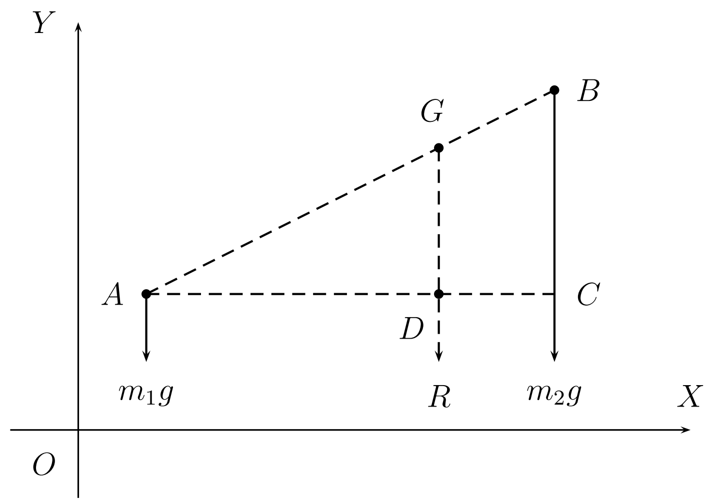
0J2 - Mechanics   Lecture Notes 7

Centre of Mass

For two particles of masses m_1 and m_2 at $A(x_1, y_1)$ and $B(x_2, y_2)$ we have two weights $m_1 g$ and $m_2 g$ as shown



The resultant weight is $R = m_1 g + m_2 g$. (We only need the magnitudes here since the weight and the vector resultant $\mathbf{R}$ are all in the same (vertical) direction.

The line of action of the resultant passes through G on the line AB. Let the coordinates of G be $(\bar{x}, \bar{y})$, so the line of action of R crosses the X-axis at $(\bar{x}, 0)$.

The moment of R about the origin is thus $R\,\bar{x} = (m_1 g + m_2 g)\bar{x}$.

This must equal the sum of the moments of $m_1 g$ and $m_2 g$ about the origin so

$$(m_1 g + m_2 g)\bar{x} = m_1 g x_1 + m_2 g x_2 \quad \Rightarrow \quad \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

It can also be shown that

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}.$$

To prove this find the equation of AB. Since G lies on this line, $\bar{y}$ is the y-coordinate corresponding to the x-coordinate $\bar{x}$ on the line. (Not done here – rather long).

The line ADC on the diagram is horizontal and passes through A.

Note that the point G divides the line AB in the ratio $AG : GB$. Since triangles ADG and ACB are similar (i.e. have the same angles)

$$\frac{AG}{GB} = \frac{AD}{DC} = \frac{(\bar{x} - x_1)}{(x_2 - \bar{x})}$$

The numerator of this is

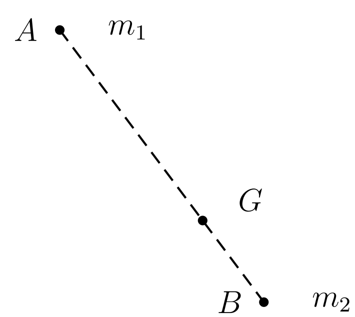
$$\bar{x} - x_1 = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} - x_1 = m_2 \left(\frac{x_2 - x_1}{m_1 + m_2} \right)$$

and the denominator is

$$x_2 - \bar{x} = x_2 - \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = m_1 \left(\frac{x_2 - x_1}{m_1 + m_2} \right)$$

and thus $\dfrac{AG}{GB} = \dfrac{m_2}{m_1}$. Therefore the point G divides the line AB in the ratio $m_2 : m_1$.

Now take the same two masses, the same distance apart but with a different orientation



By exactly the same argument as before the line of action of the resultant still goes through G, which divides the line in the ratio $m_2 : m_1$.

Since the line of action of the resultant always goes through G we can simply replace the two masses m_1 and m_2 by a single particle of mass $m_1 + m_2$ located at G.

We say the $G(\bar{x}, \bar{y})$ is the Centre of Mass (CoM) or Centre of Gravity (CoG) of the two particles.

