Calculus Notes

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Contents

1 Introduction

This document was written as an effort to devise a unified reference for my teaching. Though it is a reference, I intend for it to be self-contained. My hope is that anyone who comes accross it could read it without additional instruction, and maybe learn a thing or two about calculus along the way.

The nature of these notes also implies that they will be a perpetual work in progress.

1.1 The History of Calculus

Most sources suggest that calculus was developed independently by both Isaac Newton and Gottfried Wilhelm Leibniz in the 1600s. This is not wrong, but the full truth is also a little more complicated.

Between 350 and 50 B.C., ancient Bablyonian astronomers calculated the position of the planet Jupiter by finding the integral of a velocity function [8].

The ancient Greeks got awfully close to doing calculus too. Specifically, Archimedes wrote several works that look like what we call calculus today. Perhaps the most prominent of these is called *The Method of Mechanical Theorems*.

The *method of exhaustion*, a precursor to integral calculus, originated with Antiphon in the late 5th century B.C. It was also used by Archimedes, and was developed independently in China by Liu Hui [4].

1.2 Zeno's Paradoxes

Ancient greek philosopher Zeno of Elea (c.490-430 B.C.) was trying to figure out the nature of the universe. He thought that motion is nothing but an illusion.

These are valuable in a few ways. First, they demonstrate an early version of a proof methodology called *reductio ad absurdum*, or *proof by contradiction*. As we move further into our math career, we will learn how to write proofs, and hopefully, how to write good proofs. A natural part of this education will be reading other people's proofs, and finding the flaws in them if they are failed.

Another reason these paradoxes are valuable is because they speak to the heart of calculus. As we'll see later, the machinery of calculus will give us a way to resolve these paradoxes. So in this sense, Zeno's paradoxes give us a motivating example that helps us start thinking about *why we need calculus* at all.

1.2.1 Dichotomy Paradox

Suppose you are walking to a destination. Before you get there, you first must get halfway there. Before you get halfway there, you must get a quarter of the way there. Before you get a quarter of the way there, you must get an eighth of the way there. We can continue this subdivision process infinitely.

So, Zeno concludes, you must complete an infinite amount of tasks before you get to your destination. Zeno maintains that this is impossible, and therefore motion is an illusion.

1.2.2 Achilles and the Tortoise

Achilles, a great warrior and probably a fast runner, is racing a tortoise. Suppose Achilles is 10 times faster than the tortoise.

Because he's so confident, Achilles gives the tortoise a head-start of $100m$. By the time Achilles reaches the tortoise's starting point, it has traveled an additional $10m$. So it is now $10m$ ahead of Achilles.

By the time Achilles has travelled $110m$, the tortoise has again traveled an additional $1m$. It is now $1m$ ahead of Achilles, and $111m$ ahead of Achilles' starting point.

But by the time Achilles reaches *that* point, the turtle has traveled farther still. You can see by now that this will never end. Every time Achilles arrives at somewhere the tortoise has been, the tortoise is no longer there.

Zeno's conclusion is that Achilles can never pass the tortoise, since in order to catch him, he must first pass all the places the tortoise has been before.

1.2.3 Arrow Paradox

Consider an arrow in flight (towards a bullseye, perhaps). Now consider a moment in time, or a snapshot, if you will. The arrow is stationary at each given (durationless) instant of time. If the arrow is motionless at each instant, and time is entirely composed of such instants, then the arrow cannot be moving. At least, this is Zeno's conclusion.

2 Limits

Intuitively, the **limit** of a function is a value that the function approaches, regardless of whether the function ever truly reaches that value. In other words, it's a value which the function gets arbitrarily close to.

Definition 1 (Limit of a Function)**.** *Suppose f is a real-valued function. If the limit $b = \lim_{x\to a} f(x)$ exists, it is a real-valued number such that, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any x satisfying $0 < |x - a| < \delta$, it holds that $|f(x) - L| < \epsilon$.*

Note that this is the definition of the limit *of a function*. In math, there is a related but different *limit of a sequence*. Sequences are discussed much later, in Section 8.

2.1 Strategies for evaluating limits

The first thing we always try is direct substitution. This yields one of three results. Either it takes the form $f(a) = \frac{b}{0}$, in which case we have a likely

