Filter Coefficients to Popular Wavelets

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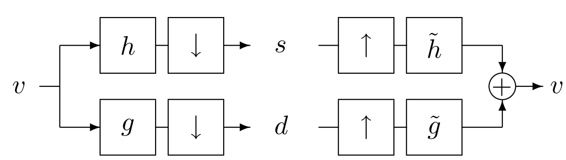
Over the last two decades, wavelets have gained a lot of popularity and become a standard tool for many disciplines. Despite all the attention, it can be difficult to obtain filter coefficients for even the most commonly used wavelets. This document is a reference, listing filters for wavelets in the Daubechies, symlets, Coiflets, and biorthogonal spline families and the CDF 9/7 wavelet. For some wavelets, a filter sequence for lifting scheme implementation is also provided.

Notes

- Wavelets are indexed by the number of vanishing moments, for example, “Daubechies 2” has 2 vanishing moments and 4-tap filters.
- Wavelets can have more than one name; for example, “Symlet 2” is also known as “Daubechies 2.”
- There are different conventions for filter scale factors.

Background

Let h and g be the wavelet decomposition (analysis) filters, where h is a lowpass filter and g is a highpass filter. Let the dual filters $\tilde{h}$ and $\tilde{g}$ be the wavelet reconstruction (synthesis) filters. One stage of decomposition followed by reconstruction is



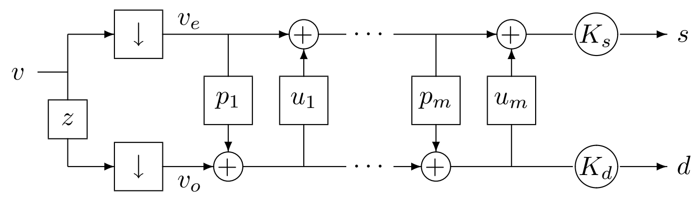
The wavelet filters h, g, $\tilde{h}$, $\tilde{g}$ must satisfy the perfect reconstruction conditions,

$$\begin{aligned} h(z)\tilde{h}(z) + g(z)\tilde{g}(z) &= 2, \\ h(z)\tilde{h}(-z) + g(z)\tilde{g}(-z) &= 0. \end{aligned}$$

Scaling the filters by some scale factors α, β and shifting by some even integers $2j$, $2k$

$$\begin{aligned} h'(z) &= \alpha z^{2j} h(z), & g'(z) &= \beta z^{2k} g(z), \\ \tilde{h}'(z) &= \alpha^{-1} z^{-2j} \tilde{h}(z), & \tilde{g}'(z) &= \beta^{-1} z^{-2k} \tilde{g}(z), \end{aligned}$$

preserves the perfect reconstruction conditions. Exchanging the primal filters h, g with the dual filters $\tilde{h}$, $\tilde{g}$ also produces a valid wavelet.



Any FIR (compact support) wavelet transform can be expressed as a lifting scheme [2]. The lifting scheme analysis is described with a sequence of "predict" and "update" filters, denoted $p_1, p_2, \ldots$ for predict filters and $u_1, u_2, \ldots$ for update filters. After the filtering steps, x_e is multiplied by K_s and x_o is multiplied by K_d. For the inverse transform, undo the K_s and K_d scale factors, change additions to subtractions, and perform the filtering steps in the reverse order.

1 Daubechies' Maximally Flat Wavelets

1.1 Daubechies 2

Daubechies 2 is an orthogonal wavelet with two vanishing moments.

$$\begin{aligned} h(z) &= h_{-2}z^2 + h_{-1}z + h_0 + h_1 z^{-1}, & \tilde{h}(z) &= h_1 z + h_0 + h_{-1}z^{-1} + h_{-2}z^{-2}, \\ g(z) &= -h_1 z^2 + h_0 z - h_{-1} + h_{-2}z^{-1}, & \tilde{g}(z) &= h_{-2}z - h_{-1} + h_0 z^{-1} - h_1 z^{-2}, \end{aligned}$$

$$h_{-2} = \frac{1+\sqrt{3}}{4\sqrt{2}},\ h_{-1} = \frac{3+\sqrt{3}}{4\sqrt{2}},\ h_0 = \frac{3-\sqrt{3}}{4\sqrt{2}},\ h_1 = \frac{1-\sqrt{3}}{4\sqrt{2}}.$$

The dual filters are $\tilde{h}(z) = h(z^{-1})$ and $\tilde{g}(z) = g(z^{-1})$.

The filters for lifting scheme implementation are

$$\begin{aligned} p_1(z) &= -\sqrt{3} \\ u_1(z) &= \tfrac{1}{4}(\sqrt{3}-2)z + \tfrac{1}{4}\sqrt{3} \\ p_2(z) &= z^{-1} \end{aligned}$$

with $K_s = \frac{\sqrt{3}+1}{\sqrt{2}}$ and $K_d = \frac{\sqrt{3}-1}{\sqrt{2}}$.

