Classical Mechanics and Relativity

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Contents

Year 1 Mechanics: treated objects as point masses, concerned with translation only

Year 2 Mechanics: looking at extended bodies which leads to rotation

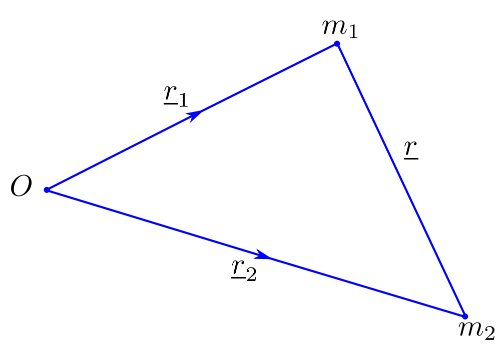
Translation	**Rotation**
Force $\underline{F}$	Torque $\underline{\tau}$
Momentum $\underline{p}$	Angular Momentum $\underline{L}$
Mass, velocity $\underline{F} = \frac{\mathrm{d}p}{\mathrm{d}t}$	Momentum of inertia, angular velocity $\underline{\tau} = \frac{\mathrm{d}\underline{L}}{\mathrm{d}t}$
$F = 0 \Rightarrow \underline{p} = \text{constant}$	$\underline{\tau} = 0 \Rightarrow \underline{L} = \text{constant}$

Part I

Motion of Extended Body

The fall analysis can be very complex so it is simplified by separating the motion into the motion of the center of mass and the rotation about the center of mass of the rest of the object.

1 Reduced Mass



Consider the external forces $\underline{F}_1^{ext}$ and $\underline{F}_2^{ext}$ acting on masses 1 and 2. If the force on m_1 due to m_2 is $\underline{F}_{12}$, then according to Newton's 3rd law, the force om m_2 due to m_1, is

$$\underline{F}_{21} = -\underline{F}_{12} \equiv -\underline{F}$$

Equations of motion for the two bodies

$$m_1\ddot{\underline{r}}_1 = \underline{F}_1^{ext} + \underline{F} \tag{1}$$
$$m_2\ddot{\underline{r}}_2 = \underline{F}_2^{ext} + \underline{F} \tag{2}$$

To makes solving these equations easier, we will simplify the situation. Center of mass is given by,

$$\underline{R} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

and the relative position is given by,

$$\underline{r} = \underline{r}_1 - \underline{r}_2$$

Given these definitions, and adding 1 and 2 gives

$$\begin{aligned} m_1\ddot{\underline{r}}_1 + m_2\ddot{\underline{r}}_2 &= \underline{F}_1^{ext} + \underline{F}_2^{ext} + \underline{F} - \underline{F} \\ (m_1+m_2)\ddot{\underline{R}} &= \underline{F}_1^{ext} + \underline{F}_2^{ext} \\ M\ddot{\underline{R}} &= \underline{F}_{tot}^{ext} \end{aligned}$$

So the system behaves like a point mass M at coordinate R under the external force $\underline{F}_{tot}^{ext}$. If $\underline{F}_{tot}^{ext} = 0$ then the center of mass acceleration is zero and hence the momentum of the center of mass is constant.

$$\begin{aligned} M\ddot{\underline{R}} &= 0 \\ \Rightarrow M\dot{\underline{R}} &= \text{constant} \end{aligned}$$

Also from (1) and (2),

$$\begin{aligned} \ddot{\underline{r}}_1 &= \frac{\underline{F}_1^{ext}}{m_1} + \frac{\underline{F}}{m_1} \\ \ddot{\underline{r}}_2 &= \frac{\underline{F}_2^{ext}}{m_2} - \frac{\underline{F}}{m_2} \end{aligned}$$

Supposing the only external force is gravity, then

$$\frac{\underline{F}_1^{ext}}{m_1} = \frac{\underline{F}_2^{ext}}{m_2}$$

this is the gravitational acceleration.

$$\begin{aligned} \Rightarrow \ddot{\underline{r}} &= \ddot{\underline{r}}_1 - \ddot{\underline{r}}_2 = F\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{F}{\mu} \\ F &= \mu\ddot{\underline{r}} \end{aligned}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$. This is typically called the reduce mass. So the motion of a two body system can be separated into,

1. Center of mass of the system which moves under the influence of the of the net external force and
2. Relative motion which can be described as the motion of a single particle whose mass is the reduced mass, μ, under the influence of the internal force

2 Center of Mass of Many Body System

For two bodies;

$$R_{cm} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

For many bodies;

$$R_{cm} = \frac{\sum_k^n m_k \underline{r}_k}{\sum_k^n m_k} = \frac{1}{M}\sum_k m_k \underline{r}_k$$

where M is the total mass of the system. For a continuous mass distribution;

$$R_{cm} = \frac{\displaystyle\int \underline{r}\mathrm{d}m}{\displaystyle\int \mathrm{d}m} = \frac{\displaystyle\int \underline{r}\rho(\underline{r})\mathrm{d}V}{\displaystyle\int \rho(\underline{r})\mathrm{d}V}$$

where $\rho(\underline{r})$ = mass density at the point r.

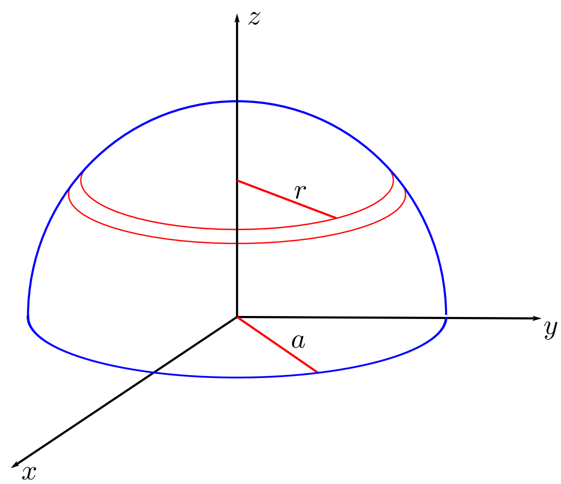
2.1 Finding Center of Mass

Use symmetry where possible.

- If the mass distribution has an axis of symmetry, then the center of mass will lie on that axis.
- If there is more than one axis of symmetry, then the center of mass must lie on both axes, therefor must lie at the point where the axes intersect.
- If ρ is constant, then for a sphere, for example, the center of mass must lie at the center of the sphere.
  - If $\rho = \rho(r)$ then CM is still at the center,
  - If $\rho = \rho(z)$ then CM lies on the z axis.

Ex

A solid hemisphere, of uniform density , ρ, and radius a. Find the location of the CM.



By symmetry, $x_{cm} = y_{cm} = 0$, so just need to calculate z_{cm}.

Consider the hemisphere as a stack of disks, radius r, with height $\to 0$,

$$\begin{aligned} \mathrm{d}m &= \rho \mathrm{d}V \\ &= \rho\pi r^2 \mathrm{d}z \end{aligned}$$

Rewrite new variable, r, in terms of z, $r^2 = a^2 - z^2$

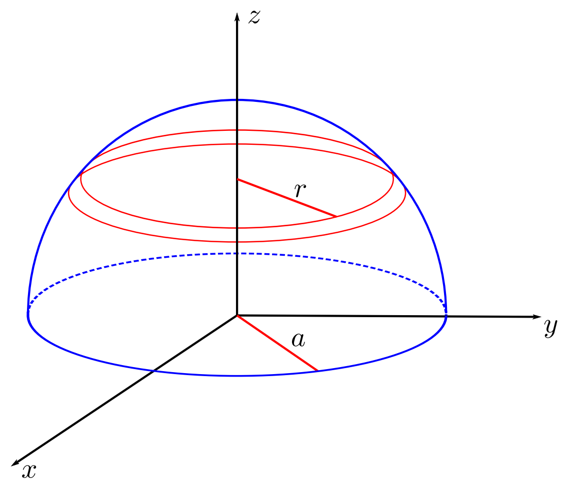
$$\begin{aligned} \Rightarrow \mathrm{d}m &= \rho\pi(a^2 - z^2)\mathrm{d}z \\ z_{cm} &= \frac{\displaystyle\int z\mathrm{d}m}{\displaystyle\int \mathrm{d}m} \\ &= \frac{\displaystyle\int_0^a \rho\pi z(a^2 - z^2)\mathrm{d}z}{\displaystyle\int_0^a \rho\pi(a^2 - z^2)\mathrm{d}z} \end{aligned}$$

$$= \frac{\displaystyle\int_0^a za^2 - z^3 \mathrm{d}z}{\displaystyle\int_0^a a - z^2 \mathrm{d}z}$$

$$= \frac{\left[\frac{a^2z^2}{2} - \frac{z^4}{4}\right]_0^a}{\left[a^2z - \frac{z^3}{3}\right]_0^a} = \frac{\frac{a^4}{4}}{\frac{2}{3}a^3} = \frac{3}{8}a$$

Ex

Consider a hollow hemisphere,



Mass per unit area$= \sigma$,

$$\begin{aligned}
\Rightarrow \mathrm{d}m &= \sigma \mathrm{d}A \\
x_{cm} &= y_{cm} = 0 \\
\mathrm{d}A &= 2\pi r a \mathrm{d}\theta \\
&= 2\pi r a^2 \sin(\theta)\mathrm{d}\theta \\
\mathrm{d}m &= \sigma 2\pi a^2 \sin(\theta)\mathrm{d}\theta \\
z &= a\cos(\theta)
\end{aligned}$$

$$\begin{aligned}
z_{cm} &= \frac{\displaystyle\int z \mathrm{d}m}{\displaystyle\int \mathrm{d}m} \\
&= \frac{\displaystyle\int \sigma 2\pi a^2 \sin(\theta) a\cos(\theta)\mathrm{d}\theta}{\displaystyle\int \sigma 2\pi a^2 \sin(\theta)\mathrm{d}\theta} \\
&= \frac{a}{2}\frac{\displaystyle\int_0^{\frac{\pi}{2}} 2\sin(\theta)\cos(\theta)\mathrm{d}\theta}{\displaystyle\int_0^{\frac{\pi}{2}} \sin(\theta)\mathrm{d}\theta}
\end{aligned}$$

$$= \frac{a}{2}\frac{\int_0^{\frac{\pi}{2}} \sin(2\theta)\mathrm{d}\theta}{\int_0^{\frac{\pi}{2}} \sin(\theta)\mathrm{d}\theta} = \frac{a}{2}$$

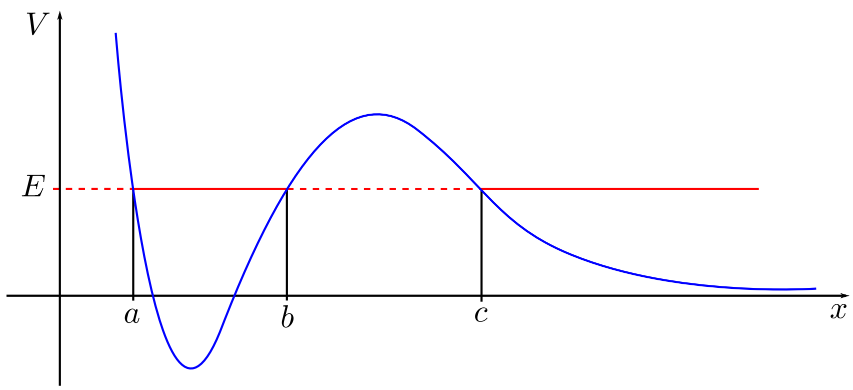
Part II

Harmonic Oscillations and Motion

Motion in a conservative potential, if the potential depends only on position, $V = V(r)$, then $E = T + V =$ constant. In one dimension this gives

$$E = \frac{p^2}{2m} + V(x) = \text{constant}$$

3 Types of Motion



For a particle with energy E there are three types of region,

1. bounded motion, $a < x < b$,
2. excluded from the region, $b < x < c$,
3. unbounded, can travel so can travel to ∞, $x > c$.

3.1 Bounded Motion

Consider a particle in the region $a < x < b$.

- $x = b$, $V = E$ so $T = 0$ and $V = 0$
  - $\frac{\mathrm{d}V}{\mathrm{d}x} > 0$, $F = -\frac{\mathrm{d}V}{\mathrm{d}t} < 0$ so the force pushes back towards a.
- $x = a$, $V = E$ so $T = 0$ and $V = 0$
  - $\frac{\mathrm{d}V}{\mathrm{d}x} < 0$, $F = -\frac{\mathrm{d}V}{\mathrm{d}t} > 0$ so the force pushes towards b.

$\therefore$ Bound particle will oscillate continually between these limits. The nature of the oscillation will depend on the shape of the potential inside the bounded region.

3.2 Equilibrium

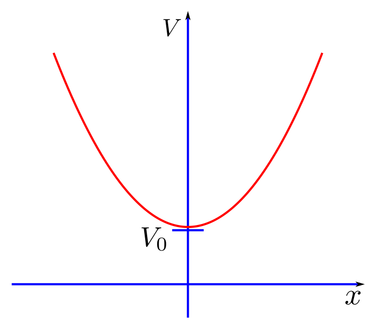
The point A, is an equilibrium position for a particle if that particle when released from rest at that position doesn't move so remains there, ie the point where the net force on an object is zero. Since $F = -\frac{\mathrm{d}V}{\mathrm{d}x} = 0$. So the equilibrium position is the stationary point in the potential function $V(x)$.

Let $V(x)$ have stationary point at x_0. For small displacement from x_0, we expand,

$$V(x) = V_0 + \left[\frac{\mathrm{d}V}{\mathrm{d}x}\right]_{x=x_0}(x-x_0) + \left[\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right]_{x=x_0}\frac{(x-x_0)^2}{2} + \mathcal{O}(x-x_0)^3$$

Therefore, near the equilibrium point,

$$\begin{aligned} V(x) &\approx \left[\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right]_{x=x_0}\frac{(x-x_0)^2}{2} + V_0 \\ \frac{\mathrm{d}^2V}{\mathrm{d}x^2} &= \text{constant} = k \end{aligned}$$



$$\begin{aligned} \frac{\mathrm{d}^2V}{\mathrm{d}x^2} &\neq k \\ F(x) &= -k(x-x_0) \\ F &\propto -\text{displacement} \\ &= \text{resulting force} \end{aligned}$$

So small displacement from the equilibrium will give oscillations about the equilibrium,

$$F = m\ddot{x} = -kx$$

This is the simple harmonic oscillator. So it there is a sufficiently small displacement if any system from the equilibrium, that will produce SHM.

3.2.1 Solutions

$$\begin{aligned} \ddot{x} &= -\frac{kx}{m} \\ \Rightarrow x(t) &= Ae^{i\omega t} + Be^{-i\omega t} \\ &\text{or} \\ x(t) &= C\cos(\omega t) + D\sin(\omega t) \end{aligned}$$

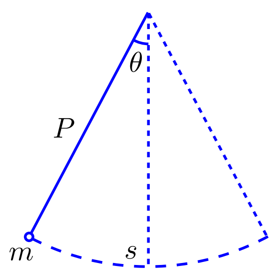
Unstable equilibrium

$$\begin{aligned}\frac{\mathrm{d}^2V}{\mathrm{d}x^2} &< 0 \rightarrow \ddot{x} = \frac{k}{m}x \\ x(t) &= Ae^{\omega t} + Be^{-\omega t}\end{aligned}$$

which corresponds to exponential decrease with time.

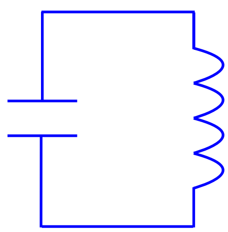
Examples

- Simple Pendulum



$$\ddot{\theta} = -\frac{g}{l}\theta$$
$$\ddot{s} = -\frac{g}{l}s$$

- weight on a string
- LC circuit



$$\ddot{v} = -\frac{C}{L}v$$

4 Damping

In reality, dissipative forces are usually present. These will convert the energy of motion of this oscillator into other forms. And so the oscillations will decay. The simplest type is one that is proportional to velocity,

$$F = -b\ddot{x}$$

Then the equation of motion becomes

$$m\ddot{x} + b\dot{x} + kx = 0$$

Ex

Mass attached to a light plate that is immersed in a viscous fluid;

- when the mass moves up, the force points downward
- when the mass moves down, the force points upward
- when the mass is at rest there is no net force

$$m\ddot{x} + b\dot{x} + kx = 0$$

Solution, try $x = e^{pt}$

$$\begin{aligned}
(mp^2 + bp + k)e^{pt} &= 0 \\
mp^2 + bp + k &= 0 \\
p &= -\frac{b}{2m} - \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}} \\
\Rightarrow x(t) &= Ae^{p_1 t} + Be^{p_2 t}
\end{aligned}$$

where

$$p_1 = -\frac{b}{2m} - \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

and

$$p_2 = -\frac{b}{2m} + \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

Ideal SHM $m\ddot{y} = ky = 0$. With damping $m\ddot{y} + b\dot{y} + ky = 0$. So the damping force,

$$F_{\text{damp}} = -b\frac{dy}{dt}$$

The general solution is given by,

$$y(t) = Ae^{p_1 t} + Be^{p_2 t}$$

where $p_{1,2} = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$. Note, the frequency of the undamped oscillator $\omega_0 = \sqrt{\frac{k}{m}}$. So

$$\begin{aligned}
p_{1,2} &= -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}} \\
&= -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \omega_0^2}
\end{aligned}$$

If $b = 0$, $p_{1,2} = \pm i\omega_0$

$$\Rightarrow y(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$

as expected. Get A and B from the initial conditions, e.g. at $t = 0$, $y = 10\text{m}$, $\frac{\mathrm{d}y}{\mathrm{d}t} = v = 2.6\text{ms}^{-1}$

$$\begin{aligned}
y(0) &= A + B = 10 \\
y'(0) &= p_1 A + p_2 B = 2.6
\end{aligned}$$

then solve these simultaneously for A and B.

4.1 Behavior of Damped Oscillator

There are two qualitatively different types of behavior, depending on the conditions. The transition case is a special case that is useful and important for many applications.

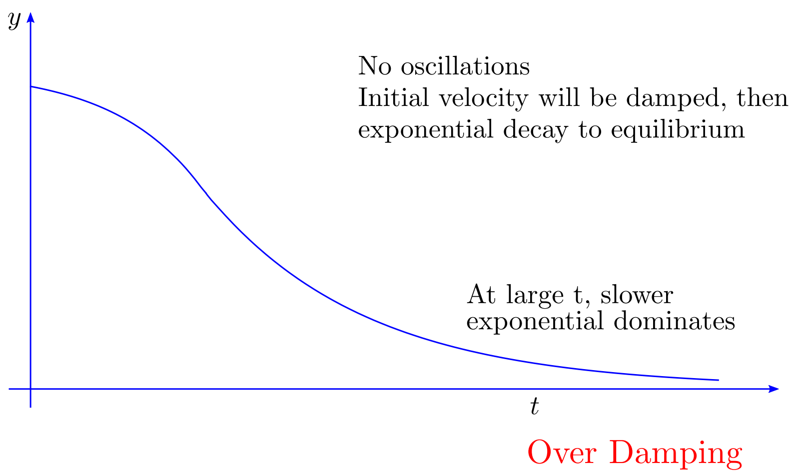
- Consider the case where

$$\frac{b^2}{4m^2} > \frac{k}{m}$$

This means that both $p_1$and p_2 are real and negative. So $y = Ae^{p_1 t} + Be^{p_2 t}$, which is that sum of two decaying exponentials.

