# 0F2 Vectors Lecture Notes
# by Dr. Mike Simon

(with added diagrams by Dr. Holly Barker)

**Note**: This course is also known as MATH19842.

**Timetable**: Review sessions are in Engineering Building A (Lecture theatre A) on Tuesdays at 5PM. Tutorials are on Monday afternoons and may be led by myself or one of the other members of teaching staff on the course. See your timetable for room/time details. There are no traditional lectures for this course. The content is delivered asynchronously via online videos to be watched before the review session.

**Assessment**: Coursework test 1 on this topic will take place on Wednesday 16th March (worth 10%), and half of the exam at the end of the course will be on vectors.

**Syllabus**

1. Introduction
2. Basic concepts of vectors
3. Addition of vectors
4. Position vectors
5. Cartesian coordinates
6. Scalar product (or dot product)
7. Lines and planes
8. Further 3-D geometry
9. Vector product (or cross product)

# 1 Introduction

## 1.1 What is a vector?

Most quantities in engineering can be divided into two types: 1) **scalars** and 2) **vectors**.

A **scalar** is a quantity which can be described by a single number (positive, negative or zero). If we ignore the sign, that gives us the **magnitude**, which is positive (or zero) and shows us how big the scalar is. An example of a physical quantity which is a scalar is the volume of tea that my mug will hold.

A **vector** is a quantity which has both a **magnitude** and also a specific **direction** in space. The magnitude (also known as the *modulus* or *length*) of a vector is positive (or zero). An example of a physical quantity which is a vector is the force needed to hold this pen stationary in the air.

*Note*: Often the vectors we are interested in are physical quantities which can be thought of in **2 or 3 dimensions**. However there are occasions where we can generalise the results and imagine vectors in $n$-dimensional space.

## 1.2 When do we use vectors?

If we talk about the **speed** of a car (e.g. 60 miles per hour) this is just a number (positive or zero) in appropriate units; this is a **scalar**.

However, it may be important to consider also the direction the car is travelling. In that case we talk about the **velocity** of the car (e.g. 60 miles per hour heading due west); velocity is a **vector** which includes the direction as well as the speed.

There are many other examples in both engineering and the sciences of the use of vectors.

# 2 Basic concepts of vectors

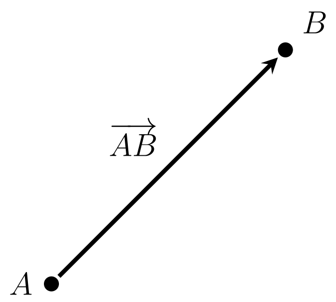
A vector can be represented by drawing a line with a direction arrow on it, referred to as a *directed line segment*. The *directed line segment* has two important characteristics:-

1. The length of the line represents the magnitude of the vector *given some appropriate scale.*

2. The direction of the line (arrow) represents the direction of the vector *given some appropriate orientation.*

## 2.1 General Notation

More generally an arbitrary vector quantity is represented as follows as a directed line between two points $A$ and $B$:




It is **extremely** important that vectors are written in such a way as to distinguish them from scalars. Various notations are used:

1. For a vector between two endpoints $A$ and $B$ in the direction from $A$ to $B$, the notation $\overrightarrow{AB}$ is used.

2. Often in textbooks, vectors are indicated by using a bold typeface such as $\mathbf{a}$.

3. In handwritten text, vector quantities are often underlined such as $\underline{a}$.

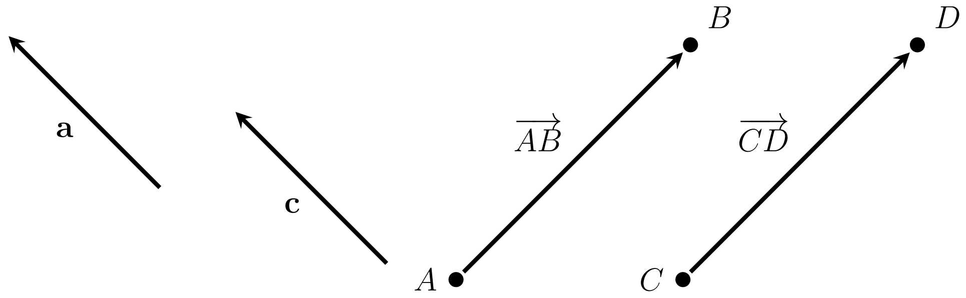
In this, the three notations are alternatives for the same vector, and so

$$\overrightarrow{AB} = \mathbf{a} = \underline{a}.$$

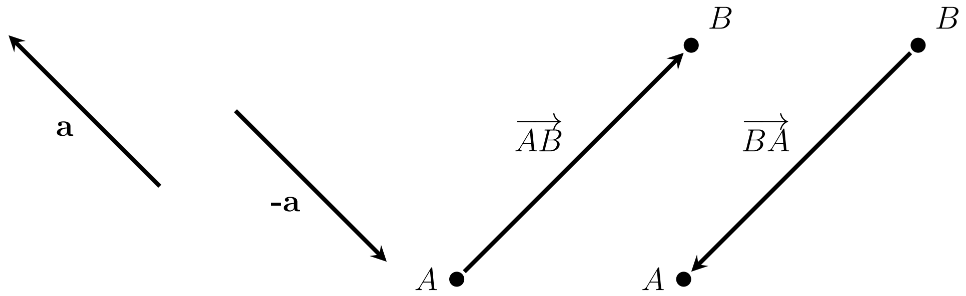
## 2.2 Two fundamental ideas

The following ideas are essential:-

1. Two vectors are *equal* if they have the same magnitude and direction, regardless of where they are. In this diagram, we have $\mathbf{a} = \mathbf{c}$ and $\overrightarrow{AB} = \overrightarrow{CD}$.