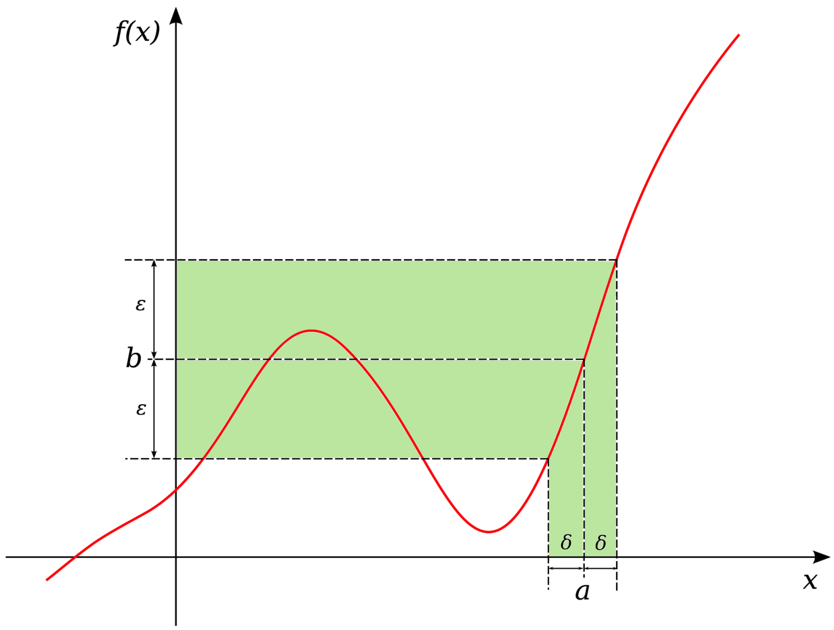

Figure 1: An illustration of the epsilon-delta definition of the limit (Definition 2) [10]

asymptote; it takes the form $f(a) = b$, in which case we have found the limit; or it takes the form $f(a) = \frac{0}{0}$, which we call indeterminate.

Suppose we've gotten an indeterminate form. This means we need to rewrite the limit somehow. We will see three strategies to deal with the indeterminate form: factoring, conjugates, and trig identities.

2.1.1 Factoring

Recall your most recent Algebra class. You probably worked on factoring expressions. Factoring is one of the ways we can simplify an expression, and by doing so, we might just be able to simplify it into a determinate form.

Example 2.1. *Find*

$$\lim_{x \to -4} f(x)$$

where f is defined as:

$$f(x) = \frac{3x + 12}{x^2 + x - 12} \tag{1}$$

Answer: *To solve this example, we first factor a 3 out of the numerator, and then factor the polynomial in the denominator. This gives us the following:*

$$f(x) = \frac{3(x+4)}{(x+4)(x-3)}$$

And by cancelling out the $x+4$, we simplify the function to:

$$f(x) = \frac{3}{x-3}$$

At this point, we can just find the limit by substitution: $f(-4) = -\frac{3}{7}$

2.1.2 Conjugates

Sometimes our indeterminate expression will include radicals (square roots). In this case, it's usually in our best interest to get rid of them if possible.

2.1.3 Trigonometric Identities

I hope you remember your trigonometry! For this kind of manipulation, we need to make use of the well-studied trigonometric functions, and the relationships between these functions. Since these aren't trigonometry notes, I wont spend much time going over these identities. If you need to review them, see an appropriate external source.

One particularly useful identity is the Pythagorean Identity:

Definition 2 (Pythagorean Identity)**.**

$$sin^2(x) + cos^2(x) = 1 \tag{2}$$

Let's try using this identity to solve the following example problem:

Example 2.2. *Find*

$$\lim_{x \to \frac{\pi}{2}} f(x)$$

where

$$f(x) = \frac{cot^2(x)}{1 - sin(x)}$$

Answer: *First note that*

$$cot(x) = \frac{cos(x)}{sin(x)}$$

By plugging this in, we obtain:

$$f(x) = \frac{cos^2(x)}{sin^2(x)[1 - sin(x)]}$$

Now we can use the pythagorean identity to substitute $cos^2(x)$ for $1 - sin^2(x)$...

$$\frac{1 - sin^2(x)}{sin^2(x)[1 - sin(x)]}$$

The final tricky step of solving this one is to factor the numerator as a difference of squares, which is to say that it takes the form $a^2 - b^2 = (a+b)(a-b)$*:*

$$\frac{[1 - sin(x)][1 + sin(x)]}{sin^2(x)[1 - sin(x)]}$$

Make the last cancellation and we obtain:

$$\frac{1 + sin(x)}{sin^2(x)}$$

Now we can find the limit by direct substitution:

$$\frac{1 + sin(\frac{\pi}{4})}{sin^2(\frac{\pi}{4})} = 2$$

Before moving on, we're going to talk about one more technique that is not included in Figure **??**.

Theorem 1 (Squeeze Theorem)**.** *Suppose* f*,* g*,* h *are continuous functions. Suppose that the interval* I *contains* a*, and that for all* x *in* I*,* $x \neq a$*, the following is true:*

$$g(x) \leq f(x) \leq h(x) \tag{3}$$

Also suppose that

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L \tag{4}$$

Then

$$\lim_{x \to a} f(x) = L \tag{5}$$

We can use this theorem to find $\lim_{x \to a} f(x)$, as long as we can find the appropriate functions g and h.

3 Continuity

In your math education so far, you've mostly seen only continuous functions. Informally, we can think of these as functions which "can be drawn without lifting the pencil."