Ex

Release an object from rest



- Consider the special case where $\frac{b^2}{4m^2} = \frac{k}{m}$

$$\Rightarrow p_1 = p_2 = -\frac{b}{2m}$$

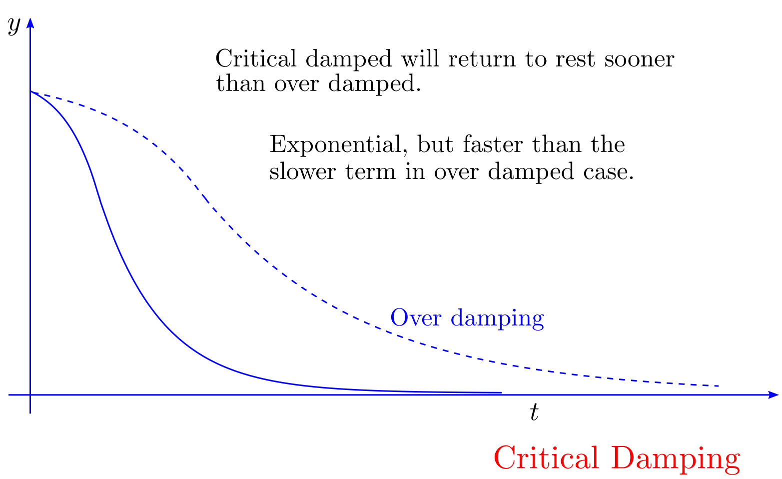
So the solution has the form

$$y(t) = (A + B)e^{-\frac{b}{2m}t}$$

Qualitatively this has the same sort of behavior.

Ex

Release an object from rest



- Consider the case where

$$\frac{b^2}{4m^2} < \frac{k}{m}$$

So $p_1$and p_2 are complex

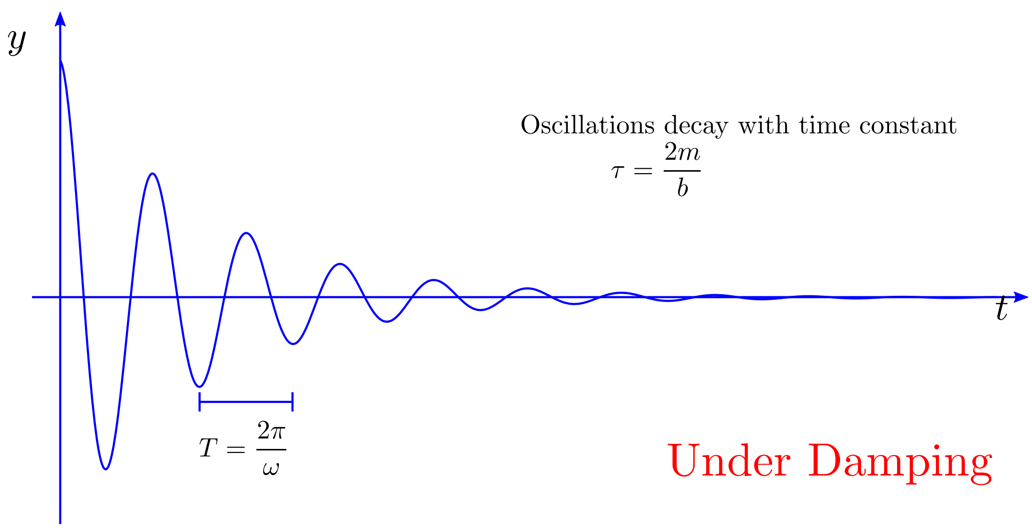
$$\begin{aligned} p_1 &= p_2 = -\frac{b}{2m} \pm i\omega \\ \Rightarrow y(t) &= Ae^{p_1 t} + Be^{p_2 t} \\ &= e^{-\frac{b}{2m}t}\left(Ae^{i\omega t} + Be^{-i\omega t}\right) \end{aligned}$$

We can rewrite this as

$$\begin{aligned} y(t) &= e^{-\frac{b}{2m}t}(C\sin(\omega t) + D\cos(\omega t)) \\ &= e^{-\frac{b}{2m}t}(C^2+D^2)^{\frac{1}{2}}\left(\frac{C}{(C^2+D^2)^{\frac{1}{2}}}\cos(\omega t) + \frac{D}{(C^2+D^2)^{\frac{1}{2}}}\sin(\omega t)\right) \end{aligned}$$

Now let $\frac{C}{(C^2+D^2)^{\frac{1}{2}}} = \cos(\alpha)$, and $\frac{D}{(C^2+D^2)^{\frac{1}{2}}} = \sin(\alpha)$, and $(C^2+D^2)^{\frac{1}{2}} = E$

$$\begin{aligned} y(t) &= e^{-\frac{b}{2m}t}E\left(\cos(\alpha)\cos(\omega t) + \sin(\alpha)\sin(\omega t)\right) \\ &= Ee^{-\frac{b}{2m}t}\cos(\omega t - \alpha) \end{aligned}$$



4.1.1 Frequency of Damped Oscillations

From above

$$\omega^2 = \frac{k}{m} - \frac{b^2}{4m^2} = \omega_0^2 - \frac{b^2}{4m^2}$$

So unless the system is very heavily damped $\left(\frac{b^2}{4m^2} \approx \omega_0^2 \text{ ie close to critical damping}\right)$ we usually find that the shift in frequency is not very large, $\omega \approx \omega_0$.

4.1.2 Decay of Damped Oscillations

Convenient to write

$$e^{-\frac{bt}{2m}} = e^{-\frac{\lambda}{\tau}t} \qquad T = \frac{2\pi}{\omega}$$

Then, for each cycle of oscillations, the amplitude decreases by a factor of $e^{-\lambda}$, where

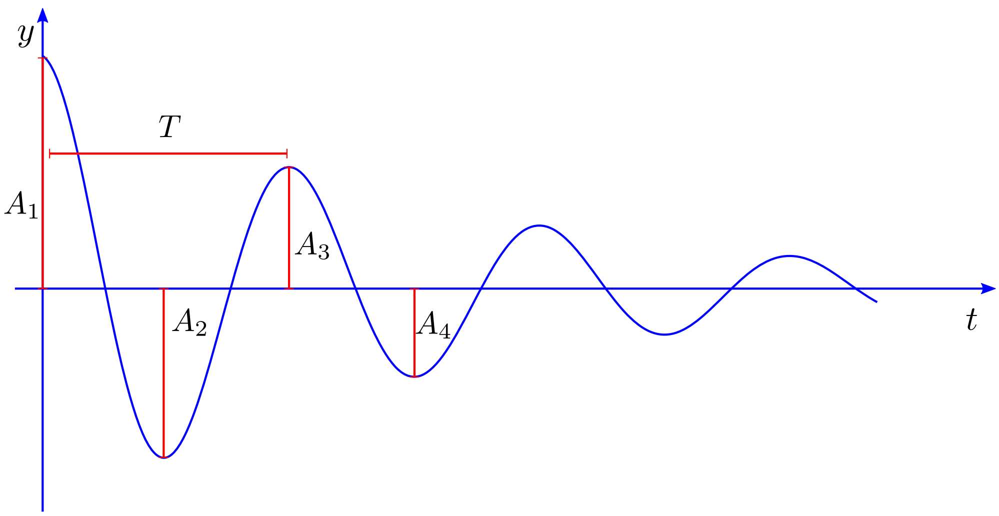
$$\begin{aligned}\frac{b}{2m} &= \frac{\lambda}{\frac{2\pi}{\omega}} \\ \lambda &= \frac{b}{2m}\frac{2\pi}{\omega}\end{aligned}$$

4.2 Q Factor

$$Q = 2\pi \times \frac{\text{Energy stored in the oscillator}}{\text{Energy dissipated in one cycle}}$$

This is a dimensionless quantity that relates the loss of energy to the cycles made by the oscillator.

$$\text{Energy} \propto (\text{Amplitude})^2$$



This is the fraction of energy lost per cycle (change in energy)

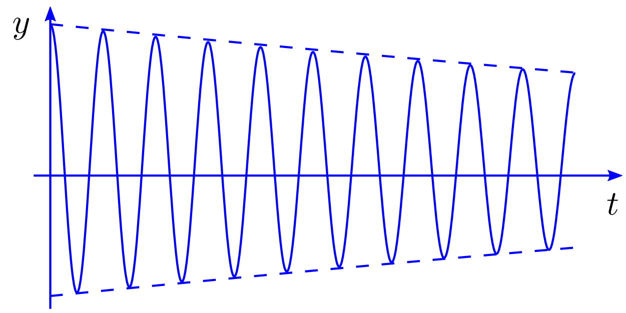
$$\begin{aligned}\frac{\Delta E}{E} &= \frac{A_3}{A_1}\frac{A_1^2 - A_3^2}{A_1^2} \qquad \frac{A_3}{A_1} = e^{-\lambda} \\ \frac{\Delta E}{E} &= 1 - \frac{A_3^2}{A_1^2} = 1 - e^{-2\lambda} \\ \therefore Q &= 2\pi\frac{\Delta E}{E} = \frac{2\pi}{1 - e^{-2\lambda}}\end{aligned}$$

When damping is very light, $\lambda \ll 1$

$$\begin{aligned}1 - e^{-2\lambda} &\approx 2\lambda \\ \Rightarrow Q &\approx \frac{\pi}{\lambda} = \frac{m\omega}{b}\end{aligned}$$

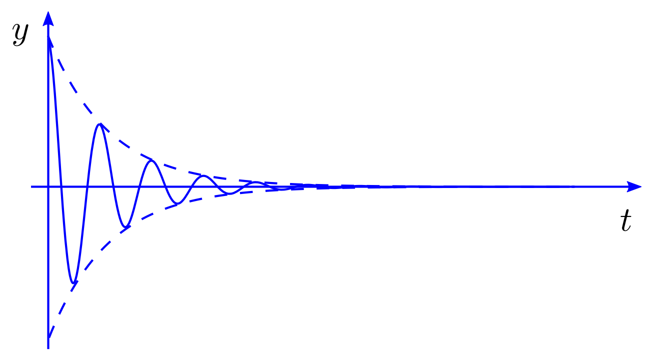
4.2.1 Large Q

- Light damping
- Little energy lost per cycle



4.2.2 Small Q

- Heavy damping
- Lots of energy lost per cycle



5 Forced Damped Simple Harmonic Oscillations

- Consider the system with
  - Restoring force $= ky$
  - Damping force $= b\dot{y}$
  - Periodic driving force $= F_0 \cos(\omega t)$

Equation of motion

$$\begin{aligned} m\ddot{y} &= -ky - b\dot{y} + F_0 \cos(\omega t) \\ m\ddot{y} + ky + b\dot{y} &= F_0 \cos(\omega t) \end{aligned}$$

This is a second order, non homogeneous differential equation.

5.0.3 General Solution

$$\text{General Solution} = \text{Particular Integral} + \text{Complimentary Function}$$

5.1 Why study this example?

There are many examples of other forced systems, but many of these are not sinusoidal in their forces. Study the sinusoidal forced oscillations because,

- its the easiest to study
- it does have many important applications
- any other force can be rewritten as a set of interfering sinusoidal waves.

This uses the fact that this equation is linear. So if y_1 and y_2 are solutions of the equation

$$\begin{aligned} m\ddot{y}_1 + ky_1 + b\dot{y}_1 &= A\cos(\omega_1 t) \\ m\ddot{y}_2 + ky_2 + b\dot{y}_2 &= A\sin(\omega_2 t) \end{aligned}$$

then $y = y_1 + y_2$ is the solution of the following equation which is the same operator being driven by the sum of those two forces,

$$m\ddot{y} + ky + b\dot{y} = A\cos(\omega_1 t) + B\sin(\omega_2 t)$$

5.1.1 Solving the Equation

Replace $\cos(\omega t)$ with $e^{i\omega t}$,

$$\Rightarrow m\ddot{y} + b\dot{y} + ky = F_0 e^{i\omega t}$$

Solve these two equations together then take the real part of y as the solution.

Particular integral Try $y = Ae^{i\omega t}$

$$\begin{aligned} A(-m\omega^2 + ib\omega + k)e^{i\omega t} &= F_0 e^{i\omega t} \\ A &= \frac{F_0}{m}\frac{1}{-\omega^2 + \frac{k}{m} + \frac{i\omega b}{m}} \\ A &= \frac{F_0}{m}\frac{1}{(\omega_0^2 - \omega^2) + i\omega\frac{b}{m}} \end{aligned}$$

Rewrite in the form $A = A_0 e^{i\phi}$, so

$$A_0 = \sqrt{c^2 + a^2}, \qquad \tan\phi = \frac{b}{c}$$

Giving

$$A_0 = \frac{F_0}{m}\frac{1}{\left((\omega_0^2 - \omega^2) + \frac{\omega^2 b^2}{m^2}\right)^{\frac{1}{2}}}$$

and

$$\phi = \tan^{-1}\left(\frac{-\omega b}{m\left(\omega_0^2 - \omega^2\right)}\right)$$

Convention dictates this is written as phase angle $\alpha = -\phi$, giving

$$y(t) = A_0 e^{i(\omega t + \phi)} = A_0 e^{i(\omega t - \alpha)}$$

So the real part of the solution

$$y(t) = A_0 \cos(\omega t - \alpha)$$

$-\alpha$ is the phase difference between the applied force and the response of the oscillator to it.

Complimentary function This is the solution of $m\ddot{y} + b\dot{y} + ky = 0$, from unforced example. Since the complimentary function decays with time, it describes a transient response to initial conditions. For large times, $CF \to 0$ and only PI is left. Therefore we will ignore CF and consider the steady state behavior described by PI.

5.1.2 Constant Force $(b = 0)$

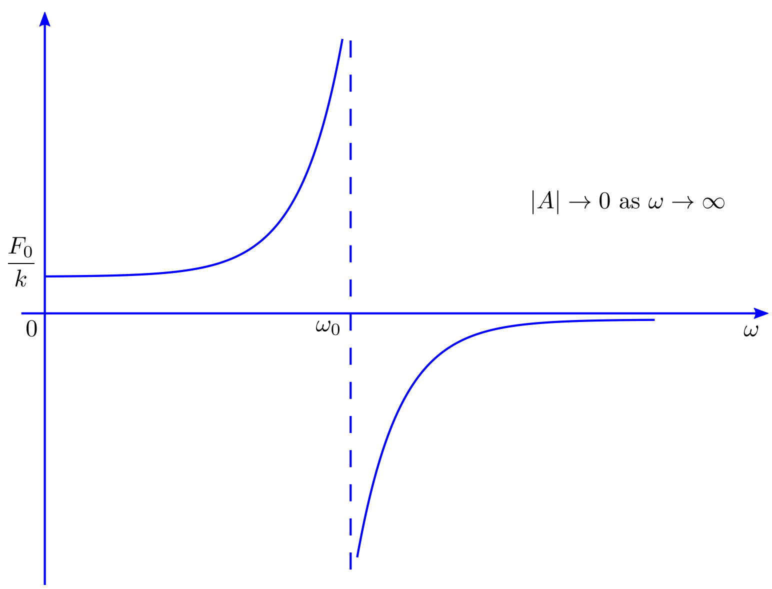
$$A_0 = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k} \qquad \text{(Hoches Law)}$$

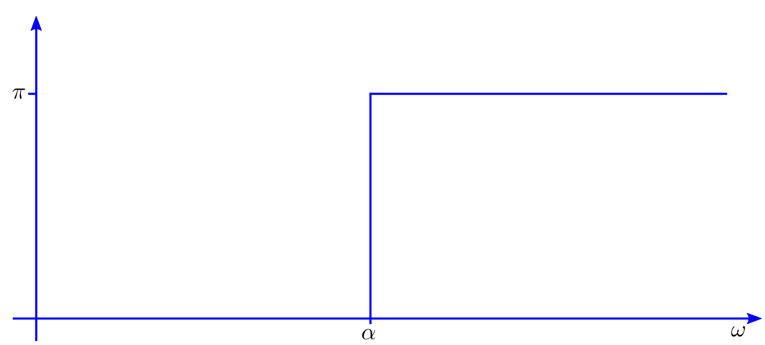
5.1.3 Without Damping $(b = 0)$

$$\begin{aligned} A_0 &= \frac{F_0}{m(\omega_0^2 - \omega^2)} \to \infty \qquad \text{when } \omega = \omega_0 \\ \alpha &= \tan^{-1}(0) = 0, \pm\pi \end{aligned}$$

For

$$\begin{aligned} \omega &< \omega_0,\ A > 0,\ \alpha = 0 \qquad \text{oscillates in phase with the force} \\ \omega &> \omega_0,\ A < 0,\ \alpha = \pi \qquad \text{oscillates in antiphase} \end{aligned}$$





5.2 Forced Oscillations with Damping

Resonant frequency

$$A_0 = \frac{F}{m}\left(\left(\omega_0^2 - \omega^2\right)^2 + \frac{\omega^2 b^2}{m^2}\right)^{-\frac{1}{2}}$$

Maximum when there is a turning point,

$$\begin{aligned} \frac{\partial A}{\partial \omega} &= 0 \\ \Rightarrow \omega_{\text{res}}^2 &= \omega_0^2 - \frac{b^2}{2m^2} \end{aligned}$$

So the resonant frequency of the damped oscillator is $< \omega_0$. Though is the damping is light, the difference is very small.

$$\begin{aligned} \text{Resonant Frequency} &\neq \text{Natural Frequency of damped oscillator} \\ \omega_{\text{res}}^2 &= \omega_0^2 - \frac{b^2}{2m^2} \\ \omega_{\text{damp}}^2 &= \omega_0^2 - \frac{b^2}{4m^2} \end{aligned}$$

The resonant frequency is the lowest. The damping force is proportional to the velocity, so increasing the velocity means that the damping force increases too, so the higher frequencies provide more force.

5.2.1 Amplitude of Resonance

$$A_{\text{res}} = \frac{F_0}{m}\left(\frac{1}{\left(\omega_0^2 - \omega_{\text{res}}^2\right)^2 + \frac{\omega_{\text{res}}^2 b^2}{m^2}}\right)^{\frac{1}{2}}$$

Substitute for ω_{res} from above,

$$\begin{aligned} A_{\text{res}} &= \frac{F_0}{m}\left(\frac{1}{\frac{b^2}{m^2}\left(\omega_0^2 - \frac{b^2}{4m^2}\right)}\right)^{\frac{1}{2}} \\ A_{\text{res}} &= \frac{F_0}{b\omega_{\text{damp}}} \end{aligned}$$

As expected, the heavier damping leads to smaller amplitude at resonance.

