Significance of Commuting Operators

We showed in the lecture that if 2 operators shared a complete set of eigenfunctions, i.e. if every wavefunction can be written as a linear combination of functions which are eigenfunctions of both operators, then the two operators will commute:

$$\left[\hat{A},\hat{B}\right]\Psi = 0 \qquad \text{for any function } \Psi$$

The significance of the two operators sharing such a set of eigenfunctions is that it is possible to know precise values for both observables at the same time.

What we did not show was the converse: the fact that the two operators commute means that they must have a set of common eigenfunctions. The argument is slightly more subtle than the previous one.

Consider two commuting operators $\hat{A}$ and $\hat{B}$. Let $\hat{A}$ have a complete set of eigenfunctions ψ_n. Now let us evaluate the commutator applied to one of these eigenfunctions:

$$\begin{aligned} \left[\hat{A},\hat{B}\right]\psi_n = \hat{A}\hat{B}\psi_n - \hat{B}\hat{A}\psi_n &= 0 \\ \therefore \hat{A}\hat{B}\psi_n &= \hat{B}\hat{A}\psi_n \\ \hat{A}\left(\hat{B}\psi_n\right) &= a_n\hat{B}\psi_n \end{aligned}$$

Therefore $\hat{B}\psi_n$ is an eigenfunction of $\hat{A}$ with eigenvalue a_n.

There are now 2 cases we should consider:

1. The eigenfunctions ψ_n of $\hat{A}$ are non-degenerate, i.e. all of the eigenvalues are different.

   In this case it follows that since $\hat{B}\psi_n$ is an eigenfunction of $\hat{A}$ with the same eigenvalue, a_n, as ψ_n, $\hat{B}\psi_n$ must be the same eigenfunction as ψ_n itself. However, it could be ψ_n multiplied by a constant, as that would not affect the eigenvalue equation. So in general:

   $$\hat{B}\psi_n = \text{constant} \times \psi_n$$

   Thus ψ_n is also an eigenfunction of $\hat{B}$, and so the operators $\hat{A}$ and $\hat{B}$ do have a set of common eigenfunctions.

2. Some of the eigenfunctions ψ_n are degenerate, i.e. have same eigenvalue.

   Let us say that ψ_j and ψ_k have the same eigenvalue, a:

   $$\begin{aligned} \hat{A}\psi_j &= a\psi_j \\ \hat{A}\psi_k &= a\psi_k \end{aligned}$$

   Then the statement

   $$\hat{A}\left(\hat{B}\psi_j\right) = a\hat{B}\psi_j$$

does not imply that $\hat{B}\psi_j = \text{constant} \times \psi_j$, i.e. ψ_j need not be an eigenfunction of $\hat{B}$. What it does mean is

$$\hat{B}\psi_j \quad = \quad c_j\psi_j + c_k\psi_k$$

i.e. that $\hat{B}\psi_j$ must be a linear combination of the eigenfunctions of $\hat{A}$ with this eigenvalue. However, since any linear combination of these eigenfunctions is an equally good eigenfunction of $\hat{A}$, we may simply find the combinations of ψ_j and ψ_k which *are* eigenfunctions of $\hat{B}$ and use these as eigenfunctions of $\hat{A}$ in place of ψ_j and ψ_k.

Hence the fact that the two operators commute does indeed mean that the eigenfunctions of one operator are eigenfunctions of the other, i.e. that they share a set of eigenfunctions.