Semivariogram Models

Below is a list of commonly used valid parametric semivariogram models. These models are isotropic, i.e., $h = ||\mathbf{h}||$. In each example below, the vector parameter $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)'$ is such that $\theta_i \geq 0$, $i = 1, 2, 3$. In each case, θ_1 represents the nugget.

a) Linear: (valid in $\mathbb{R}^d$, $d \geq 1$)

$$\gamma(h, \boldsymbol{\theta}) = \begin{cases} 0 & \text{if } h = 0, \\ \theta_1 + \theta_2 \cdot h & \text{if } h \neq 0 \end{cases}$$

b) Power: (valid in $\mathbb{R}^d$, $d \geq 1$) Spatial correlation does not level off for large lag distances.

$$\gamma(h, \boldsymbol{\theta}) = \begin{cases} 0 & \text{if } h = 0, \\ \theta_1 + \theta_2 \cdot h^{\theta_3} & \text{if } h \neq 0 \end{cases}$$

where $\theta_3 < 2$.

c) Pure Nugget (white noise): (valid in $\mathbb{R}^d$, $d \geq 1$)

$$\gamma(h, \boldsymbol{\theta}) = \begin{cases} 0 & \text{if } h = 0, \\ \theta_1 & \text{if } h \neq 0 \end{cases}$$

d) Linear Bounded: (valid in $\mathbb{R}^1$)

$$\gamma(h, \boldsymbol{\theta}) = \begin{cases} 0 & \text{if } h = 0, \\ \theta_1 + \frac{\theta_2}{\theta_3} \cdot h & \text{if } 0 < h \leq \theta_3, \\ \theta_1 + \theta_2 & \text{if } h > \theta_3 \end{cases}$$

e) Circular: (valid in $\mathbb{R}^1$ and $\mathbb{R}^2$)

$$\gamma(h, \boldsymbol{\theta}) = \begin{cases} 0 & \text{if } h = 0, \\ \theta_1 + \theta_2 \left(1 - \frac{2}{\pi} \arccos\left(\frac{h}{\theta_3}\right) + \frac{2}{\pi} \frac{h}{\theta_3} \sqrt{1 - \left(\frac{h}{\theta_3}\right)^2} \right) & \text{if } 0 < h \leq \theta_3, \\ \theta_1 + \theta_2 & \text{if } h > \theta_3 \end{cases}$$

f) Spherical: (valid in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$) Nearly linear at the origin.

$$\gamma(h, \boldsymbol{\theta}) = \begin{cases} 0 & \text{if } h = 0, \\ \theta_1 + \theta_2 \left(\frac{3}{2} \cdot \frac{h}{\theta_3} - \frac{1}{2} \cdot \left(\frac{h}{\theta_3}\right)^3 \right) & \text{if } 0 < h \leq \theta_3, \\ \theta_1 + \theta_2 & \text{if } h > \theta_3 \end{cases}$$

g) Rational Quadratic: (valid in $\mathbb{R}^d$, $d \geq 1$)

$$\gamma(h, \boldsymbol{\theta}) = \begin{cases} 0 & \text{if } h = 0, \\ \theta_1 + \theta_2 \left(\frac{h^2}{1+\frac{h^2}{\theta_3}} \right) & \text{if } h \neq 0 \end{cases}$$

h) Exponential: (valid in $\mathbb{R}^d$, $d \geq 1$)

$$\gamma(h, \boldsymbol{\theta}) = \begin{cases} 0 & \text{if } h = 0, \\ \theta_1 + \theta_2 \left(1 - \exp\left(-\frac{h}{\theta_3} \right) \right) & \text{if } h \neq 0 \end{cases}$$

i) Gaussian: (valid in $\mathbb{R}^d$, $d \geq 1$) Parabolic near origin.

