

# **Markowitz Portfolio Theory I**

(Jing Li, Miami University)

## Return

1. Consider a risky asset such as common stock. The (daily) return is defined as (assuming no dividend payment)

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} \approx \log(P_t) - \log(P_{t-1}) \quad (1)$$

2. Return is random since price  $P$  varies over time
3. The asset is risky because price  $P$  and return  $r$  cannot be predicted without error

## Mean and Variance

1. Statistically speaking, return follows a distribution, such as a bell-shaped one
2. The mean value (expected return) measures central tendency

$$\mu = E(r) \tag{2}$$

3. Variance and standard deviation measure dispersion

$$\sigma^2 = var(r) \tag{3}$$

$$\sigma = sd(r) \tag{4}$$

4. We use historical data to estimate  $\mu$  and  $\sigma$

# Portfolio

1. Suppose the mean return for asset 1 is 4 percent; mean return for asset 2 is 5 percent
2. Consider a portfolio for which we buy 200 dollar of asset 1, and 800 dollar of asset 2.
3. The return for this two-asset portfolio is

$$\frac{200(0.04) + 800(0.05)}{1000} = 0.2(0.04) + 0.8(0.05) = 0.048$$

where the weights for the two assets are  $0.2 = \frac{200}{1000}$  and  $0.8 = \frac{800}{1000}$ .

4. In general, a portfolio is a combination of  $n$  assets. Let  $w_i$  be the weight for the  $i$ -th asset, and  $\mu_i$  be the mean return for the  $i$ -th asset. The mean return of portfolio is

$$\mu_p = E \left( \sum w_i r_i \right) = \sum w_i E(r_i) = \sum w_i \mu_i \quad (5)$$

5. Exercise: compute the mean return of a portfolio when we short sell 1000 dollar of asset 1, and buy 2000 dollar of asset 2 (hint:  $w_1 = -1, w_2 = 2$ )
6. Exercise: (true or false) mean return of a two-asset portfolio is bounded between  $\mu_1$  and  $\mu_2$

# Risk of Portfolio

1. The risk of portfolio is measured by the variance of its return

$$\sigma_p^2 = \text{var} \left( \sum w_i r_i \right) = \sum_i w_i^2 \sigma_i^2 + \sum_i \sum_j w_i w_j \sigma_{ij} \quad (6)$$

where  $\sigma_{ij}$  denotes covariance. Note we apply this property of variance

$$\text{var}(ax + by) = a^2 \text{var}(x) + b^2 \text{var}(y) + 2ab\text{cov}(x, y) \quad (7)$$

2. According to the definition of correlation (coefficient)

$$\rho_{ij} \equiv \frac{\sigma_{ij}}{\sigma_i \sigma_j} \Rightarrow \sigma_{ij} = \rho_{ij} \sigma_i \sigma_j \quad (8)$$

we can rewrite the variance of portfolio return as

$$\sigma_p^2 = \sum_i w_i^2 \sigma_i^2 + \sum_i \sum_j w_i w_j \rho_{ij} \sigma_i \sigma_j \quad (9)$$

3. Exercise: find  $\sigma_p^2$  when  $n = 2, \rho_{12} = 1$
4. In short, risk of a portfolio depends on variance of individual asset and covariance

## Diagram of $\mu_p$ against $\sigma_p$

1. Note that the mean and variance of portfolio return change when weights of individual assets vary
2. In other words, mean return of portfolio is an implicit function of standard deviation of portfolio return
3. It is instructive to draw a plot of  $\mu_p$  against  $\sigma_p$ , called mu-sigma diagram, in order to show how return and risk are related for portfolio

## Example

We use R codes below to compute mean and standard deviation for return of asset 1

```
> p1 = c(100, 110, 108, 120, 117)
> p1lag = c(NA, 100, 110, 108, 120)
> r1 = (p1 - p1lag)/p1lag
> mean(r1, na.rm=T)
[1] 0.04198232
> sd(r1, na.rm=T)
[1] 0.07360071
```

The mean return during the five-day period is  $\mu_1 = 0.04198232$ , and standard deviation of return is  $\sigma_1 = 0.07360071$

