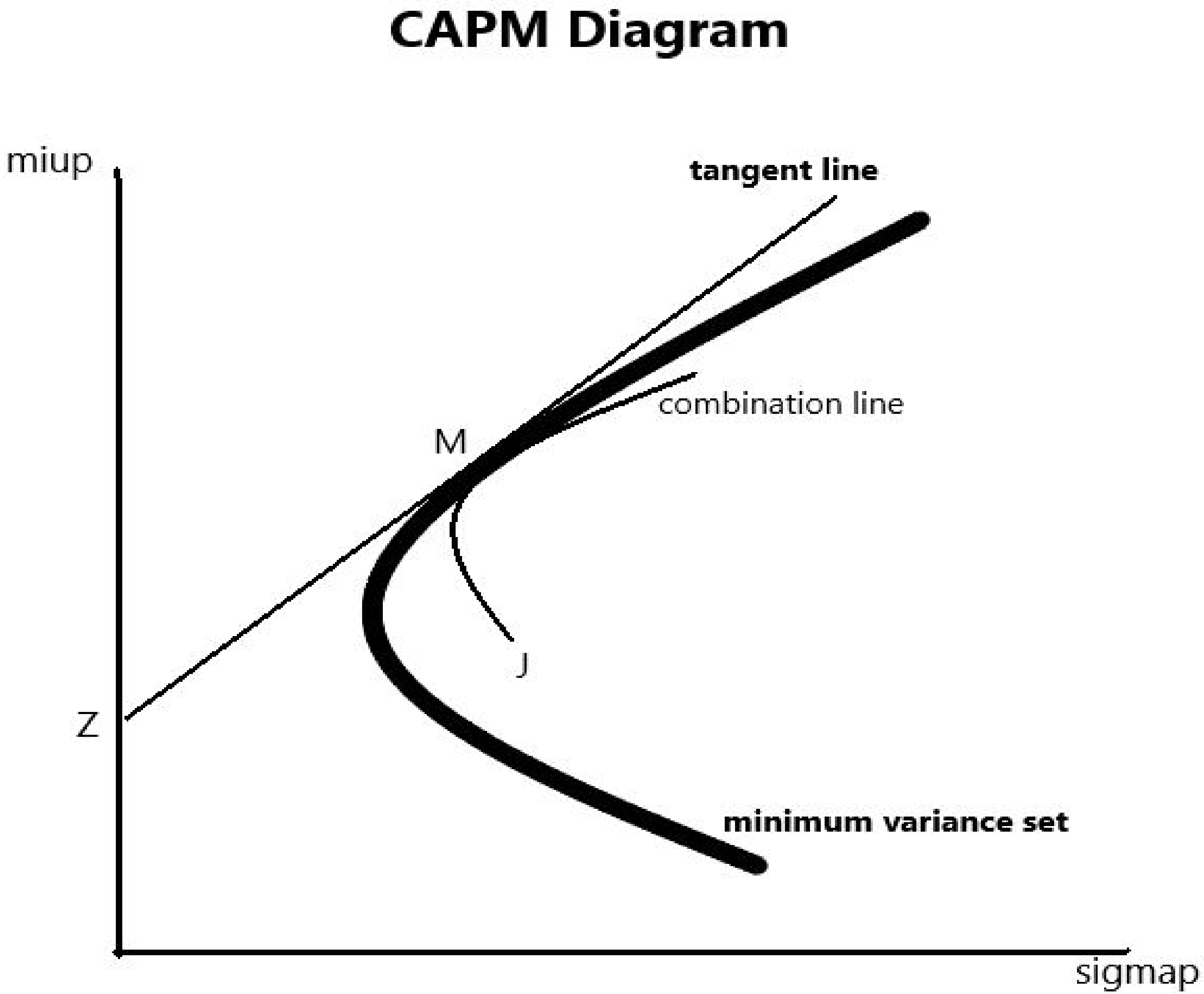
Capital Asset Pricing Model (CAPM) I

(Jing Li, Miami University)

Big Picture of CAPM



Main Ideas of CAPM

1. Derived from MPT, Capital Asset Pricing Model (CAPM) specifies the expected return for an asset given its risk measured by beta
2. CAPM requires several assumptions such as
   (a) Every investor uses MPT, and their portfolios are all mean-variance optimal
   (b) Every investor agrees on distributions of returns of all assets
   (c) Every investor has unlimited access to the same risk free asset, and they can borrow and lend freely
3. Under those assumptions, the key implications of CAPM are
   (a) The efficient frontier becomes a straight line called Capital Market Line (CML)
   (b) Prices of assess keep adjusting until the tangent portfolio between CML and MVS becomes the market portfolio (collection of all traded assets such as stocks, bonds, real estates, international assets)
   (c) Return of any assets is linearly related to beta. This linear relationship can be visualized as Security Market Line (SML)

A Property of Minimum Variance Set (Bullet)

Let M denote any portfolio on the bullet-shaped minimum variance set (MVS). Define the beta factor for asset J as the ratio of covariance between returns of J and M over the variance of return of M:

$$\beta_J \equiv \frac{\sigma_{JM}}{\sigma_M^2} \tag{1}$$

Next we will show that the μ_J and β_J have a linear relation

Proof—step 1

1. First, note that the combination line of J and M must be tangent to MVS at M. If not, then the combination line would penetrate MVS, then M would not have the smallest variance given the expected return, a contradiction!

