Hierarchical Modularity: The Description of Multi-Level Complex Systems as Nested Coupled Fokker-Planck Equations

Santiago Núñez-Corrales
e-Science Research Program
Costa Rica Institute of Technology
snunez@itcr.ac.cr
Eric Jakobsson
National Center for Supercomputing Applications
University of Illinois at Urbana-Champaign
jake@ncsa.uiuc.edu

Complex systems, viewed as networks of interacting components giving rise to nonlinearity and emergent behaviour, can be modeled as stochastic systems inherently determined by the choice of adequate parameters. Fokker-Planck Equations (FPE) are a proper formalism for analysis of stochastic systems where their solution is equivalent to finding the probability density of values in macroscopic variables used to finely determine related dynamics in detail. Beginning with Einsteins work on Brownian motion, we know that all problems that can be represented as diffusion, or mass action, can be represented as FPEs. Since Schrödingers equation is a diffusion equation in 3 plus one dimensions (x, y, z, ict) we know that the FPE description pertains down to the level of quantum phenomena. Since the distribution of galactic clusters has the pattern characteristic of a solution to an FPE (locally rough but globally smooth), it appears that FPEs describe phenomena on the largest possible scale.

We hypothesize that FPEs are universal descriptors at all scales of organization of

natural systems, with different levels of description having characteristic core theoretical descriptions. At many of those levels FPEs are not only appropriate, but more correctly describe the phenomena under scrutiny. We illustrate this point by showing that the FPEs corresponding to the classic Hodgkin-Huxley description of nerve excitation correctly describe the behavior of a nerve cell at threshold, whereas the original classical H-H description produces incorrect behavior. We revisit the butterfly effect by translating a deterministic chaotic model into the corresponding FPEs.

We explore the connection between multi-level complex systems and their descriptions in terms of nested coupled FPEs, where the parameters are stochastic and their average values can be mapped to other FPEs at a more disaggregate level of description of reality. Also, an approach based on the application of functionals is discussed both for stochastic variables and parameters as a means of unifying distinct theories into a single mathematical corpus. The application of this method is discussed in the context of the strong characterization of parameters at different theories as fundamental in Nature or deducible from a theory at a lower level of description. Finally we note that the underlying physics of the quantum computers of the future will lead directly to a description of natural systems as FPEs rather than deterministic differential equations.

1 Introduction

Nature as a whole exhibits a wide variety of admissible theoretical descriptions based on experimental observation. The development of ontologies based on the scientific method –supported by constantly-improving mathematical tools– has lead to several formal descriptions where each one of those seems suited to a particular level in the structure of the universe depending on a desired focus. Nonetheless, there is still much work to be done in providing a single general, multiscale framework for complex systems, one able to deal with the inherent stochastic nature of reality, would provide valuable insight into the formal properties of dynamic system and the interpretation of their parameters and constants at each level of the integrated theory as fundamental or artificial.

Nevertheless, mathematical artifacts remain that despite their role of providing essential operational scaffolding, make extremely difficult to obtain a reasonable interpretation at the interface of phenomena described by different theories. The complexity in reconciling macroscopic behaviour with the delicate selection of microscopic parameters (e.g. entropy and the arrow of time, conservation or loss of information in black holes) is one of many examples for the need of rethinking our theoretical approaches to an integrated theory of Nature.

1.1 The General Problem of the Particulate Nature of Systems and Continuum Descriptions of System Dynamics

The line of argument in this subsection follows closely Jakobsson (2011). Consider the defining relationship for a derivative:

$$\frac{dx}{dy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x}{\Delta y} \tag{1}$$

as applied to the dynamics of a system variable b evolving in time t:

$$\frac{db}{dt} = \lim_{\substack{\Delta b \to 0 \\ \Delta t \to 0}} \frac{\Delta b}{\Delta t} \tag{2}$$

where b is some quantifiable attribute of a system and t is time. Now there is a fundamental problem with the equation above. We can imagine Δt becoming arbitrarily small, because time is not quantized, as far as it is known. But Δb can not become arbitrarily small, because the physical quantities in the systems we simulate are generally composed by subunits of a finite and definable size. The size could be as small as an atom or a molecule, or as large as an organism, a community of organisms, a financial entity, or even of astronomical size, such as stars and galaxies. What can be done after realizing Δb cannot go to zero? There are essentially two choices:

1. Pretend that Δb can go to zero and use a continuous differential equation model to describe the system. This is often done, and will generally capture a lot of the behavior of the system. However, such a model will deviate from the behavior of the real system at thresholds, when small changes in the state of the system will cause large changes in the subsequent trajectory. A prime example of such deviation is in dynamic population models of the predator-prey type (Leslie and Gower, 1960). The difference couldnt be more dramatic. Stochastic versions of predator-prey models predict that species extinctions are more likely in small populations, whereas deterministic models do not. With respect to this critically important aspect of ecology, stochastic models must be used (Drake and Lodge, 2004). In this paper we will suggest that in many other systems stochastic behaviour is fundamental to obtain correct approximations of the evolution of dynamic systems.

2. Follow the path suggested by Einsteins classic Brownian motion paper of 1905 [3]. Here Einstein showed the correspondence between diffusion and random walk. In these papers Einstein showed that the microscopic phenomenon underlying the continuum diffusion equation was a random walk, or *drunkards walk*. In the most straightforward extension to modeling systems, this path means that the continuous variable b in the expression above is replaced by an integer representing the number of subunits in the state b:

$$\Delta N_b = \Delta t \cdot R\left(\frac{db}{dt}\right) \tag{3}$$

where ΔN_b is the change in the number of subunits in state b during the time interval Δt and R is a random number drawn from a distribution

weighted according to $\frac{db}{dt}$, interpreted as a change function. This new relationship is an example of a Fokker-Planck equation broadly defined, in that it combines a deterministic bias and a random walk to describe the system dynamics.

In this paper we assert that, because of the universality of the particulate nature of complex systems, the Fokker-Planck description of systems is always the more rigorously correct. The continuum description is at best an approximation, and at worst will give qualitatively incorrect results, such as in the ecological example given above. Therefore, although the continuum description may often be more computationally efficient, one should always consider the possibility that it will give wrong answers, and one should use the Fokker-Planck description whenever feasible. Additionally, at a more profound philosophical level, we present the case of measurement in quantum computing where the Fokker-Planck equation is intrinsically defined into quantum mechanics via the probability density function spanned by the eigenstates of the system. Therefore, computing with quantum rules –the most strict version of the Physical Church-Turing hypothesis- can be understood as a process where a global stochastic result comes from decoherence due to environmental random noise.

2 The Fokker-Planck Equation

Stochastic systems are characterized by the combination of a statistically deterministic macrostate with an ensemble of microstates disturbed by random noise (e.g. Browinian motion) [10]. Suppose that a particular system is made up of a large amount of particles (e.g. proportional to Avogadro's number). It stochastic nature is evident from the facts that (1) the solution of coupled equations for all particles is not feasible and (2) even if finding a solution were feasible, the initial values for all particles in the system are unknown. Moreover, the dynamics of systems identical in every respect except in their initial values differ, being the most general state a *Gibbs ensemble* in thermodynamics jargon.

2.1 Derivation of the Fokker-Planck Equation

Consider a Gibbs ensemble where $x(t)$ is a variable describing a macroscopic property of the system and $\Gamma(t)$ a stochastic force such that the differential equation

$$F[x(t)] = \Gamma(t) \tag{4}$$

describes its dynamics, where $F[\cdot]$ is a functional of $x(t)$. $\Gamma(t)$ is a force that depends on the force $h(t)$ resulting from the interaction of all particles of the system with a single particle, described in terms of both a dampening, continuous force $h_c(t)$ and a fluctuating force $h_f(t)$. The force

$$h(t) = h_c(t) + h_f(t) \tag{5}$$

is hence stochastic and therefore

$$\Gamma(t) = g(h_c(t), h_f(t)) \tag{6}$$

will be defined as a function g of both dampening and fluctuations in the system. $\Gamma(t)$ is usually called a *Langevin force* for which

$$\langle \Gamma(t) \rangle = 0 \tag{7}$$

$$\langle \Gamma(t)\Gamma(t') \rangle = 0, \quad |t - t'| \geq \tau_0 \tag{8}$$

$$\langle \Gamma(t)\Gamma(t') \rangle = q\delta(t - t') \tag{9}$$

$$\delta(x) = \lim_{a \to \infty} \frac{1}{a\sqrt{\pi}} e^{-\frac{x^2}{a^2}} \tag{10}$$

$$\tag{11}$$

for finite particle interaction time τ_0 and relaxation time τ for $x(t)$ such that $\tau_0 << \tau$ The $\Gamma(t)$ follow a Gaussian distribution with *δ-correlation* or *white noise*.