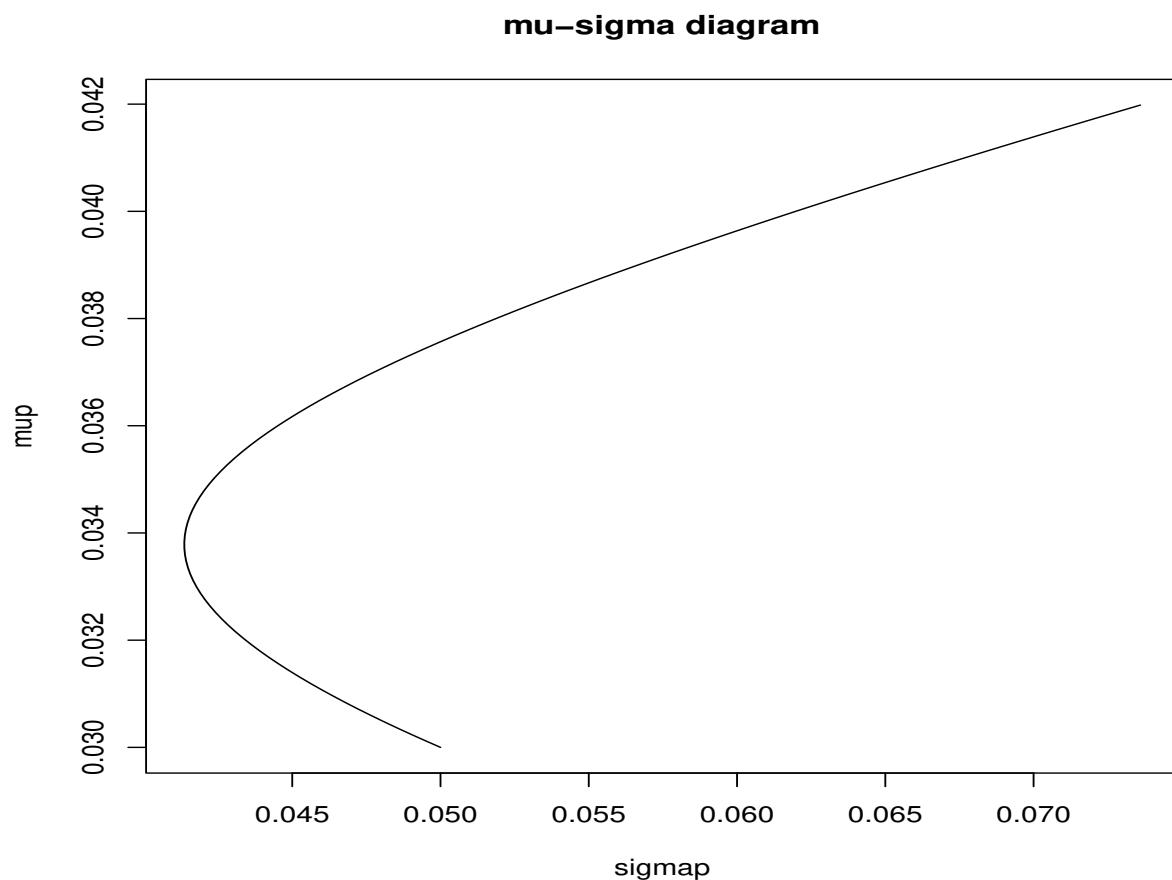
## mu-sigma diagram for a portfolio of two uncorrelated risky assets

Now consider a portfolio that consists of asset 1 and asset 2 ( $\mu_2$  is 0.03,  $\sigma_2$  is 0.05). Assume the two assets are uncorrelated ( $\rho_{12} = 0$ ). When  $w_1$  changes from 0 to 0.01, 0.02...until 1, the mu-sigma diagram for this two-uncorrelated-asset portfolio is

```
> mu1 = mean(r1, na.rm=T); sig1 = sd(r1, na.rm=T)
> mu2 = 0.03; sig2 = 0.05
> rho12 = 0
> w1 = seq(0,1,0.01)
> w2 = 1 - w1
> mup = w1*mu1 + w2*mu2
> sigp = sqrt(w1^2*sig1^2+w2^2*sig2^2+2*w1*w2*rho12*sig1*sig2)
> plot(sigp,mup,type="l",xlab="sigmap",ylab="mup",main="mu-sigma diagram")
```

Note that shorting is not allowed here since  $0 \leq w_i \leq 1$ ,  $i = 1, 2$ .

# **mu-sigma diagram for a portfolio of two uncorrelated assets with no shorting**



In class exercise: find the two individual assets in that diagram

## **Exercise**

1. Modify the R codes and draw mu-sigma diagram when  $\rho_{12} = 1$
2. Use math to explain why you see two straight lines with a kink when  $\rho_{12} = -1$
3. How to modify R codes to allow shorting asset 1? How does the shorting affect the mu-sigma diagram when  $\rho_{12} = 0$

# Expected Utility

1. Let  $r_p$  denote the portfolio return, and  $f$  utility function
2. Consider a second-order Taylor expansion of utility around  $\mu_p$  :

$$f(r_p) \approx f(\mu_p) + f'(\mu_p)(r_p - \mu_p) + \frac{1}{2}f''(\mu_p)(r_p - \mu_p)^2 \quad (10)$$

Essentially we are using a quadratic function to approximate the utility function

3. Based on this quadratic approximation we can show the expected utility is

$$E(f(r_p)) \approx f(\mu_p) + f'(\mu_p)E(r_p - \mu_p) + \frac{1}{2}f''(\mu_p)E(r_p - \mu_p)^2 = f(\mu_p) + \frac{1}{2}f''(\mu_p)\sigma_p^2 \quad (11)$$

