MATH19872, Mathematics 0D2

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Contents

Chapter 1

Interpolation

Frequently we make a measurement f at discrete locations in time or space x as in Figure 1.1. For example,

- An oscilloscope may sample a voltage in a circuit every 1μs
- A survey team may measure the elevation of some topography at intervals of a few hundred metres
- A weather balloon may be sent up to measure the atmospheric conditions once every 24 hours
- The government may collect daily covid data like the number of people tested positive [1]

But what if we want to know the value of our quantity in between these discrete measurements? For example, what if we want to know the atmospheric conditions every hour, rather than every 24 hours? One way of doing this would be to make measurements more frequently, but this may not be practical. Instead, we can approximate the value in between our measurements, by a technique called *interpolation*.

[1] The data are freely available at https://coronavirus.data.gov.uk/

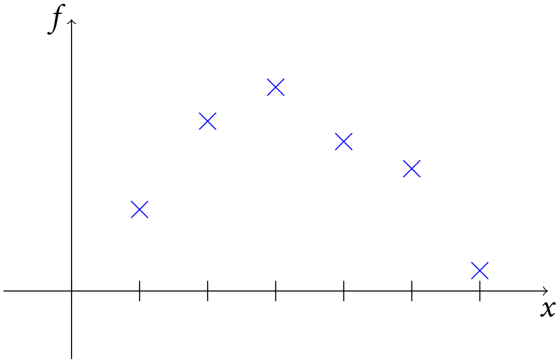

Figure 1.1: A measurement of f at discrete locations x.

1.1 Linear interpolation

One of the simplest methods of interpolation is to assume that our quantity varies linearly between the measured values. Geometrically, this means we connect our measurements with straight lines and use these to determine the value of our quantity in between measurements as in Figure 1.2.

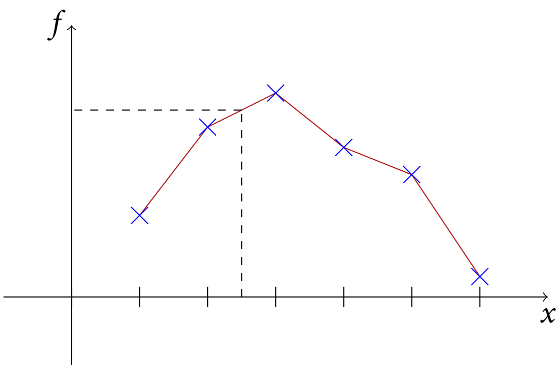

Figure 1.2: Linear interpolation with discrete measurements.

Suppose that two measurements are taken at locations x_0 and x_1. We denote the value of these measurements f_0 and f_1, respectively, as in Figure 1.3.

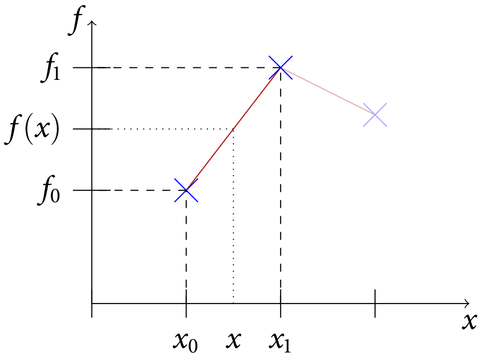

Figure 1.3: Linear interpolation between the discrete measurements, where $f(x) = f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0)$, using the given data points.

We wish to approximate the value of

$$f(x), \qquad \text{where } x_0 \leq x \leq x_1.$$

To do this we construct the straight line connecting the known points (x_0, f_0) and (x_1, f_1). This line has the equation

$$f(x) = f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0). \tag{1.1}$$

To find the value of $f(x)$ we simply substitute x into (1.1).

Example 1.1.1: Verify that (1.1) is indeed a straight line which passes through the known points (x_0, f_0) and (x_1, f_1). That is, show that $f(x) = mx + c$ for some constants m and c, and show that $f(x_0) = f_0$ and $f(x_1) = f_1$. (This means that when we interpolate our function at an x-value where we already have a measurement, our answer is exactly that measurement.)

Example 1.1.2: Given that

$$\sin(0.2) \approx 0.1987 \quad \text{and} \quad \sin(0.3) \approx 0.2955,$$

estimate the value of $\sin(0.26)$ by linear interpolation. (Note: angles are given in radians unless otherwise specified.)

Example 1.1.3: The height of the tide is measured against a harbour wall every hour, and the depth of water is recorded:

Hours after midnight x	Depth $f(x)$ (m)
0	1.5
1	1.9
2	2.5
3	3.4
4	3.8
5	3.9
6	4.0

Estimate the depth of the water at 3.6 hours after midnight

Interpolating x given $f(x)$

We can also use linear interpolation to find x given $f(x)$. For instance, we know that $\sin(0.2) \approx 0.1987$ and $\sin(0.3) \approx 0.2955$, we want to know for which angle x (still given in radians) such that $\sin(x) = 0.2500$. To do this, we rearrange our linear interpolation equation (1.1) to make x the subject of the formula:

$$x = x_0 + \frac{x_1 - x_0}{f_1 - f_0}\left(f(x) - f_0\right). \tag{1.2}$$

(note that this looks just like (1.1), but with x and f swapped).

Example 1.1.4: A boat can sail from the harbour only when the depth of water is greater than 3m. By linearly interpolating the tide data given in example 1.1.3, find the earliest time at which the boat can sail.

1.2 Polynomial interpolation

The linear interpolation we have looked at so far is a special case of *polynomial interpolation*. In polynomial interpolation, we construct a polynomial of degree n so that it passes through $n+1$ of our known measurements. We can then evaluate that polynomial at a value of x to find the interpolated value.

In the previous section we looked at linear interpolation, where our interpolating function is a polynomial of degree $n = 1$, a straight line. We used two measurements f_0 and f_1 to construct this interpolating function.

We can also choose our interpolating function to be a polynomial of degree $n = 2$, a quadratic,

$$f(x) = ax^2 + bx + c.$$

In this case we need three measurements, f_0, f_1 and f_2 (taken at x_0, x_1 and x_2 respectively) to construct the interpolating quadratic.

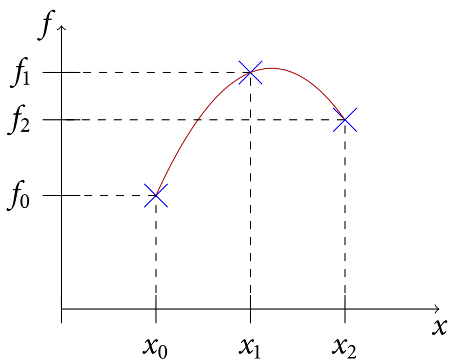

Figure 1.4: Quadratic interpolation using the data (x_0, f_0), (x_1, f_1) and (x_2, f_2).

The equation for the interpolating quadratic is

$$\begin{aligned} f(x) = {} & \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 \\ & + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 \\ & + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2. \end{aligned} \tag{1.3}$$

Example 1.2.1: Verify that $f(x_0) = f_0$, $f(x_1) = f_1$ and $f(x_2) = f_2$, i.e. that the interpolating polynomial passes through the measured values.

Example 1.2.2: Find the quadratic interpolating polynomial for

$$\begin{aligned} f(0) &= 6.4 \\ f(1) &= 4.7 \\ f(2) &= 1.2 \end{aligned}$$

and evaluate it at $x = 0.25$.

The formula for quadratic interpolation (1.3) looks quite complicated. A helpful trick that can help pick up mistakes in the use of this formula is to note that the coefficients multiplying f_0, f_1 and f_2 in (1.3) should always sum to 1. That is,

$$\begin{aligned} & \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \\ & + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \\ & + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\ & = 1. \end{aligned}$$

Chapter 2

Numerical differentiation

Given an explicit expression for a function, for example

$$f(x) = x^2, \tag{2.1}$$

we can differentiate this to find the derivative of our function,

$$\frac{df}{dx} = 2x. \tag{2.2}$$

But sometimes we may not have an algebraic expression because:

- Our function may not have an explicit form and we can only evaluate or measure it numerically (see chapter 5),
- Our function may be defined by a set of discrete measurements, such as the tide data in example 1.1.3.

In these cases we have to evaluate the derivative numerically.

2.1 The first derivative

Given a function $f(x)$, the derivative f' at a point $x = x_0$ can be approximated using a *finite-difference formula*

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h}, \tag{2.3}$$

where h is a (small) 'step size'.

The expression (2.3) is in fact the gradient of the *chord* to the curve, shown in Figure 2.1 as a blue dashed line, which we can see is

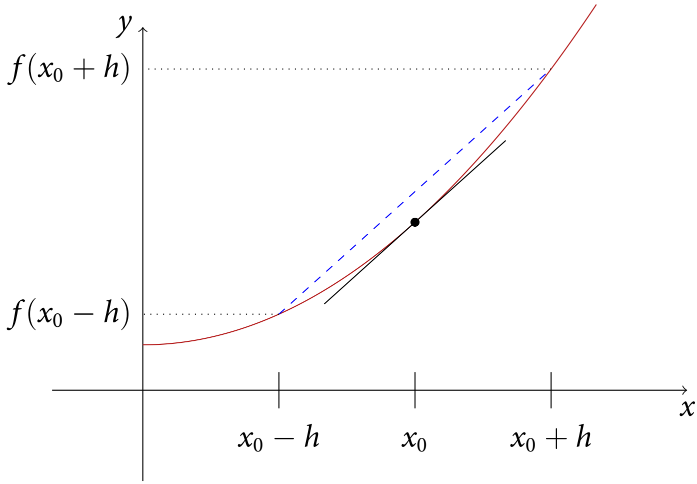

Figure 2.1: The derivative of f at x_0 is approximated using $f(x_0 + h)$ and $f(x_0 - h)$.

a good approximation to the actual gradient of the curve at $x = x_0$ (black solid line). If $f(x)$ comes from tabulated data (as in example 2.1.2), h is determined by the interval at which the values are tabulated. If we have an algebraic expression for the function $f(x)$, we can choose any value of h, since we can evaluate f at any value of x. Decreasing the step size h makes the approximation more accurate, so in this case we usually set h to be a very small number, perhaps 10^{-6}. When choosing a small value of h, we must be careful of 'roundoff error': if h is *too* small, the difference between $f(x_0 + h)$ and $f(x_0 - h)$ may be too small to represent accurately on a calculator. When using finite-difference formulae, make sure to use all of the digits displayed on the calculator.

Example 2.1.1: Let

$$f(x) = e^{\sin x}.$$

Working to 8 decimal places, estimate $f'(x)$ when $x = 1$, with

1. $h = 0.1$
2. $h = 0.001$
3. $h = 10^{-7}$

and compare with the exact value.

Example 2.1.2: On the basis of the following data, estimate $f'(x)$ at the point where $x = 5.3$.

x	$f(x)$
5.0	2.2804
5.1	2.3013
5.2	2.3221
5.3	2.3428
5.4	2.3633
5.5	2.3837
5.6	2.4039

2.2 The second derivative

Given a function $f(x)$, the second derivative f'' at a point $x = x_0$ can be approximated by another finite-difference formula,

$$f''(x_0) \approx \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}, \tag{2.4}$$

where h is again the step size. As with the first derivative, decreasing h will make this approximation more accurate, but values h that are too small will be less accurate due to roundoff error.

Example 2.2.1: On the basis of the following data, estimate $f''(5.3)$.

x	$f(x)$
5.0	2.2804
5.1	2.3013
5.2	2.3221
5.3	2.3428
5.4	2.3633
5.5	2.3837
5.6	2.4039

2.3 Higher-order finite difference formulae

We can construct more accurate approximations of the derivatives by making use of more values of our function. An approximation for the first derivative of a function $f'(x)$ that uses values of f at four points is

$$f'(x_0) \approx \frac{-f(x_0+2h) + 8f(x_0+h) - 8f(x_0-h) + f(x_0-2h)}{12h}. \tag{2.5}$$

Example 2.3.1: Calculate the derivative of

$$f(x) = e^{\sin x}.$$

at $x = 1$, using $h = 0.01$. Use both the two-point approximation (2.3) and the four-point approximation (2.5). Compare these approximations to the exact value.

2.4 Asymmetric finite difference formulae

The approximations discussed so far are all *centred*, meaning that f is evaluated 'symmetrically' both for values of x greater than x_0 and for values less than x_0. This is problematic if for some reason we do not know any values of f for $x > x_0$ or $x < x_0$. In these cases, we must use *one-sided* finite-difference approximations for the derivative.

If f is unknown or undefined for $x < x_0$, we can use

$$f'(x_0) \approx \frac{-3f(x_0) + 4f(x_0+h) - f(x_0+2h)}{2h}, \tag{2.6}$$

which does not evaluate f for any values of $x < x_0$. If f is unknown or undefined for $x > x_0$, we can use

$$f'(x_0) \approx \frac{3f(x_0) - 4f(x_0-h) + f(x_0-2h)}{2h}, \tag{2.7}$$

which does not evaluate f for any values of $x > x_0$.

Example 2.4.1: Using the following data, find an approximation to $f'(5.6)$.

x	$f(x)$
5.0	2.2804
5.1	2.3013
5.2	2.3221
5.3	2.3428
5.4	2.3633
5.5	2.3837
5.6	2.4039

Chapter 3

Numerical integration

Some functions can be integrated algebraically, but many cannot. For example, the integral

$$\int_0^1 xe^{-x^2}\,dx$$

can be integrated (giving $[-e^{-x^2}/2]_0^1 = (1-1/e)/2$), but the (apparently simpler) integral

$$\int_0^1 e^{-x^2}\,dx \tag{3.1}$$

cannot be written as a simple 'closed-form' algebraic expression. The geometrical interpretation of the integral (3.1) is the area under the curve $y = e^{-x^2}$, as in Figure 3. The integral evaluates to some number, but we just cannot write this number algebraically in a simple way.

