Hands-On with Jupyter Notebooks: Virtual Fields Method for Material Identification

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Learning Outcomes

1. Advanced VF Strategies

- Special VFs for direct ID
- Optimized VFs for noise robustness
- Piecewise VFs for localized identification

2. Orthotropic Material ID

- Four-parameter identification
- Engineering constants recovery
- Physical interpretation of results

3. Practical Implementation

- System conditioning assessment
- Noise sensitivity analysis
- Experimental validation procedures

4. Critical Evaluation Skills

- VF selection criteria
- Error source identification
- Method comparison capabilities

Gateway to Advanced Applications: *Ability to apply VFM to complex material characterization problems with confidence in results*

Hands-On Implementation: Jupyter Notebook Approach

Why Jupyter Notebooks?

- **Interactive environment:** combine **code, equations, and results** in one place
- **Transparent workflow:** all steps of the VFM analysis are explicit and reproducible
- **Immediate feedback:** modify parameters, re-run, and visualize outputs on the spot
- **Ideal for learning:** seamless mix of theory, simulation, and experimental data

Tools for Implementation

- **Python scientific stack**: `NumPy`, `SciPy`, `pandas`
- **Plotting & visualization**: `matplotlib`, `seaborn`
- **Symbolics**: `SymPy` for analytical derivations
- **Jupyter widgets**: interactive sliders and controls for parameter studies

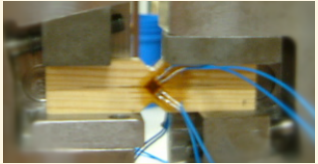
Key Advantage

A **reproducible, exploratory, and didactic platform** to implement and test the Virtual Fields Method with real experimental data or synthetic benchmarks.

Case Study: Unnotched Iosipescu Test

Introduction

Classical Iosipescu Test

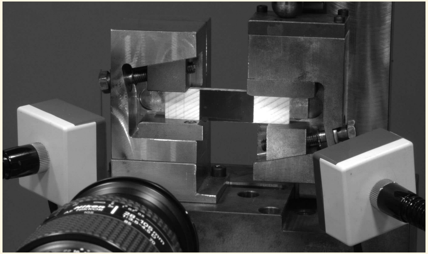


Engineering shear strain:	$\varepsilon_6 = \varepsilon_{+45^\circ} - \varepsilon_{-45^\circ}$
Nominal shear stress:	$\sigma_6 = P/A$
Stress-strain relationship:	$\sigma_6 = f_{12}(\varepsilon_6)$
Correct shear modulus:	$G_{12}^a = \sigma_6/\varepsilon_6 CS$
Apparent ultimate shear strength:	$S_{12}^a = P^{ult}/A$

where:

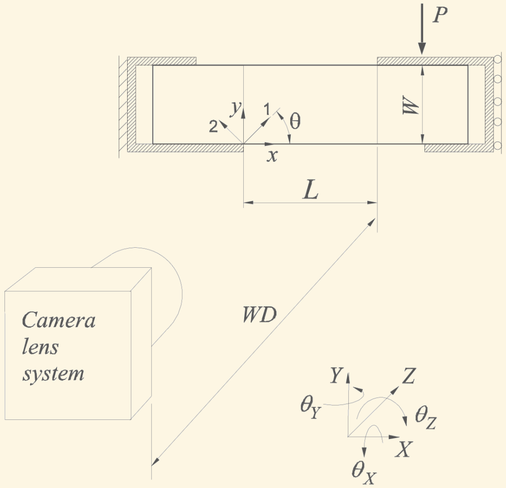
– $\varepsilon_{\pm 45^\circ}$: strains at $\pm 45^\circ$ to loading
– P: applied load
– A: cross-sectional area

Unnotched Iosipescu Test



- **Research Question**: Can all in-plane stiffness parameters Q_{ij} be simultaneously identified using the Iosipescu loading system?
- **Test design**: How can we generate the required heterogeneous strain field for multi-parameter identification?
- **VFM Implementation**: How does the selection of virtual fields (VF) influence identification robustness?

Configuration: geometry, material model and Loading



- **Design variables**:

  L and θ for optimal strain field generation

- **Material model**:

  Orthotropic with 4 unknown parameters:

  Q_{11}, Q_{22}, Q_{12}, Q_{66}

- **Global measurements**:

  Applied load (P) only

- **VF selection strategies**:
  - Manual design
  - Noise-sensitivity optimisation
  - Piecewise formulation

VFM: Orthotropic Material Model

VFM:
Hands-On
Jupyter
Notebooks

- IosipescuTest
- FE Model
- User-Defined VF
- Piecewise VF

Constitutive Relations:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix}$$

