Dynamics in 2D and 3D | 0J2 Mechanics *

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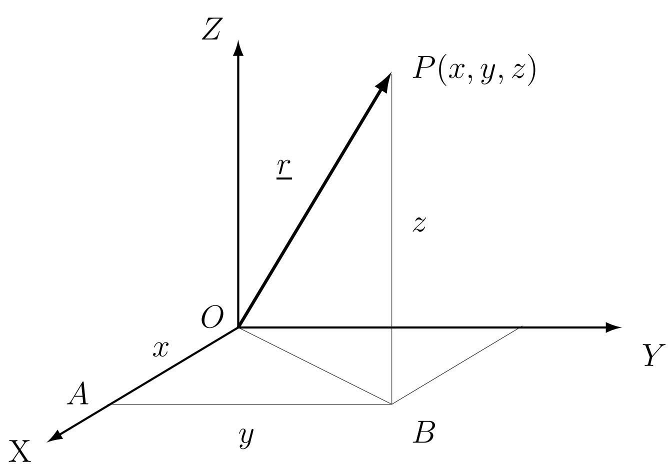
*These notes are intended to be the definitive source of material discussed in the lecture course. They are reasonably comprehensive but should be augmented by notes taken when watching the weekly videos.

2 Dynamics in 2D and 3D

In this section of the course we will generalise notions from one dimension to two and three dimensions. We will do this using vectors and vector algebra including the scalar (dot) product.

2.1 Position vectors

We will mainly work in three dimensions. The two-dimensional case is easily obtained from this by ignoring the z-coordinate. Thus, if we have a set of axes XYZ with origin O, then we can describe the position of any point P by giving its coordinates (x, y, z).



We could also specify P by giving its *position vector* which is $\vec{OP}$.

We often write $\underline{r}$ for the position vector, so that $\underline{r} = \vec{OP}$. It is easily shown that the components of the position vector of P are the same as the coordinates of P. Specifically, if $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the fundamental (Cartesian) unit vectors, then from the diagram we have

$$\begin{aligned} \vec{OB} &= \vec{OA} + \vec{AB} = x\mathbf{i} + y\mathbf{j} \\ \underline{r} = \vec{OP} &= \vec{OB} + \vec{BP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \end{aligned}$$

Thus $\underline{r} = (x, y, z) \quad \leftarrow$ the coordinates of P are the components of $\underline{r}$.

Thus

$$\begin{aligned} \underline{r} &= x\mathbf{i} + y\mathbf{j} \qquad \text{in 2D} \\ \underline{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad \text{in 3D} \end{aligned}$$

where (x, y) are the coordinates in 2D and (x, y, z) are the coordinates in 3D.

The *magnitude* (length) of $\underline{r}$ is given by

$$r^2 = x^2 + y^2 + z^2.$$

This result is true for *any* vector, not just a position vector, namely that the magnitude is the square root of the sum of the squares of its components.

The *direction* of $\underline{r}$ is given by the direction cosines $\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$, and again this is true for any vector.

Finally, since the coordinates can change in time we will write

$$\begin{aligned} \underline{r}(t) &= x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad \text{(2D)} \\ \underline{r}(t) &= x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}. \qquad \text{(3D)} \end{aligned}$$

2.2 Velocity and acceleration vectors

Velocity is the rate of change of position

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} \qquad \text{(2D)}$$

or, using the 'dot' notation

$$\underline{v} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j}$$

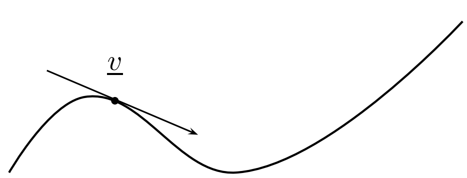
where $\dot{x}$ means $\frac{dx}{dt}$ etc.

In 3D, we often write

$$\underline{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$$

where
$$\begin{aligned} v_1 &= \dot{x} = \tfrac{dx}{dt} \quad \text{is the } x\text{-component of } \underline{v} \\ v_2 &= \dot{y} = \tfrac{dy}{dt} \quad \text{is the } y\text{-component of } \underline{v} \\ v_3 &= \dot{z} = \tfrac{dz}{dt} \quad \text{is the } z\text{-component of } \underline{v}\ . \end{aligned}$$

The direction of the velocity is always tangential to the path of the particle.



The *speed* is the magnitude (or modulus) of the velocity

$$v \;=\; |\underline{v}| \;=\; \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} \qquad \text{in 3D}$$

Acceleration is rate of change of velocity

$$\underline{a} \;=\; \frac{d\underline{v}}{dt} \;=\; \frac{d^2\underline{r}}{dt^2} \;=\; \ddot{x}\,\mathbf{i}+\ddot{y}\,\mathbf{j}+\ddot{z}\,\mathbf{k} \qquad \text{in 3D}$$

where $\ddot{x} \;=\; \dfrac{d^2x}{dt^2}$ etc.

If we write $\underline{v} \;=\; v_1\,\mathbf{i}+v_2\,\mathbf{j}+v_3\,\mathbf{k}$ then

$$\underline{a} \;=\; \frac{dv_1}{dt}\,\mathbf{i}+\frac{dv_2}{dt}\,\mathbf{j}+\frac{dv_3}{dt}\,\mathbf{k}$$

Note that acceleration and velocity may be in *different directions*.

Example 1 A particle moves in 2D. At time $t=1$ it is at the position $(1,2)$.
The velocity is given by $\underline{v} \;=\; 6t^2\mathbf{i}+(1-2t)\mathbf{j}$.
Find the acceleration and the position at time t.

Solution.

