Dr Nikita Sidorov, Dr Clifford Gilmore

sidorov@manchester.ac.uk, clifford.gilmore@manchester.ac.uk

0N1 (MATH19861)
Mathematics for Foundation Year

Lecture Notes

2021

Contents

1 Arrangements for the Course

1.1 Aims and description

AIMS OF 0N1

- A basic course in pure mathematical topics for members of the foundation year.
- Key ingredient: language of Mathematics, including specific use of English in Mathematics.

BRIEF DESCRIPTION

13 lectures: Sets. Definition, subsets, simple examples, union, intersection and complement. De Morgan's Laws. Elementary Logic; universal and existential qualifiers. Proof by contradiction and by induction.

9 lectures: Methods of proof for inequalities. Solution of inequalities containing unknown variables. Linear inequalities with one or two variables, systems of liner inequalities with two variables. Some simple problems of linear optimisation. Quadratic inequalities with one variable.

A BRIEF VERY PRAGMATIC DESCRIPTION

> The course contains all mathematics necessary for writing standard commercial time-dependent spreadsheets using the EXCEL (or a similar software package) macro language.

TEXTBOOKS:

- S Lipschutz, *Set Theory and Related Topics*. McGraw-Hill.
- J Franklin and A Daoud, *Proof in Mathematics: An Introduction*. Kew Books (Jan 2011).
- R Steege and K Bailey, *Intermediate Algebra*. (Schaum's Outlines.) McGraw-Hill.
- R. Hammack, *Book of Proof*, http://www.people.vcu.edu/~rhammack/BookOfProof/index.html

1.2 Intended learning outcomes

On completion of this unit, successful students will be able to:

1. Analyse relations between set and express them in terms of Boolean Algebra.
2. Evaluate formulae of propositional logic and identify equivalent formulae by composition of truth tables.
3. Identify tautologies of propositional logic and prove them by reduction to Fundamental Logic Identities.
4. Describe properties of various objects and systems in terms of predicate logic.
5. Prove mathematical statements using mathematical induction.
6. Determine and analyse solution sets of systems of linear inequalities in two variables.
7. Analyse mutual position of points and lines in the plane using linear inequalities.
8. Determine whether certain numbers are rational or irrational.
9. Compare arithmetic, geometric, harmonic, quadratic means of two variables.

1.3 Exam and coursework

- The exam will take place in the usual (pre-Covid, i.e., in person) format in January 2022 (70% of the total mark).
- Coursework will take place in the form of online quizzes in Weeks 3-12 (30% of the total mark).

1.4 Teaching arrangements

- All the new content will be available in advance in the form of asynchronous **videos** on the course Blackboard page. (Plus, of course, these lecture notes.)
- There will be one synchronous **review session** per week taught by Dr Gilmore, where there will be: examples complementing the videos, solutions to the exercises from the lectures and answering the students' questions from emails/Blackboard.
- There will be also one synchronous **tutorial** per week.
- Each week starting with Week 3 you will have to do an asynchronous **quiz** which will count as **coursework**.

Lecture Notes

2 Sets

2.1 Sets: Basic definitions

A *set* is any collection of objects, for example, set of numbers. The objects of a set are called the *elements* of the set.

A set may be specified by listing its elements. For example, $\{1, 3, 6\}$ denotes the set with elements 1, 3 and 6. This is called the *list form* for the set. Note the curly brackets.*

We usually use capital* letters A, B, C, etc., to denote sets.

The notation $x \in A$ means "x is an element of A".* But $x \notin A$ means "x is not an element of A".

* Typographical terms:
{ opening curly bracket
} closing curly bracket

* capital letter =
upper case letter

* Alternatively we may say "x belongs to A" or "A contains x".

Example 2.1.1

$$1 \in \{1, 3, 6\}, \quad 3 \in \{1, 3, 6\}, \quad 6 \in \{1, 3, 6\}$$

but

$$2 \notin \{1, 3, 6\}.$$

2.2 Predicate and list form of definition of a set

A set can also be specified in *predicate form**, that is by giving a distinguished property of the elements of the set (or an explicit* description of the elements in the set). For example, we can define set B by

* or *descriptive* form

* explicit = specific, definite

$$B = \{x \ : \ x \text{ is a positive integer less than } 5\}.$$

The way to read this notation is

"*B* is the set of all *x* such that *x* is a positive integer less than 5".

The curly brackets indicate a set and the colon*

$$' : '$$

is used to denote "such that", and, not surprisingly, is read "such that".

* Typographical terms:
: colon

The same set *B* can be given by listing its elements, or in *list form*:

$$B = \{ 1, 2, 3, 4 \}.$$

2.3 Equality of sets

Two sets are *equal** if they have exactly the same elements. Thus

* We also say: two sets *coincide*.

$$\{1, 2, 3, 4\} = \{x \ : \ x \text{ is a positive integer less than } 5\}.$$

In list form the same set is denoted whatever order the elements are listed and however many times each element is listed. Thus

$$\{2, 3, 5\} = \{5, 2, 3\} = \{5, 2, 3, 2, 2, 3\}.$$

Note that $\{5, 2, 3, 2, 2, 3\}$ is a set with only 3 elements: 2, 3 and 5.

Example 2.3.1

$$\{x \ : \ x \textit{ is a letter in the word GOOD} \} = \{D, G, O\}.$$

The *set* $\{2\}$ is regarded as being different from the *number* 2. A set of numbers is not a number. $\{2\}$ is a set with only one element which happens to be the number 2. But a set is not the same as the object it contains: $\{2\} \neq 2$. The statement $2 \in \{2\}$ is correct. The statement $\{2\} \in \{2\}$ is wrong.

Example 2.3.2 *The sets of letters in the words GOOD and DOG are equal.*

2.4 The empty set

The set

$$\{x \;:\; x \text{ is an integer such that } x^2 = -1\}$$

has *no* elements. This is called an *empty set**. It was said earlier that two sets are equal if they have the same elements. Thus if A and B are empty sets we have $A = B$. Mathematicians have found that this is the correct viewpoint, and this makes our first theorem.*

Theorem. *If A and B are empty sets then $A = B$.*

Proof.* The sets A and B are equal because they cannot be non-equal. Indeed, for A and B not to be equal we need an element in one of them, say $a \in A$, that does not belong to B. But A contains no elements! Similarly, we cannot find an element $b \in B$ that does not belong to A – because B contains no elements at all. □

Corollary.* *There is only one empty set, THE* empty set.*

The empty set is usually denoted by $\emptyset$.*

Thus

$$\{x \;:\; x \text{ is an integer such that } x^2 = -1\} = \emptyset.$$

* Some books call it *null set*.

* The word *theorem* means a statement that has been proved and therefore became part of mathematics. We shall also use words *proposition* and *lemma*: they are like theorem, but a proposition is usually a theorem of less importance, while lemma has no value on its own and is used as a step in a proof of a theorem.

* The word *proof* indicates that an argument establishing a theorem or other statement will follow.

* *Corollary* is something that easily follows from a theorem or a proposition.

* Notice the use of definite article THE.

* In some textbooks you will find another notation for the empty set: $\{\,\}$

2.5 A set cannot be an element of itself!

We have complete freedom of forming sets, but one rule is of absolute importance: you cannot form a set containing itself as an element:

$$A \notin A$$

for all sets A!*

This means that when we are forming a set, we assume that its elements are somehow already **given** to us; but the set itself is not made yet, it is still in the process of construction.

In particular, there is no set of all sets – because this set would contain itself as an element.

* We shall revisit this principle later in the lectures, when we shall consider the so-called *self-referential statements* and various paradoxes associated with them.

2.6 Questions from students (from past years)

*

* This section contains no compulsory material but still may be useful.

1. *My question is: Are all empty sets equal? No matter the conditions. For example is*

   $$\{x : x \textit{ is positive integer less than zero}\}$$

   equal to

   $$\{x : x \textit{ is an integer between 9 and 10}\}$$

   ANSWER. Yes, all empty sets are equal. To see that in your example, let us denote

   $$A = \{x : x \text{ is positive integer less than zero}\}$$

   and

   $$B = \{x : x \text{ is an integer between 9 and 10}\}$$

   So, I claim that $A = B$. If you do not agree with me, you have to show that A is different from B. To do so, you have to show me an element in one set that does not belong to another set. Can you do that? Can you point to an offending element if both sets have no elements whatsoever?

   Indeed, can you point to a "positive integer less than zero" which is not an "integer between 9 and 10"? Of course, you cannot, because there are no positive integers less than zero.

   Can you point to an "integer between 9 and 10" which is not a "positive integer less than zero"? Of course, you cannot, because there are no integers between 9 and 10.

   Hence you cannot prove that A is not equal to B. Therefore you have to agree with me that $A = B$.

3 Subsets; Finite and Infinite Sets

3.1 Subsets

Consider the sets A and B where $A = \{2,4\}$ and $B = \{1,2,3,4,5\}$. Every element of the set A is an element of the set B. We say that A is a *subset* of B and write $A \subseteq B$, or $B \supseteq A$. We can also say that B *contains* A.*

* Also: A is *contained* in B, A is *included* in B. The expression $B \supseteq A$ is read "B is a *superset* of A", or B *contains* A.

Notice that the word "*contains*" is used in set theory in two meanings, it can be applied to elements and to subsets: the set $\{a,b,c\}$ contains an *element* a and a *subset* $\{a\}$. Symbols used are different:

$$a \in \{a,b,c\}, \quad \{a\} \subseteq \{a,b,c\}, \quad \text{and} \quad a \neq \{a\}.$$

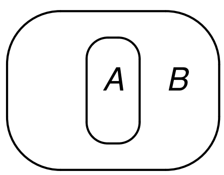


Figure 1: A diagram of $A \subseteq B$ (which is the same as $B \supseteq A$).

Figure 1 is a simple example of a *Venn diagram* for showing relationships between sets. Figure 2 is an example of a Venn diagram for three sets G, L, C of uppercase letters of the Greek, Latin and Cyrillic alphabets, respectively.

Some basic facts:

- $A \subseteq A$ for every set A. Every set is a subset of itself.

  [*Indeed every element of A is an element of A. Hence, by definition of a subset, A is a subset of A.*]

- The empty set is a subset of every set: $\emptyset \subseteq A$ for any set A.

  [*Indeed every element of $\emptyset$ is an element of A because there is no any elements in $\emptyset$.*]

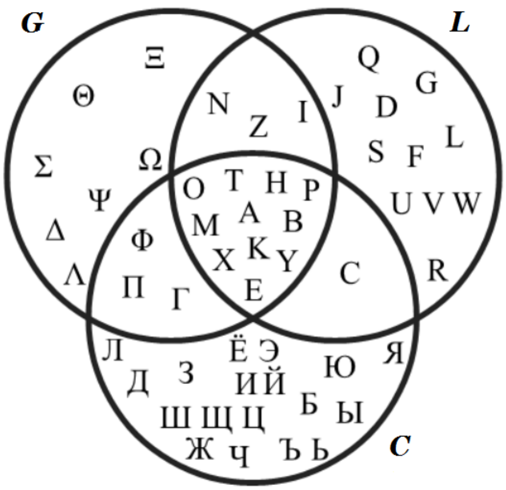


Figure 2: Venn diagram showing which uppercase letters are shared by the Greek, Latin and Cyrillic alphabets (sets G, L, C, respectively).

- If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*
- If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

* We say that $\subseteq$ is a *transitive relation* between sets. Notice that the relation $\in$ "being an element of" is not transitive. relation = connection, bond

3.2 The set of subsets of a set

Example 3.2.1 *Let $A = \{1, 2\}$. Denote by B the set of subsets of A. Then*

$$B = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

Notice that $1 \in \{1\}$ and $1 \in A$, but it is not true that $1 \in B$.

On the other hand, $\{1\} \in B$, but it is not true that $\{1\} \in A$.

Example 3.2.2 *The subsets of $\{1, 2, 3\}$ are*

$$\emptyset,\ \{1\},\ \{2\},\ \{3\},\ \{1, 2\},\ \{1, 3\},\ \{2, 3\},\ \{1, 2, 3\}.$$

Note: don't forget the empty set $\emptyset$ and the whole set $\{1, 2, 3\}$. Thus $\{1, 2, 3\}$ has 8 subsets.

Theorem. *If A is a set with n elements then A has 2^n subsets. Here,*

$$2^n = 2 \times 2 \times \cdots \times 2$$

with n factors.

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$. How many are there ways to choose a subset in A? When choosing a subset, we have to decide, for each element, whether we include this elements into our subset or not. We have two choices for the first element: 'include' and 'do not include', two choices for the second element, etc., and finally two choices for the n^{th} element:

$$2 \times 2 \times \cdots \times 2$$

choices overall. □

Another proof. When revising for the examination, prove this Theorem using the method of mathematical induction from the last lectures. □

Example 3.2.3 *In some books, the set of subsets of a set A is denoted $\mathcal{P}(A)$.* * *Stating with the empty set $\emptyset$, let us take sets of subsets:*

$$\begin{array}{rcl} \mathcal{P}(\emptyset) & = & \{\emptyset\} \\ \mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}(\{\emptyset\}) & = & \{\emptyset, \{\emptyset\}\} \\ \mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \mathcal{P}(\{\emptyset, \{\emptyset\}\}) & = & \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\ & \vdots & \end{array}$$

which have, correspondingly,

$$2^0 = 1,\ 2^1 = 2,\ 2^2 = 4,\ 2^4 = 16,\ \ldots,$$

elements. In particular, the four sets

$$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$$

(*the subsets of $\{\emptyset, \{\emptyset\}\}$*) *are all different!*

* $\mathcal{P}(A)$ is called in some books the *powerset* of A.

3.3 Proper subsets

If $A \subseteq B$ and $A \neq B$ we call A a *proper subset* of B and write $A \subset B$ to denote this.*

* If $A \subset B$, we also write $B \supset A$. Similarly, $A \subseteq B$ is the same as $B \supseteq A$

Example 3.3.1 *Let $A = \{1,3\}$, $B = \{3,1\}$, $C = \{1,3,4\}$. Then*

$A = B$ *true*

$A \subset B$ *false*

$C \subseteq A$ *false*

$A \subseteq B$ *true*

$A \subseteq C$ *true*

$C \subset C$ *false*

$B \subseteq A$ *true*

$A \subset C$ *true*

Compare with inequalities for numbers:

$2 \leqslant 2$ *true,* $1 \leqslant 2$ *true,* $2 < 2$ *false,* $1 < 2$ *true.*

A set with n elements contains $2^n - 1$ proper subsets.

3.4 Finite and infinite sets

A *finite* set is a set containing only finite number of elements. For example, $\{1,2,3\}$ is finite. If A is a finite set, we denote by $|A|$ the number of elements in A. For example, $|\{1,2,3\}| = 3$ and $|\emptyset| = 0$.

A set with infinitely many elements is called an *infinite* set. The set of all positive integers (also called *natural numbers*)

$$\mathbb{N} = \{1,2,3,\ldots,\}$$

is infinite; the dots indicate that the sequence $1,2,3$ is to be continued indefinitely.*

* indefinitely = for ever, without end

The set of all non-negative integers* is also infinite:

$$\mathbb{N}_0 = \{0,1,2,3,\ldots,\}.$$

* There is no universal agreement about whether to include zero in the set of natural numbers: some define the natural numbers to be the positive integers $\{1,2,3,\ldots\}$, while for others the term designates the non-negative integers $\{0,1,2,3,\ldots\}$. In this lecture course, we shall stick to the first one (and more traditional) convention: 0 is not a natural number.

More examples of infinite sets:

$$\begin{array}{lll} \mathbb{Z} = & \{\ldots,-2,-1,0,1,2,\ldots\} & \text{(the set of integers)} \\ & \{\ldots,-4,-2,0,2,4,\ldots\} & \text{(the set of all \textit{even} integers)} \\ & \{\ldots,-3,-1,1,3,\ldots\} & \text{(the set of all \textit{odd} integers)} \end{array}$$

$\mathbb{Q}$ denotes the set of all rational numbers (that is, the numbers of the form n/m where n and m are integers and $m \neq 0$),

$\mathbb{R}$ the set of all real numbers (in particular, $\sqrt{2} \in \mathbb{R}$ and $\pi \in \mathbb{R}$),

$\mathbb{C}$ the set of all complex numbers (that is, numbers of the form $x + yi$, where x and y are real and i is a square root of -1, $i^2 = -1$).*

* The letters $\mathbb{ABCDEFGHIJKLMOPRSTUVWXYZ}$ are called *blackboard bold* and were invented by mathematicians for writing on a blackboard instead of *bold* letters **ABC**... which are difficult to write with chalk.

They are all infinite sets. We have the following inclusions:

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

3.5 Questions from students

This section contains no compulsory material but still may be useful.

1. ```
   > When considering sets
   > {1}, {{1}}, {{{1}}}, {{{{1}}}}, ...
   > is it true that {1} is an element of {{1}},
   > but not of {{{1}}}?
   ```

   ANSWER. Yes, it is true.

2. ```
   > (c) Let U = {u, v,w, x, y, z}.
   > (i) Find the number of subsets of U.
   > (ii) Find the number of proper non-empty subsets of U.
   >
   > i think the answer of question (ii) should be 63,
   > not 62 which is given by
   > exam sample solution. how do u think about it
   ```

   ANSWER. The answer is 62: there are $2^6 = 64$ subsets in U altogether. We exclude two: U itself (because is is not proper) and the empty set (because it is not non-empty.

3.
```
>                       My question
> relates to one of the mock exam questions,
> worded slightly differently.
>
> Question: List the 8 subsets of {a,b,c,d}
>           containing {d}?
```

ANSWER. A very good question—how to *list* in a *systematic* way all subsets of a given set? I emphasise the word *systematic*, this means that if you do the same problem a week later, you get exactly the same order of subsets in the list.

There are several possible approaches, one of them is to use the principle of ordering words in a dictionary; I will illustrate it on the problem

list all subsets in the set $\{a, b, c\}$.

In my answer to that problem, you will perhaps immediately recognise the *alphabetic* order:

$$\{\}^{*}; \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}; \{b\}, \{b, c\}; \{c\}.$$

* $\{\}$ is the empty set $\emptyset$

Returning to the original question,

List the 8 *subsets of* $\{a, b, c, d\}$ *containing* $\{d\}$,

we have to add the element d to each of the sets:

$$\{d\}; \{a, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}; \{b, d\}, \{b, c, d\}; \{c, d\}.$$

4.
```
Is it true that between any two real numbers,
for example, between 2 and 3,
there are infinitely many numbers?
```

ANSWER. Is, it is true. For example, between 2 and 3 there are infinitely many numbers

$$2 < 2.1 < 2.11 < 2.111 < \cdots < 3,$$

each of them larger than all the previous ones.

4 Operations on Sets

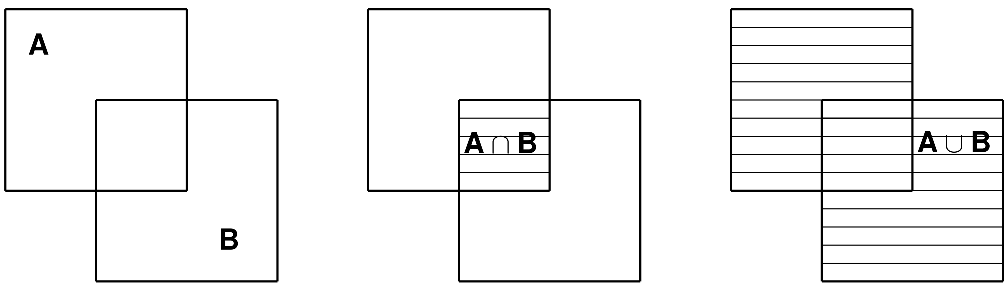


Figure 3: *Sets A and B and their intersection $A \cap B$ and union $A \cup B$.*

4.1 Intersection

Suppose A and B are sets. Then $A \cap B$ denotes the set of all elements which belong to both A and B:

$$A \cap B = \{\, x \ : \ x \in A \text{ and } x \in B \,\}.$$

$A \cap B$ is called the *intersection* of A and B.*

Example 4.1.1 *Let $A = \{1, 3, 5, 6, 7\}$ and $B = \{3, 4, 5, 8\}$, then $A \cap B = \{3, 5\}$.*

* The typographic symbol $\cap$ is sometimes called "cap". Notice that the name of a typographical symbol for an operation is not necessary the same as the name of operation. For example, symbol *plus* is used to denote *addition* of numbers, like $2 + 3$.

4.2 Union

$A \cup B$ denotes the set of all elements which belong to A or to B:

$$A \cup B = \{\, x \ : \ x \in A \textbf{ or } x \in B \,\}.$$

$A \cup B$ is called the *union* of A and B.*

* The typographic symbol $\cup$ is sometimes called "cup".

Notice that, in mathematics, **or** is usually understood in the *inclusive* sense: elements from $A \cup B$ belong to A or to B or to both A and B; or, in brief, to A **and/or** B. In some human languages, the connective* 'or' is understood in the *exclusive* sense: to A or to B, but *not* both A and B. **We will always understand 'or' as inclusive 'and/or'.** In particular, this means that

$$A \cap B \subseteq A \cup B.$$

* "Connective" is a word like 'or', 'and', 'but', 'if', ...

Example 4.2.1 *Let* $A = \{1, 3, 5, 6, 7\}$, $B = \{3, 4, 5, 8\}$*, then*

$$A \cup B = \{1, 3, 4, 5, 6, 7, 8\}.$$

If A and B are sets such that $A \cap B = \emptyset$, that is, A and B have no elements in common, we say that A is *disjoint* from B, or that A and B are *disjoint** (from each other).

* Or that A and B do not intersect.

Example 4.2.2 $A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$*. Here A and B are disjoint.*

4.3 Universal set and complement

In any application of set theory all the sets under consideration will be subsets of a background set, called the *universal set*. For example, when working with real numbers the universal set is the set $\mathbb{R}$ of real numbers. We usually denote the universal set by U.

U is conveniently shown as a "frame" when drawing a Venn diagram.

All the sets under consideration are subsets of U and so can be drawn inside the frame.

Let A be a set and U be the universal set. Then A' (called the *complement** of A and pronounced "A prime") denotes the set of all elements in U which do *not* belong to A:

$$A' = \{\, x \;:\; x \in U \text{ and } x \notin A \,\}.$$

* Notice that the complement A' is sometimes denoted $\neg A$ and pronounced "not A", or $\overline{A}$ (pronounced "A bar"), or A^c ("A compliment")

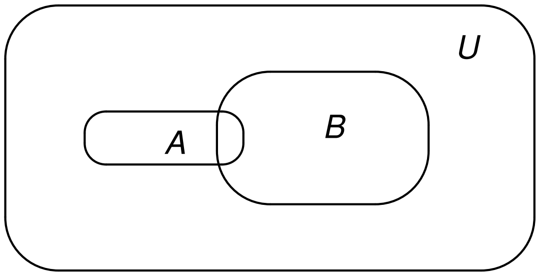


Figure 4: *The universal set U as a 'background' set for sets A and B.*

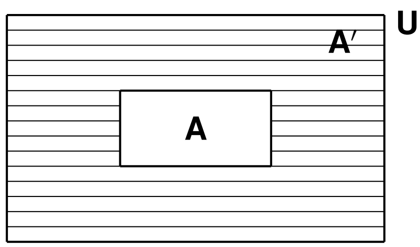


Figure 5: *The shaded area is the complement A' of the set A.*

Example 4.3.1 *Let $U = \{a, b, c, d, e, f\}$, $A = \{a, c\}$, $B = \{b, c, f\}$, $C = \{b, d, e, f\}$. Then*

$$\begin{aligned} B \cup C &= \{b, c, d, e, f\}, \\ A \cap (B \cup C) &= \{c\}, \\ A' &= \{b, d, e, f\} \\ &= C, \\ A' \cap (B \cup C) &= C \cap (B \cup C) \\ &= \{b, d, e, f\} \\ &= C. \end{aligned}$$

It will be convenient for us to modify predicate notation: instead of writing

$$\{\, x \;:\; x \in U \text{ and } x \text{ satisfies } \ldots \}$$

we shall write

$$\{\, x \in U \;:\; x \text{ satisfies } \ldots \}$$

Example 4.3.2

$$\{\, x \in \mathbb{Z} \;:\; x^2 = 4 \,\} = \{\, -2, 2 \,\}.$$

4.4 Relative complement

If A and B are two sets, we define the *relative complement of B in A* as

$$A \smallsetminus B = \{\, a \in A \;:\; a \notin B \,\}.$$

Example 4.4.1 *If*

$$A = \{\, 1, 2, 3, 4 \,\}$$

and

$$B = \{\, 2, 4, 6, 8 \,\}$$

then

$$A \smallsetminus B = \{\, 1, 3 \,\}.$$

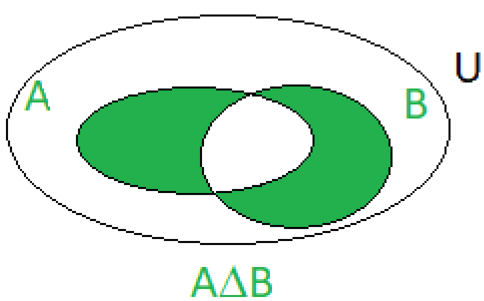
This operation can be easily expressed in terms of intersection and taking the complement:

$$A \smallsetminus B = A \cap B'.$$

4.5 Symmetric difference

The *symmetric difference* of sets A and B is defined as

$$A \triangle B = (A \smallsetminus B) \cup (B \smallsetminus A).$$



Sets A, B, and $A \triangle B$.

