Complex Numbers

IB Course Notes

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1 Introduction

Girolamo Cardano (1501-76) sought to find a formula that gives the solution to the general cubic equation

$$x^3 + ax^2 + bx + c = 0$$

The first step in doing so is to make the Tchirnhaus transformation $x = y - \frac{a}{3}$. Now we have

$$\left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c = 0$$

That is,

$$y^3 + \left(b - \frac{a^2}{3}\right)y + \left(\frac{2a^3}{27} - \frac{ab}{3} + c\right) = 0$$

By letting $p = b - \frac{a^2}{3}$ and $q = \frac{2a^3}{27} - \frac{ab}{3} + c$, we obtain the *canonical form of the cubic equation*:

$$y^3 + py + q = 0$$

Our next step is to make the substitution $y = v + w$, and then we must have

$$(v+w)^3 + p(v+w) + q = 0$$

or, with the terms collected, the equivalent form

$$(3vw + p)(v+w) + (v^3 + w^3 + q) = 0$$

Notice now that if we can select v and w to satisfy

$$vw = -\frac{p}{3} \tag{1}$$
$$v^3 + w^3 = -q \tag{2}$$

then our cubic will be solved!

We see that equation (1) can be rewritten and substituted into (2) to give

$$v^3 - \frac{p^3}{27v^3} = -q$$

which is a quadratic in v^3. We see this clearly if we rewrite it as

$$\left(v^3\right)^2 + q\left(v^3\right) - \frac{p^3}{27} = 0$$

Then the quadratic formula tells us (using the symmetry of v and w) that

$$v^3 = -\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$$
$$w^3 = -\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$$

or

$$v = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$
$$w = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

It then follows

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

which is the famous Cardano formula!

Consider what happens when we apply this formula to the historical example of $x^3 = 15x + 4$:

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Notice how we see the appearance $\sqrt{-121}$ - the square root of a negative number! Up until this point in history, square roots of negatives were easily dismissed. For example, equations such as $x^2+1=0$, it could simply be dismissed as having no solutions. But this cubic started to cause problems. This cubic was known to have three *real* roots: $y=4$, $y=-2\pm\sqrt{3}$. But how could it be that Cardano's formula gives a "meaningless" solution to an equation who's roots were known?

As such, the task arose to reconcile the formal solution $x=\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$ with the known solutions of $x=4,-2\pm\sqrt{3}$. It wasn't until 30 years later that a hydraulic engineer by the name of Rafael Bombelli (1526-73) made the wild leap to write

$$\sqrt[3]{2+\sqrt{-121}}=a+\sqrt{-b} \quad \text{and} \quad \sqrt[3]{2-\sqrt{-121}}=a-\sqrt{-b}$$

He then proceeded to solve for a and b using known real methods, and eventually showed that $a=2$, $b=1$. Thus

$$\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}=(2+\sqrt{-1})+(2-\sqrt{-1})=4$$

and it was shown that the so called "meaningless" solution Cardano arrived at was equivalent to the known root given by inspection.

2 Basics of Complex Numbers

Using the notation developed by Euler, we define the number i as

$$i = \sqrt{-1}$$

where i is our fundamental complex unit. From here, we can define a general complex number z to be

$$z = a + bi$$

where $a, b \in \mathbb{R}$. We call this representation of a complex number the *Cartesian* form. This is just one of three ways that we will represent comple numbers in this course.

We denote the set of complex numbers by $\mathbb{C}$ where we say

$$\mathbb{C} = \{z : z = a + bi, \quad a, b \in \mathbb{R}\}$$

A interesting point to notice that by taking $b = 0$, we have $z = a$, and so z becomes a real number. It follows that $\mathbb{R} \subset \mathbb{C}$ - i.e. that the set of real numbers is a subset of the set of complex numbers. We can extend this idea so as to write:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

A complex number $z = a + bi$ is said to have real part a and complex (or imaginary) part b. We write

$$\Re(z) = a \quad \text{and} \quad \Im(z) = b$$

to denote the real and complex parts of z respectively.

The arithmetic of complex numbers follows in the logical way: once the operation has been performed, we group together real and imaginary parts to form a new complex number. Given two complex numbers $z = a + bi$ and $w = x + yi$, we have

$$\begin{aligned} z \pm w &= (a + bi) \pm (x + yi) = (a \pm x) + (b \pm y)i \\ zw &= (a + bi)(x + yi) = ax + ayi + bxi + byi^2 = (ax - by) + (bx + ay)i \\ \frac{z}{w} &= \frac{a + bi}{x + yi} \end{aligned}$$

You may notice that the expression for the ratio of two complex numbers does not take the standard form of a complex number that we defined above. This is not an appealing form for the number to take, and so we want to find a way to fix it. Namely, we want to removed the imaginary part from the denominator. To do this, we make use of the *difference of two squares*:

$$\frac{z}{w} = \frac{a + bi}{x + yi} = \frac{a + bi}{x + yi} \times \frac{x - yi}{x - yi} = \frac{(a + bi)(x - yi)}{x^2 + y^2}$$

where the cross terms in xyi cancel. This is the form we will seek to express the ratio of two complex numbers in going forward.

So, for example, if we take $z = 1 + 2i$ and $w = -2 + 3i$, we find

$$\begin{aligned} z + w &= (1 - 2) + (2 + 3)i = -1 + 5i \\ z - w &= (1 + 2) + (2 - 3)i = 3 - i \\ zw &= (1 + 2i)(-2 + 3i) = -2 + 3i - 4i + 6i^2 = -8 - i \\ \frac{z}{w} &= \frac{1 + 2i}{-2 + 3i} = \frac{(1 + 2i)(-2 - 3i)}{13} = \frac{-2 - 3i - 4i + 6}{13} = \frac{4 - 7i}{13} \end{aligned}$$

An interesting problem (and one that we will spend a lot time discussing later on) is to find the roots of complex numbers. For example, if we say $z = a + bi$, how do we find a w such that $w^2 = z$. To do so, we start be writing $w = x + yi$. It is important to realise that two complex numbers are equal if and only if both their real and imaginary parts are equal. As such, we write

$$\begin{aligned} z &= w^2 \\ a + bi &= (x + yi)^2 \\ &= (x^2 - y^2) + i(2xy) \end{aligned}$$

and obtain the system of equations

$$a = x^2 - y^2$$
$$b = 2xy$$

from equating the real and imaginary parts of the LHS and RHS.

