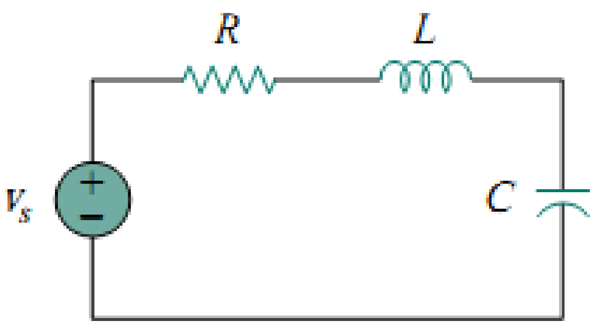
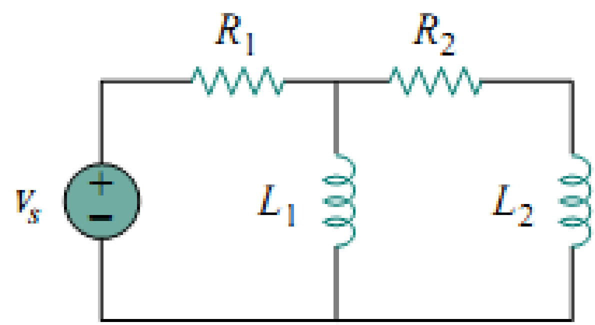
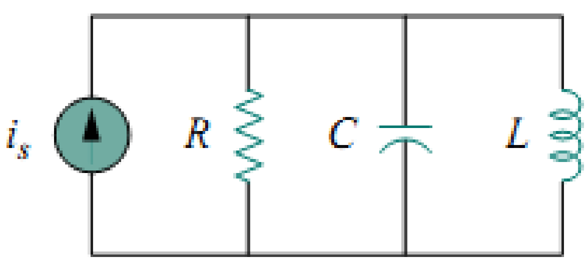
Second Order Circuits

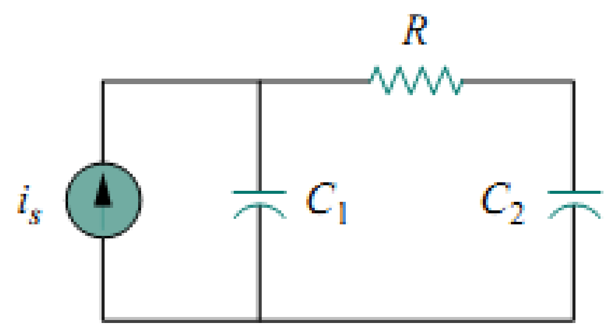
Introduction

A second-order circuit is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements.









Response of RLC Circuits

The complete response of the circuit is the sum of the natural response and the forced response.

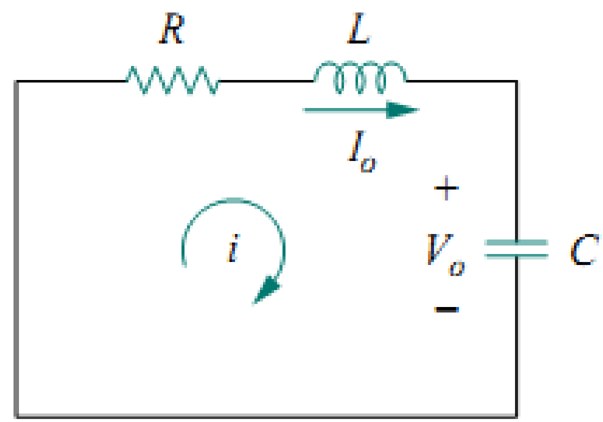
$$x(t) = x_f(t) + x_n(t)$$

The forced response or steady-state response is the behavior of the circuit a long time after an external excitation is applied.

The natural response or transient response is the circuit's temporary response that will die out with time.