Aside: In practice the CoM and CoG are the same. If the gravitational constant g were different at A and B they would not be the same but this never happens in practice.

(End of aside).

Special Case – Equal masses

If the masses are equal, i.e. $m_2 = m_1$, then the point G divides the line AB in the ratio $m_2 : m_1 \quad \Rightarrow \quad m_1 : m_1$.
In other words it divides it equally so the CoM G is at the mid-point of the line AB.

We can also see this from the coordinates of G:

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_1 x_2}{m_1 + m_1} = \frac{(x_1 + x_2)}{2}.$$

Similarly

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{m_1 y_1 + m_1 y_2}{m_1 + m_1} = \frac{(y_1 + y_2)}{2}.$$

so clearly $(\bar{x}, \bar{y})$ is the mid point of $A(x_1, y_1)$ and $B(x_2, y_2)$.

1.1 Generalisation

If we now add mass m_3 at point (x_3, y_3) we find that the new centre of mass G has coordinates

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}.$$

For n particles the result is

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{M}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{M},$$

where $M = \sum_{i=1}^{n} m_i$ which is the total mass of all the particles.

(The symbol $\sum_{i=1}^{n}$ means the sum over all values of i from 1 to n.)

In 3D we would also have a z-coordinate of the centre of mass given by

$$\bar{z} = \frac{\sum_{i=1}^{n} m_i z_i}{M}.$$

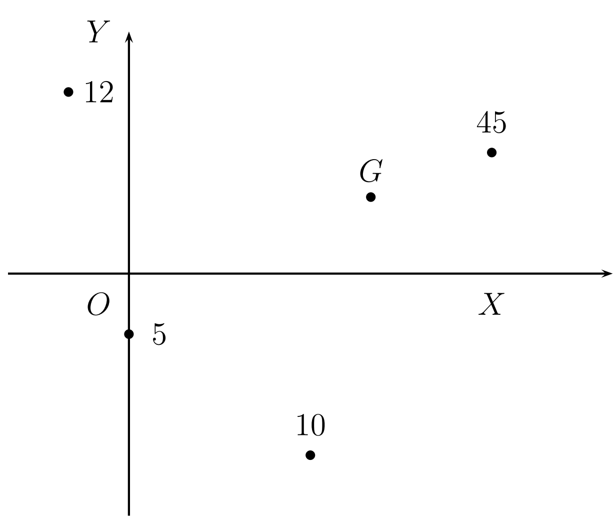
Example:

Four particles with different masses are located at different points in a 2D plane. They are:

1. Mass 45 kg at (6,2)
2. Mass 12 kg at (-1,3)
3. Mass 10 kg at (3,-3)
4. Mass 5 kg at (0,-1)

Find the centre of mass G of this system.

so the Centre of Mass is at $(4, 1.264)$.

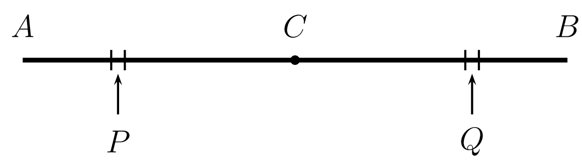


Note that G is closest to the 45 kg mass since this is the largest.

1.2 2D laminar bodies

Since any 2D body (i.e. a lamina) can be regarded as made up of a large number of particles we can use the above results to find the centre of mass.

Centre of mass of a heavy uniform rod



Divide the rod into small equal segments. Let P and Q be two of these, equal distance from the centre of the rod C. Since the masses of the segments are equal the centre of mass of these two segments is at the mid-point i.e. at C.

Similarly for all other pairs of segments, so the centre of mass of the whole rod is at C.