Substituting $y = \frac{b}{2x}$, we obtain

$$\begin{aligned} a &= x^2 - \frac{b^2}{4x^2} \\ \Rightarrow \quad 0 &= x^4 - ax^2 - \frac{b^2}{4} \\ \Rightarrow \quad 0 &= \left(x^2 - \frac{a}{2}\right)^2 - \frac{a^2}{4} - \frac{b^2}{4} \\ \Rightarrow \quad x^2 &= \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 + b^2} \end{aligned}$$

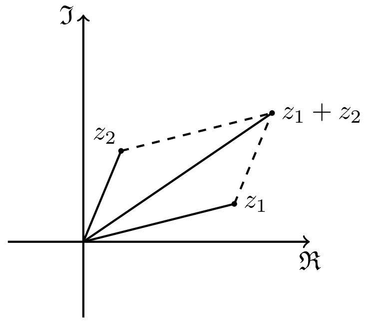
and we can plug this into our earlier equations to get an expression for y.

As you can see, this "brute force" method is rather painstaking, but don't fret! We will uncover far more efficient methods for evaluating roots as we progress through the course.

The important point to take away from this exercise is that when solving equations involving complex numbers, it is necessary that we solve both the real and imaginary parts of the equation separately.

2.1 Argand diagrams and modulus-argument form

Where we ca represent a point $\mathbf{x} \in \mathbb{R}^2$ on an $x - y$ grid, we can represent a number $z \in \mathbb{C}$ as a point on a special set of axes called an *argand diagram*. An argand diagram is a geometric representation of complex numbers, established by direct analogy to the standard Cartesian plane: the x-axis representing the real component and the y-axis the imaginary component of a complex number. In such a diragram, we represent a complex number $z = a + bi$ by a vector $\mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix}$. Representing complex numbers, we see that addition of complex numbers corresponds directly to the addition of their vectors on the argand diagram:



It can be useful to think of a complex number $z = a + bi$ as a point $(a, b) \in \mathbb{R}^2$. In this way, we have a nice pictorial view of the manifestation of a complex number. We can actually define a very natural one-to-one mapping (a *bijection*) between $\mathbb{R}^2$ and $\mathbb{C}$ as $(a, b) \mapsto a + bi$. It is in this way that people often become fooled into thinking that these spaces are the same - but they most defininitely are not.
Although it can be useful to think of $\mathbb{C}$ as $\mathbb{R}^2$ in some ways, it is very important that we realise these *are not* the same thing. It becomes very nuanced very quickly to discuss why these two spaces are different. So much so, that even the use of the word "spaces" brings about much ambiguity: do we mean metric spaces, vector spaces or even groups? As such, we won't waste our time on a discussion of how they differ, rather we will settle on an agreement that we can think of $\mathbb{R}^2$ and $\mathbb{C}$ as the same as long as it is useful, provided we understand that they most certainly are not the same thing.

Two fundamental definitions that come about from an argand diagram is that of the *modulus* and *argument* of a complex number. We define the modulus, $|z|$, of a complex number $z = a + bi$ to be

$$|z| = \sqrt{a^2 + b^2}$$

and the argument, $\arg z$, to be the angle the complex number makes with positive $\Re$-axis.

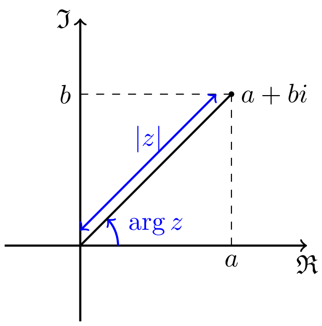
How we compute the argument depends on how we choose to define it. For the time being, we will define the argument as the angle measure anticlockwise from the positive $\Re$-axis with $\arg z \in [0, 2\pi)$. As such, we find that the formula for $\arg z$ depends on where we are in the complex plane. For the simplest case that our complex number z is such that $\Re(z), \Im(z) > 0$, we have

$$\arg z = \arctan\left(\frac{b}{a}\right)$$

For other places in the complex plane, we have

$$\begin{aligned}
(\Re(z) > 0, \Im(z) > 0), &\quad \arg z = \arctan\left(\frac{b}{a}\right) \\
(\Re(z) > 0, \Im(z) > 0), &\quad \arg z = \arctan 2\pi - \left(\frac{b}{a}\right) \\
(\Re(z) < 0, \Im(z) > 0), &\quad \arg z = \pi - \arctan\left(\frac{b}{a}\right) \\
(\Re(z) < 0, \Im(z) < 0), &\quad \arg z = \pi + \arctan\left(\frac{b}{a}\right)
\end{aligned}$$

These definitions become intuitively obvious when we consider the number z on an argand diagram:



Now that we have these definitions, we can take a look at the second way we will represent complex numbers: in *modulus-argument* form. Notice that we can use some simple right-angled triangle geometry to say

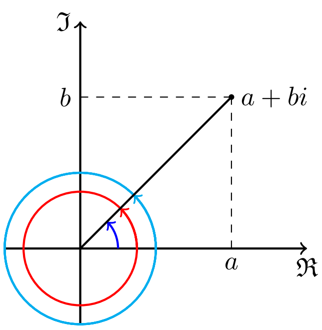
$$a = |z| \sin(\arg x)$$
$$b = |z| \cos(\arg z)$$

As such, if we use the notation $\theta = \arg z$ and $r = |z|$, we have

$$z = r(\cos\theta + i \sin\theta)$$

This is an incredibly powerful way for us to represent complex numbers, and will often be preferable over the Cartesian form we have used thus far.

Before we move on, we need to quickly discuss the range of our argument. Notice earlier that we specified $\arg z \in [0, 2\pi)$. Why did we do this? The reason stems from the periodicity of the sine and cosine functions. Because $\cos\theta = \cos(\theta + 2\pi)$ (and the same for sine), the function $f(\theta) = r(\cos\theta + i \sin\theta)$ takes the same value for multiple values of θ. This means that any complex number can be represented by multiple values of θ by encircling the origin any number of times. We plot this below:



where each angle (moving radially outwards) is 2π more than its predecessor. We can see here that each argument defines the same complex number.

This room for variation in the argument of a complex number can cause problems from time-to-time. As such, it is on our best interest to pin our arguments down to some range. There are two common ways of doing this, and we will use both in this course. The first restricts $\arg z \in [0, 2\pi)$, and the second restricts $\arg z \in (-\pi, \pi]$, where we interpret the negative range as being an angle below the positive $\Re$-axis. It makes no difference which range we pick, provided we are consistent with our choice. Of course, if a question specifies a choice of range, then clearly that is the one you should use!

If you want to read further about how we handle multi-valuedness in the complex plane, please read the *branch cuts* section of the appendices.