1.2 Daubechies 3

Orthogonal wavelet with three vanishing moments

$$
\begin{aligned}
h(z) &= h_{-3}z^3 + h_{-2}z^2 + h_{-1}z + h_0 + h_1 z^{-1} + h_2 z^{-2},\\
g(z) &= -h_2 z^3 + h_1 z^2 - h_0 z + h_{-1} - h_{-2} z^{-1} + h_{-3} z^{-2},
\end{aligned}
$$

$$
\begin{aligned}
h_{-3} &= \sqrt{2}\left(1+\sqrt{10}+\sqrt{5+2\sqrt{10}}\right)/32, & h_{-2} &= \sqrt{2}\left(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}\right)/32,\\
h_{-1} &= \sqrt{2}\left(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}}\right)/32, & h_0 &= \sqrt{2}\left(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}}\right)/32,\\
h_1 &= \sqrt{2}\left(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}\right)/32, & h_2 &= \sqrt{2}\left(1+\sqrt{10}-\sqrt{5+2\sqrt{10}}\right)/32.
\end{aligned}
$$

$\tilde{h}(z) = h(z^{-1})$ and $\tilde{g}(z) = g(z^{-1})$.

The filters for lifting scheme implementation are

$$
\begin{aligned}
u_1(z) &= \alpha\\
p_1(z) &= \beta z + \beta'\\
u_2(z) &= \gamma + \gamma' z^{-1}\\
p_2(z) &= \delta
\end{aligned}
$$

with scale factors $K_s = \zeta$ and $K_d = 1/\zeta$ where

$$
\begin{aligned}
r_0 &= h_{-1} - h_3 \cdot h_{-2}/h_2\\
r_1 &= h_1 - h_2 \cdot h_0/h_2\\
s_1 &= h_0 - h_{-2}r_1/r_0 - h_2 \cdot r_0/r_1\\
t &= -h_3/h_{-2} \cdot {s_1}^2
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= h_3/h_1 & &\doteq -0.412\ 286\ 595\ 0\\
\beta &= h_2/r_1 & &\doteq -1.565\ 136\ 279\ 6\\
\beta' &= h_{-2}/r_0 & &\doteq 0.352\ 387\ 657\ 6\\
\gamma &= r_1/s_1 & &\doteq 0.028\ 459\ 089\ 6\\
\gamma' &= r_0/s_1 & &\doteq 0.492\ 151\ 844\ 9\\
\delta &= -h_3/h_{-2} \cdot {s_1}^2 & &\doteq -0.389\ 620\ 390\ 0\\
\zeta &= s_1 & &\doteq 1.918\ 202\ 946\ 2
\end{aligned}
$$

1.3 Daubechies 4, 5, 6, 7, 8, 9

$$h(z) = \sum_k h_k z^{-k}, \qquad g(z) = zh(-z^{-1}), \qquad \tilde{h}(z) = h(z^{-1}), \qquad \tilde{g}(z) = g(z^{-1}).$$

Daubechies 4

$$\begin{aligned}
h_0 &\doteq 0.230\ 377\ 813\ 309,\\
h_1 &\doteq 0.714\ 846\ 570\ 553,\\
h_2 &\doteq 0.630\ 880\ 767\ 940,\\
h_3 &\doteq -0.027\ 983\ 769\ 417,\\
h_4 &\doteq -0.187\ 034\ 811\ 719,\\
h_5 &\doteq 0.030\ 841\ 381\ 836,\\
h_6 &\doteq 0.032\ 883\ 011\ 667,\\
h_7 &\doteq -0.010\ 597\ 401\ 785
\end{aligned}$$

Daubechies 5

$$\begin{aligned}
h_0 &\doteq 0.160\ 102\ 397\ 974,\\
h_1 &\doteq 0.603\ 829\ 269\ 797,\\
h_2 &\doteq 0.724\ 308\ 528\ 438,\\
h_3 &\doteq 0.138\ 428\ 145\ 901,\\
h_4 &\doteq -0.242\ 294\ 887\ 066,\\
h_5 &\doteq -0.032\ 244\ 869\ 585,\\
h_6 &\doteq 0.077\ 571\ 493\ 840,\\
h_7 &\doteq -0.006\ 241\ 490\ 213,\\
h_8 &\doteq -0.012\ 580\ 751\ 999,\\
h_9 &\doteq 0.003\ 335\ 725\ 285
\end{aligned}$$

