Dynamics in 1D | 0J2 Mechanics *

John B. Parkinson
School of Natural Sciences, University of Manchester
john.b.parkinson@manchester.ac.uk.

Version 1.0 | 8 January 2021

*These notes are intended to be the definitive source of material discussed in the lecture course. They are reasonably comprehensive but should be augmented by notes taken when watching the weekly videos.

1 Forces and motion in 1D

Mechanics describes the effect of forces on bodies. It is normally divided into two parts:-

Statics: There is no motion, so the forces on the body balance out — called equilibrium.

Examples are a bridge, car parked on a hill, building, book lying on a table etc. etc.

Dynamics: This deals with systems in motion.

Examples are cars moving along roads, projectiles, planetary motion, pendulums etc. etc.

We shall consider both statics and dynamics and for each we shall consider

1. Principles: e.g. Newton's laws.
2. Practice: e.g. the equations describing the motion of a bouncing ball.

Both aspects are important.

1.1 Motion in one dimension

We consider particles (with no size) moving along a straight line (i.e. in 1D). We only need one coordinate, x, for the position of the particle.

If the particle starts at x_0 then the change in position, called the *displacement*, is $x - x_0$.

Since we are working in 1D all variables will be scalars not vectors. In general these can also change with time, t. We shall mainly use the following variables

$$\begin{array}{lrcl} \text{Position} & x(t) & & \\ \text{Displacement} & s(t) & = & x(t) - x_0 \\ \text{Velocity} & v(t) & = & \dfrac{dx}{dt} = \dfrac{ds}{dt} \\ \text{Acceleration} & a(t) & = & \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} \end{array}$$

In 1D *speed* is the positive value of the velocity, and is the same as velocity if the velocity is positive.

Worked example

A particle moves in 1D with velocity given by

$$v(t) \;=\; 5 + 2\cos t \quad \mathrm{ms}^{-1}$$

At $t \;=\; 0$ it is at $x \;=\; 10$. Find

(i) The maximum magnitude of the acceleration

(ii) The position at time $t \;=\; \pi/2$

(iii) The average speed from $t = 0$ to $t \;=\; \pi/2$.

Solution

(i) Here $v \;=\; 5 + 2\cos t$ so $a \;=\; \dfrac{dv}{dt} \;=\; -2\sin t$

Maximum $|a|$ is when $\sin t \;=\; \pm 1$. Therefore maximum $|a| \;=\; 2$.

(ii) $\dfrac{dx}{dt} \;=\; v \;=\; 5 + 2\cos t$ so integrating

$$x \;=\; 5t + 2\sin t + k$$

where k is the constant of integration.

$x = 10$ at $t = 0$ so $10 \;=\; 0 + 0 + k \;\Rightarrow\; k \;=\; 10$. Hence

$$x \;=\; 5t + 2\sin t + 10 \quad \mathrm{ms}^{-1}$$

At $t \;=\; \pi/2$, $x \;=\; 5\pi/2 + 2 + 10 \;=\; 19.854$ m.

(iii) The change in position from the starting point, the displacement, is $x - 10 \;=\; 9.854$.

The time taken to get to here is $\pi/2$ so the average speed is $\frac{9.854}{\pi/2} \;=\; 6.274\ \mathrm{ms}^{-1}$.

1.2 Constant acceleration

There are two cases to consider.

a) Zero acceleration If the acceleration is zero then $a \;=\; 0$ so $\dfrac{dv}{dt} \;=\; 0$.

Integrating: $v(t) \;=\; u$ where u is the constant of integration. This tells us that the velocity is constant.

Also $\dfrac{dx}{dt} = u$ so integrating again $x = k + ut$ where k is the constant of integration.
If $x = x_0$ at $t = 0$ then $k = x_0$ so

$$\boxed{x = x_0 + ut}$$

or, using the displacement, $s = x - x_0$, we have

$$\boxed{\mathrm{s} = \mathrm{ut}}.$$

Remember that u is a constant.

This is the equation for *uniform motion* (i.e. with no acceleration, constant velocity) in one-dimensional dynamics.

Much more important is
b) Constant acceleration Let the acceleration be constant with value a. Then

$$\begin{aligned}\frac{dv}{dt} &= a\\ v &= \int a\,dt = k + at\end{aligned}$$

where k is the constant of integration.

If $v = u$ at $t = 0$ then $k = u$ so

$$\boxed{\mathrm{v} = \mathrm{u} + \mathrm{at}} \qquad (1)$$

However, $v = \dfrac{dx}{dt}$ so

$$\begin{aligned}\frac{dx}{dt} &= u + at\\ x &= \int (u + at)\,dt = c + ut + \frac{at^2}{2}\end{aligned}$$

where c is the constant of integration.

If $x = x_0$ at $t = 0$ then $c = x_0$ and so

$$\boxed{x = x_0 + ut + \tfrac{1}{2}at^2} \qquad (2)$$

From (1) we have $v - u = at$ so $\frac{1}{2}at^2 = \frac{1}{2}(v - u)t$.
Substituting in (2) gives

$$x = x_0 + ut + \frac{1}{2}(v - u)t$$

$$\boxed{x = x_0 + \tfrac{1}{2}(u+v)t} \qquad (3)$$

We need one final equation. From (1) $v = u + at$ so $t = \dfrac{v-u}{a}$.
Substitute in (3)

$$x - x_0 = \frac{1}{2}(u+v)\frac{(v-u)}{a}$$

$$\begin{aligned} \therefore \quad 2a(x - x_0) &= (u+v)(v-u) \\ &= v^2 - u^2 \end{aligned}$$

$$\text{so} \qquad \boxed{v^2 = u^2 + 2a(x - x_0)} \qquad (4)$$

These four equations describe motion in 1D with constant acceleration.

