Calculating the Transmission Coefficient for a Potential Barrier

Consider a simple barrier potential:

$$\begin{aligned} V(x) &= 0 & x < 0 \\ &= V_0 & 0 \le x \le a \\ &= 0 & x > a \end{aligned}$$

For a particle approaching the barrier from $x < 0$, with $E < V_0$, we may write the equations and solutions in the three regions as:

$$x < 0: \qquad \frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \Rightarrow \psi = Ae^{ikx} + Be^{-ikx}$$

$$0 \le x \le a: \qquad \frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \Rightarrow \psi = Ce^{-Kx} + De^{Kx}$$

$$x > a: \qquad \frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \Rightarrow \psi = Fe^{ikx}$$

where

$$k = \frac{\sqrt{2mE}}{\hbar}, \qquad K = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

We are interested in the transmission coefficient:

$$T = \frac{\text{transmitted flux}}{\text{incident flux}} = \frac{v|F|^2}{v|A|^2}$$

So to obtain this we need to relate A to F.

The various normalisation constants are related by the requirements that ψ and ψ' must be continuous across the two boundaries:

$$x = 0: \text{Continuity of } \psi \Rightarrow \; A + B = C + D \qquad (1)$$

$$\text{Continuity of } \psi' \Rightarrow \; ik(A - B) = -K(C - D)$$

$$\Rightarrow \; A - B = \frac{iK}{k}(C - D) \qquad (2)$$

$$x = a: \text{Continuity of } \psi \Rightarrow \; Ce^{-Ka} + De^{Ka} = Fe^{ika} \qquad (3)$$

$$\text{Continuity of } \psi' \Rightarrow \; -K\left(Ce^{-Ka} - De^{Ka}\right) = ikFe^{ika}$$

$$\Rightarrow \; \frac{-ik}{K}Fe^{ika} = \left(Ce^{-Ka} - De^{Ka}\right) \qquad (4)$$

The conditions at $x = 0$ allow us to write A in terms of C and D. Adding (1) and (2) we obtain:

$$A = \frac{C}{2k}(k + iK) + \frac{D}{2k}(k - iK) \qquad (5)$$

The conditions at $x = a$ meanwhile allow us to write C and D in terms of F. By adding and subtracting (4) to/from (3) we obtain:

$$D = \frac{1}{2} F e^{ika} e^{-Ka} \left(1 + \frac{ik}{K}\right) = \frac{F}{2K}(K + ik)e^{ika} e^{-Ka}$$
$$C = \frac{1}{2} F e^{ika} e^{Ka} \left(1 - \frac{ik}{K}\right) = \frac{F}{2K}(K - ik)e^{ika} e^{Ka}$$

and substituting these into (5) gives:

$$A = \frac{F}{4kK}(K - ik)(k + iK)e^{ika} e^{Ka} + \frac{F}{4kK}(K + ik)(k - iK)e^{ika} e^{-Ka}$$
$$A = \frac{F}{4kK}\left(i(K - ik)^2 e^{Ka} - i(K + ik)^2 e^{-Ka}\right)e^{ika}$$
$$A = \frac{iF}{4kK}\left((K - ik)^2 e^{Ka} - (K + ik)^2 e^{-Ka}\right)e^{ika}$$

We can therefore calculate the transmission coefficient T :

$$T = \frac{F^* F}{A^* A} = \frac{16k^2 K^2}{\left((K + ik)^2 e^{Ka} - (K - ik)^2 e^{-Ka}\right)\left((K - ik)^2 e^{Ka} - (K + ik)^2 e^{-Ka}\right)}$$

However, the expression is a lot simpler if we consider the case $Ka >> 1$ (so very high or very thick barrier). In this case the relation between A and F simplifies to:

$$A = \frac{iF}{4kK}(K - ik)^2 e^{Ka} e^{ika}$$

giving:

$$T = \frac{F^* F}{A^* A} = \frac{16k^2 K^2}{(K + ik)^2 e^{Ka} (K - ik)^2 e^{Ka}} = \frac{16k^2 K^2}{\left(K^2 + k^2\right)^2} e^{-2Ka}$$

and substituting for k and K we finally obtain:

$$T = 16 \frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2Ka}$$