

02-28 L2 MG, Polya

Friday, February 28, 2025 11:32 AM

L2 MG

Setup: $X_0 = 0$, $\{X_n, \mathcal{F}_n\}_{n=0}^{\infty}$ MG, $E(X_n^2) < \infty$ $\forall n$.
 $\{X_n^2, \mathcal{F}_n\}_{n=0}^{\infty}$ submg. $D_k = X_k - X_{k-1}$, $A_n = \sum_{k=1}^n D_k^2$, $n \geq 1$
 $X_n^2 = M_n + A_n$, $\{M_n, \mathcal{F}_n\}$ MG, $A_0 = 0$, $A_n \in \mathcal{F}_{n-1}$, $A_n \perp \text{as.}$
Prob. $E(A_\infty) \leq E(\sup_{n \geq 0} |X_n|^2) \leq 4E(A_\infty)$

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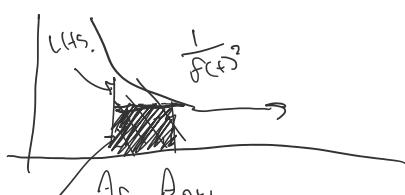
Convergence of series

$X_n \xrightarrow[n \rightarrow \infty]{a.s.}$ on $\{A_\infty < \infty\}$

SLLN: Let $f(t)$, $t \geq 1$ be increasing. $\int_1^\infty \frac{dt}{f(t)^2} < \infty$

Then $\frac{X_n}{f(A_n)} \xrightarrow[n \rightarrow \infty]{a.s.} 0$ on $\{A_\infty = \infty\}$ Proved test t.h.
upper bound.

should draw picture.



$$\frac{A_{n+1} - A_n}{f^2(A_{n+1})} \leq \int_{t=A_n}^{A_{n+1}} \frac{dt}{f(t)^2}$$

rewrite AS $\left\{ \sum_{n=1}^{\infty} P_n = \infty \right\}$ omega by otregar
it is the same.

BC. II, $B_n \in \mathcal{F}_n$, $n \geq 1$ then $\{B_n \text{ i.o.}\} = \left\{ \sum_{n=1}^{\infty} P_{\mathcal{F}_n}(B_n) = \infty \right\}$

BC. II⁺ $\sum_{k=1}^{\infty} \mathbb{1}_{B_k}$ $\xrightarrow[n \rightarrow \infty]{a.s.} 1 \text{ or } \left\{ \sum_{k=1}^{\infty} P_k = \infty \right\} \cap P_n$.

thus P_k MG. diff. P_k is measurable in \mathcal{F}_{k-1}

Proof wts A_n $X_n = \sum_{k=1}^n \mathbb{1}_{B_k} - P_{k-1}$

thus $P_{k-1} - P_k = 0$.

$\# \left(\mathbb{D}_k^2 \right)$ what is the variance of Bernoulli $P(1-p)$.

$$= P_k(1-P_k) \therefore A_n = \sum_{k=1}^n P_k(1-P_k)$$

$\left\{ \sum_{k=1}^{\infty} P_k = \infty \right\} \stackrel{?}{=} \left\{ A_\infty = \sum_{k=1}^{\infty} P_k(1-P_k) = \infty \right\}$ something smaller

think this.

take $P_k = 1$

$$\{ \infty \} \rightarrow \{ 0 \}$$

$$\{\infty\} \rightarrow \{0\}$$

$$\{A_{\infty} < \infty\} \wedge \left\{ \sum_{k=1}^{\infty} P_k = \infty \right\}$$

What happens here? need to use X_n w/

$$\frac{X_n}{\sum_{k=1}^{\infty} P_k} = \frac{\sum_{k=1}^n (B_k - P_k)}{\sum_{k=1}^n P_k} \rightarrow 0$$

$$\frac{X_n}{\sum_{k=1}^n P_k} \xrightarrow[\substack{\text{finite} \\ n \rightarrow \infty}]{} 0. \quad (\text{B.C. II+})$$

What happens if $\{A_{\infty} = \infty\}$

use SLN:

$$f(t) = t \quad \int_1^{\infty} \frac{dt}{t^2} < \infty. \quad \text{need finite} \quad f(A_n) = A_n.$$

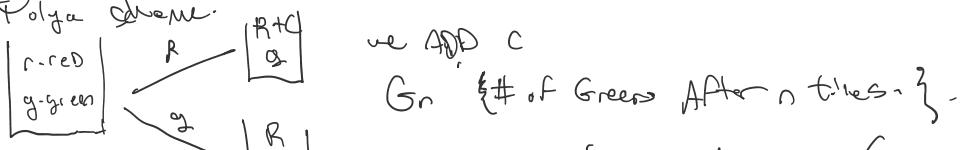
$$\frac{X_n}{\sum_{k=1}^n P_k (1 - P_k)} \xrightarrow[\substack{\text{As.} \\ n \rightarrow \infty}]{} 0 \text{ on } \sum_{k=1}^{\infty} P_k (1 - P_k) = \infty.$$

so divide by something bigger.