We usually work with the *displacement* $s = x - x_0$. The four equations then become

$$\begin{aligned} v &= u + at && (1) \\ s &= ut + \frac{1}{2}at^2 && (2) \\ s &= \frac{1}{2}(u+v)t && (3) \\ v^2 &= u^2 + 2as && (4) \end{aligned}$$

We also need an alternative form of (2) using v instead of u.
From (1) $u = v - at$, and substituting in (2) gives

$$\begin{aligned} s &= (v - at)t + \frac{1}{2}at^2 \\ s &= vt - \frac{1}{2}at^2 && \text{(2a)} \end{aligned}$$

In these equations the t is the time, s is the displacement, u is the velocity at $t = 0$, v is the velocity at time t and a is the acceleration.

Notes
1. Each equation omits one of the five: t, s, u, v, a.
2. If the acceleration is negative it is usually called the *deceleration*.
3. a and u (and x_0) are constants.
4. It may be necessary to use more than one of these equations.

1.3 Worked examples (constant acceleration)

Example 1 A car travels from rest at a constant acceleration of 6.05 ms^{-2}. How long will it take to travel 1 km and what will be the velocity at the end.

Solution Here $a = 6.05, \quad u = 0 \quad$ and $s = 1000$ m.
We need t and v. Use (4) to find v:

$$\begin{aligned} v^2 &= u^2 + 2as \\ &= 0 + 2 \times 6.05 \times 1000 \\ &= 12100 \\ \therefore \quad v &= \sqrt{12100} = 110 \text{ ms}^{-1} \end{aligned}$$

Now use (1) to find t

$$\begin{aligned} v &= u + at \\ 110 &= 0 + 6.05t \\ \therefore \quad t &= \frac{110}{6.05} = 18.18 \text{ s} \end{aligned}$$

Example 2 A ball is rolled along the ground. The initial speed is 16 ms^{-1} and the deceleration is 4 ms^{-2}. How far will it travel?

Solution Here $u = 16, \quad v = 0 \quad$ and $a = -4$.
We need s so use (4):

$$0^2 = 16^2 + 2 \times (-4) \times s$$

Hence $s = \dfrac{16^2}{8} = 32$ m.

Example 3 A stone is dropped down a (dry!) well and strikes the bottom after 3 s. Neglecting air resistance, at what speed does it land and how deep is the well. Take the acceleration due to gravity $g = 9.81 \text{ ms}^{-2}$.

Solution Measuring downwards we have $u = 0, \quad a = g = 9.81 \quad$ and $t = 3$.
We need v and s.
Using (1) $v = u + at = 30 \text{ ms}^{-1}$.
Using (2) $s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2} \times 9.81 \times 9 = 44.15$ m.

Example 4 A coin is tossed vertically upwards with initial speed u ms^{-1}. At what time does it reach the highest point and what is the highest point? (Take $g = 9.81 \text{ ms}^{-2}$.)

Solution At the highest point $v = 0$. Measuring upwards we have $a = -9.81$.

Using (1): $v = u + at$ so $0 = u - 9.81t$. Therefore $t = \dfrac{u}{9.81}$.

Now use (3): $s \;=\; \frac{1}{2}(u+v)t \;=\; \frac{1}{2}(u+0)t \;=\; \frac{1}{2}u\frac{u}{9.81} \;=\; \frac{u^2}{19.62}$ m.

We could instead use (4): $s \;=\; \dfrac{v^2-u^2}{2a} \;=\; \dfrac{u^2}{19.62}$.

1.4 Newton's laws of motion

So far we have not considered the effect of force on moving bodies. The laws which describe these effects are *Newton's laws*, as follows:

N1 A body will remain at rest *or* in a state of uniform motion in a straight line *unless* it is acted upon by an external force.

(Note: uniform motion means constant velocity.)

N2 When an external force acts on a body of constant mass then it produces an acceleration directly proportional to the force (and in the same direction).

N3 Action and reaction are equal and opposite.
This means that when a body A exerts a force on body B, the B exerts an equal and opposite force on A.

Consequences of Newton's laws

N1: This law really defines what we mean by a force. The law implies

1. if a body accelerates then the resultant force on it is non-zero,
2. if a body has no acceleration then the resultant force on it is zero,
3. a body can move with a constant velocity if no forces are acting on it.

N2: In 1D the law states that a is proportional to F, so $F = ka$, where k is the constant of proportionality. Experimentally k is always the same for a given body, and is proportional to the mass of the body, so $F = cma$, where $k = cm$.

In fact, we choose the units of force (called the 'newton') to make $c = 1$, so

$$F = ma$$

1 newton is thus the force needed to accelerate a mass of 1 kg at 1 ms^{-2}.

Note that N1 tells us that if $F = 0$ then $v =$ constant, i.e. $v = k$ (k constant).
Therefore $dv/dt = 0$. But $a = dv/dt$ so $a = 0$ as expected from N2.
This shows that N1 and N2 are consistent.

N3 No further comments needed.

Types of force

We consider three distinct types of force in this lecture course.

1. Gravity: This is the force between two bodies due to their mass.

   When one body is the earth then the force on the other body is called the force of gravity. Galileo showed that all bodies falling freely, vertically, have the same acceleration g, independent of mass m.

   $$g = 9.81\ldots \mathrm{ms}^{-2}.$$

   By N2 this means that the force of gravity must be

   $$F = mg$$

   for a body of mass m.

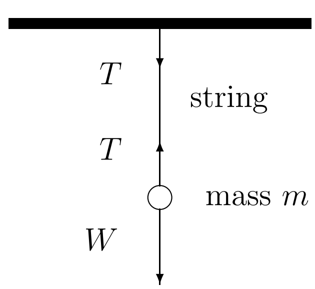
   The force of gravity on a body is often called its *weight* W. We assume that weight always acts vertically downwards.

2. Contact forces (push force): This is caused by one body pushing on another. A special case of this is *normal reaction*.

   If a book rests upon a table then the weight acts downwards. The book does not move because the table pushes upwards by an equal amount. This force is always perpendicular to the surface so it is called the normal reaction.

   Another special case is *friction* which hinders motion along a surface and therefore acts parallel to the surface.

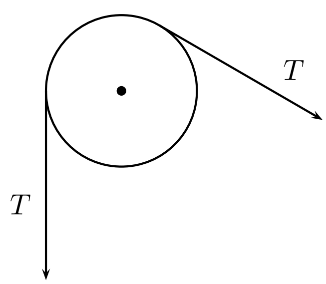
3. Forces of attachment (pull force): Typically these are forces caused by the tension in a string attached to a body. The force always acts *along* the string. The tension force is equal and opposite at the two ends of the string.