Daubechies 6

$$\begin{aligned}
h_0 &\doteq 0.111\ 540\ 743\ 350,\\
h_1 &\doteq 0.494\ 623\ 890\ 398,\\
h_2 &\doteq 0.751\ 133\ 908\ 021,\\
h_3 &\doteq 0.315\ 250\ 351\ 709,\\
h_4 &\doteq -0.226\ 264\ 693\ 965,\\
h_5 &\doteq -0.129\ 766\ 867\ 567,\\
h_6 &\doteq 0.097\ 501\ 605\ 587,\\
h_7 &\doteq 0.027\ 522\ 865\ 530,\\
h_8 &\doteq -0.031\ 582\ 039\ 317,\\
h_9 &\doteq 0.000\ 553\ 842\ 201,\\
h_{10} &\doteq 0.004\ 777\ 257\ 511,\\
h_{11} &\doteq -0.001\ 077\ 301\ 085
\end{aligned}$$

Daubechies 7

$$\begin{aligned}
h_0 &\doteq 0.077\ 852\ 054\ 085,\\
h_1 &\doteq 0.396\ 539\ 319\ 482,\\
h_2 &\doteq 0.729\ 132\ 090\ 846,\\
h_3 &\doteq 0.469\ 782\ 287\ 405,\\
h_4 &\doteq -0.143\ 906\ 003\ 929,\\
h_5 &\doteq -0.224\ 036\ 184\ 994,\\
h_6 &\doteq 0.071\ 309\ 219\ 267,\\
h_7 &\doteq 0.080\ 612\ 609\ 151,\\
h_8 &\doteq -0.038\ 029\ 936\ 935,\\
h_9 &\doteq -0.016\ 574\ 541\ 631,\\
h_{10} &\doteq 0.012\ 550\ 998\ 556,\\
h_{11} &\doteq 0.000\ 429\ 577\ 973,\\
h_{12} &\doteq -0.001\ 801\ 640\ 704,\\
h_{13} &\doteq 0.000\ 353\ 713\ 800
\end{aligned}$$

Daubechies 8

$$\begin{aligned}
h_0 &\doteq 0.054\ 415\ 842\ 243,\\
h_1 &\doteq 0.312\ 871\ 590\ 914,\\
h_2 &\doteq 0.675\ 630\ 736\ 297,\\
h_3 &\doteq 0.585\ 354\ 683\ 654,\\
h_4 &\doteq -0.015\ 829\ 105\ 256,\\
h_5 &\doteq -0.284\ 015\ 542\ 962,\\
h_6 &\doteq 0.000\ 472\ 484\ 574,\\
h_7 &\doteq 0.128\ 747\ 426\ 620,\\
h_8 &\doteq -0.017\ 369\ 301\ 002,\\
h_9 &\doteq -0.044\ 088\ 253\ 931,\\
h_{10} &\doteq 0.013\ 981\ 027\ 917,\\
h_{11} &\doteq 0.008\ 746\ 094\ 047,\\
h_{12} &\doteq -0.004\ 870\ 352\ 993,\\
h_{13} &\doteq -0.000\ 391\ 740\ 373,\\
h_{14} &\doteq 0.000\ 675\ 449\ 406,\\
h_{15} &\doteq -0.000\ 117\ 476\ 784
\end{aligned}$$

Daubechies 9

$$\begin{aligned}
h_0 &\doteq 0.038\ 077\ 947\ 364,\\
h_1 &\doteq 0.243\ 834\ 674\ 613,\\
h_2 &\doteq 0.604\ 823\ 123\ 690,\\
h_3 &\doteq 0.657\ 288\ 078\ 051,\\
h_4 &\doteq 0.133\ 197\ 385\ 825,\\
h_5 &\doteq -0.293\ 273\ 783\ 279,\\
h_6 &\doteq -0.096\ 840\ 783\ 223,\\
h_7 &\doteq 0.148\ 540\ 749\ 338,\\
h_8 &\doteq 0.030\ 725\ 681\ 479,\\
h_9 &\doteq -0.067\ 632\ 829\ 061,\\
h_{10} &\doteq 0.000\ 250\ 947\ 115,\\
h_{11} &\doteq 0.022\ 361\ 662\ 124,\\
h_{12} &\doteq -0.004\ 723\ 204\ 758,\\
h_{13} &\doteq -0.004\ 281\ 503\ 682,\\
h_{14} &\doteq 0.001\ 847\ 646\ 883,\\
h_{15} &\doteq 0.000\ 230\ 385\ 764,\\
h_{16} &\doteq -0.000\ 251\ 963\ 189,\\
h_{17} &\doteq 0.000\ 039\ 347\ 320
\end{aligned}$$

2 Symlets

Symlets are Daubechies' approximately symmetry wavelets, orthogonal wavelets where the scaling function is close to symmetric.

$$h(z) = \sum_k h_k z^{-k}, \qquad g(z) = zh(-z^{-1}), \qquad \tilde{h}(z) = h(z^{-1}), \qquad \tilde{g}(z) = g(z^{-1}).$$

