Preliminary Exam: Probability, August 2025.
Modality: In-person.
Time: 10:00am - 3:00pm, Friday, August 22, 2025.
Place: **A102 Wells Hall.**

Your goal should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.
The exam consists of 6 main problems, each with several steps designed to help you with the overall solution.

Important: If you cannot solve a certain part of a problem, you still may use its conclusion in a later part!

Please make sure to apply the following guidelines:

1. On each page you turn in, write your assigned code number. Don't write your name on any page.

2. Start each problem on a new page.

Problem 1.

Let $N \sim \text{Poisson}(\lambda), \lambda > 0$.

a. Show the steps to calculate $\varphi_N(t) = E(e^{itN}), t \in \mathcal{R}$.

b. (i) Let $\{W, W_k\}_{k=1,2,...}$ be i.i.d. and assume that $\{W_k\}$ and N are independent. Let $X = \sum_{k=0}^{N} W_k$, $X = 0$ if $N = 0$. Calculate $E(e^{itX} | \sigma\{N\})$.

(ii) Calculate $\varphi_X(t) = E(e^{itX}), t \in \mathcal{R}$.

c. For each $n = 1, 2, \ldots$: $N_n \sim Poisson\ (n)$, $\{ W_{n,k}\}_{k \geq 1}$ are i.i.d, and assume that N_n and $\{ W_{n,k}\}_{k \geq 1}$ are independent. The distribution of $W_{n,1}$ is given by $P\left(W_{n,1} = \frac{1}{\sqrt{n}}\right) = \frac{1}{2} = \left(W_{n,1} = -\frac{1}{\sqrt{n}}\right)$. Finally, let $X_n = \sum_{k=0}^{N_n} W_{n,k}$.

(i) Calculate $\varphi_{X_n}(t) = E(e^{itX_n}), t \in \mathcal{R}$.

(ii) Prove that X_n converge in distribution as $n \to \infty$, and identify the limit distribution.

Problem 2.

a. (i) Let X be a symmetric random variable (i.e. $X = -X$ in distribution) with $P(X = 0) = 0$. Prove that $X = \epsilon \cdot |X|$, a.s., where $|X|, \epsilon$ are independent, and $P(\epsilon = 1) = \frac{1}{2} = P(\epsilon = -1)$.

(ii) Let X, Y be symmetric, independent, and squared integrable random variables with $P(X = 0) = P(Y = 0) = 0$. Let $\mathcal{H}$ be the σ-algebra $\mathcal{H} = \sigma\{|X|, |Y|\} = \sigma\{X^2, Y^2\}$. Prove that $E_{\mathcal{H}}(XY) = 0$. Also, calculate $E_{\mathcal{H}}[(X+Y)^2]$.

b. Let $\{B_t : t \geq 0\}$ is standard Brownian motion. Let $Q_n = \sum_{k=1}^{n} D_{k,n}^2$, where $D_{k,n} = B_{t_{k,n}} - B_{t_{k-1,n}}$, $\Delta_{n,k} = t_{k,n} - t_{k-1,n}$, $1 \leq k \leq n$, and $0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = 1$, $n = 1, 2, \ldots$.

(i) Calculate: $E(Q_n)$ and $Var(Q_n)$.

(ii) Denote $\Delta_n = \max_{1 \leq k \leq n} \{\Delta_{k,n}\}$. Prove that:
If $\Delta_n \xrightarrow[n\to\infty]{} 0$ then $Q_n \xrightarrow[n\to\infty]{} 1$ in probability.

c. We continue with the setup of part b. Assume that $\{t_{k,n}\}_{0 \leq k \leq n} \subset \{t_{k,n+1}\}_{0 \leq k \leq n+1}$, $n = 1, 2, \ldots$, where $\{t_{k,n+1}\}_{0 \leq k \leq n+1}$ is a refinement of $\{t_{k,n}\}_{0 \leq k \leq n}$ by an addition of one point.

Let $\{\mathcal{H}_n\}_{n=1,2,\ldots}$ be a filtration defined by $\mathcal{H}_n = \sigma(\cup_{m=n}^{\infty}\{D_{k,m}^2, 1 \leq k \leq m\})$. Observe that the sequence of σ-algebras $\{\mathcal{H}_n\}_{n=1,2,\ldots}$ is decreasing in n.