2. A vector which has the same magnitude as the vector $\mathbf{a}$ but has the *opposite* direction is denoted by $-\mathbf{a}$. By this logic, we may write $\overrightarrow{BA} = -\overrightarrow{AB}$




Note that the above idea of the equality of vectors does not depend upon *location in space*, and therefore these vectors are sometimes referred to as *free* vectors, meaning free from a specific location.
*For example, we have the same vector if we travel 40 miles north from Manchester as if we travel 40 miles north from London.*

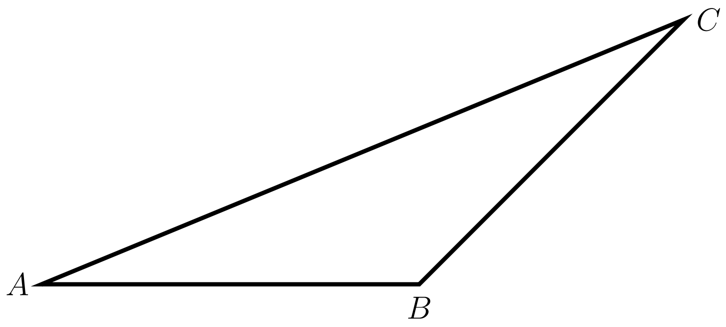
## 2.3 Magnitude only

If we wish to refer to just the magnitude (length) of a vector we write this using modulus signs such as $|\overrightarrow{AB}|$ or $|\underline{a}|$ or $|\mathbf{a}|$.

Later on it will also be useful to have vectors which we think of as being only a direction!

# 3 Addition of vectors

**Example**: Suppose an aircraft starts at point $A$ and flies 200 miles due East to point $B$, and then turns and flies 250 miles North-East to point $C$. The end result is equivalent to flying in a straight line from point $A$ to point $C$; this is most easily seen geometrically.




Therefore vectors are said to satisfy the *triangle law of addition*. That is, since travelling from $A$ to $B$ and then $B$ to $C$ is equivalent to travelling directly from $A$ to $C$, we write

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

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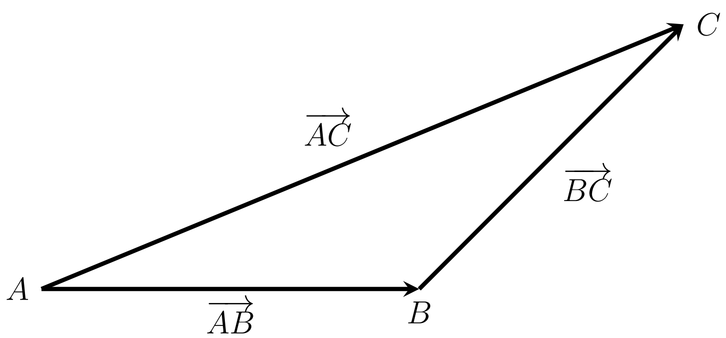


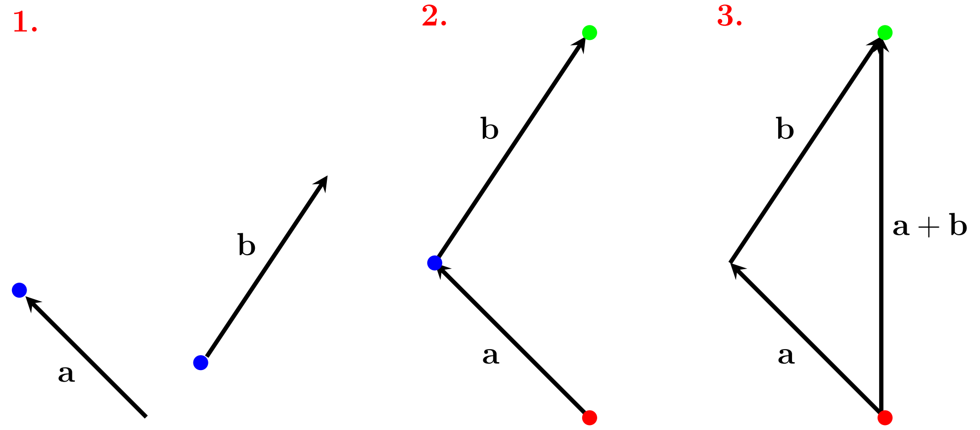

## 3.1 Definition of vector addition

In the general situation if we want to add the vector $\mathbf{b}$ to $\mathbf{a}$, we copy the vector $\mathbf{b}$ in the place where its *tail* coincides with the *head* of $\mathbf{a}$.
Then $\mathbf{a}$ and $\mathbf{b}$ form two sides of a triangle; the third side is the vector sum $\mathbf{a} + \mathbf{b}$.

*Note*: If you think of the vector as an *arrow*, the *head* is the sharp forward end and the *tail* has the feathers; so for a vector, the head is the end to which the arrow points and tail is the opposite end.)

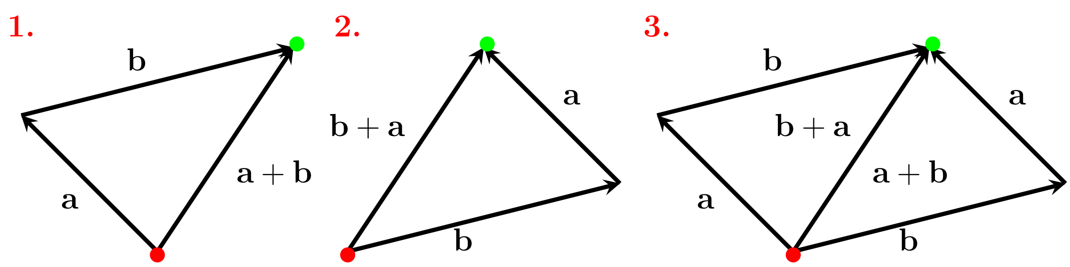
*Also note*: this particular sort of vector copying can be thought of a moving it in a special way where the start and end points change but the direction and length are unchanged - we talk about this as a *translation*.

The image below shows the procedure. Stage 1 shows two vectors $\mathbf{a}$ and $\mathbf{b}$, where the head of $\mathbf{a}$ and the tail of $\mathbf{b}$ are indicated by blue dots. Stage 2 shows the vector $\mathbf{b}$ copied so that the tail of $\mathbf{b}$ has moved onto the head of $\mathbf{a}$. The red dot (at the tail of $\mathbf{a}$) indicates the start point of the vector sum, and the green dot (at the head of $\mathbf{b}$) indicates the end point of the vector sum. Stage 3 shows the vector sum $\mathbf{a}+\mathbf{b}$ – the vector along the third side of the triangle.