An example is a particle of mass m suspended by a string from a ceiling. Weight $W = mg$ acts downwards. Tension in the string is a force T upwards acting on the mass m *and* a force T downwards acting on the ceiling.

1.5 Pulleys

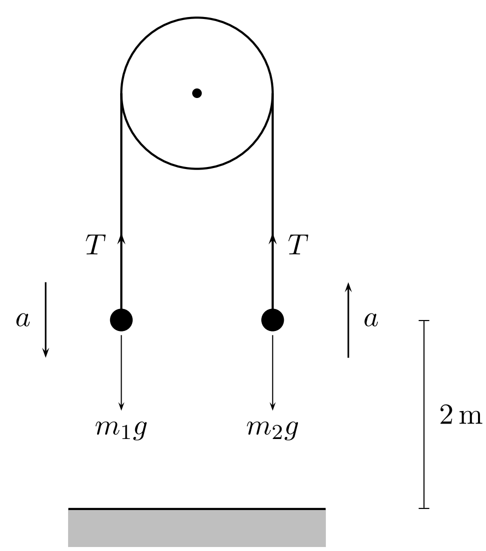
A pulley is a smooth (i.e. frictionless) disc, about which a string can pass. The magnitude of the tension (i.e. force) T in the string is the same on both sides of the pulley. Clearly the direction is different.



Alternatively, the disc may be free to rotate without friction about its centre. Again there is no change in the magnitude of T, only direction.

Example 1 Particles of masses m_1 and m_2 kg, with $m_1 > m_2$ are attached to either end of a *light inelastic* string which passes over a pulley. The two particles are initially at rest, at a height 2 m above a table. Find the acceleration of m_1, and the velocity with which it hits the table.

Solution. Let the tension in the string be T.



Apply N2 to m_1 particle (measure distances *downwards*)

$$m_1 g - T = m_1 a \qquad (1)$$

Apply N2 to m_2 particle (*upwards*)

$$T - m_2 g = m_2 a \qquad (2)$$

Adding (1) and (2) gives $(m_1 - m_2)g = (m_1 + m_2)a$, so

$$a = \frac{m_1 - m_2}{m_1 + m_2} g$$

Now $u = 0$, $s = 2$, $a =$ as above, and we want v.

$v^2 = u^2 + 2as$ gives $v^2 = 0 + 2\,\dfrac{m_1 - m_2}{m_1 + m_2} g \times 2$ so

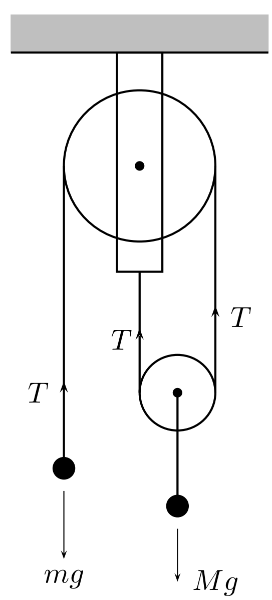
$$v = 2\sqrt{\frac{m_1 - m_2}{m_1 + m_2}\, g}$$

N.B. This system is not in equilibrium which means that the total force on each particle is not zero and the acceleration is also not zero. Thus $T \neq m_1 g$ and $T \neq m_2 g$. We shall discuss systems in equilibrium in the statics section of the course later.

Example 2

Consider two pulleys connected as shown. What mass m is needed is needed to balance the mass M? (Assume that the small moving pulley has negligible mass.)

The centre of the large pulley is fixed and so is the end of the string which is below the large pulley.



Solution. If the two masses are balanced then there is no acceleration. Since $F = ma$ there is no total force F on either mass.

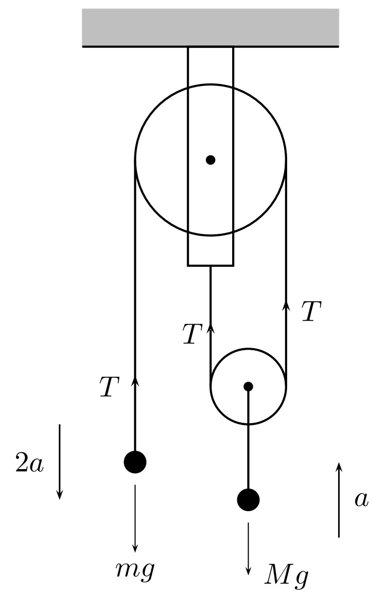
For mass m this means that $T - mg = 0$ so $T = mg$.
For mass M this means that $2T - Mg = 0$ so $2mg - Mg = 0$ and so $m = \frac{1}{2}M$.

Example 3

Find the acceleration of mass M in the same configuration as Example 2 but with $m = 3M/4$.

Here, if mass M moves a distance x upwards, then mass m moves a distance $2x$ downwards.

Hence, if the acceleration of M is $a = \frac{d^2x}{dt^2}$ upwards then the acceleration of m is $2a = 2\frac{d^2x}{dt^2}$ downwards.



Solution.

Using N2 for mass m: $mg - T = m(2a)$ (1) (downwards).
Using N2 for mass M: $2T - Mg = Ma$ (2) (upwards).

$2 \times (1) + (2)$ gives: $2mg - Mg = 2m(2a) + Ma = 4ma + Ma$.

$$\therefore \quad a = \frac{2mg - Mg}{4m + M}$$

But $m = \dfrac{3M}{4}$ so

$$a = \frac{\frac{3}{2}Mg - Mg}{3M + M} = \frac{\frac{1}{2}Mg}{4M} = \frac{1}{8}g\,.$$

1.6 Work and power

If a force is applied to a body, which then moves, we say the force **does work**.
In 1D, if the force is *constant* with magnitude F, and the body moves a distance x, the work done is

$$\boxed{W = Fx}$$

The unit of work is the *joule* (symbol J).

$$1 \text{ joule} = 1 \text{ newton metre (Nm)},$$

which is the work done by a force of 1 newton moving through a distance of 1 metre.