2. It follows that the slops of MVS and combination line are equal at M, or they share the same tangent line shown in the CAPM Diagram

3. Suppose that tangent line intersects the vertical axis at point Z (risk-free asset such as T-bill)

Proof—step 2

1. Note that Z is risk-free since it has zero standard deviation. How much is β_Z?
2. The slope of tangent line is

$$\frac{\mu_M - \mu_Z}{\sigma_M - \sigma_Z} = \frac{\mu_M - \mu_Z}{\sigma_M - 0} = \frac{\mu_M - \mu_Z}{\sigma_M} \tag{2}$$

3. Let p be a portfolio consisting of M and J, and w be the "extra" weight of J:

$$p = wJ + (1-w)M \tag{3}$$

$$\mu_p = w\mu_J + (1-w)\mu_M \tag{4}$$

$$\sigma_p = \sqrt{w^2\sigma_J^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{JM}} \tag{5}$$

4. It follows that

$$\frac{d\mu_p}{dw} = \mu_J - \mu_M \tag{6}$$

$$\frac{d\sigma_p}{dw} = \frac{\sigma_{JM} - \sigma_M^2}{\sigma_M} \quad when\ w = 0 \tag{7}$$

Proof—step 3

1. The chain rule and last two equations imply that the slope of combination line $\frac{d\mu_p}{d\sigma_p}$ at M (when $w = 0$) is

$$\frac{d\mu_p}{dw} = \frac{d\mu_p}{d\sigma_p}\frac{d\sigma_p}{dw} \Rightarrow \frac{d\mu_p}{d\sigma_p} = \frac{\frac{d\mu_p}{dw}}{\frac{d\sigma_p}{dw}} = \frac{\mu_J - \mu_M}{\frac{\sigma_{JM}-\sigma_M^2}{\sigma_M}} = \frac{\mu_J - \mu_M}{\beta_J\sigma_M - \sigma_M} \tag{8}$$

