L03 - 08-30 lecture

Friday, August 30, 2024 11:30 AM

Lebesgue measure.

We have a measure μ on $\Omega = (0,1]$

$\mathcal{a} = \{ \bigcup_{i=1}^{n} (a_i, b_i) \}$

Goal Extend μ to a measure on $\sigma\{\mathcal{a}\}$

Borel. sometimes $\mathcal{B}$

The smallest σ Algebra of $\mathcal{a}$

there is a Bigger Algebra

Lebesgue $\subset$ Algebra $\supset$ Borel

Borel - sigma Algebra that contains All open sets

$(a,b] \in \mathcal{a}$, $\bigcup_{n=1}^{\infty} (a, b-\frac{1}{n}] = (a,b)$

The minimum σ Algebra contain open set.

① Existence? yes!

② unique? yes!

Dynkin soviet → cornell

"start with Existence?"

① WTS: V_1 and V_2 are extension measures on $(\Omega, \mathcal{B})$

then, $V_1(A) = V_2(A)$. $\forall A \in \mathcal{B}$

we Assume $V_1(A) = V_2(A)$, $A \in \mathcal{a}^*$

<u>Def</u> π - system = collection of subset closed under intersection $\cap$

$\Omega \in \pi$ $A, B \in \pi$ then $A \cap B \in \pi$

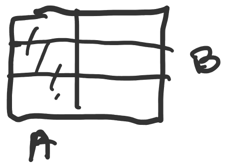
$\Omega \cap A = A$ if $A \subset \Omega$

Def λ-system: 1) $\Omega \in \lambda$

2) $A, B \in \lambda \wedge A \subset B \Rightarrow B \setminus A \in \lambda$

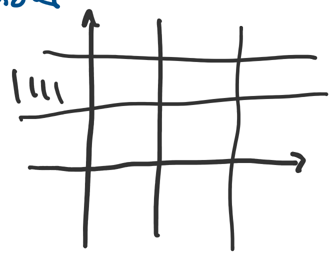
3) $A_n \in \lambda$, $n = 1, 2, \ldots$ $A_n \uparrow A$ $\left\{ \begin{array}{l} A_n \subset A_{n+1} \\ \bigcup_{n=1}^{\infty} A_n = A \end{array} \right.$ $\Rightarrow A \in \lambda$

not a σ-algebra, or Algebra

$\Omega =$  $\lambda = \{\emptyset, \Omega, A, A^c, B, B^c\}$

No $A \cap B$.

Consider



to make λ-system into Algebra

if $A, B \in \lambda \Rightarrow A \cap B \in \lambda$ then λ is σ-Alg.

then If $(A \cap B) \in \lambda \Rightarrow (A^c \cap B^c)^c = (A \cup B) \in \lambda$

$A \cup B \in \lambda$ $\bigcup_{i=1}^{\infty} A_i$ $[A_1] \cup [A_1 \cup A_2] \cup [A_1 \cup A_2 \cup A_3] \ldots$

if λ system is π system then λ is σ-Alg

Dynkin said we need λ-system $\supset$ π-system

π-λ Theorem (Dynkin)

if π is a π-system then the smallest λ-system that contains π is equal To The σ-Algebra Generated By π

$$\lambda(\pi) = \sigma(\pi)$$

Easier To Prove $\lambda(\pi)$ than $\sigma(\pi)$

<u>Proof</u>

Easier To Prove $\lambda(\pi)$ than $\sigma(\pi)$

Proof

we Assume that V_1, V_2 agree on π,

$V_1(\Omega) = V_2(\Omega) < \infty$ And V_1, V_2 Are measure on $(\Omega, \sigma(\pi))$

λ system

$$\lambda \equiv \{ B \in \sigma(\pi) ; V_1(B) = V_2(B) \}$$

wts: λ is a λ-system. then use Dykin.

$$\lambda(\pi) = \sigma(\pi)$$

② $A, B \in \lambda$ if $B \subset A$ then $A \setminus B \in \lambda$

$V_1(A \setminus B) = V_1(A) - V_1(B) = V_2(A) - V_2(B) = V_2(A - B)$

③ $A_i \uparrow A$, $A_i \in \lambda$, $i \geq 1$ then $A \in \lambda$

$$\Rightarrow \underbrace{V_1(A_i) \uparrow V_1(A)}_{} \;\; \& \;\; \underbrace{V_2(A_i) \uparrow V_2(A)}_{} \quad = $$

The Proff is By cardiotory

Goal: Prove Existance of Lebesgue measure a To $\sigma(a) = B$ (Proff in The Book)

We have Measure μ on (Ω, a)

Define a measure Based on All Ω

Define $\mu^*(A) \equiv \inf \{ \sum_{i=1}^{\infty} \mu(A_i) : \bigcup_{i=1}^{\infty} A_i \supset A,\ A_i \in a \}$

Outer measure

we Cover All a with minimum

Properties of outer measure

1) $\mu^*(\phi) = 0$

2) if $E \subset F \Rightarrow \mu^*(E) \leq \mu^*(F)$

Every cover in E is in F + more

3) $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \subset \Omega$, $\Rightarrow \mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

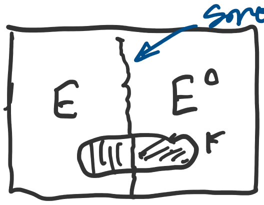
3) $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \subset \Omega$, $\Rightarrow$ $\mu^*(A) \le \sum_{i=1}^{\infty} \mu^*(A_i)$

true for All subset of omega

Def $E \subset \Omega$ is called Measurable if

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \cap E^c)$$

smooth enough



we can find cover with Each part F.

Conclusion $\mathcal{a}^* = \{A \text{ measurable}\}$

tasks 1) $\mathcal{a} \subset \mathcal{a}^*$ Prove $\mathcal{a}$ is measurable

2) $\mathcal{a}^*$ is σ - Algebra.

Legbegue Algebra $\rightarrow$ $\mathcal{a}^* \supset \mathcal{B} = \sigma(\mathcal{a})$

3) μ^* is a measure on $(\Omega, \mathcal{a}^*)$

4) if $\mu^*(E) = 0$ then $E \in \mathcal{a}^*$

Says $\mu^*(F) \overset{\le}{=} \cancel{\mu^*(F \cap E)}^{0} + \mu^*(F \cap E^c)$

$\uparrow$ set is smaller or Equal

$\therefore$ $\mu^*(F) = \mu^*(F \cap E^c)$