The acceleration is the differential of the velocity so

$$\underline{a} \;=\; \frac{d\underline{v}}{dt} \;=\; (12t)\,\mathbf{i}-2\,\mathbf{j}.$$

Now, writing the position vector as $\underline{r} \;=\; x\,\mathbf{i}+y\,\mathbf{j}$, the velocity is the differential of the position so

$$\underline{v}=\frac{d\underline{r}}{dt}=\frac{dx}{dt}\,\mathbf{i}+\frac{dy}{dt}\,\mathbf{j}. \;=\; 6t^2\,\mathbf{i}+(1-2t)\,\mathbf{j}$$

Components must agree so

$$\frac{dx}{dt}=6t^2 \quad \text{and} \quad \frac{dy}{dt}=1-2t$$

Integrating these gives

$$x=2t^3+k_1 \quad \text{and} \quad y=t-t^2+k_2$$

where k_1 and k_2 are the constants of integration.

At $t=1$, $\quad x=1$ and $y=2$ so

$$1=2+k_1 \quad \text{and} \quad 2=1-1+k_2$$

Thus $k_1 = -1$ and $k_2 = 2$ so the position at time t is

$$\underline{r} = (2t^3 - 1)\,\mathbf{i} + (t - t^2 + 2)\,\mathbf{j}$$

or in coordinate form $(x, y) = (2t^3 + 1, t - t^2 + 2)$.

Summary

The relation between position $\underline{r}$, velocity $\underline{v}$ and acceleration $\underline{a}$ for general motion in 2D or 3D using vectors is as shown below.

$$\boxed{\begin{array}{ccccc} & \text{differentiate} & & \text{differentiate} & \\ & \longrightarrow & & \longrightarrow & \\ \underline{r} & & \underline{v} & & \underline{a} \\ & \longleftarrow & & \longleftarrow & \\ & \text{integrate} & & \text{integrate} & \end{array}}$$

Note: when integrating we need extra information to determine the constants of integration. For example we might know the values of $\underline{r}$ and/or $\underline{v}$ at $t = 0$.

2.3 Constant acceleration

Just as in 1D the case of constant acceleration is important and we shall treat it in detail. We shall work in 2D but the generalisation to 3D will be obvious.

Let

$$\underline{a} \;=\; a_1\,\mathbf{i} + a_2\,\mathbf{j} \qquad (\mathbf{a})$$

Here a_1 and a_2 are the components of the acceleration and are constants.

Now $\underline{a} = \frac{dv_1}{dt}\,\mathbf{i} + \frac{dv_2}{dt}\,\mathbf{j}$ where v_1 and v_2 are the components of the velocity $\underline{v}$.

Integrating

$$v_1 = a_1 t + u_1, \qquad v_2 = a_2 t + u_2,$$

where u_1 and u_2 are constants of integration. Clearly at $t = 0$ $v_1 = u_1$ and $v_2 = u_2$ so u_1, u_2 are the values of v_1 and v_2 at $t = 0$.

Thus our final expression for $\underline{v}$ is

$$\underline{v} \;=\; (a_1 t + u_1)\,\mathbf{i} + (a_2 t + u_2)\,\mathbf{j} \qquad (\mathbf{v})$$

Also $\underline{v} \;=\; \dfrac{d\underline{r}}{dt}$ and writing $\underline{r} \;=\; x\,\mathbf{i} + y\,\mathbf{j}$ we have

$$\frac{dx}{dt} = v_1 = a_1 t + u_1, \qquad \frac{dy}{dt} = v_2 = \; a_2 t + u_2.$$

Integrating again gives

$$x = a_1\frac{t^2}{2} + u_1 t + c_1, \qquad y = a_2\frac{t^2}{2} + u_2 t + c_2,$$

where c_1 and c_2 are constants of integration. Clearly these are just the values of x and y at $t = 0$.

$$\begin{aligned} \text{Hence}\quad \underline{r} &= x\,\mathbf{i} + y\,\mathbf{j} \\ &= (c_1 + u_1 t + a_1\frac{t^2}{2})\,\mathbf{i} + (c_2 + u_2 t + a_2\frac{t^2}{2})\,\mathbf{j} \\ &= (c_1\,\mathbf{i} + c_2\,\mathbf{j}) + (u_1\,\mathbf{i} + u_2\,\mathbf{j})t + (a_1\,\mathbf{i} + a_2\,\mathbf{j})\frac{t^2}{2} \end{aligned}$$

so

$$\underline{r} \;=\; \underline{c} + \underline{u}t + \frac{1}{2}\underline{a}t^2 \qquad (\mathbf{x})$$

$$\begin{aligned} \text{where}\quad \underline{c} &= c_1\,\mathbf{i} + c_2\,\mathbf{j} && \text{is the initial position} \\ \underline{u} &= u_1\,\mathbf{i} + u_2\,\mathbf{j} && \text{is the initial velocity} \\ \underline{a} &= a_1\,\mathbf{i} + a_2\,\mathbf{j} && \text{is the constant acceleration} \end{aligned}$$

All three of these are constant vectors.

Since $\underline{c}$ is the initial position, $\underline{r} - \underline{c}$ is the change in position, i.e. the displacement $\underline{s}$, we can rewrite $(\mathbf{x})$ as

$$\boxed{\underline{s} \;=\; \underline{u}t + \tfrac{1}{2}\underline{a}t^2} \qquad (1)$$

Clearly this is a generalisation of the 1D formula $s = ut + \frac{1}{2}at^2$.

In 3D the equations $(\mathbf{a})$, $(\mathbf{v})$ and $(\mathbf{x})$ are exactly the same (except that the vectors now have three components instead of two).

Notes

1. If we have motion in 3D *with constant acceleration* then all the motion takes place in a single plane. This is the plane containing the velocity vector at the start (or any other time) and the acceleration vector. The motion is therefore effectively in 2D. e.g. a projectile. The motion of a simple pendulum is also in a single plane, although the acceleration in not constant.