VFM Equation:

$$Q_{11} \int_S \varepsilon_1 \varepsilon_1^* \,\mathrm{d}S + Q_{22} \int_S \varepsilon_2 \varepsilon_2^* \,\mathrm{d}S \\ + Q_{12} \int_S (\varepsilon_1 \varepsilon_2^* + \varepsilon_2 \varepsilon_1^*) \,\mathrm{d}S + Q_{66} \int_S \varepsilon_6 \varepsilon_6^* \,\mathrm{d}S = \int_{L_f} T_i u_i^* \,\mathrm{d}l$$

Key Requirement

Need **4 independent virtual fields** to solve for 4 unknown parameters

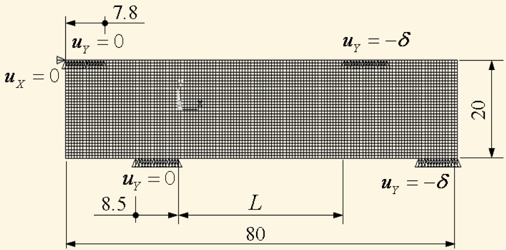
Engineering Constants:

$$E_1 = Q_{11} - \frac{Q_{12}^2}{Q_{22}} \quad \wedge \quad \nu_{12} = \frac{Q_{12}}{Q_{22}} \quad \wedge \quad E_2 = Q_{22} - \frac{Q_{12}^2}{Q_{11}} \quad \wedge \quad G_{12} = Q_{66}$$

Data Generation: FE Model

Material and Geometric Parameters:

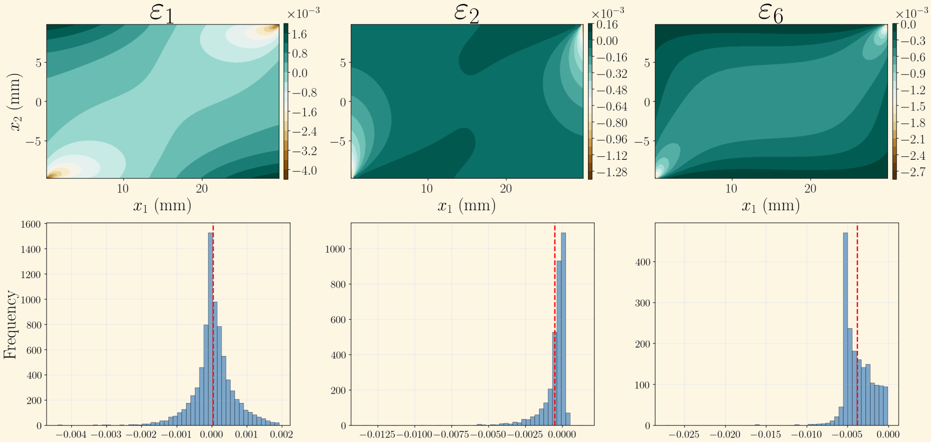
Parameter	Value
Off-axis angle, θ	0°
Length of interest region, L	34 mm
Width, W	20 mm
Thickness, t	5 mm
Young's modulus, E_1	15.1 GPa
Young's modulus, E_2	1.91 GPa
Poisson's ratio, ν_{12}	0.471
Shear modulus, G_{12}	1.109 GPa
Prescribed displacement, u_y	-0.5 mm



Computed Stiffness Matrix:

$$\mathbf{Q} = \begin{bmatrix} 15.54 & 0.93 & 0.0 \\ 0.93 & 1.97 & 0.0 \\ 0.0 & 0.0 & 1.109 \end{bmatrix} \text{ GPa}$$

FE Strain Field



Key Observations:

- **Shear dominance**: ε_6 is the primary strain component
- **Gauge section uniformity**: Relatively constant strain in central region
- **Load introduction effects**: Strain concentrations near loading points
- **Heterogeneity**: Complex strain distribution requires careful VF selection

VFs Selection Strategies

Key Challenges for Iosipescu Test:

1. Unknown Load Distribution:
- Complex contact conditions
- Non-uniform stress at loading points
- Need to filter unknown external work

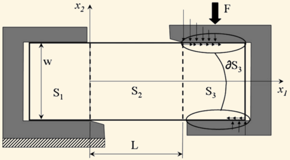
2. Four Parameter Identification:
- Q_{11}, Q_{22}, Q_{12}, Q_{66}
- Linear independence requirement
- Well-conditioned system needed

3. Strain Field Heterogeneity:
- Load introduction effects
- Edge effects
- Noise sensitivity

4. Strategy for Designing Virtual Fields:
- Eliminate unknown load contributions
- Provide sensitivity to each parameter
- Minimize noise amplification
- Respect kinematic admissibility

- **Manual VFs**: Linear system of equations
- **Optimized VFs**: Minimize noise effects
- **Piecewise VFs**: Localized approach