2. Now we equate $\frac{d\mu_p}{d\sigma_p}$ with $\frac{\mu_M-\mu_Z}{\sigma_M}$

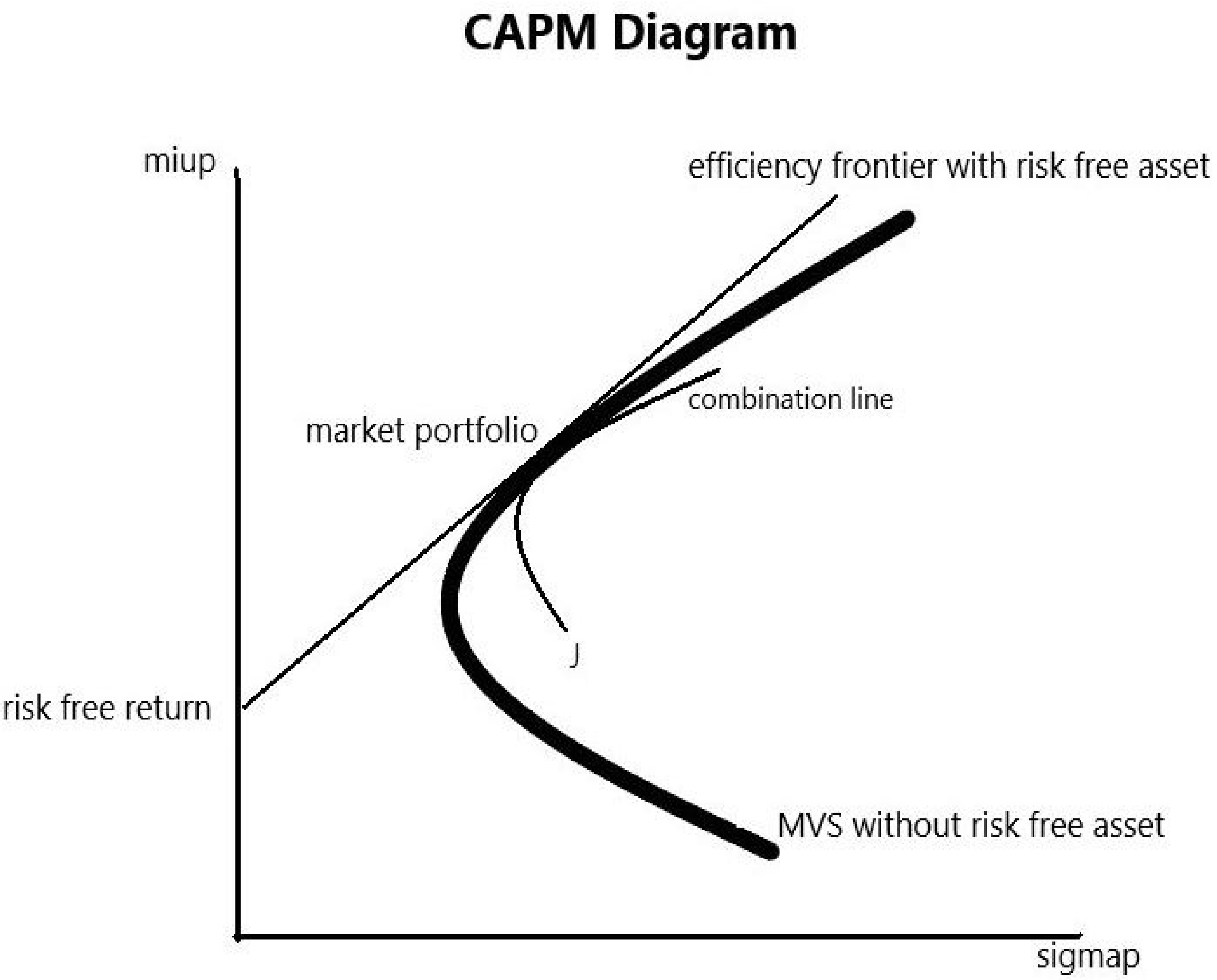
$$\frac{\mu_J - \mu_M}{\beta_J\sigma_M - \sigma_M} = \frac{\mu_M - \mu_Z}{\sigma_M} \tag{9}$$

After simplification we have

$$\mu_J = \mu_Z + \beta_J(\mu_M - \mu_Z) \tag{10}$$

The proof is complete

New Efficiency Frontier when adding a risk free asset



Remarks

1. Recall that the efficient frontier is the upper portion of bullet-shaped MVS if all assets are risky
2. Z is on the vertical axis, and its standard deviation is 0. Thus we call Z a risk-free asset or zero-beta asset. The return of Z is risk free return (denoted by r_f)
3. We already proved that

$$\textit{Combination line of a risk free asset and a risky asset is a straight line} \tag{11}$$

4. With inclusion of a risk free asset, the new efficiency frontier is the combination line of risk free asset and M, or the new efficiency frontier is that tangent line. People call that tangent line Capital Market Line (CML)

CAPM

The Capital Asset Pricing Model states that

1. Price of risky assets adjust when the risk free return changes. At equilibrium, M is the Market Portfolio, which is the portfolio for all risky assets (stocks, derivatives, real estates, foreign exchange, etc) and the weight for each risky asset is given by its value over total value of all risky assets

2. Denoting the risk free rate by r_f. CAPM implies the following pricing formula for any asset J

$$\mu_J = r_f + \beta_J(\mu_M - r_f) \tag{12}$$

$$\beta_J = \frac{\sigma_{JM}}{\sigma_M^2} \tag{13}$$

Risk Premium

Consider (12) as a decomposition

$$\mu_J = r_f + \texttt{risk premium}, \tag{14}$$

$$\texttt{risk premium} \equiv \beta_J(\mu_M - r_f) \tag{15}$$

1. So CAPM implies that the expected return for an asset is the sum of risk-free return r_f plus risk premium $\beta_J(\mu_M - r_f)$.

2. The risk premium is a product of two factors: $\mu_M - r_f$ is the risk premium for the market portfolio; β_J measures the <u>systematic risk</u> that is specific to the asset and cannot be diversified in a portfolio

3. According to CAPM, $\beta_J = \frac{\sigma_{JM}}{\sigma_M^2}$, so the risk of an asset stems from its <u>covariance</u> with the market portfolio

Limitations of CAPM

Criticisms of CAPM include

1. Assumptions such as everyone has unlimited access to risk free asset are unrealistic
2. More importantly, CAPM cannot be empirically tested because there is no way to obtain data for the market portfolio—collection of all assets.

Estimating Beta

1. Each asset has its own beta

2. The asset-specific beta can be estimated by a time series simple regression

$$r_{jt} = \alpha_J + \beta_J r_{mt} + u_{jt}, \quad (t = 1, 2, \ldots T) \tag{16}$$

   where r_{jt} is the return time series for asset J, and r_{mt} is the return time series for market portfolio. Most often we use SP500 as proxy for the market portfolio.