2. Often we can treat the components separately as 1D problems so that a 2D or 3D problem becomes two or three separate 1D problems. This is especially true for the constant acceleration case above.

Other vector formulae for the constant acceleration case

Starting with (**x**) $\underline{r} \;=\; \underline{c} + \underline{u}t + \frac{1}{2}\underline{a}t^2$
we can differentiate with respect to time to get $\underline{v}$

$$\underline{v} \;=\; \frac{d\underline{r}}{dt} \;=\; 0 + \underline{u} + \underline{a}t$$

so $$\boxed{\underline{v} \;=\; \underline{u} + \underline{a}t} \qquad (2)$$

which corresponds to $v \;=\; u + at$ in 1D.

Now consider

$$\begin{aligned}\underline{v}\cdot\underline{v} &= (\underline{u} + \underline{a}t)\cdot(\underline{u} + \underline{a}t)\\ &= \underline{u}\cdot\underline{u} + 2\underline{u}\cdot\underline{a}t + \underline{a}\cdot\underline{a}t^2\end{aligned}$$

but from (1)

$$2\underline{s}\cdot\underline{a} = 2\underline{u}\cdot\underline{a}t + \underline{a}\cdot\underline{a}t^2$$

so $$\boxed{\underline{v}\cdot\underline{v} \;=\; \underline{u}\cdot\underline{u} + 2\,\underline{s}\cdot\underline{a} \quad \text{or} \quad v^2 \;=\; u^2 + 2\,\underline{a}\cdot\underline{s}} \qquad (3)$$

corresponding to $v^2 \;=\; u^2 + 2as$ in 1D.

Finally, from (2) $\underline{v}t \;=\; \underline{u}t + \underline{a}t^2$ so

$$\begin{aligned}\tfrac{1}{2}\underline{v}t &= \tfrac{1}{2}\underline{u}t + \tfrac{1}{2}\underline{a}t^2\\ &= \tfrac{1}{2}\underline{u}t + (\underline{s} - \underline{u}t)\\ &= \underline{s} - \tfrac{1}{2}\underline{u}t\end{aligned}$$

so $$\boxed{\underline{s} \;=\; \tfrac{1}{2}(\underline{u} + \underline{v})t} \qquad (4)$$

which corresponds to $s = \frac{1}{2}(u + v)t$ in 1D.

As noted above, we often treat a 2D or 3D problem as two or three separate 1D problems, so in this case these vector forms are not needed.

2.4 Projectiles I

An important application is the motion of things thrown or dropped which move under the action of gravity. These are called projectiles and the study of their motion is called *ballistics*.

The motion is in 3D but we usually choose the plane in which motion occurs to be the XY-plane so the motion is effectively in 2D. In this plane we choose axes so the the Y-axis is *upwards*, and the X-axis is to the right.

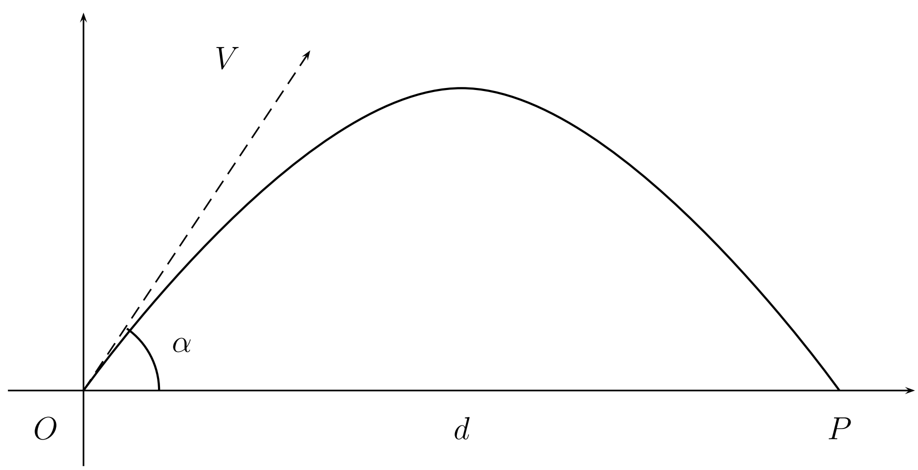
The acceleration due to gravity is constant and is downwards so

$$\underline{a} = -g\,\mathbf{j}.$$

We shall discuss this by means of an example which has most of the important features.

Example: A stone is thrown from the origin with speed V at an angle α above the horizontal. Assuming the ground is horizontal, how far does it travel before it lands?

Let the horizontal distance travelled be d, and let the time of flight be T.



$$\begin{aligned} \text{Initially}\quad \underline{r} &= \underline{0} = 0\,\mathbf{i} + 0\,\mathbf{j} \\ \text{and}\quad \underline{v} &= V\cos\alpha\,\mathbf{i} + V\sin\alpha\,\mathbf{j} \end{aligned}$$

We will do this problem by splitting the motion into horizontal and vertical parts. Each of these is 1D and we can use the usual 1D formulae for motion with constant acceleration.

Horizontally   acceleration $a = 0$
initial velocity $u = V\cos\alpha$
final velocity $v = u + at = V\cos\alpha$
distance travelled $s = \left(\frac{u+v}{2}\right)t = ut$
and $s = d$ and $t = T$ so $d = (V\cos\alpha)T$ (1)

Vertically (upwards)   acceleration $= -g$
initial velocity $u = V\sin\alpha$
distance travelled $s = 0$ since it reaches the ground at P which is at same height as O.

Using the formula $s = ut + \frac{1}{2}at^2$ gives

$$0 = (V\sin\alpha)T + \frac{1}{2}(-g)T^2 \qquad (2)$$

rearranging this gives

$$T\left(\frac{1}{2}\,gT - V\sin\alpha\right) = 0$$

so either $T = 0$ (this is the start) *or* $T = \dfrac{2V}{g}\sin\alpha$ (this is the end).

