MODULE *SelfRefPuzzle*

A self-referential logic puzzle, based on

`https://davecturner.github.io/2018/10/22/kitty-grundman-self-referential-puzzle.html`

EXTENDS *Naturals*, *FiniteSets*

VARIABLES X, S We define X and S as variables so that we could use TLC

$$
\begin{array}{l}
\mathit{Puzzle} \triangleq \\
\quad \land\ X \in 1 \,..\, 10 \\
\quad \land\ S \in [1 \,..\, 10 \to \text{BOOLEAN}\ ] \\
\quad \land\ \text{LET}\ \mathit{Statements} \triangleq \text{DOMAIN}\ S \quad \text{The statement indices} \\
\qquad\quad \mathit{IsTrue}(x) \triangleq S[x] \\
\qquad\quad \mathit{IsFalse}(x) \triangleq \neg S[x] \\
\qquad\quad \mathit{True} \triangleq \{i \in \mathit{Statements} : \mathit{IsTrue}(i)\ \} \\
\qquad\quad \mathit{False} \triangleq \{i \in \mathit{Statements} : \mathit{IsFalse}(i)\} \\
\qquad\quad \mathit{IsEven}(x) \triangleq x\%2 = 0 \\
\qquad\quad \mathit{IsOdd}(x) \triangleq x\%2 \neq 0 \\
\qquad\quad a \otimes b \triangleq \neg(a \equiv b) \quad \text{Exclusive or} \\
\quad \text{IN}\quad \land \neg\forall\, i \in \mathit{Statements} : \mathit{IsTrue}(i) \\
\qquad\quad \land \neg\forall\, i \in \mathit{Statements} : \mathit{IsFalse}(i) \\
\qquad\quad \land\ S[1] = \text{LET}\ \mathit{sum}[s \in \text{SUBSET}\ \mathit{Nat}] \triangleq \text{IF}\ s = \{\}\ \text{THEN}\ 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{ELSE}\ \text{LET}\ x \triangleq \text{CHOOSE}\ x \in s : \text{TRUE} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{IN}\quad x + \mathit{sum}[s \setminus \{x\}] \\
\qquad\qquad\qquad\quad \text{IN}\quad X = \mathit{sum}[\mathit{False}] \\
\qquad\quad \land\ S[2] = (X < \mathit{Cardinality}(\mathit{False}) \land \mathit{IsTrue}(10)) \\
\qquad\quad \land\ S[3] = ((\mathit{Cardinality}(\mathit{True}) = 3) \otimes \mathit{IsFalse}(1)) \\
\qquad\quad \land\ S[4] = (1 \,..\, 3 \subseteq \mathit{False} \lor \mathit{IsTrue}(9)) \\
\qquad\quad \land\ S[5] = (\mathit{IsOdd}(X) \otimes \mathit{IsTrue}(7)) \\
\qquad\quad \land\ S[6] = \text{LET}\ \mathit{Odds} \triangleq \{i \in \mathit{Statements} : \mathit{IsOdd}(i)\} \\
\qquad\qquad\qquad\quad \text{IN}\quad \mathit{Cardinality}(\mathit{Odds} \cap \mathit{False}) = 2 \\
\qquad\quad \land\ S[7] = \mathit{IsTrue}(X) \\
\qquad\quad \land\ S[8] = \text{LET}\ \mathit{Evens} \triangleq \{i \in \mathit{Statements} : \mathit{IsEven}(i)\} \\
\qquad\qquad\qquad\quad \text{IN}\quad \mathit{Evens} \subseteq \mathit{True} \lor \mathit{Evens} \subseteq \mathit{False} \\
\qquad\quad \land\ S[9] = \text{LET}\ \mathit{First}(s) \triangleq \text{CHOOSE}\ x \in s : \forall\, y \in s : y \geq x \\
\qquad\qquad\qquad\quad \text{IN}\quad (X = 3 * \mathit{First}(\mathit{True})) \lor \mathit{IsFalse}(4) \\
\qquad\quad \land\ S[10] = (\mathit{IsEven}(X) \lor \mathit{IsTrue}(6))
\end{array}
$$

To get TLC to solve the puzzle for us, we have to put it in the form of a temporal specification and an invariant. A counterexample of the invariant would be the solution. To verify its uniqueness, we then let TLC check that the solution is an invariant (it is).

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\begin{array}{l}
\mathit{Spec} \triangleq \mathit{Puzzle} \land \Box[\text{UNCHANGED}\ \langle X, S\rangle]_{\langle X, S\rangle} \\
\mathit{Invariant} \triangleq X \notin 1 \,..\, 10
\end{array}
$$

$\land\ S = \langle$FALSE, FALSE, TRUE, TRUE, FALSE, TRUE, TRUE, FALSE, TRUE, FALSE$\rangle$
$\land\ X = 9$