(i) Find $E_{\mathcal{H}_{n+1}}(Q_n)$. Which type of process is $\{Q_n, \mathcal{H}_n\}, n \geq 1$?

Hint: The assumption $\{t_{k,n}\}_{0 \leq k \leq n} \subset \{t_{k,n+1}\}_{0 \leq k \leq n+1}$ implies that all the intervals $\{(t_{k-1,n}, t_{k,n}] : 1 \leq k \leq n\}$ are contained in $\{(t_{k-1,n+1}, t_{k,n+1}] : 1 \leq k \leq n+1\}$ **except** one denoted by $(t_{k_*,n}, t_{k_*+1,n}]$, $0 \leq k_* \leq n-1$ that satisfies

$$(t_{k_*,n}, t_{k_*+1,n}] = (t_{k_*,n+1}, t_{k_*+1,n+1}] \cup (t_{k_*+1,n+1}, t_{k_*+2,n+1}].$$

(ii) Prove that if $\Delta_n \xrightarrow[n\to\infty]{} 0$ then $Q_n \xrightarrow[n\to\infty]{} 1$, almost surely.

Problem 3

Let $(X, Y) \in \mathbb{R}^2$be a random vector. The characteristic function (c.f.) of (X, Y) is defined as

$\varphi_{(X,Y)}(s,t) = E\left(e^{i(sX+tY)}\right), (s,t) \in \mathbb{R}^2$. It is known that if the characteristics functions of the two random vectors (X, Y) and (U, V) are identical then $(X, Y) = (U, V)$ in distribution, where $(U, V) \in \mathbb{R}^2$

a. Let $\varphi_X(s), \varphi_Y(t), s, t \in \mathbb{R}$ be the cf. of X, Y, respectively. Prove that

(i) If X, Y are independent, then $\varphi_{(X,Y)}(s,t) = \varphi_X(s) \cdot \varphi_Y(t), s, t \in \mathbb{R}$.

(ii) I f $\varphi_{(X,Y)}(s,t) = \varphi_X(s) \cdot \varphi_Y(t), s, t \in \mathbb{R}$ then X, Y are independent.

b. Let $X \in \mathbb{R}^3$ satisfy $X = \sum_{m=1}^{n} X_m$, where $\{X_m \in \mathbb{R}^3 : m = 1, \ldots, n\}$ are i.i.d. The distribution of $X_1 \in \mathbb{R}^3$ is given by $P(X_1 = \varepsilon_k) = p_k, k = 1, 2, 3$, where $\sum_{k=1}^{3} p_k = 1, p_k \geq 0$, $\varepsilon_1 = (1,0,0), \varepsilon_2 = (0,1,0)$, and $\varepsilon_3 = (0,0,0)$.

(i) Find the c.f. of X_1 , namely $\varphi_{X_1}(t) = E\left(e^{it\cdot X_1}\right), t = (t_1, t_2, t_3) \in \mathbb{R}^3$,
$X_1 = (X_{1,1}, X_{1,2}, X_{1,3})$ and $t \cdot X_1 = \sum_{k=1}^{3} t_k X_{1,k}$,
where $X_{1,k} \in \mathbb{R}$, $k = 1, 2, 3$.
Also, how is $X_{1,3}$ distributed?

(ii) Find the c.f. of X, namely $\varphi_X(t) = E\left(e^{it\cdot X}\right), t \in \mathbb{R}^d$.

c. Let $X_n \in \mathbb{R}^3$ satisfy $X_n = \sum_{m=1}^{n} X_{n,m}$, $n = 1, 2, \ldots$ where $\{X_{n,m} \in \mathbb{R}^3 : m = 1, \ldots, n\}$ are i.i.d. The distribution of $X_{n,1} \in \mathbb{R}^3$ is given by $P(X_{n,1} = \varepsilon_k) = p_{n,k}, k = 1, 2, 3$, where $\sum_{k=1}^{3} p_{n,k} = 1, p_{n,k} \geq 0$, $\{\varepsilon_k, k = 1, 2, 3\}$ are as in part b.
Assumption: $n \cdot p_{n,k} \xrightarrow[n\to\infty]{} \lambda_k$ and $0 < \lambda_k < \infty, k = 1, 2$.

Hint: The assumption does not hold for $k = 3$.