Finally using (1)

$$\begin{aligned} d &= V\cos\alpha\left(\frac{2V}{g}\sin\alpha\right) \\ &= \frac{2V^2}{g}\sin\alpha\cos\alpha \;=\; \frac{V^2}{g}\sin(2\alpha). \qquad \square \end{aligned}$$

2.5 Projectiles II

Instead of resolving into components, we can construct a solution to the example above using the vector equation

$$\underline{s} = \underline{u}t + \frac{1}{2}\underline{a}t^2$$

with displacement vector $\underline{s} = x\,\mathbf{i} + y\,\mathbf{j}$. Here, the initial velocity is

$$\underline{u} = V\cos\alpha\,\mathbf{i} + V\sin\alpha\,\mathbf{j}$$

and the acceleration is downwards so $\underline{a} = -g\,\mathbf{j}$. Thus

$$x\,\mathbf{i} + y\,\mathbf{j} \;=\; (V\cos\alpha\,\mathbf{i} + V\sin\alpha\,\mathbf{j})t + \frac{1}{2}(-g\,\mathbf{j})t^2 \qquad (3)$$

At time $t = T$, $x = d$ and $y = 0$ so

$$d\,\mathbf{i} \;=\; (V\cos\alpha\,\mathbf{i} + V\sin\alpha\,\mathbf{j})\,T - \frac{1}{2}g\,\mathbf{j}\,T^2\,.$$

Comparing the $\mathbf{i}$ coefficients on both sides gives

$$d = V\cos\alpha\,T \qquad (1)$$

and comparing the $\mathbf{j}$ coefficients on both sides gives

$$0 = V\sin\alpha\,T - \frac{1}{2}g\,T^2 \qquad (2)$$

Since (1) and (2) match the equations derived earlier the solution is the same as before.

Other quantities of interest:

1. The horizontal distance travelled d is called the *range*.

2. The *time of flight* is $T = \frac{2V}{g} \sin\alpha$.

3. At the *highest point* of the flight when $y = h$, the vertical velocity is zero. Thus using $v^2 = u^2 + 2as$ gives

$$0 = (V \sin\alpha)^2 - 2gh \quad \text{so} \quad h = \frac{V^2}{2g} \sin^2\alpha \,.$$

4. The *time taken* to reach the greatest height is $\frac{1}{2}T$. (left as an exercise.)

5. For a given V the range $d = \frac{V^2}{g} \sin(2\alpha)$ is largest when $\sin(2\alpha) = 1$.

Since the corresponding angle satisfies $2\alpha = 90°$ the *angle for maximum range* is given by

$$\alpha \;=\; 45° \,.$$

The actual *maximum range* is given by $\frac{V^2}{g}$.

6. The equation of the path is a *parabola*. To show this from (3) we have

$$x = V \cos\alpha\, t \quad \text{and} \quad y = V \sin\alpha\, t - \frac{1}{2} g\, t^2,$$

$$\text{so} \quad t = \frac{x}{V \cos\alpha}$$

$$\begin{aligned} \text{and} \quad y &= (V \sin\alpha) \frac{x}{V \cos\alpha} - \frac{1}{2} g \left(\frac{x}{V \cos\alpha} \right)^2 \\ &= (\tan\alpha) x - \frac{g}{2V^2 \cos^2\alpha} x^2 \\ &= Ax - Bx^2, \qquad (4) \end{aligned}$$

where

$$A = \tan\alpha \qquad \text{and} \qquad B = \frac{g}{2V^2} \sec^2\alpha \,.$$

Equation (4) is the equation of a parabolic curve that passes through the origin.

Note that

$$y \;=\; -B(x - \frac{A}{2B})^2 + \frac{A^2}{4B}.$$

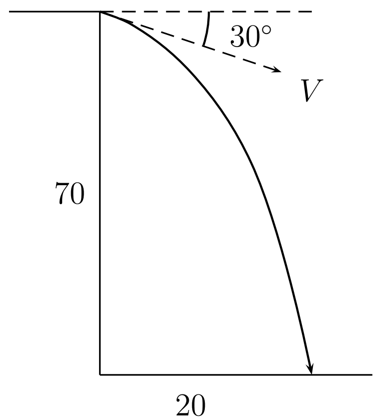
is the same as $y = -Bx^2$ if we shift the origin by replacing

$$y \longrightarrow y - \frac{A^2}{4B} \quad \text{and} \quad x \longrightarrow x - \frac{A}{2B}. \quad \heartsuit$$

2.6 Projectiles III

Two simple example problems will be considered next. (We will take $g = 9.81$ ms^{-2}.)

Example 1 A stone is thrown from the top of a cliff of height 70 m at an angle of 30° *below* the horizontal. It hits the sea 20 m from the base of the cliff. Find the initial speed of the stone, the time of flight and the angle at which it strikes the sea.



Solution Let the initial speed be V. Let the time of flight be T.

Horizontally $a = 0$, $u = V\cos 30° = V\dfrac{\sqrt{3}}{2}$, $s = 20$.

Using $s = ut + \frac{1}{2}at^2$ gives

$$20 = V\frac{\sqrt{3}}{2}T + 0 \quad \text{thus} \quad T = \frac{40}{\sqrt{3}V}. \qquad (1)$$

Vertically (downwards) $a = g$, $u = V\sin 30° = \dfrac{1}{2}V$, $s = 70$.

$$70 = (\frac{1}{2}V)T + \frac{1}{2}gT^2. \qquad (2)$$

Substituting (1) for T into (2) gives

$$\begin{aligned}
70 &= \frac{1}{2}V\left(\frac{40}{\sqrt{3}V}\right) + \frac{1}{2}g\left(\frac{40}{\sqrt{3}V}\right)^2 \\
70 &= \frac{20}{\sqrt{3}} + \frac{800 \times 9.81}{3V^2} \\
V^2 &= \frac{7848}{3 \times 58.45} = 44.76 \quad \text{thus} \quad V = 6.69\,\text{ms}^{-1}.
\end{aligned}$$

Also from (1)

$$T = \frac{40}{\sqrt{3} \times 6.69} = 3.452 \text{ seconds.}$$

The *angle of entry* to the sea is determined by the horizontal and vertical components of the velocity v_1 and v_2.

