Voronoi Diagrams

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Introduction

The year, 1853. The industrial revolution has built London into the largest city in the world, but much of its infrastructure did not grow at the same time.

Tragedy strikes. A cholera outbreak decimates 10% of the population of Soho, a region of London. Current physicians still subscribe to “Miasma Theory”, or the belief that sickness spread via foul emanations from the soil and air [1]. Enter John Snow, a surgeon and physician. John argued that from his work as a surgeon, that cholera was a disease of the gut, and that the cause must be transmitted by ingestion, and spread to others via the ingestion of faecal matter [2]. But how to figure out how an entire area was being affected?

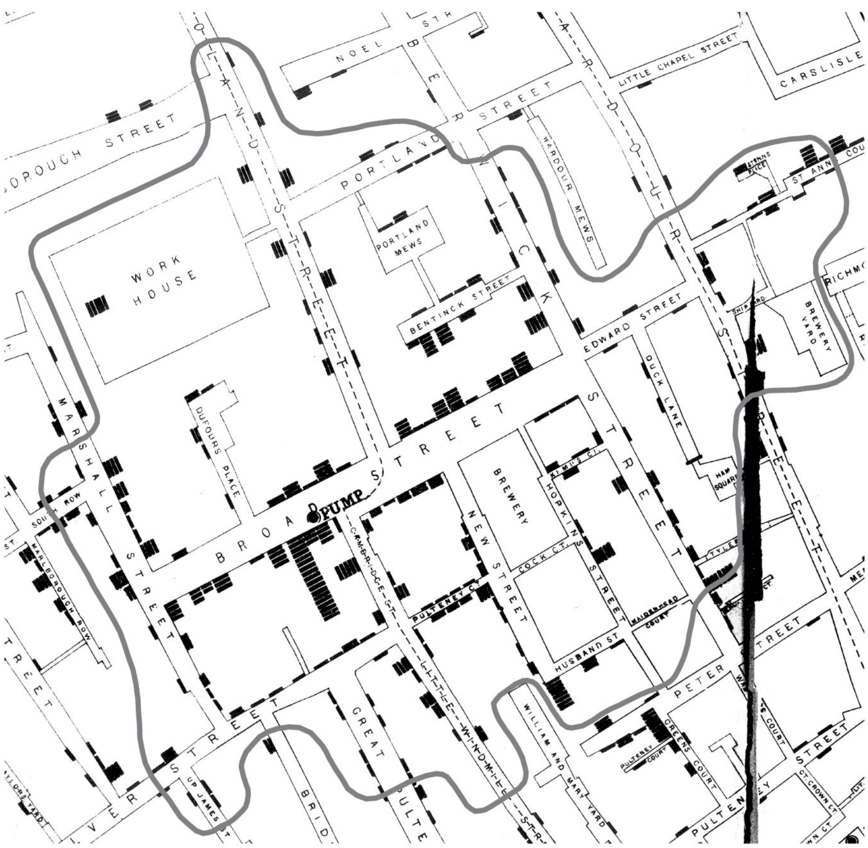

Figure 1: John Snow’s Diagram of the Outbreak

Snow mapped out all of the cholera cases in an area, and realized that they were all closest to a single water pump in the neighborhood. He proposed that this pump must be contaminated, and now Snow is considered to be the father of “epidemiology”, the study and analysis of the patterns of disease conditions in populations [3].

Formalization

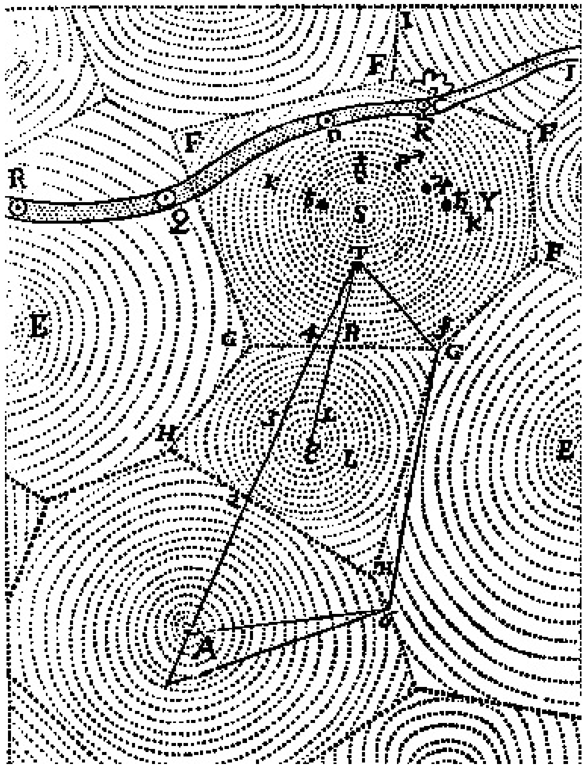

Figure 2: Analysis of Gravitational Effects on Star Clusters

Snow wasn't the first to use the idea of equidistance to analyze groupings, in fact the first recorded use was Descartes in 1644, where similar techniques were used to analyze the gravitational effects of stars [4]. Dirichlet described the diagram formally in 1850 [5], but the diagrams themselves are referred to as "Voronoi Diagrams" to credit Georgy Feodosevich Voronoy and his extension of the concept to n-dimensions in 1908 [6, 7].

A **Voronoi Diagram** is the partitioning of a plane with n points into convex polygons such that each polygon contains exactly one generating point and every point in a given polygon is closer to its generating point than to any other [8].