**Question**: Is $\mathbf{b}+\mathbf{a}$ equal to $\mathbf{a}+\mathbf{b}$?
**Answer: Yes!**

We can demonstrate this by sketching a parallelogram using two vectors $\mathbf{a}$ and $\mathbf{b}$. Stage 1 shows the addition $\mathbf{a}+\mathbf{b}$ following the procedure above. Stage 2 shows the addition $\mathbf{b}+\mathbf{a}$ following the procedure above. Stage 3 has both diagrams put together. We see that the vectors form a parallelogram, and $\mathbf{b}+\mathbf{a}=\mathbf{a}+\mathbf{b}$.




There is a mathematical word for the fact that you can change the order of the vectors without changing the sum - we say that vector addition is *commutative.* This means that

$$\begin{aligned}\overrightarrow{OP}+\overrightarrow{PQ}=\overrightarrow{PQ}+\overrightarrow{OP},&\\ \underline{c}+\underline{d}=\underline{d}+\underline{c}, &\qquad \text{and}\\ \mathbf{b}+\mathbf{a}=\mathbf{a}+\mathbf{b}.&\end{aligned}$$

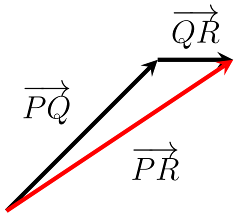
**Another Question**: Is $\underline{a}+(\underline{b}+\underline{c})$ equal to $(\underline{a}+\underline{b})+\underline{c}$?
**Answer: Yes!**

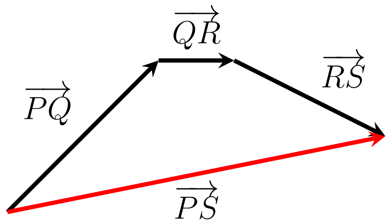
*There is an example sheet question on this for you to do.*

The mathematical word for the fact that the way in which you add three vectors does not matter is to say that vector addition is *associative.*

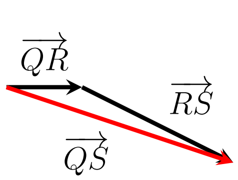
This means that if I wish to add together $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ and $\overrightarrow{RS}$, I can either

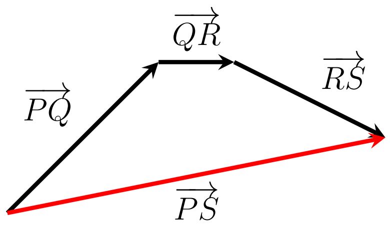
1. Use the equation $(\overrightarrow{PQ}+\overrightarrow{QR})+\overrightarrow{RS}$. Following the brackets, I should add $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ first to get $\overrightarrow{PR}$, and then add $\overrightarrow{RS}$ to that:




2. Use the equation $\overrightarrow{PQ}+(\overrightarrow{QR}+\overrightarrow{RS})$. Again, following the brackets, I should add $\overrightarrow{QR}$ and $\overrightarrow{RS}$ first to get $\overrightarrow{QS}$, and then add that to $\overrightarrow{PQ}$:



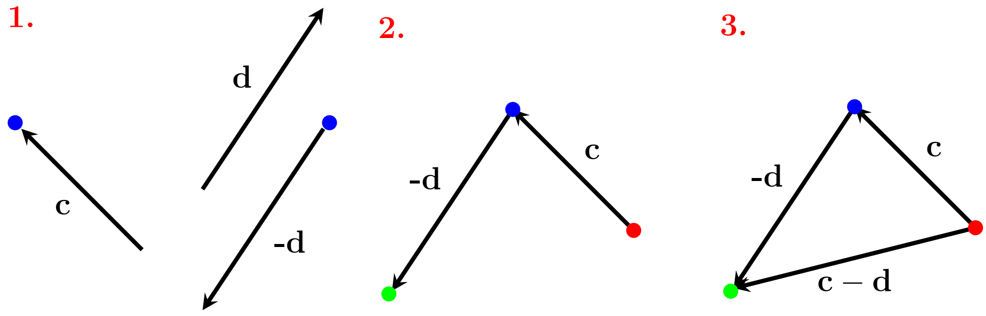

Either method gives the same result.

## 3.2 Subtraction of a vector

This is simply done by adding the negative of the vector.

$$\begin{aligned}\underline{a}-\underline{b} &= \underline{a}+(\underline{-b}), \qquad \text{and}\\ \mathbf{c}-\mathbf{d} &= \mathbf{c}+(-\mathbf{d}).\end{aligned}$$

The diagram below demonstrates the subtraction $\mathbf{c}-\mathbf{d}$. Stage 1 shows the vectors $\mathbf{c}$, $\mathbf{d}$ and $-\mathbf{d}$, with blue dots on the head of $\mathbf{c}$ and the tail of **-d** (the two vectors to be added). Stage 2 shows the vector **-d** copied so that the tail of **-d** has moved onto the head of **c**. The red dot (at the tail of **c**) indicates the start point of the vector sum, and the green dot (at the head of **-d**) indicates the end point of the vector sum. Stage 3 shows the vector sum $\mathbf{c} + (\mathbf{-d}) = \mathbf{c} - \mathbf{d}$ – the vector along the third side of the triangle.




A special case is where you subtract a vector from itself:-

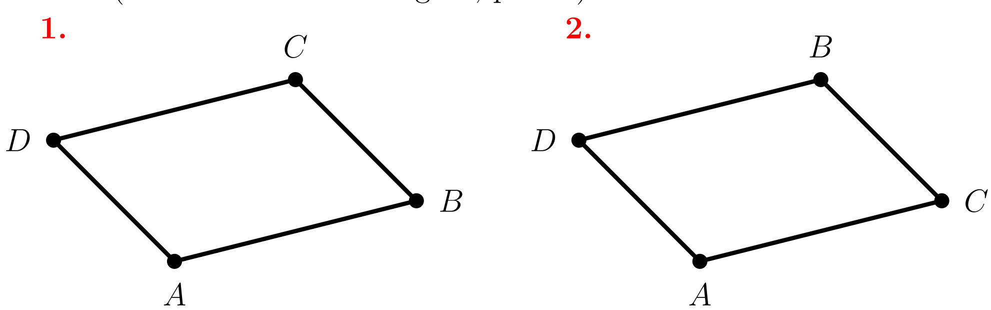
$$\underline{a} - \underline{a} = \underline{0}.$$

*Note* 1: The right-hand side is **not** just 0 (which is a scalar); it is the **zero vector** which is a vector but has no magnitude. In textbooks it is often written **0** in bold (so that $\mathbf{a} - \mathbf{a} = \mathbf{0}$).
*Note* 2: The zero vector $\mathbf{0}$ is such that $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$ for any vector **a**.

