Reconocimiento de Patrones

Version 2022-2

LDA, QDA, Mahalanobis

[Capítulo 4]

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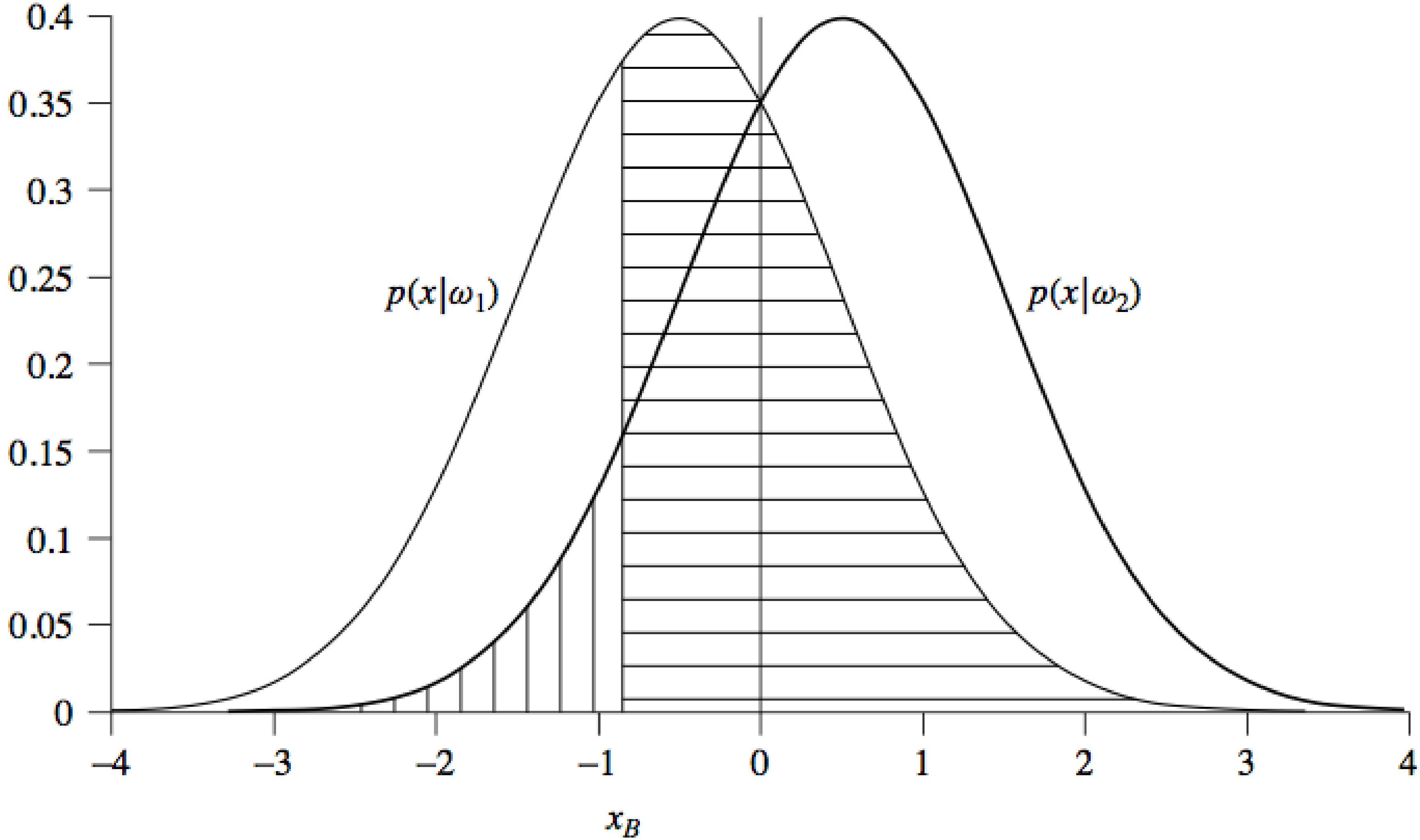
Clasificador de Bayes: $\boldsymbol{x}$ es clasificado como clase j si

