MTH 331 - Homework 4 Draft

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Exercise 1:

Assume towards a contradiction, that $\sqrt{2}$ is rational. Then it can be written

$$\sqrt{2} = \frac{a}{b}$$

for some integers $a, b > 0$. We are not assuming $\frac{a}{b}$ is in lowest terms.

Squaring both sides gives

$$2 = \frac{a^2}{b^2} \quad \Rightarrow \quad a^2 = 2b^2.$$

So a^2 is even, which means a must be even. $a = 2k$ for some integer k. Substituting back,

$$(2k)^2 = 2b^2 \quad \Rightarrow \quad 4k^2 = 2b^2 \quad \Rightarrow \quad b^2 = 2k^2.$$

Thus b^2 is even, so b is even as well. In other words, both a and b are divisible by 2.

Define $a_1 = a/2$ and $b_1 = b/2$. Then

$$\sqrt{2} = \frac{a}{b} = \frac{a_1}{b_1}.$$

The exact same reasoning shows a_1, b_1 are both even, and so on for $a_2, b_2, \ldots$

This leads to an infinite descent. That cannot happen, since by the well-ordering principle, the natural numbers must eventually reach a smallest element.

Therefore our assumption was wrong, and $\sqrt{2}$ is irrational.

Exercise 2: Logical equivalences and examples

We want to show

$$(P \wedge Q \Rightarrow R) \equiv (P \wedge \sim R \Rightarrow \sim Q).$$

Starting with $P \wedge Q \Rightarrow R$, we rewrite it as

$$\sim (P \wedge Q) \vee R.$$

This expands to

$$(\sim P \vee \sim Q) \vee R.$$

This is the same as: if P is true and R is false, then Q must also be false. Which is equivalent to

$$(P \wedge \sim R) \Rightarrow \sim Q.$$

So the equivalence is proved.

(a) Nonzero rational $\times$ irrational $=$ irrational. Suppose r is a nonzero rational and s is irrational. If rs were rational, then $s = (rs)/r$ would also be rational, which is a contradiction. So the product must be irrational.

(b) Nonzero integer $\times$ noninteger $=$ noninteger. This statement is false. Example: $m = 2$, $x = \frac{1}{2}$. Then m is an integer, x is noninteger, but $mx = 1$ is an integer. Dividing an integer by another integer does not always give an integer.

(c) Integer $+$ noninteger $=$ noninteger. If m is an integer and $m + x$ were also an integer, then $x = (m + x) - m$ would be an integer, which contradicts the assumption that x is noninteger. Therefore the sum of an integer and a noninteger is always noninteger. For instance, $1 + \frac{1}{2} = \frac{3}{2}$.

(a) is true, (b) is false (counterexample $2 \cdot \frac{1}{2} = 1$), (c) is true.