L26 - 10-28

Monday, October 28, 2024 11:30 AM

Kolm. 0-1 Law

Let $\{X_i\}_{i\geq 1}$ be IND $\mathcal{T} = \{$tail events$\}$

Aside

$\mathcal{F}_{[n,\infty)} = \sigma\{X_n, X_{n+1}, \dots\}$ $n \geq 1$

$\mathcal{T} \equiv \bigcap_{n=1}^{\infty} \mathcal{F}_{[n,\infty)}$

then $\mathcal{T}$ is trivial ie. $P(A) \in \{0,1\}$, $A \in \mathcal{T}$

ch 1

Proof $\mathcal{F}_1, \mathcal{F}_2, \dots$ IND then $n_1 < n_2 < \dots$

$\mathcal{F}_{[1,n_1]} = \sigma\{\mathcal{F}_1, \dots, \mathcal{F}_{n_1}\}$ $\mathcal{F}_{[n_1+1, n_2]} = \sigma\{\mathcal{F}_{n_1+1}, \dots, \mathcal{F}_{n_2}\}$

then $\mathcal{F}_{[1,n_1]} \perp\!\!\!\perp \mathcal{F}_{[n_1+1, n_2]}$ By pi-lambda system theorem.

$\mathcal{F}_n \equiv \sigma\{X_1, \dots, X_n\}$, $\underbrace{\mathcal{F}_{[n,\infty)} = \sigma\{X_{n+1}, \dots\}}$ ARE IND.

SAME PROOF.

$\sigma\{\bigcup_{m=n+1}^{\infty} \mathcal{F}_{[n,m]}\}$ Lebesgue-system $\supset$ Pi system

then IND By π-λ system

$\sigma\{X_{n+1}, \dots\} \supset \mathcal{T}$ $\Rightarrow$ $\mathcal{F}_n, \mathcal{T}$ IND $n \geq 1$

$\Rightarrow \sigma\{\bigcup_{n=1}^{\infty} \mathcal{F}_n\}, \mathcal{T}$ are ind.

$\Rightarrow \sigma\{X_1, X_2, \dots\} \supset \mathcal{T}$ $\therefore$ $\mathcal{T} \perp\!\!\!\perp \mathcal{T}$

$\therefore$ $P(A \cap A) = P(A)P(A)$

$P(A) = P(A)^2$

Since $0 \leq P(A) \leq 1$

Ex: $S_n = \sum_{k=1}^{n} X_k$, $\{X_k\}_{k\geq 1}$ IND.

$\{S_n \text{ Converges}\} \in \mathcal{T}$

$P(\{S_n \text{ Converges}\}) \in \{0,1\}$ Either it wont Converge or will.

Kolgomorov Maximal Inequality.

"cubonian."

More than 4 Days, your wasting time

$\{X_i\}$, i = i, ..., n I.D. centered. $E(X_i) = 0$ $E(X_i^2) < \infty$ $1 \leq i \leq n$.

More than 4 Days, your wasting time

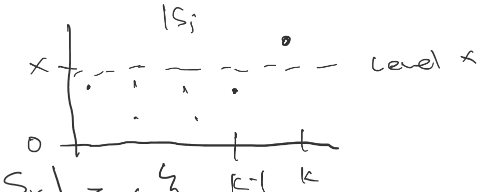
$\{X_i\}_{1 \le i \le n}$ IND. centered. $E(X_i) = 0$, $E(X_i^2) < \infty$ $1 \le i \le n$.

Then $P(\max_{1 \le k \le n} |S_n| \ge x) \le \dfrac{E(S_n^2)}{x^2}$ $(E(S_k) = 0,\ E(S_k^2) = Var(S_k),\ 1 \le k \le n)$

stronger than chebychev. $S_k = \sum_{i=1}^{k} X_i$

Proof: $A_k = \{|S_k| \ge x,\ |S_j| < x,\ j = 1, \dots, k-1\}$ $k = 1, \dots, n$

First time cross.



