L39 - 12-02 Poisson Convergence

Monday, December 2, 2024 11:37 AM

Ex of CLT $\{Y_k\}_{k=1}$ InD

$P(Y_k = 1) = 1/k \qquad P(Y_k = 0) = 1 - \frac{1}{k} \qquad Y_k \sim Ber(p = 1/k)$

$S_n = \sum_{k=1}^{n} Y_k \qquad E(S_n) = \sum_{k=1}^{n} E(Y_k) = \sum_{k=1}^{n} \frac{1}{k} \sim \log(n)$

$Var(S_n) = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k^2} \sim \log(n)$

$$\frac{S_n - (\log n)}{\sqrt{\log n}} \Rightarrow N(0,1).$$

write interms of triangle Array.

$\left\{ X_{n,k} = \frac{Y_k - 1/k}{\sqrt{\log(n)}} \right\}_{1 \le k \le n} \qquad \Sigma\, 1/k = \log n.$

Lindeberg Condition check

$\sum_{k=1}^{n} E\left(X_{n,k}^2 ; |X_{n,k}| > \varepsilon \right) \xrightarrow[n\to\infty]{} 0, \quad \varepsilon > 0$

this is Empty set

$\left| Y_k - \frac{1}{k} \right| \le 1 \qquad$ so $\qquad |X_{n,k}| < \frac{1}{\sqrt{\log(n)}} \xrightarrow[n\to\infty]{} 0$

If $\sup_{1 \le k \le n} |X_{n,k}| \le C_n$ and $C_n \xrightarrow[n\to\infty]{} 0$ then L. condition holds.

Example 2 with truncation

$\{X_{n,m}\}_{1 \le m \le n}$ iid for each n.

$P\left(X_{n,1} = \pm \frac{1}{\sqrt{n}}\right) = \frac{1}{2} - \frac{1}{2n^2}$

$P\left(X_{n,1} = \pm \frac{4^k}{\sqrt{n}}\right) = \frac{1}{2n^2 \cdot 2^k}, \quad k = 1, 2, 3 \ldots \qquad \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$

$P(X_{n,1} = \pm \sqrt{n}) = \frac{1}{2n^2 \cdot 2^k}$, $k = 1, 2, 3 \ldots$ $\quad \sum_{k=1} 2^k = 1$

$S_n = \sum_{k=1}^{n} X_{n,k}$

this Disappeared.

what is $E|X_{n,1}| = \frac{1}{\sqrt{n}}\left(1 - \frac{1}{n^2}\right) + \underbrace{\sum_{k=1}^{\infty} \frac{\overset{2^k}{\cancel{4^k}}}{\sqrt{n}} \cdot \frac{1}{n^2 \cdot \cancel{2^k}}}_{\frac{1}{\sqrt{n}\, n^2} \sum_{k=1}^{\infty} 2^k} = \infty$

$E|X_{n,1}| = \infty$

$\Rightarrow E(X_{n,1}^2) = \infty$

Aside

$\sqrt{E(x^2)} > E|X|$ By Jensen, !

thus need truncation.

$Y_{n,m} = X_{n,m} \mathbb{1}_{|X_{n,m}| \le 1/\sqrt{n}}$ $\quad 1 \le m \le n.$

$$Y_{n,m} = \begin{cases} 0 & \text{w.p } 1/n^2 \\ \pm \frac{1}{\sqrt{n}} & \text{w.p } \frac{1}{2} - \frac{1}{2n^2} \end{cases}$$

$E(Y_{n,m}) = 0$

$\text{var} = E(Y^2_{n,m}) = \frac{1}{n}\left(\frac{1}{2} - \frac{1}{2n^2}\right) = \frac{1}{n} - \frac{1}{n^3}.$

$\sum_{m=1}^{n} \text{Var}(Y_{n,m}) = 1 - \frac{1}{n^2} \xrightarrow[n\to\infty]{} 1$,

if $T_n = \sum_{m=1}^{n} Y_{nm}$

$T_n \Rightarrow N(0,1)$

we want : $S_n \Rightarrow N(0,1)$

wts $P(S_n \neq T_n) \xrightarrow[n\to\infty]{} 0.$

$\left(S_n - T_n \xrightarrow[n\to\infty]{P} 0\right)$

$P(S_n \neq T_n) \le P\left(\bigcup_{m=1}^{n} \{X_{nm} \neq Y_{n,m}\}\right).$

$$\leq n \cdot P(X_{n,1} \neq Y_{n,1})$$

$$\leq n \cdot \frac{1}{n^2} = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$$

Domain of Attraction By Levy. $\frac{S_n - A_n}{b_n} \Rightarrow N.$

DOA of Normal Law.

