# Monte Carlo Simulation

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## Big Picture

1. Monte Carlo Simulation is widely used in statistics and finance to show the effect of randomness or uncertainty
2. Example 1 illustrates mean, variance, and insurance
3. Example 2 illustrates uncertainty of investment

# Example 1

1. This example shows how to use mean and variance to make decisions in the presence of uncertainty

2. Suppose there is a simple game offered by a Las Vegas casino (or sports betting): you pay 2 dollar to roll a fair die; the casino pays you 6 dollars if you get the number 6, and pays 1 dollar otherwise. Do you want to play this game?

3. The return $r$ is random: you lose $2-1=1$ dollar with probability 5/6, and you win $6-2=4$ dollars with probability 1/6. The expected return (mean value), a weighted average, is

   $$E(r) = (-1)\left(\frac{5}{6}\right) + (4)\left(\frac{1}{6}\right) = -\frac{1}{6} \tag{1}$$

   So on average you lose 1/6 dollar, or equivalently, the casino makes a profit of 1/6 dollar from you.

# Simulation

Next we run simulation, pretending you play this game ten thousand times

```
set.seed(12345)
i=1
return = rep(-1,10000)
while (i<=10000) {
 die = sample(1:6, 1)
 if (die==6) return[i] = 4
 i = i+1
}
mean(return)
[1] -0.17
var(return)
[1] 3.461446
```

## Remarks

1. The average return from 10000 plays (sample mean $\bar{r}$) is $-.17$, very close to the population mean $-\frac{1}{6}$. This finding is implied by the Law of Large Number (LLN)
2. LLN also implies that the sample variance $s^2 = 3.461446$ is close to population variance $\sigma^2$
3. So based on the expected return, a rational person should not play this game because there is an expected loss.
4. Now suppose a different game is offered by Casino B. You pay 2 dollars to roll two dice. The casino pays you 31 dollars when you get 12, and 1 dollar otherwise. Is the game better or worse? Now we are comparing two games. Comparing investment opportunities is essentially the same
5. This time we just run simulation, because we dislike doing probability math, and we know the simulation result should be close to true answer thanks to LLN

# Simulating the game of Casino B

```
set.seed(12345)
i=1
return = rep(-1,100000)
while (i<=100000) {
 dice1 = sample(1:6,1)
 dice2 = sample(1:6,1)
 if (dice1+dice2==12) return[i] = 29
 i = i+1
}
mean(return)
[1] -0.1714
var(return)
[1] 24.17166
```

## Remarks

1. We see that the mean return of this new game $-0.1714$ is almost the same as the first game $-0.17$. Actually you can compute the population mean of the second game return, which is the same as the first game. So in terms of expected return, the two games are the same, and you should be indifferent.

2. However, the variance of return for the second game 24.17166 is much greater than the first game 3.461446. You may wonder, does variance matters?

3. Variance matters if we switch our focus from return to utility

## Utility

1. Consider a log utility function $U = \log(r)$ that satisfies diminishing marginal utility

$$\frac{d^2 \log(r)}{dr^2} = -\frac{1}{r^2} < 0 \tag{2}$$

2. Or without using calculus, compare the change in utility (marginal utility) when wealth changes from 2 to 1, and from 2 to 3

```
log(3)-log(2)
[1] 0.4054651
log(1)-log(2)
[1] -0.6931472
```

This implies that the decrease in utility 0.6931472 when losing one dollar is greater than the increase in utility 0.4054651 when winning one dollar. Variance matters because of this asymmetry

## Simulating utility

Next we rerun simulation and compare change in utility

```
set.seed(12345)
i=1
muA = rep(log(1)-log(2),100000); muB = rep(log(1)-log(2),100000)
while (i<=100000) {
 die = sample(1:6,1)
 if (die==6) muA[i] = log(6)-log(2)
 dice1 = sample(1:6,1)
 dice2 = sample(1:6,1)
 if (dice1+dice2==12) muB[i] = log(31)-log(2)
 i = i+1
}
mean(muA)
[1] -0.3958226
mean(muB)
[1] -0.5949695
```

## Variance matters (for most people)

1. So the second game leads to a greater average loss in utility 0.5949695 than the first game 0.3958226. As a result, a rational person should prefer the first game (which has the same average return as the second game, but smaller variance in return).