Construction

Voronoi Diagrams are typically constructed using computers since it becomes too time consuming once there are a large number of points; however it is possible to construct Voronoi Diagrams by hand.

Straight Edge

Firstly, each point must be connected to each of its nearest points by a line segment without any line segments intersecting. This step is known as Delauny Triangulation and is often thought of as the most difficult step in constructing Voronoi Diagrams by hand. There are several ways to connect the sites properly. One way is by brute force, every site can be connected to each other and the intersecting line segments can each be compared with the longer being deleted. Another method is to construct a test triangle of sites and draw their circumcircle, rejecting the triangle if the circle contains any other point.

The easiest method however is using human intuition to guess, but this does not work well with larger number of points. Once the sites have been connected according to the conditions, the center points of each connecting line segment is found. Then a perpindicular bisector is drawn at each of these midpoints and if everything has been done right then there should be three lines intersecting at a point. The final step is to retrace the outline of each Voronoi cell along the perpindicular bisectors. In the end there will be one point in every cell, where every point in that cell in closer to the enclosed point than any other marked point.

Computer Applet

It can be shown that the Voronoi Diagram of a set of N points in the plane can be computed in $O(N\log(N))$ time. There are many programs available to use online for constructing Voronoi Diagrams, such as Alex Beutel's online applet [10]. This online applet allows the user to add points wherever they want, add a random set of N points, move the points for a specific speed function, and save the set of points for future reference.

Applications

The three most common uses of Voronoi Diagrams can be explained as geographical and physical examples.

Nearest site problem

The nearest site problem can be used for simplified consumer behavior analysis. Let's choose an example that's close to home: Coffee. By mapping the locations of all of the Tim Horton's in Abbotsford, we are able to analyze the Voronoi neighborhoods that each one covers. From this, we can notice that the new

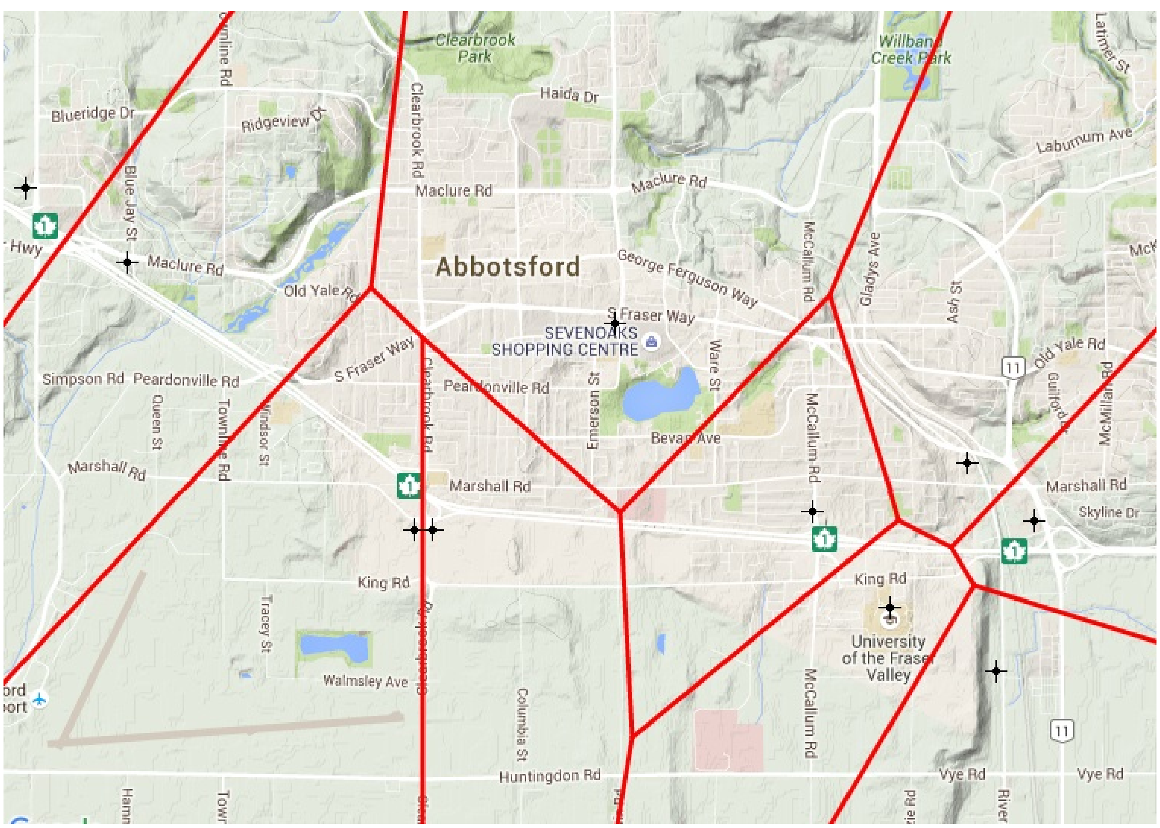

Figure 3: Tim Horton's Locations in Abbotsford

location that is opening on Clearbrook Road does not help in reducing the size of any of the neighborhoods, and in addition, the Voronoi neighborhoods in East Abbotsford are significantly smaller than those in West Abbotsford.

Closest pair

Voronoi Diagrams can also be used to find the closest pair of two things. This can be useful in collision detection for example. Instead of calculating the distance between every set of two points individually, by constructing a Voronoi Diagram, the distance between neighbors can simply be analyzed and give the same results.

Largest empty circle

The last is the so called "Toxic Waste Dump" Application, where you need to find the furthest location away from a set of points. This is given by the voronoi vertex that is the furthest distance away from its generating sites.

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