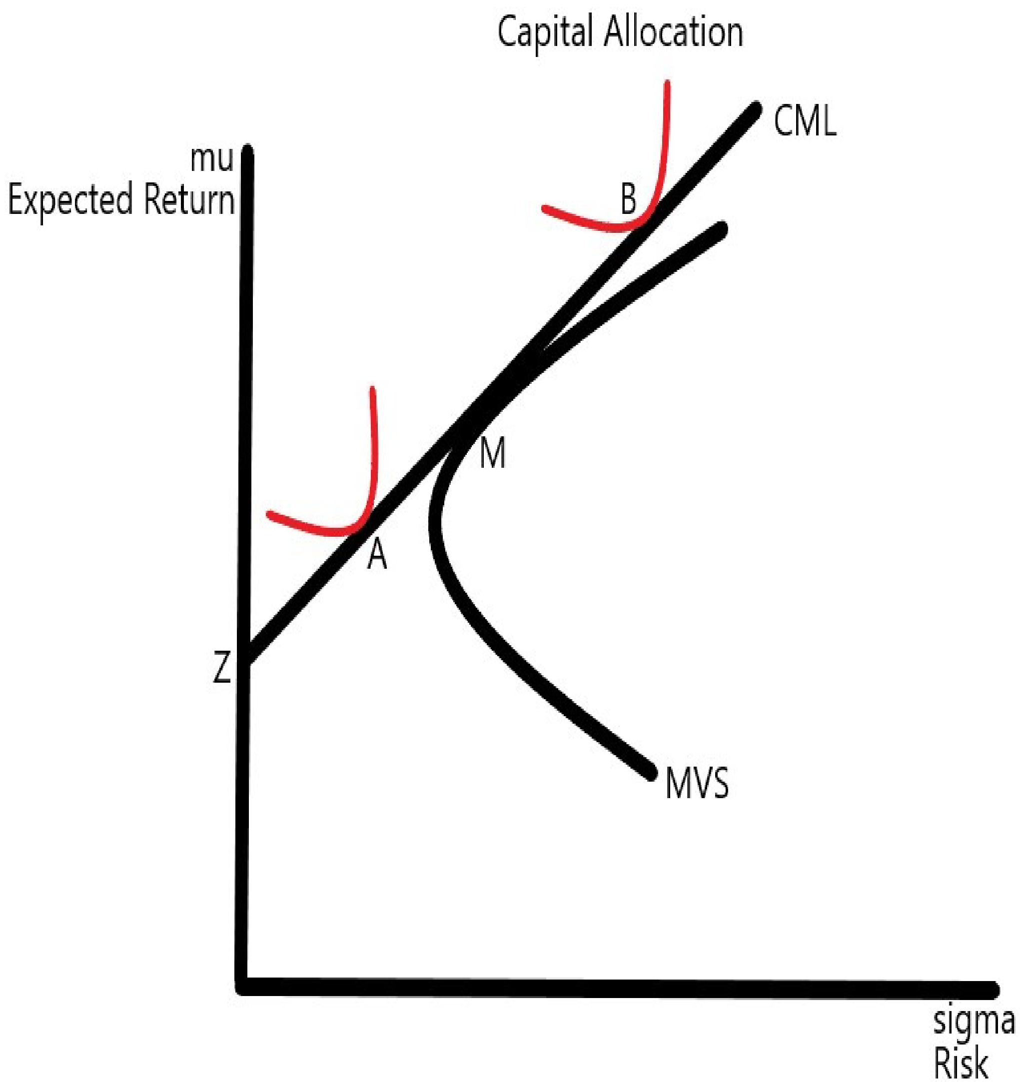
3. Equivalently, we can estimate beta using formula (13): $\beta_J = \frac{\sigma_{JM}}{\sigma_M^2}$

4. We run the regression separately for each asset to estimate its own beta

5. Instead of using whole sample, we could estimate beta using rolling windows—for instance, consider window size of 100. Then first window is $t = 1, 2, \ldots 100$, second window is $t = 2, 3, \ldots 101$, third window is $t = 3, 4, \ldots 102$, and so on

6. Rolling-window regression is able to capture time-evolving beta (risk)

Capital Allocation



Capital Allocation

1. According to CAPM, investor does not need to find the optimal portfolio—at equilibrium, the market portfolio is the optimal portfolio
2. An investor only needs to determine the proportion of capital invested in risk free asset and the market portfolio. That decision depends on red indifference curve (utility function of the investor)
3. At point A, the weight for risk-free asset is positive. So investor lends some capital and uses the remaining capital to buy shares of market portfolio
4. At point B, the investor is less risk averse. So the weight for risk-free asset is negative. The investor borrows some capital to buy more shares of the market portfolio.

