Properties of Fourier Transform

The properties of the Fourier transform are summarized below. The properties of the Fourier expansion of periodic functions discussed above are special cases of those listed here. In the following, we assume $\mathcal{F}[x(t)]=X(j\omega)$ and $\mathcal{F}[y(t)]=Y(j\omega)$.

- **Linearity**

$$\mathcal{F}[ax(t)+by(t)]=a\mathcal{F}[x(t)]+b\mathcal{F}[y(t)]$$

- **Time shift**

$$\mathcal{F}[x(t\pm t_0)]=X(j\omega)e^{\pm j\omega t_0}$$

Proof: Let $t'=t\pm t_0$, i.e., $t=t'\mp t_0$, we have

$$\begin{aligned}\mathcal{F}[x(t\pm t_0)] &= \int_{-\infty}^{\infty} x(t\pm t_0)e^{-j\omega t}dt=\int_{-\infty}^{\infty} x(t')e^{-j\omega(t'\mp t_0)}dt' \\ &= e^{\pm j\omega t_0}\int_{-\infty}^{\infty} x(t')e^{-j\omega t'}dt'=X(j\omega)e^{\pm j\omega t_0}\end{aligned}$$

- **Frequency shift**

$$\mathcal{F}^{-1}[X(j\omega\pm\omega_0)]=x(t)e^{\mp j\omega_0 t}$$

Proof: Let $\omega'=\omega\pm\omega_0$, i.e., $\omega=\omega'\mp\omega_0$, we have

$$\begin{aligned}\mathcal{F}^{-1}[X(j(\omega\pm\omega_0))] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j(\omega\pm\omega_0))e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega')e^{j\omega(\omega'\mp\omega_0)}d\omega' \\ &= e^{\mp j\omega_0 t}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega')e^{j\omega'}d\omega'=x(t)e^{\mp j\omega_0 t}\end{aligned}$$

- **Time reversal**

$$\mathcal{F}[x(-t)]=X(-\omega)$$

Proof:

$$\mathcal{F}[x(-t)] = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt$$

Replacing t by $-t'$, we get

$$\mathcal{F}[x(-t)] = -\int_{\infty}^{-\infty} x(t')e^{j\omega t'}dt' = \int_{-\infty}^{\infty} x(t')e^{j\omega t'}dt' = X(-\omega)$$

- **Even and Odd Signals and Spectra**

  If the signal $x(t)$ is an even (or odd) function of time, its spectrum $X(j\omega)$ is an even (or odd) function of frequency:

$$\text{if } \; x(t) = x(-t) \quad \text{then} \quad X(j\omega) = X(-j\omega)$$

and

$$\text{if } \; x(t) = -x(-t) \quad \text{then} \quad X(j\omega) = -X(-j\omega)$$

Proof: If $x(t) = x(-t)$ is even, then according to the time reversal property, we have

$$X(j\omega) = \mathcal{F}[x(t)] = \mathcal{F}[x(-t)] = X(-\omega)$$

i.e., the spectrum $X(j\omega) = X(-\omega)$ is also even. Similarly, if $x(t) = -x(-t)$ is odd, we have

$$X(j\omega) = \mathcal{F}[x(t)] = \mathcal{F}[-x(-t)] = -X(-\omega)$$

i.e., the spectrum $X(j\omega) = -X(-\omega)$ is also odd.

- **Time and frequency scaling**

$$\mathcal{F}[x(at)] = \frac{1}{a}X(\frac{\omega}{a}) \quad \text{or} \quad \mathcal{F}[ax(at)] = X(\frac{\omega}{a})$$

Proof: Let $u = at$, i.e., $t = u/a$, where $a > 0$ is a scaling factor, we have

$$\mathcal{F}[x(at)]=\int_{-\infty}^{\infty}x(at)e^{-j\omega t}dt=\int_{-\infty}^{\infty}x(u)e^{-j\omega u/a}d(u/a)=\frac{1}{a}X(\frac{\omega}{a})$$

Note that when $a<1$, time function $x(at)$ is stretched, and $X(j\omega/a)$ is compressed; when $a>1$, $x(at)$ is compressed and $X(j\omega/a)$ is stretched. This is a general feature of Fourier transform, i.e., compressing one of the $x(t)$ and $X(j\omega)$ will stretch the other and vice versa. In particular, when $a\rightarrow 0$, $x(at)$ is stretched to approach a constant, and $X(j\omega/a)/a$ is compressed with its value increased to approach an impulse; on the other hand, when $a\rightarrow\infty$, $ax(at)$ is compressed with its value increased to approach an impulse and $X(j\omega/a)$ is stretched to approach a constant.