The Natural Response

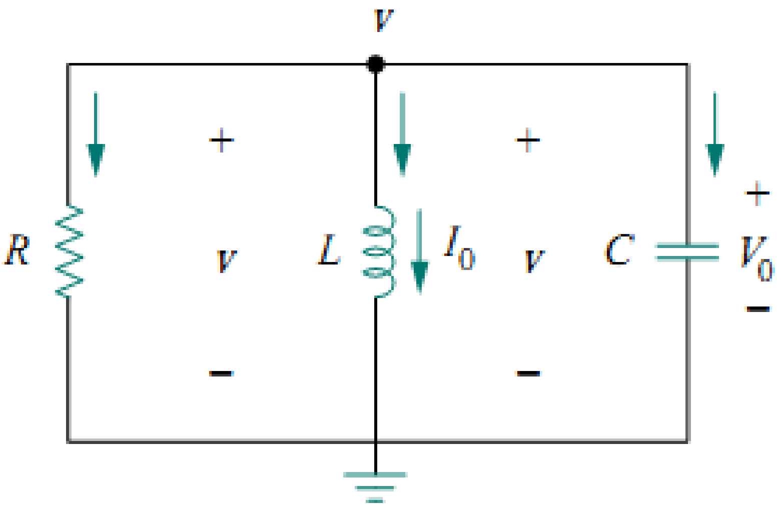
Series RLC Circuit



$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0$$

$$\boxed{\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0}$$

Parallel RLC Circuit



$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v\,dt + C\frac{dv}{dt} = 0$$

$$\boxed{\frac{d^2 v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0}$$

General Equation

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

$$\frac{d^2 v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$$\frac{d^2 x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^{\ 2}x = 0$$

Series RLC Circuit

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

Parallel RLC Circuit

$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

General Solution

Overdamped Case ($\alpha > \omega_0$)

$$\boxed{x = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

$$\frac{dx}{dt} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^{\ 2}}$$

Critically Damped Case ($\alpha = \omega_0$)

$$\boxed{x = e^{-\alpha t}\left(A_1 t + A_2\right)}$$

$$\frac{dx}{dt} = e^{-\alpha t}\left(A_1 - \alpha(A_1 t + A_2)\right)$$

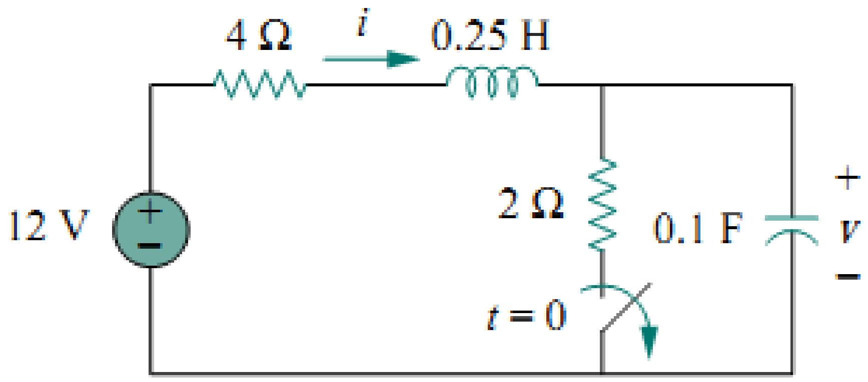
Underdamped Case ($\alpha < \omega_0$)

$$\boxed{x = e^{-\alpha t}\left(A_1 \cos \omega_d t + A_2 \sin \omega_d t\right)}$$

$$\frac{dx}{dt} = e^{-\alpha t}\left((A_2 \omega_d - A_1 \alpha) \cos \omega_d t - (A_1 \omega_d + A_2 \alpha) \sin \omega_d t\right)$$

$$\omega_d = \sqrt{\omega_0^{\ 2} - \alpha^2}$$

Finding Initial Values



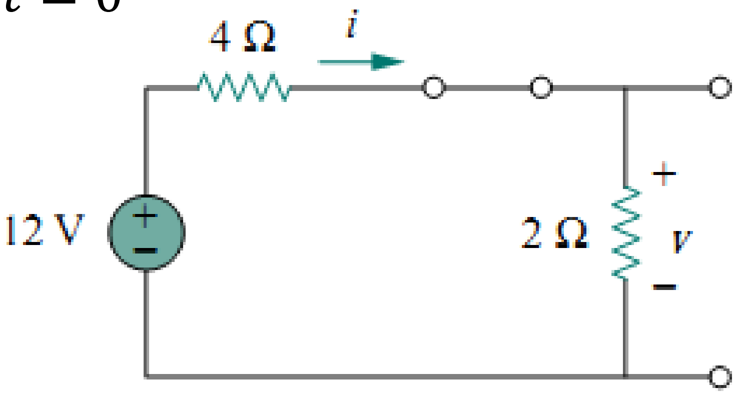
Find: (a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)dt$, $dv(0^+)/dt$.