Capital Asset Pricing Model (CAPM) II

(Jing Li, Miami University)

Security Market Line (SML)

Using the fact that

$$\beta_Z = 0 \tag{17}$$

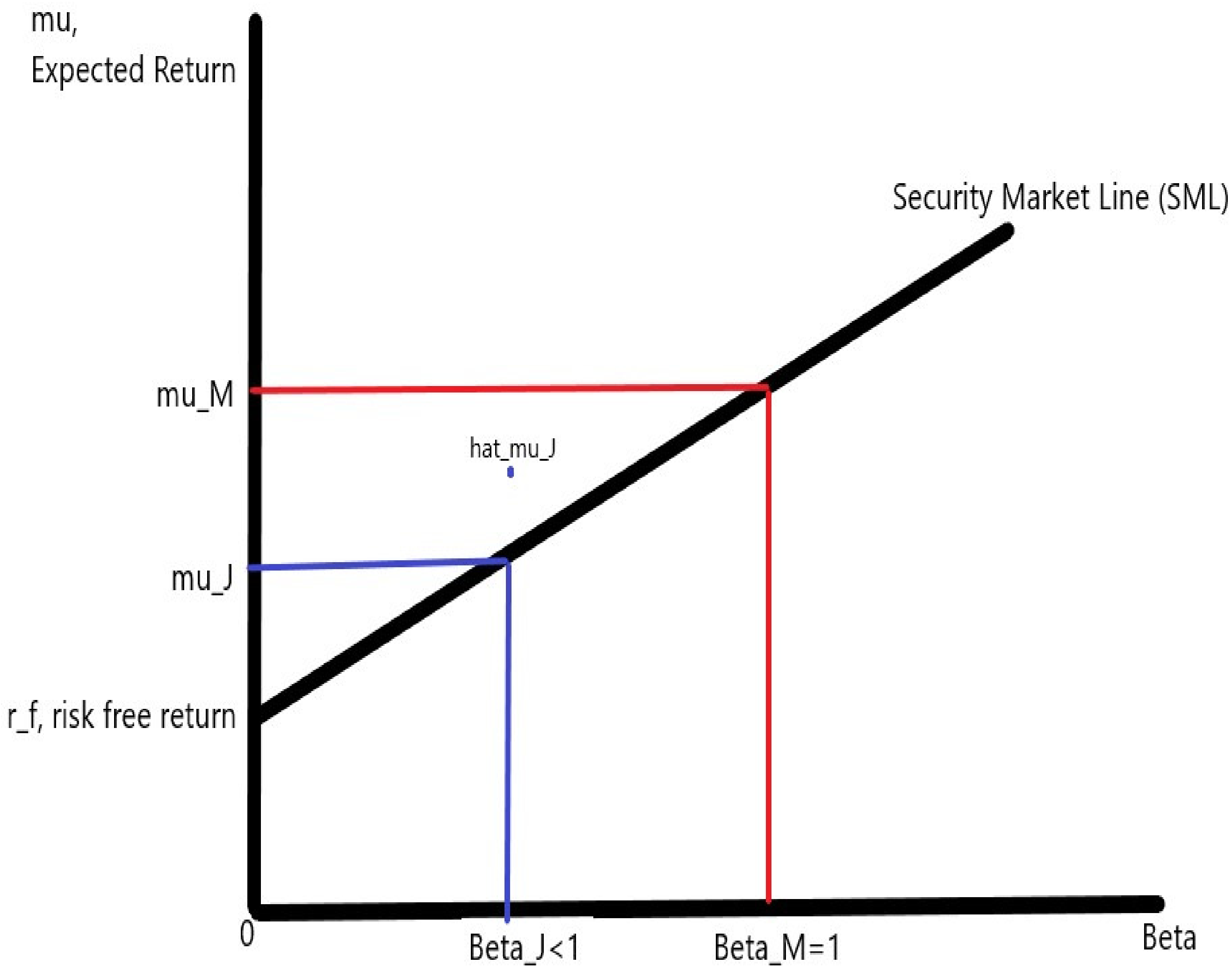
$$\beta_M = 1 \tag{18}$$

we can rewrite $\mu_J = r_f + \beta_J(\mu_M - r_f)$ as

$$\mu_J = intercept + slope\beta_J \tag{19}$$

where $intercept = r_f$, $slope = \frac{\mu_M - r_f}{\beta_M - \beta_Z}$. This equation motivates a straight line called the security market line in the mu-beta space (not mu-sigma space)

SML


mu,
Expected Return
Security Market Line (SML)
mu_M
hat_mu_J
mu_J
r_f, risk free return
0
Beta_J<1
Beta_M=1
Beta

Remarks

1. A line is determined by two points. We first locate the point for the risk free asset on the vertical axis with $\beta = 0, \mu = r_f$. We then locate the point for the market portfolio with $\beta_M = 1, \mu = \mu_M$. Connecting those two points produces the security market line (SML)

2. CAPM states that the expected return for a risky asset can be determined by its beta. For instance, the blue lines show how to find expected return for an asset with $\beta_J < 1$. Because SML line is upward-sloping, its expected return is less than market portfolio $\mu_J < \mu_M$.

