L35 - 11-18 Lindeburg-Feller, Charactistic Functions

Monday, November 18, 2024 11:33 AM

Lindberg - Feller CLT

triangular Array $\{X_{n,k}\}_{1 \le k \le, n \ge 1}$

$E(X_{n,k}) = 0$, $\sum_{k=1}^{m} E(X_{n,k}^2) = 1$, $\{X_{n,k}\}_{1 \le k \le n}$ Row ind.

on Friday;

$\{X_k\}_{k \ge 1}$ iid. $E(X_k) = 0$, $E(X_k^2) = 1$,

$X_{n,k} = \frac{X_k}{\sqrt{n}}$ $1 \le k \le n$.

to get above. "how can we generalize.
- Remove iid. constraint.

Really need Linberg condition.

if $L_n(\varepsilon) = \sum E(X_{n,k}^2 ; X_{n,k} > \varepsilon) \xrightarrow[n\to\infty]{} 0 \;\forall \varepsilon > 0$
then $S_n \Rightarrow Z \sim N(0,1)$ ($S_n = \sum_{n=1}^{\infty} X_n$) "is it?"

thereom 3.4.5 Part (ii) Lindeburg-Feller

$\cancel{n} \cdot E(\frac{X^2}{\cancel{n}} ; |X_k| > \sqrt{n}\varepsilon) \xrightarrow[n\to\infty]{} 0$

Because $EX^2 \to 1$ this goes to 0. As $n \to \infty$.

Linberg Condition $\Rightarrow \max_{1 \le k \le n} E(X_{n,k}^2) \xrightarrow[n\to\infty]{} 0$ D Part (i)

$\limsup \max_{1 \le k \le n} E(X_{n,k}^2) = \limsup E(X_{n,k^*}) \xrightarrow[n\to\infty]{} \varepsilon^2 \longrightarrow 0 \quad \forall \varepsilon > 0$

$E(X_{n,k^*}^2) = E(X_{n,k^*}^2, |X_{n,k^*}| > \varepsilon) + E(X_{n,k^*}^2 ; |X_{n,k^*}| \le \varepsilon)$

$\le L_n(\varepsilon) + \varepsilon^2$

By Linberg

0

$E(X_{n,k}^2) = \sigma_{n,k}^2$ $sd(X_{nk}) = \sigma_{nk}$

$f \in C_B^{(3)}\mathbb{R}$ Z, Z_i iid $Z \sim N(0,1)$

$\left| E(f(\sum_{k=1}^{n} X_{n,k}) - f(\sum \sigma_{n,k} Z_k)) \right|$ A

$\sum_{k=1}^{n} \sigma_{n,k} Z_k \sim N(0,1)$.

Replace one By one like Friday.

A $\le \sum_{k=1}^{n} \left| E f(T_{n,k} + X_{n,k}) - f(T_{n,k} + \sigma_{n,k} Z_k) \right|$

$T_{n,k} = \underbrace{\sum_{m=1}^{k-1} X_{n,m}}_{\text{Before } X_{n,k}} + \underbrace{\sum_{m=k+1}^{n} \sigma_{n,m} Z_m}_{\text{After } X_{n,k}}$

$\le C \cdot \sum_{k=1}^{n} E(X_{n,k}^2 ; |X_{n,k}| > \varepsilon) + \overbrace{\cancel{\sum_{k=1}^{n} E(|X_{n,k}|^2 ; |X_{n,k}| < \varepsilon)}}^{\text{All less than } 1} + \sum_{k=1}^{n} \sigma_{n,k}^3 E(|Z_k|^3)$

Can Replace

$\le C L_n(\varepsilon) + \varepsilon$

$+ \max_{1 \le k \le n}\{\sigma_{n,k}\} \cancel{\left[\sum_{k=1}^{n} \sigma_{n,k}^2\right]} E(|Z|)^3 \to 0$ As $n \to \infty$ By B

$\xrightarrow[n\to\infty]{} \varepsilon$

Example where $L_n(\varepsilon) \xrightarrow[n\to\infty]{} 0 \;\forall \varepsilon > 0$