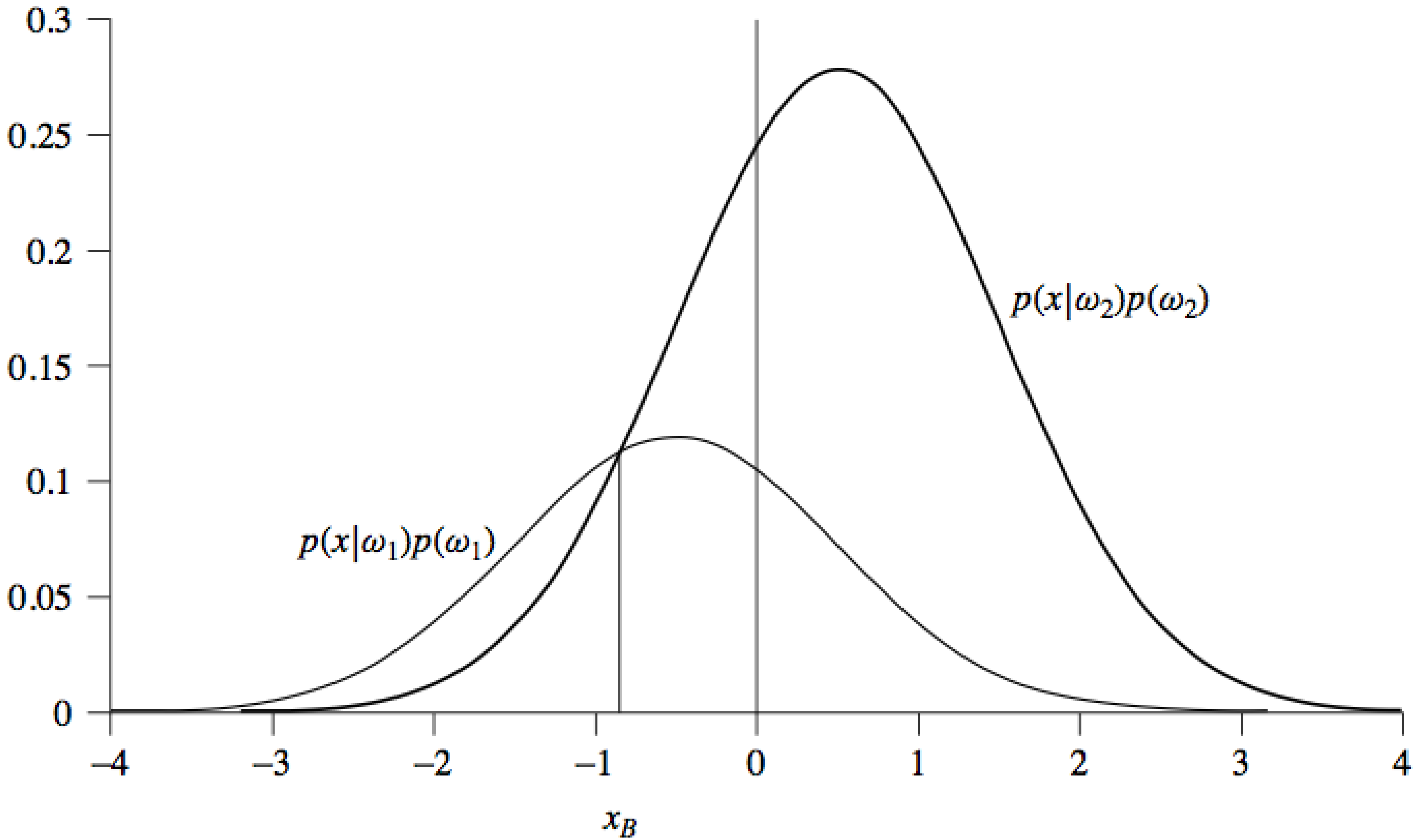
$$p(\omega_j|\boldsymbol{x}) > p(\omega_k|\boldsymbol{x}) \quad k = 1, \ldots, C; k \neq j$$

Usando el teorema de Bayes:

$$p(\omega_i|\boldsymbol{x}) = p(\omega_i)\frac{p(\boldsymbol{x}|\omega_i)}{p(\boldsymbol{x})}$$

$$p(\boldsymbol{x}|\omega_j)p(\omega_j) > p(\boldsymbol{x}|\omega_k)p(\omega_k) \quad k = 1, \ldots, C; k \neq j$$


0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
–4
–3
–2
–1
0
1
2
3
4
$p(x|\omega_1)$
$p(x|\omega_2)$
x_B



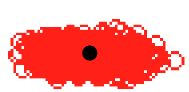
0.3
0.25
0.2
0.15
0.1
0.05
0
$p(x|\omega_2)p(\omega_2)$
$p(x|\omega_1)p(\omega_1)$
−4
−3
−2
−1
0
1
2
3
4
x_B

Para distribuciones Gaussianas

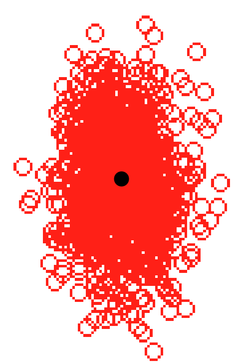
$$p(\boldsymbol{x}|\omega_i) = \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}_i|^{1/2}} \exp\left\{ -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^T \boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i) \right\}$$

Estimador de Matriz de Covarianza: $\hat{\boldsymbol{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{x}_i - \boldsymbol{m})(\boldsymbol{x}_i - \boldsymbol{m})^T$