Having that $\Gamma(t)$ changes by definition from system to system, we may ask for the probability of measuring $x(t)$ and finding the value in the interval $[x, x+dx]$. The latter can be interpreted as the ratio of the number of systems for which $x(t) \in [x, x + dx]$ and the total amount of systems. The fact that $x(t)$ is continuous enables us to look for the *probability density* or *probability distribution* $W(x, t)$ of the system. Noting that the probability of finding $x(t)$ in $[x, x + dx]$ is given by

$$P(x < x(t) < x + dx) = dx \cdot W(x, t) \tag{12}$$

the equation describing the dynamics of $W(x, t)$ becomes

$$\frac{dW}{dt} = \left(-\frac{\partial}{\partial x}\right)^v D^{(v)}(x) W(x, t) \tag{13}$$

known as the *Kramers-Moyal* expansion. If $x(t)$ is a discrete variable, the coefficients $D^{(v)}$ do not generally vanish and a detailed approximation may be required. If otherwise $x(t)$ is continuous and the force $\Gamma(t)$ is Gaussian and δ-correlated, all coefficients $D^{(v)}$ for $v \geq 3$ vanish. Thus, we arrive at the *Fokker-Planck Equation* (FPE)

$$\frac{dW}{dt} = \left[-\frac{\partial}{\partial x} D^{(1)}(x) + \frac{\partial^2}{\partial x^2} D^{(2)}(x)\right] W(x, t) \tag{14}$$

also called a forward Kolmogorov equation. It is a second-order, parabolic differential equation with a diffusion coefficient $D^{(2)}(x)$ and a drift coefficient $D^{(1)}(x)$ both of which can be also dependent on time. By solving the FPE, a probability density function is found and average macroscopic values can be obtained by integration across the ensemble. Its application is both possible for

systems close to equilibrium or far from it. The FPE can be generalized for N variables $x_1(t), x_2(t), \cdots, x_N(t) = \bar{x}(t)$ as

$$\frac{dW}{dt} = \left[-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} D_i^{(1)}(x) + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} D_{ij}^{(2)}(x) \right] W(x,t) \tag{15}$$

where the drift vector $D_i^{(1)}$ and the diffusion tensor $D_{ij}^{(2)}$ normally depend on the N variables.

3 Nerve Excitation and the Hodgkin-Huxley Model

In this section we create a Fokker-Planck description of an excitable membrane that corresponds to the classical continuum Hodgkin-Huxley description [7], and show that the Fokker-Planck version gives the observed all-or-none behavior of the membrane in response to a threshold stimulus, while the continuum description does not. The Hodgkin-Huxley equations describe the generation of action potentials as due to the voltage-dependent increase and decrease in specific sodium and potassium conductance according to the equations:

$$Cond_{Na} = Cond_{max,Na} m^3 h \tag{16}$$

$$Cond_K = Cond_{max,K} n^4 \tag{17}$$

$$\frac{dm}{dt} = \alpha_m (1-m) + \beta_m m \tag{18}$$

$$\frac{dh}{dt} = \alpha_h (1-h) + \beta_h h \tag{19}$$

$$\frac{dn}{dt} = \alpha_n (1-n) + \beta_n n \tag{20}$$

where $\alpha_\{m,h,n\}$ and $\beta_\{m,h,n\}$ are empirical functions of voltage, $Cond_\{Na,K\}$ are the specific conductances of the membrane to sodium and potassium, respectively and $Cond_{max,\{Na,K\}}$ are the maximum specific conductances the membrane can achieve to sodium and potassium, respectively.