To evaluate integrals such as (3.1) we must resort to numerical techniques for evaluating the area under the curve. This is known as *numerical integration* or *quadrature*. Numerical methods can only find approximations to the exact value of the integral, but often these approximations can be made very accurate.

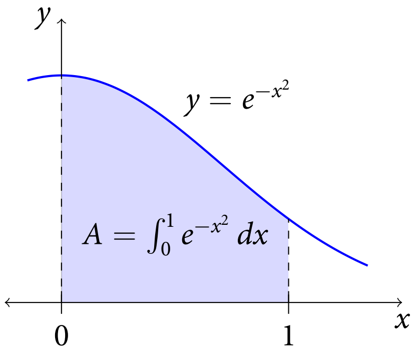

Figure 3.1: The integral $\int_0^1 e^{-x^2} dx$ is the area under the curve $y = e^{-x^2}$, i.e., the shaded region.

3.1 The trapezium rule

Suppose we wish to evaluate the integral

$$\int_a^b f(x)\,dx. \tag{3.2}$$

The concept of the *trapezium rule* is that we split the domain from a to b into some number of strips n, each of width h, where

$$h = (b-a)/n. \tag{3.3}$$

We approximate the curve $f(x)$ by making the top edge of each strip into a straight line, so the strip becomes a trapezium[1] (Figure 3.1 and 3.1). The area of a trapezium is

$$\text{width} \times \text{average height} = h \times \frac{\text{height on left} + \text{height on right}}{2}.$$

[1]Note that this procedure is closely linked to linear interpolation (section 1.1). One way of viewing the trapezium rule is that we linearly interpolate our integrand $f(x)$ between equally spaced points $a, a+h, \dots, b$ and then integrate (algebraically) this linear interpolation.

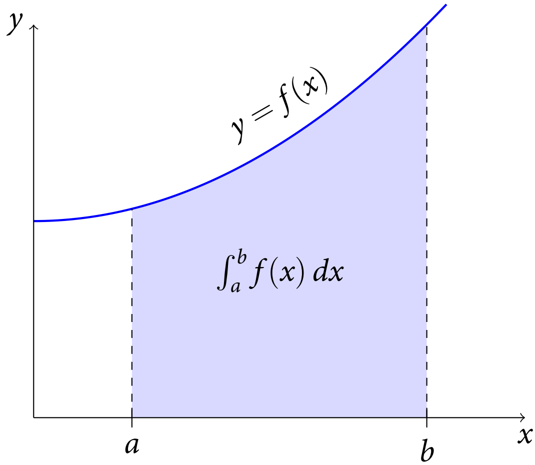

Figure 3.2: How do we approximate the integral $\int_a^b f(x)dx$ numerically?

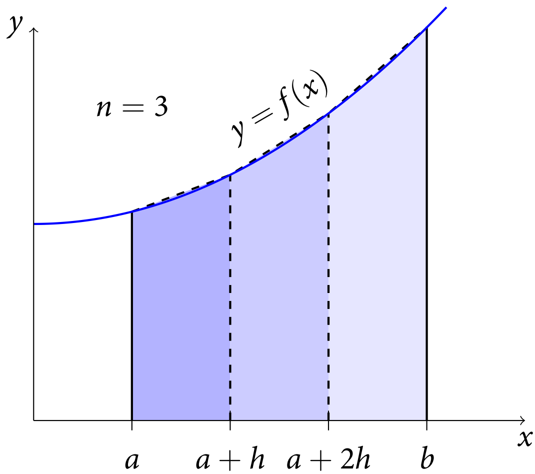

Figure 3.3: Each shaded region under the curve is approximated by a trapezoid or trapezium.

For the first trapezium, this area is

$$\frac{h}{2}\left(f(a)+f(a+h)\right)$$

For the second trapezium, this area is

$$\frac{h}{2}\left(f(a+h)+f(a+2h)\right)$$

For the third trapezium, this area is

$$\frac{h}{2}\left(f(a+2h)+f(b)\right)$$

The sum of these areas is

$$\frac{h}{2}\left[f(a)+2f(a+h)+2f(a+2h)+f(b)\right]$$

This sum of trapezium areas is an approximation for the integral we originally wanted,

$$\frac{h}{2}\left[f(a)+2f(a+h)+2f(a+2h)+f(b)\right] \approx \int_a^b f(x)\,dx.$$

More generally, splitting the domain from a to b into n slices, this *trapezium rule* for calculating the approximate value of an integral is

$$\begin{aligned}\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a)+2f(a+h)+2f(a+2h)+\ldots\right.\\ \left.+2f(a+(n-1)h)+f(b)\right]\end{aligned} \tag{3.4}$$

where h is given by (3.3).

Example 3.1.1: Use the trapezium rule with 4 strips to estimate

$$\int_0^1 e^{-x^2}\,dx.$$

Example 3.1.2: Use the trapezium rule to estimate

$$\int_0^\pi \sqrt{\sin x}\, dx,$$

using 4 strips. Repeat the calculation with 8 strips.

3.2 Simpson's rule

We have seen that the trapezium rule is a method of numerical integration based on a *linear interpolation* of the integrand $f(x)$. *Simpson's rule* is another method of numerical integration based on quadratic interpolation of the integrand. Simpson's rule is a little more complicated than the trapezium rule, but is often more accurate.

To use Simpson's rule we first divide our domain as before into n strips. For Simpson's rule **we require that** n **is even**, which allows us to 'pair up' adjacent strips, as in Figure 3.4.

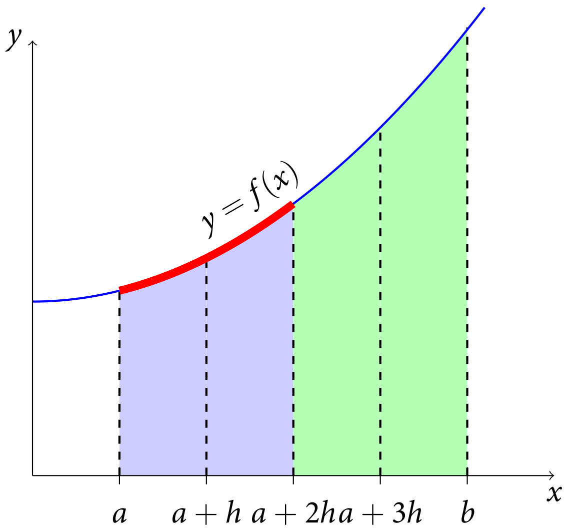

Figure 3.4: The Simpson's rule by approximating the curve by a qudratic function.

Within each pair of strips, our integrand $f(x)$ is approximated by its interpolating quadratic (see section 1.2). Recalling the formula for the interpolating quadratic (1.3), we need to evaluate $f(x)$ at three values of x. For the first pair of strips (shaded in blue, with $a \leq x \leq a+2h$) we have

$$\begin{aligned} f_0 &= f(a) & f_1 &= f(a+h) & f_2 &= f(a+2h) \\ x_0 &= a & x_1 &= a+h & x_2 &= a+2h. \end{aligned}$$

Substituting these values into our equation for quadratic interpolation (1.3), we find an approximation for $f(x)$ in the region of our first pair of strips,

$$\begin{aligned} f(x) \approx Q(x) = & \frac{(x-(a+h))(x-(a+2h))}{2h^2} f(a) \\ & - \frac{(x-a)(x-(a+2h))}{h^2} f(a+h) \\ & + \frac{(x-a)(x-(a+h))}{2h^2} f(a+2h). \end{aligned} \tag{3.5}$$

This approximation is shown as a thick red line above in Figure 3.4. To approximate the area of the first pair of strips we integrate (3.5) between $x = a$ and $x = a + 2h$ and obtain[2]

$$\int_a^{a+2h} f(x)\,dx \approx \frac{h}{3}\left(f(a) + 4f(a+h) + f(a+2h)\right). \tag{3.6}$$

Similarly, an approximation for the area under the second (green) pair of strips is

$$\int_{a+2h}^{b} f(x)\,dx \approx \frac{h}{3}\left(f(a+2h) + 4f(a+3h) + f(b)\right). \tag{3.7}$$

[2]try calculating this integral yourself – it is not difficult, but is rather lengthy!

Adding these together, our integral is approximated by

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + f(b)\right).$$

This is Simpson's rule approximation to our integral for $n = 4$ strips. For general n the formula is

$$\begin{aligned}\int_a^b f(x)\,dx \approx \frac{h}{3}\,[f(a) &+ 4f(a+h) \\ &+ 2f(a+2h) + 4f(a+3h) \\ &+ 2f(a+4h) + 4f(a+5h) \\ &+ \ldots \\ &+ 2f(b-2h) + 4f(b-h) \\ &+ f(b)]\,. \end{aligned} \tag{3.8}$$

Note the pattern of coefficients of $f(a), f(a+h), \ldots$:

$$1, 4, 2, 4, 2, 4, \ldots, 2, 4, 1$$

- The number of strips n is even, so the number of terms to add is odd
- The first and last coefficients $f(a)$ and $f(b)$ are 1
- Coefficients of intermediate terms alternate between 2 and 4
- The second term and the penultimate term are 4

Example 3.2.1: Use Simpson's rule with 4 strips to approximate

$$\int_0^1 e^{-x^2}\,dx.$$

Compare the answer to the approximation found using the trapezium rule in example 3.1.1.

Example 3.2.2: Use Simpson's rule to estimate

$$\int_0^{\pi} \sqrt{\sin x}\, dx,$$

using 4 strips. Repeat the calculation with 8 strips. Compare these estimates to those found in example 3.1.2 using the trapezium rule.

3.3 Accuracy of numerical integration methods

In the last four examples we have calculated two example integrals,

$$\int_0^1 e^{-x^2}\, dx \qquad \text{and} \qquad \int_0^{\pi} \sqrt{\sin x}\, dx,$$

using both the trapezium rule and Simpson's rule. A sketch in Figure 3.5 of these functions shows the areas under the curve that we are calculating when performing these integrals.

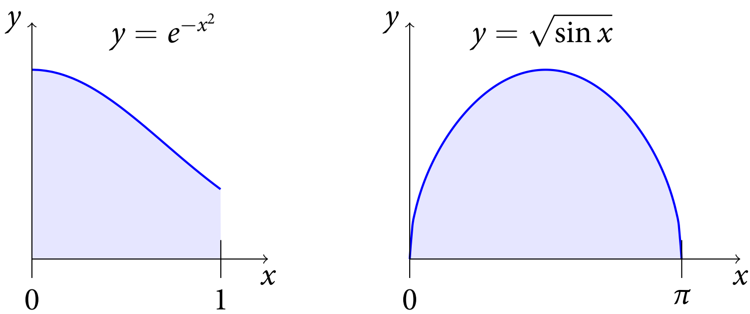

Figure 3.5: The two integrals $\int_0^1 e^{-x^2} dx$ (left) and $\int_0^{\pi} \sqrt{\sin x} dx$ (right).

We now extend these results to a larger number of strips n, with the aim of calculating a more accurate approximation. We look first at the results for

$$\int_0^1 e^{-x^2}\, dx. \tag{3.9}$$

n	Trapezium rule	Error	Simpson's rule	Error
4	0.742984098	0.003840035	0.746855380	0.000031247
8	0.745865615	0.000958518	0.746826121	0.000001988
16	0.746584597	0.000239536	0.746824257	0.000000125
32	0.746764255	0.000059878	0.746824141	0.000000008
64	0.746809164	0.000014969	0.746824133	0.000000000
128	0.746820391	0.000003742	0.746824133	0.000000000
256	0.746823197	0.000000936	0.746824133	0.000000000
512	0.746823899	0.000000234	0.746824133	0.000000000

Note that:

- as the number of strips n increases, the estimates of both methods get more accurate
- The error from Simpson's rule is smaller than that of the trapezium rule, and gets smaller more quickly as n increases
- For this integral, the answer obtained from Simpson's rule is accurate to 8 digits with $n = 64$.

We now look at the results for

$$\int_0^{\pi} \sqrt{\sin x}\, dx.$$

n	Trapezium rule	Error	Simpson's rule	Error
4	2.10628	0.29001	2.28477	0.11151
8	2.29391	0.10237	2.35646	0.03983
16	2.36010	0.03618	2.38217	0.01412
32	2.38349	0.01279	2.39129	0.00499
64	2.39176	0.00452	2.39451	0.00177
128	2.39468	0.00160	2.39566	0.00062
256	2.39572	0.00057	2.39606	0.00022
512	2.39608	0.00020	2.39620	0.00008
1024	2.39621	0.00007	2.39625	0.00003

- as the number of strips n increases, the estimates of both methods get more accurate: we say that the estimate *converges* to the true value
- The error from Simpson's rule is smaller than that of the trapezium rule.
- For this integral, the answer obtained from Simpson's rule is accurate to only 5 digits with $n = 512$.

Our second integral (3.3) is in some sense 'more difficult' than the first (3.9), since it takes a greater number of strips to estimate to some accuracy. Why is this? We can get an idea from the sketches of the two functions, above. While e^{-x^2} is a 'smooth' function, $\sqrt{\sin x}$ is less smooth, in the sense that it becomes very steep at the two endpoints of integration $x = 0$ and $x = \pi$. In fact the *gradient* of the function becomes infinite at these endpoints (in the same way as it does at $x = 0$ in a graph of $\sqrt{x}$). This makes the function $\sqrt{\sin x}$ difficult to approximate closely by trapeziums, or by quadratic interpolation in Simpson's rule. More sophisticated numerical integration techniques can be used for troublesome integrals such as (3.3) – for example, a method called *Chebyshev-Gauss quadrature* calculates the solution of (3.3) to 8 digits of accuracy using only $n = 6$ 'slices'.