4. Exercise: how to generalize this result to include skewness and kurtosis?

## Mean-Variance Optimization

From (11) it is evident that for a risk-averse investor whose concave utility satisfies  $f'' < 0$ , maximizing expected utility is the same as minimizing variance of portfolio return for a given target mean return of portfolio

$$\max E(f(r_p)) = \min \sigma_p^2 \quad \text{for given } \mu_p \quad (12)$$

Alternatively, maximizing expected utility is the same as maximizing mean return of portfolio for a given variance

$$\max E(f(r_p)) = \max \mu_p \quad \text{for given } \sigma_p^2 \quad (13)$$

This problem is called Mean-Variance Optimization

## Markowitz Portfolio Theory (MPT)

1. MPT aims to solve the mean variance optimization problem on the right hand side of (12). That is, MPT tries to find an optimal portfolio (finding optimal weights) that has the smallest variance (risk) for a given mean return
2. For instance, an investor may seek an optimal portfolio with a target mean return  $\mu_p = 0.04$ , and the investor wants to minimizes the variance of portfolio return
3. Using math, MPT is about solving

$$\min_{w_i} \sigma_p^2 = \sum_i w_i^2 \sigma_i^2 + \sum_i \sum_j w_i w_j \sigma_{ij} \quad (14)$$

*subject to*

$$\sum w_i \mu_i = \mu_p = r^{target} \quad (15)$$

$$\sum w_i = 1 \quad (16)$$

$$w_i \geq 0, \forall i, \quad (if\ no\ shorting) \quad (17)$$

## Lagrangian and First Order Condition (FOC)

1. We can set up the Lagrangian for this constrained optimization problem

$$L \equiv \sigma_p^2 - \lambda_1 \left( \sum w_i \mu_i - r^{target} \right) - \lambda_2 \left( \sum w_i - 1 \right) \quad (18)$$

2. The weight of  $i$ -th asset satisfies the first order condition of

$$\frac{\partial L}{\partial w_i} = 0 \Rightarrow \frac{\partial \sigma_p^2}{\partial w_i} = \lambda_1 \mu_i + \lambda_2 \quad (19)$$

3. At the optimum, the marginal increase in the portfolio variance due to the  $i$ -th asset is proportional to its mean return

## Covariance Matters

From (6) we can show

$$\frac{\partial \sigma_p^2}{\partial w_i} = 2w_i\sigma_i^2 + 2 \sum_j w_j \sigma_{ij} \quad (20)$$

Note that  $w_i$  is small for a portfolio consisting of many assets (big  $n$ ). Thus, the main contribution of the  $i$ -th asset to the portfolio variance is through its covariance with other assets. In light of this, **the risk of an individual risky asset is mainly about its covariance rather than variance.**

# Matrix Algebra

With inner product and quadratic form we can write

$$\mu_p = \sum w_i \mu_i = (w_1, \dots, w_n) \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} \quad (21)$$

$$\sigma_p^2 = \sum_i w_i^2 \sigma_i^2 + \sum_i \sum_j w_i w_j \sigma_{ij} = (w_1, \dots, w_n) \begin{pmatrix} \sigma_1^2 & \dots & \dots \\ \vdots & \sigma_{ij} & \vdots \\ \dots & \dots & \sigma_n^2 \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \quad (22)$$

The square matrix in the middle of (22) is called variance-covariance matrix

## Matrix Form of MPT

With matrix notation, the optimization problem of MPT can be expressed as

$$\min_W \sigma_p^2 = W' \Sigma W \quad (23)$$

*subject to*

$$W' \mu = r^{target} \quad (24)$$

$$W' 1 = 1 \quad (25)$$

where  $W$  is a column vector of  $w_1, w_2, \dots, w_n$ ;  $\Sigma$  is the variance-covariance matrix for returns of  $n$  assets;  $\mu$  is a column vector of  $\mu_1, \mu_2, \dots, \mu_n$ ;  $1$  on the left hand side of (25) is a column vector of  $n$  ones. Solving this problem is one example of quadratic programming (QP). We can add the nonnegativeness constraint  $0 \leq W \leq 1$  if shorting is not allowed.

# Example 1 of Applying Quadratic Programming to Mean Variance Optimization

```
> library(quadprog)
> miu1 = 5; s1 = 4; miu2=8; s2 = 5; miu3=7; s3 = 4.5
> rho12 = -0.5; rho13 = -0.3; rho23 = 0.4
> s12=rho12*s1*s2; s13=rho13*s1*s3; s23=rho23*s2*s3
> miuv = c(miu1, miu2, miu3)
> sigma = matrix(c(s1^2,s12,s13,s12,s2^2, s23, s13,s23,s3^2), nrow=3, ncol=3)
> targetmiu = 7
>
> Dmat = 2*sigma
> dvec = c(0,0,0)
> Amat = matrix(c(1,1,1,miu1,miu2,miu3), nrow=3, ncol=2)
> bvec = c(1, targetmiu)
```

## Optimal Weights

```
> ou = solve.QP(Dmat,dvec,Amat,bvec,meq=2)
> cat("Optimal weight is ", ou$solution, "\n")
Optimal weight is  0.2636189 0.5272379 0.2091432
> cat("Min-sigmap is ", sqrt(ou$value), "\n")
Min-sigmap is  2.748954
>
> library(tseries)
> op = portfolio.optim(t(miu), pm=7, covmat=sigma)
> op$pw
[1] 0.2636189 0.5272379 0.2091432
```