Symlet 2

$$\begin{aligned} h_0 &\doteq 0.482\ 962\ 913\ 145, \\ h_1 &\doteq 0.836\ 516\ 303\ 737, \\ h_2 &\doteq 0.224\ 143\ 868\ 042, \\ h_3 &\doteq -0.129\ 409\ 522\ 551 \end{aligned}$$

Symlet 3

$$\begin{aligned} h_0 &\doteq 0.332\ 670\ 552\ 951, \\ h_1 &\doteq 0.806\ 891\ 509\ 313, \\ h_2 &\doteq 0.459\ 877\ 502\ 119, \\ h_3 &\doteq -0.135\ 011\ 020\ 010, \\ h_4 &\doteq -0.085\ 441\ 273\ 882, \\ h_5 &\doteq 0.035\ 226\ 291\ 882 \end{aligned}$$

Symlet 4

$$\begin{aligned} h_0 &\doteq 0.032\ 223\ 100\ 604, \\ h_1 &\doteq -0.012\ 603\ 967\ 262, \\ h_2 &\doteq -0.099\ 219\ 543\ 577, \\ h_3 &\doteq 0.297\ 857\ 795\ 606, \\ h_4 &\doteq 0.803\ 738\ 751\ 807, \\ h_5 &\doteq 0.497\ 618\ 667\ 633, \\ h_6 &\doteq -0.029\ 635\ 527\ 646, \\ h_7 &\doteq -0.075\ 765\ 714\ 789 \end{aligned}$$

Symlet 5

$$\begin{aligned} h_0 &\doteq 0.019\ 538\ 882\ 735, \\ h_1 &\doteq -0.021\ 101\ 834\ 025, \\ h_2 &\doteq -0.175\ 328\ 089\ 908, \\ h_3 &\doteq 0.016\ 602\ 105\ 765, \\ h_4 &\doteq 0.633\ 978\ 963\ 458, \\ h_5 &\doteq 0.723\ 407\ 690\ 402, \\ h_6 &\doteq 0.199\ 397\ 533\ 977, \\ h_7 &\doteq -0.039\ 134\ 249\ 302, \\ h_8 &\doteq 0.029\ 519\ 490\ 926, \\ h_9 &\doteq 0.027\ 333\ 068\ 345 \end{aligned}$$

Symlet 6

$$\begin{aligned} h_0 &\doteq -0.007\ 800\ 708\ 325, \\ h_1 &\doteq 0.001\ 767\ 711\ 864, \\ h_2 &\doteq 0.044\ 724\ 901\ 771, \\ h_3 &\doteq -0.021\ 060\ 292\ 512, \\ h_4 &\doteq -0.072\ 637\ 522\ 786, \\ h_5 &\doteq 0.337\ 929\ 421\ 728, \\ h_6 &\doteq 0.787\ 641\ 141\ 030, \\ h_7 &\doteq 0.491\ 055\ 941\ 927, \\ h_8 &\doteq -0.048\ 311\ 742\ 586, \\ h_9 &\doteq -0.117\ 990\ 111\ 148, \\ h_{10} &\doteq 0.003\ 490\ 712\ 084, \\ h_{11} &\doteq 0.015\ 404\ 109\ 327 \end{aligned}$$

Symlet 7

$$\begin{aligned} h_0 &\doteq 0.010\ 268\ 176\ 709, \\ h_1 &\doteq 0.004\ 010\ 244\ 872, \\ h_2 &\doteq -0.107\ 808\ 237\ 704, \\ h_3 &\doteq -0.140\ 047\ 240\ 443, \\ h_4 &\doteq 0.288\ 629\ 631\ 752, \\ h_5 &\doteq 0.767\ 764\ 317\ 003, \\ h_6 &\doteq 0.536\ 101\ 917\ 092, \\ h_7 &\doteq 0.017\ 441\ 255\ 087, \\ h_8 &\doteq -0.049\ 552\ 834\ 937, \\ h_9 &\doteq 0.067\ 892\ 693\ 501, \\ h_{10} &\doteq 0.030\ 515\ 513\ 166, \\ h_{11} &\doteq -0.012\ 636\ 303\ 403, \\ h_{12} &\doteq -0.001\ 047\ 384\ 889, \\ h_{13} &\doteq 0.002\ 681\ 814\ 568 \end{aligned}$$

Symlet 8

$$
\begin{aligned}
h_0 &\doteq 0.001\ 889\ 950\ 333,\\
h_1 &\doteq -0.000\ 302\ 920\ 515,\\
h_2 &\doteq -0.014\ 952\ 258\ 337,\\
h_3 &\doteq 0.003\ 808\ 752\ 014,\\
h_4 &\doteq 0.049\ 137\ 179\ 674,\\
h_5 &\doteq -0.027\ 219\ 029\ 917,\\
h_6 &\doteq -0.051\ 945\ 838\ 108,\\
h_7 &\doteq 0.364\ 441\ 894\ 835,\\
h_8 &\doteq 0.777\ 185\ 751\ 701,\\
h_9 &\doteq 0.481\ 359\ 651\ 258,\\
h_{10} &\doteq -0.061\ 273\ 359\ 068,\\
h_{11} &\doteq -0.143\ 294\ 238\ 351,\\
h_{12} &\doteq 0.007\ 607\ 487\ 325,\\
h_{13} &\doteq 0.031\ 695\ 087\ 811,\\
h_{14} &\doteq -0.000\ 542\ 132\ 332,\\
h_{15} &\doteq -0.003\ 382\ 415\ 951
\end{aligned}
$$