(Note: 1 newton = 1 $\text{kg}\,\text{ms}^{-2}$, so 1 joule = 1 $\text{kg}\,\text{m}^2\text{s}^{-2}$.)

The distance moved is actually the displacement $s = x - x_0$. For convenience we will assume a starting position $x_0 = 0$ so that $s = x$.

Aside If the force is not constant then the work is calculated by dividing the displacement into small sections of length δx at position x. The work done in this small section is the force at x, $F(x)$ times the length

$$\delta W = F(x)\delta x$$

Adding all the sections gives the total work

$$W = \Sigma F(x)\,\delta x\,.$$

where Σ denotes summation. In the limit that the sections become infinitesimally small then $\delta x \to dx$ and $\Sigma \to \int$ so that

$$W = \int F(x)\,dx\,.$$

We shall only consider constant forces in 0J2 so this will not be needed. End of aside

The **power** P is defined by

$$\boxed{P = Fv},$$

where F is the force on a body, and v is its velocity.

This definition applies even if the force and/or velocity are changing. If the force is *constant* then $W = Fx$ and

$$\frac{dW}{dt} = F\frac{dx}{dt} = Fv = P$$

so in this case the power is the 'rate of doing work'.

(One of the reasons why power is important in mechanics is that, for example, a car engine working at a fixed rate—at a fixed r.p.m.—generates (approximately) a fixed power; the force the engine generates will however vary with the speed of the car. As a car goes up a steep hill at constant power, it will slow down. As the velocity decreases the force produced by the engine will increase, until it is sufficient to maintain a constant (lower) velocity.)

The unit of power is joules/sec; this also has the name 'watt' (symbol W).

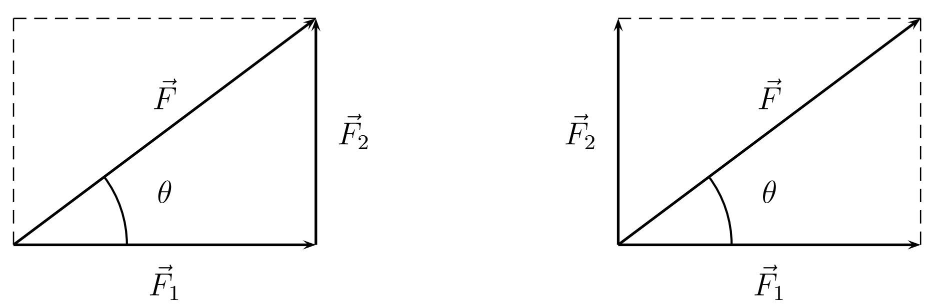
$$\begin{aligned} 1 \text{ watt} &= 1 \text{ joule/sec} = 1 \text{ Js}^{-1} \\ &= 1 \text{ Nms}^{-1} \\ &= 1 \text{ kg}\,\text{m}^2\text{s}^{-3} \end{aligned}$$

(Note: 1 kW = 1000W.)

1.7 Motion on an inclined plane

Before discussing an example of work and power, we need to consider how a force can be split into two separate forces in different directions. This is called *resolving forces* into components.

Forces are vectors and can be split (resolved) into two perpendicular separate forces in more convenient directions. Consider the vector sum of forces $\vec{F_1}$ and $\vec{F_2}$ using the triangle law. Let the sum be $\vec{F}$.



The second picture is equivalent to the first because vectors can be moved parallel to themselves without affecting their value.

Clearly the *magnitudes* of the three vectors satisfy

$$F_1 = F\cos\theta \quad \text{and} \quad F_2 = F\sin\theta$$

Notes
1. When we move *through* the angle θ we get a cosine
2. When we move *away from* the angle θ we get a sine.

Reversing this process we can now replace vector $\vec{F}$ by the two vectors $\vec{F_1}$ and $\vec{F_2}$ arranged as in the second figure.

Example 1 A car of mass 800 kg moves at a constant speed of 36 km/hr up a 1 in 8 incline. The top of the hill is a horizontal distance of 128 m from the car's initial position. Find

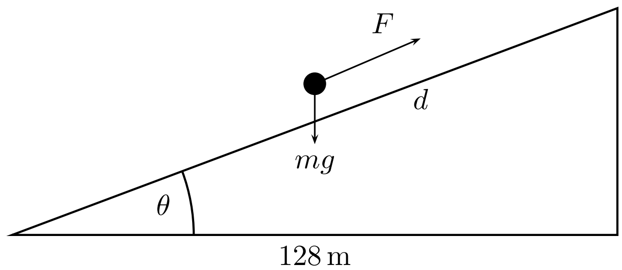
1. The work done in getting to the top of the hill.

2. The power at which the car works.

(Neglect friction, air resistance etc. Take $g = 9.81\,\mathrm{ms}^{-2}$.)

Solution
For a 1 in 8 slope:

$$\tan\theta = \frac{1}{8}, \quad \cos\theta = \frac{8}{\sqrt{65}},$$
$$\sin\theta = \frac{1}{\sqrt{65}}$$

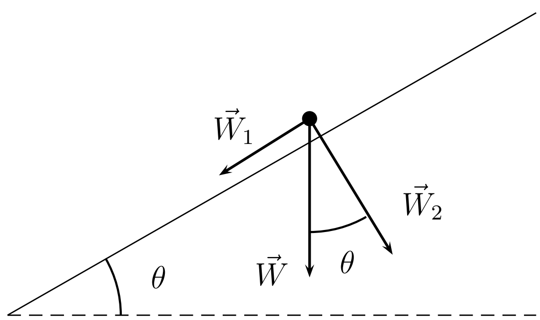


Let the distance moved be d, so $d\cos\theta = 128$ and

$$d = \frac{128}{\cos\theta} = \frac{128\sqrt{65}}{8} = 16\sqrt{65}$$

Let the force of the engine of the car be F. This acts parallel to the slope.

Now consider the weight (force of gravity) $\vec{W}$ in this example. This is a vertical force, with magnitude mg. However we can split it into two forces parallel ($\|$) and perpendicular ($\perp$) to the plane.