It can be seen (check!) that

$$A \triangle B = \{\, x : x \in A \text{ or } x \in B, \text{ but } x \notin A \cap B \,\}$$

and also that

$$A \triangle B = (A \cup B) \smallsetminus (A \cap B).$$

Example 4.5.1 *If* $A = \{\, 1, 2, 3, 4, 5 \,\}$ *and* $B = \{\, 4, 5, 6, 7 \,\}$ *then*

$$A \triangle B = \{\, 1, 2, 3, 6, 7 \,\}.$$

Please notice that the symmetric difference of sets A and B does not depend on the universal set U to which they belong; the same applies to conjunction $A \wedge B$, disjunction $A \vee B$, and relative complement $A \smallsetminus B$; it is the complement A' where we have to take care of the universal set.

5 Boolean Algebra

When dealing with sets, we have operations $\cap$, $\cup$ and $'$. The manipulation of expressions involving these symbols is called *Boolean algebra* (after George Boole, 1815–1864). The identities of Boolean algebra* are as follows. (A, B and C denote arbitrary sets all of which are subsets of U.)

* Or "laws" of Boolean algebra.

$$\left.\begin{aligned} A \cap B &= B \cap A \\ A \cup B &= B \cup A \end{aligned}\right\} \quad \text{commutative laws} \qquad (1)$$

$$\left.\begin{aligned} A \cap A &= A \\ A \cup A &= A \end{aligned}\right\} \quad \text{idempotent laws} \qquad (2)$$

$$\left.\begin{aligned} A \cap (B \cap C) &= (A \cap B) \cap C \\ A \cup (B \cup C) &= (A \cup B) \cup C \end{aligned}\right\} \quad \text{associative laws} \qquad (3)$$

$$\left.\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \end{aligned}\right\} \quad \text{distributive laws} \qquad (4)$$

$$\left.\begin{aligned} A \cap (A \cup B) &= A \\ A \cup (A \cap B) &= A \end{aligned}\right\} \quad \text{absorbtion laws} \qquad (5)$$

identity laws:

$$\begin{matrix} A \cap U = A & \qquad & A \cup U = U \\ A \cup \emptyset = A & \qquad & A \cap \emptyset = \emptyset \end{matrix} \qquad (6)$$

complement laws:

$$\begin{matrix} (A')' = A & \qquad & A \cap A' = \emptyset & \qquad & U' = \emptyset \\ & & A \cup A' = U & & \emptyset' = U \end{matrix} \qquad (7)$$

$$\left.\begin{aligned} (A \cap B)' &= A' \cup B' \\ (A \cup B)' &= A' \cap B' \end{aligned}\right\} \quad \text{De Morgan's laws} \qquad (8)$$

We shall prove these laws in the next lecture. Meanwhile, notice similarities and differences with laws of usual arithmetic. For example, multiplication is distributive with respect to addition:

$$a \times (b + c) = (a \times b) + (a \times c),$$

but addition is not distributive with respect to multiplication: it is NOT TRUE that

$$a + (b \times c) = (a + b) \times (a + c).$$

Notice also that the idempotent laws are not so alien to arithmetic as one may think: they hold for zero,

$$0 + 0 = 0, \qquad 0 \times 0 = 0.$$

5.1 Proof of Laws of Boolean Algebra by Venn diagrams

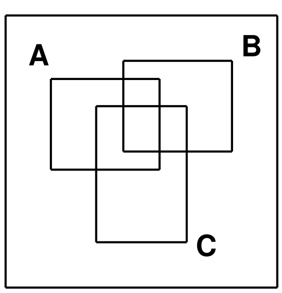
The identities in (1)–(8) of the previous lecture are called the *laws of Boolean algebra*. Several of them are obvious* because of the definitions of $\cap, \cup$ and $'$. The others may be verified* by drawing Venn diagrams. For example, to verify that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

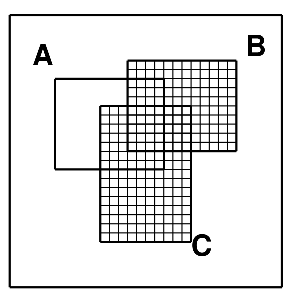
we draw the following diagrams.

* obvious = evident, self-evident

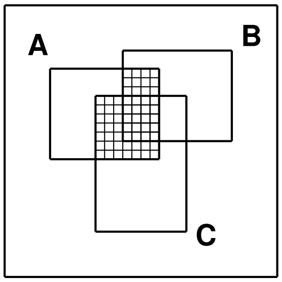
* to verify
= to check, to confirm, to validate



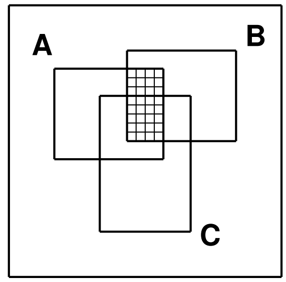
(a) **A**, **B**, **C**



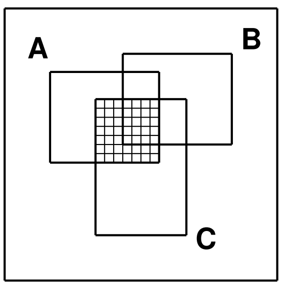
(b) $\mathbf{B} \cup \mathbf{C}$



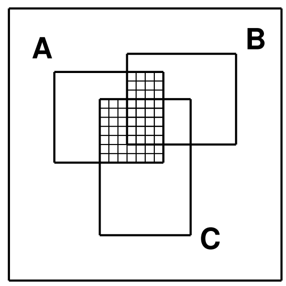
(c) $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C})$



(d) $\mathbf{A} \cap \mathbf{B}$



(e) $\mathbf{A} \cap \mathbf{C}$



(f) $(\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$

Equality holds* because diagrams (c) and (f) are the same.

* holds = is true

Because of the associative laws in (1) of the previous lecture, we can write $A \cap B \cap C$ and $A \cup B \cup C$ with unambiguous* meanings. But we *must not write* $A \cap B \cup C$ or $A \cup B \cap C$ without brackets. This is because, in general*

$$A \cap (B \cup C) \neq (A \cap B) \cup C,$$

$$A \cup (B \cap C) \neq (A \cup B) \cap C.$$

(Give your examples!)

* unambiguous = unmistakable, definite, clear
ambiguous = vague, unclear, uncertain

* A good example when the use of a word in mathematics is different from its use in ordinary speech. In the usual language "in general" means "as a rule", "in most cases". In mathematics "in general" means "sometimes". For example, in mathematics the phrases "Some people are more than 100 years old" and "In general, people are more than 100 years old" are the same.

5.2 Questions from Students

*

* This section contains no compulsory material but still may be useful.

1. Which of the following sets is finite?

(A) $\{1,2\} \cap \mathbb{R}$ (B) $\{x \in \mathbb{R} : x^2 < 9\}$ (C) $[0,1] \cap [\frac{1}{2}, \frac{3}{2}]$

A student wrote:

```
> what is the definition of finite and infinite sets?
> because the question that
> you gave us today confused me:
> I think all answers could be correct,
> for example, answer b is  -3<x<3 and
> I think it is correct.
```

ANSWER. A finite set is a set containing only finite number of elements. For example, 1,2,3 is finite. A set with infinitely many elements is called an infinite set.

The set that you mentioned,

$$\{x \in \mathbb{R} : -3 < x < 3\}$$

is infinite: there are infinitely many real numbers between -3 and 3.

For example, take a real number which has decimal expansion

$$1.2345\ldots$$

No matter how you continue write more digits after the decimal point (and this can be done in infinitely many ways), you will have a number which is bigger than -3 and smaller that 3. Therefore the set

$$\{x \in \mathbb{R} : -3 < x < 3\}$$

is not finite.

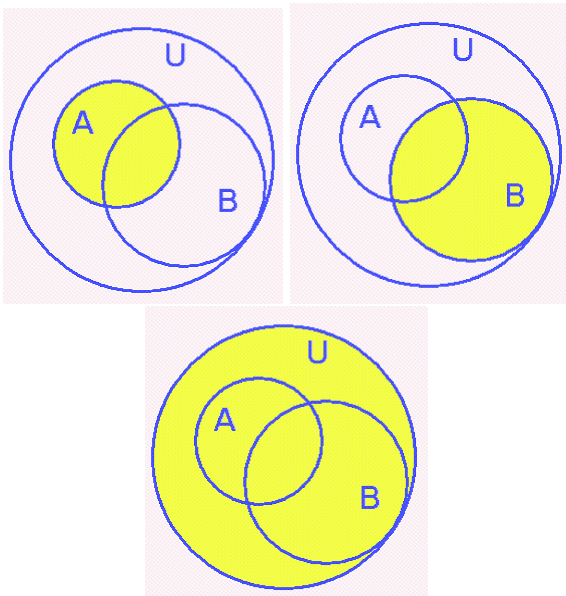
However, the set

$$\{x \in \mathbb{Z} : -3 < x < 3\}$$

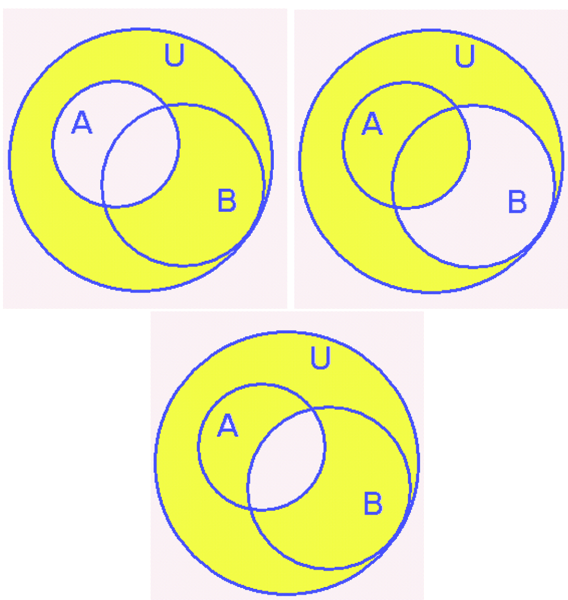
is finite, it equals $\{-2, -1, 0, 1, 2\}$ and therefore has 5 elements.

2.
```
> sorry to disturb you I have got one more question
> Given that A and B are intersecting sets,
> show following on venn
> diagram: A’, AUB’, A’UB’, and A’nB’
>can you please do these in the lecture
```

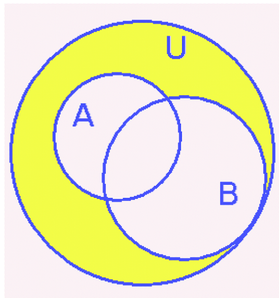
ANSWER is the following sequence of Venn diagrams:



Sets A, B, U.



Sets A', $A \cup B'$, $A' \cup B'$.



Set $A' \cap B'$.

3. `> Dear Sir,`

   `> Can you please help me with the following question?`

   [6 marks] Let

   $$A = \{x \in R : x^4 + x > 2\}$$
   $$B = \{x \in R : x^3 < 1\}$$

   and

   $$C = \{x \in R : x^8 > 1\}.$$

   (i) Prove that $A \cap B \subseteq C$.

   ```
   > Can you say that A and B are disjoint as they
   > do not meet?
   > And therefore the Empty Set is a subset of C
   ```

   ANSWER: It would be a valid argument if A and B were indeed disjoint. But they are not; one can easily see that -2 belongs to both A and B.

   A correct solution: Assume $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since $x \in A$, it satisfies

   $$x^4 + x > 2.$$

   Since $x \in B$, it satisfies

   $$x^3 < 1$$

   which implies $x < 1$ which is the same as $1 > x$. Adding the left hand sides and the right hand sides of inequalities $x^4 + x > 2$ and $1 > x$, one gets

   $$\begin{array}{rrcl} & x^4 + x & > & 2 \\ + & 1 & > & x \\ = & & & \\ & x^4 + x + 1 & > & 2 + x \end{array}$$

   which simplifies as

   $$x^4 > 1.$$

Both parts of this inequality are positive, therefore we can square it and get

$$x^8 > 1.$$

But this means that $x \in C$. Hence $A \cap B \subseteq C$.

4.

```
> Say for eg you have a situation whereby you have
>
> A U A'U B
>
> Does this simply to A U U (which is U) or A U B?
> Because i no A U A' is Union but i get confused
> when simplifying these when you have A'U B. is
> it Union or is it B?
```

ANSWER: You are mixing the union symbol $\cup$ and letter U used to denote the universal set. The correct calculation is

$$A \cup A' \cup B = (A \cup A') \cup B = U \cup B = U,$$

I set it in a large type to emphasise the difference between symbol $\cup$ and letter U. The answer is U, the universal set.

5.

```
> Was just wandering about a note I took in your
> lecture that doesn't seem right.
> I might have copied it down wrong but I wrote:
>
> A = 'Any integer'   B = 'Any Real Number
>
> A union B = any integer
>
> Was just wandering wether that should be,
> A union B = any real number
```

ANSWER: Of course, you are right: if $A = \mathbb{Z}$ and $B = \mathbb{R}$ then

$$A \cup B = B \text{ and } A \cap B = A.$$

I believe I gave in my lecture both equalities and also a general statement:

If $A \subseteq B$, then $A \cup B = B$ and $A \cap B = A$.

6 Set Theory

6.1 Proving inclusions of sets

To prove the property $A \subseteq B$ for particular* sets A and B we have to prove that every element of A is an element of B (see definition of $\subseteq$). Sometimes this is clear.* But if not proceed as in the next examples.

* particular = individual, specific

* clear = obvious, self-evident

Example 6.1.1 *Let*

$$A = \{x \in \mathbb{R} : x^2 - 3x + 2 = 0\}.$$

Prove that $A \subseteq \mathbb{Z}$.

Solution. Let $x \in A$. Then

$$\begin{aligned} x^2 - 3x + 2 &= 0, \\ (x-1)(x-2) &= 0, \\ x &= \begin{cases} 1 & \text{or} \\ 2 \end{cases} \\ x &\in \mathbb{Z}. \end{aligned}$$

6.2 Proving equalities of sets

To prove $A = B$ for particular sets A and B we have to prove $A \subseteq B$ and then $B \subseteq A$.

Recall that a *segment* $[a, b]$ of the real line $\mathbb{R}$ is defined as the set*

$$[a, b] = \{\, x \in \mathbb{R} : a \leqslant x \leqslant b \,\}.$$

* Typographical symbols:
[opening square bracket
] closing square bracket

Example 6.2.1 *Let* $A = [1, 2]$ *and*

$$B = [0, 2] \cap [1, 3].$$

Prove that $A = B$.*

* prove that . . .
= show that . . . , demonstrate that . . .

Solution. We first prove that

$$[1,2] \subseteq [0,2] \cap [1,3].$$

Let $x \in [1,2]$. Then $1 \leqslant x \leqslant 2$. Hence $0 \leqslant x \leqslant 2$ and $1 \leqslant x \leqslant 3$. Hence $x \in [0,2]$ and $x \in [1,3]$. Hence

$$x \in [0,2] \cap [1,3],$$

and, since x is an arbitrary* element of $[1,2]$, this means that

$$[1,2] \subseteq [0,2] \cap [1,3].$$

* arbitrary = taken at random

Now we prove that

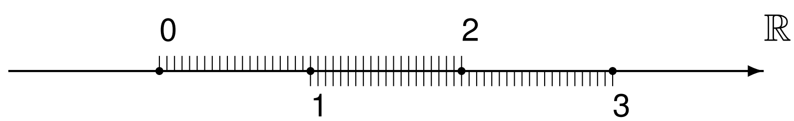
$$[0,2] \cap [1,3] \subseteq [1,2].$$

Let $x \in [0,2] \cap [1,3]$. Then $x \in [0,2]$ and $x \in [1,3]$. Hence* $0 \leqslant x \leqslant 2$ and $1 \leqslant x \leqslant 3$. Therefore $x \geq 1$ and $x \leqslant 2$. For this reason $1 \leqslant x \leqslant 2$. Consequently, $x \in [1,2]$.

* hence
= therefore, for this reason, thus, consequently, so

Comment: In a lecture, an alternative* method could be used for solving a similar problem. It is based on a graphic representation of segments $[a,b]$ on the real line $\mathbb{R}$.

* alternative = other, another, different



One can immediately see* from this picture that

$$[0,2] \cap [1,3] = [1,2].$$

* see = observe, notice

Similarly, an *interval* $]a,b[$ of the real line $\mathbb{R}$ (or (a,b), which is more popular in most countries – France being an exception) is defined as the set

$$]a,b[= \{ x \in \mathbb{R} : a < x < b \}.$$

Example 6.2.2 *Notice that*

$$[0,1] \cap [1,2] = \{1\}$$

while

$$]0,1[\cap]1,2[= \emptyset.$$

Do not mix notation $\{a, b\}$**,** $[a, b]$**,** $]a, b[$ **!**

Example 6.2.3 *Please notice that if $a > b$ then*

$$[a, b] =]a, b[= \emptyset.$$

Indeed, in that case there are no real numbers x which satisfy

$$a \leqslant x \leqslant b \text{ or } a < x < b.$$

6.3 Proving inclusions and equalities of sets by Boolean Algebra

Inclusions of sets can be proven from Laws of Boolean Algebra.

Indeed it is easy to prove by Venn Diagrams that*

* Recall that *iff* is an abbreviation for "if and only if".

Theorem 6.1

$$A \subseteq B \textbf{ iff } A \cap B = A \textbf{ iff } A \cup B = B$$

This theorem makes it possible to prove many inclusions of set by calculating in Boolean Algebra.

Example 6.3.1 *Prove that if $A \subseteq B$ then*

$$A \cup C \subseteq B \cup C.$$

Solution. $A \subseteq B$ is the same as

$$A \cup B = B.$$

Now we compute:

$$\begin{aligned}
(A \cup C) \cup (B \cup C) &= ((A \cup C) \cup B) \cup C \\
&\quad \text{(by associativity of } \cup) \\
&= (A \cup (C \cup B)) \cup C \\
&\quad \text{(by associativity of } \cup) \\
&= (A \cup (B \cup C)) \cup C \\
&\quad \text{(by commutativity of } \cup) \\
&= ((A \cup B) \cup C) \cup C \\
&\quad \text{(by associativity of } \cup) \\
&= (B \cup C) \cup C \\
&\quad \text{(by the observation above)} \\
&= B \cup (C \cup C) \\
&\quad \text{(by associativity of } \cup) \\
&= B \cup C \\
&\quad \text{(by the idempotent law for } \cup).
\end{aligned}$$

Hence

$$A \cup C \subseteq B \cup C.$$

6.4 Additional Problems: Some problems solved with the help of Venn diagrams

*

Venn diagrams can be used to solve problems of the following type.

* This section contains no compulsory material but still may be useful.

Example 1. 100 people are asked about three brands of soft drinks called A, B and C.

(i) 18 like A *only* (not B and not C).

(ii) 23 like A but not B (and like C or don't like C).

(iii) 26 like A (and like or don't like other drinks).

(iv) 8 like B and C (and like A or don't like A).

(v) 48 like C (and like or don't like other drinks).

(vi) 8 like A and C (and like or don't like B).

(vii) 54 like one and only one of the drinks.

Find how many people like B and find how many people don't like any of the drinks.

For solution, we draw a Venn diagram. Let

a be number of people liking A only

b be number of people liking B only

c be number of people liking C only

d be number of people liking A and B but *not* C

e be the number of people who like A and C, but not B.

f be the number of people who like B and C, but not A.

g be the number of people who like all tree products A, B, and C.

h be number of people liking *none* of the drinks,

as shown on the Venn diagram below.

From (i)–(vii) we get

(i) $a = 18$

(ii) $a + e = 23$

(iii) $a + d + e + g = 26$

(iv) $f + g = 8$

(v) $c + e + f + g = 48$

(vi) $e + g = 8$

(vii) $a + b + c = 54$

We also have

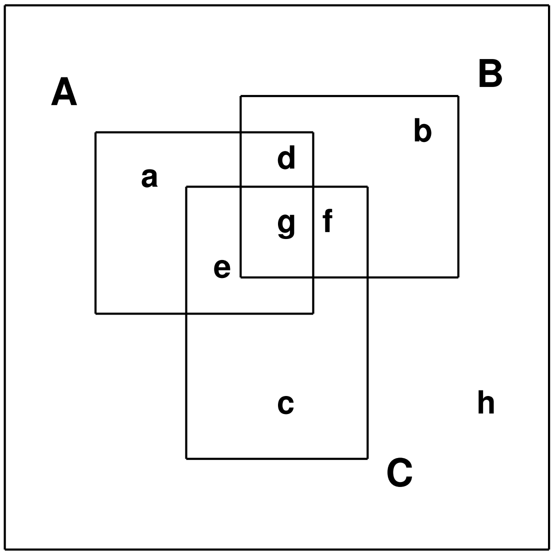
(viii) $a + b + c + d + e + f + g + h = 100$

Now (i) gives* $a = 18$, (ii) gives $e = 5$, (vi) gives $g = 3$, (iii) gives $d = 0$, (iv) gives $f = 5$, (v) gives $c = 35$, (vii) gives $b = 1$, (viii) gives $h = 33$.

* gives = yields

Therefore the number of people who like B is

$$b + d + f + g = 9,$$



and the number of people who like none is $h = 33$. □

Example 2. X and Y are sets with the following three properties.

(i) X' has 12 elements.

(ii) Y' has 7 elements.

(iii) $X \cap Y'$ has 4 elements.

How many elements in $X' \cap Y$?

(A) 6 (B) 8 (C) 9

ANSWER. (C).

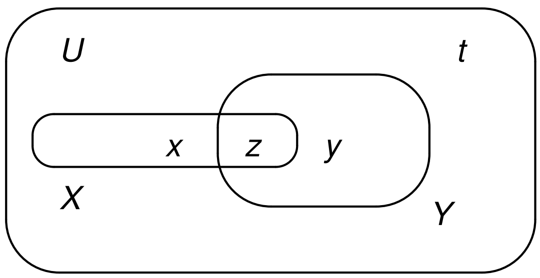
BRIEF SOLUTION. Denote $x = |X \cap Y'|$, $y = |X' \cap Y|$ (this is what we have to find), $z = |X \cap Y|$, $t = |(X \cup Y)'|$ (make a Venn diagram!), then

$$\begin{aligned} |X'| = y + t &= 12 \\ |Y'| = x + t &= 7 \\ |X \cap Y'| = x &= 4 \end{aligned}$$

Excluding unknowns, we find $t = 3$ and $y = 9$. □

DETAILED SOLUTION. Recall that we use notation $|A|$ for the number of elements in a finite set A.

Denote $x = |X \cap Y'|$, $y = |X' \cap Y|$ (this is what we have to find), $z = |X \cap Y|$, $t = |(X \cup Y)'|$, see a Venn diagram below.



Then

$$\begin{aligned} |X'| = y + t &= 12 \\ |Y'| = x + t &= 7 \\ |X \cap Y'| = x &= 4 \end{aligned}$$

So we have a system of three equations:

$$\begin{aligned} y + t &= 12 \\ x + t &= 7 \\ x &= 4 \end{aligned}$$

Excluding unknowns, we find $t = 3$ and $y = 9$.

This last step can be written in more detail. Substituting the value $x = 3$ from the third equation into the second equations, we get

$$4 + t = 7,$$

which solves as $t = 3$. Now we substitute this value of t in the first equation and get

$$y + 3 = 12;$$

solving it, we have $y = 9$. □

6.5 Questions from students

*

* This section contains no compulsory material but still may be useful.

1. ```
   > A survey was made of 25 people to ask about
   > their use of products A and B. The following infor-
   > mation was recorded: 14 people used only one of the
   > products; 9 people did not use B ; 11 people
   > did not use A.
   > (i) How many people used A?
   > (ii) How many people used both products?
   ```
   ANSWER. A solution is straightforward: denote

   a number of people using A but not B

   b number of people using B but not A

   c number of people using both A and B

   d number of people not using any product

   (it is useful to draw a Venn diagram and see that a, b, c, d correspond to its 4 regions).
   Then

   - "14 people used only one of the products" means a+b = 14
   - "9 people did not use B" means a+d = 9
   - "11 people did not use A" means $b + d = 11$
   - Finally, $a + b + c + d = 25$.

   Thus you have a system of 4 linear equations with 4 variables:

   $$\begin{aligned} a + b &= 14 \\ a + d &= 9 \\ b + d &= 11 \\ a + b + c + d &= 25 \end{aligned}$$

   and it is easy to solve; I leave it you to work out details. Answer: $a = 6$, $b = 8$, $c = 8$, $d = 3$.
   ```

7 Propositional Logic

7.1 Statements

A *statement* (or *proposition*) is a sentence which states or asserts* something. It is either true or false. If true, we say that the statement has *truth value* $\mathbb{T}$. If false, it has *truth value* $\mathbb{F}$.

* assert = state, claim

Example 7.1.1 • *"London is the capital of England" has truth value* $\mathbb{T}$*.*

• $2 \times 2 = 5$ *has truth value* $\mathbb{F}$*.*

• *"Are you asleep?" is not a statement.* □

Mathematically we do not distinguish between statements which make the same assertion, expressed differently. For example, "The capital of England is London" is regarded as equal to "London is the capital of England".

We use p, q, r, ... to denote statements.