5.2.2 Amplitude and Q Factor

When there is light damping,

$$\begin{aligned} Q &\gtrsim \frac{m\omega_{\text{damp}}}{b} \\ b &= \frac{m\omega_{\text{damp}}}{Q} \\ \Rightarrow A_{\text{res}} &= \frac{F_0 Q}{m\omega_{\text{damp}}^2} \\ A_{\text{res}} &\propto Q \end{aligned}$$

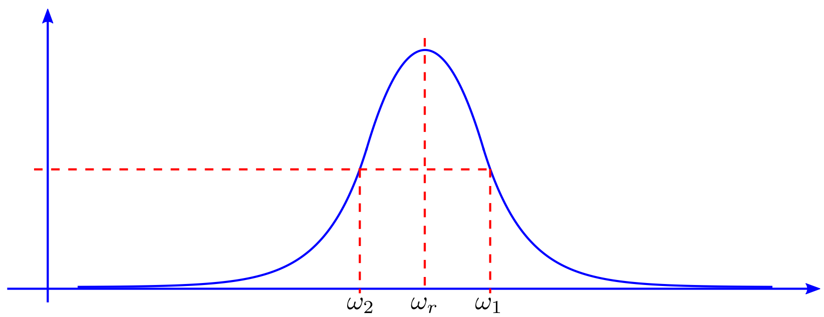
Also, for light damping,

$$\begin{aligned} \omega_{\text{damp}}^2 &\approx \omega_0^2 = \frac{k}{m} \\ \Rightarrow A_{\text{res}} &= \frac{F_0}{k} Q \end{aligned}$$

So, the amplitude at resonance is the Quality factor times the constant displacement caused by a force of the same magnitude with no oscillations.

5.3 Width of Resonance

Define the width as the full width at half maximum of $|A|^2$.



$$A^2 = \frac{F_0^2}{m^2} \frac{1}{\left(\omega_0^2 - \omega^2\right)^2 + \frac{b^2\omega^2}{m^2}}$$

We want ω for which the denominator is doubled, ie has twice the resonant value,

$$\left(\omega_0^2 - \omega^2\right)^2 + \frac{b^2\omega^2}{m^2} = 2 \times \left(\frac{b^2\omega_0^2}{m^2} - \frac{b^2}{4m^2}\right)$$

Consider light damping, so $\frac{b^2}{4m^2} \ll \omega_0^2$

$$\begin{aligned} \left(\omega_0^2 - \omega^2\right)^2 + \frac{b^2\omega^2}{m^2} &\approx \frac{2b^2\omega_0^2}{m^2} \\ \left(\omega_0^2 - \omega^2\right)^2 &= \frac{b^2\omega^2}{m^2} + \frac{b^2}{m^2}\left(\omega_0^2 - \omega^2\right) \end{aligned}$$

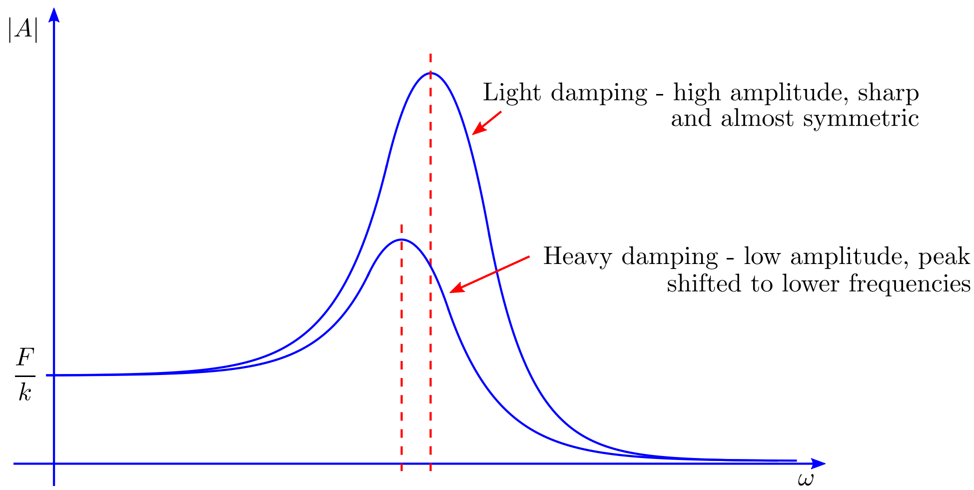
Near resonance $\omega_0^2 - \omega^2 \ll \omega_0^2$

$$\begin{aligned} \left(\omega_0^2 - \omega^2\right)^2 &\approx \frac{b^2\omega_0^2}{m^2} \\ \omega_0^2 - \omega^2 &= \pm\frac{b\omega_0}{m} \\ \left(\omega_0 - \omega\right)\left(\omega_0 + \omega\right) &= \pm\frac{b\omega_0}{m} \end{aligned}$$

As $\omega_0 \approx \omega \to \omega_0 + \omega = 2\omega_0$

$$\begin{aligned} 2\omega_0\left(\omega_0 - \omega\right) &\approx \pm\frac{b\omega_0}{m} \\ \frac{\omega_0 - \omega}{\omega_0} &= \pm\frac{b}{2m\omega} = \pm\frac{1}{2Q} \end{aligned}$$

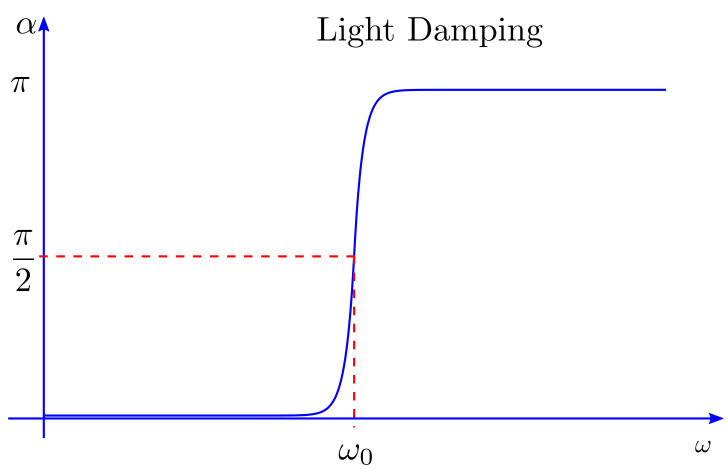
This gives the half width, so the full width at half maximum (FWHM) frequency is given by $\frac{1}{Q}$. So light damping, ie a large Q value, gives a narrow amplitude with a large resonance. As the damping gets heavier, the width and the amplitude decrease.

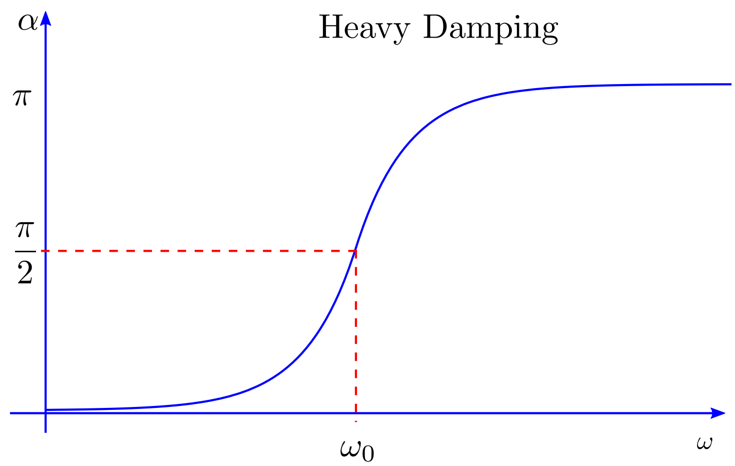


5.4 Phase of Oscillator

$$\alpha = \tan^{-1}\left(\frac{b\omega}{m\left(\omega_0^2 - \omega^2\right)}\right)$$

The tan $\to \infty$ when $\omega \to \omega_0$ so, $\alpha\left(\omega_0\right) \to \frac{\pi}{2}$. So the phase of the oscillator does not depend of the damping. We still have the phase switch, but it is softened by the damping.





Part III

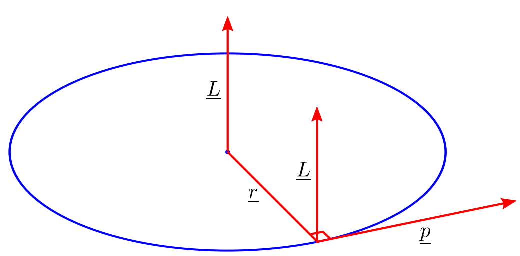
Rotational Motion

6 Angular Momentum

The angular momentum of an object, relative to the origin of co-ordinates is,

$$\underline{L} = \underline{r} \times \underline{p}$$

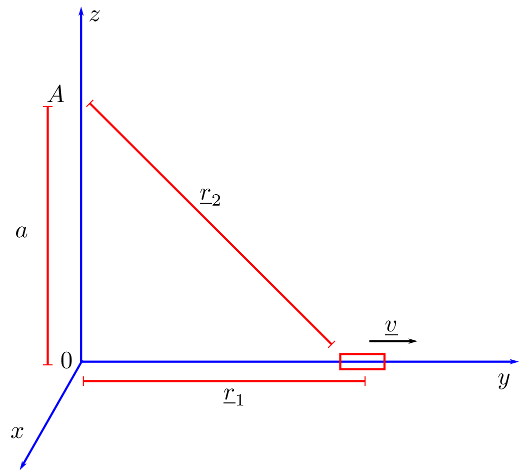
where $\underline{r}$ is the position vector from the origin and $\underline{p}$ is the momentum vector. This has the dimensions $[M][L^2][T^{-1}]$, but has no named standard unit. $\underline{L}$ is perpendicular to both $\underline{r}$ and $\underline{p}$, therefore, $\underline{L}$ will be normal to the plane of rotation of the object.



The direction is given by the right hand rule, there is no position in space linked with the angular momentum, just a magnitude and direction. $\underline{L}$ depends on where $\underline{r}$ and $\underline{p}$ are measured from.

<u>Ex</u>

Block standing along the axis



Relative to the origin, O,

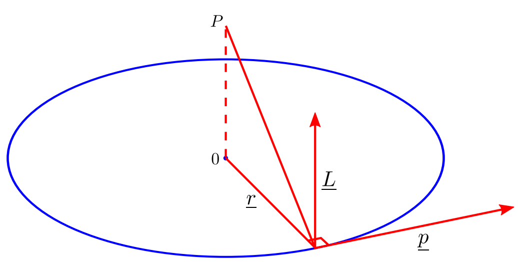
$$\begin{aligned}\underline{L} &= \underline{r}_1 \times \underline{p} \\ &= \underline{y} \times m\underline{v} = \underline{0}\end{aligned}$$

Relative to the point A,

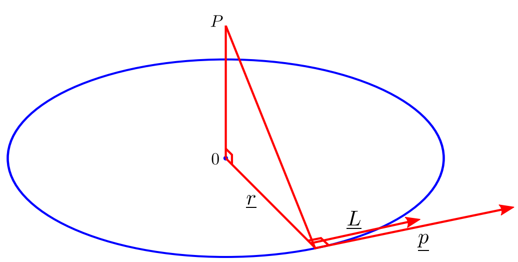
$$\begin{aligned}\underline{L} &= \underline{r}_1 \times \underline{p} \\ &= \begin{pmatrix} 0 \\ y \\ -a \end{pmatrix} \times \begin{pmatrix} 0 \\ mv \\ 0 \end{pmatrix} \\ &= mva\underline{\hat{x}}\end{aligned}$$

Note that if $\underline{v}$ is constant, then so is $\underline{L}$, $\underline{L}$ depends on the distance of closest approach, but not on the instantaneous position.

6.0.1 Conical Pendulum



Relative to O
$\underline{L}$ is parallel to the axis



Relative to P
$\underline{L}$ is perpendicular to the string
and rotates with the pendulum

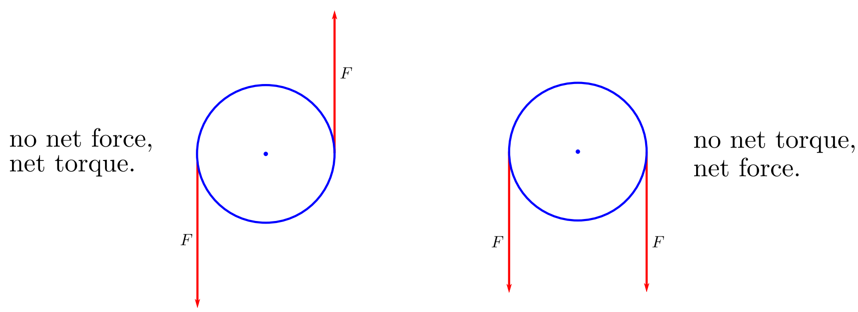
These are the same situation so the angular momentum should be able to be used to describe the motion, irrespective of the chosen co-ordinate system.

6.1 Forces and Rotation: Torque

The effect of forces on rotation is described by its torque,

$$\tau = \underline{r} \times \underline{F} = \underline{r} \times \frac{\mathrm{d}p}{\mathrm{d}t}$$

So torques are perpendicular to the forces producing it. Like the angular momentum, the magnitude of the torque depends on where it is measured from, the co-ordinate system used. So you can have a torque with no net force, and a force with no net torque.



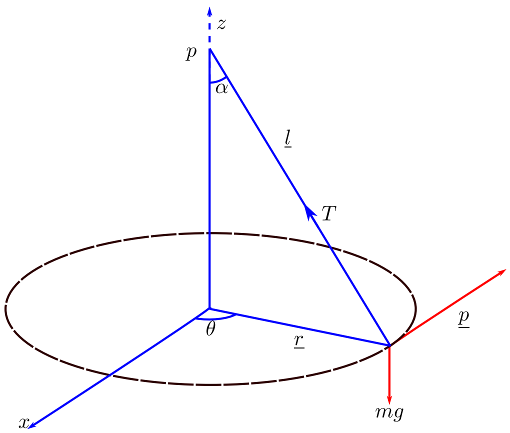
Consider the change in angular momentum,

$$\begin{aligned}
\frac{\mathrm{d}\underline{L}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\underline{r} \times \underline{p}) \\
&= \frac{\mathrm{d}\underline{r}}{\mathrm{d}t} \times \underline{p} + \underline{r} \times \frac{\mathrm{d}\underline{p}}{\mathrm{d}t} \\
&= \underline{v} \times m\underline{v} + \underline{r} \times \frac{\mathrm{d}\underline{p}}{\mathrm{d}t} \\
&= \underline{r} \times \underline{F} = \tau \\
\Rightarrow \tau &= \frac{\mathrm{d}\underline{L}}{\mathrm{d}t}
\end{aligned}$$

So the torque is the change in angular momentum per unit time.

<u>Ex</u>

Conical pendulum



Tension must balance the weight,

$$\begin{aligned} T\cos(\alpha) - mg &= 0 \\ T &= \frac{mg}{\cos(\alpha)} \end{aligned}$$

$\therefore$ the net force=horizontal component of the tension,

$$\underline{F}_r = -T\sin(\alpha)$$

1. Measure everything relative to the center of rotation, O,

$$\underline{\tau} = \underline{r} \times \underline{F} = \underline{\tau} \times \underline{F}_r = 0$$
$$\frac{\mathrm{d}\underline{L}}{\mathrm{d}t} = 0$$

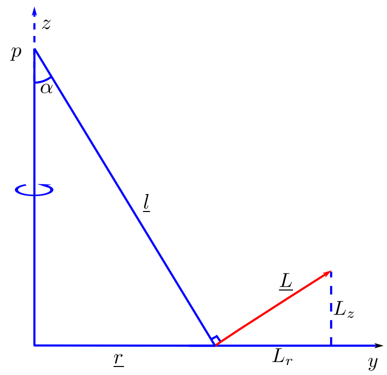
   therefore, in this case, $\underline{L}$ is constant

$$\begin{aligned} \underline{L} &= \underline{r}\underline{p} \\ &= \underline{l}\sin(\alpha)p\hat{\underline{z}} \end{aligned}$$

2. Measure relative to the pivot, P,

$$|\underline{L}| = |\underline{l} \times \underline{p}| = lp = lmr\omega = \text{constant}$$

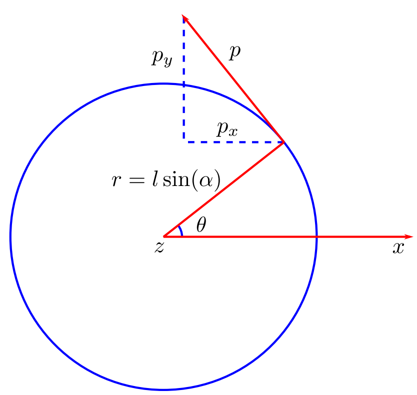
   Direction of $\underline{L}$,



Vertical component $L_z = L\sin(\alpha) = \text{constant}$

Radial component $L_r = L\cos(\alpha)$, direction changes with r.

Components of $\underline{L}$,



$$\begin{array}{ll} x = l\sin(\alpha)\cos(\theta) & p_x = -p\sin(\theta) \\ y = l\sin(\alpha)\sin(\theta) & p_y = p\cos(\theta) \\ z = -l\cos(\alpha) & p_z = 0 \end{array}$$

$$\therefore \underline{L} = \underline{r} \times \underline{p} = pl \begin{pmatrix} \cos(\alpha)\cos(\theta) \\ \cos(\alpha)\sin(\theta) \\ \sin(\alpha) \end{pmatrix}$$

Pendulum rotates with a constant angular velocity,

$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \omega \\ \frac{\mathrm{d}\underline{L}}{\mathrm{d}t} &= \frac{\mathrm{d}\underline{L}}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}t} \\ &= pl \begin{pmatrix} -\cos(\alpha)\sin(\theta) \\ \cos(\alpha)\cos(\theta) \\ 0 \end{pmatrix} \end{aligned}$$

For circular motion,

$$\begin{aligned} |\underline{F}| &= mr\omega^2 \\ &= p\omega \end{aligned}$$

This acts towards the center of rotation,

$$\begin{aligned} \underline{F} &= p\omega \begin{pmatrix} -\cos\theta \\ -\sin\theta \\ 0 \end{pmatrix} \\ \Rightarrow \underline{\tau} &= \underline{r} \times \underline{F} \\ &= p\omega l \begin{pmatrix} -\cos\alpha\sin\theta \\ \cos\alpha\cos\theta \\ 0 \end{pmatrix} \end{aligned}$$

Hence, demonstrated again, in this case, $\tau = \frac{\mathrm{d}\underline{L}}{\mathrm{d}t}$

$$\textsc{Torque} \quad = \quad \textsc{Rate of change of angular momentum}$$

Relative to the pivot point, P,

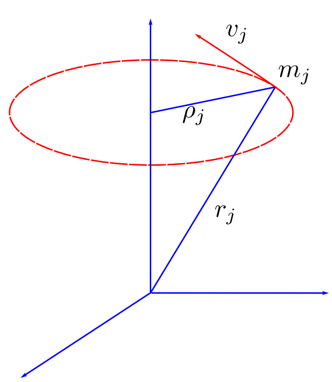
- angular momentum precesses with the pendulum
- torque from the net force causes the precession.

