MTH 331 — Homework #2

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September 17, 2025

Exercise 2: Implication is not associative

We can compare the statements

$$P \Rightarrow (Q \Rightarrow R) \quad \text{and} \quad (P \Rightarrow Q) \Rightarrow R,$$

and also relate them to $(P \wedge Q) \Rightarrow R$.

Full truth tables with intermediate columns

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow R$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	F
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	T	T	T	T
F	F	F	T	T	T	F

The last two columns differ on several rows, so

$$P \Rightarrow (Q \Rightarrow R) \quad \text{is not logically equivalent to} \quad (P \Rightarrow Q) \Rightarrow R.$$

Equivalence with $(P \wedge Q) \Rightarrow R$ by a truth table

P	Q	R	$P \Rightarrow (Q \Rightarrow R)$	$(P \wedge Q) \Rightarrow R$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

The two columns match in every row, so

$$P \Rightarrow (Q \Rightarrow R) \quad \text{is logically equivalent to} \quad (P \wedge Q) \Rightarrow R.$$

Why the equivalence is natural

$P \Rightarrow (Q \Rightarrow R)$ says: if P holds, then the inner implication $Q \Rightarrow R$ must hold. The inner implication fails only when Q is true and R is false. So the whole statement fails only in the case P and Q are both true and R is false. That is exactly when $(P \wedge Q) \Rightarrow R$ fails. This is why they are equivalent.

Is $P \Rightarrow (Q \Rightarrow R)$ equivalent to $Q \Rightarrow (P \Rightarrow R)$?

Yes. From above, $P \Rightarrow (Q \Rightarrow R)$ is equivalent to $(P \wedge Q) \Rightarrow R$. By the same reasoning with P and Q swapped, $Q \Rightarrow (P \Rightarrow R)$ is also equivalent to $(P \wedge Q) \Rightarrow R$. Since both are equivalent to the same statement, they are equivalent to each other.

Implication only form of $(P \wedge Q \wedge R) \Rightarrow S$

Use the identity $A \wedge B \equiv \neg(A \Rightarrow \neg B)$. Applying it twice,

$$(P \wedge Q \wedge R) \Rightarrow S \quad \equiv \quad \big(P \Rightarrow \neg(Q \Rightarrow \neg R)\big) \Rightarrow S,$$

which uses only $\Rightarrow$ and $\neg$. It is correct because $Q \wedge R$ is true exactly when $Q \Rightarrow \neg R$ is false.

Exercise 3: The statement $((P \Rightarrow \neg P) \Rightarrow P) \Rightarrow P$ is a tautology

Let

$$S = ((P \Rightarrow \neg P) \Rightarrow P) \Rightarrow P.$$

Truth table

Since S depends only on P, there are only two cases to check.

P	$P \Rightarrow \neg P$	$(P \Rightarrow \neg P) \Rightarrow P$	S
T	F	T	T
F	T	F	T

In both rows S is true, so the whole statement is a tautology.

Step-by-step equivalences

We can also show S is a tautology by simplifying it with standard equivalences.

$$\begin{aligned}
S &= ((P \Rightarrow \neg P) \Rightarrow P) \Rightarrow P \\
&\equiv \neg\big((P \Rightarrow \neg P) \Rightarrow P\big) \vee P && \text{rewrite outer implication} \\
&\equiv \big((P \Rightarrow \neg P) \wedge \neg P\big) \vee P && \text{negate inner implication} \\
&\equiv \big((\neg P \vee \neg P) \wedge \neg P\big) \vee P && \text{rewrite } P \Rightarrow \neg P \\
&\equiv (\neg P \wedge \neg P) \vee P && \text{idempotence of } \vee \\
&\equiv \neg P \vee P \\
&\equiv P \Rightarrow P.
\end{aligned}$$

Since $P \Rightarrow P$ is always true, S is a tautology.

Explanation in words

Work from the inside out. The statement $P \Rightarrow \neg P$ is true only when P is false, so it is equivalent to $\neg P$. That makes the middle piece $(P \Rightarrow \neg P) \Rightarrow P$ say "if $\neg P$ then P," which is just another way of saying P. Finally S becomes $P \Rightarrow P$, which is always true. This matches the everything above and confirms that S is a tautology.