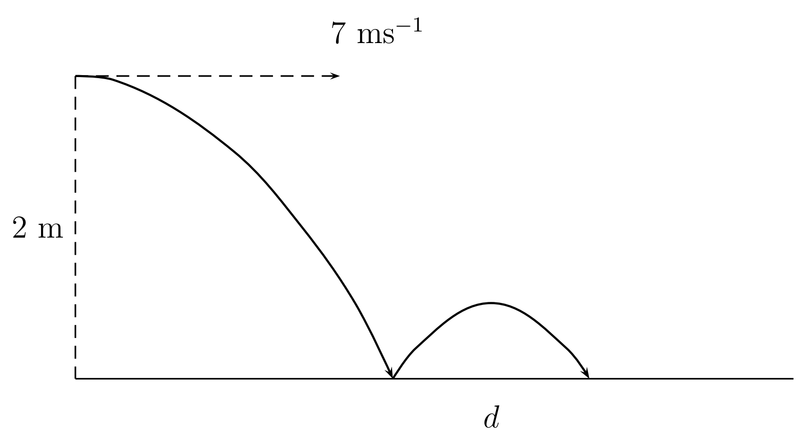
Horizontally, the velocity v_1 does not change so $v_1 = V \cos 30° = 6.96 \times 0.8660 = 6.028$.

Vertically, $v_2 = V \sin 30° + gT = 3.345 + 9.81 \times 3.452 = 36.13$. (downwards).

The angle θ with the horizontal at entry is given by $\tan\theta = v_2/v_1 = 5.993$
so $\theta = 80.5°$. (Note that it has to lie between $0°$ and $90°$).

Key point When a particle strikes a fixed surface at an angle which is not $90°$ then only the component of the velocity perpendicular to the surface changes. The component of the velocity parallel to the surface is not affected.

Example 2 A ball of mass m is thrown at a speed 7ms^{-1} in a horizontal direction at a height of 2 metres above the (horizontal) ground. The coefficient of restitution is $e = 0.5$. Find the distance between the first and second bounces.



Solution. There is no acceleration in the horizontal direction and no change in the horizontal component of the velocity at the first (or any) bounce. Therefore the horizontal velocity is always 7ms^{-1}.

We now calculate the time between the first and second bounces by considering the *vertical* motion.

1. Descent to ground.
Starting vertical velocity $u = 0$. Acceleration $a = 9.81\text{ms}^{-2}$ (downwards).

Distance $s = 2$m. Final velocity v given by

$$v^2 \; = \; u^2 + 2as \; = \; 0 + 2 \times 9.81 \times 2 \; = \; 39.24$$

so $v \; = \; \sqrt{39.24} \; = \; 6.264\text{ms}^{-1}$.

2. First bounce.
After the first bounce the vertical velocity (upwards) is $ev \; = \; 3.132\text{ms}^{-1}$.

3. Between first and second bounces.
Initial velocity (upwards) is 3.132. At the greatest height the vertical velocity is 0.
Time taken to reach greatest height is t given by

$$0 \; = \; 3.132 - 9.81t \quad \text{thus} \quad t = 0.319.$$

Next, consider the *horizontal* motion.
Since the bounces have no effect on the horizontal component of the velocity,
this is always 7ms^{-1}.

Total time between first and second bounces is $2t \; = \; 0.638$s.
Horizontal distance covered in this time is $7 \times 0.638 \; = \; 4.47$m.

2.7 Newton's laws

For a particle of mass m, Newton's second law (N2) in vector form is

$$\boxed{\vec{F} \; = \; m\,\underline{a}}$$

so the direction of the acceleration is the same as the direction of the force. Otherwise the law is the same as in 1D.

Example A force $\vec{F} \; = \; 4t\,\mathbf{i} + 2\sin t\,\mathbf{j}$ acts on a particle of mass 2 kg which is initially at $\underline{r} \; = \; (4, -1)$ with $\underline{v} \; = \; 0$. Find

(i) the velocity at a function of time

(ii) the work done by the force between $t = 0$ and $t = T$

Solution

(i) By N2 $\vec{F} \; = \; m\underline{a}$

$$\text{hence} \quad \underline{a} \; = \; \frac{1}{2}\vec{F} \; = \; 2t\,\mathbf{i} + \sin t\,\mathbf{j}$$

and since $\underline{a} = \dfrac{d\underline{v}}{dt}$ we have

$$\frac{d\underline{v}}{dt} = 2t\,\mathbf{i} + \sin t\,\mathbf{j}$$

Integrating gives

$$\underline{v} = (t^2 + c_1)\,\mathbf{i} + (-\cos t + c_2)\,\mathbf{j}$$

where c_1 and c_2 are constants of integration.
At $t = 0$

$$\underline{v} = c_1\,\mathbf{i} + (-1 + c_2)\,\mathbf{j} = 0\,\mathbf{i} + 0\,\mathbf{j}$$

Thus $c_1 = 0$ and $c_2 = 1$ and so

$$\underline{v} = t^2\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}.$$

(ii) By the principle of the equivalence of work and energy, the work done by the force must equal the change in KE. (There is no change in PE since there is no gravitational force in this problem.)

