a_{n,m} = \sum \frac{x}{n} \to x$.

$x \in \mathbb{R}$. Let $a_{n,m} = \frac{x}{n}$ (ii) $\sum_{k=1}^{n} \left|\frac{x}{n}\right| = x$

therefore e^{α} (iii) $\frac{|x|}{n} \xrightarrow[n\to\infty]{} 0$.

Aside

inequality for complex #.

$z \in \mathbb{C}$ $\quad |e^{z} - (1+z)| \le |z|^2 \quad |z| \le 1$

$e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$$|e^{z} - (1+z)| \le |z|^2 \underbrace{\left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\right)}_{\text{less than } 1}$$

Proof. $\quad |e^{a_{n,m}} - (1+a_{n,m})| \le |a_{n,m}|^2 \quad 1 \le m \le n \quad$ By (iii) $\exists N$ such ~~that~~ $|z| \le 1$

$\operatorname{Max}\{|e^{a_{n,m}}|, |1+a_{n,m}|\} \le e^{|a_{n,m}|}$

Replace By C By ~~Ass~~ assumption (ii)

$$\left|e^{\sum_{m=1}^{n} a_{n,m}} - \prod_{m=1}^{n}(1+a_{n,m})\right| \le \sum_{m=1}^{n} |a_{n,m}|^2 e^{\sum_{m=1}^{n}|a_{n,m}|}$$

Product $\therefore$ use theorem 1

$$\le \sup_{1\le m\le n} |a_{n,m}| \left(\sum_{m=1}^{n} |a_{n,m}|\right)$$

e^{a} $\qquad e^{a}$

Weak Convergence.

Def $\quad X_n \Rightarrow X \quad$ (X_n Converge in Distribution)

if $\quad F_{X_n} \Rightarrow F_X$ $\qquad F_Y(y) = P(Y \le y),\ y \in \mathbb{R}$

if $\quad F_{X_n}(y) \longrightarrow F_X(y), \quad \forall y \in \mathbb{R}$ s.t. $\qquad$ CDF $\Rightarrow$ $F(y) \xrightarrow[y\to\infty]{} 1$, $F(y) \xrightarrow[y\to-\infty]{} 0$, F is RCLL

$F(y) = F(y-)$

if $E g(X_n) \longrightarrow E g(x), \ \forall g$ Bounded. $g: \mathbb{R} \to \mathbb{R}$ is continuous. $\quad$ limit from the Right. $\quad$ Jump

Probability measure on Real line

$\mu_X(B) = \{P(X \in B)\}$ $\qquad B$ - Borel set