The variables m, h, n and are sometimes referred to as *gates*, in particular the sodium activation and inactivation gates and the potassium activation gates respectively. Although there was no molecular basis for understanding these variables when the Hodgkin-Huxley equations were formulated, biophysicists have subsequently discovered the molecular bases. The variables refer to the fraction of individual selective protein ion channels that are open and closed. The openings and closings are probabilistic. The m and the n gates correspond to a narrowing or widening of the channel pore at the intracellular end, while the h gate corresponds to the occlusion of the channel by an intracellular peptide

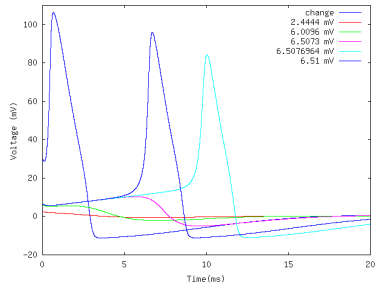

Figure 1: Deterministic Hodgkin-Huxley equations response to repeated stimuli near threshold.

ball that is tethered to the channel protein. The history of how these new understandings is well reviewed in [6].

Undergraduate student Santiago Santana in the Jakobsson Lab embodied these understandings of the molecular bases for excitability in a Fokker-Planck version of the Hodgkin-Huxley equations. In this modeling a patch of membrane with hundreds of individual sodium and potassium ion channels is the system. For each channel the m, h, n and gates opened and closed probabilistically with the opening and closing probabilities given by the and variables; i.e.

$$m_{closed} \underset{\beta_m}{\overset{\alpha_m}{\longleftrightarrow}} m_{open} \tag{21}$$

$$h_{closed} \underset{\beta_h}{\overset{\alpha_h}{\longleftrightarrow}} h_{open} \tag{22}$$

$$n_{closed} \underset{\beta_n}{\overset{\alpha_n}{\longleftrightarrow}} n_{open} \tag{23}$$

Specifically, for each time step every closed gate opened with a probability $\alpha\Delta t$ and every open gate closed with a probability $\beta\Delta t$. One of the differences between the deterministic and Fokker-Planck models is that in the deterministic model the resting potential is absolutely stationary while in the Fokker-Planck version there is resting noise, with the membrane potential fluctuating by a few tenths of a millivolt, consistent with experimental observation. Even more important for function is the response to a stimulus at the threshold of action potential firing.

Fig.1 shows the response of the deterministic Hodgkin-Huxley equations to repeated stimuli near threshold (a current pulse producing a sudden increment of voltage between 6 and 7 millivolts depolarized relative to resting potential; i.e., a change in the initial conditions for the simulation). It is seen that for small changes in the initial condition the later trajectories diverge, a sign of a system exhibiting deterministic chaos.

On the other hand, Fig.2 shows the behavior of the Fokker-Planck version of the Hodgkin-Huxley model. In these simulations the model is repeatedly stim-

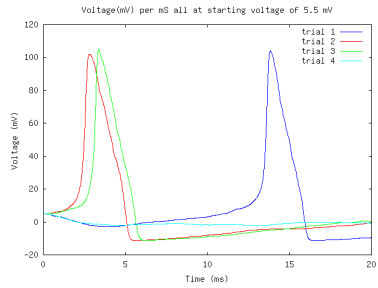

Figure 2: Response of the Fokker-Planck version of the Hodgkin-Huxley equations.

ulated by a jump of 5.5 mv. The threshold jump is slightly lower in this model than in the deterministic version. This is because of the reduced stability of the resting state due to noise. The nerve sometimes fires an action potential (3 times out of 4 in this case), and sometimes not, (1 time out of 4 in this case). The action potentials that do fire are almost exactly the same size (the experimentally observed all-or-none law), albeit with variable latencies after the stimulus. By contrast the amplitudes of the spikes in the deterministic model are variable, depending on small changes in the stimulus. Both the variable latency and the invariant size of the action potentials in response to repeated stimuli at threshold are observed in single nerve experiments [14]. In an earlier study combining experiment and simulation [4] it was shown that aperiodic responses to periodic stimuli in squid axons were due to noise of channel openings and closings rather than deterministic chaos. At that time, computer power limitations prevented using explicit probabilities for individual channel opening and closing. Therefore the noise was represented by a sum of periodic functions with amplitude and frequency distribution to correspond to experimentally measured membrane noise. This strategy can in general be applied to any system where the number of individual particles contributing to the noise makes computing them individually prohibitive.