- **Complex Conjugation**

$$\mbox{if}\;\;\mathcal{F}[x(t)]=X(j\omega),\quad \mbox{then}\;\mathcal{F}[x^*(t)]=X^*(-j\omega)$$

Proof: Taking the complex conjugate of the inverse Fourier transform, we get

$$x^*(t)=[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega]^*=\frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(j\omega)e^{-j\omega t}d\omega$$

Replacing ω by $-\omega'$ we get the desired result:

$$x^*(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(-\omega')e^{j\omega' t}d\omega'=\mathcal{F}^{-1}[X^*(-\omega)]$$

We further consider two special cases:

- If $x(t)=x^*(t)$ is real, then

$$\mathcal{F}[x(t)]=X(j\omega)=X_r(j\omega)+jX_i(j\omega)$$
$$\mathcal{F}[x^*(t)]=X^*(-\omega)=X_r(-\omega)-jX_i(-\omega)$$

i.e., the real part of the spectrum is even (with respect to frequency ω), and the imaginary part is odd:

$$\left\{\begin{array}{l}X_r(j\omega)=X_r(-j\omega)\\ X_i(j\omega)=-X_i(-j\omega)\end{array}\right.$$

- If $x(t)=-x^*(t)$ is imaginary, then

$$\mathcal{F}[x(t)] = X(j\omega) = X_r(j\omega) + jX_i(j\omega)$$

$$\mathcal{F}[-x^*(t)] = -X^*(-j\omega) = -X_r(-j\omega) + jX_i(-j\omega)$$

i.e., the real part of the spectrum is odd, and the imaginary part is even:

$$\begin{cases} X_r(j\omega) = -X_r(-j\omega) \\ X_i(j\omega) = X_i(-j\omega) \end{cases}$$

If the time signal $x(t)$ is one of the four combinations shown in the table (real even, real odd, imaginary even, and imaginary odd), then its spectrum $X(j\omega)$ is given in the corresponding table entry:

	if $x(t)$ is real	if $x(t)$ is imaginary
	X_r even, X_i odd	X_r odd, X_i even
if $x(t)$ is Even		
X_r and X_i even	$X_i = 0$, $X = X_r$ even	$X_r = 0$, $X = X_i$ even
if $x(t)$ is Odd		
X_r and X_i odd	$X_r = 0$, $X = X_i$ odd	$X_i = 0$, $X = X_r$ odd

Note that if a real or imaginary part in the table is required to be both even and odd at the same time, it has to be zero.

These properties are summarized below:

	$x(t) = x_r(t) + jx_i(t)$	$X(j\omega) = X_r(j\omega) + jX_i(j\omega)$
1	real $x(t) = x_r(t)$	even $X_r(j\omega)$, odd $X_i(j\omega)$
2	real and even $x(-t) = x_r(t)$	real and even $X_r(j\omega)$
3	real and odd $x(-t) = -x_r(t)$	imaginary and odd $X_i(j\omega)$
4	imaginary $x(t) = x_i(t)$	odd $X_r(j\omega)$, even $X_i(j\omega)$
5	imaginary and even $x(-t) = x_i(t)$	imaginary and even $X_i(j\omega)$
6	imaginary and odd $x(-t) = -x_i(t)$	real and odd $X_r(j\omega)$

As any signal can be expressed as the sum of its even and odd components, the first three items above indicate that the spectrum of the even part of a real signal is real and even, and the spectrum of the odd part of the signal is imaginary and odd.

- **Symmetry (or Duality)**

$$\text{if} \quad \mathcal{F}[x(t)] = X(j\omega), \quad \text{then} \quad \mathcal{F}[X(t)] = 2\pi\, x(-j\omega)$$

Or in a more symmetric form:

$$\text{if} \quad \mathcal{F}[x(t)] = X(f), \quad \text{then} \quad \mathcal{F}[X(t)] = x(-f)$$

Proof: As $\mathcal{F}[x(t)] = X(j\omega)$, we have

$$x(t) = \mathcal{F}^{-1}[X(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

Letting $t' = -t$, we get

$$x(-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{-j\omega t'}\,d\omega$$

Interchanging t' and ω we get:

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t')e^{-j\omega t'}\,dt' = \mathcal{F}[X(t)]$$

or

$$x(-f) = \int_{-\infty}^{\infty} X(t')e^{-j2\pi f t'}\,dt' = \mathcal{F}[X(t)]$$

In particular, if the signal is even:

$$x(t) = x(-t)$$

then we have

$$\text{if} \quad \mathcal{F}[x(t)] = X(f), \quad \text{then} \quad \mathcal{F}[X(t)] = x(f)$$

For example, the spectrum of an even square wave is a sinc function, and the spectrum of a sinc function is an even square wave.