VFs: Rigid-Body Constraints



VWP:

$$-t\int_{S_1}\boldsymbol{\varepsilon}^*:\boldsymbol{\sigma}\,dS - t\int_{S_2}\boldsymbol{\varepsilon}^*:\boldsymbol{\sigma}\,dS - t\int_{S_3}\boldsymbol{\varepsilon}^*:\boldsymbol{\sigma}\,dS + t\int_{\partial S_1}\mathbf{u}^*\cdot\overline{\mathbf{T}}\,dl + t\int_{\partial S_2}\mathbf{u}^*\cdot\overline{\mathbf{T}}\,dl + t\int_{\partial S_3}\mathbf{u}^*\cdot\overline{\mathbf{T}}\,dl = 0$$

where ∂S_i is the line boundary of surface S_i and dl the elementary line unit.

VF Selection

1. Exploit Symmetry on S_1: All contributions from S_1 vanish by assuming:

$$u_1^{*(S_1)} = u_2^{*(S_1)} = 0$$

2. Rigid-Body Motion on S_3: Impose a rigid-body-like virtual displacement on S_3:

$$u_1^{*(S_3)} = ax_2 + b \quad \wedge \quad u_2^{*(S_3)} = -ax_1 + c$$

3. Eliminate Unknown $f_1(x_1)$: To eliminate the unknown horizontal force distribution, set:

$$\therefore \quad \mathbf{u}^{*(S_3)} = \{0 \quad c\}^T \quad \wedge \quad t\int_{\partial S_3}\begin{Bmatrix}0\\c\end{Bmatrix}\cdot\begin{Bmatrix}f_1(x_1)\\f_2(x_1)\end{Bmatrix}\mathrm{d}x_1 = F\cdot c$$

4. Continuity on S_2 Boundaries:

$$u_1^{*(S_2)}(x_1=0,x_2)=0 \quad \wedge \quad u_2^{*(S_2)}(x_1=0,x_2)=0$$
$$u_1^{*(S_2)}(x_1=L,x_2)=0 \quad \wedge \quad u_2^{*(S_2)}(x_1=L,x_2)=c$$

Virtual Field Set 1: 4 Independent Manually-Selected VF

Displacement VFs:

$$\text{VF1:} \quad u_1^{*(1)} = 0, \quad u_2^{*(1)} = -x_1$$

$$\text{VF2:} \quad u_1^{*(2)} = x_1(L - x_1)x_2, \quad u_2^{*(2)} = \frac{x_1^3}{3} - \frac{Lx_1^2}{2}$$

$$\text{VF3:} \quad u_1^{*(3)} = 0, \quad u_2^{*(3)} = x_1(L - x_1)x_2$$

$$\text{VF4:} \quad u_1^{*(4)} = \frac{L}{2\pi}\sin(2\pi x_1/L), \quad u_2^{*(4)} = 0$$

Strains VFs:

$$\text{VF1:} \quad \varepsilon_1^{*(1)} = 0, \quad \varepsilon_2^{*(1)} = 0, \quad \varepsilon_6^{*(1)} = -1$$

$$\text{VF2:} \quad \varepsilon_1^{*(2)} = (L - 2x_1)x_2, \quad \varepsilon_2^{*(2)} = 0, \quad \varepsilon_6^{*(2)} = 0$$

$$\text{VF3:} \quad \varepsilon_1^{*(3)} = 0, \quad \varepsilon_2^{*(3)} = x_1(L - x_1), \quad \varepsilon_6^{*(3)} = (L - 2x_1)x_2$$

$$\text{VF4:} \quad \varepsilon_1^{*(4)} = \cos(2\pi x_1/L), \quad \varepsilon_2^{*(4)} = 0, \quad \varepsilon_6^{*(4)} = 0$$

Virtual Field Set 1: System of Equations

$\mathbf{AQ} = \mathbf{B}:$

$$\begin{bmatrix} 0 & 0 & 0 & \overline{\varepsilon_6} \\ -\overline{\varepsilon_1(L-2x_1)x_2} & 0 & -\overline{\varepsilon_2(L-2x_1)x_2} & 0 \\ 0 & -\overline{\varepsilon_2 x_1(L-x_1)} & -\overline{\varepsilon_1 x_1(L-x_1)} & -\overline{\varepsilon_6(L-2x_1)x_2} \\ \overline{\varepsilon_1 \cos(2\pi x_1/L)} & 0 & \overline{\varepsilon_2 \cos(2\pi x_1/L)} & 0 \end{bmatrix} \begin{bmatrix} Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{66} \end{bmatrix} = \begin{bmatrix} \dfrac{F}{wt} \\ \dfrac{FL^2}{6wt} \\ 0 \\ 0 \end{bmatrix}$$

where the overbar denotes spatial averaging over the domain.