Here the weight (vector) has been replaced by vector $\vec{W}_1$, parallel to the plane, and vector $\vec{W}_2$, perpendicular to the inclined plane.

The magnitude of $\vec{W}_2$ is $mg\cos\theta$ since we move *through* the angle θ.
The magnitude of $\vec{W}_1$ is $mg\sin\theta$ since we move *away from* the angle θ.

Since speed is constant there is no net force (N1).

Consider forces parallel ($\|$) to the plane

$$F - mg\sin\theta = 0$$

$\therefore \quad F \;=\; mg\sin\theta \;=\; 800 \times g \times \frac{1}{\sqrt{65}}.$

Hence work done $\quad W \;=\; Fd$

$$= \frac{800g}{\sqrt{65}} \times 16\sqrt{65} \;=\; 12800 \times g \;=\; 125568 \text{ J}.$$

Finally

$$\text{Power} \quad P = Fv$$
$$= \frac{800g}{\sqrt{65}} \times \frac{36 \times 1000}{3600} \;=\; \frac{8000 \times 9.81}{\sqrt{65}} \quad \approx \; 9734.2 \text{ watts}.$$

Remark There is an additional force in this problem, the normal reaction, $\vec{R}$.
This acts perpendicular ($\perp$) to the plane. It is equal and opposite to $\vec{W}_2$ and prevents any motion perpendicular to the slope. Since we only considered forces parallel to the plane this force was not used in the calculation.

If a body has the capacity (or ability) to do work we say it has **energy**. The energy of the body is the amount of work it can do. When the body does some work it uses up some of its energy. But if work is done on the body its energy increases.

Energy comes in many forms (heat, light, electricity, ...) but we consider only *mechanical energy*. Mechanical energy is of two types

i) kinetic energy — ability to do work by virtue of having speed,

ii) potential energy — ability to do work by virtue of position.

1.8 Kinetic Energy

Suppose a particle of mass m is accelerated from rest to velocity v in a distance x by a constant force F. Here $u = 0$, so

$$v^2 = 0 + 2ax$$

But also $F = ma$ so

$$\begin{aligned} v^2 &= 2\frac{F}{m}x \\ \therefore \quad Fx &= \frac{1}{2}mv^2. \end{aligned}$$

Force times distance is work done, so the work done in getting to speed v from speed 0 is $\frac{1}{2}mv^2$. This is called the **kinetic energy** of the particle, since if we now reverse the process the particle can do this amount of work in slowing down to rest.

$$\text{kinetic energy (KE)} = \frac{1}{2}mv^2.$$

Note: since work is measured in joules, so is energy.

1.9 Potential Energy

Potential energy is usually due to gravity. Suppose we lift a particle of mass m from height 0 to height h. (We do this infinitely slowly, so no KE is involved.) The force needed is mg, and the distance moved is h, so the work done is mgh.

Again, if we reverse the process the particle can do this amount of work in coming down (very slowly!).

$$\text{potential energy (PE) (due to gravity)} = mgh$$

Notes
1. We only ever need *changes* in height — it doesn't matter where we measure height from.
2. If h is negative $h < 0$, then so is PE.

Another type of potential energy occurs when a spring is compressed, or an elastic string is stretched. Since we have to do work to do either of these, then reversing the process can cause work to be done.

Example 1: A particle of mass 2 kg is initially 3 m above a table of height 1 m. It is moved slowly down on to the table. Find the *change* in the PE.

Solution.

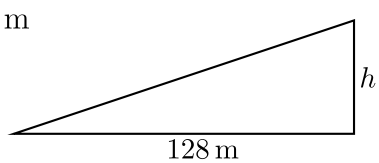
$$\begin{aligned} \text{Initial height } h_1 = 4, \quad \text{so PE } &= 2g \times 4 = 8g \\ \text{Final height } h_2 = 1, \quad \text{so PE } &= 2g \times 1 = 2g \end{aligned}$$

so change in PE is $2g - 8g = -6g$ joules.

Note: The *change* in height is -3 m,
so the change in PE is $mg \times (-3) = -6g$ as expected.

Example 2: Find the change in potential energy of the car in the earlier example, which moved up a hill.

Slope is 1 in 8; horizontal distance is 128 m



Solution.

The vertical distance is h where

$$\frac{h}{128} = \frac{1}{8} \qquad \therefore \quad h = 16\,\text{m}.$$

The change in PE is mgh. Taking $g = 9.81\,\text{ms}^{-1}$ we get

$$\text{Increase in PE} = 800 \times 9.81 \times 16 = 125568 \text{ joules.}$$

Notes
1. Increase in PE is exactly equal to work done (found earlier).
2. No change in KE since $v = 36\,\text{km/hr}$ at bottom and top.

When work is done on a body by applying a force which moves through a distance, e.g. by

— raising it (or compressing spring, stretching elastic etc.)

— accelerating it (increasing its velocity)

then its energy increases by the amount of work done.

Similarly, a body which possesses energy, either kinetic or potential, can give up that energy by doing work. Hence we say

$$\boxed{\textit{work and energy are equivalent}}$$

and they are both measured in joules.

1.10 Conservation of energy

If no work is done on a body, then its energy is unchanged. More specifically:

If the total work done by external forces acting on a body is zero, there is no change in the total mechanical energy of the body.

This is called the principle of conservation of mechanical energy.

Notes
1. The weight of a body does not count as an external force, since PE already takes this into account.
2. Although the total mechanical energy is unchanged, it can change from PE to KE or vice versa.

Example 1: A particle is dropped from rest at height h. With what velocity does it strike the ground?

We will solve this problem using two different strategies.

Solution 1 $u = 0$, $s = h$, $a = g$, so

$$\begin{aligned} v^2 &= u^2 + 2as \\ &= 0 + 2gh \\ \therefore \quad v &= \sqrt{2gh}. \end{aligned}$$

Solution 2 At start KE$= 0$, PE$= mgh$. At finish KE$= \frac{1}{2}mv^2$, PE$= 0$.