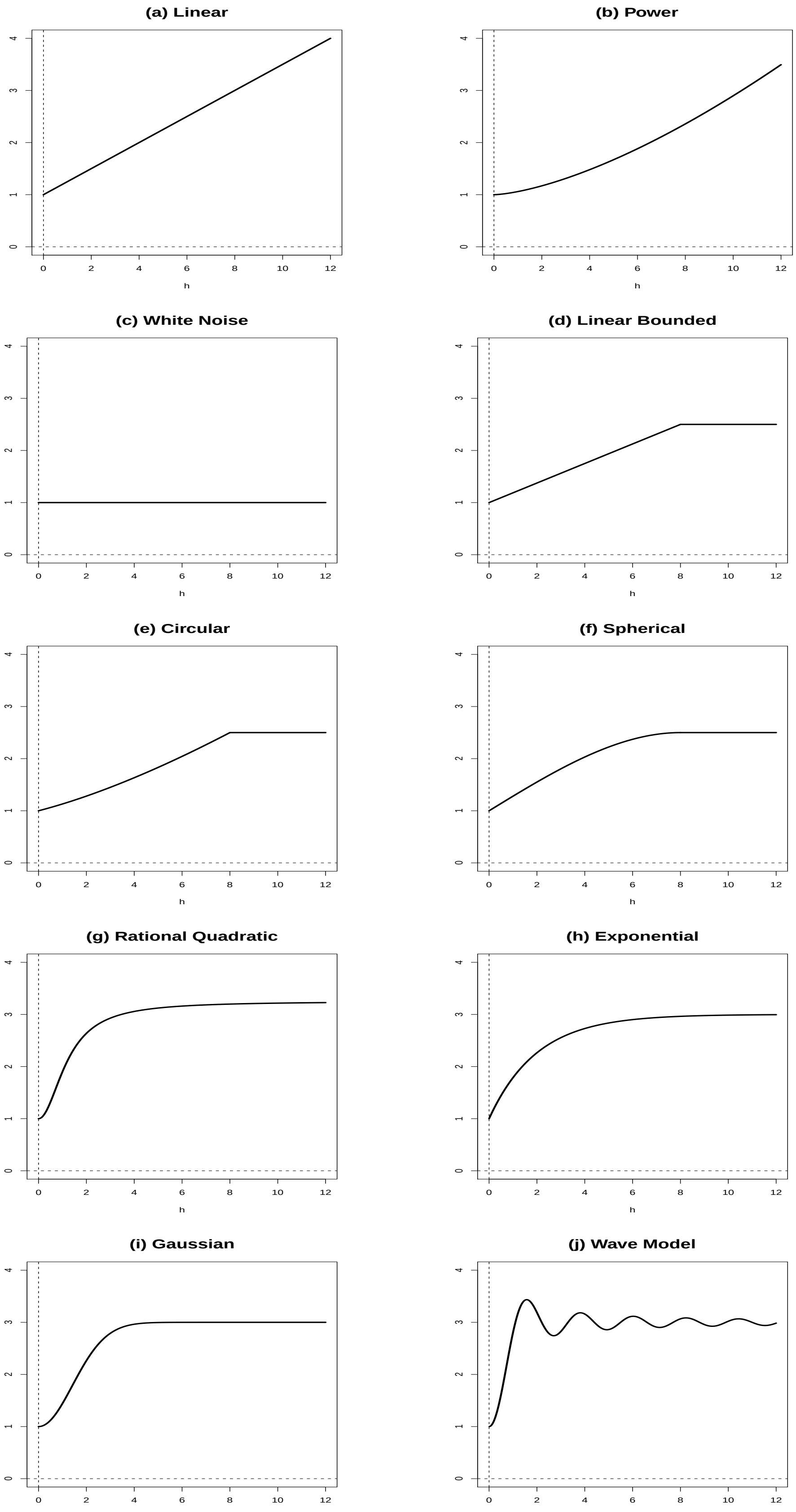
$$\gamma(h, \boldsymbol{\theta}) = \begin{cases} 0 & \text{if } h = 0, \\ \theta_1 + \theta_2 \left(1 - \exp\left(-\frac{h^2}{\theta_3^2} \right) \right) & \text{if } h \neq 0 \end{cases}$$