Virtual Field Set 1: Results

Table: Identified Stiffness Parameters - Set 1

Parameter	Computed (GPa)	Reference (GPa)	Error (%)
Q_{11}	15.54	15.54	0.053
Q_{22}	1.96	1.97	0.287
Q_{12}	0.92	0.93	0.205
Q_{66}	1.11	1.11	0.0

Key Features:

- Sinusoidal virtual field captures periodic behaviour
- Excellent accuracy for all parameters
- Matrix condition number: $\sim 10^6$
- Residual norm: $< 10^{-12}$

Virtual Field Set 2: Modified Fourth Field

Displacement VFs:

VFM: Hands-On Jupyter Notebooks
- IosipescuTest
- FE Model
- User-Defined VF
- Piecewise VF

$$\text{VF1:}\quad u_1^{*(1)} = 0, \quad u_2^{*(1)} = -x_1$$

$$\text{VF2:}\quad u_1^{*(2)} = x_1(L - x_1)x_2, \quad u_2^{*(2)} = \frac{x_1^3}{3} - \frac{Lx_1^2}{2}$$

$$\text{VF3:}\quad u_1^{*(3)} = 0, \quad u_2^{*(3)} = x_1(L - x_1)x_2$$

$$\text{VF4:}\quad u_1^{*(4)} = 0, \quad u_2^{*(4)} = x_1(L - x_1)x_2^3$$

Strains VFs:

$$\text{VF1:}\quad \varepsilon_1^{*(1)} = 0, \quad \varepsilon_2^{*(1)} = 0, \quad \varepsilon_6^{*(1)} = -1$$

$$\text{VF2:}\quad \varepsilon_1^{*(2)} = (L - 2x_1)x_2, \quad \varepsilon_2^{*(2)} = 0, \quad \varepsilon_6^{*(2)} = 0$$

$$\text{VF3:}\quad \varepsilon_1^{*(3)} = 0, \quad \varepsilon_2^{*(3)} = x_1(L - x_1), \quad \varepsilon_6^{*(3)} = (L - 2x_1)x_2$$

$$\text{VF4:}\quad \varepsilon_1^{*(4)} = 0, \quad \varepsilon_2^{*(4)} = 3x_1(L - x_1)x_2^2, \quad \varepsilon_6^{*(4)} = (L - 2x_1)x_2^3$$

Virtual Field Set 2: System of Equations

$\mathbf{AQ} = \mathbf{B}$:

$$
\begin{bmatrix}
0 & 0 & 0 & \overline{\varepsilon_6} \\
-\overline{\varepsilon_1 (L - 2x_1) x_2} & 0 & -\overline{\varepsilon_2 (L - 2x_1) x_2} & 0 \\
0 & -\overline{\varepsilon_2 x_1 (L - x_1)} & -\overline{\varepsilon_1 x_1 (L - x_1)} & -\overline{\varepsilon_6 (L - 2x_1) x_2} \\
0 & -3\overline{\varepsilon_2 x_1 (L - x_1) x_2^2} & -3\overline{\varepsilon_1 x_1 (L - x_1) x_2^2} & -\overline{\varepsilon_6 (L - 2x_1) x_2^3}
\end{bmatrix}
\begin{bmatrix}
Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{66}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\dfrac{F}{wt} \\ \dfrac{FL^2}{6wt} \\ 0 \\ 0
\end{bmatrix}
$$

where the overbar denotes spatial averaging over the domain.

Virtual Field Set 2: Results

VFM: Hands-On Jupyter Notebooks
- IosipescuTest
- FE Model
- User-Defined VF
- Piecewise VF

Table: Identified Stiffness Parameters - Set 2

Parameter	Computed (GPa)	Reference (GPa)	Error (%)
Q_{11}	15.49	15.54	0.280
Q_{22}	1.96	1.97	0.107
Q_{12}	0.98	0.93	6.228
Q_{66}	1.11	1.11	0.0

Key Features:

- Cubic virtual field in x_2 direction
- Enhanced sensitivity to through-thickness variations
- Matrix condition number: $\sim 10^7$
- Slightly higher sensitivity to noise

Virtual Field Set 3: Modified First Field

VFM: Hands-On Jupyter Notebooks
- IosipescuTest
- FE Model
- User-Defined VF
- Piecewise VF

Displacement VFs:

$$\text{VF1:} \quad u_1^{*(1)} = 0, \quad u_2^{*(1)} = -x_1^3$$

$$\text{VF2:} \quad u_1^{*(2)} = x_1(L - x_1)x_2, \quad u_2^{*(2)} = \frac{x_1^3}{3} - \frac{Lx_1^2}{2}$$

$$\text{VF3:} \quad u_1^{*(3)} = 0, \quad u_2^{*(3)} = x_1(L - x_1)x_2$$

$$\text{VF4:} \quad u_1^{*(4)} = \frac{L\sin(2\pi x_1/L)}{2\pi}, \quad u_2^{*(4)} = 0$$

