04-02

Wednesday, April 2, 2025 11:30 AM

Last time: Levy models of continuity.

Diadic Rationals Q_2 $n \to \left\{\frac{k}{2^n}\right\}$ $k = 0, \dots, 2^n$

Last step: if $n > N(\omega)$ then:

$s, t \in [0,1]$ $(s, t) < 2^{-n}$ $\Rightarrow$ (By BC) $|B_t - B_s| \leq 3(bn)^{1/2} 2^{-n/2}$

oscillation

where $b = 2(1+\varepsilon)\ln(2)$, $\varepsilon > 0$

$OSC(\delta) \equiv \sup\{|B_t - B_s| ; |t-s| < \delta, \; t, s \in Q_2, \; 0 \leq s, t \leq 1\}$

Conclusion: if $n > N(\omega)$ $2^{-n+1} <$

Then $OSC(\delta) \leq 3(bn)^{1/2} 2^{-n/2}$

$\leq 3\left[b \log_2\left(\frac{1}{\delta}\right)\right]^{1/2} \cdot (2\delta)^{1/2}$

$= \delta\left[(1+\varepsilon)\delta \log\left(\frac{1}{\delta}\right)\right]^{1/2}$

??? note: $\log \frac{1}{\delta} < n$

$-\log \delta < n$

$\log \delta > n$

Final Result:

$$\overline{\lim_{\delta \to 0}} \frac{OSC(\delta)}{\sqrt{\delta \log(\frac{1}{\delta})}} \leq \delta \sqrt{1 + \cancel{\varepsilon}}, \quad a.s.$$

Abitrary ε.

$= \delta$

talking About small #'s.

then we get:

$$|B_t - B_s| \leq \delta |t-s|^{1/2} \left(\log\left(\frac{1}{|t-s|}\right)\right)^{1/2} \qquad |t-s| < \delta(\omega).$$

Define for Diriac. Rationals. but say "Now we Extend to All Rational"

$\hookrightarrow Q_2$

By uniform continuity.

Define $B(t) = \lim_{\substack{t_n \to t \\ t_n \in Q_2}} B(t_n)$ Limit exists or violate

Another way to create Brownian motion. through Reproducing Hilbert space Kernel.

Another Construction of SBM.

$0 \leq t \leq 1$

$B(t) = \sum_{n=0}^{\infty} S_n(t) Z_n$, $\{Z_n\}$ iid $N(0,1)$

where $\{S_n(t)\}_{0 \leq t \leq 1}$.

Hilbert space $H = L^2[0,1] = \{f : [0,1] \to \mathbb{R} : \|f\|^2 = \int_0^1 f^2(x)\,dx < \infty$

Inner Product

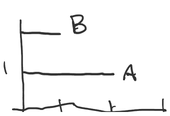
$\langle f, g \rangle = \int_0^1 f(x) g(x)\,dx$, $f, g \in H$

in Normal. All $\mathbb{R}^n$ space is describe A $MVN(\mu, \Sigma)$

$Cov(B_s, B_t) = s \wedge t$

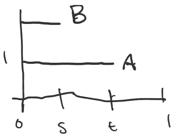
$$\int_0^{s \wedge t} 1\,dx = s \wedge t.$$

A $1_{[0,t]}(x) =: 1$



some how map

A $\quad 1_{[0,t]}(x) =: 1$



$\int_0 1\,dx = s \wedge t.$

some how map

$1_{[0,t]} \longrightarrow B(t)$

B $\quad 1[0,s]$

there exist $\{H_k\}_{1\le k}$ (infinite.), $H_k \in H$

such that $\|H_k\| = 1$, $k \geq 1$

$$\langle H_k, H_\ell \rangle = \begin{cases} 1 & \text{if } k=\ell \\ 0 & \text{if } k\neq \ell \end{cases}$$

Also if $\langle f, H_k\rangle = 0$ $k\geq 1$ then $f=0$ a.e. in $H = L^2[0,1]$

call $\{H_k\}_{k\geq 1}$ complete othonormal System (cos)

Conclusions: if $f \in H$, then $f \overset{L^2}{=} \underbrace{\sum_{k=1}^{\infty} \langle f, H_k\rangle H_k}_{A}$

L^2 implies $E[(f-A)^2] = 0$

Examples of Cos.

