02-24

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L^p, $p>1$ Doob's inequality

Let $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ be subMG ($X_n \geq 0$ a.s. $n \geq 0$)

Then $E(\max_{0\leq k\leq n} (X_k)^p) \leq E(X_n^p)\left(\frac{p}{1-p}\right)^p$ $p=2 = \left(\frac{p}{p-1}\right)^p = 4.$

Remark: For submg. which isn't positive use $(X_n^+, \mathcal{F}_n)_{n\geq 0}$

For Reg mg. " " " " $(|X_n|, \mathcal{F}_n)_{n\geq 0}$

Application: Let $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ be MG and $\sup_{n\geq 0} E(|X_n|^p) < \infty$

then ① $X_n \xrightarrow[n\to\infty]{a.s.} X$

② $E\left[|X_n - X|^p\right] \xrightarrow[n\to\infty]{} 0$ By DCT.

why? $|X_n - x|^p \leq \underbrace{2^p \max_{1\leq k\leq n} |X_n|^p}$ since X_n, X is two, 2^p. and then use The Biggest X_n

Dominating function

If we want $E|X_n - X|$ we have $\sup_{n\geq 0} E(|X_n|^p) < \infty \Rightarrow$ uniform integrability.

then $E|X_n - X| = 0$.

Application For $P=2$. Model for Population Growth. Branching Processes. $\{\xi_n\}, \{\xi_{n,i}\}_{i\geq 0, n\geq 0}$ IID

$Z_0 \equiv 1$ $\forall n$ $\{\xi_{n,i}\}_{i>1}$ iid. like # of children of Each individual

$P(\xi = 0) > 0$ $\mu = E(\xi) = \sum_{k=0}^{\infty} k P(\xi = k) < \infty$

$Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{n+1} \cdot i$ Z_n = # of People in the n^{th} Generation.

$\mathcal{F}_n = \sigma\{Z_1, \dots, Z_n\}_{n\geq 1}$

Claim $\left(\frac{Z_n}{\mu^n}, \mathcal{F}_n\right\}_{n\geq 0}$ MG.

Look At $E_{\mathcal{F}_n}\left(\frac{Z_{n+1}}{\mu^{n+1}}\right) = \frac{\mu \cdot Z_n}{\mu^{n+1}} = \frac{Z_n}{\mu^n}$

this Event is Extinction.

integers converge iff Becomes Constant.

Result if $\mu \leq 1$ then $Z_n \xrightarrow[n\to\infty]{a.s.} 0$

$\exists$ $T < \infty$ with $Z_T = 0$: Extinction.

e = P(Extinction) or $e = 1$

Proof: $Z_n = \underbrace{\frac{Z_n}{\mu^n}}_{\text{MG}} \underbrace{\mu^n}_{\text{monotone}}$

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$\mu \le 1 \Rightarrow \{Z_n, \mathcal{F}_n\}_{n \ge 0}$ is Super MG., $Z_n \ge 0 \; \forall n$

$\Rightarrow$ By MGCT $\Rightarrow Z_n \xrightarrow[n\to\infty]{a.s.} Z_\infty \equiv 0$

(2) if $\mu > 1$ then $e < 1$

How solve w/o Doob. L^p inequality

Sketch proof of (2)

Prob. Generating function NEW CONCEPT.

$\varphi(\xi)(s) = E(s^\xi) = \sum_{k=0}^{\infty} P(\xi = k) \cdot s^k, \quad 0 \le s \le 1$

Increasing & Convex

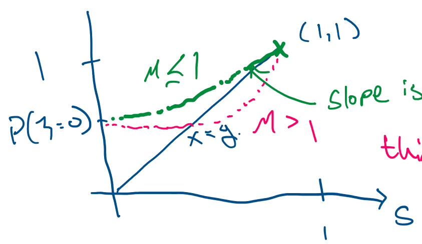
$\varphi' \ge 0 \quad \varphi'' \ge 0$

it follows $e = \varphi(e^\xi) = \sum_{k=0}^{\infty} P(\xi = k) e^k$

$\varphi(\xi)(1) = 1$

$\varphi(\xi)(0) = P(\xi = 0) > 0$

Need solution for



$P(\xi = 0) =$

slope is less than 1

this is e

Now we use Doob (in the context of $\mu > 1$)

$Var[\xi^2] = \sigma^2 < \infty, \quad \mu > 1, \quad P(Z = 0) > 0$

(1) iff $E(\xi^2) < \infty$.

$0 \le X_n = \frac{Z_n}{\mu^n} \quad n \ge 0$ is mg. we get $X_n \xrightarrow[n\to\infty]{a.s.} X$

$E_{\mathcal{F}_{n-1}}[(\underbrace{X_n - X_{n-1}}_{D_n})^2]$ like the conditional variance.

D_n - the martingale Difference.

$= E_{\mathcal{F}_{n-1}}\left(\frac{Z_n}{\mu^n} - \frac{Z_{n-1}}{\mu^{n-1}}\right)^2 = \mu^{-2n} E[Z_n - \mu Z_{n-1}]^2$

take μ^{2n} out. and leave one μ

$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_{n,i} \qquad = \mu^{-2n} \sigma^2 \cdot Z_{n-1}$

Why is this.

$E(Z_{n-1}) = \mu^{n-1}$

$E(X_n - X_{n-1})^2 = \mu^{-2n} \sigma^2 \mu^{n-1} = \mu^{-(n+1)} \sigma^2$

$\cdot E(X_n^2) =$

$\cdot\ E(X_n^2) =$

$$X_n^2 = \left[1 + \sum_{k=1}^{n} (X_k - X_{k-1})\right]^2 \Rightarrow E(X_n^2) = 1 + \sigma^2 \sum_{k=1}^{n} m^{-(k+1)}$$

$\{D_k\}_{k\geq 1}$ M.D.

$E(D_k D_m) = 0 \quad k \neq m \quad , \quad E(D_k) = 0.$

$$\sup_{n\geq 0} E(X_n^2) \leq 1 + \sigma^2 \sum_{k=1}^{\infty} m^{-(k+1)} < \infty$$

if this is finite By L^p Doob Inequality

we get $E(X_n - X)^2 \xrightarrow[n\to\infty]{} 0 \Rightarrow E|X_n - X| \xrightarrow[n\to\infty]{} 0 \Rightarrow E(X) = 1$

Note ① ↑

Kesten – STIGUM

Optimal Result: Assume $m > 1$, $\frac{Z_n}{m^n} \overset{A.S.}{\not\to} 0$ w.p. 1 $\quad n \to \infty$

iff $E(\xi \cdot \log^+(\xi)) < \infty$