Causality in Classical Discrete Symplectic Mechanics: Separability of Spekkens' Symplectomorphisms

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Abstract

In the study of the foundation of physics, some hidden variable models can be used to demystify quantum physics. In fact, some phenomena are not intrinsically linked to contextuality. We can reproduce them through, e.g., epistemic restrictions of classical mechanics. Such a theory is laid out in Spekkens' toy theory [7], and further formalised in [2]. With this theory, it can be shown that spooky phenomena such as quantum superdense coding or quantum teleportation are not so essentially quantum after all: they can be reproduced without contextuality.

Seemingly essentially quantum phenomena are further laid out in [1], where causality is studied within the quantum framework. Some results are shown that could be intuited to be inherently quantum. I set out to study one of these quantum results, and show that it has its equivalent within Spekkens' toy theory.

Keywords: hidden variable models, quantum physics, Spekkens' toy theory, quantum phenomena, symplectomorphisms, quantum causality.

*This work was done while I was affiliated to EFPL and the University of Oxford, under the supervision of Jonathan Barrett. Right now, I am affiliated to ETH Zürich.

Contents

1 Introduction

Quantum physics is weird. Among other things, two quantum particles can sometimes have no state of their own but only have a shared relation (entanglement), the value of a quantum bit cannot be copied (no-cloning theorem), and a single quantum bit can be used to send two classical bits of information (superdense coding). These phenomena tend to be associated to the weird postulates and features of quantum physics, like the possibility for two particles to influence each other instantaneously at an arbitrary distance (quantum nonlocality). One big property, revealed by the so-called Bell inequalities, is contextuality: measurements of quantum variables cannot be thought of as revealing pre-existing values.

What if they were revealing such so-called ontic (from the Greek óntos - reality) values? What if some of these weird phenomena could be reproducible in classical physics, just by posing a restriction on the state of knowledge (epistemic, from the Greek epistêmê - knowledge) one can have on a certain system? Spekkens showed in [7] that many of them can. The research program of this paper shows that quantum is less weird than it seems. Although unintuitive, phenomena like superdense coding, entanglement or quantum teleportation can be reproduced within Spekkens' framework. This program is not, however, an attempt at describing quantum physics completely via a restricted classical model. In fact, some phenomena are intrinsically linked to, e.g., contextuality. For example, the Bell inequalities would never be violated in the toy theory. Altogether, though, many phenomena are reproduced.

One field of study of physical phenomena is classical causality. As introduced by Pearl, [5], this field studies when we can say that some phenomenon happening to some object causes some other phenomenon happening to some other object. The existence of such a field in classical physics begs the question of defining and proving results about *quantum* causality. This was done in [1], and more recently in [4]. The toy theory equivalent definitions and results about quantum systems have, however, not been studied. This is what I set out to do.

2 Qudits

It is interesting to introduce quantum dits, as they are the analogue to the toy dits we are studying. They are also the only thing to tie this project, done as part of a computer science degree, to computer science.

2.1 States

Definition 1 (qudit)**.** *Let d be a natural number. Then a q-dit is an element of the Hilbert space $\mathcal{H}_d$, that is of the span of $\{|i\rangle\}_{i=0}^{d-1}$ on the scalar field $\mathbb{C}$.*

Note that this is similar to the classical notion of a digit, that we can define as the span of $\{d^i\}_{i=0}^{\infty}$ on the scalar group/field $\mathbb{Z}_d$. For $d = 2$, digits are called bits, and qdits are called qbits. There are special names for some of the bases that span $\mathcal{H}_2$. A pair of qdits $|\psi\rangle$ and $|\phi\rangle$ is noted using the tensor product, $|\psi\rangle \otimes |\phi\rangle$.[1]

The key takeaway of this definition is that quantum systems can be in some linear superposition of different states. Experimental violation of the so-called Bell inequalities proved that this superposition is more than just epistemic, i.e., the quantum cat in the state $\frac{1}{\sqrt{2}}(|dead\rangle + |alive\rangle)$ isn't just dead or alive, with no knowledge of which; it is as much dead as it is alive. What if we imagined a

[1]This is actually more than just notation, it is linked to the postulates of quantum mechanics. This notation enables entanglement, discussed later.

game where the rules make it so that systems can be in superpositions of states, but only epistemic? This is what is done in [7].