$A_{k_1} \cap A_{k_2} = \phi$ $\bigcup_{k=1}^{n} A_k = \{\max_{1 \le k \le n} |S_k| \ge x\}$

$1 \le k_1 \ne k_2 \le n$

$$E(S_n^2) \ge \sum_{k=1}^{n} \int_{A_k} S_n^2 \, dP = x^2 \sum_{k=1}^{n} P(A_k) = x^2 P(\max_{1 \le k \le n} |S_n| \ge x)$$

Because E integrate over All P-space

$S_n = S_k + (S_n - S_k)$

$S_n^2 = S_k^2 + 2 S_k (S_n - S_k) + (S_n - S_k)^2$

$$\int_{A_k} S_n^2 \ge \int_{A_k} S_k^2 + \underbrace{2 \int_{A_k} S_k (S_n - S_k)}_{0 \text{ By } \mathrm{I}} + \cancel{\int_{A_k} (S_n - S_k)^2}$$

Measureable

$A_k \in \sigma\{X_1, \dots X_k\}$

$S_k \cdot \mathbb{1}_{A_k} \in \sigma\{X_1, \dots X_k\}$

$S_n - S_k \in \sigma\{X_{k+1}, \dots, X_n\}$;

$$\int_{A_k} S_k (S_n - S_k) \, dp = E([S_k \cdot \mathbb{1}_{A_k}][S_n - S_k]) \Rightarrow S_k \mathbb{1}_{A_k} \perp\!\!\!\perp S_n - S_k$$

$$E(S_n - S_k) = E\left(\sum_{i=k+1}^{n} X_i\right) = 0$$

By

$$= E([S_k \cdot \mathbb{1}_{A_k}][S_n - S_k])$$

$$= E(S_k \mathbb{1}_{A_k}) E(S_n - S_k)$$

$$= 0 \quad \mathrm{I}$$

Paul Levy. - Giant of Prob.

Levy Maximal inequality:

$\{X_k\}_{1 \le k \le n}$ IND. $X_k \overset{\mathcal{D}}{=} -X_k$ $1 \le k \le n$.

symmetric

$X_k \overset{\mathcal{D}}{=} -X_k$, $1 \le k \le n$.

$P(X_k > 0) = P(X_k < 0)$

$P(X \ge 0) = P(X > 0) + P(X = 0) \ge 1/2$

$\{X_k\}_{1 \le k \le n}$ IND. $X_k \overset{\mathcal{D}}{=} -X_k \quad 1 \le k \le n.$

$P(X \ge 0) = P(X > 0) + P(X = 0) \ge 1/2$

By Proof By contradiction.

Claim $S_k \overset{\mathcal{D}}{=} -S_k$ is symetric,

$$\sum_{i=1}^{k} X_i \overset{\mathcal{D}}{=} -\sum_{i=1}^{k} X_i = \sum_{i=1}^{k} (-X_i)$$

"Say statement has no moments"

$\forall\ t > 0$ we have (1) $P(\max_{1 \le k \le n} S_k \ge t) \le 2 \cdot P(S_n \ge t)$

Also (2) $P(\max_{1 \le k \le n} |S_n| \ge t) \le 2 P(|S_n| \ge t)$

not Disjoint.

Proof (2) $\Rightarrow$ (1) $\quad \{\max_{1 \le k \le n} |S_n| \ge t\} = \{\max_{1 \le k \le n} \{S_k\} \ge t\} \cup \{\min_{1 \le k \le n} \{S_k\} \le -t\}$

$$P(\max_{1 \le k \le n} |S_k| \ge t) \le 2P(\{\max_{1 \le k \le n} |S_k| \ge t\})$$

$t > 0$

$$P(\max_{1 \le k \le n} S_k \ge t) = P\left[\bigcup_{k=1}^{n} \{\max_{i \le j \le k-1} S_j < t,\ S_k \ge t\}\right]$$

Replace By Bigger

$$\frac{1}{2} \cdot P(\max_{1 \le k \le n} S_k \ge t) = \sum_{k=1}^{n} P(\max_{i \le j \le k-1}\{S_j\} < t \quad S_k < t)\ (\tfrac{1}{2})$$

$P(S_n - S_k > 0) \ge 1/2$

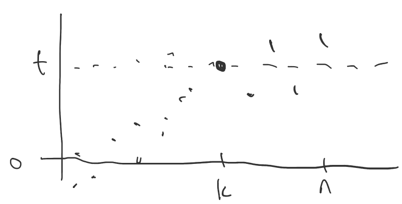
Disjoint,

$$\frac{1}{2} \cdot P(\max_{1 \le k \le n} S_k \ge t) \le \sum_{k=1}^{n} P(\max_{i \le j \le k-1}\{S_j\} < t,\ S_k < t,\ S_n - S_k \ge 0)\ P(S_n - S_k > 0)$$

$\sigma\{0$

$$\frac{1}{2} P(\max_{1 \le k \le n}\{S_k\} \ge t) \le P(S_n \ge t)$$

these Imply, $S_n \ge t$.



Kolgomorov 3 series Next.