By conservation of energy:

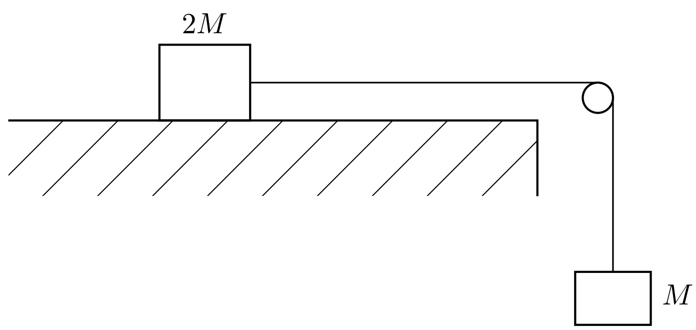
$$\begin{aligned} 0 + mgh &= \tfrac{1}{2}mv^2 \\ \therefore \quad v^2 &= 2gh \\ v &= \sqrt{2gh} \end{aligned}$$

Note: Measuring the height from a different point does not affect the answer, since conservation of energy says that

change in KE $=$ $-$ change in PE

so only the *change* in height matters.

Example 2: A block of mass $2M$ is lying on a smooth horizontal table. It is attached by a light, inelastic string, which passes over a pulley, to a mass M, as shown.



If the system is released from rest, find the velocity after M has dropped $0.8\,\text{m}$, with the $2M$ mass still on the table. (Take $g = 9.81\,\text{ms}^{-2}$).

Solution

Let the final velocity be v. Both masses have this v, so final KE is

$$\frac{1}{2}Mv^2 + \frac{1}{2}(2M)v^2 = \frac{3}{2}Mv^2$$

Initial KE is 0, so change in KE is $\frac{3}{2}Mv^2$.

Change of PE of $2M$ is zero, since it is still on the table (has not changed height).
Change of PE of M is

$$Mgh = M \times 9.81 \times (-0.8) = -7.848M$$

Total change in energy is zero so

$$\begin{aligned} \tfrac{3}{2}Mv^2 - 7.848M &= 0 \\ \therefore \quad v^2 &= \frac{2 \times 7.848}{3} = 5.232 \\ \therefore \quad v &= \sqrt{5.232} = 2.287\,\mathrm{ms}^{-1} \end{aligned}$$

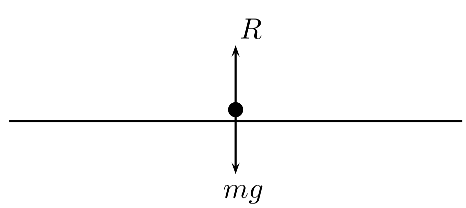
Note that we only needed the change in PE.

1.11 Reaction forces

We will consider two cases separately here.

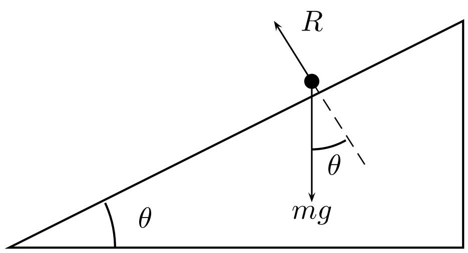
(a) Horizontal surface

When a mass m is moving, or stationary, on a horizontal surface there is no motion in the vertical direction. Since there is no vertical acceleration there is no net force in the vertical direction (Newton's 2nd law). Thus the force of gravity (weight) must be balanced by an equal and opposite force exerted by the surface on the particle. This is called the *normal reaction* R. Clearly the direction of this force is normal to the surface.



(b) Inclined surface

If it is moving (or stationary) on a surface inclined at an angle θ to the horizontal, then again the surface exerts a force R which is normal (i.e. perpendicular) to the surface, the *normal reaction*.

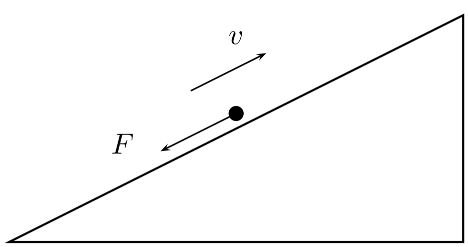


Since there is no motion in the direction normal to the surface, the total force in this direction must be zero. This means that the component of the weight in this direction must be equal and opposite to R so in this case

$$R = mg\cos\theta$$

Friction in Dynamics

If a body is moving along a surface (horizontal or inclined) which is not smooth then there is a friction force F. This always acts in the opposite direction to the velocity.



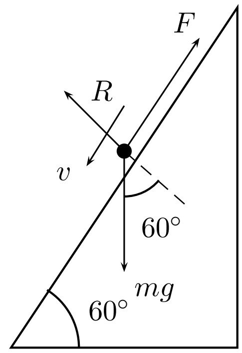
The magnitude of the friction force F is proportional to the magnitude of the normal reaction R. We use the Greek symbol μ for the constant of proportionality so

$$F = \mu R$$

and μ is called the coefficient of (dynamic) friction. Note that F is always independent of the velocity v.

There is also a coefficient of static friction if there is no motion. This is slightly different and will be discussed in a later section of the notes.

Example A particle of mass 3 kg is sliding down a surface inclined at an angle 60° to the horizontal. The coefficient of friction $\mu = 0.3$. Find the friction force and the acceleration. (Take $g = 9.81$ ms^{-2}.)



Solution

Normal to the plane ($\perp$) there is no motion, so the total force must be zero in this direction:

$$R - mg\cos 60^\circ = 0$$

$$\therefore \quad R = mg\cos 60^\circ = 3 \times 9.81 \times \frac{1}{2} \approx 14.72 \text{ N}.$$

The friction force is

$$F = \mu R \approx 0.3 \times 14.72 = 4.415 \text{ N}$$

and this acts parallel to the plane in an upwards direction.

Parallel to the plane ($\|$) we apply Newton's second law. Let the acceleration be a (downwards) then

$$mg\sin 60^\circ - F = ma$$

and using $F = \mu R = \mu mg\cos 60^\circ$ we have

$$mg\sin 60^\circ - \mu mg\cos 60^\circ = ma.$$

Hence

$$\begin{aligned} a &= g[\sin 60^\circ - \mu\cos 60^\circ] \\ &= 9.81[(\sqrt{3}/2) - 0.3(1/2)] = 7.024 \text{ ms}^{-2}. \end{aligned}$$

1.12 Conservation of momentum

If a force acts on a particle from time t_1 to time t_2 then we say that the particle receives an *impulse* I defined by

$$\boxed{I = \int_{t_1}^{t_2} F\, dt.}$$

The units of impulse are force $\times$ time i.e. N s.