2.2 Allowed Transformations

There are three types of transformations that are allowed over quantum dits. One of those is of particular interest to us: unitary transformations.

In quantum computing, it is allowed to apply unitary transformations to qdits. These transformations correspond to unitary matrices. As the name suggests, they correspond to a change of basis. These transformations conserve the structure of $\mathcal{H}_d$. Their equivalent in the toy theory are called symplectomorphisms, which we will define later.

The other allowed transformations of note are measurements and so-called discarding (taking a partial trace), but these are not of interest here.

2.3 Entanglement: a Weird Quantum Phenomenon

Entanglement is a property about groups of qdits. We know that a qudit lives in $\mathcal{H}_d$. But what set does a pair of qdits belong to? It turns out that the answer is $\mathcal{H}_d \otimes \mathcal{H}_d$. This is the span of $\{|i\rangle \otimes |j\rangle\}_{i,j \in \mathbb{Z}_d}$. However, an astute observer would see that this allows for shared states of two qdits that are not of the form $|\psi\rangle \otimes |\phi\rangle$. For example,

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right).$$

This seems really spooky. How can two systems be in a joint state that is not expressible as a product of two smaller states? How can two systems have no state of their own?

Most people would, and do, associate this phenomenon with the weirdness of noncontextuality. However, entanglement is not that linked to the latter, as the toy theory helps show. It does this by presenting something very similar to entanglement: the particles are also inseparable, and a measurmenent of one can instantly affect the state of the other.

3 Spekkens' Toy Theory

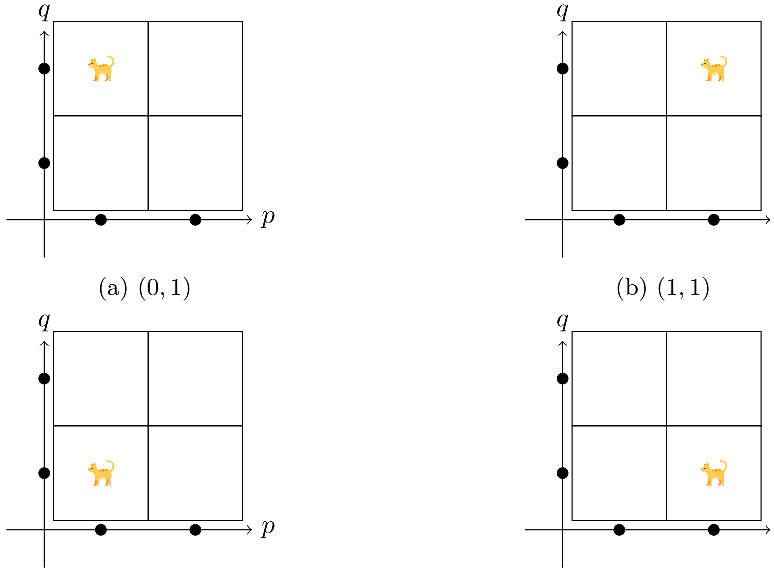
Here's a quick crash course in Spekkens' toy theory, for anyone not familiar with it. The most important thing to grasp is the difference between ontic and epistemic states. For ease of reading, we introduce Rob's cat: Rob's cat is the toy-theory equivalent of Schrödinger's cat. It can only be in one state at a time, but you can never know which. 🐈

3.1 Ontic States

Ontic states are the actual real states of objects in the toy theory. Let d be a prime number. Toy systems have a state space $\Omega = \mathbb{Z}_d$. Their phase space, $\{(q,p) | q \in \Omega \wedge p \in \Omega\}$, is Ω^2. An n-composite system has phase space $\{(q_1, p_1, q_2, p_2, \cdots, q_n, p_n\ ) | q_i \in \Omega \wedge p_i \in \Omega\ \forall 1 \leq i, j \leq n\} = \Omega^{2n}$. Let's take an example, to make it clearer.

Example 1. *For $d = 2$, ontic states can be seen as $\mathbb{Z}_2^2$, the set of four states seen in figure 1.*

In general, the phase space elements of an n-composite of elementary systems is $\mathbb{Z}_d^{2n}$.



(a) $(0, 1)$ (b) $(1, 1)$

(c) $(0, 1)$ (d) $(1, 0)$

Figure 1: The possible ontic phase space elements of a single system are $\mathbb{Z}_d^2$.

3.2 Toy Dits, Also Known as Epistemic States

We pose an artificial restriction on what an agent can know about a state.