Chapter 4

Polar Coordinates

4.1 Polar coordinates of points

The traditional rectangular *Cartesian* coordinates refer to the position of a point by (in Figure 4.1)

- x, the distance *along* to a point
- y, the distance *up* to a point

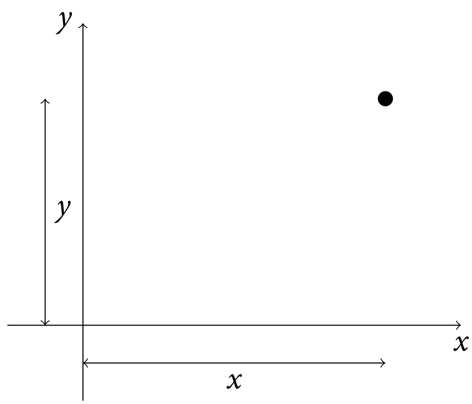

Figure 4.1: A point in the Cartesian plane.

Polar coordinates instead refer to a point using (in Figure 4.2)

- r, the distance between the point and the origin
- θ, the angle at the origin between the x-axis and the line to the point, measured anti-clockwise

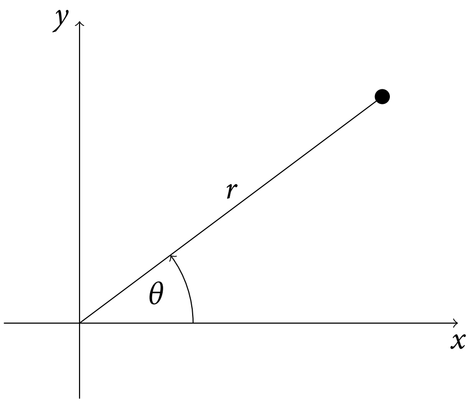

Figure 4.2: A point in polar coordiantes

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Usually r is taken to be non-negative ($r \geq 0$). Unless otherwise specified, θ is measured in radians. θ is normally taken to be between $-\pi$ and π, with values between $-\pi$ and 0 below the x-axis and values between 0 and π above the x-axis. The values taken by θ in each of the four quadrants of the plane are shown in Figure 4.3 and 4.4.

Converting from polar to Cartesian coordinates

The Cartesian and polar coordinates x, y, and r, θ are linked by

$$x = r\cos\theta \tag{4.1}$$

$$y = r\sin\theta \tag{4.2}$$

Equations (4.1) and (4.2) can be used to find x and y, if r and θ are known.

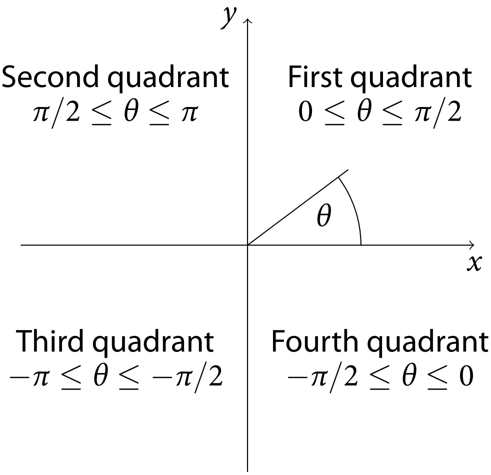

Figure 4.3: The four quadrants distinguished by the angle θ.

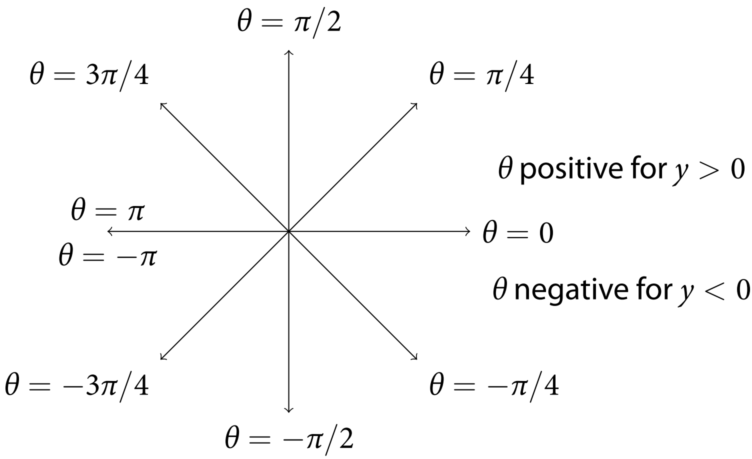

Figure 4.4: A few special lines with constant θ.

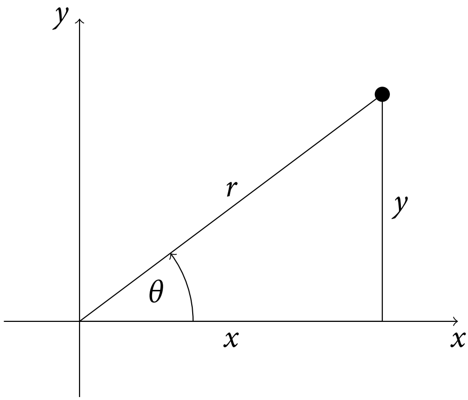

Figure 4.5: Relation between polar and Cartesian coordinates.

Converting from Cartesian to polar coordinates

If x and y are known, we can find the polar radius r and $\tan\theta$ from

$$r = \sqrt{x^2 + y^2} \tag{4.3}$$

$$\tan\theta = \frac{y}{x} \tag{4.4}$$

To find the polar angle θ from (4.6) takes a bit more thought. Simply taking $\tan^{-1}$ of (4.4) *suggests* that

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \tag{4.5}$$

but this is **not** always right, since $\tan^{-1}$ gives results only between $-\pi/2$ to $\pi/2$, whereas we know that θ can take values over a wider range, between $-\pi$ and π[1]. What *is* true, however, is the more gen-

[1]Note, for example, that (4.5) cannot distinguish between the case where $x = 1, y = 1$ (where $\theta = \pi/4$) and where $x = -1$ and $y = -1$ (where $\theta = -3\pi/4$), since in both cases $y/x = 1$.

eral statement

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + k\pi, \tag{4.6}$$

where k is either $0, 1$ or -1. We must look at which quadrant our point is in to determine the value of k:

- If our point is in the first or fourth quadrants (i.e. if $x > 0$) then

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{4.7}$$

- If our point is in the third quadrant ($x < 0, y < 0$) then

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) - \pi \tag{4.8}$$

- If our point is in the second quadrant ($x < 0, y > 0$) then

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi \tag{4.9}$$

Example 4.1.1: Point A is, in polar coordinates, $r = 6, \theta = \pi/4$. Find the Cartesian coordinates (x, y) of point A.

Example 4.1.2: Point B is $(2, 7)$ in Cartesian coordinates. Find point B in polar coordinates.

Example 4.1.3: Point C is $(-4, 1)$ in Cartesian coordinates. Find point C in polar coordinates.

Example 4.1.4: Point D is, in polar coordinates, $r = 5, \theta = -130°$. Find point D in Cartesian coordinates.

Example 4.1.5: Point E is $(0, -4)$ in Cartesian coordinates. Find point E in polar coordinates.

4.2 Polar coordinates of lines and curves

We have seen that points expressed in Cartesian coordinates (x, y) can also be expressed in polar coordinates (r, θ). Similarly *functions* which are expressed in Cartesian coordinates as $y = y(x)$ can often be expressed in polar coordinates, $r = f(\theta)$, as shown in Figure 4.6.

- Since $r \geq 0$ for polar coordinates, we (almost) always have that $f(\theta) \geq 0$.
- Each value of θ (and the corresponding value of r) represents a different point on the curve.

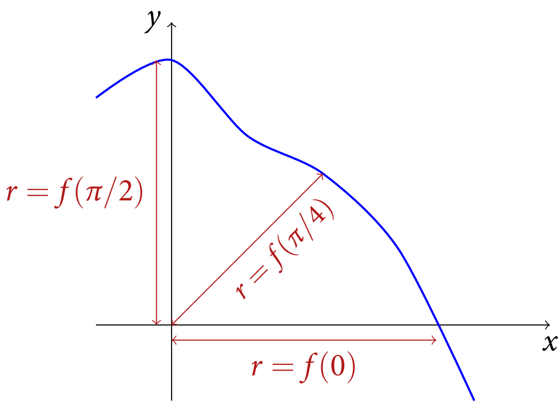

Figure 4.6: A curve represented as $r = f(\theta)$ in the polar coordinates.

We will look at the polar coordinate representation of $r = f(\theta)$ for several types of curve:

1. Circles centred on the origin
2. Circles passing through the origin
3. Straight lines

4. Archimedes' spirals
5. Logarithmic spirals
6. Ellipses
7. Cardioids

4.2.1 Circles centred at the origin

The polar equation for a circle with with centre $(0, 0)$ and radius a is

$$r = f(\theta) = a. \tag{4.10}$$

In Cartesian coordinates this is,

$$x^2 + y^2 = r^2 = a^2$$

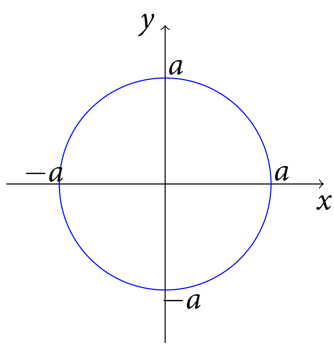

Figure 4.7: A circle centered at the origin with radius a is represented as $r = a$.

4.2.2 Circles passing through the origin

A circle passing through the origin, with centre at

$$(x_0, y_0) = (a\cos\theta_0, a\sin\theta_0)$$

is given by the polar coordinate equation

$$r = f(\theta) = 2a\cos(\theta - \theta_0), \tag{4.11}$$

where a is the radius of the circle and θ_0 is polar angle from the origin to the centre of the circle. In fact, from the equation for the circle in Cartesian coordinate

$$(x - a\cos\theta_0)^2 + (y - a\sin\theta_0)^2 = a^2,$$

we can get the corresponding equation (4.11).

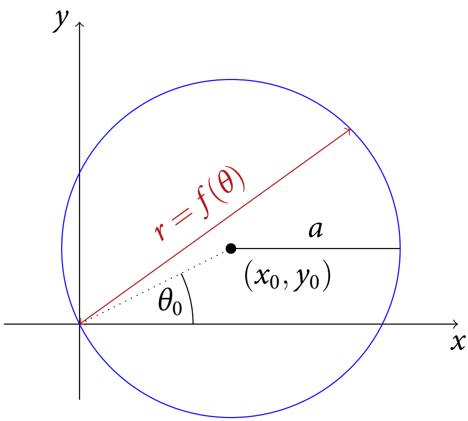

Figure 4.8: A circle passing through the origin is represented as $r = 2a\cos(\theta - \theta_0)$.

For a circle with centre $(3, 0)$ and radius 3:

$$a = 3, \qquad \theta_0 = 0,$$

$$\begin{aligned} r &= 2a\cos(\theta - \theta_0) \\ &= 6\cos(\theta). \end{aligned}$$

For a circle with centre $(-2, 2)$ and radius $\sqrt{8}$,

$$a = \sqrt{8}, \qquad \theta_0 = 3\pi/4,$$

$$\begin{aligned} r &= 2a\cos(\theta - \theta_0) \\ &= 4\sqrt{2}\cos\left(\theta - 3\pi/4\right). \end{aligned}$$

For a circle with centre $(0, -1)$ and radius 1,

$$a = 1, \qquad \theta_0 = -\pi/2,$$

$$\begin{aligned} r &= 2a\cos(\theta - \theta_0) \\ &= 2\cos\left(\theta + \pi/2\right) \\ &= -2\sin(\theta). \end{aligned}$$

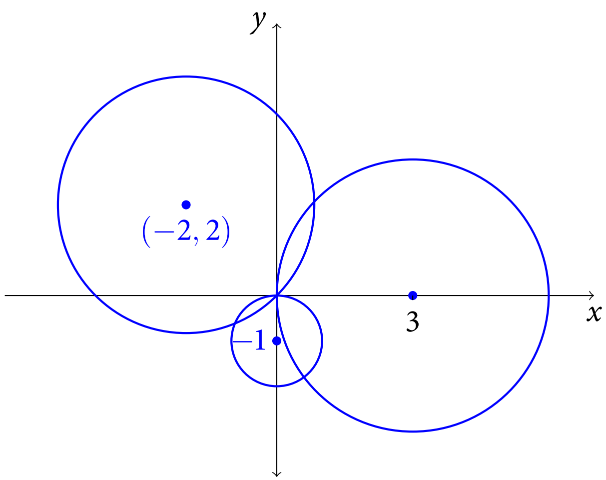

Figure 4.9: The three circles with centre $(3, 0)$, $(-2, -2)$, $(0, -1)$ and radius 3, $\sqrt{8}$, 1, respectively.

Example 4.2.1: Find the equation, in polar coordinates, of a circle passing through the origin which has its centre at the point $x = 3$, $y = 4$.