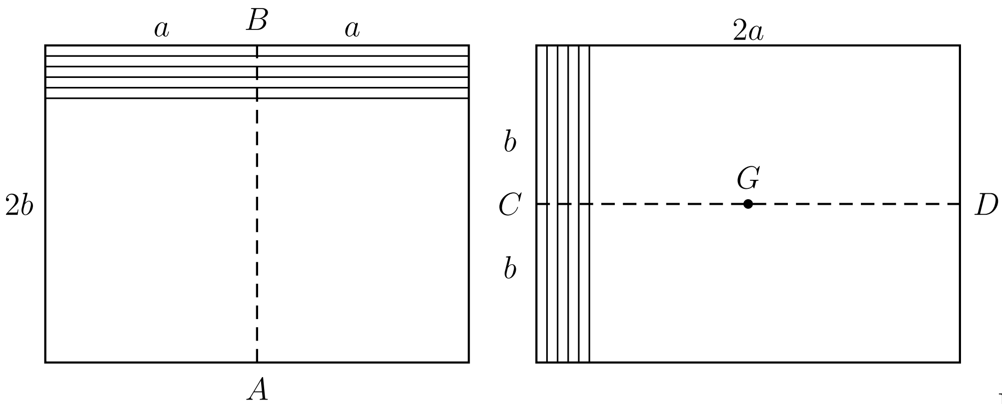
Centre of mass of symmetric 2D laminas

For the rod above we made use of the symmetry about the centre C. We can often use symmetry to find the C of M of simple 2D bodies i.e.laminas.

If there is an axis of symmetry then the C of M must lie on this axis.

If there are two axes of symmetry then the C of M will be at the point of intersection of these two axes.

Lamina 1: A rectangle with sides $2a$ and $2b$

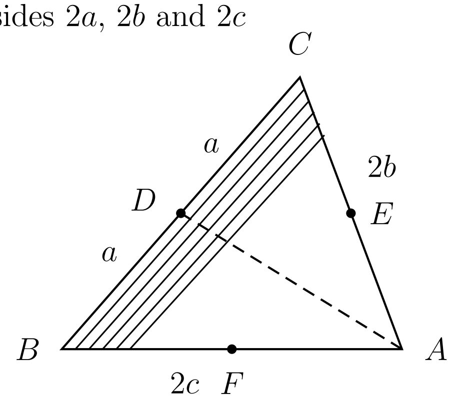


Divide the rectangle into rods of length $2a$ as shown on the left. The C of M of each rod lies at the mid-point, i.e. on the line AB, which is an axis of symmetry.

Similarly, dividing the rectangle into vertical rods of length $2b$ as shown on the right, we see that the C of M lies on the line CD, another axis of symmetry.

Therefore the C of M of the whole rectangle is at G, the centre of both lines AB and CD, in other words at the centre of the rectangle.

Lamina 2: A triangle with sides $2a$, $2b$ and $2c$



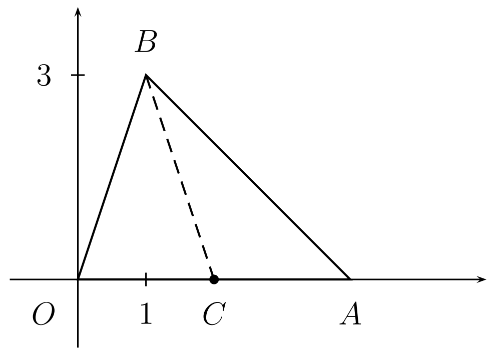
Divide into rods parallel to side BC as shown. The centre of each rod lies on the line AD where D is the mid-point of side BC. AD is called a <u>median</u> of the triangle.

Thus the C of M of the triangle lies on the median AD.

Similarly it also lies on the other two medians BE and CF. Thus the C of M of a triangle is at the point of intersection of the three medians.

It can be shown that the the medians of any triangle always intersect at a point which is 1/3 of the way from the mid-point of the side to the corner. Thus the C of M lies 1/3 of the way from D to A.

Example: Find the centre of mass of a uniform triangular lamina with corners at $O(0,0)$, $A(4,0)$ and $B(1,3)$.