3 Coiflets

$$h(z) = \sum_k h_k z^{-k}, \qquad g(z) = z h(-z^{-1}), \qquad \tilde{h}(z) = h(z^{-1}), \qquad \tilde{g}(z) = g(z^{-1}).$$

Coiflet 1

$$\begin{aligned}
h_0 &\doteq -0.072\ 732\ 619\ 513,\\
h_1 &\doteq 0.337\ 897\ 662\ 458,\\
h_2 &\doteq 0.852\ 572\ 020\ 212,\\
h_3 &\doteq 0.384\ 864\ 846\ 864,\\
h_4 &\doteq -0.072\ 732\ 619\ 513,\\
h_5 &\doteq -0.015\ 655\ 728\ 135
\end{aligned}$$

Coiflet 2

$$\begin{aligned}
h_0 &\doteq 0.016\ 387\ 336\ 464,\\
h_1 &\doteq -0.041\ 464\ 936\ 782,\\
h_2 &\doteq -0.067\ 372\ 554\ 722,\\
h_3 &\doteq 0.386\ 110\ 066\ 823,\\
h_4 &\doteq 0.812\ 723\ 635\ 450,\\
h_5 &\doteq 0.417\ 005\ 184\ 424,\\
h_6 &\doteq -0.076\ 488\ 599\ 079,\\
h_7 &\doteq -0.059\ 434\ 418\ 647,\\
h_8 &\doteq 0.023\ 680\ 171\ 946,\\
h_9 &\doteq 0.005\ 611\ 434\ 819,\\
h_{10} &\doteq -0.001\ 823\ 208\ 871,\\
h_{11} &\doteq -0.000\ 720\ 549\ 445
\end{aligned}$$

Coiflet 3

$$\begin{aligned}
h_0 &\doteq -0.003\ 793\ 512\ 864,\\
h_1 &\doteq 0.007\ 782\ 596\ 427,\\
h_2 &\doteq 0.023\ 452\ 696\ 142,\\
h_3 &\doteq -0.065\ 771\ 911\ 282,\\
h_4 &\doteq -0.061\ 123\ 390\ 003,\\
h_5 &\doteq 0.405\ 176\ 902\ 410,\\
h_6 &\doteq 0.793\ 777\ 222\ 626,\\
h_7 &\doteq 0.428\ 483\ 476\ 378,\\
h_8 &\doteq -0.071\ 799\ 821\ 619,\\
h_9 &\doteq -0.082\ 301\ 927\ 107,\\
h_{10} &\doteq 0.034\ 555\ 027\ 573,\\
h_{11} &\doteq 0.015\ 880\ 544\ 864,\\
h_{12} &\doteq -0.009\ 007\ 976\ 137,\\
h_{13} &\doteq -0.002\ 574\ 517\ 689,\\
h_{14} &\doteq 0.001\ 117\ 518\ 771,\\
h_{15} &\doteq 0.000\ 466\ 216\ 960,\\
h_{16} &\doteq -0.000\ 070\ 983\ 303,\\
h_{17} &\doteq -0.000\ 034\ 599\ 773
\end{aligned}$$

Coiflet 4

$$
\begin{aligned}
h_0 &\doteq 0.000\ 892\ 313\ 669,\\
h_1 &\doteq -0.001\ 629\ 492\ 013,\\
h_2 &\doteq -0.007\ 346\ 166\ 328,\\
h_3 &\doteq 0.016\ 068\ 943\ 965,\\
h_4 &\doteq 0.026\ 682\ 300\ 156,\\
h_5 &\doteq -0.081\ 266\ 699\ 681,\\
h_6 &\doteq -0.056\ 077\ 313\ 317,\\
h_7 &\doteq 0.415\ 308\ 407\ 030,\\
h_8 &\doteq 0.782\ 238\ 930\ 921,\\
h_9 &\doteq 0.434\ 386\ 056\ 491,\\
h_{10} &\doteq -0.066\ 627\ 474\ 263,\\
h_{11} &\doteq -0.096\ 220\ 442\ 034,\\
h_{12} &\doteq 0.039\ 334\ 427\ 123,\\
h_{13} &\doteq 0.025\ 082\ 261\ 845,\\
h_{14} &\doteq -0.015\ 211\ 731\ 528,\\
h_{15} &\doteq -0.005\ 658\ 286\ 687,\\
h_{16} &\doteq 0.003\ 751\ 436\ 157,\\
h_{17} &\doteq 0.001\ 266\ 561\ 929,\\
h_{18} &\doteq -0.000\ 589\ 020\ 756,\\
h_{19} &\doteq -0.000\ 259\ 974\ 552,\\
h_{20} &\doteq 0.000\ 062\ 339\ 034,\\
h_{21} &\doteq 0.000\ 031\ 229\ 876,\\
h_{22} &\doteq -0.000\ 003\ 259\ 680,\\
h_{23} &\doteq -0.000\ 001\ 784\ 985
\end{aligned}
$$