3. Then we can compare the historical average return to the expected return. In the diagram, asset J's historical average return is greater than the expected return predicted by CAPM $\hat{\mu}_J > \mu_J$. It could be less than the expected return predicted by CAPM.

Alpha

1. The alpha of an asset or portfolio is a measurement of over-performance or under-performance
2. Alpha is defined as the difference between the historical average return and the expected return predicted by CAPM

$$\alpha_J \equiv \hat{\mu}_J - \mu_J = \hat{\mu}_J - r_f - \beta_J(\mu_M - r_f) \tag{20}$$

3. A positive alpha implies that an asset over-performs relative to the prediction of CAPM
4. In SML diagram, alpha is the vertical gap between actual return and expected return

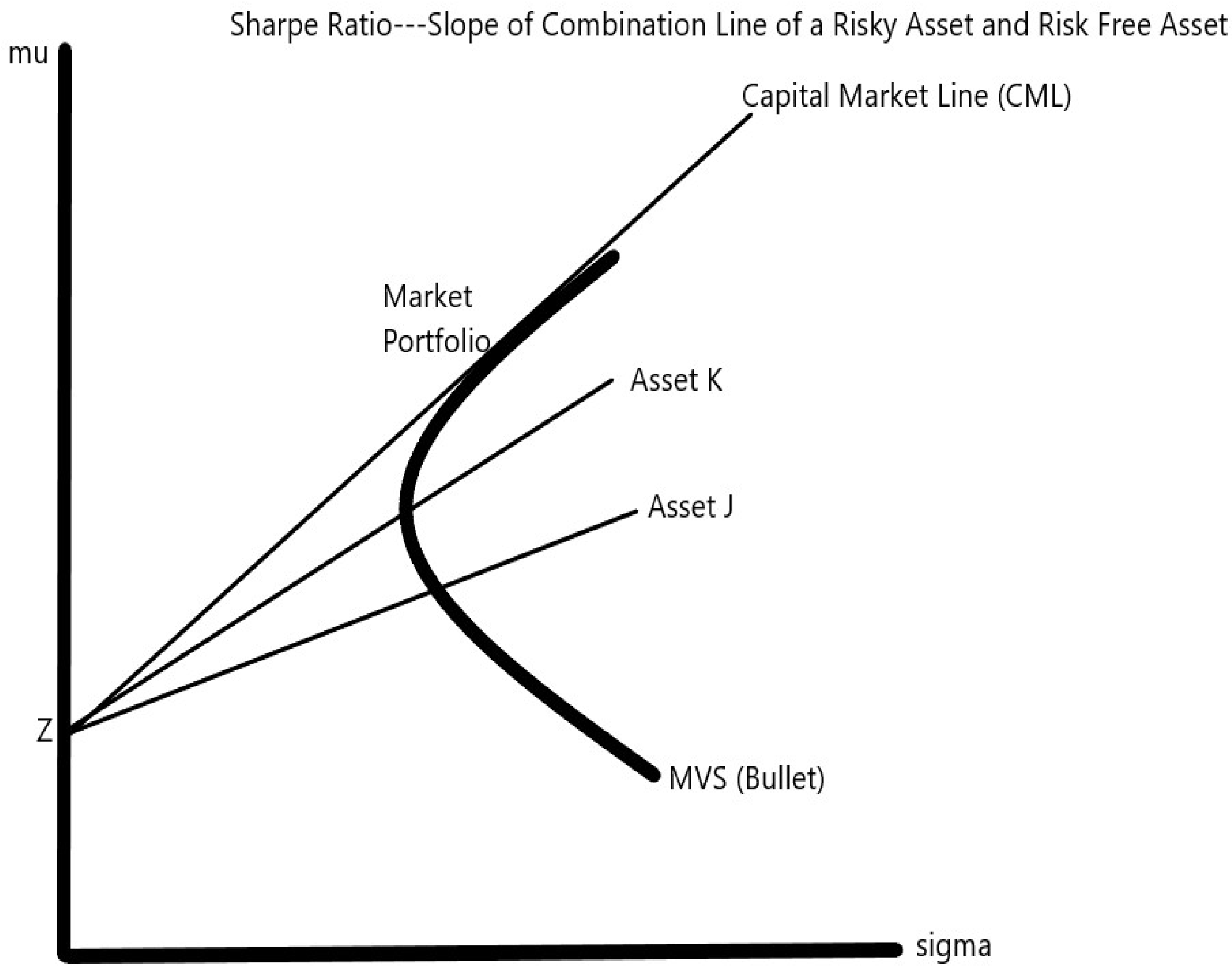
Sharpe Ratio

Another measurement of performance is Sharpe Ratio (SR), which is defined as

$$SR_J \equiv \frac{\mu_J - r_f}{\sigma_J} \tag{21}$$

In the mu-sigma diagram, Sharpe Ratio is the slope of combination line connecting the risk free asset and asset J. Sharpe Ratio measures the risk-adjusted award