4.2.3 Straight lines

Straight lines (except those that are parallel to the y-axis) can be written in Cartesian coordiantes as

$$y = mx + c \tag{4.12}$$

with constant gradient m and offset c. Substituting (4.1) and (4.2) into this we find

$$\begin{aligned} y &= mx + c \\ r\sin\theta &= mr\cos\theta + c \\ r\sin\theta - mr\cos\theta &= c. \end{aligned}$$

Rearranging to make r the subject of the formula, the polar coordinate form of a straight line is

$$r = \frac{c}{\sin\theta - m\cos\theta} \tag{4.13}$$

For example, in the line $y = x + 2$ we have $m = 1$, $c = 2$. Thus,

$$\begin{aligned} r &= \frac{c}{\sin\theta - m\cos\theta} \\ &= \frac{2}{\sin\theta - \cos\theta}. \end{aligned}$$

A horizontal line might have the equation $y = -1$, so $m = 0$, $c = -1$.

$$r = \frac{-1}{\sin\theta - m\cos\theta} = \frac{-1}{\sin\theta}.$$

A vertical line $x = a$ cannot be represented in the form $y = mx+c$, but is represented in polar coordinates by

$$r = \frac{a}{\cos\theta}$$

For example the line $x = 3$,

$$r = \frac{3}{\cos\theta}.$$

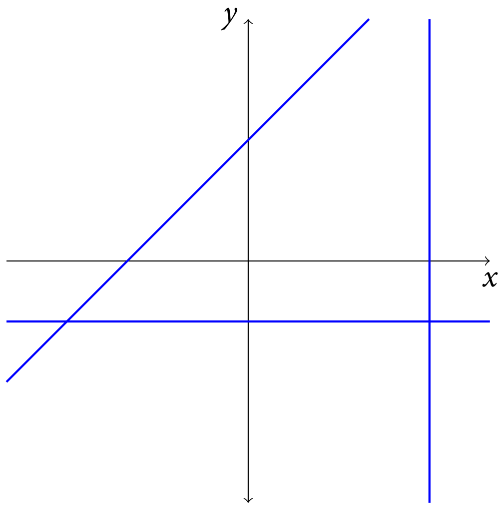

Figure 4.10: The three stright lines: $y = x + 2, y = -2$ and $x = 3$.

A half line through the origin is represented in polar coordinates by

$$\theta = c,$$

for example the line $\theta = -\pi/4$

Example 4.2.2: Find the polar form of the equation for the straight line $y = 2x - 5$.

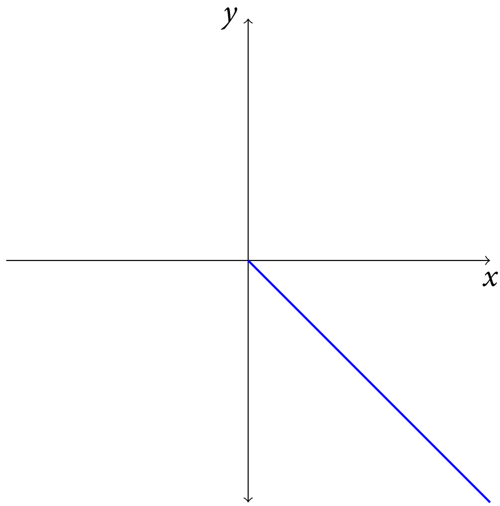

Figure 4.11: A half line $y = -x$ with $x > 0$, or $\theta = -\pi/4$ in polar coordinate.

4.2.4 Archimedes' spirals

An *Archimedes' spiral* is a spiral with a constant distance between the 'arms'. It is described in polar coordinates by the equation

$$r = k(\theta - \theta_0), \tag{4.14}$$

where, as with all definitions of spirals, angles are measured in radians rather than degrees. An Archimedes' spiral has several properties:

- The distance between the arms (measured radially) is $2k\pi$
- The curve comes out of the origin at polar angle θ_0
- If k is positive, the spiral curves anticlockwise as it moves away from the origin

- If k is negative, the spiral curves clockwise as it moves away from the origin

For example, the equation $r = \dfrac{1}{5}\left(\theta - \dfrac{\pi}{2}\right)$ defines an Archimedes' spiral. Note that

- $k = 1/5, \theta_0 = \pi/2$.
- k positive so spiral is anticlockwise coming away from origin
- The spiral leaves the origin at angle $\theta_0 = \pi/2$, i.e. along positive y-axis

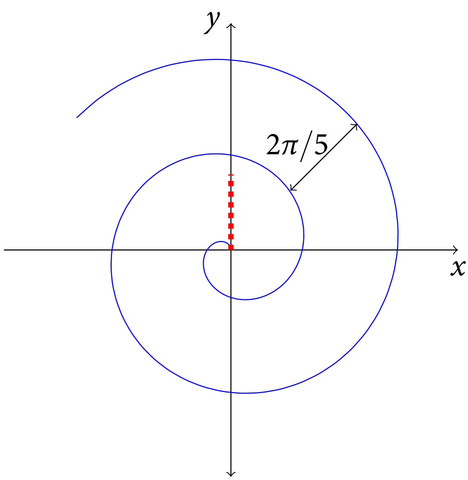

Figure 4.12: The Archimedes's spiral $r = \frac{1}{5}(\theta - \frac{\pi}{2})$.

Consider the Archimedes's spiral $r = -2\,(\theta - \pi)$:

- $k = -2, \theta_0 = \pi$.
- k negative so spiral is clockwise coming away from origin
- Curve leaves origin at angle π, i.e. along negative x-axis

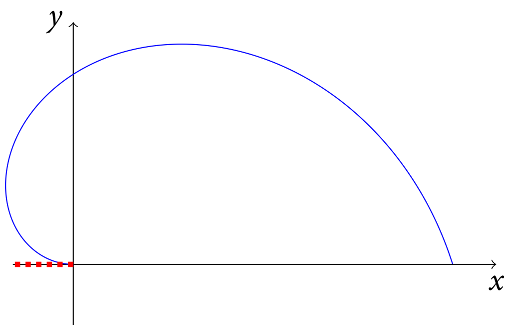

Figure 4.13: The Archimedes's spiral $r = -2(\theta - \pi)$.

Example 4.2.3: Find the equation of the spiral, with constant distance between the arms, sketched below. The coordinates given are three intersections of the spiral with the axes.

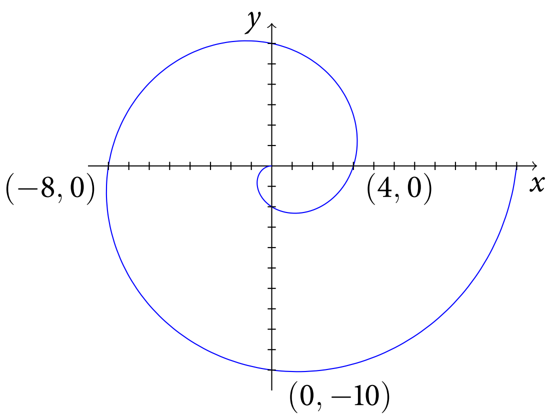

Figure 4.14: The sprial in Example 4.2.3.

4.2.5 Logarithmic spirals

A *logarithmic* spiral is one where the spiral makes a constant angle with the line to the origin. It is described by the equation

$$r = e^{k(\theta-\theta_0)}$$

(Remember that for spirals, radians rather than degrees must be used.)

- At any point on the spiral, the angle between the spiral and a line to the origin is $\cot^{-1} k$
- The curve never reaches the origin ($r > 0$)
- If k is positive, the spiral curves anticlockwise as it moves away from the origin
- If k is negative, the spiral curves clockwise as it moves away from the origin

For the logarithmic spiral $r = \exp\left(\dfrac{\theta - \pi/2}{5}\right)$, we have $k = 1/5$ and $\theta_0 = \pi/2$, $\alpha = \cot^{-1}(1/5) = \tan^{-1}(5) = 1.37$ radians $= 78.7°$.

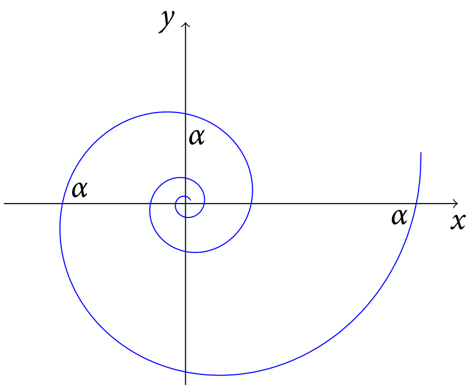

Figure 4.15: The logarithmic spiral $r = \exp\left(\frac{\theta-\pi/2}{5}\right)$.

4.2.6 Ellipses

The Cartesian equation for an ellipse centred on the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{4.15}$$

where a and b are the *semi-axes* of the ellipse.

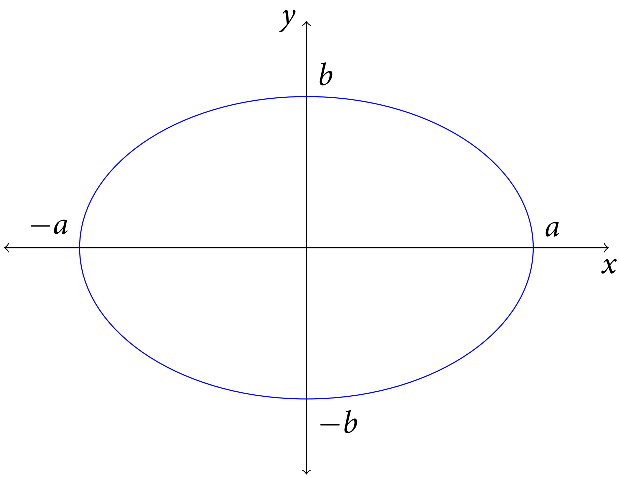

Figure 4.16: The ellipse $x^2/a^2 + y^2/b^2 = 1$.

The semi-axes a and b are generalisation of a circle's radius to an ellipse. The larger of a and b is the *semi-major axis* and is half the largest diameter. The smaller of a and b is the *semi-minor* axis and is half the smallest diameter.

The polar coordinate form of an ellipse is obtained by converting x and y in (4.15) to their polar coordinate forms, using (4.1) and (4.2),

$$\frac{r^2 \cos^2\theta}{a^2} + \frac{r^2 \sin^2\theta}{b^2} = 1,$$

which we can rearrange to make

$$r = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \tag{4.16}$$

We sketch three ellipses (shown in Figure 4.2.6):

$$a = 4, \qquad b = 3, \qquad r = \frac{12}{\sqrt{9\cos^2\theta + 16\sin^2\theta}},$$

$$a = 1, \qquad b = 2, \qquad r = \frac{2}{\sqrt{4\cos^2\theta + \sin^2\theta}},$$

and ddding a constant to θ rotates the ellipse:

$$a = 3, \qquad b = 2, \qquad r = \frac{6}{\sqrt{4\cos^2(\theta - \pi/4) + 9\sin^2(\theta - \pi/4)}}.$$

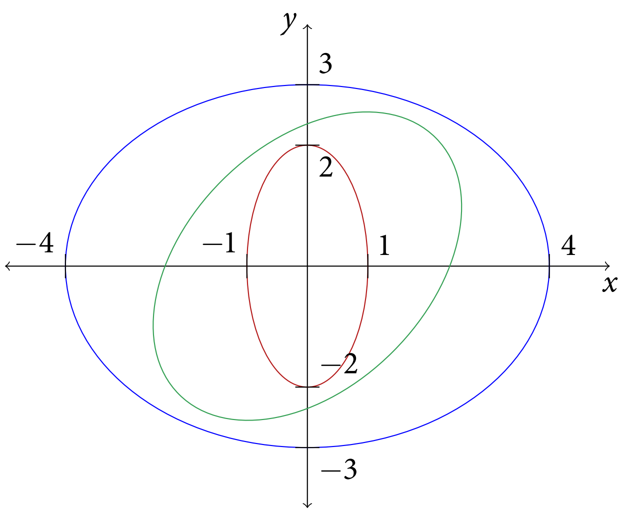

Figure 4.17: Three different ellipses.

4.2.7 Other shapes

Cardioids

A *cardioid* is defined by the function

$$r = a\,(1 - \cos\theta)\,.$$

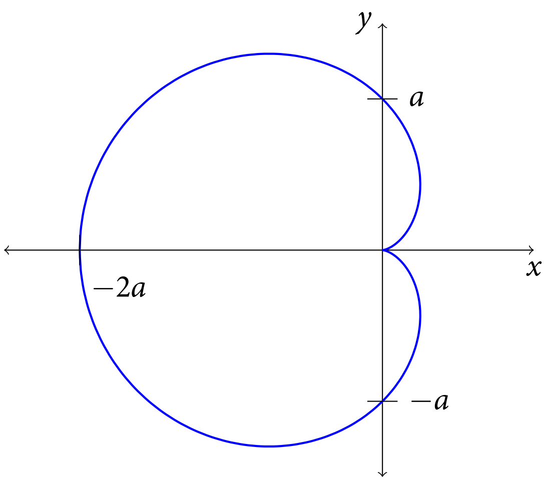

Figure 4.18: The cardiod $r = a(1 - \cos\theta)$.

Example 4.2.4: Where does the cardioid

$$r = 2\,[1 + \cos\left(\theta - \pi/4\right)]$$

intersect the circle

$$r = 3?$$

Chapter 5

Numerical solution of equations

We begin with an example motivated by the previous chapter on polar coordinates. Where does the spiral $r = \theta$intersect the circle $r = \cos\theta$? As well as the intersection at the origin, a sketch suggests that there is one further intersection in the first quadrant ($0 \leq \theta \leq \pi/2$).

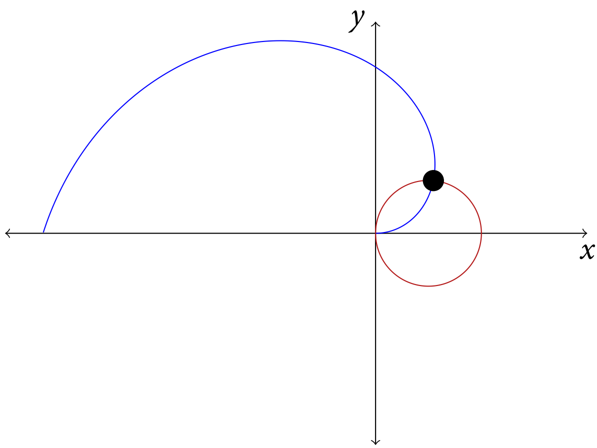

Figure 5.1: The solution of the equation $\theta = \cos\theta$ characterised by the intersection of the two curves $r = \theta$ and $r = \cos\theta$ in polar coordinates.