Although $\underline{L}$ and $\underline{\tau}$ depend on where they are measured from, $\tau = \frac{\mathrm{d}\underline{L}}{\mathrm{d}t}$ is universal and they both describe the same motion of the pendulum. So always choose the co-ordinate system to make the problem as simple as possible.

7 Rigid Body Rotation

7.1 Rotation About a Fixed Axis

Consider rotation about the z-axis.



For any element, j, of the body,

$$\begin{aligned} \underline{L}_z &= \left(\underline{r}_j \times \underline{p}_j\right) \hat{\underline{z}} \\ &= \rho_j m_j v_j \end{aligned}$$

Using $v_j = \rho_j \omega$,

$$\underline{L}_z = m_j \rho_j^2 \omega$$

Since this is a rigid body that is rotating,

- the distance, ρ, of any element from the axis of rotation remains constant
- the angular velocity of rotation, ω, is the same for all elements.

Therefore, for the whole body,

$$\underline{L}_z = \left(\sum_{j=1}^{n} m_j \rho_j \right) \omega$$

Let,

$$I = \sum_{j=1}^{n} m_j \rho_j^2$$

so we can then rewrite

$$\underline{L}_z = I\omega$$

where I is the moment of inertia of the body about the z-axis (I is to rotation what m is to translation).

$$p = mv \qquad \rightarrow \qquad L = I\omega$$

I depends on the distribution of the mass in the body and where the axis of rotation is. For a continuous body, the individual elements become infinitely small and the equation becomes,

$$\begin{aligned} I &= \int \rho^2 \mathrm{d}m \\ &= \int (y^2 + x^2) \mathrm{d}m \end{aligned}$$

or,

$$I = \int \delta(r)(x^2 + y^2)\, \mathrm{d}x \mathrm{d}y \mathrm{d}z$$

where $\delta(r)$ is the density at the point $\underline{r}$. (For the rotation about the z-axis). This is then simplest if the object is symmetric about the axis of rotation.

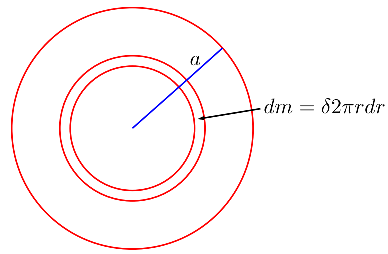
<u>Ex</u>

A ring of mass m, radius a is rotating about the axis through the center perpendicular to the plane of the ring. All of the mass is at a distance a from the axis of rotation,

$$I = a^2 m$$

<u>Ex</u>

Consider a uniform disk. Subdivide into multiple concentric rings.



$$
\begin{aligned}
I &= \int r^2 \mathrm{d}m \\
&= \int_0^a \delta 2 m r^3 \mathrm{d}r \\
&= \frac{\delta\pi}{2} a^4
\end{aligned}
$$

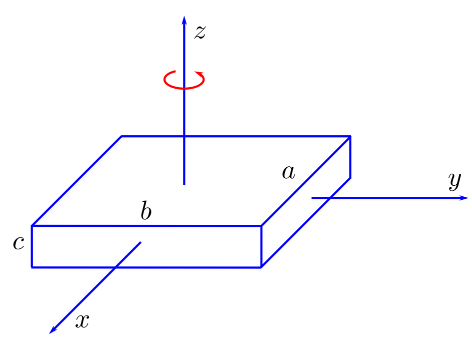
Since $m = \delta\pi a^2$,

$$\Rightarrow I = \frac{ma^2}{2}$$

So the value of I is half of that for the uniform ring.

Ex

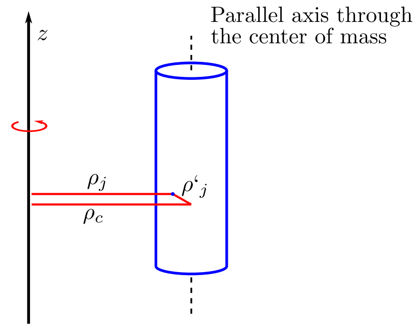
Consider a cuboid with the axis of rotation through the center.



$$
\begin{aligned}
M &= \delta abc \\
\mathrm{d}M &= \delta \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
\Rightarrow I &= \delta \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} \left(x^2 + y^2\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \delta abc \left(\frac{a^2}{12} + \frac{b^2}{12}\right) \\
&= \frac{M}{2}\left(a^2 + b^2\right)
\end{aligned}
$$

7.2 Parallel Axis Theorem

This is for use when the axis of rotation does not pass through the center of mass.



Take an element j of the body. The vector from the axis to j is $\rho_j = \rho_c + \rho'_j$. Then

$$\begin{aligned} I &= \sum_j m_j \rho_j^2 \\ &= \sum_j m_j \left(\underline{\rho}_c + \underline{\rho}'_j\right)^2 \\ &= \sum_j m_j \left(|\underline{\rho}_c|^2 + |\underline{\rho}'_j|^2 + 2\underline{\rho}_c \underline{\rho}'_j\right) \\ &= M\rho_c^2 + \sum_j m_j \underline{\rho}'^2_j + 2\sum_j m_j \underline{\rho}_c \underline{\rho}'_j \end{aligned}$$

- $M\rho_c^2$ is the moment of inertia of mass M at a perpendicular distance ρ_c
- $\sum_j m_j \rho'^2_j$ is the moment of inertia of the body about the axis through the center of mass.

Using $\underline{\rho}'_j = -\underline{\rho}_c + \underline{\rho}_j$,

$$\begin{aligned} \sum_j m_j \rho_c \rho'_j &= -\sum_j m_j \rho_c^2 + \sum_j m_j \rho_c \rho_j \\ &= -M\rho_c^2 + \rho_c \sum_j m_j \rho_j \\ &= -M\rho_c^2 + \rho_c (M\rho_c) \\ &= 0 \end{aligned}$$

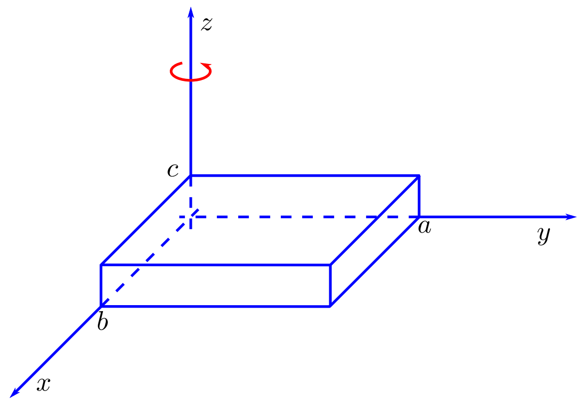
Therefore,

$$I = M\rho_c^2 + I_0$$

where I_0 is the moment of inertia for rotation about the parallel axis through the center of mass.

<u>Ex</u>

Rotate a cuboid about one edge.



$$
\begin{aligned}
I &= \int x^2 + y^2 \mathrm{d}m \\
&= \rho \int_0^b \int_0^c x^2 + y^2 \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \frac{M}{3}(a^2 + b^2)
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
I_0 \text{ about CM} &= \frac{M}{12}(a^2 + b^2) \\
\text{CM} &= \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right) \\
\Rightarrow \text{Total mass} \times \rho_c^2 &= M\left(\frac{a^2}{4} + \frac{b^2}{4}\right) \\
\Rightarrow I &= \frac{M}{3}(a^2 + b^2)
\end{aligned}
$$

7.3 Perpendicular Axis Theorem

Consider a rigid body rotating about a fixed axis (z), we write,

$$L = I\omega$$

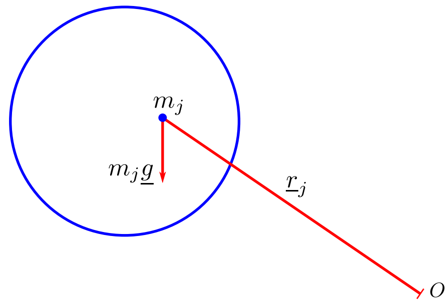
If a torque is applied parallel to the axis,

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} I\omega = I\frac{\mathrm{d}\omega}{\mathrm{d}t} = I\alpha$$

where $\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t}$ which is the angular acceleration. Consider the kinetic energy of the rotating object,

$$
\begin{aligned}
E_k &= \sum_j \frac{1}{2} m_j v_j^2 \\
&= \frac{1}{2} \sum_j m_j \rho_j^2 \omega^2 \qquad \text{(using } v = \rho\text{)} \\
T = E_k &= \frac{1}{2} I\omega^2
\end{aligned}
$$

7.4 Torques Due to Gravity



Torque on element j,

$$\tau_j = \underline{r}_j \times m_j \underline{g}$$

Therefore, on the whole body,

$$\begin{aligned}\tau &= \sum_j \tau_j \\ &= \sum_j r_j \times m_j \underline{g} \\ &= \left(\sum_j m_j r_j\right) \times \underline{g} \\ &= M\underline{R} \times \underline{g}\end{aligned}$$

where $\underline{R}$ is the vector to the center of mass.

$$\therefore \tau = \underline{R} \times M\underline{g}$$

7.5 Simple Pendulum

Consider the torque about the pivot P,

$$\begin{aligned}\tau &= \underline{l} \times \underline{F} \\ &= \underline{l} \times (\underline{g}m + I) \\ &= \underline{l} \times m\underline{g} \\ \tau &= -lmg \sin\theta\end{aligned}$$

So,

$$\begin{aligned}\underline{L} &= \underline{r} \times \underline{p} \\ &= ml^2\dot{\theta} = I\dot{\theta} = I\omega\end{aligned}$$

Since $\tau = \frac{\mathrm{d}\underline{L}}{\mathrm{d}t}$,

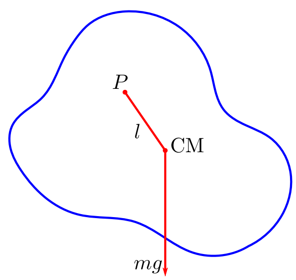
$$\begin{aligned}-lmg\sin\theta &= \frac{\mathrm{d}}{\mathrm{d}t}(ml^2\dot{\theta}) \\ &= ml^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\end{aligned}$$

Using the small angle approximation, $\theta \ll 1$, $\sin\theta \approx \theta$,

$$\begin{aligned} \Rightarrow l\ddot{\theta} + g\theta &= 0 \\ \theta(t) &= a\cos(\omega t + \phi) \end{aligned}$$

where $\omega^2 = \frac{g}{m}$.

8 General Rigid Body Pendulum



The only net torque is that due to gravity,

$$\tau = -mgl\sin\theta = \frac{\mathrm{d}\underline{L}}{\mathrm{d}t} = I\ddot{\theta}$$

From the parallel axis theorem,

$$I = I_0 + Ml^2$$

where I_0 is the moment of inertia about the center of mass, and I is for the whole body. It is convenient to write,

$$I_0 = M\kappa^2$$

where $\kappa = \sqrt{\frac{I_0}{M}}$ which is the "radius of gyration". So,

$$\begin{aligned} I &= I_0 + ml^2 \\ &= M(\kappa^2 + l^2) \end{aligned}$$

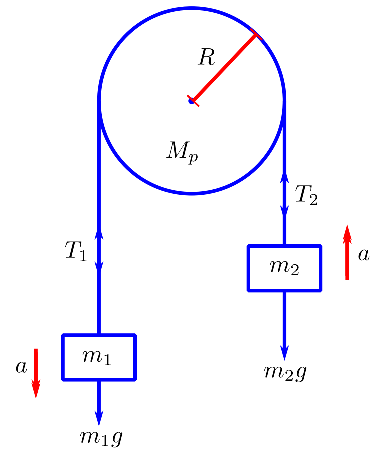
Putting this back into the equation of motion,

$$\begin{aligned} -Mgl\sin\theta &= M(\kappa^2 + l^2)\ddot{\theta} \\ -gl\sin\theta &= (\kappa^2 + l^2)\ddot{\theta} \\ \ddot{\theta} + \frac{gl}{\kappa^2 + l^2}\theta &= 0 \\ \omega^2 &= \frac{gl}{\kappa^2 + l^2} \end{aligned}$$

This is the same form as the simple pendulum with κ describing the effect of distributing the mass rather than concentrated at a point. For a simple pendulum, $\kappa = 0$, so $\omega^2 = \frac{g}{l}$ as before.

8.1 Atwood's Machine

This machine was designed to show the acceleration due to gravity down so that, knowing how much it had slowed, the value of g could be measured without having perfectly accurate timing.



- Massless pulley

$$
\begin{aligned}
F &= ma \\
(m_1 - m_2)g &= a(m_a + m_2) \\
a &= \frac{m_1 - m_2}{m_1 + m_2} g
\end{aligned}
$$

  This relies on the unphysical assumption that the pulley is massless. To be accurate, the mass of the pulley needs to be taken into account.

- Massive pulley

  - Linear motion of masses

$$
\begin{aligned}
F &= ma \\
m_1 g - T_1 &= m_1 a \qquad (3) \\
m_2 g - T_2 &= m_2 a \qquad (4)
\end{aligned}
$$

  - Rotation of the pulley

$$
\begin{aligned}
\tau &= \frac{\mathrm{d}\underline{L}}{\mathrm{d}t} \\
T_1 R - T_2 R &= \tau \\
\underline{L} &= I\alpha
\end{aligned}
$$

    Assume the rope does not slip

$$
\begin{aligned}
v &= R\omega \\
a &= R\alpha \qquad (5)
\end{aligned}
$$

    Eliminate the unwanted variables, T,

$$
\begin{aligned}
(3) + (4) \qquad (m_1 - m_2)g - (T_1 - T_2) &= (m_1 + m_2)a \\
\text{sub}(5) \qquad (m_1 - m_2)g - \frac{I\alpha}{R} &= (m_1 + m_2)a \\
(m_1 - m_2)g - \frac{Ia}{R^2} &= (m_1 + m_2)a \\
a &= \frac{(m_1 - m_2)}{(m_1 + m_2 + \frac{I}{R^2})} g
\end{aligned}
$$

Ex

If the pulley is a uniform disk,

$$
\begin{aligned}
I &= \frac{1}{2} M_p R^2 \\
a &= g \frac{(M_1 - M_2)}{\left(M_1 + M_2 + \frac{M_p}{2}\right)}
\end{aligned}
$$

9 Motion Including Both Rotation and Translation

Consider an object that is both moving with linear translation and is rotating.

- Angular momentum

$$
\underline{L} = \sum_j r_j \times m_j \underline{r}_j
$$

if R is the center of mass,

$$
\underline{R} = \frac{\sum_j m_j r_j}{\sum_j m_j}
$$

r_j' is the position of the element j relative to the center of mass. Then

$$
\begin{aligned}
\underline{r}_j &= \underline{R} + \underline{r}_j' \\
\therefore \underline{L} &= \sum_j \left(\underline{R} + \underline{r}_j'\right) \times m_j \left(\dot{\underline{R}} + \dot{\underline{r}_j'}\right) \\
&= \underline{R} \times \sum_j m_j \dot{\underline{R}} + \sum_j m_j r_j \times \dot{\underline{R}} + \underline{R} \times \sum_j m_j \dot{\underline{r}}_j + \sum_j m_j \underline{r}_j' \times \dot{\underline{r}}_j'
\end{aligned}
$$

Using

$$
\begin{aligned}
\sum_j m_j \underline{r}_j' &= \sum_j m_j (\underline{r}_j - \underline{R}) \\
&= \sum_j m_j \underline{r}_j - \underline{R} \sum_j m_j \\
&= M\underline{R} - \underline{R}M = 0
\end{aligned}
$$

Also

$$
\begin{aligned}
\sum_j m_j \dot{\underline{r}}_j' &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_j m_j \underline{r}_j' = 0 \\
\Rightarrow \underline{L} &= \underbrace{R \times M \dot{\underline{R}}}_{a} + \underbrace{\sum_j \underline{r}_j' \times m \dot{\underline{r}}_j'}_{b}
\end{aligned}
$$

Here, a is the angular momentum due to the center of mass, and b is the angular momentum due to the rotation of the object about the center of mass. If the axis of rotation maintains a steady direction, remains parallel to the z-axis, e.g. a rolling wheel, then,

$$
\begin{aligned}
L_z &= (R \times M\underline{v})_z + \left(\sum_j m_j \underline{r}_j' \times \dot{\underline{r}}_j' \right)_z \\
&= (R \times M\underline{v})_z + I_0 \omega
\end{aligned}
$$

- Similarly, we can factor the torque,

$$\tau_z = \left(\underline{R} \times \underline{F}\right)_z + I_0\alpha$$

where $\left(\underline{R} \times \underline{F}\right)_z$ is the torque on the center of mass due to an extended force and $I_0\alpha$ is the torque about the center of mass.

- And the kinetic energy in the same way,

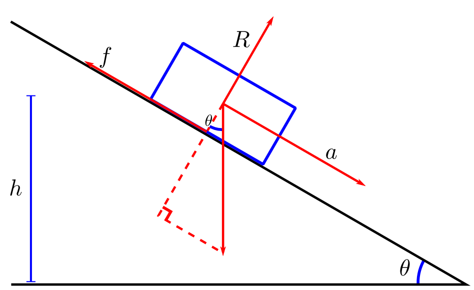
$$\begin{aligned} T = E_k &= \frac{1}{2}\sum_j m_j \left(\underline{\dot{r}}_j\right)^2 \\ &= \frac{1}{2}\sum_j m_j \left(\underline{\dot{r}}'_j + \underline{\dot{R}}\right)^2 \qquad \underline{\dot{R}} = \underline{v} \\ &= \frac{1}{2}\sum_j m_j \underline{\dot{r}}'^2_j + \left(\sum_j m_j \underline{\dot{r}}'_j\right)\underline{v} + \frac{1}{2}\sum_j m_j v^2 \\ \therefore T &= \frac{1}{2} I_0\omega^2 + \frac{1}{2} M v^2 \end{aligned}$$

So the kinetic energy is the component from the regular motion of the body added to the component provided by the rotation of the body about the center of mass.