2.2 Converting between Cartesian and modulus-argument forms

Converting between these two forms of a complex number is simply a matter of using the definitions and considering where in the complex plane we lie. Let's consider some examples:

Example 1. *Write the complex number $z = 1 + i$ in modulus-argument form with* $\arg z \in [0, 2\pi)$

For the modulus, we find

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

For the argument, we have $\Re(z), \Im(z) > 0$ and so we can simply find

$$\arg z = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

thence

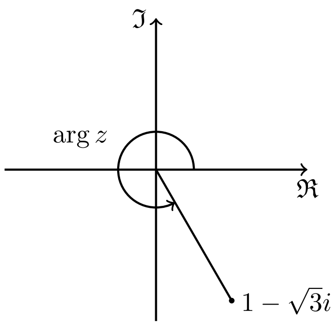
$$z = r\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Example 2. *Write the complex number $z = 1 - \sqrt{3}i$ in modulus-argument form with* $\arg z \in [0, 2\pi)$.

In this case, we find

$$r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$$

For the argument, we need to think a little bit more as we no longer lie in the first quadrant. Consider the position of z in the complex plane:



We see that the argument is going to be given by $2\pi + \arctan\left(\frac{-\sqrt{3}}{1}\right)$ and so

$$\arg z = \frac{5\pi}{3}$$

within the given range. It follows

$$z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

Example 3. *Write the complex number $z = 3\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$ in Cartesian form.*

Converting from modulus-argument to Cartesian is considerably easier than the other way around. All that is required is for us to evaluate the sine and cosine terms. Since

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$
$$\sin\frac{\pi}{6} = \frac{1}{2}$$

It follows

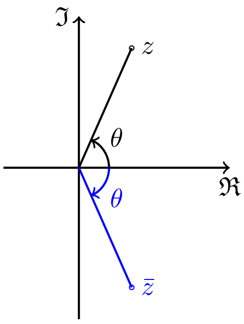
$$z = 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$$

2.3 Complex conjugates

For any number $z \in \mathbb{C}$, we have an associated number called the *complex conjugate* of z, denoted by $\bar{z}$ or z^*.

The definition of the complex conjugate is simple enough: given a complex number $z = a + bi$, the conjugate $\bar{z}$ is given by $\bar{z} = a - bi$, i.e. it has the same real part but the negative imaginary part.

Consider what this corresponds to geometrically:



We see from the diagram that the angle $\bar{z}$ makes with the positive $\mathfrak{R}$-axis is the same measured clockwise as the angle that z makes with the same line measure anticlockwise. In other words, if we choose $\arg z \in (-\pi, \pi]$, we have $\arg z = -\arg \bar{z}$. We can thus think of the complex conjugate as a reflection of z in the $\mathfrak{R}$-axis.

Complex conjugates are going to become exceptionally useful in the later parts of this topic when we start to look at the Fundamental Theorem of Algebra and solving polynomials over $\mathbb{C}$.

For now, let's just examine some properties of the complex conjugate:

Lemma 1. *We have the following:*

1.
$$(z^*)^* = z$$

2.
$$(z_1 \pm z_2)^* = z_1^* \pm z_2^*$$

3.
$$(z_1 z_2)^* = z_1^* z_2^*$$

4.
$$\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$$

5.
$$zz^* = |z|^2$$

6.
$$\frac{1}{z} = \frac{z^*}{|z|^2}$$

Proof. Throught this proof, define $z = a + bi$, $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$.

1. By definition $z^* = a - bi$, and so
$$(z^*)^* = (a-bi)^* = a + bi = z$$

2. We know $z_1 \pm z_2 = (a_1 \pm a_2) + (b_1 \pm b_2)i$. Thus
$$\begin{aligned}(z_1 \pm z_2)^* &= ((a_1 \pm a_2) + (b_1 \pm b_2)i)^* \\ &= (a_1 \pm a_2) - (b_1 \pm b_2)i \\ &= (a_1 - b_1 i) \pm (a_2 - b_2 i) \\ &= z_1^* \pm z_2^*\end{aligned}$$

3. We know $z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$, and so
$$\begin{aligned}(z_1 z_2)^* &= ((a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i)^* \\ &= (a_1 a_2 - b_1 b_2) - (a_1 b_2 + a_2 b_1)i \\ &= (a_1 - b_1 i)(a_2 - b_2 i) \\ &= z_1^* z_2^*\end{aligned}$$

4. For this proof, we use the previous result. Let $\frac{z_1}{z_2} = z_3$, then
$$\begin{aligned} & z_1 = z_2 z_3 \\ \Rightarrow \quad & z_1^* = z_2^* z_3^* \\ \Rightarrow \quad & z_3^* = \frac{z_1^*}{z_2^*}\end{aligned}$$

5. We find
$$\begin{aligned} zz^* &= (a+bi)(a-bi) \\ &= a^2 - abi + abi + b^2 \\ &= a^2 + b^2 \\ &= |z|^2\end{aligned}$$

6. Finally, we have
$$zz^* = |z|^2 \quad \Rightarrow \quad \frac{1}{z} = \frac{z^*}{|z|^2}$$

□

We can now prove an important result that is used frequently in many branches of mathematics:

Lemma 2 (Triangle Inequality)**.** *For all complex numbers $z_1, z_2 \in \mathbb{C}$, we have*
$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof. We have
$$\begin{aligned}|z_1 + z_2|^2 &= (z_1 + z_2)(z_1 + z_2)^* \\ &= z_1 z_1^* + z_1 z_2^* + z_1^* z_2 + z_2 z_2^* \\ &= |z_1|^2 + z_1 z_2^* + (z_1 z_2^*)^* + |z_2|^2 \\ &= |z_1|^2 + 2\Re(z_1 z_2^*) + |z_2|^2 \\ &\leq |z_1|^2 + 2|z_1 z_2^*| + |z_2|^2 \\ &= |z_1|^2 + 2|z_1||z_2^*| + |z_2|^2 \\ &= (|z_1| + |z_2|^2)\end{aligned}$$

□

2.4 Complex numbers and circles

Consider the equation for a circle of radius r centred on the origin in $\mathbb{R}^2$:

$$x^2 + y^2 = r^2$$

It is possible to parameterise this curve by defining

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

To see this, simply note

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

and so we arrive back at our Cartesian equation for a circle.

Now take a look at the modulus-argument form of a complex number:

$$z = r(\cos\theta + i\sin\theta)$$

Comparing to the Cartesian form of the same complex number (which way call $x + yi$) we see

$$\begin{aligned} \Re(z) &= x = r\cos\theta \\ \Im(z) &= y = r\sin\theta \end{aligned}$$

It thus follows that, in the complex plane, the equation $f(\theta) = r(\cos\theta + i\sin\theta)$ defines a circle of radius r centred at the origin!

2.5 Exponential form

Rarely is an area of mathematics studied without the mention of the legend that is Leonhard Euler. Fortunately, the study of complex numbers is no exception.