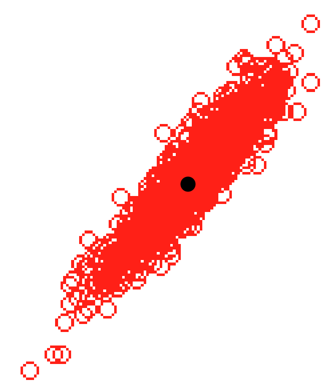
Ejemplos de Σ y μ en 2D:



$$\begin{matrix} 6.0057 & -0.1020 \\ -0.1020 & 1.0632 \end{matrix}$$



$$\begin{matrix} 7.7947 & -0.7772 \\ -0.7772 & 43.1459 \end{matrix}$$



$$\begin{matrix} 15.1951 & 21.8267 \\ 21.8267 & 37.6734 \end{matrix}$$

Clasificador de Bayes

$$p(\omega_j|\boldsymbol{x}) > p(\omega_k|\boldsymbol{x}) \quad k = 1, \ldots, C; k \neq j$$

$$p(\boldsymbol{x}|\omega_j)p(\omega_j) > p(\boldsymbol{x}|\omega_k)p(\omega_k) \quad k = 1, \ldots, C; k \neq j$$

$$\log\{p(\boldsymbol{x}|\omega_j)p(\omega_j)\} > \log\{p(\boldsymbol{x}|\omega_k)p(\omega_k)\} \quad k = 1, \ldots, C; k \neq j$$

Para distribuciones Gaussianas:

$$p(\boldsymbol{x}|\omega_i) = \frac{1}{\underline{(2\pi)^{p/2}}\,\underline{|\boldsymbol{\Sigma}_i|^{1/2}}} \exp\left\{\underline{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^T \boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i)}\right\}$$

$$\begin{aligned}\log(p(\boldsymbol{x}|\omega_i)\,p(\omega_i)) &= \log(p(\boldsymbol{x}|\omega_i)) + \underline{\log(p(\omega_i))} \\ &= \underline{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^T \boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i)} - \underline{\frac{1}{2}\log(|\boldsymbol{\Sigma}_i|)} \\ &\quad \underline{-\frac{p}{2}\log(2\pi)} + \underline{\log(p(\omega_i))}\end{aligned}$$

$$\log\{p(\boldsymbol{x}|\omega_j)p(\omega_j)\} > \log\{p(\boldsymbol{x}|\omega_k)p(\omega_k)\} \quad k = 1, \ldots, C; k \neq j$$

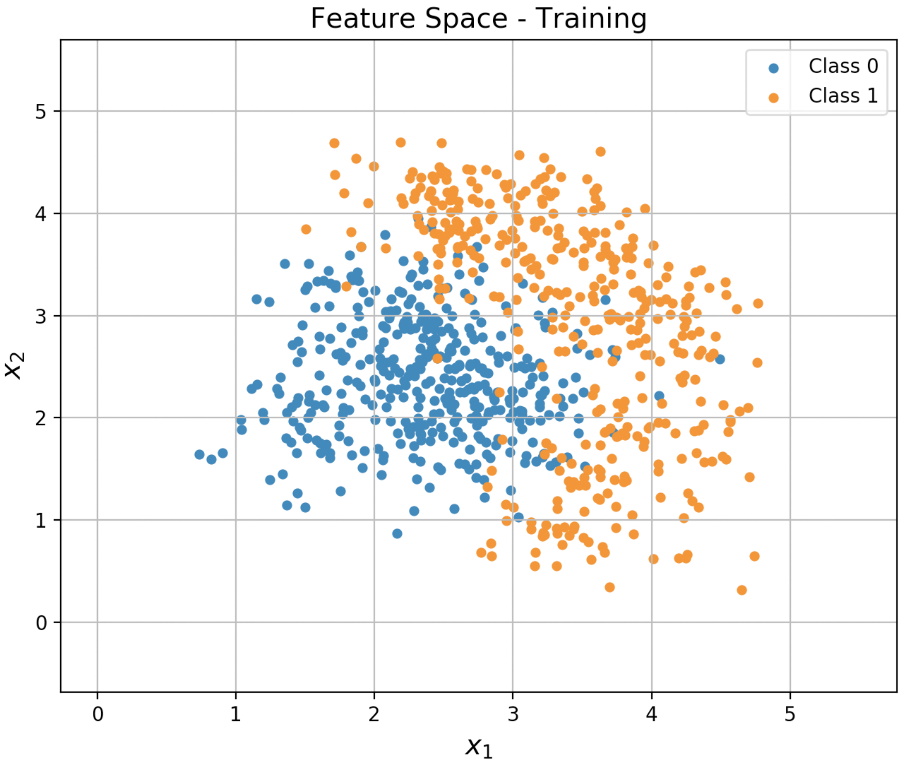
$$\begin{aligned}\log(p(\boldsymbol{x}|\omega_i)\,p(\omega_i)) &= \log(p(\boldsymbol{x}|\omega_i)) + \log(p(\omega_i)) \\ &= -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^T\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i) - \frac{1}{2}\log(|\boldsymbol{\Sigma}_i|) \\ &\quad \cancel{-\frac{p}{2}\log(2\pi)} + \log(p(\omega_i))\end{aligned}$$

Este término es constante, se puede eliminar de la desigualdad.

