L16 - 10-02 independence

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$Z \sim N(0,1)$

$P(Z > x) \le \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x} \wedge 1, \quad x > 0$ min 1

$(P(Z > x) \le e^{-\frac{x^2}{2}}, \; x > 0)$

$P(Z > 0)$

$$\int_x^\infty e^{-\frac{y^2}{2}} dy = \int_{z=0}^\infty e^{-\frac{(x+z)^2}{2}} dz \le \left[\int_{z=0}^\infty e^{-xz} dz \right] e^{-\frac{x^2}{2}}$$

$y = x$ change of variable $y = x + z$, $dy = dz$

$$= \frac{e^{-xz}}{x} \Big|_{z=0}^\infty = \frac{1}{x}$$

$x > 0 \cdot t > 0$

$P(Z > x)$

$= P(tZ > tx)$

$= P(e^{tZ} > e^{tx})$ Markov sol. $P(X > a) \le \frac{E(X)}{a}$

$\le E(e^{tZ}) e^{-tx}$

$= e^{t^2/2 - tx} \Big|_{t = x}$ optimal t. wrt x small. $x = t$.

$= e^{-x^2/2}$

$$\underbrace{\frac{1}{\sqrt{2\pi}} \left(\frac{1}{x} - \frac{1}{x^3} \right) e^{-\frac{x^2}{2}}}_{} \overset{\text{Max } 0.}{\wedge\, 0} \le P(Z > x) \le \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{x^2}{2}}}{x} \wedge 1, \quad x > 0$$

$$\frac{1}{x} e^{-\frac{x^2}{2}} = (1) e^{-\frac{x^2}{2}} +$$

$$\int_{y=x}^\infty e^{-y^2/2} dy \ge \int_{y=x}^\infty \left(1 - \frac{3}{y^4}\right) e^{-y^2/2} dy. = \left(\frac{1}{x} - \frac{1}{x^3} e^{-\frac{x^2}{2}} \right)$$

$M = E(X)$

$\sigma^2 = V(X)$

$P(|X| > a) \le \frac{E(X)}{a}$ Markov

$P(|X - M| > a) \le \frac{\sigma^2}{a^2}$ chebychev.

X X

$M-M$ M $M+a$

$P(X > M + a) \le ? \quad \frac{\sigma^2}{\sigma^2 + a^2}$

Assume Y with $E(Y) = 0$; $V(Y) = \sigma^2$

$P(Y > a) \le P(|Y + b|^2 > (a+b)^2)$

$$\le \frac{E[(Y+b)^2]}{(a+b)^2} = \frac{E(Y^2) + b^2 + \cancel{bE(Y)}^{0}}{\cancel{\sigma^2 + b^2}\,(a+b)^2} = \frac{\sigma^2 + b^2}{(a+b)^2\;\cancel{\sigma^2 + b^2}} \overset{?}{=} \frac{\sigma^2}{\sigma^2 + a^2}$$

$b^* = \arg\min \left(\frac{a^2 + b^2}{(a+b)^2} \right)$

Independence. Chapter 2.1 Durrett

Let $\mathcal{F}_\alpha \subset \mathcal{F}$, $\alpha \in I$

① we say $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ are independent if $P(\bigcap_{i=1}^n A_i) = \prod_{i=1}^n P(A_i)$

$A_i \in \mathcal{F}_{\alpha_i} \quad i = 1, \cdots, n$

② Let $\{X_\alpha\}_{\alpha \in I}$ Be collection of Random Variables r.v.

Every $X_\alpha : \Omega \to \mathbb{R}$

X_α in $\mathcal{F} | \mathcal{B}(\mathbb{R})$ measurable

$\sigma(X_\alpha) \subset \mathcal{F}$, $\alpha \in I$.

we say $\{X_\alpha\}_{\alpha \in I}$ are Independent if $\{\sigma\{X_\alpha\}\}_{\alpha \in I}$ Are Independent

$\mathcal{A} \subset \mathcal{F}$

theorem | if $\{\mathcal{A}_i\}$. . . Are independent and $\mathcal{A}_i$ is π-system closed under Intersection dinkin.

$\mathcal{a} \subset \mathcal{F}$

<u>theorM</u> | if $\{\mathcal{a}_i\}_{i=1,\dots,n}$ Are independent and $\mathcal{a}_i$ is π-system ← closed under Intersection Dinkin.

then $\{\sigma\{\mathcal{a}_i\}\}_{1 \le i \le n}$ Are independent as well. theorem 2.1.3. Durrett.

Ex: we have 2 R.V.S: X, Y

we know $P(X \le x, Y \le y) = P(X \le x) P(Y \le y)$, $x, y \in \mathbb{R}$

Does it mean that (X, Y) Are IND? YES.

$\{X < x\}_{x \in \mathbb{R}}$ not σ-Alg but it is a π system.

$\{1 < X < 3\}$ $\quad \{X \le x_1\} \cap \{X \le x_2\} = \{X \le x_1 \wedge x_2\}$ ← min.

Proof:

Let $\mathcal{L} = \{A \in \mathcal{F} : P(A \cap \bigcap_{i=2}^{n} A_i) = P(A) \prod_{i=2}^{n} P(A_i) \quad \forall A_i \in \mathcal{a}_i \quad 2 \le i \le n\}$

Claim λ system, ⟶
1. $\Omega \in \mathcal{L}$
2. $A \subset B$, $A \in \mathcal{L}$, $B \in \mathcal{L} \Rightarrow B \setminus A \in \mathcal{L}$
3. $B_i \in \mathcal{L}$, $B_i \uparrow B \Rightarrow B \in \mathcal{L}$

$\mathcal{L} \supset \mathcal{a}_1$ → a π system

Dynkin says $\mathcal{L} \supset \sigma\{\mathcal{a}_1\}$.

$\Rightarrow \sigma\{\mathcal{a}_1\}, \mathcal{a}_2, \dots, \mathcal{a}_n\}$ IND.

Then Replace $\mathcal{a}_2$ can show IND.

Ex. Let $\{\mathcal{F}_i\}_{i=1,2,3}$ be IND. σ-Alg.

are $\sigma\{\mathcal{F}_1, \mathcal{F}_2\}, \mathcal{F}_3$ IND? **YES** should follow from Dynkyn

$\mathcal{F}_1 \cap \mathcal{F}_2 = \{A \cap B : A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$.

observe. $\mathcal{F}_1 \cap \mathcal{F}_2$, $\mathcal{F}_3$ are IND.

$\Rightarrow \sigma\{\mathcal{F}_1 \cap \mathcal{F}_2\}, \mathcal{F}_3$ are IND.

Let $\{X_1, X_2, X_3, X_4, X_5\}$ be IND r.v.

Then $X_1 + X_2$, $e^{X_3} \cdot \sin(X_4 + X_5)$ Are IND.