Consider the modulus-argument form of $z \in \mathbb{C}$:

$$z = r(\cos\theta + i\sin\theta)$$

What Euler managed to do was arrive at the remarkable equivalence:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Which truly is an incredible result. To see this, we will consider the Taylor series of some relevant functions. If you don't know what a Taylor series is, please see the appendices.

The Taylor series for the sine, cosine and exponential functions are as follows:

$$\begin{aligned} \cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \\ \sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\ e^z &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n \end{aligned}$$

Now consider what happens when we let $z = i\theta$:

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{1}{n!}(i\theta)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n)!}(i\theta)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(i\theta)^{2n+1} \quad \text{(split the sum into even and odd terms)} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n)!}\theta^{2n} i^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\theta^{2n+1} i^{2n} i \\ &= \left(\sum_{n=0}^{\infty} \frac{1}{(2n)!}\theta^{2n}\right) + i\left(\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\theta^{2n+1}\right) \quad (i^{2n} = 1) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

and so we see (after multiplying by r) that

$$r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Why is this form so useful? Well, for a start, both multiplication and division become much easier in exponential form. To see this, let $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

so we see that these operations are far easier to manipulate.

We have been slightly hand-wavy here regarding our definitions and the properties of the exponential function. In this course we do not dwell for long on rigorous proof, but it may be of interest to some of you and I find that it helps promote understanding and to develop the mindset necessary for future study. As such, in the appendices is a proof of the necessary properties of the exponential function that we will use without justification in this course.

Recall that the sine and cosine functions are odd and even, respecively:

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$

These properties land us at a very interesting result which allow us to express the sine and cosine functions in terms of complex exponential functions. We know that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

But notice how it follows that

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

by the odd and even properties of the functions. This means that if we add these two equations, we will obtain

$$2\cos\theta = e^{i\theta} + e^{-i\theta}$$

and so

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Similarly, we can subtract the two equations and divide by $2i$ to yield

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

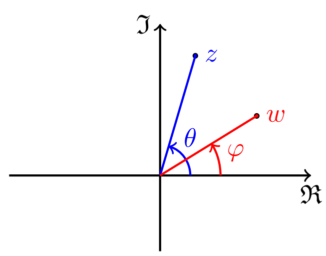
and so we see that we are able to express our familiar (real-valued) trigonometric functions in terms of complex-valued exponentials!

2.6 Multiplying complex numbers

Here we look to point out a very interesting property of complex number. Let us consider two numbers, $z, w \in \mathbb{C}$ defined by

$$z = e^{i\theta}$$
$$w = e^{i\varphi}$$

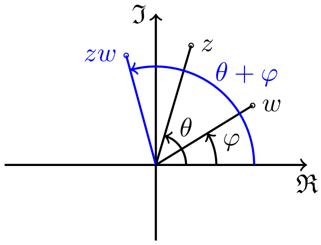
with $\theta, \varphi \in [0, 2\pi)$. Let's plot these two numbers on an argand diagram:



Consider now what happens when we multiply these two numbers:

$$zw = e^{i\theta}e^{i\varphi} = e^{i\theta+i\varphi} = e^{i(\theta+\varphi)}$$

We see that we've ended up with a new complex number whose argument is the sum of the arguments of z and w. In this way, we see that multiplication corresponds to rotation in the complex plane!



There's nothing special about the exponential form with regard to this result. We can easily derive the same result using the modulus-argument form. Let $z = \cos\theta + i\sin\theta$ and $w = \cos\varphi + i\sin\varphi$, then

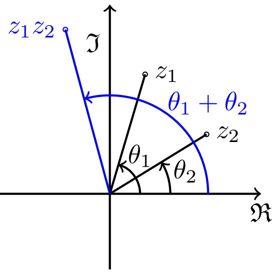
$$\begin{aligned} zw &= (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi) \\ &= \cos\cos\varphi - \sin\theta\sin\varphi + i(\cos\theta\sin\varphi + \sin\theta\cos\varphi) \\ &= \cos(\theta+\varphi) + i\sin(\theta+\varphi) \end{aligned}$$

where we have made use of the compound angle formulae in going from the penultimate to the final line.

Notice that we have only considered complex numbers with unit modulus in the above discussion. This was only done to emphasise the rotation aspect of complex number multiplication. Consider now letting $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then

$$z_1z_2 = r_1r_2e^{i\theta_1}e^{i\theta_2} = (r_1r_2)e^{i(\theta_1+\theta_2)}$$

We see that upon multiplying the complex numbers, we multiply their moduli. Geometrically, this will correspond to scaling the "length" of the first complex number by a scale factor corresponding to the modulus of the second:



3 Solving Equations with Complex Numbers

3.1 Introduction to complex variables

So far in our studies, we have only over tried to solve equations over $\mathbb{R}$, i.e. we have only looked for *real solutions* to equations.

Now we expand our search to solutions over $\mathbb{C}$. When we do this, we encounter a rather amazing theorem (stated without proof):

Theorem 1 (Fundamental Theorem of Algebra)**.** *Given any integer $n \geq 1$ and any set of complex numbers $a_0, \cdots, a_n$, such that $a_n \neq 0$, the polynomial equation*

$$a_0 + a_1 z + \cdots + a_n z^n = 0$$

has exactly n roots (counted with multiplicity) over $\mathbb{C}$.

This is a rather remarkable statement. No analogous result holds if we restrict $a_0, \cdots, a_n$ to be real numbers. For example, the equation $x^2+1=0$ has no solutions over the real numbers. But if we allow ourselves to look for roots over the complex numbers, then we are guaranteed to find two of them as the degree of our polynomial is 2.

What does "counted with multiplicity" mean? Well, put simply, it just says that if we have a repeated root (e.g. $(x-1)^2=0$), we count that as 2 roots. More explicitly, we say that a root $z=z_0$ of the equation $f(z)=0$ has *multiplicity* m if $(z-z_0)^m$ divides $f(z)$, but $(z-z_0)^{m+1}$ does not.

3.2 Examples of complex variable equations

Example 4. *Solve the quadratic equation*

$$5x^2 + 6x + 3 = 0$$

Firstly, we notice

$$\Delta = 36 - 4(3)(5) = 24 < 0$$

and so we would normally accept that there are no solutions to this quadratic. However, now that we are dealing with complex variables, we know that there are.