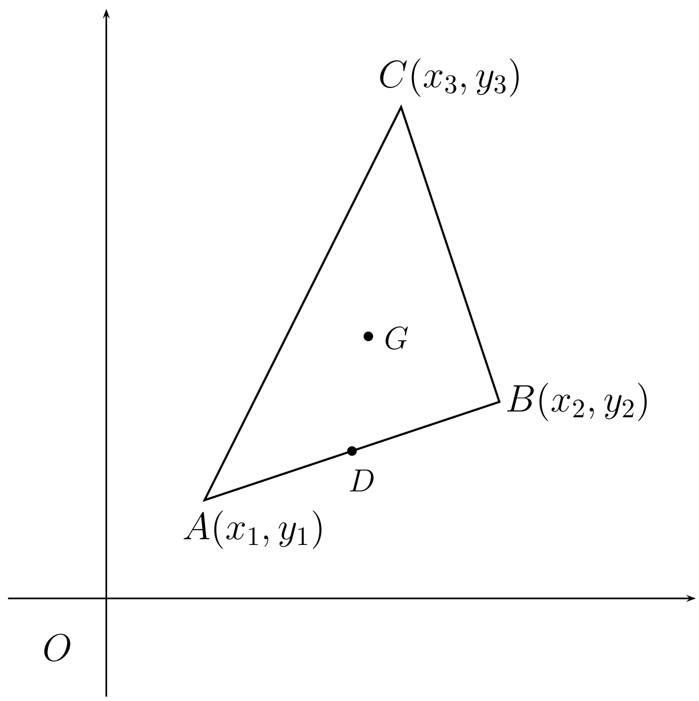


Let C be the mid-point of OA. The coordinates are $(2,0)$. The line BC is a median.

so the coordinates of the C of M are $(5/3, 1)$.

An alternative way of finding the point of intersection of the medians is as follows.

Let the coordinates of the the three corners be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$



D is the mid point of AB so it has coordinates $D(\frac{(x_1+x_2)}{2}), \frac{(y_1+y_2)}{2})$ and position vector

$$\vec{OD} = \frac{(x_1 + x_2)}{2}\mathbf{i} + \frac{(y_1 + y_2)}{2}\mathbf{j}$$

Also

$$\begin{aligned}\vec{DC} &= \left(x_3 - \frac{(x_1 + x_2)}{2}\right)\mathbf{i} + \left(y_3 - \frac{(y_1 + y_2)}{2}\right)\mathbf{j} \\ &= \frac{(2x_3 - x_1 - x_2)}{2}\mathbf{i} + \frac{(2y_3 - y_1 - y_2)}{2}\mathbf{j}\end{aligned}$$

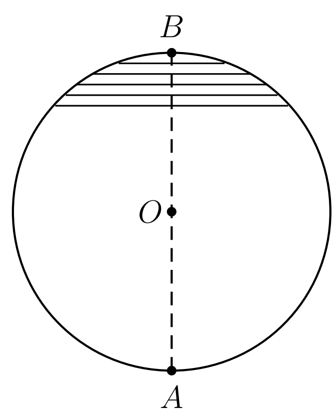
Let the centre of mass be $G(\bar{x}, \bar{y})$, where G is 1/3 of the way from D to C.

This means that the position vector of G is

$$\begin{aligned}\vec{OG} &= \vec{OD} + (1/3)\,\vec{DC} \\ &= \frac{(x_1+x_2)}{2}\mathbf{i} + \frac{(y_1+y_2)}{2}\mathbf{j} + (1/3)\left(\frac{(2x_3-x_1-x_2)}{2}\,\mathbf{i} + \frac{(2y_3-y_1-y_2)}{2}\,\mathbf{j}\right) \\ &= \frac{(x_1+x_2+x_3)}{3}\mathbf{i} + \frac{(y_1+y_2+y_3)}{3}\mathbf{j}\end{aligned}$$

so the coordinates of G are the averages of the coordinates of the three corners.

Lamina 3: A circle.

