MTH 331 – Homework 7

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Exercise 1

Consider the relation R on $\mathbb{Q}$ defined by

$$R = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x - y \in \mathbb{Z}\}.$$

In other words, xRy means that x and y differ by an integer.

(a) R is an equivalence relation

Reflexive. Take any $x \in \mathbb{Q}$. Then

$$x - x = 0,$$

and 0 is an integer. So $(x, x) \in R$, and R is reflexive.

Symmetric. Now suppose $(x, y) \in R$. Then $x - y$ is an integer; write $k = x - y$. Then

$$y - x = -(x - y) = -k.$$

The integers are closed under taking negatives, so $-k \in \mathbb{Z}$, which means $y - x \in \mathbb{Z}$ and hence $(y, x) \in R$. So R is symmetric.

Transitive. Finally, suppose $(x, y) \in R$ and $(y, z) \in R$. Then $x - y$ and $y - z$ are both integers. Let $m = x - y$ and $n = y - z$. Then

$$x - z = (x - y) + (y - z) = m + n.$$

Since the integers are closed under addition, $m + n$ is an integer. So $x - z \in \mathbb{Z}$ and $(x, z) \in R$. This shows R is transitive.

R is reflexive, symmetric, and transitive, so R is an equivalence relation on $\mathbb{Q}$.

(b) Infinitely many equivalence classes

For $x \in \mathbb{Q}$, the equivalence class of x is

$$[x] = \{y \in \mathbb{Q} : x - y \in \mathbb{Z}\}.$$

So $[x]$ consists of all rationals that differ from x by an integer.

To see that there are infinitely many different classes, it is convenient to pick one representative from the interval $[0, 1)$ for each class. Let $r, s \in \mathbb{Q} \cap [0, 1)$ with $r \neq s$. Then $r - s$ is a rational number with

$$-1 < r - s < 1.$$

The only integer between -1 and 1 is 0. Since $r \neq s$, we have $r - s \neq 0$, so $r - s \notin \mathbb{Z}$. That means r and s are not related, so $[r] \neq [s]$.

There are infinitely many rational numbers in $[0, 1)$, and each one gives a distinct equivalence class. Therefore R has infinitely many equivalence classes.

(c) Examples

Here are examples (or non-examples) for each type of partition.

I. A partition of $\mathbb{Z}$ (or $\mathbb{N}$) into finitely many infinite sets.
Fix an integer $k \geq 2$ and look at congruence classes mod k. On $\mathbb{Z}$, define

$$A_j = \{n \in \mathbb{Z} : n \equiv j \pmod{k}\}, \quad j = 0, 1, \dots, k-1.$$

Each A_j is infinite, the sets A_j are pairwise disjoint, and together they cover all of $\mathbb{Z}$. So $\{A_0, \dots, A_{k-1}\}$ is a partition of $\mathbb{Z}$ into finitely many infinite sets.

II. A partition of $\mathbb{Z}$ (or $\mathbb{N}$) into infinitely many finite sets.
An easy example is the partition of $\mathbb{Z}$ into singletons:

$$P = \{\{n\} : n \in \mathbb{Z}\}.$$

Each set $\{n\}$ has one element, so it is finite. The sets are pairwise disjoint and their union is all of $\mathbb{Z}$. There are infinitely many of them, so this gives a partition into infinitely many finite sets. (We could also group elements into pairs, e.g. $\{2n, 2n+1\}$, and get a similar example.)

III. A partition of $\mathbb{Z}$ (or $\mathbb{N}$) into finitely many finite sets.
This cannot happen. Suppose, for contradiction, that

$$\mathbb{Z} = B_1 \cup \cdots \cup B_m$$

for some finite collection of finite sets $B_1, \dots, B_m$. A finite union of finite sets is finite, so the right-hand side would be finite. But $\mathbb{Z}$ is infinite, so this is impossible. The same argument shows there is no such partition of $\mathbb{N}$ either.

IV. A partition of $\mathbb{N}$ (or $\mathbb{Z}$) into infinitely many infinite sets.
Here is a standard example on $\mathbb{N} = \{1, 2, 3, \dots\}$. Every positive integer can be written uniquely as

$$n = 2^k m,$$

where $k \geq 0$ and m is odd. For each $k \geq 0$ define

$$S_k = \{n \in \mathbb{N} : n = 2^k m \text{ for some odd } m\}.$$

Each S_k is infinite (there are infinitely many odd numbers m), the sets S_k are pairwise disjoint (the exponent k of 2 in this factorization is unique), and their union is all of $\mathbb{N}$. So $\{S_0, S_1, S_2, \dots\}$ is a partition of $\mathbb{N}$ into infinitely many infinite sets.

Exercise 2

Recall that if R is an equivalence relation on a set A, then the set of its equivalence classes

$$P_R = \{[a] : a \in A\}$$

is a partition of A. Conversely, if P is a partition of A, we can define a relation R_P on A by

$$R_P = \bigcup_{B \in P} (B \times B).$$

(a) The relation R_P for a specific partition

Let $A = \{0, 1, 2, 3, 4, 5\}$ and

$$P = \big\{\{0\}, \{1, 2\}, \{3, 4, 5\}\big\}.$$

By definition,

$$R_P = \bigcup_{B \in P} (B \times B).$$

We can write this out block by block:

$$\{0\} \times \{0\} = \{(0, 0)\},$$

$$\{1, 2\} \times \{1, 2\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\},$$

$$\{3, 4, 5\} \times \{3, 4, 5\} = \{(3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}.$$

Putting these together,

$$\begin{aligned} R_P = {} & \{(0, 0)\} \\ & \cup \{(1, 1), (1, 2), (2, 1), (2, 2)\} \\ & \cup \{(3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}. \end{aligned}$$

So two elements of A are related by R_P exactly when they lie in the same part of the partition P. For instance, 1 and 2 are related, and 3 and 5 are related, but 0 and 1 are not, and 2 and 3 are not.

(b) In general, R_P is an equivalence relation

Now let A be any set and let P be a partition of A. This means:

- Each $B \in P$ is a nonempty subset of A,
- The sets in P are pairwise disjoint, and
- Every element of A lies in at least one $B \in P$.

Define

$$R_P = \bigcup_{B \in P} (B \times B).$$

We check the three properties.

Reflexive. Let $a \in A$. Since P is a partition, a belongs to some block $B \in P$. Then $(a, a) \in B \times B$, so $(a, a) \in R_P$.

Symmetric. Suppose $(a, b) \in R_P$. Then $(a, b) \in B \times B$ for some $B \in P$, which means a and b are both in B. It follows that $(b, a) \in B \times B$ as well, so $(b, a) \in R_P$.

Transitive. Suppose $(a, b) \in R_P$ and $(b, c) \in R_P$. Then there are blocks $B_1, B_2 \in P$ such that

$$(a, b) \in B_1 \times B_1 \quad \text{and} \quad (b, c) \in B_2 \times B_2.$$

So $a, b \in B_1$ and $b, c \in B_2$. In particular, $b \in B_1 \cap B_2$. But different blocks in a partition are disjoint, so if $B_1 \cap B_2$ is nonempty, we must have $B_1 = B_2$. Call this common block B. Then $a, c \in B$, which means $(a, c) \in B \times B \subseteq R_P$.

Therefore R_P is reflexive, symmetric, and transitive, so it is an equivalence relation on A.