Ejemplo

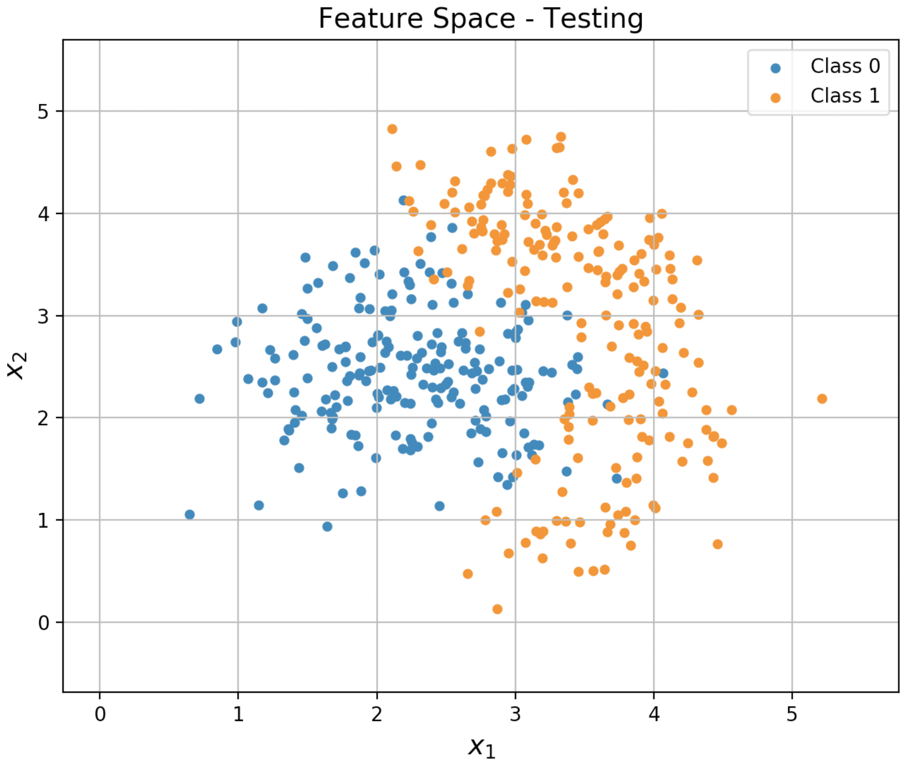
(X,d)

TRAINING



(X_t,d_t)

TESTING



LDA

$$\log\{p(\boldsymbol{x}|\omega_j)p(\omega_j)\} > \log\{p(\boldsymbol{x}|\omega_k)p(\omega_k)\} \quad k = 1, \ldots, C; k \neq j$$

$$\begin{aligned} \log(p(\boldsymbol{x}|\omega_i)\,p(\omega_i)) &= \log(p(\boldsymbol{x}|\omega_i)) + \log(p(\omega_i)) \\ &= -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^T\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i) - \cancel{\frac{1}{2}\log(|\boldsymbol{\Sigma}_i|)} \\ &\quad - \cancel{\frac{p}{2}\log(2\pi)} + \log(p(\omega_i)) \end{aligned}$$

En LDA (Análisis Discriminante Lineal) se supone que $\Sigma_i = \Sigma$ (es constante) .

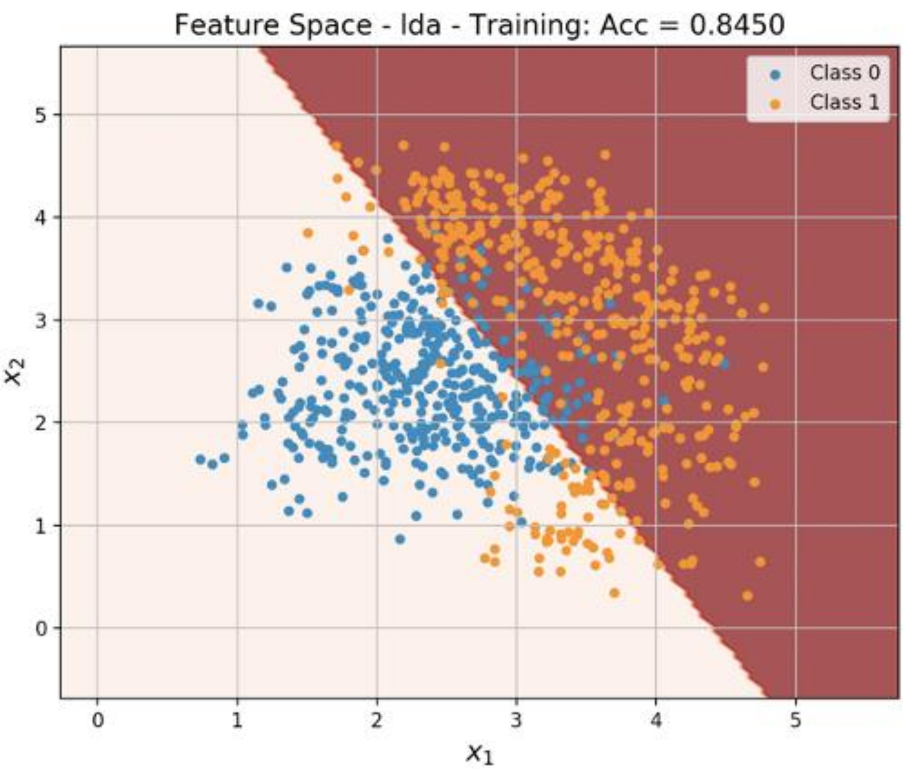
Σ se calcula a partir de datos de entrenamiento. Una buena estimación es el promedio de las matrices de covarianza individuales: $\Sigma = (\Sigma_1+\Sigma_2)/2$