- **Multiplication theorem**

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)d\omega$$

Proof:

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} x(t)[\frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(j\omega)e^{-j\omega t}d\omega]dt$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(j\omega)[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt]d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)d\omega$$

- **Parseval's equation**

In the special case when $y(t) = x(t)$, the above becomes the Parseval's equation (Antoine Parseval 1799):

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega)d\omega = \text{Total energy in } x(t)$$

where

$$S_X(j\omega) \triangleq |X(j\omega)|^2$$

is the energy density function representing how the signal's energy is distributed along the frequency axes. The total energy contained in the signal is obtained by integrating $S(j\omega)$ over the entire frequency axes.

The Parseval's equation indicates that the *energy* or *information* contained in the signal is reserved, i.e., the signal is represented equivalently in either the time or frequency domain with no energy gained or lost.

- **Correlation**

The *cross-correlation* of two real signals $x(t)$ and $y(t)$ is defined as

$$R_{xy}(t) \triangleq \int_{-\infty}^{\infty} x(\tau)y(\tau - t)d\tau = \int_{-\infty}^{\infty} x(t+\tau)y(\tau)d\tau$$

Specially, when $x(t) = y(t)$, the above becomes the *auto-correlation* of signal $x(t)$

$$R_x(t) \triangleq \int_{-\infty}^{\infty} x(\tau)x(\tau-t)d\tau$$

Assuming $\mathcal{F}[x(t)] = X(j\omega)$, we have $\mathcal{F}[x(t-\tau)] = X(j\omega)e^{-j\omega\tau}$ and according to multiplication theorem, $R_x(\tau)$ can be written as

$$\begin{aligned} R_x(\tau) &: \int_{-\infty}^{\infty} x(t)x(t-\tau)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)X^*(j\omega)e^{j\omega\tau}d\omega \\ &: \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 e^{j\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega)e^{j\omega\tau}d\omega = \mathcal{F}^{-1}[S_X(j\omega)] \end{aligned}$$

i.e.,

$$\mathcal{F}[R_x(t)] = S_X(j\omega)$$

that is, the auto-correlation and the energy density function of a signal $x(t)$ are a Fourier transform pair.

- **Convolution Theorems**

The convolution theorem states that convolution in time domain corresponds to multiplication in frequency domain and vice versa:

$$\mathcal{F}[x(t) * y(t)] = X(j\omega)\, Y(j\omega) \qquad (a)$$

$$\mathcal{F}[x(t)\; y(t)] = X(j\omega) * Y(j\omega) \qquad (b)$$

Proof of (a):

$$\begin{aligned} \mathcal{F}[x(t) * y(t)] &: \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau]e^{-j\omega t}dt \\ &: \int_{-\infty}^{\infty} x(\tau)[\int_{-\infty}^{\infty} y(t-\tau)e^{-j\omega t}dt]d\tau \\ &: \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}[\int_{-\infty}^{\infty} y(t-\tau)e^{-j\omega(t-\tau)}d(t-\tau)]d\tau \\ &: X(j\omega)\, Y(j\omega) \end{aligned}$$

Proof of (b):

$$\mathcal{F}[x(t)\;y(t)] : \int_{-\infty}^{\infty} x(t)\;y(t)e^{-j\omega t}dt$$

$$: \int_{-\infty}^{\infty}[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega')e^{j\omega' t}d\omega']\;y(t)e^{-j\omega t}dt$$

$$: \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega')[\int_{-\infty}^{\infty} y(t)e^{j\omega' t}e^{-j\omega t}dt]d\omega'$$

$$: \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega')[\int_{-\infty}^{\infty} y(t)e^{-j(\omega-\omega')t}dt]\,d\omega'$$

$$: \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega')Y(j(\omega-\omega'))d\omega' = X(j\omega) * Y(j\omega)$$