4 Chaos, Noise and the Butterfly Effect

Chaos is generally defined as a property of deterministic dynamic systems, in which very small changes in initial conditions give rise to large changes in subsequent dynamic trajectories; i.e., the famous *Butterfly Effect*. *Noise* on the other hand, is a property of systems whose significant variables cannot all be individually precisely described, and which therefore must include some randomness in the description. The properties of a noisy system and a chaotic system, with respect to initial conditions, are very different. If one simulates a noisy system repeatedly using the same initial conditions, one will get different dynamic trajectories. If one changes the initial conditions slightly, one will get a slightly

different distribution of trajectories. There is no butterfly effect; i.e., a slight change in initial conditions leads to a slight change in the outputs. If one simulates a deterministic chaotic system repeatedly using the same initial conditions, one will get exactly the same trajectory. However, as noted above, small changes in the initial conditions may lead to large later changes in the trajectoriesthe butterfly effect.

The latter is particularly put in evidence by models that despite of having random variations in their input parameters do not show extreme sensitivity to them. Concretely, climate models exhibit this behaviour with isolated exceptions. As long as some sensitivity function can be computed, stochastic variations of parameters can be accounted for.

4.1 The Stochastic Lorentz Equation

The butterfly effect was postulated in the context of turbulent fluid flow, with application to weather prediction (Lorenz, 1963). The equations of turbulent fluid flow used were

$$\frac{dx}{dt} = \sigma(y - x) \tag{24}$$

$$\frac{dy}{dt} = x(\rho - z) - y \tag{25}$$

$$\frac{dz}{dt} = xy - \beta z \tag{26}$$

where σ is the Prandtl number and ρ is the Rayleigh number. The system exhibits chaotic behavior, specifically aperiodic dynamics, for $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$.

Corresponding noisy, or Fokker-Planck, equations can be written as

$$\frac{dx}{dt} = (\sigma(y - x))(1 + a_x \cdot grn(0, v_x)) \tag{27}$$

$$\frac{dy}{dt} = (x(\rho - z) - y)(1 + a_y \cdot grn(0, v_y)) \tag{28}$$

$$\frac{dz}{dt} = (xy - \beta z)(1 + a_z \cdot grn(0, v_z)) \tag{29}$$

where $grn(\phi, \psi)$ is a random number chosen from a Gaussian (normal) distribution with mean (bias) ϕ and variance ψ and $a_i, i = x, y, z$ is the noise amplitude for each variable.

Both the deterministic chaotic and the noisy versions of the equations imply unpredictable dynamics, but for different reasons. In the former case the unpredictability would lie in the inability in the real world to define the initial conditions precisely. In the latter case, the postulate about the real world is that there are many variables that cannot be specified in detail, which must be

accounted for because their influence in the aggregate is significant but can only accounted for by a noise function [11]. In the former description, the flapping of a single butterflys wings in the Pacific Northwest might indeed alter the weather in the Midwest a week later. In the latter description, the flapping of the butterflys wings is but one of many individually small unpredictable influences on the atmospheric dynamics that, in the aggregate, will have a significant impact on the weather.

Undoubtedly there are some systems that are well described by deterministic chaos, for example a hinged pendulum in stagnant air. In that case the random fluctuations in air molecules impinging on the pendulum will be so small compared to gravity that deterministic chaos will be a good description. On the other hand, we suggest that atmospheric dynamics is not well described by deterministic chaos, because it is a system where many unspecifiable variables will combine in the aggregate modify the dynamics.

4.2 Formal Derivation of Stochastic Dynamic Equations

A general procedure adding noise to dynamic equations exists [13]. Given a complex dynamic system, let its defining equations be

$$\frac{d\bar{x}(t)}{dt} = M(\bar{x}, \bar{p}) \tag{30}$$

where $\bar{x}(t)$ is a vector of finite dimension, $\bar{p}$ is a vector of arbitrary parameters and M represents the model. We can assume that the variables $\bar{x}(t)$ define a trajectory in the space-state. Its probability density function $f(\bar{x})$ evolves according to the Liouville equation of probability conservation

$$\frac{df}{dt} + \nabla \cdot (Mf) = 0 \tag{31}$$

The equation of the state becomes

$$d\bar{X}(t) = Mdt + \xi(\bar{X}(t))dW(\bar{X}, t) \tag{32}$$

where now $\bar{X}$ is a stochastic variable dependent upon time t, $W(\bar{X}, t)$ describes a Markov process and for times t, t', $W(\bar{X}, t') - W(\bar{X}, t)$ follows a Gaussian distribution with mean 0 and variance $|t - t'|$. $\xi(\bar{X}(t))$ is a square matrix of (possibly variable) coefficients that, without loss of generality, will be assumed here to be equal to a constant value of ξ times the identity matrix. A probability density function can be found that translates the effects of stochastic terms in the state space as a form of diffusion. The Fokker-Planck equation can be formulated for the system