7.2 Conjunction

If p and q are statements then "p and q" is a new statement called the *conjunction* of p and q and written $p \wedge q$. According to mathematical convention,* $p \wedge q$ has truth value $\mathbb{T}$ when both p and q have truth value $\mathbb{T}$, but $p \wedge q$ has truth value $\mathbb{F}$ in all other cases.* Here is the *truth table*:

* convention = custom, agreement

* The typographical symbol $\wedge$ is called *wedge*. It is used not only in logic, but in some other areas of mathematics as well, with a completely different meaning.

p	q	$p \wedge q$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

Example 7.2.1 • *Suppose p is "2 is even" and q is "5 is odd". Then* $p \wedge q$ *is "2 is even and 5 is odd". Since p has truth value* $\mathbb{T}$ *and q has truth value* $\mathbb{T}$*,* $p \wedge q$ *has truth value* $\mathbb{T}$ *(1st row of the table).*

- *"3 is odd and 2 is odd" has truth value $\mathbb{F}$ (see 2nd row of the truth table).*

- *If we know q is true but $p \wedge q$ is false we can deduce that p is false (the only possibility in the truth table).* $\square$

$p \wedge q$ is sometimes expressed without using "and". For example, "Harry is handsome, but George is rich" is the same, mathematically, as "Harry is handsome and George is rich".

7.3 Disjunction

"p or q" is called the *disjunction* of p and q and written $p \vee q$. Truth table:

p	q	$p \vee q$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

In effect, this table tells us how "or" is used in mathematics: it has the meaning of "and/or" ("inclusive" meaning of "or").*

Note that $p \vee q$ is true if at least one of p and q is true. It is only false when both p and q are false.*

Example 7.3.1

- *Suppose p is "4 is odd" and q is "5 is odd". Then $p \vee q$ is "4 is odd or 5 is odd". Since p has truth value $\mathbb{F}$ and q has truth value $\mathbb{T}$, $p \vee q$ has truth value $\mathbb{T}$ (3rd row of truth table).*

- *"$3 > 4$ or $5 > 6$" has truth value $\mathbb{F}$ (see 4th row of truth table).*

* The typographical symbol $\vee$ is called "vee"

* In Computer Science, the exclusive version of "or" is also used, it is usually called XOR (for eXclusive OR) and is denoted $p \oplus q$. Its truth table is

p	q	$p \oplus q$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

7.4 Negation

The statement obtained from p by use of the word "not" is called the *negation** of p and is written $\sim p$. For example, if p is "I like coffee" then $\sim p$ is " I don't like coffee". The truth value of $\sim p$ is the opposite of the truth value of p.

* Symbols sometimes used to denote negation: $\neg p$, $\overline{p}$. $\sim p$ is sometimes called "the opposite of p"

p	$\sim p$
$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$

Example 7.4.1 *"2 is odd" is false, but "2 is not odd" is true.*
☐

7.5 Conditional

Suppose p and q are statements. The statement "If p then q", denoted $p \to q$, is called a *conditional statement*.* The truth values to be given to $p \to q$ are open to some debate but the mathematical convention is as follows.

* There is a huge number of ways to express "if p then q", for example

- p implies q
- p leads to q
- p yields q
- q follows from p
- q is a consequence of p
- q is a necessary condition for p
- p is a sufficient condition for q
- q is true provided p is true
- p entails q

p	q	$p \to q$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$

YOU MUST WORK ACCORDING TO THIS TABLE WHETHER YOU LIKE IT OR NOT!

The convention is that $p \to q$ is true when p is false, regardless of the truth value of q. Rough explanation: when p is false there is nothing wrong with $p \to q$ because it means "if p then q" and so makes an assertion only when p is true.

Another explanation: some conditional statements can be thought of as statements of *promise*. For example:

if I have no cold, I'll come to class.

Here p is “I have no cold” and q is “I'll come to class”. If p is false, that is, if I have cold, you would agree that I have kept my promise even if I have not come to class (in which case q is false).*

* This example is expanded at the end of this lecture.

Perhaps the most surprising is the third row of the table. You may think of it as the principle of the absolute priority of Truth: Truth is Truth regardless of how we came to it or from whom we heard it. This is because our statements are about the world around us and are true if they describe the world correctly.* For a statement to be true, it is not necessary to receive it from a source of authority or trust.

* This is why in the literature, our rule for implication is sometimes called *material implication*: it is about material world.

Statements of promise also give a good explanation. Returning to the phrase *if I have no cold, I'll come to class*, you would agree that if I have cold (p is $\mathbb{F}$) but nevertheless came to class (q is $\mathbb{T}$), I have kept my promise and told the truth; hence $\mathbb{F} \to \mathbb{T}$ is $\mathbb{T}$.

Example 7.5.1

Suppose p is “$4 > 1$” and q is “$3 = 5$”. Then $p \to q$ is “If $4 > 1$ then $3 = 5$”. This is false because p is true and q is false (see 2nd row of truth table).

- *“If $3 = 5$ then $2 = 0$” is true (see 4th row of truth table).*
- *“If $3 = 5$ then $2 = 2$” is true (see 3rd row of truth table).*
- *If $p \to q$ has truth value $\mathbb{F}$ we can deduce that p is true and q is false (only the second row of the truth table gives $p \to q$ false).* □

Statements of the form $p \to q$ usually arise only when there is a “variable” or “unknown” involved.

Example 7.5.2 *“If $x > 2$ then $x^2 > 4$” is a true statement, whatever the value of x. For example, when $x = 3$, $x^2 = 9 > 4$ and when $x = 4$, $x^2 = 16 > 4$. The statement is regarded as true, by convention, for values of x which do not satisfy $x > 2$. For numbers like $x = -1$ we do not care whether $x^2 > 4$ is true.* □

The following example illustrates different expression of $p \to q$ in English. Let p be "$x > 2$ and q be "$x^2 > 4$". Then all of the following expresses $p \to q$.

- If $x > 2$ then $x^2 > 4$.
- $x^2 > 4$ if $x > 2$.
- $x > 2$ implies $x^2 > 4$.
- $x > 2$ only if $x^2 > 4$.
- $x > 2$ is sufficient condition for $x^2 > 4$.
- $x^2 > 4$ is necessary condition for $x > 2$.

7.6 Questions from students

*

* This section contains no compulsory material but still may be useful.

1. ```
   > I'm having a bit of trouble with the propositional
   > logic conditional statement.
   > Surely if p implies q and p is false but q is true,
   > the statement that p implies q is false?
   > I know you said we would have trouble with this but
   > i've found it difficult to trust my own logical
   > reasoning when working out subsequently more
   > complex compound statements. Could you suggest a
   > more logical way of approaching this concept?
   ```

   ANSWER. I expand my example with interpretation of implication as *promise*.I am using here a large fragment from Peter Suber's paper *Paradoxes of Material Implication*,
   http://www.earlham.edu/~peters/courses/log/mat-imp.htm.

   It is important to note that material implication does conform to some of our ordinary intuitions about implication. For example, take the conditional statement,

   > "If I am healthy, I will come to class."

   We can symbolize it, $H \to C$. The question is: when is this statement false? When will I have broken my promise?

   There are only four possibilities:
   ```

H	C	$H \to C$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$

In case #1, I am healthy and I come to class. I have clearly kept my promise; the conditional is true.

In case #2, I am healthy, but I have decided to stay home and read magazines. I have broken my promise; the conditional is false.

In case #3, I am not healthy, but I have come to class anyway. I am sneezing all over you, and you're not happy about it, but I did not violate my promise; the conditional is true.

In case #4, I am not healthy, and I did not come to class. I did not violate my promise; the conditional is true.

But this is exactly the outcome required by the material implication. The compound is only false when the antecedent* is true and the consequence is false (case #2); it is true every other time.

* In the conditional statement $H \to C$, the first term H is called *antecedent*, the second C *consequent*.

Many people complain about case #4, when a false antecedent and a false consequent make a true compound. Why should this be the case?

If the promise to come to class didn't persuade you, here's an example from mathematics.

> "If n is a perfect square, then n is not prime."

I hope you'll agree that this is a true statement for any n. Now substitute 3 for n:

> "If 3 is a perfect square, then 3 is not prime."

As a compound, it is still true; yet its antecedent and consequent are both false.

Even more fun is to substitute 6 for n:

> "If 6 is a perfect square, then 6 is not prime."

it is a true conditional, but its antecedent is false and consequent is true.

```
> Unfortunately, case #4 seemed perfectly logical to me. It was case #3
> which I found illogical. If I told you that I would come to class IF I
> was not sick, and yet I came to class despite being sick, surely my
> promise was not honoured? If I had said I MAY not come to class if I am
> sick then I would always be honouring my promise so long as I came to
> class when I was well... Is this a more appropriate way to think about
> it? Would I have problems using the 'may' component?
```

ANSWER. An excellent question. I wish to emphasise:

Propositional Logic is designed for communication with machines, it gives only very crude description of the way how natural human language. Such constructions as “I MAY” are too subtle for Propositional Logic to capture their meaning.

> **Therefore we have to live with rules of material implication as they are: they present a best possible compromise between language for people and language for machines.**

Logical constructions of the kind “I MAY” are studied in a more sophisticated branch of logic, *Modal Logic*. I simply copy the following description of Modal Logic from *Wikipedia*:

> A *modal logic* is any system of formal logic that attempts to deal with modalities. Traditionally, there are three ‘modes’ or ‘moods’ or ‘modalities’ of the copula to be, namely, *possibility, probability*, and *necessity*. Logics for dealing with a number of related terms, such as *eventually, formerly, can, could, might, may, must*, are by extension also called modal logics, since it turns out that these can be treated in similar ways.

But we are not studying Modal Logics in our course. However, they are taught in Year 4 of School of Mathematics.

8 Propositional Logic, Continued

8.1 Converse

Notice that $p \to q$ and $q \to p$ are different; $q \to p$ is called the *converse* of $p \to q$.

Example 8.1.1 *Let p = "$x > 2$" and q = "$x^2 > 4$". Then $p \to q$ is "If $x > 2$ then $x^2 > 4$" – TRUE. But $q \to p$ is "If $x^2 > 4$ then $x > 2$". This is FALSE (for $x = -3$, for example).* $\square$

8.2 Biconditional

"p if and only if q" is denoted by $p \leftrightarrow q$ and called the *biconditional* of p and q. The truth table is as follows.*

p	q	$p \leftrightarrow q$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$

So, if p and q are both true or both false then $p \leftrightarrow q$ is true: otherwise it is false.

The biconditional $p \leftrightarrow q$ can be expressed as

"p if and only if q"

or

"p is a necessary and sufficient condition for q".

Example 8.2.1

"$x > 2$ if and only if $x + 1 > 3$"

is the same as

"For $x > 2$ it is necessary and sufficient that $x + 1 > 3$".

$p \leftrightarrow q$ may be thought of as a combination of $p \to q$ and $q \to p$.

* Notice that, in mathematical literature and blackboard writing, the expression "if and only if" is sometimes abbreviated "iff".

8.3 XOR

Excluded OR, or *XOR* $p \oplus q$ is defined by the truth table

p	q	$p \oplus q$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

It is the exclusive version of "or" (as opposed to inclusive "or" $\vee$). XOR is widely used in computer programming and Computer Science. Its name is an abbreviation of e**X**clusive **OR**.

Read more on XOR in Section 9.3.

8.4 Compound statements and truth tables

The symbols $\wedge$, $\vee$, $\sim$, $\rightarrow$, $\leftrightarrow$, and $\oplus$ are called *connectives*.

*Compound** statements may be built up from statements

* compound = complex, composite

$$p, q, r, \ldots$$

by means of connectives. We use brackets for punctuation as in

$$(p \rightarrow q) \leftrightarrow (\sim r \wedge q).$$

We take the convention that $\sim$ applies only to the part of the expression which comes immediately after it. Thus $\sim r \wedge q$ means $(\sim r) \wedge q$, which is not the same as $\sim(r \wedge q)$.

The truth value of a compound statement involving statements $p, q, r, \ldots$ can be calculated from the truth values of $p, q, r, \ldots$ as follows.

Example 8.4.1 *Find the truth table of**

$$\sim(p \rightarrow (q \vee r)).$$

* Some typographic terminology: in the expression

$$\sim\underline{(}p \rightarrow (q \vee r)\underline{)}$$

the first opening bracket and the last closing bracket (they are underlined) *match* each other. This is another pair of *matching* brackets:

$$\sim(p \rightarrow \underline{(}q \vee r\underline{)}).$$

See more in Section 8.7.

Solution (We take 8 rows because there are 3 variables $p, q, r, \ldots$ each with two possible truth values.)

p	q	r	$q \vee r$	$p \to (q \vee r)$	$\sim (p \to (q \vee r))$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$

Please always write the rows in this order, it will help you to easier check your work for errors.

(We get each of the last 3 columns by use of the truth tables for $\vee$, $\to$ and $\sim$.)

This can also be set out as follows.

$\sim$	$(p$	$\to$	$(q$	$\vee$	$r))$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

(The truth values for p, q, r (8 possibilities) are entered first:

$\sim$	$(p$	$\to$	$(q$	$\vee$	$r))$
	$\mathbb{T}$		$\mathbb{T}$		$\mathbb{T}$
	$\mathbb{T}$		$\mathbb{T}$		$\mathbb{F}$
	$\mathbb{T}$		$\mathbb{F}$		$\mathbb{T}$
	$\mathbb{T}$		$\mathbb{F}$		$\mathbb{F}$
	$\mathbb{F}$		$\mathbb{T}$		$\mathbb{T}$
	$\mathbb{F}$		$\mathbb{T}$		$\mathbb{F}$
	$\mathbb{F}$		$\mathbb{F}$		$\mathbb{T}$
	$\mathbb{F}$		$\mathbb{F}$		$\mathbb{F}$

Then the other columns are completed in order 5, 3, 1.)

□

Example 8.4.2 *Find the truth table of $p \wedge (\sim q \to p)$.*

Solution

p	q	$\sim q$	$\sim q \to p$	$p \wedge (\sim q \to p)$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$

or

p	$\wedge$	$(\sim$	q	$\to$	$p)$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

□

8.5 Tautologies

The statements

$$p \vee \sim p \text{ and } (p \wedge (p \to q)) \to q$$

have the following truth tables.

p	$\sim p$	$p \vee \sim p$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q)) \to q$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$

Only $\mathbb{T}$ occurs in the last column. In other words, the truth value of the statement is always $\mathbb{T}$, regardless of the truth values of its components $p, q, r, \ldots$. A statement with this property is called a *tautology*.

Example 8.5.1

(i) Let p = "It is raining". Then $p \vee \sim p$ is "Either it is raining or it is not raining". This is true regardless of whether it is raining or not.

(ii) Let p = "$x > 2$" and q = "$y > 2$". Then

$$(p \wedge (p \to q)) \to q$$

is "If $x > 2$, and $x > 2$ implies $y > 2$, then $y > 2$". This is true because

$$(p \wedge (p \to q)) \to q$$

is a tautology: the meanings of p and q are not important. □

We can think of tautologies as statements which are true for entirely logical reasons.

8.6 Contradictions

A statement which is always $\mathbb{F}$ regardless of the truth values of its components $p, q, r, \ldots$ is called a *contradiction*. (Only $\mathbb{F}$ occurs in the last column of the truth table.)

Example 8.6.1 *$p \wedge \sim p$. It is raining and it is not raining.* □

8.7 Matching brackets: a hard question

*

*This section contains no compulsory material but still may be useful.

This is a continuation of discussion in started in a marginal comment on Page 51.

It is obvious that a sequence of brackets

$$(\ (\ )\ (\ (\ (\ )\ )\ )\ )$$

they properly match each other and correspond to a valid algebraic expression, for example

$$((a+b)\times(c+((d\div e)+f)))$$

or

$$((a\vee b)\wedge(c\vee((d \implies e)\vee f))),$$

while brackets

$$(\ (\ )\ )\ )\ (\ )\ (\ (\ )$$

do not match each other properly.

Problem (non-compulsory and hard). Formulate a simple and easy to use rule which allows to distinguish between “correct” and “incorrect” combinations of brackets.

8.8 Questions from students

*

*This section contains no compulsory material but still may be useful.

1.
```
When drawing truthtables, i found that there are
2 types of u can do, is the correct method putting
in T or F values underneath each of the symbols
or I have seen in our notes that the answer can
still be found without finding out each symbol
and by breaking up the particular question.

for example the question ~q --> ( p ---> q)
can a truth table be written in the exam as this:

p q  ~q  (p-->q)  ~q-->(p-->q)
```

```
T T   F     F           T
T F   T     T           T

etc. Will you be given full marks for this method
or must u include values for each symbol?
```

My answer: either way of composing truth tables is valid, can be used in the exam and be given full marks.

But please, try to write in a neat and comprehensible way, so that table looks like a table and is not stretched diagonally all over page.

2.
```
> can I have a simple English sentence illustrates
> this statement p -> (q -> p)?
```

   My answer: quite a number of English sentences built around an expression "even without" or "even if" belong to this type. For example, "the turkey is good, even without all the trimmings":

   p is "the turkey is good"

   q is "without all the trimmings".

   The statement $p \to (q \to p)$ becomes

   "the turkey is good, and for that reason, even without all the trimmings, the turkey is still good".

3.
```
There is an example in the note which is :

Given that p V q is T and q V r is F ,
2. Which of the following statements is a tautology?
(A) (P --> q) V (~ p --> q)
(B) (p ^ q) V (~ p ^ ~ q)
(C) (q --> p) V (~ p --> ~ q)

The mentioned answer is A

But when we apply the truth table we find C True as well.
```

   My answer: Your question refers to Question 2 on Page **??**, but the sentence

```
Given that p V q is T and q V r is F ,
```

   is from Question 1 and has no relation to Question 2 – perhaps this is the reason for your misunderstanding.

9 Logically equivalent statements

9.1 Logical equivalence: definitions and first examples

Let X and Y be two statements built up from the same components $p, q, r, \ldots$. If the truth value of X is the same as the truth value of Y for every combination of truth values of $p, q, r, \ldots$ then X and Y are said to be *logically equivalent*. In other words X and Y are logically equivalent if the final columns of their truth tables are the same.

Example 9.1.1

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
			$*$			$**$

Columns $*$ and $**$ are the same, i.e.[*] for every choice of truth values for p and q, $\sim(p \wedge q)$ and $\sim p \vee \sim q$ have the same truth values. Thus $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent. □

[*] i.e. = that is,

If X and Y are logically equivalent statements we write $X \equiv Y$.

Example 9.1.2 $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

A particular case of this is shown by taking

p = "You are French" and

q = "You are a woman".

Then

$\sim(p \wedge q)$ = "You are not a French woman" and

$\sim p \vee \sim q$ = "Either you are not French or you are not a woman". □

9.2 Boolean algebra, revisited

The logical equivalence

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

is analogous to the set theory identity

$$(A \cap B)' = A' \cup B'.$$

In fact it is remarkable that if we replace $\cap$ by $\wedge$, $\cup$ by $\vee$, $'$ by $\sim$, U by $\mathbb{T}$ (to denote a tautology) and $\emptyset$ by $\mathbb{F}$ (to denote a contradiction) then all the rules of Boolean algebra turn into logical equivalences.

$$\left.\begin{array}{lcl} \mathrm{p} \wedge \mathrm{q} & \equiv & \mathrm{q} \wedge \mathrm{p} \\ \mathrm{p} \vee \mathrm{q} & \equiv & \mathrm{q} \vee \mathrm{p} \end{array}\right\} \quad \text{commutative laws} \qquad (1)$$

$$\left.\begin{array}{lcl} \mathrm{p} \wedge \mathrm{p} & \equiv & \mathrm{p} \\ \mathrm{p} \vee \mathrm{p} & \equiv & \mathrm{p} \end{array}\right\} \quad \text{idempotent laws} \qquad (2)$$

$$\left.\begin{array}{lcl} \mathrm{p} \wedge (\mathrm{q} \wedge \mathrm{r}) & \equiv & (\mathrm{p} \wedge \mathrm{q}) \wedge \mathrm{r} \\ \mathrm{p} \vee (\mathrm{q} \vee \mathrm{r}) & \equiv & (\mathrm{p} \vee \mathrm{q}) \vee \mathrm{r} \end{array}\right\} \quad \text{associative laws} \qquad (3)$$

$$\left.\begin{array}{lcl} \mathrm{p} \wedge (\mathrm{q} \vee \mathrm{r}) & \equiv & (\mathrm{p} \wedge \mathrm{q}) \vee (\mathrm{p} \wedge \mathrm{r}) \\ \mathrm{p} \vee (\mathrm{q} \wedge \mathrm{r}) & \equiv & (\mathrm{p} \vee \mathrm{q}) \wedge (\mathrm{p} \vee \mathrm{r}) \end{array}\right\} \quad \text{distributive laws} \qquad (4)$$

$$\left.\begin{array}{lcl} \mathrm{p} \wedge \mathrm{p} \vee \mathrm{q}) & \equiv & \mathrm{p} \\ \mathrm{p} \vee (\mathrm{p} \wedge \mathrm{q}) & \equiv & \mathrm{p} \end{array}\right\} \quad \text{absorbtion laws} \qquad (5)$$

$$\begin{array}{lcl} \mathrm{p} \wedge \mathbb{T} \equiv \mathrm{p} & \qquad & \mathrm{p} \vee \mathbb{T} \equiv \mathbb{T} \\ \mathrm{p} \vee \mathbb{F} \equiv \mathrm{p} & & \mathrm{p} \wedge \mathbb{F} \equiv \mathbb{F} \end{array} \qquad (6)$$

$$\begin{array}{lll} \sim(\sim \mathrm{p}) \equiv \mathrm{p} \qquad & \mathrm{p} \wedge \sim \mathrm{p} \equiv \mathbb{F} \qquad & \sim\mathbb{T} \equiv \mathbb{F} \\ & \mathrm{p} \vee \sim \mathrm{p} \equiv \mathbb{T} & \sim\mathbb{F} \equiv \mathbb{T} \end{array} \qquad (7)$$

$$\left.\begin{array}{lcl} \sim(\mathrm{p} \wedge \mathrm{q}) & \equiv & \sim \mathrm{p} \vee \sim \mathrm{q} \\ \sim(\mathrm{p} \vee \mathrm{q}) & \equiv & \sim \mathrm{p} \wedge \sim \mathrm{q} \end{array}\right\} \quad \text{De Morgan's laws} \qquad (8)$$

They may all be proved by means of truth tables as we did for

$$\sim(p \wedge q) \equiv \sim p \vee \sim q.$$

Similarly:

$$\mathrm{p} \to \mathrm{q} \equiv \sim \mathrm{p} \vee \mathrm{q} \tag{9}$$

$$(\mathrm{p} \leftrightarrow \mathrm{q}) \equiv (\mathrm{p} \to \mathrm{q}) \wedge (\mathrm{q} \to \mathrm{p}) \tag{10}$$

$$\mathrm{p} \oplus \mathrm{q} \equiv (\mathrm{p} \wedge \sim \mathrm{q}) \vee (\sim \mathrm{p} \wedge \mathrm{q}) \tag{11}$$

We call (1)–(8) the **fundamental logical equivalences**. Rules 9, 10, and 11 enable us to rewrite $\to$, $\leftrightarrow$, and $\oplus$ entirely in terms of $\wedge$, $\vee$ and $\sim$. Expressions involving $\wedge$, $\vee$ and $\sim$ can be manipulated by means of rules (1)–(8).

Example 9.2.1 *Simplify* $\sim p \vee (p \wedge q)$.

$$\begin{aligned} \sim p \vee (p \wedge q) &\overset{by\ (4)}{\equiv} (\sim p \vee p) \wedge (\sim p \vee q) \\ &\overset{by\ (1)}{\equiv} (p \vee \sim p) \wedge (\sim p \vee q) \\ &\overset{by\ (7)}{\equiv} T \wedge (\sim p \vee q) \\ &\overset{by\ (1)}{\equiv} (\sim p \vee q) \wedge T \\ &\overset{by\ (6)}{\equiv} \sim p \vee q. \end{aligned}$$

To determine whether or not statements X and Y are logically equivalent we use truth tables. If the final columns are the same then $X \equiv Y$, otherwise $X \not\equiv Y$.

If we are trying to prove $X \equiv Y$ we can either use truth tables or we can try to obtain Y from X by means of fundamental logical equivalences (1)–(10).

Example 9.2.2 *Prove that* $\sim q \to \sim p \equiv p \to q$.

We could use truth tables or proceed as follows

$$\begin{aligned}
\sim q \to \sim p &\overset{\text{by (9)}}{\equiv} \sim\sim q \vee \sim p \\
&\overset{\text{by (7)}}{\equiv} q \vee \sim p \\
&\overset{\text{by (1)}}{\equiv} \sim p \vee q \\
&\overset{\text{by (9)}}{\equiv} p \to q.
\end{aligned}$$

9.3 "Either or" and "neither nor"

I am frequently asked by students whether an expression of everyday language "either p or q" is expressed in Propositional Logic by the "exclusive or" connective $\oplus$. My answer is yes, it is so in most cases, but not always; you have to look at the context where the expression "either or" is used. There is an additional difficulty: if "either p or q" is understood as $p \oplus q$, that is "either p, or q, but not both", then the expression of everyday language "neither p nor q" is *NOT* the negation of "either p or q". Sketch Venn diagrams and see it for yourselves.

Example 9.3.1 *Let p means "John lives in Peterborough" and q means "John lives in Queensferry".*

Then $p \oplus q$ means

> *"John lives either in Peterborough, or in Queensferry, but not in both".*

The negation $\sim(p \oplus q)$ is

> *"John either does not live in Peterborough or in Queensferry, or lives in both of them",*

while

> *"John lives neither in Petersborough nor in Queensferry"*

is $\sim(p \vee q)$.

