

BMEG 802 – Advanced Biomedical Experimental Design and Analysis

Probability

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Lecture Objectives

- Point Probabilities
 - a. axioms
 - b. definitions (complement, mutually exclusive, joint, marginal, conditional)
 - c. rules (addition, subtraction, multiplication)
- Continuous Probabilities
 - a. normal distribution (univariate and multivariate)
 - b. joint, marginal, conditional

POINT PROBABILITIES

- axioms
- definitions (complement, mutually exclusive, joint, marginal, conditional)
- rules (addition, subtraction, multiplication)

Notation

- S = sample space (all possible outcomes)
- $p(A)$ = probability of event A
- $A \cup B$ = union of events A and B
- $A \cap B$ = intersection of events A and B
- $p(B|A)$ = probability of B given A
- $p(A')$ or $p(A^C)$ or $p(\bar{A})$ = complement probability of $p(A)$

Axioms

- The probability of an event is a non-negative real number

$$P(E) \geq 0$$

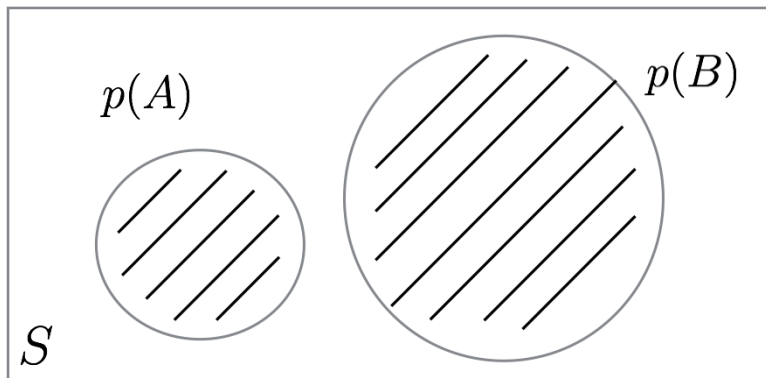
- The probability that at least one of the elementary events in the entire space will occur is 1

$$P(\Omega) = 1.$$

- Any countable sequence of disjoint sets (synonymous with mutually exclusive events) E_1, E_2, \dots satisfies

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$

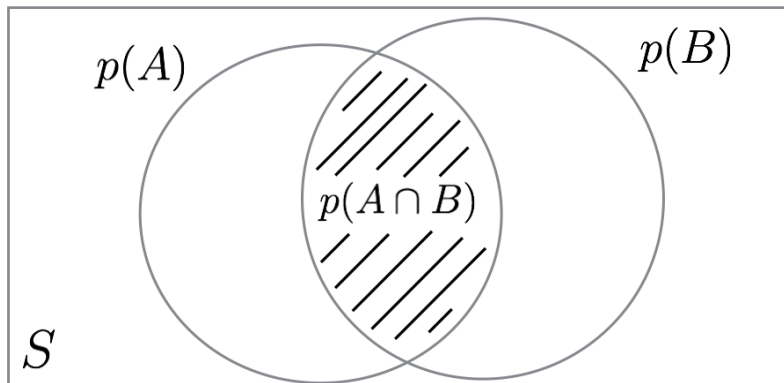
Mutually Exclusive (Disjoint Probability)