Definition 3 (Continuity)**.** *A function is* continuous *at* $x = a$ *if and only if all of the following conditions hold:*

1. *The function is defined at* a*, i.e.,* $f(a)$ *equals a real number*
2. *The limit* $\lim_{x \to a} f(x)$ *exists*
3. $\lim_{x \to a} f(x) = f(a)$

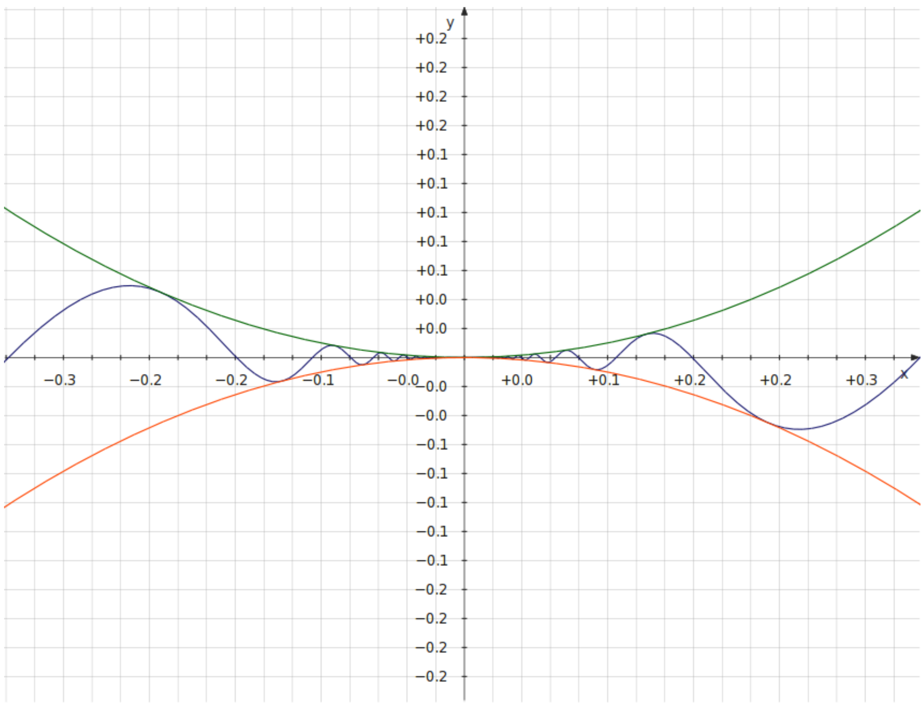

Figure 2: An illustration of the squeeze theorem [7].

Why do we care about continuity? That's a little hard to explain, but the truth is that continuity is one of the most important concepts in all of math. One reason that the "pure mathematicians" would give is because we can prove a lot of things about a function if we know it is continuous. A more applied math perspective might suggest that we care about continuity because most real world functions are continuous. Position, velocity, and acceleration, the functions we discussed in the Introduction, all must be continuous by nature. If the position of an object with respect to time was *discontinuous*, then that object would literally be teleporting.

3.1 Discontinuity

If continuous functions are functions which can be drawn without lifting the pencil, discontinuities occur when you are forced to pick up the pencil. They can be classified into three categories: removable discontinuities, jump discontinuities, and essential discontinuities.

We'll visit each of these one by one in order to get the full picture of what discontinuity can look like.

3.1.1 Removable Discontinuities

Consider the following function:

$$f(x) = \begin{cases} x^2, & \text{for } x < 1 \\ 0, & \text{for } x = 1 \\ 2 - x, & \text{for } x > 1 \end{cases} \tag{6}$$

This function has a *removable discontinuity* at $x = 1$. This basically means that it is continuous everywhere except for at $x = 1$.

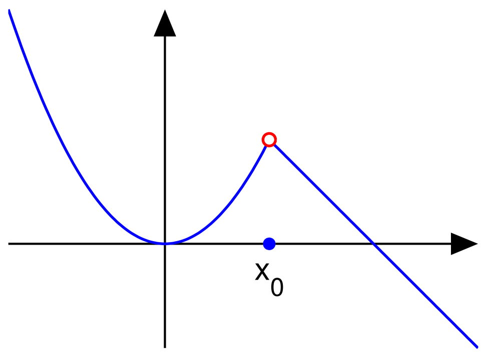

Figure 3: A removable discontinuity [2].

3.1.2 Jump Discontinuities

A jump discontinuity occurs when both one-sided limits exist, but the left limit does not match the right limit. Formally, f has a jump discontinuity at x_0 if:

$$\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x) \tag{7}$$

3.1.3 Essential Discontinuities

An essential discontinuity occurs when either of the one-sided derivatives don't exist.

3.2 Continuity Results

What is continuity good for? That's the question we'll begin answering in this section. After all, it probably seems like a pointless property so far. But trust me, continuity is *really* important.