Example If a force $F = (4+2t)$ N acts from $t=1$ to $t=3$ find the impulse.

Solution

Here the force is varying in time and the formula gives

$$I = \int_1^3 (4+2t)\,dt = [4t+t^2]_1^3 = (12+9)-(4+1) = 16 \text{ N s}$$

Special case If the force F is *constant* then

$$I = \int_{t_1}^{t_2} F\,dt = F\int_{t_1}^{t_2} dt = F(t_2-t_1) = FT$$

where $T = t_2 - t_1$ is the total time. End of special case.

Next, by Newton's 2nd law $F = ma = m\dfrac{dv}{dt}$.

Thus, if the force is not constant but the mass m is, then we can integrate both sides with respect to t to give

$$\int_{t_1}^{t_2} F\,dt = m\int_{t_1}^{t_2} \frac{dv}{dt}\,dt$$

$$I = m[v]_{t_1}^{t_2} = mv_2 - mv_1$$

where v_1 is the velocity at t_1 and v_2 is the velocity at t_2 .

Next, the *momentum* of a particle is defined to the product of its mass and its velocity:

$$\boxed{M = mv}$$

Hence, in the constant mass case, we have the important relation

$$I = mv_2 - mv_1 \quad \text{so} \quad \text{Impulse} = \text{change in momentum.}$$

Impulses are frequently used where we have a large force acting for a small time. Examples are (a) the blow of a hammer on a nail, and (b) the collision of two snooker balls.

Example A hammer of mass 0.8 kg is moving at 12 ms^{-1} when it strikes a nail and comes to rest. What is the impulse on the hammer? If the impulse lasts for 0.05 seconds, what is the average force of the hammer on the nail?

Solution

$$I = \text{change in momentum} = 0 - 0.8\times 12 = -9.6 \text{ Ns.}$$

Let the average force on the *hammer* be f. Since impulse = force × time for a constant force we have

$$f \times 0.05 \;=\; -\,9.6 \;=\; -\,192 \text{ N}.$$

The force on the *nail* is equal and opposite to that on the hammer (N3) so the average force on the nail is 192 N.

Key point Since the force of body A on body B is equal and opposite to the force of B on A, and the time is the same for both, the *impulses are equal and opposite.*

Collisions of particles

When two particles collide the impulses are equal and opposite so the change in momentum of one is equal and opposite to the change in momentum of the other. Thus there is no change in the total momentum, and we have

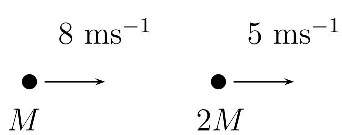
In a given direction, if no external forces act on a system then the total momentum of the system, in that direction, remains unchanged.

This is called the principle of conservation of linear momentum.

Example 1 A railway truck of mass M and velocity 8 ms^{-1} moves in a straight line to a truck of mass $2M$ moving at 5 ms^{-1} in the same direction. On impact they couple together. Find the final velocity v.

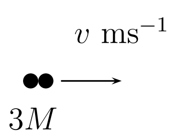
Solution

Before



$$\text{Momentum} \;=\; 8M + 5(2M) \;=\; 18M.$$

After



$$\text{Momentum} \;=\; v(3M).$$

By conservation of momentum $\quad 3Mv \;=\; 18M. \qquad \therefore \quad v \;=\; 6 \text{ ms}^{-1}.$

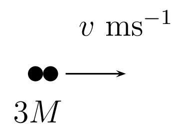
Example 2 What happens if they are initially moving in opposite directions?

Solution

Before

Momentum $= 8M - 5(2M) = -2M$.

After



By conservation of momentum $3Mv = -2M. \quad \therefore \quad v = -2/3 \text{ ms}^{-1}$.

Let's consider the change in kinetic energy in Example **1**.

$$\text{Before:} \quad \text{KE} = \frac{1}{2}M8^2 + \frac{1}{2}(2M)5^2 = 57M \quad \text{joules} \tag{1}$$

$$\text{After:} \quad \text{KE} = \frac{1}{2}(3M)6^2 = 54M \quad \text{joules} \tag{2}$$

Clearly (1) is not the same as (2). The PE is the same before and after.

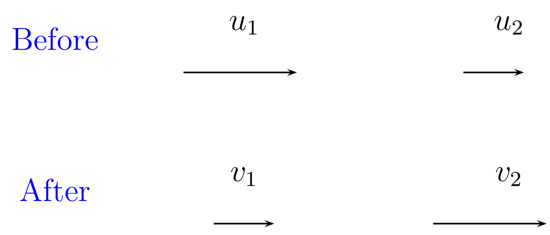
This shows that the mechanical energy is *not conserved* in collisions, or in general when we have impulses. Typically some energy is lost in sound (the 'bang 'of the collision) and some energy is converted to heat.

Note that, the impulse during the collision is the change in momentum of *one* of the bodies. The other one receives an equal and opposite impulse. In Example **1**, considering only the mass M, the momentum before the collision is $8M$, whereas the momentum after the collision is $6M$. This implies that the impulse on mass M is $-2M$ Newton seconds.

1.13 Law of restitution

In the two previous examples we assumed that the particles joined together at the collision and then moved with the same velocity. In general this does not happen and after the collision the particles will have *different* velocities.

Suppose we have two bodies moving with velocities u_1 and u_2 before the collision.
Here we measure velocities in the same direction (to the right, say). If the bodies are moving in opposite directions then u_2 will be negative. Let the two velocities after the collision be v_1 and v_2, again measured to the right.