The capacitor voltage is always continuous ➡ $v(0^+) = v(0^-)$

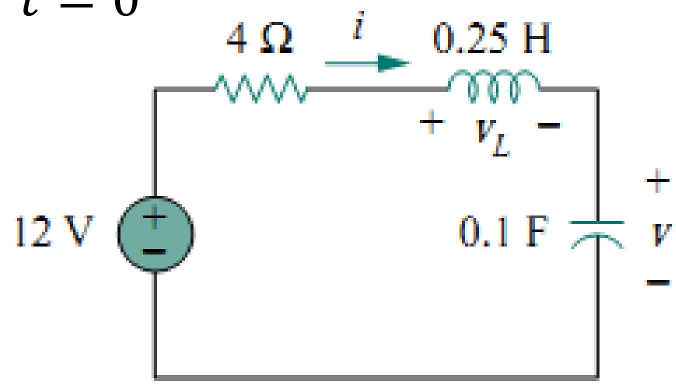
The inductor current is always continuous ➡ $i(0^+) = i(0^-)$

Equivalent circuit

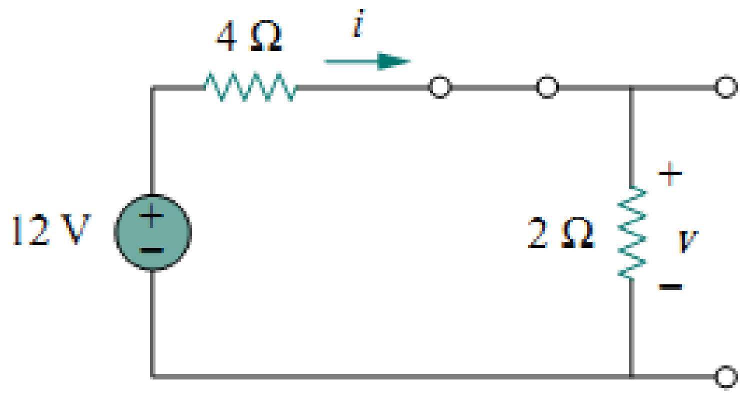
$t = 0^-$



$t = 0^+$



$t = 0^-$

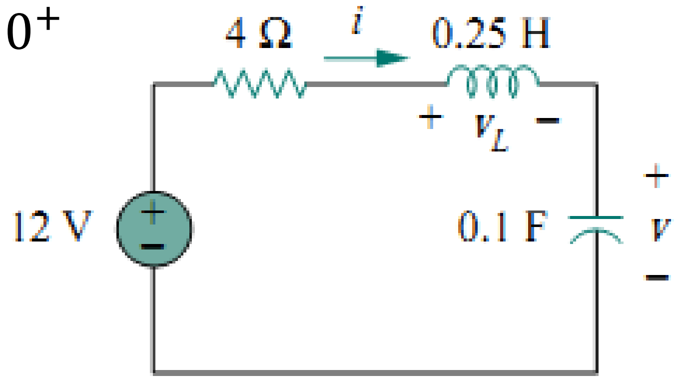


$$i(0^-) = \frac{12}{4+2} = 2\text{ A}, \qquad v(0^-) = 2i(0^-) = 4\text{ V}$$

As the inductor current and the capacitor voltage cannot change abruptly

$$i(0^+) = i(0^-) = 2\text{ A}, \qquad v(0^+) = v(0^-) = 4\text{ V}$$

$t = 0^+$



The same current flows through both the inductor and capacitor

$$i_C(0^+) = i(0^+) = 2\text{ A}$$

$C\, dv/dt = i_C,\ dv/dt = i_C/C,$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20\text{ V/s}$$