Ejemplo

LDA

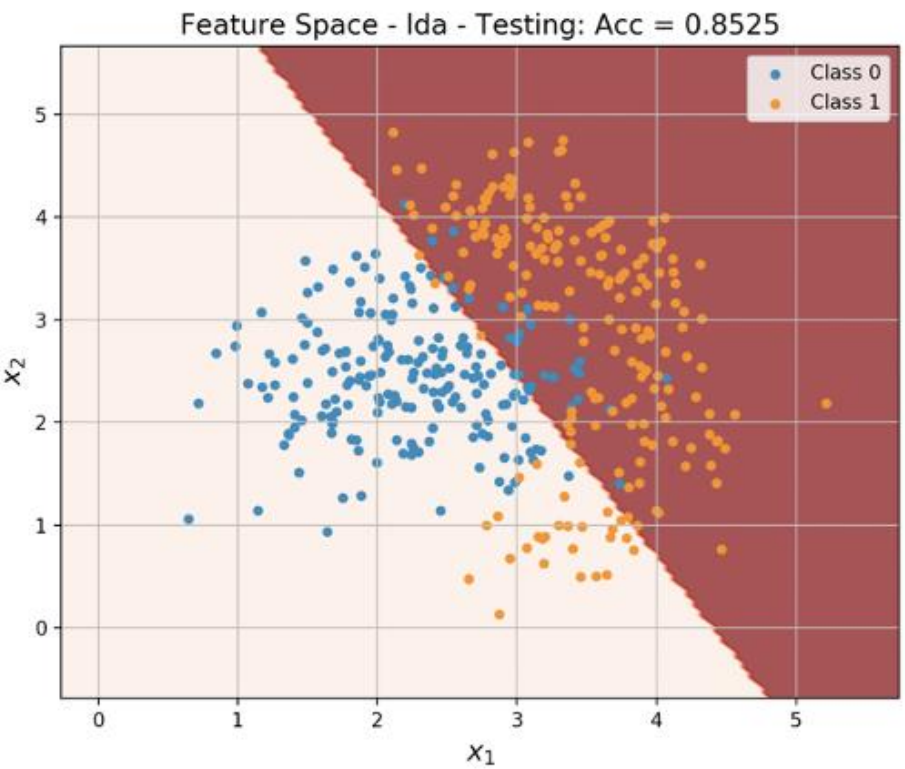
(X,d)

TRAINING



(X_t,d_t)

TESTING



MAHALANOBIS

$$\log\{p(\boldsymbol{x}|\omega_j)p(\omega_j)\} > \log\{p(\boldsymbol{x}|\omega_k)p(\omega_k)\} \quad k = 1, \ldots, C; k \neq j$$

$$\begin{aligned}\log(p(\boldsymbol{x}|\omega_i)\,p(\omega_i)) &= \log(p(\boldsymbol{x}|\omega_i)) + \log(p(\omega_i)) \\ &= -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu}_i)^T \boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x} - \boldsymbol{\mu}_i) - \frac{1}{2}\log(|\boldsymbol{\Sigma}_i|) \\ &\quad \cancel{- \frac{p}{2}\log(2\pi)} + \cancel{\log(p(\omega_i))}\end{aligned}$$