Experimentally, we find that for any two colliding bodies the 'separation speed' is proportional to the 'approach speed'.

$$\begin{aligned}
\text{approach speed} &= \text{difference in velocity before the collision} = u_2 - u_1 < 0 \\
\text{separation speed} &= \text{difference in velocity after the collision} = v_2 - v_1 > 0
\end{aligned}$$

The constant of proportionality has the symbol e and is called the *coefficient* of restitution. Since the two bodies are moving towards each other before and away from each other after, we write

$$\boxed{(v_2 - v_1) = -e\,(u_2 - u_1)}$$

Notes

1. All velocities are measured in the same direction.
2. e is a positive number and $0 \le e \le 1$.
3. If $e = 1$ then we say the collision is 'perfectly elastic'. In this case energy IS conserved.
4. If $e = 0$ then then we say the collision is 'perfectly inelastic'. This is the case where $v_2 = v_1$ and the particles coalesce, as in Example **1**.

It is straightforward to show that kinetic energy *is conserved* in the collision if $e = 1$.

Proof: If $e = 1$ then $v_2 - v_1 = -(u_2 - u_1)$ so $\quad v_2 + u_2 = v_1 + u_1$ (1)

Next, using conservation of momentum, we have

$$\begin{aligned}
m_1 u_1 + m_2 u_2 &= m_1 v_1 + m_2 v_2 \\
\therefore \quad m_2(u_2 - v_2) &= m_1(v_1 - u_1) \qquad (2)
\end{aligned}$$

Multiplying (2) by (1) gives

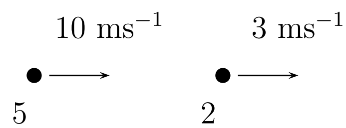
$$\begin{aligned}
m_2(u_2^2 - v_2^2) &= m_1(v_1^2 - u_1^2) \\
\therefore \quad m_1 u_1^2 + m_2 u_2^2 &= m_1 v_1^2 + m_2 v_2^2 \\
\therefore \quad \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2. \quad \heartsuit
\end{aligned}$$

Thus the kinetic energy is the same before and after.

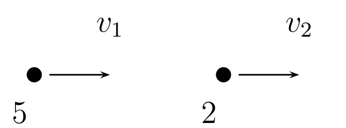
Key point: for perfectly elastic ($e = 1$) collisions energy is conserved as well as momentum. For other collisions ($e < 1$) only momentum is conserved.

Example A particle of mass 5 kg travelling at 10 ms^{-1} strikes a particle of mass 2 kg travelling at 3 ms^{-1} in the same direction. If the coefficient of restitution $e = 0.5$, find the velocities after impact.

Before



After



Here $m_1 = 5$, $m_2 = 2$, $u_1 = 10$, $u_2 = 3$.

Using the law of restitution

$$v_2 - v_1 = -e(u_2 - u_1) = -0.5 \times (-7) = 3.5 \qquad (1)$$

Using conservation of momentum

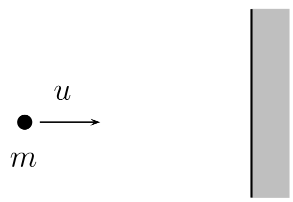
$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2 \\ 5 \times 10 + 2 \times 3 &= 5v_1 + 2v_2 \\ 5v_1 + 2v_2 &= 56 \qquad (2) \end{aligned}$$

From (1) $v_2 = v_1 + 3.5$ so

$$\begin{aligned} 5v_1 + 2(v_1 + 3.5) &= 56 \\ 7v_1 &= 49 \\ \therefore \quad v_1 &= 7 \text{ ms}^{-1}. \\ \text{and so} \quad v_2 &= 10.5 \text{ ms}^{-1}. \end{aligned}$$

Notice that m_1 slows down and m_2 speeds up.

1.14 Rebound velocity



If a particle of mass m strikes a *fixed* surface, then we cannot use conservation of momentum, since we cannot talk about the 'momentum' of a surface!

In the diagram the surface is a wall, and we know that the wall does not move. In fact the impulse at the collision causes the particle to bounce off the wall, and the law of restitution applies.

Let u be the velocity of approach to the wall (left to right) and v be the velocity away from the wall after impact. Note that v is the velocity from right to left!

Thus we have

$$\begin{aligned} u_1 &= u, \quad u_2 = 0 \\ \text{and} \quad v_1 &= -v, \quad v_2 = 0 \end{aligned}$$

u_2 and v_2 being zero since the wall does not move.

Using the law of restitution

$$\begin{aligned} v_2 - v_1 &= -e(u_2 - u_1) \\ 0 + v &= -e(0 - u) \end{aligned}$$

That is $\boxed{\text{v = eu}}$.

Example 1 A particle of mass 3 kg travelling at 4 ms^{-1} strikes a wall. If the coefficient of restitution is $e = 0.2$ find the velocity after the impact and the energy lost in the collision.

Solution
Before: $u = 4$ (towards the wall)
After: $v = +e\,u = 0.8\ \text{ms}^{-1}$ (away from the wall)

The energy (KE) before $= \frac{1}{2} 3 \times 4^2 = 24$ joules.
The energy after $= \frac{1}{2} 3 \times (0.8)^2 = 0.96$ joules.

So the energy lost is 23.04 joules.

Key point: The velocity before impact means the velocity immediately before impact. Likewise the velocity after impact means the velocity immediately after impact. If the velocity of the particle is changing, i.e. it is accelerating or decelerating then we need the velocity *at impact*.

Example 2 A ball of mass m is dropped from rest at a height h above the (horizontal) ground. The coefficient of restitution is $e = 1/3$. Find the height of the first bounce.

Solution The key idea is to break the calculation into three distinct stages:

1. Descent to ground

By conservation of energy, the ball hits the ground with a velocity u where

$$\tfrac{1}{2}mu^2 = mgh \quad \text{so} \quad u = \sqrt{2gh} \text{ (downwards).}$$

2. Collision

Using the law of restitution the velocity after impact is

$$v = e\,u = (1/3)\sqrt{2gh} \text{ (upwards).}$$

3. Ascent to the top of first bounce

It now rises to a height h_2 where, by conservation of energy,

$$mgh_2 = \frac{1}{2}mv^2 = \frac{1}{2}m\,\frac{1}{9}(2gh)$$

Thus $\quad h_2 = \dfrac{1}{9}\,h.$