$\sum_{k=1}^{n} E(|X_{n,k}|^{2+\delta}) \xrightarrow[n\to\infty]{} 0$ $\delta > 0$

Claim $L_n(\varepsilon) \le \varepsilon^{-\delta} \sum_{k=1}^{n} E(|X_{n,k}|^{2+\delta}) \xrightarrow[n\to\infty]{} 0$

$\varepsilon^{\delta} L_n(\varepsilon) \le \sum_{k=1}^{n} E(|X_{n,k}|^2 \varepsilon^{\delta} ; |X_{n,k}| > \varepsilon)$ mult by ε^{δ}

$|X_{n,k}|^{\delta} > \varepsilon^{\delta}$. Replace by.

$\le \sum_{k=1}^{n} E(|X_{n,k}|^{2+\delta} ; \cancel{|X_{n,k}| > \varepsilon}) \xrightarrow[n\to\infty]{} 0$ By Assumption.

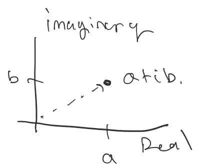
Characteristic Function (Fourier transform) Ch. 3.3.

Def Let X be r.v.

$\varphi_X(t) = E(e^{itx}) \quad t \in \mathbb{R}$

Def. C.F. Let X be r.v. then C.F. $\varphi(t) = Ee^{itX} = E\cos tX + iE\sin tX$

$e^{itx} = \cos(tx) + i\sin(tx)$



Complex # Plane.

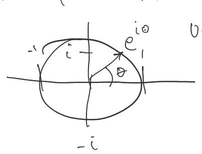
$|a+ib| = \sqrt{a^2+b^2} \implies |a+ib|^2 = a^2+b^2 = (a+ib)\overline{(a+ib)} = a^2+b^2$ | Def magnitude.

Conjugate: $\overline{a+ib} = a - ib$ | Def Conjugate

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$|e^{i\theta}|^2 = \cos^2(\theta) + (i)^2\sin^2(\theta) = 1$ | Important identity.

$|e^{i\theta}| = 1.$



unit circle

Moment Generating Function. (MGF) | Relation to MGF.

$\psi_X(t) = E(e^{tX})$

Properties

(a) $\varphi(0) = 1$ | Part of theorem 3.3.1

(b) $\overline{\varphi(t)} = \varphi(-t)$ Conjugate.

(c) if $X \perp\!\!\!\perp Y$ then $\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$; $t \in \mathbb{R}$ | theorem 3.3.2

Aside

Characteristic Function of a vector.

$\varphi_{(X,Y)}(s,t) = E e^{i(sX+tY)}$, $s,t \in \mathbb{R}^2$

if X, Y IND $\iff \varphi_{(X,Y)}(s,t) = \varphi_X(s) \cdot \varphi_Y(t)$

Example

$X \backslash Y$	0	1	2	$P(X=x)$
0	1/9	0	2/9	1/3
1	2/9	1/9	0	1/3
2	[illegible]	[illegible]	[illegible]	[illegible]

Dis $X+Y \overset{\mathcal{D}}{\implies} \tilde{X}+\tilde{Y}$, $P(\tilde{X}=k) = 1/3$, $k = 0,1,3$ if $\tilde{X} \perp\!\!\!\perp \tilde{Y}$

$P(\tilde{Y}=k) = 1/3$, k ...

k	0	1	2	3	4
$P(X+Y=k)$	1/9	2/9	1/3	2/9	1/9

$X \overset{\mathcal{D}}{=} \tilde{X}$ [illegible]

0	[illegible]		[illegible]	
1	2/9	1/9	0	1/3
2	0	2/9	1/9	1/3
$P(Y=y)$	1/3	1/3	1/3	

$P(X+Y=k)$ 1/9 1/9 1/3 1/9 1/9

$X \overset{\mathcal{D}}{=} \tilde{X}$
$Y \overset{\mathcal{D}}{=} \tilde{Y}$ SAME Distribution.

Rewrite $\varphi_{\tilde{X}+\tilde{Y}}(t) = \varphi_{\tilde{X}}(t)\varphi_{\tilde{Y}}(t)$