Notice that $\sim(p \oplus q)$ is not logically equivalent to $\sim(p \vee q)$.

9.4 Problems

Problem 9.1 *Prove all Fundamental Logical Equivalences (1) – (11) by computing truth tables.*

Problem 9.2 *Use Fundamental Logical Equivalences (1) – (11) to prove logical equivalences involving XOR $\oplus$;*

(i) $p \oplus \mathbb{F} \equiv p$

(ii) $p \oplus p \equiv \mathbb{F}$

(iii) $p \oplus q \equiv q \oplus p$ *(commutativity)*

(iv) $(p \oplus q) \oplus r \equiv p \oplus (q \oplus r)$ *(associativity)*

and logical equivalences involving XOR and the negation:

(v) $p \oplus \sim p \equiv \mathbb{T}$

(vi) $p \oplus \mathbb{T} \equiv \sim p$

Problem 9.3 *Use Fundamental Logical Equivalences (1) – (11) to prove tautologies involving conditional $\rightarrow$:*

(i) $p \rightarrow p$

(ii) $(p \wedge q) \rightarrow p$

(iii) $(p \wedge q) \rightarrow q$

(iv) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow q)$

(v) $((p \rightarrow q) \wedge (\sim p \rightarrow q)) \rightarrow q$

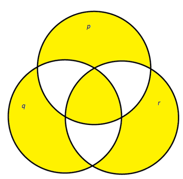
Problem 9.4 *Prove, by constructing truth table and by drawing Venn diagrams, that $\sim(p \oplus q))$ is not logically equivalent to $(p \vee q)$. (See Section 9.3 for discussion.)*

9.5 Solutions

9.2(iv). This tautology deserves attention because its proof is long, while the Venn diagram is highly symmetric.

$$
\begin{aligned}
(p \oplus q) \oplus r \quad &\equiv \quad \underbrace{((p \oplus q) \wedge \sim r)} \vee \underbrace{(\sim (p \oplus q) \wedge r)} \\
&\equiv \quad \underbrace{(((p \wedge \sim q) \vee (\sim p \wedge q)) \wedge \sim r)}_{\text{we simplify this first}} \vee \underbrace{(\sim ((p \wedge \sim q) \vee (\sim p \wedge q)) \wedge r)} \\
&\equiv \quad \underbrace{[p \wedge \sim q \wedge \sim r] \vee [\sim p \wedge q \wedge \sim r]} \vee ((\sim (p \wedge \sim q) \wedge \sim (\sim p \wedge q)) \wedge r) \\
&\equiv \quad [p \wedge \sim q \wedge \sim r] \vee [\sim p \wedge q \wedge \sim r] \vee (\underbrace{((\sim p \vee \sim\sim q) \wedge (\sim\sim p \vee \sim q))}_{\text{and now we work with that bit}} \wedge r) \\
&\equiv \quad [p \wedge \sim q \wedge \sim r] \vee [\sim p \wedge q \wedge \sim r] \vee (\underbrace{((\sim p \vee q) \wedge (p \vee \sim q))}_{\text{and now we work here}} \wedge r) \\
&\equiv \quad [p \wedge \sim q \wedge \sim r] \vee [\sim p \wedge q \wedge \sim r] \vee (\underbrace{((\sim p \wedge p) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim q))} \wedge r) \\
&\equiv \quad [p \wedge \sim q \wedge \sim r] \vee [\sim p \wedge q \wedge \sim r] \vee (\underbrace{\mathbb{F} \vee (\sim p \wedge \sim q) \vee (q \wedge p) \vee \mathbb{F}} \wedge r) \\
&\equiv \quad [p \wedge \sim q \wedge \sim r] \vee [\sim p \wedge q \wedge \sim r] \vee (\underbrace{(\sim p \wedge \sim q) \vee (q \wedge p))} \wedge r) \\
&\equiv \quad [p \wedge \sim q \wedge \sim r] \vee [\sim p \wedge q \wedge \sim r] \vee [\sim p \wedge \sim q \wedge r] \vee [p \wedge q \wedge r]
\end{aligned}
$$

Please observe that we get a disjunction of four conjunctions (enclosed in square brackets) each of which contains even number (zero or two) of negations. If you do a similar calculation with $p \oplus (q \oplus r)$, you will get the same expression. The Venn diagram for the both $(p \oplus q) \oplus r$ and $p \oplus (q \oplus r)$ is very symmetric:



9.6 Self-referential statements

A *self-referential statement* is a statement that refers to itself or its own referent. The most famous example of a self-referential sentence is the liar sentence: "This sentence is not true".

Exercise. Which of the statements is True:

(A) $2 \times 2 = 3$

(B) $2 \times 2 = 5$

(C) all statements are False.

"Smallest natural number which cannot be expressed by less than hundred characters"

The sentence contains 81 characters.

"Epimenides, a Cretan, said: "All Cretans are liars.""

Figure 6: A self-referential graffiti

9.7 Questions from students

1. ```
   > In the exam are we going to receive a formula
   > sheet with rules of boolean algebra?
   ```
   ANSWER. Yes, you are. NS
   ```

10 Predicate Logic

10.1 Predicaes

Many mathematical sentences involve "unknowns" or "variables".

Example 10.1.1 *(i) $x > 2$ (where x stands for an unknown real number).*

(ii) $A \subseteq B$ (where A and B stand for unknown sets).

Such sentences are called *predicates*. They are not statements because they do not have a definite truth value: the truth value depends on the unknowns.

Example 10.1.2

(i) $x > 2$ is $\mathbb{T}$ for $x = 3, 3\frac{1}{2}$, etc., $\mathbb{F}$ for $x = 2, -1$, etc.

(ii) $A \subseteq B$ is $\mathbb{T}$ for $A = \{1, 2\}$, $B = \mathbb{R}$.
$A \subseteq B$ is $\mathbb{F}$ for $A = \{1, 2\}$, $B = \{2, 3, 4\}$.

We can write $p(x)$, $q(x)$, ... for predicates involving an unknown x, $p(x, y)$, $q(x, y)$, ... when there are unknowns x and y, $p(A, B)$, $q(A, B)$, ... when there are unknowns A and B, etc.

Example 10.1.3 *(i) Let $p(x)$ denote the **predicate** $x > 2$. Then $p(1)$ denotes the **statement** $1 > 2$ (truth value $\mathbb{F}$) while $p(3)$ denotes the **statement** $3 > 2$ (truth value $\mathbb{T}$).*

(ii) Let $p(x, y)$ denote $x^2 + y^2 = 1$. Then $p(0, 1)$ denotes $0^2 + 1^2 = 1$ (true) while $p(1, 1)$ denotes $1^2 + 1^2 = 1$ (false).

10.2 Compound predicates

The logical connectives $\wedge$, $\vee$, $\sim$, $\rightarrow$, $\leftrightarrow$, $\oplus$ can be used to combine predicates to form compound predicates.

Example 10.2.1 *(i) Let $p(x)$ denote $x^2 > 5$ and let $q(x)$ denote "x is positive". Then $p(x) \wedge q(x)$ denotes the predicate "$x^2 > 5$ and x is positive".*

(ii) Let $p(x, y)$ denote $x = y^2$. Then $\sim p(x, y)$ denotes $x \neq y^2$.

(iii) Let $p(A, B)$ denote $A \subseteq B$ and let $q(A)$ denote $A \cap \{1, 2\} = \emptyset$. Then $q(A) \rightarrow p(A, B)$ denotes the predicate "If $A \cap \{1, 2\} = \emptyset$ then $A \subseteq B$". □

We can calculate truth values as follows.

Example 10.2.2 *Let $p(x, y)$ denote $x > y$ and let $q(x)$ denote $x < 2$. Find the truth value of the predicate*

$$\sim(p(x, y) \wedge q(x))$$

when $x = 3$ and $y = 1$.

Solution. We need to find the truth value of the statement

$$\sim(p(3, 1) \wedge q(3)).$$

Now $p(3, 1)$ is $\mathbb{T}$ and $q(3)$ is $\mathbb{F}$. Therefore $p(3, 1) \wedge q(3)$ is $\mathbb{F}$. Therefore

$$\sim(p(3, 1) \wedge q(3))$$

is $\mathbb{T}$. □

11 Quantifiers

11.1 Universal quanifier

Many statements in mathematics involve the phrase "for all" or "for every" or "for each": these all have the same meaning.

Examples.

(i) For every x, $x^2 \geqslant 0$.

(ii) For all A and B, $A \cap B = B \cap A$. □

If $p(x)$ is a predicate we write $(\forall x)p(x)$ to denote the statement "For all x, $p(x)$". Similarly, $(\forall x)(\forall y)p(x, y)$ denotes "For all x and all y, $p(x, y)$".

Examples.

(i) Let $p(x)$ denote $x^2 \geqslant 0$. Then $(\forall x)p(x)$ denotes "For every x, $x^2 \geqslant 0$" or x, "For each x, $x^2 \geqslant 0$".

(ii) Let $p(A, B)$ denote $A \cap B = B \cap A$. Then

$$(\forall A)(\forall B)p(A, B)$$

denotes "For all A and B, $A \cap B = B \cap A$". □

When we write $(\forall x)p(x)$ we have in mind that x belongs to some universal set U. The truth of the statement $(\forall x)p(x)$ may depend on U.

Example. Let $p(x)$ denote $x^2 \geqslant 0$. Then $(\forall x)p(x)$ is true provided that the universal set is the set of all real numbers, but $(\forall x)p(x)$ is false if $U = \mathbb{C}$ because $i^2 = -1$.

Usually the universal set is understood from the context. But if necessary we may specify it:

"For every real number x, $x^2 \geqslant 0$"

may be denoted by $(\forall x \in \mathbb{R})p(x)$ instead of $(\forall x)p(x)$.

If $p(x)$ is a PREDICATE then

$(\forall x)p(x)$ is a STATEMENT.

$(\forall x)p(x)$ is **true** if $p(x)$ is true for every $x \in U$, whereas*

* whereas = while

$(\forall x)p(x)$ is **false** if $p(x)$ is false for at least one $x \in U$.

Similar remarks apply to $(\forall x)(\forall y)p(x, y)$, etc.

Examples.

(i) Let $p(x)$ denote $x^2 \geqslant 0$ where $U = \mathbb{R}$. Then $(\forall x)p(x)$ is true.

(ii) The statement "For every integer x, $x^2 \geqslant 5$" is false. Here $U = \mathbb{Z}$ but there is at least one $x \in \mathbb{Z}$ for which $x^2 \geqslant 5$ is false, e.g. $x = 1$.

(iii) Let $p(x, y)$ denote

"If $x \geqslant y$ then $x^2 \geqslant y^2$",

where $U = \mathbb{R}$. Then $(\forall x)(\forall y)p(x, y)$ is false. Take, for example, $x = 1$ and $y = -2$. Then $p(x, y)$ becomes

"If $1 > -2$ then $1 > 4$".

Here $1 > -2$ is $\mathbb{T}$ but $1 > 4$ is $\mathbb{F}$. From the truth table for $\rightarrow$ we see that "If $1 > -2$ then $1 > 4$" is $\mathbb{F}$. Hence $(\forall x)(\forall y)p(x, y)$ is $\mathbb{F}$.

(iv) "For all x and all y, if $x \geqslant y$ then $2x \geqslant 2y$" is $\mathbb{T}$. $\square$

The symbol $\forall$ is called the *universal quantifier*: it has the meaning "for all", "for every" or "for each".

11.2 Existential quantifier

We now also study $\exists$, the *existential quantifier*: it has the meaning "there is (at least one)", "there exists" or "for some".

Examples.

(i) Let $p(x)$ denote $x^2 \geqslant 5$, where $U = \mathbb{R}$. Then $(\exists x)p(x)$ denotes

"There exists a real number x such that $x^2 \geqslant 5$".

This can also be expressed as

"$x^2 \geqslant 5$ for some real number x".

(ii) The statement

"There exist sets A and B for which $(A \cap B)' = A' \cap B'$"

may be denoted by

$$(\exists A)(\exists B)p(A, B)$$

where $p(A, B)$ denotes the predicate $(A \cap B)' = A' \cap B'$, or

$$(\exists A)(\exists B)((A \cap B)' = A' \cap B').$$

If $p(x)$ is a PREDICATE then $(\exists x)p(x)$ is a STATEMENT.

$(\exists x)p(x)$ is **true** if $p(x)$ is true for at least one $x \in U$, whereas

$(\exists x)p(x)$ is **false** if $p(x)$ is false for all $x \in U$.

Examples.

(i) Let $U = \mathbb{R}$. The statement $(\exists x)x^2 \geqslant 5$ is $\mathbb{T}$ because $x^2 \geqslant 5$ is $\mathbb{T}$ for at least one value of x, e.g. $x = 3$.

(ii) Let $p(x)$ denote $x^2 < 0$, where $U = \mathbb{R}$. Then $(\exists x)p(x)$ is $\mathbb{F}$ because $p(x)$ is $\mathbb{F}$ for all $x \in U$.

(iii) $(\exists x)(\exists y)(x + y)^2 = x^2 + y^2$ (where $U = \mathbb{R}$) is $\mathbb{T}$: take $x = 0$, $y = 0$ for example. $\square$

Statements may involve both $\forall$ and $\exists$.

Example. Consider the following statements.

(i) Everyone likes all of Beethoven's symphonies.

(ii) Everyone likes at least one of Beethoven's symphonies.

(iii) There is one Beethoven's symphony which everyone likes.

(iv) There is someone who likes all of Beethoven's symphonies.

(v) Every Beethoven's symphony is liked by someone.

(vi) There is someone who likes at least one of Beethoven's symphonies.

If we let $p(x, y)$ denote the predicate "x likes y" where x belongs to the universal set of all University of Manchester students and y belongs to the universal set of all Beethoven's symphonies then the statements become:

(i) $(\forall x)(\forall y)p(x, y)$

(ii) $(\forall x)(\exists y)p(x, y)$

(iii) $(\exists y)(\forall x)p(x, y)$

(iv) $(\exists x)(\forall y)p(x, y)$

(v) $(\forall y)(\exists x)p(x, y)$

(vi) $(\exists x)(\exists y)p(x, y)$

All have different meanings: in particular, $(\forall x)(\exists y)$ is not the same as $(\exists y)(\forall x)$. $\square$

Example 11.2.1 *Consider the statements*

(i) $(\forall x)(\exists y)x < y$ *and*

(ii) $(\exists y)(\forall x)x < y$

where $U = \mathbb{R}$*.*

Statement (i) is true but statement (ii) is false. Note that (i) states that whatever number x we choose we can find a number y which is greater than x (e.g. $y = x + 1$). But (ii) states that there is a number y which is simultaneously greater than **every** *number x: this is impossible because, with $x = y$, $x < y$ does not hold.* $\square$

11.3 Questions from Students

*

* This section contains no compulsory material but still may be useful.

1. ```
   > I can not differentiate the true from the false
   > when it comes to different arrangements of
   > quantifiers or variables after the quantifier.
   >
   > For example:
   > Let the Universal set be Z.
   >
   > (i) For all x there exists an integer y such that y^2=x.
   >
   > (ii) For all y there exists an integer x such that y^2=x.
   >
   > Which one of those statements is true?
   > which one is false?
   > are they both false or true?
   ```

   ANSWER: (ii) is true, (i) is false.

   Why (i) is false? If it is true, then, since it is true for all x, it has to be true for $x = 2$. So let us plug $x = 2$ into the statement:

   For $x = 2$ there exists an integer y such that $y^2 = x$.

   but this is the same as to say

   there exists an integer y such that $y^2 = 2$.

   But this obviously false – there is no such integer y.

   Why is (ii) true? Because, for every y, we can set $x = y^2$.

   For example,

   - for $y = 1$ there exists an integer x such that $1^2 = x$ (indeed, take $x = 1$);
   - for $y = 2$ there exists an integer x such that $y^2 = x$ (indeed, take $x = 4$);
   - for $y = 3$ there exists an integer x such that $y^2 = x$ (indeed, take $x = 9$);
   - for $y = 4$ there exists an integer x such that $y^2 = x$ (indeed, take $x = 16$);
   - for $y = 5$ there exists an integer x such that $y^2 = x$ (indeed, take $x = 25$).
   ```

12 Logical equivalences

Statements can be formed from predicates by means of a mixture of connectives and quantifiers.

Examples.

(i) Let $p(x, y)$ denote $x < y$ and let $q(y)$ denote $y \neq 2$. Then
$$(\forall x)(\exists y)(p(x, y) \wedge q(y))$$
denotes

> "For all x there exists y such that $x < y$ and $y \neq 2$".

(This is $\mathbb{T}$).

(ii) Let $p(x)$ denote $x > 2$ and let $q(x)$ denote $x^2 > 4$. Then
$$(\forall x)(p(x) \to q(x))$$
denotes

> "For all x, if $x > 2$ then $x^2 > 4$".

(True).

(iii) Let $p(x)$ denote $x > 2$ and let $q(x)$ denote $x < 2$. Then we may form
$$((\exists x)p(x) \wedge (\exists x)q(x)) \to (\exists x)(p(x) \wedge q(x)).$$
This is $\mathbb{F}$ because
$$(\exists x)p(x) \wedge (\exists x)q(x)$$
is $\mathbb{T}$ but
$$(\exists x)(p(x) \wedge q(x))$$
is $\mathbb{F}$. $\mathbb{T} \to \mathbb{F}$ gives $\mathbb{F}$.

As in propositional logic, we say that two statements X and Y are **logically equivalent**, and write $X \equiv Y$, if X and Y have the same truth value for purely logical reasons.

Example. $\sim\sim (\exists x)p(x) \equiv (\exists x)p(x)$. We don't need to know the meaning of $p(x)$. □

Fundamental logical equivalence (6) of propositional logic is

$$\sim\sim p \equiv p.$$

This can be applied to predicate logic to show that

$$\begin{aligned} \sim\sim(\exists x)p(x) &\equiv (\exists x)p(x), \\ \sim\sim(\forall x)(\exists y)p(x,y) &\equiv (\forall x)(\exists y)p(x,y), \end{aligned}$$

etc. We can use all of the fundamental logical equivalences (1)–(10) in this way, plus two additional equivalences:

(11) $\sim(\forall x)p(x) \equiv (\exists x) \sim p(x)$.

(12) $\sim(\exists x)p(x) \equiv (\forall x) \sim p(x)$.

Example of (11). Let U be the set of all University of Manchester students. Let $p(x)$ denote "x is British". Then $\sim(\forall x)p(x)$ denotes

> "It is not true that every University of Manchester student is British"

and $(\exists x) \sim p(x)$ denotes

> "There is a University of Manchester student who is not British".

These are logically equivalent. □

Example of (12). Let $U = \mathbb{Z}$. Let $p(x)$ denote $x^2 = 2$. Then $\sim(\exists x)p(x)$ denotes

> "It is false that there exists $x \in \mathbb{Z}$ such that $x^2 = 2$"

and $(\forall x) \sim p(x)$ denotes

> "For all $x \in \mathbb{Z}$, $x^2 \neq 2$".

These are logically equivalent. □

Example. Prove that

$$\sim(\forall x)(\forall y)(p(x,y) \to q(x,y)) \equiv (\exists x)(\exists y)(p(x,y) \wedge \sim q(x,y)).$$

Solution.

$$\begin{aligned}
\sim(\forall x)(\forall \mathbf{y})(\mathbf{p}(\mathbf{x},\mathbf{y}) \to \mathbf{q}(\mathbf{x},\mathbf{y})) &\overset{\text{by (11)}}{\equiv} (\exists x) \sim(\forall \mathbf{y})(\mathbf{p}(\mathbf{x},\mathbf{y}) \to \mathbf{q}(\mathbf{x},\mathbf{y})) \\
&\overset{\text{by (11)}}{\equiv} (\exists x)(\exists y) \sim(p(x,y) \to q(x,y)) \\
&\overset{\text{by (9)}}{\equiv} (\exists x)(\exists y) \sim(\sim p(x,y) \vee q(x,y)) \\
&\overset{\text{by (8)}}{\equiv} (\exists x)(\exists y)(\sim\sim p(x,y) \wedge \sim q(x,y)) \\
&\overset{\text{by (7)}}{\equiv} \equiv (\exists x)(\exists y)(p(x,y) \wedge \sim q(x,y))
\end{aligned}$$

□

Perhaps, the very first line in this solution needs a comment: we apply rule

(11) $\sim(\forall x)p(x) \equiv (\exists x) \sim p(x)$

with the formula

$$p(x) = (\forall \mathbf{y})(\mathbf{p}(\mathbf{x},\mathbf{y}) \to \mathbf{q}(\mathbf{x},\mathbf{y}))$$

highlighted by use of a boldface font.

12.1 Questions from Students

*

* This section contains no compulsory material but still may be useful.

1.
```
> Since I can not use symbols,
> A=for all and E=there exists.
>
> If there was a statement like this:
>
> ~((Ax)(Ey)(p(x,y)^(Ey)~q(y)))
>
> If I want to simplify this,
> I multiply the negation inside the brackets,
> but I am not sure of what would happen,
> will the negation be multiplied by both (Ax)(Ex) ?
```

```
> as it will be
> ~(Ax)~(Ey) ~((p(x,y)^(Ey)~q(y))??
```

ANSWER. I am afraid it works differently. Here is a sequence of transformations:

$$\begin{aligned}
\sim((\forall x)(\exists y)(p(x,y)\wedge(\exists y)\sim q(y))) &\equiv (\exists x)\sim(\exists y)(p(x,y)\wedge(\exists y)\sim q(y)) \\
&\equiv (\exists x)(\forall y)\sim(p(x,y)\wedge(\exists y)\sim q(y)) \\
&\equiv (\exists x)(\forall y)(\sim p(x,y)\vee\sim(\exists y)\sim q(y)) \\
&\equiv (\exists x)(\forall y)(\sim p(x,y)\vee(\forall y)\sim\sim q(y)) \\
&\equiv (\exists x)(\forall y)(\sim p(x,y)\vee(\forall y)q(y)).
\end{aligned}$$

2. I refer to Example (iii) in this Lecture.

> Let $p(x)$ denote $x > 2$ and let $q(x)$ denote $x < 2$. Then we may form
>
> $$((\exists x)p(x)\wedge(\exists x)q(x)) \to (\exists x)(p(x)\wedge q(x)).$$
>
> This is $\mathbb{F}$ because
>
> $$(\exists x)p(x)\wedge(\exists x)q(x)$$
>
> is $\mathbb{T}$ but
>
> $$(\exists x)(p(x)\wedge q(x))$$
>
> is $\mathbb{F}$. $\mathbb{T}\to\mathbb{F}$ gives $\mathbb{F}$.

MY PROBLEM. I entirely accept that

$$(\exists x)(p(x)\wedge q(x))$$

is $\mathbb{F}$. I can find no value of x for which $p(x)$ i.e. $x > 2$ is true and for which $q(x)$ i.e. $x < 2$ is also true for that same value of x. If we let $x = 3$ then $x > 2$ which is $3 > 2$ is $\mathbb{T}$ but $x < 2$ which is $3 < 2$ is $\mathbb{F}$.

From the conjunction truth table for $\wedge$ we see that $\mathbb{T}\wedge\mathbb{F}$ gives $\mathbb{F}$.

If we let $x = 1$ then $x > 2$ which is $1 > 2$ is $\mathbb{F}$ but $x < 2$ which is $1 < 2$ is $\mathbb{T}$. Again from the truth table for $\wedge$ we see that $\mathbb{F}\wedge\mathbb{T}$ gives $\mathbb{F}$.

So far so good but now we come to

$$(\exists x)p(x)\wedge(\exists x)q(x)$$

and the brackets appear to produce an unexpected result. I read this as the logical statement that

> "there exists some value of x for which $x > 2$ is true *and* there exists some value of x for which $x < 2$ is true".

But the normal method of testing by giving x a value of, let us say 1, gives us $(\exists x)p(x)$ which is $1 > 2$ which is $\mathbb{F}$ *and* $(\exists x)q(x)$ which is $1 < 2$ which is $\mathbb{T}$.

From the truth table for $\wedge$ we see that $\mathbb{F} \wedge T$ gives $\mathbb{F}$.

Testing by giving x a value of 3 gives us $(\exists x)p(x)$ which is $3 > 2$ which is $\mathbb{T}$ *and* $(\exists x)q(x)$ which is $3 < 2$ which is $\mathbb{F}$.

From the truth table for $\wedge$ we see that $\mathbb{T} \wedge \mathbb{F}$ also gives $\mathbb{F}$.

I am therefore unable to identify a value of x in the two predicates $p(x)$ and $q(x)$ for which

$$(\exists x)p(x) \wedge (\exists x)q(x)$$

gives $\mathbb{T}$ as stated in example (iii).

The only possibility I can see is that, because of the brackets, I should read the logical statement as being that

> "there exists some value of x for which $x > 2$ is true (it is true for the value $x = 3$) *and* separately there exists some potentially different value of x for which $x < 2$ is true (it is true for the value $x = 1$)".

Only then can I get $\mathbb{T} \wedge T$ which is the required condition in the $\wedge$ truth table to give a result of $\mathbb{T}$.

ANSWER. Your problem disappears if

$$(\exists x)p(x) \wedge (\exists x)q(x)$$

is replaced by

$$(\exists x)p(x) \wedge (\exists y)q(y)$$

which says exactly the same.