If we try to equate the two values of r,

$$\theta = \cos\theta \tag{5.1}$$

and try to solve for θ, we cannot find a solution for θ algebraically. But the solution does exist, and we can find an approximation to it numerically. Tabulating some selected values of θ and $\cos\theta$,

θ	$\cos\theta$
0.5	0.877583
0.6	0.825336
0.7	0.764842
0.8	0.696707
0.9	0.621610
1.0	0.540302

we see that when $\theta = 0.7$, we have $\cos\theta > \theta$ (the circle is outside the spiral). On the other hand, when $\theta = 0.8$, we have $\cos\theta < \theta$ (the circle is inside the spiral). This indicates that our solution lies somewhere between 0.7 and 0.8, as in Figure 5.

Looking in more detail at values of θ between 0.7 and 0.8,

θ	$\cos\theta$
0.7	0.764842
0.72	0.751806
0.74	0.738469
0.76	0.724836
0.78	0.710914
0.8	0.696707

we see that the solution must lie between $\theta = 0.72$ and $\theta = 0.74$. We could theoretically tabulate values of $\cos\theta$ between $\theta = 0.72$ and $\theta = 0.74$ and find a more accurate approximation again, but this quickly gets tedious.

This chapter is about more efficient numerical methods for solving equations, such as (5.1), that do not have a straightforward alge-

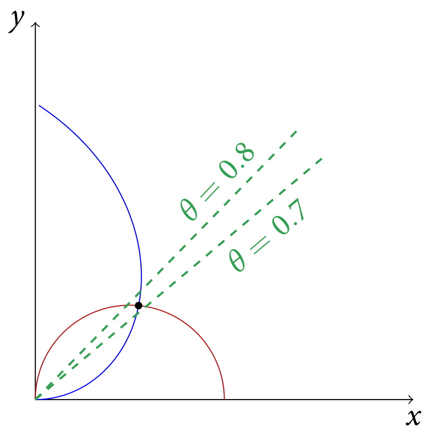

Figure 5.2: The configuration suggests that the desired solution θ is between 0.7 and 0.8 (in radian).

braic solution. These methods are often known as *root finding* techniques. In a sense, each is a method of ‘trial and error’ like the tabulation used above, but they provide very good ‘guesses’ so that an accurate solution can be found quickly.

Most root-finding techniques involve re-casting the equation in the form $f(x) = 0$. For example, our equation $\cos\theta = \theta$ (where we are solving for θ, not x) would be re-cast as

$$f(\theta) = \cos\theta - \theta = 0$$

(we could equally write $\theta - \cos\theta = 0$). The function $f(x)$ is known as the *residual*, and our we aim to find a solution such that the residual function is zero.

Most root-finding techniques also require an initial guess for a root, or a range in which the root lies. Often the easiest way to find this initial guess is to sketch the residual function.

For example, suppose we want to solve

$$2^x + \sin x = 5.$$

We write this as as residual function,

$$f(x) = 2^x - 5 + \sin x = 0,$$

and plot, and we see that there is one root, close to $x = 2$.

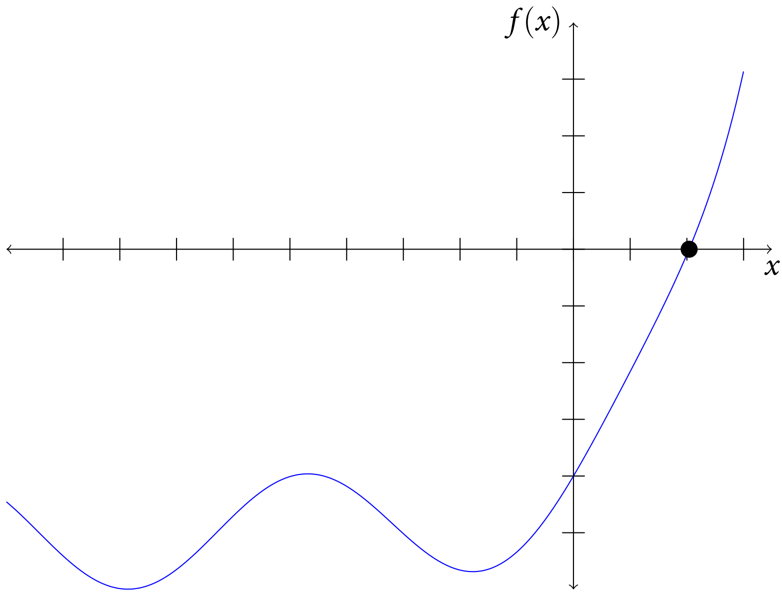

Figure 5.3: A polot of the function $f(x) = 2^x - 5 + \sin x$ suggestion the solution $f(x) = 0$ is close to 2.

If instead we want to solve

$$\sin x = -e^x,$$

we write this as

$$f(x) = \sin x + e^x = 0,$$

and plot, we see that there are several roots (in fact, infinitely many), close to $x = -0.5, -3, -6.5, -9.5, \ldots$.

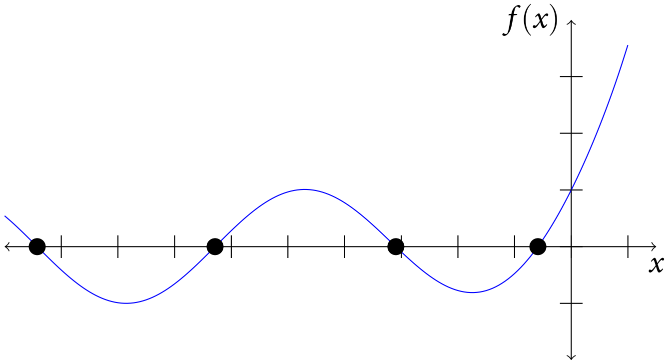

Figure 5.4: A plot of the function $f(x) = \sin x + e^x$ suggests serveral solutions on the negative axis.

Sometimes sketches can be tricky to interpret. It is not immediately clear from a sketch whether

$$f(x) = e^x + x^2 - 1 = 0 \tag{5.2}$$

has no roots, one repeated root, or two roots:

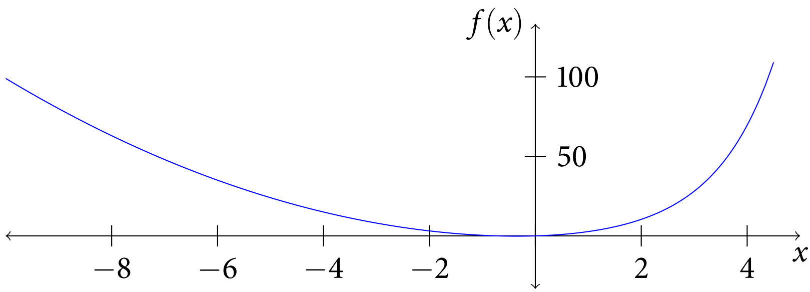

Figure 5.5: The plot of the function $f(x) = e^x + x^2 - 1$ may not clearly indicate the number of solutions to the equation $f(x) = 0$.

A closer inspection is needed to reveal the two roots: One root is

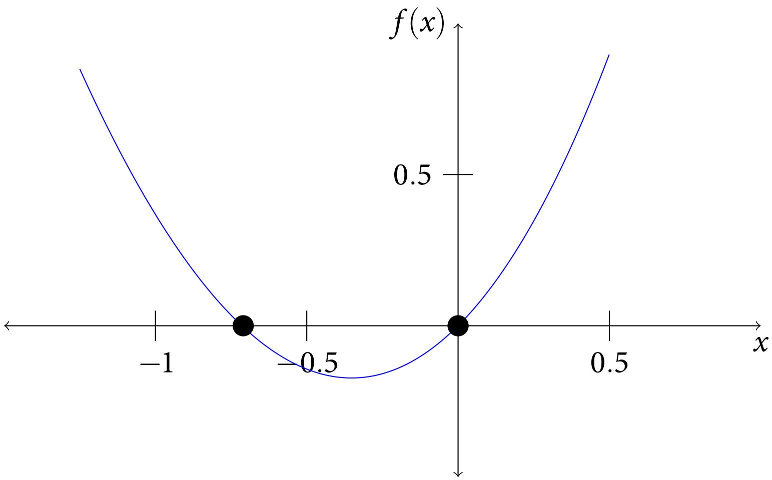

Figure 5.6: The plot of the function $f(x) = e^x + x^2 - 1$ on a smaller interval indicates two roots for the equation $f(x) = 0$.

close to $x = -0.75$ and the other is close to $x = 0$. (In this case we can easily verify that $x = 0$ is an exact solution).

We will study three methods for finding roots of equations accurately: the bisection method, the rule of false position, and the Newton-Raphson (or Newton's) method. Each method requires an initial guess or initial interval, often most easily obtained from the sketch.

5.1 The bisection method

For the bisection method, we first need to find two values of x, which we denote x_1 and x_2, that define an interval that we expect our root to lie in.

The bisection method requires that $f(x_1)$ has a different sign to $f(x_2)$ (here, $f(x_1) < 0$ and $f(x_2) > 0$). If we can find two guesses x_1 and x_2 that have this property, then whatever the form of the function f, there must be a root between x_1 and x_2.

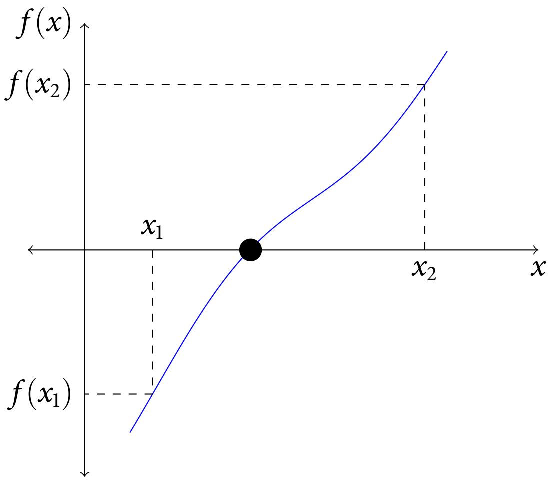

Figure 5.7: Initial configuration for bisection method: $f(x_1)$ and $f(x_2)$ have opposite signs, with one root between them.

More formally, if

1. we can find an x_1 and x_2 such that $f(x_1)$ has a different sign to $f(x_2)$, and

2. the function f has no 'jumps' or discontinuities (we say that f is *continuous*) between x_1 and x_2

then a mathematical theorem known as the *intermediate value theorem* tells us that there is an x_r, with $x_1 < x_r < x_2$, such that $f(x_r) = 0$. This value x_r is exactly the value we want to find.

The bisection method starts by evaluating f at the midpoint of x_1 and x_2, i.e.

$$f(x_3) = f\left(\frac{x_1 + x_2}{2}\right).$$

If, as in the sketch, $f(x_3)$ is the same sign as $f(x_2)$, then we know that the root must lie within the new, smaller, interval between x_1 and

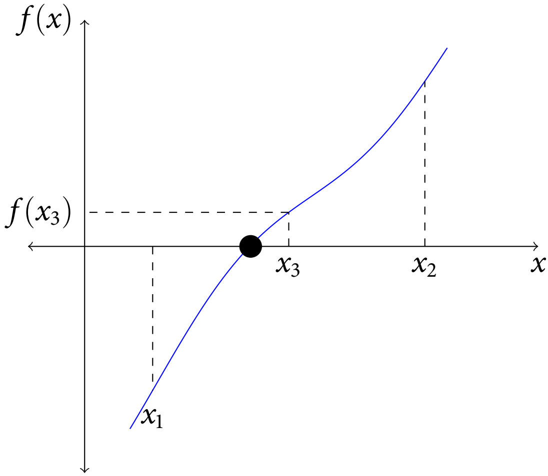

Figure 5.8: The first step of the bisection method, by checking the sign at $x_3 = (x_1 + x_2)/2$.

x_3 (because $f(x_1)$ must have a different sign to $f(x_3)$). On the other hand, if $f(x_3)$ is the same sign as $f(x_1)$, then our root must lie between x_3 and x_2, because $f(x_3)$ has a different sign to $f(x_2)$. In either case, we have halved the size of the interval that we know the root must lie in.

We can then repeat the process, evaluating f at the midpoint of our new interval:

$$f(x_4) = f\left(\frac{x_1 + x_3}{2}\right).$$

Here $f(x_4)$ is the same sign as $f(x_1)$, so we know the root must like between x_4 and x_3. We can repeat the process again, calculating

$$f(x_5) = f\left(\frac{x_4 + x_3}{2}\right),$$

and so on, at each step halving the size of the interval that our root

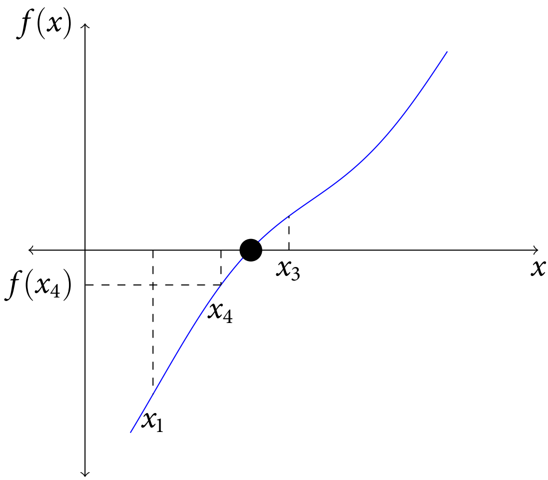

Figure 5.9: The second step of the bisection method, by checking the sign at $x_4 = (x_1 + x_3)/2$.

must be in. Once this interval is small enough for the accuracy we desire, we simply take the midpoint of our final interval as our best approximation for the root x_r,

$$x_r \approx \frac{x_{n-1} + x_n}{2}$$

Methods like the bisection method that apply the same step repeatedly are known as *iterative* methods, and each step is called an *iteration*.