## Remarks

1. The optimal weights for the three risky assets are

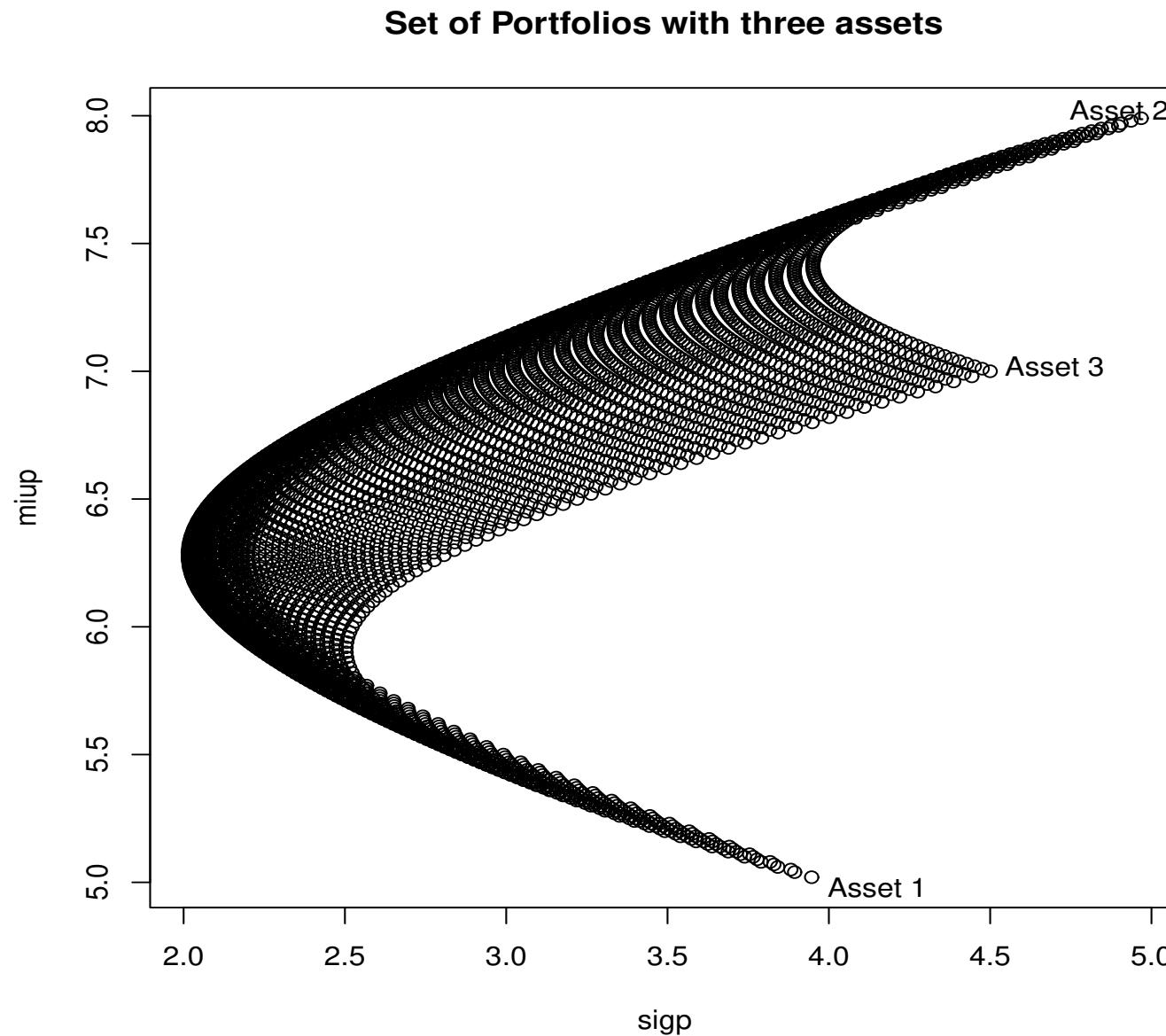
$$w_1 = 0.2636189, w_2 = 0.5272379, w_3 = 0.2091432$$

2. With those optimal weights, we get the optimal portfolio, whose standard deviation is  
 $\sigma_p = 2.748954$
3. Notice that the optimal portfolio has the same mean return as asset 3. However, risk of the portfolio is smaller than asset 3  $2.748954 < 4.5 = \sigma_3$ .
4. Lesson is, **diversification can reduce risk**. Do not put all eggs in one basket (do not put all investment in one asset)
5. The **portfolio.optim** function in the **tseries** packages produces the same optimal weights

## **Markowitz Portfolio Theory II—Efficiency Frontier**

(Jing Li, Miami University)