3. `For real numbers x and y, let p(x,y) denote the predicate x < y. In the statement`
   `(Ax)(Ay)(p(x,y) v p(y,x))`
   `For answer B does this mean the predicate p(y,x) is y < x because y and x have switched positions?`

   ANSWER. Yes, it does, $p(y, x)$ means $y < x$.

4. `I wanted to know if you could provide me with specific examples for when (Ax)(Ey) p(x,y) is logically equivalent to (Ey(Ax)p(x,y)?`

   ANSWER. I cannot provide you with specific, because these two statements are not logical equivalent. It is like asking: "when are you alive?" I am either alive or not, and the word "when" cannot be used in a question.

By definition, two statements of Predicate Logic are logically equivalent if they are simultaneously true of false for purely logical reasons, regardless of their meaning, regardless of concrete interpretation of predicates involved, regardless of choice of universal sets to which they are applied. For example, if the universal set is the set of real numbers and $p(x, y)$ has meaning $x = y$, then the both statements $(\forall x)(\exists y)p(x, y)$ and $(\exists y)(\forall x)p(x, y)$ are true; if the universal set is the set of natural numbers and the predicate $p(x, y)$ has meaning $x < y$, then $(\forall x)(\exists y)p(x, y)$ is $\mathbb{T}$ while $(\exists y)(\forall x)p(x, y)$ is $\mathbb{F}$. This second example automatically makes the two particular sentences NOT elementary equivalent.

13 Inequalities

In this and next lectures we shall study, in more detail, properties of *inequality*, or *order relation*, $x \leqslant y$ on the set **R** of real numbers.

$x \leqslant y$ is read

"x is less or equal y"

or

"x is at most y."

13.1 Basic properties of inequalities

Let $x, y, z \in \mathbb{R}$ be arbitrary real numbers. Then

- $x \leqslant x$;
- $x = y$ if and only if $x \leqslant y$ and $y \leqslant x$;
- if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$;
- $x \leqslant y$ or $y \leqslant x$.

It is a useful exercise to rewrite these properties in formal logical notation:

- $(\forall x)(x \leqslant x)$;
- $(\forall x)(\forall y)(x = y \leftrightarrow (x \leqslant y \wedge y \leqslant x))$;
- $(\forall x)(\forall y)(\forall z)((x \leqslant y \wedge y \leqslant z) \rightarrow x \leqslant z)$;
- $(\forall x)(\forall y)(x \leqslant y \vee y \leqslant x)$.

Some additional notation:

- If $x \leqslant y$, we write $y \geq x$.
- if $x \leqslant y$ and $x \neq y$, we write $x < y$
- if $x \geq y$ and $x \neq y$, we write $x > y$

13.2 Intervals and segments

Let $a, b \in \mathbb{R}$ with $a \leqslant b$.

By definition,

- *interval* $]a, b[$ (or (a, b)) is the set
$$]a, b[= \{\, x : a < x < b \,\}.$$
- *segment* $[a, b]$ is the set
$$[a, b] = \{\, x : a \leqslant x \leqslant b \,\}.$$
- *semi-closed intervals* are sets
$$[a, b[= \{\, x : a \leqslant x < b \,\}.$$
and
$$]a, b] = \{\, x : a < x \leqslant b \,\}.$$

The numbers a and b are called the *endpoints* of segments, intervals, semi-closed intervals

$$]a, b[\,, [a, b], [a, b[\,, \,]a, b],$$

and the number $b - a$ is called their *length*.

We also define

- positive-directed *ray* $[a, +\infty[$ is the set
$$[a, +\infty[= \{\, x : a \leqslant x \,\}.$$
- negative directed *ray* $]-\infty, a]$ is the set
$$]-\infty, a] = \{\, x : x \leqslant a \,\}.$$
- half-lines $]-\infty, a[$ and $]a, +\infty[$ are sets
$$]-\infty, a[= \{\, x : x < a \,\}.$$
and
$$]a, +\infty[= \{\, x : a < x \,\}.$$

14 Operations over Inequalities

14.1 Formal properties of real numbers

It is time for us to make list of some properties of real numbers. Let a, b, c be arbitrary real numbers.

Addition

R1 $a + b$ is a unique real number (Closure Law)

R2 $a + b = b + a$ (Commutative Law)

R3 $a + (b + c) = (a + b) + c$ (Associative Law)

R4 $a + 0 = 0 + a = a$ (Identity Law)

R5 $a + (-a) = (-a) + a = 0$ (Inverse Law)

Multiplication

R6 $a \cdot b$ is a unique real number (Closure Law)

R7 $a \cdot b = b \cdot a$ (Commutative Law)

R8 $a \cdot (b \cdot c) = (a \cdot b)\cdot$ (Associative Law)

R9 $a \cdot 1 = 1 \cdot a = a$ (Identity Law)

R10 $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$ for $a \neq 0$ (Inverse Law)

R11 $a \cdot (b + c) = a \cdot b + a \cdot c$ (Distributive Law)

Inequality

R12 $a \leqslant a$ (Reflexive Law)

R13 $a = b$ iff* $a \leqslant b$ and $b \leqslant a$; (Antisymmetric Law)

R14 If $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$ (Transitive Law)

R15 $a \leqslant b$ or $b \leqslant a$ (Total Order Law)

R16 If $a \leqslant b$ then $a + c \leqslant b + c$

R17 If $a \leqslant b$ and $0 \leqslant c$ then $a \cdot c \leqslant b \cdot c$

* Recall “iff” = “if and only if”

14.2 Properties of strict inequality

We need two types of inequality, $\leqslant$ and $<$ because they allow to express the negations of each other:

$$\begin{aligned} \sim(a \leqslant b) &\leftrightarrow b < a \\ \sim(a < b) &\leftrightarrow b \leqslant a \end{aligned}$$

R*12 It is never true that $a < a$ (Anti-reflexive Law)

R*13 One and only one of the following is true:

$$a < b \qquad a = b \qquad b < a$$

(Antisymmetric + Total Order Law)

R*14 If $a < b$ and $b < c$ then $a < c$ (Transitive Law)

R*15 $a < b$ or $a = b$ or $b < a$ (Total Order Law)

R*16 If $a < b$ then $a + c < b + c$

R*17 If $a < b$ and $0 < c$ then $a \cdot c < b \cdot c$

On a number of occasions I have been asked by students:

> *How can we claim that* $2 \leqslant 3$ *if we already know that* $2 < 3$*?*

Propositional Logic helps:

$$a \leqslant b \equiv (a < b) \vee (a = b),$$

in particular,

$$2 \leqslant 3 \equiv (2 < 3) \vee (2 = 3).$$

Obviously, $2 < 3$ is $\mathbb{T}$, $2 = 3$ is $\mathbb{F}$, therefore their disjunction $2 \leqslant 3$ has truth value $T \vee F \equiv \mathbb{T}$.

14.3 Inequalities can be added

Theorem 14.1 *If $a \leqslant b$ and $c \leqslant d$ then*

$$a + c \leqslant b + d.$$

PROOF.

1. $a + c \leqslant b + c$ [R16]
2. $c + b \leqslant d + b$ [R16]
3. $b + c \leqslant b + d$ [R2]
4. $a + c \leqslant b + d$ [R14 applied to 1. and 2.]

□

14.4 Proofs can be re-used

Theorem 14.2 *If $0 \leqslant a \leqslant b$ and $0 \leqslant c \leqslant d$ then*

$$a \cdot c \leqslant b \cdot d.$$

PROOF.

1. $a \cdot c \leqslant b \cdot c$ [R17]
2. $c \cdot b \leqslant d \cdot b$ [R17]
3. $b \cdot c \leqslant b \cdot d$ [R2]
4. $a \cdot c \leqslant b \cdot d$ [R14 applied to 1. and 2.]

□

Corollary 14.3 *For all $0 \leqslant x \leqslant y$,*

$$x^2 \leqslant y^2.$$

PROOF. In the theorem above, set $a = c = x$ and $b = d = y$.
□

Theorem 14.4 *If $a < b$ and $c < d$ then*

$$a + c < b + d.$$

PROOF.

1. $a + c < b + c$ [R*16]
2. $c + b < d + b$ [R*16]
3. $b + c < b + d$ [R2]
4. $a + c < b + d$ [R*14 applied to 1. and 2.]

□

More on proving inequalities (and on proof in mathematics in general) is the next lecture.

15 Methods of Proof

*

* Recommended additional (but not compulsory) reading: *Book of Proof* by Richard Hammack, Chapter 4.

15.1 Statements of the form $(\forall \mathbf{x})\mathbf{p}(\mathbf{x})$

To prove $(\forall x)p(x)$ is $\mathbb{T}$ we must prove that $p(x)$ is $\mathbb{T}$ for all $x \in U$. (The method will vary.)

Theorem 15.1 *For all real numbers x,*

$0 \leqslant x$ *if and only if* $-x \leqslant 0$.

In formal logical notation, this theorem reads as

$$(\forall x)(0 \leqslant x \leftrightarrow -x \leqslant 0)$$

PROOF. We will prove first that if $0 \leqslant x$ then $-x \leqslant 0$

1. $0 \leqslant x$ (given)
2. $0 + (-x) \leqslant x + (-x)$ (R16)
3. $-x \leqslant 0$ (algebra)

Now we prove that if $-x \leqslant 0$ then $0 \leqslant x$.

1. $-x \leqslant 0$ (given)
2. $-x + x \leqslant 0 + x$ (R16)
3. $0 \leqslant x$ (algebra)

So we proved both

$$0 \leq x \rightarrow -x \leqslant 0$$

and

$$-x \leqslant 0 \rightarrow 0 \leqslant x,$$

hence proved

$$0 \leqslant x \leftrightarrow -x \leqslant 0$$

for all real numbers x. □

15.2 Change of sign in an inequality

Theorem 15.2 *For all real numbers x and y,*

$x \leqslant y$ if and only if $-y \leqslant -x$.

In formal logical notation, this theorem reads as

$$(\forall x)(x \leqslant y \leftrightarrow -y \leqslant -x)$$

PROOF. We will prove first that if $x \leq y$ then $-y \leqslant -x$

1. $x \leqslant y$ (given)
2. $x + [-x - y] \leqslant y + [-x - y]$ (R16)
3. $-y \leqslant -x$ (algebra)

The proof of the implication in other direction, from right to left, is left as an exercise – use the previous theorem as a hint. □

The statement of this theorem is frequently written as

$$(\forall x)(x \leqslant y \leftrightarrow -x \leq -y)$$

and is read as

If we change the signs of the both sides of the inequality, we change its direction.

Recall that a real number a is called

positive if $0 < a$,

negative if $a < 0$,

non-negative if $0 \leqslant a$,

non-positive if $a \leqslant 0$.

15.3 Squares are non-negative

Theorem 15.3 * *For all real numbers x,*

$$0 \leqslant x^2.$$

* In formal language: $(\forall x)(0 \leqslant x^2)$.

PROOF. We shall write this proof in a less formal way.*

* We ar using here "case-by-case" proof, which we will discuss in more detail in Section 16.4.

If x is non-negative, then x^2 is non-negative by Corollary 14.3.

If x is negative, then $x \leqslant 0$ and $0 \leqslant -x$ by Theorem 15.1, so $-x$ is non-negative. Now, by Corollary 14.3 again, $0 \leqslant (-x)^2 = x^2$. □

Remark. Observe that if a statement $(\forall x)p(x)$ about real numbers is true then it remains true if we substitute for x an arbitrary function or expression $x = f(y)$.

For example, since $(\forall x)(x^2 \geq 0)$ is $\mathbb{T}$, the following statements are also true for all x and y:

$$(y+1)^2 \geq 0; \quad (x+y)^2 \geq 0; \quad \sin^2 x \geq 0$$

and therefore

$$(\forall y)((y+1)^2 \geq 0);$$
$$(\forall x)(\forall y)((x+y)^2 \geq 0);$$
$$(\forall x)(\sin^2 x \geq 0).$$

Example 15.3.1 *Prove that the statement*

"For all real numbers y, $y^2 + 2y + 3 > 0$"

is true.

PROOF. Let y be an arbitrary real number. We can rewrite

$$\begin{aligned} y^2 + 2y + 3 &= (y^2 + 2y + 1) + 2 \\ &= (y+1)^2 + 2. \end{aligned}$$

But we know from the previous remark that, for all y,

$$(y+1)^2 \geq 0.$$

Therefore

$$(y+1)^2 + 2 \geq 0 + 2 = 2 > 0.$$

Hence

$$y^2 + 2y + 3 = (y+1)^2 + 2 > 0,$$

and the statement is true. □

Example 15.3.2 *Prove that the statement*

> *"For all real numbers x and y, $x^2 + y^2 \geqslant 2xy$"*

is true.

PROOF. We know that

$$(x-y)^2 \geqslant 0;$$

after opening the bracket, we have

$$x^2 - 2xy + y^2 \geqslant 0.$$

After we add $2xy$ to the both sides of this inequality, we get

$$x^2 + y^2 \geqslant 2xy.$$

□

15.4 Counterexamples

To prove $(\forall x)p(x)$ is $\mathbb{F}$* we must show that there exists at least one $x \in U$ such that $p(x)$ is $\mathbb{F}$ for this x. Such a value of x is called a **counterexample** to the statement $(\forall x)p(x)$.

* We also say: "disprove" $(\forall x)p(x)$; "refute" $(\forall x)p(x)$.

Example 15.4.1 *Prove that*

> *"For all real numbers x, $x^2 - 3x + 2 \geq 0$"*

is false.

PROOF. Note that

$$x^2 - 3x + 2 = (x - 1)(x - 2).$$

If $1 < x < 2$ then $x - 1$ is positive: $x - 1 > 0$, and $x - 2$ is negative: $x - 2 < 0$, so their product $(x - 1)(x - 2)$ is negative:

$$(x - 1)(x - 2) < 0.$$

Thus any number x with $1 < x < 2$ is a counterexample: the statement is false. For a concrete* value of x, we can take $x = 1\frac{1}{2}$. One counterexample is enough: we do not have to show that

$$x^2 - 3x + 2 \geq 0$$

is false for all x. □

* concrete = specific, "existing in reality or in real experience; perceptible by the senses".

Example 15.4.2 *Prove that the statement*

"For all sets A, B and C,

$$A \cap (B \cup C) = (A \cap B) \cup C''$$

is false.

PROOF. We try to find a counterexample by experiment. Try $A = \emptyset$, $B = \emptyset$, $C = \{1\}$. Then

$$A \cap (B \cup C) = \emptyset$$

but

$$(A \cap B) \cup C = \{1\}.$$

Thus $A = \emptyset$, $B = \emptyset$, $C = \{1\}$ gives a counterexample: the statement is false. □

Example 15.4.3 *Prove that*

$$(\forall x)(0 \leqslant x^3)$$

is false.

PROOF. For a counterexample, you can take $x = -1$. □

Remark **One** counterexample is enough to prove that a statement is false.

15.5 Statements of the form $(\forall \mathbf{x})(\mathbf{p}(\mathbf{x}) \to \mathbf{q}(\mathbf{x}))$

An example is

"For all x, if $x > 2$ then $x^2 > 4$".

In practice such a sentence is often expressed as

"If $x > 2$ then $x^2 > 4$"

where the phrase "For all x" is taken as obvious. However, in symbols, we should write

$$(\forall x)(p(x) \to q(x)).$$

Notice that an expression

"If $A \subseteq B$ then $A \cup B = B$"

is shorthand* for

* shorthand = abbreviation

"For all A and all B, if $A \subseteq B$ then $A \cup B = B$",

written as

$$(\forall A)(\forall B)(p(A, B) \to q(A, B))$$

where $p(A, B)$ denotes $A \subseteq B$ and $q(A, B)$ denotes $A \cup B = B$.

To prove that $(\forall x)(p(x) \to q(x))$ is $\mathbb{T}$ we need to prove that $p(x) \to q(x)$ is $\mathbb{T}$ for each element x of U. The truth table for $\to$ shows that $p(x) \to q(x)$ is automatically $\mathbb{T}$ when $p(x)$ is $\mathbb{F}$. Therefore we only need to prove that $p(x) \to q(x)$ is $\mathbb{T}$ for elements x of U such that $p(x)$ is $\mathbb{T}$. We take an arbitrary value of x for which $p(x)$ is $\mathbb{T}$ and try to deduce that $q(x)$ is $\mathbb{T}$. (The method will vary.) It then follows that $(\forall x)(p(x) \to q(x))$ is $\mathbb{T}$.

Example 15.5.1 *Prove the statement*

"If $x \in]1,2[$ then $x^2 - 3x + 2 < 0$".

PROOF. Note that

$$x^2 - 3x + 2 = (x - 1)(x - 2).$$

If $x \in]1,2[$ then $1 < x < 2$, hence $x - 1 > 0$ is positive and $x - 2 < 0$ is negative, and their product $(x - 1)(x - 2)$ is negative. □

To prove that $(\forall x)(p(x) \to q(x))$ is $\mathbb{F}$ we have to show that there exists $x \in U$ such that $p(x) \to q(x)$ is $\mathbb{F}$. The truth table of $\to$ shows that $p(x) \to q(x)$ can only be $\mathbb{F}$ when $p(x)$ is $\mathbb{T}$ and $q(x)$ is $\mathbb{F}$. Thus we have to show that there exists $x \in U$ such that $p(x)$ is $\mathbb{T}$ and $q(x)$ is $\mathbb{F}$. This will be a counterexample to $(\forall x)(p(x) \to q(x))$.

Example 15.5.2 *Prove that the statement*

"If x is a real number such that $x^2 > 4$ then $x > 2$"

is false.

PROOF. Let $x = -3$. Then $x^2 > 4$ is $\mathbb{T}$ but $x > 2$ is $\mathbb{F}$. Thus $x = -3$ is a counterexample: the statement is false. □

15.6 Contrapositive

*

* Recommended reading: *Book of Proof* by Richard Hammack, Chapter 5.

By the method of truth tables we can prove*

* Do that as an exercise!

$$p \to q \equiv \sim q \to \sim p.$$

Alternatively, we can prove this from Fundamental Logical equivalences:

$$\begin{aligned} p \to q &\equiv \sim p \vee q \\ &\equiv q \vee \sim p \\ &\equiv \sim(\sim q) \vee \sim p \\ &\equiv \sim q \to \sim p. \end{aligned}$$

Here, at the last step we apply the definition of conditional:

$$\sim X \vee Y \equiv X \to Y$$

with $X = \sim q$ and $Y = \sim p$.

$\sim q \to \sim p$ is called the **contrapositive** of $p \to q$. It follows that

$$(\forall x)(p(x) \to q(x)) \equiv (\forall x)(\sim q(x) \to \sim p(x)).$$

$$(\forall x)(\sim q(x) \to \sim p(x))$$

is called the **contrapositive** of

$$(\forall x)(p(x) \to q(x)).$$

To prove a statement $p \to q$ or $(\forall x)(p(x) \to q(x))$ it is enough to prove the contrapositive. Sometimes this is easier.

Example 15.6.1 *Prove the statement*

> *"If x is an integer such that x^2 is odd then x is odd".*

The contrapositive is

> *"If x is an integer such that x is not odd then x^2 is not odd".*

However "not odd" is the same as "even". So the contrapositive is

> *"If x is an even integer then x^2 is even".*

This statement is much easier to prove: if x is even, $x = 2u$ for some integer u. but then

$$x^2 = (2u)^2 = 2^2 \cdot u^2 = 2 \cdot (2u^2)$$

is also even. $\square$

16 Methods of Proof, Continued

16.1 Converse

A conditional statement $q \to p$ is called the **converse** of $p \to q$.

Similarly,

$$(\forall x)(q(x) \to p(x))$$

is called the **converse** of

$$(\forall x)(p(x) \to q(x)).$$

The converse is NOT equivalent to the original statement.

Example 16.1.1 *Let p be "You got full marks" and let q be "You passed the exam".*

$p \to q$ is "If you got full marks you passed the exam".

The contrapositive $\sim q \to \sim p$ is

"If you did not pass the exam you did not get full marks".

The converse $q \to p$ is

"If you passed the exam you got full marks".

$\sim q \to \sim p$ is equivalent to $p \to q$, but $q \to p$ is not.

Example 16.1.2 *The statement*

"If $x > 2$ then $x^2 > 4$"

is true, but the converse

"If $x^2 > 4$ then $x > 2$"

*is false.** □

* Indeed, give a counterexample!

16.2 Inequalities for square roots

Theorem 16.1 *If* $0 \leqslant x$, $0 \leqslant y$ *and*

$$x^2 < y^2$$

then

$$x < y$$

PROOF. The contrapositive to

$$x^2 < y^2 \to x < y$$

is

$$\sim(x < y) \to \sim(x^2 < y^2)$$

or

$$(y \leqslant x) \to (y^2 \leqslant x^2)$$

But this is a theorem proved in Corollary 14.3. □

By setting in Theorem 16.1 $u = x^2$ and $v = y^2$, we have the following important inequality for square roots:

Corollary 16.2 *If* $0 \leqslant u$, $0 \leqslant v$ *and*

$$u < v$$

then

$$\sqrt{u} < \sqrt{v}.$$

I leave proving the following version of that result as an exercise to the reader.

Theorem 16.3 *If*

$$0 \leqslant u \leqslant v$$

then

$$\sqrt{u} \leqslant \sqrt{v}.$$

16.3 Statements of the form $(\forall \mathbf{x})(\mathbf{p}(\mathbf{x}) \leftrightarrow \mathbf{q}(\mathbf{x}))$

We make use of the logical equivalence

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p).$$

Thus to prove that $p \leftrightarrow q$ is $\mathbb{T}$ it is sufficient to prove two things

(i) $p \to q$ is $\mathbb{T}$

(ii) $q \to p$ is $\mathbb{T}$.

To prove that $p \leftrightarrow q$ is $\mathbb{F}$ it is sufficient to prove that either $p \to q$ is $\mathbb{F}$ **or** $q \to p$ is $\mathbb{F}$.

Similarly to prove that $(\forall x)(p(x) \leftrightarrow q(x))$ is $\mathbb{T}$ we usually proceed in TWO STEPS.

(i) We prove $(\forall x)(p(x) \to q(x))$.

(ii) We prove (the converse) $(\forall x)(q(x) \to p(x))$.

In order to prove (i) we follow the method described in II above: we take an arbitrary x such that $p(x)$ is $\mathbb{T}$ and try to deduce that $q(x)$ is $\mathbb{T}$. Than to prove (ii) we take an arbitrary x such that $q(x)$ is $\mathbb{T}$ and try to deduce that $p(x)$ is $\mathbb{T}$.

To prove that $(\forall x)(p(x) \leftrightarrow q(x))$ is $\mathbb{F}$ we prove that

$$(\forall x)(p(x) \to q(x)) \text{ is } \mathbb{F}$$

or

$$(\forall x)(q(x) \to p(x)) \text{ is } \mathbb{F}.$$

Example 16.3.1 *Prove that*

$$(\forall x \in \mathbf{R})(x \geq 0 \leftrightarrow x^3 \geq 0)$$

16.4 Case-by-case proofs

*

It is easy to check that this is a tautology:*

$$((P \to Q) \wedge (\sim P \to Q)) \to Q$$

* Compare with Section 15.3.

* Check it!

Therefore, to prove $(\forall x)Q(x)$, it suffices to prove

$$(\forall x)(P(x) \to Q(x)) \wedge (\forall x)(\sim P(x) \to Q(x))$$

More generally, we have a tautology*

* Check it!

$$((P' \vee P'') \wedge (P' \to Q) \wedge (P'' \to Q)) \to Q$$

Therefore, to prove $(\forall x)Q(x)$, it suffices to prove

$$(\forall x)(P'(x) \vee P''(x)) \wedge (\forall x)(P'(x) \to Q(x)) \wedge (\forall x)(P''(x) \to Q(x))$$

Example 16.4.1 *Prove that, for all real numbers x,*

if $x \neq 0$ then $x^2 > 0$.

16.5 Absolute value

For a real number x, we define its *absolute value** as

* Another term used: *module*.

$$|x| = \begin{cases} x & \text{if } 0 \leqslant x \\ -x & \text{if } x < 0 \end{cases}$$

For example, $|-2| = 2$, $|3| = 3$.

The following theorem immediately follows from this definition.

Theorem 16.4 *For all real numbers x,*

$$|-x| = |x| \;\; \textit{and} \;\; x \leqslant |x|.$$

Geometric interpretation: $|a-b|$ is the distance between the points a and b on the real line.

Example 16.5.1 *Prove that, for all real numbers x and y,*

$$|x+y| \leqslant |x| + |y|.$$

PROOF. It is an example of a case-by-case proof.