9.1 Sliding vs. Rolling

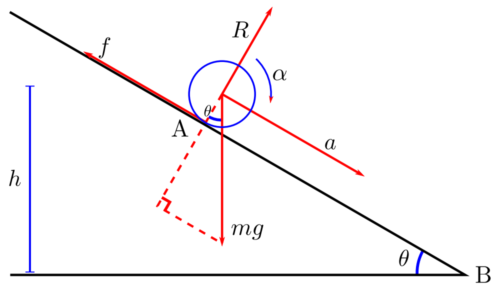
9.1.1 Using Equations of Motion

- Sliding without rolling



$$\begin{aligned} \text{Friction, } f &= \mu \times \text{normal reaction force} \\ f &= \mu m g \cos\theta \\ \therefore F = ma &= mg\sin\theta - f \\ a &= g\left(\sin\theta - \mu\cos\theta\right) \end{aligned}$$

- Rolling without sliding



$$
\begin{aligned}
F = ma &= mg\sin\theta - f \\
\tau &= I_0\alpha = Rf \\
\Rightarrow f &= \frac{I_0}{R}\alpha \\
\Rightarrow ma &= mg\sin\theta - \frac{I_0}{R}\alpha
\end{aligned}
$$

Using $v = R\omega$, $a = R\alpha$

$$mg\sin\theta - \frac{I_0}{R^2}a = ma$$

Using $I_0 = m\kappa^2$

$$
\begin{aligned}
\Rightarrow m\left(g\sin\theta - \frac{\kappa^2}{R^2}a\right) &= ma \\
a &= \frac{g\sin\theta}{\left(1+\frac{\kappa^2}{R^2}\right)}
\end{aligned}
$$

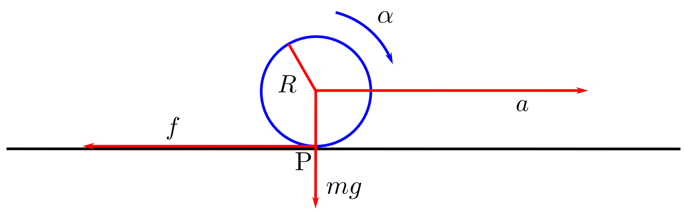
9.1.2 Using Energy

	Sliding (no friction)	Rolling without sliding
At A :	$E_k + E_p = 0 + mgh$	$E_k + E_p = 0 + mgh$
At B :	$E_k + E_P = \frac{1}{2}mv^2$	$E_k + E_p = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$
	$\Rightarrow mgh = \frac{1}{2}mv^2$	$\Rightarrow mgh = \frac{1}{2}mv^2 = \frac{1}{2}T\omega^2$
	$v^2 = 2gh$	$mgh = \frac{1}{2}mv^2 + \frac{1}{2}m\kappa^2\frac{v^2}{R^2}$
		$v^2 = \frac{2gh}{\left(1+\frac{\kappa^2}{R^2}\right)}$

More complicated example is when an object is sliding and rolling.

Ex

Consider a bowling ball projected horizontally along the floor.



Initially the ball slides along the floor, and eventually stops sliding and rolls. How to describe this motion?

The forces present that must be considered are the weight and the reaction from the floor which must cancel each other, and friction from the floor. This means that there is only one resultant force acting on the ball. The ball is initially both rolling and sliding. The friction produces linear deceleration and produces an angular acceleration.

Consider the point of contact between the ball and the floor,

- Relative to the center of mass, P has velocity

$$v_p = -R\omega$$

- Relative to the floor, it has velocity at any given moment in time of

$$\begin{aligned} v_p' &= v_c + v_p \\ &= v_c - R\omega \end{aligned}$$

When $v_c = R\omega$, $v_p' = 0$, so the point of contact of the ball with the floor is no longer moving with respect to the floor. So if the point of contact is stationary, it cannot be sliding, then the friction must be zero. At this point, the ball will switch motion with constant velocity.

10 Impulsive Forces and Torques

An impulse is a force that acts over a very short period of time. We can work out the effect by using,

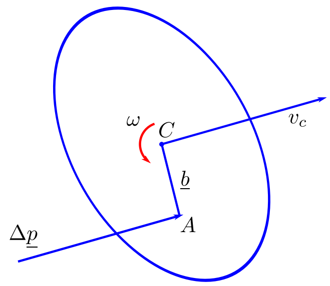
$$\Delta \underline{p} = \int \underline{F} \mathrm{d}t = \underline{F} \Delta t$$

This well also produce an abrupt change in angular momentum,

$$\begin{aligned} \Delta \underline{L} &= \int \underline{\tau} \mathrm{d}t = \int \underline{r} \times \underline{F} \mathrm{d}t \\ &= \underline{r} \times \int \underline{F} \mathrm{d}t = \underline{r} \times \Delta \underline{p} \end{aligned}$$

10.1 Free Body

So if a rigid body is subject to an impulse, at the point A,



Impulse is applied perpendicular to the point CA (C is the center of mass). The center of mass moves with velocity v_c, given by,

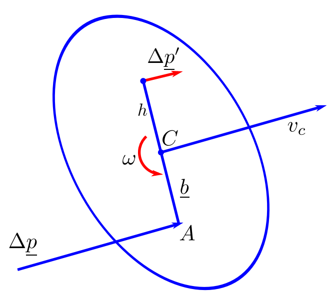
$$v_c = \frac{\Delta \underline{p}}{m}$$

The body then rotates about the center of mass with angular velocity ω,

$$\begin{aligned} \Delta\tau &= b\Delta\underline{p} \\ &= I_0\omega \\ &= m\kappa^2\omega \\ \Rightarrow \omega &= \frac{\Delta\tau}{I_0} \\ &= \frac{b\Delta\underline{p}}{m\kappa^2} \end{aligned}$$

10.2 Behavior When Object Has a Pivot

Consider point O on the opposite side of the center of mass.



If O is free to move, its velocity immediately after the impulse is applied is given by $v_0 = v_c - \omega h$. Suppose instead that O is a pivot, so is fixed in space. Then $v_0 = 0$ and so

$$\begin{aligned} 0 &= v_c - \omega h \\ v_c &= \omega h \end{aligned}$$

(Note that this ω is not the same as for the free object). To keep O fixed in space, there will be an impulse reaction $\Delta\underline{p}'$ at the point O acting on the pivot.

10.2.1 Removing Impulse Reaction

The system can be set up in such a way that the impulse $\Delta \underline{p}'$ is zero. Equations of motion;

$$\begin{aligned} \Delta p + \Delta p' &= m v_c \qquad && \Rightarrow \text{Linear} \\ b\Delta p - h\Delta p' &= m\kappa^2 \omega \qquad && \Rightarrow \text{Rotational} \end{aligned}$$

Using $v_c = \omega h$,

$$b\Delta p - h\Delta p' = m\kappa^2 \frac{v_c}{h} \tag{6}$$

From (6),

$$v_c = \frac{\Delta p + \Delta p'}{m}$$

Sub into (7),

$$\Rightarrow \Delta p' = \left(\frac{bh - \kappa^2}{\kappa^2 + h^2} \right) \Delta p$$

So if $bh = \kappa^2 = \frac{I_0}{m}$, ie $b = \frac{\kappa^2}{h}$, then the reaction force is zero, $\Delta p' = 0$. The point at which the reaction force is zero is called the "center of percussion" or "sweet spot".

10.2.2 Distance of Sweet Spot from Pivot

$$\begin{aligned} l &= b + h \\ &= \frac{h^2 + \kappa^2}{h} \end{aligned}$$

This has been seen before when a pendulum, when the distance from the center of mass to the pivot was also h, has frequency,

$$\begin{aligned} \omega^2 &= \frac{gh}{\kappa^2 + h^2} \\ \omega &= \sqrt{\frac{g}{l}} \end{aligned}$$

So measuring ω gives the distance to the sweet spot l.

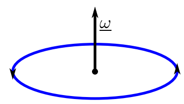
11 Angular Velocity Vector

This is used when the axis of rotation is not fixed. Define the angular velocity vector,

$$\underline{\omega} = \left(\frac{\mathrm{d}\theta_x}{\mathrm{d}t}, \frac{\mathrm{d}\theta_y}{\mathrm{d}t}, \frac{\mathrm{d}\theta_z}{\mathrm{d}t} \right)$$

where $\frac{\mathrm{d}\theta_x}{\mathrm{d}t}$ is the rate of rotation about the x-axis. So for an object rotating in a plane as the examples we have so far considered, $\underline{\omega}$ is parallel to the axis of rotation and perpendicular to the plane of rotation. So for the simplest case,

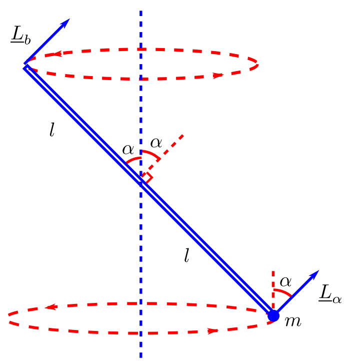
$$\underline{L} = I\underline{\omega}$$



But this is *not* true in general.

11.0.3 Rotating Skew Rod

Two masses of mass m, connected by a light rigid rod of length $2l$ is rotating about the center of mass at an angle to the axis of rotation α.



$$\begin{aligned} \underline{L} &= \underline{r_a} \times \underline{p_a} + \underline{r_b} \times \underline{p_b} \\ &= \underline{L_a} + \underline{L_b} \end{aligned}$$

Relative to the center of mass,

$$\begin{aligned} |\underline{L_a}| &= |\underline{L_b}| = ml^2\omega\cos\alpha \\ |\underline{L}| &= 2ml^2\omega\cos\alpha \end{aligned}$$

as $\underline{L_a}$ and $\underline{L_b}$ are parallel. $\underline{L}$ turns with the rod, and is perpendicular to the rod and at an angle α to th axis of rotation. But the angular velocity component has only one component, it is parallel to the axis of rotation. So the angular momentum vector is not parallel to the angular velocity.

So in general, how are the angular momentum and the angular velocity vector related?

As before, consider a rotating rigid body,

$$\underline{L} = \sum_j \underline{r}_j \times m_j \underline{\dot{r}}_j$$

as this is a rigid body,

$$\underline{\dot{r}}_j = \underline{\omega} \times \underline{r}_j$$

so,

$$\underline{L} = \sum_j \underline{r}_j \times m_j \left(\underline{\omega} \times \underline{r}_j\right)$$

and rearranging gives

$$\underline{L} = \underline{\underline{I}} \times \underline{\omega}$$

As before, consider a rotating rigid body,

$$\underline{L} = \sum_j \underline{r}_j \times m_j \underline{\dot{r}}_j$$

As this is a rigid body,

$$\underline{\dot{r}}_j = \underline{\omega} \times \underline{r}_j$$

So

$$\underline{L} = \sum_j \underline{r}_j \times m_j \left(\underline{\omega} \times \underline{r}_j \right)$$

Rearranging gives

$$\underline{L} = \underline{\underline{I}} \times \underline{\omega}$$

where $\underline{L}$ and $\underline{\omega}$ are vectors and $\underline{\underline{I}}$ is a tensor, meaning this becomes,

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Diagonal elements of the tensor have the form,

$$\begin{aligned} I_{xx} &= \sum_j m_j \left(y_j^2 + z_J^2 \right) \\ &= \text{moments of inertia} \end{aligned}$$

Non-diagonal elements have the form

$$\begin{aligned} I_{xy} &= -\sum_j m_j x_j y_j \\ &= -\sum_j m_j y_j x_j \\ &= I_{yx} \end{aligned}$$

These are the moments of inertia.

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

This can be simplified.

11.1 Principle Axis

For any object you can find a set of mutually perpendicular axes for which this inertia takes a particular simple form,

$$\underline{\underline{I}} = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

So the tensor only has non-zero elements on the diagonal of the matrix. So

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_{11} & \omega_1 \\ I_{22} & \omega_2 \\ I_{33} & \omega_3 \end{pmatrix}$$

These are the principle axes of the object. So if the axis of rotation is one of the principle axes,$\underline{L} = I\underline{\omega}$ as before. For a symmetric body, the principle axes are the symmetry axes if that body. We can also write the rotational kinetic energy in terms of the principle axes,

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2}\underline{\omega_0 L} \\ &= \frac{1}{2} I_{11}\omega_1^2 + \frac{1}{2} I_{22}\omega_2^2 + \frac{1}{2} I_{33}\omega_3^2 \end{aligned}$$

12 Conservation of Angular Momentum

$$\tau = \frac{\mathrm{d}\underline{L}}{\mathrm{d}t}$$

Therefore, when $\tau = 0$,

$$\frac{\mathrm{d}\underline{L}}{\mathrm{d}t} = 0, \qquad \underline{L} = \text{constant}$$

But since $\underline{\omega}$ need not be parallel to the angular momentum, the axis of rotation need nit remain constant.

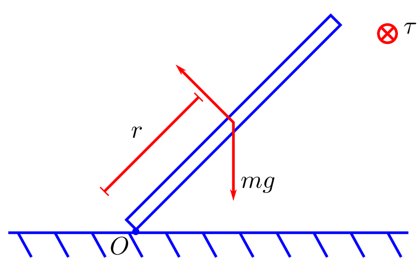
12.1 Torques on Rotating Bodies

$$\underline{\tau} = \frac{\mathrm{d}\underline{L}}{\mathrm{d}t}$$

So in some instant, $\Delta\underline{L} = \underline{\tau}\Delta\underline{L}$. This means that any change in $\underline{L}$will be parallel to $\underline{\tau}$.

12.1.1 Tilted Spinning Wheel

Consider a wheel rolling into the page with an angular velocity ω.



Rotating about a symmetry axis,

$$\underline{L} = I\underline{\omega}$$

The torque about O is given by,

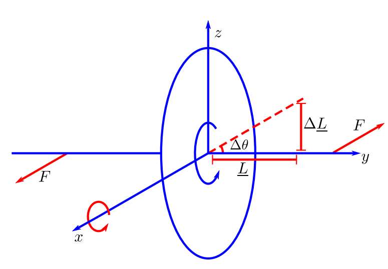
$$\underline{\tau} = R \times mg$$

which acts into the page. Therefore, the change in angular momentum ΔL must be in the same direction,

$$\Delta L = \Delta\left(I\underline{\omega}\right) = I\Delta\underline{\omega}$$

($\Delta\omega$ into the page). Also note $\Delta\underline{\omega}$ is perpendicular to $\underline{\omega}$, magnitude stays the same while the direction changes, so the wheel rotates about the vertical axis. The wheel cannot topple as this would change the vertical component of $\underline{L}$, while torque is horizontal.

12.1.2 Torque on a Spinning Wheel



Wheel spinning about an axle in the y-direction. Apply small horizontal forces to the axis of rotation in the $\pm x$ direction of equal magnitude and equal distances from the wheel, $\pm l$. There is no net force, so the center of mass does not move,

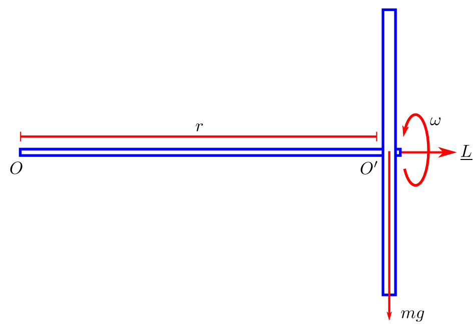
$$\tau = \sum \underline{r} \times \underline{F} = 2lF\underline{\hat{z}}$$

The wheel rotates about the x axis so that the angular momentum gains a z component, ΔL in the z direction. The torque is perpendicular to $\underline{L}$, so there is a change in direction of $\underline{L}$, but not the magnitude,

$$\Delta \underline{L}_z = L\Delta\theta$$

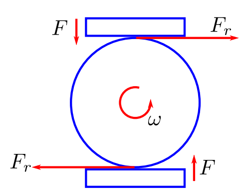
12.2 Gyroscope

Flywheel on a light axle OO', Pivoted completely freely at O. The wheel rotates about OO'. About the pivot, $\tau = \underline{r} \times \underline{F}$



1. When $\omega = 0$, (not spinning) the gyroscope behaves like a pendulum with a pivot at O and mass all in the flywheel at the end of the light axle.

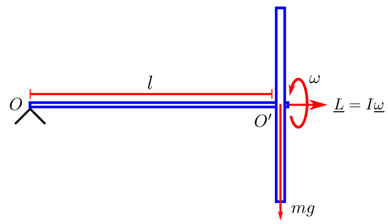
2. When the wheel is spinning, torque is parallel to the angular velocity, e.g. brake pads.



The friction, f_r, acts to produce a torque opposed to $\underline{\omega}$, so

$$\begin{aligned} \Delta \underline{L} &= I\Delta\omega = \tau\Delta t \\ \Rightarrow \frac{\Delta\omega}{\Delta t} &\propto \tau \propto \frac{1}{I} \\ \frac{\Delta\omega}{\Delta t} &= \frac{\tau}{I} \end{aligned}$$

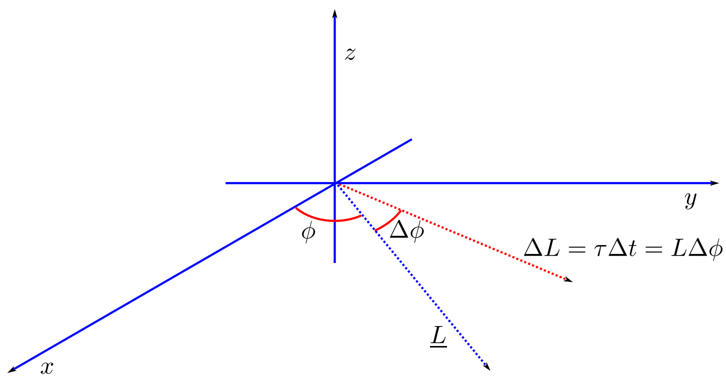
3. When the wheel is spinning, the torque is perpendicular to $\underline{L}$.



The torque about the pivot O is given by,

$$\begin{aligned} \tau &= \underline{l} \times m\underline{g} \\ \Rightarrow \tau &= \frac{\mathrm{d}\underline{L}}{\mathrm{d}t} \end{aligned}$$

so is perpendicular to $\underline{L}$. Because the torque is perpendicular to $\underline{L}$, it changes its direction and not the magnitude, and so the system starts to rotates. As the torque is always perpendicular to $\underline{L}$, the system continues to rotate, it precesses about the vertical axis.



$$\begin{aligned} \tau\Delta t &= \underline{L}\Delta\phi \\ \Rightarrow \frac{\Delta\phi}{\Delta t} &= \Omega = \frac{\tau}{L} \end{aligned}$$

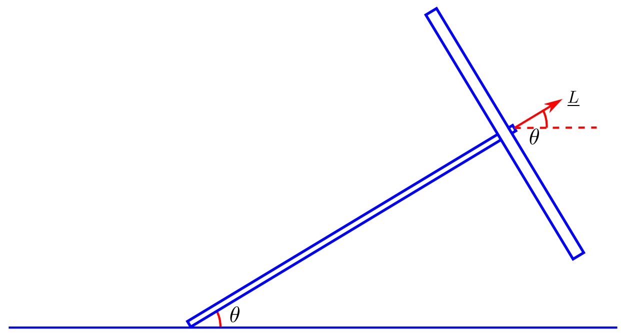
Because the axis is horizontal,

$$\begin{aligned} \underline{\tau} &= \underline{l} \times m\underline{g} \\ |\tau| &= lmg \\ \Rightarrow \Omega &= \frac{\tau}{L} = \frac{lmg}{I\omega} \end{aligned}$$

Rewrite $I = m\kappa^2$,

$$\Omega = \frac{lg}{\omega\kappa}$$

If the axis is not horizontal, there will be precession with a given frequency as the angular momentum is in the same direction as the axle.



$$\underline{\tau} = \underline{\Omega} \times \underline{L}$$

This is the special case when th axis is already rotating at the correct speed, so that when the flywheel spins, the motion of the axis is already set. If the flywheel is set spinning, then the axis released from rest, the axis must accelerate from zero velocity up to the required velocity.

