# L33 - 11-13 Uniform Continuous, Helly, Tightness

Wednesday, November 13, 2024 11:30 AM

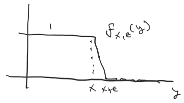
we say $X_n \Rightarrow X$ if (if and only if) $\forall f \in C_B(\mathbb{R})$ we have $Ef(X_n) \to Ef(X)$ | theorem 3.2.3

Enough to check for uniformly continuous.

we can restrict: $\forall f$ uniformly continuous and bounded $f: \mathbb{R} \to \mathbb{R}$

Def: uniformly continuous

$\forall \epsilon > 0, \exists \delta < 0$ s.t. $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon$ $\forall x_1, x_2 \in \mathbb{R}$.





All Derivative Exits.

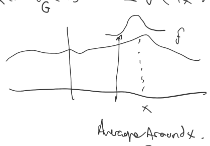
we can restrict: $\forall f \in C_B^\infty(\mathbb{R})$ we have $Ef(X_n) \xrightarrow[n\to\infty]{} Ef(X)$

All Derivative are Bound.

Step 1 Let $f \in C_B(\mathbb{R})$. How can approximate $f$ by $g \in C_B^\infty(\mathbb{R})$?

$T_x \sim N(x, \sigma^2)$ $\sigma > 0$

$\forall x \in \mathbb{R}$ $f_G(x) = Ef(T_x) \in C_B^\infty(\mathbb{R})$





Moving Average?

step 2. Let $X$ be a r.v. $X, Y$ ind $Y \sim N(0,1)$

Claim: $Ef(X + \sigma Y) = E(f_\sigma(X))$ (use Fubini)

$$\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f(x+\sigma y)\, dF_X(x)\, \cancel{dF_Y(y)}$$ ← we know $\frac{1}{2\pi}e^{-y^2/2}dy\, dF(x)$

change of variable.

interpret first.

$$\int_{y=-\infty}^{\infty} f(x+\sigma y)\frac{1}{2\pi}e^{-y^2/2}dy.$$

$$f(u)\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(u-x)^2}{2\sigma^2}}$$ change of variable

Bounded because f is ~~~~

Step 3 Claim. if $X_n + \sigma Y \Rightarrow X + \sigma Y$ $\forall \sigma > 0$ then $X_n \Rightarrow X$

"uniformly continuous Bounded in $\mathbb{R}$"

$f \in UC_B(\mathbb{R})$

$|E(f(W+\sigma Y)) - Ef(W)| \xrightarrow[\sigma\to 0]{} 0$ and Rate of convergence doesn't depend on $W$.

$\leq E|f(W+\sigma Y) - f(W)|; \sigma|Y| \leq \delta) + E(|f(W+\sigma Y) - f(W)|; \sigma|Y| > \delta)$

then $f(W+\sigma Y) - f(W)$ $\quad + 2CP(|Y| > \frac{\delta}{\sigma}) \xrightarrow[\sigma\to 0]{} \epsilon$ Note: $|f(x)| \leq C$ $x \in \mathbb{R}$

small Diff in distance

$Ef(X_n + \sigma Y) \sim Ef(X + \sigma Y)$

$\downarrow$ if $\sigma$ is small $\downarrow$

$Ef(X_n)$ $\quad Ef(X)$ "Doesn't Depend on W"

the Diff is $3 \times \epsilon$.

"The key is use smooth function $Y \sim N(0,1)$

Helly selection theorem.

Theorem 3.2.6 Helly selection theorem

$\forall$ sequence of Distributions $F_n$, $\exists F_{n_k}$ and $F \in C\uparrow^+$

s.t. $\lim_{k\to\infty} F_{n_k}(y) = F(y)$ $\forall$ continuity point $y$ in $F$

Let $f_n$ be CDF

RCLL nondecreasing.

Not symmetric means Left Limit

Claim $\exists n_k$ s.t. $F_{n_k}(x) \xrightarrow[k\to\infty]{} F(x)$, if $F(x) = F(x-)$

Proof: take $\mathbb{Q} = \{q_k\}_{k\geq 1}$ rational number countable & Dense Points.