## 3.3 Parallelograms

Parallelograms have been used in previous examples. A parallelogram $ABCD$ is a special *quadrilateral* where opposite sides (*e.g.* $AB$ and $CD$) are parallel and of equal length (and the same will apply to $AC$ and $BD$).
As a result, the vectors for opposite sides of the parallelogram are equal. That is, for the parallelogram $ABCD$, we have $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{BC} = \overrightarrow{AD}$.

*Note*: the *order* that we list the vertices of the parallelogram matters – a parallelogram called $ABCD$ has sides $AB$, $BC$, $CD$ and $DA$ (shown in the below figure, part 1) and a parallelogram called $ACBD$ has sides $AC$, $CB$, $BD$ and $DA$ (shown in the below figure, part 2)

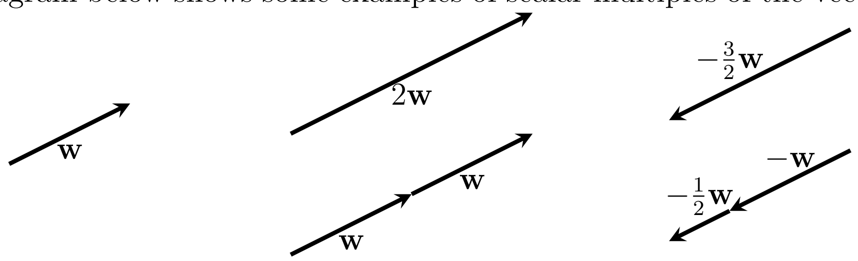


## 3.4 Multiplying a vector by a scalar

If we add $\underline{a}$ to itself, we get a vector which is in the same direction as $\underline{a}$ but twice as long; we call this vector $2\underline{a}$.
Similarly, for any scalar $k$ and vector $\underline{a}$ we can form the vector $k\underline{a}$, which is in the same direction as $\underline{a}$ but has magnitude $k$ times bigger (*i.e.* $k \times |\underline{a}|$).

The vector $k\underline{a}$ is said to be a *scalar multiple* of the vector $\underline{a}$. If we want to *divide* a vector by a scalar $k$ we simply multiply by $\frac{1}{k}$ so that $\underline{a}/k = \frac{1}{k}\underline{a}$.

The diagram below shows some examples of scalar multiples of the vector $\mathbf{w}$.




**Rules for scalar multiplication:**

$$\begin{aligned} k(\underline{a}+\underline{b}) &= k\underline{a}+k\underline{b}, \\ (k+l)\underline{a} &= k\underline{a}+l\underline{a}, \qquad \text{and} \\ k(l\underline{a}) &= (kl)\underline{a}. \end{aligned}$$

## 3.5 Unit vectors

A vector which has a magnitude 1 is said to be a *unit vector*.

Recall that a vector $\underline{a}$ has magnitude $|\underline{a}|$; we can therefore create a unit vector in the direction of $\underline{a}$ by dividing $\underline{a}$ by the scalar $|\underline{a}|$. This unit vector is often denoted by $\hat{\underline{a}}$. Therefore

$$\hat{\underline{a}} = \frac{1}{|\underline{a}|} \times \underline{a} = \frac{\underline{a}}{|\underline{a}|}.$$

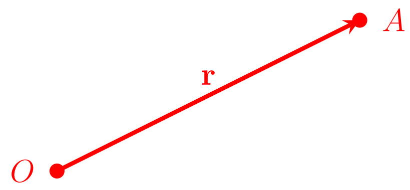
(This is a special case of scalar multiplication.)

## 4 Position Vectors

**Free** vectors have no specific location. However, some vectors represent a specific position. The letter $O$ is generally used to denote a fixed *origin* in space.

Then the vector $\overrightarrow{OA}$ is called the **position vector** $\mathbf{r} = \overrightarrow{OA}$ of $A$ relative to $O$.

This *displacement* is unique and **cannot** be represented by any other line of equal length and direction. In other words, for position vectors we have specified a starting point in space as well as direction and magnitude.



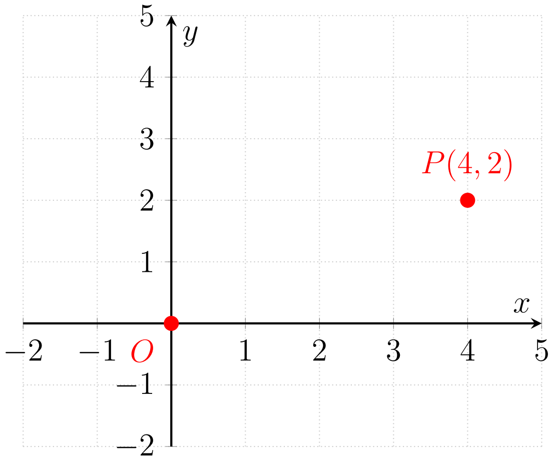

Note that in some problems you are told where to put the origin; however, there are situations where you can *choose* where to put the origin, and sometimes certain choices will make the solution easier than other choices.

# 5 Cartesian Coordinates

Usually we want to define vectors in two-dimensional $(2 - D)$ or in three-dimensional $(3 - D)$ space. Therefore it is natural to use Cartesian coordinates, where we have axes at right-angles to each other.

## 5.1 Coordinates in 2-D

Any point $P$ in the plane $(2 - D$ space) can be defined in terms of its $x$ and $y$ coordinates.