$L\, di/dt = v_L,\ di/dt = v_L/L$

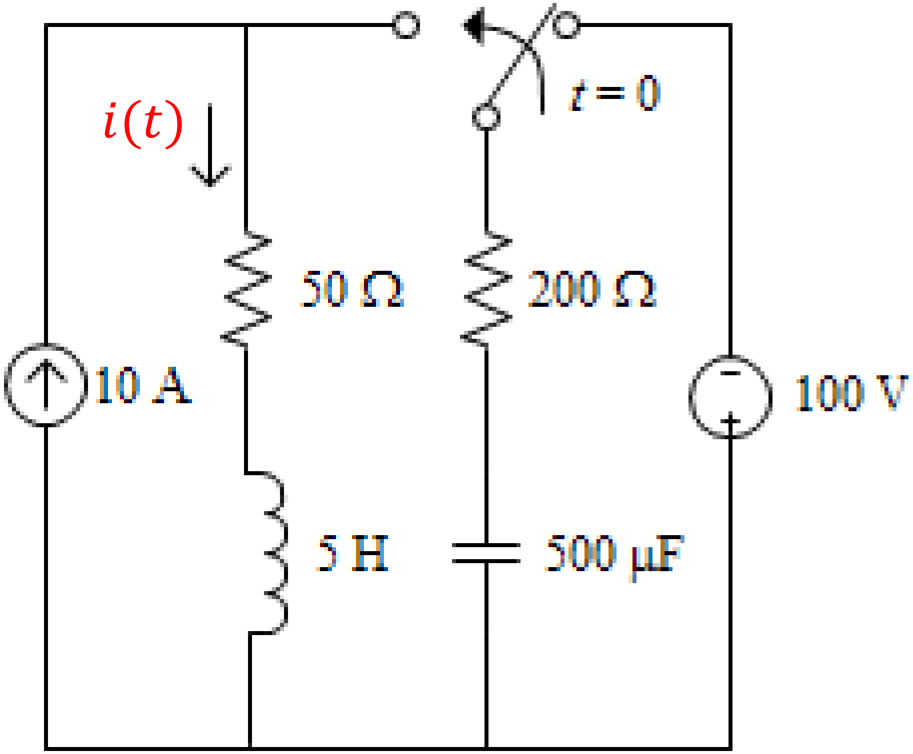
$$-12 + 4i(0^+) + v_L(0^+) + v(0^+) = 0$$

$$v_L(0^+) = 12 - 8 - 4 = 0$$

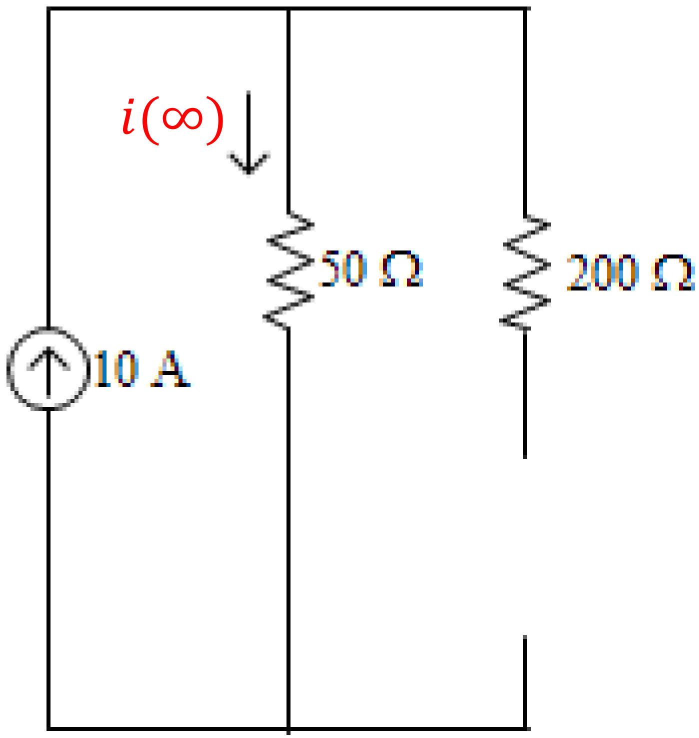
$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0\text{ A/s}$$

Example

1 Find $i(t)$ for $t > 0$



The forced response



$t \to \infty$

$$i(\infty) = 10\ \mathrm{A}$$

$$\boxed{i_f(t) = i(\infty) = 10\ \mathrm{A}}$$

The natural response

$$\alpha = \frac{R}{2L} = \frac{(200 + 50)}{2\ (5)} = 25$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{5(500 \cdot 10^{-6})}} = \frac{100}{5} = 20$$