- **Time Derivative**

$$\mathcal{F}[\frac{d}{dt}x(t)] = j\omega\;X(j\omega)$$

Proof: Differentiating the inverse Fourier transform $X(j\omega)$ with respect to t we get:

$$\frac{d}{dt}x(t) : \frac{d}{dt}[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\frac{d}{dt}e^{j\omega t}d\omega$$

$$: \frac{1}{2\pi}\int_{-\infty}^{\infty}[j\omega X(j\omega)]e^{j\omega t}d\omega = \mathcal{F}^{-1}[j\omega X(j\omega)]$$

Repeating this process we get

$$\mathcal{F}[\frac{d^n}{dt^n}x(t)] = (j\omega)^n X(j\omega)$$

- **Time Integration**

First consider the Fourier transform of the following two signals:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}, \qquad sgn(t) = \begin{cases} -1/2 & t < 0 \\ 1/2 & t > 0 \end{cases} = u(t) - \frac{1}{2}$$

$$\frac{d}{dt}u(t) = \delta(t), \qquad \frac{d}{dt}sgn(t) = \frac{d}{dt}[u(t) - \frac{1}{2}] = \delta(t)$$

According to the time derivative property above

$$X(j\omega) = \mathcal{F}[x(t)] = \frac{1}{j\omega}\mathcal{F}[\frac{d}{dt}x(t)]$$

we get

$$\mathcal{F}[u(t)] = \frac{1}{j\omega}\mathcal{F}[\frac{d}{dt}x(t)] = \frac{1}{j\omega}\mathcal{F}[\delta(t)] = \frac{1}{j\omega}$$

and

$$\mathcal{F}[sgn(t)] = \frac{1}{j\omega}\mathcal{F}[\frac{d}{dt}sgn(t)] = \frac{1}{j\omega}\mathcal{F}[\delta(t)] = \frac{1}{j\omega}$$

Why do the two different functions have the same transform?

In general, any two function $f(t)$ and $g(t) = f(t) + c$ with a constant difference c have the same derivative $d\ f(t)/dt$, and therefore they have the same transform according the above method. This problem is obviously caused by the fact that the constant difference c is lost in the derivative operation. To recover this constant difference in time domain, a delta function needs to be added in frequency domain. Specifically, as function $sgn(t)$ does not have DC component, its transform does not contain a delta:

$$\mathcal{F}[sgn(t)] = \frac{1}{j\omega}$$

To find the transform of $u(t)$, consider

$$u(t) = sgn(t) + \frac{1}{2}$$

and

$$\mathcal{F}[u(t)] = \mathcal{F}[sgn(t)] + \mathcal{F}[\frac{1}{2}] = \frac{1}{j\omega} + \pi\delta(\omega)$$

The added impulse term $\pi\delta(\omega)$ directly reflects the constant $c = 1/2$ in time domain.

Now we show that the Fourier transform of a time integration is

$$\mathcal{F}[\int_{-\infty}^{t} x(\tau)d\tau] = \frac{1}{j\omega}\, X(j\omega) + \pi X(0)\delta(\omega)$$

Proof:

First consider the convolution of $x(t)$ and $u(t)$:

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)d\tau$$

Due to the convolution theorem, we have

$$\mathcal{F}[\int_{-\infty}^{t} x(\tau)d\tau] = \mathcal{F}[x(t)*u(t)] = X(j\omega)[\frac{1}{j\omega}+\pi\delta(\omega)] = \frac{1}{j\omega}\, X(j\omega)+\pi X(0)\delta(\omega)$$

- **Frequency Derivative**

$$\mathcal{F}[tx(t)] = j\frac{d}{d\omega}X(j\omega)$$

Proof: We differentiate the Fourier transform of $x(t)$ with respect to ω to get

$$\begin{aligned}\frac{d}{d\omega}X(j\omega) &= \frac{d}{d\omega}[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt] = \int_{-\infty}^{\infty} x(t)\frac{d}{d\omega}e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(t)(-jt)e^{-j\omega t}dt\end{aligned}$$

i.e.,

$$\mathcal{F}[-jtx(t)] = \frac{d}{d\omega}X(j\omega)$$

Multiplying both sides by j, we get

$$j\frac{d}{d\omega}X(j\omega) = \int_{-\infty}^{\infty} tx(t)e^{-j\omega t}dt = \mathcal{F}[tx(t)]$$

Repeating this process we get

$$\mathcal{F}[t^n x(t)] = j^n\frac{d^n}{d\omega^n}X(j\omega)$$

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