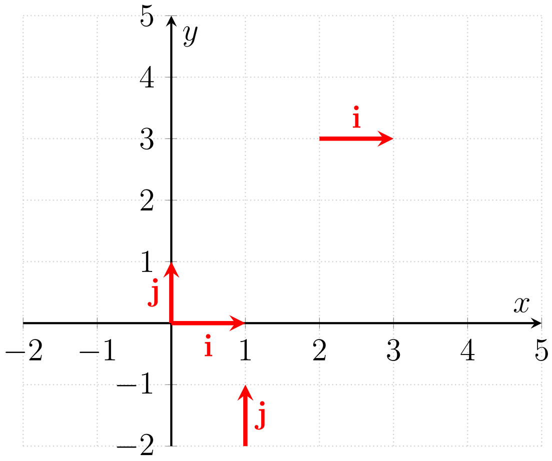



*Note*: In writing points in this way we have already decided on the position of the origin (which becomes the unique point where $x = y = 0$) and on the $x$ and $y$ directions.nOften there are obvious, sensible choices for these, (*such as*) along the horizontal for $x$ and vertically upwards for $y$.

## 5.2 Basis vectors

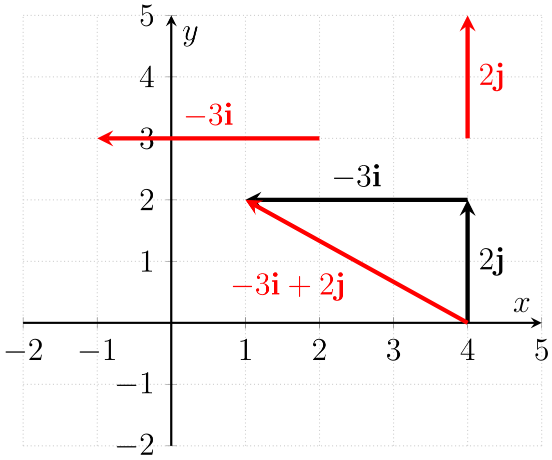
Cartesian coordinates make use of unit vectors in the directions parallel to the $x$- and $y$-axes, known as **basis vectors**. (We need two of them in 2-D.) A unit vector in the $x$-direction is usually denoted by $\mathbf{i}$ and a unit vector in the $y$-direction is usually denoted by $\mathbf{j}$.
*Note*: These are both of unit length and so $|\mathbf{i}| = |\mathbf{j}| = 1$.

Then $a\mathbf{i}$ and $b\mathbf{j}$ denote vectors of length $a$ and $b$ along the $x$ and $y$ directions, respectively.
If $a$ is negative then $a\mathbf{i}$ is a vector of length $|a|$ in the direction of the unit vector $-\mathbf{i}$.

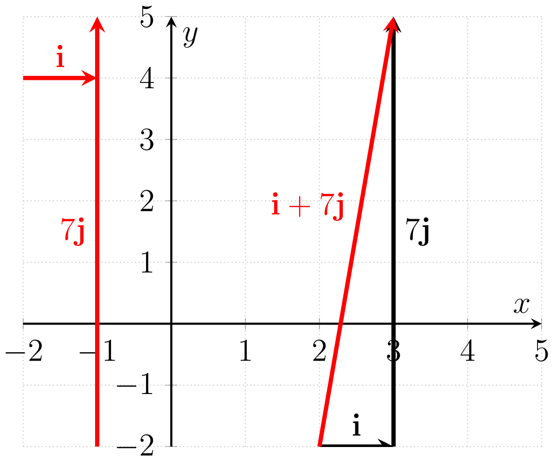



**Example:**

Draw the vectors $-3\mathbf{i}$ and $2\mathbf{j}$. Then using the triangle law of addition we find $-3\mathbf{i} + 2\mathbf{j}$. This is shown in the diagram below.




Similarly we could draw the vector $\mathbf{i}+7\mathbf{j}$ by first drawing the two separate vectors $\mathbf{i}$ and $7\mathbf{j}$ and then using the triangle law to add them together.



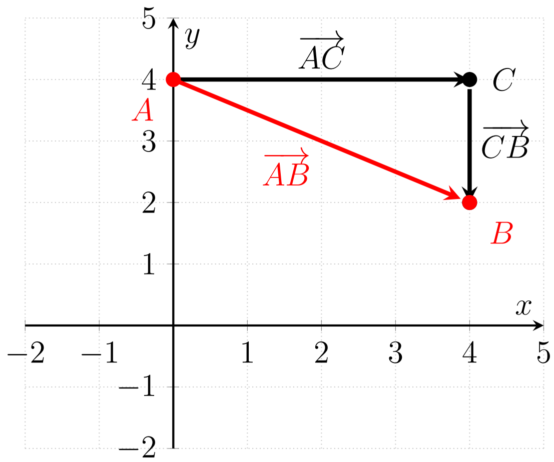
### 5.2.1 Expressing vectors in terms of basis vectors

Consider the vector $\overrightarrow{AB}$. This can be regarded as coming from the sum of two vectors; one in the $x$ direction and the other in the $y$ direction.

In particular, we can add in a third point $C$ such that $\overrightarrow{AC} = a\mathbf{i}$ and $\overrightarrow{CB} = b\mathbf{j}$. Then from the triangle law of vector addition

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AC} + \overrightarrow{CB} \\ &= a\mathbf{i} + b\mathbf{j}.\end{aligned}$$

The diagram below shows an example of this. The vector $\overrightarrow{AB}$ may be written as the sum of the vectors $\overrightarrow{AC} = 4\mathbf{i}$ and $\overrightarrow{CB} = -2\mathbf{j}$. Thus $\overrightarrow{AB} = 4\mathbf{i} - 2\mathbf{j}$.




In this same way, any vector in the $x - y$ plane can be expressed as a **linear combination of basis vectors** in the $x$ and $y$ directions.

## 5.3 Row and column vector notation

This is a useful alternative way to write vectors which are expressed in Cartesian coordinates. Thus the vector $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$ can instead be written as

$$\begin{pmatrix} a, & b \end{pmatrix} \qquad \text{or as} \qquad \begin{pmatrix} a \\ b \end{pmatrix}$$

in row or column vector notation respectively.

**Examples:**

a) The vector $\mathbf{t} = -\mathbf{i} - 7\mathbf{j}$ can be written as

$$\begin{pmatrix} -1 \\ -7 \end{pmatrix}$$

in column vector notation.

b) The vector $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j}$ can be written as

$$( \ -3\,, \quad 2 \ )$$

in row vector notation.

c) The vector $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$ can be written as

$$( \ 2\,, \quad -4 \ ) \qquad \text{or as} \qquad \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

in row or column vector notation respectively.