Coiflet 5

$$
\begin{aligned}
h_0 &\doteq -0.000\ 212\ 080\ 840,\\
h_1 &\doteq 0.000\ 358\ 589\ 688,\\
h_2 &\doteq 0.002\ 178\ 236\ 358,\\
h_3 &\doteq -0.004\ 159\ 358\ 782,\\
h_4 &\doteq -0.010\ 131\ 117\ 521,\\
h_5 &\doteq 0.023\ 408\ 156\ 788,\\
h_6 &\doteq 0.028\ 168\ 028\ 974,\\
h_7 &\doteq -0.091\ 920\ 010\ 569,\\
h_8 &\doteq -0.052\ 043\ 163\ 181,\\
h_9 &\doteq 0.421\ 566\ 206\ 733,\\
h_{10} &\doteq 0.774\ 289\ 603\ 730,\\
h_{11} &\doteq 0.437\ 991\ 626\ 216,\\
h_{12} &\doteq -0.062\ 035\ 963\ 969,\\
h_{13} &\doteq -0.105\ 574\ 208\ 714,\\
h_{14} &\doteq 0.041\ 289\ 208\ 754,\\
h_{15} &\doteq 0.032\ 683\ 574\ 270,\\
h_{16} &\doteq -0.019\ 761\ 778\ 945,\\
h_{17} &\doteq -0.009\ 164\ 231\ 163,\\
h_{18} &\doteq 0.006\ 764\ 185\ 449,\\
h_{19} &\doteq 0.002\ 433\ 373\ 213,\\
h_{20} &\doteq -0.001\ 662\ 863\ 702,\\
h_{21} &\doteq -0.000\ 638\ 131\ 343,\\
h_{22} &\doteq 0.000\ 302\ 259\ 582,\\
h_{23} &\doteq 0.000\ 140\ 541\ 150,\\
h_{24} &\doteq -0.000\ 041\ 340\ 432,\\
h_{25} &\doteq -0.000\ 021\ 315\ 027,\\
h_{26} &\doteq 0.000\ 003\ 734\ 655,\\
h_{27} &\doteq 0.000\ 002\ 063\ 762,\\
h_{28} &\doteq -0.000\ 000\ 167\ 443,\\
h_{29} &\doteq -0.000\ 000\ 095\ 177
\end{aligned}
$$

4 Biorthogonal Spline Wavelets

Godavarthy [3] shows how to algorithmically construct wavelets in this family. Given N and $\tilde{N}$ such that $M = \frac{1}{2}(N + \tilde{N}) - 1$ is integer, the filters for Spline $N.\tilde{N}$ are

$$
\begin{aligned}
h(z) &= \sqrt{2}\, z^{\lfloor N/2 \rfloor} \left(\frac{1 + z^{-1}}{2} \right)^{N}, \\
\tilde{h}(z) &= \sqrt{2}\, z^{\lceil \tilde{N}/2 \rceil} \left(\frac{1 + z^{-1}}{2} \right)^{\tilde{N}} \sum_{n=0}^{M} \binom{M+n}{n} (-4)^{-n} (z - 2 + z^{-1})^{n}, \\
g(z) &= z^{-1} \tilde{h}(-z), \qquad \tilde{g}(z) = z h(-z),
\end{aligned}
$$

where $\lfloor \cdot \rfloor$ denotes the floor function. The filters for particular N and $\tilde{N}$ are listed below:

$$
h(z) = \sum_k h_k z^{-k}, \qquad \tilde{h}(z) = \sum_k \tilde{h}_k z^{-k}, \qquad g(z) = z^{-1}\tilde{h}(-z), \qquad \tilde{g}(z) = z h(-z).
$$

Spline 1.3

$$
\begin{aligned}
h_0 &= \tfrac{1}{2}\sqrt{2}, & \tilde{h}_{-3} &= -\tfrac{1}{16}\sqrt{2}, \\
h_1 &= \tfrac{1}{2}\sqrt{2} & \tilde{h}_{-2} &= \tfrac{1}{16}\sqrt{2}, \\
& & \tilde{h}_{-1} &= \tfrac{1}{2}\sqrt{2}, \\
& & \tilde{h}_{0} &= \tfrac{1}{2}\sqrt{2}, \\
& & \tilde{h}_{1} &= \tfrac{1}{16}\sqrt{2}, \\
& & \tilde{h}_{2} &= -\tfrac{1}{16}\sqrt{2}
\end{aligned}
$$