(i) Prove that $\varphi_{X_n}(t) \xrightarrow[n\to\infty]{} \varphi_{X_\infty}(t), t \in \mathbb{R}^3$, where $X_\infty \in \mathbb{R}^3$ is a random vector.
Hint. $\sum_{k=1}^{3} a_k p_{n,k} = 1 + \sum_{k=1}^{2} (a_k - 1) p_{n,k}$ if $a_3 = 1$.

(ii) What is the relationship between the first 2 coordinates of X_∞? What is the distribution of each of the 3 coordinates of X_∞? Justify your answer.

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Problem 4

Let X be a random variable that satisfy $\sum_{k=1}^{\infty} P(|X| \geq a_k) < \infty$, where $\{a_k\}_{k\geq 1}$ is a **non-negative** and **non-decreasing** sequence of real numbers that satisfy

(i) $a_k \xrightarrow[k\to\infty]{} \infty$, and

(ii) there exists $C < \infty$ so that for each $n \geq 1$, $\sum_{k=n}^{\infty} a_k^{-2} \leq C n a_n^{-2}$.

Let $\{X, X_k\}_{k\geq 1}$ be an i.i.d. sequence of random variables, and denote

$Y_k = X_k \cdot 1_{\{|X_k| \leq a_k\}}$, $k \geq 1$. Prove the following:

a. (i) $\frac{\sum_{k=1}^{n} X_k \cdot 1_{\{|X_k| > a_k\}}}{a_n} \xrightarrow[n\to\infty]{} 0$, a.s.

(ii) If $\frac{\sum_{k=1}^{n} Y_k - E(Y_k)}{a_n} \xrightarrow[n\to\infty]{} 0$, a.s. , then $\frac{\sum_{k=1}^{n} X_k - E(Y_k)}{a_n} \xrightarrow[n\to\infty]{} 0$, a.s.

b. (i) Let $a_0 = 0$. Prove: $\sum_{n=1}^{\infty} a_n^{-2} E(Y_n^2) = \sum_{k=1}^{\infty} E(X^2 \cdot 1_{\{a_{k-1} < |X| \leq a_k\}}) \cdot (\sum_{n=k}^{\infty} a_n^{-2})$.

Hint: $(0, a_n] = \cup_{k=1}^{n} (a_{k-1}, a_k]$

(ii) Use the result of part (i) to prove that

$\sum_{n=1}^{\infty} a_n^{-2} E(Y_n^2) \leq C \sum_{k=1}^{\infty} k P(a_{k-1} < |X| \leq a_k)$

c. (i) $\sum_{k=1}^{\infty} \frac{Y_k - E(Y_k)}{a_k}$ converge a.s.

Hint: $\sum_{k=1}^{\infty} k P(a_{k-1} < |X| \leq a_k) = \sum_{k=0}^{\infty} P(|X| > a_k)$ by summation by parts.

(ii) $\frac{\sum_{k=1}^{n} X_k - E(Y_k)}{a_n} \xrightarrow[n\to\infty]{} 0$, a.s.

Problem 5.

Let $\{X_k, \mathcal{F}_k\}_{k=0,1,\dots}$ be a supermartingale sequence of random variables where $X_0 = 1$. Next, we define a sequence of stopping times:

$$T_{-1} = T_0 = 0, T_1 = \inf\{m > 0: X_m \leq 0\},$$
$$T_{2k} = \inf\{m > T_{2k-1}: X_m \geq 1\}, k \geq 1, \text{ and}$$
$$T_{2k+1} = \inf\{m > T_{2k}: X_m \leq 0\}, k \geq 1.$$

Each interval $[T_{2k}, T_{2k+1}], k \geq 0$ represents a down-crossing of the interval $[0, 1]$ by $\{X_k\}_{k=0,1,\dots}$ as $X_{T_{2k}} \geq 1, X_m > 0$ if $T_{2k} < m < T_{2k+1}, and\ X_{T_{2k+1}} \leq 0$.

For each $n \geq 0$ let $D_n(X)$ denotes the number of **down-crossings** of the interval $[0, 1]$ by $\{X_k\}, 0 \leq k \leq n$. A formal definition of $D_n(X)$ is

$D_n(X) = \max\{k \geq 1: T_{2k-1} \leq n\}$, ($D_n(X) = 0$ if no such k exists.)

Prove:

a. (i) Let $Y_k = \min\{X_k, 1\}, k \geq 0$. Then $\{Y_k, \mathcal{F}_k\}_{k=0,1,\dots}$ is a supermartingale, and $D_n(X) = D_n(Y)$, a.s. where $D_n(Y)$ denotes the number of down-crossings of the interval $[0, 1]$ by $\{Y_k\}, 0 \leq k \leq n$.