At $t = 0$, $\underline{v} = 0$ so $v = \underline{0}$ and $\mathrm{KE} = \frac{1}{2}mv^2 = 0$.
At $t = T$, $\underline{v} = T^2\,\mathbf{i} + (1 - \cos T)\,\mathbf{j}$ so

$$v^2 = \underline{v} \cdot \underline{v} = T^4 + (1 - \cos T)^2$$

$$\text{thus} \quad \mathrm{KE} = \frac{1}{2}mv^2 = \frac{1}{2} \times 2 \times v^2 = v^2 = T^4 + (1 - \cos T)^2$$

and this is the work done.

2.8 Momentum and energy conservation

Momentum in vector form is $m\,\underline{v}$.

The rate of change of momentum, assuming mass m is constant is

$$\frac{d}{dt}(m\underline{v}) = m\frac{d\underline{v}}{dt} = m\underline{a} = \vec{F},$$

so if $\vec{F} = 0$ then momentum is *conserved.*

Kinetic energy is given by

$$\mathrm{KE} = \frac{1}{2}m\underline{v} \cdot \underline{v} = \frac{1}{2}mv^2,$$

where v is the magnitude of vector $\underline{v}$, which is the same as in 1D.

Note: the scalar product of a vector with itself gives the magnitude of the vector:

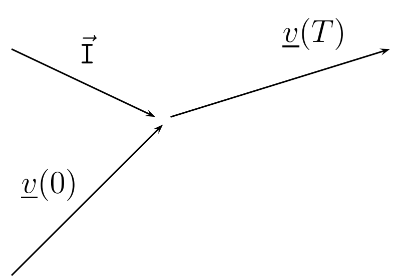
$$\underline{v} \cdot \underline{v} \;=\; (v_x, v_y, v_z) \cdot (v_x, v_y, v_z) \;=\; v_x^2 + v_y^2 + v_z^2 \;=\; v^2$$

Impulse is now a vector quantity

$$\begin{aligned}
\vec{\mathtt{I}} &= \int_0^T \vec{F}\, dt \\
&= \int_0^T m\underline{a}\, dt \\
&= m \int_0^T \frac{d\underline{v}}{dt}\, dt \\
&= m[\underline{v}(T) - \underline{v}(0)]
\end{aligned}$$

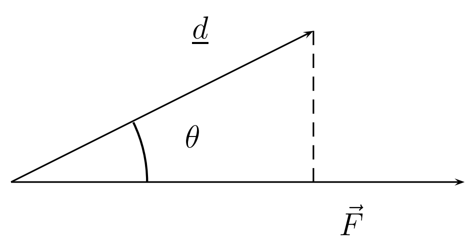
Hence $\vec{\mathtt{I}}$ gives the change in momentum, just as in 1D.

However, the direction of $\vec{\mathtt{I}}$ is not in general the same as that of $\underline{v}$,



$\underline{v}(0)$, $\underline{v}(T)$ and $\vec{\mathtt{I}}$ can be in three different directions!

Work done is force × distance moved in the direction of the force.



For displacement $\underline{d}$ and force $\vec{F}$ we have $W = Fd\cos\theta$ or

$$\boxed{W \;=\; \vec{F} \cdot \underline{d}}$$

Note This formula is true for constant forces. If the force is not constant then it is usually better to measure the work done as the change in KE.

Power is now defined as

$$P = \vec{F} \cdot \underline{v}$$

and for constant forces

$$\frac{dW}{dt} = \vec{F}.\frac{d}{dt}(\underline{d}) = \vec{F}.\underline{v} = P$$

so the power is the rate of doing work, as in 1D.

Example (revisited) A force $\vec{F} = 4t\,\mathbf{i} + 2\sin t\,\mathbf{j}$ acts on a particle of mass 2 kg which is initially at $\underline{r} = (4, -1)$ with $\underline{v} = 0$. Find

(iii) the power of $\vec{F}$ at time T

(iv) the position at time $t = 2$.

Solution

(iii) From part (i) we know that

$$\underline{v} = t^2\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}.$$

$$\begin{aligned} \text{Power} &= \vec{F} \cdot \underline{v} \\ &= (4T\,\mathbf{i} + 2\sin T\,\mathbf{j}) \cdot (T^2\,\mathbf{i} + (1 - \cos T)\,\mathbf{j}) \\ &= 4T^3 + 2\sin T(1 - \cos T). \end{aligned}$$

(iv) Since the velocity is the differential of position, we have

$$\frac{d\underline{r}}{dt} = t^2\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}\,.$$

Integrating gives

$$\underline{r} = (t^3/3 + d_1)\,\mathbf{i} + (t - \sin t + d_2)\,\mathbf{j}$$

where d_1 and d_2 are the constants of integration.

We know that at $t = 0$, $\underline{r} = 4\mathbf{i} - \mathbf{j}$.
Therefore

$$0 + d_1 = 4 \quad \text{and} \quad 0 - 0 + d_2 = -1$$

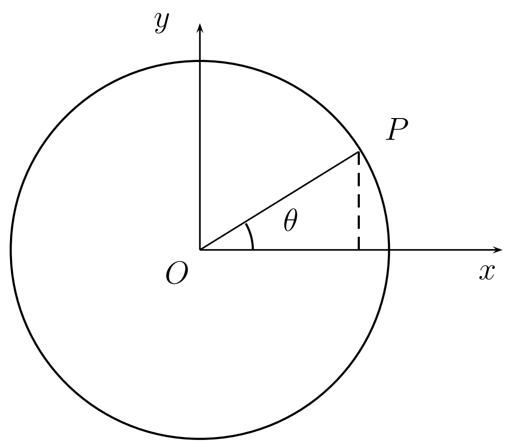
so $d_1 = 4$ and $d_2 = -1$. Hence

$$\underline{r} = (t^3/3 + 4)\,\mathbf{i} + (t - \sin t - 1)\,\mathbf{j},$$

and when $t = 2$, we have $\underline{r} = 20/3\,\mathbf{i} + (1 - \sin 2)\,\mathbf{j} \approx\ 6.666\mathbf{i} + 0.0907\mathbf{j}$.