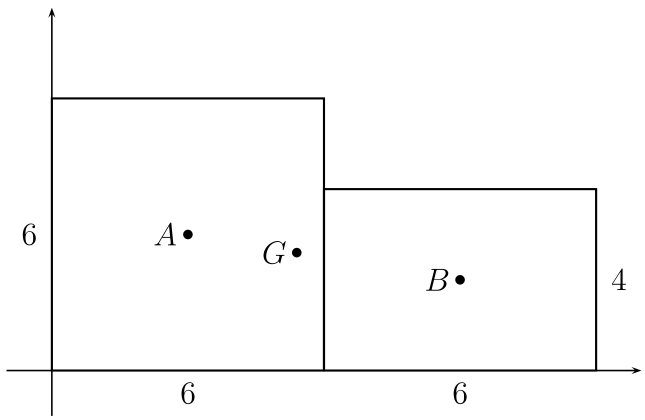


Divide into rods as shown. The centre of each rod lies on the line AOB which is the diameter of the circle perpendicular to the rods. O is the centre of the circle. Clearly the centre of mass of the whole must lie on AOB. An exactly similar argument would apply if we chose a different orientation of the rods, e.g. vertical, and the centre of mass of the circle must lie on a different diameter. Since it lies on two different diameters (and in fact on any diameter) the centre of mass of the whole circle must lie at the point where all the diameters intersect. i.e. at the centre of the circle O. (as expected!)

1.3 Composite bodies

Two or more symmetric bodies joined together can be treated by finding the centre of mass, and also the mass, of each element and then thinking of each element as a particle of that mass at the centre of mass of the element. We then find the centre of mass of the 'particles'.

Example: A uniform lamina consists of a square of side 6 m with corners at $(0,0)$, $(6,0)$, $(6,6)$ and $(0,6)$ joined to a rectangle with corners $(6,0)$, $(12,0)$, $(12,4)$ and $(6,4)$ as shown. The mass/unit area is ρ in both parts. Find the position of the centre of mass.

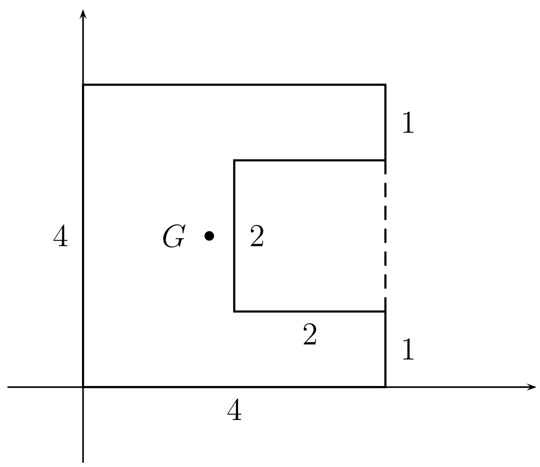


Area of the square is 36, so mass is 36ρ and the centre of mass is at $A(3,3)$.
Area of the rectangle is 24, so mass is 24ρ and the centre of mass is at $B(9,2)$.
Treat these as two particles. The centre of mass is at $G(\bar{x},\bar{y})$ where

so the centre of mass is at $G(5.4, 2.6)$.

Cutouts can be regarded in the same way but with one piece having negative mass.

Example 1: A uniform lamina consists of a square of side 4 m with corners at $(0,0)$, $(4,0)$, $(4,4)$ and $(0,4)$ has a square of side 2 m with corners at $(2,1)$, $(4,1)$, $(4,3)$ and $(2,3)$ removed as shown. The mass/unit area is ρ. Find the centre of mass.