$$\alpha > \omega_0 \quad \Rightarrow \quad \text{Overdamped}$$

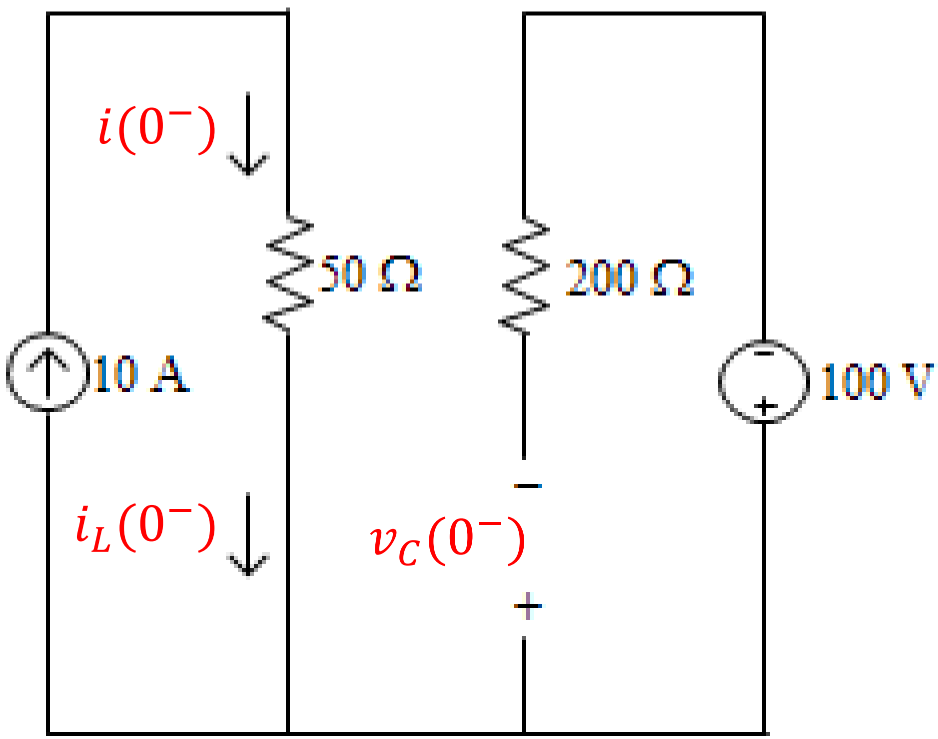
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -25 \pm \sqrt{625 - 400} = -25 \pm 15$$