Spline 1.5

$$
\begin{aligned}
h_0 &= \tfrac{1}{2}\sqrt{2}, & \tilde{h}_{-5} &= \tfrac{3}{256}\sqrt{2}, \\
h_1 &= \tfrac{1}{2}\sqrt{2} & \tilde{h}_{-4} &= -\tfrac{3}{256}\sqrt{2}, \\
& & \tilde{h}_{-3} &= -\tfrac{22}{256}\sqrt{2}, \\
& & \tilde{h}_{-2} &= \tfrac{22}{256}\sqrt{2}, \\
& & \tilde{h}_{-1} &= \tfrac{1}{2}\sqrt{2}, \\
& & \tilde{h}_{0} &= \tfrac{1}{2}\sqrt{2}, \\
& & \tilde{h}_{1} &= \tfrac{22}{256}\sqrt{2}, \\
& & \tilde{h}_{2} &= -\tfrac{22}{256}\sqrt{2}, \\
& & \tilde{h}_{3} &= -\tfrac{3}{256}\sqrt{2}, \\
& & \tilde{h}_{4} &= \tfrac{3}{256}\sqrt{2}
\end{aligned}
$$

Spline 2.2 (also CDF 5/3)

$$
\begin{aligned}
h_{-1} &= \tfrac{1}{4}\sqrt{2}, & \tilde{h}_{-2} &= -\tfrac{1}{8}\sqrt{2}, \\
h_0 &= \tfrac{1}{2}\sqrt{2}, & \tilde{h}_{-1} &= \tfrac{1}{4}\sqrt{2}, \\
h_1 &= \tfrac{1}{4}\sqrt{2} & \tilde{h}_{0} &= \tfrac{3}{4}\sqrt{2}, \\
& & \tilde{h}_{1} &= \tfrac{1}{4}\sqrt{2}, \\
& & \tilde{h}_{2} &= -\tfrac{1}{8}\sqrt{2}
\end{aligned}
$$

Spline 2.4

$$
\begin{aligned}
h_{-1} &= \tfrac{1}{4}\sqrt{2}, & \tilde{h}_{-4} &= \tfrac{3}{128}\sqrt{2}, \\
h_0 &= \tfrac{1}{2}\sqrt{2}, & \tilde{h}_{-3} &= -\tfrac{6}{128}\sqrt{2}, \\
h_1 &= \tfrac{1}{4}\sqrt{2} & \tilde{h}_{-2} &= -\tfrac{16}{128}\sqrt{2}, \\
& & \tilde{h}_{-1} &= \tfrac{38}{128}\sqrt{2}, \\
& & \tilde{h}_{0} &= \tfrac{90}{128}\sqrt{2}, \\
& & \tilde{h}_{1} &= \tfrac{38}{128}\sqrt{2}, \\
& & \tilde{h}_{2} &= -\tfrac{16}{128}\sqrt{2}, \\
& & \tilde{h}_{3} &= -\tfrac{6}{128}\sqrt{2}, \\
& & \tilde{h}_{4} &= \tfrac{3}{128}\sqrt{2}
\end{aligned}
$$

Spline 2.6

$$\begin{array}{rclrcl} h_{-1} &=& \frac{1}{4}\sqrt{2}, & \tilde{h}_{-6} &=& -\frac{5}{1024}\sqrt{2}, \\ h_0 &=& \frac{1}{2}\sqrt{2}, & \tilde{h}_{-5} &=& \frac{10}{1024}\sqrt{2}, \\ h_1 &=& \frac{1}{4}\sqrt{2} & \tilde{h}_{-4} &=& \frac{34}{1024}\sqrt{2}, \\ && & \tilde{h}_{-3} &=& -\frac{78}{1024}\sqrt{2}, \\ && & \tilde{h}_{-2} &=& -\frac{123}{1024}\sqrt{2}, \\ && & \tilde{h}_{-1} &=& \frac{324}{1024}\sqrt{2}, \\ && & \tilde{h}_{0} &=& \frac{700}{1024}\sqrt{2}, \\ && & \tilde{h}_{1} &=& \frac{324}{1024}\sqrt{2}, \\ && & \tilde{h}_{2} &=& -\frac{123}{1024}\sqrt{2}, \\ && & \tilde{h}_{3} &=& -\frac{78}{1024}\sqrt{2}, \\ && & \tilde{h}_{4} &=& \frac{34}{1024}\sqrt{2}, \\ && & \tilde{h}_{5} &=& \frac{10}{1024}\sqrt{2}, \\ && & \tilde{h}_{6} &=& -\frac{5}{1024}\sqrt{2} \end{array}$$

