Year 2: Observing the Universe

This is a supplementary lecture note on attenuation of low frequency radio waves by the Earth's ionosphere. It replaces the material entitled "Physics of the Ionosphere" in the main lecture notes.

The ionosphere is partially ionized by the UV radiation field of the Sun, and thus contains a population of free electrons – i.e. is a conducting medium. This plasma renders the Earth's atmosphere opaque to electromagnetic radiation at frequencies less than the so-called "plasma frequency".

A rigorous treatment of the propagation of radiation through plasmas can be obtained by solving Maxwell's equations, and is beyond the scope of this course. The aim of this supplementary lecture note is to clarify, via a simplified treatment, the key physics.

Plasma Frequency

Consider an electromagnetic wave of frequency ν traveling in the z-direction through an ionized medium. The electric field vector of the wave varies with time, t, as follows:

$$E = E_0 \, sin(2\pi\nu t)$$

If this field is in (say) the x-direction then an electron in the plasma will experience a force $-eE$, where e is the charge of the electron. The equation of motion of the electron will be:

$$m_e \ddot{x} = -eE = -eE_0 \, sin(2\pi\nu t)$$

Integrating the equation of motion, and re-arranging gives:

$$4\pi^2\nu^2 m_e x = eE_0 sin(2\pi\nu t)$$

We can therefore write:

$$\ddot{x} = -(4\pi^2\nu^2)x$$

i.e. the electrons oscillate with simple harmonic motion at a frequency driven by the incident radiation.

The oscillation of the electrons can be expressed in terms of a net polarization of the plasma. Defining P as the dipole moment per unit volume, N as the number density of electrons, we can write:

$$P = -Nex = -\frac{Ne^2 E}{4\pi^2\nu^2 m_e}$$

From dielectric theory we know that $P = \epsilon_0(\epsilon_r - 1)E$, where ϵ_0 is the permittivity of free space and ϵ_r is the relative permittivity of the dielectric. We therefore obtain an expression for the relative permittivity of the plasma:

$$\epsilon_r = 1 - \frac{Ne^2}{4\pi^2\nu^2 m_e \epsilon_0} = 1 - \left(\frac{\nu_p}{\nu}\right)^2$$

where ν_p is known as the "plasma frequency":

$$\nu_p(\mathrm{Hz}) = \frac{\omega_p}{2\pi} = \sqrt{\frac{Ne^2}{4\pi^2 m_e \epsilon_0}} = 8.98\sqrt{N(\mathrm{m}^{-3})}$$

The ionosphere comprises various layers of differing states of ionization. Knowledge of the detailed structure of ionosphere is not required for this course. Typically, the number density of electrons in the ionosphere is $N \simeq 6 \times 10^{10}\mathrm{m}^{-3}$, which gives a typical plasma frequency of $\nu_p = 2.2\mathrm{MHz}$, which corresponds to a wavelength of $\lambda_p = c/\nu_p = 136\mathrm{m}$.

Refractive Index

In an isotropic medium of relative permeability μ_r and relative permittivity ϵ_r, the phase velocity of an electromagnetic wave is given by:

$$v_{\mathrm{ph}} = \frac{c}{\mu_r \epsilon_r} = \frac{c}{n}$$

where c is the speed of light in vacuo and n is the refractive index of the medium. A reasonable approximation for the Earth's ionosphere is $\mu_r = 1$. We can therefore obtain an expression for the refractive index of the ionosphere:

$$n = \sqrt{\epsilon_r} = \sqrt{1 - \left(\frac{\nu_p}{\nu}\right)^2}$$

Phase and Group Velocity

Limiting our attention for now to $\nu > \nu_p$, the refractive index of the ionosphere is clearly less than unity, $n < 1$, and the phase velocity of electromagnetic radiation exceeds the speed of light: $v_{\mathrm{ph}} = c/n > c$.

However information is not actually transported at this unphysical speed. Formally, v_{ph} is the phase velocity of an infinite monochromatic wave. Energy and information are transported at the group velocity, v_{g}, which is the velocity of a wave of finite bandwidth. You can see from the frequency dependence of the refractive index above that each frequency component of the wave group will travel at a different phase velocity. This causes the radio beam to become dispersed. The dispersion suffered by an electromagnetic wave travelling through a plasma is described by the dispersion relation:

$$\omega^2 = \omega_p^2 + k^2 c^2$$

where $\omega = \nu/2\pi$, $\omega_p = \nu_p/2\pi$, and $k = 2\pi/\lambda$. In terms of ω and k, the phase velocity is simply:

$$v_{\mathrm{ph}} = \frac{c}{n} = \frac{\omega}{k}$$

However the group velocity, i.e. the velocity that corresponds to the beat frequency of the wave group, is given by:

$$v_{\mathrm{g}} = \frac{d\omega}{dk} = \frac{kc^2}{\omega} = cn$$

Therefore dispersion causes the wave group to travel slower (and thus energy and information to travel slower) than the speed of light in vacuo:

$$v_{\mathrm{g}} = cn = c\sqrt{1 - \left(\frac{\nu_p}{\nu}\right)^2}$$

Plasma Frequency as a Cut-off Frequency

Consider a beam of radio waves of small bandwidth $\Delta\nu$ around a frequency ν propagating through the Earth's atmosphere in a direction normal to the Earth's surface.

At $\nu > \nu_p$ the radio wave group travels slower than the speed of light in vacuo, and the ionosphere is transparent.

At $\nu = \nu_p$ the refractive index of the ionosphere and the group velocity of the radio wave beam are both zero, and the radio beam is reflected.

At $\nu < \nu_p$ the refractive index is imaginary, and the amplitude of the radio wave beam exponentially declines – the ionosphere is therefore opaque at $\nu < \nu_p$.

This last point can be shown by writing the electric field of the radio wave in complex notation:

$$E = E_0 \exp[i(kz - \omega t)] = E_0 \exp\left[i\left(\frac{\omega n z}{c} - \omega t\right)\right]$$

at $\omega < \omega_p$ the refractive index is:

$$n = i\sqrt{\left(\frac{\omega_p}{\omega}\right)^2 - 1}$$

Substituting the refractive index into the expression for the electric field then gives the result that the amplitude of the field declines exponentially:

$$E = E_0 \exp\left[-\frac{\omega z}{c}\sqrt{\left[\left(\frac{\omega_p}{\omega}\right)^2 - 1\right]} - i\omega t\right]$$

The plasma frequency is therefore a cut-off frequency below which the atmosphere is opaque to electromagnetic radiation.