$$p(A \cup B) = p(A) + p(B)$$

$$0.7 = 0.4 + 0.3$$

Joint Probability

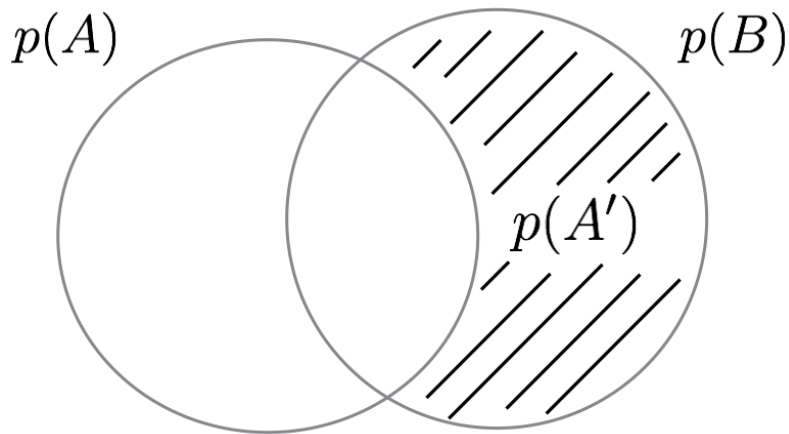


$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

$$0.5 = 0.4 + 0.3 - 0.2$$

- the probability of two events occurring simultaneously

Complement Probability



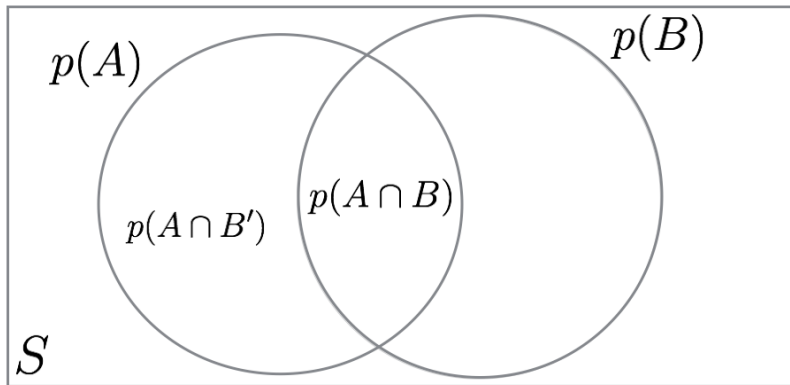
$$p(A') = 1 - p(A)$$

Marginal Probability

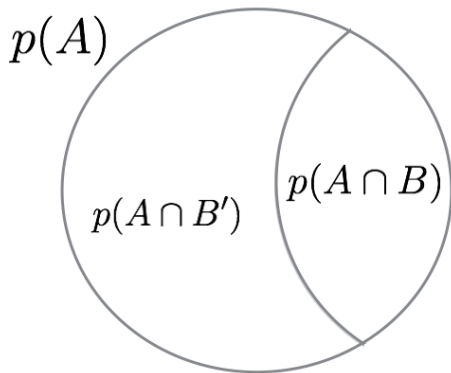
H \ L	Red	Yellow	Green	Marginal probability P(H)
Not Hit	0.198	0.09	0.14	0.428
Hit	0.002	0.01	0.56	0.572
Total	0.2	0.1	0.7	1

- Probability of a single event occurring (hit), independent of other events (light)
- e.g., probabilities of getting in an accident at an intersection irrespective of lights
- note: joint probabilities in each cell

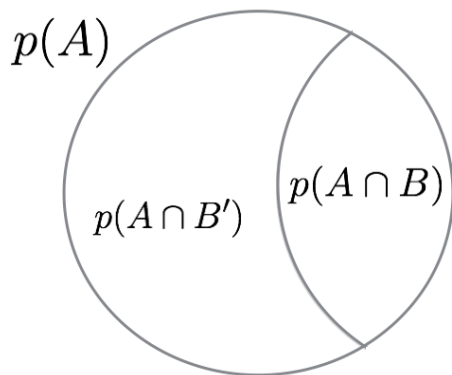
Marginal Probability



Marginal Probability

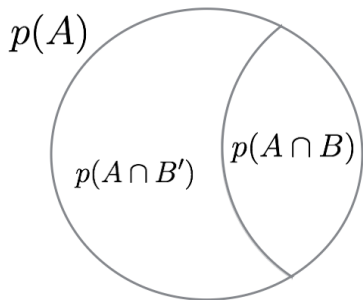


Marginal Probability



$$p(A \cap B') = p(A) - p(A \cap B)$$

Marginal Probability



$$p(A \cap B') = p(A) - p(A \cap B)$$

The marginal $p(A)$ or $p(B)$ is found by summing their disjoint parts.