CASE 1: $x \geqslant 0$ and $y \geqslant 0$. Then $x + y \geqslant 0$ and by definition of absolute value we have

$$|x| = x, \; |y| = y \;\text{ and }\; |x + y| = x + y,$$

hence

$$|x + y| = |x| + |y|$$

and therefore

$$|x + y| \leqslant |x| + |y|.$$

CASE 2: $x < 0$ and $y < 0$. Then $x + y < 0$ and by definition of absolute value we have

$$|x| = -x, \; |y| = -y \;\text{ and }\; |x + y| = -(x + y),$$

hence

$$|x + y| = |x| + |y|$$

and therefore

$$|x + y| \leqslant |x| + |y|.$$

CASE 3: $x \geqslant 0$ and $y < 0$. We have two subcases:

SUBCASE 3.1: $x + y \geqslant 0$

Then by definition of absolute value we have

$$|x| = x, \; |y| = -y \;\text{ and }\; |x + y| = x + y,$$

hence in view of Theorem 16.4

$$|x + y| = x + y \leqslant |x| + |y|.$$

SUBCASE 3.2: $x + y < 0$

Then by definition of absolute value we have

$$|x| = x, \; |y| = -y \;\text{ and }\; |x + y| = -(x + y),$$

hence in view of Theorem 16.4

$$|x + y| = -x - y \leqslant |-x| + |-y| = |x| + |y|.$$

CASE 4: $x < 0$ and $y \geqslant 0$.The proof is similar to that of Case 3, you simply have to swap x and y. $\square$

17 Proof by contradiction

Suppose we want to prove some statement q. *Assume* that q is false, i.e. assume $\sim q$ is true. Try to deduce from $\sim q$ a statement which we know is definitely false. But a true statement cannot imply a false one. Hence $\sim q$ must be false, i.e. q must be true.

The same can be formulated differently: notice that

$$(\sim q \to \mathbb{F}) \to q$$

is a tautology* Therefore if we prove

* I leave its proof to you as an exercise.

$$\sim q \to \mathbb{F},$$

q will follow.

17.1 An example: proof of an inequality

I will illustrate a proof by contradiction by showing a proof of an inequality which is perhaps hard to prove by any other method.

Theorem 17.1 *Suppose x is a positive real number. Then*

$$x + \frac{1}{x} \geqslant 2.$$

Remark. In formal logical notation, it means proving

$$(\forall x \in \mathbb{R}) \left(x > 0 \to x + \frac{1}{x} \geqslant 2 \right).$$

PROOF. Consider some arbitrary positive real number x. Let $P(x)$ be statement

$$x + \frac{1}{x} \geqslant 2.$$

We want to prove that $P(x)$ is $\mathbb{T}$. By the way of contradiction, it suffices to prove that

$$\sim P(x) \to \mathbb{F}$$

is true.

So we assume that $\sim P(x)$ is $\mathbb{T}$, that is,

$$x + \frac{1}{x} < 2$$

is $\mathbb{T}$. Since x is positive, we can multiply the both sides of this inequality by x and get

$$x^2 + 1 < 2x,$$

which can be rearranged as

$$x^2 - 2x + 1 < 0$$

and then as

$$(x - 1)^2 < 0.$$

But squares cannot be negative – a contradiction. Hence our assumption that

$$x + \frac{1}{x} < 2$$

was false, which means that

$$x + \frac{1}{x} \geqslant 2$$

for all positive real numbers x. □

Later, when we shall study inequalities in more detail, we will frequently use proofs by contradiction; they are quite useful in case of inequalities, and for a simple reason: the negation of the inequality $a \leqslant b$ is the inequality $b < a$.

17.2 A basic quadratic inequality

To analyse harmonic and geometric means in later lectures,, we shall need a basic inequality about quadratic expressions.

Theorem 17.2 *Assume that $a, b > 0$ are positive real numbers. Then*

$$4ab \leqslant (a + b)^2.$$

If, in addition, $a \neq b$, we have a strict inequality:

$$4ab < (a + b)^2.$$

Proof. Assume the contrary, that the negation of the desired inequality

$$4ab \leqslant (a+b)^2$$

is true, that is,

$$(a+b)^2 < 4ab.$$

Open brackets:

$$a^2 + 2ab + b^2 < 4ab$$

and add $-4ab$ to the the both parts of the inequality:

$$a^2 + 2ab + b^2 - 4ab < 4ab - 4ab.$$

Simplify:

$$a^2 - 2ab + b^2 < 0$$

and rearrange:

$$(a-b)^2 < 0.$$

This is a contradiction because squares are non-negative by Theorem 15.3. □

We still have to do the "in addition" part of the theorem and prove the strict inequality

$$4ab < (a+b)^2.$$

in the case of $a \neq b$. But we have proved

$$4ab \leqslant (a+b)^2;$$

if the strict inequality does not hold, then

$$4ab = (a+b)^2,$$

which can be easily rearranged as

$$4ab = a^2 + 2ab + b^2,$$

$$0 = a^2 - 2ab + b^2,$$

$$0 = (a-b)^2,$$

and we get $a = b$ in contradiction to our assumption $a \neq b$.
□ □

17.3 Proofs of irrationality of $\sqrt{2}$

Recall that a real number x is *rational* if it can be written as a ratio of two integers

$$x = \frac{m}{n}$$

with $n \neq 0$, and that the set of all rational numbers is denoted by $\mathbb{Q}$. Real numbers which are not rational are called *irrational*.

Here, we will consider a proof of irrationality of $\sqrt{2}$, a classical mathematical theorem, often seen as one of the most important results in the history of mathematics. This will help us to discuss general approaches to proofs by contradiction, see Section 17.4 for more detail.

Theorem 17.3 *$\sqrt{2}$ is irrational.*

PROOF. Assume the contrary, that $\sqrt{2}$ is rational. It means that it can be written as

$$\sqrt{2} = \frac{a}{b}$$

where a and b are integers. Since $\sqrt{2}$ is positive, we can also assume that a and b are positive. Also, we can assume that a and b are not both even – otherwise we can cancel factor 2 from the numerator and denominator of $\frac{a}{b}$ and repeat it until one of a or b becomes odd.

By squaring both parts of the equation, we get

$$2 = \frac{a^2}{b^2}$$

and

$$2b^2 = a^2,$$

i.e., $a^2 = 2b^2$. Hence a^2 is even. This implies that a is even, since if a were odd, so would be a^2. Thus, $a = 2a_1$ for some natural a_1. But now

$$a^2 = (2a_1)^2 = 4a_1^2,$$

and

$$4a_1^2 = 2b^2.$$

Cancelling factor 2 from the both sides of this equality, we have

$$2a_1^2 = b^2,$$

whence b^2 is even, which implies that b is even. So both a and b are even – a contradiction, because we assumed that a is odd or b is odd.

17.4 Proof by contradiction: a discussion

*

This may sound as a paradox, but proofs by contradiction could be much easier than direct proofs. And here are reasons for that:

* Material of this section is not compulsory.

- Students frequently complain that they do not know where to start a proof. Here, you know where to start: by assuming the contrary to what you wish to prove.
- You know where to go – to a contradiction of some sort;
- Moreover, it does not matter what kind of contradiction you eventually get: as we already know, all contradictions are logically equivalent.

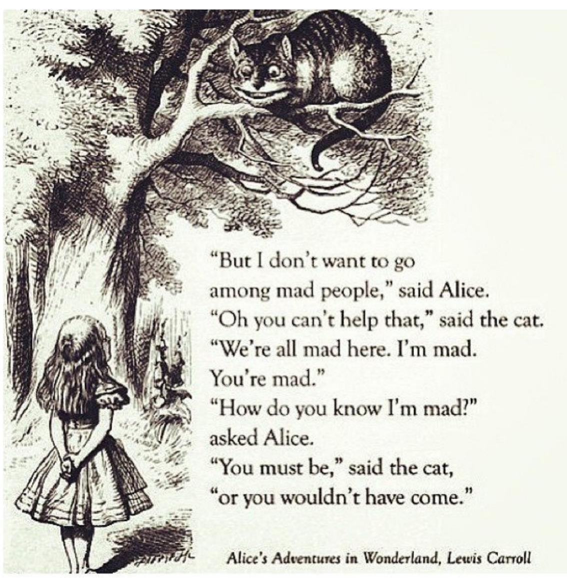
This was illustrated in Section 17.3: three proofs of irrationality of $\sqrt{2}$ started exactly at the same point:

> Assume the contrary, that $\sqrt{2}$ is rational. It means that it can be written as
>
> $$\sqrt{2} = \frac{a}{b}$$
>
> where a and b are integers. Since $\sqrt{2}$ is positive, we can also assume that a and b are positive.

But then the proofs went three different ways: Geometry, Algebra, and Number Theory, each one leading to a contradiction.

The famous conversation between Alice and the Cheshire Cat in *Alice in Wonderland* is very relevant hear:

> *"Would you tell me, please, which way I ought to go from here?"*
> *"That depends a good deal on where you want to get to," said the Cat.*
> *"I don't much care where–" said Alice.*
> *"Then it doesn't matter which way you go," said the Cat.*
> *"–so long as I get SOMEWHERE," Alice added as an explanation.*
> *"Oh, you're sure to do that," said the Cat, "if you only walk long enough."*



Lewis Carroll, the author of Alice in Wonderland, was one of the first mathematical logicians at the time when this branch of mathematics was still very young; his real name was Charles Dodgson. The set theory was also non-existent at his time, and in his book on logic he talks about classes rather than sets, and in a very peculiar way:

> *'Classification', or the formation of Classes, is a Mental Process, in which we imagine that we have put together, in a group, certain things. Such a group is called a '**Class**'.*
>
> *As this Process is entirely* Mental*, we can perform it whether there* is*, or* is not*, an* existing *Thing* [in that Class – AB]*. If there* is*, the Class is said to be '*Real*'; if not, it is said to be '**Unreal**', or '**Imaginary**'*.

For us, all 'Imaginary Classes' are just the empty sets, and, for us, all empty sets are equal; for Lewis Carroll (aka Charles Dodgson), the class of real roots of the equation $x^2 = -1$ and the class of flying pigs would be different.

However, what is disturbing about Proof 1 of irrationality of $\sqrt{2}$ is that we are talking about, and analysing, a *non-existent object* ('Thing'): a square with the side of integer length such that its area is twice the area of another square with the side of integer length. In Proofs 2 and 3 we analysed non-existent fractions of integers which equal $\sqrt{2}$, and manipulated them, replacing non-existent fractions by other fractions, which were also non-existent, but had smaller denominators.

It is like studying flying pigs, replacing, in the process, one flying pig by another one – of smaller weight.

Proofs from contradiction are Wonderland of mathematics; doing them, you have to be prepared to meet creatures no less strange than Cheshire Cat or Mad Hatter.

17.5 A few words about abstraction

*

* Material in this section is not compulsory and can be skipped.

Mathematicians adore *Alice in Wonderland* because the essence of mathematical abstraction is captured in another famous episode:

> 'All right,' said the Cat; and this time it vanished quite slowly, beginning with the end of the tail,

> and ending with the grin, which remained some time after the rest of it had gone.
>
> 'Well! I've often seen a cat without a grin,' thought Alice; 'but a grin without a cat! It's the most curious thing I ever saw in my life!'

I have already had a chance to tell you that statements of Propositional Logic have no meaning, they have only truth values. This is why "2+2=5" implies "London is the capital of Britain" – because the former is $\mathbb{F}$, the latter is $\mathbb{T}$. The meaning of the statements is irrelevant. The truth value of a statement is 'a grin without a cat' left after the meaning of the statement vanished.

It is easy to check that the following statement of Propositional Logic is a tautology*:

$$(p \to q) \vee (q \to p)$$

it takes truth value $\mathbb{T}$ regardless of the meaning of p and q. For example, if we take

p is "there is life on Mars"

and

q is "today is Wednesday",

the compound statement $(p \to q) \vee (q \to p)$ remains $\mathbb{T}$ regardless of the day of the week.

There is nothing unusual in that, exactly the same is happening in arithmetic: numbers $1, 2, 3, \ldots$ have no meaning, but have '*numerical values*', and can be compared by their values and operated according to them. In arithmetic, the sentence

> 'The number of cats in London is larger than the number of books in the town of Winesburg, Ohio',

is fully legitimate, even if cats in London have no connection whatsoever with books on the other side of Atlantic. Even more: we can take the number of cats in London and multiply it by the number of books in Winesburg, Ohio.

* I leave its proof as an exercise for the reader.

Or another example: I can claim that

> 'The number of my children is less than the number of Jupiter's moons',

despite the fact that the two numbers have no relation to each other whatsoever.

And notice that no-one would claim that arithmetic was absurd or counter-intuitive; over the history, people got used, and stopped paying attention, to the level of *mathematical abstraction* present in ordinary prime school arithmetic. Propositional logic (manipulation with truth values $\mathbb{T}$ and $\mathbb{F}$) is arithmetic of formal logical thinking. It is much younger than arithmetic of numbers, but we have to get used to it, too, because of its tremendous importance for all things electronic, IT, computing in our lives.

17.6 Wonderland of Mathematics

To illustrate the power of proofs from contradiction, I give an example which shows that sometimes we can easily prove by contradiction a statement which otherwise is very hard to comprehend.

Theorem 17.4 $\log_2 3$ *is an irrational number.*

PROOF. Assume, the contrary, that $\log_2 3$ is not irrational. Then $\log_2 3$ is a rational number, that is,

$$\log_2 3 = \frac{m}{n}$$

for integers m and n, with $n \neq 0$. By definition of logarithm, it means that

$$2^{\frac{m}{n}} = 3.$$

Since $2 < 3$, we conclude that $\frac{m}{n} > 0$. Now we may assume that $m > 0$ and $n > 0$ are natural numbers. But then

$$2^m = 3^n$$

for $m, n \in \mathbb{N}$, and 2^m and 3^n are also natural numbers. But one of them is even, the other one is odd. We reached an obvious contradiction which completes our proof.

You would perhaps agree that this proof is very simple and very natural – but it also is a wonder.

And now is something completely different: a proof by contradiction which leads to a very paradoxical situation. Indeed, look for yourself:

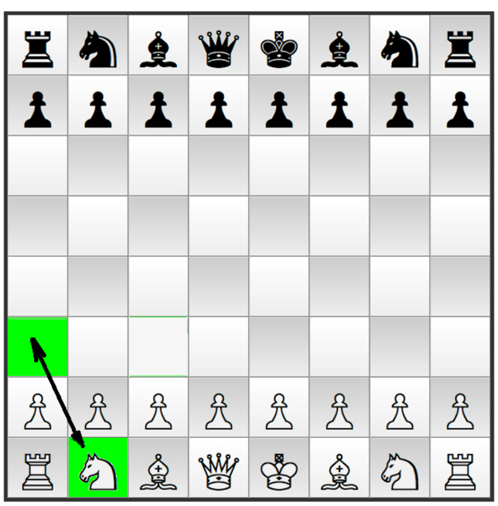
Example 17.6.1 *The game of "double chess" follows all the usual rules of chess, with one exception: both players are allowed to make two moves in a row.*

Prove that White has a strategy which ensures a draw or a win.

Proof A proof is deceptively simple: assume that White has no such strategy.

Then Black has a winning strategy.

But White may use the property that Knight can jump over other pieces, in the first two moves of the game, move a Knight forth and back, returns the chessboard into the pre-game state:



That way, White yields the first move to Black, in effect, changing his own color to Black.

But Black has a winning strategy, hence White, which has become Black, also has a winning strategy – a contradiction. $\square$

This is what mathematicians call "*a pure proof of existence*": it says nothing whatsoever about the actual strategy! We have forced White into the ridiculous situation that he must to react to the whole optimal strategy of Black – without even knowing whether Black's strategy brings victory or just a draw.

And here is another slightly paradoxical situation when a proof by contradiction provides some insight by not a total knowledge:

Theorem 17.5 *There are two irrational real numbers r and s such that r^s is rational.*

PROOF. We now definitely know that $\sqrt{2}$ is irrational, so consider the pair of numbers $r = s = \sqrt{2}$. If $r^s = \sqrt{2}^{\sqrt{2}}$ is rational, we are done.

But if $\sqrt{2}^{\sqrt{2}}$ is irrational, take

$$r = \sqrt{2}^{\sqrt{2}} \quad \text{and } s = \sqrt{2},$$

then, by properties of exponentiation, we have

$$r^s = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$$

which is rational.

Remark. If fact, $\sqrt{2}^{\sqrt{2}}$ is known to be irrational. This is a consequence of a very deep result by Gelfond and Schneider.

In this proof, we got two options for irrational numbers r and s – we know that in one of them r^s is irrational, but we do not know in which one.

17.7 Problems

Problem 17.1 *Using the fact that $\sqrt{2}$ is irrational, prove that if r is a rational number then $r + \sqrt{2}$ is irrational.*

Problem 17.2 *Using the fact that $\sqrt{2}$ is irrational, prove that, for every integer $k \neq 2$, the number $\sqrt{k} - \sqrt{2}$ is also irrational.*

Problem 17.3 *Prove the tautology*

$$(p \to q) \vee (q \to p).$$

Problem 17.4 *In a certain English city, two local football clubs, A and B, face each other in a derby**. *The sum of salaries of players in team A is bigger than the sum of salaries of team B, and the sum of salaries of foreign players (in both teams taken together) is bigger than the sum of salaries of British players.*

* Derby: a sports event between two rival teams in the same area

Could it happen that there are no foreign players in team A?

17.8 Some more challenging problems

[1]

Problem 17.5 *Prove that the product of three consecutive positive integers is never a cube of an integer. (You may need some results about inequalities from later lectures.)*

Problem 17.6 *Investigate this question: can the product of 4 consecutive integers be a 4^{th} power of an integer?**

* I do not know the answer, but the problem appears to be accessible.

Problem 17.7 *Prove that** *the point of the form $(\cos\theta, \sin\theta)$ cannot lie strictly inside (that is, inside, but not on the sides) of the triangle with the vertices $(0,0)$, $(1,0)$, and $(0,1)$.*

* You may wish to return to this question after learning more about inequalities.

[1] These are not examinable.

18 Harmonic, geometric, and arithmetic means

18.1 Averaging and mixing

A few examples of how “mixing” leads to “averaging”.

Example 18.1.1 *A rectangular sheet of paper of dimensions a by b, where $a < b$, is cut and rearranged, without holes and overlaps, as a square of side c. Then $a < c < b$.*

Proof The area of the square, c^2, is the same as the area ab of the rectangle. If $c \leqslant a$ then $c < b$ and

$$c^2 = c \cdot c < ab,$$

a contradiction.

Similarly, if $b \leqslant c$ then $a < c$ and

$$ab < c \cdot c = c^2,$$

again a contradiction. □

Example 18.1.2 *Two jars with salt solutions of concentrations p% and q%, with $p < q$, are emptied into a third jar. We assume that both jars were not empty, that is, both contained some amount of solution. Then the concentration of salt in the third jar, r%, satisfies the same inequality, $p < r < q$.*

Proof Let the volumes of solutions in the first and in the second jar be U and V. Then the amount of salt in both solutions is $pU + qV$, and amount of salt after mixing of solutions is $r(U + v)$. Obviously,

$$pU + qV = r(U + V).$$

If $q \leqslant r$, then

$$pU + qV < qU + qV = q(U + V) \leqslant r(U + V),$$

a contradiction.

If $r \leqslant p$, then

$$r(U+V) \leqslant p(U+V) = pU + PV < pU + qV.$$

also a contradiction.

Hence $p < r < q$. $\square$

Example 18.1.3 *Two cisterns of different shape and sizes are positioned at different levels above the ground and connected by a pipe with a valve, initially closed. The cisterns are filled with water to levels $h_1 < h_2$ above the ground and then valve is opened. The water now is at the shared level h above the ground in the both cisterns. Of course, $h_1 < h < h_2$.*

Example 18.1.4 *Two cyclist started at the same time on a route from A to B and back. The first cyclist was cycling from A to B with average speed u km/h, and on way from B to A with average speed v km/h, where $u < v$. The second cyclist had average speed w km/h over the whole route, A to B to A. They returned to A simultaneously. In that case, $u < w < v$.*

Why? Because if $w < u$, then $w < u < v$, then the second cyclist is always behind the first one.

If $v < w$, then $u < v < w$, and the second cyclist is always ahead of the first one.

Hence $u \leqslant w$ and $w \leqslant v$, and $u \leqslant w \leqslant v$. $\square$

18.2 Arithmetic mean

Example 18.2.1 *John and Mary are married. This tax year, John's income tax increased by £40, and Mary's income tax increased by £60. Between them, what is the average increase in income tax?*

Solution.

$$\frac{\pounds 40 + \pounds 60}{2} = \pounds 50.$$

This is an example of an arithmetic mean. For real numbers a and b, their *arithmetic mean* is

$$\frac{a+b}{2}.$$

More generally, the arithmetic mean of n numbers

$$a_1, a_2, \ldots, a_n$$

is

$$\frac{a_1 + a_2 + \cdots + a_n}{n}.$$

18.3 Harmonic mean

18.3.1 Example.

> A car traveled from city A to city B with speed 40 miles per hour, and back with speed 60 miles per hour. What was the average speed of the car on the round trip?

Many students give an almost instant answer: 50 mikes per hour, that is, the arithmetic men of the two speeds:

$$50 = \frac{60+40}{2}.$$

But this answer immediately collapses into absurdity if we slightly change the problem: what would happen if the speed of the car on its way back from B to A was 0 miles per hour? Will the average speed be

$$\frac{60+0}{2} = 30 \text{ mph?}$$

But the car will never return!

This suggests that the arithmetic mean is not a solution to this problem.

18.3.2 A simpler example.

Let us make a problem a bit more concrete by assuming that we know the distance from A to B.

Example 18.3.1 *The distance between A and B is 120 miles. A car traveled from A to B with speed 40 miles per hour, and back with speed 60 miles per hour. What was the average speed of the car on the round trip?*

□

This result shows that speeds are averaging not by the law of arithmetic mean! So let us look at this example in more detail.

Example 18.3.2 *The distance between A and B is d miles. A track traveled from A to B with speed u miles per hour, and back with speed v miles per hour. What was the average speed of the car on the round trip?*

Solution. It took

$$\frac{d}{u} \text{ hours}$$

for a truck to get from A to B and

$$\frac{d}{v} \text{ hours}$$

to get back. Therefore it took

$$\frac{d}{u} + \frac{d}{v}$$

hours to make the round trip of $2d$ miles. Hence the average speed on the entire round trip was

$$\frac{2d}{\frac{d}{u} + \frac{d}{v}} = \frac{2}{\frac{1}{u} + \frac{1}{v}}$$

miles per hour. Please observe:

- The result does not depend on the distance d.
- the expression
$$\frac{2}{\frac{1}{u}+\frac{1}{v}}$$
does not look at all as the arithmetic mean of u and v.

What we get is the *harmonic mean*: it is defined for positive real numbers $a, b > 0$ as

$$\frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

A simple algebraic rearrangement allow to write the harmonic mean in a bit more compact form:

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b}.$$

This form is more preferable because it allows one of a or b be non-negative: if $a > o$ and $b = 0$

$$\frac{2a \cdot 0}{a+0} = 0,$$

thus resolving the paradox with the zero speed on the way back.

18.4 Geometric mean

Example. In an epidemics, the daily number of new cases had grow up by factor of 4 over November and by factor of 9 over December. What was the average monthly growth in the daily number in new cases over the two months?

Solution. Assume that daily number of new cases was equal R at the beginning of November, then at the beginning of December it was $4 \cdot R$, and at the end of December it became equal $9 \cdot 4 \cdot R = 36R$.

The average monthly growth is the coefficient k such that, it it were equally applied to November and to December, it would produce the same outcome: that is, R at the beginning of November, $k \cdot R$ at the beginning of December and $k \cdot k \cdot R$ at the end of December, which means that

$$k \cdot k \cdot R = 36R,$$

$$k^2 = 36$$

and

$$k = \sqrt{36} = 6.$$

Observe that the result is different from the arithmetic mean of 4 and 9 (which equals $6\frac{1}{2}$). To see why this is happening we need to take a look at the same problem in "general notation":

> In an epidemics, the daily number of new cases had grow up by factor of a over November and by factor of b over December. What was the average monthly growth in the daily number in new cases over the two months?

The same argument gives us

$$k \cdot k \cdot R = a \cdot b \cdot R,$$

$$k^2 = ab$$

and

$$k = \sqrt{ab}.$$

For positive real numbers a and b, the quantity $\sqrt{ab}$ is called the *geometric mean* of a and b.

More generally, the *geometric mean of n positive numbers*

$$a_1, a_2, \ldots, a_n$$

is

$$\sqrt[n]{a_1 a_2 \cdots a_n}.$$

18.5 Comparing the three means

Theorem 18.1 *For all positive real numbers a and b,*

$$\frac{2ab}{a+b} \leqslant \sqrt{ab} \leqslant \frac{a+b}{2}.$$

Our proof of this theorem will be based on a simpler inequality of Theorem 17.2.

Proof. We shall prove the two inequalities

$$\frac{2ab}{a+b} \leqslant \sqrt{ab}$$

and

$$\sqrt{ab} \leqslant \frac{a+b}{2}$$

separately but by the same method, in both cases starting from the inequality of Theorem 17.2:

$$4ab \leqslant (a+b)^2.$$

(A) Proof of

$$\frac{2ab}{a+b} \leqslant \sqrt{ab}.$$

We start with

$$4ab \leqslant (a+b)^2.$$

divide the both sides of the inequality by the positive number $(a+b)^2$:

$$\frac{4ab}{(a+b)^2} \leqslant 1,$$

then multiply the both sides by $ab > 0$:

$$\frac{4a^2b^2}{(a+b)^2} \leqslant ab,$$

and extract the square roots from the both (positive!) sides of the inequality (Theorem 16.3 on Page 93):

$$\frac{2ab}{a+b} \leqslant \sqrt{ab}.$$

(B) Proof of

$$\sqrt{ab} \leqslant \frac{a+b}{2}$$

is even simpler. Again, we start with Theorem 17.2

$$4ab \leqslant (a+b)^2$$

and, using the same Theorem 15.3, take the square roots of both parts:

$$2\sqrt{ab} \leqslant a+b,$$

and divide by 2:

$$\sqrt{ab} \leqslant \frac{a+b}{2}.$$

□

18.6 Where are the three means used?

*

This section is written in response to a question from a student:

> *Could you elaborate a good way to know when to use which type of mean?*

This is the basic principle: in every particular situation a *mean* is a number that can be used in place of each number in a set, for which the *net effect* will be the same as that of the original set of numbers. What determines which mean to use is the way in which the numbers act together to produce that net effect.