# Set of Portfolios Consisting of Three Risky Assets



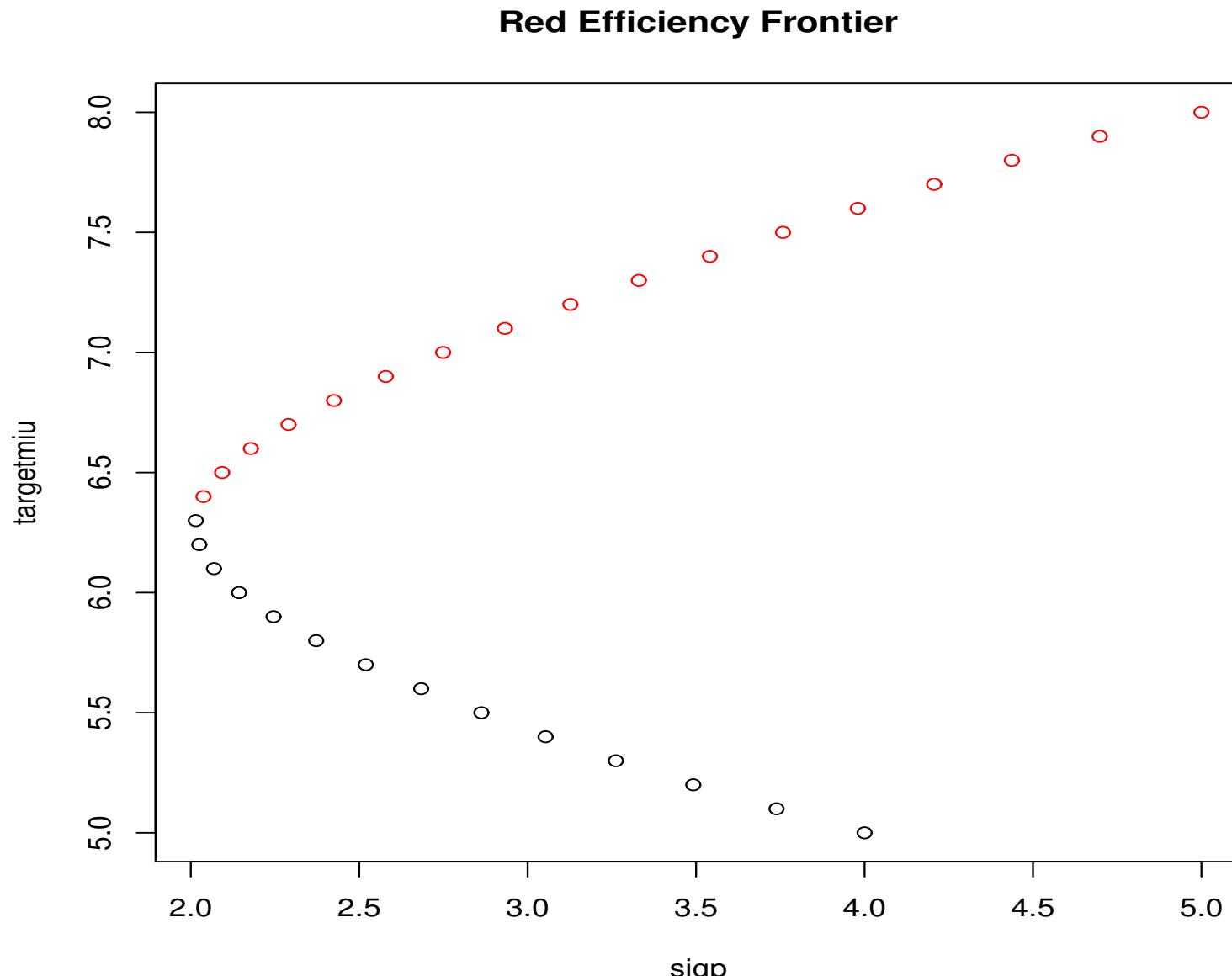
## Minimum Variance Set and Efficiency Frontier

1. Previous diagram shows a set of all possible portfolios with  $n$  assets,  $n = 3$ . Next we use **portfolio.optim** to find the optimal (minimum variance) portfolio for given target expected return (border of the bullet)
2. The set of all minimum variance portfolios is minimum variance set (MVS). MVS looks like the border of bullet in the mu-sigma diagram.
3. The head of MVS is the global minimum variance portfolio (GMVP)
4. Efficiency Frontier is the upper portion of MVS that lies above GMVP

## Example of Efficiency Frontier

```
targetmiu = seq(5,8,0.1)
sigp = rep(0, length(targetmiu))
for (i in 1:length(targetmiu)) {
  op = portfolio.optim(t(miuv), pm=targetmiu[i], covmat=sigma)
  sigp[i] = sqrt(t(op$pw) %*% sigma %*% op$pw)
}
plot(sigp, targetmiu, main="Efficiency Frontier")
```

# MVS (Bullet) and Efficiency Frontier (Red)



## Remarks

1. We let the target expected return range from 5 to 8
2. The GMVP has  $\sigma_p = 2.014758$  and  $\mu_p = 6.3$

```
> min(sigp)
[1] 2.014758
> targetmiu[which.min(sigp)]
[1] 6.3
```

3. The efficiency frontier is the upper red portion of the bullet that lies above GMVP
4. The lower black portion is irrational and is not efficiency frontier, since we could have obtained a higher mean return for a given risk  $\sigma_p$
5. For a target return of, say, 7.5, the  $\sigma_p$  of corresponding optimal portfolio is

```
> sigp[targetmiu==7.5]
[1] 3.757844
```