Knowledge Balance Principle. An agent can never know more than half the information available about the ontic state of a system.

Example 2. *For $d = 2$, there are 5 possible epistemic states, as seen in figure 2.*

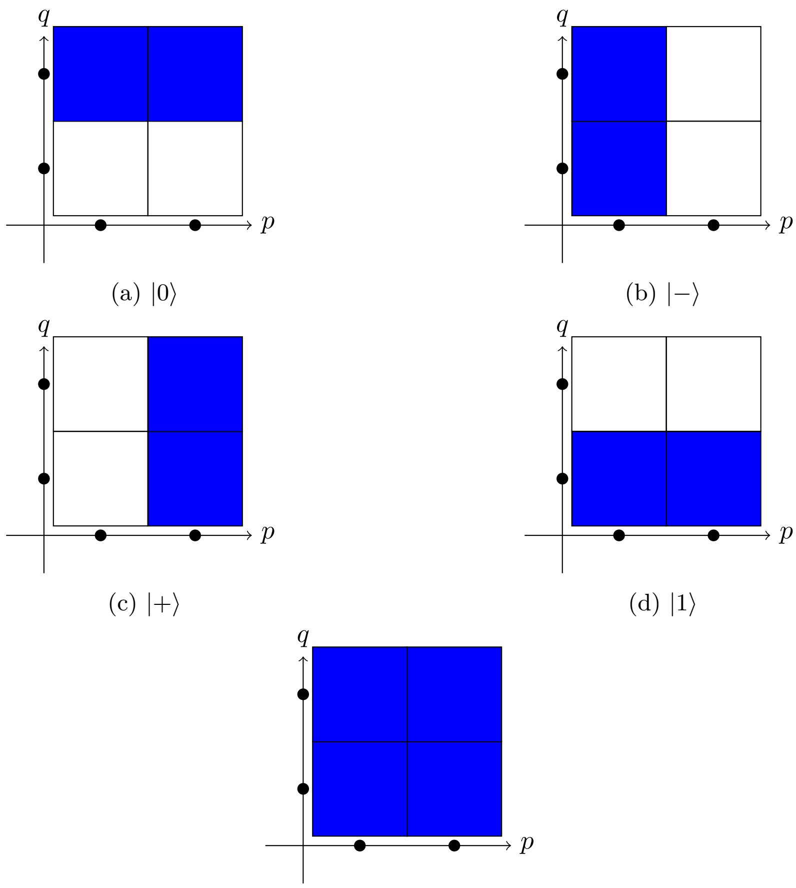
The blue represents where the cat could be. With this representation, the cat is never in two states at the same time, but it is never known exactly where the cat is.

3.3 Symplectomorphisms

It is allowed to apply gates to the underlying ontic state (where the cat is, in figure 1) that change the epistemic state (what Rob can know about his cat's position), as long as these transformations preserve the knowledge balance principle. For example, it would not be okay to apply a function that maps every ontic state to the $(0, 0)$ ontic state: after applying this function, an agent could know the state of the system with certainty! It has been argued by [2] that the way to do this is to only allow those functions that preserve a specific kind of structure, called *symplectic*. The allowed symplectomorphisms have been somewhat formalised in [2], but not quite as clearly as here. We will need the following definition to prove results about it.

Definition 2 (Symplectomorphism)**.** *Let $s : \Omega^{2n} \to \Omega^{2n}$, for $n > 0$. Then s is a symplectomorphism when:*

1. $s(x) = Sx^T + a^T, a \in \Omega^{2n}, S$ *a* $2n \times 2n$ *matrix.*



(a) $|0\rangle$ (b) $|-\rangle$ (c) $|+\rangle$ (d) $|1\rangle$ (e) $\frac{I}{2}$

Figure 2: The possible epistemic states of a single system.

2. S is non-singular, and

3. $S^T J_n S = J_n$,

Where $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, *and*

$$J_n = I_n \otimes J_1 = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \cdots \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \\ \vdots & & \ddots \end{pmatrix}.$$

We can write x_A and x_B for the inputs of A and B respectively, and y_C and y_D analogously for the output.

Remark 1. *The definition of J_n is very similar to the one found in typical articles, which is* $\begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} = J_1 \otimes I_n$.