For the rest of the problem, we assume that $X_k = Y_k$, namely $X_k \leq 1, k \geq 0$.

(ii) Is $T_{2D_n - 1}$ a stopping time ? explain.

b. For a, b integers denote $[a, b) = \{a, a + 1, \dots, b - 1\}, a < b$. For each $m \geq 1, \omega \in \Omega$ we define the random variable H_m by

$$\begin{cases} H_m(\omega) = 1, & \text{if there is k} \geq 0 \text{ so that } m - 1 \in [T_{2k}(\omega), T_{2k+1}(\omega)). \\ H_m(\omega) = 0, & \text{otherwise.} \end{cases}$$

Prove the following:

(i) $\{H_k, \mathcal{F}_k\}_{k\geq 1}$ is predictable, i.e. $H_m \in \mathcal{F}_{m-1}, m \geq 1$.
Hint: Prove first that the event $\{m - 1 \in [T_{2k}, T_{2k+1})\}$ belongs to $\mathcal{F}_{m-1}$ for each $k \geq 0$.

(ii) The gambling systems $(H \cdot X)_{n\geq 0}$ and $((1 - H) \cdot X)_{n\geq 0}$ are both supermartingales. Recall that a gambling system is defined by $(H \cdot X)_n = \sum_{m=1}^{n} H_m(X_m - X_{m-1}), n \geq 1$, and $(H \cdot X)_0 = 0$.

c. (i) $-D_n \geq (H \cdot X)_{T_{2D_n - 1}}$, a.s. Also, $(H \cdot X)_{T_{2D_n-1}} \geq (H \cdot X)_n$, a.s.

(ii) $E(D_n) \leq E(X_0 - X_n) = 1 - E(X_n)$.
Hint. Find the smallest upper bound that you can for $E((1 - H) \cdot X)_n$.
Also, look at $E[((1 - H) \cdot X)_n]$ +E$[(H \cdot X)_n]$.

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Problem 6.

Let $\{X_k\}_{k=1,2,...}$ be uncorrelated random variables. Also, there exists a constant $0 < M < \infty$ so that $0 < X_k < M$ a.s., $k \geq 1$. Assume that $\sum_{k=1}^{\infty} E(X_k) = \infty$.

Denote $S_n = \sum_{k=1}^{n} X_k, n \geq 1$. The goal of the problem is to prove that $\frac{S_n}{E(S_n)} \xrightarrow[n\to\infty]{} 1$, a.s. by proving the following steps:

a. (i) If the result holds for $M = 1$, then the result holds for any $0 < M < \infty$.

For the rest of the problem, we assume without loss of generality that $M = 1$.

(ii) $Var(S_n) \leq E(S_n), n \geq 1$. Use this to show that $\frac{S_n}{E(S_n)} \xrightarrow[n\to\infty]{} 1$ in probability.

b. Let $\{a_k\}_{k=1,2,...}$ be a strictly increasing sequence of positive integers so that $a_k \xrightarrow[k\to\infty]{} \infty$, and $\frac{a_{k+1}}{a_k} \xrightarrow[k\to\infty]{} 1$. Let $\{n_k\}_{k=1,2,...}$ be a strictly increasing sequence of positive integers so that

$$a_k \leq E(S_{n_k}) \leq 1 + a_k, k \geq 1.$$

If $\frac{S_{n_k}}{E(S_{n_k})} \xrightarrow[k\to\infty]{} 1$, a.s. then $\frac{S_n}{E(S_n)} \xrightarrow[n\to\infty]{} 1$, a.s.

Hint: show first that $\frac{E(S_{n_{(k+1)}})}{E(S_{n_k})} \xrightarrow[k\to\infty]{} 1$. Then work with $n_k \leq n < n_{k+1}$.

c. (i) We continue with the notations of part b. Let $\delta > 0$. Then for each $k \geq 1$
$P(|S_{n_k} - E(S_{n_k})| > \delta(1 + a_k)) \leq \frac{1}{\delta^2(1+a_k)}$

(ii) Let $a_k = k^3, k \geq 1$. Prove that $\frac{S_n}{E(S_n)} \xrightarrow[n\to\infty]{} 1$, a.s.