Ladder Operator Solution of the Harmonic Oscillator

Solving the Schrödinger equation for the harmonic oscillator potential is straightforward but slightly long-winded. However, it is possible to find the energy eigenvalues without actually solving the differential equation. The method is quite simple, and a nice introduction to more sophisticated approaches to quantum mechanics, so this handout is provided for anyone who is interested. The method described here will not be needed for any assessment in this course though.

We start from the time-independent Schrödinger equation in operator form:

$$\hat{H}\psi = E\psi \,,$$

where E is the energy eigenvalue and $\hat{H}$ is the hamiltonian:

$$\begin{aligned} \hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 x^2 \\ &= \left(\frac{\hat{p}^2}{2m\omega} + \frac{1}{2}m\omega x^2\right)\omega \,. \end{aligned}$$

We see that this contains 2 quadratic terms. For 2 numbers b and c we could write:

$$b^2 + c^2 = (b + ic)(b - ic) \,.$$

Since operators do not in general commute, for 2 operators $\hat{B}$ and $\hat{C}$ we instead find:

$$\begin{aligned} (\hat{B} + i\hat{C})(\hat{B} - i\hat{C}) &= \hat{B}^2 + \hat{C}^2 + i(\hat{C}\hat{B} - \hat{B}\hat{C}) \\ \therefore \qquad \hat{B}^2 + \hat{C}^2 &= (\hat{B} + i\hat{C})(\hat{B} - i\hat{C}) + i[\hat{B}, \hat{C}] \,. \end{aligned}$$

So, using this we rewrite the hamiltonian as:

$$\hat{H} = \left(\frac{\hat{p}^2}{2m\omega} + \frac{1}{2}m\omega x^2\right)\omega = \left(\frac{\hat{p}}{\sqrt{2m\omega}} + ix\sqrt{\frac{m\omega}{2}}\right)\left(\frac{\hat{p}}{\sqrt{2m\omega}} - ix\sqrt{\frac{m\omega}{2}}\right)\omega + \frac{i}{2}[\hat{p}, x]\omega \,.$$

We know that $[\hat{p}, x] = -i\hbar$, so if we introduce 2 new operators:

$$\begin{aligned} \hat{A}^\dagger &= \frac{\hat{p}}{\sqrt{2m\omega}} + ix\sqrt{\frac{m\omega}{2}} \,, \\ \hat{A} &= \frac{\hat{p}}{\sqrt{2m\omega}} - ix\sqrt{\frac{m\omega}{2}} \,, \end{aligned}$$

we can rewrite the hamiltonian in a compact form:

$$\hat{H} = \hat{A}^\dagger \hat{A}\omega + \frac{\hbar\omega}{2} \,.$$

We need just one more thing: the commutators of A, $A^\dagger$ with each other and the hamiltonian. A little algebra gives us:

$$\begin{aligned} [\hat{A}, \hat{A}^\dagger] &= i[\hat{p}, x] = \hbar \,, \\ \left[\hat{H}, \hat{A}\right] &= [\omega\hat{A}^\dagger\hat{A}, \hat{A}] + [\frac{\hbar\omega}{2}, \hat{A}] \\ &= \omega(\hat{A}^\dagger[\hat{A}, \hat{A}] + [\hat{A}^\dagger, \hat{A}]\hat{A}) \\ &= -\hbar\omega\hat{A} \,, \\ \left[\hat{H}, \hat{A}^\dagger\right] &= \hbar\omega\hat{A}^\dagger \,. \end{aligned}$$

So, how does all this help?

Consider an eigenfunction ϕ_j with energy eigenvalue E_j:

$$\hat{H}\phi_j = E_j\phi_j\,.$$

From above we have

$$\begin{aligned}
[\hat{H}, \hat{A}]\phi_j &= -\hbar\omega\hat{A}\phi_j \\
\hat{H}\hat{A}\phi_j - \hat{A}\hat{H}\phi_j &= -\hbar\omega\hat{A}\phi_j \\
\hat{H}\hat{A}\phi_j - E_j\hat{A}\phi_j &= -\hbar\omega\hat{A}\phi_j \\
\hat{H}\hat{A}\phi_j &= (E_j - \hbar\omega)\hat{A}\phi_j\,,
\end{aligned}$$

and so $\hat{A}\phi_j$ is also an eigenfunction of $\hat{H}$ with eigenvalues $E_j - \hbar\omega$. Thus $\hat{A}$ converts one eigenfunction into another with a lower energy eigenvalue (or, if you prefer, it lowers the energy of the particle by $\hbar\omega$). It is therefore referred to as a "lowering operator"

We can similarly show that $\hat{A}^\dagger$ has the opposite effect, and increases the energy by $\hbar\omega$. It is therefore referred to as a "raising operator". The two are collectively known as "ladder operators", as they allow us to step up and down through the "ladder" of eigenvalues.

The well is infinitely deep, so will contain an infinite number of eigenvalues. Hence we can apply the raising operator as many times as we like and the results will make sense. However, there must be some "ground state" where lowering the energy further would result in a negative eigenvalue. Since energy eigenvalues cannot be negative, then for this ground state ϕ_0 the only solution of the equation:

$$\hat{H}\hat{A}\phi_0 = (E_0 - \hbar\omega)\hat{A}\phi_0$$

is:

$$\hat{A}\phi_0 = 0\,.$$

Using this, we may now solve the Schrödinger equation:

$$\begin{aligned}
\hat{H}\phi_0 = E_0\phi_0 &= \omega\hat{A}^\dagger\hat{A}\phi_0 + \frac{1}{2}\hbar\omega\phi_0 = \frac{1}{2}\hbar\omega\phi_0 \\
\therefore \qquad E_0 &= \frac{1}{2}\hbar\omega
\end{aligned}$$

and having obtained the ground state energy, we just apply the raising operator to find the other energy eigenvalues:

$$\begin{aligned}
E_1 &= E_0 + \hbar\omega = \frac{3}{2}\hbar\omega\,, \\
E_2 &= E_1 + \hbar\omega = \frac{5}{2}\hbar\omega\,, \\
&\text{etc.},
\end{aligned}$$

and thus obtain the general result:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega\,.$$