Example 3 (Symplectomorphism)**.** *Let $d = 2, n_A = n_B = n_C = n_D = 1$. As seen in [2], the allowed rotations (S in $Sx^T + a^T$) are*

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The possible displacements (a in $Sx^T + a^T$) are

$$\begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

These combine in a total of $6 \times 4 = 24$ *possible symplectomorphisms for $d = 2, n_A = n_B = n_C = n_D = 1$.*

4 A Causal Study of Symplectomorphisms

4.1 Context

First of all, let us define some notation tools.

Definition 3 $(\cdot|\cdot)$**.** *For two vectors v, w, we define $v|w$ as the vector containing first all the elements of v, and then all the elements of w.*

We consider the case of figure 3, where $\mathrm{card}(A) = n_A, \mathrm{card}(B) = n_B, \mathrm{card}(C) = n_C, \mathrm{card}(D) = n_D$. In this case, d can be any arbitrary prime. We choose d prime so that $\mathbb{Z}_d^*$ can be a group. This case is the equivalent of a quantum unitary transformation from two qdits A and B to two qdits C and D. Thanks to this definition, we can define a notion of causality.

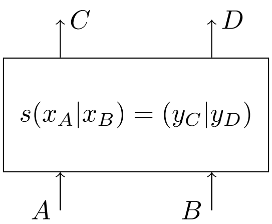


Figure 3: a symplectomorphism in its natural habitat

Definition 4 $(A \not\rightsquigarrow D)$**.** *We say that A **does not cause** D when $\forall x_A, x'_A \in \Omega^{2n_A}, x_B \in \Omega^{2n_B}$,*

$$y_D(x_A|x_B) = y_D(x'_A|x_B).$$

4.2 Motivation

In this section, it is shown that $A \not\rightsquigarrow D \wedge B \not\rightsquigarrow C \implies s = s_{AC} \oplus s_{BD}$, with preservation of symplectomorphism structure. Graphically, this means that $A \not\rightsquigarrow D \wedge B \not\rightsquigarrow C$ implies what figure 4 describes.

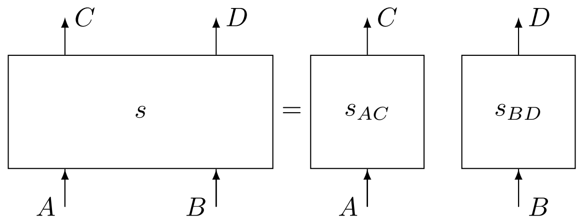


Figure 4: Theorem 1

This result is the toy theory analogue to a result from [4] (see figure 3), applying to causal quantum theory. It is the equivalent result, with symplectomorphisms replaced by unitaries. As a bonus, we also get that $A \rightsquigarrow D$ implies a certain structure on s. Graphically, see Figure 5. The latter is also reminiscent of a result in quantum causality [1], although we see that in this case the symplectomorphic structure is not preserved. But what is a symplectomorphism? Let us propose a definition, somewhat further formalising the defintion given in [2].

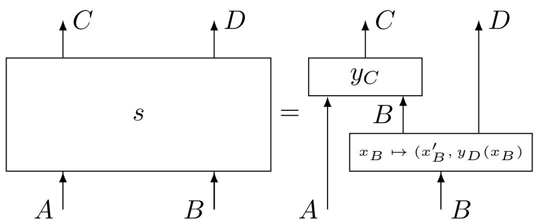


Figure 5: Corollary 1

x'_B is some part of x_B. Can this be a symplectomorphism (allowing, e.g., ancillary states)?

In our case, we will consider a symplectomorphism that corresponds to Figure 3. In other words, let $s : \Omega^{2n_A} \times \Omega^{2n_B} \to \Omega^{2n_C} \times \Omega^{2n_D}$ be a symplectomorphism. We can write $s(x_A|x_B) = (y_C(x_A|x_B)|y_D(x_A|x_B))$, and $s(x) = Sx + a, a \in \Omega^{2n_C} \times \Omega^{2n_D}$.

Note that we can write S as a block matrix. $S = \begin{pmatrix} S_{AC} & S_{BC} \\ S_{AD} & S_{BD} \end{pmatrix}$. With this, we can write

$$\begin{cases} y_C(x_A|x_B) = S_{AC}x_A^T + S_{BC}x_B^T + a_C^T \\ y_D(x_A|x_B) = S_{AD}x_A^T + S_{BD}x_B^T + a_D^T \end{cases},$$

where | stands for **concatenation**.

