L18 - 10-07

Monday, October 7, 2024 11:33 AM

WLLN

$\forall \delta > 0 \ \exists M > 0$ s.t. $xP(|X| > x) < \delta$, $x > M$

Weak Law of Large numbers

Let $\{X, X_k\}_{k \geq 1}$ be iid Assume $x P(|X| > x) \xrightarrow[x \to \infty]{} 0$

Markov: $x P(|X| > x) \leq \cancel{x} \frac{E(|X|)}{\cancel{x}}$ can't use.

$$\underbrace{x \cdot 1_{\{|X| \geq x\}}}_{} \leq |X| \quad \text{a.s.}$$

$\downarrow$ a.s. $x \to \infty$ 0

Dominate convergence theorem

By DCT $E(X 1_{(X \geq x)}) \xrightarrow[x \to \infty]{} 0$

$\Rightarrow x P(X > x) \xrightarrow[x \to \infty]{} 0$

take $M_n = E(X; |X| \leq n)$, $n = 1, 2, \ldots$

then $\frac{S_n}{n} - M_n \xrightarrow[n \to \infty]{P} 0$, "truncation import tool"

Corallory: Khinching theorem

if $E|X| < \infty$ then $\frac{S_n}{n} \xrightarrow[n \to \infty]{P} E(X)$

Proof $M S_n M_n$ (2) $\xrightarrow[n \to \infty]{} E(X)$ is $E(X 1_{\{|X| \leq n\}}) \xrightarrow[n \to \infty]{} E(X)$

$\frac{S_n}{n} - M_n \xrightarrow[n \to \infty]{P} 0$ $\qquad |X| \cdot 1_{\{|X| \leq n\}} < |X|$

$\downarrow n \to \infty$

X

$M_n \xrightarrow[n \to \infty]{P} E(X)$

Partial sum $S_n = \sum_{k=1}^{n} X_k$ $n = 1, 2, \ldots$

Proof of WLLN

$X_{n,k} = X_k \cdot 1_{\{|X_k| \leq n\}}$, $k = 1, 2, \ldots, n$

$\{X_{n,k}\}_{1 \leq k \leq n}$ are iid, $X_{n,k} \overset{D}{=} X \cdot 1_{\{|X| \leq n\}}$

$S_n' = \sum_{k=1}^{n} X_{n,k}$ "need to cut ~~or~~ cant use chebychev"

$$P\left(\left|\frac{S_n}{n} - M_n\right| > \varepsilon\right) \leq P\left(\left|\frac{S_n}{n} - M_n\right|, \cancel{S_n = S_n'}\right) + P\left(\left|\cancel{\frac{S_n}{n} - M_n}\right| > \varepsilon, S_n \neq S_n'\right)$$

intersection

Remove to make larger.

$P(S_n \neq S_n') \leq P\left(\bigcup_{k=1}^{n} \{|X_k| > n\}\right) \leq P(|X| > n) \xrightarrow[n \to \infty]{} 0$ ← Given

All the same dist.

$P(S_n \neq S_n) = 0$

$$P\left(\left|\frac{S_n'}{n} - M_n\right| > \varepsilon\right) \leq \varepsilon^{-2} V\left(\frac{S_n'}{n}\right) = \varepsilon^{-2} \frac{n \cdot V(X_{n,1})}{n^2} \leq \frac{\varepsilon^{-2} E(X_{n,1}^2)}{n}$$

$$\frac{E(X_{n,1}^2)}{n} = \frac{E|X_{n,1}|^2}{n} = \int_0^\infty 2x P(|X_{n,1}| \geq x)\, dx$$

use Fubini

if $Y \geq 0$ then $E[Y^2] = \int_{y=0}^{\infty} y^2 \, dF_Y(y) = \int_{y=0}^{n} 2y P(Y \geq y)\, dx$

Which is bigger

$\overbrace{P(|X_{n,1}| > x) = P(|X| > x)}$ $- P(|X| > n)$ $x > 0$

$\leq \int_0^n \frac{2x P(|X| \geq x)\, dx}{n} \xrightarrow[n \to \infty]{} 0$ Again by the given $xP(|X| > x) \xrightarrow[x \to \infty]{} 0$

$$= \int_0^M \frac{2x\, P(|X| \geq x)\, dx}{n} + \int_M^n 2x P(|\bar{X}| \geq x)\, dx \qquad n \geq M$$

$$\leq \frac{2M^2}{n} \xrightarrow[n\to\infty]{} 0. \qquad \wedge\!\!| \quad \frac{\delta(n-M)}{n} \quad \text{The } \limsup_{n\to\infty}.$$

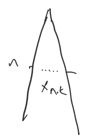
$$\hookrightarrow 1.$$

$$\limsup_{n\to\infty} \int_M^n \frac{2x P(|\bar{X}| \geq x)\, dx}{n} \leq \delta$$

We got $\limsup_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu_n\right| < \varepsilon\right) \leq \varepsilon^{-2}\delta$.

a Weak Law for triangle Arrays.

Proof is the same



$1 \leq k \leq n$ serves double Role.

WLLN for triangular arrays

$\{X_{n,k}\}_{k=1,\dots,n,\ n=1,2,\dots}$ IND for each n

Let $b_n > 0$, $n \geq$, $b_n \xrightarrow[n\to\infty]{} \infty$

Assume: (i) $\sum_{k=1}^{n} P(|X_{n,k}| > b_n) \xrightarrow[n\to\infty]{} 0$

chebyshev for truncated (ii) $b_n^{-2} \sum_{k=1}^{n} \mathbb{E}(\bar{X}_{n,k}^2 : |X_{n,k}| \leq b_n) \xrightarrow[n\to\infty]{} 0$

$\Rightarrow \frac{S_n - a_n}{b_n} \xrightarrow[n\to\infty]{} 0$ ← this is center.

second moment > Var. where $a_n = \sum_{k=1}^{n} \mathbb{E}[X_{n,k} : |\bar{X}_{n,k}| \leq b_n]$ like indicator

St. petersburg. Paradox (Example 2.2.7 Durrett)

Dealt with $EX = \infty$

$P(X = 2^k) = 2^{-k} \quad k = 1, 2, \dots$

$S_n = \sum_{k=1}^{n} X_k$ "don't Expect nice

$\therefore \frac{S_n}{n \log_2(n)} \xrightarrow[n\to\infty]{P} 1$

Exercise. Hard. unfair fair games

Weak Law for triangular Arrays

$\{X_{n,k}\}$, $1 \leq k \leq n$ Be iid.

Let $b_n > 0$ with $b_n \to \infty$.

Let $\bar{X}_{n,k} = X_{n,k} \mathbb{1}(|X_{n,k}| \leq b_n)$.

(i) $\sum_{k=1}^{n} P(|X_{n,k}| > b_n) \xrightarrow[n\to\infty]{} 0$

(ii) $b_n^{-2} \sum_{k=1}^{n} E\bar{X}_{n,k}^2 \xrightarrow[n\to\infty]{} 0$

if $S_n = X_1 + \dots + X_n$ and $a_n = \sum_{n=1}^{n} E\bar{X}_{n,k}$ then

$$\frac{S_n - a_n}{b_n} \xrightarrow[n\to\infty]{P} 0$$