## Insights

1. Efficiency frontier is upward-sloping, implying that **rising return must come with rising risk**, or equivalently, **a higher return is expected to compensate a higher risk**
2. A rational investor should not ask for a return in the lower portion of MVS

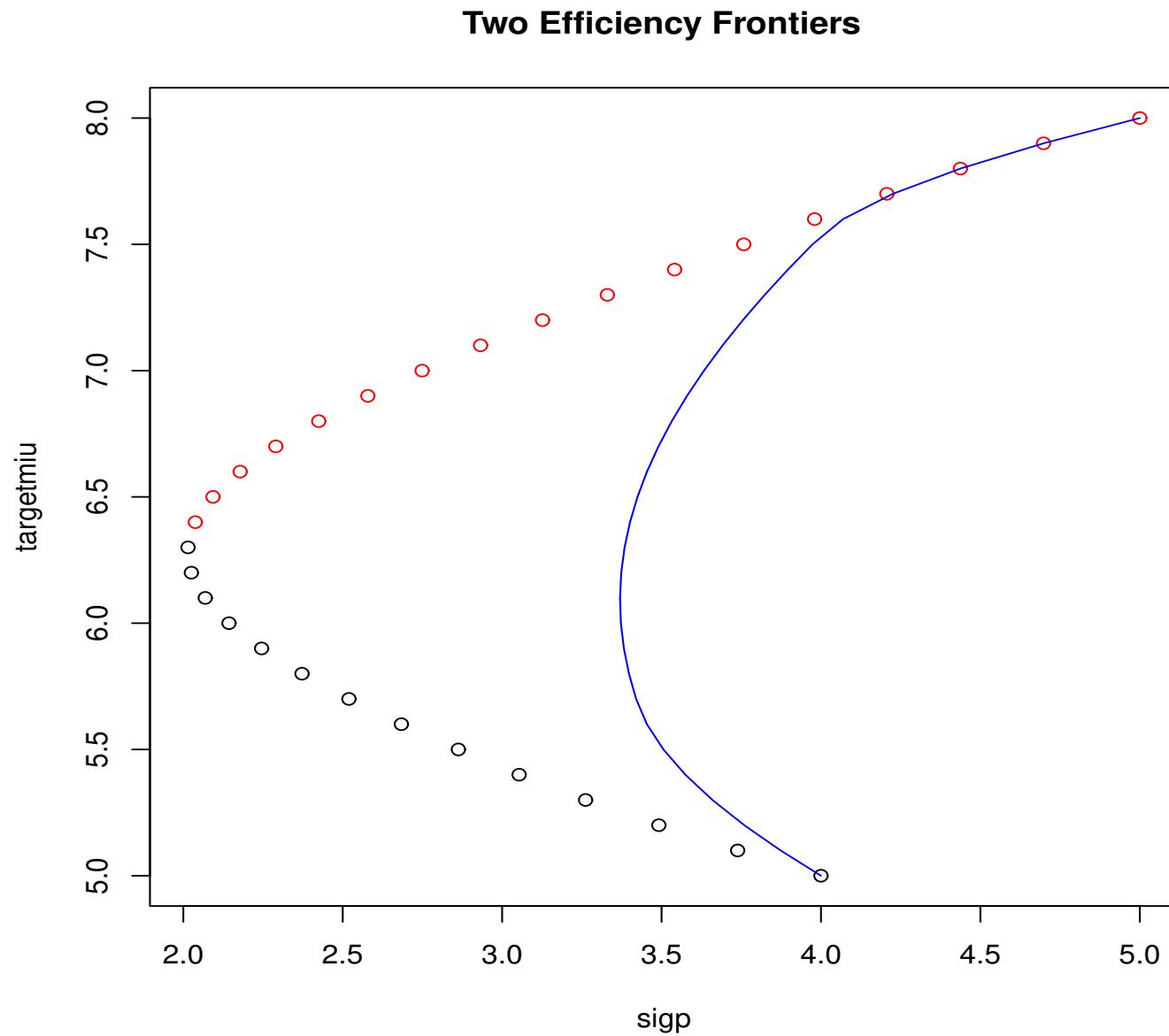
## **Exercise**

Please compare the old efficiency frontier to the new one when

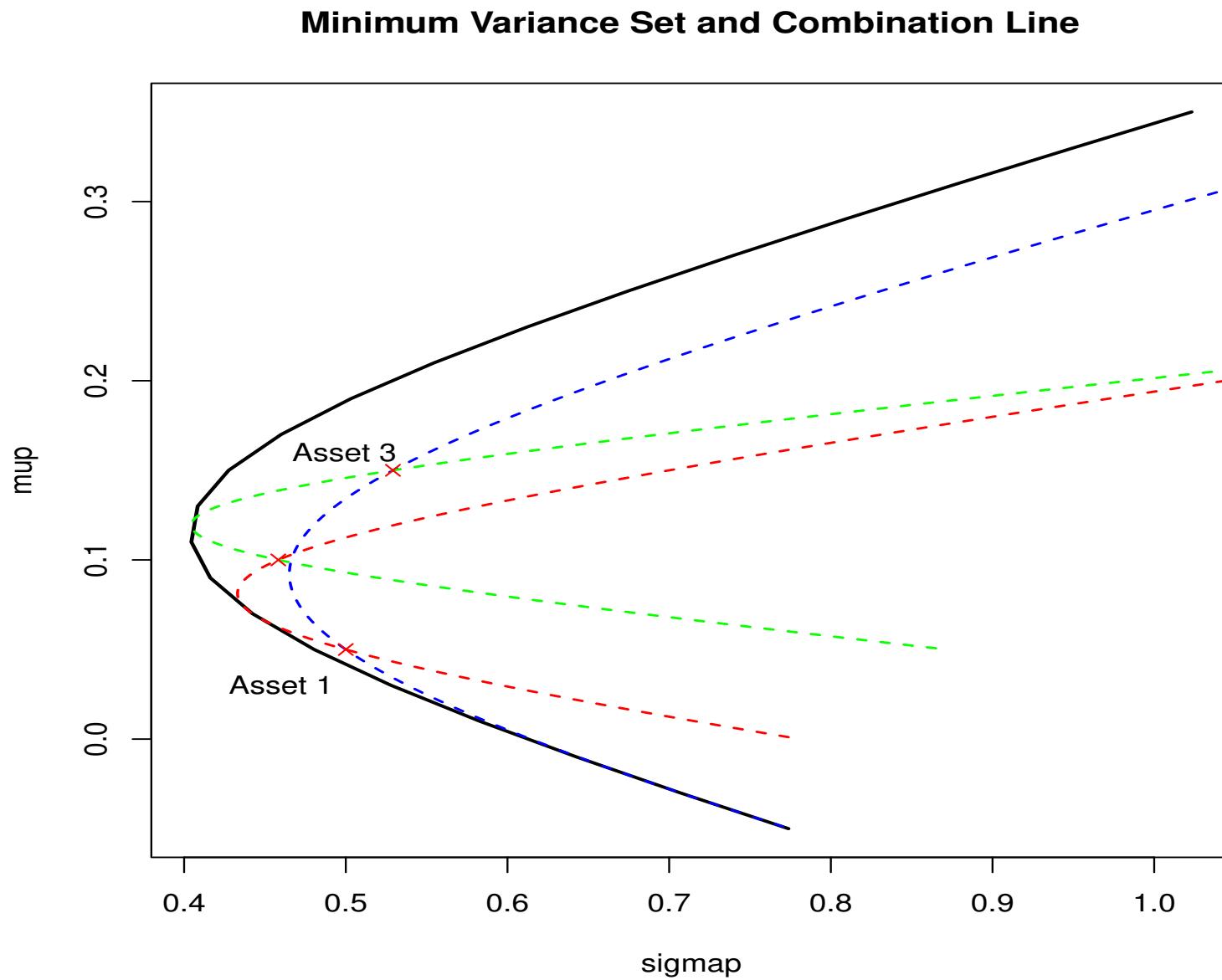
$\rho_{12} = 0.5; \rho_{13} = 0.3; \rho_{23} = 0.4$

Comment on the difference. What is the intuition behind the difference of two efficiency frontiers?

# Two Efficiency Frontiers



# Complete Picture



## Codes-1

```
miu1 = 0.05; miu2=0.10; miu3=0.15
miuv = c(miu1, miu2, miu3)
sigma = matrix(c(0.25,0.15,0.17,0.15,0.21, 0.09, 0.17,0.09,0.28),nrow=3,nc