Theorem 2 (Intermediate Value Theorem)**.** *Suppose f is a real valued function, and that the interval $[a, b]$ is in the domain of f. Let u be some value in the interval $[f(a), f(b)]$. Then there is some $c \in [a, b]$ such that $f(c) = u$.*

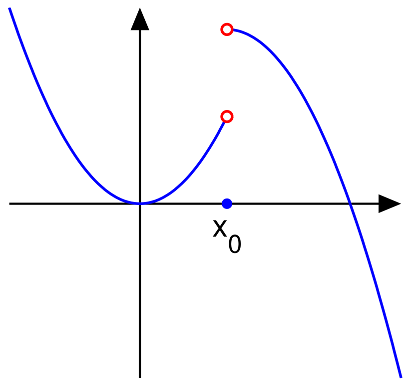

Figure 4: A jump discontinuity [6].

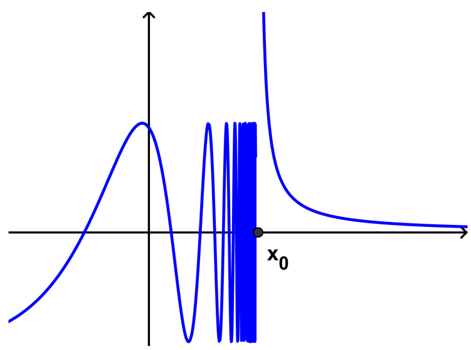

Figure 5: An essential discontinuity occurs at x_0 because the left-sided limit does not exist, and the right-sided limit is infinite [3].

Theorem 3 (Mean Value Theorem)**.** *Suppose f is a continuous function on the interval $[a, b]$, and differentiable on the interval (a, b). Then there exists a point $c \in (a, b)$ such that the tangent at c is parallel to the secant line between the points $(a, f(a))$ and $(b, f(b))$.*

4 Differentiation

4.1 Rate of Change

Definition 4 (Average Rate of Change Formula)**.** *The average rate of change of the function $f(x)$ between $x = a$ and $x = b$ is given by:*

$$\frac{f(b) - f(a)}{b - a} \tag{8}$$

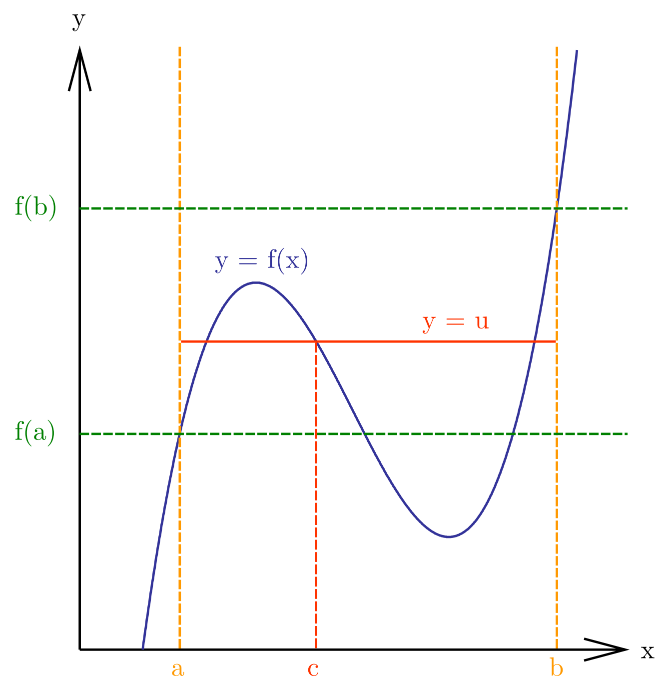

Figure 6: The intermediate value theorem [5].

4.2 The Derivative

Intuitively, we can think of the derivative of a function as its *slope*. You should be familiar with the notion of slope when it comes to linear functions of the form $y = mx + b$. For non-linear functions, the slope changes depending on where you are.

Notation There are a couple of different notations in use for derivatives, so it's important to take a moment and discuss them. The *derivative* of a function $f(x)$ is written in one of two ways: $f'(x)$ or $\frac{d}{dx}f(x)$. Sometimes, we'll leave the function argument out of this notation and just write f', or $\frac{df}{dx}$. It all means essentially the same thing.[1]

Definition 5 (Derivative)**.** *Let f be a real-valued function. The derivative of f, denoted f' is given by:*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{9}$$

The above equation may look daunting to you, so it's worth taking a moment to try and understand it. Ignoring the limit for a moment, we should interpret

[1] f' is called Lagrange notation, while $\frac{df}{dx}$ is called Leibniz notation.