En el Clasificador Mahalanobis se asume $p(w_i) = p$ y

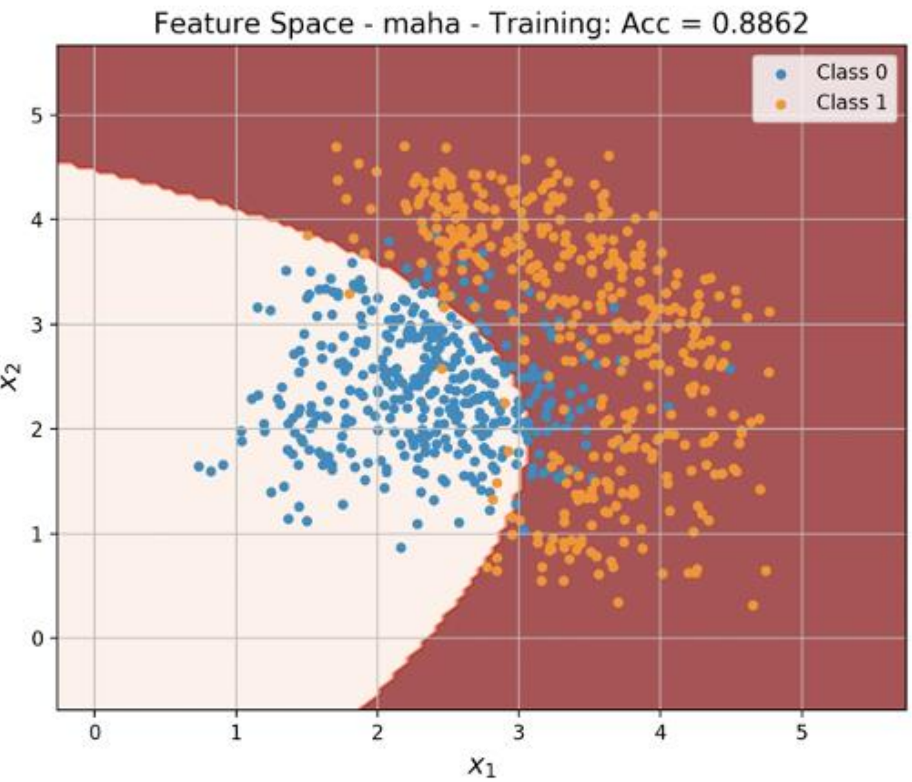
las matrices Σ_i son distintas.

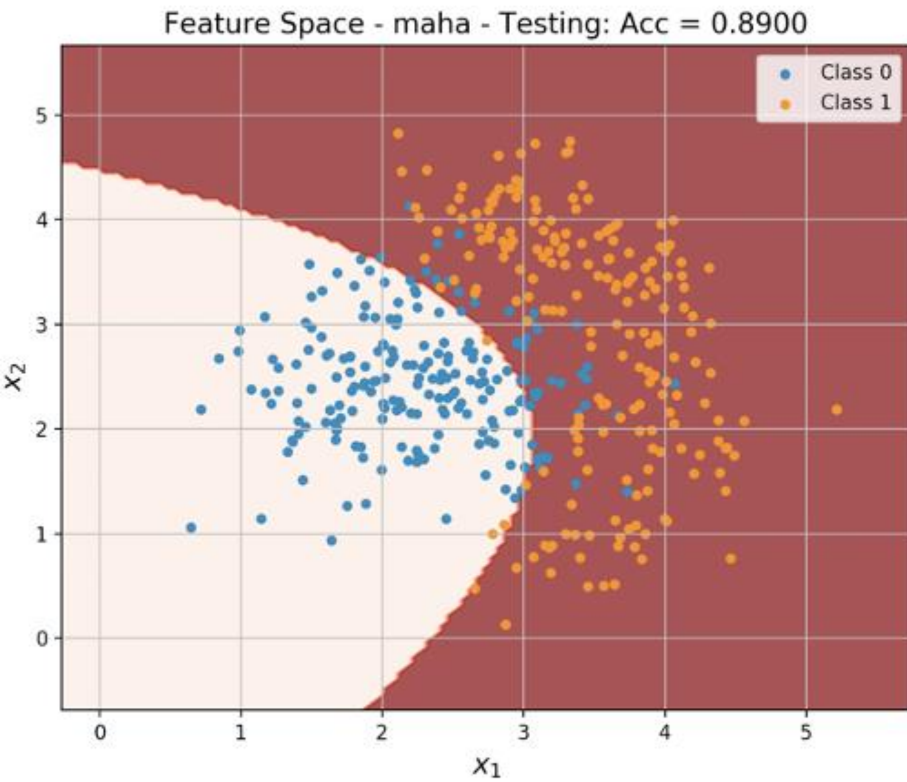
Ejemplo

Mahalanobis

(X,d) TRAINING

(X_t, d_t) TESTING





MAHALANOBIS-0

$$\log\{p(\boldsymbol{x}|\omega_j)p(\omega_j)\} > \log\{p(\boldsymbol{x}|\omega_k)p(\omega_k)\} \quad k = 1, \ldots, C; k \neq j$$

$$\begin{aligned}\log(p(\boldsymbol{x}|\omega_i)\, p(\omega_i)) &= \log(p(\boldsymbol{x}|\omega_i)) + \log(p(\omega_i)) \\ &= -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu}_i)^T \boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x} - \boldsymbol{\mu}_i) \cancel{- \frac{1}{2}\log(|\boldsymbol{\Sigma}_i|)} \\ &\quad \cancel{- \frac{p}{2}\log(2\pi)} + \cancel{\log(p(\omega_i))}\end{aligned}$$

En el Clasificador Mahalanobis se asume $p(w_i) = p$.