We proceed just as we always would with a quadratic equation:

$$x = \frac{-6 \pm \sqrt{36 - 4(3)(5)}}{10} = \frac{-6 \pm \sqrt{-24}}{10}$$

Now we use the fact that $\sqrt{-1} = i$ to write

$$x = \frac{-6 \pm \sqrt{24}i}{10}$$

and so, we find our roots to be

$$x = -\frac{3}{5} \pm \frac{\sqrt{6}}{5}i$$

Example 5. *Solve the quartic equation*

$$x^4 + 12x^2 - 64 = 0$$

Notice that we can write this equation as

$$(x^2)^2 + 12(x^2) - 64 = 0$$

and so we proceed to factor as we would for any quadratic equation. Doing so yields

$$(x^2+16)(x^2-4) = 0$$

We can then factorise the difference of two squares in the second bracket to obtain

$$(x^2+16)(x+2)(x-2) = 0$$

So we see that we have two real solutions when $x = \pm 2$. For the other two roots, we need to solve

$$x^2 + 16 = 0$$

Rearranging and square rooting gives

$$x = \pm 4i$$

and so our solutions are $x = \pm 2, \pm 4i$.

Example 6. *Solve the quartic equation*

$$x^4 - 3x^3 - x^2 + 13x - 10 = 0$$

This isn't as simple as our previous quartic and, as such, we will have to use the factor theorem. Define

$$f(x) = x^4 - 3x^3 - x^2 + 13x - 10$$

Then we use our normal trial and improvement strategy:

$$\begin{aligned}
&f(0) = -10 \neq 0 \\
&f(1) = (1)^4 - 3(1)^3 - (1)^2 + 13(1) - 10 = 1 - 3 - 1 + 13 - 10 = 0
\end{aligned}$$

and so we have found a factor of $(x - 1)$. We use polynomial long division to find:

$$\begin{array}{rr}
 & x^3 - 2x^2 \quad - 3x + 10 \\ \cline{2-2}
x - 1 \big) & x^4 - 3x^3 \quad - x^2 + 13x - 10 \\
 & -x^4 \ + x^3 \qquad\qquad\qquad\qquad\quad \\ \cline{2-2}
 & -2x^3 \quad - x^2 \qquad\qquad\quad \\
 & 2x^3 - 2x^2 \qquad\qquad\quad \\ \cline{2-2}
 & -3x^2 + 13x \qquad \\
 & 3x^2 \quad - 3x \qquad \\ \cline{2-2}
 & 10x - 10 \\
 & -10x + 10 \\ \cline{2-2}
 & 0
\end{array}$$

and thus we can write

$$f(x) = (x - 1)(x^3 - 2x^2 - 3x + 10)$$

Defining

$$g(x) = x^3 - 2x^2 - 3x + 10$$

and continuing with our trial and improvement:

$$\begin{aligned}
g(-1) &= (-1)^3 - 2(-1)^2 - 3(-1) + 10 = 10 \neq 0 \\
g(2) &= (2)^3 - 2(2)^2 - 3(2) + 10 = 4 \neq 0 \\
g(-2) &= (-2)^3 - 2(-2)^2 - 3(-2) + 10 = 0
\end{aligned}$$

We carry out the long division:

$$\begin{array}{rr}
 & x^2 - 4x \quad + 5 \\ \cline{2-2}
x + 2 \big) & x^3 - 2x^2 - 3x + 10 \\
 & -x^3 - 2x^2 \qquad\qquad\quad \\ \cline{2-2}
 & -4x^2 - 3x \qquad \\
 & 4x^2 + 8x \qquad \\ \cline{2-2}
 & 5x + 10 \\
 & -5x - 10 \\ \cline{2-2}
 & 0
\end{array}$$

and so we obtain

$$f(x) = (x - 1)(x + 2)(x^2 - 4x + 5)$$

Applying the quadratic formula to our quadratic gives

$$x = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm \frac{1}{2}\sqrt{-4} = 2 \pm i$$

and so, we finally obtain

$$f(x) = (x - 1)(x + 2)(x - 2 + i)(x - 2 - i)$$

thence, solving $f(x) = 0$, gives the solutions

$$x = 1, -2, 2 \pm i$$

3.3 The complex conjugate theorem

A particularly convenient property of complex conjugates is that if we find a polynomial equation (with real coefficients) that $z = z_0$ satisfies, then $z = z_0$ will also satisfy it! This makes life a lot easier for us, as it means that finding one complex root for a polynomial equation gives us another without having to do any work at all!

Let's state and prove this theorem formally:

Theorem 2 (Complex Conjugate Root Theorem)**.** *Consider the polynomial equation $P(z) = 0$ where $P(z)$ is of degree n and is given by*

$$P(z) = a_0 + \cdots + a_n z^n = \sum_{i=0}^{n} a_i z^i$$

with $a_0, \cdots, a_n \in \mathbb{R}$. If z_0 is such that $P(z_0) = 0$, then $P(z_0^*) = 0$ also.

Proof. Since z_0 is a root of our equation, we clearly have that

$$P(z_0) = \sum_{i=0}^{n} a_i z_0^i = 0$$

Then,

$$\begin{aligned} P(z_0^*) &= \sum_{i=0}^{n} a_i (z_0^*)^i \\ &= \sum_{i=0}^{n} a_i^* (z_0^*)^i \\ &= \sum_{i=0}^{n} a_i^* (z_0^i)^* \\ &= \sum_{i=0}^{n} \left(a_i z_0^i\right)^* \\ &= \left(\sum_{i=0}^{n} a_i z_0^i\right)^* \\ &= 0^* \\ &= 0 \end{aligned}$$

□

Notice that this proof relies on the fact that the a_i are all real, as we use the fact that $a_i = a_i^*$ in going from the first to the second line in the chain of steps.

A direct consequence of this theorem is that any real polynomial of odd degree *must* have at least one real root. If this isn't immediately apparent, think about the fact that every complex root must come in a pair.

Geometrically, this makes perfect sense. Consider, for example, the quadratic equation $z^2 - 2z + 2 = 0$. We can solve this equation using the quadratic formula to obtain:

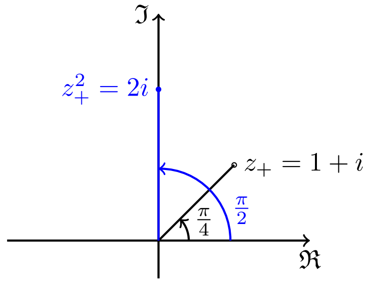
$$\begin{aligned} z_\pm &= \frac{2 \pm \sqrt{4 - 8}}{2} \\ &= \frac{2 \pm 2i}{2} \\ &= 1 \pm i \end{aligned}$$

To interpret the equation geometrically, it would be wise to rewrite $z_\pm$ as

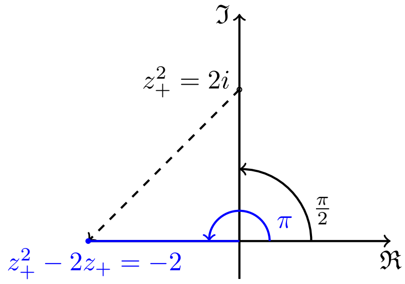
$$z_\pm = \sqrt{2} e^{\pm i \frac{\pi}{4}}$$

where we restrict $\arg z \in (-\pi, \pi]$. Now let's consider the sequence of operations that are applied to the complex number (starting with z_+).