For example, if you were self-employed and had, over year 2017, monthly incomes $I_1, I_2, \ldots, I_{12}$, then you note that your total income over the year is found by *adding* the monthly numbers; so if you add them up and divide by the number of months, the resulting *arithmetic* mean

$$I_{\text{mean}} = \frac{I_1 + \cdots + I_{12}}{12}$$

* This section uses bits from several anonymous Internet sources, in particular, postings on `http://mathforum.org`: I would love being able to attribute them to particular authors. The images are from Wikipedia.

is the amount of income you could have had on *each* of those months, to get the same total.

If you have several successive price markups, say by 5% (or, which is the same, by factor 1.05) and then by 6% (that is, by factor 1.06), and want to know the mean markup, you note that the net effect is to first *multiply* by 1.05 and then by 1.06, equivalent to a single markup of $1.05 \times 1.06 = 1.113$; taking the square root of this, you get $\sqrt{1.113} \approx 1.055$. This means that if you had *two* markups of 5.5% each, you would get the same result. This is the *geometric* mean. In general, you use it where the product is an appropriate "total".

Another example is when you combine several enlargements of a picture: the average of two enlargements, of 125% and 175% of the original, is the enlargement by factor

$$\sqrt{1.25 \times 1.75} = 1.48,$$

that is of 148% of the original. Notice a difference in terminology with price markups – it is traditional; the terminology for computer graphics was created by computer programmers, who knew mathematics better, and were more honest to their customers, than traders; of the latter, many would love to have their customers to believe that two consecutive markups of 10% make a markup of 20%, and not 21% (which it is, because $1.1 \times 1.1 = 1.21$.

If you want the mean speed of a car that goes the same distance (not time! – for example, doing several runs on the same circuit) at each of several speeds $v_1, \ldots, v_n$, then the net effect of all the driving (the total time taken) is found by dividing the common distance l by each speed v_i to get the time for that leg of the trip, and then adding up those times:

$$\frac{l}{v_1} + \cdots + \frac{l}{v_n}.$$

The constant speed v that would take the same total time for the whole trip of total length nl is the *harmonic* mean of the speeds.

$$\frac{nl}{v} = \frac{l}{v_1} + \cdots + \frac{l}{v_n},$$

or, after simplification,

$$v = \frac{n}{\frac{1}{v_1} + \cdots + \frac{1}{v_n}},$$

or

$$\frac{1}{v} = \frac{\frac{l}{v_1} + \cdots + \frac{l}{v_n}}{n} :$$

the reciprocal* of the mean speed is the arithmetic mean of reciprocals of speeds on each leg.

* The *reciprocal* of a positive real number v is $\frac{1}{v}$.

However, if travelled on n consecutive days for fixed time T each day, with average speeds $v_1, \ldots, v_n$, what is *added* are distances $l_k = v_k T$, travelled at k-th day, for each $k = 1, 2, \ldots, n$, and the total distance travelled is

$$l = v_1 T + \cdots l_n T,$$

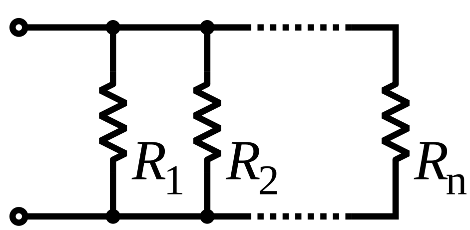
and the mean speed is

$$\begin{aligned} v_{\text{mean}} &= \frac{l}{nT} \\ &= \frac{v_1 T + \cdots + v_n T}{nT} \\ &= \frac{v_1 + \cdots + v_n}{n} \end{aligned}$$

is the *arithmetic mean of* the speeds.

Another example is combining resistances in a parallel electrical circuit: what is added are currents I_k through k-th resistor, which are proportional to reciprocals of this resistances R_k because voltage V on each resistance is the same: by Ohm's Law,

$$I_k = \frac{V}{R_k}.$$



Therefore the total current I can be found as

$$I = \frac{V}{R_1} + \cdots + \frac{V}{R_n}$$

and then the total resistance R can be found from

$$\frac{V}{R} = I = \frac{V}{R_1} + \cdots + \frac{V}{R_n},$$

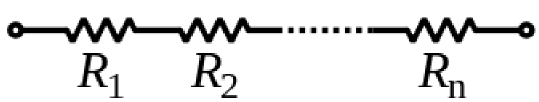
or, after cancelling V from the both parts of the equation.

$$\frac{1}{R} = \frac{1}{R_1} + \cdots + \frac{1}{R_n}$$

and the mean resistance R_{mean} equals

$$R_{\text{mean}} = \frac{n}{\frac{1}{R_1} + \cdots + \frac{1}{R_n}}.$$

If resistors are connected consecutively (a series circuit),



it is voltages are added up, while the current is constant, and I leave you as an exercise to check that in that case the mean resistance (that is, the resistance of the same number of identical resistors that you would have used to achieve the same effect) is the *arithmetic mean*.

In summary, you use the

- arithmetic mean when numbers just add up;
- geometric mean when numbers multiply together;
- harmonic mean when the reciprocals of the numbers add up.

18.7 Advanced problems

The material in this section is not compulsory. The reason for its existence is a request from students: some students ask for more advanced, or harder, problems. Here are some of such problems.

The first problems, 18.1 to 18.9, require only basic arithmetic and some understanding of inequalities.

Problem 18.1 *A hiker walked for* 3.5 *hours covering, in each one hour long interval of time, exactly* 2 *miles. Does it necessarily follow that his average speed over his hike was* 3 *miles per hour?*

Problem 18.2 *The front tyres of a car get worn out after* $15,000$ *miles, the back ones after* $25,000$ *miles. When they have to be swapped to achieve the longest possible run?*

Problem 18.3 *A paddle-steamer takes five days to travel from St Louis to New Orleans, and seven days for the return journey. Assuming that the rate of flow of the current is constant, calculate how long it takes for a raft to drift from St Louis to New Orleans.*

Problem 18.4 *A train carriage is called* overcrowded *if there are more than* 60 *passengers in it. On a Friday 19:00 train from London Euston to Manchester Piccadilly, what is higher: the percentage of overcrowded carriages or the percentage of passengers travelling in overcrowded carriages?*

Problem 18.5 *The average age of* 11 *players in a football team on the field is* 22 *years. During the game, one player got a red card. The average age of his teammates left of the field is* 21 *years. What is the age of the player who got the red card?*

Problem 18.6 20 *people sit around a big table. The age of each of them is the arithmetic mean of the ages of his/her two neighbours. Prove that all of them have the same age.*

Problem 18.7 *Gulnar has an average score of* 87 *after* 6 *tests. What does Gulnar need to get on the next test to finish with an average of* 78 *on all* 7 *tests?*

Problem 18.8 *Ms Fontaine, a teacher of French at a school, teaches two groups of students. In the following table you can see the lists of groups with the end of term marks. Can Ms Fontaine transfer students from one group to another in such a way that the mean marks in both groups will increase?*

Group A				Group B		
1	Altasan	31		1	Armitage	36
2	Barnard	46		2	Burns	49
3	Cable	52		3	Chiu	31
4	Debonis	51		4	Dolaslan	35
5	Edmond	32		5	Edelbaum	48
6	Fryer	41		6	Gardiner	32
7	Huang Jin	59		7	Klymchuk	35
8	Kuber	32		8	Leyland	47
9	Marsh	44		9	Peterson	35
10	Wiscons	54		10	Walter	40
Arithmetic mean		**44.2**		**Arithmetic mean**		**38.8**
Last year		**44.4**		**Last year**		**39.2**

The Headmaster expects the mean marks to grow from one year to another. Ms Fontaine cannot change marks, but she can transfer students from one group to another. Can she make the mean marks in the both groups higher than they were last year?

Problem 18.9 *Place these numbers in increasing order:*

$$222^2, \quad 22^{22}, \quad 2^{222}.$$

The following problems involve a bit of school level algebra.

Problem 18.10 *Prove that if*

$$0 < a_1 < a_2 < \cdots < a_8 < a_9$$

then

$$\frac{a_1 + a_2 + \cdots + a_9}{a_3 + a_6 + a_9} < 3.$$

Problem 18.11 *Without using Theorem 18.1, give a direct proof of an inequality for harmonic and arithmetic means:*

$$\frac{2ab}{a+b} \leqslant \frac{a+b}{2}$$

for all $a > 0$ and $b > 0$.

Problem 18.12 *Prove the inequality between the* quadratic mean *and the arithmetic mean:*

$$\frac{a+b}{2} \leqslant \sqrt{\frac{a^2+b^2}{2}}.$$

Problem 18.13 *Prove that, for all $x \geqslant 0$,*

$$1 + x \geqslant 2\sqrt{x}.$$

Problem 18.14 *Prove that,for all $x > 0$ and $y > 0$,*

$$\frac{1}{x} + \frac{1}{y} \geqslant \frac{4}{x+y}.$$

Problem 18.15 *Prove that if the product of two positive numbers is bigger than their sum, then the sum is bigger than 4.**

* HINT: *Use Problem 18.14 or Theorem 17.1.*

Problem 18.16 *If you ask junior school children: what is bigger,*

$$\frac{2}{3} \ \textit{or}\ \frac{4}{5},$$

they perhaps will not be able to answer. But if you ask them: what is better, 2 bags of sweets for 3 kids of 3 bags for 4 kids, they will immediately give you the correct answer.

Indeed there is an easy line of reasoning which leads to this conclusion. Let us treat fractions not as numbers but descriptions of certain situations: $\frac{2}{3}$ means 2 bags, 3 kids, $\frac{3}{4}$ means 3 bags, 4 kids. How to get situation $\frac{3}{4}$ from $\frac{2}{3}$? The fourth kid comes, bringing with him a bag. He has more for him compared with his three friends, who have 2 bags for 3, and of course 3 kids will benefit if the fourth one shares with them his bag.

This argument amounts to claiming (correctly) that

$$\frac{2}{3} < \frac{2+1}{3+1} < \frac{1}{1}$$

What we see here is a version of the ***Mediant Inequality:***

if $a, b, c, d > 0$ and

$$\frac{a}{b} < \frac{c}{d}$$

then

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

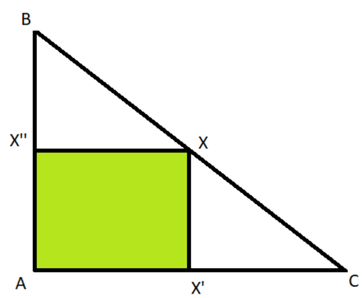
Prove it.

The expression

$$\frac{a+c}{b+d}$$

is called the **mediant** of $\frac{a}{b}$ and $\frac{c}{d}$; it makes sense and is used only for positive numbers a, b, c, d.

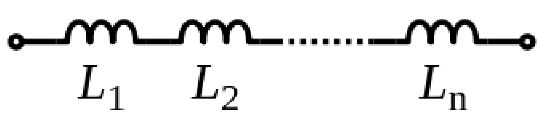
Problem 18.17 *How you have to choose point X on the hypothenuse BC of a rightangled triangle $\triangle ABC$ so that the area of the inscribed rectangle $AX'XX''$ is maximal possible?*



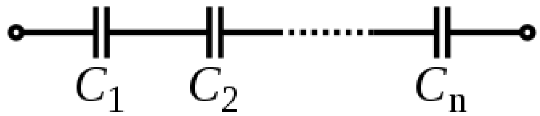
And a problem on means in electrical engineering – for those students who know school level physics.

Problem 18.18 *In each of the following circuits, find the mean inductance or capacity, that, is, inductance or capacities of n identical inductors (respectively, capacitors) which produce the same effect.*

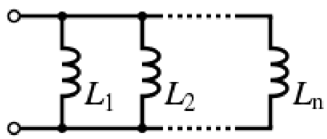
(a) *The mean inductance of non-coupled inductors in series:*



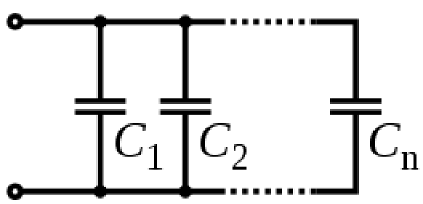
(b) *The mean capacitance of capacitors in series:*



(c) The mean inductance of non-coupled inductors in parallel:



(b) The mean capacitance of capacitors in parallel:



18.8 Solutions to advanced problems

18.1 Assume that the hiker walks for half an hour with speed 4 m/h, then rests for half an hour, etc. Then in any given hour he will advance by exactly 2 miles, and, since he walks for 4 half an hour intervals, he will cover 8 miles. Hence his average speed is $8 \div 3.5 > 2$ m/h. □

18.2 ANSWER: After 9,375 miles – it will ensure ensure the run of 18,750 miles, the harmonic mean of 15,000 and 25,000. □

18.3 ANSWER: 35 days. □

18.4 Let us paint overcrowded carriages red. In each carriage, increase or decrease the number of passengers so it becomes exactly 60. Now each carriage has the same number of passengers, and the percentage of red carriages equals the percentage of passengers in red carriages. But in order to achieve that, we removed some passengers from red carriages and added passengers to other carriages. Hence, prior to this change, the percentage of passengers in red carriages was higher then the percentage of red carriages. □

18.5 ANSWER: 22. □

18.6 Consider an oldest person: his two neighbours have to have the same age as him/her. Continue applying the same arguments around the table. □

18.7 SOLUTION 1. What follows are hints provided, one after another, by the Khan Academy website[2]

Hint 1: Since the average score of the first 6 tests is 87, the sum of the scores of the first 6 tests is $6 \times 87 = 522$.

Hint 2: If Gulnar gets a score of x on the 7th test, then the average score on all 7 tests will be: $\frac{522+x}{7}$.

Hint 3: This average needs to be equal to 78 so: $\frac{522+x}{7} = 78$.

[2] Khan Academy. http://www.khanacademy.org/about. Last Accessed 14 Apr 2011.

Hint 4: $x = 24$. □

SOLUTION 2. And here is how the same problem would be solved by the "steps" or "questions" method as it was taught in schools half a century ago, in 1950–60s.

Question 1: How many points in total did Gulnar get in 6 tests? Answer: $6 \times 87 = 522$.

Question 2: How many points in total does Gulnar need to get in 7 tests? Answer: $7 \times 78 = 546$.

Question 3: How many points does Gulnar need to get in the 7th test? Answer: $546 - 522 = 24$. □

SOLUTION 3. There is a quicker solution which requires a bit better understanding of averages.[3]

Question 1: How many "extra" – that is, above the requirement – points did Gulnar get, on average, in 6 tests? Answer: $87 - 78 = 9$.

Question 2: How many "extra" points does Gulnar have? Answer: $9 \times 6 = 54$.

Question 3: How many points does Gulnar need to get in the last test? Answer: $78 - 54 = 24$. □

18.8 To increase the mean mark in both groups, it is necessary to move from Group A to Group B students with marks which are higher than the mean mark in Group B but lower than the mean mark in Group A; these students are Fryer and Marsh. If they are moved from from Group A to Group B, the mean mark in Group A becomes

$$\frac{44.2 \times 10 - 41 - 44}{8} = 44.625 > 44.4,$$

and in Group B

$$\frac{38,8 \times 10 + 41 + 44}{12} = \frac{473}{12} = 39.4166\cdots > 39.2,$$

that is, higher than the last year mean marks. □

18.9 They are already in the increasing order:

$$222^2 < 22^{22} < 2^{222}.$$

Indeed $222^2 < 1000^2$ contains at most 9 digits, while $22^{22} > 10^{22}$ contains at least 22 digits. Similarly, $22^{22} < 100^{22}$ contains at most 44 digits, while

$$2^{222} > 2^{220} = 2^{4\times 55} = (2^4)^{55} = 16^{55} > 10^{55}$$

contains at least 55 digits. As you can see, the problem can me solved by mental arithmetic. ***And where would you place*** 222^{22}***?*** □

18.10 □

18.11 □

18.12 □

18.13 Substitute $x = u^2$. □

18.14 □

18.15 SOLUTION 1. By Problem 18.14, $\frac{1}{x} + \frac{1}{y} \geqslant \frac{4}{x+y}$, which is equivalent to $\frac{x+y}{xy} \geqslant \frac{4}{x+y}$. But since $xy > x + y$, we have $1 > \frac{x+y}{xy} \geqslant \frac{4}{x+y}$, hence $x + y > 4$. □

SOLUTION 2. The inequality $xy > x + y$ can be rearranged as $xy - x - y > 0$, and,after adding 1 to the both sides becomes $xy - x - y + 1 > 1$. The left-hand side can be factorised: $(x - 1)(y - 1) > 1$. Now replace the variables: set $u = x - 1$ and $v = y - 1$ (check that $u > 0$ and $v > 0$!). We have a new problem equivalent to our original problem: given positive numbers u and v such that $uv > 1$, prove that $u + v > 2$. Notice now that $uv > 1$ is equivalent to $v > \frac{1}{u}$ and $u + v > u + \frac{1}{u} > 2$ by Theorem 17.1. □

18.16 Since all a, b, c, d are positive, the inequality $\frac{a}{b} < \frac{c}{d}$ is equivalent to $ad < bc$.

To prove $\frac{a}{b} < \frac{a+c}{b+d}$, we can replace it by an equivalent inequality $a(b+d) < b(a+c)$ (that is, the two inequalities are true or false simultaneously), which is equivalent to $ab + ad < ab + bc$, which is equivalent to the one we already know: $ad < bc$.

The other inequality, $\frac{a+c}{b+d} < \frac{c}{d}$, can be done in a similar way. □

18.17 ANSWER: X should be the midpoint of the hypothenuse. □

[3] Proposed by John Baldwin.

19 Inequalities in single variable

19.1 Linear inequalities in single variable

We shall look at inequalities of the form

$$ax + b > cx + d$$

$$ax + b \geqslant cx + d$$

$$ax + b \leqslant cx + d$$

$$ax + b < cx + d$$

where x is are unknown (variable) and a, b, c, d are real coefficients. This inequalities are called *linear inequality in single variable* because they involve only linear functions of the same variable.

The **solution set** of an inequality with the unknown x is the set of all real numbers x for which it is true.

Two inequalities are called **equivalent** if they have the same solution set.

Theorem 19.1 *The solution sets of an inequality*

$$ax + b \leqslant cx + d$$

is either empty, or equal to the set of all real numbers $\mathbb{R}$*, or a ray.*

Similarly, the solution set of an inequality

$$ax + b < cx + d$$

is either empty, or equal to the set of all real numbers $\mathbb{R}$*, or a half-line.*

Example 19.1.1

- *The inequality*

  $$x + 1 \leqslant x - 1$$

  has no solution.

- *Every real number is a solution of the inequality*

$$x - 1 \leqslant x + 1.$$

- *The inequality*

$$2x - 1 \leqslant x + 1$$

  can be rearranged, by adding $-x$ to the both sides, as

$$x - 1 \leqslant 1$$

  and then, by adding 1 to the both sides, as

$$x \leqslant 2.$$

  Hence the solution set is the ray

$$\{\, x : x \leqslant 2 \,\} =]-\infty, 2].$$

- *Similarly, the inequality*

$$x - 1 \leqslant 2x + 1$$

  has the solution set $[-2, +\infty[$, a ray of another direction.

- *The same examples remain valid if we replace $\leqslant$ by $<$ and the rays by half-lines.*

19.2 Quadratic inequalities in single variable

In this lecture, we consider inequalities involving quadratic functions such as

$$ax^2 + bx + c > 0,$$

$$ax^2 + bx + c \geqslant 0,$$

$$ax^2 + bx + c \leqslant 0,$$

$$ax^2 + bx + c < 0.$$

19.2.1 Simplifying the quadratic function

We assume that $a \neq 0$ (for otherwise we would have just a linear inequalities of the kind $bx + c \geq 0$, etc.). We can divide the inequalities by a – of course, taking into account the sign of a and changing the directions of inequalities appropriately, so that

if a > 0,

$$ax^2 + bx + c > 0 \text{ becomes } x^2 + \frac{b}{a} + \frac{c}{a} > 0$$

$$ax^2 + bx + c \geq 0 \text{ becomes } x^2 + \frac{b}{a} + \frac{c}{a} \geq 0$$

$$ax^2 + bx + c \leqslant 0 \text{ becomes } x^2 + \frac{b}{a} + \frac{c}{a} \leqslant 0$$

$$ax^2 + bx + c < 0 \text{ becomes } x^2 + \frac{b}{a} + \frac{c}{a} < 0$$

if a < 0,

$$ax^2 + bx + c > 0 \text{ becomes } x^2 + \frac{b}{a} + \frac{c}{a} < 0$$

$$ax^2 + bx + c \geq 0 \text{ becomes } x^2 + \frac{b}{a} + \frac{c}{a} \leqslant 0$$

$$ax^2 + bx + c \leqslant 0 \text{ becomes } x^2 + \frac{b}{a} + \frac{c}{a} \geq 0$$

$$ax^2 + bx + c < 0 \text{ becomes } x^2 + \frac{b}{a} + \frac{c}{a} > 0,$$

so we can assume, after changing notation

$$\frac{b}{a} \text{ back to } b \text{ and } \frac{c}{a} \text{ back to } c,$$

and without loss of generality, that we are dealing with one of the inequalities

$$x^2 + bx + c > 0,$$

$$x^2 + bx + c \geq 0,$$

$$x^2 + bx + c \leqslant 0,$$

$$x^2 + bx + c < 0.$$

19.2.2 Completion of squares: examples

Example 19.2.1 *Now consider two quadratic functions*

$$f(x) = x^2 + 4x + 3$$

and

$$g(x) = x^2 + 4x + 5.$$

Obviously,

$$f(x) = x^2 + 4x + 3 = x^2 + 4x + 4 - 1 = (x + 2)^2 - 1$$

and

$$g(x) = x^2 + 4x + 5x^2 + 4x + 4 + 1 = (x + 2)^2 + 1.$$

Now it becomes obvious that the function

$$g(x) = (x + 2)^2 + 1$$

takes only positive values (because, for all real x*,* $(x+2)^2 \geq 0$ *and* $(x + 2)^2 + 1 \geq 1 > 0$*), hence inequalities*

$$x^2 + 4x + 5 \leqslant 0$$

and

$$x^2 + 4x + 5 < 0$$

have no solution, while

$$x^2 + 4x + 5 > 0$$

and

$$x^2 + 4x + 5 \geq 0$$

have the whole real line $\mathbb{R}$ *as solution sets.*

The behaviour of the quadratic function

$$f(x) = (x + 2)^2 - 1$$

is different. We can use the formula

$$u^2 - v^2 = (u + v)(u - v)$$

and factorise

$$\begin{aligned} f(x) &= (x+2)^2 - 1 \\ &= [(x+2)+1] \cdot [(x+2)-1] \\ &= (x+3)(x+1). \end{aligned}$$

We can see now that

$$\begin{array}{lcl} \textit{if} & x < -3 & \textit{then } (x+3)(x+1) > 0 \\ \textit{if} & x = -3 & \textit{then } (x+3)(x+1) = 0 \\ \textit{if} & -3 < x < -1 & \textit{then } (x+3)(x+1) < 0 \\ \textit{if} & -1 < x & \textit{then } (x+3)(x+1) > 0 \end{array}$$

This allows us to solve every inequality

$$\begin{array}{lcl} x^2+4x+3 > 0 & : & x \in]-\infty, -3[\, \cup\,]-1, +\infty[\\ x^2+4x+3 \geq 0 & : & x \in]-\infty, -3] \cup [-1, +\infty[\\ x^2+4x+3 \leqslant 0 & : & x \in [-3, -1] \\ x^2+4x+3 < 0 & : & x \in]-3, -1[\end{array}$$

19.2.3 Completion of square: general case

As we can see, the crucial step of the previous examples is *completion of square*, rewriting a quadratic function $x^2 + bx + c$ as

$$x^2 + bx + c = (x+e)^2 + d$$

where$(x+e)^2$ is always non-negative for all real x, while d is a constant that can be negative, zero, or positive.