Strains VFs:

$$\text{VF1:} \quad \varepsilon_1^{*(1)} = 0, \quad \varepsilon_2^{*(1)} = -3x_1^2, \quad \varepsilon_6^{*(1)} = 0$$

$$\text{VF2:} \quad \varepsilon_1^{*(2)} = (L - 2x_1)x_2, \quad \varepsilon_2^{*(2)} = 0, \quad \varepsilon_6^{*(2)} = 0$$

$$\text{VF3:} \quad \varepsilon_1^{*(3)} = 0, \quad \varepsilon_2^{*(3)} = x_1(L - x_1), \quad \varepsilon_6^{*(3)} = (L - 2x_1)x_2$$

$$\text{VF4:} \quad \varepsilon_1^{*(4)} = \cos(2\pi x_1/L), \quad \varepsilon_2^{*(4)} = 0, \quad \varepsilon_6^{*(4)} = 0$$

Virtual Field Set 3: System of Equations

$\mathbf{AQ} = \mathbf{B}$:

$$
\begin{bmatrix}
0 & 0 & 0 & 3\overline{\varepsilon_6 x_1^2} \\
-\overline{\varepsilon_1 (L - 2x_1) x_2} & 0 & -\overline{\varepsilon_2 (L - 2x_1) x_2} & 0 \\
0 & -\overline{\varepsilon_2 x_1 (L - x_1)} & -\overline{\varepsilon_1 x_1 (L - x_1)} & -\overline{\varepsilon_6 (L - 2x_1) x_2} \\
\overline{\varepsilon_1 \cos(2\pi x_1 / L)} & 0 & \overline{\varepsilon_2 \cos(2\pi x_1 / L)} & 0
\end{bmatrix}
\begin{bmatrix} Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{66} \end{bmatrix}
=
\begin{bmatrix} \dfrac{FL^2}{wt} \\ \dfrac{FL^2}{6wt} \\ 0 \\ 0 \end{bmatrix}
$$

where the overbar denotes spatial averaging over the domain.

Note: First VF: cubic in x_1, highlighting lengthwise variation

Virtual Field Set 3: Results

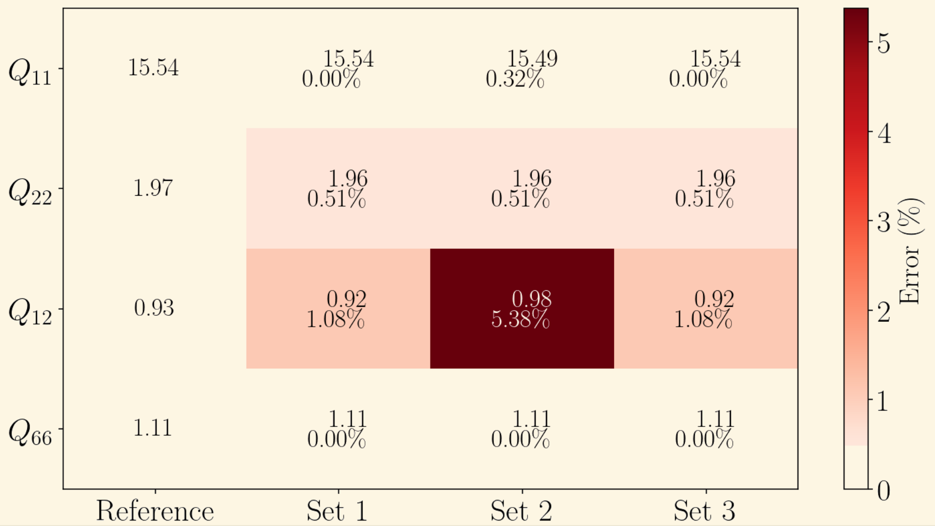
Table: Identified Stiffness Parameters - Set 3

Parameter	Computed (GPa)	Reference (GPa)	Error (%)
Q_{11}	15.54	15.54	0.053
Q_{22}	1.96	1.97	0.284
Q_{12}	0.92	0.93	0.205
Q_{66}	1.11	1.11	0.004

Key Features:

- Cubic virtual field in x_1 direction
- Enhanced sensitivity to length-wise variations
- Matrix condition number: $\sim 10^5$
- Good numerical stability