$\forall n \geq 1$ $\quad 0 \leq F_n(q_1) \leq 1$,

select $(1,k), k \geq 1$ s.t. $F_n$

$(1,k) \in \{1, 2, \ldots\}$

$(1,k) \uparrow$ as $k\uparrow$

$0 \leq F(q_2) \leq 1$ $\quad F_{(1,k)}(q_1) \longrightarrow \tilde{F}(q_1)$

$0 \leq F_{(1,k)}(q_2) \leq 1$ $\quad F_{(2,k)}(q_2) \xrightarrow[k\to\infty]{} \tilde{F}(q_2)$

$F$ $\vdots$

$F_{(n,k)}(q_n) \to \tilde{F}_{(n,k)}(q_n)$

And so on.

Find $\{n_k\}_{k\geq 1}$ so that $F_{n_k}(q_n) \xrightarrow[k\to\infty]{} \tilde{F}(q_n)$, $n \geq 1$

And so on.  $F_{(n,k)}(q_n) \to \tilde{F}_{(n,k)}(q_n)$

Find $\{r_k\}_{k \geq 1}$ so that $F_{r_k}(q_n) \xrightarrow[k\to\infty]{} \tilde{F}(q_n)$, $n \geq 1$

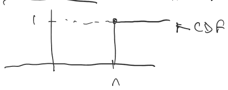
through cantor diagonization method
select the first from first. subseq
second form second subseq
⋮
$n^{th}$ from $n^{th}$ subseq.

$F_{(n,n)}(q_n) \xrightarrow[n\to\infty]{} \tilde{F}(q_m)$

$\tilde{F}(x) = \inf_{q_m > x} \{\tilde{F}(q_m)\}$ for $x \in \mathbb{R}$.

$F_{(n,n)}$ becomes $F_{n_k}(x) \xrightarrow[k\to\infty]{} F(x)$, $F$ non desc, RCLL.

Example $P(X_n = n)$




$F_n = F_{X_n}$  Not CDF. $F_n(\infty) \neq 1$

$\lim_{n\to\infty} F_n(x) = 0$  $x \in \mathbb{R}$

"how force $F$ to be proper CDF"

we are looking for condition that $F(-\infty) = 0$, $F(+\infty) = 1$

Condition: we say $\{F_{X_n}\}_{n\geq 1}$ is Tight iff $\forall \epsilon > 0$, $\exists M > 0$ s.t. $P(|X_n| \geq M) < \epsilon$, $n \geq 1$
"same $M$ for all $n$"

"sometimes called uniformly tightness"

Result: if $F_n \Rightarrow F$, $\{F_n, F\}_{n\geq 1}$ are CDFs

then $\{F, F_n\}_{n\geq 1}$ are tight

Find $M$ s.t. $P(|X| > M) < \epsilon$ & $P(X = M) = P(X = -M) = 0$

then we get
$2\epsilon$ $\left\{ \begin{array}{l} F_n(M) \xrightarrow[n\to\infty]{} F(M) \\ F_n(-M) \xrightarrow[n\to\infty]{} F(-M) \end{array} \right\}$ if Diff is $\epsilon$.

if $n \geq N(\epsilon)$ then $P(|X_n| > M) < 2\epsilon$.

$\{F_k\}_{k<n}$ "There is a finite # below $N$"
"Then take maximum between the two"

Next time
CLT  3 moment is finite, 2 moment is finite.
Linberg Feller CLT

theorem 3.2.7
Def tight  if $\forall \epsilon > 0$, $\exists M_\epsilon$ s.t. $\limsup_{n\to\infty} 1 - F_n(M_\epsilon) + F_n(-M_\epsilon) \leq \epsilon$

theorem 3.2.8  if $\exists \varphi \geq 0$ s.t. $\varphi(x) \to \infty$ as $|x| \to \infty$ and
$C = \sup_n \int \varphi(x)\, dF_n(x) < \infty$ then $F_n$ is tight.