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CP600 Practical Algorithm Design

Assignment #1

Problem 1

Express the function $\frac{n^3}{1000} - 100n^2 - 100n + 3$ in terms of Θ-notation

There must be two constants c_1 and c_2 such that for all $n \geq n_0$ and the value of $f(n)$ will be between $c_1 g(n)$ and $c_2 g(n)$.

$$f(n) = \frac{n^3}{1000} - 100n^2 - 100n + 3$$

$$g(n) = n^3$$

$$c_1 n^3 \leq \frac{n^3}{1000} - 100n^2 - 100n + 3 \leq c_2 n^3$$

Find c_2:

$$c_2 n^3 \geq \frac{n^3}{1000} - 100n^2 - 100n + 3$$

$$c_2 n^3 \geq \frac{n^3}{1000} + 3 \qquad n = 1$$

$$c_2 \geq \frac{1}{1000} + 3$$

$$c_2 = 4$$

Find c_1:

$$c_1 n^3 \leq \frac{n^3}{1000} - 100n^2 - 100n + 3$$

$$c_1 n^3 \leq \frac{n^3}{1000} - 100n^2 - 100n$$

$$c_1 \leq \frac{1}{1000} - \frac{100}{n} - \frac{100}{n^2} \qquad n = 10^6$$

$$c_1 \leq \frac{1}{1000} - \frac{1}{10000} - \frac{1}{1000000000}$$

$$c_1 = \frac{1}{2000}$$

$\therefore f(n) = \Theta(n^3)$ because there exists c_1 and c_2 with $n \geq n_0$

Problem 2

Show that for any real constants a and b, where $b > 0$, $(n+a)^b = \Theta(n^b)$

There must be two constants c_1 and c_2 such that for all $n \geq n_0$ and the value of $f(n)$ will be between $c_1 g(n)$ and $c_2 g(n)$

$$0 \leq c_1 n^b \leq (n+a)^b \leq c_2 n^b$$

Since $n+a \leq 2n$ when $|a| \leq n$, and $n+a \geq \frac{1}{2}n$ when $|a| \leq \frac{n}{2}$

Thus, when $n \geq 2|a|$

$$0 \leq \frac{n}{2} \leq n+a \leq 2n$$

As $b > 0$ we can add exponent without breaking inequality

$$0 \leq \left(\frac{n}{2}\right)^b \leq (n+a)^b \leq (2n)^b$$

$$0 \leq \frac{1}{2^b} n^b \leq (n+a)^b \leq 2^b n^b$$

Therefore $(n+a)^b = \Theta(n^b)$ because there exists $c_1 = \frac{1}{2^b}$, $c_2 = 2^b$ and $n_0 = 2|a|$

Problem 3

Rank the following function in increasing order of growth using O-notation:

$f_1(n) = n^2 \lg n$

$f_2(n) = n(\lg n)^2$

$f_3(n) = \sum_i^n 2^i$

$f_4(n) = \lg(\sum_{i=0}^n 2^i)$

Using mathmatical induction to prove

$\sum_{i=0}^n 2^i = O(2^n)$, find c and n_0 such that

$$\sum_{i=0}^n 2^i \leq c2^n$$

Basis:

$S(0) = \sum_{i=0}^0 2^i = 2^0 = 1 \leq c1$ iff $c \geq 1$

Hypothesis:

Assume that $S(k) = \sum_{i=0}^k 2^i \leq c2^k$

Induction:

Show that $S(k+1) = \sum_{i=0}^{k+1} 2^i \leq 2^{k+1}$ is true

$$\begin{aligned} S(k+1) &= \sum_{i=0}^{k+1} 2^i \\ &= \sum_{i=0}^{k} 2^i + 2^{k+1} \\ &\leq c2^k + 2^{k+1} \\ &= \left(\tfrac{1}{2} + \tfrac{1}{c}\right)c2^{k+1} \quad \text{since } \left(\tfrac{1}{2} + \tfrac{1}{c}\right) \leq 1 \end{aligned}$$

Claim:

Since $S(k+1)$ holds for any positive integer k, we claim that $\sum_{i=0}^n 2^i = O(2^n)$

We can now rank the functions using O-notation (least growth to greatest)

$f_4 = O(\lg 2^n)$, $f_2 = O(n(\lg n)^2)$, $f_1 = O(n^2 \lg n)$, $f_3 = O(2^n)$

Problem 4

Use mathematical induction to prove the following

a) $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} = \Theta(n^3)$

Basis:

$S(1) = c_1 \cdot 1 \leq \frac{k(k+1)(2k+1)}{6} = \frac{6}{6} = 1 \leq c_2 \cdot 1$

iff $c_1 \leq 1$ and $c_2 \geq 1$

Hypothesis:

Assume that $S(k) = c_1 k^3 \leq \frac{k(k+1)(2k+1)}{6} \leq c_2 k^3$

Induction:

$S(k+1) = c_1(k+1)^3 \leq \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \leq c_2(k+1)^3$

$= c_1(k+1)^3 \leq \frac{(k+1)(k+2)(2k+3)}{6} \leq c_2(k+1)^3$

$= c_1(k^3+3k^2+3k+1) \leq \frac{k^3}{3} + \frac{3k^2}{2} + \frac{13k}{6} + 1 \leq c_2(k^3+3k^2+3k+1)$

$c_1 = \frac{1}{5}, c_2 = 1$

$\therefore$ Since $S(k+1)$ holds for any positive integer k and there exists c_1 and c_2 we claim $\frac{n(n+1)(2n+1)}{6} = \Theta(n^3)$

b) $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} = \Theta(n^4)$

Basis:

$$S(1) = c_1 \cdot 1 \le \frac{k^2(k+1)^2}{4} = \frac{4}{4} = 1 \le c_2 \cdot 1$$

iff $c_1 \le 1$ and $c_2 \ge 1$

Hypothesis:

Assume that $S(k) = c_1 k^4 \le \frac{k^2(k+1)^2}{4} \le c_2 k^4$

Induction:

$$S(k+1) = c_1(k+1)^4 \le \frac{(k+1)^2((k+1)+1)^2}{4} \le c_2(k+1)^4$$

$$c_1(k+1)^4 \le \frac{(k+1)^2(k+2)^2}{4} \le c_2(k+1)^4$$

$$c_1(k^4 + 4k^3 + 6k^2 + 4k + 1) \le \frac{k^4}{4} + \frac{3k^3}{2} + \frac{13k^2}{4} + 3k + 1 \le c_2(k^4 + 4k^3 + 6k^2 + 4k + 1)$$

$c_1 = \frac{1}{5}$, $c_2 = 1$

$\therefore$ Since $S(k+1)$ holds for any positive integer k and there exists c_1 and c_2 we claim

$$\frac{n^2(n+1)^2}{4} = \Theta(n^4)$$