Sharpe Ratio



Maximizing Sharpe Ratio

1. From the diagram it is evident that the market portfolio has the greatest Sharpe Ratio since the capital market line (efficient frontier with risk-free asset) is steeper than all other combination lines.

2. This fact suggests that we can rank different portfolios or assets by ranking their Sharpe Ratios. For instance, in the diagram asset K is preferred over asset J since K has greater Sharpe Ratio (steeper combination line)

3. We may try to find the optimal portfolio (consisting of k assets) that maximizes Sharpe Ratio. This optimization problem differs from MPT by taking into account of risk free asset.

Proof of market portfolio having greatest Sharpe Ratio

Recall CAPM

$$
\begin{aligned}
& \mu_J = r_f + \beta_J(\mu_M - r_f) && (22) \\
\Rightarrow \quad & \frac{\mu_J - r_f}{\sigma_J} = \frac{\rho\sigma_J\sigma_M}{\sigma_J\sigma_M}\frac{\mu_M - r_f}{\sigma_M} && (23) \\
\Rightarrow \quad & SR_J = \rho SR_M && (24) \\
\Rightarrow \quad & SR_J \leq SR_M && (25)
\end{aligned}
$$

The last result follows since the correlation $-1 \leq \rho \leq 1$

Finding Tangency Portfolio

1. In reality, an investor is unlikely to hold market portfolio that includes all traded assets
2. More realistically, an investor may be interested in finding a tangency portfolio, denoted by T, that consists of, say, only three risky assets.
3. Mathematically, finding T is equivalent to solving following constrained optimization problem of maximizing Sharpe Ratio

$$\max_{w} \frac{\mu_p - r_f}{\sigma_p} \quad s.t. \quad \sum w_i = 1 \tag{26}$$

where $\mu_p = \sum w_i \mu_i$ is the expected return of portfolio, $\sigma_p = \sqrt{w'\Sigma w}$ is the standard deviation of portfolio return, and r_f is the risk-free return. The analytical solution is

$$w = \frac{\Sigma^{-1}(\mu - r_f)}{1'\Sigma^{-1}(\mu - r_f)} \tag{27}$$

where w is vector of optimal weights, Σ is the variance-covariance matrix of individual return, μ is the column vector of individual expected return, and 1 is a column vector of ones.

Finding Tangency Portfolio-Grid Search

1. Deriving solution (27) involves matrix calculus and tedious algebra
2. Alternatively, we can try grid search in a three step procedure
   (a) applying **portfolio.optim** in **tseries** package to obtain the standard deviation of mean-variance optimal portfolio for a range of targeted expected return
   (b) computing Sharpe Ratio for each optimal portfolio (each point on the bullet)
   (c) locating the one with th highest Sharpe Ratio