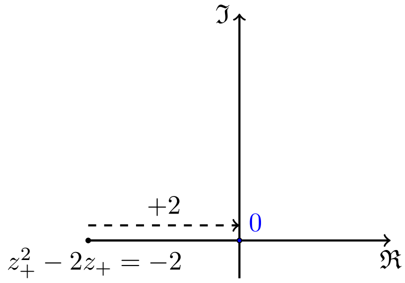
First, we square z_+:



Next, we add $-2z_+$ to z_+^2:

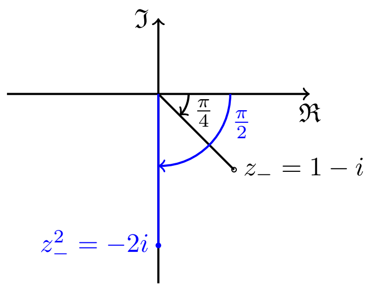


Finally, we add 2 to obtain

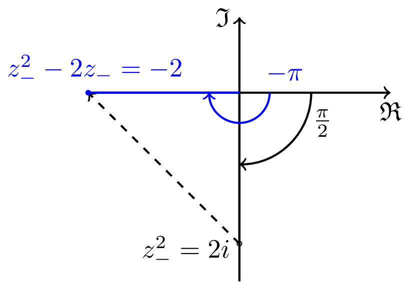


As we end up at the origin, we can see geometrically how $z = 1 + i$ is a solution to our quadratic. Now, let's apply the same sequence of operations to $z = 1 - i$ and see what happens.

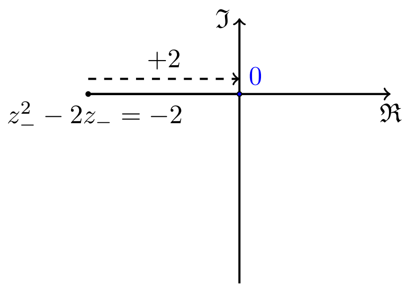
First, we square z_-:



Next, we add $-2z_-$ to z_-^2:



Finally, we add 2 to obtain



and so we again end up at the origin!

This example provides a nice pictorially representation of how if a complex number satisfies a (real) polynomial equation, then its complex number will also satisfy the same equation.

3.4 Square roots of complex numbers: revisited

Consider trying to solve the equation $z^2 = 1 + i$. Earlier, we discussed a "brute force" method in which we let $z = x + yi$, found z^2 and then solved a system of equations obtained from equating real and complex parts of the RHS and LHS. Although this method will indeed work, we can make our lives far easier by writing our complex number in a more convenient form.

First, let's write $1 + i$ in exponential form. We find $\arg 1 + i = \frac{\pi}{4}$ and $|1 + i| = \sqrt{2}$. Hence $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$. Now we want to find our solutions for z.

We have a quadratic and so we know there will be two solutions for z. This time, instead of writing $z = x + yi$, we will write $z = re^{i\theta}$ and then we try to solve

$$r^2 e^{i2\theta} = \sqrt{2}e^{i\frac{\pi}{4}}$$

Equating the moduli, we find that $r = 2^{1/4}$. For the arguments, we first define the range to be $\theta \in [0, 2\pi)$ and then equate

$$2\theta = \frac{\pi}{4}$$

However, since $\theta \in [0, 2\pi)$, it follows that $2\theta \in [0, 4\pi)$. By the periodicity of arguments, we also have

$$2\theta = \frac{\pi}{4} + 2\pi$$

The question to ask now is, why can't we say

$$2\theta = \frac{\pi}{4} + 4\pi$$

The answer is simple, it is because $\frac{\pi}{4} + 4\pi \notin [0, 4\pi)$. This is an important point to note. When we solve equations such as these, we need to consider the periodicity of the argument. However, we need to be clear on the range our argument can fall into, and thus stop when we leave this set of values.

So, we now have

$$2\theta = \frac{\pi}{4}$$
$$2\theta = \frac{9\pi}{4}$$

and so

$$\theta = \frac{\pi}{8}, \quad \theta = \frac{9\pi}{8}$$

From which we obtain our solutions:

$$z_1 = \sqrt[4]{2}e^{i\frac{\pi}{8}}$$
$$z_2 = \sqrt[4]{2}e^{i\frac{9\pi}{8}}$$

3.5 De Moivre's Theorem

We now come to a very beautiful theorem. To prove the result that we are about to state, we will use the principle of mathematical induction. It would be useful to attempt this proof as an exercise before looking at the given proof below:

Theorem 3 (De Moivre's Theorem)**.** *For any $\theta \in \mathbb{C}$, and $n \in \mathbb{Z}$ we have*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof. Since $\sin 0 = 0$ and $\cos 0 = 1$, the result is seen to be true for $n = 0$. Now we split into positive and negative n.

We start with positive n. For $n = 1$, we see the result is trivially true. Now assume the result to hold for $n = k$:

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$$

and let $n = k + 1$. Then

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)\\ &= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)\\ &= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \sin\theta\cos k\theta)\\ &= \cos(k+1)\theta + i\sin(k+1)\theta\end{aligned}$$

In going from the second to the third line we have used our induction hypothesis, and in going from the third to the fourth we have used our compound angle formulae. So if the formula is true for $n = k$, it is also true for $n = k + 1$. As it true for $n = 1$, then it is true for all $n \geq 1$.

Now, we look to prove for negative n. For $n = -1$, we have

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-1} &= \frac{1}{\cos\theta + i\sin\theta}\\ &= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}\\ &= \cos\theta - i\sin\theta\\ &= \cos(-\theta) + i\sin(-\theta)\end{aligned}$$

where we have made use of the odd and evenness of the sine and cosine functions, respectively. So we see the result holds for $n = -1$. Assume the result to hold for $n = -k$ and let $n = -(k + 1)$:

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-(k+1)} &= (\cos\theta + i\sin\theta)^{-k}(\cos\theta + i\sin\theta)^{-1}\\ &= (\cos(-k\theta) + i\sin(-k\theta)(\cos(-\theta) + i\sin(-\theta))\\ &= (\cos(-k\theta)\cos(-\theta) - \sin(-k\theta)\sin(-\theta)) + i(\sin(-k\theta)\cos(-\theta) + \sin(-\theta)\cos(-k\theta))\\ &= \cos(-(k+1)\theta) + i\sin(-(k+1)\theta)\end{aligned}$$

and so the result is shown to also hold for $n \leq -1$.