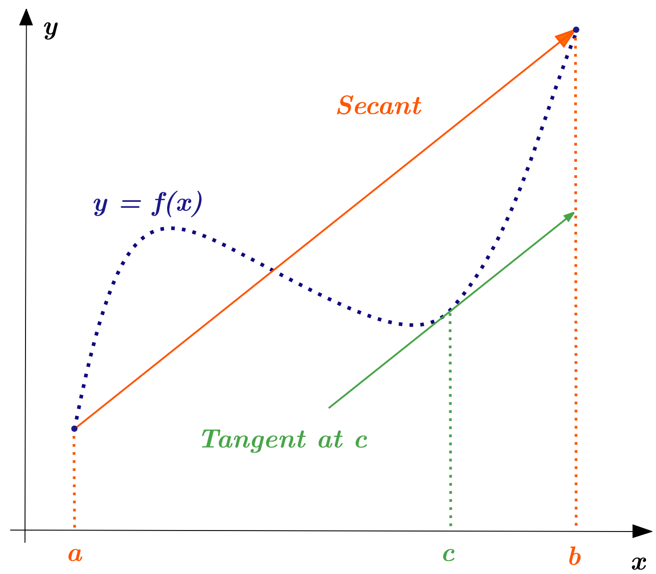

Figure 7: The mean value theorem [1].

the right-hand side (RHS) expression as a "rate of change" function, like we saw in Definition 4. Specifically, this is the average rate of change between x and $x + h$.

By taking the limit of this average rate of change as $h \to 0$, we get the *instantaneous* rate of change at x.

4.3 Differentiability

Believe it or not, not all functions are differentiable. For example, consider the absolute value function $f(x) = |x|$, depicted in Figure 8. What is the slope of this function at $x = 0$?

Of course, it doesn't really make sense to talk about the slope of this function at $x = 0$. The slope is different depending on which direction you move! For this reason, we would say that the function is *not differentiable* at $x = 0$.

Any time you see a "corner" in the graph like this, it is probably non-differentiable at that point. But of course, this is a somewhat hand-wavey definition. We need to formalize it a little more. Lucky for us, we already have all the tools we need in order to prove whether a function is differentiable or not!

Recall Definition 4.2. For a function to be differentiable, the limit in this definition just has to exist. Take our example from earlier, can you show why the limit below doesn't exist? [2]

[2] Hint: look at the left and right limits

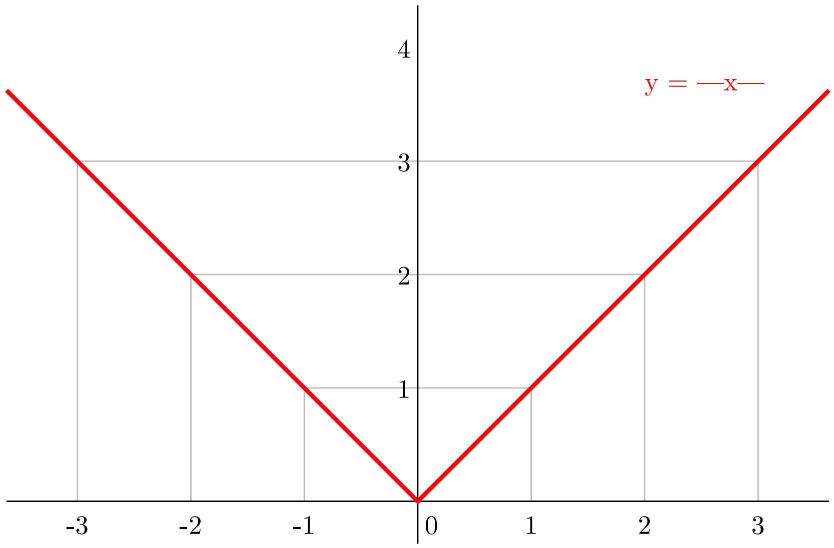

Figure 8: $y = |x|$ is not differentiable at $x = 0$ [9].

$$\lim_{h \to 0} \frac{|x+h| - |x|}{h} \tag{10}$$

4.4 Calculating the Derivative

One of the most important properties of the derivative is that it is a *linear operator*. If that term sounds complicated or confusing, you don't need to remember it, you just need to remember what it means. And it means two things: First, you can "distribute" the operation like this:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

It also means that you can "pull out" scaling factors:

$$\frac{d}{dx}[a \times f(x)] = a \times \frac{d}{dx} f(x)$$

4.4.1 Differentiating Polynomials

Polynomial functions are, in a sense, the easiest to differentiate. I'll teach you just one rule, and then we'll have everything we need to start differentiating polynomial functions! Here's the rule:

$$\frac{d}{dx} x^k = kx^{k-1}$$

Pretty straightforward, right? So, for example, this means that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. I suggest taking a moment to look at these two graphs, and recalling how we described the derivative as the slope of the original function. For $x < 0$, the $f'(x)$ is negative, and for $x > 0$ it is positive. The magnitude of $f'(x)$ also grows as we move away from $x = 0$, which is exactly what we would expect by examining the graph $f(x)$.