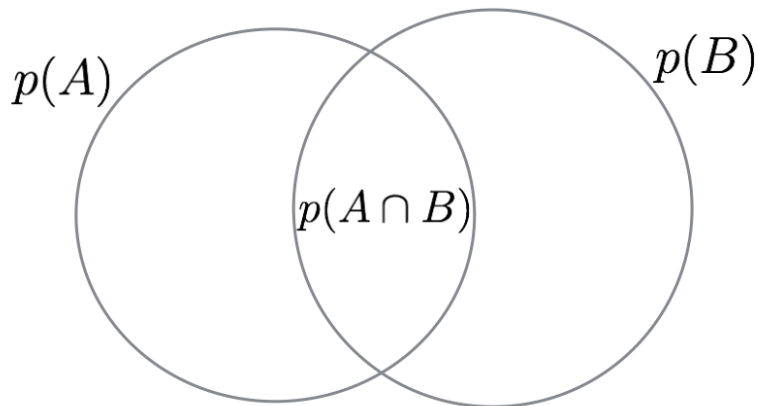
$$p(A) = p(A \cap B) + p(A \cap B'), \text{ and similarly}$$

$$p(B) = p(A \cap B) + p(A' \cap B)$$

Conditional Probability

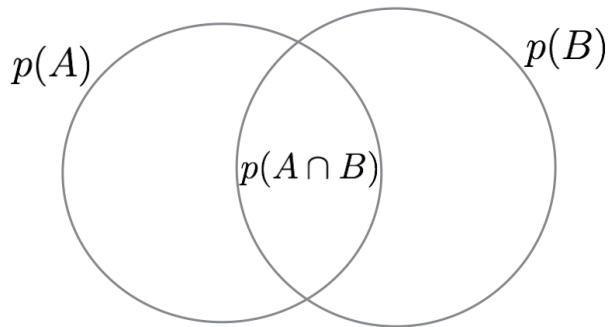
- $p(\text{accepted}) = 0.3$
- $p(\text{funding}|\text{accepted}) = 0.43$
- $p(\text{funding} \cap \text{accepted}) = p(\text{funding}|\text{accepted}) \cdot p(\text{accepted})$
- $p(\text{funding} \cap \text{accepted}) = 0.43 \cdot 0.3 = 0.13$
- Probability that an event occurs given that another specific event *has already* occurred

Conditional Probability



$$p(A \cap B) = p(B|A) \cdot p(A)$$
$$p(B|A) = \frac{p(A \cap B)}{p(A)}$$

Conditional Probability



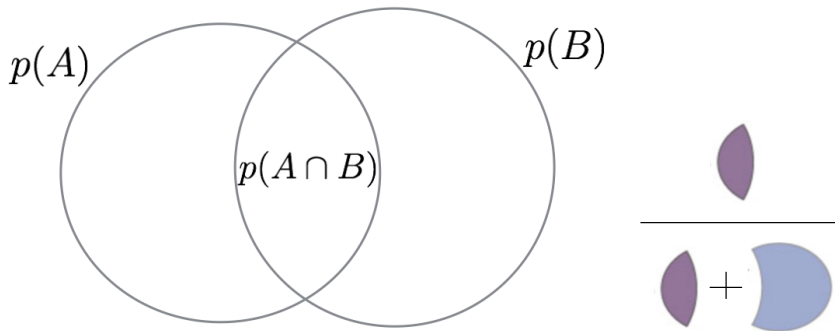
$$p(A \cap B) = p(B|A) \cdot p(A)$$

$$p(B|A) = \frac{p(A \cap B)}{p(A)}$$

$$p(A \cap B) = p(A|B) \cdot p(B) \text{ (in terms of B)}$$

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$

Conditional Probability



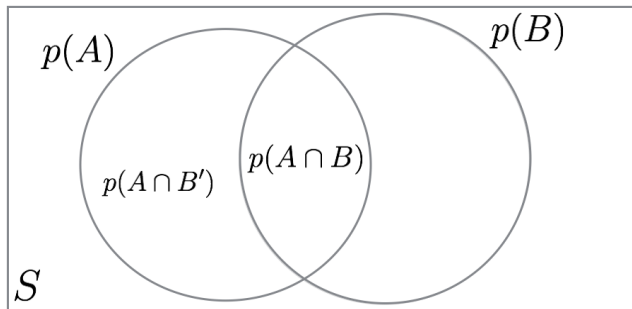
$$p(A \cap B) = p(B|A) \cdot p(A)$$

$$p(B|A) = \frac{p(A \cap B)}{p(A)}$$

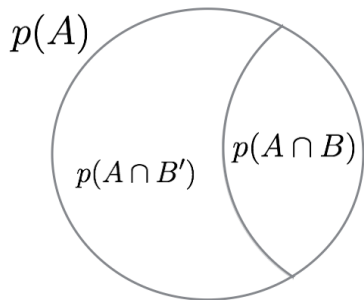
$$p(A \cap B) = p(A|B) \cdot p(B) \text{ (in terms of B)}$$

