

MODULE *SelfRefPuzzle*

A self-referential logic puzzle, based on

<https://davecturner.github.io/2018/10/22/kitty-grundman-self-referential-puzzle.html>

EXTENDS *Naturals*, *FiniteSets*

VARIABLES  $X, S$  We define  $X$  and  $S$  as variables so that we could use TLC

$Puzzle \triangleq$

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   $\wedge X \in 1 \dots 10$ 
   $\wedge S \in [1 \dots 10 \rightarrow \text{BOOLEAN}]$ 
   $\wedge \text{LET } Statements \triangleq \text{DOMAIN } S$  The statement indices
     $IsTrue(x) \triangleq S[x]$ 
     $IsFalse(x) \triangleq \neg S[x]$ 
     $True \triangleq \{i \in Statements : IsTrue(i)\}$ 
     $False \triangleq \{i \in Statements : IsFalse(i)\}$ 
     $IsEven(x) \triangleq x \% 2 = 0$ 
     $IsOdd(x) \triangleq x \% 2 \neq 0$ 
     $a \otimes b \triangleq \neg(a \equiv b)$  Exclusive or
  IN
     $\wedge \neg \forall i \in Statements : IsTrue(i)$ 
     $\wedge \neg \forall i \in Statements : IsFalse(i)$ 
     $\wedge S[1] = \text{LET } sum[s \in \text{SUBSET } Nat] \triangleq \text{IF } s = \{\} \text{ THEN } 0$ 
      ELSE LET  $x \triangleq \text{CHOOSE } x \in s : \text{TRUE}$ 
        IN  $x + sum[s \setminus \{x\}]$ 
      IN  $X = sum[False]$ 
     $\wedge S[2] = (X < \text{Cardinality}(False) \wedge IsTrue(10))$ 
     $\wedge S[3] = ((\text{Cardinality}(True) = 3) \otimes IsFalse(1))$ 
     $\wedge S[4] = (1 \dots 3 \subseteq False \vee IsTrue(9))$ 
     $\wedge S[5] = (IsOdd(X) \otimes IsTrue(7))$ 
     $\wedge S[6] = \text{LET } Odds \triangleq \{i \in Statements : IsOdd(i)\}$ 
      IN  $\text{Cardinality}(Odds \cap False) = 2$ 
     $\wedge S[7] = IsTrue(X)$ 
     $\wedge S[8] = \text{LET } Evens \triangleq \{i \in Statements : IsEven(i)\}$ 
      IN  $Evens \subseteq True \vee Evens \subseteq False$ 
     $\wedge S[9] = \text{LET } First(s) \triangleq \text{CHOOSE } x \in s : \forall y \in s : y \geq x$ 
      IN  $(X = 3 * First(True)) \vee IsFalse(4)$ 
     $\wedge S[10] = (IsEven(X) \vee IsTrue(6))$ 

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To get TLC to solve the puzzle for us, we have to put it in the form of a temporal specification and an invariant. A counterexample of the invariant would be the solution. To verify its uniqueness, we then let TLC check that the solution is an invariant (it is).

$Spec \triangleq Puzzle \wedge \Box[\text{UNCHANGED } \langle X, S \rangle]_{\langle X, S \rangle}$

$Invariant \triangleq X \notin 1 \dots 10$  Taunt TLC into proving us wrong

$\wedge S = \langle \text{FALSE}, \text{FALSE}, \text{TRUE}, \text{TRUE}, \text{FALSE}, \text{TRUE}, \text{TRUE}, \text{FALSE}, \text{TRUE}, \text{FALSE} \rangle$   
 $\wedge X = 9$