4.4.2 Chain Rule

Theorem 4 (Chain Rule)**.** *Suppose we have two differentiable functions f and g. Then the derivative of their composition, $h(x) = (f \circ g)(x)$ is:*

$$h'(x) = f'(g(x))g'(x) \tag{11}$$

4.4.3 Product Rule

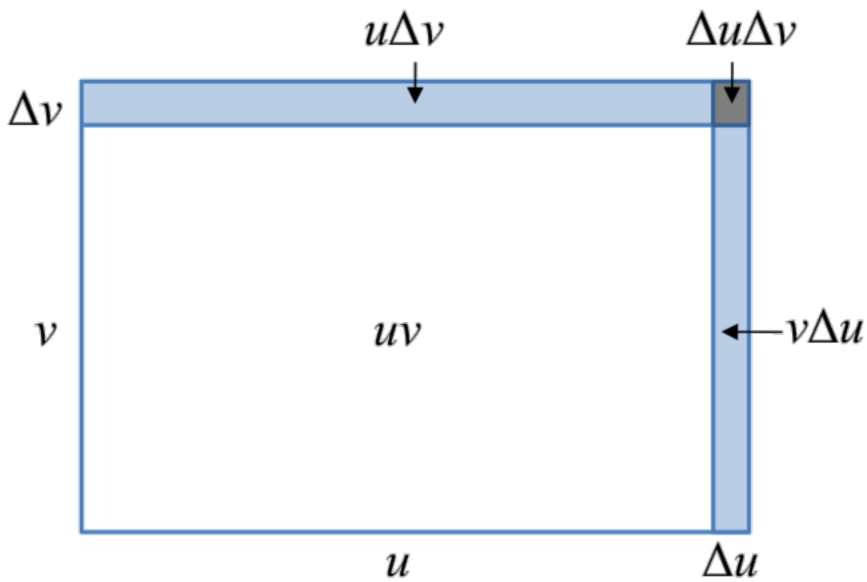

Figure 9: A geometric depiction of the product rule [11].

Theorem 5 (Product Rule)**.** *Suppose we have two differentiable functions f and g. Then the derivative of their product, $h(x) = f(x)g(x)$ is:*

$$h'(x) = f(x)g'(x) + f'(x)g(x) \tag{12}$$

A useful mnemonic to remember this rule is ”one-dee-two plus two-dee-one”. I don’t know why, but this phrase has been stuck in my head ever since I took AP Calculus, so it might work for you too.

4.4.4 Quotient Rule

In the last section, section 4.4.3, we saw what happens to the derivative of h when h is a product of two differentiable functions. But what if h is a quotient of two functions?

Theorem 6 (Quotient Rule)**.** *Suppose we have two differentiable functions f and g. Then the derivative of their product, $h(x) = f(x)/g(x)$ is:*

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \tag{13}$$

Or, a bit less formally, "low dee-high minus high dee-low all over the square of what's below."

4.5 Differentiation Trigonometric Functions

I recommend memorizing at least the first few lines of the following table:

Function	Derivative
$sin(x)$	$cos(x)$
$cos(x)$	$-sin(x)$
$tan(x)$	$sec^2(x)$
$cot(x)$	$-csc^2(x)$
$sec(x)$	$sec(x)tan(x)$
$csc(x)$	$-csc(x)cot(x)$

5 Limits Revisited

Now that we know a little more, it's time we return to limits.

Remember Figure **??**? Suppose we got the indeterminate form (D), and we couldn't find an equivalent form using E, F, or G. Don't worry, these aren't the only options! There's another way we can rewrite the limit in an equivalent form: it's called L'Hopital's Rule!

Theorem 7 (L'Hopital's Rule)**.** *If , then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{14}$$

This is extremely useful when $\frac{f(a)}{g(a)}$ is indeterminate, because $\frac{f'(a)}{g'(a)}$ may not be!

6 Integration

When you learned addition, you learned subtraction soon afterwards. When you learned multiplication, you learned division soon afterwards. These operations are opposites in the sense that they "undo eachother." Well, now you've learned a new operation called the derivative, so it's time to learn the operation that "undoes" the derivative.

6.1 The Antiderivative

Definition 6 (Anti-derivative)**.** *The antiderivative of a function f is a differentiable function F such that $f = F'$.*

In other words, every function is the derivative of its antiderivative. For example, $F(x) = x^2$ is the antiderivative of $f(x) = 2x$.

But wait a minute, how do we know that the antiderivative of $f(x) = 2x$ is x^2 and not x^2+10? After all, differentiating x^2+10 will get us back to $f(x)$! It's true, and in fact, they're both valid antiderivatives. In fact, every function has infinitely many antiderivatives, since we can add any constant to $F(x)$ without changing its derivative. So normally, we'd write the antiderivative of $f(x) = 2x$ as $F(x) = x^2 + C$.

This makes sense graphically as well. We can shift the entire graph up and down without changing the slope in any way.

Example 6.1. *Suppose you travel at a velocity of $v(t) = t(9-t)$. What is your position $s(t)$ between time $t = 0$ and $t = 9$?*

6.2 The Integral as a Concept

Suppose you're in a car, but you can't see out the windows. You can only see the speedometer. Using calculus, you can determine how far you've traveled at any time[3].

That might not make sense yet, so let's consider a simple example:

Example 6.2 (Constant speed area under curve)**.** *Suppose you travel at a constant $v(t) = 10m/s$. How far have you traveled after 5 seconds?*

If you answered 50 meters to the problem above, you're right. You probably multiplied 5 by 10, realizing this is a simple unit conversion problem.