### 5.3.1 Adding/subtracting 2-D vectors in this notation

The addition and subtraction of vectors in Cartesian coordinates is very straightforward and simply involves adding (or subtracting) the respective $\mathbf{i}$ and $\mathbf{j}$ components.

For example, if $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{s} = f\mathbf{i} + g\mathbf{j}$, then

$$\begin{aligned} \mathbf{r} + \mathbf{s} &= (a + f)\mathbf{i} + (b + g)\mathbf{j} \qquad \text{and} \\ \mathbf{r} - \mathbf{s} &= (a - f)\mathbf{i} + (b - g)\mathbf{j}. \end{aligned}$$

It is equally easy if the vectors are expressed in row or column vector notation; for example

$$\mathbf{r} + \mathbf{s} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} a + f \\ b + g \end{pmatrix}.$$

in column vector notation, and

$$\mathbf{r} - \mathbf{s} = ( \ a\,, \quad b \ ) - ( \ f\,, \quad g \ ) = ( \ a - f\,, \quad b - g \ )$$

in row vector notation.

**Examples:**
Calculate $\mathbf{r} + \mathbf{s}$ and $\mathbf{r} - \mathbf{s}$ in the following cases.

1. Let $\mathbf{r} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{s} = -4\mathbf{i} + 3\mathbf{j}$. Then

$$\begin{aligned} \mathbf{r} + \mathbf{s} &= (2 + (-4))\mathbf{i} + (1 + 3)\mathbf{j} = -2\mathbf{i} + 4\mathbf{j} \qquad \text{and} \\ \mathbf{r} - \mathbf{s} &= (2 - (-4))\mathbf{i} + (1 - 3)\mathbf{j} = 6\mathbf{i} - 2\mathbf{j}. \end{aligned}$$

2. Let

$$\mathbf{r} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \text{and} \quad \mathbf{s} = \begin{pmatrix} 2 \\ 4 \end{pmatrix};$$

then

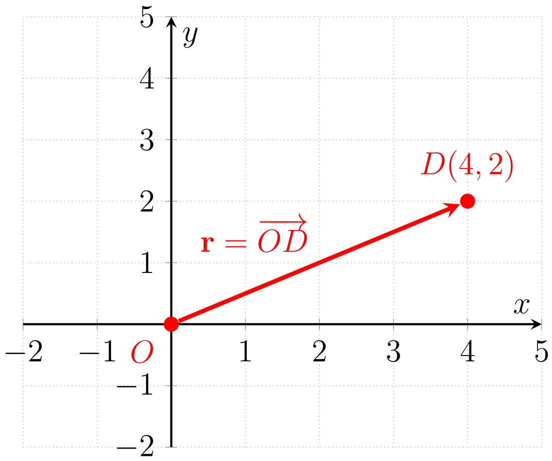
$$\mathbf{r} + \mathbf{s} = \begin{pmatrix} 1 + 2 \\ (-3) + 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ and } \mathbf{r} - \mathbf{s} = \begin{pmatrix} 1 - 2 \\ (-3) - 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \end{pmatrix}.$$

## 5.4 Position vectors in 2D

When we set up coordinates in the $2-D$ plane, we choose the *origin*, denoted $O$, which becomes the point $(0,0)$. Every vector in the plane is then taken to start at $O$. Then the vector from $O$ to the point $P=(a,b)$ is known as the *position vector* of $P$.

The position vector of $P$ with coordinates $(a,b)$ is $\mathbf{r}=\overrightarrow{OP}=a\mathbf{i}+b\mathbf{j}$. As mentioned before, position vectors are fixed vectors in that their origin (starting point) is specified.

In the example in the diagram below, the position vector of the point $D$ which has coordinates $(4,2)$ is $\mathbf{r}=\overrightarrow{OD}=4\mathbf{i}+2\mathbf{j}$.




**Examples:**

State the position vectors of the points P, Q, R S with respective coordinates, $(-3,1),(2,2),(0,5)$ and $(-1,-1)$.

**Answers:**

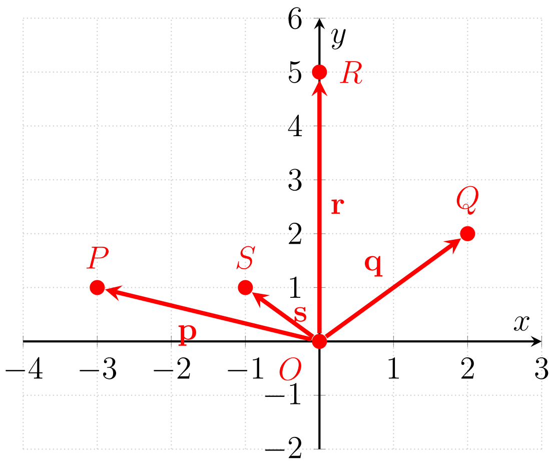
$\mathbf{p}=\overrightarrow{OP}=(-3)\mathbf{i}+1\mathbf{j}=-3\mathbf{i}+\mathbf{j}$,

$\mathbf{q}=\overrightarrow{OQ}=2\mathbf{i}+2\mathbf{j}$,

$\mathbf{r}=\overrightarrow{OR}=0\mathbf{i}+5\mathbf{j}=5\mathbf{j}$ and

$\mathbf{s}=\overrightarrow{OS}=(-1)\mathbf{i}+(-1)\mathbf{j}=-(\mathbf{i}+\mathbf{j})$.

These vectors are sketched on the below diagram.




**Question:**
From the above points, find the vector $\overrightarrow{PQ}$.

$a)$ Directly from the diagram $\overrightarrow{PQ} = 5\mathbf{i} + \mathbf{j}$.