Comparison of Virtual Field Sets

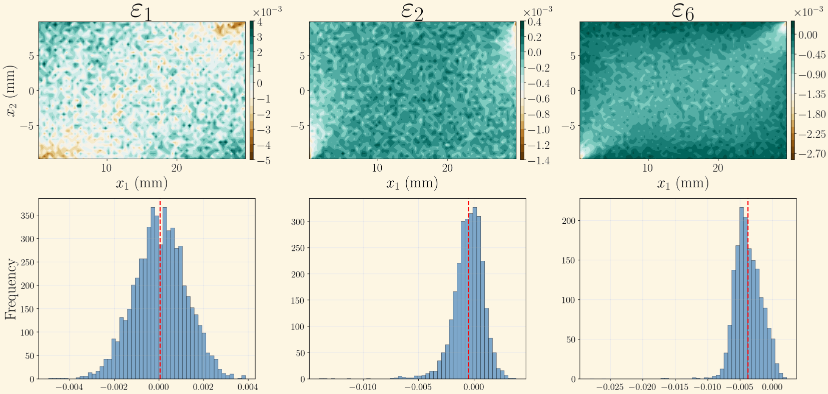


FE Noisy Strain Fields

Noise Analysis Parameters

Noise amplitude (std dev): **1.0×10^{-4}**

Monte Carlo iterations: **30**



Orthotropic VFM — Noise Analysis Summary

	Param	Ref Value	Mean ± Std	CV%	Err%
Set 1	Q_{11}	15.54	15.62 ± 0.32	2.02	0.54
	Q_{22}	1.97	1.98 ± 0.04	2.21	0.72
	Q_{12}	0.93	0.93 ± 0.18	19.16	0.09
	Q_{66}	1.11	1.11 ± 0.00	0.26	0.00
Set 2	Q_{11}	15.54	15.60 ± 0.53	3.38	0.41
	Q_{22}	1.97	1.98 ± 0.06	2.93	0.85
	Q_{12}	0.93	0.95 ± 0.58	60.60	2.48
	Q_{66}	1.11	1.11 ± 0.00	0.26	0.00
Set 3	Q_{11}	15.54	15.62 ± 0.32	2.02	0.54
	Q_{22}	1.97	1.98 ± 0.04	2.24	0.79
	Q_{12}	0.93	0.93 ± 0.18	19.16	0.09
	Q_{66}	1.11	1.11 ± 0.00	0.31	0.08

Orthotropic VFM — Noise Analysis Summary

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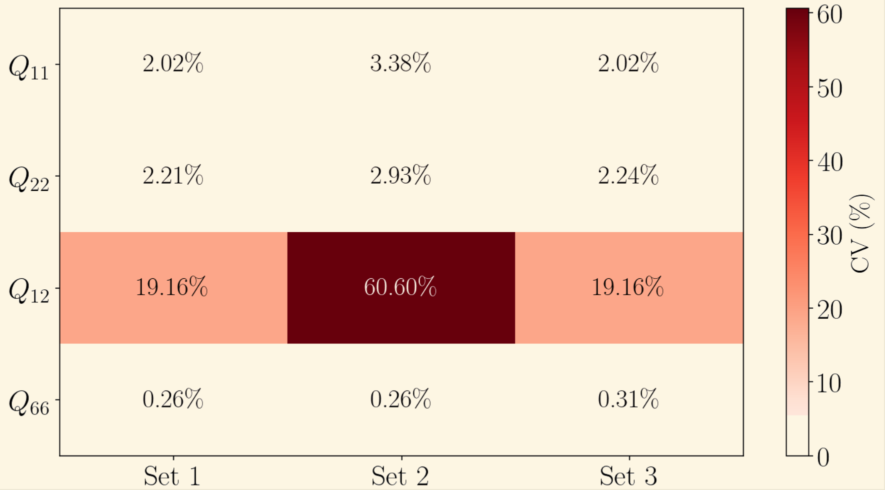
- Piecewise VF



Virtual Field Selection Guidelines

Set 1 - Original Formulation

- **Best for:** General applications, balanced sensitivity to all parameters
- **Advantages:** Sinusoidal field captures periodic behavior, proven reliability

Set 2 - Through-Thickness Enhanced

- **Best for:** Applications with significant x_2 (thickness) variations
- **Advantages:** Enhanced sensitivity to transverse effects, good for thick specimens

Set 3 - Length-wise Enhanced

- **Best for:** Applications with significant x_1 (length) variations
- **Advantages:** Best numerical conditioning, excellent for longitudinal effects

Piecewise VFs

VFM:
Hands-On
Jupyter
Notebooks

- IosipescuTest
- FE Model
- User-Defined VF
- Piecewise VF

- Bilinear shape functions with 4-noded elements
- Virtual fields expanded as: $\mathbf{u}^* = \mathbf{N}\hat{\mathbf{u}}^{*(e)}$
- Key advantages:
  - More flexible for complex geometries
  - Easier boundary condition implementation

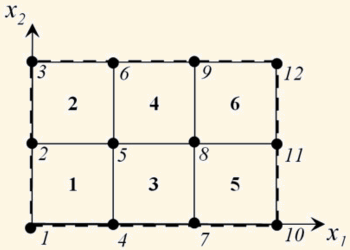
Mesh Configuration

Implementation Parameters:

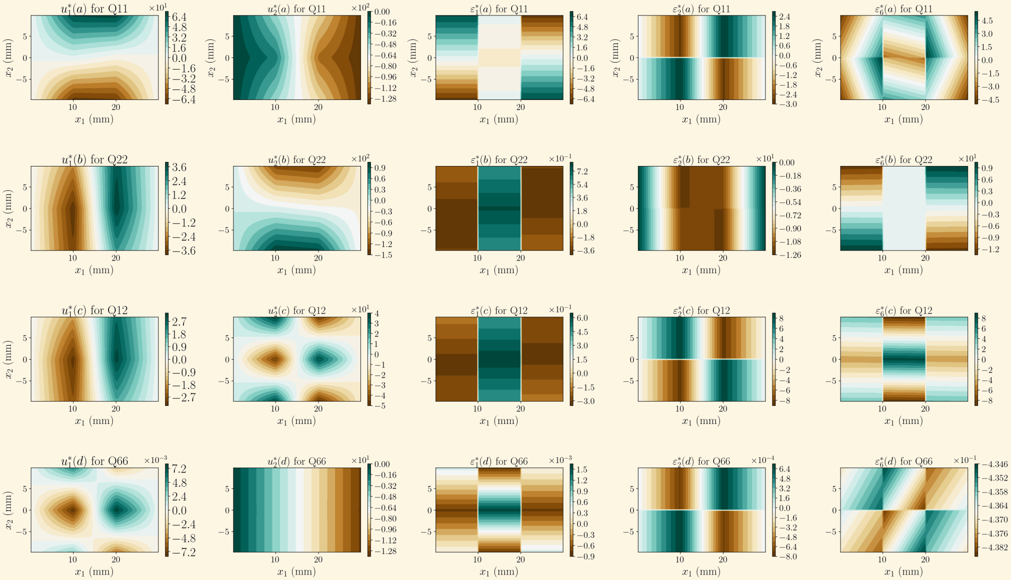
- Elements: $3 \times 2 = 6$ elements
- Nodes: 12 total nodes
- DOFs $\rightarrow 2(m+1)(n+1)$: 24 (2 per node)
- Element size: $L_{el} = L/3$, $w_{el} = w/2$

Constraint Conditions:

- Left boundary: $u_1^* = u_2^* = 0$
- Right boundary: $u_1^* = 0$, $u_2^* =$ constant
- Total constraints: 4n + 3 = 11



VFs Visualisation



Piecewise VFM Results & Sensitivity Analysis

Identification Results (mesh: 2×3):

Param	Piecewise	Ref.	Error (%)
Q_{11}	**15.51**	15.54	0.20
Q_{22}	**1.96**	1.97	0.41
Q_{12}	**0.98**	0.93	6.27
Q_{66}	**1.11**	1.11	0.00

Noise Sensitivity (η/Q):

1. Q_{66}: 2.51 (most stable)
2. Q_{11}: 22.06 (good)
3. Q_{22}: 40.85 (moderate)
4. Q_{12}: 87.79 (highest)

Physical Interpretation:

- ✓ Q_{66}: Direct shear measurement — highly stable
- ✓ Q_{11}: Strong bending signature — excellent accuracy
- $\sim$ Q_{22}: Limited by small ε_2 strain levels
- $\times$ Q_{12}: Coupling effects challenging to separate

Key Observations:

- Convergence in 2-3 iterations
- Excellent for Q_{11} and Q_{66}
- Higher η/Q ratio indicates noise sensitivity

Test Limitation: Iosipescu not optimal for Q_{22} and Q_{12} due to low transverse strain levels

Piecewise VFM Results: Mesh Sensitivity

Table: VFM piecewise identification: mesh convergence study

Mesh	**Elem.**	**Nodes**	Q_{11} **(GPa)**	Q_{22} **(GPa)**	Q_{12} **(GPa)**	Q_{66} **(GPa)**	**Err.** Q_{11} **(%)**	**Err.** Q_{22} **(%)**	**Err.** Q_{12} **(%)**	**Err.** Q_{66} **(%)**
Ref.	—	—	15.54	1.97	0.93	1.11	—	—	—	—
3×2	6	12	15.51	1.96	0.98	1.11	0.20	0.41	6.27	0.00
5×4	20	30	15.55	1.96	0.93	1.11	0.11	0.43	0.34	0.01
7×6	42	56	15.57	1.92	0.93	1.11	0.21	2.29	0.17	0.27
10×8	80	99	15.57	1.92	0.92	1.11	0.24	2.06	0.34	0.24

- Optimal mesh: 5×4 (best accuracy-to-cost ratio)
- Q_{11} and Q_{66} show excellent stability ($< 0.5\%$ error)
- Q_{22} converges to ∼1.92 GPa with finer meshes

