L31 - 11-08 Convergence In Distribution

Friday, November 8, 2024 10:24 AM

① Def : $X_n \Rightarrow X$ (convergence in Dist.) if $P(X_n \le x) \xrightarrow{n\to\infty} P(X \le x)$, $\forall x \in \mathbb{R}$ s.t. $P(X = x) = 0 \Leftrightarrow F_X(x-) = F_X(x)$

Def ② $X_n \Rightarrow X$ if $Eg(X_n) \xrightarrow[n\to\infty]{} Eg(X)$, $g \in C_B(\mathbb{R})$

RCLL — Right continuous left limit.

$C_B(\mathbb{R})$ Continuous Bounded on Real line

<u>theorem A</u> if $X_n \Rightarrow X$ (By Def 1) then, $\exists (\Omega, \mathcal{F}, P)$ (Prob space) and r.v.s Y_n, Y s.t.

$Y_n \overset{\mathcal{D}}{=} X_n$, $n > 1$, $Y \overset{\mathcal{D}}{=} X$ and $Y_n \xrightarrow[n\to\infty]{as} Y$

then to go from Def ① to Def ② use theorem A

Assume ①, we use the theorem

$$Eg(Y_n) \xrightarrow[n\to\infty]{} Eg(Y) \quad \text{“this is Enough”}$$

$Y_n \xrightarrow[n\to\infty]{a.s.} Y$ so we conclude that $g(Y_n) \xrightarrow{a.s.} g(Y)$ By Continuity.

by DCT $Eg(Y_n) \xrightarrow[n\to\infty]{} E(g(Y))$

$|g(x)| \le C < \infty$, $x \in \mathbb{R}$.

we get ②

Let Y be a r.v. and $F(y) = P(Y \le y)$, $y \in \mathbb{R}$ (if is nice.)

We need $F^{-1}(p)$, $0 \le p \le 1$

Def: $F^{-1}(p) = \sup\{y : F(y) < p\}$ $0 \le p \le 1$

<u>step 1</u>
Claim Let $U \sim \text{Uniform}(0,1)$

$F^{-1}(U) \overset{\mathcal{D}}{=} Y$ ← Result. $\qquad P(U \le x) = x$, $0 \le x \le 1$

$P(F^{-1}(u) < y) = P(u < F(y)) = F(y) = P(Y \le y)$, $y \in \mathbb{R}$

<u>step 2</u> if $F_n(y) \xrightarrow[n\to\infty]{} F(y)$ (CDF), $F(y) = F(y-)$ For All y where no Jump

then $F_n^{-1}(u) \xrightarrow[n\to\infty]{} F^{-1}(u)$, $0 < u < 1$

$F_n^{-1}(U)$

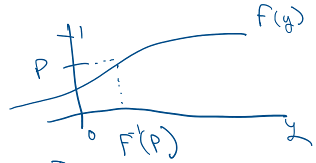
$U \sim \text{uniform}(0,1)$: $F_n^{-1}(U) \xrightarrow[n\to\infty]{a.s.} F^{-1}(U)$

$$\underset{Y_n}{\overset{\mathcal{D}}{\parallel}} \xrightarrow[n\to\infty]{as} \underset{Y}{\overset{\mathcal{D}}{\parallel}}$$

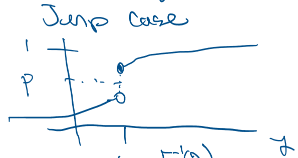
Proof from Def ② to Def ①

three cases

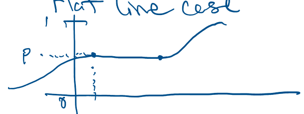
nice case CDF.