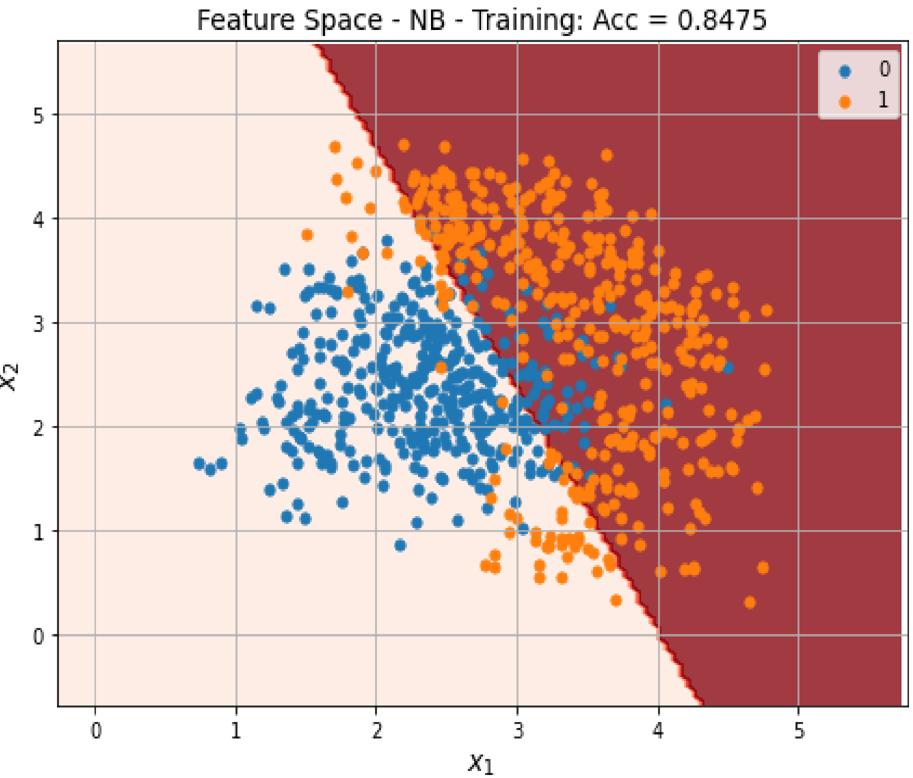
Hay una variante de Mahalanobis en la que se supone que $\Sigma_i = \Sigma$.

Ejemplo

Mahalanobis-0 $(\Sigma_i = \Sigma)$

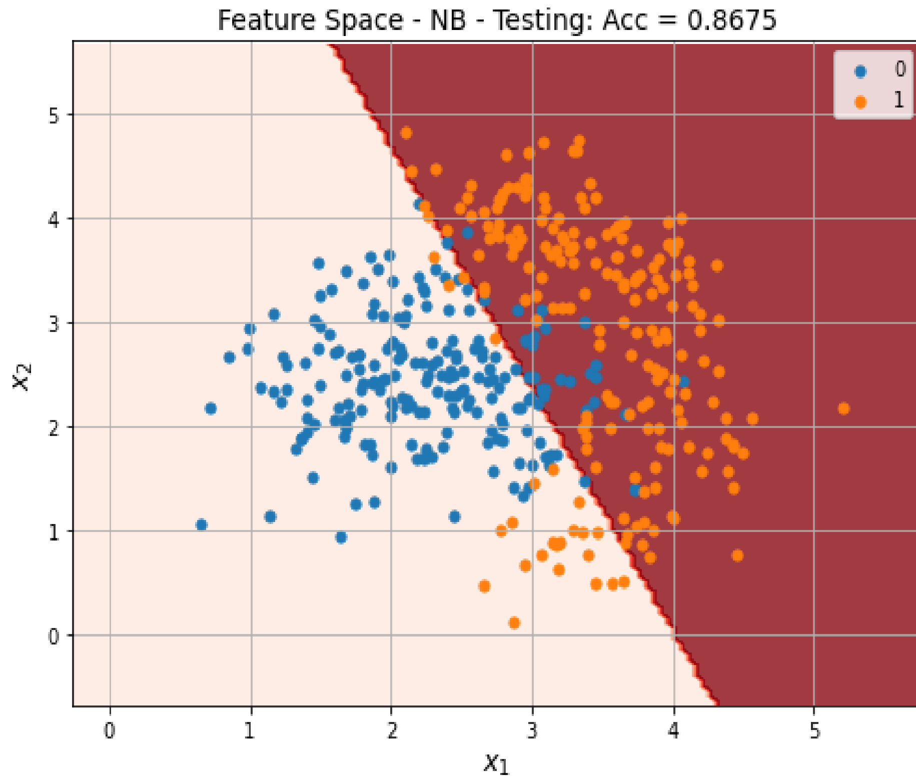
(X, d)