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$

Conditional Probability Complements

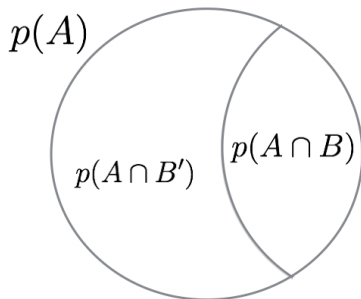


Conditional Probability Complements



$$p(A \cap B') = p(B'|A) \cdot p(A)$$

Conditional Probability Complements



$$p(A \cap B') = p(B'|A) \cdot p(A)$$

Other friendly complements:

$$p(A' \cap B) = p(B|A') \cdot p(A')$$

$$p(A' \cap B') = p(B'|A') \cdot p(A')$$

Probability Rules

- **Rule of Subtraction:** The probability that A will occur is equal to 1 minus the probability that A will NOT occur.

$$p(A) = 1 - P(A')$$

- **Rule of Multiplication:** The probability that Events A and B both occur is equal to the probability that Event A occurs times the probability that Event B occurs, given that A has occurred.

$$p(A \cap B) = p(A) \cdot p(B|A)$$

- **Rule of Addition:** The probability that Event A or Event B occurs is equal to the probability that Event B occurs minus the probability that both Events A and B occur.

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

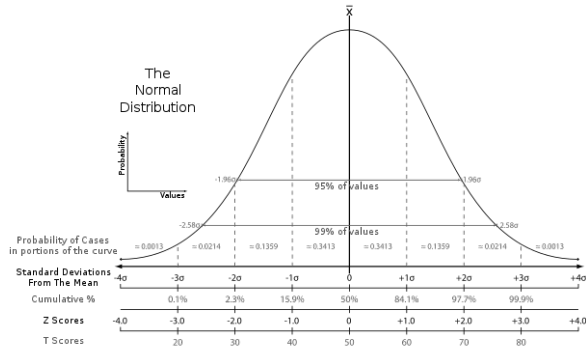
Note: we can redefine the addition rule given that $p(A \cap B) = p(A) \cdot p(B|A)$, such that:

$$p(A \cup B) = p(A) + p(B) - p(A) \cdot p(B|A)$$

CONTINUOUS PROBABILITIES

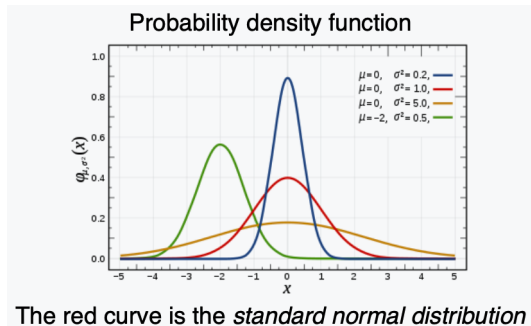
- Identical concepts to point probabilities!
- Normal distribution (univariate and multivariate)
- joint, marginal, conditional

Normal Distribution (Univariate)



- The probability that some value of x will occur
- Think of a histogram

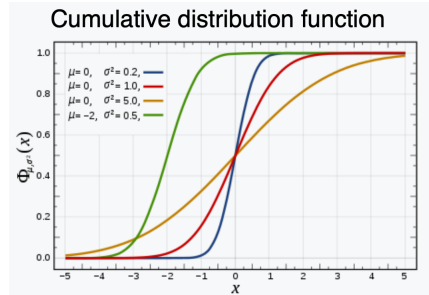
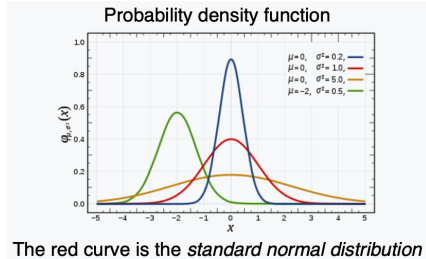
Normal Distribution - Probability Density Function



$$f(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\mathcal{N}(\mu, \sigma)$$

Normal Distribution - Cumulative Density Function



$$\Phi(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{x-\mu}{\sigma\sqrt{2}} \right) \right]$$