$$\frac{df}{dt} + \nabla \cdot (Mf) = \nabla^2(Kf) \tag{33}$$

All stochastic variables are interestingly related by the quantity $K = \frac{\xi^2}{2}$. We can now find the steady-state solution of $f(\bar{X}, \bar{p})$ as

$$\nabla \cdot (Mf) = \nabla^2 (Kf) \tag{34}$$

It is worth mentioning that a direct advantage of adding stochastic noise as described above is that the Loiuville equation is linear, making the corresponding Fokker-Planck equation smooth and well-behaved. This implies that even non-linear sensitivities to small perturbations can be easily accounted for in a simple probability density function. Hence, sensitivity also is a macroscopic assessment of the roughness or smoothness of the trajectory of variables of interest in the system, being galactic cluster formation a particular case [9].

5 Functionals and the Fokker-Plank Equation

In all phenomena we have explored up to know, the connection between one system and another occurs when a parameter in a theory is an average of a macrostate in a stochastic system at a lower level (e.g. quantum mechanics and nerve excitation). Two issues remain in the path of unifying different theories with an FPE. One of them is the continuity of our definition for the δ function, which is not effective for discrete scenarios. The second problem is that given a microstate variable $x(t)$ for a system A and a second system B for which a microstate variable $y(t)$ expressed as a functional of the average for the macrostate of $x(t)$

$$y(t) = \Omega(\langle x(t) \rangle) \tag{35}$$

needs to preserve the structure of the contributions of the probability density function B to those of the structure of the probability density function in A. That is, to find whether the FPE at one level corresponds to a sub-model of an exponential manifold at a higher level.

5.1 Use of Distributed Approximating Functionals for the δ function

A distributed approximation functional (DAF) is a mapping from a definite set of continuous L_2-functions to itself accurate to a given tolerance [15]. The power of DAF resides in their ability to provide an analytical representation of a function and its derivatives in terms of a discrete set of values of the function. Beyond its computational attractive (see [8] for a usual numerical approach), it maps elegantly into discrete phenomena where continuity is not appropriate.

By using the family of even Hermite polynomials H_{2n}, the delta function for a difference φ can be re-expressed as a functional on M and σ as

$$\delta(\varphi, M, \sigma) = \frac{1}{\sigma} e^{\frac{-\varphi}{2\sigma^2}} \cdot \sum_{n=0}^{M} \left(\frac{-1}{4}\right)^n \frac{1}{\sqrt{2\pi} n!} H_{2n} \frac{\varphi}{\sqrt{2\pi}\sigma} \tag{36}$$

for both limits

$$\lim_{M\to\infty} \delta(\varphi, M, \sigma) = \delta(\varphi) \tag{37}$$

and

$$\lim_{\sigma\to 0} \delta(\varphi, M, \sigma) = \delta(\varphi) \tag{38}$$

Appropriate selections of M and σ allow proper modeling of discrete grid elements for obtaining approximation at adequate points of a function

$$f(x_j) \approx f_M(x_j) \approx \Delta \sum k\delta(x_j - x_k) f(\delta_k) \tag{39}$$

5.2 Projecting Fokker-Planck Equations into Sub-Manifolds

Brigo and Pistone in [1] revisit a fundamental question: Can we define a diffusion Y_t whose density is the projected density of another diffusion X_t? The answer is affirmative only when Y_t and X_t belong to the same exponential manifold. An example of one of such mappings is

$$R : p \in (M) \mapsto \sqrt{p} \in \mathcal{L}^2(\mu) \tag{40}$$

where μ is a reference measure. R is C^∞ (continuously smooth). It is interesting to note that the above definition of the δ function assumes an exponential manifold, and defines a sub-model for each combination of M and σ. Therefore, for such manifolds the Fokker-Planck equation is universal and a fundamental constant of nature can be seen as a value that belongs to all sub-manifolds for every mapping.