4.3 Existence of a no-cloning theorem

To justify the interpretation of lemma 2 given in figure 6, it is good to think about whether copy gates of ontic states exist. It seems that they do not.

Definition 5 (Copy Gate)**.** *A copy gate is a symplectomorphism* $s : \Omega^{2n} \times \Omega^{2n} \to \Omega^{2n} \times \Omega^{2n}$, $\exists x_B \forall x_A s(x_A|x_B) = (x_A|x_A)$.

I have, as of right now, not found any symplectomorphism that satisfies this property (including while relaxing the definition to allow ancillaries). It could be interesting to prove that such a symplectomorphism does not exist. The main problem seems to be the thirds rule of the definition of a symplectomorphism: preservation of symplectic form.

This would indicate that no-copy theorems are not inherently linked to contextuality, but note that it would have no use in practise, as it is impossible to prepare a particle in a specific ontic state in the toy theory. The knowledge balance principle of [7] forbids this.

4.4 Main Original Contribution

Lemma 1. *A few useful results about our symplectomorphism.*

1. *If $n_A = n_C$, equation* 3 *in the definition of a symplectomorphism can be rewritten as the block matrix equality*

$$\begin{pmatrix} S_{AC}^T J_{n_A} S_{AC} + S_{AD}^T J_{n_B} S_{AD} & S_{AC}^T J_{n_A} S_{BC} + S_{AD}^T J_{n_B} S_{BD} \\ S_{BC}^T J_{n_A} S_{AC} + S_{BD}^T J_{n_B} S_{AD} & S_{BC}^T J_{n_A} S_{BC} + S_{BD}^T J_{n_B} S_{BD} \end{pmatrix} = \begin{pmatrix} J_{n_A} & 0 \\ 0 & J_{n_B} \end{pmatrix}.$$

2. $n_A < n_C \iff n_B > n_D$.

Proof.

1. I leave the proof of this as an algebra exercise to the avid reader that you are. Have fun!
2. $n = n_A + n_B = n_C + n_D$ by definition, which is equivalent to $n_A - n_C = n_B - n_D$. So $n_A < n_C$ is equivalent to $n_B > n_D$.

$\square$

Lemma 2 (Equivalence of $A \not\rightsquigarrow D$)**.** *If and only if $A \not\rightsquigarrow D$, then $S_{AD} = 0$.*

A way to restate lemma 2 graphically is figure 6, where the pink dot stands for duplication (a copy gate). This sparks discussion about the existence of such a gate, that we partake in posthaste.

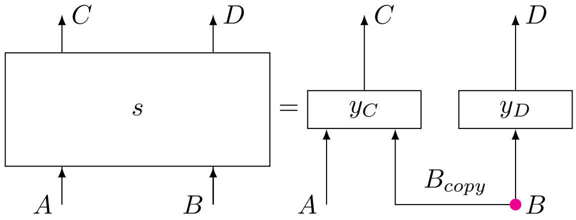


Figure 6: If and only if $A \not\rightsquigarrow D$.

Proof. " $\Longrightarrow$ " (Reductio ad absurdium.) Suppose there is $2n_C < i \leq 2n_C + 2n_D, 0 < j \leq 2n_A, S_{ij} \neq 0$, such that $S_{ij} \neq 0$. Consider $e_j = \begin{pmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 \end{pmatrix}$ with a 1 only at the j-th position. Consider the i-th component of $s(e_j) = Se_j^T + a^T$. Compare it to the i-th component of $s(0)$, which is a_i. This is

$$S_{ij} + a_i \neq s(0).$$

We have found $x_A = e_j, x'_A = 0$ such that $s(x_A|0) \neq s(x'_A|0)$. This is in contradiction with $A \not\rightsquigarrow D$.

" $\Longleftarrow$ " Suppose $S_{AD} = 0$. Then $y_D(x_A|x_B) = S_{BD} x_B^T + a_D^T$. This implies that $A \rightsquigarrow D$. $\square$

We may note that lemma 2 gives us one of the three necessary conditions for y_D being a symplectomorphism:

Corollary 1. *If $A \not\rightsquigarrow D$, we may overload y_D and write it only as a function of x_B, that is, $y_D(x_B) := S_{BD}x_B^T + a_D^T$, where S_{BD} is a matrix from the phase space of B to that of D.*

Proof. Trivial using Lemma 2. $\square$

However, S_{BD} is not necessarily invertible in this case. It is also not necessarily the case that $S_{BD}^T J_{n_B} S_{BD} = J_{n_B}$. For example, if $n_A = n_C$, by lemma 1 this is only the case if $S_{BC} J_{n_A} S_{BC} = 0$, i.e. by lemma 2 only if $B \not\rightsquigarrow D$. Let us search for causality conditions that separate s by conserving its symplectomorphic properties. In this search, we will be interested in sufficient conditions for causality. This leads us to the following proposition.