See Probability and Statistics Primer

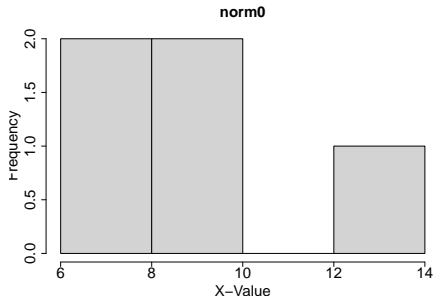
Sampling from a Normal Distribution (N = 5)

```
norm0 <- rnorm(5, mean=10, sd=2)
```

```
norm0
```

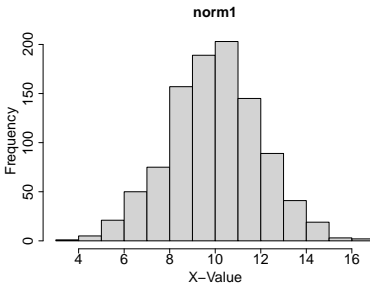
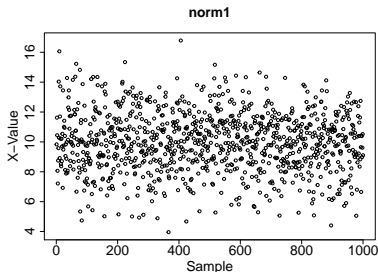
```
## [1] 12.749536  8.415900  6.050497  6.741884  9.619732
```

```
hist(norm0, main="norm0", xlab="X-Value", ylab="Frequency",  
      cex.lab=2.0, cex.axis=2.0, cex.main=2.0)
```



Sampling from a Normal Distribution (N = 1000)

```
norm1 <- rnorm(1000, mean=10, sd=2)
par(mar=c(5,5,5,5))
plot(norm1, main="norm1", xlab="Sample",
     ylab="X-Value", cex.lab=2.0, cex.axis=2.0, cex.main=2.0)
hist(norm1, main="norm1", xlab="X-Value", ylab="Frequency",
     cex.lab=2.0, cex.axis=2.0, cex.main=2.0)
```

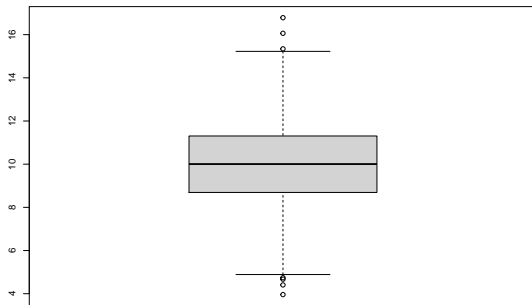


Summary of Sampled Data

```
summary(norm1)
```

##	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
##	3.958	8.692	10.005	9.989	11.301	16.785

```
boxplot(norm1)
```



Fit the Data

```
install.packages("fitdistrplus")
```

```
library(fitdistrplus)
```

```
## Loading required package: MASS
```

```
## Loading required package: survival
```

```
FIT <- fitdist(norm1, "norm")
```

```
FIT
```

```
## Fitting of the distribution ' norm ' by maximum likelihood
```

```
## Parameters:
```

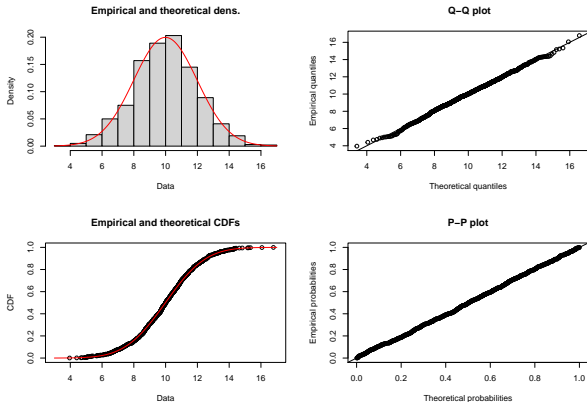
```
##      estimate Std. Error
```

```
## mean 9.989210 0.06318720
```

```
## sd    1.998155 0.04468004
```

Fit the Data

```
plot(FIT)
```



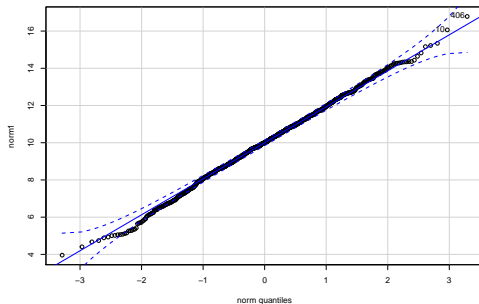
Q-Q (Quantile-Quantile) compares quantiles (divide data into n parts),
P-P (Probability-Probability) compares CDF. Both test 'normality'