combination.line = function(muv, Sig) {
  w1 = seq(-2,2,0.02);  w2 = 1-w1
  mupf = w1*muv[1] + w2*muv[2]
  sigpf = sqrt(w1^2*Sig[1,1]+w2^2*Sig[2,2]+2*w1*w2*Sig[1,2])
  return(cbind(mupf,sigpf))
}
cl12 = combination.line(miu[-3],sigma[-3,-3])
cl13 = combination.line(miu[-2],sigma[-2,-2])
cl23 = combination.line(miu[-1],sigma[-1,-1])
```

## Codes-2

```
library(tseries)
minimumvarianceset = function(muv, Sig, targetrange) {
  targetmiu = seq(targetrange[1],targetrange[2],0.02)
  sigp = rep(0, length(targetmiu))
  for (i in 1:length(targetmiu)) {
    op = portfolio.optim(t(muv), pm=targetmiu[i],covmat=Sig, shorts=TRUE)
    sigp[i] = sqrt(t(op$pw)%*%Sig%*%op$pw)
  }
  return(cbind(targetmiu,sigp))
}
mvs = minimumvarianceset(miuv,sigma,c(min(cl13[,1]),max(cl13[,1])))
sigv = sqrt(c(sigma[1,1],sigma[2,2],sigma[3,3]))
```