2.9 Motion in a circle at constant speed

Suppose a particle moves in 2D on a circle of radius R centred at O.



When the angle of the position vector $\vec{OP}$ is θ to the x-axis the coordinates are $(R\cos\theta,\ R\sin\theta)$, so the position vector is given by

$$\underline{r} \;=\; R\cos\theta\,\mathbf{i} + R\sin\theta\,\mathbf{j}.$$

Since θ is the angle we say that the rate of change $\frac{d\theta}{dt}$ is the *angular velocity*. The units are radians/sec. We assume here that the angular velocity is *constant* and equal to a value ω (Greek letter omega).

$$\text{so} \quad \frac{d\theta}{dt} = \omega \qquad \text{(constant)}$$

Integrating this expression gives

$$\theta = \omega\, t + \phi$$

where ϕ (Greek letter phi) is the constant of integration, sometimes called the *phase*. We typically choose θ to be 0 at $t = 0$ so $\phi = 0$.

We now have

$$\underline{r} \;=\; R\cos(\omega t)\,\mathbf{i} + R\sin(\omega t)\,\mathbf{j}.$$

Differentiating gives the velocity

$$\underline{v} \;=\; -\,\omega R\sin(\omega t)\,\mathbf{i} + \omega R\cos(\omega t)\,\mathbf{j}.$$

The *speed* v is the magnitude of the velocity vector so

$$\begin{aligned} v^2 &= [-\omega R \sin(\omega t)]^2 + [\omega R \cos(\omega t)]^2 \\ &= \omega^2 R^2 \sin^2(\omega t) + \omega^2 R^2 \cos^2(\omega t) \\ &= \omega^2 R^2 [\sin^2(\omega t) + \cos^2(\omega t)] \\ &= \omega^2 R^2. \end{aligned}$$

Thus

$$\boxed{v \ = \ \omega R}$$

Clearly v is constant so for circular motion. Thus constant angular velocity means constant speed and vice versa.

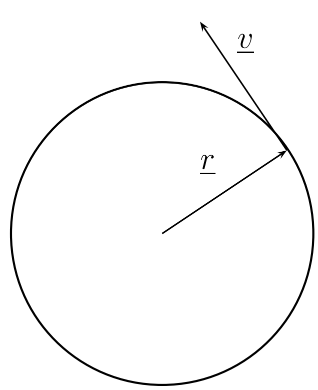
The *period* is the time taken for one complete revolution. We use the symbol τ (Greek letter tau) for this. The distance is $2\pi R$ and the speed is ωR so the time taken is $2\pi R/\omega R$ and so the period is

$$\boxed{\tau \ = \ \frac{2\pi}{\omega}}$$

We expect the direction of the velocity $\underline{v}$ to be the tangent to the circle, i.e. perpendicular to the radius. We can show this by calculating the dot product of the velocity and position

$$\begin{aligned} \underline{r} \cdot \underline{v} &= [R \cos(\omega t)\,\mathbf{i} + R \sin(\omega t)\,\mathbf{j}] \cdot [-\omega R \sin(\omega t)\,\mathbf{i} + \omega R \cos(\omega t)\,\mathbf{j}] \\ &= -\omega R^2 \cos(\omega t) \sin(\omega t) + \omega R^2 \cos(\omega t) \sin(\omega t) \\ &= 0 \end{aligned}$$

so $\underline{v}$ is perpendicular to $\underline{r}$ and so the direction of $\underline{v}$ is the tangent to the circle.



Example A student sits on the equator. If the radius of the earth is 6400 km what is the speed of the student in km/hr (relative to the centre of the earth)?

Solution The earth does one revolution of 2π radians in 24 hours.

Therefore the angular velocity is

$$\omega \;=\; \frac{2\pi}{24} \;=\; \frac{\pi}{12} \quad \text{radians per hour}$$

$$\begin{aligned} \text{speed} \quad v \;=\; \omega R \;&=\; \frac{\pi}{12} \times 6400 \text{ km/hr} \\ &=\; \frac{1600\pi}{3} \;=\; 1676 \text{ km/hr.} \qquad \heartsuit \end{aligned}$$

We can also compute the acceleration in the general case. Thus,

$$\begin{aligned} \underline{r} \;&=\; R\cos(\omega t)\,\mathbf{i} + R\sin(\omega t)\,\mathbf{j} \\ \underline{v} \;&=\; -\,\omega R\sin(\omega t)\,\mathbf{i} + \omega R\cos(\omega t)\,\mathbf{j} \\ \text{so} \quad \underline{a} \;=\; \frac{d\underline{v}}{dt} \;&=\; -\,\omega^2 R\cos(\omega t)\,\mathbf{i} - \omega^2 R\sin(\omega t)\,\mathbf{j} \\ &=\; -\,\omega^2[R\cos(\omega t)\,\mathbf{i} + R\sin(\omega t)\,\mathbf{j}] \\ &=\; -\,\omega^2\underline{r}. \end{aligned}$$

This implies that the acceleration is directed towards the centre of the circle.

The force needed to generate this acceleration is called the *centripetal force*

$$\vec{F} = m\underline{a} = -m\omega^2\underline{r}$$

and is also directed towards the centre of the circle. Sometimes it is wrongly called the centrifugal force which would be a force *away* from the centre.

Finally, the magnitude of the acceleration is given by

$$a \;=\; |\underline{a}| \;=\; |-\omega^2\underline{r}| \;=\; \omega^2 R$$

since $|\underline{r}|$ is always R wherever it is on the circle.