Testing Normality - QQ plot

```
library("car")
```

```
## Loading required package: carData
```

```
qqPlot(norm1)
```



```
## [1] 406 10
```

Testing Normality - Shapiro-Wilk Test (Univariate)

```
library("car")  
shapiro.test(norm1)
```

```
##
```

```
##  Shapiro-Wilk normality test
```

```
##
```

```
## data:  norm1
```

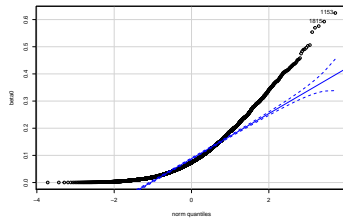
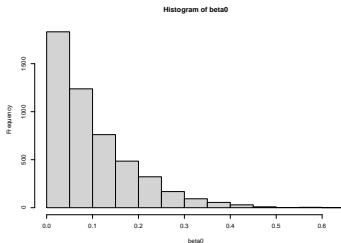
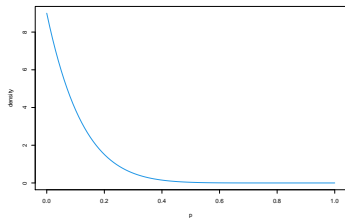
```
## W = 0.99843, p-value = 0.5085
```

p-value > 0.05 = normally distributed (we'll get more into p-values next class)

Testing Normality - Beta Distribution

```
beta0 <- rbeta(5000, 1, 9)
p = seq(0,1, length=100)
plot(p, dbeta(p, 1, 9), ylab="density", type="l", col=4)
hist(beta0)
qqPlot(beta0)
```

```
## [1] 1153 1815
```



Testing Normality - Beta Distribution

```
shapiro.test(beta0)
```

```
##
```

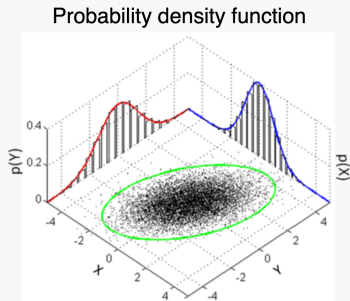
```
## Shapiro-Wilk normality test
```

```
##
```

```
## data:  beta0
```

```
## W = 0.87013, p-value < 2.2e-16
```

Normal Distribution (Multivariate)



Many sample points from a multivariate normal distribution with

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 3/5 \\ 3/5 & 2 \end{bmatrix}, \text{ shown along with the 3-sigma}$$

ellipse, the two marginal distributions, and the two 1-d histograms.

e.g., X = height, Y = weight

Normal Distribution (Bivariate PDF)

$$f_{\mathbf{X}}(x_1, \dots, x_k) = \frac{\exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right)}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}} \quad (\text{bivariate, } k = 2)$$

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_X^2 & r\sigma_X\sigma_Y \\ r\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}.$$

OR

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)} \left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2r\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 \right]}$$

r = correlation between X and Y

$\boldsymbol{\Sigma}$ is positive definite

Normal Distribution (Bivariate PDF)

```
install.packages("bivariate")
```

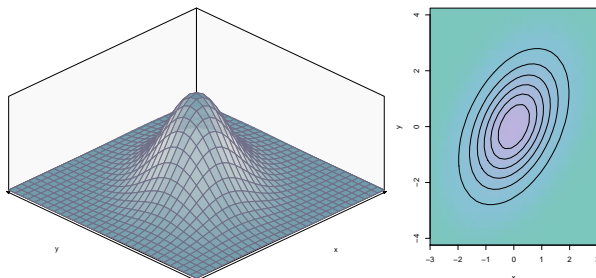
```
library(bivariate)
```

```
f <- nbvpdf (0, 0, 1, 1.414, 0.424)
```

```
F <- nbvcdf (0, 0, 1, 1, 0)
```