2. (Optional Math Proof) Consider a Taylor expansion of the utility function around the expected return $\mu = E(r)$

$$U(r) \approx U(\mu) + (r - \mu)U'(\mu) + \frac{1}{2}(r - \mu)^2 U''(\mu) \tag{3}$$

   Taking expectation leads to

$$E(U(r)) \approx U(\mu) + \frac{1}{2} var(r) U''(\mu) \tag{4}$$

   So holding $\mu$ equal, for a utility function satisfying $U'' < 0$ (diminishing marginal utility), a greater variance in return leads to less expected utility

## Risk averse

1. If utility function is linear, then $U'' = 0$, and the variance is irrelevant for expected utility; by contrast, if the utility function is concave, then $U'' < 0$, and Eq (4) shows that higher variance reduces expected utility

2. Later we will learn that variance of return measures risk (or uncertainty in general), which implies that a person with concave utility is risk averse. Variance matters for a risk-averse person.

3. For instance, the log utility function $U = \log(r)$ satisfies $U'' = -\frac{1}{r^2} < 0$. So the person with log utility prefers having two dollars for sure (no risk) than having 1 dollar with probability 0.5 and 3 dollars with probability 0.5

$$\log(2) > \frac{1}{2}\log(1) + \frac{1}{2}\log(3) \tag{5}$$

```
log(2)
[1] 0.6931472
0.5*log(1)+0.5*log(3)
[1] 0.5493061
```

## Insurance

There are many unavoidable risks (house catches fire, car gets stolen...). Then a risk averse person would like to pay a premium to buy an insurance policy so that the uncertainty can be eliminated. For instance, as long as the insurance costs less than 0.5857864 (called maximum insurance premium), then buying the insurance makes a risk-averse person better off than facing the risk of losing one dollar with probability of 0.5

```
0.5*log(1)+0.5*log(2)
[1] 0.3465736
log(2-0.5857864)
[1] 0.3465736
```

The maximum risk premium 0.5857864 (denoted by $\pi$) is obtained by solving the following equation

$$\log(2-\pi) = \frac{1}{2}\log(1) + \frac{1}{2}\log(2) \tag{6}$$

## Example 2

1. Suppose you are interested in an index fund that tracks the performance of a specific market index
2. For instance, SP 500 Index (Standard & Poor's 500) is a stock market index that tracks the performance of 500 of the largest publicly traded companies in the U.S
3. Example 2 shows how to use simulation to measure the uncertainty associated with buying and holding SP 500 index fund

# Data

1. The daily close SP500 index is available at FRED
   `https://fred.stlouisfed.org/series/SP500`

2. We download into R the data from 2015-02-09 to current day with these functions

```
library(tseries)
library(quantmod)
symbols = c("SP500")
getSymbols(symbols, src = "FRED")
class(SP500)
[1] "xts" "zoo"
head(SP500)
SP500
2015-02-09 2046.74
2015-02-10 2068.59
2015-02-11 2068.53
...
```

## Simulating the return of buy-and-sell-after-one-year strategy

Let us focus on two kinds of uncertainties: (1) when to buy the index fund? (2) when to sell the index fund (or when to exit the investment). The simulation below compute the return (ra.sp) of buying the index fund (sp) at <u>any day</u> starting from 2015-02-09, and sell it after 252 trading days (approximately one year)

```
sp = coredata(SP500)
n = nrow(sp)
j = 1
ra.sp=NULL
while (j+252<=n) {
 ra.sp = c(ra.sp, (sp[j+252]-sp[j])/sp[j])
 j = j + 1
}
```

## Distribution of return

```
quantile(ra.sp, prob=c(0.05,0.5,0.95),na.rm=T)
5%         50%        95%
-0.1221714  0.1302969  0.3422119
```