Comparison: Manual VF Sets vs Piecewise

Table: Performance Comparison (All Methods)

Method	Q_{11} (GPa)	Q_{22} (GPa)	Q_{12} (GPa)	Q_{66} (GPa)
Set 1 (Manual)	15.54	1.96	0.92	1.11
Set 2 (Manual)	15.49	1.96	0.98	1.11
Set 3 (Manual)	15.54	1.96	0.92	1.11
Piecewise (5×4)	**15.55**	**1.96**	**0.93**	**1.11**
Reference	15.54	1.97	0.93	1.11

Advantages of Piecewise Approach:

- Systematic and automated (no manual virtual field derivation required)
- Excellent agreement with reference values (all parameters within 0.5%)
- Comparable accuracy to manual VF sets while being generalizable
- Enables systematic mesh refinement for convergence studies
- More intuitive boundary condition implementation

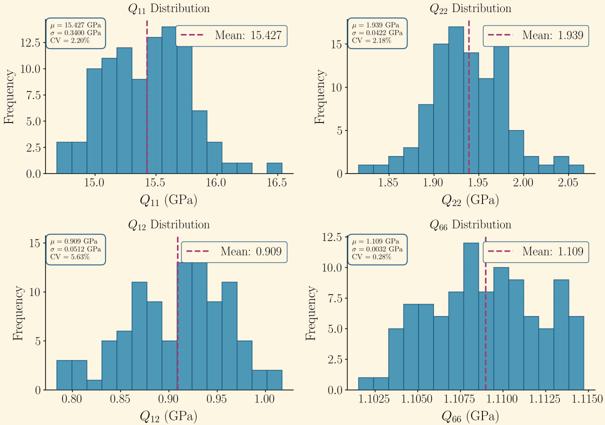
Virtual Fields Method: Monte Carlo Noise Sensitivity Analysis

Analysis Configuration

- Material: Wood (orthotropic)
- Mesh: 5×4 elements (20 total)
- Noise amplitude: $\sigma = 10^{-3}$
- Monte Carlo iterations: 100
- Virtual fields: Piecewise special

Key Findings:

- Q_{66} (shear stiffness) shows highest robustness (CV = 0.29%)
- Q_{12} exhibits highest sensitivity to noise (CV = 5.63%)
- All parameters within 2.0% error from reference values
- Method demonstrates good stability under measurement noise



Param	Mean (GPa)	Std (GPa)	CV (%)	Ref (GPa)	Error (%)
Q_{11}	15.427	0.340	2.20	15.536	0.70
Q_{22}	1.939	0.042	2.18	1.965	1.32
Q_{12}	0.909	0.051	5.63	0.926	1.82
Q_{66}	1.109	0.003	0.29	1.109	0.00

Piecewise VF: Summary and Implementation

Key Characteristics and Advantages:

- Continuous virtual displacements across element boundaries
- Element-wise constant strain distribution (discontinuous virtual strains)
- Automatic satisfaction of equilibrium requirements
- Robust numerical performance with scalable mesh refinement
- Compatible with standard FE software frameworks

Computational Benefits:

- Assembly procedure similar to standard FEM
- Direct nodal constraint application
- Natural handling of complex geometries and boundaries

Implementation Guidelines:

- Start with coarse mesh ensuring sufficient DOFs $>$ constraints
- Rule of thumb: DOFs $\approx 2 \times$ number of constraints
- Monitor conditioning of optimization matrix
- Validate with known reference cases

Google Colab Notebooks & GitHub Repository

Manual Virtual Fields:

- Orthotropic | Open in Colab
  Exercise Open in Colab
- With Noise | Open in Colab
  Exercise Open in Colab

Piecewise Virtual Fields:

- Orthotropic | Open in Colab
- With Noise Open in Colab

https://github.com/josexavier3/CISM-C2516_Image-Based-Mechanics_pyVFM.git

Summary

Technical Mastery:

- Successfully identified material parameters from a single test configuration
- Demonstrated alternative strategies for VF selection
- Established criteria for robustness against noise

Methodological Insights:

- Full-field measurements provide comprehensive information
- VF selection is a critical step in the identification process
- Optimisation enhances performance in the presence of noise
- Direct identification reduces the overall computational cost
- The VFM enables efficient material characterisation

Next Steps

Apply VFM principles to your specific material systems and experimental configurations!

Thank you for your attention!
Questions and Discussion

Hands-On with Jupyter Notebooks: Virtual Fields Method for Material Identification

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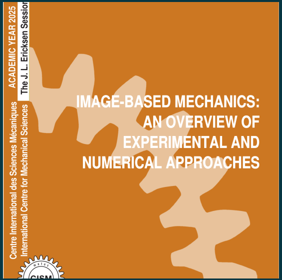

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