Example 5.1.1: Use the bisection method to find a numerical solution to

$$f(x) = 2^x - 5 + \sin x = 0,$$

to four digits of accuracy. Start with the interval $x_1 = 1$ to $x_2 = 3$.

Solution 5.1.1: We first calculate $f(x_1)$ and $f(x_2)$,

$$f(x_1) = f(1) = -2.1585\ldots \qquad \text{and}$$
$$f(x_2) = f(3) = 3.1411\ldots,$$

so $f(x_1)$ and $f(x_2)$ have different signs, as required. We then calculate $x_3 = (x_1 + x_2)/2 = 2$, and evaluate

$$f(x_3) = f(2) = -0.0907\ldots.$$

This has the same sign as $f(x_1)$, so we know the root must lie between x_3 and x_2. We evaluate f at the midpoint of this interval,

$$f(x_4) = f\left(\frac{x_3 + x_2}{2}\right) = 1.25533\ldots.$$

We list each iteration of the method in a table:

i	x_i	$f(x_i)$	Lower bound	Upper bound
1	1	-2.15853		
2	3	3.14112	1	3
3	2	-0.0907	2	3
4	2.5	1.25533	2	2.5
5	2.25	0.534902	2	2.25
6	2.125	0.212351	2	2.125
7	2.0625	0.0586249	2	2.0625
8	2.03125	-0.0165604	2.03125	2.0625
9	2.04688	0.0208984	2.03125	2.04688
10	2.03906	0.00213597	2.03125	2.03906
11	2.03516	-0.00722042	2.03516	2.03906
12	2.03711	-0.00254428	2.03711	2.03906
13	2.03809	-0.000204671	2.03809	2.03906
14	2.03857	0.000965521	2.03809	2.03857
15	2.03833	0.000380393	2.03809	2.03833

At iteration 15 we know that the root lies between $2.03809\ldots$ and

$2.03833\ldots$, and so to four digits is 2.038. Our most accurate guess for the root is

$$x_r \approx \frac{x_{13} + x_{15}}{2} = 2.03821\ldots$$

but we can only be sure that the first four digits of this are correct.

The bisection method is a very *robust* method of finding a root, meaning that once the procedure is started (with f a continuous function and $f(x_1)$ and $f(x_2)$ of different signs), a root will be always found to some accuracy, within a known number of iterations. However, if our starting interval is too large, there may be more than one root within this interval. In this case the bisection method will converge to one of these roots, but it is difficult to predict which one this will be. Another difficulty occurs for functions like (5.2), where there are two roots very close together. In this case it may be difficult to find an initial point where $f(x) < 0$, since f is negative only for quite a small region of the domain. An extreme case of this is a function with a double root, for example $f(x) = x^2$, where we cannot find a starting point with $f(x) < 0$; in this case the bisection method cannot be used. As is often the case with root-finding, a sketch is useful to find the approximate value of the root before a root-finding method is used.

5.2 Rule of false position

Although the bisection method is very reliable, it is not always the fastest method of root finding. Looking at the table in the previous example, we see that $x_3 = 2$ was actually quite a good approximation to the root, much closer to the converged value than $x_4 = 2.5$. We could have spotted that $f(x_3)$ was much smaller in magnitude than $f(x_2)$, suggesting that the root was much closer to x_3 than x_2. However, the bisection method ignores this information, and always chooses the next guess (x_4 here) to be midway between x_3 and x_2.

The rule of false position is another root-finding technique, similar to the bisection method. In the bisection method, we choose the next guess midway between the previous two guesses. For the rule of false position, we instead use the previous two guesses to linearly interpolate the function f, as in section 1.1, and use the root of this linear interpolation as the next guess. From the linear interpola-

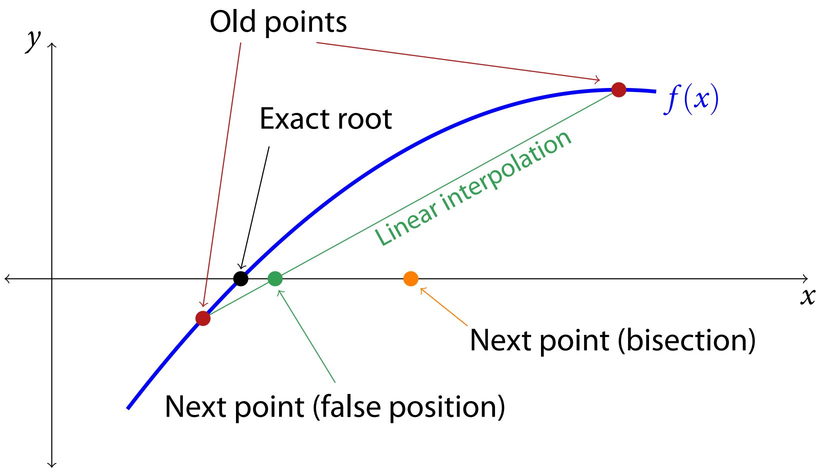

Figure 5.10: The fule of false position.

tion formula (1.1), the straight line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by

$$y = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

Equating this to zero, to find the root, we rearrange to find

$$x = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}. \tag{5.3}$$

When using the rule of false position, we use the value of x given by (5.3) to determine the next value of x at each stage.

The rule of false position converges in a different way to the bisection method. In the bisection method, the root is guaranteed to be in an interval that halves in size for each iteration of the method, and so both the upper and lower endpoints of this interval converge to the true solution. In the rule of false position, usually only *one* of the endpoints converges to the desired solution, and this is indicated by f tending to zero at this endpoint. The rule of false position can be quicker to converge than the bisection method, but for some functions can be very much slower – this means that it is less robust than the bisection method.

Example 5.2.1: Use the rule of false position to find a numerical solution to the equation

$$f(x) = 2^x - 5 + \sin x = 0,$$

to six digits of accuracy. Start with the interval $x_1 = 1$ to $x_2 = 3$.

Solution 5.2.1: We first calculate $f(x_1)$ and $f(x_2)$,

$$\begin{aligned} f(x_1) &= f(1) = -2.1585\ldots \qquad \text{and} \\ f(x_2) &= f(3) = 3.1411\ldots, \end{aligned}$$

so $f(x_1)$ and $f(x_2)$ have different signs, as required. We then calculate the next x value from (5.3)

$$\begin{aligned} x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \\ &= 1.81459\ldots, \end{aligned}$$

and evaluate f at this next x value,

$$f(x_3) = f(1.81459) = -0.511967\ldots.$$

This has the same sign as $f(x_1)$, so we know the root must lie between

x_3 and x_2. So from (5.3) we calculate

$$\begin{aligned} x_4 &= \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \\ &= 1.98072\ldots, \end{aligned}$$

and so on. We list each iteration of the method in a table:

i	x_i	$f(x_i)$	Lower bound	Upper bound
1	1	-2.15853		
2	3	3.14112	1	3
3	1.81459	-0.511967	1.81459	3
4	1.98072	-0.13594	1.98072	3
5	2.02301	-0.0362196	2.02301	3
6	2.03414	-0.00964532	2.03414	3
7	2.0371	-0.00256797	2.0371	3
8	2.03789	-0.000683647	2.03789	3
9	2.0381	-0.000181998	2.0381	3
10	2.03815	-4.84506e-05	2.03815	3
11	2.03817	-1.28982e-05	2.03817	3

By iteration 11 we can see that x_i is tending to a value near $2.03817\ldots$. This value is accurate to 6 digits, and is reached more quickly than in the bisection method. However, unlike the bisection method, the interval between the upper and lower bounds (in which the root is *guaranteed* to exist) does not shrink quickly.

5.3 The Newton-Raphson method

The Newton-Raphson method (or Newton's method) is a very rapid technique for finding roots of functions. It differs from the bisection method and the rule of false position in several important ways

- The Newton-Raphson method makes use of not only the value of the residual function $f(x)$, but also its derivative $f'(x)$.

- Whereas the bisection and false position methods start with an initial interval defined by two points, the Newton-Raphson method starts with only one initial guess for a root.

- The Newton-Raphson method can converge much more quickly than either the bisection method or the rule of false position. The convergence of the Newton-Raphson method is *quadratic*, which means that the number of correct digits can double for each iteration taken.

The Newton-Raphson method works as follows. Suppose we have a function $f(x)$ with a root x_r, such that $f(x_r) = 0$. We make an initial guess for that root x_1. We can improve our initial guess by taking the point P on our curve at $x = x_1$, and drawing a tangent line to the curve through this point (the red dashed line). This tangent curve intersects the x-axis at some point x_2, and we use this as our improved guess. We then start the procedure again, creating the

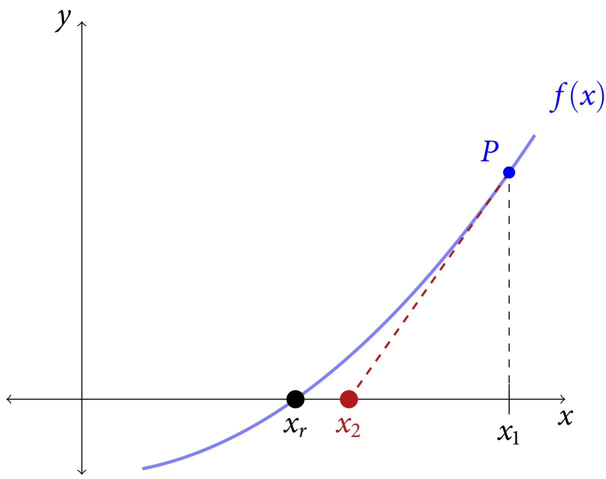

Figure 5.11: The first step to find x_2, given by the intersection of the tangent line of f at x_1 and the x-axis.

(green dashed) tangent line that touches our curve at the point Q at $x = x_2$, and use this to find an improved guess, x_3. After two iter-

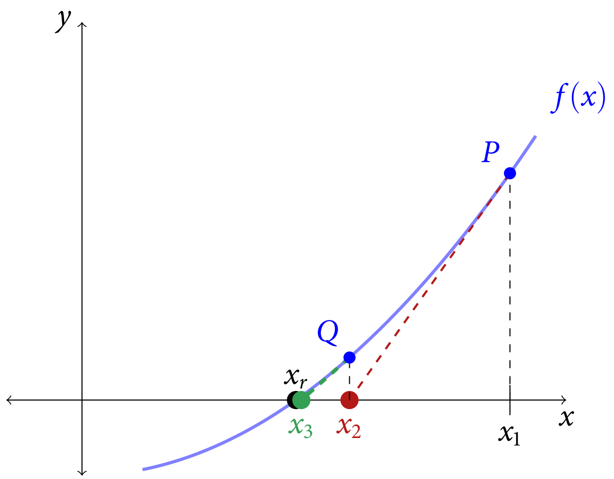

Figure 5.12: The second step to find x_3, given by the intersection of the tangent line of f at x_2 and the x-axis.

ations we have an approximation x_3 that is much closer to the true root x_r than our initial guess x_1. How do we convert this procedure into an equation? From our starting guess x_1, the point P is at co-ordinates $(x_1, f(x_1))$. Our tangent line must pass through this point, and have the same gradient as f at the tangent point. This gradient is $f'(x_1)$. The equation for our tangent line is therefore

$$y = mx + c = f'(x_1)x + c.$$

We find the constant c by requiring that our line passes through the point P, so that

$$y = f(x_1) = mx_1 + c = f'(x_1)x_1 + c,$$

which implies that

$$c = f(x_1) - f'(x_1)x_1.$$

Thus

$$y = f'(x_1)(x - x_1) + f(x_1).$$

To find the point x_2 we set $y = 0$ in this equation, and rearrange to obtain

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

More generally, at step i of the Newton-Raphson method, we can find the approximation at step $i + 1$ with

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}. \tag{5.4}$$

Example 5.3.1: Use the Newton-Raphson method to solve the equation

$$f(x) = 2^x - 5 + \sin x = 0,$$

to 12 digits of accuracy. Start with the initial guess $x_1 = 3$.

Example 5.3.2: Use the Newton-Raphson method to solve the same equation as in the previous example,

$$f(x) = 2^x - 5 + \sin x = 0.$$

Start with the initial guess $x_1 = -1.77$.

This example illustrates the problem with the Newton-Raphson method, that without a sufficiently good initial guess, it may not converge. Nonetheless, its rapid rate of convergence for a good initial guess means that it is used very widely in scientific computer programmes.

Newton's method forms the basis of the algorithms used internally by calculators and computers to evaluate functions such as the square root. If we define

$$f(x) = x^2 - a, \tag{5.5}$$

where x is a constant, then solving $f(x) = 0$ is equivalent to finding $x = \sqrt{a}$. Applying (5.4) to our equation (5.5), we find

$$x_{i+1} = x_i - \frac{x_i^2 - a}{2x_i} = \frac{1}{2}\left(x_i + \frac{a}{x_i}\right). \tag{5.6}$$

This special case of the Newton-Raphson rule is known as *Heron's method* of finding the square root. Choosing, for example, $a = 2$ and an initial guess $x_i = 1$, we find

$$\begin{aligned} x_2 &= 1.5 \\ x_3 &= 1.4156666666667\ldots \\ x_4 &= 1.4142156862745\ldots \\ x_5 &= 1.4142135623741\ldots. \end{aligned}$$

Our approximation x_5 matches the true value of $\sqrt{2}$ for all 14 digits shown.