Area of the large square is 16, so mass is 16ρ and the centre of mass is at $(2,2)$.
Area of the cut out square is 4, so 'mass' is -4ρ and the centre of mass is at $B(3,2)$.
Treat these as two particles. The centre of mass is at $G(\bar{x},\bar{y})$ where

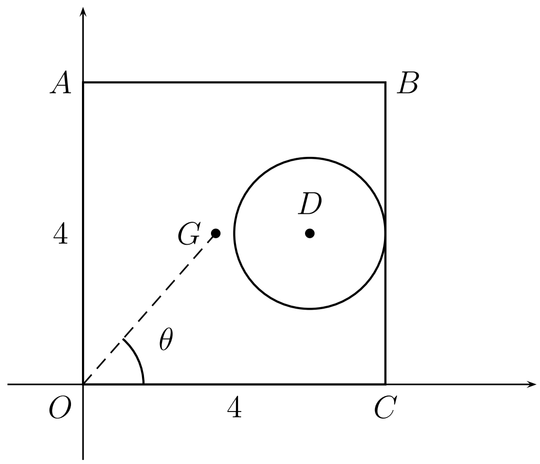
$$\bar{x} \;=\; \frac{m_1x_1+m_2x_2}{m_1+m_2} \;=\; \frac{16\rho\times 2+(-4\rho)\times 3}{16\rho+(-4\rho)} \;=\; \frac{5}{3}\,.$$

Similarly

$$\bar{y} \;=\; \frac{m_1y_1+m_2y_2}{m_1+m_2} \;=\; \frac{16\rho\times 2+(-4\rho)\times 2}{16\rho+(-4\rho)} \;=\; 2\,.$$

so the centre of mass is at $G(5/3,2)$.

Example 2: A uniform lamina consists of a square of side 4 m with corners at $O(0,0)$, $A(0,4)$, $B(4,4)$ and $C(4,0)$ has a circle of radius 1 m with centre at $D(3,2)$ removed as shown. The mass/unit area is ρ. Find the centre of mass G.



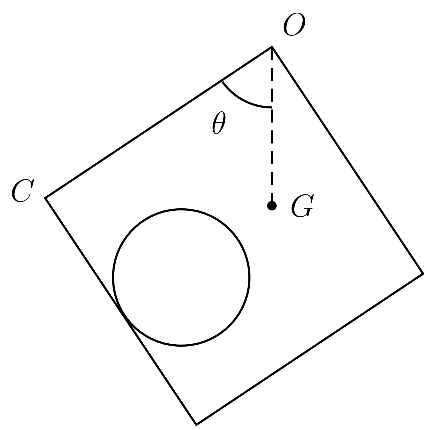
Area of the square is 16, so mass is 16ρ and the centre of mass is at $(2,2)$.
Area of the cut out circle is $\pi r^2 = \pi$, so 'mass' is $-\pi\rho$ and the centre of mass is at $D(3,2)$.
Treat these as two particles. The centre of mass is at $G(\bar{x},\bar{y})$ where

so the centre of mass is at $G(1.7557, 2)$.

Hanging Bodies

If a body is freely suspended at some point P so as to hang in equilibrium under gravity then the centre of mass G must be directly below P. If this were not so then the moment of the weight about P would not be zero and the body would turn. (G could be directly above P but this would not be stable). Hence we can find the equilibrium position of a hanging body.

Example: The body in the previous example is freely suspended at point O. Find the angle that the line OC will make with the vertical.



G is directly below O.

In the previous diagram OC is along the X-axis and G has the coordinates $(1.7557, 2)$. Thus the angle θ between OG and OC is given by

$$\tan\theta = \frac{2}{1.7557} \quad \Rightarrow \quad \theta = 48.7^\circ$$

N.B. The material from here on will not be examined and will only be covered in the lectures if there is time.

Centres of Mass (Gravity) in 2D using integration

If a uniform lamina in 2D has straight edges we can always divide it into triangular regions. We can then calculate the Centre of Mass (CoM) for each of these and their masses. We then treat them as a set of particles and combine them using the method for composite bodies to find the CoM of the whole lamina.

However if one or more of the edges is curved this does not work and we have to use integration to find the CoM.

(The special case of the circle that we did earlier only works because of the high degree of symmetry of the circle.)