$$s_1 = -10 \qquad s_2 = -40$$

$$\boxed{i_n(t) = A_1 e^{-10t} + A_2 e^{-40t}\ \text{A}}$$

The initial values

$t = 0^-$

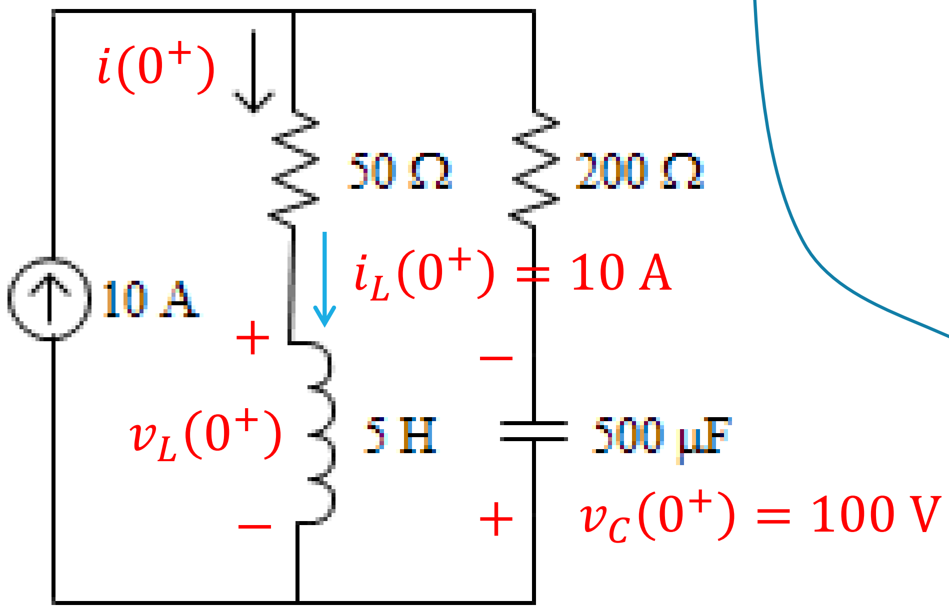


$$i(0^-) = 10\ \text{A}$$

$$i_L(0^-) = i(0^-) = 10\ \text{A}$$

$$v_C(0^-) = 100\ \text{V}$$

$t = 0^+$

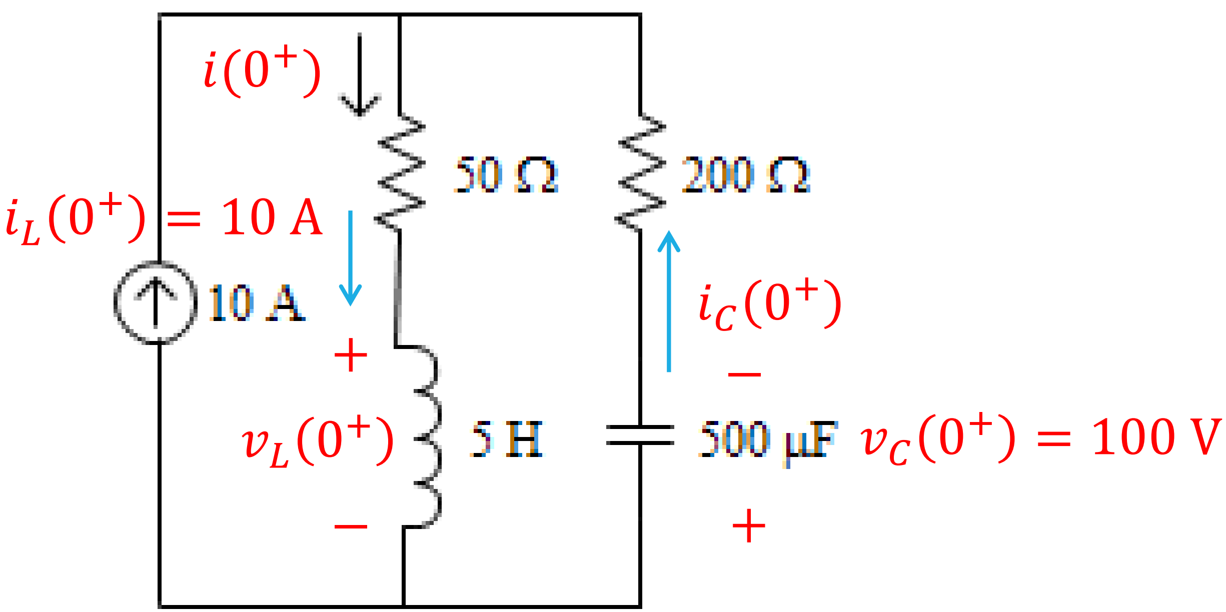


$$i_L(0^+) = i_L(0^-) = 10 \text{ A}$$

$$v_C(0^+) = v_C(0^-) = 100 \text{ V}$$

$$\boxed{i(0^+) = i_L(0^+) = 10 \text{ A}}$$

$$\frac{di(0^+)}{dt} = \frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$$



$$50\ i(0^+) + v_L(0^+) + v_C(0^+) + 200\ i_C(0^+) = 0$$

$$50\ (10) + v_L(0^+) + 100 + 200\ (0) = 0$$

$$v_L(0^+) = -600\ \text{V}$$

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{-600}{5}$$

$$\boxed{\frac{di(0^+)}{dt} = -120\ \text{A/s}}$$

The complete response

$$i(t) = i_f(t) + i_n(t)$$

$$\boxed{i(t) = 10 + A_1 e^{-10t} + A_2 e^{-40t} \text{ A}}$$

$$\frac{di(t)}{dt} = -10\, A_1 e^{-10t} - 40\, A_2 e^{-40t} \text{ A/s}$$

$\boxed{t = 0^+}$

$$\left.\begin{aligned} i(0^+) &= 10 + A_1 + A_2 \\ 10 &= 10 + A_1 + A_2 \end{aligned}\right\} \quad A_1 + A_2 = 0 \qquad (1)$$

$$\left.\begin{aligned} \frac{di(0^+)}{dt} &= -10\, A_1 - 40\, A_2 \\ -120 &= -10\, A_1 - 40\, A_2 \end{aligned}\right\} \quad A_1 + 4\, A_2 = 12 \qquad (2)$$

From equation (1) and (2) : $A_1 = -4 \quad A_2 = 4$

$$\boxed{i(t) = 10 - 4\,e^{-10t} + 4\,e^{-40t}\ \mathrm{A} \qquad (t > 0)}$$

The additional note

$$i(t) = \begin{cases} 10\ \mathrm{A} & t < 0 \\ 10 - 4\,e^{-10t} + 4\,e^{-40t}\ \mathrm{A} & t > 0 \end{cases}$$