Properties of Commutators

Although in the lectures our main interest in commutators of 2 operators will be whether they are zero or non-zero (whether the operators commute or not), it can be useful, even for this limited purpose, to be able to do some algebra with them. We will use this later when we look at angular momentum, for example.

There are a few algebraic properties of commutators which can be useful. Starting from the definition of the commutator of 2 operators:

$$\left[\hat{A},\hat{B}\right]=\hat{A}\hat{B}-\hat{B}\hat{A},$$

we can obtain the following:

$$\begin{aligned}\left[\hat{B},\hat{A}\right] &= \hat{B}\hat{A}-\hat{A}\hat{B}\\ &= -\left[\hat{A},\hat{B}\right],\end{aligned}$$

and, of course

$$\left[\hat{A},\hat{A}\right]=0.$$

Also,

$$\begin{aligned}\left[\hat{A},\hat{B}+\hat{C}\right] &= \hat{A}\left(\hat{B}+\hat{C}\right)-\left(\hat{B}+\hat{C}\right)\hat{A}\\ &= \hat{A}\hat{B}-\hat{B}\hat{A}+\hat{A}\hat{C}-\hat{C}\hat{A}\\ &= \left[\hat{A},\hat{B}\right]+\left[\hat{A},\hat{C}\right],\end{aligned}$$

$$\begin{aligned}\left[\hat{A}\hat{B},\hat{C}\right] &= \hat{A}\hat{B}\hat{C}-\hat{C}\hat{A}\hat{B}\\ &= \hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B}+\hat{A}\hat{C}\hat{B}-\hat{C}\hat{A}\hat{B}\\ &= \hat{A}\left[\hat{B},\hat{C}\right]+\left[\hat{A},\hat{C}\right]\hat{B},\end{aligned}$$

and

$$\left[\hat{A},\left[\hat{B},\hat{C}\right]\right]+\left[\hat{B},\left[\hat{C},\hat{A}\right]\right]+\left[\hat{C},\left[\hat{A},\hat{B}\right]\right]=0.$$

These relations can be useful where you know the commutator(s) of 2 (or more) operators and need to find whether particular combinations of them commute or not, allowing you to express the unknown commutator as a combination of known ones.