Part IV

Fictitious Forces and Non-Inertial Frames

Consider observers in 2 different frames:

- an inertial frame (non accelerating)
- a frame with constant linear acceleration $\underline{A}$

In the inertial frame, the object has acceleration $\underline{a}$ due to the force $\underline{F}$ applied on it,

$$\underline{F} = m\underline{a}$$

In the accelerating frame, the object appears to have acceleration $\underline{a}'$ where

$$\underline{a}' = \underline{a} - \underline{A}$$

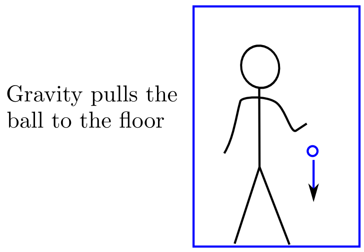
So to the non inertial observer, the object appears to be subject to an extra force,

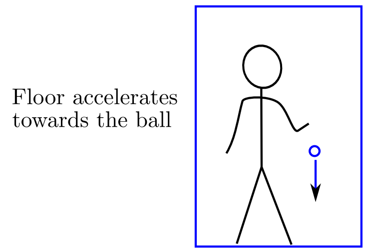
$$\begin{aligned} \underline{F}' &= m\underline{a}' \\ &= m(\underline{a} - \underline{A}) \\ &= m\underline{a} - m\underline{A} \\ &= \underline{F_a} - \underline{F_A} = \underline{F} + \underline{F}_{\text{inertial}} \end{aligned}$$

where $\underline{F}_{\text{inertial}}$ is the apparent force due to the acceleration of the frame.

<u>Ex</u>

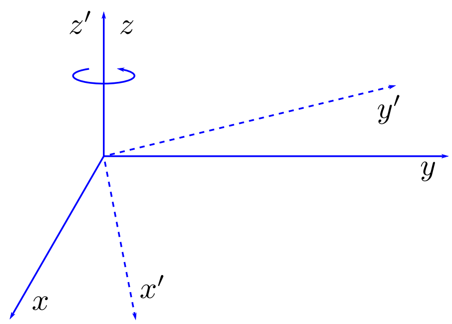
Rocket on launchpad | **Rocket in deep space accelerating at** 1g



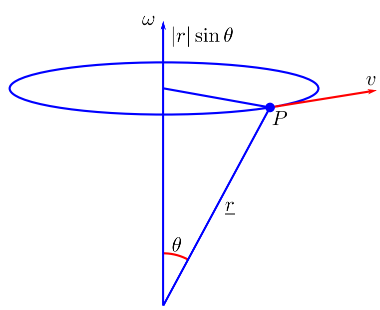


The results are indistinguishable to the astronaut, only an external observer can tell the difference. This is the basis of Einstein's Equivalence Principle.

13 Rotating Reference Frame



Let (x, y, z) be inertial co-ordinates and (x', y', z') be co-ordinates in frame rotating about z with angular velocity ω. In this case, $z = z'$ as (x', y') rotate in the $x - y$ plane.



Point P, stationary in the rotating frame has velocity $\underline{v} = \underline{\omega} \times \underline{v}$ in the inertial frame.

How do we relate changes in the vector $\underline{A}$ in the two frames?

- $\Delta\underline{A}_i$ =change seen by the inertial observer
- $\Delta\underline{A}_r$ =change seen in the rotating frame
- $(\underline{\omega} \times \underline{A})\,\Delta t$ =change the inertial observer would see if $\underline{A}$ was fixed in the rotating frame.

So the total change seen in the inertial frame is given by,

$$\Delta\underline{A}_i = \Delta\underline{A}_r + \underline{\omega} \times \underline{A}\Delta t$$

So the rate of change is given by,

$$\left(\frac{\mathrm{d}\underline{A}}{\mathrm{d}t}\right)_i = \left(\frac{\mathrm{d}\underline{A}}{\mathrm{d}t}\right)_r + \underline{\omega} \times \underline{A}$$

This can then be applied to the velocity and the acceleration.

1. Velocity
   Let $\underline{A}$ =position, $\underline{r}$, of the object

$$\begin{aligned} \Rightarrow \left(\frac{\mathrm{d}\underline{A}}{\mathrm{d}t}\right)_i &= \underline{v}_i = \text{velocity in the inertial frame} \\ \left(\frac{\mathrm{d}\underline{A}}{\mathrm{d}t}\right)_r &= \underline{v}' = \text{velocity in the rotating frame} \\ \Rightarrow \underline{v} &= \underline{v}' + \underline{\omega} \times \underline{r} \end{aligned}$$

2. Acceleration
   Let $\underline{A}$ =velocity, $\underline{v}$, of the object

$$\left(\frac{\mathrm{d}\underline{v}}{\mathrm{d}t}\right)_i = \left(\frac{\mathrm{d}\underline{v}}{\mathrm{d}t}\right)_r + \underline{\omega} \times \underline{r}$$

We want to relate $\underline{a} = \frac{\mathrm{d}\underline{V}}{\mathrm{d}t}$ with $\underline{a}' = \frac{\mathrm{d}\underline{V}'}{\mathrm{d}t}$. Substitute from above,

$$\begin{aligned} \underline{v} &= \underline{v}' + \underline{\omega} \times \underline{r} \\ \Rightarrow \left(\frac{\mathrm{d}\underline{V}}{\mathrm{d}t}\right)_i &= \underline{a} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\underline{v}' + \underline{\omega} \times \underline{r}\right)_i + \underline{\omega} \times \left(\underline{\omega} \times \underline{r}\right) \end{aligned}$$

So if a real external force exists,

$$\underline{F}_{\text{ext}} = m\underline{a}$$

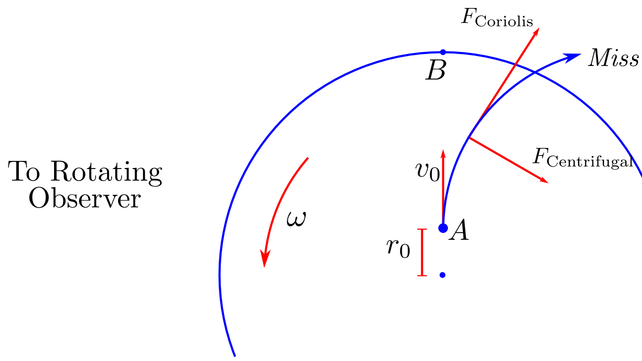
then the apparent force in the rotating frame is given by,

$$\begin{aligned} \underline{F}' &= m\underline{a}' \\ &= \underline{F}_{\text{ext}} \underbrace{-2m\left(\underline{\omega} \times \underline{v}'\right)}_{\text{"Colriolis" Force}} \underbrace{-m\omega \times \left(\underline{\omega} \times \underline{r}\right)}_{\text{"Centrifugal" Force}} \end{aligned}$$

These two forces, the Coriolis and Centrifugal forces, are fictional internal forces.

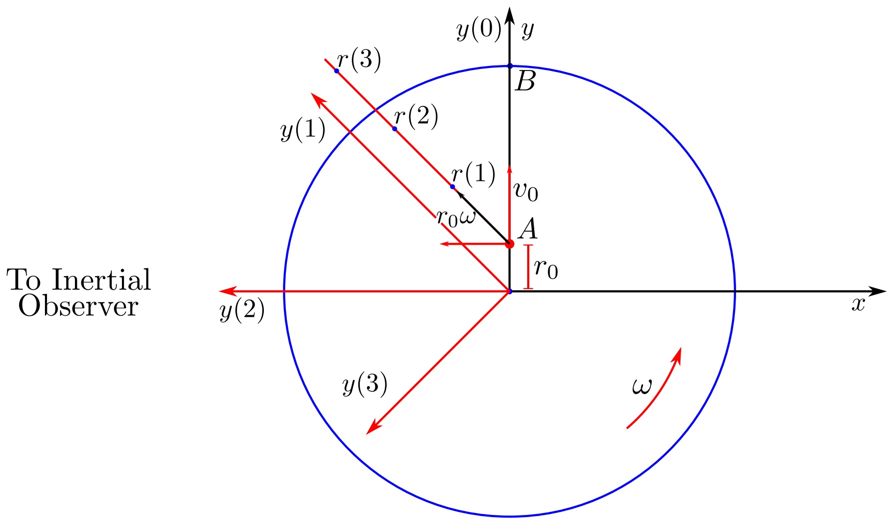
- Coriolis force
  $-2m\left(\underline{\omega} \times \underline{r}\right)$ acts perpendicular to $\underline{v}'$ so deflects the motion sideways. This depends on the $\underline{v}'$ but not the position.
- Centrifugal force
  $-m\underline{\omega} \times \left(\underline{\omega} \times \underline{r}\right)$ acts radially outwards and depends on the position but not the velocity.

13.1 Illustration



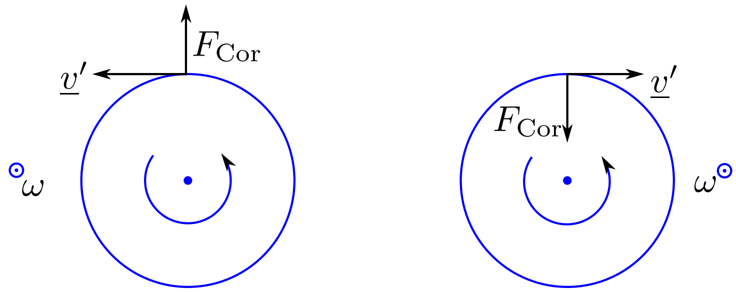
Person at A throws a ball to the person at B (radially outwards). To the rotating observer, the ball is subject to inertial forces:

- $-m\omega^2\underline{r}$ (radially outwards)
- $-2m\omega \times \underline{v}^{'}$ (perpendicular to $\underline{v}'$).



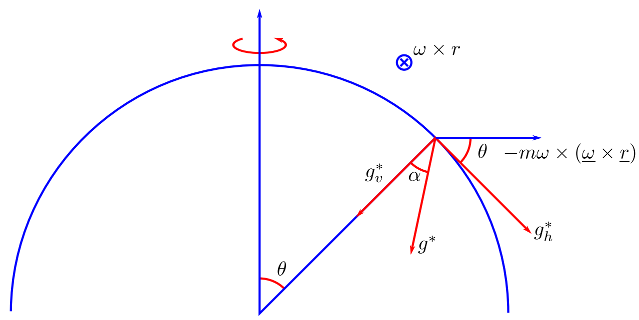
To the inertial observer (outside the system) the ball travels in a straight line. Relative to the rotating axis, the ball appears to move to the right.

14 Direction of Coriolis Force



14.1 Variation of g with Latitude

Consider a plumb-line suspended at the point P on the earth's surface.



If the earth did not rotate, it would hang down directly towards the center of the planet. On a rotating planet, however, this is not true.

Rotating frame: weight hangs stationary, so can ignore the effect of the Coriolis force,

$$F = mg - m\underline{\omega} \times (\underline{\omega} \times \underline{r})$$

In the example of the earth, $r_{\text{earth}} \approx 6.4 \times 10^6$m and $\omega \approx \frac{2\pi}{1 \text{ day}} = 7.27 \times 10^{-5}\text{s}^{-1}$. Let $\underline{g}* =$ acceleration of the object which can then be resolved into g_h^* (horizontal) and g_v^* (vertical),

$$\begin{aligned} \Rightarrow mg_h^* &= F_{\text{cent}} \cos\theta \\ &= -m\omega^2 r \sin\theta \cos\theta \\ mg_v^* &= gm - mr\omega^2 \sin\theta\cos\theta \\ &= m(g - r\omega^2 \sin^2\theta) \end{aligned}$$

- At one of the poles, $\theta = 0$, so $g_h^* = 0$, $g_v^* = g$ (max g_v^*)
- At the equator, $\theta = \frac{\pi}{2}$, so $g_h^* = 0$, $g_v^* = g - \omega^2 r = g - 0.034$

This means that g_v^* varies by about 0.35% with latitude.

Angle between g and $g^* = \alpha$

$$\begin{aligned} \alpha \approx \tan\alpha &= \frac{g_h^*}{g_v^*} \\ &= \frac{\omega^2 r \sin\theta\cos\theta}{g - \omega^2 r \sin\theta\sin\theta} \\ &\approx \frac{\omega^2 r \sin 2\theta}{2g} \end{aligned}$$

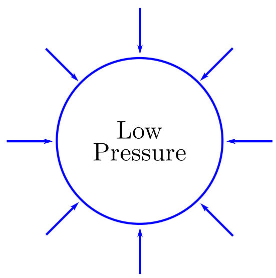
α_{max} occurs when $\sin 2\theta = 1$, $\theta = \frac{\pi}{4}$

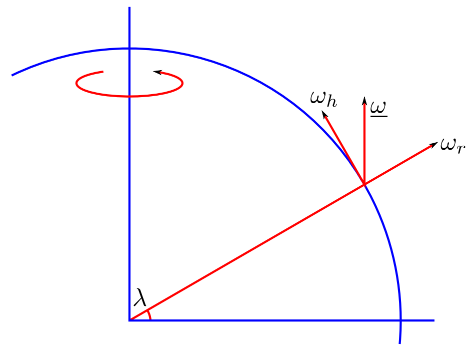
$$\alpha_{\text{max}} = \frac{\omega^2 r}{2g} \approx 1.7 \times 10^{-3}\text{rad}$$

Ex

Circulation of the atmosphere.

For a non-rotating planet, the air would flow radially from the high pressure to the low pressure region. However, the rotation of the earth means that this does not happen.





Resolve $\underline{\omega}$ into horizontal and vertical components. The Coriolis force acts on the moving air,

$$\begin{aligned}
\underline{F}_{\text{Cor}} &= -2m(\underline{\omega} \times \underline{v}') \\
&= -2m\left(\underline{\omega}_h + \underline{\omega}_v\right) \times \underline{v}' \\
&= \underbrace{-2m\underline{\omega}_h \times \underline{v}'}_{A} + \underbrace{2m\underline{\omega}_v \times \underline{v}'}_{B}
\end{aligned}$$

Here A is the force generated perpendicular to the surface and B produces a force parallel to the surface but perpendicular to the velocity.

Horizontal component of $\underline{F}_{\text{Cor}}$,

$$\begin{aligned}
(F_{\text{Cor}})_H &= -2m\underline{\omega}_v \times \underline{v}' \\
&= -2m\omega_v v' \\
&= -2m\omega v' \sin\lambda
\end{aligned}$$

To note from this, the magnitude of the horizontal Coriolis force depends on the latitude you're at, but not the direction of the wind, $\underline{v}'$.

- $(\underline{F}_{\text{Cor}})_H$ deflects motion to the right if in the northern hemisphere ($\lambda > 0$)
- $(\underline{F}_{\text{Cor}})_H$ deflects motion to the left if in the southern hemisphere ($\lambda < 0$)

The result of this is that in the northern hemisphere, a storm will form a circulating system, spiraling in the anti-clockwise direction. Similarly, for the southern hemisphere, the storm will develop spinning in the other direction, clockwise.

Part V

Motion Under Central Conservative Forces - Orbits

A conservative force is one where the work done in moving between A and B does not depend on which route is taken. This also means that, moving around a closed path, $A \to B \to A$, the total work done is zero. This leads to the conclusion that that total mechanical energy $= E_k + E_p =$ constant. This however is a particular case and does not always apply. It can be shown that this is true if the curl of the force is zero,

$$\underline{\nabla} \times \underline{F} = 0$$

and hence if

$$\underline{F} = -\underline{\nabla}\phi$$

where ϕ is a scalar potential. A central force is one that depends only on the radial distance from the source,

$$\underline{F} = f(\underline{r})\underline{\hat{r}}$$

15 Properties of a Central Force

1. Conservative

$$\begin{aligned} \underline{F}(r) &= f(r)\underline{\hat{r}} = -\frac{\mathrm{d}V(r)}{\mathrm{d}r}\underline{\hat{r}} \\ \therefore \underline{F}(r) &= -\underline{\nabla}V(r) \end{aligned}$$

So any central force must be conservative.

2. Can also assume that the angular momentum is constant

$$\begin{aligned} \underline{\tau} &= \underline{r} \times \underline{F} \\ &= \underline{r} \times \underline{\hat{r}} f(r) = 0 = \frac{\mathrm{d}\underline{L}}{\mathrm{d}t} \\ \Rightarrow \underline{L} &= \text{Constant} \end{aligned}$$

Therefore the motion is confined to a plane and this can be reduced to a 2D system,

$$L = \underline{r} \times m\underline{\dot{r}}$$

So both $\underline{r}$ and $\underline{\dot{r}}$ are perpendicular to the angular momentum. So if the direction of L is fixed, since it is constant, the plane of motion must also be fixed.