Spline 3.3

$$\begin{array}{rclrcl} h_{-1} &=& \frac{1}{8}\sqrt{2}, & \tilde{h}_{-4} &=& \frac{3}{64}\sqrt{2}, \\ h_0 &=& \frac{3}{8}\sqrt{2}, & \tilde{h}_{-3} &=& -\frac{9}{64}\sqrt{2}, \\ h_1 &=& \frac{3}{8}\sqrt{2}, & \tilde{h}_{-2} &=& -\frac{7}{64}\sqrt{2}, \\ h_2 &=& \frac{1}{8}\sqrt{2} & \tilde{h}_{-1} &=& \frac{45}{64}\sqrt{2}, \\ && & \tilde{h}_{0} &=& \frac{45}{64}\sqrt{2}, \\ && & \tilde{h}_{1} &=& -\frac{7}{64}\sqrt{2}, \\ && & \tilde{h}_{2} &=& -\frac{9}{64}\sqrt{2}, \\ && & \tilde{h}_{3} &=& \frac{3}{64}\sqrt{2} \end{array}$$

Cohen-Daubechies-Fauraue 4.4

This wavelet is often also called the CDF 9/7 wavelet (where 9 and 7 denote the number of filter taps). It is used by the FBI for fingerprint compression and one of the wavelets selected for the JPEG2000 image format.

$$\begin{aligned} h(z) &= h_3(z^3+z^{-3}) + h_2(z^2+z^{-2}) + h_1(z+z^{-1}) + h_0 \\ g(z) &= g_4(z^4+z^{-4}) + g_3(z^3+z^{-3}) + g_2(z^2+z^{-2}) + g_1(z+z^{-1}) + g_0 \\ \tilde{h}(z) &= \tilde{h}_4(z^4+z^{-4}) + \tilde{h}_3(z^3+z^{-3}) + \tilde{h}_2(z^2+z^{-2}) + \tilde{h}_1(z+z^{-1}) + \tilde{h}_0 \\ \tilde{g}(z) &= \tilde{g}_3(z^3+z^{-3}) + \tilde{g}_2(z^2+z^{-2}) + \tilde{g}_1(z+z^{-1}) + \tilde{g}_0 \end{aligned}$$

$$\begin{array}{rcrrcrrcrrcr} h_0 &\doteq& 0.788\ 485\ 616\ 614, & g_0 &\doteq& 0.852\ 698\ 679\ 009, & \tilde{h}_0 &=& g_0, & \tilde{g}_0 &=& h_0, \\ h_1 &\doteq& 0.418\ 092\ 273\ 333, & g_1 &\doteq& -0.377\ 402\ 855\ 613, & \tilde{h}_1 &=& -g_1, & \tilde{g}_1 &=& -h_1, \\ h_2 &\doteq& -0.040\ 689\ 417\ 620, & g_2 &\doteq& -0.110\ 624\ 404\ 418, & \tilde{h}_2 &=& g_2, & \tilde{g}_2 &=& h_2, \\ h_3 &\doteq& -0.064\ 538\ 882\ 646, & g_3 &\doteq& 0.023\ 849\ 465\ 019, & \tilde{h}_3 &=& -g_3 & \tilde{g}_3 &=& -h_3, \\ && & g_4 &\doteq& 0.037\ 828\ 455\ 507 & \tilde{h}_4 &=& g_4 && & \end{array}$$

The filters for lifting scheme implementation are

$$\begin{aligned} p_1(z) &= \alpha(z+1) \\ u_1(z) &= \beta(1+z^{-1}) \\ p_2(z) &= \gamma(z+1) \\ u_2(z) &= \delta(1+z^{-1}) \end{aligned}$$

with scale factors $K_s = \zeta$ and $K_d = 1/\zeta$ where

$$\begin{aligned} r_0 &= \tilde{h}_0 - 2\tilde{h}_4 \cdot \tilde{h}_1/\tilde{h}_3 \\ r_1 &= \tilde{h}_2 - \tilde{h}_4 - \tilde{h}_4 \cdot \tilde{h}_1/\tilde{h}_3 \\ s_0 &= \tilde{h}_1 - \tilde{h}_3 - \tilde{h}_3 \cdot r_0/r_1 \\ t_0 &= r_0 - 2r_1 \end{aligned}$$

$$\begin{aligned} \alpha &= \tilde{h}_4/\tilde{h}_3 &\doteq& -1.586\ 134\ 342\ 060 \\ \beta &= \tilde{h}_3/r_1 &\doteq& -0.052\ 980\ 118\ 573 \\ \gamma &= r_1/s_0 &\doteq& 0.882\ 911\ 075\ 531 \\ \delta &= s_0/t_0 &\doteq& 0.443\ 506\ 852\ 044 \\ \zeta &= t_0 &\doteq& 1.149\ 604\ 398\ 860 \end{aligned}$$

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