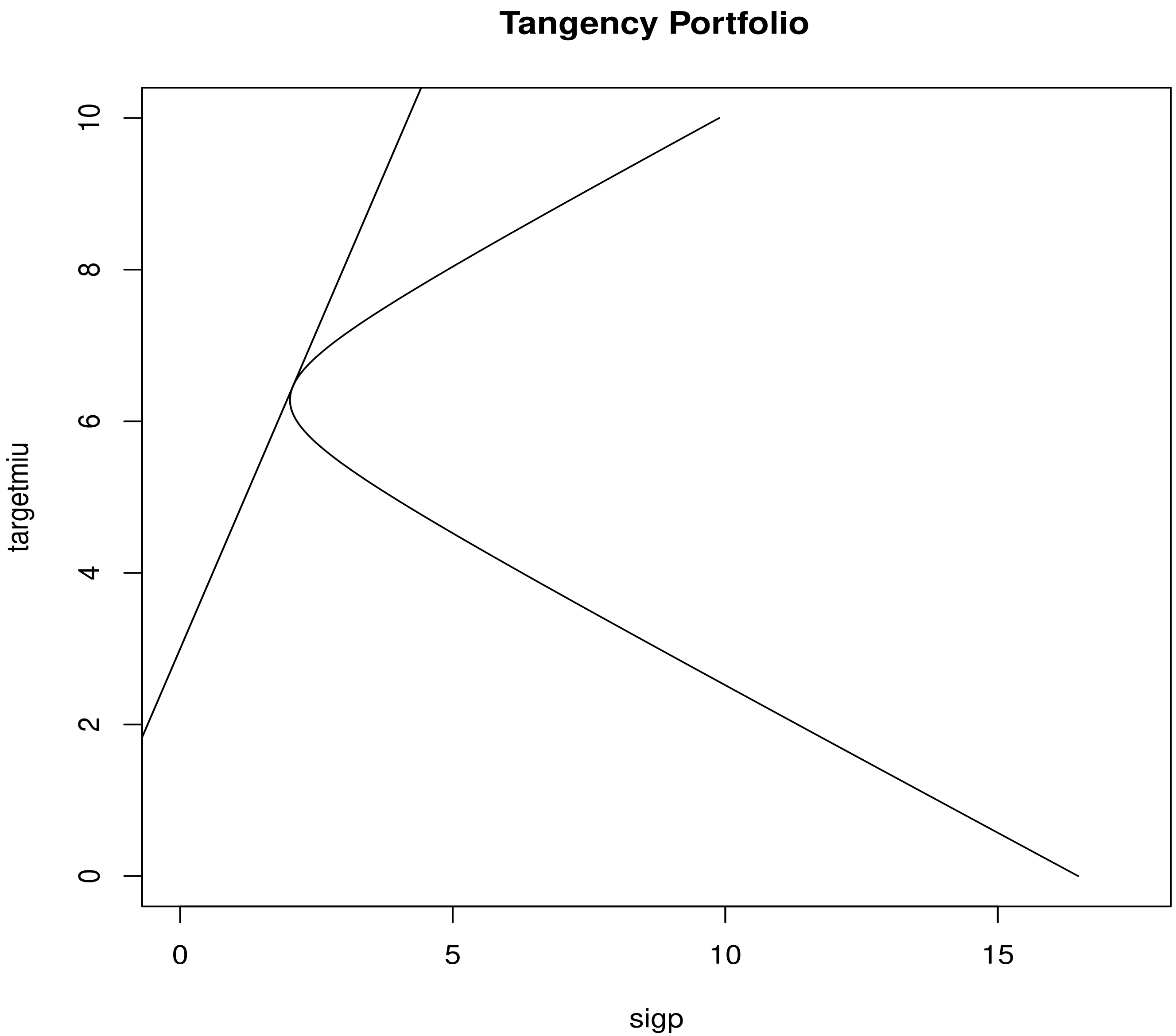
R codes of finding tangency portfolio of three risky assets

```
library(tseries)
miu1 = 5;s1 = 4; miu2=8; s2 = 5; miu3=7; s3 = 4.5; rho12 = -0.5; rho13 = -
s12=rho12*s1*s2; s13=rho13*s1*s3; s23=rho23*s2*s3
miuv = c(miu1, miu2, miu3)
sigma = matrix(c(s1^2,s12,s13,s12,s2^2, s23, s13,s23,s3^2),nrow=3,ncol=3,b

# analytical solution when risk-free return is 3
rf = 3
w = solve(sigma)%*%(miuv-rf)/sum(solve(sigma)%*%(miuv-rf))
w
cat("Expected Return of Tangency Portfolo is ", t(w)%*%miuv, "\n")
cat("Standard deviation of Tangency Portfolo return is ", sqrt(t(w)%*%sigm
```

R codes of grid search

```
# grid search
targetmiu = seq(0,10,0.01)
sigp = rep(0, length(targetmiu))
for (i in 1:length(targetmiu)) {
op = portfolio.optim(t(miuv), pm=targetmiu[i],covmat=sigma, shorts=T)
sigp[i] = sqrt(t(op$pw)%*%sigma%*%op$pw)
}
sharpe.r = (targetmiu-rf)/sigp
cat("Expected Return of Tangency Portfolo is ", targetmiu[which.max(sharpe
op = portfolio.optim(t(miuv), pm=targetmiu[which.max(sharpe.r)],covmat=sig
op$pw

a = rf
b = (t(w)%*%miuv-rf)/sqrt(t(w)%*%sigma%*%w)
plot(sigp, targetmiu, xlim=c(0,max(sigp)+1), main="Tangency Portfolio",typ
abline(a,b)
```

Drawing tangency portfolio



HW5 (See syllabus for due date)

1. (2 points) The risk-free rate is 3%, the expected return of market portfolio is 9%, and the stock's beta is 1.2. What is the expected return on the stock according to the CAPM?
2. (1 points) Suppose the actual return of that stock is 8%. Does that stock under or over-perform?
3. (2 points) Compute the alpha for that stock, and label that stock in the SML diagram