6 Quantum Computing and Fokker-Planck Equation

Richard Feynman in 1982 and in a further series of lectures posed a strong question: is the universe by some means computable? [5]. Due to the discrete operation of Turing Machines, no computer can perfectly simulate classical physics. Quantum mechanics can be used for computing consistently with the structure of nature. Given that the probability function of the Schrödinger equation describes quantum systems, quantum information under measurements from the classical world is ultimately determined by a Fokker-Planck equation in (x, y, z, ict). At the most fundamental level known to us, information is not only quantum, but stochastically quantum.

6.1 Measurement of Quantum Bits in the Presence of Noise

In a more profound aspect of quantum information, measurements and decoherence can be modeled as projection operators into subspaces of the general Hilbert space describing the system. Decoherence in particular occurs due to interaction with the environment (thermal noise). The interferometry experiment developed by Sidles [12] is illuminating in this regard. Suppose that an apparatus is devised with a state defined by two interacting qubits. The input state contains an input qubit $|i\rangle$ and a initial qubit $|s\rangle$ and the output state will contain an output qubit $|o\rangle$ and a final qubit $|e\rangle$. The experiment, using photons, restricts possible unitary operations to a phase shift θ that affects the z component of its polarization. For the two separate channels, a sequence of measurements by photodiodes A and B in each respective channel is performed.

$$|e\rangle = ABAB \cdots AB|i\rangle \tag{41}$$

After developing the equations for the state of the system after measurements (see [12] for details), the probability of observing polarization $z = \pm 1$ conditional to z_0 (initial polarization) in time t has a probability density function described by the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = r\theta^2 \frac{\partial}{\partial z}\left(D(z)\frac{\partial P}{\partial z} - V(z)P\right) \tag{42}$$

where r is the photon flux, $D(z) \equiv \frac{(1-z^2)^2}{2}$ becomes a diffusion coefficient and $V(z) \equiv 2z(1-z^2)$ is a drift velocity. Further addition of Langevin fields in each component of space can transform the above FPE in

$$\frac{\partial P}{\partial t} = r\theta^2 \frac{\partial}{\partial z}\left(D(z)\frac{\partial P}{\partial z} - V(z)P\right) + (S_x + S_y)\frac{\partial}{\partial z}\left((1-z^2)\frac{\partial P}{\partial z}\right) \tag{43}$$

where $S_{x,y}$ is the strength of the signal in those components. From Eq. 43 we see that despite its effect in terms of decoherence, transitions induced by white noise are not affected by the Quantum Zeno effect. It is a significant aspect in terms of simplifying quantum hardware design when accounting for noise and spontaneous transitions.

6.2 The Stochastic Nature of Quantum Algorithms

Quantum computers can perform tasks in ways no classical computer can achieve, such as the application of a function to a whole ensemble of superpositions and performing clever measurements that exploit the statistical structure of the problem at hand [5].

Stochastic analysis is inherent to the construction of algorithms that match real hardware operation conditions. The no cloning theorem prevents us from

setting initial states with arbitrary precision, requiring accurate knowledge of the probability density function of the system. Additionally, measurements will be bound by noise in the system, producing errors that, if not correctly addressed, will render the computation useless. In terms of hardware design, the challenge remains at ensuring that noise from the Quantum Zeno effect is accounted for. The process of simulating physical nature with arbitrary precision will thus need specification of the system by means of multi-level Fokker-Planck equations describing its dynamics and those of the computing apparatus simultaneously.

7 Conclusions

We explored the connection between the Fokker-Plank equation and various systems and formal descriptions of diverse structures in natural systems that are closely related. Mathematically, FPEs are a powerful device for modeling hierarchical systems where stochastic dynamics define their evolution. A first direction of future research is a rigorous construction of a hierarchical model based on the FPE approach and determining the relationship between their probability density functions by means of computer simulations.

A second direction of research within Cosmology is the connection between Fokker-Planck equations and the arrow of time. The Second Law of Thermodynamics does not specify *how fast entropy grows*. Stochastic systems do have a metric of how fast entropy grows, the Kolmogorov-Sinai entropy which is an entropy rate [2]. If, as we propose in this paper, all natural systems at all scales can be understood in terms of Fokker-Planck equations, FPEs may provide essential information about preservation of information at the most extreme events of physical reality.

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