Proposition 1. *If $A \not\rightsquigarrow D$ (resp. $B \not\rightsquigarrow C$), then $n_A \leq n_C$ (resp. $n_A \geq n_C$), .*

Proof. Suppose $A \not\rightsquigarrow D$. Then, by lemma 2, $S_{AD} = 0$. Note that

$$\begin{aligned}
\operatorname{rank}(S) &= \operatorname{rank}\begin{pmatrix} S_{AC} & S_{BC} \\ S_{AD} & S_{BD} \end{pmatrix} \\
&\leq \operatorname{rank}\begin{pmatrix} S_{AC} \\ S_{AD} \end{pmatrix} + \operatorname{rank}\begin{pmatrix} S_{BC} \\ S_{BD} \end{pmatrix} \\
&= \operatorname{rank}\begin{pmatrix} S_{AC} \\ 0 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} S_{BC} \\ S_{BD} \end{pmatrix}.
\end{aligned}$$

By the rank theorem, $\operatorname{rank}\begin{pmatrix} S_{BC} \\ S_{BD} \end{pmatrix} \leq 2n_B$. Crucially, $\operatorname{rank}\begin{pmatrix} S_{AC} \\ 0 \end{pmatrix} \leq \operatorname{rank}(S_{AC}) \leq 2n_C$. We then have

$$\operatorname{rank}(S) \leq 2n_B + 2n_C.$$

But $\operatorname{rank}(S) = 2n = 2n_B + 2n_A$. This implies

$$2n_A \leq 2n_C \implies n_A \leq n_C.$$

The proof for $B \not\rightsquigarrow C$ is the same, with $S_{BC} = 0$.

$\square$

Corollary 2. *$A \not\rightsquigarrow D$ and $B \not\rightsquigarrow C \implies n_A = n_C$ (which is the same as $n_B = n_D$).*

Proof. By applying proposition 1, lemma 1.2, as well as the fact that $n_A + n_B = n_C + n_D$. $\square$

This leads us to our main result:

Theorem 1. *$A \not\rightsquigarrow D$ and $B \not\rightsquigarrow C \implies y_C, y_D$ are symplectomorphisms.*

Proof. Throughout, we suppose that $n_B = n_D$, which is given to us by corollary 2.

1. The first condition is satisfied by corollary 1.
2. S is block triangular, so its determinant is given by $\det(S_{AC})\det(S_{BD})$. It is nonzero, so both of those have to be nonzero. Thus, the second condition for being symplectomorphisms is satisfied.
3. By lemma 1, we see that $\begin{cases} S_{AC}^T J_{n_A} S_{AC} = J_{n_A} \\ \\ S_{BD}^T J_{n_B} S_{BD} = J_{n_B}, \end{cases}$, which completes our proof.

$\square$

5 Conclusion

This original result is the first theorem about causal structure of Spekkens' symplectomorphisms. It lays the foundation for many other theorems in foundations of physics. While the question of the existence of a no-cloning theorem remains open, I proved that some of the causal structure of quantum physics in [1] is also present in the toy theory of [2].

5.1 Future work

Sabilizer quantum theory. The links between these results and stabilizer quantum theory have yet to be studied, but there is a strong link between the latter and Spekkens' toy theory, as shown in [3]. Stabilizer notation has even been developed for Spekkens' toy theory [6]. Another further exploration would be to see that the definition of the symplectic form J_n can be written as an element of the Pauli group, which can help us see the symplectomorphisms as a stabilizer group.

Other quantum causal properties. There are many quantum causal properties proved in [1]. The interpretation of lemma 2 made in figure 6 is a potential branching path from this.

Continuous symplectic classical mechanics. In a typical physics degree, classical mechanics is explored with continuous degrees of freedom. It could be interesting to generalise the result of this paper to continuous degrees of freedom. Generalising these result could help us understand the underlying causal structure of the classical world.

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