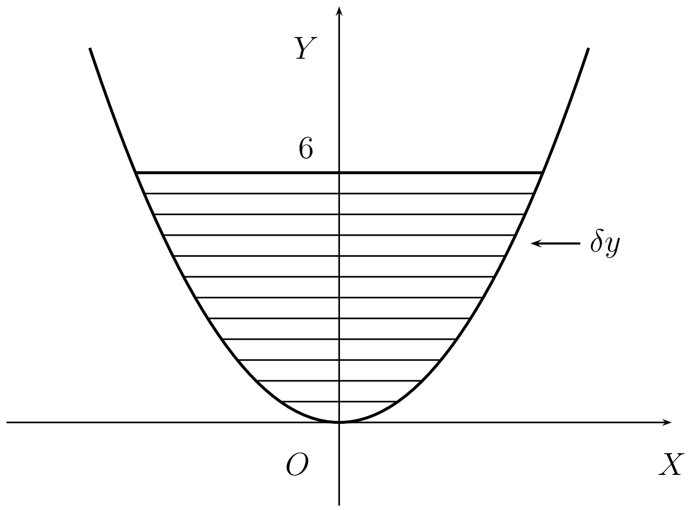
The method is rather similar to the method we used for the rectangle and triangle.
We divide the lamina into small strips of equal width. The centre of mass of each strip is the mid-point. We then combine these to find the CoM of the whole lamina.

The difference now is that each strip has infinitesimal width and there are infinitely many of them. The combination (summation) of these is then done by integrating.

We shall show the method by means of two examples.

1. parabola
2. semicircle

Find C of M $(\bar{x}, \bar{y})$ of a uniform lamina bounded by the parabola $y = x^2$ and the line $y = 6$.



Let the mass per unit area be ρ.

Divide into horizontal strips of thickness δy.

The strip whose Y-coordinate is y stretches from $-\sqrt{y}$ to $\sqrt{y}$ and so has a length $2\sqrt{y}$. It has an area $2\sqrt{y}\delta y$ and so a mass $\delta m = 2\rho\sqrt{y}\delta y$.

This strip has its centre of mass at $(0, y)$

Clearly all the strips have a C of M on the Y-axis, so the C of M of the whole lies on the Y-axis so $\bar{x} = 0$.

To find $\bar{y}$ we use the formula

$$\bar{y} = \frac{\Sigma_i y_i m_i}{\Sigma_i m_i}$$

where $m_i = \delta m$, the mass of the strip
and $y_i = y$, the Y-coordinate of the strip.

This gives

$$\bar{y} = \frac{\Sigma\, y \times 2\rho\sqrt{y}\,\delta y}{\Sigma\, 2\rho\sqrt{y}\,\delta y} = \frac{\Sigma\, y \times \sqrt{y}\,\delta y}{\Sigma\, \sqrt{y}\,\delta y}$$

We now take the limit δy becomes infinitesimal and replace $\delta y \to dy$ and $\Sigma \to \int$.

Consider the denominator first:

$$\Sigma \sqrt{y}\, \delta y \to \int \sqrt{y}\, dy \;=\; \int y^{1/2}\, dy$$

and the limits are the starting and finishing values of y: 0 and 6.

Hence the denominator is

$$\int_0^6 y^{1/2}\, dy \;=\; \left[\frac{2y^{3/2}}{3}\right]_0^6 \;=\; \frac{2 \times 6\sqrt{6}}{3} \;=\; 4\sqrt{6}$$

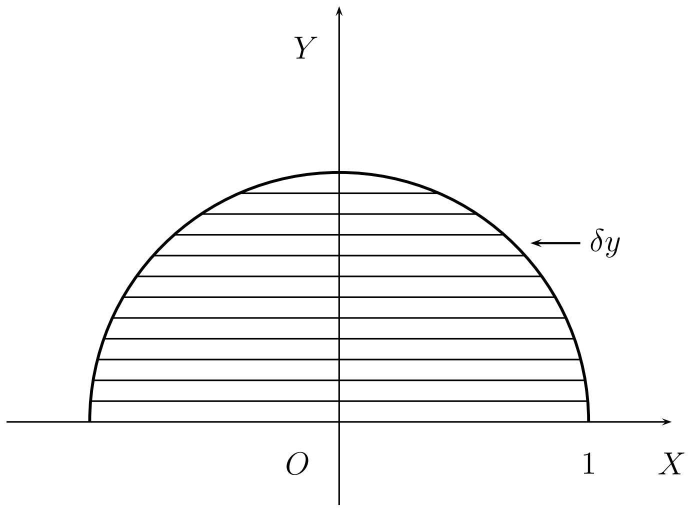
The numerator is exactly the same but with an extra y