Try to visualize this problem by graphing $v(t)$. Because it's a constant function, the graph is a straight horizontal line. Solving this problem was analogous to finding the area of a rectangle which was 10 units tall and 5 units long. In other words, solving this problem was the same as finding the area under $v(t)$, between our two limits.

Let's try a slightly more difficult problem.

Example 6.3 (Stepwise area under curve)**.** *Suppose your velocity is given by:*

$$v(t) = \begin{cases} 1 & t < 2 \\ 3 & t >= 2 \end{cases} \tag{15}$$

How far have you traveled between $t = 0$ and $t = 5$?

[3]Assuming you have access to an accurate and precise clock as well

This problem is really similar to the last one, except for at $t = 2$. We can solve it the same way if we just break it down into two sub-problems: first, how far did you travel in the first two seconds, then, how far did you travel between $t = 2$ and $t = 5$? Clearly, we should be able to add these two distances together and get the final result.

Let's interpret this problem graphically too. We had to find the area under a complicated curve, so we broke it up into two simple rectangles, whose area we know.

6.2.1 Negative Area

Suppose your car goes in reverse at some point. This would mean that the velocity function is negative during that time. In these regions, the area between the curve and the x-axis is subtracted from our total.

It's natural that we would want to subtract this area if our integral represents position or distance.

6.3 The Fundamental Theorem of Calculus

Suppose you have a graph which represents the velocity of your car, v. Then $s(t)$ represents how far you've traveled by time t. As we saw earlier $s(t)$ can be interpreted as the area under the graph v between 0 and t, and it can be written as[4]:

$$s(T) = \int_0^T v(t)dt \tag{16}$$

Now I have a simple question: what is the derivative of $s(t)$? Well, since s represents position, we would expect its derivative to be velocity. And that's exactly right, the derivative of this function gets us back to the function v!

Theorem 8 (FTC Part 1)**.** *Suppose functions F and f are such that*

$$F(x) = \int_a^x f(t)dt \tag{17}$$

Then $F'(x) = f(x)$ over $[a, b]$

We can see this graphically as well. Consider a little nudge to the input t, and how that changes the expression in equation 6.3. A nudge of dt adds additional area under the curve. This area is approximated by a rectangle which has a width of dt and a height of $v(t)$. We can write the amount of newly added area as $ds = v(t)dt$. Rearranging terms in this equation gives us $v(t) = \frac{ds}{dt}$.

Theorem 9 (FTC Part 2)**.**

$$\int_a^b f(x)dx = F(b) - F(a) \tag{18}$$

[4]We call x a "dummy variable" in situations like these. It still represents time, but can be ignored.

This crucial theorem relates the antiderivative, from section 6.1, to the integral.

Remember back when we defined the antiderivative, and we said that we can add any constant to it? Notice that, in the FTC, these constants would cancel out entirely.

Definition 7 (The Indefinite Integral)**.** *The indefinite integral of a function $f(x)$ is the set of all antiderivatives of that function.*

Definition 8 (The Definite Integral)**.** *Let $[a, b]$ be an interval in the domain of f. Subdivide this interval into n intervals of equal width Δx, and from each interval i, choose a point x_i^*. Then the definite integral of f on the interval $[a, b]$ is defined as:*

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=0}^{n} f(x_i^*)\Delta x \tag{19}$$

As a point of interest, we may note that this is not the *only* way to define the integral. This is called the Riemann integral. There is an equivalent definition called the Darboux integral, but for now, that is beyond the scope of these notes.

6.4 Computing the Integral

Computing an indefinite integral is easy. It's just the anti-derivative added to an arbitrary constant.

To compute a definite integral we need to make use of Equation 18.

6.4.1 U-Substitution

In this section we'll discuss one of the most useful techniques for solving integrals. This technique is often referred to as u-substitution. Essentially, this technique is way to reverse the chain rule.

Suppose you were asked to solve the following integral:

$$\int 2xcos(x^2)dx$$

This is quite a bit more complicated than the integrals we're used to. When you see a complicated integral like this, your first step should always be to try to *simplify the integrand*. The good news is that we can use u-substitution to do exactly this. Let's see how.

First, let $u = x^2$. This will simplify the integrand in a very helpful way: our cosine function will take a single-variable argument instead of an exponent.

Next, we need to relate an infinitesimal change in u to an infinitesimal change in x. Don't be scared, this is essentially just taking the integral of x^2. This step yields

$$\frac{du}{dx} = 2x$$

Or, equivalently:

$$dx = \frac{du}{2x}$$

.