Chapter 6

Areas, lengths and volumes

6.1 Areas inside polar curves

The area underneath a curve $y = f(x)$ specified in Cartesian coordinates, between $x = a$ and $x = b$, is

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx. \tag{6.1}$$

This equation for the area is specific to Cartesian coordinates.

As we have seen in chapter 4, curves can also be specified in polar coordinates, $r = f(\theta)$. In this case the area of a sector between $\theta = a$ and $\theta = b$ is given by (as in Figure 6.1)

$$A = \frac{1}{2}\int_a^b r^2\,d\theta = \frac{1}{2}\int_a^b f(\theta)^2\,d\theta. \tag{6.2}$$

Example 6.1.1: Find the area inside a circle of radius R.

To find the area *between* two curves $r = r_{\text{outer}}(\theta)$ and $r = r_{\text{inner}}(\theta)$ we subtract the inner area from the whole area:

$$A = \frac{1}{2}\left[\int_a^b r_{\text{outer}}^2\,d\theta - \int_a^b r_{\text{inner}}^2\,d\theta\right]. \tag{6.3}$$

The appropriate limits for the integral a and b are often where the

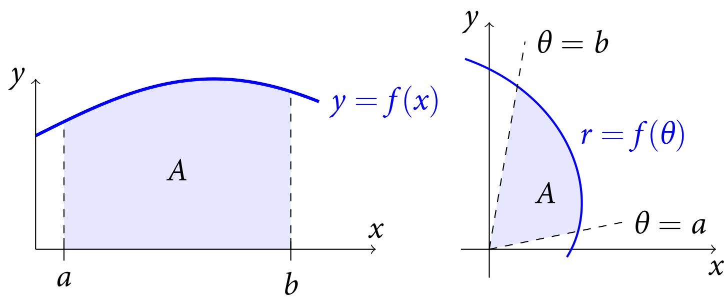

Figure 6.1: Area under a curve in Cartesian coordinates (left) and inside a curve in polar coordinates (right).

two curves cross, and in this case are found by equating r_{outer} and r_{inner}.

Example 6.1.2: Find the area of the circle

$$r_2 = 2\cos\theta$$

that lies outside the circle

$$r_1 = 1.$$

6.2 Volumes of solids of revolution

Rotations about the x-axis

Consider the curve $y = f(x)$ between $x = a$ and $x = b$ (Figure 6.2). The curve can be rotated around the x-axis to produce a solid of revolution (Figure 6.2).

What is the volume of this shape? The solid can be represented as a series of discs at position x, radius $f(x)$ and thickness dx. The volume of a single disc is $\pi f(x)^2\,dx$.

The volume of the whole solid of revolution is the sum of the vol-

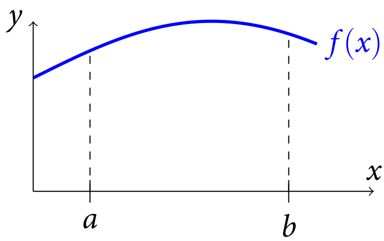

Figure 6.2: A curve $y = f(x)$ between $x = a$ and $x = b$.

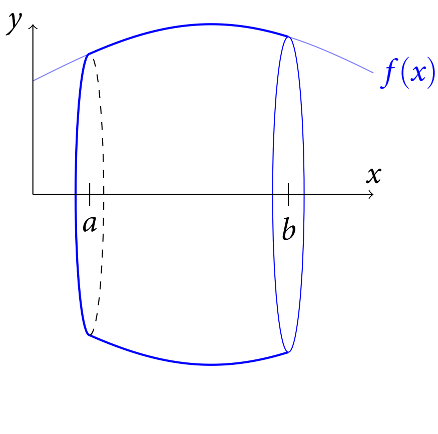

Figure 6.3: The curve $y = f(x)$ is rotated around the x-axis to produce a solid of revolution.

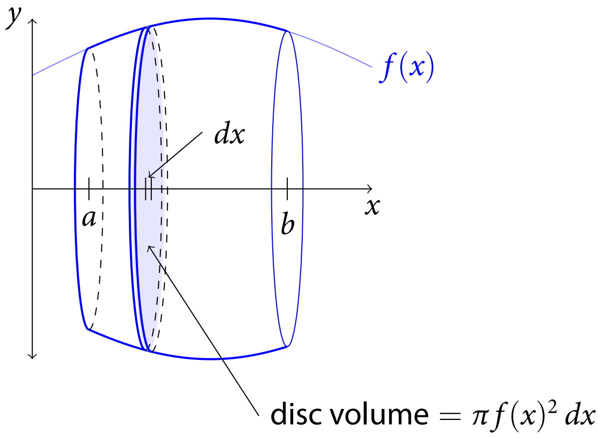

Figure 6.4: The volume of the solid of revolution is the sum of the discs.

ume of these discs, or

$$V = \int_a^b \pi f(x)^2\, dx. \tag{6.4}$$

Example 6.2.1: The semi-circle

$$y = \sqrt{a^2 - x^2} \qquad \text{for} \qquad -a \le x \le a,$$

when rotated about the x-axis, forms a sphere of radius a. Calculate the volume of this sphere.

Example 6.2.2: The curve

$$y = x(1 - x) \qquad \text{for} \qquad 0 \le x \le 1,$$

when rotated about the x-axis. Calculate the volume of the solid of revolution formed.

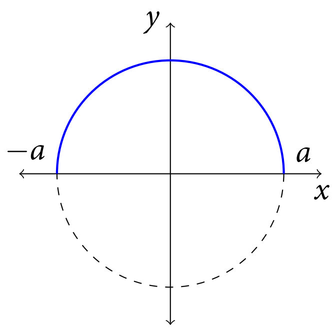

Figure 6.5: The semi-circle $y = \sqrt{a^2 - x^2}$ is rotated about the x-axis, to form a sphere of radius a in Example 6.2.1.

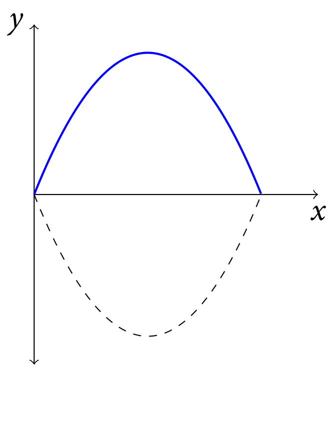

Figure 6.6: The curve $y = x(1-x)$ is rotated about the x-axis, to form a solid in Example 6.2.2.

As with the area inside a polar curve, to find the volume of a shape specified between two curves, we subtract the volume of the inner shape from that of the outer shape

$$V = \pi \left[\int_a^b f_{\text{outer}}(x)^2 \, dx - \int_a^b f_{\text{inner}}(x)^2 \, dx \right]. \tag{6.5}$$

Example 6.2.3: The area between the curves

$$y = x \qquad \text{and} \qquad y = x^{1/2} \qquad \text{for} \qquad 0 \le x \le 1$$

is revolved around the x-axis. Find the volume swept out.

Rotations about the y-axis

If the area between a curve $y = f(x)$ and the x-axis is rotated a full turn about the y-*axis* (Figure 6.2, the volume swept out is given by

$$V = 2\pi \int_a^b x f(x) \, dx. \tag{6.6}$$

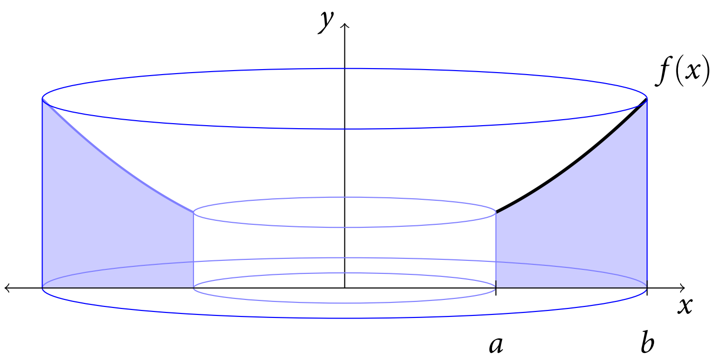

Figure 6.7: The volume by rotating the curve around the y-axis.

Why this expression? The volume of revolution can composed out of many thin 'rings', of radius x, height $f(x)$ and thickness dx. The perimeter of each 'ring' is $2\pi x$ and the volume of each ring is approximately the product of the perimeter, height and thickness, $2\pi x f(x)\, dx$. Summing over each ring for infinitesimally small dx gives the integral (6.6).

Example 6.2.4: The curve

$$y = 1 + x^2 \qquad \text{for} \qquad 1 \leq x \leq 2$$

is revolved around the y-axis. Find the volume of this solid of revolution.

6.3 Arc length

Curves in Cartesian coordinates

Suppose we have a curve described in Cartesian coordinates,

$$y = f(x)$$

between two points A and B, at $x = a$ and $x = b$ respectively.

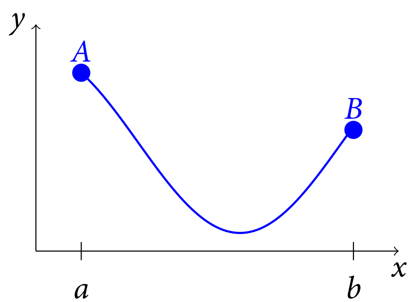

Figure 6.8: How to find the arc-length of the curve $y = f(x)$?

The length measured along the curve is called the *arc-length*. How do we find the arc-length between points A and B as in Figure 6.3?

Let's consider dividing this curve into many small segments, each of width δx The change in $f(x)$ over this segment is δy, and the length

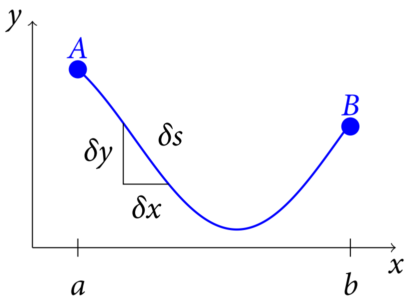

Figure 6.9: The arc-length of the curve is the sum of small segments.

of this segment is δs. If δx is sufficiently small, our segment is nearly a straight line, and

$$\delta s \approx \sqrt{(\delta x)^2 + (\delta y)^2}, \tag{6.7}$$

by Pythagoras' theorem. If the segment is nearly a straight line, its gradient is given by df/dx, and so

$$\delta y \approx \delta x \, df/dx. \tag{6.8}$$

Substituting (6.8) into (6.7) we find that

$$\delta s \approx \sqrt{1 + (df/dx)^2} \, \delta x.$$

Summing up the length δs of each segment making up the curve, the total arc length between A and B is

$$S = \int_a^b \sqrt{1 + (df/dx)^2} \, dx. \tag{6.9}$$

Example 6.3.1: Find the arc-length along the curve

$$y = f(x) = \frac{1}{3}(2x-1)^{3/2}$$

between $x = 1$ and $x = 2$. Compare this with the straight line distance between these two points.

Example 6.3.2: A semicircle of radius 1 is given by the equation

$$y = f(x) = \sqrt{1-x^2} \qquad \text{for} \qquad -1 \leq x \leq 1.$$

From the equation of the perimeter of a circle, $l = 2\pi r$, we expect this arc-length of this semicircle to be π. Verify that this is true.

Curves in polar coordinates

If we have a curve specified in polar coordinates, $r = f(\theta)$, the equation for the arc-length of a curve between $\theta = \theta_1$ and $\theta = \theta_2$ is

$$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta. \tag{6.10}$$

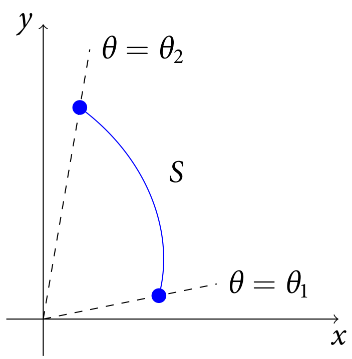

Figure 6.10: The arc-length of a curve in polar coordinates.

Example 6.3.3: A semi-circle of radius 1 is specified in polar coordinates simply by

$$r = 1 \qquad \text{for} \qquad 0 \leq \theta \leq \pi.$$

Calculate the arc-length of this semi-circle.

Example 6.3.4: Find the arc-length of the spiral $r = e^{\theta/5}$ between $\theta = 0$ and $\theta = 4\pi$.

6.4 Surface areas of solids of revolution

We have seen that a curve $y = f(x)$ can be rotated about the x-axis to form a solid of revolution, and have calculated the volume of shape. But what is the curved surface area of a solid of revolution?

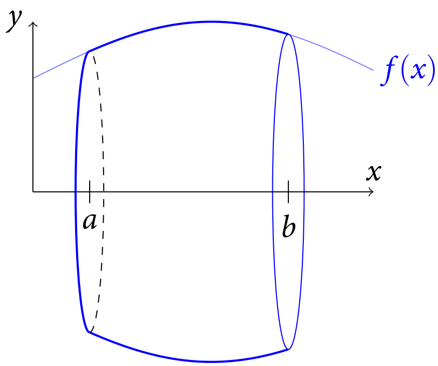

Figure 6.11: The surface of a solid by rotating the curve $y = f(x)$ around x-axis.