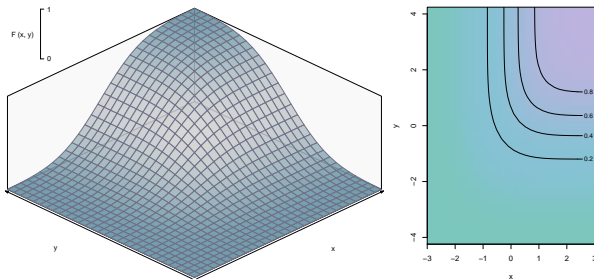
```
plot (f)
```

```
plot (f, FALSE)
```



Normal Distribution (Bivariate CDF)

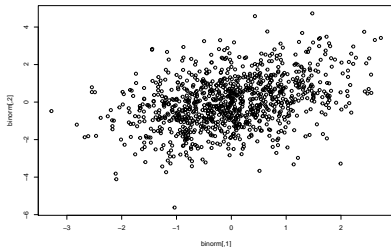
```
library(bivariate)
f <- nbvpdf (0, 0, 1, 1.414, 0.424)
F <- nbvcdf (0, 0, 1, 1.414, 0.424)
plot (F)
plot (F, FALSE)
```



Sample from a Bivariate Normal

```
install.packages("MASS")
```

```
library("MASS")  
my_n1 <- 1000  
my_mu1 <- c(0, 0)  
my_Sigma1 <- matrix(c(1, 0.6, 0.6, 2), ncol = 2)  
binorm <- mvrnorm(n = my_n1, mu = my_mu1, Sigma = my_Sigma1)  
plot(binorm)
```

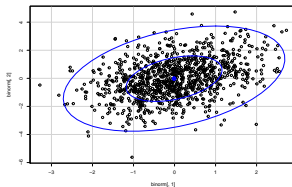


Confidence Ellipse

```
install.packages("car")
```

```
library(car)
```

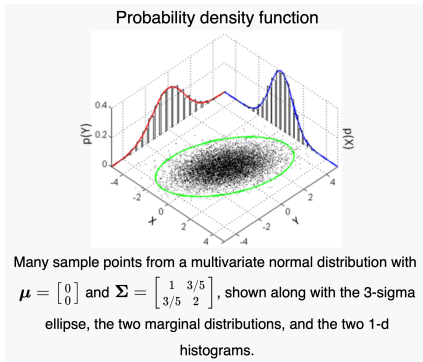
```
dataEllipse(binorm[,1], binorm[,2], levels=c(0.5, 0.975))
```



Eigendecomposition on Covariance Matrix

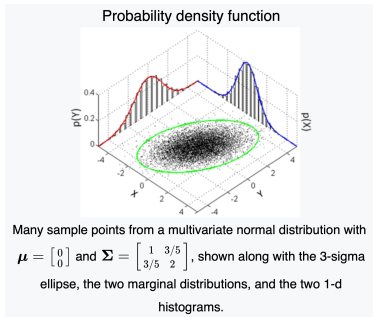
Square root of eigenvalues = principle axes

Joint Probability



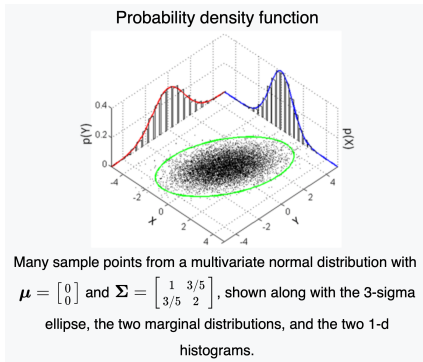
- The bivariate normal distribution IS an example of a Joint Distribution
 - $p(X \cap Y)$
- We have already defined its PDF and CDF
- e.g., what is the probability you are x cm tall and weigh y kg?

Marginal Probability



- Represented as the univariate normal distributions on the 'walls'
- Simply drop terms from the mean vector and covariance matrix related to the variable you want to marginalize out
- $p(X) = \mathcal{N}(\mu_X, \sigma_X^2)$ and $p(Y) = \mathcal{N}(\mu_Y, \sigma_Y^2)$
- e.g., what is the probability you weigh y kg?

Conditional Probability



$$p(X \mid Y = y) = \mathcal{N}\left(\mu_X + \frac{\sigma_X}{\sigma_Y} r(y - \mu_Y), (1 - r^2)\sigma_X^2\right).$$

- e.g., probability that you are x cm tall given you weigh y kg
- $(1 - r^2)\sigma_X^2$ is the variance, so take square root to find SD