Combining, we see that we have the stated result for $n \in \mathbb{Z}$. □

It is interesting to note that a form of the result also holds for all rational n (which can be proved using Euler's formula). However, raising complex numbers to a non-integer causes problems with multivaluedness and so we only end up with one of the *possible* values, rather than an exact equality.

Now that we have this theorem, we can use it to obtain some interesting angle identities:

Example 7. *Find an expression for* $\sin 3\theta$ *in terms of* $\sin\theta$ *only.*

To do this, we first use De Moivre's theorem to say

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

Next, we binomially expand the LHS:

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

Now that we have this expression for the LHS, we can equate the complex part of this with the RHS of our earlier expression to obtain

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

However, the question explicitly asked for the answer to be in terms of $\sin\theta$ only, so

$$\begin{aligned}3\cos^2\theta\sin\theta - \sin^3\theta &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

From which we finally deduce,

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

Note that we could also have equated the real parts to obtain

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta\\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

We can do this for as high a power as we want. Let's consider one more example:

Example 8. *Find an expression for* $\cos^5 5\theta$ *as a sum of* $\cos\theta$, $\cos 3\theta$ *and* $\cos 5\theta$.

We proceed in the same way:

$$\begin{aligned}(\cos\theta + i\sin\theta)^5 &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta\\ &= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)\end{aligned}$$

De Moivre says

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

and we can equate to find

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

We then look to rewrite in terms of $\cos\theta$:

$$\begin{aligned}\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2\\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta\\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

Thus we obtain

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

which we rearrange to get

$$\cos^5\theta = \frac{1}{16}\big(\cos 5\theta + 20\cos^3\theta - 5\cos\theta\big)$$

We know from the previous example that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \quad \Rightarrow \quad \cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$$

and so

$$\begin{aligned}\cos^5\theta &= \frac{1}{16}\big(\cos 5\theta + 20\cos^3\theta - 5\cos\theta\big)\\ &= \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 15\cos\theta - 5\cos\theta)\\ &= \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)\end{aligned}$$

which is what we wanted.

3.6 Roots of unity

The *nth roots of unity* are the roots of the equation $z^n = 1$ for $n \in \mathbb{N}$. Since this is a polynomial of order n, there are n roots of unity.

Deriving a general formula for an nth root of unity is simple enough. Our first step is to specify a range for our arguments, which we take to be $[0, 2\pi)$. Next, we write $1 = e^{i0}$ (the argument of 1 is 0) and $z = re^{i\theta}$, then

$$z^n = 1 \quad \equiv \quad r^n e^{in\theta} = e^{i0}$$

Comparing the moduli gives $r = 1$. For the arguments, we note that since $\theta \in [0, 2\pi)$, it follows that $n\theta \in [0, 2n\pi)$ and so

$$n\theta = 0, 2\pi, 4\pi, \cdots, 2(n-1)\pi$$

Hence

$$\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \cdots, \frac{2(n-1)\pi}{n}$$

We then arrive at our expression for the nth roots of unity:

$$z = \exp\left(i\frac{2k\pi}{n}\right), \quad k = 0, 1, \cdots, n-1$$

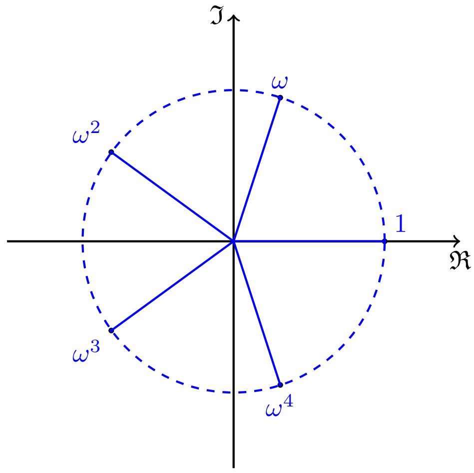
Let's consider the geometric implications of this result. We have discussed the relationship between complex numbers and circles. Since our roots of unity (by necessity) have unit modulus, it follows that they must all lie on the unit circle.

We are by this point aware that multiplication of complex numbers corresponds to a rotation. As such, squaring, cubing etc. corresponds to a doubling, tripling etc. of the angle the original complex number makes the the positive $\Re$-axis.

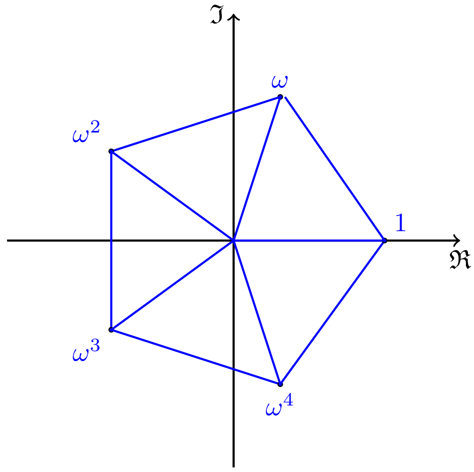
Consider now the nth roots of unity and take $\omega = \exp\left(i\frac{2\pi}{n}\right)$. We can *generate* the $n-1$ other roots of unity by simply raising this root to different integer powers. But, when we raise a complex number to a power, we rotate it by that many times its argument. So, for example, if we consider the 5th roots of unity and write $\omega = \exp\left(i\frac{2\pi}{5}\right)$, $\omega^3 = \exp\left(i\frac{6\pi}{5}\right)$ is a rotation of ω by $\frac{4\pi}{5}$ about the origin.

It then follows that each of our roots of unity must be equally spaced around the unit circle, with each making an angle of $\frac{2\pi}{n}$ with its adjacent roots.

Consider now the 5th roots of unity. Each will make an angle of $\frac{2\pi}{5}$ with its adjacent roots, and each will lie on the unit circle. As such, they will look like:



Notice what happens if we connect the points representing each root with a straight line to its neighbour:



We see that we have formed a pentagon!

This result is not unique to $n = 5$. In fact, the nth roots of unity, when plotted on an argand diagram, will form the vertices of a regular n-gon.

Before we consider some examples, we will first state and prove an interesting result:

Lemma 3. *If* $\omega = \exp\left(i\frac{2\pi}{n}\right)$, *then*

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$$

Proof. We provide two proofs of this results:

1. Since the equation $\omega^n = 1$ is equivalent to $\omega^n - 1 = 0$, we use our knowledge of the sum and product of roots. The sum of the roots to this equation is the coefficient of ω^{n-1}, which is 0. Thus

$$1 + \omega + \cdots + \omega^{n-1} = 0$$

2. For the second proof, we use the fact that we can factor the expression $\omega^n - 1$ as

$$\omega^n - 1 = (\omega - 1)(1 + \omega + \omega^2 + \cdots + \omega^{n-1}) = 0$$

   Since $\neq 1$, we arrive at the result.