We can split the shape into many thin discs, as we did when calculating the volume. Each disc has perimeter $2\pi f(x)$ and width dx in the x-direction. However, the arc-length of $f(x)$ over the width of the disc dx is

$$\sqrt{1 + (df/dx)^2}\, dx,$$

(see section 6.3). The surface area of each disc is the product of arc length and perimeter,

$$2\pi f(x)\,\sqrt{1+(df/dx)^2}\,dx.$$

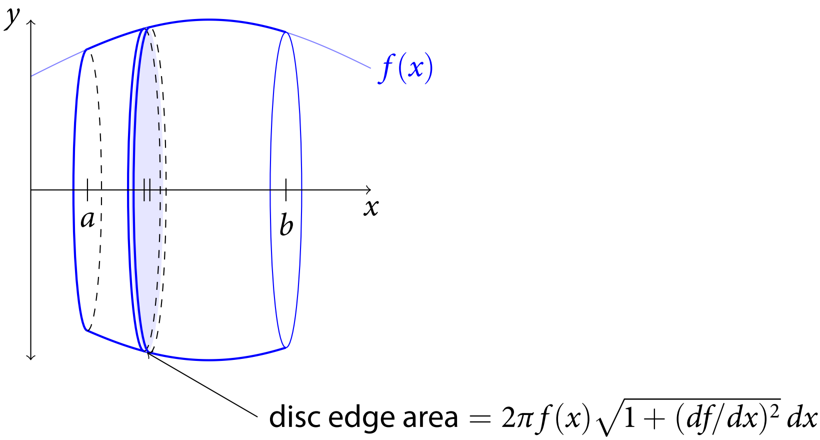

Figure 6.12: The surface area is obtained by integrating the disc edge area.

Integrating over each disc gives us the curved surface area of the solid of revolution between $x = a$ and $x = b$,

$$A = 2\pi \int_a^b f(x)\sqrt{1+(df/dx)^2}\,dx. \qquad (6.11)$$

Note that this curved surface area does not include the area of the circular 'ends' of the solid of revolution at $x = a$ and $x = b$.

Example 6.4.1: When the line

$$y = \frac{r}{h}x \qquad \text{for} \qquad 0 \le x \le h$$

(for constants r and h) is rotated about the x-axis, the solid of revolution formed is a cone of height h and base radius r. Find the curved surface area of the cone.

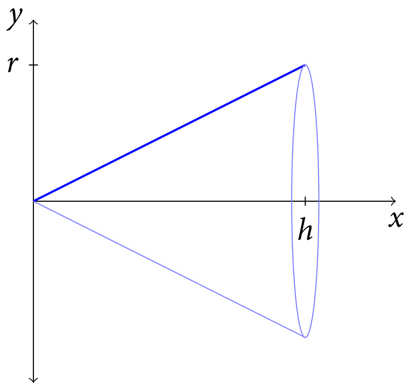

Figure 6.13: The line $y = rx/h$ is rotated to produce a solid of revolution (a cone) in Example 6.4.1.

Example 6.4.2: Calculate the curved surface area when the curve

$$y = \frac{2}{3}x^3,$$

taken between $x = 0$ and $x = 1$, is rotated once about the x-axis.

Chapter 7

Recurrence relations and reduction formulae

7.1 Recurrence relations

A recurrence relation is an equation that enables a term in a sequence of numbers to be calculated from previous terms. If we have a sequence $u_0, u_1, u_2, \ldots$, where the terms are referred to by u_n (for $n \geq 0$), a example of a recurrence relation linking these terms is

$$u_n = 2u_{n-1} - 1.$$

If an initial value u_0 is also provided, the recurrence relation can be used to calculate subsequent values in the sequence. For example if

$$u_0 = 2, \tag{7.1}$$
$$u_n = 2u_{n-1} - 1, \tag{7.2}$$

we can use (7.2) with $n = 1$ to calculate $u_1 = 2u_0 - 1 = 3$, then use it again with $n = 2$ to calculate $u_2 = 2u_1 - 1 = 5$, then $u_3 = 9$ and so on.

Sometimes a term in the sequence may depend on more than one previous term, for example

$$u_n = nu_{n-1} - u_{n-2}. \tag{7.3}$$

In this case, more than one initial value is required. If we specify $u_0 = 1$ and $u_1 = 3$, (7.3) with $n = 2$ gives

$$u_2 = 2u_1 - u_0 = 5,$$

With $n = 3$ we then find

$$u_3 = 3u_2 - u_1 = 12,$$

then $u_4 = 43$, etc.

Example 7.1.1: Find u_1, u_2, u_3, u_4, u_5 for

$$\begin{aligned} u_n &= 2u_{n-1} - 4 \\ u_0 &= 3. \end{aligned}$$

Example 7.1.2: Find u_2, u_3, u_4, u_5, u_6 for the Fibonacci sequence

$$\begin{aligned} u_n &= u_{n-1} + u_{n-2} \\ u_0 &= 0, \\ u_1 &= 1. \end{aligned}$$

Finding the general expression for the nth term

Sometimes it is possible to write down a simple expression for the nth term, for any n. For example, if

$$\begin{aligned} u_n &= nu_{n-1} \\ u_0 &= 1, \end{aligned}$$

then $u_2 = 2$, $u_3 = 6$, $u_4 = 24$. For general n,

$$u_n = n! \qquad (n \text{ factorial}).$$

A general expression for the nth term can be difficult (or even impossible) to find. However, once found, the general expression is usually more useful than the original recurrence relation, since we can use the general expression to evaluate any term in the sequence without having to evaluate all of the previous terms.

Example 7.1.3: Suppose we have a recurrence relation and initial condition

$$\begin{aligned} u_n &= 2(6-n)u_{n-1} \\ u_0 &= 1. \end{aligned}$$

Describe the behaviour of this sequence for $n \geq 0$.

Example 7.1.4: Describe the behaviour of the sequence u_n $(n \geq 0)$ when

$$\begin{aligned} u_n &= 15 - u_{n-1} \\ u_0 &= 4. \end{aligned}$$

Example 7.1.5: Find u_1, u_2, u_3, u_4, u_5 for

$$\begin{aligned} u_n &= \frac{-3u_{n-1}^2 + 5u_{n-1} + 2}{2} \\ u_0 &= 0. \end{aligned}$$

7.2 Reduction formulae

A particular class of recurrence relations, called *reduction formulae*, can simplify the process of integrating certain functions. Suppose we want to evaluate the integral

$$\int_0^{\pi/2} (\sin x)^n \; dx,$$

for some integer $n \geq 0$. For $n = 0$, $n = 1$, $n = 2$ the integral is straightforward enough, but for larger n it becomes more difficult. Reduction formulae sometimes provide an easy way of working out these integrals for large n.

Let's define our integral as I_n,

$$I_n = \int_0^{\pi/2} (\sin x)^n \; dx. \tag{7.4}$$

We can think of I_n as a sequence, $I_0, I_1, I_2, \ldots$ where each term is defined by (7.4). Our aim will be to calculate a recurrence relation that links successive terms of this sequence, which allows relatively straightforward computation of I_n for any n. We can rewrite (7.4) as

$$I_n = \int_0^{\pi/2} (\sin x)(\sin x)^{n-1}\, dx,$$

and integrate this by parts (see appendix A):

$$\begin{aligned}
I_n &= \int_0^{\pi/2} (\sin x)^n \; dx \\
&= \int_0^{\pi/2} \overbrace{(\sin x)}^{dv/dx}\, \overbrace{(\sin x)^{n-1}}^{u} \; dx \\
&= \left[\overbrace{(-\cos x)}^{v}\, \overbrace{(\sin x)^{n-1}}^{u} \right]_0^{\pi/2} - \int_0^{\pi/2} -(\cos x)^2 (n-1)(\sin x)^{n-2}\, dx \\
&= -\int_0^{\pi/2} -(\cos x)^2 (n-1)(\sin x)^{n-2}\, dx \\
&= (n-1) \int_0^{\pi/2} \left(1 - (\sin x)^2\right) (\sin x)^{n-2}\, dx \\
&= (n-1)\left(I_{n-2} - I_n\right).
\end{aligned}$$

Rearranging by collecting all the terms in I_n on the left hand side, and dividing by n, we find a recurrence relation for I_n,

$$I_n = \frac{n-1}{n} I_{n-2}. \tag{7.5}$$

This recurrence relation is called a reduction formula, because it allows the integral I_n to be reduced to an integral with a smaller vale of the parameter n. We can use the reduction formula repeatedly to evaluate I_n for any n. For example,

$$I_6 = \frac{5}{6} I_4 = \frac{5}{6}\frac{3}{4} I_2 = \frac{5}{6}\frac{3}{4}\frac{1}{2} I_0 = \frac{5}{16} I_0.$$

In our example we can straightforwardly calculate initial conditions for $n = 0$ and $n = 1$,

$$\begin{aligned} I_0 &= \int_0^{\pi/2} 1\,dx = \pi/2 \\ I_1 &= \int_0^{\pi/2} \sin x\,dx = 1, \end{aligned}$$

which, with the reduction formula (7.5), allow us to calculate $n_2 = \pi/4$, $n_3 = 2/3$, $n_4 = 3\pi/16$, $n_5 = 8/15$, $n_6 = 5\pi/32$ etc.

A summary of the technique for reduction formulae is

1. Integrate by parts to write I_n in terms of I_{n-1} and/or I_{n-2}
2. Evaluate the integrals I_0 and/or I_1 as required.
3. Use the results from step 2 to write I_n in terms of I_0 and/or I_1.

Example 7.2.1: Find a reduction formula for

$$I_n = \int_0^{\pi/4} (\sec\theta)^n \, d\theta.$$

Hence, find

$$I_4 = \int_0^{\pi/4} (\sec\theta)^4 \, d\theta.$$

Example 7.2.2: Find a reduction formula for

$$I_n = \int_1^{e} (\log x)^n \, dx.$$

(Note that $\log$ is the natural logarithm, base e.) Hence, find

$$I_5 = \int_1^e (\log x)^5 \; dx.$$

Hint: $\int \log x \, dx = x \log x - x + c$. (This can be proved by integrating by parts, setting $u = \log x$ and $dv/dx = 1$.)

Example 7.2.3: Find a reduction formula satisfied by the integral

$$I_n = \int_0^1 \left(x^2 + 1\right)^n \; dx.$$

Hence, evaluate the integral

$$\int_0^1 \left(x^2 + 1\right)^5 \; dx.$$

The examples we have seen so far have all been integrals of the form

$$\int_a^b f(x)^n \, dx,$$

for some function $f(x)$. The integration by parts has involved splitting the term $f(x)^n$ into two, either,

$f(x)^2 \times f(x)^{n-2}$	in example (7.2.1),
$f(x) \times f(x)^{n-1}$	in example (7.2.2), or
$1 \times f(x)^n$	in example (7.2.3).

Reduction formulae can be found for many integrals of the form $\int f(x)^n \, dx$ by splitting in one of these three ways, then integrating by parts. However it may not be immediately obvious which method of splitting is most useful for a particular integrand – some trial and error may be required.

Reduction formulae can also be found for other forms of integral. For example, if the integrand is a trigonometric function ($\sin$, $\cosh$, $\exp$, etc.) times a *monomial* x^n,

$$I_n = \int_a^b f(x)\, x^n \, dx$$

then this can be integrated by parts, using

$$u = x^n \qquad \frac{dv}{dx} = f(x). \tag{7.6}$$

We will give an example of the simplest problem of this type

Example 7.2.4: Find a recurrence relation for

$$I_n = \int_0^1 e^x x^n \, dx$$

and hence evaluate

$$\int_0^1 e^x x^5 \, dx.$$

We integrate by parts, using (7.6),

$$\begin{aligned} I_n &= \int_0^1 e^x x^n \, dx \\ &= \left[e^x x^n\right]_0^1 - n \int_0^1 e^x x^{n-1} \, dx \\ &= e - nI_{n-1}, \end{aligned}$$

and so our recurrence relation is

$$I_n = e - nI_{n-1}.$$

We need one initial condition, I_0, which we evaluate as

$$I_0 = \int_0^1 e^x \, dx = [e^x]_0^1 = e - 1.$$

Applying the recurrence, we find

$$\begin{aligned} I_0 &= e - 1 \\ I_1 &= 1 \\ I_2 &= e - 2 \\ I_3 &= 6 - 2e \\ I_4 &= 9e - 24 \\ I_5 &= 120 - 44e \approx 0.39560. \end{aligned}$$

Integration by parts without limits

All the reduction formulae we have found so far have been for definite integrals, *i.e.* those with limits,

$$\int_a^b f(x) \, dx.$$

Reduction formulae can also be found for indefinite integrals,

$$\int f(x) \, dx.$$

The procedure is very similar to that for definite integrals.

Example 7.2.5: Find a reduction formula for

$$I_n = \int (\cos x)^n \, dx,$$

and hence evaluate

$$I_5 = \int (\cos x)^5 \, dx.$$

Appendix A

Integration by parts

An important step in deriving a reduction formula is integration by parts. This is taught primarily in MATH19821 Mathematics 0C1, but in this appendix we will briefly cover the method. Suppose we have two functions $u(x)$ and $v(x)$. Then from the product rule of differentiation,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrating both sides of this equation with respect to x, we find

$$\int_a^b \frac{d}{dx}(uv)\ dx = \int_a^b u\frac{dv}{dx}\,dx + \int_a^b v\frac{du}{dx}\,dx. \qquad \text{(A.1)}$$

But integration is simply the reverse of differentiation, so the left hand side of this simplifies, to give

$$[uv]_a^b = \int_a^b u\frac{dv}{dx}\,dx + \int_a^b v\frac{du}{dx}\,dx.$$

Rearranging, we find

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\frac{du}{dx}\,dx. \qquad \text{(A.2)}$$

This means that if we have some integral

$$\int_a^b f(x)\,dx \tag{A.3}$$

and we can find functions $u(x)$ and $v(x)$ such that

$$f(x) = u\frac{dv}{dx} \tag{A.4}$$

then, from (A.2), we can write

$$\int_a^b f(x)\,dx = [uv]_a^b - \int_a^b v\frac{du}{dx}\,dx. \tag{A.5}$$

Why would we want to do this? Well, sometimes $\int_a^b v\frac{du}{dx}\,dx$ is much easier to evaluate than our original integral (A.3). In this case, using (A.5) may be an easy way to calculate (A.3).

Example A.0.1: Use integration by parts to evaluate

$$\int_0^\pi x\sin(x)\,dx.$$