MODULE *SelfRefPuzzle*

A self-referential logic puzzle, based on

`https://davecturner.github.io/2018/10/22/kitty-grundman-self-referential-puzzle.html`

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\begin{array}{l}
\textsc{extends}\ \mathit{Naturals},\ \mathit{FiniteSets} \\[6pt]
\textsc{variables}\ X,\ S \quad \text{We define } X \text{ and } S \text{ as variables so that we could use TLC} \\
\mathit{Puzzle} \triangleq \\
\quad \land\ X \in 1 \,..\, 10 \\
\quad \land\ S \in [1 \,..\, 10 \rightarrow \textsc{boolean}\,] \\
\quad \land\ \textsc{let}\ \begin{array}[t]{lcl}
\mathit{Statements} & \triangleq & \textsc{domain}\ S \quad \text{The statement indices} \\
\mathit{IsTrue}(x) & \triangleq & S[x] \\
\mathit{IsFalse}(x) & \triangleq & \neg S[x] \\
\mathit{True} & \triangleq & \{s \in \mathit{Statements} : \mathit{IsTrue}(s)\} \\
\mathit{False} & \triangleq & \{s \in \mathit{Statements} : \mathit{IsFalse}(s)\} \\
\mathit{IsEven}(x) & \triangleq & x \% 2 = 0 \\
\mathit{IsOdd}(x) & \triangleq & x \% 2 \neq 0 \\
a \otimes b & \triangleq & \neg(a \equiv b) \quad \text{Exclusive or}
\end{array} \\
\quad\ \ \textsc{in}\ \ \begin{array}[t]{l}
\land\ \neg\forall\, s \in \mathit{Statements} : \mathit{IsTrue}(s) \\
\land\ \neg\forall\, s \in \mathit{Statements} : \mathit{IsFalse}(s) \\
\land\ S[1] = \textsc{let}\ \mathit{sum}[s \in \textsc{subset}\ \mathit{Nat}] \triangleq \textsc{if}\ s = \{\}\ \textsc{then}\ 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textsc{else}\ \textsc{let}\ x \triangleq \textsc{choose}\ x \in s : \textsc{true} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ \textsc{in}\ \ x + \mathit{sum}[s \setminus \{x\}] \\
\qquad\qquad\quad \textsc{in}\ \ X = \mathit{sum}[\mathit{False}] \\
\land\ S[2] = (X < \mathit{Cardinality}(\mathit{False}) \land \mathit{IsTrue}(10)) \\
\land\ S[3] = ((\mathit{Cardinality}(\mathit{True}) = 3) \otimes \mathit{IsFalse}(1)) \\
\land\ S[4] = (1 \,..\, 3 \subseteq \mathit{False} \lor \mathit{IsTrue}(9)) \\
\land\ S[5] = (\mathit{IsOdd}(X) \otimes \mathit{IsTrue}(7)) \\
\land\ S[6] = \textsc{let}\ \mathit{Odds} \triangleq \{s \in \mathit{Statements} : \mathit{IsOdd}(s)\} \\
\qquad\qquad\quad \textsc{in}\ \ \mathit{Cardinality}(\mathit{Odds} \cap \mathit{False}) = 2 \\
\land\ S[7] = \mathit{IsTrue}(X) \\
\land\ S[8] = \textsc{let}\ \mathit{Evens} \triangleq \{s \in \mathit{Statements} : \mathit{IsEven}(s)\} \\
\qquad\qquad\quad \textsc{in}\ \ \mathit{Evens} \subseteq \mathit{True} \lor \mathit{Evens} \subseteq \mathit{False} \\
\land\ S[9] = \textsc{let}\ \mathit{First}(s) \triangleq \textsc{choose}\ x \in s : \forall\, y \in s : y \geq x \\
\qquad\qquad\quad \textsc{in}\ \ (X = 3 * \mathit{First}(\mathit{True})) \lor \mathit{IsFalse}(4) \\
\land\ S[10] = (\mathit{IsEven}(X) \lor \mathit{IsTrue}(6))
\end{array}
\end{array}
$$

To get TLC to solve the puzzle for us, we have to put it in the form of a temporal specification and an invariant. A counterexample of the invariant would be the solution. To verify its uniqueness, we then let TLC check that the solution is an invariant (it is).

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\begin{array}{lcl}
\mathit{Spec} & \triangleq & \mathit{Puzzle} \land \Box[\textsc{unchanged}\ \langle X,\ S\rangle]_{\langle X,\, S\rangle} \\
\mathit{Invariant} & \triangleq & X \notin 1 \,..\, 10 \quad \text{Taunt TLC into proving us wrong}
\end{array}
$$

$\land\ S = \langle \textsc{false}, \textsc{false}, \textsc{true}, \textsc{true}, \textsc{false}, \textsc{true}, \textsc{true}, \textsc{false}, \textsc{true}, \textsc{false} \rangle$
$\land\ X = 9$