MTH 331 – Homework 4 Final Draft

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Exercise 1

Assume, towards a contradiction, that $\sqrt{2}$ is rational. Then it can be written

$$\sqrt{2} = \frac{a}{b}$$

for some integers $a, b > 0$. We are not assuming $\frac{a}{b}$ is in lowest terms.

Squaring both sides gives

$$2 = \frac{a^2}{b^2} \quad \Rightarrow \quad a^2 = 2b^2.$$

So a^2 is even, which means a must be even. This is because if a were odd, then a^2 would also be odd. Hence $a = 2k$ for some integer k. Substituting back,

$$(2k)^2 = 2b^2 \quad \Rightarrow \quad 4k^2 = 2b^2 \quad \Rightarrow \quad b^2 = 2k^2.$$

Thus b^2 is even, so b is even as well. In other words, both a and b are divisible by 2.

Define $a_1 = a/2$ and $b_1 = b/2$. Then

$$\sqrt{2} = \frac{a}{b} = \frac{a_1}{b_1}.$$

Applying the same reasoning again, we find that a_1 and b_1 are also even. We can therefore define $a_2 = a_1/2$ and $b_2 = b_1/2$, and in general,

$$a_{n+1} = \frac{a_n}{2}, \qquad b_{n+1} = \frac{b_n}{2}.$$

This process repeats indefinitely, producing an infinite sequence $a_0, a_1, a_2, \ldots$ of positive integers, each smaller than the one before it.

By the well-ordering principle, the set $\{a_0, a_1, a_2, \ldots\}$ must have a least element. However, it cannot, since each a_{n+1} is less than a_n. This contradiction shows that our original assumption was false.

Therefore, $\sqrt{2}$ is irrational.

Exercise 2: Logical equivalences and examples

We want to show

$$(P \wedge Q \Rightarrow R) \ \equiv \ (P \wedge \sim R \Rightarrow \sim Q).$$

Starting with $P \wedge Q \Rightarrow R$, we can use the equivalence $A \Rightarrow B \equiv (\sim A \vee B)$:

$$P \wedge Q \Rightarrow R \ \equiv \sim (P \wedge Q) \vee R.$$

This simplifies to

$$(\sim P \vee \sim Q) \vee R.$$

Now start with the second statement and apply the same steps:

$$(P \wedge \sim R \Rightarrow \sim Q) \ \equiv \sim (P \wedge \sim R) \vee \sim Q.$$

This becomes

$$(\sim P \vee R) \vee \sim Q.$$

Since both expressions simplify to a disjunction of the same three parts $(\sim P, \sim Q, R)$ in some order, they are logically equivalent.

This logical equivalence can be used to prove the following statements.

(a) Nonzero rational $\times$ irrational = irrational. Let r be a nonzero rational and s be irrational. Suppose, for contradiction, that rs is rational. Then $s = (rs)/r$. Since a rational number divided by a nonzero rational number is rational, this would make s rational — a contradiction. Therefore the product must be irrational.

(b) Nonzero integer $\times$ noninteger = noninteger. This statement is false. Example: $m = 2$, $x = \frac{1}{2}$. Then m is an integer, x is noninteger, but $mx = 1$ is an integer. The step that fails here is the assumption that multiplying a noninteger by a nonzero integer always produces another noninteger. This shows the conclusion of the logical equivalence does not hold for this case.

(c) Integer + noninteger = noninteger. If m is an integer and x is noninteger, assume for contradiction that $m+x$ is an integer. Then $x = (m+x)-m$ would also be an integer, contradicting the assumption that x is noninteger. So the sum of an integer and a noninteger is always noninteger. For example, $1 + \frac{1}{2} = \frac{3}{2}$.

(a) is true, (b) is false (counterexample $2 \cdot \frac{1}{2} = 1$), and (c) is true.