15.1 Plane Polar Co-ordinates

$$\underline{r} = r\underline{\hat{r}}$$

where $\underline{\hat{r}} = \underline{i}\cos\theta + \underline{j}\sin\theta$ and $\underline{\hat{\theta}} = -\underline{i}\sin\theta + \underline{j}\cos\theta$. From this, it can be shown that

$$\begin{aligned} \frac{\mathrm{d}\underline{\hat{r}}}{\mathrm{d}t} &= \dot{\theta}\underline{\hat{\theta}} \\ \frac{\mathrm{d}\underline{\hat{\theta}}}{\mathrm{d}t} &= -\underline{\hat{r}}\dot{\theta} \end{aligned}$$

Hence

$$\underline{v} = \frac{\mathrm{d}\underline{v}}{\mathrm{d}t} = \dot{r}\underline{\hat{r}} + r\frac{\mathrm{d}\underline{\hat{r}}}{\mathrm{d}t} = \dot{r}\underline{\hat{r}} + r\dot{\theta}\underline{\hat{\theta}}$$

and

$$\begin{aligned} \underline{a} &= \frac{\mathrm{d}\underline{v}}{\mathrm{d}t} = \ddot{r}\underline{\hat{r}} + 2\dot{r}\dot{\theta}\underline{\hat{\theta}} + r\ddot{\theta}\underline{\hat{\theta}} - r(\dot{\theta})^2\underline{\hat{r}} \\ \therefore \underline{a} &= \underbrace{\left(\ddot{r} - r(\dot{\theta})^2\right)\underline{\hat{r}}}_{\text{Radial acceleration}} + \underbrace{\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\underline{\hat{\theta}}}_{\text{Tangential acceleration}} \end{aligned}$$

Since $\underline{F}$ is purely radial, $\underline{F} = m\underline{a}$ can be separated into two terms,

1. Radial
$$f(r) = m\left(\ddot{r} - r\dot{\theta}^2\right)$$
2. Tangential
$$0 = m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)$$

The tangential equation is a consequence of the angular momentum being constant. Consider the tangential velocity

$$\begin{aligned} v_\theta &= r\dot{\theta} \\ |\underline{L}| = l &= mrv_\theta = mr^2\dot{\theta} \\ \therefore \frac{\mathrm{d}l}{\mathrm{d}t} &= m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) \\ &= mr\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) \end{aligned}$$

This is just the tangential equation of motion multiplied by r,

$$\frac{\mathrm{d}l}{\mathrm{d}t} = mr\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0$$

15.2 Energy Equation

$$\begin{aligned} E &= \frac{1}{2}mv^2 + V(r) \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) \end{aligned}$$

Using $mr^2\dot{\theta}^2 = \frac{l^2}{mr^2}$ (l= constant)

$$\Rightarrow E = \frac{1}{2}m\dot{r} + \frac{l^2}{2mr^2} + V(r)$$

This is the radial energy equation. This looks like a one dimensional equation of motion,

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$$

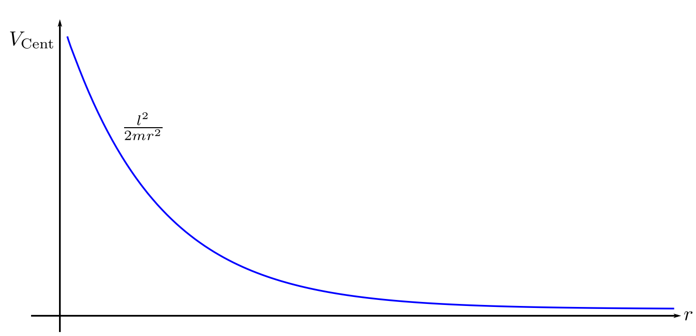
where the effective potential is given by

$$V_{\text{eff}} = \frac{l^2}{2mr^2} + V(r)$$

and $\frac{l^2}{2mr^2}$ corresponds to the force,

$$F = -\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{l^2}{2mr^3} = mr\dot{\theta}^2$$

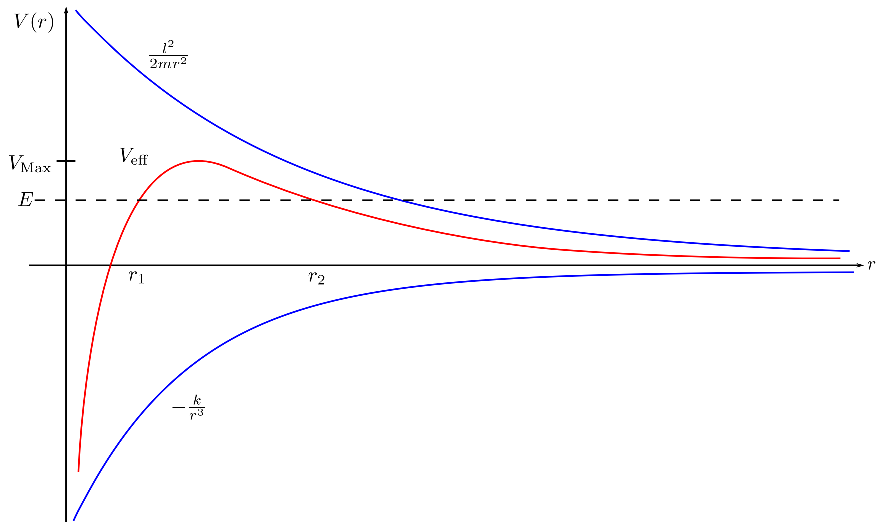
This angular momentum term is effectively a repulsive force equal to the (effective) centrifugal force, so the term $\frac{l^2}{2mr^2}$ is known as the centrifugal potential.



Since l is constant, for small r, a large v_θ is required, and so a large kinetic energy is present.

Ex

Consider a short range attractive force, for example, the potential given by $V(r) = -\frac{k}{r^3}$, so $V_{\text{eff}} = \frac{l^2}{2mr^2} - \frac{k}{r^3}$



- If the energy of an object is greater than $V_{\max}$, then it is not bound by this potential and is free to move anywhere.
- If the energy is less than $V_{\max}$, the object could be bound in $r < r_1$.
- If the energy is greater than zero and still less that $V_{\max}$, it could be bound in the region $r > r_2$.

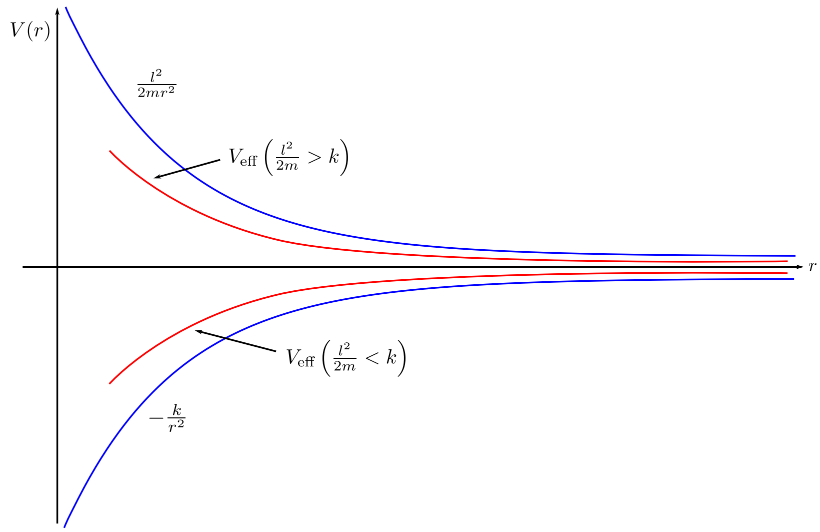
Ex

Consider an inverse cube force (force is inverse cube so potential is inverse square $F = \frac{\mathrm{d}v}{\mathrm{d}r}$),

$$V(r) = -\frac{k}{r^2}$$

so

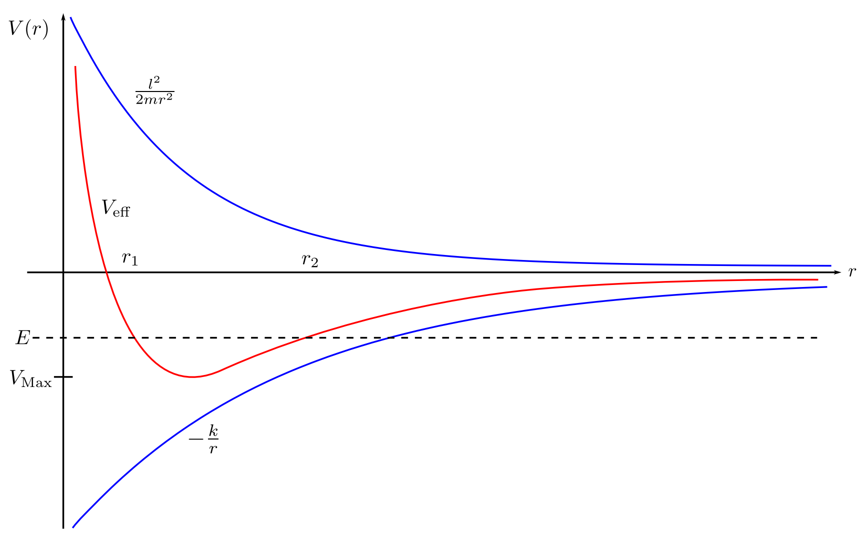
$$V_{\text{eff}} = \frac{1}{r^2}\left(\frac{l^2}{2m} - k\right)$$



So this force is either attractive or repulsive depending on the relative strength of k. This means there is no stable orbits for an inverse cube potential.

Ex

Consider the inverse square force.



V_{eff} has minimum, $E_{\text{min}} = -\frac{mk^2}{2l^2}$ which occurs at a radius $r_{\text{min}} = \frac{l^2}{mk}$. Therefore, for a given energy less than zero, there is also a maximum angular momentum,

$$l_{\text{max}} = \sqrt{\frac{mk^2}{2|E|}}$$

16 Orbital Shapes

Looking at the inverse square force,

- when the energy is $E_{\min}$, the orbit will be circular as there is only one possible value of r, so the radius must be constant,
- for $E_{\min} < E < 0$ (still bound) the orbit will change between r_1 and r_2. In three dimensions, this would mean the radius oscillates between these limits, so the shape of the orbit must be an ellipse,
- if $E > 0$, the object is unbound, but still with some minimum value of r, a distance of closest approach. In general, the shape in three dimensions would be a hyperbola. If $E = 0$, the shape would be a parabola.

16.1 Orbit Equation

The radial equation used the fact that l =constant to eliminate the angular dependence from the energy equation. The orbit equation uses the same principle, that l is constant, to eliminate time from the equation, to relate r and θ.

This gives the equation

$$r\,(1 - \epsilon\cos\theta) = r_0$$

where $r_0 = \frac{l^2}{mk}$ which is the radius that a circular orbit for the particular values of l, m, and k would have, and $\epsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$ which is the eccentricity and characterises the shape of the orbit. This is derived by starting with the equation in plane polars, then making the 2 substitutions,

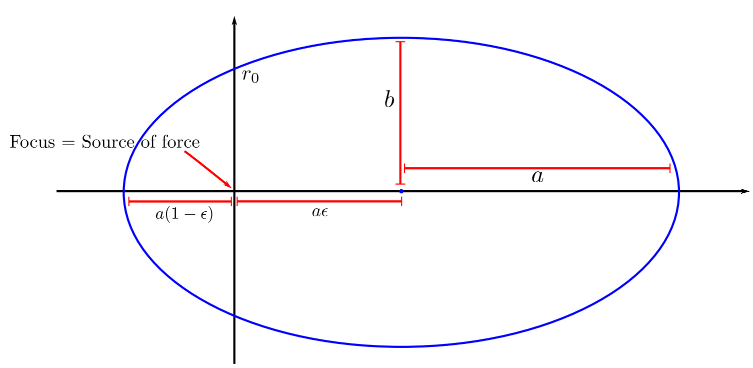
$$\begin{aligned} u &= \frac{1}{r} \rightarrow \frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} \\ \Rightarrow \dot{r} &= \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = -r^2\dot{\theta}\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{l}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta} \end{aligned}$$

16.2 Bound Orbits

Bound orbits occur when the total energy, $E < 0$ so that also the eccentricity, $\epsilon < 1$. This can be written as

$$\frac{(x - a\epsilon)^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{Elliptical orbit}$$

where $a = \frac{r_0}{1-\epsilon^2} = -\frac{k}{2E}$ and $b^2 = r_0 a = -\frac{l^2}{2mE}$.



- semi-major axis, a, is determined by the energy, E,

$$\begin{aligned} a &= -\frac{k}{2E} \\ E &= -\frac{k}{2a} \end{aligned}$$

- semi-latus rectum, r_0, is defined by the angular momentum,
- the eccentricity, ϵ, is defined by
$$\epsilon^2 = 1 - \frac{r_0}{a}$$
- the semi-minor axis is given by
$$b^2 = -\frac{l^2}{2mE}$$

So when $E = E_{\text{min}}$, $\epsilon = 0$ so the semi-major axis is the same as the semi-minor axis, $a = b = r_0$so there is a circular orbit.

16.3 Conservation Laws

If we know that the energy, E, and the angular momentum, l, are constant at any point in the orbit, we can calculate the shape of the orbit. If we know E at any point, we can work out $|\underline{v}|$ at any other point,

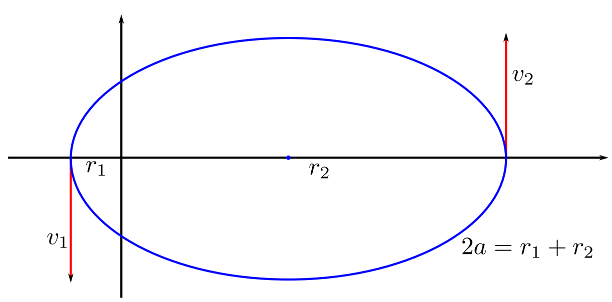
$$E = \frac{1}{2}mv^2 + V(r)$$

If we know $|\underline{v}|$ when r is perpendicular to the radius, then we can calculate l,

$$l = mvr$$

Ex

Show that $E = -\frac{k}{2a}$



1. Conversation of energy

$$\begin{aligned} E &= \frac{1}{2}mv_1^2 - \frac{k}{r_1} \\ &= \frac{1}{2}mv_2^2 - \frac{k}{r_2} \end{aligned} \tag{5}$$

2. Conservation of angular momentum (v perpendicular to r at this point)

$$\begin{aligned} l &= mv_1r_1 \\ &= mv_2r_2 \\ v_2 &= \frac{r_1}{r_2}v_1 \end{aligned} \tag{6}$$

Substitute (6) into (5)

$$\begin{aligned}
\frac{1}{2}mv_1^2 - \frac{k}{r_1^2} &= \frac{1}{2}mv_2^2 - \frac{k}{r_2^2} \\
\frac{1}{2}mv_1^2 - \frac{k}{r^2} &= \frac{1}{2}m\left(\frac{r_1}{r_2}v_1\right)^2 - \frac{k}{r_2^2} \\
\frac{1}{2}v_1^2 m\left(1 - \frac{r_1^2}{r_2^2}\right) &= k\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \\
\frac{1}{2}m\left(\frac{r_2^2 - r_1^2}{r_2^2}\right)v_1^2 &= k\left(\frac{r_2 - r_1}{r_1 r_2}\right) \\
\frac{1}{2}m\frac{(r_2 - r_1)(r_2 + r_1)}{r_2}v_1^2 &= k\left(\frac{r_2 - r_1}{r_1}\right) \\
\frac{1}{2}m\frac{(r_2 + r_1)}{r_2}v_1^2 &= \frac{k}{r_1} \\
\frac{1}{2}mv_1^2 &= k\frac{r_2}{r_1}\frac{1}{r_1 + r_2}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow E &= k\frac{r_2}{r_1}\frac{1}{r_2 + r_1} - \frac{k}{r_1} = -\frac{k}{r_1 + r_2} \\
\Rightarrow E &= -\frac{k}{2a} \\
a &= -\frac{k}{2E} \qquad \text{As required.}
\end{aligned}$$

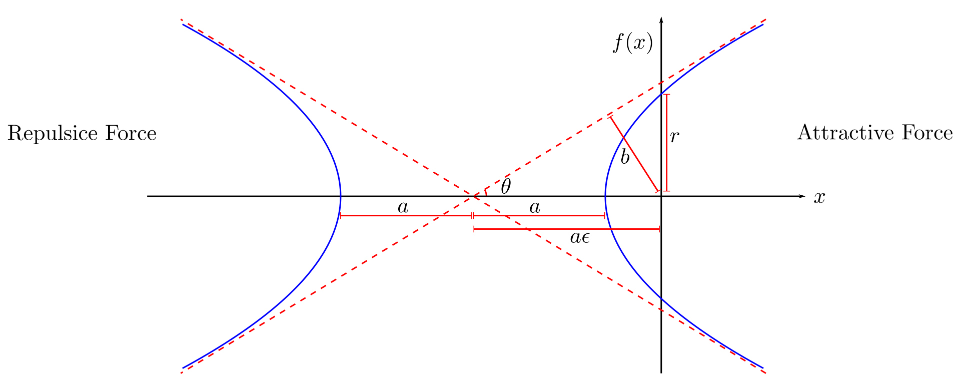
So energy $E > 0$, and so the eccentricity $\epsilon > 1$.

16.4 Unbound Orbits

- For $\epsilon > 1$, the orbit equation can be written as

$$\frac{(x + \epsilon a)^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{(Hyperbola)}$$

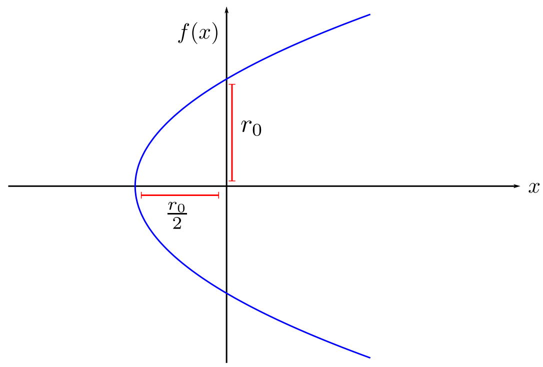
where $a = \frac{r_0}{\epsilon^2 - 1} = \frac{k}{2E}$ and $b^2 = r_0 a = \frac{l^2}{2mE}$. This describes a hyperbola with focus at the point $x = y = 0$.



Here b =impact parameter and θ =scattering angle= $\cos^{-1}\left(\frac{1}{\epsilon}\right)$. As $r \to \infty$, $r = \frac{r_0}{1-\epsilon\cos\theta} \to 0$. So $\epsilon\cos\theta \to 1$.

- When $E = 0$, so the object is just unbound, $\epsilon = 1$, so

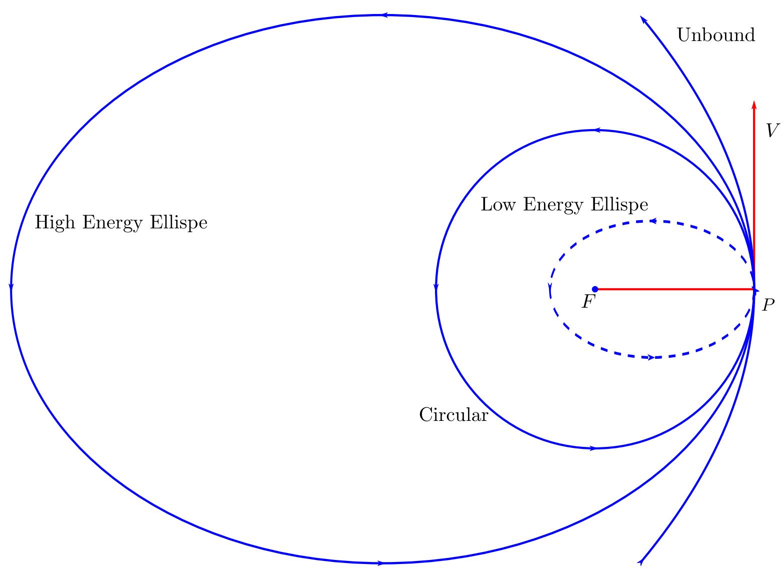
$$\begin{aligned} r(1-\cos\theta) &= r_0 \\ r(1-x) &= r_0 \\ \Rightarrow y^2 &= r_0^2 + 2r_0x \qquad \text{(Parabola)} \end{aligned}$$



This is the case when the object has just enough energy to escape capture, $E_k \to 0$, $r \to \infty$.