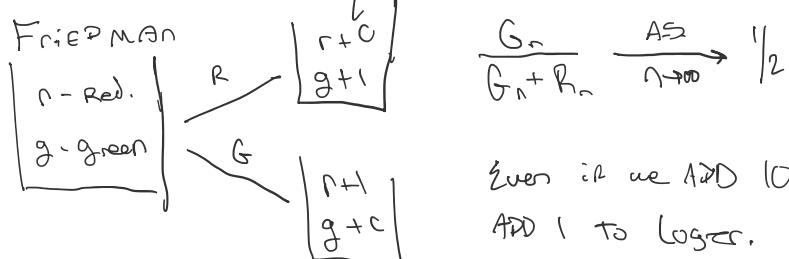
$$\frac{X_n}{\sum_{k=1}^n P_k} \xrightarrow[\substack{\text{As.} \\ n \rightarrow \infty}]{} 0.$$

Where do you use it? Polya scheme.

Polya scheme.



$$\frac{G_n}{G_n + R_n}, F_n \xrightarrow[\substack{\text{As.} \\ n \rightarrow \infty}]{} M_6 \quad \frac{G_n}{G_n + R_n} \xrightarrow[\substack{\text{As.} \\ n \rightarrow \infty}]{} X \sim \text{Beta}\left(\frac{g}{c}, \frac{r}{c}\right)$$



Even if we ADD 10 to one
ADD 1 to other.

$$G_n = \# \text{ green after } n \text{ times.}$$

$$R_n = \# \text{ red after } n \text{ times}$$

$$g_n = \frac{G_n}{R_n + G_n} \quad - R_n = n^{\text{th}} \text{ select is green}$$

$$D. \quad \approx 2.1$$

$$g_n = \frac{B_n}{B_n + G_n} \quad - R_n = n^{\pi} \text{ select is green}$$

$$r_n = \frac{R_n}{B_n + G_n} \quad D_n = \sum_{k=1}^n \frac{1}{B_k}$$

what is condition probability $P_{\tilde{\Omega}_{n-1}}(B_n) = g_n$

what can say about g_n $g_n \geq \frac{g}{g + c + n(c+1)}$ $\sim \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$
 $\therefore \sum_{n=1}^{\infty} g_n > \infty$ As.
 ↑↑↑
 comes when
 passes phase

$$\frac{D_n}{\sum_{k=1}^n g_{k-1}} \xrightarrow[n \rightarrow \infty]{\text{As.}} 1 \quad \text{since BC II +} \quad \frac{\sum_{k=1}^n \frac{1}{B_k}}{\sum_{k=1}^n P_k} \xrightarrow[n \rightarrow \infty]{\text{As.}} 1$$

Result $\hat{g}_n \xrightarrow[n \rightarrow \infty]{\text{as.}} 1_L$.

Dob's inequality L^p for $p=2$.

Another version with same setup $\boxed{1}$

$$E(\sup_{n \geq 0} |x_n|) \leq 3 \cdot E(\sqrt{A_\infty}) \quad \text{Proof is in the book.}$$

Application...

corollary let $\{x_n, \tilde{x}_n\}_{n \geq 0}$ L^2 M.G.

if $E(\sqrt{A_\infty}) < \infty$ then $\tilde{x}_n \xrightarrow[n \rightarrow \infty]{\text{as.}} x$ by MGCT

we only need $\sup_{n \geq 0} E|x_n| \leq E(\sup_{n \geq 0} |x_n|) <$

2. $E|x_n - x| \xrightarrow[n \rightarrow \infty]{} 0$ By DCT.

Extension of Wald First Equation

let $\{\zeta_k\}$ be iid $E(\zeta_k) = 0$. then T is st. wrt \tilde{x}_n .

$E(T) < \infty \Rightarrow T < \infty$ a.s.

then $E(S_T) = 0$. w/w first eq.

Version: we also assume $E(\zeta^2) < \infty$, but

now $E(\sqrt{T}) < \infty$, then $E(S_T) = 0$

$\Rightarrow T < \infty$.

Always true. $S = \sum_{i=1}^T x_i$

$\{S_{T \wedge n}\}$ M.g. $E(S_{T \wedge n}) = 0$.

$$S_{T \wedge n} \xrightarrow[n \rightarrow \infty]{\text{as.}} S_T$$

solve using A_∞

$$A_n = \sum_{k=1}^n E_{\tilde{x}_k} (\zeta_k^2) = \sigma^2 (n \wedge T)$$

$$A_{\infty \wedge T} = A_T = \sigma^2 T \Rightarrow E(\sqrt{A_\infty}) < \infty$$

$$\begin{aligned} A_{\infty \wedge T} &= A_T = \sigma^2 T \Rightarrow E(\sqrt{A_\infty}) < \infty \\ \Rightarrow E(S_T) &= \lim_{n \rightarrow \infty} E(S_{T \wedge n}) < \infty \quad \text{Borel-Cantelli lemma} \\ &\quad - \text{demonstrated.} \end{aligned}$$