

MG For BM. (Ω, \mathcal{F}, P)

Def. of MG in continuous time

$\{X_t, \mathcal{F}_t\}_{t \geq 0}$ is MG if ① $E|X_t| < \infty$
 ② $X_t \in \mathcal{F}_t$
 ③ $E_{\mathcal{F}_t}(X_{t+h}) = X_t \quad \forall t \geq 0, h > 0$

We want to study MG Associated with BM.

$M(x, t), x \in \mathbb{R}, t \geq 0$ w/ natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ or canonical filtration.

$M(x, t) \quad x \in \mathbb{R}, t \geq 0$.

Question When is $\{M(B_t^x, t), \mathcal{F}_t\}_{t \geq 0}$ a MG?

$$E_x M(B_{t+h}, t+h | \mathcal{F}_t) = M(B_t^x, t) \text{ a.s. } \forall x \in \mathbb{R}, t \geq 0, h > 0$$

By markov Property.

$$\begin{aligned} \Psi(B_t^x) &= E_{B_t^x} [M(B_h, t+h)] \\ &= M(B_t^x, t) \end{aligned} \quad \begin{array}{l} \text{this is the} \\ \text{formula} \end{array} \quad \forall x \in \mathbb{R}, t \geq 0, h > 0$$

$$\Psi(y) = E_y [M(B_h, t+h)]$$

Example Famous MG Exponential

$$M(x, t) = e^{\theta x - \theta^2 \frac{t}{2}} \quad \theta \in \mathbb{R}, \theta \text{ constant.}$$

$$E_x(M(B_h, t+h)) = E_x(e^{\theta B_h - \theta^2(t+h)/2})$$

$$= e^{-\theta^2(t+h)/2} \cdot E_x(e^{\theta B_h})$$

$$= e^{-\theta^2(t+h)/2} \cdot E(e^{\theta(x + \sqrt{h} Z)})$$

$$B_h \sim N(x, h)$$

$$B_h \stackrel{d}{=} x + \sqrt{h} Z \sim N(0, 1)$$

$$= e^{-\theta^2(t+h)/2} \underbrace{E(e^{\theta(x+\sqrt{h}\tilde{Z})})}_{e^{\theta x} e^{\theta^2 h/2}}$$

$$e^{-\frac{\theta^2 t}{2}} e^{-\frac{\theta^2 h}{2}} = M(x, t)$$

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$\left(e^{\theta B_t^+ - \frac{\theta^2 t}{2}}, \tilde{f}_t \right)_{t \geq 0}$ is mg wrt P_x $x \in \mathbb{R}$

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$$\frac{\partial}{\partial \theta} E_x[\mu(B_n, t+h; \theta)] = \mu(x; t; \theta)$$

$$E_x\left[\frac{\partial}{\partial \theta} \mu(B_n + t+h; \theta)\right] = \frac{\partial}{\partial \theta} \mu(x, t; \theta) \quad \forall \theta$$

$\{\mu(B_t^+, \theta, \tilde{f}_t)\}$ is mg.

$\frac{\partial}{\partial \theta} (\mu(B_t^+, \cdot, \theta), \tilde{f}_t)$ Mb.

Continue to take Derivatives $\frac{\partial^2}{\partial \theta^2} \mu(\cdot, \theta)$

$$M(\cdot) = e^{\theta x - \frac{\theta^2 t}{2}}$$

$$\frac{\partial}{\partial \theta} M(\cdot) = M(\cdot)(x - \theta t)$$

$$\stackrel{\theta=0}{\Rightarrow} \frac{\partial}{\partial \theta} M(0)|_{\theta=0} = x$$

$$\Rightarrow \{B_t^+, \tilde{f}_t\}_{t \geq 0}$$

Endless Brownian motion is mg.

$$\frac{\partial^2}{\partial \theta^2} = M(x) \cdot (x - \theta t)^2 \Big|_{\theta=0} = x^2$$

$$\Rightarrow (B_t^2 - t \mathbb{F}_t)_{t \geq 0}.$$

$$\frac{\partial^3}{\partial \theta^3} = M(x)(x - \theta t)^3 + M(x) 2(x - \theta t)x - t M(0)(x - \theta t)$$

$$\Rightarrow \underset{\theta=0}{B_t^3 - 3t B_t, \mathbb{F}_{t \geq 0}} \} M_6.$$

M6. What we get: 1, B_t , $B_t^2 - t$, $B_t^3 - 3t B_t$, $B_t^4 - 6t B_t^2 + 3t^2$, ..

w.r.t. \mathbb{F}_t PDE

In the book let $M(x, t)$ satisfy $\frac{\partial M}{\partial t} + \frac{1}{2} \frac{\partial^2 M}{\partial x^2} = 0 \quad \forall x \in \mathbb{R}, t \geq 0$

then $M(B_t, t)$ is M6.

The book proved $\frac{\partial}{\partial t} E_x(M(B_t, t)) = 0$

$$\Leftrightarrow E_x M(B_t, t) = M(x, 0) \quad \forall t \geq 0, x \in \mathbb{R}.$$

Starting.
 $P_t(x, y) = \mathbb{P}_{M(x, t)}(Y)$, it turns out $\frac{\partial P_t}{\partial t} = \frac{1}{2} \frac{\partial^2 P_t}{\partial y^2} \quad \forall x \in \mathbb{R}$.

τ s.t. $\tau \leq t$ a.s. $E_y \underset{\mathbb{F}_\tau}{\mathbb{P}}(M(B_\tau, \tau)) = M(B_p, \tau)$