02-12 upcoarse, MGCT

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Doob's.
Up coarsing Lemma

Let $\{X_n, \mathcal{F}_n\}_{n \geq 0}$ sub MG. $a < b$ fixed.

Define sequence of stop. times.

$$\tau_0 < \tau_1 < \tau_2 \qquad \begin{cases} \tau_0 = 1 \\ T_{2k-1} = \inf\{n > T_{k-2} ; X_m \leq a\} & k \geq 1 \\ T_{2k} = \inf\{m > T_{2k-1} ; X_m \geq b\}, \end{cases}$$

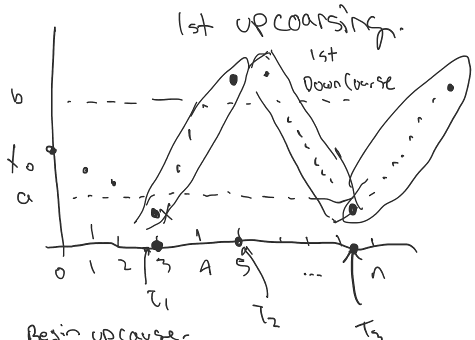
$U_n^{a,b} = \sup\{k : T_{2k} \leq n\}$ # of upcoarsing of $\{X_k\}$ of $[a,b]$. obsen.

then $(b-a)E(U_n^{a,b}) \leq E(X_n - a)^+ - \underbrace{E(X_0 - a)^+}_{\text{Drop it}} \leq E(X_n^+) + |a|$

subMG Apply +  still subMG.

1st upcoarsing. 2nd upcoarsing.

Doob inequality



we can finish with
- Downcoarsing.
- upcoarse } incomplete.

Begin upcause. End upcoarse. stopping time.

$$H_m = \begin{cases} 1 & \text{if } T_{2k-1} < m < T_{2k} \\ 0 & \text{otherwise} \end{cases}$$

complement $\{\tau_{2k} < m-1\} \in \mathcal{F}_n$

Formal Proof $H_m = \sum_{k \geq 1} \mathbb{1}_{\{T_{2k} \geq m\}} - \mathbb{1}_{\{T_{2k-1} > m\}}$.

H_m is known at $m-1$, "predictable"

if we are between τ_1 τ_2
During upcoarsing. this equal 1.

$(b-a)U_n^{a,b} \leq (H \cdot X)_n$

issue if we Down course, we reenter market.

So Define $Y_k = a + (X_k - a)^+ = \begin{cases} X_k & \text{if } X_k > a. \\ a & \text{if } X_k \le a. \end{cases}$

$U_n^{a,b}(X) = U_n^{a,b}(Y)$ and $\{Y_k, \mathcal{F}_k\}_{k \ge 0}$ is sub MG as well.

$\therefore$

$(b-a) U_n^{a,b} \le (H \cdot Y)_n$ a.s.

Define $K_m = 1 - H_m$. $K_m \in \mathcal{F}_{m-1}$ ← Predictable. , $m \ge 1$

$(H \cdot Y)_n + (K \cdot Y)_n = Y_n - Y_0$ a.s.

$H_m + K_m = 1$. $\{(K \cdot Y)_m, \mathcal{F}_m\}$ sub MG.

$(b-a) E[U_n^{a,b}] \le E[H \cdot Y]_n + E[K \cdot Y)_n = E[Y_n - Y_0]$ $\boxtimes$

$E(K \cdot Y)_n \ge 0$
$(K \cdot Y)_0 = 0$

Martingale Convergence theorem

Let $\{X_n, \mathcal{F}_n\}_{n \ge 0}$ be sub MG (super MG)

Assume $\sup_n E(X_n^+) < \infty$ $\left(\sup_n \{E(X_n^-)\} < \infty \right)$

then ① $X_n \xrightarrow[n \to \infty]{a.s.} X$, ② $E|X| < \infty$.

EX1 $S_n = \sum_{k=1}^n \varepsilon_k$ $n \ge 1$ $\{\varepsilon_k\}_{k \ge 1}$ iid $P(\varepsilon = \pm 1) = 1/2$

$S_0 = 0$

$T = \inf\{n : S_n = 1\}$ At most 1

observe $\{S_{T \wedge n}\}_{n \ge 0}$ is sub MG. $\sup_n \{E(S_{T \wedge n}^+)\} \le 1$

By MGCT: $S_{T \wedge n} \xrightarrow[n \to \infty]{a.s.} S$ claim: $T < \infty$ a.s.

$\xrightarrow[n \to \infty]{a.s.} 1$

$$\xrightarrow[n\to\infty]{a.s.} 1$$

However $EX_n \to X$? $\{S_{T\wedge n}\}_{n\ge 0}$ is not UI.

$$E(S_{T\wedge n}) = 0 \quad \forall n$$

$$E(S) = E(1) = 1$$

Example 2. $\{Z_k\}_{1,2,\ldots}$ iid $N(0,1)$ $\mathcal{F}_n$ is natural filtration $\sigma\{Z_1, \ldots, Z_n\}$ $n \ge 1$

$$S_n = \sum_{k=1}^{n} Z_k, \quad n \ge 1 \qquad S_0 = 0.$$

$$Y_n = e^{S_n - \frac{n}{2}} \quad n \ge 0$$

Note: $E(e^{Z}) = e^{(1/2)s^2} = e^{1/2}$

$$E_{\mathcal{F}_n}(Y_n) = E\left(\left(e^{S_{n-1} - \frac{n-1}{2}}\right)\left(e^{Z_n - 1/2}\right)\right)$$

$$= Y_{n-1} \cancel{E}\left(e^{Z_n - \frac{1}{2}}\right) \to 1$$

$\{Y_n, \mathcal{F}_n\}_{n \ge 1}$ is MG.

$Y_n \ge 0$ By MGCT $\Rightarrow$ $Y_n \xrightarrow[n\to\infty]{a.s.} Y$

$E(Y_n) = 1$, $n \ge 1$

$$Y_n = e^{S_n - \frac{n}{2}} = e^{n\left(\frac{S_n}{n} - 1/2\right)}$$

By WLLN. $\frac{S_n}{n} \xrightarrow{P} 0$

$\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s.} 0$ By SLLN.

$\{Y_n\}_{n \ge 1}$ is not UI