$b)$ Use the triangle law of addition. First note that $\overrightarrow{PO} = -\overrightarrow{OP} = -(-3\mathbf{i} + \mathbf{j}) = 3\mathbf{i} - \mathbf{j}$. Then

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = (3\mathbf{i} - \mathbf{j}) + (2\mathbf{i} + 2\mathbf{j}) = (3+2)\mathbf{i} + (-1+2)\mathbf{j} = 5\mathbf{i} + \mathbf{j}.$$

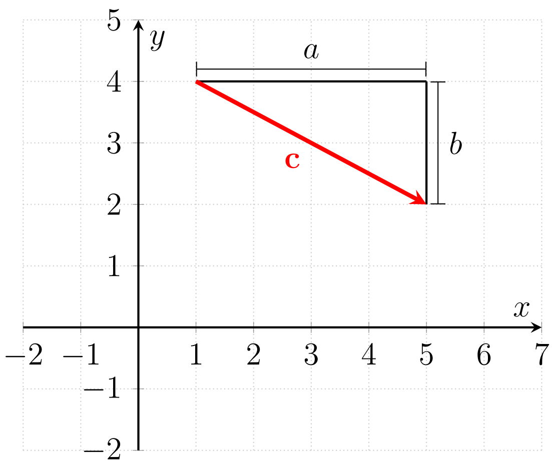
## 5.5 The modulus of a 2-D vector

The modulus of a vector is simply its length. Pythagoras' theorem shows that, for $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$,

$$|\mathbf{r}| = \sqrt{a^2 + b^2}.$$

In the diagram below, the vector $\mathbf{c}$ has length

$$|\mathbf{c}| = \sqrt{a^2 + b^2} = \sqrt{4^2 + 2^2} = \sqrt{20} \approx 4.47$$



**Examples:**
Find the modulus of the vectors
$\mathbf{s} = \mathbf{i} + 2\mathbf{j}, \mathbf{t} = 5\mathbf{i} - 3\mathbf{j}$ and $\mathbf{u} = -2\mathbf{i} + 7\mathbf{j}$.
**Answers:**
$|\mathbf{s}| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$,
$|\mathbf{t}| = \sqrt{5^2 + (-3)^2} = \sqrt{25+9} = \sqrt{34}$, and
$|\mathbf{u}| = \sqrt{(-2)^2 + 7^2} = \sqrt{4+49} = \sqrt{53}$.

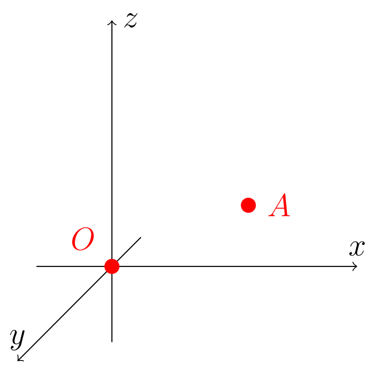
*Note* 1: Resist the temptation to press the buttons on your calculator at this stage - anything your calculator produces as $\sqrt{5}$ will only be an approximation whereas $\sqrt{5}$ is the exact result for $|\mathbf{s}|$.

*Note* 2: The modulus is positive unless the vector itself is $\mathbf{0}$ - only the zero vector has zero modulus (and therefore has components zero in both the $\mathbf{i}$ and $\mathbf{j}$ directions).

*Note* 3: Different vectors can have the same modulus; for instance, all the vectors $6\mathbf{i} + 7\mathbf{j}$, $-6\mathbf{i} + 7\mathbf{j}$, $6\mathbf{i} - 7\mathbf{j}$, $-6\mathbf{i} - 7\mathbf{j}$, $7\mathbf{i} + 6\mathbf{j}$, $7\mathbf{i} - 6\mathbf{j}$,
$7\mathbf{i} + 6\mathbf{j}$, $-7\mathbf{i} - 6\mathbf{j}$, $9\mathbf{i} + 2\mathbf{j}$, $9\mathbf{i} - 2\mathbf{j}$, $-9\mathbf{i} + 2\mathbf{j}$, $-9\mathbf{i} - 2\mathbf{j}$, $2\mathbf{i} + 9\mathbf{j}$,
$2\mathbf{i} - 9\mathbf{j}$, $-2\mathbf{i} + 9\mathbf{j}$ and $-2\mathbf{i} - 9\mathbf{j}$ have the same modulus $\sqrt{85}$. This means that all the points represented by these position vectors are the same distance from the origin, and that means that they must all lie on a circle.

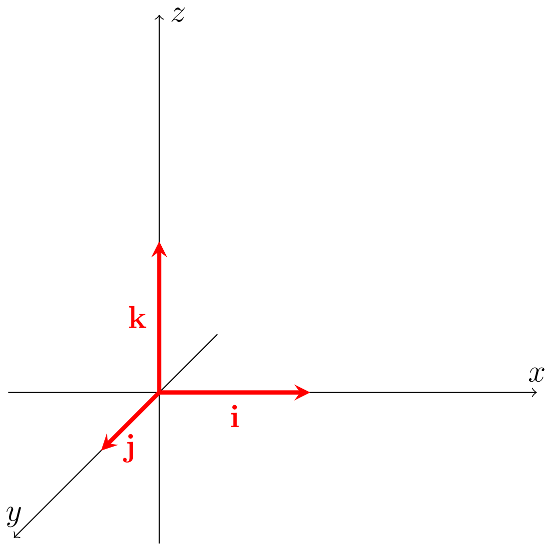
## 5.6 Cartesian coordinates in 3-D

Any point in three-dimensional space can be defined in terms of its $x, y$ and $z$ coordinates.




The three axes in the $x, y$ and $z$ directions are mutually perpendicular; that is, there is a right angle (*i.e.* $90^o$ or $\frac{\pi}{2}$ radians) between any two of the axes.

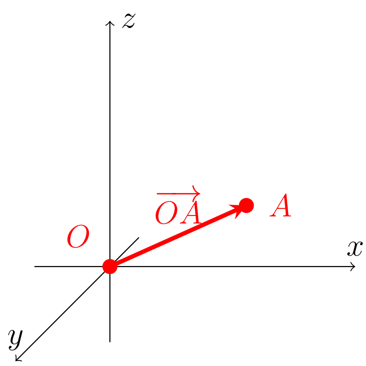
The unit vector in the $z$ direction is denoted by $\mathbf{k}$, as shown in the diagram below.




Therefore in the natural extension of the $2-D$ case, the point $P$ having coordinates $(a, b, c)$ has position vector

$$\overrightarrow{OP} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

The diagram below shows the point $A(4, 1, 2)$ and the position vector $\overrightarrow{OA} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.