$$\int_0^6 y^{3/2}\, dy \;=\; \left[\frac{2y^{5/2}}{5}\right]_0^6 \;=\; \frac{2 \times 36\sqrt{6}}{5} \;=\; \frac{72\sqrt{6}}{5}$$

Finally

$$\bar{y} \;=\; \frac{\text{numerator}}{\text{denominator}} \;=\; \frac{72\sqrt{6}}{5 \times 4\sqrt{6}} \;=\; \frac{72}{20} \;=\; 3.6$$

Find C of M $(\bar{x}, \bar{y})$ of a uniform lamina which lies above the X-axis and the within the semicircle $y^2 + x^2 \ = \ 1$. (The method is the same as in the previous example.)



Let the mass per unit area be ρ.

Divide into horizontal strips of thickness δy.

The strip whose Y-coordinate is y stretches from $-\sqrt{1-y^2}$ to $+\sqrt{1-y^2}$ and so has a length $2\sqrt{1-y^2}$. It has an area $2\sqrt{1-y^2}\delta y$ and so a mass $\delta m \ = \ 2\rho\sqrt{1-y^2}\delta y$.

This strip has its centre of mass at $(0, y)$

Clearly all the strips have a C of M on the Y-axis, so the C of M of the whole lies on the Y-axis so $\bar{x} \ = \ 0$.

As before, to find $\bar{y}$ we use the formula $\bar{y} \ = \ \dfrac{\Sigma_i y_i m_i}{\Sigma_i m_i}$

where $m_i \ = \ \delta m$, the mass of the strip
and $y_i \ = \ y$, the Y-coordinate of the strip.

This gives

$$\bar{y} \ = \ \frac{\Sigma\, y \times 2\rho\sqrt{1-y^2}\,\delta y}{\Sigma\, 2\rho\sqrt{1-y^2}\,\delta y} \ = \ \frac{\Sigma\, y \times \sqrt{1-y^2}\,\delta y}{\Sigma\, \sqrt{1-y^2}\,\delta y}$$

We now take the limit δy becomes infinitesimal and replace $\delta y \to dy$ and $\Sigma \to \int$.

Consider the denominator first:

$$\Sigma \sqrt{1-y^2}\,\delta y \to \int \sqrt{1-y^2}\,dy$$

and the limits are the starting and finishing values of y: 0 and 1.

Hence the denominator is

$$\int_0^1 \sqrt{1-y^2}\,dy$$

This definite integral is done by substitution. Put $y = \sin z$ then $\sqrt{1-y^2} = \sqrt{1-\sin^2 z} = \cos z$. Also

$$dy = \frac{dy}{dz}\,dz = \cos z\,dz.$$

And finally we need to change the limits.
When $y = 0$, $z = 0$ and when $y = 1$, $z = \pi/2$.

The denominator is now

$$\int_0^{\pi/2} \cos z\ \cos z\,dz = \frac{1}{2}\int_0^{\pi/2} [\cos 2z + 1]\,dz$$

$$= \frac{1}{2}\left[\frac{\sin 2z}{2} + z\right]_0^{\pi/2} = \frac{1}{2}\{[0+\pi/2] - [0+0]\} = \pi/4$$

The numerator is exactly the same but with an extra $y = \sin z$

$$\int_0^{\pi/2} \cos z\ \cos z\ \sin z\,dz$$

$$= \left[\frac{-\cos^3 z}{3}\right]_0^{\pi/2} = 0 - [-1/3] = 1/3$$

Finally

$$\bar{y} = \frac{\text{numerator}}{\text{denominator}} = \frac{4}{3\pi}$$