Needs a lot of closely spaced observations to assess the behavior near the origin.

j) Hole Effect (Wave): (valid in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$)

$$\gamma(h, \boldsymbol{\theta}) = \begin{cases} 0 & \text{if } h = 0, \\ \theta_1 + \theta_2 \left(1 - \frac{\theta_3}{h} sin\left(\frac{h}{\theta_3} \right) \right) & \text{if } h \neq 0 \end{cases}$$

- Models (a) and (b) are not $2nd$ order stationary. All others are $2nd$ order stationary since $\gamma(h)$ is bounded by the sill.
- For models (d), (e), and (f), the sill is reached, *i.e.* the spatial covariance is exactly zero for locations farther apart than the range.
- For models (g), (h), and (i), the sill is only reached asymptotically.
- The most popular semivariograms used in practice are spherical, exponential, and Gaussian.
- Covariances can be negative, as in model (j), but positive and negative autocorrelations cannot change arbitrarily in a $2nd$ order stationary process. This is useful for processes with cyclical or periodic variability.
- To see the covariogram, just turn page 3 upside down (except for (a) and (b) whose covariograms do not exist).
- Models (f) and (h) (spherical and exponential) behave linearly near the origin.
- Different parameterizations may be given in different texts, so read carefully! For example, the Gaussian model can also be written as $\gamma(h) = \theta_1 + \theta_2(1 - \exp\{-3\frac{h^2}{\alpha^2}\})$, where α represents the practical range.
- The exponential model is the continuous time analog of an AR(1) time series covariance structure.

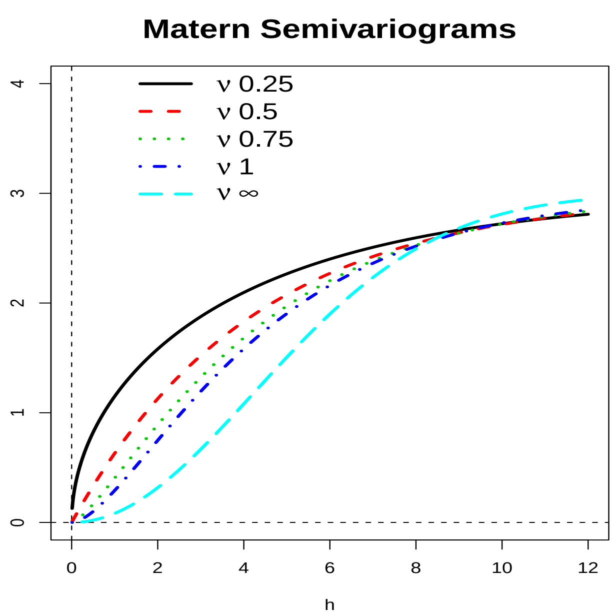

(a) Linear
(b) Power
(c) White Noise
(d) Linear Bounded
(e) Circular
(f) Spherical
(g) Rational Quadratic
(h) Exponential
(i) Gaussian
(j) Wave Model
h
0 2 4 6 8 10 12
0 1 2 3 4

Stationary Matérn Covariance

$$C_{\boldsymbol{\theta}}(\mathbf{h}) = \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)}\left(\frac{2\sqrt{\nu}||\mathbf{h}||}{\rho}\right)^{\nu} K_{\nu}\left(\frac{2\sqrt{\nu}||\mathbf{h}||}{\rho}\right),$$

where

- The parameter vector is $\boldsymbol{\theta} = (\sigma^2, \rho, \nu)'$.
- Γ is the Gamma function.
- K_ν is the modified Bessel function of the second kind of order ν.
- σ^2 is the variance.
- ρ controls the spatial range.
- ν controls the smoothness.
- When $\nu = 1/2$, this is the exponential model.
- When $\nu = \infty$, this is the Gaussian model.



Advantage: The behavior near the origin is controlled by ν, so the near origin behavior can be estimated from the data rather than assuming that it takes a particular form.
Disadvantage: Estimating ν can be computationally difficult, and it also requires closely spaced observations to estimate it well.