Criterion of Paul Levy.

X_1, X_2, iid. $\exists a_n, b_n \in \mathbb{R}$: $\frac{S_n - a_n}{b_n} \Rightarrow N(0,1)$

iff

$$\frac{y^2 P(|X| > y)}{E(X^2; |X| \leq y)} \xrightarrow[y\to\infty]{} 0$$

if Does not hold forget About truncation

Berry - Eseen

$X, X_1, X_2, \ldots$ iid. $E(X) = 0$, $Var(X) = 1$ Assume $E|X^3| < \infty$

" $\frac{S_n}{\sqrt{n}} \Rightarrow N(0,1)$". need stronger

Thm. $\sup_{-\infty < x < \infty} | F_n(x) - \Phi(x) | \leq \frac{3E(X^3)}{\sqrt{n}}$ " says $\frac{1}{\sqrt{n}}$ too Big "

$\Phi(x) = P(Z \leq x)$, $Z \sim N(0,1)$

$F_n(x) = P(\frac{S_n}{\sqrt{n}} \leq x)$

"convergence is uniform."

"not Pointwise"

Poisson convergence. section 6 ?

$Y \sim$ Poisson(λ) $\lambda > 0$, $P(Y = k) = \frac{e^{-\lambda} \lambda^k}{}$ $k = 0, 1, 2, \ldots$

$Y \sim \text{Poisson}(\lambda) \quad \lambda > 0, \qquad P(Y = k) = \dfrac{e^{-\lambda}\lambda^k}{k!} \quad k = 0,1,2,\ldots$

Law of Rare Events.

$\text{Bin}(n, \frac{\lambda}{n}) \underset{n \text{ large}}{\sim} \text{Poisson}(\lambda)$ "Binomial converge in ~~Dist~~ to Poison"

theorem; $\{X_{n,m}\}_{1 \le m \le n}$ are InD,

$X_{n,m} \sim \text{Ber}(P_{n,m})$, ie $P(X_{n,m} = 1) = P_{n,m}$, $P(X_{n,m} \overset{0}{=}) = 1 - P_{n,m}$

Assume: ① $\sum_{m=1}^{n} P_{n,m} \xrightarrow[n\to\infty]{} \lambda \ge 0$

② $\max_{1 \le m \le n} \{P_{n,m}\} \xrightarrow[n\to\infty]{} 0$

Then $S_n = \sum_{m=1}^{n} X_{n,m} \Rightarrow \text{Poisson}(\lambda)$ "want to C.F."

ζ " is positive

Enough to check

If $P(S_n = k) \xrightarrow[n\to\infty]{} \dfrac{e^{-\lambda}\lambda^k}{k!}$, $k = 0,1,2\ldots$ Then $\sum_{k=0}^{\infty} \left| P(S_n = k) - \dfrac{e^{-\lambda}\lambda^k}{k!} \right| \xrightarrow[n\to\infty]{} 0$

if

$f_{X_n}(x) \xrightarrow[n\to\infty]{} f_X(x) \quad \forall x \in \mathbb{R}$.

Then $X_n \Rightarrow X$

"total variation"

in Fact $\int_{-\infty}^{\infty} |f_{X_n}(x) - f_X(x)|\,dx \xrightarrow[n\to\infty]{} 0$

C.F. of Bern: $1 - P_{n,m} + P_{n,m} e^{it}$.

$= 1 + P_{n,m}(e^{it} - 1)$.

λ?

$\varphi_{S_n}(t) = \prod_{m=1}^{n} (1 + P_{n,m}(e^{it} - 1)) \longrightarrow e^{\lambda(e^{it} - 1)} = \varphi_{\text{poison}}^{(\lambda)}$.

Lemma $\prod(1+a_{n,m}) \longrightarrow e^{\lambda}$

if (1) $\sum_{m=1}^{n} a_{n,m} \rightarrow \lambda$

(2) $\sup_{n} \sum_{m=1}^{n} |a_{n,m}| < \infty$

(3) $\max_{1 \le m \le n} |a_{n,m}| \xrightarrow[n\to\infty]{} 0$

Based on Lemma. $\frac{1}{3}$