## Codes-3

```
plot(mvs[,2],mvs[,1], xlab="sigmap",ylab="mup",type="l",lwd=2, main="Minim  
lines(cl12[,2],cl12[,1],col="red",lty=2,lwd=1.5)  
lines(cl13[,2],cl13[,1], col="blue",lty=2,lwd=1.5)  
lines(cl23[,2],cl23[,1], col="green",lty=2,lwd=1.5)  
points(sigv, miuv,col="red",pch=4)  
text(sigv[1]-0.04, miuv[1]-0.02,"Asset 1")  
text(sigv[3]-0.03, miuv[3]+0.01,"Asset 3")
```

## What if there is a risk-free asset

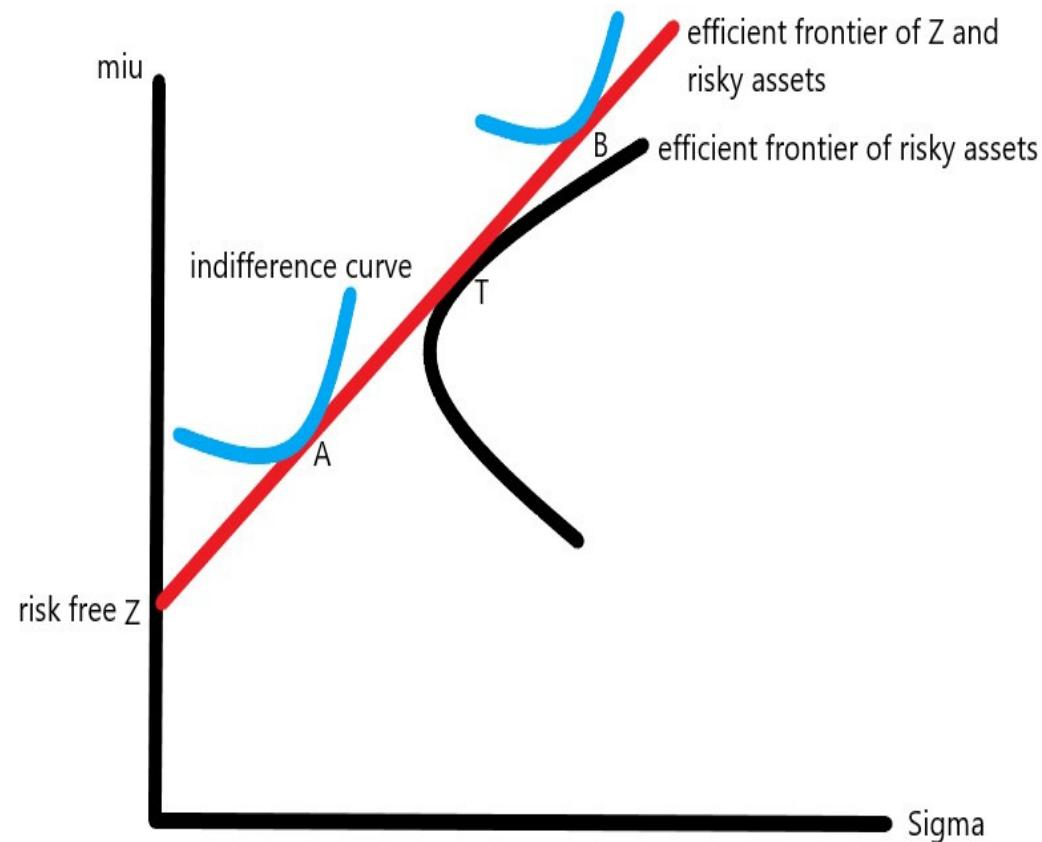
1. Let  $Z$  denote a risk free asset such as Treasury Bill. Its expected return is  $\mu_Z$  and its standard deviation is  $\sigma_Z = 0$
2. In the miu-sigma diagram, the risk free asset  $Z$  is located on the vertical axis
3. The combination line of  $Z$  and any risky asset is a straight line
4. Proof. Consider a portfolio that consists of  $Z$  and a risky asset  $T$ . The expected return and standard deviation of the portfolio is

$$\mu_{ZT} = w\mu_T + (1 - w)\mu_Z \quad (26)$$

$$\sigma_{ZT} = \sqrt{w^2\sigma_T^2 + (1 - w)^2\sigma_Z^2 + 2w(1 - w)\sigma_{ZT}} = \sqrt{w^2\sigma_T^2 + (1 - w)^20 + 2w(1 - w)0} = w\sigma_T \quad (27)$$

where we use the fact that for a constant  $var(c) = 0$ ,  $cov(c, Y) = 0$ . The  $\mu_{ZT}$  and  $\sigma_{ZT}$  are linearly related since both are linear functions of  $w$ .

# Tobin's Separation Theorem



## Remarks

1. The upper portion of the black bullet is the efficient frontier for risky assets
2. When there is a risk-free asset  $Z$ , the efficient frontier becomes the red straight line
3. The blue line represents investor's indifference curve. Its tangent point with the red straight line determines whether the investor borrows (point B) or lends (point A) money
4. Exercise: Let  $\mu_Z = 2, \mu_T = 6, \sigma_T = 5$ . Use R to draw the combination line of  $Z$  and  $T$ .

## **Homework 4 (see syllabus for due date)**

Please do exercise on page 28