$\square$

So, it turns out the rum of the roots of unity is equal to 0. This is a rather fascinating and unexpected result. The reason for this is in a sense intuitive when we consider the position of the points on the argand diagram. Since each point is equally spaced around the perimetre of the circle, it follows that their centre of mass will lie at the origin of said circle. As such, it would make sense for the roots to sum to 0.

Let's now consider a few examples.

Example 9. *Solve the equation* $z^4 = 1$.

Our first step is to write $z = e^{i\theta}$ (we set $r = 1$ as we know it has unit modulus). Then we find

$$e^{i4\theta} = 1 = e^{i0}$$

Since $4\theta \in [0, 8\pi)$, we list the possible values of 4θ as

$$4\theta = 0, 2\pi, 4\pi, 6\pi$$

and so

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

Substituting these in, we find the four roots to be

$$\begin{aligned} z_1 &= 1 \\ z_2 &= e^{i\frac{\pi}{2}} \\ z_3 &= e^{i\pi} \\ z_4 &= e^{i\frac{3\pi}{2}} \end{aligned}$$

We can be asked to express these roots in any of the three forms we have met. As such, we could also write

$$\begin{aligned} z_1 &= 1 \\ z_2 &= i \\ z_3 &= -1 \\ z_4 &= -i \end{aligned}$$

in Cartesian form.

Example 10. *Solve the equation* $z^5 = 32$.

We can also use our knowledge of solving for roots of unity to solve other equations such as the one above. In this case, we write $z = re^{i\theta}$ and so

$$r^5 e^{i5\theta} = 32e^{i0}$$

Comparing moduli tells us $r^5 = 32$, or simply that $r = 2$. For the arugments, we know that $5\theta \in [0, 10\pi)$, and so

$$5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$$

Hence

$$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$$

Let's express these in modulus-argument form:

$$\begin{aligned} z_1 &= \cos(0) + i\sin(0) \\ z_2 &= \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) \\ z_3 &= \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) \\ z_4 &= \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right) \\ z_5 &= \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right) \end{aligned}$$

Example 11. *Solve the equation* $z^3 = 8i$.

There's nothing stopping us from solving equations such as the above, i.e. ones where we look for the nth roots of a pure complex number.

Let's write $z = re^{i\theta}$ and $8i = 8e^{i\frac{\pi}{2}}$, then

$$z^3 = r^3 e^{i3\theta} = 8e^{i\frac{\pi}{2}}$$

We can see that $r = 2$, and that

$$3\theta = \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 4\pi$$

Simplifying gives

$$3\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$$

or

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6}$$

Putting this all together, we find

$$\begin{aligned} z_1 &= 2e^{i\frac{\pi}{6}} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i \\ z_2 &= 2e^{i\frac{5\pi}{6}} 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i \\ z_3 &= 2e^{i\frac{9\pi}{6}} = 2\left(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right) = -2i \end{aligned}$$

3.7 Solving geometric problems using complex numbers

Example 12. *An equilateral triangle has its centroid at the origin and one of its vertices at the point (2,0). Find the coordinates of the other vertices.*

We could approach this using standard geometric techniques, but that would be quite arduous. Instead, we look to put some of our newly acquired knowledge to the test.

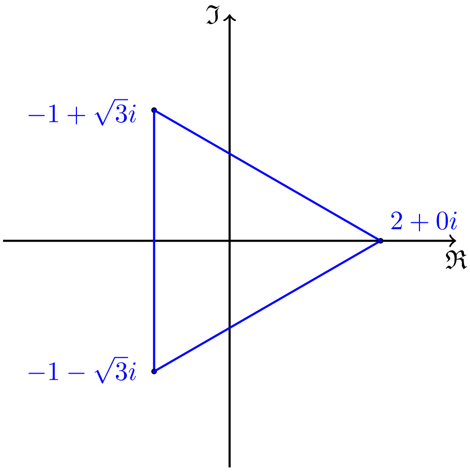
A triangle is the shape formed by joining the points on an argand diagram corresponding to the 3rd roots of unity. However, since our first vertex is at (2,0), we instead consider the 3rd roots of 8. Doing so, we find

$$\begin{aligned} z_1 &= 2 \\ z_2 &= 2e^{i\frac{2\pi}{3}} \\ z_2 &= 2e^{i\frac{4\pi}{3}} \end{aligned}$$

We can convert these into Cartesian form:

$$\begin{aligned} z_1 &= 2 \\ z_2 &= -1 + \sqrt{3}i \\ z_3 &= -1 - \sqrt{3}i \end{aligned}$$

and so we find that our vertices will be at (2,0), $(-1, \sqrt{3})$ and $(-1, -\sqrt{3})$:



Example 13. *If we rotate the point* $(1,3)$ *by* $\frac{3\pi}{4}$ *radians about the origin, at what point do we end up?*

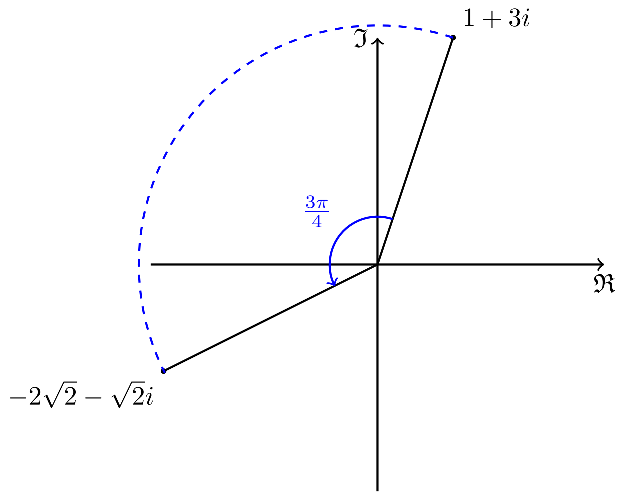
This is another example where complex numbers can simplify a problem. We notice that $\frac{3\pi}{4} = \frac{6\pi}{8}$, and is corresponds to the argument of an 8th root of unity. As such, we can evaluate this root of unit and then multiply the complex number $1+3i$ (corresponding to (1,3)) by the root to rotate it by that angle. We find

$$e^{i\frac{3\pi}{4}} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

Then

$$(1+3i)\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -2\sqrt{2} - \sqrt{2}i$$

Thus we end up at the point $(2\sqrt{2}, -\sqrt{2})$:


$1+3i$
$\Im$
$\frac{3\pi}{4}$
$\Re$
$-2\sqrt{2}-\sqrt{2}i$