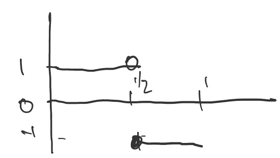
1) $\cos(2\pi k x)$, $\sin(2\pi k x)$, $k \geq 1$

we will use this.

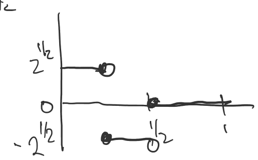
2) HAAR system $\quad (0 \leq x \leq 1)$

$H_0(x) \equiv 1$

$$H_1(x) = \begin{cases} 1, & 0 < x < \frac{1}{2} \\ -1, & 1/2 \leq x \leq 1 \end{cases} \longrightarrow$$

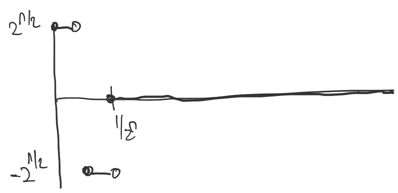


H_k



$$H_2(x) = \begin{cases} 2, & 0 \leq x \leq 1/4 \\ -2, & 1/4 \leq x < 1/2 \\ 0, & 1/2 < x \end{cases}$$

with $2^n \leq k < 2^{n+1}$



$\int_0^{1/2^n} 2^n\,dx = 1$

$$H_3(x) = \begin{cases} 0 & 0 < x < 1/2 \\ 2 & 1/2 \leq x < \frac{3}{4} \\ -2 & \frac{3}{4} < x < 1 \end{cases}$$

Type equation here.

$\langle H_i(x), H_j(x)\rangle = 0$

$$1_{[0,t]} \overset{L^2}{=} \sum_{k=0}^{\infty} \langle 1_{[0,t)}, H_k\rangle H_k.$$

$$1_{[0,t]} = \sum_{k=0}^{\infty} \left[\int_0^t H_k(x)\,dx\right] H_k$$

$S_k(t) \quad \times \quad S_k(s)$

$$\Big\langle \sum_{k=0}^{\infty}\Big[\int_0^t H_k(x)\,dx\Big] H_k ,\ \sum_{m=0}^{\infty}\Big(\int_0^s H_m(x)\,dx\Big) H_m \Big\rangle = \sum \Big(\int_0^t H_k(x)\,dx\Big)\Big(\int_0^s H_k(x)\,dx\Big)$$

$\downarrow \qquad\qquad \downarrow$

$1_{[0,t]} \qquad 1_{[0,s]} \quad = \quad t \wedge s.$

Percival equality - say we can Do term By ter...

$S_n(t) = \int_0^t H_n(x)\,dx, \quad 0 \leq t \leq 1, \quad n \geq 0,1,2,\ldots$

$\sum^{\infty} \langle\ (\ )\ \rangle \quad - \quad D(\ )$

$S_n(t) = \int_0^t H_n(x)\,dx, \quad 0 \le t \le 1, \quad n \ge 0, 1, 2, \dots$

$$\sum_{n=0}^{\infty} S_n(t) Z_n = B(t), \quad n \ge 0, \quad 0 \le t \le 1$$

constant For Fixed t.   Gaussian. iid.

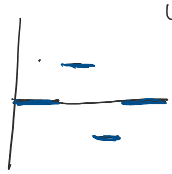
By kolmogorov thm from 881 kolgo AS

therefore Converges

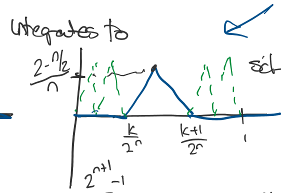
$$\|S_n\|^2 = \sum_{n=0}^{\infty} (S_n(t))^2 = t < 1$$

$H_k \longrightarrow Z_k$

where is the continuity?



the Beauty of It is The support is mutally Exclusive



$$\sum_{k=2^n}^{2^{n+1}-1} S_k(t) \le \frac{2^{-n/2}}{2}. \quad 0 \le t \le 1$$

$$\sum_{n=0}^{N} S_n(t) Z_n \in C[0,1]$$

$$|Z_k(\omega)| \le c(\omega)\sqrt{\log(k)} \quad k \ge 2$$

prove by tail.

$$\sum_k P(Z > 2\sqrt{\log k}) \lesssim \sum_k e^{-(\log(k))2} = \sum_k k^{-2} < \infty$$

STD normal