### 5.6.1 The modulus of a 3-D vector

For a vector expressed as $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, the modulus of $\mathbf{r}$ can again be found using Pythagoras' theorem.

$$|\mathbf{r}| = \sqrt{a^2 + b^2 + c^2}.$$

**Example:**
Points $A$, $B$ and $C$ have coordinates $(1, -3, 2), (-3, -2, -1)$ and $(4, 0, 2)$, respectively.

*a*) Find the position vectors of $A$, $B$ and $C$.

*b*) Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

*c*) Find $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$.

**Answer:**

*a*) $\overrightarrow{OA} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$
$\overrightarrow{OB} = -3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
$\overrightarrow{OC} = 4\mathbf{i} + 0\mathbf{j} + 2\mathbf{k} = 4\mathbf{i} + 2\mathbf{k}$.

*b*)

$$\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} \\
&= -(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) + (-3\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \\
&= \Big(-1 + (-3)\Big)\mathbf{i} + \Big(-(-3) + (-2)\Big)\mathbf{j} + \Big(-2 + (-1)\Big)\mathbf{k} \\
&= -4\mathbf{i} + \mathbf{j} - 3\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC} = -\overrightarrow{OB} + \overrightarrow{OC} \\
&= -(-3\mathbf{i} - 2\mathbf{j} - \mathbf{k}) + 4\mathbf{i} + 0\mathbf{j} + 2\mathbf{k} \\
&= \Big(-(-3) + 4\Big)\mathbf{i} + \Big(-(-2) + 0\Big)\mathbf{j} + \Big(-(-1) + 2\Big)\mathbf{k} \\
&= 7\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}
\end{aligned}$$

*c*) $|\overrightarrow{AB}| = \sqrt{(-4)^2 + 1^2 + (-3)^2} = \sqrt{16 + 1 + 9} = \sqrt{26}$
and $|\overrightarrow{BC}| = \sqrt{7^2 + 2^2 + 3^2} = \sqrt{49 + 4 + 9} = \sqrt{62}$.

**Another Example:**
Let $\underline{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and let $B$ be the point $B(-2, 3, 1)$; find the point $A$ such that

$$\overrightarrow{AB} = \underline{u}.$$

**Answer:**
We know that $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Then

$$\begin{aligned}
\overrightarrow{OA} &= -2\mathbf{i} + 3\mathbf{j} + \mathbf{k} - \overrightarrow{AB} \\
&= -2\mathbf{i} + 3\mathbf{j} + \mathbf{k} - \mathbf{u} \qquad \textit{by assumption} \\
&= -2\mathbf{i} + 3\mathbf{j} + \mathbf{k} - \Big(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}\Big) \\
&= -4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}.
\end{aligned}$$

## 5.7 Equal vectors and parallel vectors

Equal vectors
Suppose that the vectors $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{s} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$ are equal (*i.e.* $\mathbf{r} = \mathbf{s}$). That means that

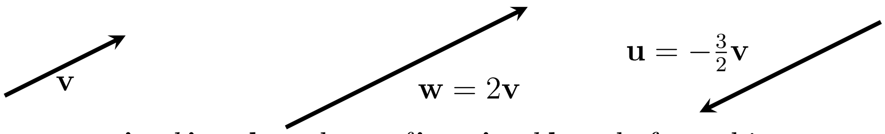
$$\mathbf{0} = \mathbf{r} - \mathbf{s} = (a - f)\mathbf{i} + (b - g)\mathbf{j} + (c - h)\mathbf{k}.$$

Now remember that the components of the zero vector $\mathbf{0}$ are all zero, which therefore means that $a - f = b - g = c - h = 0$.
In this way we see that

$$\mathbf{r} = \mathbf{s} \quad \underline{\text{if and only if}} \quad a = f, b = g \text{ and } c = h.$$

Parallel vectors
Two vectors $\mathbf{r}$ and $\mathbf{s}$ are parallel if there exists a scalar $m$ such that

$$\mathbf{r} = m\mathbf{s}$$




If we have $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{s} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$ as before, this means they will be parallel (*i.e.* the vectors $(a, b, c)$ and $(f, g, h)$ will be parallel) when

$$a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \mathbf{r} = m\mathbf{s} = mf\mathbf{i} + mg\mathbf{j} + mh\mathbf{k}.$$

This therefore requires $a = mf, b = mg$ and $c = mh$, and so

$$\frac{a}{f} = \frac{b}{g} = \frac{c}{h} \quad (= m).$$

**Example:**
$\mathbf{r} = (2, 4, 8)$ and $\mathbf{s} = (3, 6, 12)$ are parallel because $\mathbf{r} = \frac{2}{3}\mathbf{s}$:-

$$\frac{a}{f} = \frac{2}{3} = \frac{b}{g} = \frac{4}{6} = \frac{c}{h} = \frac{8}{12}.$$

## 5.8 Unit vectors

We already have unit vectors denoted by $\mathbf{i}$ in the $x$ direction, $\mathbf{j}$ in the $y$ direction and $\mathbf{k}$ in the $z$ direction.

Sometimes we adopt the notation where a 'hat' over a vector signifies that it is a unit vector; thus we sometimes call our three basis vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$.

In general we might want to *create* a unit vector in the direction of a given vector $\mathbf{v}$. As already mentioned (section 3.7) we do this by making $\hat{\mathbf{v}} = k\mathbf{v}$, with $k$ chosen so that $|\hat{\mathbf{v}}| = 1$.

Now as $|\hat{\mathbf{v}}| = |k\mathbf{v}| = k|\mathbf{v}|$, this therefore requires $k = 1/|\mathbf{v}|$ and so we see that the unit vector in the direction of $\mathbf{v}$ is $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$.

Hence a unit vector $\underline{\hat{r}} = \hat{\mathbf{r}}$, in the direction of $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, is given by

$$\begin{aligned}
\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} &= \frac{1}{\sqrt{a^2+b^2+c^2}}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \\
&= \frac{a}{\sqrt{a^2+b^2+c^2}}\mathbf{i} + \frac{b}{\sqrt{a^2+b^2+c^2}}\mathbf{j} + \frac{c}{\sqrt{a^2+b^2+c^2}}\mathbf{k}.
\end{aligned}$$