TRAINING



(X_t, d_t)

TESTING



QDA

$$\log\{p(\boldsymbol{x}|\omega_j)p(\omega_j)\} > \log\{p(\boldsymbol{x}|\omega_k)p(\omega_k)\} \quad k = 1, \ldots, C; k \neq j$$

$$\begin{aligned}
\log(p(\boldsymbol{x}|\omega_i)\, p(\omega_i)) &= \log(p(\boldsymbol{x}|\omega_i)) + \log(p(\omega_i)) \\
&= -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu}_i)^T \boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x} - \boldsymbol{\mu}_i) - \frac{1}{2}\log(|\boldsymbol{\Sigma}_i|) \\
&\quad \cancel{- \frac{p}{2}\log(2\pi)} + \log(p(\omega_i))
\end{aligned}$$

En QDA (Análisis Discriminante Cuadrático) se supone que Σ_i y $p(w_i)$ son diferentes.

Ejemplo

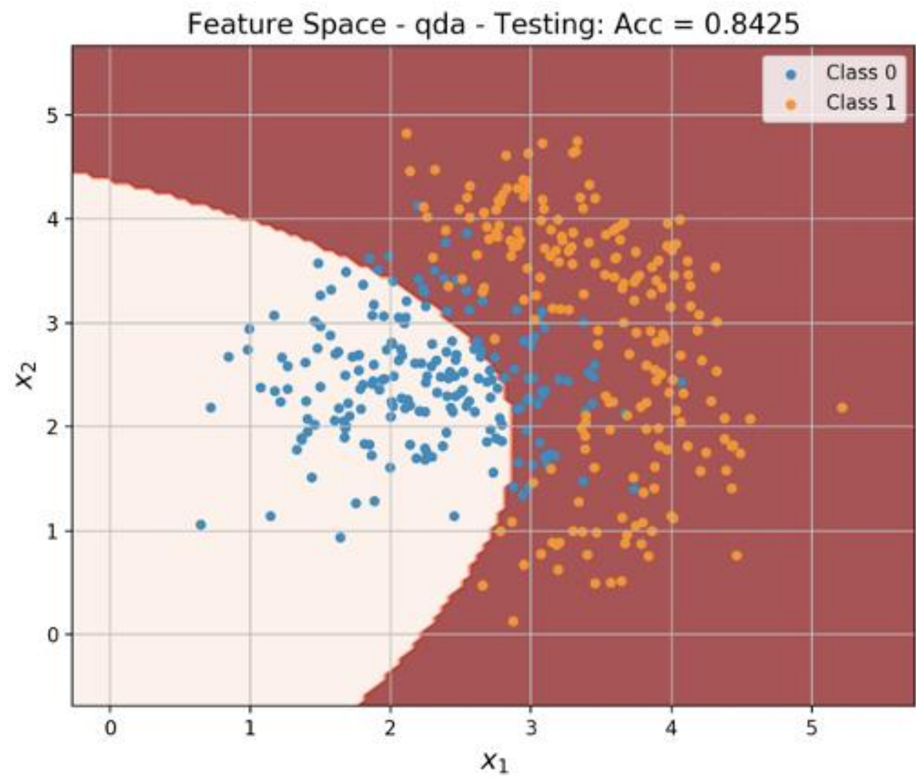
QDA

(X,d)

TRAINING

(X_t,d_t)

TESTING



$$\log(p(\boldsymbol{x}|\omega_i)) = -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^T\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i) - \frac{1}{2}\log(|\boldsymbol{\Sigma}_i|) - \frac{p}{2}\log(2\pi) + \log(p(\omega_i))$$

$\Sigma_i = \Sigma = \text{cte}$ cte

LDA $$= -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^T\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i) - \cancel{\frac{1}{2}\log(|\boldsymbol{\Sigma}_i|)} - \cancel{\frac{p}{2}\log(2\pi)} + \log(p(\omega_i))$$

$\Sigma_i = \Sigma = \text{cte}$ cte   *$p(w_i) = cte$*

Mahalanobis $$= -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^T\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i) - \cancel{\frac{1}{2}\log(|\boldsymbol{\Sigma}_i|)} - \cancel{\frac{p}{2}\log(2\pi)} + \cancel{\log(p(\omega_i))}$$

cte

QDA $$= -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^T\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i) - \frac{1}{2}\log(|\boldsymbol{\Sigma}_i|) - \cancel{\frac{p}{2}\log(2\pi)} + \log(p(\omega_i))$$