We can easily get a formula expressing e and d in terms of b and c. For that purpose, open brackets in the previous formula:

$$x^2 + bx + c = x^2 + 2ex + e^2 + d.$$

We can cancel x^2 from the both sides of the equation and get an equality of linear functions:

$$bx + c = 2ex + (e^2 + d).$$

Hence

$$b = 2e \text{ and } c = e^2 + d.$$

Substituting $e = \frac{b}{2}$ into $c = e^2 + d$, we see that

$$e = \frac{b}{2} \text{ and } d = c - \frac{b^2}{4}.$$

Hence

$$\begin{aligned} x^2 + bx + c &= \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) \\ &\quad \text{which is traditionally written as} \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b^2}{4} - c\right) \end{aligned}$$

As we discovered, the behaviour of solutions sets of inequalities

$$x^2 + bx + c > 0,$$
$$x^2 + bx + c \geq 0,$$
$$x^2 + bx + c \leqslant 0,$$
$$x^2 + bx + c < 0$$

on which of the following is true:

$$\frac{b^2}{4} - c > 0$$

$$\frac{b^2}{4} - c = 0$$

$$\frac{b^2}{4} - c < 0$$

In the literature, usually a slightly different form of this expression is used, which, however, has the same sign:

$$\Delta = b^2 - 4c = 4 \cdot \left(\frac{b^2}{4} - c\right);$$

Δ is called the *discriminant* of the quadratic function

$$y = x^2 + bx + c.$$

20 Linear inequalities in two variables

20.1 Two variables: equations of lines

Every line in the plane with coordinates x and y has an equation of the form

$$ax + by + c = 0.$$

This equation can be rearranged to one of the forms

$$x = C$$

(vertical lines),

$$y = C$$

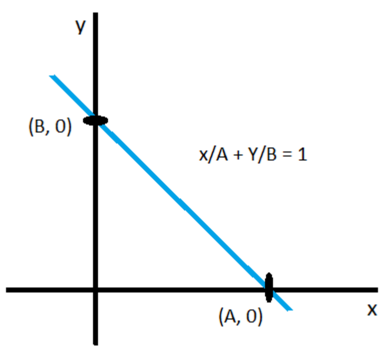
(horizontal lines),

$$y = Cx$$

(lines passing through the origin $O(0,0)$), or

$$\frac{x}{A} + \frac{y}{B} = 1$$

(the so-called **intercept equations**). In the latter case, the points $(A, 0)$ and $(0, B)$ are intersection points of the line with the x-axis and y-axis, respectively, (and are called **intercepts**), and the line given by an intercept equation is easy to plot.



Intercepts

Example 20.1.1 *Equation of a straight line*

$$2x + 3y = 6$$

rewritten in terms of intercepts becomes

$$\frac{x}{3} + \frac{y}{2} = 6.$$

20.2 Linear inequalities in two variables

We shall look at inequalities of the form

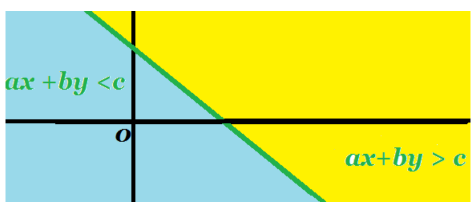
$$ax + by > c$$
$$ax + by \geq c$$
$$ax + by \leqslant c$$
$$ax + by < c$$

where x and y are unknowns (variables) and a, b, c are real coefficients.

Notice that linear inequalities in single variable are special cases of linear inequalities in two variables: if $b = 0$, we have

$$ax > c, \quad ax \geq c, \quad ax \leqslant c, \quad ax < c.$$

The solution set of a linear inequality in two variables x and y is the set of all pairs (x, y) of real numbers which satisfy the inequality. It is natural to represent (x, y) as a point with coordinates x and y in the plane $\mathbb{R}^2$.



The line

$$ax + by = c$$

divides the plane in two *halfplanes*: the one is the solution set of the inequality

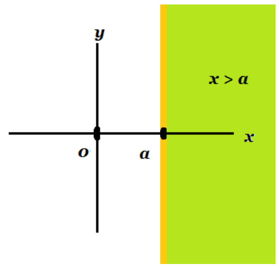
$$ax + by > c$$

another one is the solution set of the inequality

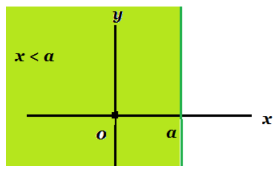
$$ax + by < c$$

The line $ax + by = c$ itself is the *border line* between the two halflines, it *separates* them.

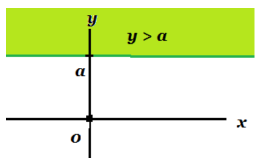
Here is a sample of some more common linear inequalities.



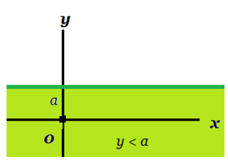
$x > a$



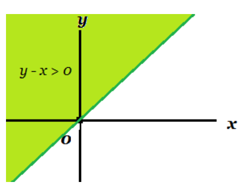
$x < a$



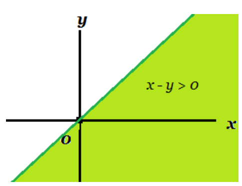
$y > a$



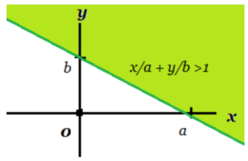
$y < a$



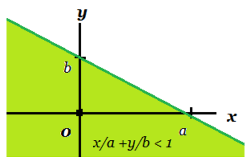
$y - x > 0$



$x - y > 0$



$$\frac{x}{a} + \frac{y}{b} > 1$$



$$\frac{x}{a} + \frac{y}{b} < 1$$

20.3 Systems of simultaneous linear inequalities in two variables

The solution set of a system of inequalities in two variables

$$ax + by > c$$

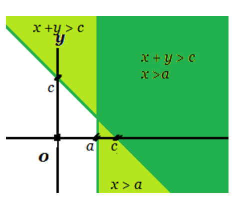
$$dx + ey > f$$

is the inter section of two halfplanes, the solution set of the inequality

$$ax + by > c$$

and of the inequality

$$dx + ey > f$$



The solution set of the system of inequalities $x > a$ and $x + y > c$.

Solution sets of systems of several simultaneous inequalities are intersections of halfplanes. In the examples above in this section halfplanes were *open*, they corresponded to strict inequalities

$$ax + by > c$$

or

$$ax + by < c;$$

and did not contained the border line

$$ax + by + c = 0.$$

Non-strict inequality

$$ax + by \geqslant c$$

or

$$ax + by \leqslant c;$$

correspond to *closed* halfplanes which contain the border line

$$ax + by + c = 0.$$

A system of simultaneous inequalities could combine strict and non-strict inequalities, and the the correspondent solution sets contain some parts of their borders but not others. Try to sketch the solution set of the system

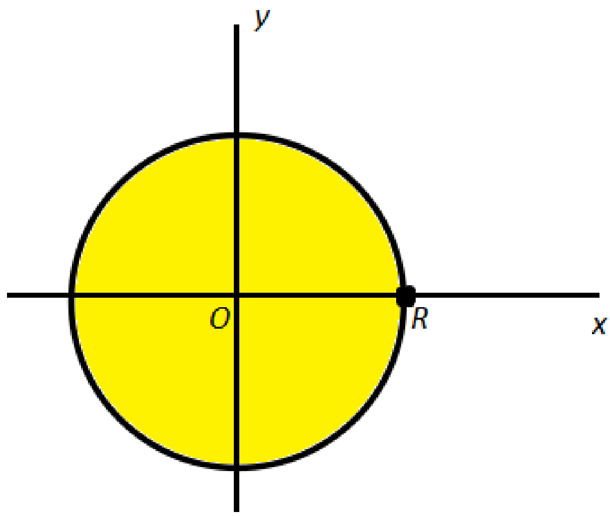
$$\begin{array}{rcl} x & > & 1 \\ x + y & \geqslant & 2 \end{array}$$

and you will see it for yourselves.

20.4 Some quadratic inequalities in two variables

20.4.1 Parabolas

20.4.2 Circles and disks



The solution set of the inequality

$$x^2 + y^2 \leqslant R^2$$

is the circle of radius R centered at the origin $O(0, 0)$.

20.5 Questions from students and some more advanced problems

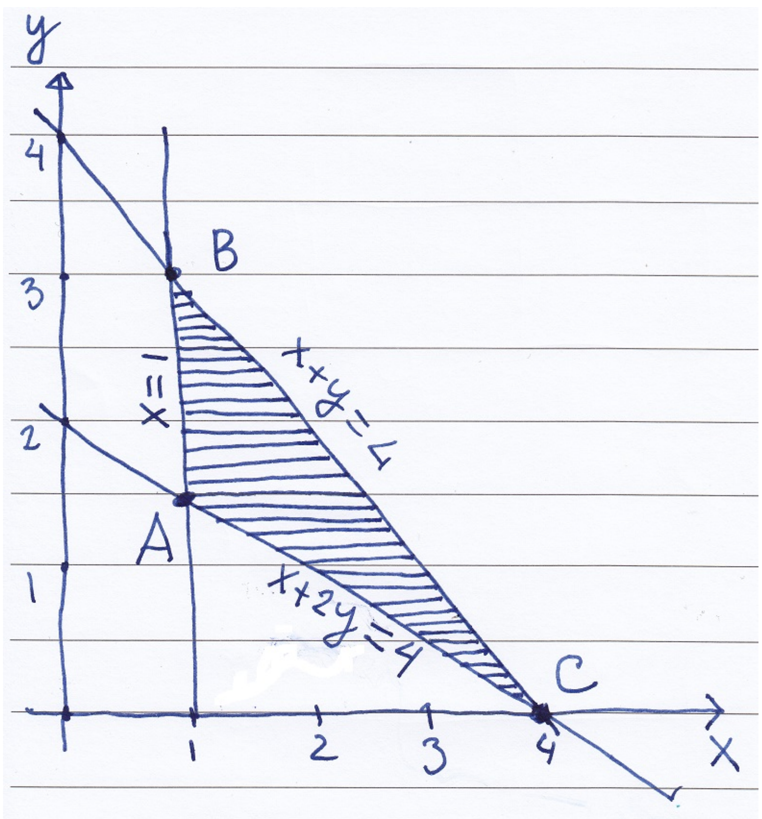
One of the students asked me:

```
> Are we allowed to take a plain sheet
> of graph paper into the 0N1 exam in January?
```

The answer is **NO**. But ruled paper of examination notebooks suffices for crude sketches. Below is an example of such sketch. As you can see, nothing difficult. Actually, it illustrates a problem: the triangle *ABC* is formed by lines

$$\begin{aligned} x &= 1 \\ x+y &= 4 \\ x+2y &= 4 \end{aligned}$$

and therefore points **inside** of the triangle are solutions of the system of simultaneous inequalities



$$\begin{aligned} x &\lessgtr 1 \\ x+y &\lessgtr 4 \\ x+2y &\lessgtr 4 \end{aligned}$$

where, in each case, $\lessgtr$ stands for one of the symbols $<$ and $>$.

Determine which of the signs $<$ or $>$ have to be put in the inequalities.

21 Convexity

21.1 Convexity

A set S in the plane is called **convex** if, with any two points $A, B \in S$, it contains the segment $[A, B]$ connecting the points.

Examples

- A rectangle $[a, b] \times [c, d]$;
- A disc $\{(x, y) : (x - a)^2 + (y - b)^2 \leqslant R^2\}$;
- The set $y > x^2$.

Theorem 21.1 *Intersection of two convex sets is convex.*

PROOF. Assume A and B are convex subsets of $\mathbb{R}^2$. First, notice that if $A \cap B = \emptyset$, there is nothing to prove. Same goes for the case when their intersection is a single point.

Assume now that $(x, y), (x', y') \in A \cap B$. Then by our assumption, the segment $[(x, y), (x', y')]$ lies inside both A and B, whence it lies inside $A \cap B$. Since $(x, y), (x', y')$ are arbitrary, this proves the theorem.

Corollary 21.2 *If $A_1, \ldots, A_n$ are convex, then $A_1 \cap \cdots \cap A_n$ is convex as well.*

PROOF. By induction (last lecture).

Theorem 21.3 *Half planes are convex.*

PROOF. It is obvious from the definition that if a set is convex, any translation and any rotation of this set is convex as well.

Thus, it suffices to show that the half-plane $H = \{(x, y) : y > 0\}$ is convex. (Because any half-plane can be obtained from H via at most one rotation and at most one translation.)

Suppose (x_1, y_1) and (x_2, y_2) lie in H. Without loss of generality we may assume that $0 < y_1 < y_2$.

Hence any point (x_3, y_3) of the segment $[(x_1, y_1), (x_2, y_2)]$ has $y_1 \leqslant y_3 \leqslant y_2$. In particular, $y_3 > 0$, i.e., $(x_3, y_3) \in H$.

Theorem 21.4 *The solution set of a system of homogeneous linear inequalities is convex.*

PROOF. The solution set of each inequality is a half plane, which is convex by Theorem 21.3. The intersection of a finite number of convex sets is convex by Corollary 21.2.

This is no longer true if inequalities are not linear (for example, quadratic): the solution set of

$$y \geqslant x^2$$

is convex, but of

$$y < x^2$$

is not (check!).

Corollary 21.5 *If a system of homogeneous linear inequalities has two distinct solution then it has infinitely many solutions.*

PROOF. Assume (x_1, y_1) and (x_2, y_2) are two distinct solutions.Since the set of solutions is convex, the whole segment $[(x_1, y_1), (x_2, y_2)]$ lies in the set of solutions. This is an infinite set.

22 Principle of Mathematical Induction

22.1 Formulation of the Principle of Mathematical Induction

*

Let $p_1, p_2, p_3, \ldots$ be an infinite sequence of statements, one statement p_n for each positive integer n. For example,

p_1 is " $9^1 - 1$ is divisible by 8"

p_2 is " $9^2 - 1$ is divisible by 8"

p_3 is " $9^3 - 1$ is divisible by 8"

so for each positive integer n, p_n is the statement

p_n is " $9^n - 1$ is divisible by 8".

Suppose that we have the following information

(1) p_1 is true.

(2) The statements

$$p_1 \to p_2, \quad p_2 \to p_3, \quad p_3 \to p_4, \quad p_4 \to p_5 \ldots$$

are all true, i.e.

$$p_k \to p_{k+1}$$

is true for each positive integer k.

Then we can deduce

p_1 is true and $p_1 \to p_2$ is true implies p_2 is true,

p_2 is true and $p_2 \to p_3$ is true implies p_3 is true,

p_3 is true and $p_3 \to p_4$ is true implies p_4 is true,

that is,

$p_1, p_2, p_3, \ldots$ are *all* true i.e.

p_n is true for all n.

* Recommended additional (but not compulsory) reading: Richard Hammack, *Book of Proof*, Chapter 10.

22.2 Examples

Example 22.2.1 *Prove, by induction on n, that*

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for every positive integer n.

Solution. For each positive integer n, p_n denotes the statement

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

In particular,

$$\begin{array}{llrcll}
p_1 & \text{is} & 1 & = & 1^2 & \mathbb{T} \\
p_2 & \text{is} & 1+3 & = & 2^2 & \mathbb{T} \\
 & & \vdots & & \vdots & \\
p_k & \text{is} & 1+3+\cdots+(2k-1) & = & k^2 & \\
p_{k+1} & \text{is} & 1+3+\cdots+(2k-1)+(2k+1) & = & (k+1)^2 &
\end{array}$$

- p_1 is the statement "$1 = 1^2$" which is clearly true.
- Suppose the statement p_n is true for $n = k$, i.e.

  $$1 + 3 + \cdots + (2k - 1) = k^2.$$

  Add $(2(k + 1) - 1) = 2k + 1$ to both sides:

  $$\begin{aligned} 1 + 3 + \cdots + (2k - 1) + (2k + 1) &= k^2 + (2k + 1) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2. \end{aligned}$$

  But this is the statement p_n for $n = k$ as required.

Hence, by mathematical induction, p_n is true for all n. $\square$

Example 22.2.2 ***(Examination of January 2007).*** *Let p_1 denote the statement*

$$\frac{1}{2} = 1 - \frac{1}{2};$$

furthermore, for each positive integer n, let p_n denote the statement

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}\,.$$

Prove, by induction, that p_n is true for all n.

Solution. BASIS OF INDUCTION is the statement p_1,

$$\frac{1}{2} = 1 - \frac{1}{2}\,;$$

it is obviously true.

INDUCTIVE STEP: We need to prove that $p_k \to p_{k+1}$ for all k. To do that, assume that p_k is true, that is,

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}\,.$$

Form this identity, we need to get p_{k+1}. This is achieved by adding

$$\frac{1}{2^{k+1}}$$

to the both sides of the equality p_k:

$$\left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}\right) + \frac{1}{2^{k+1}} = \left(1 - \frac{1}{2^k}\right) + \frac{1}{2^{k+1}}\,.$$

But the righthand side simplifies as

$$\begin{aligned}
1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} &= 1 - \frac{2}{2 \cdot 2^k} + \frac{1}{2^{k+1}} \\
&= 1 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}} \\
&= 1 - \left(\frac{2}{2^{k+1}} - \frac{1}{2^{k+1}}\right) \\
&= 1 - \frac{2-1}{2^{k+1}} \\
&= 1 - \frac{1}{2^{k+1}}
\end{aligned}$$

and the result of this rearrangement is

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}\,,$$

which is exactly the statement p_{k+1}. This completes the proof of the inductive step. □

23 Mathematical Induction: Examples with briefer solutions

23.1 The sum of arithmetic progression

Example 23.1.1 *Prove by induction on n that*

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$$

for every positive integer n.

Solution.

Let p_n be the statement "$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$".

p_1 is the statement "$1 = \frac{1}{2} \times 1 \times 2$". This is clearly true.

Suppose p_n is true for $n = k$, i.e. $1+2+\cdots+k = \frac{1}{2}k(k+1)$. Then

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= (1 + 2 + \cdots + k) + (k+1) \\ &= \frac{1}{2}k(k+1) + (k+1) \\ &= \frac{1}{2}k(k+1) + \frac{1}{2}2(k+1) \\ &= \frac{1}{2}(k+1)(k+2). \end{aligned}$$

Thus

$$1 + 2 + \cdots + k + (k+1) = \frac{1}{2}(k+1)((k+1)+1).$$

Therefore p_n is true for $n = k+1$. By induction, p_n is true for all n. □

23.2 A historic remark

*

* Material of this section is not compulsory

There is a famous legend about Carl Friedrich Gauss (1777–1855), one of the greatest mathematicians of all time.

The story goes that, in school, at the age of 8, his teacher set up a task to his class: add up the first 100 natural numbers,

$$1 + 2 + 3 + 4 + \cdots + 9 + 100.$$

It is frequently claimed that the teacher used this trick many times to keep the class busy for long periods while he took a snooze.

Unfortunately for the teacher, young Gauss instantly produced the answer: 5050. He observed that if the same sum is written in direct and reversed orders:

$$\begin{array}{ccccccccccccc} S & = & 1 & + & 2 & + & 3 & + & \cdots & + & 99 & + & 100 \\ S & = & 100 & + & 99 & + & 98 & + & \cdots & + & 2 & + & 1 \end{array}$$

then each of 100 columns at the RHS sums up to 101:

$$\begin{array}{ccccccccccccc} S & = & 1 & + & 2 & + & 3 & + & \cdots & + & 99 & + & 100 \\ \underline{S} & = & \underline{100} & + & \underline{99} & + & \underline{98} & + & \cdots & + & \underline{2} & + & \underline{1} \\ 2S & = & 101 & + & 101 & + & 101 & + & \cdots & + & 101 & + & 101 \end{array}$$

and therefore

$$2S = 100 \times 101$$

and

$$S = 50 \times 101 = 5050.$$

Of course, we can repeat the same for arbitrary positive integer n:

$$\begin{array}{ccccccccccccc} S = & 1 & + & 2 & + & 3 & + & \cdots & + & n-1 & + & n \\ S = & n & + & n-1 & + & n-2 & + & \cdots & + & 2 & + & 1 \end{array}$$

Then each of $1n$ columns at the RHS sums up to $n+1$, and therefore

$$2S = n \cdot (n+1)$$

and

$$S = \frac{n(n+1)}{2},$$

thus proving the formula

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$$

for every positive integer n – without the use of mathematical induction.

Many problems which can be solved by mathematical induction can also be solved by beautiful tricks like that, each trick specifically invented for a particular problem. But mathematical induction has the advantage of being a general method, applicable, with some slight modification, to a vast number of problems.

Figure 7: German 10-Deutsche Mark Banknote (1993; discontinued). Source: WIKIPEDIA.

If this clever summation was the only mathematical achievement of little Carl, he would not be known to us, the unit for measurement of a magnetic field (in the centimeter / gram / second system) would not be called *gauss*, and his portrait would not be on banknotes—see Figure 7. But Gauss did much more in mathematics, statistics, astronomy, physics.

Remarkably, WIKIPEDIA gives the names of Gauss' teacher, J. G . Büttner, and the teaching assistant, Martin Bartels. Perhaps Carl's teachers were not so bad after all – especially after taking into consideration that Bartels (1769–1836) later became a teacher of another universally acknowledged genius of mathematics, Nikolai Lobachevsky (1792–1856).

If you find this story interesting, please consider a career in teaching of mathematics.

The humanity needs you.

23.3 Mathematical induction in proofs of inequalities

Example 23.3.1 *Prove, by induction on n, that $n < 2^n$ for every positive integer n.*

Solution. BASIS OF INDUCTION, $n = 1$:

$$1 < 2^1$$

is obviously true.

INDUCTIVE STEP. Assume that, for some $k > 1$,

$$k < 2^k$$

is true. Since $1 < k$ by assumption, we also have

$$1 < 2^k.$$

Add the two inequalities together:

$$k + 1 < 2^k + 2^k = 2^{k+1}.$$

This proves the inductive step. □

Example 23.3.2 *Let $x \geqslant -1$ and n a natural number. Prove that*

$$(1 + x)^n \geqslant 1 + nx.$$

Solution. BASIS OF INDUCTION, $n = 1$:

$$1 + x \geqslant 1 + x$$

is true.

INDUCTIVE STEP. Assume that, for some $k > 1$,

$$(1 + x)^k \geqslant 1 + kx$$

is true. Since $x \geqslant -1$, we have $1 + x \geqslant 0$ and we can multiply the both sides of the inequality by $1 + x$:

$$(1+x)^k(1+x) \geqslant (1+kx)(1+x).$$

But

$$(1+x)^k(1+x) = (1+x)^{k+1},$$

while

$$\begin{aligned}(1+kx)(1+x) &= 1 + x + kx + kx^2 \\ &= (1+(k+1)x) + kx^2 \\ &\geqslant 1 + (k+1)x \quad (\text{since} \quad kx^2 \geqslant 0).\end{aligned}$$

Combining these equality and inequality together, we get

$$(1+x)^{k+1} \geqslant 1 + (k+1)x,$$

which proves the inductive step. □

23.4 A paradox

As a warning of one frequent mistake, consider the following "proof" of an obviously false statement.

Example 23.4.1 *Prove by induction on n, that in any sequence of n real numbers*

$$\{ x_1, x_2, x_3, \ldots, x_n \}$$

are equal:

$$x_1 = x_2 = \cdots = x_{n-1} = x_n.$$

Solution.

BASIS OF INDUCTION is obvious: when $n = 1$, the sequence of one number x_1 is just this number, and the number is equal to itself: $x_1 = x_1$.

INDUCTIVE STEP. Assume that the statement is true for $n = k$, and consider a sequence of $k + 1$ numbers:

$$x_1, x_2, \ldots, x_k, x_{k+1}.$$

In the sequence of *first* k numbers all numbers are equal by inductive assumption:

$$x_1 = x_2 = \cdots = x_k.$$

In the sequence of last k numbers all numbers are also equal by inductive assumption:

$$x_2 = \cdots = x_k = x_{k+1}.$$

Hence

$$x_1 = x_2 = \cdots = x_k = x_{k+1},$$

which proves the inductive step.

It appears that we proved the statement. WHAT'S THE CATCH?

EXPLANATION. Let us look at this "proof" with a bit more attention.

Let p_n be the statement

> In any sequence of n real numbers
>
> $$\{\, x_1, x_2, x_3, \ldots, x_n \,\}$$
>
> are equal:
>
> $$x_1 = x_2 = \cdots = x_{n-1} = x_n.$$

We need to prove

$$p_1,\ p_1 \to p_2,\ p_2 \to p_3,\ \ldots, p_k \to p_{k+1}, \ldots$$

We proved p_1 (basis of induction). Let us look at the proof of $p_1 \to p_2$:

> In the sequence x_1, x_2, consider the sequence x_1 formed by the first number and sequence x_2 formed by the last number. Then ...

And then nothing follows – the two numbers x_1 and x_2 are not related to each other. We cannot prove $p_1 \to p_2$. However, the proof of inductive step is valid for proving

$$p_2 \to p_3,\ p_3 \to p_4, \ldots$$

Indeed let us look at proof of

$$p_2 \to p_3$$

> Given sequence x_1, x_2, x_3, we have, by the inductive assumption,
>
> $$x_1 = x_2 \text{ and } x_2 = x_3.$$
>
> Hence $x_1 = x_2 = x_3$.

The same arguments work for longer sequences: what matters is that the two sequences $x_1, \ldots, x_k$ and $x_2, \ldots x_{k+1}$ have terms (numbers) in common.

So our conclusion is that just the first inductive step

$$p_1 \to p_2$$

of infinitely many steps that fails – and this destroys the entire proof. It is like a chain: one link is broken means the whole chain is broken. □

23.5 Problems

Problem 23.1 *Prove that, for all natural numbers n,*

$$1 \times 1! + 2 \times 2! + \cdots + n \times n! = (n+1)! - 1.$$

Problem 23.2 *Prove that for every integer $n > 1$*

$$1^1 \cdot 2^2 \cdot 3^3 \cdots n^n < n^{n(n+1)/2}.$$

Problem 23.3 *For which natural numbers n we have this inequality:*

$$2^n > n^3?$$

Problem 23.4 *Prove that, for all natural numbers n,*

$$3^n > n \cdot 2^n.$$

Problem 23.5 *Prove that, for all natural numbers n,*

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

Problem 23.6 *Prove that, for all integers n,*

$$\sum_{k=n}^{2n} k = 3 \sum_{k=1}^{n} k,$$

that is,

$$n + (n+1) + \cdots + (2n-1) + 2n = 3 \cdot (1 + 2 + \cdots + n).$$