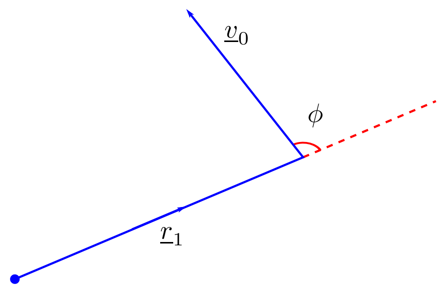
16.5 Family of Related Forces

Launch an object from the point P with velocity v perpendicular to FP.



$$\begin{array}{lcc} \text{Low } v: \quad P & = & \text{Perigee of ellipse} \\ | & & \downarrow \\ | & & \text{Circle} \\ | & & \downarrow \\ | & & \text{Apogee of ellipse} \\ \downarrow & & \downarrow \\ \text{High } v: & & \text{Hyperbola - unbound orbit} \end{array}$$

16.6 Calculating an Orbit



$$V(r) = -\frac{k}{r} = -\frac{GMm}{r}$$

Given the initial position and the velocity at that position, $\underline{r}$, and $\underline{v}_0$, find the orbit.

1. Is $E > 0$ or $E < 0$?
   $\frac{1}{2}mv^2 > \frac{GMm}{r_1}$ or $\frac{1}{2}mv^2 < \frac{GMm}{r_1}$
   Lets assume $E < 0$ (bound).

2. Orbital angular momentum
   $$|l| = (\underline{r}_j \times m\underline{v}_0) = r_j m \underline{v}_0 \sin\phi$$
   At perigee and apogee, the position and velocity are perpendicular
   $$\begin{aligned} l &= mv_1r_1 = mv_2r_2 \\ \Rightarrow \frac{1}{r_1} &= \frac{mv_1}{l} \qquad \frac{1}{r_2} = \frac{mv_2}{l} \end{aligned}$$

3. Energy
   $$\begin{aligned} E &= \frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = -\frac{GMm}{2a} \\ \Rightarrow E &= \frac{1}{2}mv_1^2 - \frac{m_1v_1}{l}GMm \\ &= \frac{1}{2}mv_1^2 - \frac{GMm^2v_1}{l} \end{aligned}$$
   This gives a quadratic for $v_1(v_2)$ so can find $r_1(r_2)$ from this.

4. Eccentricity
   $$\left.\begin{aligned} r_1 &= a(1-\epsilon) \\ r_2 &= a(1+\epsilon) \end{aligned}\right\} \epsilon$$

5. Angle of the orbit from the orbit equation
   $$\begin{aligned} r_j(1-\epsilon\cos\theta_j) &= r_0 \\ r_0 &= a(1-\epsilon^2) \end{aligned}$$

16.7 Perturbed Circular Orbit

Consider an object in a circular orbit that is subject to some impulse (force applied for a very short amount of time.

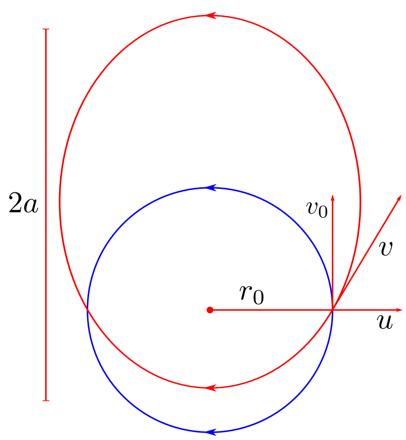
16.7.1 Radial Impulse

The impulse, I, does not change the angular momentum, l, of the orbit, since the torque about $F = 0$. It does change the radial velocity O to $u = \frac{\Delta F}{m}$. Therefore, the energy of the object in the orbit will be increased,

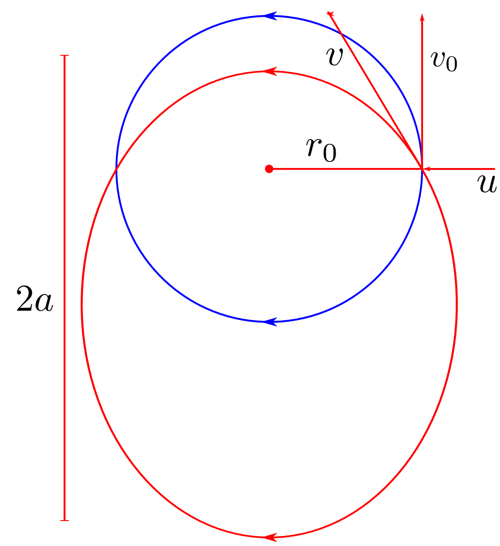
$$E_k = \frac{1}{2}m(v^2 + u^2) - \frac{GMm}{r}$$

So now $E > E_{\text{min}}$, so the orbit is no longer circular.

Outward Impulse

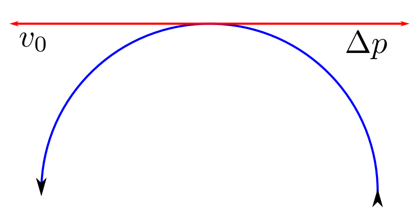


Inward Impulse



In both cases, the energy is the same as is the size of the orbit, $2a$, and so the shape, ie the eccentricity, ϵ, is the same.

16.7.2 Tangential Impulse



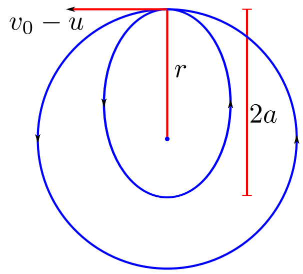
Impulse is anti-parallel to the velocity.

$$\begin{aligned} \therefore v &= v' = v_0 - u \\ \Rightarrow E &= \frac{1}{2}m(v_0 - u)^2 - \frac{GMm}{r} \end{aligned}$$

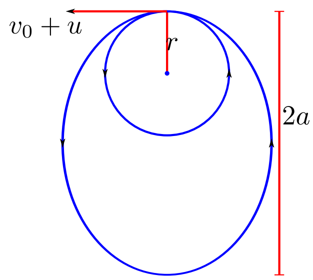
So the energy decreases, and the angular momentum also decreases

$$l \to l' = mr(v_0 - u)$$

The new orbit must be elliptical with a semi-major axis, $a < r$.



If the impulse had increased the velocity, then the new velocity must be $v_0 + u$.



Part VI

Special Relativity

17 Postulates

1. The laws of physics are the same in all inertial frames
2. The speed of light (in a vacuum) has the same value in all inertial frames.

18 Lorentz Transformation

If x, y, z and t are the co-ordinates in the inertial frame S, and x', y', z' and t' are the co-ordinates in the frame S' moving relative to S with velocity v in the positive x direction. Then

$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt)$$
$$y' = y \qquad y = y'$$
$$z' = z \qquad z = z'$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad t = \gamma\left(t' + \frac{v}{c^2}x\right)$$

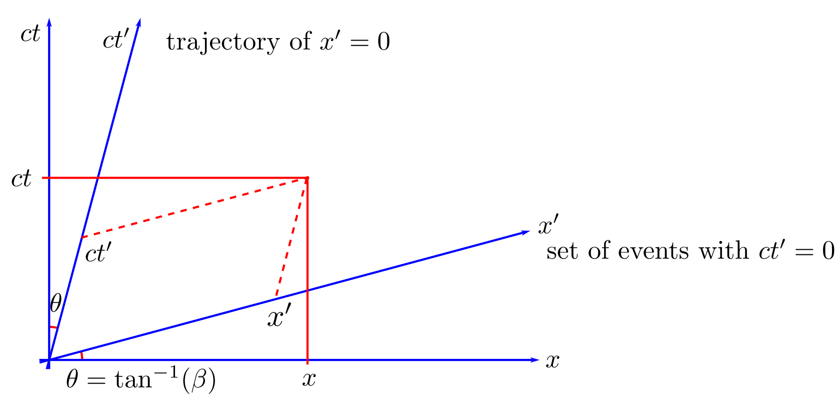
where $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1 - \dfrac{v^2}{c^2}\right)^{-\frac{1}{2}} = (1-\beta^2)^{-\frac{1}{2}}$, where $\beta = \frac{v}{c}$. We can also write this as a matrix vector,

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

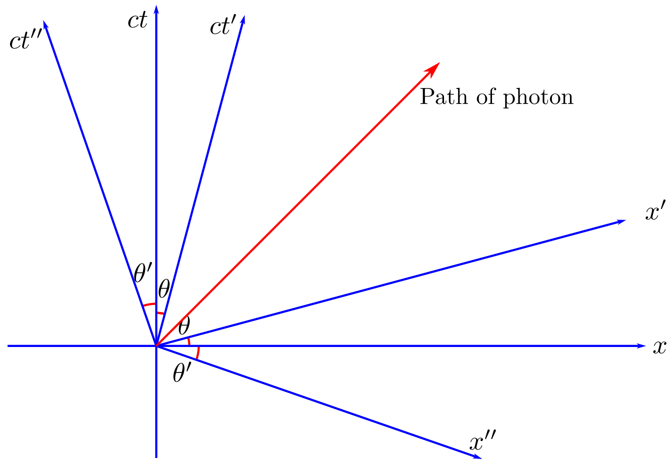
ct' is used to keep the dimensions constant. This is know as a "4 vector".

18.1 Space-Time Diagrams

These are a way of representing how events appear in different frames. Only the x and ct co-ordinates are plotted.



So the events with co-ordinates (x, ct) is S has co-ordinates (x', ct') in S'. The lines joining this point to the x' and ct' axes must be parallel to the other, ct' or x' respectively, axis. Reversing the velocity reverses the angle.



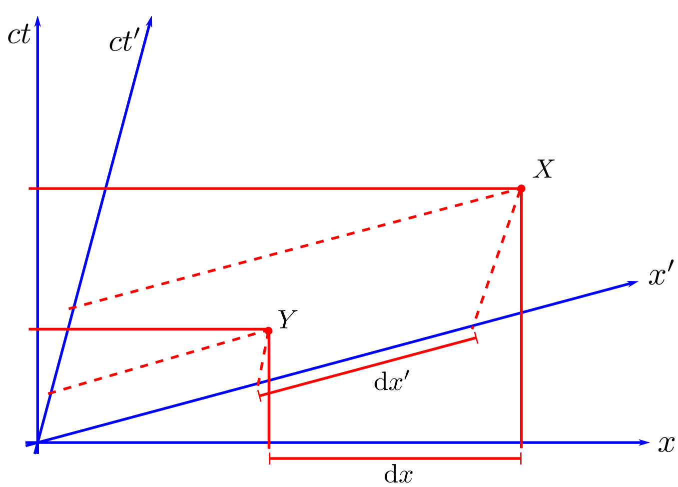
The path of a photon, since it travels with velocity c, in all frames must be mid way between the axis for all frames,

$$x = ct$$
$$x' = ct'$$
$$x'' = ct''$$

18.2 Space-Time Interval

An invariant quantity is one that has the same value in all inertial frames. One of these quantities is the speed of light. Another is the interval, $\mathrm{d}S$, between 2 events where

$$(\mathrm{d}S)^2 = c^2(\mathrm{d}t)^2 - (\mathrm{d}x)^2$$



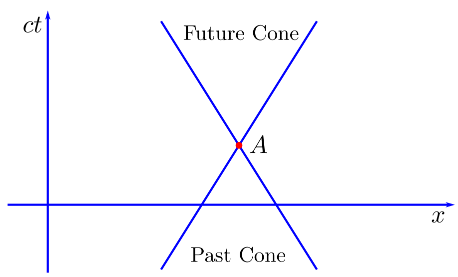
In S'

$$\begin{aligned}
(\mathrm{dS'})^2 &= c\mathrm{d}t'^2 - \mathrm{d}x'^2 \\
&= \gamma^2 \left(c\mathrm{d}t - \beta \mathrm{d}\right)^2 - \gamma^2 \left(\mathrm{d}x - \beta c\mathrm{d}t\right)^2 \\
&= \gamma^2 \left(c^2\mathrm{d}t^2 + \beta^2\mathrm{d}x^2 - 2\beta c\mathrm{d}t\mathrm{d}x - \mathrm{d}x^2 - \beta^2 c^2\mathrm{d}t^2 + 2\beta c\mathrm{d}t\mathrm{d}x\right) \\
&= \gamma^2 c^2\mathrm{d}t^2 \left(1-\beta^2\right) - \gamma^2\mathrm{d}x^2\left(1-\beta^2\right) \\
&= c^2\mathrm{d}t^2 - \mathrm{d}x^2 \\
&= \left(\mathrm{d}S\right)^2
\end{aligned}$$

$\therefore$ $\mathrm{d}S$ is the same in both frames. This is the Lorentz Invariance.

19 Intervals and Causality

No influence can travel faster than the speed of light, c. This means that for any event there can be defined light cones which contain all events that this event might have influenced and all events that might have influenced this event. These are all the possible events where there can be a causal connection.



Future Cone Region in space-time where event A may influence what happens

Past Cone Region in space-time which contains all the events that could have had an influence on A.

Events outside these regions have no causal connection with what happened at A. The condition for an event to be on one of these cones is if the separation in space is $\mathrm{d}r$ in the instant of time $\mathrm{d}t$, then

$$|\mathrm{d}r| \leq c|\mathrm{d}t|$$

as measured in any reference frame. Expressed as an invariant interval ,

$$\mathrm{d}S^2 = c^2\mathrm{d}t^2 - \mathrm{d}r^2 \geq 0$$

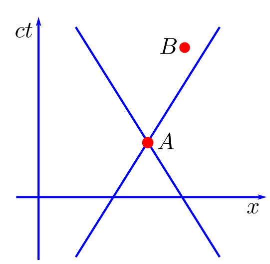
So this interval is invariant, so is the same in all reference frames.

19.1 $\mathrm{d}S^2 > 0$ "Time-like" Interval

A signal with speed less than c can connect these two events, A and B. The invariant distance is time-like,

$$\mathrm{d}\tau = \left(c^2\mathrm{d}t^2 - \mathrm{d}r^2\right)^{\frac{1}{2}}$$

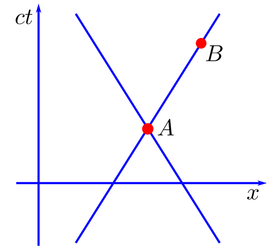
This is known as the proper time.



19.2 $\mathrm{d}S^2 = 0$ "Light-like" Interval

Only something traveling at the speed of light could connect these two events,

$$\mathrm{d}\tau = 0$$

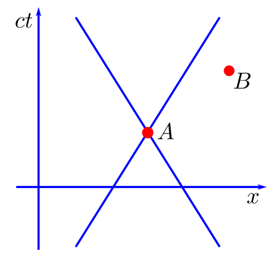


19.3 $\mathrm{d}S^2 < 0$ "Space-like" Interval

Nothing can connect these two events as it would have to travel faster than light,

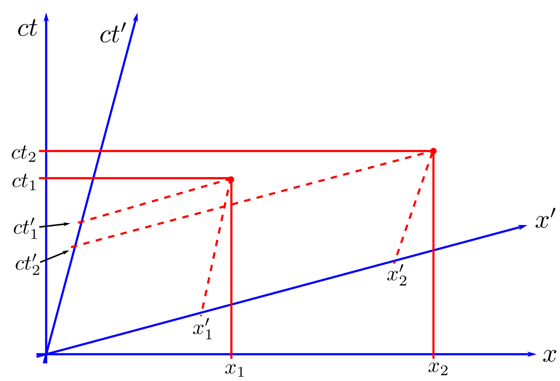
$$\mathrm{d}\sigma = \left(\mathrm{d}r^2 - c\mathrm{d}t^2\right)^{\frac{1}{2}}$$

This is the proper distance.



20 Causality and Time Order

Two observers need not agree on the time order of events.



Since the events are outside each other's event cone, there is no problem here as the two events are not causally connected.

Let

$$\begin{aligned}
\mathrm{d}t &= t_2 - t_1 > 0 \\
c\mathrm{d}t' &= ct'_2 - ct'_2 \\
&= \gamma\left(ct_2 - \beta x_2\right) - \gamma\left(ct_1 - \beta x_1\right) \\
\Rightarrow c\mathrm{d}t' &= \gamma\left(c\mathrm{d}t - \beta\mathrm{d}x\right)
\end{aligned}$$

If $t'_1 > t'_2$ then $\mathrm{d}t' < 0$,

$$\begin{aligned}
\Rightarrow \beta\mathrm{d}x &> c\mathrm{d}t \\
\mathrm{d}x &> c\mathrm{d}t
\end{aligned}$$

So this pair are space-like separated.

21 Energy and Momentum

Energy and momentum transform between frames in the same way as for position and time,

$$\left.\begin{array}{rcl} p'_x c &=& \gamma\left(p_x c - \beta E\right) \\ p'_y c &=& p_y c \\ p'_z c &=& p_z c \\ E' &=& \gamma\left(E - \beta p_x c\right) \end{array}\right\} \begin{array}{c}\text{Energy momentum} \\ \text{4 vector}\end{array}$$

As these transform in exactly the same way as $(ct, \underline{r})$, the combination of corresponding to $\mathrm{d}S^2$ must also be invariant.

$$P^2 = E^2 - p^2c^2$$

is also invariant, independent of the frame that it is measured in. In the center of mass frame of a particle, $p = o$, so

$$P^2 = E_{cm}^2$$

For a single object, $E_{cm} = mc^2$, therefore,

$$E^2 - p^2c^2 = m^2c^4$$

in any frame.

- Photon, $E = pc$, $E^2 - p^2c^2 = 0$, $\quad\therefore$ photon is massless
- Multiple-particle system, $E = \sum_i E_j$, $\underline{p} = \sum_i \underline{p}_i$

$$\Rightarrow E^2 - p^2c^2 = E_{cm}^2$$

If we combine these into one object, then

$$E_{cm}^2 = \text{mass} \times c^2$$

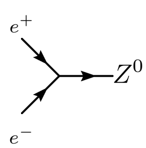
then we still write

$$E^2 - p^2c^2 = m^2c^4$$

where m^2c^4 is the invariant mass of the system.

<u>Ex</u>

Create Z^0 boson in e^+e^- collision



$$\begin{array}{rcl} m_z c^2 &\approx& 90\text{GeV} \\ E^2 - p^2c^2 &=& m_z^2 c^2 \end{array}$$

When symmetric beams are produced,

$$\begin{array}{rcl} E_+ = E_- = E_e & \Rightarrow & \sum E = 2E_e \\ \underline{p}_+ = -\underline{p}_- & \Rightarrow & \sum \underline{p} = 0 \\ \Rightarrow E_{cm} &=& 2E_e = m_z c^2 \\ E_e &=& \dfrac{m_z c^2}{2} \end{array}$$

21.1 Fixed Target

A positron beam is fired at a stationary hydrogen target. We know that

$$\begin{aligned} m_z^2c^4 &= \left(E_+ + m_ec^2\right)^2 - p_+^2c^2 \\ &= E_+^2 + 2m_ec^2E_+ + m_e^2c^4 - p_+^2c^2 \\ &= 2m_ec^2E_+ + 2m_e^2c^4 \end{aligned}$$

$$m_z^2c^4 \approx 90^2, \qquad \begin{aligned} m_e^2c^4 &= \left(0.5\times10^{-3}\right)^2 \\ \Rightarrow 90^2 &\approx 2m_ec^2E_+ \\ E_+ &\approx 8.1\times10^6\text{GeV} = 8.1\text{PeV} > 1000\times\text{LHC beam energy} \end{aligned}$$