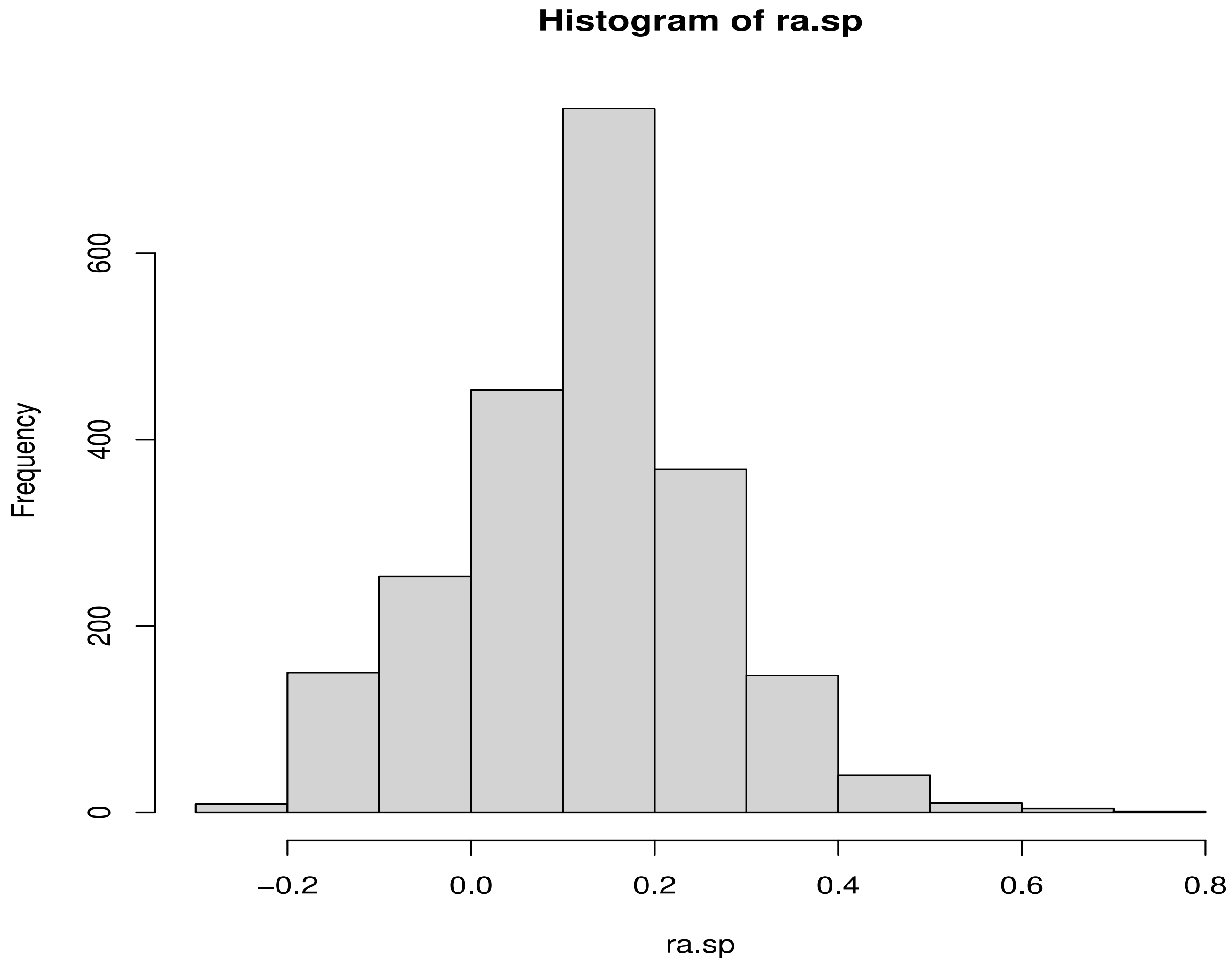
1. The results shown above may change if using most recent data
2. There are many one-year-long investment opportunities. The starting date is random, so is return. Put differently, we obtain a distribution of return, and each return is computed as
$$ra.sp = \frac{sp_{t+252} - sp_t}{sp_t} \tag{7}$$
3. The 5th, 50th, and 95th percentiles of return are shown above. They are called downside, median, and upside in finance, see Table 2.4 on page 46 of CJ book

## Confidence interval

1. The 90% confidence interval for the return is $(-0.1221714,\ 0.3422119)$
2. It provides a range of potential returns for the buy-and-sell-after-one-year strategy
3. It is unlikely to see a return less than negative 12 percent. It is also unlikely to see a return greater than 34 percent. If you invest 10000 dollars, you can be 90 percent confident that after one year you will have something between 8800 and 13400 dollars

# Histogram

A complete picture of the return distribution is its histogram



## Homework 1 (see syllabus for due date)

For this homework, you need to show me both the R codes and results.

1. (2.5 points) Please use R to run simulation and report the expected return (mean value) and variance of return for a new game: you pay 2 dollars to roll three dices. The casino pays you 5 dollars if the sum of three dices is 10. Otherwise, the casino pays you 1 dollar

2. (2.5 points) Please use R to run a simulation that produces the 95 percent confidence interval for the return of buy-SP500-index-fund-and-sell-after-FIVE-year strategy