Since $v \;=\; \omega R$ we can also write

$$\boxed{a \;=\; \omega v \;=\; \frac{v^2}{R}}$$

Example Find the magnitude of the acceleration of the student in the previous example.

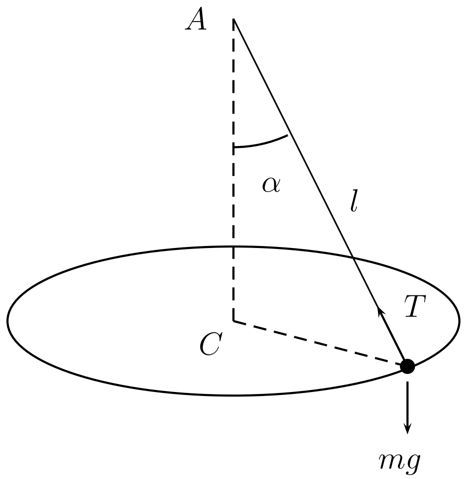
Solution we have $\omega = \dfrac{\pi}{12}$ km/hr, and $R = 6400$ km,

$$\begin{aligned} \text{so} \quad a \;&=\; \omega^2 R \\ &=\; \frac{\pi^2}{144} \times 6400 \quad \text{km/hr}^2 \\ &=\; 438.6 \quad \text{km/hr}^2. \end{aligned}$$

(We could convert to ms^{-2} by multiplying by $\frac{1000}{3600^2}$.)

2.10 A conical pendulum

This consists of a light inelastic string attached at one end to a fixed point A. The other end is attached to a particle of mass m which is moving in a *horizontal* circle whose centre O is directly below A.



Notation

Let the length of the string be l. Let the angle with the vertical be α.
Let the tension in the string be T. Let the speed be v.

The radius of the circle is $R = l \sin \alpha$
so the angular velocity $\omega = \frac{v}{R} = \frac{v}{l \sin \alpha}$.

Here the tension T does two jobs. The vertical component balances the weight mg so that there is no vertical motion. The horizontal component acts towards C and provides the centripetal force necessary for the circular motion.

Vertically

$$T \cos \alpha = mg \quad \text{so} \quad T = \frac{mg}{\cos \alpha} .$$

Clearly we could never have $\alpha = 90°$ which would require T to be infinite!

Horizontally (towards the centre C)

$$\text{force} = T \sin \alpha \qquad \text{so from N2 the acceleration is } a = \frac{T \sin \alpha}{m} \ .$$

But $a = \omega^2 R = \omega^2 l \sin\alpha$ (from above)

$$\text{thus} \quad \omega^2 l \sin\alpha = \frac{T\sin\alpha}{m} \quad \textit{and}$$
$$\omega^2 = \frac{T}{ml} = \frac{1}{ml}\,\frac{mg}{\cos\alpha}$$

$$\text{Hence} \quad \boxed{\omega^2 = \frac{g}{l\cos\alpha}}.$$

Notice that this result is independent of m!

This gives the relation between ω, l and α for the conical pendulum. We can also calculate

$$v = \omega R = l\sin\alpha\sqrt{\frac{g}{l\cos\alpha}} \quad \text{and} \quad \tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l\cos\alpha}{g}}.$$

Example A conical pendulum has string of length 2 m. If the pendulum makes 1 revolution per second, find the angle the string makes with the vertical.

Solution

One revolution $= 2\pi$ radians.

Therefore the angular velocity $\omega = 2\pi$ radians/sec.

Let the angle the string makes with the vertical be α.

From the above

$$\omega^2 = \frac{g}{l\cos\alpha}$$

$$\text{thus} \quad \cos\alpha = \frac{g}{\omega^2 l} = \frac{9.81}{(2\pi)^2 \times 2} = 0.1242$$

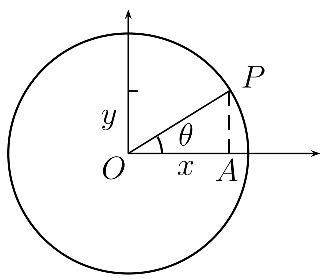
$$\text{thus} \quad \alpha = 1.4462 \text{ radians} = 82.86^\circ \qquad \heartsuit$$

where we have taken $g = 9.81$ ms^{-2}.

Note that this particle is going rather fast. This results in a high value for α, so the string is getting close to horizontal.

2.11 Simple harmonic motion

Recall the case of motion in a circle at constant speed



The position vector is $\underline{r} = x\,\mathbf{i} + y\,\mathbf{j}$, where $x = R\cos\theta$ and $y = R\sin\theta$, with a constant angular velocity ω. Next, consider the horizontal coordinate, that is, the motion of the point A along the x-axis as the point P travels around the circle.

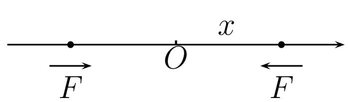
$$\begin{aligned} x &= R\cos(\omega t) \\ \frac{dx}{dt} &= -\omega R\sin(\omega t) \\ \frac{d^2x}{dt^2} &= -\omega^2 R\cos(\omega t) \end{aligned}$$

so clearly

$$\boxed{\frac{d^2x}{dt^2} \;=\; -\,\omega^2 x}$$

This is the equation of *simple harmonic motion* in 1D.

General case A particle of mass m moves in 1D and has position x at time t.



Suppose that the force acting upon the particle is given by $F = \ kx$, where k is a positive constant. Note that F always acts towards the origin.

Since $ma = F$ we have $m\dfrac{d^2x}{dt^2} = -kx$ and $\dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x$.

Since k is positive and so is m we can set $\frac{k}{m} = \omega^2$ to get the equation above.

Simple harmonic motion occurs, for example, when a particle moves up and down on an elastic string or spring. It also describes the motion of a simple pendulum provided the angle of swing is small. You have to love SHM ...)

It arises *everywhere* in engineering situations.