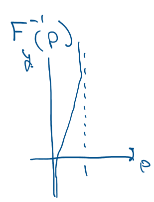
Jump case



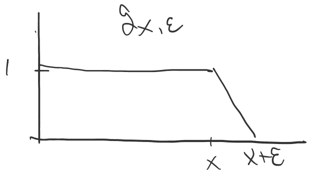
Flat line case



The First point



take $x \in \mathbb{R}$ s.t. $F_X(x) = F_X(x-)$ $\quad (P(X = x) = 0)$



$$P(X_n \le x) \le E\, g_{x,\varepsilon}(X_n) \overset{\swarrow \text{ integral}}{} \cancel{\le P(X \le x+\varepsilon)} \xrightarrow[n\to\infty]{} E\, g_{x,\varepsilon}(X) \le P(X \le x+\varepsilon)$$

$$P(X_n \le x) \le \int_0^{x+\varepsilon} g_{x,\varepsilon}(y)\, dF_{X_n}(y)$$

Note

$$1 = \int_0^{x} g_{x,\varepsilon}(y)\, dF_X(y)$$

$$\boxed{\overline{\lim_{n\to\infty}}\, P(X_n \le x) \le P(X \le x \cancel{+\varepsilon})}$$

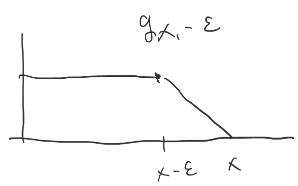
$$P(X \le x + \varepsilon) \xrightarrow[\varepsilon \to 0]{} P(X \le x)$$

$P(X = x) = 0.$

...

we need $\underline{\lim_{n\to\infty}}\, P(X_n \ge x) \ge P(X < x) \uparrow P(X \le x)$

$\underline{\lim} = \overline{\lim}$ the equality.



Both $g_{x,\varepsilon}$ $g_{x,-\varepsilon}$ Are uniformly continuous.

Continuity. F is continuous at x

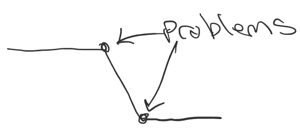
$\forall \varepsilon > 0$, $\exists \delta < 0$ s.t. $|f(x+u) - f(x)| < \varepsilon$, $|u| < \delta(x)$

uniformly continuous mean use Any $0 < \delta \cancel{>} \delta(x)$, $x \in \mathbb{R}$ $\le$

Enough to check $\overset{\text{Def}}{②}$ for g Bounded & uniformly Cont.

Also have infinite Derivatives.

Says



Examples ① $X_n \equiv a_n$ (constant), $X = a$

$X_n \Rightarrow$ iff $a_n \xrightarrow[n\to\infty]{} a$

② $X_n \xrightarrow[n\to\infty]{a.s.} X$ then we have Also. $X_n \Rightarrow X$

③ $Y_n \sim \text{Geometric}\left(\frac{\lambda}{n}\right)$, $n \ge 1$

$$\frac{Y_n}{n} \Longrightarrow Y \sim \exp(\lambda)$$

$Z \sim \text{Geometric}(p)$, $0 < p < 1$

$P(Z > k) = (1-p)^k$, $k = 1, 2 \ldots$

$P(\frac{Y_n}{n} > t) = P(Y_n > nt) = \left[\left(1 - \frac{\lambda}{n}\right)^n\right]^t \xrightarrow[n\to\infty]{} e^{-\lambda t} = P(Y > t)$, $t \geq 0$

Example 4 uniform on N.

$\{X_i\}_{i\geq 1}$ iid $P(X_i = k) = \frac{1}{N}$, $k = 1, 2, \ldots, N$

First time there is a tie

$T_N = \min\{n \geq 2 : \exists\, i < n \text{ with } X_i = X_n\}$.

$P(T_N > n) = \dfrac{N(N-1)\cdots(N-n-1)}{N^n} = \prod_{m=2}^{n}\left(1 - \frac{m-1}{N}\right)$

$n > N$ tie for sure

$2+3+4\ldots$

Look at $X_1, \ldots X_n$

we are interested in $P\left(\frac{T_N}{\sqrt{N}} > y\right) = P(T_n > \sqrt{N}\cdot y) = \prod_{m=2}^{\sqrt{N}y}\left(1 - \frac{m-1}{N}\right) \xrightarrow[N\to\infty]{} e^{-\frac{y^2}{2}}$

$y \geq 0$

$P\left(\frac{T_N}{\sqrt{N}} \leq y\right) \xrightarrow[y\to\infty]{} 1 - e^{-\frac{y^2}{2}}$ $y \geq 0$