Now we can return to our original integral, substituting $u = x^2$ and $dx = \frac{du}{2x}$. As a result, we obtain a surprisingly simple integral which we can solve immediately:

$$\int 2xcos(x^2)dx = \int cos(u)du = sin(u) + C$$

All that's left to do is write our solution in terms of x. We know $u = x^2$, so we plug that back in to obtain our final solution:

$$sin(x^2) + C$$

6.4.2 Integration By Parts

Integration by parts is a way to find the integral of a product of functions. In this sense, it is kind of the reverse of the product rule (which was covered in Section 4.4.3).

The formula to remember is below:

$$\int udv = uv - \int vdu \tag{20}$$

Equivalently, for definite integrals:

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu \tag{21}$$

6.5 Applications of the Integral

6.5.1 Average Value of a Function

The average value of f on the closed interval $[a, b]$ is given by

$$\bar{f}_{a,b} = \frac{1}{b-a}\int_a^b f(x)dx \tag{22}$$

Example 6.4. *What is the average value of e^x on the interval $[3, 8]$?*

6.6 Multiple Integrals

So far, we've seen integrals of the form $\int_a^b f(x)dx$. We saw that we can interpret this the area under the *curve* $f(x)$ on the interval $[a, b]$. In this section, we'll add a dimension. Let $f(x, y)$ be a two-variable function, which takes inputs x and y. In this situation, f gives a *surface* over the two-dimensional x, y coordinate plane. If we integrate this function[5], we'll get a *volume under the surface* instead of an *area under the curve*.

We could integrate this function with respect to a single variable. For example, $g(y) = \int_a^b f(x,y)dx$ is a function of y which gives the area under $f(x, y_0)$ for each y_0.

We could also take the integral of $g(y)$ with respect to y. We'll write this integral as $I = \int_c^d g(y)dy$. So far, this is all stuff we should be comfortable with.

But in reality, I've just introduced you to something new. Remember how $g(y)$ itself was defined as an integral? We just took the integral of an integral!

$$I = \int_c^d g(y)dy = I = \int_c^d \big(\int_a^b f(x,y)dx\big)dy = \int_c^d \int_a^b f(x,y)dxdy$$

This is our first introduction to the *double integral*. We have specified a rectangular region in the domain of f, and written an expression for the *volume*

7 Differential Equations

In Algebra, we found solutions to equations. Problems like these often began with "Solve for x", where x is the *variable* we are solving for. Finding a solution means finding a number/value which x can take on while preserving the equality.

In *differential equations*, our solutions are not variables, but instead they are *functions*.

Differential equations often relate a function to one or more of its derivatives.

Unfortunately, differential equations are often really hard to solve. For the most basic ones, we resort to a method that is aptly named "guess and check". Let's see how this works in practice.

Example 7.1.

$$f'(x) = f(x) \tag{23}$$

Solving this equation amounts to finding a function that is equal to its derivative. There is a very specific function for which this is true: e^x. *So our solution to this differential equation is* $f(x) = e^x$

Example 7.2.

$$f''(x) = f'(x) \tag{24}$$

Given the last example (Example 7.1), we might think this solution is also $f(x) = e^x$. *And you'd be on the right track, that is a solution.*

But there are actually more solutions to this differential equation. Like with many other differential equations, there are infinitely many solutions! The key insight here is to notice that we can add any constant to our function without changing either its first or second derivative.

For this reason, our solutions take the form $f(x) = e^x + C$.

[5]With respect to both variables

The above example showed us that there may be infinitely many solutions to a differential equation, in much the same way that there were infinitely many solutions to our indefinite integrals in Section 6. For this reason, differential equations often come with an *initial value.*

8 Infinite Sequences and Series

Definition 9 (Sequence)**.** *A sequence is "an enumerated collection of objects in which repetitions are allowed and order matters."*

For us, the "objects" in question are all going to be numbers.

Definition 10 (Limit of a Sequence)**.** *A sequence of real numbers (a_k) converges to a real number L if, for all $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$,*

$$|a_n - L| < \epsilon$$

It may be worth examining the differences between this definition and the definition of a *function's limit* in Definition 2.

By way of example, the limit of $a_n = 1/n$ is zero.

8.1 Power Series

A power series is an infinite series of the form

$$\sum_{i=1}^{\infty} a_n (x-c)^n \tag{25}$$

Any polynomial can be represented by a power series.

8.2 Taylor Series

Suppose f is a real-valued function that is infinitely differentiable. If you need a refresher on differentiability, you can visit Section 4.3.

The Taylor Series expansion of this function can be written as:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \tag{26}$$

Let's see some examples.

The Taylor series of a polynomial is quite simply the polynomial itself.

Example 8.1. *The Taylor Series of e^x is given by:*

$$f(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... \tag{27}$$

Or, equivalently, since $f^{(n)}(0) = 1$ always when $f(x) = e^x$, we could say the following

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{28}$$

9 Euler's Method

Euler's method is a technique for solving the differential equations we saw in Section 7.

Aside: Numerical vs. Analytic approaches Euler's method is called a *numerical method*, as opposed to an *analytic method*. We are used to analytic methods already. These are techniques which involve symbolic manipulation, like the product rule, chain rule, etc.

Now to discuss the method itself:

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